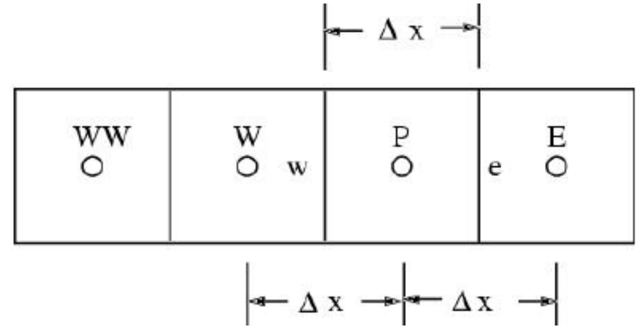
- Higher Order methods for hyperbolic systems
- Monotonicity as a measure for hyperbolic systems
- Flux limiters for bounded solutions

Higher Order Schemes

- Neither Upwind Difference Scheme (UDS) nor Central Difference Scheme (CDS) are satisfactory
- Upwind too dissipative, CDS is dispersive, explicit CDS is unstable!
- Variety of higher-order schemes
    - FROMM
    - Beam Warming
    - QUICK

Taylor Series Basis



- First order UDS

$$\phi_e = \phi_P$$

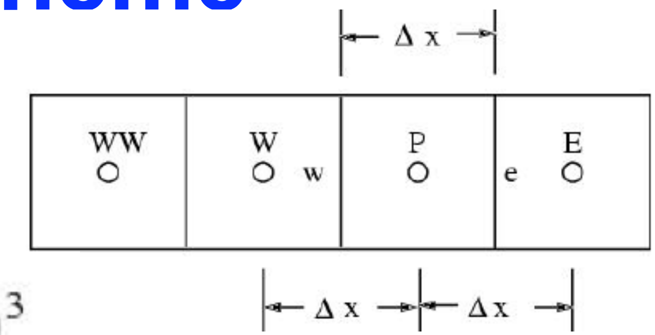
- think of this as truncation of Taylor Series

$$\phi(x) = \phi_P + (x - x_P)\frac{\partial \phi}{\partial x} + \frac{(x - x_P)^2}{2!}\frac{\partial^2 \phi}{\partial x^2} + O(\Delta x)^3$$

- What if we truncated to higher order?

- Note we are still using upwinded differencing!

Second-Order Scheme



- Taylor Series about P

$$\phi(x) = \phi_P + (x - x_P)\frac{\partial \phi}{\partial x} + \frac{(x - x_P)^2}{2!}\frac{\partial^2 \phi}{\partial x^2} + O(\Delta x)^3$$

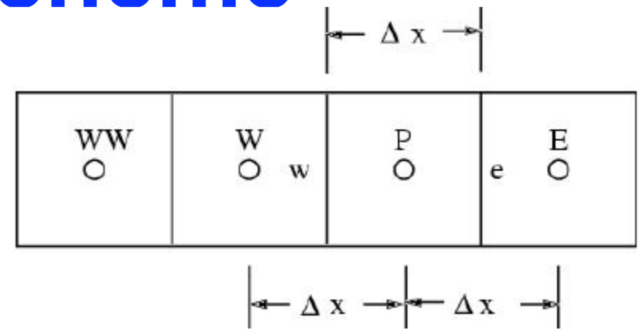
- Truncate series after second term

$$\phi_e = \phi_P + \frac{\Delta x}{2}\frac{\partial \phi}{\partial x}$$

$\boxed{\text{Truncation error : } O(\Delta x^2)}$

- Several ways to approximate gradient
  - can write gradient @ P using forward, backward or central difference
  - gradient @ P should at least be order (Δx)

Basis of FROMM Scheme



- Central for the derivative @ P

$$\frac{\partial \phi}{\partial x} = \frac{\phi_E - \phi_W}{2\Delta x}$$

Truncation error : $O(\Delta x^2)$

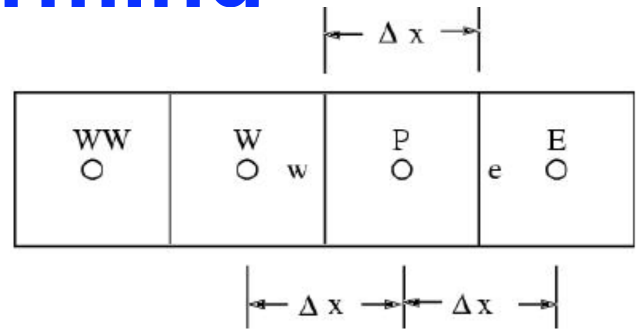
- Thus

$$\phi_e = \phi_P + \frac{\Delta x}{2}\frac{\partial \phi}{\partial x} \qquad \phi_e = \phi_P + \frac{(\phi_E - \phi_W)}{4}$$

- Add and subtract $\phi_P/4$

$$\phi_e = \phi_P + \frac{(\phi_P - \phi_W)}{4} + \frac{(\phi_E - \phi_P)}{4}$$

Basis of Beam-Warming



- Second order scheme. Upwinded gradient

$$\frac{\partial \phi}{\partial x} = \frac{\phi_P - \phi_W}{\Delta x}$$

Truncation error : $O(\Delta x)$

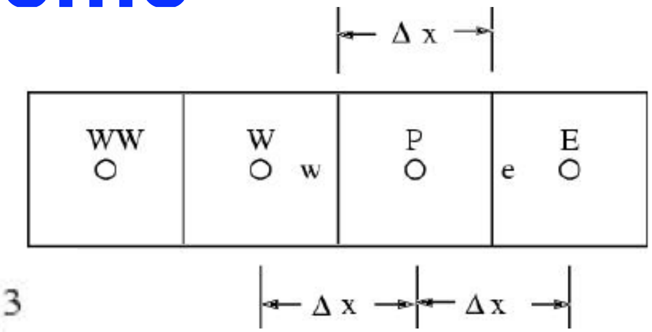
- Thus

$$\phi_e = \phi_P + \frac{\Delta x}{2}\frac{\partial \phi}{\partial x} \qquad \frac{\partial \phi}{\partial x} = \frac{\phi_P - \phi_W}{\Delta x}$$

- Combining

$$\phi_e = \phi_P + \frac{(\phi_P - \phi_W)}{2}$$

Third-Order Scheme



- Taylor Series about P

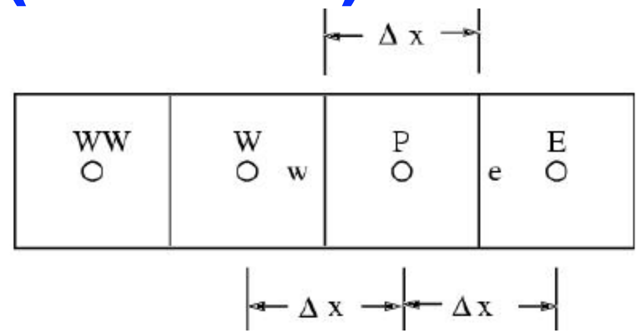
$$\phi(x) = \phi_P + (x - x_P)\frac{\partial \phi}{\partial x} + \frac{(x - x_P)^2}{2!}\frac{\partial^2 \phi}{\partial x^2} + O(\Delta x)^3$$

- Need to write first gradient @ P at least second-order accurate

- Need to write second gradient @ P at least first order accurate

Quadratic Upwind Interpolation for Convective Kinetics (QUICK)

- QUICK (central for first and second derivatives



$$\frac{\partial \phi}{\partial x} = \frac{(\phi_E - \phi_W)}{2\Delta x} + O(\Delta x^2)$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{(\phi_E + \phi_W - 2\phi_P)}{(\Delta x)^2} + O(\Delta x^2)$$

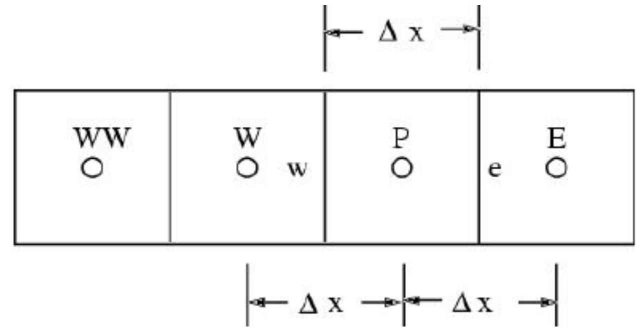
- Combining

$$\phi_e = \phi_P + \frac{(\phi_E - \phi_W)}{4} + \frac{(\phi_E + \phi_W - 2\phi_P)}{8}$$

$O(\Delta x^3)$ accurate

QUICK

- Rearranging



$$\phi_e = \frac{(\phi_E + \phi_P)}{2} - \frac{(\phi_E + \phi_W - 2\phi_P)}{8}$$

Central difference (linear)

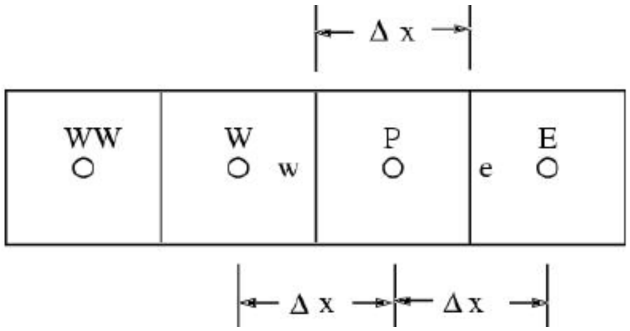
Curvature term

$$\phi_e = \frac{1}{2}(\phi_E + \phi_P) - C(\phi_E + \phi_W - 2\phi_P)$$

Curvature factor
C=1/8

ALL Schemes Together

- Combined representation



$$\phi_e = \phi_P + \frac{(1-\kappa)}{4}(\phi_P - \phi_W) + \frac{(1+\kappa)}{4}(\phi_E - \phi_P)$$

κ = -1 Beam Warming scheme

κ = 0 Fromm scheme

κ = 1/2 QUICK

κ = 1 Central difference scheme

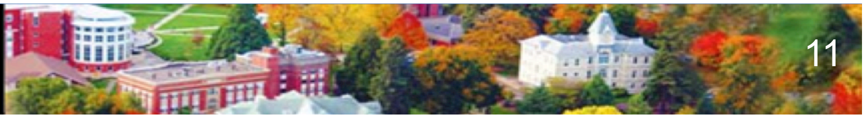
Higher-Order Schemes

- The schemes we saw provide higher-order accuracy; however, if used with explicit Euler (Forward Difference) in time, are unconditionally unstable!

- For steady convection problems, these schemes provide spatial oscillations

- Ways to counter these issues

  - Use implicit schemes for time discretization
  - Add extra terms from model equations to counter negative diffusion
  - Add even higher-order dissipation terms



Added Dissipation Schemes

- Want to keep truncation error (Δx^2) if using second-order schemes

- Add fourth-order term as added artificial dissipation

$$(\text{constant})\Delta x^3 \frac{\partial^4 \phi}{\partial x^4}$$

Note that artificial term is $O(\Delta x^3)$ – preserves $O(\Delta x^2)$ error

Added Dissipation Schemes

- Corresponding face value for CDS

$$\phi_e = \frac{\phi_P + \phi_E}{2} + \varepsilon_e^{(4)}(\phi_{EE} - 3\phi_E + 3\phi_P - \phi_W)$$

- Near shocks and discontinuities need to added more dissipation

$$(\text{constant})\Delta x^2 \frac{\partial^2 \phi}{\partial x^2}$$

Destroys second-order accuracy of scheme – reduces to first-order near shocks

Monotonicity

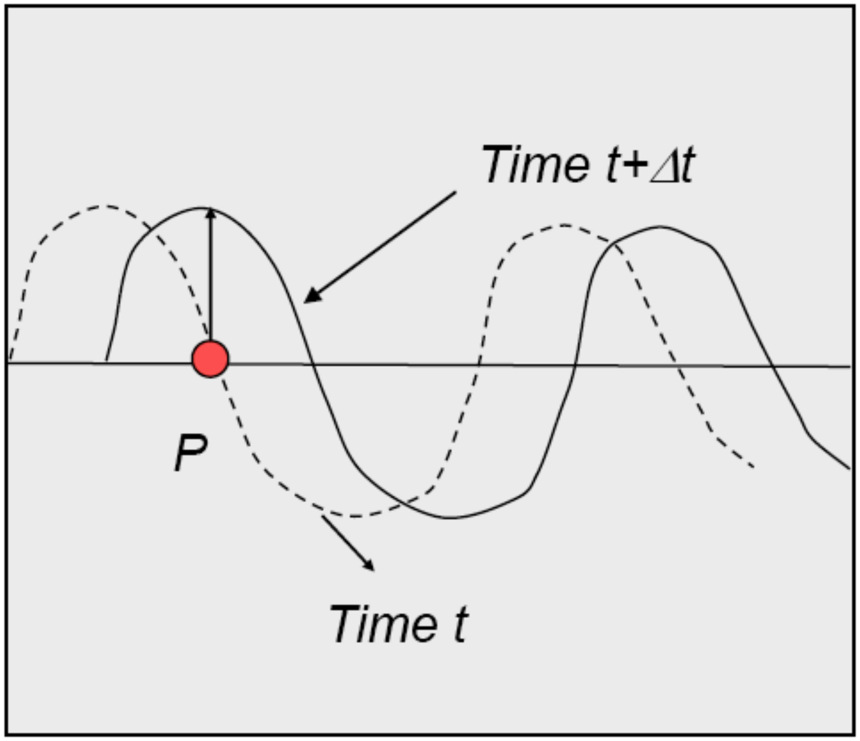
- Symmetry of discretization and positivity of coeffs. was used as measure for physically correct behavior of schemes for parabolic and elliptic problems

$$a_P = \sum_{nb} a_{nb}; \quad a_{nb} > 0$$

- This ensures bounded solutions

- For hyperbolic systems, boundedness is not a good (or sufficient) measure

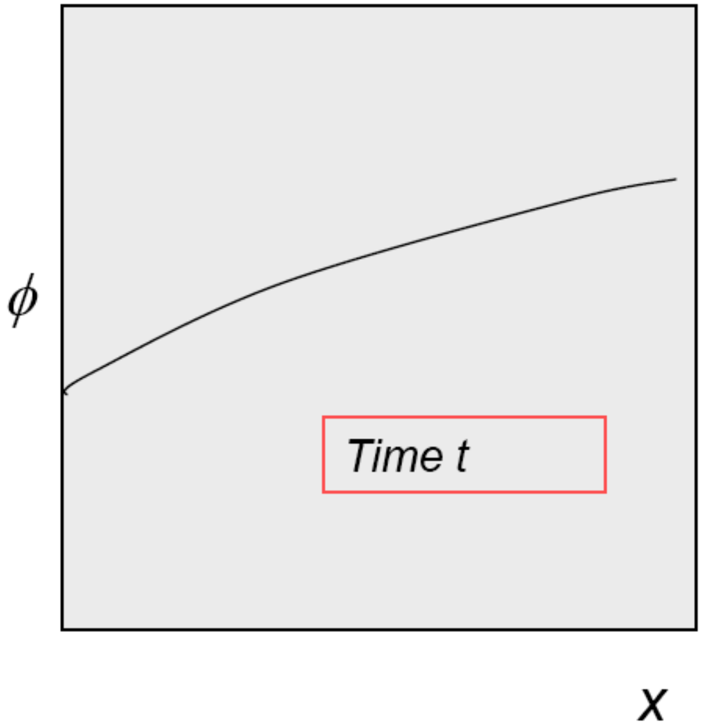
Wave Transport

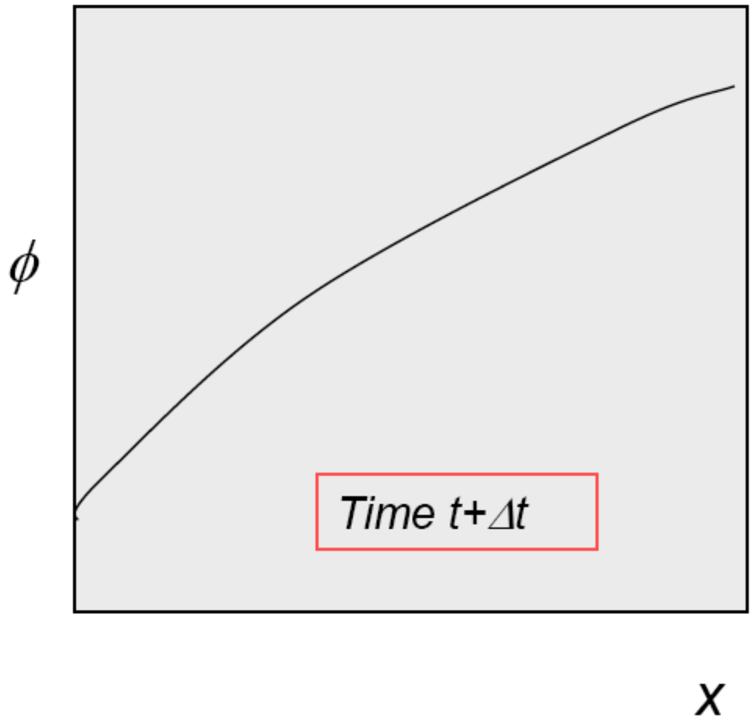


- As wave passes through point P, value @ P can exceed both the old time-level value and also the neighbor values!

- What properties should we impose on our numerical scheme for proper solution of the hyperbolic system?

Monotonicity Preservation





If $\phi(x,t)$ is monotonic in x, then $\phi(x,t+\Delta t)$ is also monotonic in x

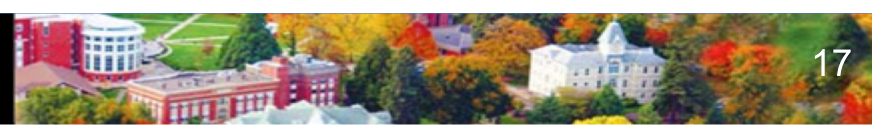
No new maxima or minima are created

Godunov Theorem

- Sergei Godunov

- A consistent numerical scheme for the **linear wave equation** that is monotonicity preserving can at most be first order accurate

- key word is “linear”. Can come up with non-linear higher-order schemes



Total Variation

-For a hyperbolic system it can shown that the total variation (TV) does not increase with time

$$TV(\phi) = \int \left| \frac{\partial \phi}{\partial x} \right| dx$$

- For a traveling wave, the total variation will actually remain constant in the absence of diffusion and source terms in an infinite domain

- In presence of a diffusion term (and in absence of a source term), TV cannot increase

Total Variation

- TV is measure of wiggliness

$$TV(\phi) = \int \left| \frac{\partial \phi}{\partial x} \right| dx$$

- High frequency sine wave has more wiggliness and total variation than a low frequency one

- Corresponding discrete concept

$$TV = \sum |\phi_P - \phi_W|$$

Total Variation Diminishing (TVD) Scheme

- A numerical scheme is called total variation diminishing (TVD) if

$$TV(\phi) \leq TV(\phi^0)$$

Current time value

Old time value

A TVD Scheme is also monotonicity preserving (can be shown)

Local Extremum Diminishing (LED) Scheme

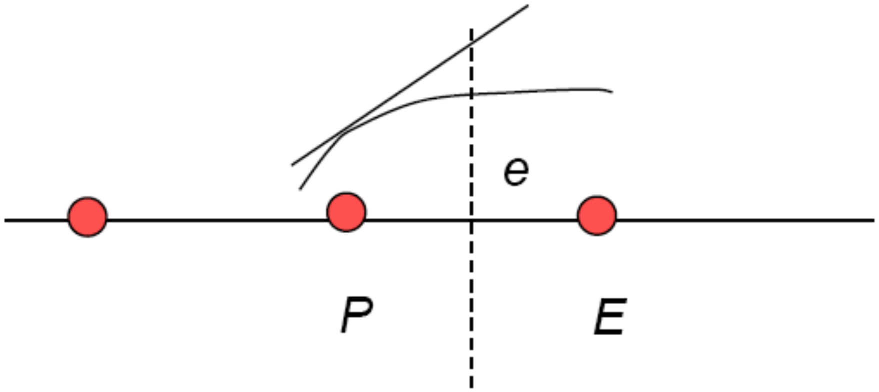
- TVD is a “weak” condition. Since total variation decreases, it is still possible to create locally small wiggles even though TV decreases overall

- A more stringent concept is that of LED

    - Create no new extrema
    - do not amplify existing extrema
    - LED schemes are TVD
    - this is done by looking at local gradients and “limiting” fluxes to ensure LED property
    - LED schemes can suffer from “clipping errors”

LED Example

-



- A second order scheme would obtain face value as

$$\phi_e = \phi_P + \frac{\Delta x}{2}\left(\frac{\partial \phi}{\partial x}\right)_P$$

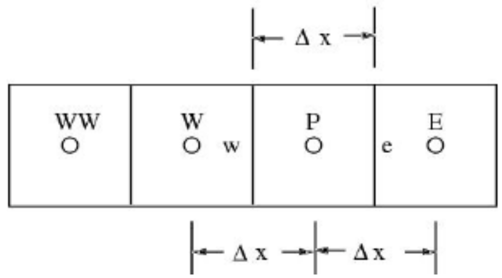
- this can cause local “overshoot”

LED with Limiters

-To avoid local creation of extrema, we employ a limiting function called “**limiter**” that ensures non-wiggly solution

- the face value is found as

$$\phi_e = \phi_P + \Psi(r_e)\frac{\Delta x}{2}\left(\frac{\partial \phi}{\partial x}\right)_P$$



- using an upwind evaluation

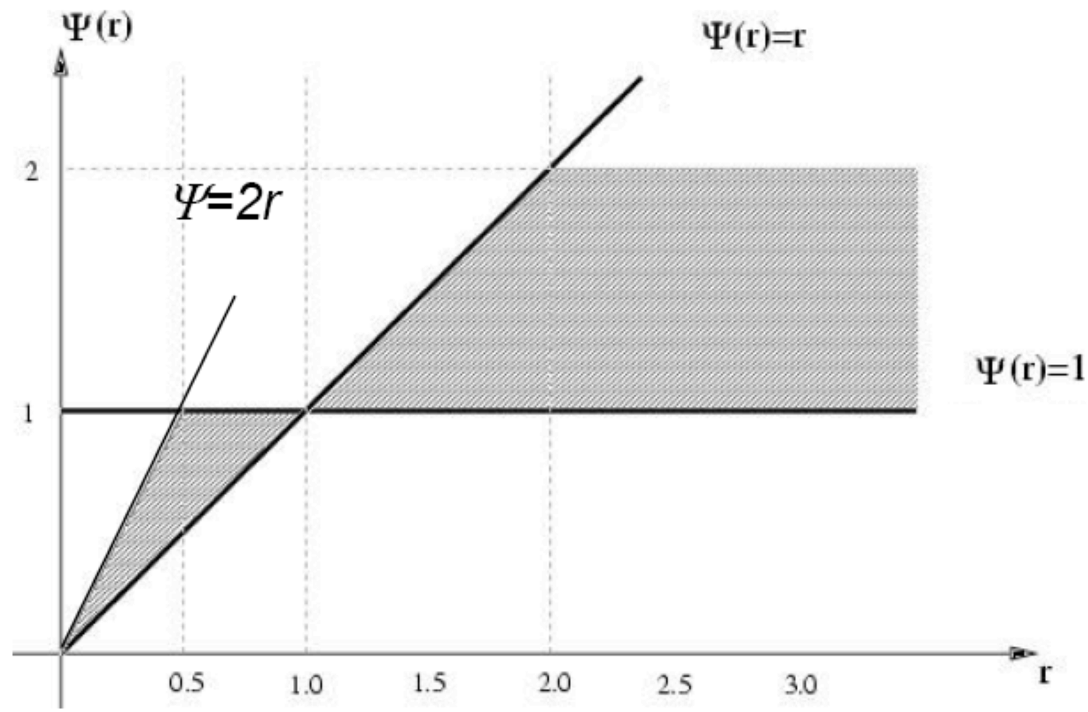
$$\phi_e = \phi_P + \Psi(r_e)\frac{\Delta x}{2}\frac{(\phi_P - \phi_W)}{\Delta x} \qquad r_e = \frac{\phi_E - \phi_P}{\phi_P - \phi_W}$$

Limiters

- limiter function chooses gradient adaptively to avoid creating new extrema

- to be LED scheme, it is possible to show that the limiter should occupy the gray region

- also it is desirable to have it pass through (1,1)



$$r_e = \frac{\phi_E - \phi_P}{\phi_P - \phi_W}$$

Downwind cell gradient

Upwind cell gradient