The Wave Equation with Respect to a String

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Abstract. In this paper, we investigate the wave equation and implement it for a string. We first give the general overview of the wave equation and derive a model applying to the string. We will then make a finite difference approximation of it allowing for compilation on FORTRAN. Finally, we plot the results for time steps 1, 30, 200, 500, 700, and 900 and discuss the difference in behavior.

Introduction

We begin our study of wave equations by simulating one-dimensional waves on a string, say on a guitar or violin. Let the string in the deformed state coincide with the interval $[0, L]$ on the x axis and let $y(x,t)$ be the displacement at time t in the y direction of a point initially at x. The displacement function y is governed by the mathematical model,

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \qquad x \in (0, L), t \in (0, T]$$

$$y(x,0) = I(x) \qquad x \in [0, L]$$

$$\frac{\partial}{\partial t} y(x,0) = 0 \qquad x \in [0, L]$$

$$y(0,t) = 0 \qquad t \in (0, T]$$

$$y(L,t) = 0 \qquad t \in (0, T]$$

The constant c and the function $I(x)$ must be prescribed. Eqn. (1) is known as the one-dimensional wave equation. Since this PDE contains a second-order derivative in time, we need two initial conditions. The condition, eqn. (2) specifies the initial shape of the string. $I(x)$ and eqn. (3) expresses that the initial velocity of the string is zero. In addition, PDEs need boundary conditions, given here as eqn. (4) and eqn. (5). These two conditions specify that the string is fixed at the ends, i.e., that the displacement y is zero. The solution $y(x,t)$ varies in space and time and describes waves that move with velocity v to the left and right [1].

Derivation of the Model and Finite Difference Approximation

From Newton's second law,

$$m\frac{d^2 y}{dt^2} = \sum F_{net}$$

$$\because \ \mu = \frac{dm}{dx} \ \Rightarrow \ m = \mu \Delta x$$

$$\therefore \ \mu \Delta x \frac{d^2 y}{dt^2} = \sum F_{net}$$

Where, $\mu \equiv density$.

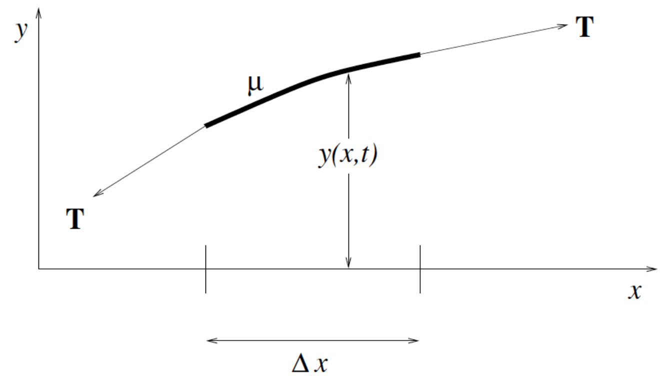

Figure 1. Graphical representation of the derivation of the model

From figure 1, the net force F_{net} is,

$$F_y = T \sin\theta_i - T \sin\theta_{i-1}$$

Here, $\sin\theta \approx \tan\theta$, so,

$$F_y = T \tan\theta_i - T \tan\theta_{i-1}$$

$$\because \quad \tan\theta_i = \frac{y_{i+1} - y_i}{\Delta x} \quad \text{and} \quad \tan\theta_{i-1} = \frac{y_i - y_{i-1}}{\Delta x}$$

$$\Rightarrow F_y = T\left[\frac{y_{i+1} - y_i}{\Delta x} - \frac{y_i - y_{i-1}}{\Delta x}\right] = T\left[\frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x}\right]$$

So,

$$\mu \Delta x \frac{d^2 y}{dt^2} = T\left[\frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x}\right]$$

$$\therefore \quad \frac{d^2 y}{dt^2} = \frac{T}{\mu}\left[\frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2}\right]$$

Comparing this to eqn. (1), this suggests,

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2}$$

Where, $v = \sqrt{\frac{T}{\mu}}$

Introduce a grid in space-time [2],

$$x_i = (i-1)\Delta x, \quad i = 1, \dots, n$$

$$t_l = l\Delta t, \quad l = 0, 1, \dots$$

Thus, the central difference approximations are,

$$\frac{\partial^2 y}{\partial x^2}(x_i, t_l) \approx \frac{y(i+1, lt) - 2y(i, lt) + y(i-1, lt)}{(\Delta x)^2}$$

And,

$$\frac{\partial^2 y}{\partial t^2}(x_i, t_l) \approx \frac{y(i, lt+1) - 2y(i, lt) + y(i, lt-1)}{(\Delta t)^2}$$

Now insert this back into eqn. (1),

$$\frac{y(i, lt+1) - 2y(i, lt) + y(i, lt-1)}{(\Delta t)^2} = v^2 \left[\frac{y(i+1, lt) - 2y(i, lt) + y(i-1, lt)}{(\Delta x)^2}\right]$$

Solve for $y(i, lt+1)$,

$$\Rightarrow \; y(i, lt+1) = \; 2y(i, lt) - y(i, lt-1) + v^2 \left(\frac{\Delta t}{\Delta x}\right)^2 [y(i+1, lt) - 2y(i, lt) + y(i-1, lt)]$$

Let, $r^2 \equiv v^2 \left(\frac{\Delta t}{\Delta x}\right)^2$, where r^2 is the Courant number,

$$\Rightarrow \; y(i, lt+1) = \; 2y(i, lt) - y(i, lt-1) + r^2[y(i+1, lt) - 2y(i, lt) + y(i-1, lt)]$$

If we take $r = 1$, the equation then reduces to,

$$y(i, lt+1) = y(i+1, lt) + (i-1, lt) - y(i, lt-1)$$

Solution for Different Time Steps

We now compile the numerical results for the wave equation in FORTRAN and do it for different time steps. We first start with time steps 1, and 30.

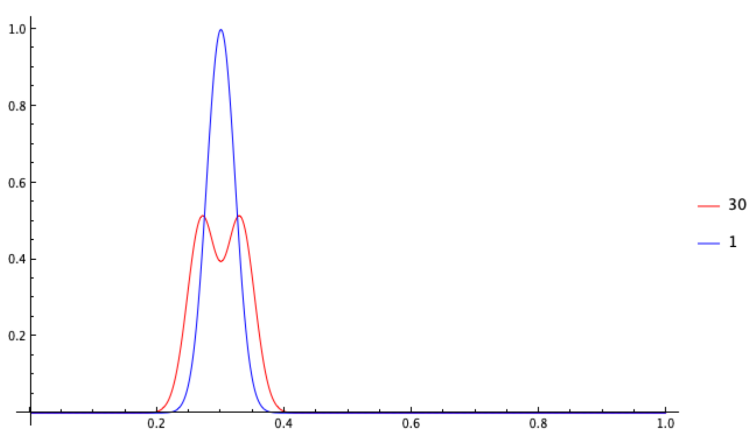

Figure 2. Solution for time steps 1, and 30

From figure 2, we consider a fixed point on the string and see how the amplitude evolves with time. see how the behavior of the graph evolves with the different time steps. At time step 1, the graph should be a gaussian function $y(x) = e^{-\alpha(x-x_0)^2}$ with $\alpha = 1000$, and $x_0 = 0.3$. We now look at the behavior of the graph for time steps 200, 500, 700, and 900.

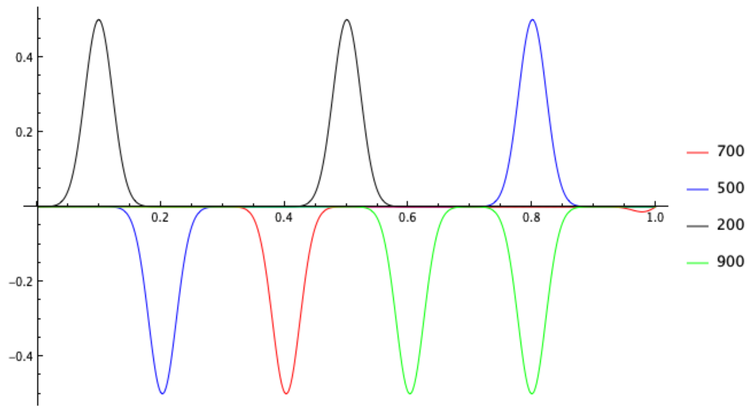

Figure 3. Solution for time steps 200, 500, 700, and 900

From figure 3, see how the behavior of the graph evolves with the different time steps. Furthermore, we produce a contour plot and 3D list plot of t with respect to x on figure 4.

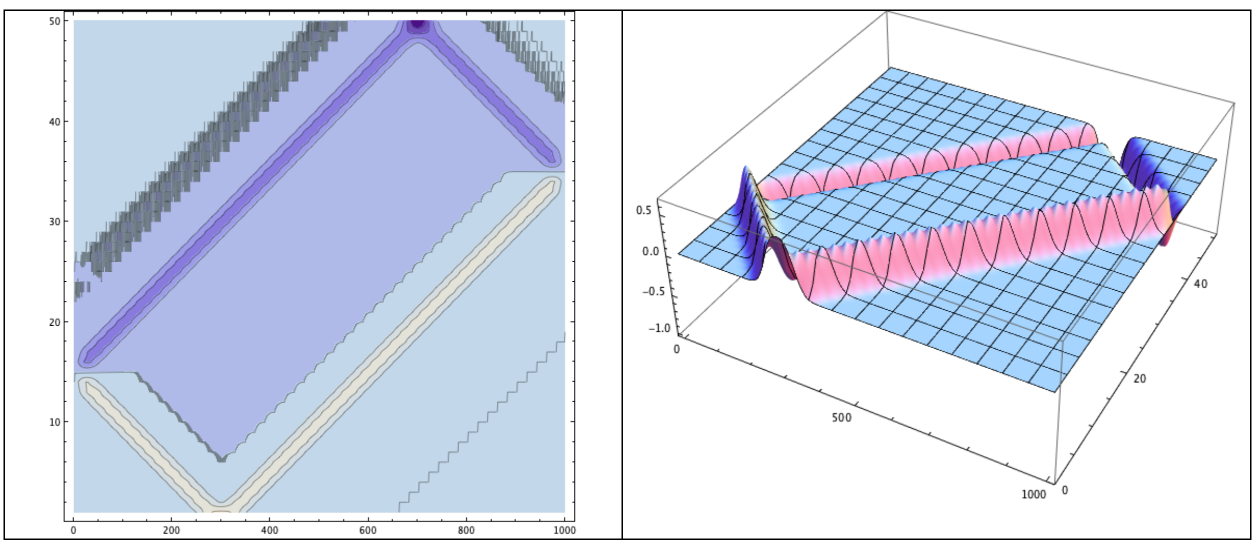

Figure 4. Contour plot (left) and 3D list plot (right) of t with respect to x

Discussion

From this paper, we have derived the model of the string and numerically solved the wave equation for different time steps. This was done so that we could utilize Fortran for the compiled data. We also showed how to express the wave equation in terms of the finite difference approximation. For the string model we consider a fixed point on the string and see how the amplitude evolves with the different time steps. Furthermore, we included a contour plot at a 3D list plot of t with respect to x given on figure 4.

References:

[1] H. P. Langtangen, Finite difference methods for wave equations, 2016.
[2] K.-A. Lie, The Wave Equation in 1D and 2D, 2005.