

$$\left(i \hbar \frac{\partial}{\partial t} - q \varphi' \right) \psi' = \left[\frac{(-i \hbar \nabla - q A')^2}{2m} + V(r) \right] \psi' \quad (5.49)$$

i.e., using the new gauge, the Schrodinger equation takes exactly the same form (i.e. the physics law remains the same).

5.2. Example: Landau levels and quantum Hall effect

5.2.1. 2D electron gas

Now, we consider free electrons moving in 2D.

$$H = \frac{(-i \hbar \nabla)^2}{2m} = \frac{(-i \hbar \partial_x)^2}{2m} + \frac{(-i \hbar \partial_y)^2}{2m} \quad (5.50)$$

Here, we have no potential energy, only kinetic energy. The eigenwavefunctions and eigenenergy for this Hamiltonian are very simple.

The eigen-wavefunctions are 2D plane waves

$$\psi = e^{i k_x x + i k_y y} \quad (5.51)$$

and the corresponding eigenenergy is

$$\epsilon = \frac{P^2}{2m} = \frac{P_x^2 + P_y^2}{2m} = \frac{\hbar^2 k_x^2 + \hbar^2 k_y^2}{2m} = \frac{\hbar^2}{2m} (k_x^2 + k_y^2) \quad (5.52)$$

Key conclusion: The eigenenergy can be any positive real number.

5.2.2. 2D electron gas + uniform magnetic field

Now we apply a uniform B field (B is along the z-direction). As will shown below, the eigen-energy will now become discrete energy levels (similar to a harmonic oscillator). Here, eigen-energy can only some discrete values.

For an uniform B field along z, we can represent it using the following scalar and vector potential

$$\varphi = 0 \quad (5.53)$$

and

$$A = (0, Bx, 0) \quad (5.54)$$

It is easy to verify that

$$\vec{E} = -\nabla \varphi - \frac{\partial \vec{A}}{\partial t} = 0 \quad (5.55)$$

and

$$\vec{B} = \nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & Bx & 0 \end{vmatrix} = (\partial_x Bx) \hat{z} = B \hat{z} \quad (5.56)$$

Note: This choice of φ and A are not the unique choice. We can choose any gauge, but the final result would be the same, since our Schrodinger equation doesn't depends on gauge choices.

Now the Hamiltonian changes into

$$H = \frac{(-i \hbar \partial_x - q A_x)^2}{2m} + \frac{(-i \hbar \partial_y - q A_y)^2}{2m} = \frac{(-i \hbar \partial_x)^2}{2m} + \frac{(-i \hbar \partial_y - q Bx)^2}{2m} \quad (5.57)$$

The static Schrodinger equation is

$$\frac{(-i \hbar \partial_x)^2}{2m} \psi(x, y) + \frac{(-i \hbar \partial_y - q Bx)^2}{2m} \psi(x, y) = \epsilon \psi(x, y) \quad (5.58)$$

First, we notice that

$$[H, -i\hbar\partial_y] = 0 \quad (5.59)$$

This is because H doesn't contain any y , so it commutes with $-i\hbar\partial_y$.

This commutator implies that we can find common eigenstates of H and $-i\hbar\partial_y$

$$\frac{(-i\hbar\partial_x)^2}{2m}\psi(x, y) + \frac{(-i\hbar\partial_y - qBx)^2}{2m}\psi(x, y) = \epsilon\psi(x, y) \quad (5.60)$$

and

$$-i\hbar\partial_y\psi(x, y) = k_y\hbar\psi(x, y) \quad (5.61)$$

The solution of the second equation is easy to find

$$\psi(x, y) = f(x) e^{ik_y y} \quad (5.62)$$

If we put this solution back to the first equation, we get

$$\frac{(-i\hbar\partial_x)^2}{2m}[f(x) e^{ik_y y}] + \frac{(-i\hbar\partial_y - qBx)^2}{2m}[f(x) e^{ik_y y}] = \epsilon[f(x) e^{ik_y y}] \quad (5.63)$$

We multiply both sides with $e^{-ik_y y}$

$$\frac{(-i\hbar\partial_x)^2}{2m}f(x) + e^{-ik_y y} \frac{(-i\hbar\partial_y - qBx)^2}{2m}[f(x) e^{ik_y y}] = \epsilon f(x) \quad (5.64)$$

$$\frac{(-i\hbar\partial_x)^2}{2m}f(x) + f(x) e^{-ik_y y} \frac{(-i\hbar\partial_y - qBx)^2}{2m} e^{ik_y y} = \epsilon f(x) \quad (5.65)$$

In the second term on the l.h.s., we cannot simply cancel $e^{-ik_y y}$ and $e^{ik_y y}$. This is because one of them is before ∂_y , while the other is after ∂_y .

We know that we cannot switch ∂_y and $e^{ik_y y}$, because ∂_y does not commute with y .

Here, we utilize the fact that $-i\hbar\partial_y e^{ik_y y} = k_y\hbar e^{ik_y y}$

$$\frac{(-i\hbar\partial_x)^2}{2m}f(x) + f(x) e^{-ik_y y} \frac{(\hbar k_y - qBx)^2}{2m} e^{ik_y y} = \epsilon f(x) \quad (5.66)$$

$$\frac{(-i\hbar\partial_x)^2}{2m}f(x) + \frac{(\hbar k_y - qBx)^2}{2m}f(x) = \epsilon f(x) \quad (5.67)$$

$$\left[\frac{(-i\hbar\partial_x)^2}{2m} + \frac{(\hbar k_y - qBx)^2}{2m} \right] f(x) = \epsilon f(x) \quad (5.68)$$

$$\left[\frac{(-i\hbar\partial_x)^2}{2m} + \frac{q^2 B^2 \left(x - \frac{\hbar k_y}{qB}\right)^2}{2m} \right] f(x) = \epsilon f(x) \quad (5.69)$$

How, we define

$$x_0 = \frac{\hbar k_y}{qB} \quad (5.70)$$

and

$$k = \frac{q^2 B^2}{m} \quad (5.71)$$

$$\left[\frac{(-i\hbar\partial_x)^2}{2m} + \frac{k}{2}(x - x_0)^2 \right] f(x) = \epsilon f(x) \quad (5.72)$$

This equation is the static Schrodinger equation for a harmonic oscillator (with equilibrium position at $x = x_0$). For a harmonic oscillator, we know that the eigenenergy is

$$\epsilon_n = \hbar\omega(n + 1/2) \quad (5.73)$$

where $\omega = \sqrt{k/m}$

Here, our k is $\frac{q^2 B^2}{m}$, so we found that

$$\omega = \sqrt{k/m} = \sqrt{\frac{q^2 B^2}{m^2}} = \frac{q B}{m} \quad (5.74)$$

This frequency is known as the “cyclotron frequency” and the eigenenergies are

$$\epsilon_n = \hbar \omega (n + 1/2) = \frac{q \hbar B}{m} \left(n + \frac{1}{2} \right) \quad (5.75)$$

The eigenenergy can no longer take arbitrary positive values. Only some discrete values are not allowed. Each of these discrete energy level is known as a **Landau level**

5.2.3. cyclotron frequency

The cyclotron frequency has a simple physical meaning. In classical mechanics, if we have a charged particle moving in a uniform B field, the trajectory is a circle. This is because the Lorentz force is always perpendicular to velocity, which means that the acceleration is always perpendicular to the velocity, i.e. a circular motion.

If the velocity of the particle is v , the Lorentz force is

$$F = q v B \quad (5.76)$$

For a circular motion, the acceleration is

$$a = v^2 / r \quad (5.77)$$

We know that $F = m a$, and thus

$$q v B = m v^2 / r \quad (5.78)$$

$$\frac{q B}{m} = \frac{v}{r} \quad (5.79)$$

We found that neither v nor r is a universal number. However, their ratio is independent of v or r . Once we fixed the B field, because q/m can never change for a particle (ignore relativistic effects), v/r is always a constant.

Q: what is v/r for a circular motion?

A: it is the angular frequency ω

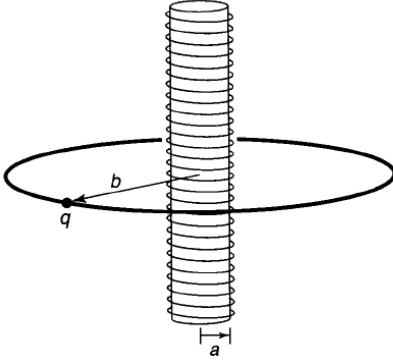
$$\omega = \frac{q B}{m} \quad (5.80)$$

This is the cyclotron frequency.

Quantum mechanics told us, once we have a periodic motion with angular momentum ω , if it is a harmonic oscillator, we shall expect eigenenergy $\epsilon = \hbar \omega (n + 1/2)$. This is exactly what happened here for Landau levels..

5.3. Example: particles moving around a solenoid.

Consider a charged particle moving around a solenoid.



For an infinite long solenoid, there is B field inside the solenoid, but the B field outside the cylinder is zero. Now consider a charged particle with charge q . We put the particle on a ring, which circles around the solenoid, i.e. the particle can only move on this circular ring.

By changing the electric current that passes through the solenoid, we can change the field inside the solenoid, but the B field outside will always remain zero. Now, we ask the question whether the motion of the charged particle will change when we change the B field inside the solenoid.

In classical mechanics, the motion of the particle does NOT depend on the solenoid, because the B field is always zero for any point on the ring. In classical mechanics, if $B = 0$, it means that the particle cannot see the B field and thus the solenoid doesn't matter at all, as far as the particle is concerned.

However, in quantum mechanics, this is not the case. It turns out that the particle does know the magnetic field inside the solenoid, although the field strength for any point on the ring is zero.

5.3.1. vector potential

In quantum mechanics, the magnetic field couples to the motion of a particle via vector potential \vec{A} . This is very different from classical mechanics, where what matters is \vec{B} , instead of \vec{A} . If the motion of the particle only depends on \vec{B} , then the solenoid doesn't matter, because $B = 0$ for the ring (this is what happens in classical mechanics). If the motion of the particle relies on \vec{A} , things would be different, because when $B = 0$, \vec{A} may not be zero.

Q: what is \vec{A} for a solenoid

A: It is not zero on the ring, if the solenoid has a nonzero current.

Consider the following integral along the ring:

$$\oint d\vec{r} \cdot \vec{A} = \int \int d\sigma (\nabla \times \vec{A}) = \int \int d\sigma B = \Phi \quad (5.81)$$

Here, we used Stokes' theorem and the fact that the curl of A is B . Φ is the magnetic flux inside the solenoid.

Conclusion: the integral $\oint d\vec{r} \cdot \vec{A}$ equals to the magnetic flux inside the solenoid. Although $B = 0$ outside the solenoid, A is NOT zero (if $A = 0$, the integral would be zero, which is impossible, because we know that the magnetic flux is nonzero).

We can choose the gauge such that

$$\vec{A} = \frac{\Phi}{2\pi r} \hat{\phi} \quad (5.82)$$

Here, we use the cylindrical coordinate, and $\hat{\phi}$ is the unit vector that is tangential to the ring.

It is easy to verify that

$$\vec{A} = \frac{\Phi}{2\pi r} (-\sin \varphi, \cos \varphi, 0) = \frac{\Phi}{2\pi r} \left(-\frac{y}{r}, \frac{x}{r}, 0 \right) = \frac{\Phi}{2\pi} \left(-\frac{y}{r^2}, \frac{x}{r^2}, 0 \right) = \frac{\Phi}{2\pi} \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0 \right) \quad (5.83)$$

5.3.2. Hamiltonian

$$\begin{aligned}
H &= \frac{(-i\hbar\partial_x - qA_x)^2}{2m} + \frac{(-i\hbar\partial_y - qA_y)^2}{2m} + \frac{(-i\hbar\partial_z - qA_z)^2}{2m} \\
&= \frac{\left(-i\hbar\partial_x + \frac{q\Phi}{2\pi} \frac{y}{x^2+y^2}\right)^2}{2m} + \frac{\left(-i\hbar\partial_y - \frac{q\Phi}{2\pi} \frac{x}{x^2+y^2}\right)^2}{2m} + \frac{(-i\hbar\partial_z)^2}{2m} \\
&= -\frac{\hbar^2\left(\partial_x + i\frac{q\Phi}{2\pi\hbar} \frac{y}{x^2+y^2}\right)^2}{2m} - \frac{\hbar^2\left(\partial_y - i\frac{q\Phi}{2\pi\hbar} \frac{x}{x^2+y^2}\right)^2}{2m} - \frac{\hbar^2\partial_z^2}{2m}
\end{aligned}$$

Start with the first term:

$$\begin{aligned}
\left(\partial_x + i\frac{q\Phi}{2\pi\hbar} \frac{y}{x^2+y^2}\right)^2 \psi(x, y, z) &= \left(\partial_x + i\frac{q\Phi}{2\pi\hbar} \frac{y}{x^2+y^2}\right) \left(\partial_x + i\frac{q\Phi}{2\pi\hbar} \frac{y}{x^2+y^2}\right) \psi(x, y, z) \\
&= \partial_x^2 \psi(x, y, z) + i\frac{q\Phi}{2\pi\hbar} \frac{y}{x^2+y^2} \partial_x \psi(x, y, z) + \partial_x \left[i\frac{q\Phi}{2\pi\hbar} \frac{y}{x^2+y^2} \psi(x, y, z) \right] + \left(i\frac{q\Phi}{2\pi\hbar} \frac{y}{x^2+y^2} \right)^2 \psi(x, y, z) \\
&= \partial_x^2 \psi(x, y, z) + i\frac{q\Phi}{2\pi\hbar} \frac{y}{x^2+y^2} \partial_x \psi(x, y, z) + \\
&\quad i\frac{q\Phi}{2\pi\hbar} \left(\partial_x \frac{y}{x^2+y^2} \right) \psi(x, y, z) + i\frac{q\Phi}{2\pi\hbar} \frac{y}{x^2+y^2} \partial_x \psi(x, y, z) - \left(\frac{q\Phi}{2\pi\hbar} \right)^2 \frac{y^2}{(x^2+y^2)^2} \psi(x, y, z) \\
&= \partial_x^2 \psi(x, y, z) + i\frac{q\Phi}{\pi\hbar} \frac{y}{x^2+y^2} \partial_x \psi(x, y, z) - i\frac{q\Phi}{2\pi\hbar} \frac{2xy}{(x^2+y^2)^2} \psi(x, y, z) - \left(\frac{q\Phi}{2\pi\hbar} \right)^2 \frac{y^2}{(x^2+y^2)^2} \psi(x, y, z) \\
&= \partial_x^2 \psi(x, y, z) + i\frac{q\Phi}{\pi\hbar} \frac{y}{x^2+y^2} \partial_x \psi(x, y, z) - i\frac{q\Phi}{\pi\hbar} \frac{xy}{(x^2+y^2)^2} \psi(x, y, z) - \left(\frac{q\Phi}{2\pi\hbar} \right)^2 \frac{y^2}{(x^2+y^2)^2} \psi(x, y, z)
\end{aligned} \tag{5.85}$$

Similarly, the second term is

$$\begin{aligned}
\left(\partial_y - i\frac{q\Phi}{2\pi\hbar} \frac{x}{x^2+y^2}\right)^2 \psi(x, y, z) &= \partial_y^2 \psi(x, y, z) - i\frac{q\Phi}{\pi\hbar} \frac{x}{x^2+y^2} \partial_y \psi(x, y, z) + i\frac{q\Phi}{\pi\hbar} \frac{xy}{(x^2+y^2)^2} \psi(x, y, z) - \left(\frac{q\Phi}{2\pi\hbar} \right)^2 \frac{x^2}{(x^2+y^2)^2} \psi(x, y, z)
\end{aligned} \tag{5.86}$$

As a result,

$$\begin{aligned}
H &= -\frac{\hbar^2\left(\partial_x + i\frac{q\Phi}{2\pi\hbar} \frac{y}{x^2+y^2}\right)^2}{2m} - \frac{\hbar^2\left(\partial_y - i\frac{q\Phi}{2\pi\hbar} \frac{x}{x^2+y^2}\right)^2}{2m} - \frac{\hbar^2\partial_z^2}{2m} + V(\vec{r}) \\
&= -\frac{\hbar^2}{2m} (\partial_x^2 + \partial_y^2 + \partial_z^2) - \frac{\hbar^2}{2m} i\frac{q\Phi}{\pi\hbar} \frac{y}{x^2+y^2} \partial_x - \frac{\hbar^2}{2m} i\frac{q\Phi}{\pi\hbar} \frac{x}{x^2+y^2} \partial_y + \frac{\hbar^2}{2m} \left(\frac{q\Phi}{2\pi\hbar} \right)^2 \frac{x^2+y^2}{(x^2+y^2)^2} \\
&= -\frac{\hbar^2}{2m} (\partial_x^2 + \partial_y^2 + \partial_z^2) - i\frac{\hbar}{2m} \frac{q\Phi}{\pi} \frac{y\partial_x - x\partial_y}{x^2+y^2} + \frac{1}{2m} \left(\frac{q\Phi}{2\pi} \right)^2 \frac{x^2+y^2}{(x^2+y^2)^2} \\
&= -\frac{\hbar^2\nabla^2}{2m} - i\frac{\hbar q\Phi}{2\pi m r^2} (y\partial_x - x\partial_y) + \frac{1}{2m} \left(\frac{q\Phi}{2\pi r} \right)^2
\end{aligned} \tag{5.87}$$

$$H\psi(x, y, z) = -\frac{\hbar^2\nabla^2}{2m} \psi(x, y, z) - i\frac{\hbar q\Phi}{2\pi m r^2} (y\partial_x - x\partial_y) \psi(x, y, z) + \frac{1}{2m} \left(\frac{q\Phi}{2\pi r} \right)^2 \psi(x, y, z) \tag{5.88}$$

Convert to the cylindrical coordinate,

$$\nabla^2 \psi(r, \varphi, z) = \frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial \psi(r, \varphi, z)}{\partial r} \right] + \frac{1}{r^2} \frac{\partial^2 \psi(r, \varphi, z)}{\partial \varphi^2} + \frac{\partial^2 \psi(r, \varphi, z)}{\partial z^2} \tag{5.89}$$

Because the particle can only move along the ring, r and z are fixed and thus the wavefunction only depends on φ , independent of r and z .

$\psi(r, \varphi, z) = \psi(\varphi)$. Therefore, $\partial_z \psi = \partial_r \psi = 0$

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial \psi}{\partial r} \right] + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \varphi^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \varphi^2} = \frac{1}{r^2} \frac{d^2 \psi}{d\varphi^2} \quad (5.90)$$

For the second term,

$$\begin{aligned} (y \partial_x - x \partial_y) \psi(x, y, z) &= -\hat{z} \cdot (x, y, 0) \times (\partial_x, \partial_y, \partial_z) \psi(x, y, z) = -\hat{z} \cdot \vec{r} \times \nabla \psi(x, y, z) \\ &= -\hat{z} \cdot \vec{r} \times \nabla \psi(x, y, z) \end{aligned} \quad (5.91)$$

We know that

$$\nabla \psi = \frac{\partial \psi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \psi}{\partial \varphi} \hat{\varphi} + \frac{\partial \psi}{\partial z} \hat{z} \quad (5.92)$$

$$\vec{r} = r \hat{r} \quad (5.93)$$

$$\begin{aligned} \vec{r} \times \nabla \psi(x, y, z) &= r \hat{r} \times \left(\frac{\partial \psi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \psi}{\partial \varphi} \hat{\varphi} + \frac{\partial \psi}{\partial z} \hat{z} \right) = r \hat{r} \times \frac{1}{r} \frac{\partial \psi}{\partial \varphi} \hat{\varphi} + r \hat{r} \times \frac{\partial \psi}{\partial z} \hat{z} \\ &= \frac{\partial \psi}{\partial \varphi} \hat{r} \times \hat{\varphi} + r \frac{\partial \psi}{\partial z} \hat{r} \times \hat{z} \end{aligned} \quad (5.94)$$

$$\hat{z} \cdot \vec{r} \times \nabla \psi(x, y, z) = \frac{\partial \psi}{\partial \varphi} \hat{z} \cdot (\hat{r} \times \hat{\varphi}) + r \frac{\partial \psi}{\partial z} \hat{z} \cdot (\hat{r} \times \hat{z}) = \frac{\partial \psi}{\partial \varphi} \quad (5.95)$$

$$(y \partial_x - x \partial_y) \psi(x, y, z) = -\hat{z} \cdot \vec{r} \times \nabla \psi(x, y, z) = -\frac{\partial \psi}{\partial \varphi} = -\frac{d\psi}{d\varphi} \quad (5.96)$$

Therefore,

$$\begin{aligned} H \psi(x, y, z) &= -\frac{\hbar^2 \nabla^2}{2m} \psi(x, y, z) - i \frac{\hbar q \Phi}{2\pi m r^2} (y \partial_x - x \partial_y) \psi(x, y, z) + \frac{1}{2m} \left(\frac{q \Phi}{2\pi r} \right)^2 \psi(x, y, z) \\ &= -\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{d^2 \psi}{d\varphi^2} + i \frac{\hbar q \Phi}{2\pi m r^2} \frac{d\psi}{d\varphi} + \frac{1}{2m} \left(\frac{q \Phi}{2\pi r} \right)^2 \psi \end{aligned} \quad (5.97)$$

The Schrodinger equation

$$H \psi(\varphi) = E \psi(\varphi) \quad (5.98)$$

So we have

$$-\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{d^2 \psi}{d\varphi^2} + i \frac{\hbar q \Phi}{2\pi m r^2} \frac{d\psi}{d\varphi} + \frac{1}{2m} \left(\frac{q \Phi}{2\pi r} \right)^2 \psi = E \psi(\varphi) \quad (5.99)$$

$$-\frac{d^2 \psi}{d\varphi^2} + i \frac{q \Phi}{\pi \hbar} \frac{d\psi}{d\varphi} + \left(\frac{q \Phi}{2\pi \hbar} \right)^2 \psi = \frac{2m E r^2}{\hbar^2} \psi \quad (5.100)$$

$$-\frac{d^2 \psi}{d\varphi^2} + i \frac{q \Phi}{\pi \hbar} \frac{d\psi}{d\varphi} = \left[\frac{2m E r^2}{\hbar^2} - \left(\frac{q \Phi}{2\pi \hbar} \right)^2 \right] \psi \quad (5.101)$$

Define $\beta = \frac{q \Phi}{2\pi \hbar}$ and $\epsilon = \frac{2m E r^2}{\hbar^2} - \beta^2$

$$-\frac{d^2 \psi}{d\varphi^2} + 2i\beta \frac{d\psi}{d\varphi} = \epsilon \psi \quad (5.102)$$

So,

$$\frac{d^2 \psi}{d\varphi^2} - 2i\beta \frac{d\psi}{d\varphi} + \epsilon \psi = 0 \quad (5.103)$$

The solution for this second order differential equation is

$$\psi(\varphi) = A e^{i\lambda \varphi} \quad (5.104)$$

$$-\lambda^2 \psi + 2\lambda\beta \psi + \epsilon \psi = 0 \quad (5.105)$$