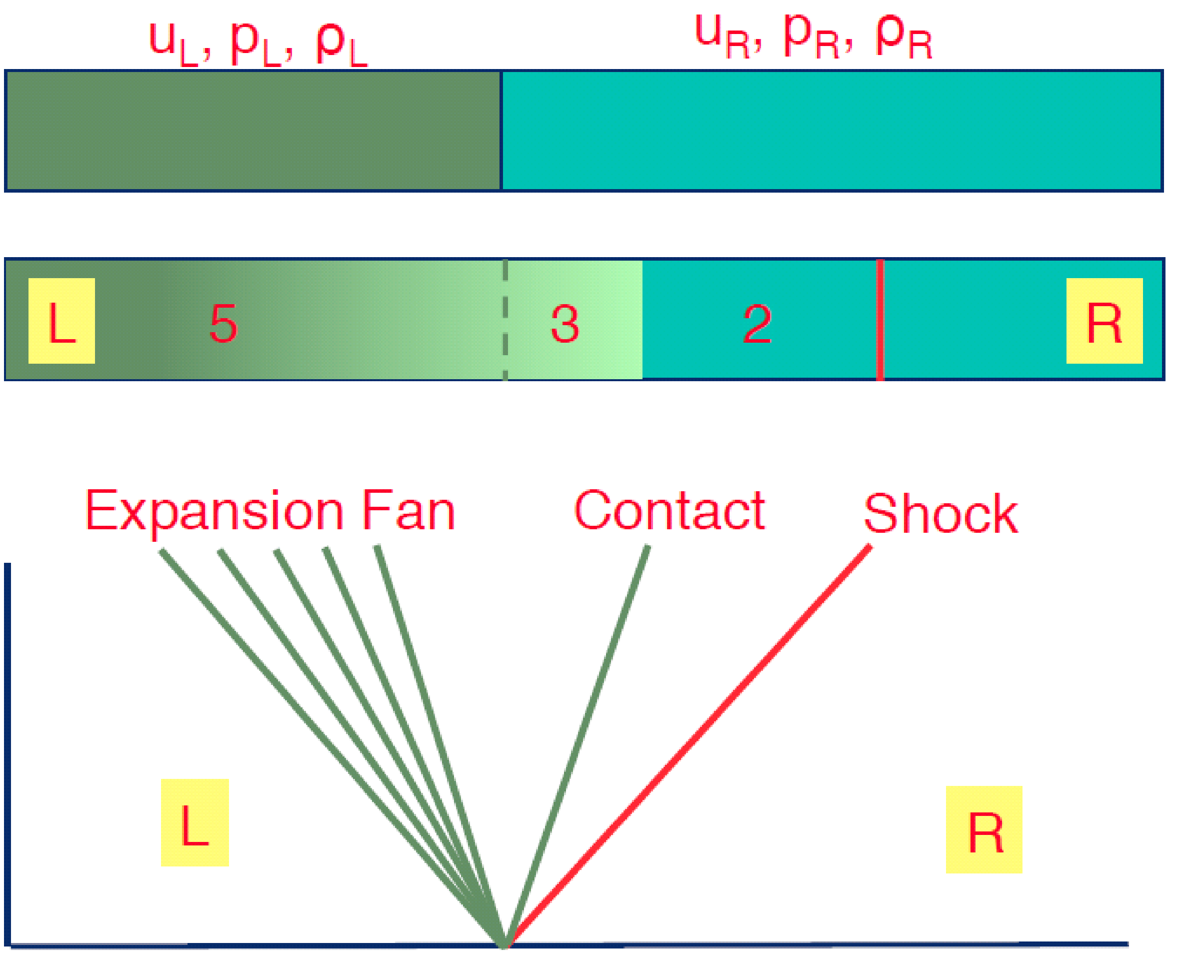
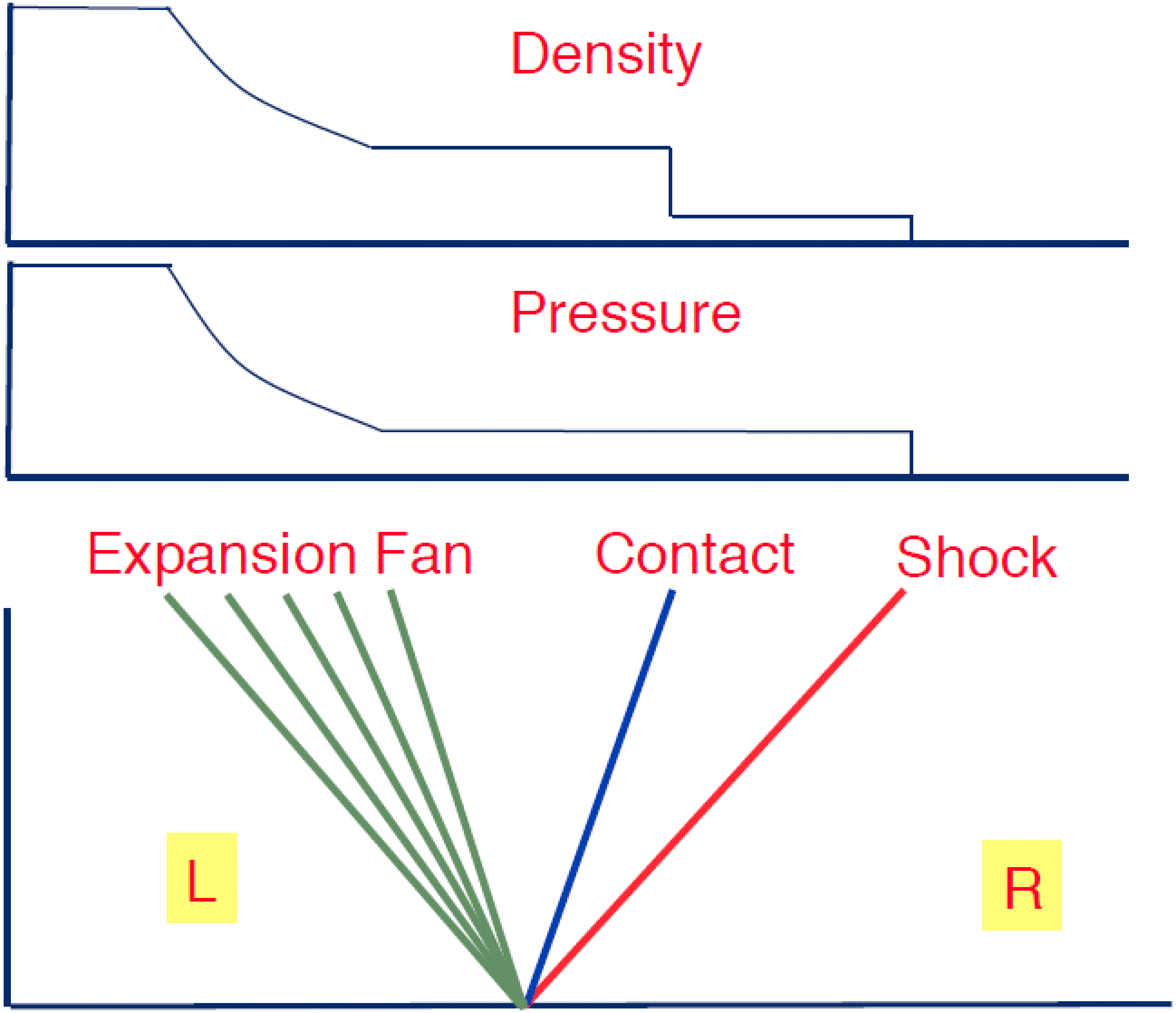
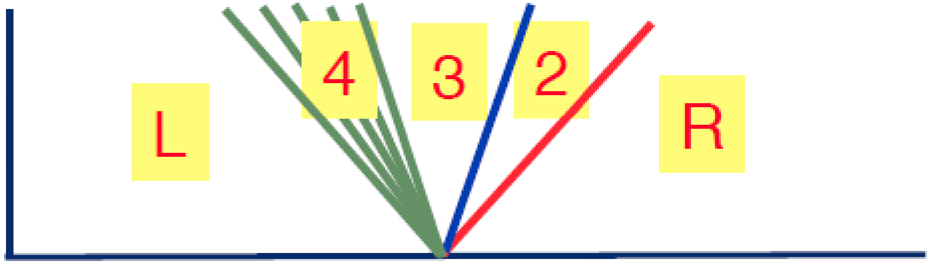
The Shock Tube Problem



The Euler Equations in 1D



The Euler Equations in 1D



$$\alpha = \frac{\gamma+1}{\gamma-1}$$

given: $p_L, \rho_L, u_L,$ $\qquad$ p_R, ρ_R, u_R

$$c_L = \sqrt{\gamma p_L/\rho_L} \qquad c_R = \sqrt{\gamma p_R/\rho_R}$$

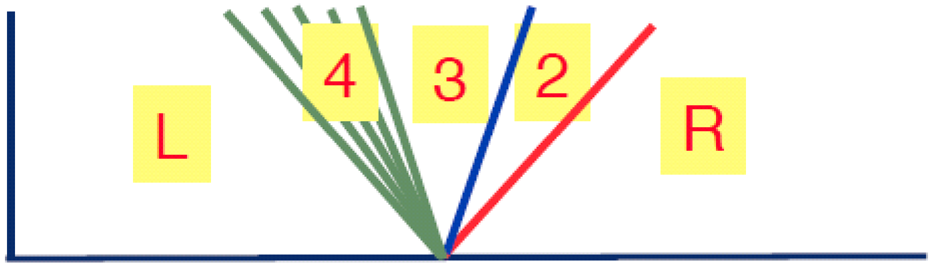
Consider the case $p_L > p_R$:

Shock separates R and 2

Contact discontinuity separates 2 and 3

Expansion fan separates 3 and L

Exact Solution



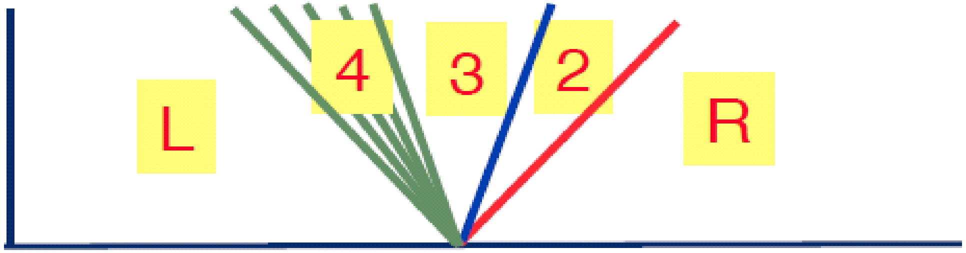
given: $p_L, \rho_L, u_L,$ p_R, ρ_R, u_R

The Rankine-Hugoniot conditions give a nonlinear relation for the pressure jump across the shock $P = \dfrac{p_2}{p_R}$

$$P = \frac{p_L}{p_R}\left[1 - \frac{(\gamma-1)(c_R/c_L)(P-1)}{\sqrt{2\gamma\left(2\gamma+(\gamma+1)(P-1)\right)}}\right]^{2\gamma/(\gamma-1)}$$

which can be solved by iteration

Exact Solution



The speed of the shock is

$$s_{shock} = u_R + c_R\left(\frac{\gamma-1+(\gamma+1)P}{2\gamma}\right)^{1/2}$$

$$x_{shock} = x_0 + s_{shock}t$$

The speed of the contact is

$$s_{contact} = u_3 = u_2 = u_L + \frac{2c_L}{\gamma-1}\left(1-\left(P\frac{p_R}{p_L}\right)^{\frac{\gamma-1}{2\gamma}}\right)$$

$$x_{contact} = x_0 + s_{contact}t$$

The left hand side of the fan moves with speed

$$s_{fL} = -c_L$$

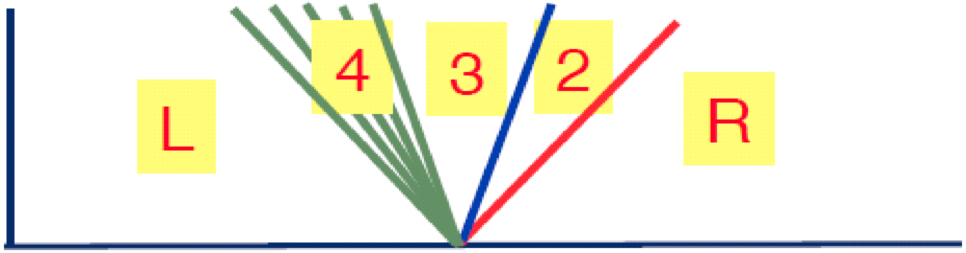
$$x_{fL} = x_0 + s_{fL}t$$

The right hand side of the fan moves with speed

$$s_{fR} = u_2 - c_L$$

$$x_{fR} = x_0 + s_{fR}t$$

The Euler Equations in 1D



The speed of the shock is

$$s_{shock} = u_R + c_R \left(\frac{\gamma - 1 + (\gamma + 1) P}{2\gamma} \right)^{1/2}$$

$$x_{shock} = x_0 + s_{shock} t$$

The speed of the contact is

$$s_{contact} = u_3 = u_2 = u_L + \frac{2 c_L}{\gamma - 1} \left(1 - \left(P \frac{p_R}{p_L} \right)^{\frac{\gamma - 1}{2\gamma}} \right)$$

$$x_{contact} = x_0 + s_{contact} t$$

The left hand side of the fan moves with speed

$$s_{fL} = -c_L$$

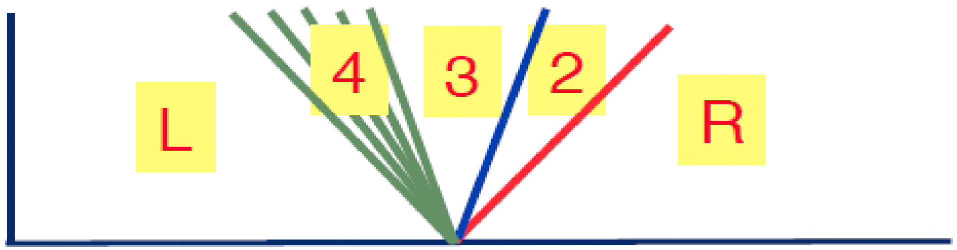
$$x_{fL} = x_0 + s_{fL} t$$

The right hand side of the fan moves with speed

$$s_{fR} = u_2 - c_L$$

$$x_{fR} = x_0 + s_{fR} t$$

Exact Solution



Left uniform state

given:

$p_L, \rho_L, u_L,$

$$c_L = \sqrt{\gamma p_L / \rho_L}$$

In the expansion fan (4)

$$\rho_4 = \rho_4 \left(\frac{1 + (s - x)}{c_L \alpha t} \right)^{\frac{2\gamma}{\gamma - 1}}$$

$$u_4 = u_L + \frac{2}{\gamma + 1} \left(\frac{s - x}{t} \right)$$

$$p_4 = p_4 \left(\frac{1 + (s - x)}{c_L \alpha t} \right)^{\frac{2\gamma}{\gamma - 1}}$$

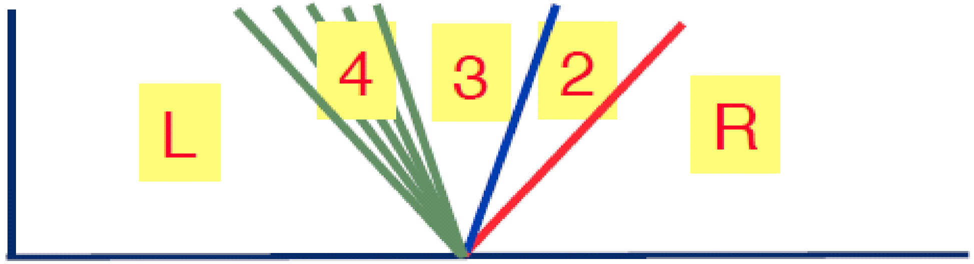
Behind the contact (3)

$$p_3 = p_2$$

$$u_3 = u_2$$

$$\rho_3 = \rho_L \left(P \frac{p_R}{p_L} \right)^{1/\gamma}$$

Exact Solution



$$\alpha = \frac{\gamma+1}{\gamma-1}$$

Behind the contact (3)

$$p_3 = p_2$$

$$u_3 = u_2$$

$$\rho_3 = \rho_L \left(P \frac{p_R}{p_L} \right)^{1/\gamma}$$

Behind the shock (2)

$$\rho_2 = \left(\frac{1+\alpha P}{\alpha + P} \right) \rho_R$$

$$p_2 = P\, p_R$$

$$u_2 = u_L + \frac{2c_L}{\gamma - 1} \left(1 - \left(P \frac{p_R}{p_L} \right)^{\frac{\gamma-1}{2\gamma}} \right)$$

Right uniform state

given:

$$p_R, \rho_R, u_R$$

$$c_R = \sqrt{\gamma\, p_R / \rho_R}$$

Exact Solution

The speed of sound is given by

$$c = \sqrt{\frac{\gamma p}{\rho}}$$

The Mach number is defined as the ratio of the local velocity over the speed of sound

$$Ma = \frac{u}{c}$$

$Ma < 1$ subsonic

$Ma > 1$ supersonic

Exact Solution

Test case:

Shocktube problem of G.A. Sod, JCP 27:1, 1978

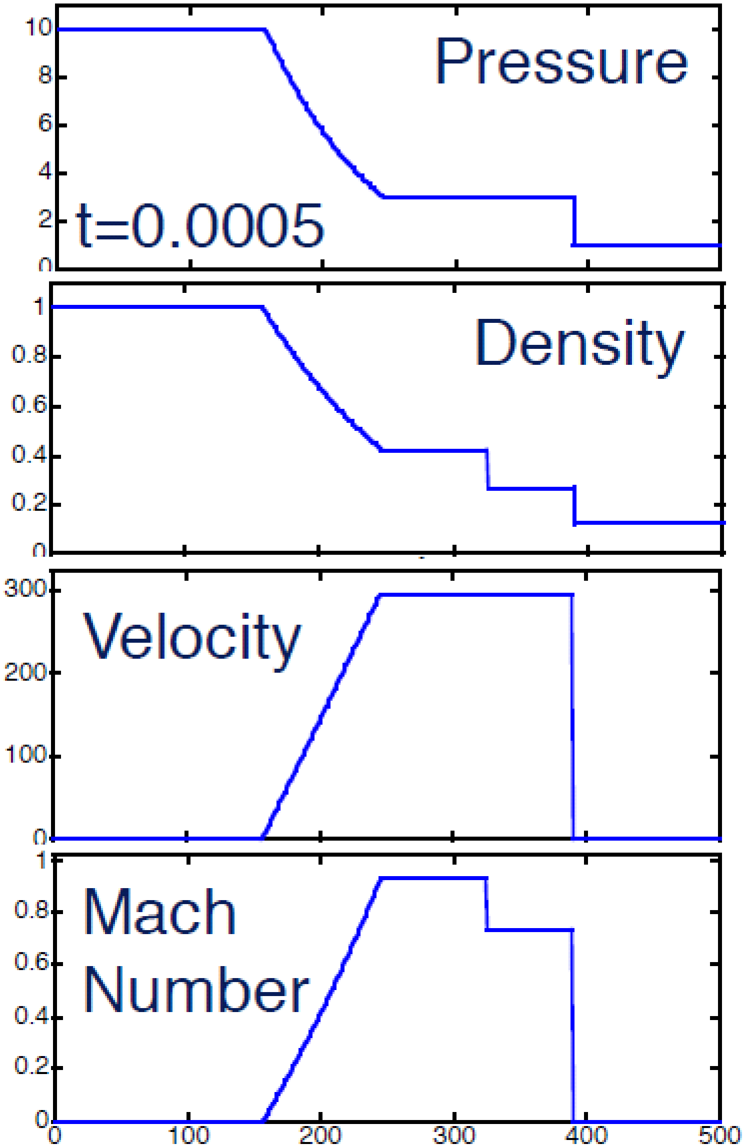
$$p_L = 10^5; \quad \rho_L = 1.0; \quad u_L = 0$$

$$p_R = 10^4; \quad \rho_R = 0.125; \quad u_R = 0$$

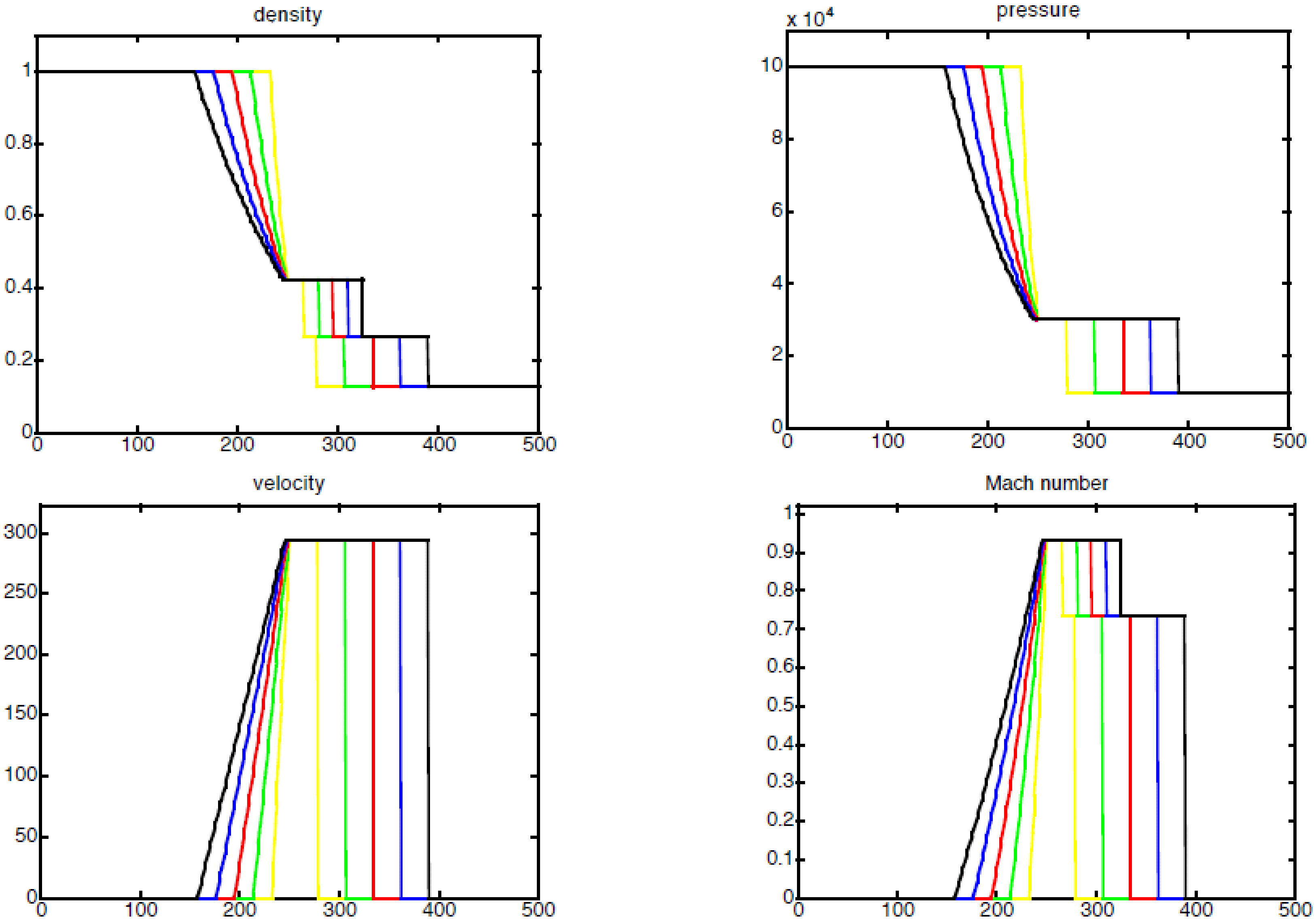
$$t_{final} = 0.005$$

$$x_0 = 0.5$$

Subsonic case



Exact Solution



Exact Solution

Test case:

Shocktube problem of G.A. Sod, JCP 27:1, 1978

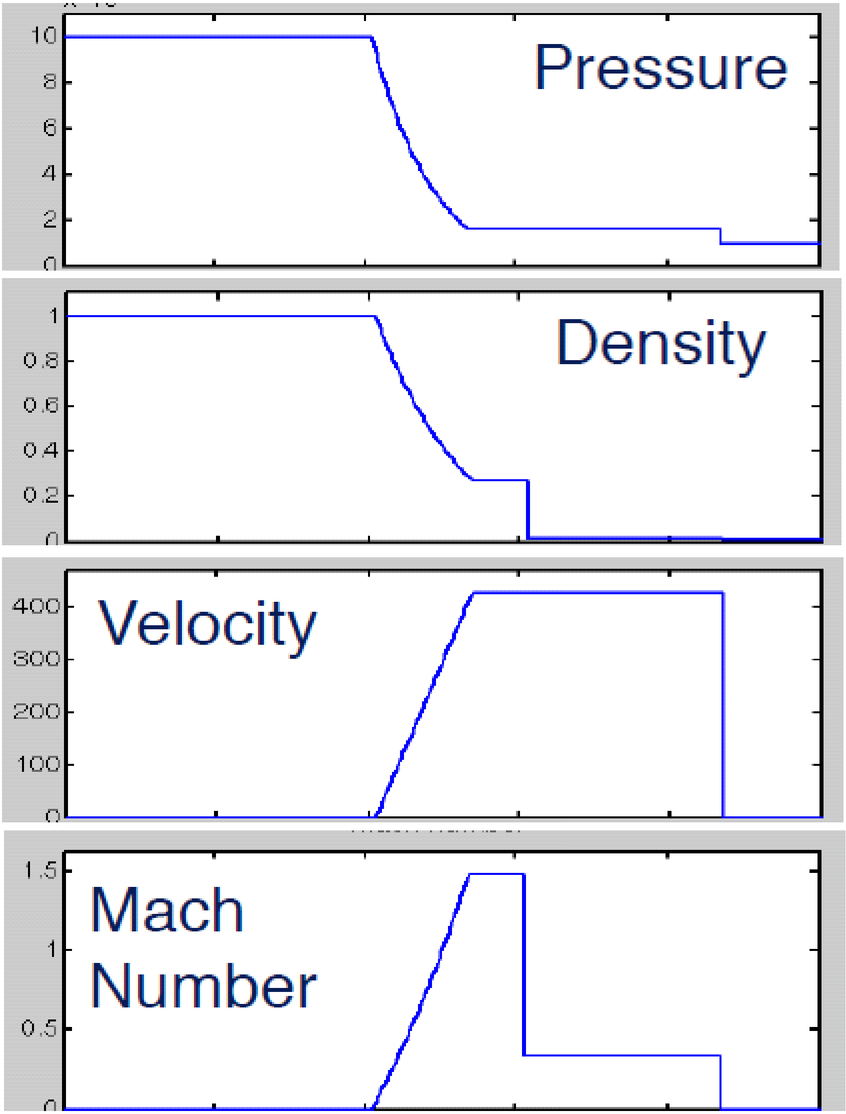
$$p_L = 10^5; \quad \rho_L = 1.0; \quad u_L = 0$$

$$p_R = 10^4; \quad \rho_R = 0.01; \quad u_R = 0$$

$$t_{final} = 0.0025$$

$$x_0 = 0.5$$

Supersonic case



Shock-Tube Problem
Numerical Solution

Solve using
second order
Lax-Wendroff

Lax Wendroff with Artificial Viscosity

For the fluid-dynamic system of equations (Euler equations):

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ \rho u \\ \rho E\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\rho u \\ \rho u^2+p \\ \rho u(E+p/\rho)\end{pmatrix}=0$$

where $E=e+u^2/2;\ \ p=(\gamma-1)\rho e$

Add the artificial viscosity to RHS:

$$=\frac{\partial}{\partial x}\left(\alpha h^2\rho\begin{vmatrix}0 \\ 1 \\ u\end{vmatrix}\left|\frac{\partial u}{\partial x}\right|\frac{\partial u}{\partial x}\right)$$

Lax Wendroff with Artificial Viscosity

Solutions of the 1D Euler equation using Lax-Wendroff

$$\mathbf{f}^{*}_{j+\frac{1}{2}} = 0.5\left(\mathbf{f}^{n}_{j} + \mathbf{f}^{n}_{j+1}\right) - 0.5\,\frac{\Delta t}{h}\left(\mathbf{F}^{n}_{j+1} - \mathbf{F}^{n}_{j}\right)$$

$$\mathbf{f}^{n+1}_{j} = \mathbf{f}^{n}_{j} - \frac{\Delta t}{h}\left(\mathbf{F}^{*}_{j+\frac{1}{2}} - \mathbf{F}^{*}_{j-\frac{1}{2}}\right)$$

With an artificial viscosity term added to the corrector step

$$\mathbf{F}' = \mathbf{F} - \alpha h^{2}\rho \begin{bmatrix} 0 \\ 1 \\ u \end{bmatrix} \left|\frac{\partial u}{\partial x}\right| \frac{\partial u}{\partial x}$$

Lax Wendroff with Artificial Viscosity

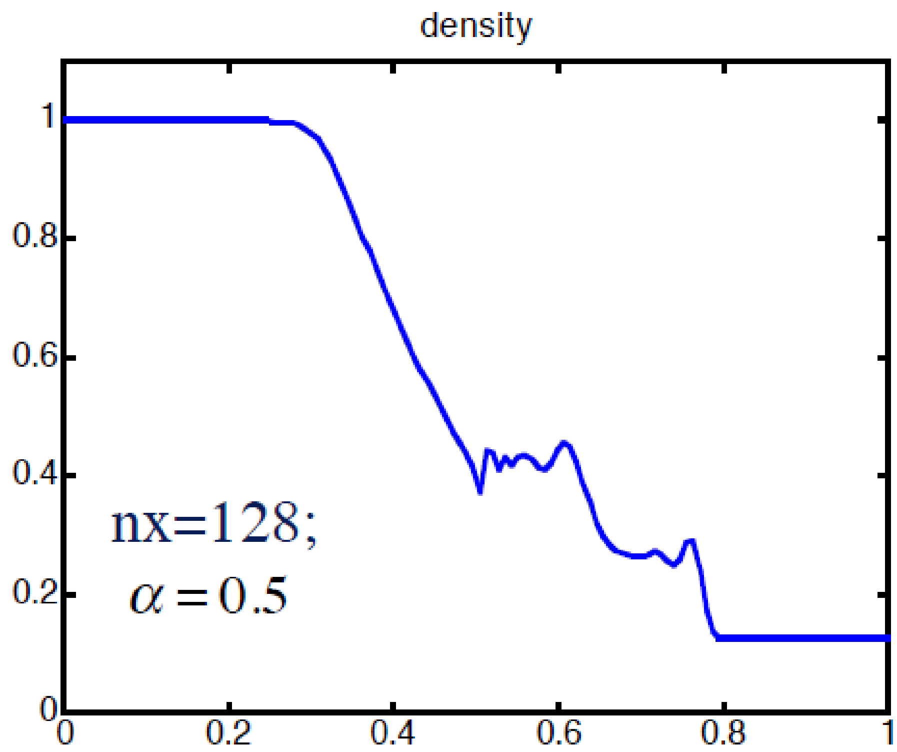
Test case:

Shocktube problem of
G.A. Sod, JCP 27:1, 1978

$$p_L = 10^5; \quad \rho_L = 1.0; \quad u_L = 0$$

$$p_R = 10^4; \quad \rho_R = 0.125; \quad u_R = 0$$

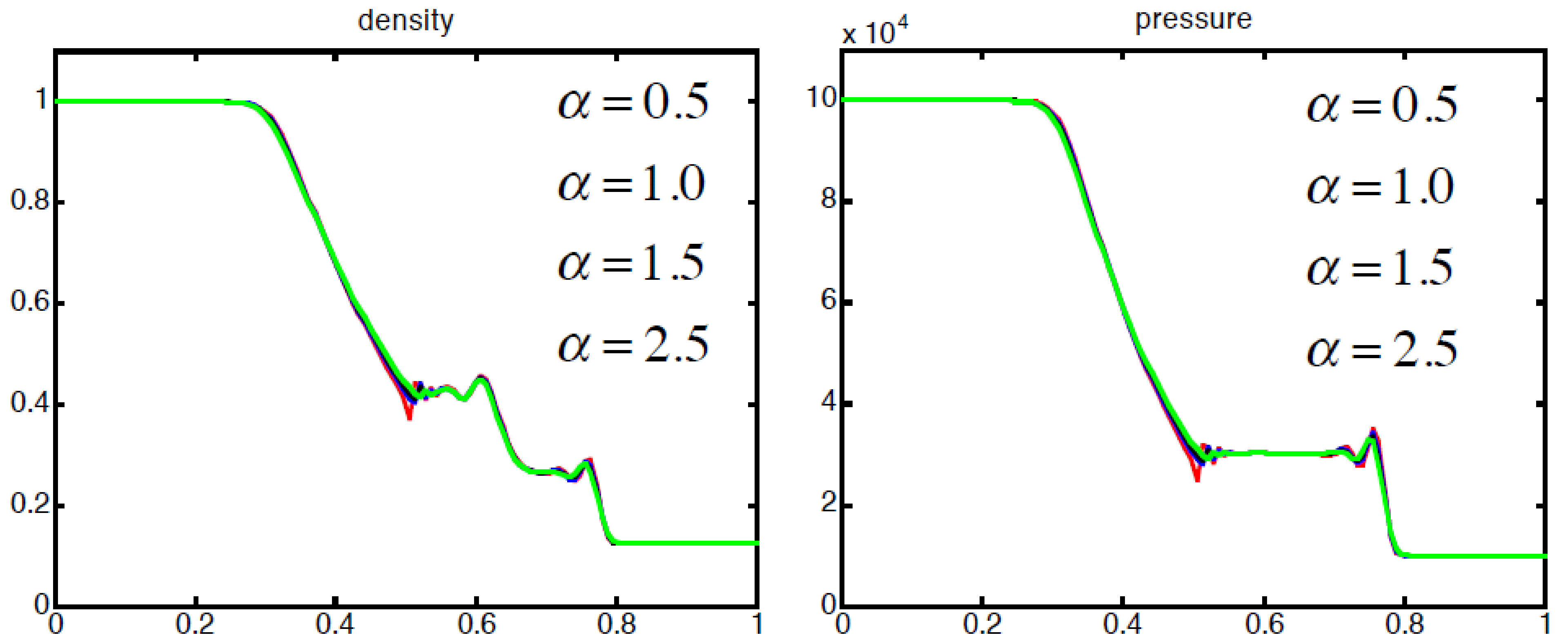
Final time: 0.005



Lax Wendroff with Artificial Viscosity

Effect of α

nx=128;



Lax Wendroff with Artificial Viscosity

Effect of resolution

nx=64;
nx=128;
nx=256;
time=0.5

$$\alpha = 1.5$$