# Finite-Difference Schemes for 1D Advection

## ME 667 Computational Fluid Dynamics

Consider the advection equation:

$$\frac{\partial \phi}{\partial t} + a\frac{\partial \phi}{\partial x} = 0; \qquad a > 0; \quad \phi(x, 0) = \phi_0(x) \tag{1}$$

We use uniform time-step and grid spacing to define the following schemes:

1. **Explicit Upwind**

$$\frac{\phi_k^{n+1} - \phi_k^n}{\Delta t} + a\left(\frac{\phi_k^n - \phi_{k-1}^n}{\Delta x}\right) \tag{2}$$

2. **Lax Friedrichs**

$$\phi_k^{n+1} = \frac{1}{2}\left(\phi_{k-1}^n + \phi_{k+1}^n\right) - \frac{a\Delta t}{2\Delta x}\left(\phi_{k+1}^n - \phi_{k-1}^n\right) \tag{3}$$

3. **Lax-Wendroff**

$$\phi_k^{n+1} = \phi_k^n - \frac{a\Delta t}{2\Delta x}\left(\phi_{k+1}^n - \phi_{k-1}^n\right) + \frac{(a\Delta t)^2}{2\Delta x^2}\left(\phi_{k+1}^n - 2\phi_k^n + \phi_{k-1}^n\right) \tag{4}$$

4. **Beam-Warming**

$$\phi_k^{n+1} = \phi_k^n - \frac{a\Delta t}{2\Delta x}\left(3\phi_k^n - 4\phi_{k-1}^n + \phi_{k-2}^n\right) + \frac{(a\Delta t)^2}{2\Delta x^2}\left(\phi_k^n - 2\phi_{k-1}^n + \phi_{k-2}^n\right) \tag{5}$$

5. **QUICK**

$$\phi_k^{n+1} = \phi_k^n - \frac{a\Delta t}{\Delta x}\left(\phi_{k+1/2}^{n+1} - \phi_{k-1/2}^{n+1}\right) \tag{6}$$

where

$$\phi_{k+1/2}^{n+1} = \frac{\phi_{k+1}^{n+1} + \phi_k^{n+1}}{2} - \frac{\phi_{k+1}^{n+1} + \phi_{k-1}^{n+1} - 2\phi_k^{n+1}}{8} \tag{7}$$

$$\phi_{k-1/2}^{n+1} = \frac{\phi_k^{n+1} + \phi_{k-1}^{n+1}}{2} - \frac{\phi_k^{n+1} + \phi_{k-2}^{n+1} - 2\phi_{k-1}^{n+1}}{8} \tag{8}$$

Note that following finite-volume notation, $k + 1/2$ and $k - 1/2$ can be thought of as $e$ and $w$ faces around a control volume $P$ centered at the node $k$.

6. **Essentially Non-Oscillatory Scheme (ENO) (see reference Jiang & Peng (2000)**

Let $x_k$ be the discretization with *uniform* grid spacing $\Delta x$. We introduce the divided difference notation as follows

$$\phi_k = \phi(x_k), \quad \Delta^+\phi_k = \phi_{k+1} - \phi_k, \quad \Delta^-\phi_k = \phi_k - \phi_{k-1} \tag{9}$$

The derivative $\phi_x(x_k)$ is approximated based on either the left-biased stencil $x_k, k = i-3, ...., i+2$ or the right-biased stencil $[x_k, k = i-2, ...., i+3]$ depending on the upwind direction of the flow. So, for example, if the flow is from left to right (which is the case in our problem, note that $a$ (the speed) is unity) we choose the left biased stencil (Figure 1) to evaluate $\phi_x(x_k)$. This will be denoted as $\phi_{x,i}^-$. Similarly the right-biased stencil based derivative is denoted as $\phi_{x,i}^+$. The third order ENO approximation to $\phi_x(x_k)$ is given as[1]

[1] It will be helpful if you derive each differentiation first to make it clear for yourself. For example $\Delta^-\phi_k = \phi_k - \phi_{k-1}$; $\Delta^+\Delta^-\phi_k = \Delta^+(\phi_k - \phi_{k-1}) = (\phi_{k+1} - \phi_k) - (\phi_k - \phi_{k-1})$ and so on.

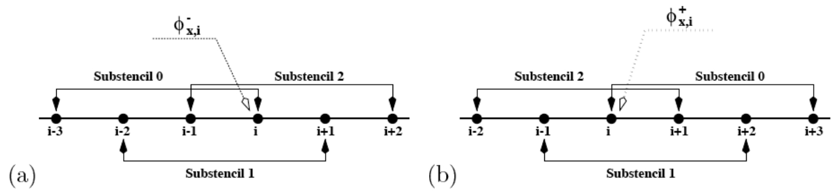


Figure 1: **Left and right-biased stencils.**

$$\phi^-_{x,i} = \begin{cases} \phi^{-,0}_{x,i} & \text{if } |\Delta^-\Delta^+\phi_{i-1}| < |\Delta^-\Delta^+\phi_i| \text{ and} \\ & \quad |\Delta^-\Delta^-\Delta^+\phi_{i-1}| < |\Delta^+\Delta^-\Delta^+\phi_{i-1}|; \\ \phi^{-,2}_{x,i} & \text{if } |\Delta^-\Delta^+\phi_{i-1}| > |\Delta^-\Delta^+\phi_i| \text{ and} \\ & \quad |\Delta^-\Delta^-\Delta^+\phi_i| > |\Delta^+\Delta^-\Delta^+\phi_i|; \\ \phi^{-,1}_{x,i} & \text{otherwise.} \end{cases}$$

where $\phi^{-,s}_{x,i}$ $s = 0, 1, 2$ is given as below.

$$\begin{aligned} \phi^{-,0}_{x,i} &= \frac{1}{3}\frac{\Delta^+\phi_{i-3}}{\Delta x} - \frac{7}{6}\frac{\Delta^+\phi_{i-2}}{\Delta x} + \frac{11}{6}\frac{\Delta^+\phi_{i-1}}{\Delta x}, \\ \phi^{-,1}_{x,i} &= -\frac{1}{6}\frac{\Delta^+\phi_{i-2}}{\Delta x} + \frac{5}{6}\frac{\Delta^+\phi_{i-1}}{\Delta x} + \frac{1}{3}\frac{\Delta^+\phi_i}{\Delta x}, \\ \phi^{-,2}_{x,i} &= \frac{1}{3}\frac{\Delta^+\phi_{i-1}}{\Delta x} + \frac{5}{6}\frac{\Delta^+\phi_i}{\Delta x} - \frac{1}{6}\frac{\Delta^+\phi_{i+1}}{\Delta x}, \end{aligned}$$