The Euler Equations in 1D

The Euler equations for 1D flow:

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u(E + p/\rho) \end{pmatrix} = 0$$

where $E = e + u^2/2$

Define $h = e + p/\rho; \quad H = h + u^2/2 = E + p/\rho;$

Ideal Gas: $p = \rho R T; \quad e = e(T); \quad c_v = de/dT$

$$h = h(T); \quad c_p = dh/dT$$

$$R = c_p - c_v; \quad \gamma = c_p/c_v; \quad p = (\gamma - 1)\rho e$$

The Euler Equations in 1D

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or

$$\frac{\partial}{\partial t}\mathbf{f}+\frac{\partial}{\partial x}\mathbf{F}=0$$

where

$$\mathbf{f}=\begin{pmatrix}\rho \\ \rho u \\ \rho E\end{pmatrix}; \qquad \mathbf{F}=\begin{pmatrix}\rho u \\ \rho u^2+p \\ \rho u(E+p/\rho)\end{pmatrix}$$

The Euler Equations in 1D

Predictor-corrector method: Limiting the variables

Predictor step $$f_j^{n+1/2} = f_j^n - \frac{\Delta t}{2h}\left(F_{j+1/2}^n - F_{j-1/2}^n\right)$$

Variables $$f_{j+1/2}^L = f_j^{n+1/2} + \frac{1}{2}\Psi^L\left(f_j^{n+1/2} - f_{j-1}^{n+1/2}\right)$$

$$f_{j+1/2}^R = f_{j+1}^{n+1/2} - \frac{1}{2}\Psi^R\left(f_{j+1}^{n+1/2} - f_j^{n+1/2}\right)$$

Find: $$F_{j+1/2}^{n+1/2} = F\left(\left(f^L\right)_{j+1/2}^{n+1/2}, \left(f^R\right)_{j+1/2}^{n+1/2}\right)$$

Final step $$f_j^{n+1} = f_j^n - \frac{\Delta t}{h}\left(F_{j+1/2}^{n+1/2} - F_{j-1/2}^{n+1/2}\right)$$

The Euler Equations in 1D

The Euler equations can be solved using the flux limited high order methods described earlier by finding the fluxes using solutions to the Riemann problem

$$F_{j+1/2}^{n+1/2} = F\left(\left(f^L\right)_{j+1/2}^{n+1/2}, \left(f^R\right)_{j+1/2}^{n+1/2}\right)$$

In principle we can solve this problem, the Riemann problem, exactly by assuming constant states and then integrate the fluxes over the time step. In the last lecture we did so for the case when the fluids are initially at rest.

The Euler Equations in 1D

Although the Riemann problem can be solved for the general case when the fluids are not stationary, the solution is expensive (involving solving a nonlinear equation for the pressure ratio across the shock) and the only information that we need from the solution is the flux across the cell boundary.

Therefore, usually we use approximate Riemann solvers. Different approximations are possible but generally the result is a complex process. Here we will only outline it briefly

The Euler Equations in 1D

The Roe approximate Riemann solver was one of the first method to compute the fluxes in a “simpler” way.

It is based on approximating the Euler equation by a linear equation

$$\frac{\partial \mathbf{f}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{f}}{\partial x} = 0$$

whose fluxes can be found analytically. The linearization is done in such a way that the correct wave speed is preserved. The jump in the solution across a shock can be written in terms of the eigenvectors:

$$\mathbf{f}_R - \mathbf{f}_L = \sum_{p=1}^{3} \alpha_p R_p$$

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The Euler Equations in 1D

Roe introduced the following matrix A

$$A=\begin{bmatrix} 0 & 1 & 0 \\ \dfrac{\gamma-3}{2}\bar{u}^2 & (3-\gamma)\bar{u} & \gamma-1 \\ \dfrac{\gamma-1}{2}\bar{u}^3-\bar{u}\bar{H} & \bar{H}-(\gamma-1)\bar{u}^2 & \gamma\bar{u} \end{bmatrix}$$

where the average variables are defined by

$$\bar{u}=\frac{\sqrt{\rho_L}u_L+\sqrt{\rho_R}u_R}{\sqrt{\rho_L}+\sqrt{\rho_R}};\quad \bar{H}=\frac{\sqrt{\rho_L}H_L+\sqrt{\rho_R}H_R}{\sqrt{\rho_L}+\sqrt{\rho_R}}$$

The Euler Equations in 1D

The fluxes are given by

$$F_{i+1/2} = \frac{1}{2}\left(F^L + F^R\right) - \frac{1}{2}\sum_{p=1}^{3}\left|\lambda_p\right|\alpha_p R_p$$

where the eigenvalues are the same as for the original

$$\lambda_1 = \left|\bar{u} - \bar{c}\right|; \quad \lambda_2 = \left|\bar{c}\right|; \quad \lambda_3 = \left|\bar{u} + \bar{c}\right|;$$

and the scaled eigenvectors are given by

$$R_1 = \begin{pmatrix} 1 \\ \bar{u} - \bar{c} \\ \bar{H} - \bar{u}\,\bar{c} \end{pmatrix}; \quad R_1 = \begin{pmatrix} 1 \\ \bar{u} \\ \frac{1}{2}\bar{u}^2 \end{pmatrix}; \quad R_1 = \begin{pmatrix} 1 \\ \bar{u} + \bar{c} \\ \bar{H} + \bar{u}\,\bar{c} \end{pmatrix};$$

The Euler Equations in 1D

The alphas are given by

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \frac{\bar{u}}{4\bar{c}}\left(2+(\gamma-1)\frac{\bar{u}}{\bar{c}}\right) & -\frac{1}{2\bar{c}}\left(1+(\gamma-1)\frac{\bar{u}}{\bar{c}}\right) & \frac{\gamma-1}{2}\frac{1}{\bar{c}^2} \\ 1-\frac{\gamma-1}{2}\frac{\bar{u}^2}{\bar{c}^2} & (\gamma-1)\frac{\bar{u}}{\bar{c}^2} & -(\gamma-1)\frac{\bar{u}}{\bar{c}^2} \\ -\frac{\bar{u}}{4\bar{c}}\left(2-(\gamma-1)\frac{\bar{u}}{\bar{c}}\right) & \frac{1}{2\bar{c}}\left(1-(\gamma-1)\frac{\bar{u}}{\bar{c}}\right) & \frac{\gamma-1}{2}\frac{1}{\bar{c}^2} \end{bmatrix} \begin{bmatrix} \Delta\rho_{j+1/2} \\ \Delta\rho u_{j+1/2} \\ \Delta\rho E_{j+1/2} \end{bmatrix}$$

Where the jumps across the cell boundary are:

$$\Delta\rho_{j+1/2} = \rho^R_{j+1/2} - \rho^L_{j+1/2}; \quad \Delta\rho u_{j+1/2} = \rho u^R_{j+1/2} - \rho u^L_{j+1/2}; \quad \Delta\rho E_{j+1/2} = \rho E^R_{j+1/2} - \rho E^L_{j+1/2}$$

and (again)

$$\bar{u} = \frac{\sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R}{\sqrt{\rho_L}+\sqrt{\rho_R}}; \quad \bar{H} = \frac{\sqrt{\rho_L}H_L + \sqrt{\rho_R}H_R}{\sqrt{\rho_L}+\sqrt{\rho_R}}; \quad \bar{c}^2 = (\gamma-1)\left(\bar{H}-\frac{1}{2}\bar{u}^2\right)$$

The Euler Equations in 1D

The fluxes are given by

$$F^1_{i+1/2} = \frac{1}{2}\left(\left(\rho u\right)^R_{j+1/2} + \left(\rho u\right)^L_{j+1/2}\right) - \frac{1}{2}\lambda_1\alpha_1 - \frac{1}{2}\lambda_2\alpha_2 - \frac{1}{2}\lambda_3\alpha_3$$

$$F^2_{i+1/2} = \frac{1}{2}\left(\left(\gamma-1\right)\left(\rho E\right)^R_{j+1/2} - \frac{1}{2}\left(\gamma-3\right)\left(\left(\rho u\right)^R_{j+1/2}\right)^2 \Big/ \left(\rho\right)^R_{j+1/2} + \left(\gamma-1\right)\left(\rho E\right)^L_{j+1/2} - \frac{1}{2}\left(\gamma-3\right)\left(\left(\rho u\right)^L_{j+1/2}\right)^2 \Big/ \left(\rho\right)^L_{j+1/2}\right)$$

$$- \frac{1}{2}\lambda_1\alpha_1\left(\bar{u} - \bar{c}\right) - \frac{1}{2}\lambda_2\alpha_2\bar{u} - \frac{1}{2}\lambda_3\alpha_3\left(\bar{u} + \bar{c}\right)$$

$$F^3_{i+1/2} = \frac{1}{2}\left(\left(\rho u\right)^R_{j+1/2} H^R + \left(\rho u\right)^L_{j+1/2} H^L\right)$$

$$- \frac{1}{2}\lambda_1\alpha_1\left(\bar{h} - \bar{u}\,\bar{c}\right) - \frac{1}{2}\lambda_2\alpha_2\bar{u}^2 - \frac{1}{2}\lambda_3\alpha_3\left(\bar{h} + \bar{u}\,\bar{c}\right)$$

The Euler Equations in 1D

Time Integration

2rd order Runge-Kutta

$$f^{(1)} = f^{n} + \Delta t L\left(f^{n}\right)$$

$$f^{(2)} = f^{(1)} + \Delta t L\left(f^{(1)}\right)$$

$$f^{n+1} = \frac{1}{2}\left(f^{n} + f^{(2)}\right)$$

3rd order Runge-Kutta

$$f^{(1)} = f^{n} + \Delta t L\left(f^{n}\right)$$

$$f^{(2)} = \frac{3}{4}f^{n} + \frac{1}{4}f^{(1)} + \frac{1}{4}\Delta t L\left(f^{(1)}\right)$$

$$f^{n+1} = \frac{1}{3}f^{n} + \frac{2}{3}f^{(2)} + \frac{2}{3}\Delta t L\left(f^{(2)}\right)$$

4rd order Runge-Kutta

$$f^{(0)} = f^{n}$$

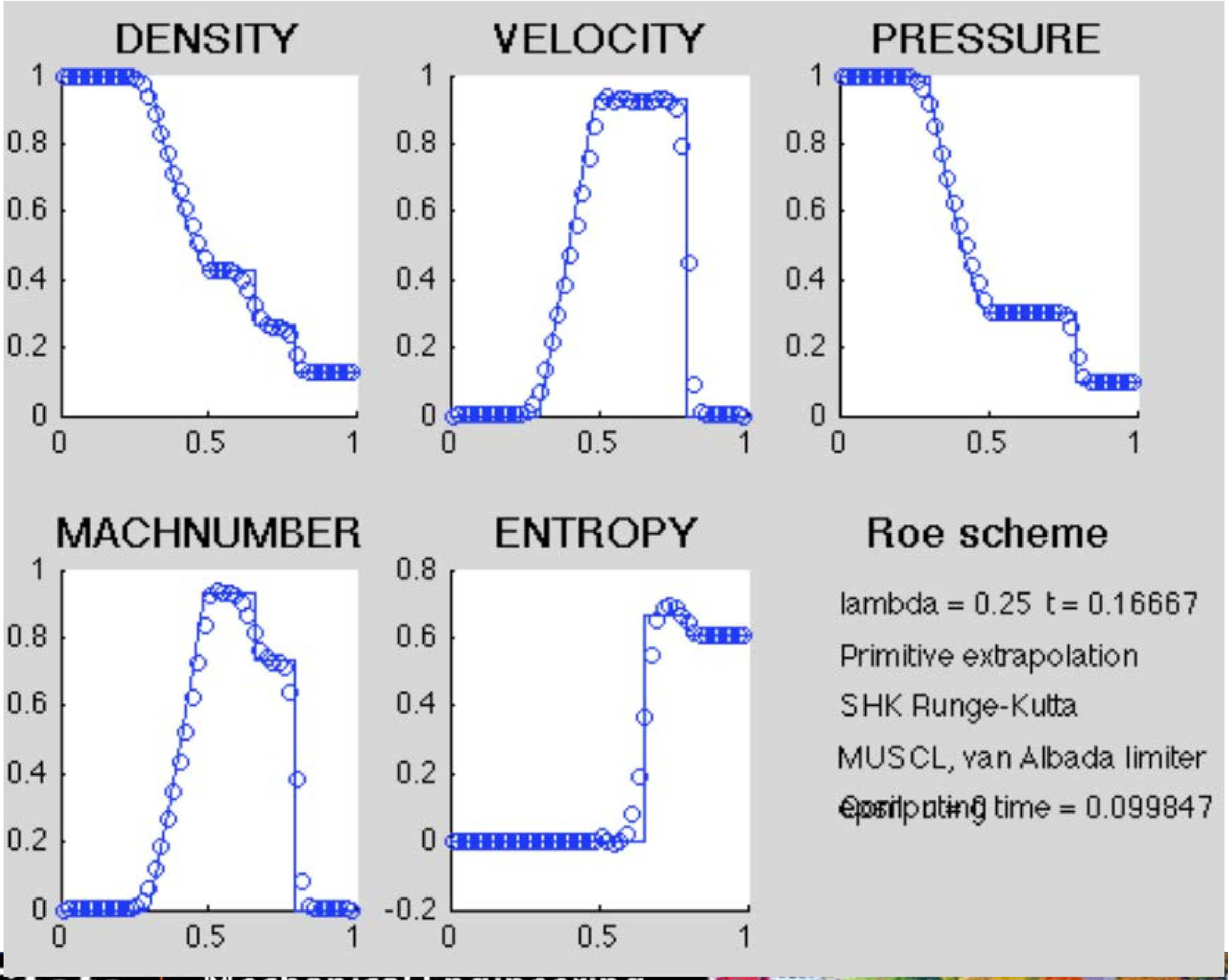
$$f^{(1)} = f^{(0)} + \frac{1}{4}\Delta t L\left(f^{n}\right)$$

$$f^{(2)} = f^{(0)} + \frac{1}{3}\Delta t L\left(f^{(1)}\right)$$

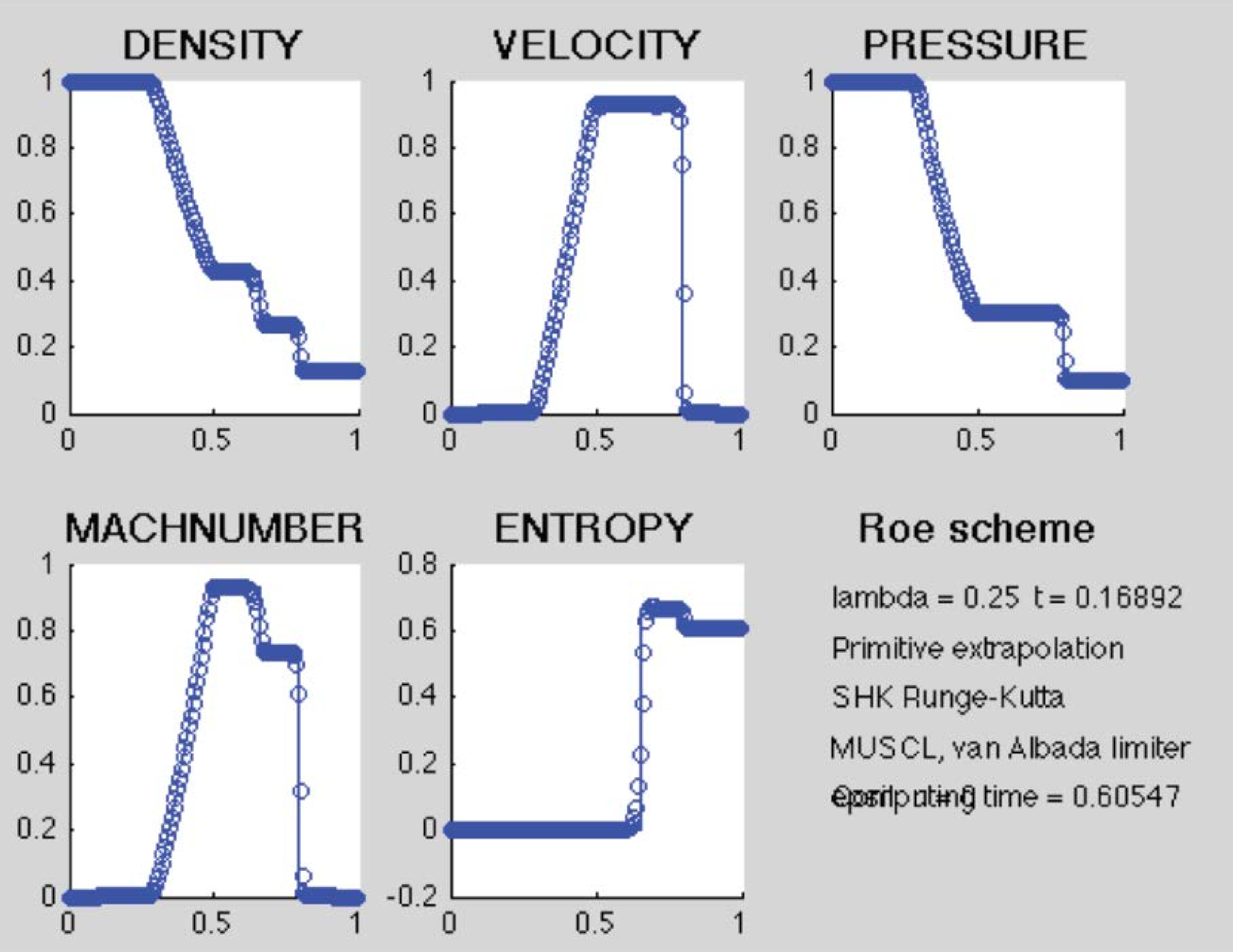
$$f^{(3)} = f^{(0)} + \frac{1}{2}\Delta t L\left(f^{(2)}\right)$$

$$f^{n+1} = f^{(0)} + \Delta t L\left(f^{(2)}\right)$$

The Euler Equations in 1D



The Euler Equations in 1D



The Euler Equations in 1D

The TVD property gives monotonic behavior for 1D scalar conservation laws but generally not for nonlinear systems or multidimensional flows. However, generally it is found that schemes that satisfy the TVD property when applied to 1D scalar equation do well for systems. When used for multidimensional flows by splitting, they are also generally found to be well behaved.

The Euler Equations in 1D

Several variants of similar methods have been proposed.

One of the better known is the AUSM (Advection Upstream Splitting Method) family of methods, originally due to Liou and Steffen (1993), where the fluxes are split into convective and a pressure part. The method and its derivatives have been used widely for many aerospace and other applications, including multiphase flows.