Chapter 3.1 - Time Value of Money

Time Value of Money

Focus of Chapter

1. Be able to compare the value of money at different points of time
    (a) Develop a method for reducing a sequence of benefits and costs to a single point in time
    (b) Make our comparisions on that basis

Time Value of money

1. Money has a time value since it can earn more money over time (AKA earning power)
2. Money has a time value where its purchasing power changes over time (AKA inflation)
3. We measure the time value of money in terms of the market interest rate which reflects both the earning and purchasing power

This isn't only just for money as well! We can associate a time-value to other things - Anything that can be exchanged for money at a margin has a time value to it

The Interest Rate

Interest is the cost of money - a cost to the borrower and an earning to the lender.
Elements of transactions involving interest:

1. Principal - amount of money burrowed/invested
2. Interest rate (i) - measure of price of money, expressed as %/Period
3. Interest Period

Cash Flow Diagram

From the perspective of the person making the transaction - upward arrows represent "income", and downward arrows represent "payments"

1. Important tool in economic analysis
2. Graphical representation of cash flows drawn on a time scale

End of Year Convention - Any cash flows occuring during the interest period are summed to a single amount and summed to a single number at the end of the interest period

1. Think about cash flows "acculiminating" throughout a period - interest rates only taken into account at the end of the period
2. This is done so banks can make financial calculations easier
3. Interest periods are typically less than 1 year

Symbols/Notations

P - Value/amount of money at present time

F - value/amount of money at future time

A - Series of consectutive, equal, end-of-period amount of money

n/N - number of interest periods (in years, months, or days)

i - interest rate per period (Compounding)

Simple Interest

This is interest being charged with respect to the **base** principal amount. **This means that the interest stays constant, regardless of the previously accumulated interest.**

$$F = P + (iP)N$$

Compound Interest/Uniform Payment Series

This is interest being charged with respect to the base principal amount **AND** previously accumulated interest that hasn't been previously withdrawn.

$$F = P(1 + i)^N$$

Also to note! All interest rates used in economic analysis are compound interest rates.

Example 1 *Paying pricipal amount and interest v. interest only*

Chapter 3.2 - Economic Equivalence

Definition 1 ***Economic Equilvalence***
Different sums of money at different times are equal in economic value (As long as interest rates are constant).

Note - This fails to be true whenever the interest rate changes between time periods

Example 2 *Find C that makes 2 transactions equivalent at $i = 10\%$. +\$500 at $t = 0$, +\$1000 at $t = 3$*

For Part A

$$500 + (1 + 0.1)^2 + 1000(1 + 0.1)^{-1}$$

For Part B

$$C + C(1 + 0.1)^1$$

Set equal to each other and solve

Chapter 3.3 - Interest Formulas for Different Types of Cash Flows

F/P Factor (Compound Amount)

To find F, given i, P, and N:

$$\begin{aligned}
&n = 0 : P \\
&n = 1 : F_1 = P(1 + i) \\
&n = 2 : F_2 = F_1(1 + i) = P(1 + i)^2 \\
&... \\
&n = N : F = P(P + i)^N
\end{aligned}$$

F/A Factor (Compound Amount Factor)

To find F, Given i, A, and N:

$$F = A(\frac{(1 + i)^n - 1}{i})$$

Note - this assumes that there are transactions occuring for every period, including on the last period.

Example 3 *Given $A = \$3000$, $N = 10$ years, and $i = 7\%$ per year, find F*

$$\begin{aligned}
A = 3000 \quad N = 10 \quad i = 7\% \\
F = A(F/A, i\%, N) \\
F = 3000(F/A, 7\%, 10) \\
F = 3000(13.8164) \\
F = \$41,449.20
\end{aligned}$$

Alternatively, using the F/A formula to find F:

$$\begin{aligned}
A = 3000 \quad N = 10 \quad i = 7\% \\
F = A(\frac{(1 + i)^n - 1}{i}) \\
F = 3000(\frac{(1 + 0.07)^10 - 1}{0.07}) \\
F = 3000(13.8164) \\
F = \$41,449.20
\end{aligned}$$

Example 4 *Derive a formula for doing consectutive deposits starting on year 0, and ending before the final period.*

To solve this, do the previous formula as normal, (assuming no deposit for the first year) but include a F/P equation for 1 year.

This still simulates the same amount of periods you're depositing money, but now you account for the final period with no deposit.

$$F = A(F/A, i\%, N)(F/P, i\%, 1)$$
$$F = 3000(F/A, 7\%, 10)(F/P, 7\%, 1)$$
$$F = \$41, 449.20(1.07)$$

Example 5 *Given 3 investment plans - Find the best plan:*

$$A - A = 2000 \quad N = 10 \quad i = 8\%$$
$$B - A = 2000 \quad N = 30 \quad i = 8\%$$
$$C - A = 2000 \quad N = 41 \quad i = 8\%$$

For Plan A, need to first do F/A formula for 1-10 years, then F/P formula for the other 31 years

$$F = 2000(F/A, 8\%, 10)(F/P, 8\%, 31)$$
$$F = 2000(14.4866)(10.8677)$$
$$F = \$314, 870$$

For Plan B, need to do F/A formula for only 31 years

$$F = 2000(F/A, 8\%, 31)$$
$$F = \$246, 691$$

For Plan C, do F/A formula for 41 years

$$F = 2000(F/A, 8\%, 41)$$
$$F = \$561, 562$$

Example 6 *Comment - Roth IRAs are a way for you to invest money - however, this is based off of your salary bracket. If you exceed a certain cap, you are blocked from investing any more into the plan itself.*

A/F Factor (Sinking Fund Factor)

To find A, Given i, F, and N

$$A = F\frac{i}{(1+i)^n - 1}$$

Note - like previous cash flow types, we assume that transactions start on the period N = 1, and continue till the last period N = N

A/P Factor (Capital Recovery Factor)

To find A, Given i, P, and N:

$$A = P(\frac{i(1+i)^n}{(1+i)^n - 1})$$

Example 7 *Find the annuity payment given:*
P = 250,000 , N = 6, i = 8%

Finding A:

$$A = P(\frac{i(1+i)^n}{(1+i)^n - 1})$$
$$A = 250,000(A/P, 8\%, 6)$$
$$A = 54,075$$

Example 8 *Deferred Loan Repayment - Continuing off of previous example:*
If the bank allows payments to occur at year 2, find A

1. Firstly, find (F/P, %, 1) - Find future value to pay 1 year before annuity payment. This is due to how the bank is continuing to collect interest on the car even though there is a grace period

$$P = 250,000(F/P, 8\%, 1)$$
$$P = 270,000$$

2. Afterwards, compute A/P

$$A = 250,000(F/P, 8\%, 1)(A/P, 8\%, 6)$$
$$A = 270,000(A/P, 8\%, 6)$$
$$A = 58,401$$

P/A Factor (Present Worth Factor)

To find P given i, A, and N:

$$P = A\left(\frac{(1+i)^n - 1}{i(1+i)^n}\right)$$

To find N given i, A, and P: Note - will have to use excel's goalseek in order to solve for N. **Set P to desired P by changing N**

Arithmetic/Linear Gradient Series

Linear Gradient Series is a cash flow series that either increases or decreases by a constant amount over n time periods.

1. A linear gradient is always comprised of 2 components - the gradient component, and the base annuity component.

P/G Factor (Present Worth Factor)

To find P given i, A, N, and G Used only with the gradient section - where you find the present worth of the linear cash flow.

1. If G ¿ 0, then you have a increasing gradient series.

2. If G ¡ 0, then you have a decreasing gradient series.

$$P = G\left[\frac{(1+i)^N - iN - 1}{i^2(1+i)^N}\right]$$

[1] [2]

[1] Note that the first cash flow in a strict linear gradient series is 0.
[2] Also, the tables don't have the future worth of a linear series, so you'll need to do it by doing P/G first, then finding F/P

Example 9 *Given:*

$$A_1 = 1000 \quad G = 250 \quad N = 5 \quad i = 12\%$$

Find P:

1. You know that gradient series are made of the base annuity and the gradient annuity - so

$$P = P_1 + P_2$$

2. P1 represents the base annuity - so:

$$P_1 = A_1(P/A, 12\%, 5)$$
$$P_1 = 1000(3.6048)$$

3. P2 represents the gradient annuity - so:

$$P_2 = 250(P/G, 12\%, 5)$$
$$P_2 = 250(6.397)$$

4. Adding P1 and P2 together:

$$P = P_1 + P_2$$
$$P = 1000(3.6048) + 250(6.397)$$
$$P = 5204$$

A/G Factor (Gradient to Equal Payment Series Factor)

Finding A given G, i, and N:

$$A = G\left[\frac{(1+i)^N - iN - 1}{i(1+i)^N - 1}\right]$$

Example 10 *Linear Gradient Example -*
Given:
A_1 *= 1000* *G = 300* *N = 6* *i = 10%*
Find:
A

1. Given that we have $A_1 = 1000$, we have our base annuity.
2. Given that we have G = 300, we have our gradient annuity.
3. Now that we have everything for the first gradient series - also, since we have our first A_1, we don't need to convert to A, however, we do need to convert our gradient annuity to A

$$1000 + 300(A/G, 10\%, 6) = A_{Barb}$$

4. Solving via looking up in tables or plugging into A/G formula:

$$A_{Barb} = 1667.08$$

Example 11 *Given:* A_1 *= 1200* *G = -200* *N = 5* *i = 10%*
Find: *F*

1. Firstly, since we don't have a F/P formula, we need to find present from gradient (P/G), when we can find the future value from the present (F/P)
2. Given that we have the base annuity and can find the future value from the annuity:

$$1200(F/A, 10\%, 5)$$

3. Finding the present value of the gradient annunity, then converting the present value to future value:

$$\begin{aligned} &= -200(P/G, 10\%, 5)(F/P, 10\%, 5) \\ &= x \end{aligned}$$

Example 12 *Given:* A_1 *= 15000* *G = -1000* *N = 12* *i = 8%*
Find: *A (Equal Payment Series)*

1. Converting Gradient Series to Equal Payment Series:

$$-1000(A/G, 8\%, 12)$$

2. Setting up Equation and solving - have first equal payment series done already

$$\begin{aligned} &= 15000 - 1000(A/G, 8\%, 12) \\ &= x \end{aligned}$$

Example 13 *Given:* A_1 *= 10000* *G = 3000* *N = 5* *i = 8%*
Find: *F*

Geometric Series

Similar to linear gradient series, but it is based off of a **percentage** rather than a fixed amount/number

$$P = \frac{A_1}{i-g}\left[1 - \left(\frac{1+g}{1+i}\right)^N\right] \quad \text{if i is not equal to g}$$
$$P = \frac{NA_1}{1+i} \quad \text{if i equals g}$$

[3]

Example 14 *Given:* $F = 100000$ $g = 6\%$ $N = 20$ $i = 8\%$
Find: A_1

1. You'd solve the problem by converting the base annuity value to present value, then convert present value to future value.
2. Setting up equation:
$$F = A_1(P/A_1, 6\%, 8\%, 20)(F/P, 8\%, 20)$$

Chapter 3.3 - Development of Formulas for Equivalence Calculations

Combining Factors

Most estimated cash flow series don't fit the factors and equations exactly. In these cases you'll be combining the engineering economy factors in order to determine either equilivalent present worth (P), future worth (F), or annual worth (A)

Shifted Uniform Series

This is a series of which whose present worth point in time is NOT t = 0, but is instead shifted to the left or to the right.

Example 15 *Reference Lecture 4 Example 1:*

To have everything balance so i = 12%

1. Not exactly describing the process, but the example goes through and continiously uses P/F to find the present worth of various items, like the annual payments and regular payments - use these to convert to present value - something like this
2. Overall - set up equations for cash flow in and out - setting these equal to each other to find x

Example 16 *Referecnce Lecture 4 Example - Cash Flows with missing payments*

1. treat problem as equal payments, but subtract one

[4]

[3] These formulas are dependent on whether i and g are equal to each other

[4] Note - check out the github page for the week 2 overview if you're confused on the concepts taught during class! This includes graphics and explainations to help explain said concepts.

What have we learned so far?

1. Time value of money
2. Economic equilvalence
3. how to look up table
4. Uniform payment series, uneven series, and gradient series.

Example 17 *Reference Lecture 4 - Practice example for quiz 1*

Find the value of x so the two cash flow shown in the diagram are equilivalent for an interest rate of 8%
(Teacher solution) Move all transactions over to the future:

$$200(F/A, 8\%, 5) - 50(F/A, 8\%, 1)$$

For the second cashflow diagram - you do the same process of moving all transactions to the future - do a recurring F/A for 5 years for x with subtracting 200+x for 1 year and 2 years respectively:

$$X(P/A, 8\%, 5) - [(200 + X)(P/F, 8\%, 3) + (200 + X)(P/F, 8\%, 4)] \quad \text{Alternatively,}$$
$$(X + 200)(F/A, 8\%, 2)(F/P, 8\%, 2)(F/P, \%, 1)$$

Set up problem by doing P/A for a constant value for 5 years:

$$200(P/A, 8\%, 5) - 50(P/A, 8\%, 1)$$

Chapter 4 - Understanding Money and its management

Ch 4 - What to expect

1. Know the different between the nominal interest rate and effective interest rate
2. The procedure for computing the effective interest rate, based on a payment period
3. How commercial loans and mortgages are structured.

Terminology

1. Ch.3
   (a) If not indicated, interest earned annually, including inflation

4.1 - Nominal and Effective Interest Rates

Definition 2 ***Nominal Interest Rate***
Interest Rate quoted based on an annual period.

Definition 3 ***Effective Interest Rate***
Actual interest earned/paid in a year or some other time period.
Note - companies/etc are required by law to give you the true effective interest rate.

Ex: Breaking down interest rate - 18% compounded monthly:

1. What does it really mean?
   (a) Interest rate per month (i) = 18%/12 = 1.5%

   (b) Number of interest periods per year = 12

2. In other words:
   (a) Bank will charge 1.5% interest every month on unpaid balance, or
   (b) You'll earn 1.5% interest each month on your remaining balance

Example 18 *If you interest \$1 for 1 year at 18% interest compounded monthly, how much would you earn?*

Formula:

$$i_a = (1 + r/M)^M - 1$$

Where:
r = nominal interest rate per year
i_a = effective annual interest rate
M = number of interest periods per year

More advanced verion:

$$i_a = (1 + r/CK)^C - 1$$

Where:
r = nominal interest rate per year
C = number of interest periods per payment period
K = number of payment periods per year CK = M = number of interest periods per year

For example, of 18% compounded monthly:

$$\begin{aligned} i_a &= (1 + 0.18/12)^1 2 - 1 \\ &= 19.56\% \end{aligned}$$

Example 19 *Lecture 4 Ch3Ch4 Example - find effective interest rate for 9% compounded quarterly:*

$$\begin{aligned} &i = 9\% \quad M = 4 \\ &r = i/M = 9/4 = 2.25\% \\ &i_a = \left(1 + \frac{0.225}{4}\right)^4 - 1 \\ &i_a = 9.31\% \end{aligned}$$

And for P = \$10,000:

$$\begin{aligned} &F = 10000(F/P, 9.31\%, 1), \quad \text{OR} \\ &F = 10000(F/P, 2.25\%, 4) \end{aligned}$$

The second approach does the same thing in the end, but you're using periods of **quarters**, not YEARS

Annual Percentage Rate (APR)

Definition 4 ***APR***
in the US, APR is expressed as the periodic interest rate times the number of compounding periods in a year (the nominal interest rate!)

Why do we need effective interest rate per payment period?

We need to do this to account for cases where payment and compounding periods differ from each other and transform so that all different cash flow types conform to the same unit of time.

Generalized Version of Compound Interest Formula:

$$i_a = (1 + r/CK)^C - 1$$

Where:
r = nominal interest rate per year
C = number of interest periods per payment period
K = number of payment periods per year CK = M = number of interest periods per year

Example 20 *Find Effective interest rate per payment period:*
9% compounded monthly Payments quarterly:

1. Know that payment periods are quarterly, so
$$K = 4$$
2. Know annual interest, so
$$r = 9\%$$
3. Know that interest compounds monthly, and there are 3 months per quarter:
$$C = 3$$
4. Plugging into compound interest equation:
$$\begin{aligned} i_a &= (1 + r/CK)^C - 1 \\ &= (1 + 9\%/(3 * 4))^3 - 1 \\ i_a &= ???? \end{aligned}$$

Example 21 *Find Effective interest rate per payment period with continious compounding:*
$C \to \infty$

Writing out formula for continious compounding:

$$\begin{aligned} i &= \lim_{C \to \infty} (1 + r/CK)^C - 1 \\ i &= e^{r/K} - 1 \end{aligned}$$

Single-Payment transactions with continious compounding - future and present worth

Finding Future Worth given Present Compounded Continiously:[5]

$$\begin{aligned} F &= P(1 + i)^N \\ &= Pe^{rN} \end{aligned}$$

Finding Present Worth given Future Compounded Continiously:

$$\begin{aligned} F &= P(1 + i)^N \\ &= Pe^{rN} \end{aligned}$$

[5]can also use the formulas used in class with continious compounding interest by doing $i_a = e^r - 1$

Equilivalence Calculations using Effective Interest Rates

1. Identify the payment period (annual,quarterly,etc)
2. Identify interest period
3. Find the effective interest rate that covers the payment periods

Case 1: When Payment Period = Compounding Period

1. Identify the number of compounding periods per year
2. Compoute the effective interest rate per year
3. Compute the payment periods per year

Example 22 *Calculating auto loan payments*

1. Price = 20,870
2. Discounts = 2,443
3. Net Sale Price = 18,427
4. Down Payment = 3,427
5. Dealer Interest Rate = 6.25%, compounded monthly
6. Length of financing = 72 months = 6 years

After taking out discounts and down payment - you're left with 15000.

1. Finding i:
$$i = 6.25/12 = 0.520833\%$$
2. Finding monthly payment:
$$A = 15,000(A/P, 0.5208\%, 72)$$
$$A = 250.37$$

Example 23 *Compute how much money you would have if you paid 3.00 every day for 30 years, assuming you earn 5% interest (nominal interest rate per year).*

1. Converting into effective interest rate:
$$i = 5\%/365 = 0.0137\%/day$$
2. Getting future worth given annuity payment:
$$F = 3(F/A, 0.0137\%, 30 * 365)$$
$$F = 76,246$$

Case 2: When payment periods differ from compounding periods

1. Identify M, K, and C
2. Compute the **effective interest rate per payment period**
   (a) For discrete compounding:
   (b) For continious compounding:
3. FInd the total number of payment periods
4. Use i and N in the appropiate equilivalence formula

Example 24 *Compounding occurs more frequently than payments are made (Discrete Case)*

A = 1,500 per quarter
N = 2 years
r = 6% compounded monthly
Find F

1. Converting interest rate to monthly:
$$i = 6\%/12 = 0.5\%/Month$$
2. Converting interest rate to quarterly:[6]
$$(1 + 0.5\%)^3 - 1 = 1.5075\%/Quarter$$
3. Getting future worth from annuity:
$$F = 1500(F/A, 1.5075\%, 8)$$

Example 25 *Compounding occurs less frequently than payments are made (Discrete Case)*

A = 500 per monthly
N = 10 years
r = 10% compounded quarterly
Find F

1. Converting interest rate to be per quarter
$$i = 10\%/4 = 2.25\%/Quarter$$
2. Splitting interest payments to be equal across each payment period[7]
3. Alternatively, describing the problem using M, K, and C

M = Compounding Periods/Year = 4
K = Payment Periods/Year = 12
C = Interest Periods/Payment Periods = 4/12 = 1/3

4. Getting monthly interest rate from C
$$i = (1 + i)^C$$
$$i = (1 + 2.5\%)^{1/3}$$
$$i = 0.26$$
5. Getting Future Worth given annuity:
$$F = (F/A,)$$

[6]Note, the 3 represents the number of interest periods in a single payment period.
[7]I'm not quite sure why we would do it this way

Chapter 4.5 Debt Management

Definition 5 ***Amortized Loan***
Loan is repaid in equal periodic amounts.

$$B_n = \textit{Remaining balance at the end of period n, with} B_0 = P$$
$$I_n = \textit{Interest Period in}$$
$$P_n$$

Example 26 *Example 4.13 - Loan Balance, principal and interest remaining - balance period*

Given:
P = 5000, 12% APR, N = 24 months
Find: Loan balance, loan

Calculating the remaining loan balance after making nth payment

Remaining-Balance Method

Alternatively, we can derive B_n[8] by computing the equivalent payments remaining after the nth payment. Thus, the balance with N-n payments remaining is:

$$B_n = A(P/A, i, N - n)$$

And the interest payment during period during n is:

$$I_n = (B_{n-1}i) = A(P/A, i, N - n + 1)i$$

Where $A(P/A, i, N - n + 1)i$ is the balance remaining at the end of period n-1 and:

$$P_n = A - I_n = A - A(P/A, i, N - n + 1)i$$
$$= A(1 - (P/A, i, N - n + 1)i)$$

Knowing the interest factor relationship (P/F,i,n) = 1-(P/A,i,n)i from table 3.6, we obtain:

$$P_n = A(P/F, i, N - n + 1)$$

Example 27 *Example 4.15 - Financing your vehicle*

Three financing options - debt, cash or lease
Payment Period - Monthly
Interest Period - Monthly
Cost = $32,508

1. Debt Financing (Find Monthly Payment)

$$A = (32508 - 4500)(A/P, \frac{5.65}{12}, 42)$$
$$A = \$736.53$$

Where the Car costs $32,508, down payment of $4,500, APR of 5.65%

2. Cash Financing

$$P = 4500 + 736.53(P/A, \frac{4.5\%}{12}, 42) - \$17,817(P/A, \frac{4.5\%}{12}, 42)$$
$$P = \$17,847$$

$$P = 31020 - 17847(P/F, 4.5\%, 42)$$
$$P = \$15,845$$

[8] B_n is the remaining balance, P_n the principal payment (Or how much of your annunity payment goes to your loan)

Home Morgage

Morgages are loans for buying property. These have mothly payments and compounding. Types of Home Morgages:

1. Fully Amortizing Mortage:
   (a) Fixed-rate mortgage
   (b) Adjustable-rate mortgage
   (c) Hybrid mortgage
2. Interest-based mortgage

The cost of mortgage:

1. Loan amount
2. Loan term
3. Payment frequency
4. Points (prepaid interest)[9]

Example 28 *Example 4.16 - Interst-only vs. Fully Amortized mortgage*

Given: P = 200000, 6.6% compounded monthly, N = 30

1. Fully Amortized payment:
$$A = 200,000(A/P, \frac{6.6}{12}, 360) = 1277.32$$
2. Five-year interest only option:
$$A = 200,000(0.0055) = 1100$$

Example 29 *Example 4.18 - Hybrid adjustable mortgage plan*

Given: Loan = 100,000, N = 30

To find annunity payments for years 1-5, do A/P for your inital loan with the term length Afterwards, if you want to find the remaining balance for the other years, take $B - n$ = A(P/A,i,term length - previous term length) - AKA for the first 5 years, the previous term length would be 60 months

Section 4.6A - Investment Basics

Right investment is a balance of liquidity, safety, and return.

1. Liquidity - should avoid being TOO liquid as you have to take into account inflation, money LOSES value
2. Safety
3. Return

Components of expected return

1. Real return
2. inflation
3. risk premium (corresponds to how much "risk" you have AKA how likely you are to lose or gain money)
4. total expected return

[9] this is where you pay a fee to lower the interest rate by a point (not a percentage! I think it's like 1/4 of a percent - this rate is about $5000 per point)