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Reanalysis of fractional quantum Hall effect for vanishing range interactions

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We study two-dimensional bosons and fermions in a magnetic field with vanishing range interactions. This model was previously proposed for fermions to illustrate the exactness and uniqueness of Laughlin's wave function as a ground state within the lowest Landau level. We show that the restriction to the lowest Landau level is not a valid approximation for these models for arbitrarily high (but finite) cyclotron frequency, since it predicts fractional quantum Hall effect (FQHE) at an incorrect filling factor. In particular, in contrast to the lowest-Landau-level theory, hard-core bosons are shown not to exhibit any FQHE. However, the set of filling factors for the FQHE are identical in the two cases, thus reaffirming the universality of the FQHE.

The fractional quantum Hall effect (FQHE) is one of the most exciting discoveries[1] in condensed-matter physics. Our fundamental understanding to this remarkable phenomenon is largely based on Laughlin's trial wave function,[2] which shows the ground state of the $\frac{1}{3}$ filled fermion system is an incompressible quantum liquid. It is then of great theoretical interest to examine possible model Hamiltonians for which Laughlin's wave function becomes the exact nondegenerate ground state. Several years ago, Haldane,[3] Trugman and Kivelson,[4] and Pokrovsky and Tapalov[5] studied FQHE for model interactions of vanishing range [of the form $\nabla^2\delta(\mathbf{r})$]. They showed the exactness and the uniqueness of the Laughlin's wave function at $\frac{1}{3}$ filling for such a model when one restricts it to the lowest Landau level (LLL). This was also found numerically for small systems.[6] The more realistic Coulomb interactions then may be considered as a perturbation to this model interaction, preserving the rigidity of the quantum Hall state. Although it is a common wisdom that projection onto the LLL is accurate at high magnetic field, it remains a question whether a system with vanishing-range interaction [of the form $\nabla^{2m}\delta(\mathbf{r})$] would show FQHE when higher Landau levels (LL's) are also considered.

Unlike fermions, which can exhibit integer QHE, noninteracting bosons cannot have QHE. The projection approach onto the LLL would suggest that there should be FQHE for bosons at $\frac{1}{2}$ filling with vanishing-range interactions, in particular for hard-core point bosons. However, the boson limit of a study of anyons in magnetic field by Ma and Zhang[7] suggests otherwise. The true ground-state properties of the hard-core bosons in magnetic fields remain unknown.

It is the purpose of this paper to examine the above open questions. We will show unambiguously that at high but finite cyclotron frequency, hard-core bosons do not exhibit FQHE, and that the $\frac{1}{3}$ Laughlin state is not the nondegenerate ground state for fermions for the $\nabla^2\delta(\mathbf{r})$ interactions, hence the system is compressible. We analyze the solution for two particles with relative angular momentum zero (for bosons) or one (for fermions), which demonstrates infinitesimal energy cost of the vanishing-range interactions. We propose trial wave functions to show this explicitly. Constructing a Jastrow-type wave function from the two-body wave function shows that this holds for many particles. Generalizing our results to interactions $\nabla^{2m}\delta(\mathbf{r})$ (m an even integer for bosons and an odd integer for fermions), we find incompressible states at corresponding filling factors $1/m$ when higher LL's are included, as opposed to $1/(m+2)$ when restricted to the LLL.[4,5] Since the same filling factors appear as FQH states in either scheme, albeit for different model interactions, the universality of FQHE is reaffirmed.

Consider first a system of two bosons in two dimensions interacting through a repulsive δ-function potential. The Hamiltonian for the relative coordinate is

$$H=\frac{1}{2\mu}\left(\mathbf{P}-\frac{q}{c}\mathbf{A}\right)^2+\lambda\delta(\mathbf{r})\,, \tag{1}$$

where $\mu=m/2$ and $q=e/2$ are the reduced mass and reduced charge, respectively, and $\lambda>0$ is the interaction strength. Hard core corresponds to $\lambda\rightarrow\infty$. However, we will see that even if λ is finite, the system is effectively hard core, a feature in two dimensions that is known for the case without the magnetic field.[8] Henceforth, we will *measure all energies in units of the cyclotron frequency* $\omega_c=eB/mc$, *and lengths in units of the magnetic length* l_B. In the symmetric gauge, the Hamiltonian conserves angular momentum l, and the eigenstates can be labeled as $\Phi_{E,l}$, with E the energy. For bosons, the angular momentum l must be even. For $\lambda=0$, the eigenenergies $E=n+\frac{1}{2}$, n is a Landau-level index. We are interested in the ground state, and the LLL eigenfunctions have the simple form, $\phi_l=z^l\exp(-|z|^2/4)$, with $z=x+iy$. Since ϕ_l vanishes at the origin for $l>0$, it remains an eigenfunction with the same energy even if $\lambda\neq 0$. However, the $l=0$ wave function is finite at the origin and hence one might expect a finite energy shift, of order 1 in the case of hard core. We now show that this is not so.

Letting $\Phi_{E,0}=f(r)\exp(-|z|^2/4)$, the Schrödinger equation for the hard-core system is

$$-\frac{1}{2}\frac{\partial^2 f}{\partial r^2}-\frac{1}{2r}\frac{\partial f}{\partial r}+\frac{r}{2}\frac{\partial f}{\partial r}=(E-\tfrac{1}{2})f\,, \tag{2}$$

subject to the boundary condition $f(r=0)=0$. The second-order differential equation (2) has two independent solutions. Close to the origin, they can be characterized by $h_1(r)$, which is regular and finite, and $h_2(r)$ which diverges logarithmically. For large r, the two possible asymptotic behaviors are the allowed solution $g_1(r)\sim r^{E'}$, with $E'=E-\frac{1}{2}$, and the forbidden solution $g_2(r)$ which diverges as $\exp(r^2/2)$. One must then pick the correct combination of $ah_1(r)+bh_2(r)$, to obtain the correct asymptotic form at the boundary. For $E'=n$, $b=0$; but otherwise $b\neq 0$. However, the hard core does not allow $b=0$ and hence $E'\neq n$. Instead, one must have $ah_1(0)+bh_2(0)=0$. Since $h_2(0)$ diverges, b must be infinitesimal, implying an infinitesimal shift in E. Note that this solution is not allowed without the interaction because of the discontinuity in the first derivative. To understand this better, consider the hard core to have a finite size R so that $f(r<R)=0$. Now we need $ah_1(R)+bh_2(R)=0$, hence $b\propto 1/\ln R$. Assuming that the amplitude of h_1 to approach g_2 asymptotically is proportional to $E'-n$, then $E'-n\propto 1/\ln R$, indicating that the energy shift due to the hard core vanishes logarithmically as the core size. In effect, the wave function is constant for all $r\neq 0$, but drops abruptly to 0 at $r=0$. This is known for the case with no magnetic field,[8] but it has been assumed in strong magnetic field that the magnetic length provides a short distance cutoff. The same result can be obtained by expanding in the complete set of Landau-level eigenstates of the noninteracting system. The solution contains each higher LL with infinitesimal amplitude compared to the lowest LL, but nevertheless when summed together satisfies the hard-core condition, and clearly demonstrates the subtlety of the meaning of "restricting" to the LLL. This will be described at the end of the paper. For soft-core interaction, λ is finite. If we assume $f(0)$ to be finite, the energy cost is also finite. This implies the energy can always be lowered by having $f(0)=0$. In effect all $\lambda>0$ systems behave identically to the hard-core system, and the explicit value of λ is an irrelevant parameter.

Instead of the exact eigenfunction, we can use an extremely simple trial wave function $f(r)=r^\eta$ for the zero angular momentum ground state, with η an arbitrary positive number.[9] The energy of this state is found to be $E=\frac{1}{2}+\eta^2/4$, and $E\rightarrow\frac{1}{2}+0_+$ as $\eta\rightarrow 0_+$. The wave function is analytic everywhere except at the origin due to the singularity of the δ-function interaction. Note that the limit $r=0$ and $\eta\rightarrow 0_+$ cannot be interchanged.

We now consider an N-boson system with hard-core interaction, $H_B=H_0+\lambda\sum_{i<j}\delta(\mathbf{r}_i-\mathbf{r}_j)$. The two-body wave function can now be used to form an N-body Jastrow-type wave function,

$$\Psi_B(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)=\prod_{i<j}|z_i-z_j|^\eta\exp\left(-\sum_k\frac{|z_k|^2}{4}\right). \tag{3}$$

It is quite natural that this state also has vanishing energy as $\eta\rightarrow 0_+$. To see this explicitly, we apply the singular gauge transformation[10] and convert (3) into an anyon wave function

$$\Psi_\eta(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)=\prod_{i<j}\frac{(z_i-z_j)^\eta}{|z_i-z_j|^\eta}\Psi_B(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)\,, \tag{4}$$

the Hamiltonian of which is

$$\begin{aligned}
H_\eta&=H_B+H'\,,\\
H'&=-\frac{1}{m}\sum_{j=1}^{N}\left(\mathbf{P}_j-\frac{e}{c}\mathbf{A}_j\right)\cdot\delta\mathbf{A}_j(\mathbf{r}_j)\\
&\quad+\frac{e^2}{2mc^2}[\delta\mathbf{A}_j(\mathbf{r}_j)]^2\,,\\
\delta\mathbf{A}_j(\mathbf{r}_j)&=-\frac{\hbar c\eta}{e}\sum_{i\neq j}\frac{z\times(\mathbf{r}_i-\mathbf{r}_j)}{|\mathbf{r}_i-\mathbf{r}_j|^2}\,.
\end{aligned} \tag{5}$$

In Eq. (5), H' is the gauge interaction. It was shown in Ref. 7 that Ψ_η is an eigenstate of H_0 with energy $\frac{1}{2}$ per particle. The δ-function interaction vanishes in the state (4). H' is of order η for small η. We see that indeed the energy of (4) for H_η, hence the energy of (3) for H_B is infinitesimally close to $N/2$, the ground-state energy of noninteracting system. From the one-component plasma analogy,[2] we know that in the limit $\eta\rightarrow 0_+$ (3) describes a droplet of unit area, *independent* of the number of particles; hence (3) is capable of describing an arbitrary particle number. Furthermore, since $\frac{1}{2}$ is a rigorous lower bound to the ground-state energy per particle, and our variational energy also shows it to be the upper bound for arbitrary particle number, there can be no chemical potential discontinuity. Thus, we conclude that hard-core bosons do not exhibit FQHE.

Consider the case now of two spinless or spin polarized fermions. The relative angular momentum must then be an odd integer. Because of the Pauli exclusion principle, the interaction in (1) is irrelevant. A nontrivial interaction is the model potential introduced in Ref. 4,

$$H_{\text{int}} = \lambda \nabla^2 \delta(\mathbf{r}) . \tag{6}$$

If we write the wave function as $\phi_F(\mathbf{r}) = z\phi(\mathbf{r})$, then $\phi(0)$ must be zero in the limit $\lambda \to \infty$. We generalize the definition of hard core and call this "hard-core" fermions. For angular momentum $l \geq 3$, this condition is automatically satisfied, but not for $l = 1$. However, we will show, similar to the boson case, the hard-core constraint does not lead to a finite energy cost. Letting

$$\phi(\mathbf{r}) = f(r)\exp(-|z|^2/4) ,$$

the Schrödinger equation for $f(r)$ is

$$-\frac{1}{2}\frac{\partial^2 f}{\partial r^2} - \frac{3}{2r}\frac{\partial f}{\partial r} + \frac{r}{2}\frac{\partial f}{\partial r} = (E - \tfrac{1}{2})f . \tag{7}$$

The two independent solutions have the same distinct asymptotic forms for large r as before, but for small r, has $h_1(r)$ regular, and $h_2(r) \propto 1/r^2$. The same analysis as the boson case then implies that if we consider a hard-core size R, the energy cost will vanish as R^2. Also, the value of λ is irrelevant as long as it is positive. Similar to the boson case, we may use a trial wave function for two fermions, $f(r) = r^\eta$. The energy of this state is $E = \frac{1}{2} + \eta^2/4$, and $E \to \frac{1}{2} + 0_+$ as $\eta \to 0_+$. Generalizing N-boson wave function (3) to N-fermion systems, we have

$$\Psi_F(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = \prod_{i<j} (z_i - z_j) \Psi_B(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) .$$

This wave function can be used to describe any fermion density less than 1, and has energy arbitrarily close to the Laughlin state for the interaction (6).

It is easy to generalize the above to general model interactions $\lambda \nabla^{2m} \delta(\mathbf{r})$. Again consider first two particles. The basic idea is that the $\delta(\mathbf{r})$ part of the interaction can always be made to give a vanishing contribution by the factor $|z|^\eta$ with infinitesimal cost in magnetic energy when higher LL's are taken into account. Writing the wave function as $|z|^\eta z^l$, we see, by integrating by parts for example, that the interaction contributes zero energy for $l \geq m$, but is finite for $l < m$. Extending to the many-body system implies incompressible states at filling factor $1/m$, as opposed to $1/(m+2)$ obtained with the LLL restriction. The difference arises because in the latter case, an extra factor of z^2 is needed to take care of the delta function. To see this, we notice that the states with zero interaction energy and kinetic energy $\frac{1}{2}$ per particle (in the limit $\eta \to 0_+$) essentially have the general form $\prod_{i<j} (z_i - z_j)^m \Psi_B(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$, with the maximum filling factor of $1/m$. Adding more particles or decreasing the magnetic flux passing through the filling $1/m$ system will cause finite energy, hence the system is incompressible at filling $1/m$.

In conclusion we have shown that an interaction $\lambda \nabla^{2m} \delta(\mathbf{r})$ leads to FQHE at filling factor $1/m$, and there can be no FQHE for $m = 0, 1$, independent of the values of the interaction strength and the cyclotron frequency. Our result shows that the conventional projection on to the lowest Landau level is not valid for the vanishing-range interactions. From a mathematical viewpoint, the two limits of $\omega \to \infty$ and interaction range $R \to 0$ cannot be interchanged, because that leads to incompressibility at different filling factors. More precisely, the $\omega \to \infty$ limit is the zero mass limit. The dimensionless parameters are the length ratio R/l_B and the energy ratio $\hbar\omega l_B^{2a}/\lambda = (\hbar^2/\mu\lambda)(\hbar c/eB)^{a-1}$, with $a = m+1$. The energy ratio is independent of the magnetic field B for $m = 0$, and it decreases as B increases for $m > 0$. Therefore the projection onto the LLL is not justified at the high-field limit in this model. Coulomb interactions in real systems have an energy scale proportional to $B^{1/2}$ compared to $\hbar\omega \propto B$, thus the projection method commonly used is applicable at high fields.[2]

Finally, we discuss the two-boson problem of δ function in terms of expansion of the eigenstates of the noninteracting Hamiltonian. This approach will show how the higher LL's are mixed to avoid the hard core but cost vanishingly small energy. This problem was studied by Prange[11] in connection to the impurity effect in QHE. We shall emphasize the result to be independent of the system size so that it is essentially applicable to many-body systems. Consider a trial wave function with relative angular momentum zero for a two-boson system,

$$\Phi_B(\mathbf{r}) = \sum_{n=0}^{M} c_n \phi_{n,0}(\mathbf{r}) , \tag{8}$$

with

$$c_n = \phi_{n,0}(0)/(E_n - E_0)$$

for $n \neq 0$, and

$$c_0 = -\sum_{n=1}^{M} c_n \phi_{n,0}(0)/\phi_{0,0}(0) .$$

$\phi_{n,0}(\mathbf{r})$ and E_n are the nth LL state and energy of the noninteracting system with angular momentum 0, and M is the highest LL index involved in the state. By construction, $\Phi_B(0) = 0$, satisfying the hard-core boundary condition. For an infinite system, $E_n - E_0 = n$, and $\phi_{n,0}(0)/\phi_{0,0}(0) = 1$. The energy of $\Phi_B(\mathbf{r})$ is $E \to \frac{1}{2} + 1/\ln M$, for large M. The mixing of the higher LL contributes very little to the energy, while they change the wave function at the origin significantly. The wave function (8) is analytic in the whole space for a fixed M, and it reaches $\phi_{0,0}(\mathbf{r})$ except at the origin. This result holds for the two-boson systems with finite area W. In that case, we do not have explicit expressions for E_n and $\phi_{n,0}(0)$ when n is comparable or larger than W. The spatial constraint at boundary pushes the amplitude $\phi_{n,0}(0)$ upward, but also increases E_n. For $n \gg W$, we should have $E \propto n^2$, and $\phi_{n,0}(0)/\phi_{0,0}(0) \propto n^{1/2}$. The energy of (8) is then of the same asymptotic form as in the finite system. A better geometry for this is the spherical surface[12] in which the eigen-energies and wave functions for a finite-size-system are known explicitly. Let the magnetic flux passing through the surface be an integer $2S$ in units of hc/e, then the energy of a single particle is $E_{n-S} = [n(n+1)$

$-S^2]/2S$, with $n=S, S+1, \ldots$, and the corresponding wave functions are spherical harmonics $Y_{n,m}(\Omega)$ in an appropriate gauge,[12] with m the azimuthal angular momentum. We consider the trial wave function for $m=0$, avoiding the δ-function interaction by replacing $\phi_{n,0}(\mathbf{r})$ by $Y_{n+S,0}(\Omega)$ in Eq. (8). The energy of that state is found to be $E \to \frac{1}{2}+1/2\ln M$, at $M \to \infty$. Therefore the energy can be infinitesimally close to $\frac{1}{2}$, independent of the system size. Similarly, one can construct a trial wave function with relative angular momentum one for a two-fermion system of $\nabla^2\delta(\mathbf{r})$ interaction in terms of expansion of the noninteracting eigenstates, and the result is $E \to \frac{1}{2}+O(1/M)$ at large M.

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