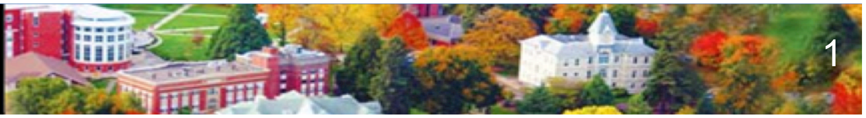
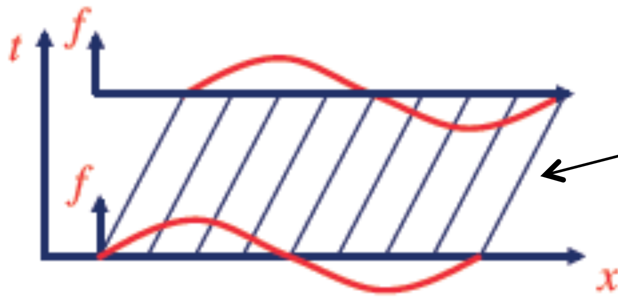
- Numerical methods for hyperbolic systems
- Upwinding and numerical dissipation
- Modified equation and accuracy/consistency



Hyperbolic Equation: Numerics

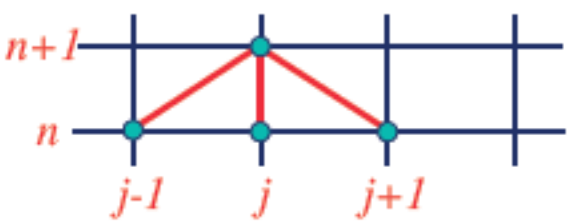
$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} = 0$$



$$\frac{dx}{dt} = U; \quad \frac{df}{dt} = 0;$$

Characteristic lines in t-x plane

$$f_j^{n+1} = f_j^n - \frac{\Delta t}{2h}U(f_{j+1}^n - f_{j-1}^n)$$



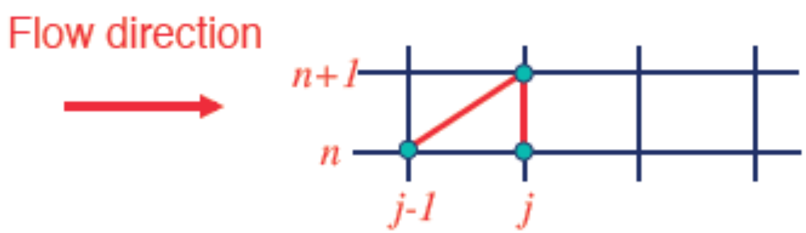
Naïve approach
Use von-Neumann analysis
To show unconditionally unstable

$$\frac{\varepsilon^{n+1}}{\varepsilon^n} = 1 - i\frac{U\Delta t}{2h}\sin kh$$

Hyperbolic Equation: Numerics

$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} = 0$$

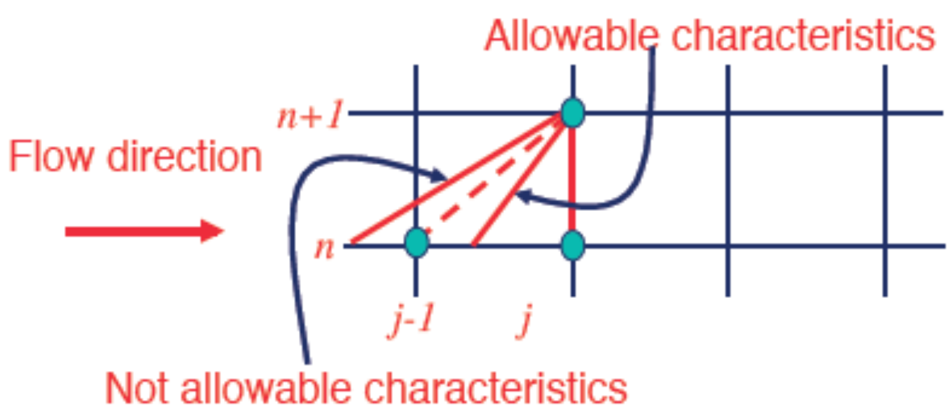
$$f_j^{n+1} = f_j^n - \frac{\Delta t}{h} U(f_j^n - f_{j-1}^n)$$



Upwind
Conditionally stable
The CFL (Courant Fredrichs Lewy number

$$\frac{U\Delta t}{h} \le 1$$

$$U\Delta t \le h$$



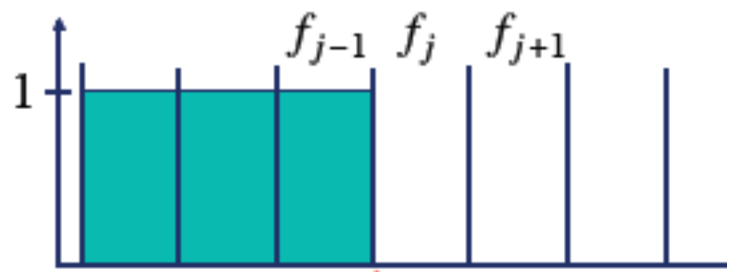
Signal has to travel
less than one grid
space in a time step

Hyperbolic Equation: FV and Flux

$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial t} + \frac{\partial U f}{\partial x} = 0 \qquad \frac{\partial f}{\partial t} + \frac{\partial F}{\partial x} = 0$$

F is flux

Initial conditions in flux format

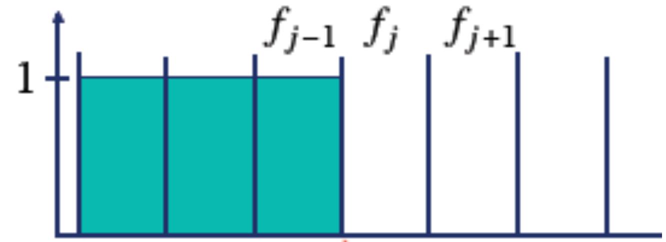


$U = 1$

"advective" flux at the faces of CV

Hyperbolic Equation: FV

$$\int_{V_j,\Delta t} \left(\frac{\partial f}{\partial t} + \frac{\partial U f}{\partial x} \right)_j^n dV\,dt \quad = \quad 0$$



$$(f_j^{n+1} - f_j^n) V_{\text{cv}} + \sum_{\text{faces of cv}} F_{\text{face}}^n \cdot A_{\text{face}} \Delta t \quad = \quad 0$$

$$(f_j^{n+1} - f_j^n) V_{\text{cv}} + (F_{j+1/2}^n - F_{j-1/2}^n) A_{\text{face}} \Delta t \quad = \quad 0$$

$$f_j^{n+1} = f_j^n - \frac{A_{\text{face}}}{V_{\text{CV}}} (F_{j+1/2}^n - F_{j-1/2}^n) \Delta t$$

$$f_j^{n+1} = f_j^n - \frac{\Delta t}{h} (F_{j+1/2}^n - F_{j-1/2}^n)$$

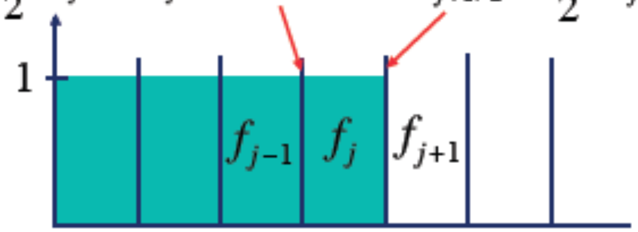
Hyperbolic Equation: FV- Central

$$f_j^{n+1} = f_j^n - \frac{\Delta t}{h}(F_{j+1/2}^n - F_{j-1/2}^n)$$

$$F_{j+1/2} = [Uf]_{j+1/2} = 0.5[Uf]_j + 0.5[Uf]_{j+1}$$

$$F_{j-1/2} = [Uf]_{j-1/2} = 0.5[Uf]_{j-1} + 0.5[Uf]_j$$

$$F_{j-1/2} = \frac{U}{2}(f_{j-1}^n + f_j^n) = 1.0 \qquad F_{j+1/2} = \frac{U}{2}(f_j^n + f_{j+1}^n) = 0.5$$



$$f_j^{n+1} = f_j^n - \frac{\Delta t}{h}(F_{j+1/2}^n - F_{j-1/2}^n) = 1.0 - 0.5(0.5 - 1) = 1.25$$

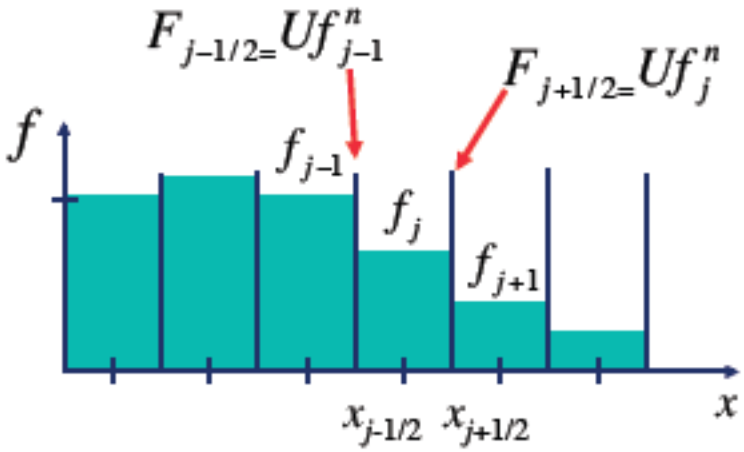
Cell j “overflows” with central scheme

Hyperbolic Equation: FV- Upwind

$$f_j^{n+1} = f_j^n - \frac{\Delta t}{h}(F_{j+1/2}^n - F_{j-1/2}^n)$$

$$F_{j+1/2} = [Uf]_{j+1/2} = [Uf]_j; \quad U > 0$$

$$F_{j-1/2} = [Uf]_{j-1/2} = [Uf]_{j-1}; \quad U > 0$$



Works fine for $\frac{U\Delta t}{h} \leq 1$

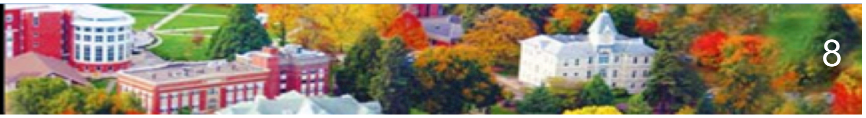
Hyperbolic Equation: FV- Upwind

$$f_j^{n+1} = f_j^n - \frac{\Delta t}{h}(F_{j+1/2}^n - F_{j-1/2}^n)$$

For any U

$$\begin{aligned} F_{j+1/2} &= \max[F_{j+1}, 0] \\ F_{j-1/2} &= \max[F_{j+1}, 0] \end{aligned}$$

$$\begin{aligned} \max[a, b] &= a \quad \text{if} \quad a > b \\ &= b \quad \text{otherwise} \end{aligned}$$



Modified Equation

Matlab example

- Explicit Upwind is conditionally stable
- Inaccurate (first order in time and space)
- Needs enormous grid resolution and very small time-step for any practical use.

Modified Equation

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} + \frac{U}{h}(f_j^n - f_{j-1}^n) = 0$$

Convert the discrete equation, for say (x_j, t_n), into a continuous differential equation by using Taylor Series expansions around (x_j, t_n)

$$f_j^{n+1} = f_j^n + \frac{\partial f}{\partial t}\Delta t + \frac{\partial^2 f}{\partial t^2}\frac{\Delta t^2}{2} + \frac{\partial^3 f}{\partial t^3}\frac{\Delta t^3}{6} + \cdots$$

$$f_{j-1}^n = f_j^n - \frac{\partial f}{\partial x}h + \frac{\partial^2 f}{\partial x^2}\frac{h^2}{2} - \frac{\partial^3 f}{\partial x^3}\frac{h^3}{6} + \cdots$$

Modified Equation: Upwind

Substitute

$$\frac{1}{\Delta t}\left\{\left[\cancel{f_j^n}+\frac{\partial f}{\partial t}\Delta t+\frac{\partial^2 f}{\partial t^2}\frac{\Delta t^2}{2}+\frac{\partial^3 f}{\partial t^3}\frac{\Delta t^3}{6}+\cdots\right]-\cancel{f_j^n}\right\}$$

$$+\frac{U}{h}\left\{\cancel{f_j^n}-\left[\cancel{f_j^n}-\frac{\partial f}{\partial x}h+\frac{\partial^2 f}{\partial x^2}\frac{h^2}{2}-\frac{\partial^3 f}{\partial x^3}\frac{h^3}{6}+\cdots\right]\right\}=0$$

$$\frac{\partial f}{\partial t}+U\frac{\partial f}{\partial x}=-\frac{\Delta t}{2}f_{tt}+\frac{Uh}{2}f_{xx}-\frac{\Delta t^2}{6}f_{ttt}-\frac{Uh^2}{6}f_{xxx}+\cdots$$

First order in space and time
This is the modified equation
Consistency?

Modified Equation: Upwind

Express higher order time derivative terms into space derivative terms. Taking further derivatives in space and in time

$$f_{tt} + Uf_{xt} = -\frac{\Delta t}{2} f_{ttt} + \frac{Uh}{2} f_{xxt} - \frac{\Delta t^2}{6} f_{tttt} - \frac{Uh^2}{6} f_{xxxt} + \cdots$$

$$+ \quad -Uf_{tx} - U^2 f_{xx} = \frac{U\Delta t}{2} f_{ttx} - \frac{U^2 h}{2} f_{xxx} + \frac{U\Delta t^2}{6} f_{tttx} + \frac{U^2 h^2}{6} f_{xxxx} + \cdots$$

$$f_{tt} = U^2 f_{xx} + \Delta t\left(\frac{-f_{ttt}}{2} + \frac{U}{2} f_{ttx} + O(\Delta t) \right)$$

$$+\Delta x\left(\frac{U}{2} u_{xxt} - \frac{U^2}{2} u_{xxx} + O(h) \right)$$

Modified Equation: Upwind

$$f_{ttt} = -U^3 f_{xxx} + O(\Delta t, h)$$

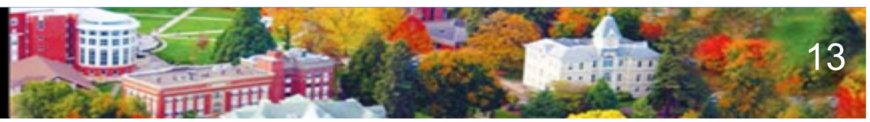
$$f_{ttx} = U^2 f_{xxx} + O(\Delta t, h)$$

$$f_{xxt} = -U f_{xxx} + O(\Delta t, h)$$

Further simplified form

$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} = \frac{Uh}{2}(1-\lambda)f_{xx} - \frac{Uh^2}{6}\left(2\lambda^2 - 3\lambda + 1\right)f_{xxx} + O\left[h^3, h^2\Delta t, h\Delta t^2, \Delta t^3\right]$$

$$\lambda = \frac{U\Delta t}{h}$$



Modified Equation: Upwind

$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} = \frac{Uh}{2}(1-\lambda)f_{xx} - \frac{Uh^2}{6}\left(2\lambda^2 - 3\lambda + 1\right)f_{xxx} + O\left[h^3, h^2\Delta t, h\Delta t^2, \Delta t^3\right]$$

$\lambda < 1$ implies that the coefficient ($Uh(1-\lambda)/2$) is positive

- Leading order truncation term has **diffusion-like** form

- Dissipative scheme

- The coefficient ($Uh(1-\lambda)/2$) if non-zero and positive, is called "**numerical viscosity**"

- Since upwind introduces numerical viscosity to advection equation, it can be used for flows with shocks. However, it is found that the discontinuities are smeared excessively owing the first-order accuracy of the scheme

Upwind Scheme

- An upwind scheme is conditionally stable

- It introduces **large amount of dissipation** (numerical or unphysical) that can overwhelm physical dissipation (if any)

- To reduce effect of the dissipation, very small grid sizes and small time-steps are needed (that can make the scheme prohibitively expensive)

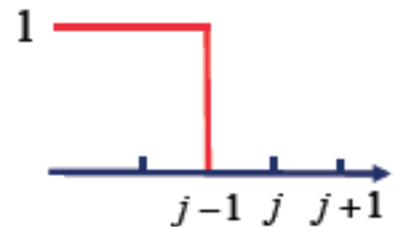
- Upwind scheme provides **bounded solution** (i.e. overshoot and undershoot are avoided); a desirable trait for many scalar advection schemes

- Upwind scheme is conservative (when applied to conservative form of equations). Note that discontinuity is smeared, but the "conservation" of a scalar over the domain is maintained

Conservative Vs Non-Conservative Schemes

-Inviscid Burger's equation (non-linear or quasi-linear)

$$\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial x} = 0 \quad \text{or} \quad \frac{\partial f}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}f^2\right) = 0$$



Non-Conservative form (FE, Upwind)

$$f_j^{n+1} = f_j^n - \frac{\Delta t}{h} f_j^n\left(f_j^n - f_{j-1}^n\right) = 0$$

Never Moves!
Unrealistic

Conservative form (FE, Upwind)

$$f_j^{n+1} = f_j^n - \frac{\Delta t}{2h}\left((f_j^n)^2 - (f_{j-1}^n)^2\right) = \frac{\Delta t}{2h}$$

$$\begin{aligned} f_j^{n+1} &= f_j^n - \frac{\Delta t}{2h}\left((f_j^n + f_{j-1}^n)(f_j^n - f_{j-1}^n)\right) \\ &= f_j^n - \frac{\Delta t}{h}\left((\frac{f_j^n + f_{j-1}^n}{2})(f_j^n - f_{j-1}^n)\right) \\ &= f_j^n - \frac{\Delta t}{h}\left((S(f_j^n - f_{j-1}^n)\right) \\ S &= \frac{f_j^n + f_{j-1}^n}{2} \end{aligned}$$

Gives the correct shock speed!

Conservative Vs Non-Conservative Schemes

- Conservative schemes are generally better than non-conservative schemes

    - while mostly true there are exceptions

- Conservative schemes can provide the correct shock speeds

- Non-conservative schemes may or may not provide the correct shock speeds

Improving Accuracy of Advection Schemes

- Leap Frog

$$\frac{\partial f}{\partial t}+U\frac{\partial f}{\partial x}=0, \quad -\infty < x < \infty, \; t \ge 0$$

$$f(x,0)=\begin{cases}1 & x<0\\ 0 & x>0\end{cases}$$

$$f_j^{n+1}=f_j^{n-1}-\frac{U\Delta t}{h}\left(f_{j+1}^n-f_{j-1}^n\right)$$

- Multilevel scheme
- needs solution at t_n, t_n+1, t_n-1
- Second order accurate in time
- second order accurate in space (central in space)
- Good for wave propagation (no damping of the wave)
- Any Issues?
- "wiggles" in solution; "dispersive errors" (errors in phase)

Improving Accuracy of Advection Schemes

- Lax Wendroff

  - Explicit addition of higher-order dissipation term for stability

$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} = 0, \quad -\infty < x < \infty, \quad t \geq 0$$

$$f(x,0) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = \frac{u^2 \Delta t}{2}\frac{\partial^2 \phi}{\partial x^2}$$

Lax-Wendroff adds a numerical dissipation explicitly

Forward Euler for time

$$f_j^{n+1} = f_j^n - \frac{U\Delta t}{2h}\left(f_{j+1}^n - f_{j-1}^n\right) + \frac{U^2 \Delta t^2}{2h^2}\left(f_{j+1}^n - 2f_j^n + f_{j-1}^n\right)$$

Central Difference
For advection

Central Difference
For added dissipation

Improving Accuracy of Advection Schemes

- Lax Wendroff Model Equation

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial t} = -u\frac{(\Delta x)^2}{6}(1 - \nu^2)\frac{\partial^3 \phi}{\partial x^3} + O(\Delta x)^3$$

- **No dissipation term** (no second derivative term!)

- Dispersive (third or odd derivative)

- **second order in space**

- dispersive errors can alter the frequency content of the signal (phase errors)

- for single frequency (smooth signals), works fine

- for sharp discontinuities, several frequencies are present (from Fourier transform). So Lax Wendroff for sharp discontinuities gives “wiggles”

- Oddity? **Steady-state solution depends on time-step**!

Improving Accuracy of Advection Schemes

- MacCormack Scheme

$$f_j^t = f_j^n - U\frac{\Delta t}{h}\left(f_{j+1}^n - f_j^n\right)$$ predictor

$$f_j^{n+1} = \frac{1}{2}\left[f_j^n + f_j^t - U\frac{\Delta t}{h}\left(f_j^t - f_{j-1}^t\right)\right]$$ corrector

- **predictor-corrector type**
- **also dispersive (wiggles present!)**

Improving Accuracy of Advection Schemes