1.06 Fractional Quantum Hall Effect and Composite Fermions

J K Jain, The Pennsylvania State University, University Park, PA, USA

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Glossary

Composite fermions Bound states of electrons and quantized vortices.

Degeneracy Multiplicity of an energy level.

Fermi sea The state in which fermions occupy all states below certain energy.

Filling factor Number of filled Landau levels.

Fractional quantum Hall effect Observation of plateaus in the Hall resistance at quantized values characterized by simple fractions.

Hall effect Observation of a voltage transverse to the direction of the current flow in the presence of a magnetic field.

Integral quantum Hall effect Observation of plateaus in the Hall resistance at quantized values characterized by integers.

Landau levels Quantized kinetic energy levels of electrons in a magnetic field.

Nomenclature

B	magnetic field
B^*	magnetic field experienced by composite fermions
e	electron charge
E	electric field
h	Planck's constant
I	current
N	number of electrons
Pf	Pfaffian
R	resistance
R_H	Hall resistance

1.06.1 Introduction

Collective quantum behavior is of fundamental interest in condensed matter physics, where a collection of interacting particles behaves in a surprising manner that would be difficult to guess from the knowledge of the properties of single particles. Superconductivity and superfluidity are two such well-known examples. During the last three decades, a new collective quantum fluid state of matter has been discovered and studied, which occurs when electrons are confined to two dimensions, cooled to near absolute zero temperature, and subjected to a strong magnetic field. The most dramatic, and unexpected, manifestation of this phase is the 'fractional quantum Hall effect' (Tsui *et al.*, 1982). The fractional quantum Hall effect (FQHE) and many other properties of this state result from the formation of a new class of topological particles, called 'composite fermions', which are bound states of electrons and quantized microscopic vortices. The composite-fermion quantum fluid provides a new paradigm for collective behavior. This chapter describes the essential phenomenology of the fractional quantum Hall effect, how the composite fermion theory resolves it, and other consequences of the existence of composite fermions.

1.06.2 Quantum Hall Effect Phenomenology

1.06.2.1 The Hall Effect

The Ohm's law is given by

$$\mathbf{J} = \sigma \mathbf{E} \tag{1}$$

where σ is the conductivity, and $\mathbf{J} = q\rho\mathbf{v}$ is the current density for particles of charge q and density ρ moving with a velocity $\mathbf{v}$. The more familiar $I = V/R$ can be obtained from this local relation. In the presence of a magnetic field, the electron current flows in a direction perpendicular to the plane containing the electric and the magnetic fields. Alternatively, passage of a current induces a voltage perpendicular to the direction of the current flow, called the Hall voltage, V_{H}, as seen in **Figure 1**. In addition to the usual (longitudinal) resistance, $R = V_{\mathrm{L}}/I$, a new resistance, called the Hall resistance, is defined as

$$R_{\mathrm{H}} = \frac{V_{\mathrm{H}}}{I} \tag{2}$$

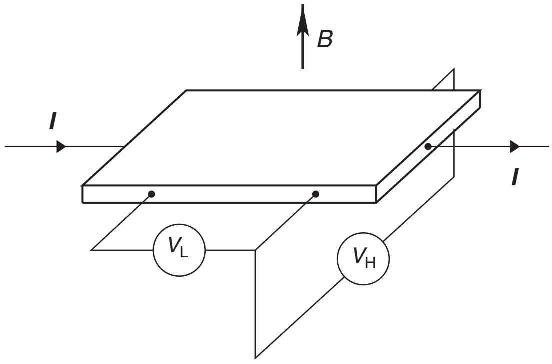

Figure 1 Schematics of magnetotransport measurements. I, $V_{\mathbf{L}}$, and $V_{\mathbf{H}}$ are the current, longitudinal voltage, and the Hall voltage, respectively. The longitudinal and Hall resistances are defined as $R_{\mathbf{L}} \equiv V_{\mathbf{L}}/I$ and $R_{\mathbf{H}} \equiv V_{\mathbf{H}}/I$.

The phenomenon has a classical origin. The Lorentz equation of electrodynamics

$$\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right) \tag{3}$$

gives the force on a particle of charge q, moving with a velocity $\mathbf{v}$, in the presence of electric and magnetic fields. A consequence of this equation is that for crossed electric and magnetic fields, say $\mathbf{E} = E\hat{y}$ and $\mathbf{B} = B\hat{z}$, the charged particle drifts in a direction perpendicular to the plane containing the two fields, with a velocity $\mathbf{v} = c(E/B)\hat{x}$.

Current density is given by $\mathbf{J} = q\rho\mathbf{v}$, where ρ is the (three-dimensional) density of particles. That produces the Hall resistivity

$$\rho_{\mathrm{H}} = \frac{E_y}{\mathcal{J}x} = \frac{B}{\rho qc} \tag{4}$$

The Hall measurement is used routinely to measure the density ρ of the mobile charges, as well as the sign of the charge carriers (i.e., whether they are electrons or holes).

1.06.2.2 Integral Quantum Hall Effect

When the Hall experiments are performed in a system in which electrons are confined to two dimensions (such confinement is achieved, for example, by constructing AlGaAs–GaAs quantum-wells to confined motion in one dimension), the behavior changes in a qualitative manner, as shown in **Figure 2**, taken from Stormer (1999). Instead of being proportional to B, now Hall resistance shows plateaus. Most prominent are plateaus on which the Hall resistance is precisely quantized at

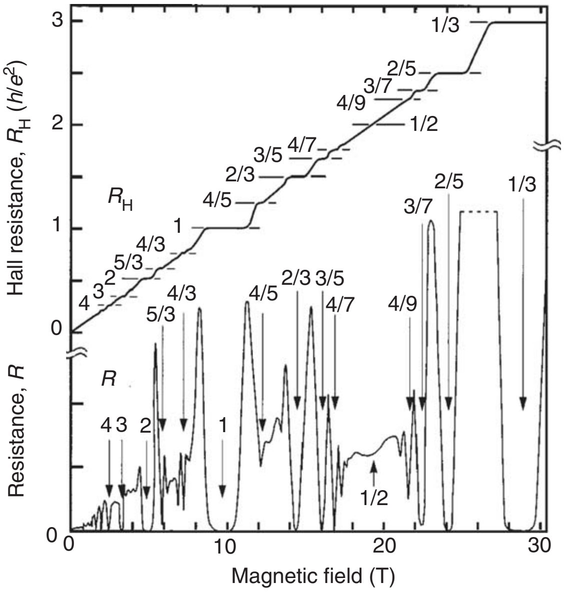

Figure 2 Hall and the longitudinal resistances for a two-dimensional electron system. The Hall resistance shows precisely quantized plateaus whereas the longitudinal resistance exhibits minima. Stormer HL (1999) Nobel lecture: The fractional quantum Hall effect. *Reviews of Modern Physics* 71: 875–889.

$$R_H = \frac{h}{ne^2} \tag{5}$$

where n is an integer, h is the Planck's constant, and e is the electron charge. This is referred to as the integral quantum Hall effect (IQHE), or the 'von Klitzing effect' (von Klitzing *et al.*, 1980). The quantization of the Hall resistance is independent of materials details, sample type, or geometry; it is also robust to variation in temperature and disorder, provided they are sufficiently small. The precision of the quantization, that is the correctness of the preceding equation, has been verified to one part in 10^7 for absolute accuracy, and to a few parts in 10^{11} for relative accuracy. The constant

$$R_K = \frac{h}{e^2} = 25813.807\ldots\Omega \tag{6}$$

is called the von Klitzing constant, and has been adopted as the unit of resistance. It should be noted that the fine structure constant

$$\alpha = \frac{e^2}{\hbar c} \tag{7}$$

also involves the same combination of the Planck's constant and the electron charge as the von Klitzing constant (the speed of light c being known much more precisely), and therefore the Hall resistance measurements also provide an accurate estimate of the fine structure constant. Concurrent with the quantized plateaus is a dissipation-less current flow in the limit of zero temperature.

1.06.2.3 Fractional Quantum Hall Effect

At lower temperatures, higher magnetic fields, and with better quality samples, plateaus are observed on which the Hall resistance is precisely quantized at

$$R_H = \frac{h}{fe^2} \tag{8}$$

where f is a fraction. This is the 'fractional quantum Hall effect' (FQHE) also known as the Tsui–Stormer–Gossard effect (Tsui *et al.*, 1982). More than 75 fractions have been observed to date, and more are being observed as the experimental conditions and the sample quality are improved. Most of the observed fractions have odd denominators, and they occur in certain sequences, with some examples given later in Equations (50)–(52). One exception to the 'odd denominator rule' is the FQHE state with $f = 5/2$.

The theory of FQHE must explain: the origin of gaps in a partially filled Landau level; the dominance of odd denominator fractions; the origin of sequences; the order of stability of fractions; the nature of state at even denominator fractions; the origin of 5/2; the role of spin; and the nature of neutral and charged excitations. We shall see below that the composite fermion theory provides a natural explanation of these facts, besides giving a microscopic theory that allows detailed quantitative comparisons with exact numerical results as well as experiment. At the same time, the composite fermion theory makes verifiable predictions (such as the existence of composite fermions; the composite fermion Fermi sea; and effective magnetic field) and unifies the fractional and the integral quantum Hall effects.

1.06.3 Electrons in a Magnetic Field: Landau Levels

The Hamiltonian for a nonrelativistic electron (relativistic effects are neglected throughout) moving in two dimensions in a perpendicular magnetic field is given by

$$H = \frac{1}{2m_{\mathbf{b}}}\left(\mathbf{p} + \frac{e\mathbf{A}}{c}\right)^2 \tag{9}$$

Here, e is defined to be a positive quantity, the electron's charge being $-e$, and $m_{\mathbf{b}}$ is the band mass of the electron. The canonical momentum is defined as $\mathbf{p} = -\mathrm{i}\hbar\boldsymbol{\nabla}$, and $\mathbf{A}$ satisfies

$$\boldsymbol{\nabla} \times \mathbf{A} = B\hat{z} \tag{10}$$

The Schrödinger equation

$$\boldsymbol{H}\Psi = E\Psi \tag{11}$$

can be solved in several gauges, but the most convenient for the fractional quantum Hall effect is the symmetric gauge:

$$\mathbf{A} = \frac{\mathbf{B} \times \mathbf{r}}{2} = \frac{B}{2}(-y, x, 0) \tag{12}$$

Choosing the unit of length as the magnetic length

$$\ell = \left(\frac{\hbar c}{eB}\right)^{1/2} \tag{13}$$

and the unit of energy as the cyclotron energy

$$\hbar\omega_{\mathbf{c}} = \hbar\frac{eB}{m_b c} \tag{14}$$

the Hamiltonian can be expressed as

$$H = \frac{1}{2}\left[-4\frac{\partial^2}{\partial z \partial \bar{z}} + \frac{1}{4}z\bar{z} - z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}\right] \tag{15}$$

Here we have defined new variables z and $\bar{z}$:

$$z = x - \mathrm{i}y = r\mathrm{e}^{-\mathrm{i}\theta}, \quad \bar{z} = x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\theta} \tag{16}$$

We further define the following sets of ladder operators:

$$b = \frac{1}{\sqrt{2}}\left(\frac{\bar{z}}{2} + 2\frac{\partial}{\partial z}\right) \tag{17}$$

$$b^\dagger = \frac{1}{\sqrt{2}}\left(\frac{z}{2} - 2\frac{\partial}{\partial \bar{z}}\right) \tag{18}$$

$$a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{\bar{z}}{2} - 2\frac{\partial}{\partial z}\right) \tag{19}$$

$$a = \frac{1}{\sqrt{2}}\left(\frac{z}{2} + 2\frac{\partial}{\partial \bar{z}}\right) \tag{20}$$

which have the property that

$$\left[a, a^\dagger\right] = 1, \left[b, b^\dagger\right] = 1 \tag{21}$$

and all the other commutators are zero. In terms of these operators, the Hamiltonian can be written as

$$H = a^\dagger a + \frac{1}{2} \tag{22}$$

This is the familiar Hamiltonian of a simple harmonic oscillator.

The angular momentum operator is defined as

$$L = -\hbar\left(b^\dagger b - a^\dagger a\right) \tag{23}$$

Exploiting the property $[H,L] = 0$, the eigenfunctions are chosen to diagonalize H and L simultaneously. The analogy to the Harmonic oscillator problem immediately gives the solution

$$H|n, m\rangle = E_n|n, m\rangle \tag{24}$$

$$E_n = \left(n + \frac{1}{2}\right) \tag{25}$$

$$|n, m\rangle = \frac{(b^\dagger)^{m+n}}{\sqrt{(m+n)!}}\frac{(a^\dagger)^n}{\sqrt{n!}}|0, 0\rangle \tag{26}$$

where n is called the Landau level index, and $m = -n, -n+1, \ldots$ is the angular momentum quantum number. The single-particle orbital at the bottom of the two ladders defined by the two sets of raising and lowering operators is

$$\langle \mathbf{r}|0, 0\rangle \equiv \eta_{0,0}(\mathbf{r}) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{4}z\bar{z}} \tag{27}$$

which satisfies

$$a|0, 0\rangle = b|0, 0\rangle = 0 \tag{28}$$

The single-particle states are especially simple in the lowest Landau level ($n = 0$):

$$\eta_{0,m} = \langle \mathbf{r}|0, m\rangle = \frac{(b^\dagger)^m}{\sqrt{m!}}\eta_{0,0} = \frac{z^m \mathrm{e}^{-\frac{1}{4}z\bar{z}}}{\sqrt{2\pi 2^m m!}} \tag{29}$$

where we have used Equations (18) and (27). Aside from the ubiquitous Gaussian factor, a general state in the lowest Landau level is simply given by a polynomial of z. In other words, apart from the Gaussian factor, the lowest Landau level wave functions are analytic functions of z.

1.06.3.1 Landau-Level Degeneracy

States with a given n but different m are degenerate. To obtain the degeneracy of a Landau level, consider a disk of radius R centered at the origin, and ask how many states lie inside it. The degeneracy can be shown to be independent of the Landau level. Taking, for simplicity, the lowest

Landau level, the eigenstate $|0, m\rangle$ has its weight located at the circle of radius $r = \sqrt{2m}l$. Thus, the largest value of m for which the state falls inside the disk is given by $m = R^2/2l^2$, which is also the total number of eigenstates in the lowest Landau level that fall inside the disk (ignoring order one corrections). Thus, the degeneracy per unit area is

$$G = \frac{1}{2\pi l^2} = \frac{eB}{hc} = \frac{B}{\phi_0} \tag{30}$$

Here we have defined the flux quantum as $\phi_0 = hc/e$; the degeneracy for each Landau level is equal to the flux quanta penetrating the sample.

1.06.3.2 Filling Factor

The filling factor is the number of occupied Landau levels, which depends on the density and the magnetic field. It is given by

$$\nu = \frac{\rho\phi_0}{B} \tag{31}$$

where ρ is the two-dimensional (2D) density of electrons. The filling factor is inversely proportional to the magnetic field. As the magnetic field is increased, each Landau level can accommodate more and more electrons, and, as a result, fewer and fewer Landau levels are occupied. Only the lowest Landau levels is occupied at sufficient high magnetic fields.

1.06.4 Integral Quantum Hall Effect Theory

The integral quantum Hall effect can be explained (Laughlin, 1981) in a model that neglects interactions between electrons. It occurs because the state of electrons at an integral filling factor is very simple: it contains a unique ground state containing an integral number of filled Landau levels, separated from excitations by the cyclotron or the Zeeman energy gap. (In other words, the state is incompressible, because to compress the ground state creates finite energy excitations.) It should be noted that the detailed explanation of the existence of the plateaus also requires a consideration of disorder-induced Anderson localization of some states.

1.06.5 The Fractional Quantum Hall Effect Problem

The phenomenon of FQHE indicates that gaps open not only at integral fillings (where their physics is straightforward), but also at many fractional filling factors. The essential goal for the theory of the FQHE is to explain the origin of these gaps. This requires a consideration of interelectron interactions, because gaps occur only at integral fillings for noninteracting electrons.

At high magnetic fields all electrons occupy the lowest Landau level. Their kinetic energy is then constant, hence irrelevant. With proper units for energy and length scales, interacting electrons in a high magnetic field are mathematically described by the Hamiltonian

$$H = \mathcal{P}_{\mathrm{LLL}}\left(\sum_{j<k}\frac{1}{r_{jk}}\right)\mathcal{P}_{\mathrm{LLL}} + V_{\mathrm{e-bg}} + V_{\mathrm{bg-bg}} \tag{32}$$

Here $r_{jk} = |z_j - z_k|$ is the distance between the electrons j and k, $\mathcal{P}_{\mathrm{LLL}}$ denotes projection into the lowest Landau-level (LLL) subspace, and the last two terms denote interaction between electrons and a uniform positive background. There are no parameters in the Hamiltonian; all energies are to be expressed in units of the Coulomb energy:

$$V_{\mathrm{C}} = \frac{e^2}{\varepsilon\ell} \tag{33}$$

where ε is the dielectric constant of the background material ($\varepsilon \approx 13$ for GaAs).

For some condensed matter systems, a nontrivial collective phenomenon can be understood as an instability of a normal state, which is the state obtained when the interaction is switched off. For the FQHE problem, switching off the interaction produces a large number of degenerate ground states; for example, for 10^9 electrons at $\nu = 2/5$ the number of degenerate ground states in the absence of interaction is $10^{7\times10^8}$. Further, there is no small parameter in the theory, which prevents a meaningful perturbation theory.

1.06.6 Composite Fermions: Basic Foundations

With the observation of many fractions, an analogy between the FQHE and the IQHE could be identified, which led to the postulation of composite fermions, and an explanation of the FQHE as the IQHE of composite fermions (Jain, 1989). The most important accomplishment of the composite fermion (CF) theory is the identification of the particles of the FQHE which can be considered as weakly interacting to a good approximation. The CF theory can be motivated by a Chern–Simons mean field theory (Jain, 1989; Lopez and Fradkin, 1991; Halperin *et al.*, 1993), but its final outcome can be stated succinctly. The eigenfunctions and eigenenergies for the ground and (low-energy) excited states of strongly interacting electrons at an arbitrary lowest Landau level filling ν are expressed in terms of the known solutions of the noninteracting electron problem at the Landau level filling ν^* as follows:

$$\Psi_\nu = \mathcal{P}_{\mathrm{LLL}} \prod_{j<k} \left(z_j - z_k\right)^{2p} \Phi_{\nu^*} \tag{34}$$

and

$$E_\nu = \frac{\left\langle \Psi_\nu \middle| \sum_{j<k} \frac{1}{r_{jk}} \middle| \Psi_\nu \right\rangle}{\langle \Psi_\nu | \Psi_\nu \rangle} + V_{\mathrm{el-bg}} + V_{\mathrm{bg-bg}} \tag{35}$$

where

$$\nu = \frac{\nu^*}{2p\nu^* \pm 1} \tag{36}$$

$$B^* = B - 2p\rho\phi_0 \tag{37}$$

Here, Φ_{ν^*} are the eigenfunctions of noninteracting electrons at ν^*, $\mathcal{P}_{LLL}$ projects the wave function to its right into the lowest Landau level, p is an integer, and B^* is an effective magnetic field. These equations define a one-to-one correspondence between the ground and excited states at filling factor ν (or magnetic field B) and ν^* (magnetic field B^*). The problem is often much simpler at ν^*, and for $\nu^* = n$ (where n is an integer), we know the exact solution for the ground and low energy states. The wave function for the ground state at $\nu = 1/(2p+1)$ was known previously (Laughlin, 1983).

Electrons capture vortices to turn into composite fermions because this is the most effective way for them to stay away from one another. A typical wave function satisfying the Pauli principle vanishes as r when two particles approach one another, r being the distance separating them, but the unprojected wave functions $\Phi \prod_{j<k}(z_j - z_k)^{2p}$ vanish as r^{2p+1}, with the Jastrow factor contributing $2p$ to the exponent and Φ contributing the rest. The Jastrow factor is very effective in keeping particles apart from one another, producing favorable correlations. It can be argued that these correlations survive projection into the lowest Landau level.

The Jastrow factor $\prod_{j<k}(z_j - z_k)^{2p}$ attaches $2p$ quantized vortices to each electron in Φ_{ν^*}. The bound state of an electron and $2p$ vortices is interpreted as a particle called the composite fermion. As composite fermions move about, the vortices bound to them produce Berry phases, which cancel part of the Aharonov Bohm phases originating from the external magnetic field. The effective magnetic field can be determined in a Berry-phase calculation. When a composite fermion, that is an electron along with its vortices, is taken in a closed loop enclosing an area A, it acquires a Berry phase

$$\Phi^* = -2\pi \frac{BA}{\phi_0} + 2\pi 2pN_{\mathrm{enc}} \tag{38}$$

where N_{enc} is the number of composite fermions inside the loop. The first term is the familiar Aharonov Bohm (AB) phase due to a charge going around in a loop. The second is the Berry phase due to the $2p$ vortices going around N_{enc} particles, with each particle contributing a phase of 2π. For uniform density states, we replace N_{enc} in Equation (38) by its average value ρA and equate the entire phase to the AB phase due to an effective magnetic field, B^*, to obtain Equation (37). Composite fermions thus sense a magnetic field B^* that is much smaller than the applied magnetic field, and can even be zero or negative. This property of composite fermions distinguishes them from electrons, and lies at the root of most of the phenomenology. Composite fermions form Landau-like levels in the reduced magnetic field, which are called Λ levels, and occupy ν^* of them.

Composite fermions represent a new class of particles realized in nature. A vortex is a topological object, because the quantum mechanical phase associated with a closed loop around a vortex is exactly 2π, independent of the shape and the size of the loop. (The topological character of vortices is implicit in the fact that we count them.) As a result, composite fermions are collective, topological particles. All fluids of composite fermions are thus topological quantum fluids. The most direct consequence of the topological quantization of the vorticity of composite fermions is the effective magnetic field, which is responsible for the FQHE and several other phenomena.

The Chern–Simons field theoretic formulation of composite fermions (Jain, 1989; Lopez and Fradkin, 1991; Halperin *et al.*, 1993) proceeds through a singular gauge transformation defined by

$$\Psi = \prod_{j<k} \left(\frac{z_j - z_k}{|z_j - z_k|} \right)^{2p} \Psi' \qquad (39)$$

under which the eigenvalue problem transforms into

$$H'\Psi' = E\Psi' \qquad (40)$$

$$H' = \left[\frac{1}{2m_b} \sum_i \left(\mathbf{p}_i + \frac{e}{c}\mathbf{A}(\mathbf{r}_i) - \frac{e}{c}\mathbf{a}(\mathbf{r}_i) \right)^2 + V \right] \qquad (41)$$

$$\mathbf{a}(\mathbf{r}_i) = \frac{2p}{2\pi}\phi_0 \sum_j{}' \nabla_i \phi_{ij} \qquad (42)$$

where $\phi_{jk} = i\ln \frac{z_j - z_k}{|z_j - z_k|}$ is the relative angle between the particles j and k. The prime denotes that j = i is to be excluded from the sum. The magnetic field corresponding to $\mathbf{a}(\mathbf{r}_i)$ is given by

$$\mathbf{b}_i = \nabla_i \times \mathbf{a}(\mathbf{r}_i) = 2p\phi_0 \sum_l{}' \delta^2(\mathbf{r}_i - \mathbf{r}_l) \qquad (43)$$

The above transformation thus amounts to attaching a point flux of strength $-2p\phi_0$ to each electron, which is how the composite fermion is modeled in this approach.

To proceed further, one makes a mean-field approximation, which amounts to spreading the point flux on each electron into a uniform magnetic field. Formally, one writes

$$\mathbf{A} - a \equiv \mathbf{A}^* + \delta\mathbf{A} \qquad (44)$$

$$\nabla \times \mathbf{A}^* = B^* \hat{z} \qquad (45)$$

The transformed Hamiltonian then becomes

$$H' = \frac{1}{2m_b} \sum_i \left(\mathbf{p}_i + \frac{e}{c}\mathbf{A}^*(\mathbf{r}_i) \right)^2 + V + V' = H_0^* + V + V' \qquad (46)$$

where V is the Coulomb interaction and V' denotes the terms containing $\delta\mathbf{A}$. The solution to H_0^* describes free composite fermions in an effective magnetic field B^*, and $V + V'$ is the effective interaction between them, which is to be treated perturbatively. When one transforms this problem into the field theoretical Langrangian, it is seen to be equivalent to the familiar Chern–Simons theory. This approach is believed to capture the topological properties of composite fermions, but does not lend itself to a systematic perturbative treatment because of the lack of a small parameter.

1.06.7 Consequences

1.06.7.1 Theory of Fractional Quantum Hall Effect

The FQHE of electrons is a manifestation of the integral QHE for composite fermions. The latter occurs because a gap opens when composite fermions fill an integral number of Λ levels, that is, when $\nu^* = n$. These fillings correspond to electron filling factors given by the sequences:

$$\nu = \frac{n}{2pn \pm 1} \qquad (47)$$

A gap here results in an FQHE plateau at $R_H = h/fe^2$, with

$$f = \frac{n}{2pn \pm 1} \qquad (48)$$

FQHE at f also implies an FQHE at the hole partner:

$$f = 1 - \frac{n}{2pn \pm 1} \qquad (49)$$

These fractions can be obtained by defining the original problem in terms of holes – rather than electrons – in the lowest Landau level, and making composite fermions by binding vortices to holes.

The composite fermion (CF) theory provides a natural explanation of many experimental facts. The fractions f given by Equations (48) and (49) are precisely the prominently observed fractions. Furthermore, the fractions appear in sequences because they are all derived from the integer sequence of the IQHE. Some of the experimental sequences are:

$$f = 1/3, 2/5, 3/7, 4/9, \ldots, 10/21 \qquad (50)$$

$$f = 2/3, 3/5, 4/7, \ldots, 10/19 \qquad (51)$$

$$f = 1/5, 2/9, \ldots, 6/25 \qquad (52)$$

The fractions have odd denominators because the vortex quantum number $2p$ is an even integer.

Different flavors of composite fermions (i.e., composite fermions carrying different numbers of vortices) occur in different filling factor regions. The filling factor range $1/2 \geq \nu \geq 1/3$ is described in terms of composite fermions carrying two vortices; the range $1/3 > \nu \geq 1/5$ in terms of composite fermions carrying four vortices (with B^* antiparallel to B for $1/3 > \nu > 1/4$); and so on. The region $1 > \nu \geq 1/2$ of electrons maps into $0 < \nu \leq 1/2$ of holes as a result of particle–hole symmetry in the lowest Landau level, and can be understood in terms of composite fermions made of holes.

1.06.7.2 Quantitative Tests against Exact Results

Exact results can be obtained for the Hamiltonian in Equation (32) for a small number of particles by a brute force diagonalization, because the dimension of the Fock space in the lowest Landau level is finite for a finite system. Typically, depending on the filling factor, 10–16 electrons can thus be studied. The exact solution gives the eigenenergies and eigenfunctions for all eigenstates. The CF explanation of the FQHE is fully confirmed by comparison to exact results (Dev and Jain, 1992; Wu *et al.*, 1993; Jain and Kamilla, 1998)

A convenient geometry is the spherical geometry, in which electrons move on the surface of a sphere, with a radial magnetic field produced by a magnetic monopole of strength Q at the center, which produces a total magnetic flux of $2Q\phi_0$. The eigenstates are conveniently labeled by the total orbital angular momentum L. The exact spectra at $\nu = 1/3$, $2/5$, and $3/7$ (the first, second, and third rows, respectively) are shown in **Figure 3** (dashes). This figure also shows the predicted energies from the CF theory, without any adjustable parameters, as dots. The ground state corresponds to one, two, or three filled Λ levels of composite fermions, the the lowest energy branch of excitations is a particle–hole pair of composite fermions. The predicted energies agree to within 0.1%, and the overlaps between the exact and the CF wave functions are greater than 99% for the numerical systems. Furthermore, the states in between the special fractions $n/(2pn \pm 1)$ are well described in terms of composite fermions at a nonintegral filling factor.

Monte Carlo methods allow determination of the thermodynamic limits of various experimentally measurable quantities, such as excitation gaps. These are often in 20–50% agreement with experiment, with the discrepancy caused primarily by disorder. An important quantity is collective modes,

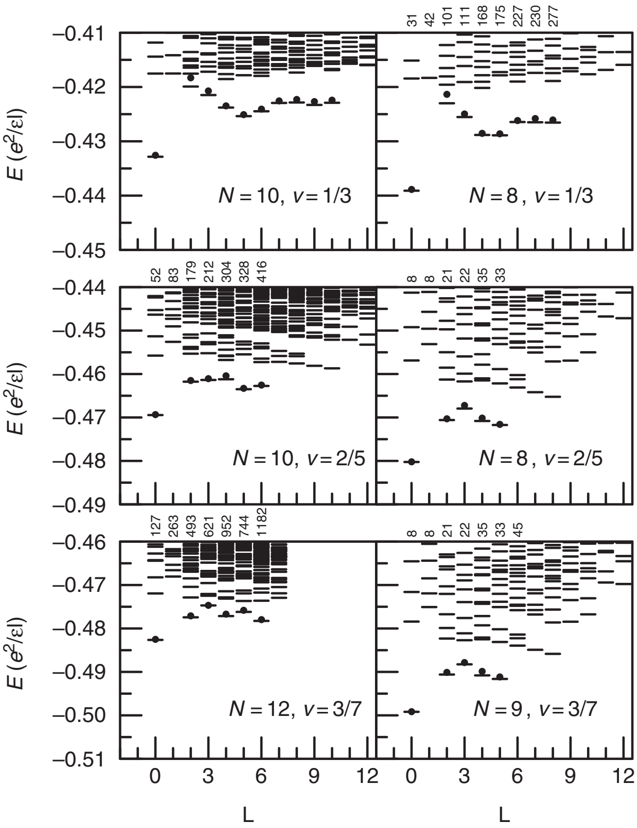

Figure 3 Comparison of spectra obtained from exact diagonalization (dashes) and CF theory (dots). The spectra in the three rows are for $\nu = 1/3$, 2/5, and 3/7, respectively. The x-axis label L is the total orbital angular momentum of the state. From Jain JK and Kamilla RK (1998) Composite fermions: Particles of the lowest Landan level. In: Heinonen O (ed.) *Composite Fermions*, ch. 1. New York: World Scientific.

which are understood as excitons of composite fermions; the experimental measurements of the energies and the dispersions (Pinczuk *et al.*, 1993; Kukushkin *et al.*, 2009) at various fractions such as 1/3, 2/5, 3/7, and 4/9 are in excellent agreement with the predictions of the composite fermion theory (Scarola *et al.*, 2000).

1.06.7.3 Composite Fermion Fermi Sea

No general principle excludes FQHE at even-denominator fractions. Such FQHE has been observed, for example, at $f=5/2$. The CF theory provides a natural explanation for why even denominator FQHE is rare: the model of noninteracting composite fermions produces only odd-denominator fractions; any even-denominator fraction must necessarily owe its existence to weak residual interactions between composite fermions, and, therefore, can be expected to be much weaker.

The nature of state at $\nu=1/2$, the simplest fraction, has been of interest. It is obtained as the $n \to \infty$ limit of the sequence $f=n/(2n+1)$. If the model of noninteracting composite fermions continues to be valid in this limit, their effective magnetic field B^* vanishes and they form a Fermi sea, called the CF Fermi sea (Halperin *et al.*, 1993; Kalmeyer and Zhang, 1992). The lack of FQHE follows because the Fermi sea has no gap to excitations. Several experiments have confirmed the formation of composite fermions in the vicinity of $\nu=1/2$ and of the CF Fermi sea at $\nu=1/2$. These include. Shubnikov de Haas oscillations of composite fermions (Du *et al.*, 1994), linear opening of the CF gap (Du *et al.*, 1993); measurement of the cyclotron resonance of composite fermions (Kukushkin *et al.*, 2007); and measurements of the CF Fermi wave vector through the semiclassical cyclotron orbit of composite fermions (Willet *et al.*, 1993; Kang *et al.*, 1993; Goldman *et al.*, 1994; Smet *et al.*, 1996)

1.06.7.4 Effective Magnetic Field

The reduced effective magnetic field, which is a direct evidence of binding of vortices to electrons and the formation of composite fermions, has been confirmed directly by numerous means. The appearance of fractional sequences that correspond to the integer sequence of the noninteracting fermions and the formation of a Fermi sea at 1/2 filled Landau level are experimental proofs of the effective magnetic field, as is the one-to-one correspondence found between the low energy spectra of the exact solutions of interacting electrons at B (from numerical diagonalization) and the exact solutions of noninteracting electrons at B^*. In addition, several experiments have measured the radius of the cyclotron orbit of the current carrying entities in the vicinity of $\nu=1/2$, and confirmed that it is determined by B^* rather than the applied magnetic field (Willet *et al.*, 1993; Kang *et al.*, 1993; Goldman *et al.*, 1994; Smet *et al.*, 1996).

1.06.7.5 Spin Physics

The spin degree of freedom is frozen when the Zeeman splitting is large compared to the interaction energy. In that limit, all FQHE states in the lowest Landau level are fully spin polarized. One might expect, by application of the Hund's maximum spin rule to electrons in the lowest Landau level, that the state would be fully polarized at all Zeeman energies. That is not the case, however. The actual state is determined by application of the Hund's rule to composite fermions.

In a model that assumes that composite fermions can be taken as nearly independent particles with an effective mass, their physics is straightforward. The IQHE of composite fermions occurs at $\nu^*=n=n_\uparrow+n_\downarrow$, where $n_\uparrow$ is the number of occupied spin-up Λ levels and $n_\downarrow$ is the number of occupied spin-down Λ levels. This fraction corresponds to electron fillings

$$\nu=\frac{n}{2pn\pm1}=\frac{n_\uparrow+n_\downarrow}{2p(n_\uparrow+n_\downarrow)\pm1} \tag{53}$$

The spin polarization of the state is given by

$$\gamma_e=\frac{n_\uparrow-n_\downarrow}{n_\uparrow+n_\downarrow} \tag{54}$$

and wave function for the FQHE state is

$$\Psi_{\frac{n}{2pn+1}}=\mathcal{P}_{\mathrm{LLL}}\Phi_{n_\uparrow,n_\downarrow}\prod_{j<k}\left(z_j-z_k\right)^{2p} \tag{55}$$

or

$$\Psi_{\frac{n}{2pn+1}}=\mathcal{P}_{\mathrm{LLL}}\Phi^*_{n_\uparrow,n_\downarrow}\prod_{j<k}\left(z_j-z_k\right)^{2p} \tag{56}$$

where

$$\Phi_{n_\uparrow,n_\downarrow}=\Phi_{n_\uparrow}\Phi_{n_\downarrow} \tag{57}$$

are wave functions of IQHE states with $n_\uparrow$ spin-up and $n_\downarrow$ spin-down Landau levels occupied.

Thus, while inclusion of spin does not give new fractions, it produces, in general, many states at a given fraction, whose spin polarizations are predicted by the CF theory. Take the example of $\nu = 4/9$, which corresponds to $n = 4$ of composite fermions. There are three possibilities: $(n_\uparrow, n_\downarrow) = (2,2)$, (3,1), and (4,0). The unpolarized state (2,2) is obtained at the lowest Zeeman energies (as application of Hund's rule to composite fermions would predict), the partially polarized state (3,1) at intermediate Zeeman energies, and the fully polarized (4,0) at large Zeeman energies. Upon increasing E_z, the ground-state spin changes discontinuously when the Λ levels of up and down spins cross one another. A quantitative determination of the energies of these states leads to a theoretical determination of the actual phase diagram of the FQHE states as a function of the Zeeman energy (Wu *et al.*, 1993; Park and Jain, 1998)

The Zeeman energy can be varied experimentally by application of a magnetic field parallel to the two-dimensional layer (tilted field experiment), or by changing the density so the FQHE state occurs at different B. The above physics has been fully confirmed by extensive experimentation (Du *et al.*, 1995; Kukushkin *et al.*, 1999).

1.06.7.6 Interacting Composite Fermions: New Fractions

1.06.7.6.1 FQHE of composite fermions

While the fractions $\nu = \dfrac{n}{2pn \pm 1}$ are obtained most immediately in the CF theory, other fractions are not ruled out. To see the physics of the next generation, fractions consider electrons in the filling factor range

$$1/3 < \nu < 2/5 \tag{58}$$

which map into composite fermions in the range

$$1 < \nu^* < 2 \tag{59}$$

The lowest Λ level is fully occupied and the second one partially occupied. (We take composite fermions to be fully spin-polarized.) No FQHE would result in this region for noninteracting composite fermions, just as non-interacting electrons in the partially filled second Landau level do not exhibit any FQHE. The weak residual interaction between composite fermions, however, can possibly cause a gap to open at certain filling factors. A natural expectation is that the strongest new CF fractions are

$$f^* = 1 + \frac{1}{3} \text{ and } f^* = 1 + \frac{2}{3} \tag{60}$$

which produce new electron fractions

$$f = \frac{4}{11} \text{ and } f = \frac{5}{13} \tag{61}$$

between the familiar fractions 1/3 and 2/5, consistent with experimental observations (Pan *et al.*, 2003). Here, composite fermions in the partially filled second Λ level capture two more vortices to turn into composite fermions of a different flavor, which condense into their own Λ levels, thereby opening a gap and producing quantum Hall effect. This physics has been confirmed by exact diagonalization studies (Chang and Jain, 2004; Wójs *et al.*, 2004), as shown in **Table 1**. Many more FQHE states can similarly be constructed.

1.06.7.6.2 Pairing

As noted above, FQHE has been observed (Willet *et al.*, 1987) at $\nu = 5/2$. Writing

$$\frac{5}{2} = 2 + \frac{1}{2} \tag{62}$$

and treating the lowest filled Landau level as inert (which, counting the spin degree of freedom, accounts for the 2 on the right hand side) shows that 5/2 corresponds to a filling of 1/2 in the second Landau level. Thus, half-filled second Landau level behaves qualitatively differently from the half-filled lowest Landau level.

The currently most promising scenario for the explanation of the 5/2 FQHE is that composite fermions form a p-wave paired state, described by a Pfaffian wave function (Moore and Read, 1991):

$$\Psi^{\mathrm{Pf}}_{1/2} = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^2 \exp\left[-\frac{1}{4}\sum_k |z_k|^2\right] \tag{63}$$

Table 1 Energies of two wave functions for 4/11: the exact Coulomb state $\Psi^{\mathrm{ex}}_{4/11}$ and the trial wave function $\Psi^{\mathrm{tr}}_{4/11} = \mathcal{P}_{LLL}\Psi_{4/3}\Phi_1^2$

E^{ex}	E^{tr}	Overlap
−0.441214	−0.44088(4)	0.99

Results are for $N = 12$ particles. The overlap is defined as $\langle \Psi^{\mathrm{tr}} | \Psi^{(0)} \rangle / \sqrt{\langle \Psi^{(0)} | \Psi^{(0)} \rangle \langle \Psi^{\mathrm{tr}} | \Psi^{\mathrm{tr}} \rangle}$, where $\Psi^{(0)}$ is a state that is very close to the exact ground state (obtained by a method called CF diagonalization). From Chang C-C and Jain JK (2004) Microscopic origin of the next-generation fractional quantum Hall effect. *Physical Review Letters* 92: 196806.

where Pf stands for Pfaffian. The Pfaffian of an antisymmetric matrix M is defined, apart from an overall normalization factor, as

$$\mathrm{Pf}\ M_{ij} = A(M_{12}M_{34}\cdots M_{N-1,N}) \qquad (64)$$

where A is the antisymmetrization operator. The usual Bardeen–Cooper–Schrieffer wave function for fully polarized electrons can be written as

$$\Psi_{\mathrm{BCS}} = \mathbf{A}[\phi_0(\mathbf{r}_1,\mathbf{r}_2)\phi_0(\mathbf{r}_3,\mathbf{r}_4)\cdots\phi_0(\mathbf{r}_{N-1},\mathbf{r}_N)] \qquad (65)$$

which is a Pfaffian. (The fully symmetric spin part is not shown explicitly.) Analogously, $\mathrm{Pf} = \dfrac{1}{z_i - z_j}$ describes a p-wave pairing of electrons (p-wave because the system is fully spin-polarized), and $\Psi_{1/2}^{\mathrm{Pf}}$ is interpreted as the p-wave paired state of composite fermions carrying two vortices. The Pfaffian wave function was originally motivated in a conformal field theory approach, which also suggests that its quasiparticles obey so-called nonabelian braiding statistics (Moore and Read, 1991).

1.06.7.7 Fractional Charge

The excitations of an FQHE state are excited composite fermions. A sole composite fermion in an otherwise empty Λ level is often called a CF quasiparticle, and a missing composite fermion from an otherwise filled Λ level is called a CF quasihole. This description has been confirmed extensively in exact diagonalization studies. In a localized representation, these represent a localized excess or deficiency of charge relative to the uniform ground state. The charge excess associated with a CF quasiparticle is the sum of the charge of an electron ($-e$) and the charge of $2p$ vortices:

$$-e^* = -e + 2pe_{\mathrm{v}} \qquad (66)$$

where e_{v} is the charge of a single vortex. The charge of the vortex can be shown to be

$$e_{\mathrm{v}} = \nu e \qquad (67)$$

The local charge of a CF-quasiparticle at $\nu = n/(2pn+1)$ therefore has the fractional value

$$-e^* = -\frac{e}{2pn+1} \qquad (68)$$

The topological quantization of the vorticity implies that this charge is precisely quantized. The fractional charge was originally predicted theoretically (Laughlin, 1983), and has been measured in shot noise experiments (de Picciotto *et al.*, 1997).

1.06.8 Open Issues

The FQHE and other phenomena in the lowest Landau level are well described by the CF theory. However, the CF theory in its simplest form is not fully satisfactory for the second Landau-level physics. One difference between the two Landau levels is indicated by the observation that while there is no FQHE at $\nu = 1/2$, an FQHE occurs at $\nu = 5/2$. Further, far fewer fractions are observed in the second Landau level, and they are much more delicate than those in the lowest Landau level. While other fractions in the second Landau level also have odd denominators (such as $2+1/3$ and $2+2/5$), their wave functions are rather different than those of the corresponding fractions in the lowest Landau level. Several imaginative approaches are currently being pursued, which have excitations with exotic character (e.g., nonabelian braiding properties). (See Chapters 1.03, 1.05 and 1.07).

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