- LED and TVD Schemes
- Flux limiters for bounded solutions
- What is the flux limiter doing?

Local Extremum Diminishing (LED) Scheme

- TVD is a “weak” condition. Since total variation decreases, it is still possible to create locally small wiggles even though TV decreases overall

- A more stringent concept is that of LED

  - Create no new extrema
  - do not amplify existing extrema
  - LED schemes are TVD
  - this is done by looking at local gradients and “limiting” fluxes to ensure LED property
  - LED schemes can suffer from “clipping errors”

LED Example

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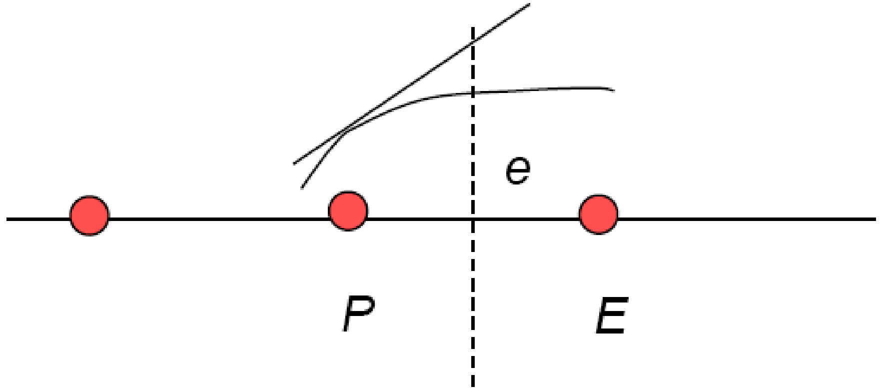


- A second order scheme would obtain face value as

$$\phi_e = \phi_P + \frac{\Delta x}{2}\left(\frac{\partial \phi}{\partial x}\right)_P$$

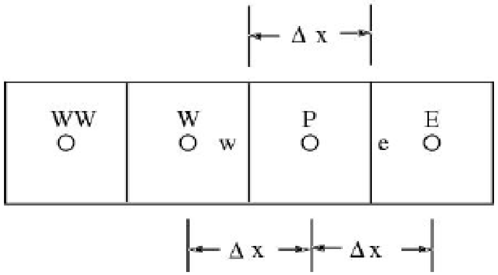
- this can cause local “overshoot”

LED with Limiters

-To avoid local creation of extrema, we employ a limiting function called "**limiter**" that ensures non-wiggly solution

- the face value is found as

$$\phi_e = \phi_P + \Psi(r_e)\frac{\Delta x}{2}\left(\frac{\partial \phi}{\partial x}\right)_P$$



- using an upwind evaluation

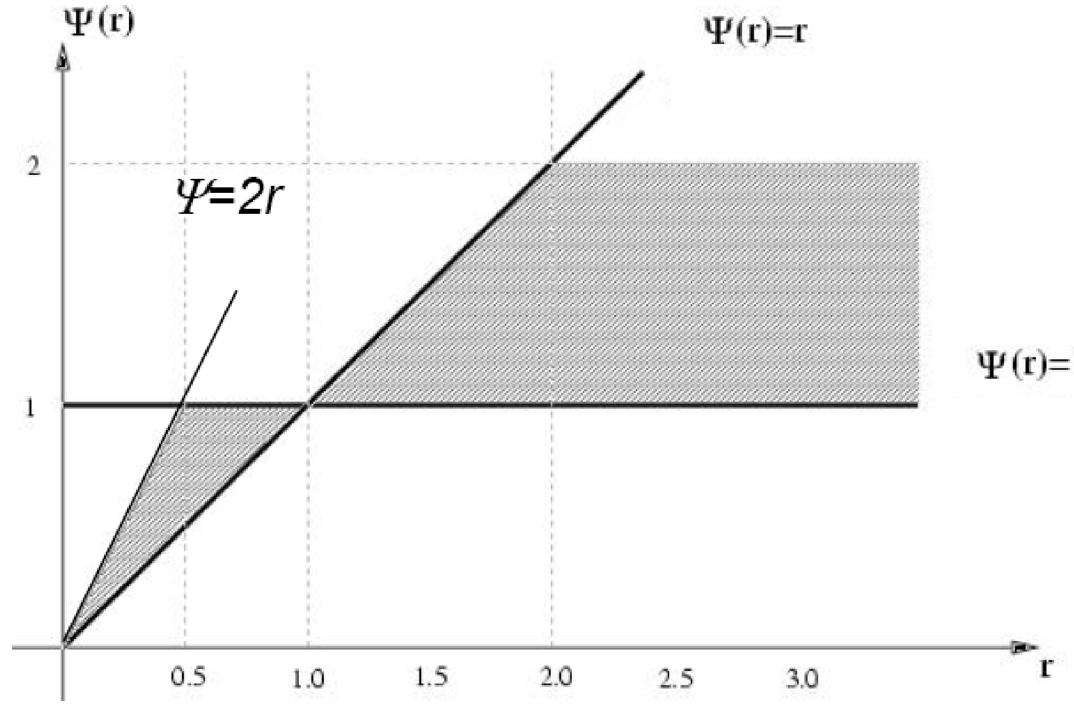
$$\phi_e = \phi_P + \Psi(r_e)\frac{\Delta x}{2}\frac{(\phi_P - \phi_W)}{\Delta x} \qquad r_e = \frac{\phi_E - \phi_P}{\phi_P - \phi_W}$$

Limiters

- limiter function chooses gradient adaptively to avoid creating new extrema

- to be LED scheme, it is possible to show that the limiter should occupy the gray region

- also it is desirable to have it pass through (1,1)



$$r_e = \frac{\phi_E - \phi_P}{\phi_P - \phi_W}$$

Downwind cell gradient

Upwind cell gradient

Higher-Order Scheme

- Consider higher-order approximation for east face scalar

-
$$\phi_e = \phi_P + \Psi(r_e)\frac{\Delta x}{2}\frac{(\phi_P - \phi_W)}{\Delta x}$$

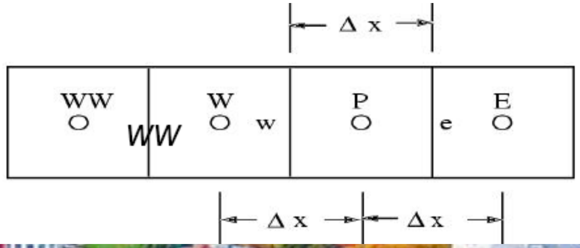
- Values of *r* can be thought of as **ratio of downwind and upwind gradient**

-
$$r_e = \frac{\phi_E - \phi_P}{\phi_P - \phi_W}$$

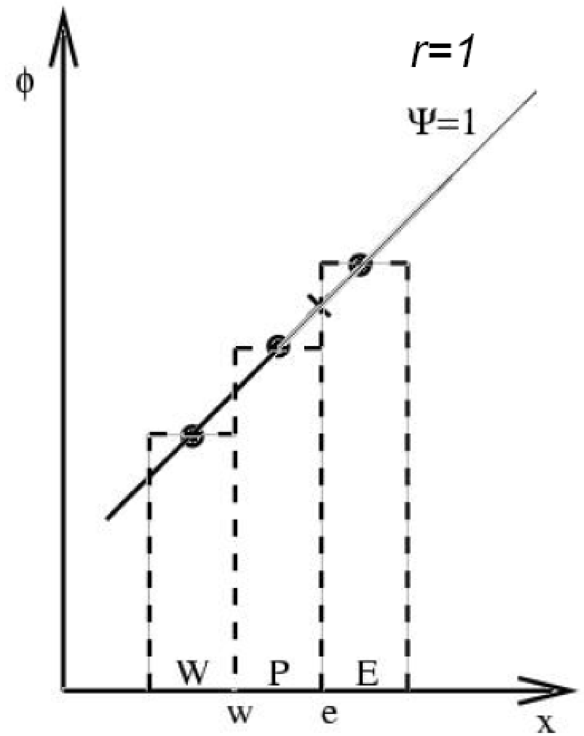
Downwind cell gradient

Upwind cell gradient

- Limiter **chooses gradient adaptively to avoid creating extrema**



Case I: Linear Variation



(a) $r = 1$

-

$$r_e = \frac{\phi_E - \phi_P}{\phi_P - \phi_W}$$

Downwind cell gradient

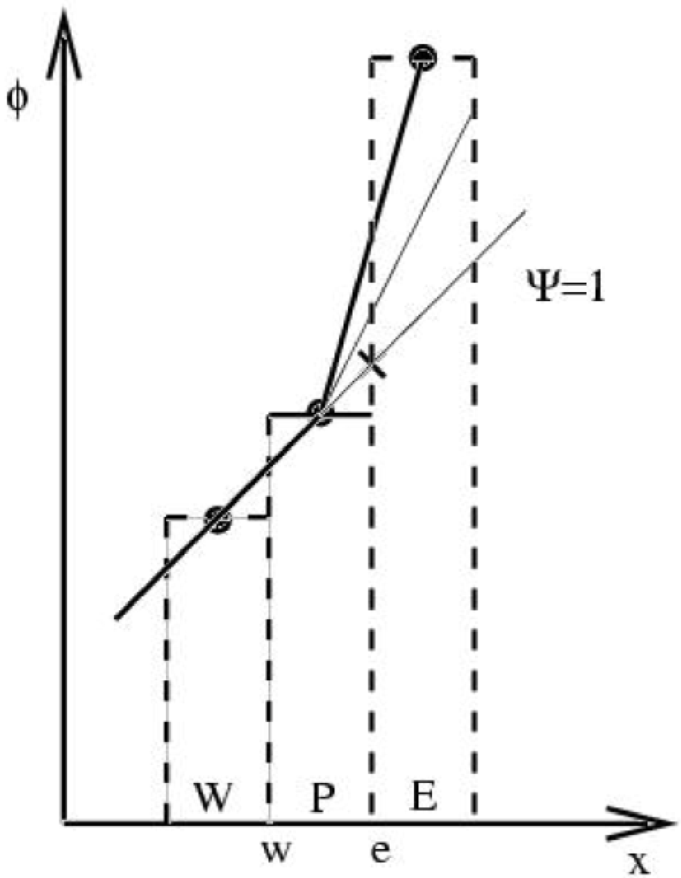
Upwind cell gradient

- If variation is a straight line on a uniform mesh, r=1

- From the limiter function range**, Ψ=1, for r=1**

- Can use either side gradient and use the right value at face e due to linear variation

Case 2: 2>r>1



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$$r_e = \frac{\phi_E - \phi_P}{\phi_P - \phi_W}$$

Downwind cell gradient

Upwind cell gradient

- r > 1 means

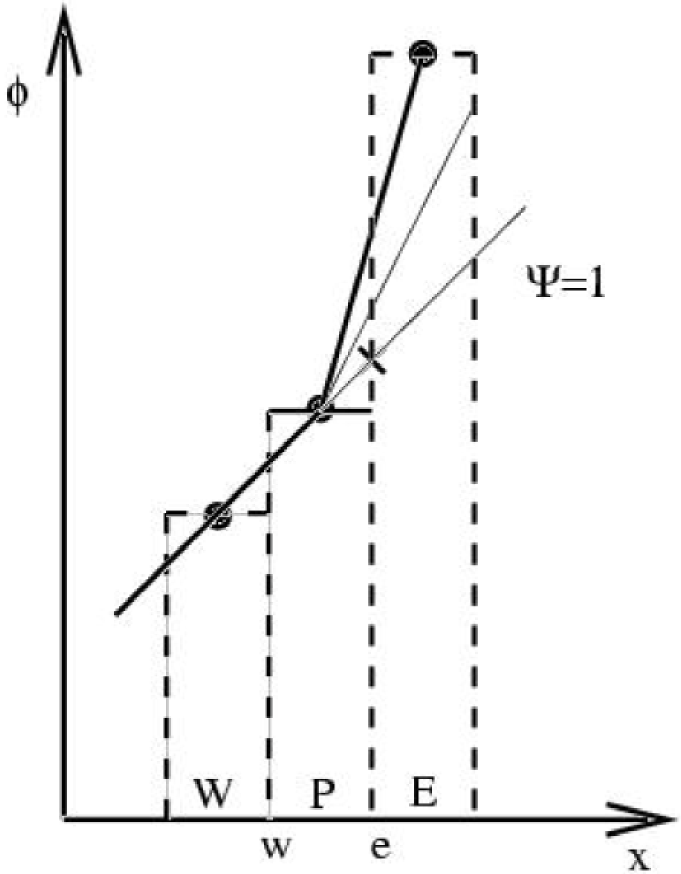
$$\left(\phi_E - \phi_P\right) > \left(\phi_P - \phi_W\right)$$

- If we use **Ψ=1,** we will not create overshoot

- We can **Ψ** use up to *r* and not create

$$\phi_e > \phi_E$$

Case 2: 2>r>1



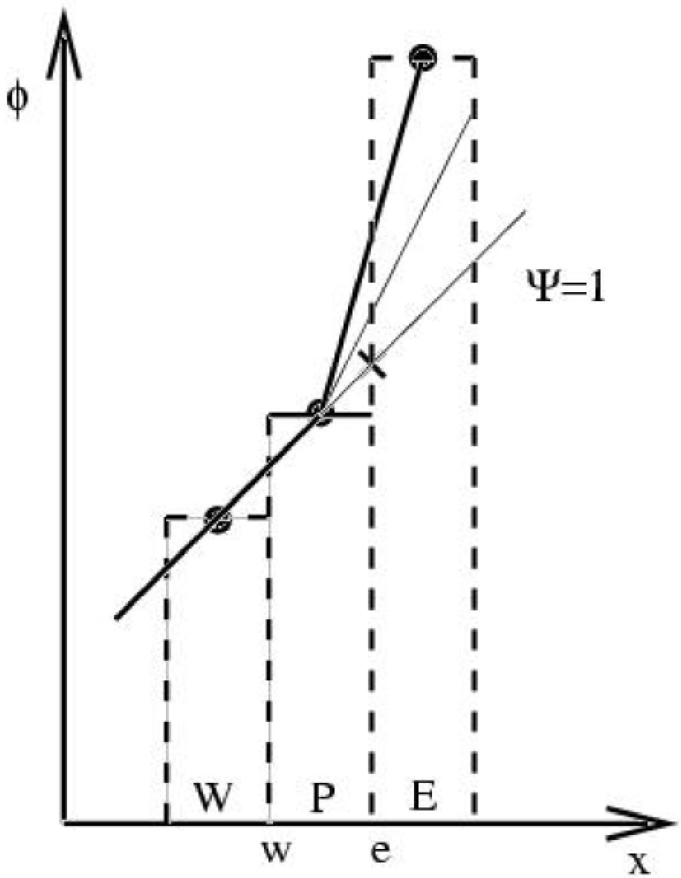
- Let $r_e > 1$; i.e. $\left(\phi_E - \phi_P\right) > \left(\phi_P - \phi_W\right)$

- If we use **$\Psi = r_e$** line

$$\phi_e = \phi_P + \Psi(r_e)\frac{\Delta x}{2}\frac{\left(\phi_P - \phi_W\right)}{\Delta x}$$
$$= \phi_P + \frac{\left(\phi_E - \phi_P\right)}{\left(\phi_P - \phi_W\right)}\frac{1}{2}\left(\phi_P - \phi_W\right)$$
$$= \phi_P + \frac{\left(\phi_E - \phi_P\right)}{2}$$
$$= \frac{1}{2}\phi_P + \frac{1}{2}\phi_E$$
$$\leq \phi_E$$

No new extrema created!

Case 2b: r>2



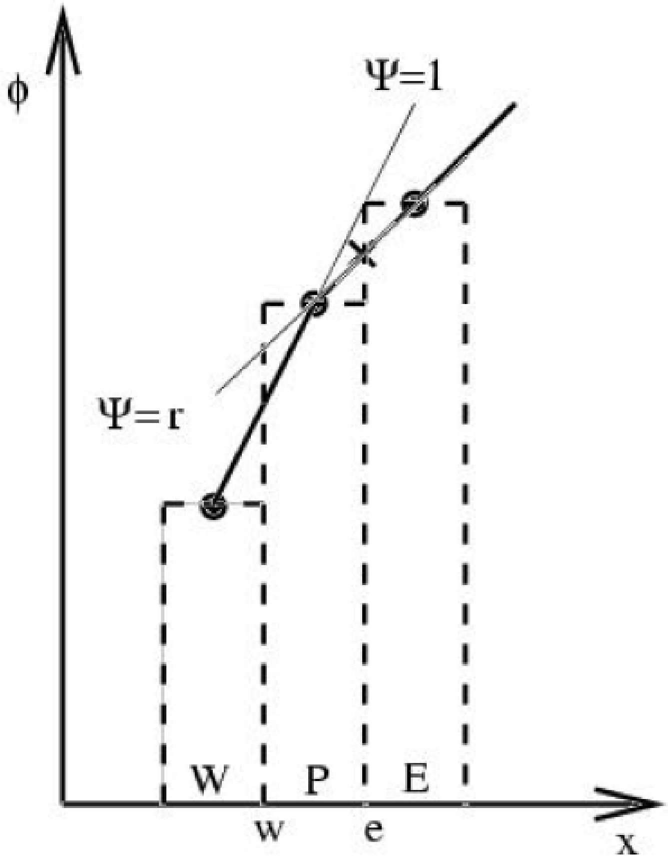
- Let $r_e > 2$; i.e. $\left(\phi_E - \phi_P\right) > \left(\phi_P - \phi_W\right)$

- If we use **$\Psi = r_e = 2$** line

$$
\begin{aligned}
\phi_e &= \phi_P + \Psi(r_e)\frac{\Delta x}{2}\frac{\left(\phi_P - \phi_W\right)}{\Delta x} \\
&= \phi_P + \left(\phi_P - \phi_W\right) \\
&= \phi_P + \frac{\left(\phi_E - \phi_P\right)}{r_e} \\
&= \left(1 - \frac{1}{r_e}\right)\phi_P + \left(\frac{1}{r_e}\right)\phi_E \\
&\leq \phi_E
\end{aligned}
$$

No new extrema created!

Case 3: 0<r<1



(c) $r < 1$

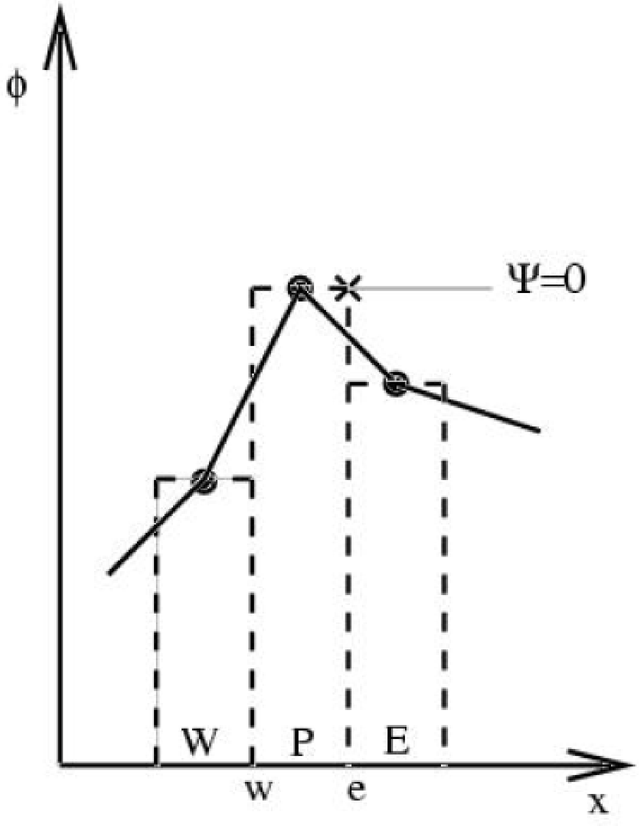
- Let $r_e < 1$; i.e. $\left(\phi_E - \phi_P\right) < \left(\phi_P - \phi_W\right)$

- If we use **$\Psi = r_e$**

$$
\begin{aligned}
\phi_e &= \phi_P + \Psi(r_e)\frac{\Delta x}{2}\frac{\left(\phi_P - \phi_W\right)}{\Delta x} \\
&= \phi_P + \frac{\left(\phi_E - \phi_P\right)}{\left(\phi_P - \phi_W\right)}\frac{1}{2}\left(\phi_P - \phi_W\right) \\
&= \phi_P + \frac{\left(\phi_E - \phi_P\right)}{2} \\
&= \frac{1}{2}\phi_P + \frac{1}{2}\phi_E \\
&\leq \phi_E
\end{aligned}
$$

No new extrema created!

Case 4: r<0



(d) $r < 0$

- Let $r_e < 0$; i.e. there is local extremum

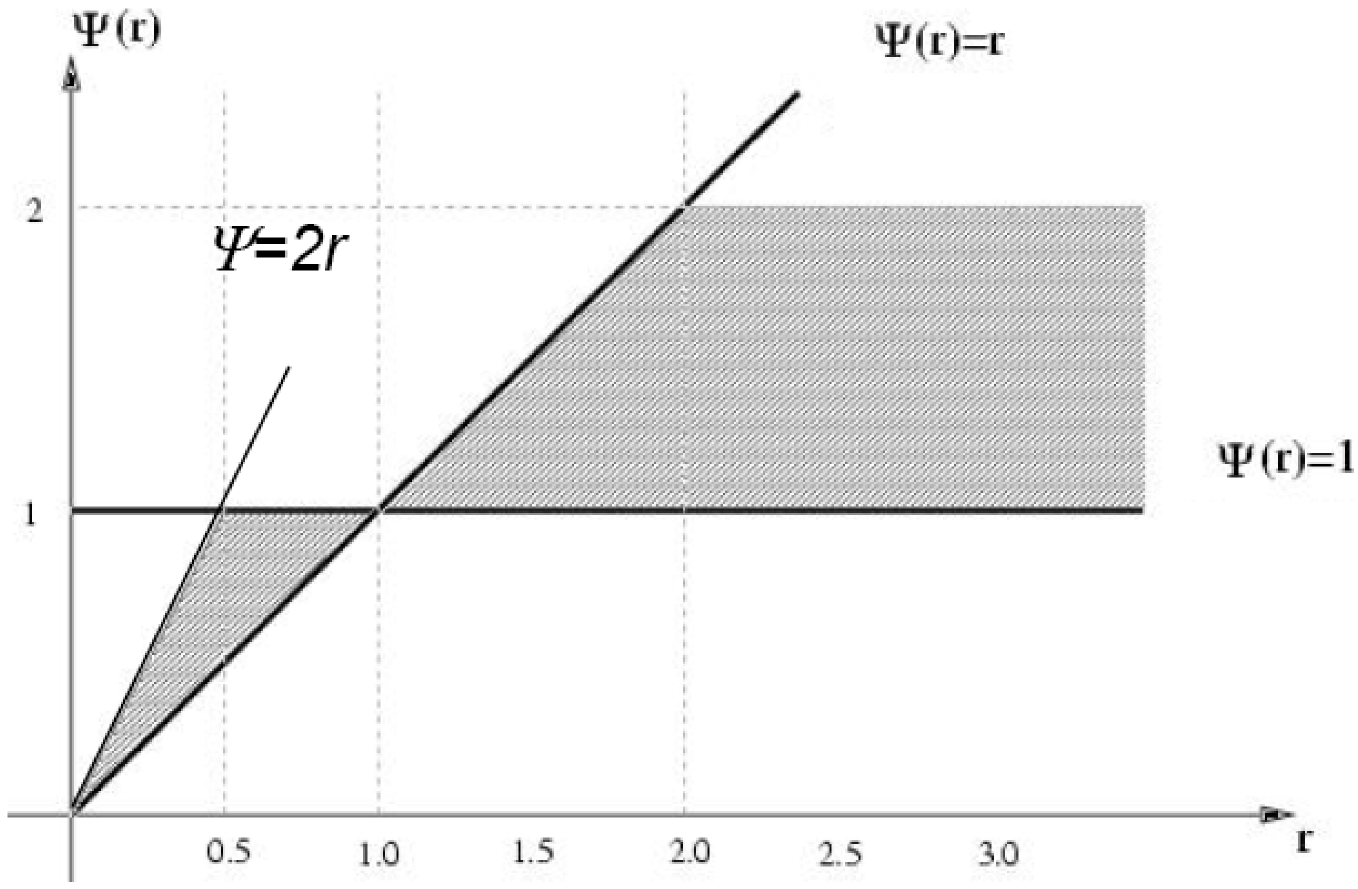
- Limiter uses **Ψ=0 for r<0**

- This gives

$$\phi_e = \phi_P$$

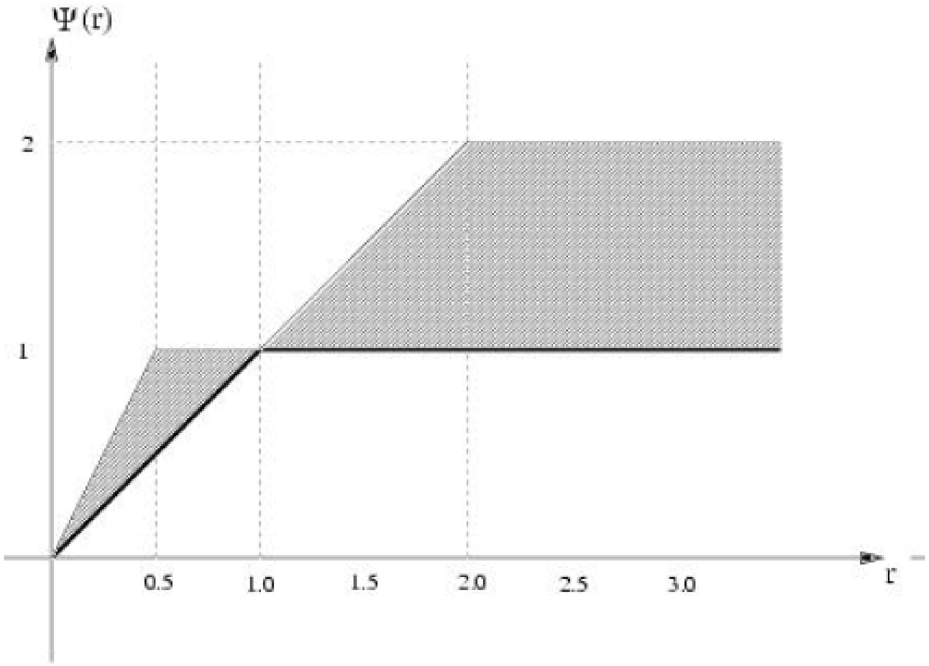
Defaults to first order upwind scheme

Limiter Function



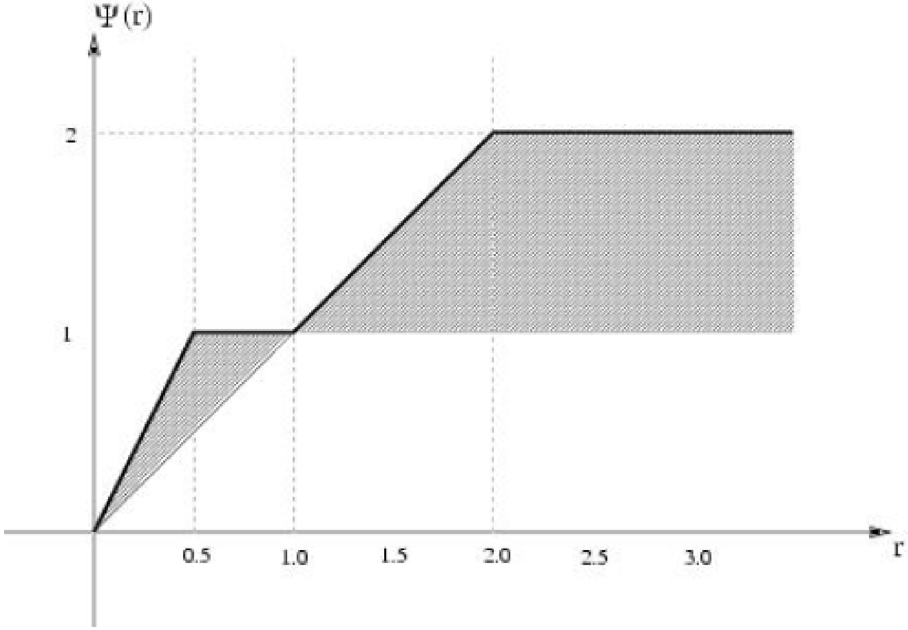
Min-Mod Limiter Function

$$\Psi(r) = \min(r, 1) \quad \text{if } r > 0$$
$$\Psi(r) = 0 \quad \text{if } r \leq 0$$



Superbee Limiter Function

$$\Psi(r) = \max[0, \min(2r, 1), \min(r, 2)]$$



Min-Mod and Superbee

- Owing to sharp variations in Ψ min-mod and superbee limiters may lead to convergence issues in an iterative scheme

- Solution: Use smoother functions where there are transitions
    - van Leer
    - van Albada
    - Quadratic
    - Cubic

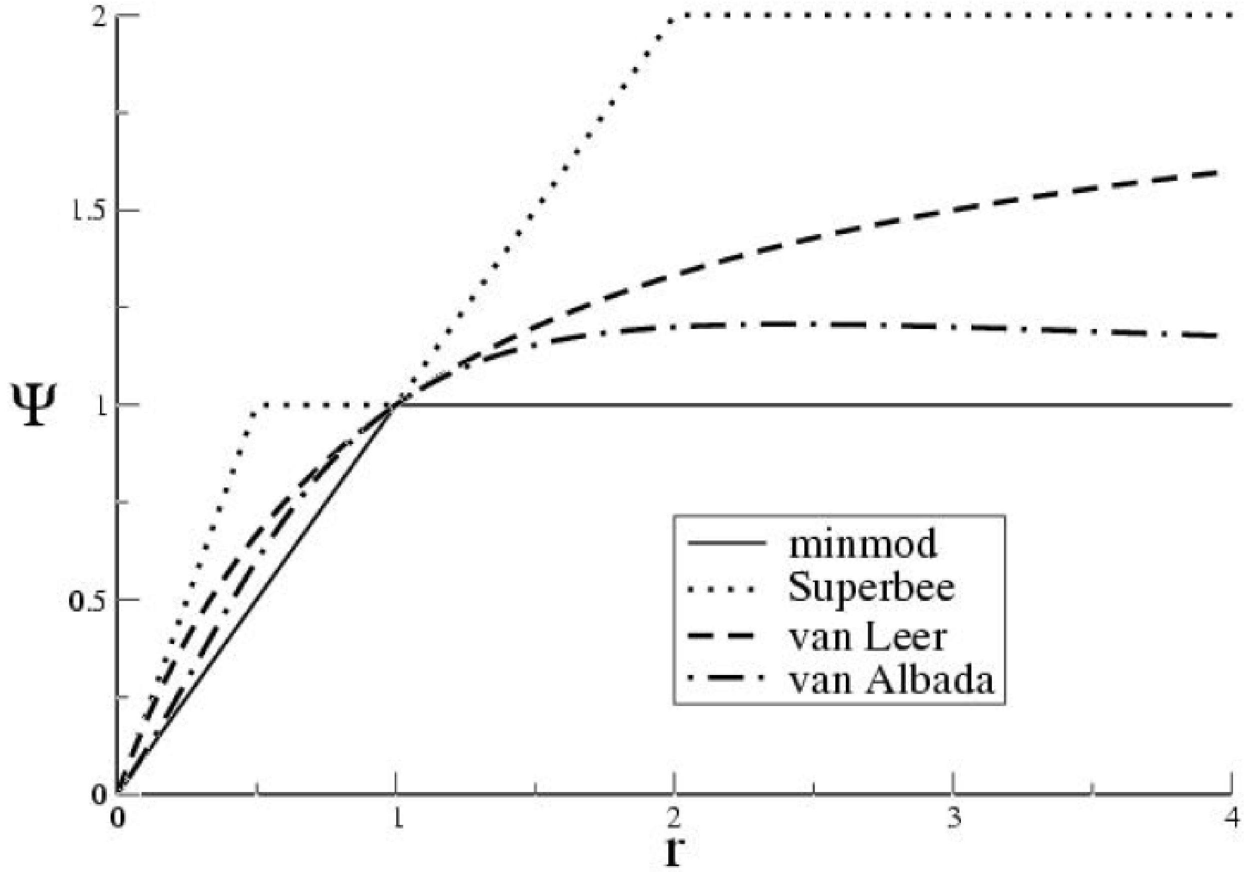
Van Leer and Albada

- van Leer

$$\Psi(r) = \frac{2r}{1+r}$$

- van Albada

$$\Psi(r) = \frac{r^2+r}{1+r^2}$$



Cubic and Quadratic

- Quadratic

$$\begin{aligned}\Psi(r) &= \frac{2r+r^2}{2+r+r^2} \qquad r \le 2 \\ &= 1 \qquad r > 2\end{aligned}$$

- cubic

$$\begin{aligned}\Psi(r) &= \frac{4r+r^3}{4+r^2+r^3} \qquad r \le 2 \\ &= 1 \qquad r > 2\end{aligned}$$

Since limiter falls somewhat outside the 2nd-order region, accuracy is somewhat less than second order for some values of r