Laplace's Equation and Potential

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Abstract. In this paper, we investigated Laplace's equation and potential. We will look and the potential of a general field, first for 4 iterations, then for 500 iterations. We will then plot a 3D list plot and compare the two. We will then do a contour plot and 3D list plot for a parallel plate capacitor's potential and discuss its behavior.

Introduction

The primary task in electrostatics is to find the electric field of a given stationary charge distribution. In principle, this purpose is accomplished by Coulomb's Law,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{r}}{r^2}\rho(\vec{r}')\,d\tau' \tag{1}$$

Unfortunately, integrals of this type can be difficult to calculate for any but the simplest charge configurations. Occasionally we can get around this by exploiting symmetry and using Gauss's law, but ordinarily, the best strategy is to calculate the potential, V,

$$\vec{V}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{r}\rho(\vec{r}')\,d\tau' \tag{2}$$

Still, even this integral is often too tough to handle analytically. Moreover, in problems involving conductors ρ itself may not be known in advance; since charge is free to move around, the only thing we control directly is the total charge (or perhaps the potential) of each conductor. In such cases, it is fruitful to recast the problem in differential form, using Poisson's equation,

$$\vec{\nabla}^2 V = -\frac{1}{\epsilon_0}\rho \tag{3}$$

Which, together with appropriate boundary conditions, is equivalent to eqn. (2). Very often, in fact, we are interested in finding the potential in a region where $\rho = 0$. In this case, Poisson's equation reduces to Laplace's equation,

$$\vec{\nabla}^2 V = 0 \tag{4}$$

This formula is so fundamental to the subject that one might almost say electrostatics is the study of Laplace's equation. In this paper we shall focus our attention to the potential in parallel plates [1].

Electrostatic Field in Vacuum

Consider a time dependent electric field in an area of space which is empty of electric charge. Maxwell's equations are reduced to Gauss's law,

$$\vec{\nabla}\cdot\vec{E}(x,y,z) = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \tag{5}$$

Together with the equation that defines the electrostatic potential,

$$\vec{E}(x,y,z) = -\vec{\nabla}V(x,y,z) \tag{6}$$

We get Laplace's equation for the function $V(x, y, z)$,

$$\vec{\nabla}^2 V(x, y, z) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{7}$$

The solution of the equation above is a boundary value problem. We are looking for the potential $V(x, y, z)$ in a region of space $\mathcal{S}$ bounded by a closed surface $\partial\mathcal{S}$. When the potential is known on $\partial\mathcal{S}$, the solution to eqn. (7) is unique and the potential and the electric field is determined everywhere in $\mathcal{S}$. For simplicity, consider the problem confined on a plane, therefore $V = V(x, y)$, thus,

$$\vec{\nabla}^2 V(x, y) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{8}$$

In order to derive a finite difference equation which approximates eqn. (7), we Taylor expand around a point (x, y) according to the equations,

$$V(x + \delta x, y) = V(x, y) + \frac{\partial V}{\partial x}\delta x + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(\delta x)^2 + \cdots \tag{9}$$

$$V(x - \delta x, y) = V(x, y) - \frac{\partial V}{\partial x}\delta x + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(\delta x)^2 + \cdots \tag{10}$$

$$V(x, y + \delta y) = V(x, y) + \frac{\partial V}{\partial y}\delta y + \frac{1}{2}\frac{\partial^2 V}{\partial y^2}(\delta y)^2 + \cdots \tag{11}$$

$$V(x, y - \delta y) = V(x, y) - \frac{\partial V}{\partial y}\delta y + \frac{1}{2}\frac{\partial^2 V}{\partial y^2}(\delta y)^2 + \cdots \tag{12}$$

By summing both sides of the above equation, taking $\delta x = \delta y$, and ignoring the terms implied by …, we obtain,

$$V(x + \delta x, y) = V(x - \delta x, y) + V(x, y + \delta y) + V(x, y - \delta y) \tag{13}$$

$$\Rightarrow \; V(x + \delta x, y) = 4V(x, y) + (\delta x)^2\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) + \cdots \tag{14}$$

$$\therefore \; V(x + \delta x, y) \approx 4V(x, y) \tag{15}$$

The second term in the second line was eliminated by using eqn. (7). We map the coordinates of the lattice points to $\mathbb{Z}\,(i, j)$ such that $x_i = (i - 1)a$ and $y_j = (j - 1)a$ where $i, j = 1, \ldots, L$. By taking $\delta x = \delta y = a$ so that $x_i \pm \delta x = x_i \pm a = (i - 1 \pm 1)a = x_{i\pm1}$ and $y_j \pm \delta y = y_j \pm a = (j - 1 \pm 1)a = y_{j\pm1}$, eqn. (15) becomes,

$$V(i, j) = \frac{1}{4}[V(i - 1, j) + V(i + 1, j) + V(i, j - 1) + V(i, j + 1)] \tag{16}$$

The equation above states that the potential at the position (i, j) is the arithmetic mean of the potential of the nearest neighbors. We will describe an algorithm which belongs to the class of "successive overrelaxation methods" (SOR) whose basic steps are:

1. Set the size L of the square lattice.
2. Flag the sites that correspond to "conductors" i.e. the sites where the potential remains fixed to the boundary condition values.
3. Choose an initial trial function for $V(x, y)$ on the vacuum sites. Of, course it is not the solution we are looking for. A good choice will lead to fast convergence of the algorithm to the true solution. A bad choice may lead to slow convergence, no convergence, or even convergence to the wrong solution. In our case, the problem is easy and the simple choice $V(x, y)$ will do.

4. Sweep the lattice and enforce eqn. (16) on each visited vacuum site. This defines the new value of the potential at this site.
5. Sweep the lattice repeatedly until two successive sweeps result in a very small change in the function $V(x, y)$.

A careful study of the above algorithm requires to test different criteria of "very small change" and test that different choices of the initial function $V(x, y)$ result in the same solution [2].

Potential of a General Field and Parallel Plate Capacitor

Now consider the potential of a general field with $L = 40$, $V_1 = 1$, and $V_2 = -1$ and compile for 4 iterations and 500 iterations. We will then compare the difference between both for this particular situation.

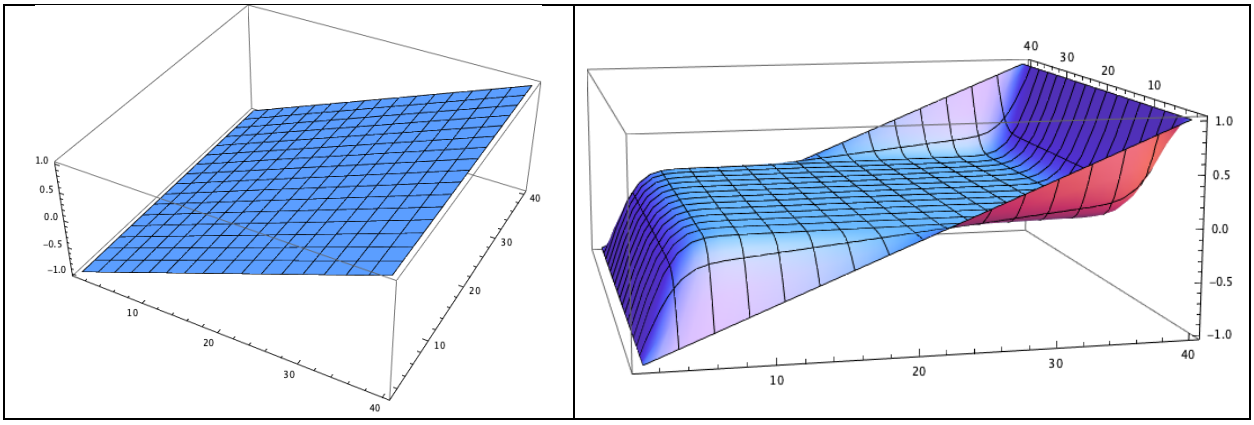

Figure 1. 3D list plot for the parallel plate potential for 4 iterations (left) and 3D list plot for the parallel plate potential for 500 iterations (right).

From figure 1, we see that as the iterations increase, the amount of detail about the potential in the system increase as well. The figure on the left has the crude general behavior of the system, but it doesn't show the different peaks and troughs as it does on the figure to the right. From this behavior, we can say that with higher amount of iterations, you will get a better description of the system. We now look at the potential for a parallel plate capacitor of $15 < L < 25$ With $V_1 = 1$ and $V_2 = -1$.

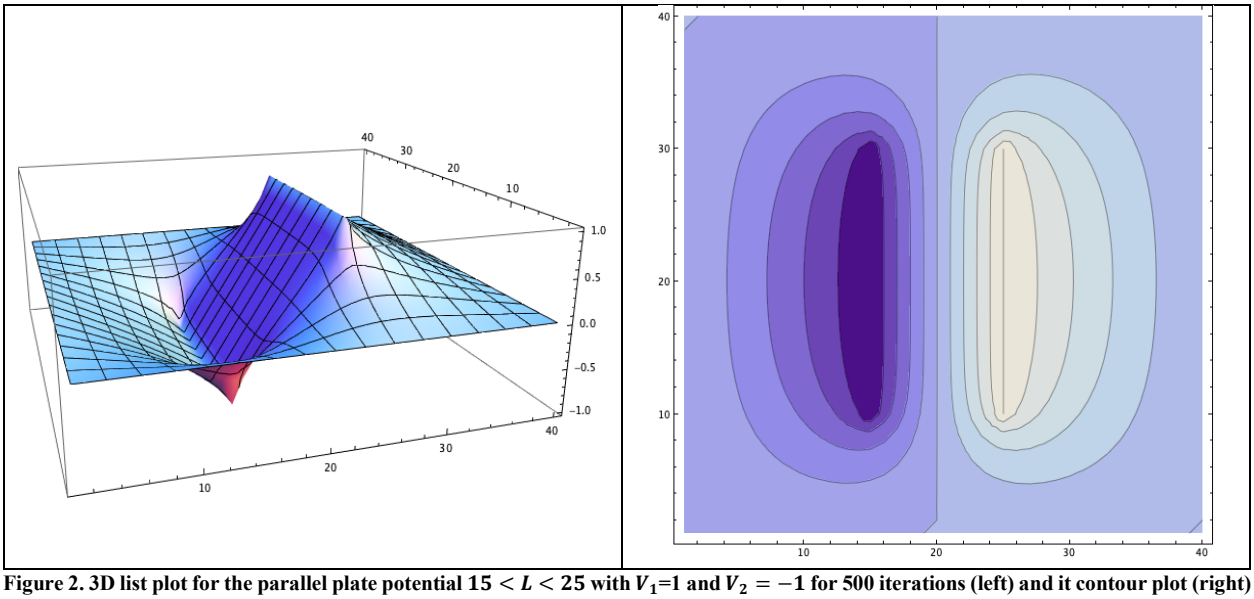

Figure 2. 3D list plot for the parallel plate potential $15 < L < 25$ with V_1=1 and $V_2 = -1$ for 500 iterations (left) and it contour plot (right)

From figure 2, we see have the 3D list plot (left) and the contour plot (right) for a parallel plate capacitor. The iterations that were used was 500. The contour plot is also included to give extra detail about the behavior of the graph for the parallel plate capacitor.

Discussion

From the paper, we obtained computer solutions to Laplace's equation using the relaxation method. This is done by starting with specified values for V at the boundary, and reasonable guesses for V on a grid of interior points. The first pass reassigns to each point the average of its nearest neighbors. The second pass repeats the process, using corrected values, and so on. After a few iterations, the numbers begin to settle down, so that subsequent passes produce negligible changes, and a numerical solution to Laplace's equation with the given boundary values is achieved [3]. We demonstrate that the amount of detail regarding the behavior of the graph is proportional to the amount of iterations. The results are given in figure 1. Furthermore, we examine the behavior of the potential for a parallel plate capacitor using this relaxation method. The results are given in figure 2.

References:

[1] D. J. Griffiths, Introduction to Electrodynamics, Pearson, **2013**.

[2] K. N. Anagnostopoulos, Computational Physics A Practical Introduction to Computional Physics and Scientific Computing, National Technical University of Athens, **2016**.

[3] E. Purcell, Electricity and Magnetism, New York: McGraw Hill, **1985**.