Math 451, Homework Set #1

Anthony Brice

January 25, 2015

Problem 1

Convert the following complex numbers into polar form and exponentiated form.

(a) $7 + 7i\sqrt{3}$.

(b) $-2 + 2i$.

(c) -16.

(d) $27i$.

We will follow the same general strategy in translating each expression. We first calculate r and θ via the following formulas. Given the complex number $a + bi$,

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ \tan\theta &= \frac{a}{b}. \end{aligned}$$

Then we apply the following for polar and exponential forms, respectively.

$$\begin{aligned} a + bi &= r\left(\cos\theta + i\sin\theta\right) \\ &= re^{i\theta}. \end{aligned}$$

(a)

$$\begin{aligned} 7 + 7i\sqrt{3} &\Rightarrow r = \sqrt{7^2 + \left(7\sqrt{3}\right)^2}, \tan\theta = \sqrt{3} \\ &\Rightarrow r = 14 \\ &\Rightarrow \theta = \frac{\pi}{3} \qquad * \\ \Rightarrow 7 + 7i\sqrt{3} &= 14\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \\ &= 14e^{i\pi/3}. \end{aligned}$$

* $\tan\theta = \sqrt{3} \Rightarrow \arctan\sqrt{3} = \theta$.

(b)

$$\begin{aligned} -2+2i \Rightarrow r &= \sqrt{(-2)^2+2^2},\ \tan\theta = -1 \\ \Rightarrow r &= 2\sqrt{2} \\ \Rightarrow \theta &= \frac{3\pi}{4} \qquad * \\ \Rightarrow -2+2i &= 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) \\ &= 2\sqrt{2}e^{3i\pi/4}. \end{aligned}$$

* $\theta \neq -\frac{\pi}{4}$. $a < 0$ and $b > 0$, so we are in the second quadrant.

(c)

$$\begin{aligned} -16 \Rightarrow r &= 16,\ \tan\theta = 0 \\ \Rightarrow \theta &= \pi \\ \Rightarrow -16 &= 16(\cos\pi + i\sin\pi) \\ &= -16 \\ &= 16e^{i\pi}. \end{aligned}$$

(d)

$$\begin{aligned} 27i &= 27i\sin\frac{\pi}{2} \qquad \dagger \\ &= 27e^{i\pi/2} \end{aligned}$$

† $\theta = \frac{\pi}{2}$ because we in the complex domain, baby.

Problem 2

Using the polar form, compute $(-2+2i)(7+7i\sqrt{3})$ and $(7+7i\sqrt{3})/(-2+2i)$. Check your answers by computing these directly (without the polar or exponential forms).

Using the polar forms computed in *Problem 1*, we will apply the following. Let

$$\begin{aligned} w &= r(\cos\alpha + i\sin\alpha) \\ z &= s(\cos\beta + i\sin\beta). \end{aligned}$$

Then

$$\begin{aligned} wz &= rs(\cos(\alpha+\beta) + i\sin(\alpha+\beta)) \\ \frac{w}{z} &= \frac{r}{s}\left(\cos(\alpha-\beta) + i\sin(\alpha-\beta)\right). \end{aligned}$$

Then

$$\begin{aligned}(-2+2i)\left(7+7i\sqrt{3}\right) &= 28\sqrt{2}\left(\cos\left(\frac{3\pi}{4}+\frac{\pi}{3}\right)+i\sin\left(\frac{3\pi}{4}+\frac{\pi}{3}\right)\right)\\ &= 28\sqrt{2}\left(\cos\frac{13\pi}{12}+i\sin\frac{13\pi}{12}\right).\end{aligned}$$

$$\begin{aligned}\frac{7+7i\sqrt{3}}{-2+2i} &= \frac{7}{\sqrt{2}}\left(\cos\left(\frac{\pi}{3}-\frac{3\pi}{4}\right)+i\sin\left(\frac{\pi}{3}-\frac{3\pi}{4}\right)\right)\\ &= \frac{7}{\sqrt{2}}\left(\cos\left(-\frac{5\pi}{12}\right)+i\sin\left(-\frac{5\pi}{12}\right)\right).\end{aligned}$$

Then we compute them directly.

$$\begin{aligned}(-2+2i)\left(7+7i\sqrt{3}\right) &= \left(-14-14i\sqrt{3}+14i-14\sqrt{3}\right)\\ &= -14(1+\sqrt{3})+14i(1-\sqrt{3})\\ &= \dots \qquad *\end{aligned}$$

* In this context, "…" means I don't know what to do.

Problem 3

Use de Moivre's theorem to compute $(-2+2i)^5$ and $(7+7i\sqrt{3})^4$. †

† De Moivre's theorem: Let $w = r(\cos\alpha + i\sin\alpha) = re^{i\alpha}$. Then $w^n = r^n(\cos(n\alpha)+i\sin(n\alpha)) = r^n e^{in\alpha}$.

$$\begin{aligned}(-2+2i)^5 &= \left(2\sqrt{2}e^{3i\pi/4}\right)^5\\ &= 12\sqrt{2}e^{-i\pi/4}. \qquad \ddagger\end{aligned}$$

‡ Add or subtract multiples of 2π when α moves outside the range.

$$\begin{aligned}\left(7+7i\sqrt{3}\right)^4 &= \left(14e^{i\pi/3}\right)^4\\ &= 38416e^{-2i\pi/3}.\end{aligned}$$

Problem 4

Find all cube roots of $27i$, and all fourth roots of -16.

$$\begin{aligned}\sqrt[3]{27i} &= \sqrt[3]{27e^{i(\pi/2+2\pi k)}}\ \forall\, k\in\{0,1,2\} \qquad \S\\ &= 3e^{i(\pi/6+2\pi k/3)}\ \forall\, k\in\{0,1,2\}.\end{aligned}$$

§ We're looking for cube roots, therefore we must have 3 of them.

$$\begin{aligned}\sqrt[4]{-16} &= \sqrt[4]{16e^{i(\pi+2\pi k)}} \ \forall\, k \in \{0,1,2,3\} \\ &= 2e^{i(\pi/4+\pi k/2)} \ \forall\, k \in \{0,1,2,3\}.\end{aligned}$$

Problem 5

Use the quadratic formula to solve $x^2 - (5-i)x + (8-1) = 0$.

$$\begin{aligned}x &= \frac{5-i \pm \sqrt{(5-i)^2 + 4(8-i)}}{2} \\ &= \frac{5}{2} - \frac{i}{2} \pm \frac{1}{2} - \frac{3i}{2}.\end{aligned}$$

Then $x = 3-2i$ and $x = 2+i$.

Problem 6

Compute the eighth roots of unity. Plot them on a graph and connect these points. What shape do you obtain?

$$\begin{aligned}\sqrt[8]{1} &= \sqrt[8]{e^{2i\pi k}} \ \forall\, k \in \{0,\ldots,7\} \\ &= e^{i\pi k/4} \ \forall\, k \in \{0,\ldots,7\}.\end{aligned}$$

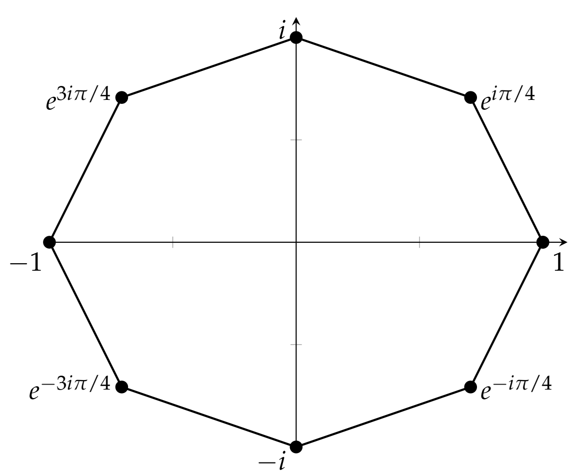

Figure 1: The eighth roots of unity, plotted on a graph and connected. Note that all roots lie on the unit circle.

We obtain an octagon, as shown in Fig. 1.

Problem 7

By using Euler's identity, show that the conjugate of $e^{i\theta}$ equals $e^{-i\theta}$.[*]

$$\begin{aligned}\overline{e^{i\theta}} &= \overline{\cos\theta + i\sin\theta} \\ &= \cos\theta - i\sin\theta && \dagger \\ &= \cos(-\theta) + i\sin(-\theta) && \ddagger \\ &= e^{-i\theta}.\end{aligned}$$

[*] Euler's identity: $e^{i\theta} = \cos\theta + i\sin\theta$. That definition is often the first question on midterms given by Dr. Sittinger.

[†] $\overline{a+b} = a - b$.

[‡] cos is an even function, so $\cos\theta$ = $\cos(-\theta)$. Similarly, sin is odd, so $-\sin\theta = \sin(-\theta)$.