Fractional quantum Hall effect on an infinite cylinder: Characterization of topological order using iDMRG

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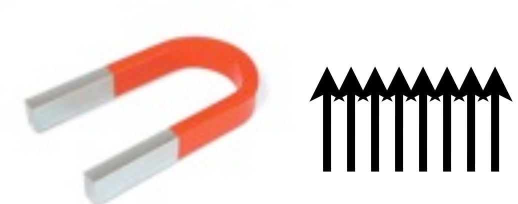
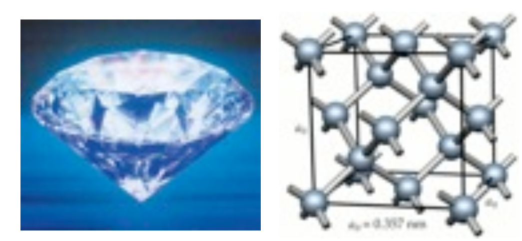
Introduction

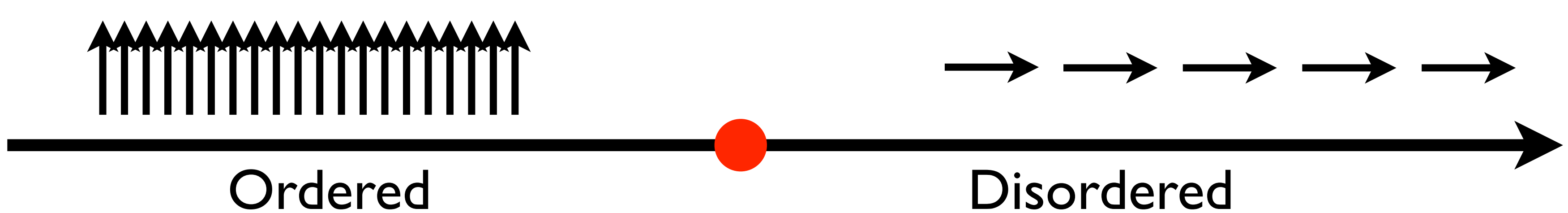
- Different phases of matter are usually understood using **local order parameters** ("symmetry breaking")
  - **Magnets**: spin rotation and TR symmetry broken



  - **Crystals**: translation and rotation symmetry broken

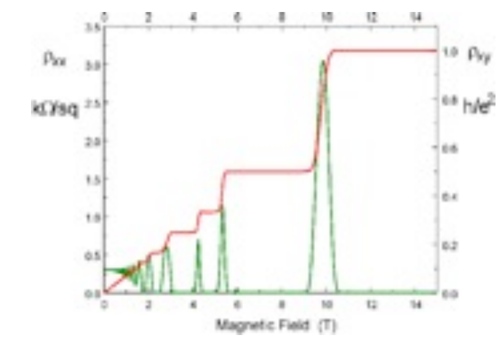


- Phase diagram of an Ising model (magnetization as oder parameter)



Introduction

- In the last years several **"new kinds" of phases** have been discovered which **cannot be described by local order parameters**
  - Quantum Hall effect [Klitzing '80]



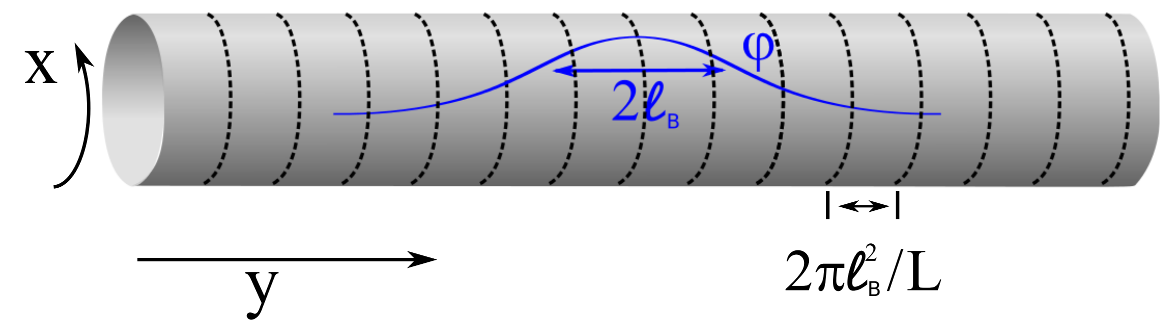
  - Fractional Quantum Hall effect [Tsui, Stormer '82, Laughlin '83]
  - Topological spin liquids [Anderson '87]
  - ...
- How to characterize topological order?
- How to tell what kind of topological order we have from **numerical simulations**?

Outline

- **Fractional quantum Hall on a cylinder**
- **Efficient simulations of infinite 1D system:** *Matrix product states* (MPS) *and density matrix renormalization group* (DMRG)
- **Characterization of topological order using MPS and the entanglement spectrum**
  - Charge distribution of a quasi particle
  - Topological entanglement entropy
  - Quantum dimensions
  - Topological spin

Fractional quantum Hall on a cylinder

- Two-dimensional electron gas in a strong magnetic field
- Lowest Landau Level on an infinite cylinder with circumference L



Landau gauge $\mathbf{A} = \ell_B^{-2}(-y, 0)$

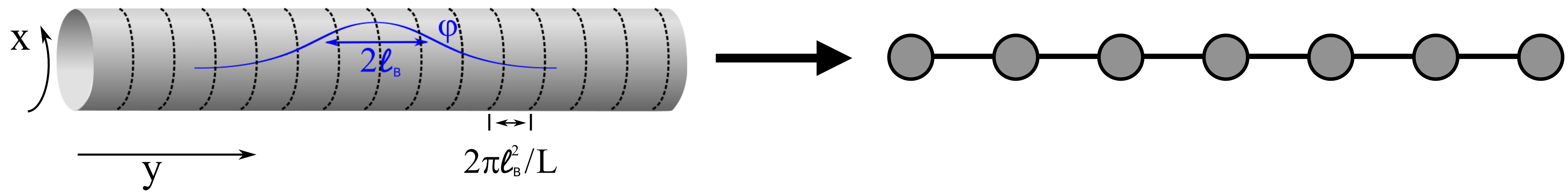
- Degenerate orbitals take the form (with $k_j = \frac{2\pi j}{L}$)

$$\varphi_j(x,y) = \frac{e^{ik_j x - \frac{1}{2\ell_B^2}(y - k_j \ell_B^2)^2}}{\sqrt{L\ell_B \pi^{1/2}}}$$

[Haldane & Rezayi ’94; Bergholtz et al.; ’05, Seidel et al. ’05]

Fractional quantum Hall on a cylinder

- Orbitals are localized at $y_j = k_j \ell_B^2$: **1D model** using an occupation number basis $|N_j\rangle, \quad N_j = 0, 1$



- Degenerate states in the orbital basis represented as

$$\ldots |\ldots 0010101011\ldots\rangle, \; |\ldots 1000101011\ldots\rangle, \ldots$$

[Haldane & Rezayi '94; Bergholtz et al.; '05, Seidel et al. '05]

Fractional quantum Hall on a cylinder

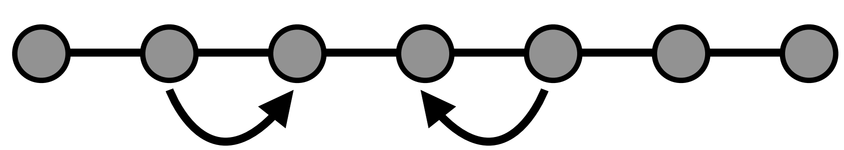
- The most general interaction allowed by symmetry is

$$\hat{H} = \sum_n \sum_{k \geq |m|} V_{km} c^\dagger_{n+m} c^\dagger_{n+k} c_{n+m+k} c_n$$

- We are using the Haldane pseudo potentials V_i (Trugman-Kivelson $\nabla^2 \delta(r)$: $V_{km} \propto e^{-\frac{1}{2}4\pi^2(k^2+m^2)/L^2}$):

  [Haldane '83; Trugman & Kivelson '85]

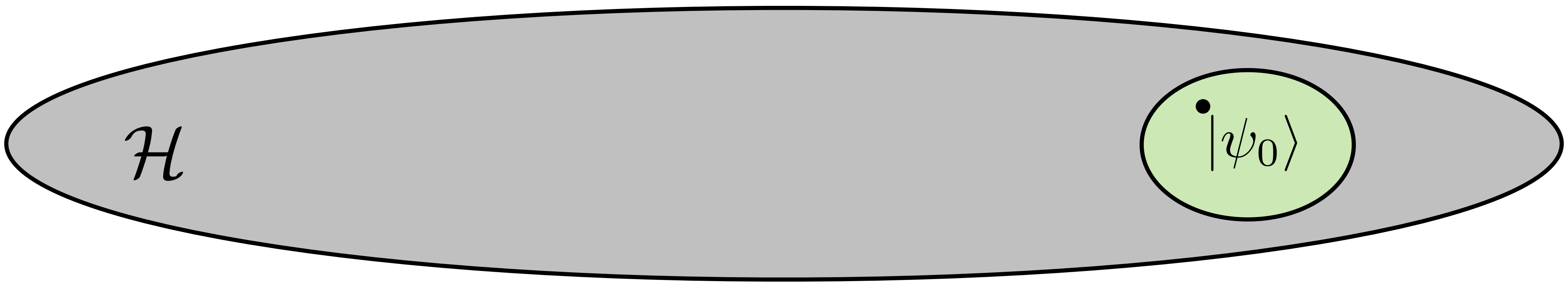
- Total charge $\hat{C} = \sum_j \hat{N}_j$ and momentum $\hat{K} = \sum_j j\hat{N}_j$ are conserved



[Haldane & Rezayi '94; Bergholtz et al.; '05, Seidel et al. '05]

Efficient simulations in 1D

- Consider local Hamiltonians acting on a Hilbert space $\mathcal{H}$:

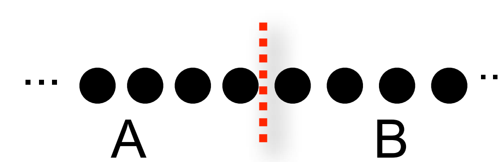


- Ground states live in a very small corner of the Hilbert space as they generically fulfill the area law (proven for 1D, gapped systems [Hastings '07])

- Efficiently constrain our simulations to this subspace: In 1D we represent the state as **matrix-product state (MPS)**

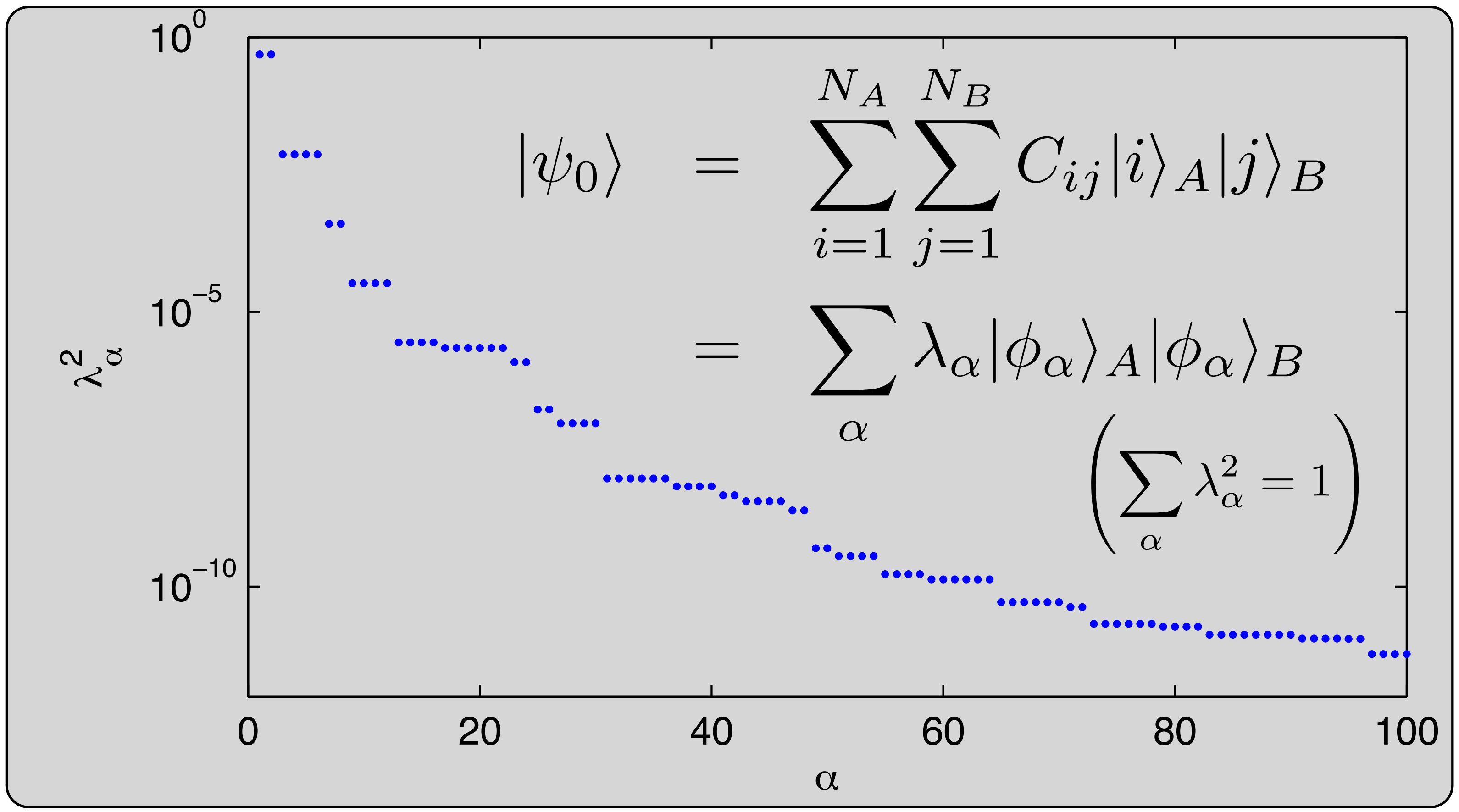
- **Main idea:** Truncate the Schmidt decomposition

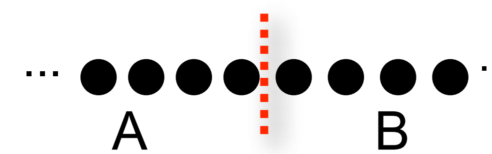
$$|\psi_0\rangle = \sum_\alpha \lambda_\alpha |\phi_\alpha\rangle_A |\phi_\alpha\rangle_B$$



Efficient simulations in 1D

- Example: Entanglement spectrum (Schmidt spectrum) of the Spin-1 Heisenberg chain $H = \sum_j \vec{S}_j \cdot \vec{S}_{j+1}$

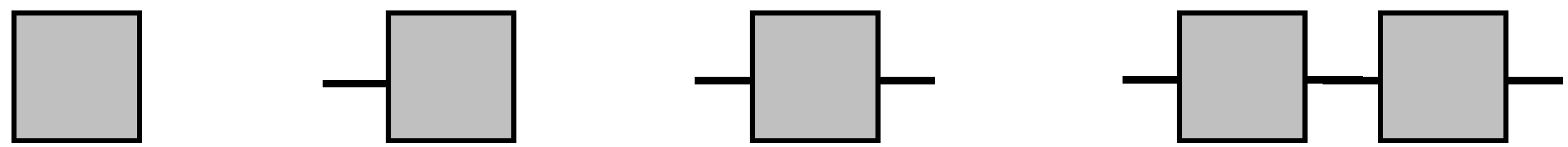




- **Schmidt values decay rapidly:** Almost the entire weight is contained in only **few important states (AREA LAW!)**

Efficient simulations in 1D

- Useful way to represent tensors



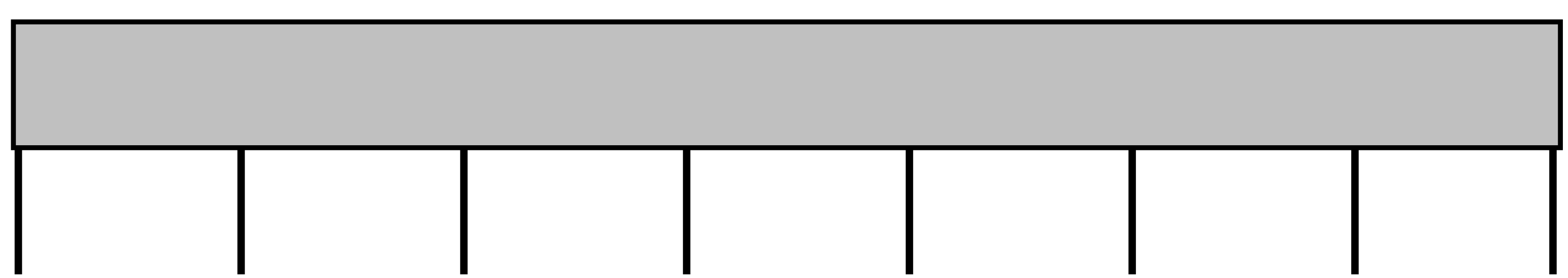
c $\quad$ c_i $\quad$ c_{ij} $\quad$ $\sum_j a_{ij} b_{jk}$

- A generic quantum state for a N site system has a d^N dimensional Hilbert space ($j_n = 1 \ldots d$)

$$|\psi\rangle = \sum_{i_1 \ldots i_N} \psi_{j_1 \ldots j_N} |j_1\rangle \cdot \cdots \cdot |j_N\rangle$$

- Coefficients are given by a rank N tensor:

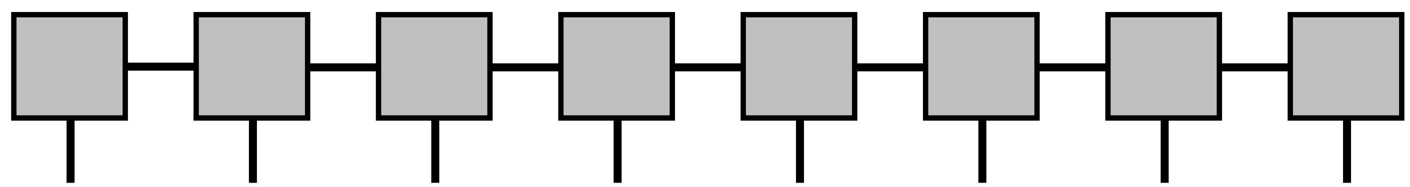
$\psi_{j_1 \ldots j_N} =$

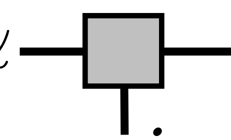


Efficient simulations in 1D

- In an **MPS** representation we write the rank-N tensor as a product of rank-3 tensors

[Fannes et al. '92]

$\psi_{j_1 \ldots j_N} =$ 

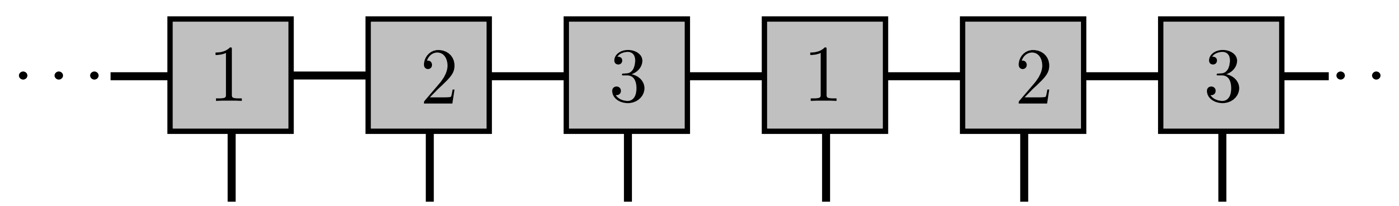
$B^{[n]j_n}_{\alpha,\beta} = \alpha$  β, j_n with dimensions $j_n = 1 \ldots d$, $\alpha, \beta = 1 \ldots \chi_n$

- Dimension χ generically grows exponentially with system size: But If we represent $|\psi\rangle$ in terms of the **Schmidt basis at each bond** we know how to truncate it!

- **Variational wave function** for which the number of parameters depends only on entanglement ($Nd\chi^2$)

Efficient simulations in 1D

- Local expectation values calculated efficiently ($\propto N\chi^3 d$)
- Operators can be expressed in terms of **matrix-product operators (MPO)** as a generalization of MPS
- For infinite systems we assume a finite unit cell such that $B^{[n]} = B^{[n+M]}$

$\psi_{\dots j_1 \dots j_N \dots} =$ 

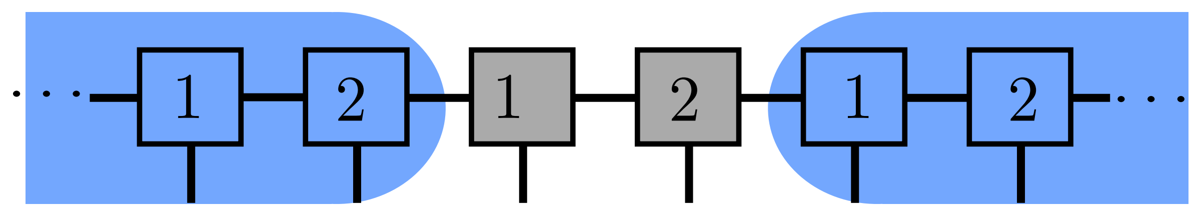
- **FQH**:Matrix dimension χ needed to represent the ground state grows $\propto \exp(L)$: This is bad - but not as bad as a $\propto \exp(L^2)$ growth :-)
- Same scaling with L as in finite system DMRG...
  [Shibata ’01; Feiguin ’08; Hu ’12; Zhao ’12;...]

Efficient simulations in 1D

- The ground state is found using an infinite system variant of the DMRG algorithm [White '92; McCulloch '08]

- **Iterative minimization of the energy**

  - Optimizing some tensors while keeping the environment fixed (e.g., Lanczos)

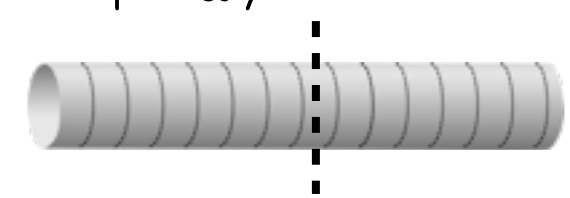


  - After each optimization, the wave function is truncated back to the original variational space of MPS's (SVD)

  - Sufficient number of iterations yields a fixed point and we can read off the tensors $B^{[1]} \ldots B^{[M]}$ which approximate the ground state

iDMRG results: Quasiparticles

- **How can we characterize topological order using iDMRG?**
- Topologically ordered states are characterized by their quasiparticle (QP) excitations: **GS degeneracy equal to the number of QP types** a [Wen '90; Kitaev '05]
- We choose a basis such that each GS corresponds to a specific QP type $\mathcal{H} = \bigoplus_a \mathcal{V}_a$: **minimally entangled states** (MES) for the bipartition of the cylinder $|\Xi_a\rangle$
[Zhang et al. '12; Grover et al. '12; Cincio & Vidal '12]

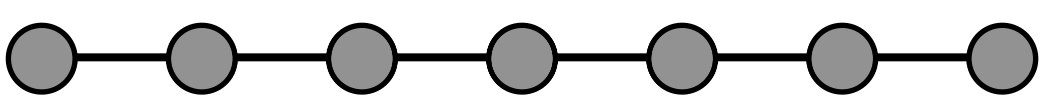


- Choose a specific "**pattern of zeros**" (e.g., ...0101...) as initial state [Wen & Wang '08; Bernevig '08]
Ergodicity: Two-site update not sufficient

iDMRG results: Quasiparticles

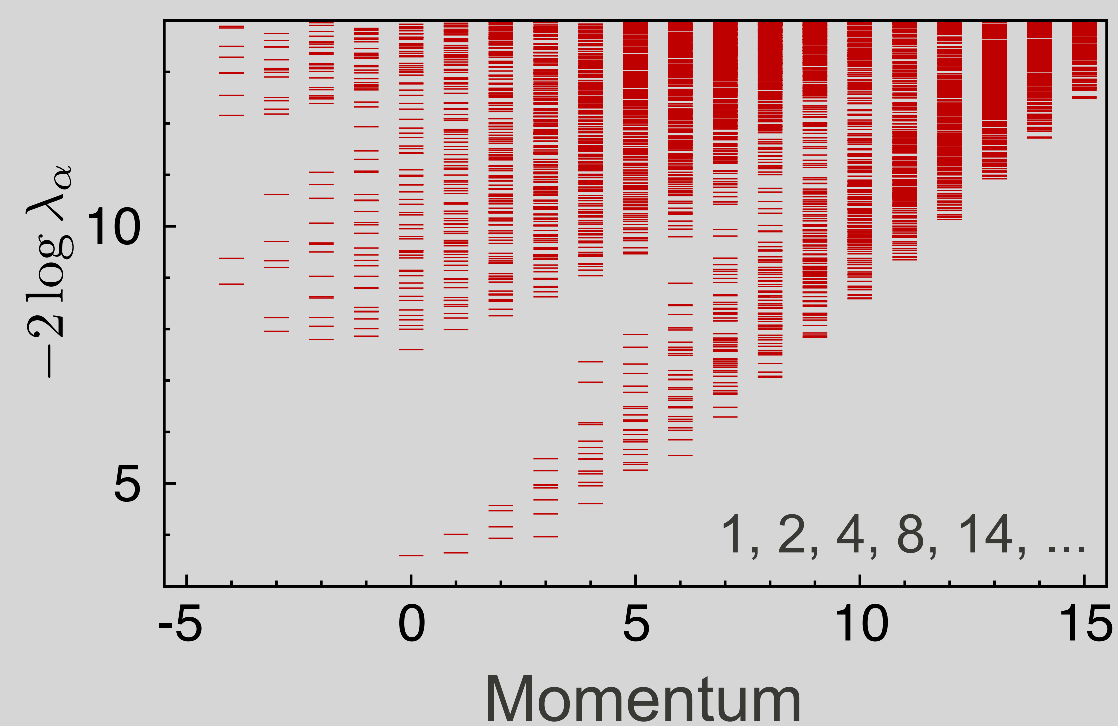
- Example: **Moore-Read phase** (six different quasi-particle sectors with different initial configurations)
  [Moore & Read '91; Fendley '06; Bergholtz '06; Fradkin '08; ...]

QP type	Charge	Seed
1	0	0110
$V_{+1/\sqrt{2}}$	$e/2$	0011
$V_{-1/\sqrt{2}}$	$e/2$	1100
ψ	0	1001
$\sigma V_{+1/2\sqrt{2}}$	$-e/4$	0101
$\sigma V_{-1/2\sqrt{2}}$	$e/4$	1010



Real space entanglement spectrum in the $\nu = 1/2$ phase:

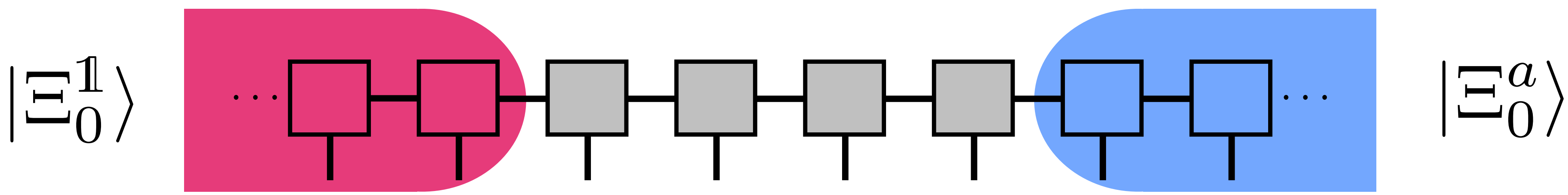
$L = 21.5\ell_B, V_1 = 1,\ V_3 = 0.65$



[Read '96; Zaletel '12; Dubail '12; Sterdyniak '12; ...]

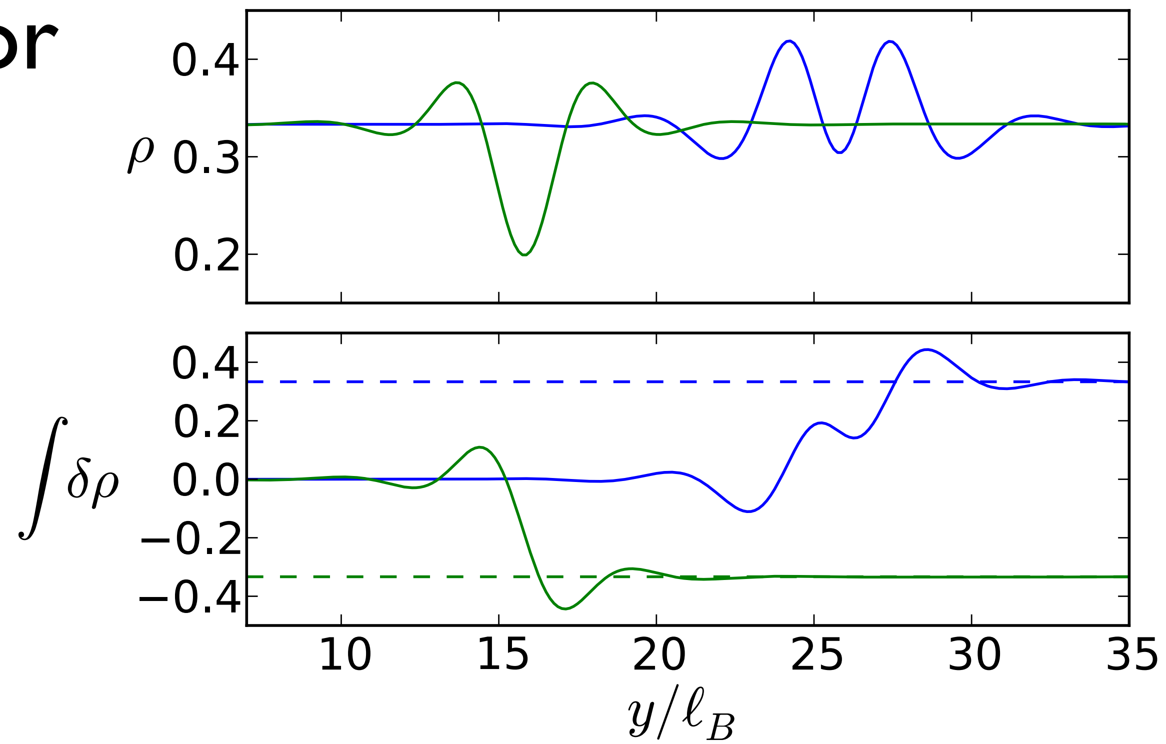
iDMRG results: Quasiparticles

- Creating a QP by glueing together two ground states with different charges at the boundary



- Example: Charge density for the $\nu = 1/3$ state

- QP charge by measuring expectation values of the charge in Schmidt states

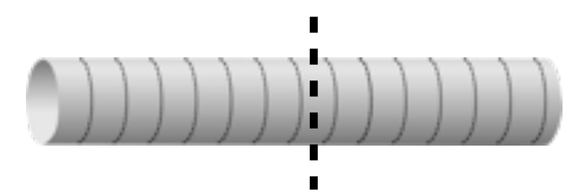


iDMRG results: Entanglement

- The entanglement entropy for a **topologically ordered ground state** $|\Xi_a\rangle$

$$S = sL - \gamma_a \quad \textbf{with} \quad \gamma_a = -\log\left(d_a \Big/ \sqrt{\sum_b d_b^2} \right)$$

- For abelian QP $d_a = 1$ and the topological entanglement entropy does not depend on a

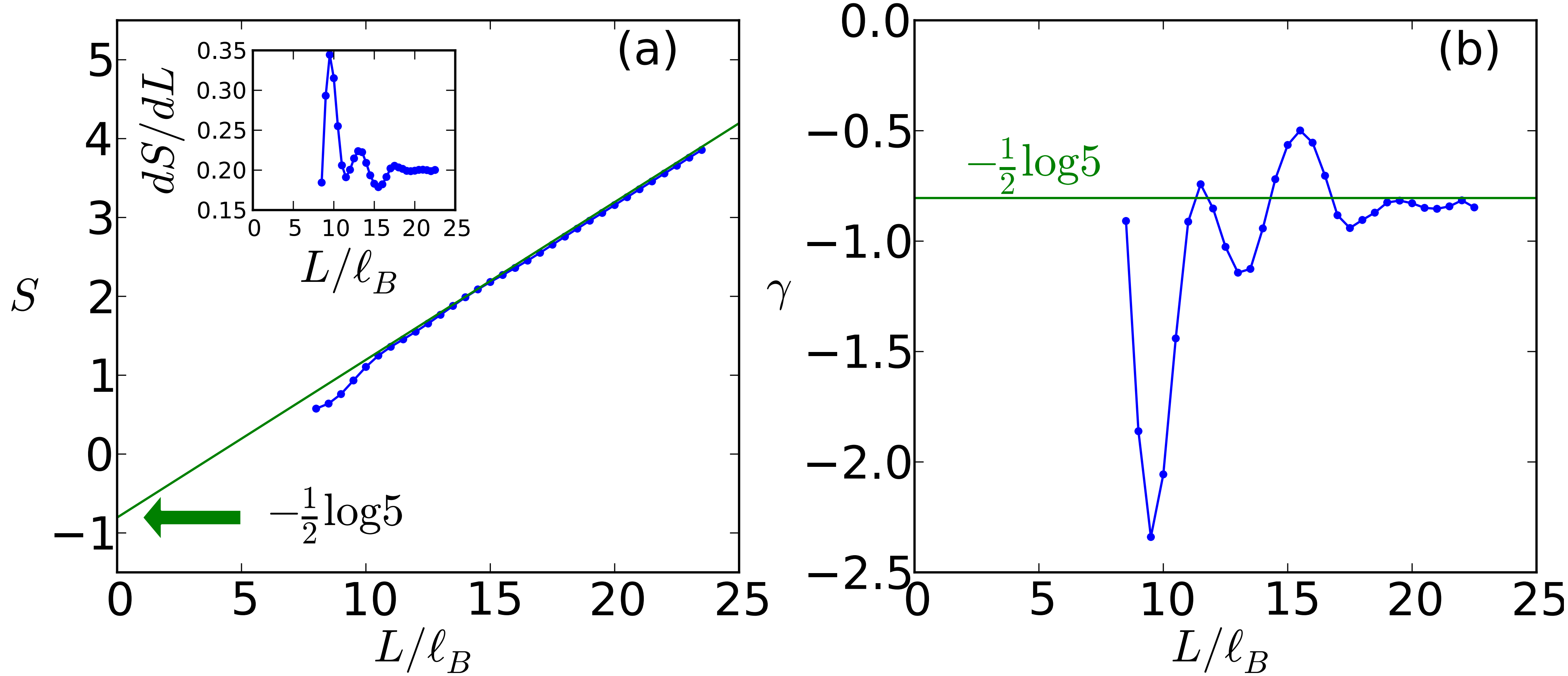


- Sum rule allows us to check if we found all ground states $\sum_i e^{-2\gamma_i} = 1$

[Kitaev & Preskill '06 ; Dong et al '08; Grover et al. '12; Cincio & Vidal. '12]

iDMRG results: Entanglement

- Example: Entanglement entropy for an (abelian) $\nu = 2/5$ state

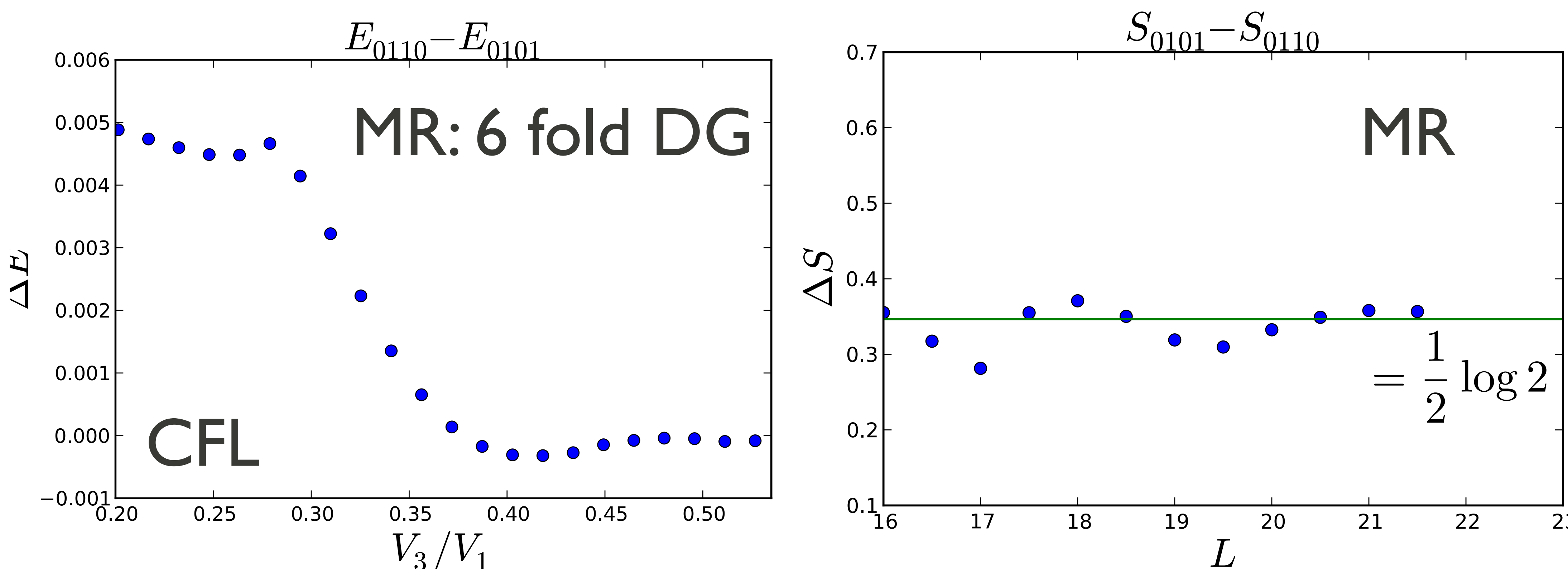


➡ Extracted topological entanglement entropy: $\gamma \approx 0.83$

[Laeuchli et al. '10]

iDMRG results: Entanglement

- Phase diagram of the $\nu = 1/2$ state by varying the Haldane pseudo potentials



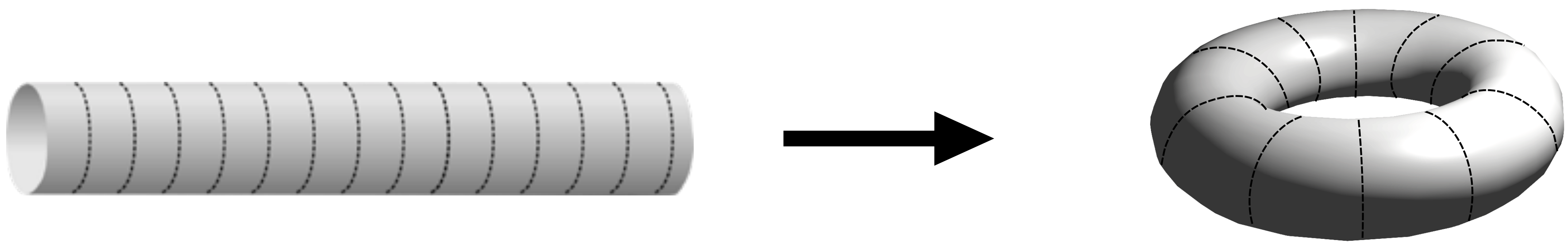
- Difference in topological entropy $\Delta S = \gamma_{\sigma V_{-1/2\sqrt{2}}} - \gamma_1$

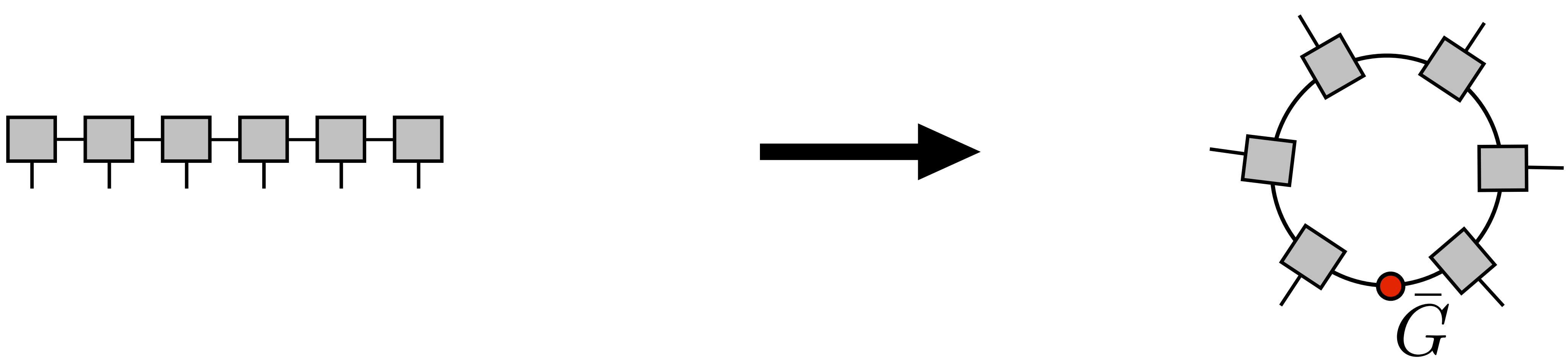
➡ Non-abelian anyons with $d_a \approx 1.43$

[Moore & Read ’91; Rezayi & Haldane ’01]

iDMRG results: From a cylinder to a torus

- Put the ground state of the infinite cylinder onto a torus (correlation length ξ is short compared to the length)

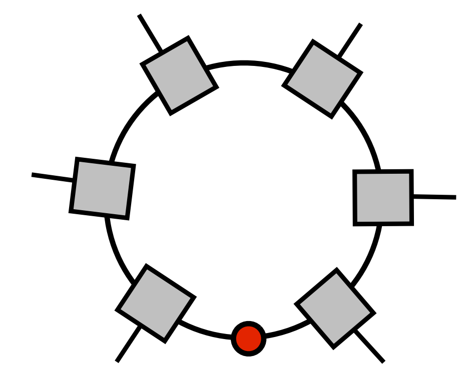




- The operator $\bar{G}$ determines the boundary conditions

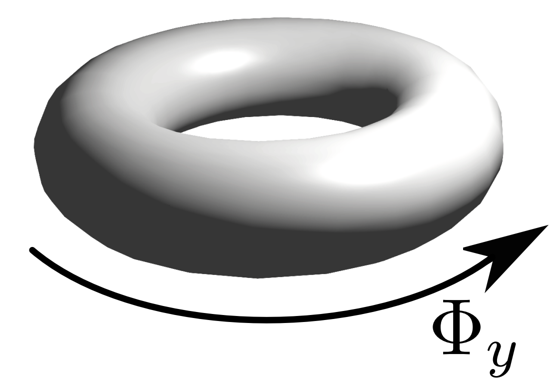
iDMRG results: From a cylinder to a torus

- Fluxes and modular parameter can be accounted for by inserting a diagonal matrix ● when connecting the two edge auxiliary bonds

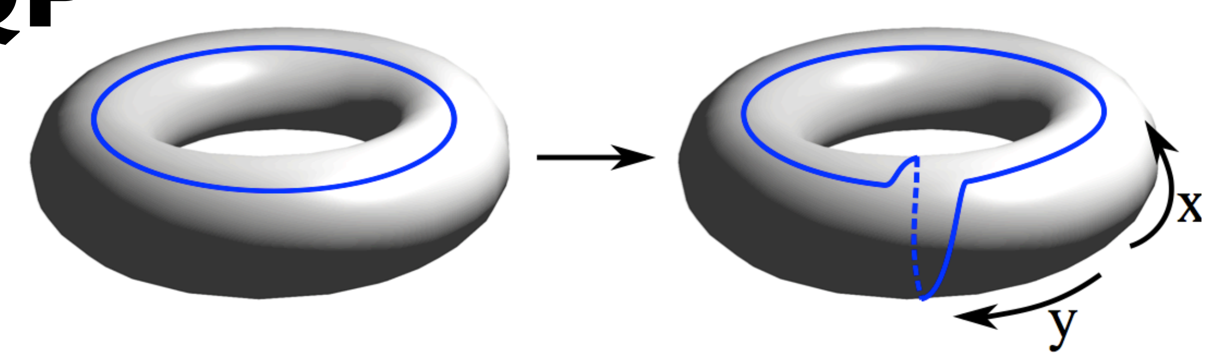


$$\bar{G} = (-1)^{(N_e-1)\frac{\bar{C}}{q}} \exp\left[-2\pi i \tau_x \frac{\bar{K}}{q} + i\Phi_y \frac{\bar{C}}{q}\right]$$

  - Inserting a flux Φ_y (Shifting between GS's): **QP charge**



  - Dehn Twist (Modular T): **Topological spin of QP**



- Numerical verifications in progress...

[Kitaev & Preskill '06 ; Dong et al '08; Grover et al. '12; Cincio & Vidal. '12]

Summary

- Simulating FQH systems using infinite MPS allows us to extract many relevant quantities
  - Clean entanglement spectra (both in real and orbital space)
  - Topological entanglement entropy and individual quantum dimensions
  - Quasiparticle charges
  - Topological spin