Math 451, Homework Set #2

Anthony Brice

February 5, 2015

Exercise 1

Apply de Moivre's theorem to $(\cos\theta + i\sin\theta)^3$ *and equate real and imaginary parts to derive trigonometric identities for* $\cos(3\theta)$ *and* $\sin(3\theta)$.

We begin by restating de Moivre's theorem: Let $w = r(\cos\alpha + i\sin\alpha) = re^{i\alpha}$. Then $w^n = r^n(\cos(n\alpha) + i\sin(n\alpha)) = r^n e^{in\alpha}$.

By expanding the polynomial we get

$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta),$$

and by de Moivre's theorem

$$(\cos\theta + i\sin\theta)^3 = \cos(3\theta) + i\sin(3\theta).$$

Then equating the real and imaginary parts of the previous two equations yields

$$\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta$$
$$\sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

Exercise 2

Show that if $\omega \neq 1$ *is an n-th root of unity, then* $1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$. *Hint: An n-th root of unity satisfies* $z^n - 1 = 0$. *Now, factor out* $z - 1$.

Consider that via synthetic division we have that

$$z^n + 1 = (z-1)\left(z^{n-1} + z^{n-2} + \cdots + z + 1\right).$$

Then

$$a + ar + ar^2 + \cdots + ar^n = \frac{a\left(1 - r^{n+1}\right)}{1 - r}$$

if $r \neq 1$.

Let $z = \omega$, $\omega \neq 1$, then

$$\omega^{n-1} + \cdots + \omega + 1 = 0.$$

Exercise 3

Given $z_1, z_2 \in \mathbb{C}$, solve $z_1 w_1 - z_2 \overline{w_2} = 1$ and $z_1 w_2 + z_2 \overline{w_1} = 0$ for w_1 and w_2. Using properties of conjugates may be useful!

Solving for w_1 and w_2,

$$\begin{array}{r} (\overline{z_1 w_1} - z_2 w_2 = 1) \cdot z_1 \\ + \, (z_2 \overline{w_1} + z_1 w_2 = 0) \cdot \overline{z_2} \\ \hline (z_1 \overline{z_1}) w_1 + (z_2 \overline{z_2}) \overline{w_1} + 0 = z_1. \end{array}$$

Therefore

$$\begin{aligned} (z_1 \overline{z_1} + z_2 \overline{z_2}) w_1 &= z_1 \\ \Rightarrow \left(|z_1|^2 + |z_2|^2 \right) \overline{w_1} &= z_1 \\ \Rightarrow w_1 &= \frac{z_1}{|z_1|^2 + |z_2|^2}. \end{aligned}$$

And from our second equation

$$\begin{aligned} w_2 &= -\frac{z_2 \overline{w_1}}{z_1} \\ &= -\frac{z_2}{z_1} \cdot \frac{z_1}{|z_1|^2 + |z_2|^2} \\ &= -\frac{z_2}{|z_1|^2 + |z_2|^2}. \end{aligned}$$

Exercise 4

First, show that any root of unity lies on the unit circle $|z| = 1$. Then, give an example of such a complex number on the unit circle which is not a root of unity. (Remark: The amazing thing is that these latter points make up "most" of the unit circle in actuality! You need not prove this here unless you want extra credit.)

Consider $z = e^{i\theta}$. Then

$$\begin{aligned} |z|^2 &= z\overline{z} \\ &= e^{i\theta} e^{-i\theta}. \end{aligned}$$

So, $|z|^2 = 1 \Rightarrow |z| = 1$.

For a root of 1, $\theta = 2\pi k/n$ for some $k, n \in \mathbb{Z}, n > 0$. Then

$$1^{1/n} = \left(e^{0i + 2\pi k i} \right)^{1/n} = e^{2\pi k i/n}.$$

Then if θ is not a $\mathbb{Q}$-multiple of π, then $e^{i\theta}$ is not a root of 1.

Exercise 5

Describe with an equation in terms of x and y the set of points equidistant from $2-3i$ and $3+i$.

We want the set of points such that

$$\sqrt{(2-x)^2+(-3-y)^2} = \sqrt{(3-x)^2+(1-y)^2}.$$

Then our set is $\{z = x+iy \mid y = x/4 - 3/8\}$.

Exercise 6

Sketch the following sets. Then state whether each set is (i) open, (ii) closed, (iii) a domain, and (iv) bounded.

(a) $|z-2+i| \leq 1$.

(b) $|2z+3| > 4$.

(c) $\operatorname{Im} z > 1$.

(d) $\operatorname{Im} z = 1$.

(e) $0 \leq \operatorname{Arg} z \leq \frac{\pi}{4}$ *with* $z \neq 0$.

(a)

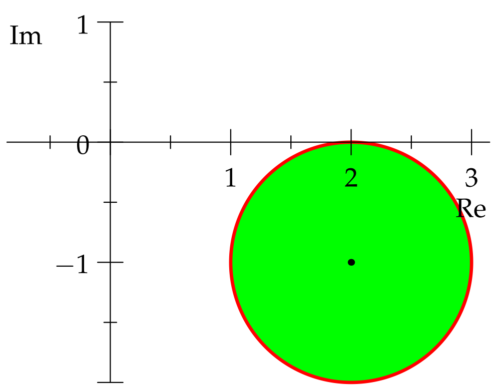

Figure 1: $|z-2+i| \leq 1$ is closed and bounded.

(b)

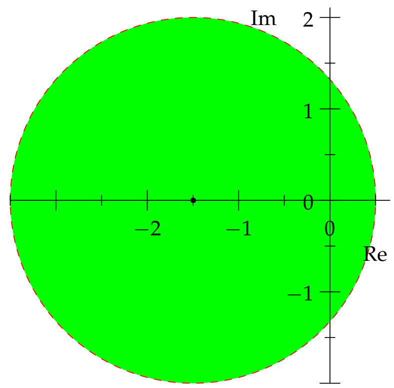

Figure 2: $|2z+3| < 4$ is bounded, and open and connected so it is a domain.

(c)

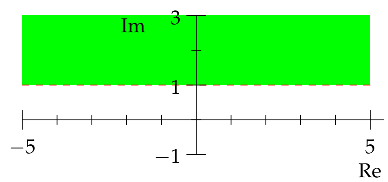

Figure 3: $\operatorname{Im} z > 1$ is open and connected, so it is a domain.

(d)

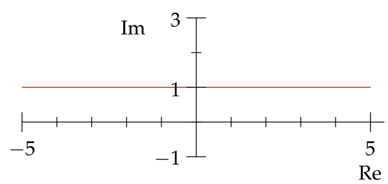

Figure 4: $\operatorname{Im} z = 1$ is closed and connected.

(e)

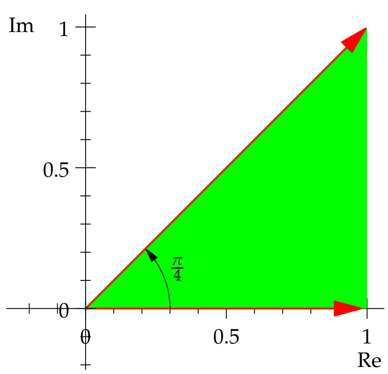

Figure 5: $0 \leq \operatorname{Arg} z \leq \frac{\pi}{4}$ with $z \neq 0$ is closed and connected