# Proactive and Adaptive Energy-Aware Programming with Hybrid Typing — Proofs

## 1. Proofs

⟨⟨⟨ **Internal vs External issue regarding : wft, abody, preservation. -Anthony** ⟩⟩⟩

**Lemma 1** (Weakening)**.**

*(1) If* $\mathtt{K} \vdash_{\mathtt{wft}} \tau$ *and* $\mathtt{K} \models \{\eta \leq \eta'\}$ *then* $\mathtt{K}, \eta \leq \eta' \vdash_{\mathtt{wft}} \tau$.

*(2) If* $\mathtt{K} \vdash \tau <: \tau'$ *and* $\mathtt{K} \models \{\eta \leq \eta'\}$ *then* $\Gamma; \mathtt{K}, \eta \leq \eta' \vdash \tau <: \tau'$.

*(3) If* $\Gamma; \mathtt{K} \vdash e : \tau$*, and* $\mathtt{K} \models \{\eta \leq \eta'\}$*, then* $\Gamma; \mathtt{K}, \eta \leq \eta' \vdash e : \tau$.

*(4) If* $\Gamma; \mathtt{K} \vdash e : \tau$*, and* $\Gamma \vdash \mathtt{y} : \tau'$*, then* $\Gamma, \mathtt{y} : \tau'; \mathtt{K} \vdash e : \tau$.

*Proof.* Each is proved by straightforward induction on the derivations of $\mathtt{K} \vdash_{\mathtt{wft}} \tau$, $\mathtt{K} \vdash \tau <: \tau'$, and $\Gamma; \mathtt{K} \vdash e : \tau$. □

**Lemma 2.** *If* $\mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \models \{\eta_2 \leq \eta_2'\}$, $\mathtt{K} \models \{\eta \leq \eta'', \eta'' \leq \eta'\}$*, and* $\mathtt{mt} \notin \mathtt{K}$*, then* $\mathtt{K}\{\eta''/\mathtt{mt}\} \models \{\eta_2\{\eta''/\mathtt{mt}\} \leq \eta_2'\{\eta''/\mathtt{mt}\}\}$.

*Proof.* Trivial. □

**Lemma 3** (Mode Substitution Perserves Type Well-Formedness)**.** *If* $\mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash_{\mathtt{wft}} \tau$, $\mathtt{K} \models \{\eta \leq \eta'', \eta'' \leq \eta'\}$*, and* $\mathtt{mt} \notin \mathtt{K}$*, then* $\mathtt{K}\{\eta''/\mathtt{mt}\} \vdash_{\mathtt{wft}} \tau\{\eta''/\mathtt{mt}\}$.

*Proof.* By induction on the derivation of $\mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash_{\mathtt{wft}} \tau$.

*Case* WF-Top $\quad T = \mathtt{Object}\langle\eta\rangle$

Trivial.

*Case* WF-MCase $\quad T = \mathbf{mcase}\langle T_1\rangle$

$\mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash_{\mathtt{wft}} T_1$

By the induction hypothesis, $\mathtt{K}\{\eta''/\mathtt{mt}\} \vdash_{\mathtt{wft}} T_1\{\eta''/\mathtt{mt}\}$. Then, by WF-MCase, $\mathtt{K}\{\eta''/\mathtt{mt}\} \vdash_{\mathtt{wft}} \mathbf{mcase}\langle T_1\{\eta''/\mathtt{mt}\}\rangle$.

*Case* WF-Class $\quad T = \mathsf{c}\langle\overline{\eta}\rangle$

$\mathbf{class}\ \mathsf{c}\ \Delta\ \mathbf{extends}\ \mathsf{c}'\ \cdots \in P \qquad \mathtt{eparam}(\Delta) = \iota' \qquad \mathtt{cons}(\Delta) = \mathtt{K}'$

$\mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \models \mathtt{K}'\{\overline{\eta}/\iota'\} \qquad \mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash_{\mathtt{wft}} \mathsf{c}'\langle\overline{\eta}\rangle$

By the induction hypothesis, $\mathtt{K}\{\eta''/\mathtt{mt}\} \vdash_{\mathtt{wft}} \mathsf{c}'\langle\overline{\eta}\rangle\{\eta''/\mathtt{mt}\}$. Lemma 2 gives us $\mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \models \mathtt{K}'\{\iota/\iota'\}\{\eta''/\mathtt{mt}\}$.

Then, by WF-Class, $\mathtt{K}\{\eta''/\mathtt{mt}\} \vdash_{\mathtt{wft}} \mathsf{c}\langle\overline{\eta}\rangle\{\eta''/\mathtt{mt}\}$.

*Case* WF-ClassDyn $\quad T = \mathsf{c}\langle ?, \overline{\eta}\rangle$

$\mathbf{class}\ \mathsf{c}\ ? \to \omega, \Omega\ \mathbf{extends}\ \mathsf{c}'\ \cdots \in P \qquad \mathtt{eparam}(? \to \omega, \Omega) = \iota' \qquad \mathtt{cons}(? \to \omega, \Omega) = \mathtt{K}'$

$\mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \models \mathtt{K}'\{\overline{\eta}/\iota'\} \qquad \mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash_{\mathtt{wft}} \mathsf{c}'\langle ?, \overline{\eta}\rangle$

Similar.

□

**Lemma 4** (Mode Substitution Perserves Subtyping)**.** *If* $\mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash \tau <: \tau'$, $\mathtt{K} \models \{\eta \leq \eta'', \eta'' \leq \eta'\}$, *and* $\mathtt{mt} \notin \mathtt{K}$, *then* $\mathtt{K}\{\eta''/\mathtt{mt}\} \vdash \tau\{\eta''/\mathtt{mt}\} <: \tau'\{\eta''/\mathtt{mt}\}$.

*Proof.* Induction on the derivation of $\mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash \tau <: \tau'$.

*Case* (S-Dynamic) $\quad \tau = \mathtt{c}\langle \mu, \overline{\eta}\rangle \quad \tau' = \mathtt{c}\langle ?, \overline{\eta}\rangle$
If $\mu = \mathtt{mt}$, then we have $\mathtt{K}\{\eta''/\mathtt{mt}\} \vdash \mathtt{c}\langle \eta'', \overline{\eta}\{\eta''/\mathtt{mt}\}\rangle <: \mathtt{c}\langle ?, \overline{\eta}\{\eta''/\mathtt{mt}\}\rangle$. If $\mu \neq \mathtt{mt}$, then we have $\mathtt{K}\{\eta''/\mathtt{mt}\} \vdash \mathtt{c}\langle \mu, \overline{\eta}\{\eta''/\mathtt{mt}\}\rangle <: \mathtt{c}\langle ?, \overline{\eta}\{\eta''/\mathtt{mt}\}\rangle$. Both cases are exactly what is needed.

*Case* (S-Mcase) $\quad \tau = \textbf{mcase}\langle T_1\rangle \qquad \tau' = \textbf{mcase}\langle T_1'\rangle$
$\mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash T_1 <: T_1'$

By the induction hypothesis, $\mathtt{K}\{\eta''/\mathtt{mt}\} \vdash T_1\{\eta''/\mathtt{mt}\} <: T_1'\{\eta''/\mathtt{mt}\}$ Then, by S-MCase, $\mathtt{K}\{\eta''/\mathtt{mt}\} \vdash \textbf{mcase}\langle T_1\{\eta''/\mathtt{mt}\}\rangle <: \textbf{mcase}\langle T_1'\{\eta''/\mathtt{mt}\}\rangle$.

*Case* (S-Exists) $\quad \tau = \exists\omega.\tau_1 \qquad \tau' = \tau_1$
$\omega = \eta_1 \leq \mathtt{mt}_1 \leq \eta_2 \quad \mathtt{K}_1, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta', \mathtt{K}_2 \models \{\eta_1 \leq \mathtt{mt}_1, \mathtt{mt}_1 \leq \eta_2\} \cup \mathtt{K}' \quad \mathtt{mt}_1 \notin \mathtt{K}'$

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*Case* (S-Class) $\quad \tau = \mathtt{c}\langle\iota\rangle \qquad \tau' = \mathtt{c}'\langle\iota\rangle$
**class** $\mathtt{c}\ \Delta$ **extends** $\mathtt{c}'\ \ldots \in P \qquad \mathtt{eparam}(\Delta) = \iota'$
$\mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' = \mathtt{cons}(\Delta)$
$\mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash_{\mathtt{wft}} \mathtt{c}'\langle\iota\rangle$

By Lemma 3 we have $\mathtt{K}\{\eta''/\mathtt{mt}\} \vdash_{\mathtt{wft}} \mathtt{c}\langle\iota\rangle\{\eta''/\mathtt{mt}\}$ and $\mathtt{K}\{\eta''/\mathtt{mt}\} \vdash_{\mathtt{wft}} \mathtt{c}'\langle\iota\rangle\{\eta''/\mathtt{mt}\}$. Lemma 2 we have $\mathtt{K}\{\eta''/\mathtt{mt}\} \models \mathtt{cons}(\Delta)\{\eta''/\mathtt{mt}\}$.

Then, by S-Class, $\mathtt{K}\{\eta''/\mathtt{mt}\} \vdash \mathtt{c}\langle\iota\{\eta''/\mathtt{mt}\}\rangle <: \mathtt{c}'\langle\iota\{\eta''/\mathtt{mt}\}\rangle$.

□

**Lemma 5.** *If* $\mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash_{\mathtt{wft}} \mathtt{c}\langle\iota\rangle$, $\mathtt{K} \models \{\eta \leq \eta'', \eta'' \leq \eta'\}$, $\mathtt{mt} \notin \mathtt{K}$, *and* $\mathtt{mtype}(\mathtt{md}, \mathtt{c}\langle\iota\rangle) = \overline{T} \to T$ *then* $\mathtt{mtype}(\mathtt{md}, \mathtt{c}\langle\iota\rangle\{\eta''/\mathtt{mt}\}) = \overline{T\{\eta''/\mathtt{mt}\}} \to T'\{\eta''/\mathtt{mt}\}$.

*Proof.* Induction on the derivation of $\mathtt{mtype}(\mathtt{md}, \mathtt{c}\langle\iota\rangle) = \overline{T} \to T$.

*Case* MT-Class

*Case* MT-Super

□

**Lemma 6.** *If* $\mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash_{\mathtt{wft}} \mathtt{c}\langle\iota\rangle$, $\mathtt{K} \models \{\eta \leq \eta'', \eta'' \leq \eta'\}$, $\mathtt{mt} \notin \mathtt{K}$ *and* $\mathtt{fields}(T) = \overline{T}\ \overline{\mathtt{fd}}$ *then* $\mathtt{fields}(\mathtt{c}\langle\iota\rangle\{\eta''/\mathtt{mt}\}) = \overline{T\{\eta''/\mathtt{mt}\}}\ \overline{\mathtt{fd}}$.

*Proof.* Induction on the derivation of $\mathtt{fields}(\mathtt{md}, \mathtt{c}\langle\iota\rangle) = \overline{T}\ \overline{\mathtt{fd}}$.

*Case* FD-Class

*Case* FD-Object

□

**Lemma 7** (Mode Substitution Preserves Typing)**.** *If* $\Gamma; \mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash e : \tau$, $\mathtt{K} \models \{\eta \leq \eta'', \eta'' \leq \eta'\}$, *and* $\mathtt{mt} \notin \mathtt{K}$, *then* $\Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}\{\eta''/\mathtt{mt}\} \vdash e\{\eta''/\mathtt{mt}\} : \tau\{\eta''/\mathtt{mt}\}$.

*Proof.* Induction on the derivation of $\Gamma; \mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash e : \tau$.

*Case* T-Var $\quad e = \mathtt{x} \quad \tau = \Gamma(\mathtt{x})$

**⟨⟨⟨ Our substitution does not effect types directly; it acts on thier parameteres. I think I need a subcase analysis here. -Anthony ⟩⟩⟩**

*Case* T-New

$$\begin{array}{ll} e = \textbf{new}\ \mathtt{c}\langle\iota\rangle & \tau = \mathtt{c}\langle\iota\rangle \\ \iota = ?, \iota' \text{ iff } \textbf{class}\ \mathtt{c}\ \Delta \cdots \in P \text{ and } \mathtt{ethis}(\Delta) = ? & \\ \iota \neq ?, \iota' \text{ iff } \textbf{class}\ \mathtt{c}\ \Delta \cdots \in P \text{ and } \mathtt{ethis}(\Delta) \neq ? & \\ \mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \models \mathtt{cons}(\Delta) & \mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash_{\mathtt{wft}} \mathtt{c}\langle\iota\rangle \end{array}$$

Using Lemmas 2 and 3 gives us $\mathtt{K}\{\eta''/\mathtt{mt}\} \models \mathtt{cons}(\Delta)\{\eta''/\mathtt{mt}\}$ and $\mathtt{K}\{\eta''/\mathtt{mt}\} \vdash \mathtt{c}\langle\iota\rangle\{\eta''/\mathtt{mt}\}$. Then, by T-New, we have $\Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}\{\eta''/\mathtt{mt}\} \vdash \textbf{new}\ \mathtt{c}\langle\iota\rangle\{\eta''/\mathtt{mt}\} : \mathtt{c}\langle\iota\rangle\{\eta''/\mathtt{mt}\}$.

*Case* T-Cast

$$\begin{array}{ll} e = (T)e_1 & \tau = T \\ \Gamma; \mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash e_1 : T' & \end{array}$$

Easy.

*Case* T-Msg

$$\begin{array}{ll} e = e_1.\mathtt{md}(\overline{e_1}) & \tau = T \\ \Gamma; \mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash e_1 : \mathtt{c}\langle\iota\rangle & \Gamma; \mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash \overline{e_1} : \overline{T} \\ \mathtt{mtype}(\mathtt{md}, \mathtt{c}\langle\iota\rangle) = \overline{T} \to T & \Gamma; \mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash \textbf{this} : T_{this} \\ \mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \models \{\mathtt{mode}(\mathtt{c}\langle\iota\rangle) \leq \mathtt{mode}(T_{this})\} & \mathtt{mode}(\mathtt{c}\langle\iota\rangle) \neq\ ? \end{array}$$

By the induction hypothesis we have,

$$\begin{array}{l} \Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}\{\eta''/\mathtt{mt}\} \vdash \underline{e_1\{\eta''/\mathtt{mt}\}} : \underline{\mathtt{c}\langle\iota\rangle\{\eta''/\mathtt{mt}}\} \\ \Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}\{\eta''/\mathtt{mt}\} \vdash \overline{e_1\{\eta''/\mathtt{mt}\}} : \overline{T\{\eta''/\mathtt{mt}\}} \\ \Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}\{\eta''/\mathtt{mt}\} \vdash \textbf{this}\{\eta''/\mathtt{mt}\} : T_{this}\{\eta''/\mathtt{mt}\}. \end{array}$$

Now, by Lemma 5 we have $\mathtt{mtype}(\mathtt{md}, \mathtt{c}\langle\iota\rangle\{\eta''/\mathtt{mt}\}) = \overline{T\{\eta''/\mathtt{mt}\}} \to T\{\eta''/\mathtt{mt}\}$.

Using Lemma 2 gives us $\mathtt{K}\{\eta''/\mathtt{mt}\} \models \{\mathtt{mode}(\mathtt{c}\langle\iota\rangle\{\eta''/\mathtt{mt}\}) \leq \mathtt{mode}(T_{this}\{\eta''/\mathtt{mt}\})\}$.

Then, by T-Msg, we have $\Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}\{\eta''/\mathtt{mt}\} \vdash e_1\{\eta''/\mathtt{mt}\}.\mathtt{md}(\overline{e_1\{\eta''/\mathtt{mt}\}}) : T\{\eta''/\mathtt{mt}\}$.

*Case* T-Field

$$\begin{array}{ll} e = e_1.\mathtt{fd}_i & \tau = T_i \\ \Gamma; \mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash e_1 : \mathtt{c}\langle\iota\rangle & \mathtt{fields}(\mathtt{c}\langle\iota\rangle) = \overline{T}\ \overline{\mathtt{fd}} \\ \Gamma; \mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash \textbf{this} : T_{this} & \mathtt{K}\eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \models \{\mathtt{mode}(\mathtt{c}\langle\iota\rangle) \leq \mathtt{mode}(T_{this})\} \\ \mathtt{mode}(\mathtt{c}\langle\iota\rangle) \neq ? & \end{array}$$

By the induction hypothesis we have,

$$\begin{array}{l} \Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}\{\eta''/\mathtt{mt}\} \vdash e_1\{\eta/\mathtt{mt}\} : \mathtt{c}\langle\iota\rangle\{\eta''/\mathtt{mt}\} \\ \Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}\{\eta''/\mathtt{mt}\} \vdash \textbf{this} : T_{this}\{\eta/\mathtt{mt}\}. \end{array}$$

Now by Lemma 6 we have $\mathtt{fields}(\mathtt{c}\langle\iota\rangle\{\eta''/\mathtt{mt}\}) = \overline{T\{\eta''/\mathtt{mt}\}}\ \overline{\mathtt{fd}}$.

Using Lemma 2 gives us $\mathtt{K}\{\eta''/\mathtt{mt}\} \models \{\mathtt{mode}(\mathtt{c}\langle\iota\rangle\{\eta''/\mathtt{mt}\}) \leq \mathtt{mode}(T_{this}\{\eta''/\mathtt{mt}\})\}$.

Then, by T-Field, we have $\Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}\{\eta''/\mathtt{mt}\} \vdash e_1\{\eta''/\mathtt{mt}\}.\mathtt{fd}_i : T_i\{\eta''/\mathtt{mt}\}$.

*Case* T-Snapshot

$$\begin{array}{ll} e = \textbf{snapshot}\ e_1\ [\eta_1, \eta_2] & \tau = \exists\omega.\mathtt{c}\langle\mathtt{mt}_1, \iota\rangle \\ \Gamma; \mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash e_1 : \mathtt{c}\langle ?, \iota\rangle & \omega = \eta_1 \leq \mathtt{mt}_1 \leq \eta_2 \end{array}$$

By the induction hypothesis, $\Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}\{\eta''/\mathtt{mt}\} \vdash e_1\{\eta''/\mathtt{mt}\} : \mathtt{c}\langle ?, \iota\rangle\{\eta''/\mathtt{mt}\}$.

Now, consider $\omega = \eta_1 \leq \mathtt{mt}_1 \leq \eta_2$: $\mathtt{mt}_1$ must be unique; hence, $(\eta_1 \leq \mathtt{mt}_1 \leq \eta_2\{\eta''/\mathtt{mt}\})$ is $\eta_1\{\eta''/\mathtt{mt}\} \leq \mathtt{mt}_1 \leq \eta_2\{\eta''/\mathtt{mt}\}$ by Lemma **??**.

Then, by T-Snapshot,

$$\Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}\{\eta''/\mathtt{mt}\} \vdash \textbf{snapshot}\ e_1\{\eta''/\mathtt{mt}\}\ [\eta_1\{\eta''/\mathtt{mt}\}, \eta_2\{\eta''/\mathtt{mt}\}] : \exists\omega\{\eta''/\mathtt{mt}\}.\mathtt{c}\langle \mathtt{mt}_1, \iota\{\eta''/\mathtt{mt}\}\rangle$$

⟨⟨⟨ **Come back to prove. -Anthony** ⟩⟩⟩

*Case* T-MCase $\quad e = \{\overline{\mathtt{m} : e_1}\}^T \qquad \tau = \textbf{mcase}\langle T\rangle$
$\Gamma; \mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash e_{1_i} : T$ for all $i \qquad \overline{\mathtt{m}} = \mathtt{modes}(P)$

Easy.

*Case* T-ElimCase $\quad e = e_1 \triangleright \eta_1 \qquad \tau = T$
$\Gamma; \mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash e_1 : \textbf{mcase}\langle T\rangle \qquad \eta_1 \in \mathtt{modes}(P)$ or $\eta_1$ appears in $\mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta'$

Easy.

*Case* T-Mode $\quad e = \mathtt{m} \quad \tau = \mathtt{modev}$
Trivial.

*Case* T-Sub $\quad e = e_1 \qquad \tau = \tau_1'$
$\Gamma; \mathtt{K}, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash e_1 : \tau_1 \qquad \mathtt{K}\eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta' \vdash \tau_1 <: \tau_1'$

By the induction hypothesis, $\Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}\{\eta''/\mathtt{mt}\} \vdash e_1\{\eta''/\mathtt{mt}\} : \tau_1\{\eta''/\mathtt{mt}\}$. Using Lemma 4 gives us $\mathtt{K}\{\eta''/\mathtt{mt}\} \vdash \tau_1\{\eta''/\mathtt{mt}\} <: \tau_1'\{\eta''/\mathtt{mt}\}$

Then, by T-Sub, we have $\Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}\{\eta''/\mathtt{mt}\} \vdash e_1\{\eta''/\mathtt{mt}\} : \tau'\{\eta''/\mathtt{mt}\}$.

⟨⟨⟨ **Finish the proof. -Anthony** ⟩⟩⟩

□

**Lemma 8** (Term Substitution Perserves Typing)**.** *If* $\Gamma, \mathtt{y} : \tau_0; \mathtt{K} \vdash e : \tau$ *and* $\Gamma; \mathtt{K} \vdash \mathtt{s} : \tau_0$ *then* $\Gamma\{\mathtt{s}/\mathtt{y}\}; \mathtt{K} \vdash e\{\mathtt{s}/\mathtt{y}\} : \tau$.

*Proof.* Easy induction on the derivation of $\Gamma, \mathtt{y} : \tau_0; \mathtt{K} \vdash e : \tau$.

□

**Lemma 9.** *If* $\mathtt{K} \vdash_{\mathtt{wft}} \mathtt{c}\langle\iota\rangle$, $\mathtt{mtype}(\mathtt{md}, \mathtt{c}\langle\iota\rangle) = \overline{T} \rightarrow T$ *and* $\mathtt{mbody}(\mathtt{md}, \mathtt{c}\langle\iota\rangle) = \overline{\mathtt{x}}.e$ *then* $\overline{\mathtt{x}} : \overline{T}; \textbf{this} : T; \mathtt{K} \vdash e : T$.

*Proof.* Induction on the derivation of $\mathtt{mbody}(\mathtt{md}, \mathtt{c}\langle\iota\rangle) = \overline{\mathtt{x}}.e$ using Lemmas 4 and 7.

*Case* MB-Class $\quad \overline{\mathtt{x}}.e = \overline{\mathtt{y}}.e_0\{\iota/\iota'\}$
$\textbf{class}\ \mathtt{c}\ \Delta\ \textbf{extends}\ \mathtt{c}'\{\overline{F}\ \overline{M}\ A\} \quad \mathtt{eparam}(\Delta) = \iota'$
$T_0\ \mathtt{md}(\overline{T_0}\ \overline{\mathtt{y}})\{\ e_0\ \} \in \overline{M}$

From T-Class and T-Method we have $\overline{\mathtt{y}} : \overline{T_0}; \textbf{this} : \mathtt{c}\langle\iota\rangle; \mathtt{K}' \vdash e_0 : T_0$. Since $\mathtt{K} \vdash_{\mathtt{wft}} \mathtt{c}\langle\iota\rangle$ we have $\mathtt{K} \models \mathtt{K}'\{\iota/\iota'\}$ and $\mathtt{K}' = \mathtt{cons}(\Delta)$ from WF-Class. Using Lemmas 1 and 7 we have $\overline{\mathtt{y}} : \overline{T_0}\{\iota/\iota'\}; \textbf{this} : \mathtt{c}\langle\iota\rangle; \mathtt{K} \vdash e_0\{\iota/\iota'\} : T_0\{\iota/\iota'\}$.

Then, by MT-Class we have $\overline{T_0}\{\iota/\iota'\} = \overline{T}$ and $T_0\{\iota/\iota'\} = T$, from which $\overline{\mathtt{x}} : \overline{T}, \textbf{this} : \mathtt{c}\langle\iota\rangle; \mathtt{K} \vdash e : T$ is immediate.

*Case* MB-Super $\quad \overline{\mathtt{x}}.e = \mathtt{mbody}(\mathtt{md}, \mathtt{c}'\langle\iota\rangle)$
$\quad$ **class** $\mathtt{c}\ \Delta$ **extends** $\mathtt{c}'\{\overline{F}\ \overline{M}\ A\} \quad \mathtt{md} \notin \overline{M}$

Immediate from the inductive hypothesis and the fact that $\mathtt{K} \vdash \mathtt{c}\langle\iota\rangle <: \mathtt{c}'\langle\iota\rangle$.

□

**Lemma 10.** *If* $\Gamma; \mathtt{K} \vdash \mathtt{obj}(\alpha, \mathtt{c}\langle\iota\rangle, \overline{v})$ *and* $\mathtt{fields}(\mathtt{c}\langle\iota\rangle) = \overline{T}\ \overline{\mathtt{fd}} = \overline{e}$ *then* $\Gamma; \mathtt{K} \vdash v_i : T_i'$.

*Proof.* Induction on the derivation of $\mathtt{fields}(\mathtt{c}\langle\iota\rangle) = \overline{\tau}\ \overline{\mathtt{fd}}$.

*Case* FD-Class $\quad$ **class** $\mathtt{c}\ \Delta$ **extends** $\mathtt{c}'\{\overline{T_0}\ \overline{\mathtt{fd}} = \overline{e_0}\ \ldots\} \quad \mathtt{eparam}(\Delta) = \iota'$
$\quad \mathtt{fields}(\mathtt{c}'\langle\iota\rangle) = \overline{T_1}\ \overline{\mathtt{fd}_1} = \overline{e_1}$

From T-Class we have $\varnothing; \mathtt{K}' \vdash \overline{e_0} : \overline{T_0}$. Since $\mathtt{K} \vdash_{\mathtt{wft}} \mathtt{c}\langle\iota\rangle$ we have $\mathtt{K} \models \mathtt{K}'\{\iota/\iota'\}$ and $\mathtt{K}' = \mathtt{cons}(\Delta)$ from WF-Class. Using Lemmas 1 and 7 we have $\Gamma; \mathtt{K} \vdash \overline{e_0}\{\iota/\iota'\} : \overline{T_0}\{\iota/\iota'\}$

Now, from T-Object we have $\overline{T_1}, \overline{T_0}\{\iota/\iota'\} = \overline{T}$ and $\overline{e_1}, \overline{e_0}\{\iota/\iota'\} = \overline{v}$. Choosing $\Gamma; \mathtt{K} \vdash v_i : T_i$ finishes the case.

*Case* FD-Object Trivial.

□

**Lemma 11** (Preservation)**.** *If* $\Gamma; \mathtt{K} \vdash e : \tau$, $e \overset{\mathtt{m}}{\Longrightarrow} e'$, *then* $\Gamma; \mathtt{K} \vdash e : \tau$.

*Proof.* By induction on the derivation of $\Gamma, \mathtt{K} \vdash e : \tau$, with a case analysis on the last rule used.

*Case* T-Var $\quad e = \mathtt{x} \quad \tau = \Gamma(\mathtt{x})$
Trival: Cannot occur.

*Case* T-New $\quad e =$ **new** $\mathtt{c}\langle\iota\rangle \qquad \tau = \mathtt{c}\langle\iota\rangle$
$\quad \iota = ?, \iota'$ iff **class** $\mathtt{c}\ \Delta \cdots \in P$ and $\mathtt{ethis}(\Delta) = ?$
$\quad \iota \neq ?, \iota'$ iff **class** $\mathtt{c}\ \Delta \cdots \in P$ and $\mathtt{ethis}(\Delta) \neq ?$
$\quad \mathtt{K} \models \mathtt{cons}(\Delta)$
Trivial.

*Case* T-Cast $\quad e = (T)e_1 \qquad \tau = T$
$\quad \Gamma; \mathtt{K} \vdash e_1 : T_1$

*Subcase* $\quad e_1 \overset{\mathtt{m}}{\Longrightarrow} e_1' \quad e' = (T)e_1'$
Easy.

*Subcase* $\quad e_1 = \mathtt{obj}(\alpha, T_1, \overline{v})$
$\quad (T)\mathtt{obj}(\alpha, T_1, \overline{v}) \overset{\mathtt{m}}{\Longrightarrow} \mathtt{obj}(\alpha, T_1, \overline{v}) \quad T_1 <: T$
$\quad e' = \mathtt{obj}(\alpha, T_1, \overline{v})$
Trivial. We have $\Gamma; \mathtt{K} \vdash \mathtt{obj}(\alpha, T_1, \overline{v}) : T_1$ from T-Cast and T-Obj. Then, by T-Sub we have $\Gamma; \mathtt{K} \vdash \mathtt{obj}(\alpha, T_1, \overline{v})$.

*Case* T-Msg $\quad e = e_1.\mathtt{md}(\overline{e_1}) \qquad \tau = T'$
$\quad \Gamma; \mathtt{K} \vdash e_1 : T \qquad \Gamma; \mathtt{K} \vdash \overline{e_1} : \overline{T} \qquad \mathtt{mtype}(\mathtt{md}, T) = \overline{T} \to T'$
$\quad \Gamma; \mathtt{K} \vdash$ **this** $: T_{this} \qquad \mathtt{K} \models \{\mathtt{mode}(T) \leq \mathtt{mode}(T_{this})\} \qquad \mathtt{mode}(T) \neq ?$

*Subcase* $\quad e_1 \overset{\mathtt{m}}{\Longrightarrow} e_1' \quad e' = e_1'.(\overline{e_1})$
Easy.

*Subcase* $\quad e_1 = o \quad e_{1_i} \overset{\mathtt{m}}{\Longrightarrow} e_{1_i}' \quad e' = o.(v_{1_i}, \ldots, e_{1_i}', \ldots, e_n)$
Easy.

*Subcase* R-Msg $\quad e_1 = o \qquad o = \mathtt{obj}(\alpha, \mathtt{c}\langle \mu, \iota\rangle, \overline{v})$

$$o.\mathtt{md}(\overline{v'}) \overset{\mathtt{m}}{\Longrightarrow} \mathbf{E}_{\mathtt{m}'}[\, e_b\{\overline{v}'/\overline{\mathtt{x}}\}\{o/\textbf{this}\}\,] \quad \mathtt{mbody}(\mathtt{md}, \mathtt{c}\langle \mu, \iota\rangle) = \overline{\mathtt{x}}.e_b \quad \mu \le \mathtt{m}$$
$$\mathtt{m}' = \mathtt{emode}(o)$$
$$e' = \mathbf{E}_{\mathtt{m}'}[\, e_b\{\overline{v}'/\overline{\mathtt{x}}\}\{o/\textbf{this}\}\,]$$

From Lemma 9 we have $\overline{\mathtt{x}} : \overline{T}, \textbf{this} : \mathtt{c}\langle \mu, \iota\rangle; \mathtt{K} \vdash \overline{\mathtt{x}}.e_b : T'$. Using Lemma 8 twice gives us $\varnothing; \mathtt{K} \vdash e_b\{\overline{v}'/\overline{\mathtt{x}}\}\{o/\textbf{this}\} : T'$.

Now, we may weaken $\varnothing$ to $\Gamma$ by Lemma 1 which gives us $\Gamma; \mathtt{K} \vdash e_b\{\overline{v}'/\overline{\mathtt{x}}\}\{o/\textbf{this}\} : T'$.

*Case* T-Field $\quad e = e_1.\mathtt{fd}_i \qquad \tau = T_i$

$$\Gamma; \mathtt{K} \vdash e_1 : T \qquad \mathtt{fields}(T) = \overline{T}\;\overline{\mathtt{fd}}$$
$$\Gamma; \mathtt{K} \vdash \textbf{this} : T_{this} \qquad \mathtt{K} \models \{\mathtt{mode}(T) \le \mathtt{mode}(T_{this})\} \quad \mathtt{mode}(T) \ne ?$$

*Subcase* $\quad e_1 \overset{\mathtt{m}}{\Longrightarrow} e_1' \quad e' = e_1'.\mathtt{fd}_i$
Easy.

*Subcase* R-Field $\quad e_1 = \mathtt{obj}(\alpha, \mathtt{c}\langle \mu, \iota\rangle, \overline{v})$

$$\mathtt{obj}(\alpha, \mathtt{c}\langle \mu, \iota\rangle, \overline{v}).\mathtt{fd}_i \overset{\mathtt{m}}{\Longrightarrow} v_i \quad \mu \le \mathtt{m}$$
$$e' = v_i$$

Lemma 10 gives $\Gamma; \mathtt{K} \vdash v_i : T_i$ which is exactly what we need.

*Case* T-Snapshot $\quad e = \textbf{snapshot}\; e_1\; [\eta_1, \eta_2] \qquad \tau = \exists\omega.\mathtt{c}\langle \mathtt{mt}, \iota\rangle$

$$\Gamma; \mathtt{K}' \vdash e_1 : \mathtt{c}\langle ?, \iota\rangle \qquad \omega = \eta_1 \le \mathtt{mt} \le \eta_2$$

*Subcase* $\quad e_1 \overset{\mathtt{m}}{\Longrightarrow} e_1' \quad e' = \textbf{snapshot}\; e_1'\; [\eta_1, \eta_2]$
Easy.

*Subcase* R-Snapshot1 $\quad \textbf{snapshot}\; o\; [\mathtt{m}_1, \mathtt{m}_2] \overset{\mathtt{m}}{\Longrightarrow} \mathtt{obj}(\alpha', \mathtt{c}\langle \textbf{check}(e_a\{o/\textbf{this}\}, \mathtt{m}_1, \mathtt{m}_2), \iota\rangle, (\mathtt{m}_1, \mathtt{m}_2), \overline{v})$

$$o = \mathtt{obj}(\alpha, \mathtt{c}\langle ?, \iota\rangle, \overline{v})$$
$$\textbf{class}\; \mathtt{c}\; \cdots\; \{\; \cdots\; A\; \} \in P$$
$$e_a = \mathtt{abody}(\mathtt{c}\langle ?, \iota\rangle)$$

From Lemma **??** we have $\textbf{this} : \mathtt{c}\langle ?, \iota\rangle; \mathtt{K}' \vdash e_a : \mathtt{modev}$. Then, by Lemma 8 we have $\varnothing; \mathtt{K}' \vdash e_a\{o/\textbf{this}\} : \mathtt{modev}$. Using Lemma 1 gives us $\Gamma; \mathtt{K} \vdash e_a\{o/\textbf{this}\} : \mathtt{modev}$.

Now, we have $\mathtt{K} = \mathtt{K}' \cup \{\mathtt{m}_1 \le \mathtt{mt}, \mathtt{mt} \le \mathtt{m}_2\}$ from T-Snapshot, from which $\mathtt{K} \models \{\mathtt{m}_1 \le \mathtt{mt}, \mathtt{mt} \le \mathtt{m}_2\}$ is immediate.

We may then apply R-Check to get $\Gamma; \mathtt{K} \vdash \textbf{check}(e_a\{o/\textbf{this}\}, \mathtt{m}_1, \mathtt{m}_2) : \mathtt{modev}$. Using Lemma (come back), we have $\Gamma; \mathtt{K} \vdash \overline{v} : \overline{T}$.

We may then apply R-Object, giving us $\Gamma; \mathtt{K} \vdash \mathtt{obj}(\alpha', \mathtt{c}\langle \textbf{check}(e_a\{o/\textbf{this}\}, \mathtt{m}_1, \mathtt{m}_2), \iota\rangle, (\mathtt{m}_1, \mathtt{m}_2), \overline{v}) : \exists \mathtt{m}_1 \le \mathtt{mt}' \le \mathtt{m}_2.\mathtt{c}\langle \mathtt{mt}', \iota\rangle$. Then, by T-Sub, we have $\Gamma; \mathtt{K} \vdash \mathtt{obj}(\alpha', \mathtt{c}\langle \textbf{check}(e_a\{o/\textbf{this}\}, \mathtt{m}_1, \mathtt{m}_2), \iota\rangle, (\mathtt{m}_1, \mathtt{m}_2), \overline{v}) : \exists \eta_1 \le \mathtt{mt} \le \eta_2.\mathtt{c}\langle \mathtt{mt}, \iota\rangle$

*Subcase* R-Snapshot2 $\quad \textbf{snapshot}\; o\; [\eta_1, \eta_2] \overset{\mathtt{m}}{\Longrightarrow} o \quad o = \mathtt{obj}(\alpha, \mathtt{c}\langle \mathtt{m}', \iota\rangle, \overline{v})$
Trivial.

*Case* T-MCase $\quad e = \{\overline{\mathtt{m} : e_1}\}^T \qquad \tau = \textbf{mcase}\langle T\rangle$

$$\Gamma; \mathtt{K} \vdash e_{1_i} : T \text{ for all } i \qquad \overline{\mathtt{m}} = \mathtt{modes}(P)$$

*Subcase* $\quad e_{1_i} \overset{\mathtt{m}}{\Longrightarrow} e_{1_i}' \quad e' = \{\mathtt{m} : v_{1_i}; \ldots; \mathtt{m} : e_{1_i}'; \ldots; \mathtt{m} : e_{1_n}\}$
Easy.

*Case* T-ElimCase $\quad e = e_1 \triangleright \eta \qquad \tau = T$
$\Gamma; \mathtt{K} \vdash e_1 : \mathbf{mcase}\langle T \rangle \qquad \eta \in \mathtt{modes}(P)$ or $\eta$ appears in $\mathtt{K}$

*Subcase* $\quad e_1 \overset{\mathtt{m}}{\Longrightarrow} e_1' \quad e' = e_1' \triangleright \eta$
Easy.

*Subcase* R-McaseProj $\quad e_1 = \{\overline{\mathtt{m} : v}\}^T \qquad \eta = \mathtt{m}_j$
$\{\overline{\mathtt{m} : v}\}^T \triangleright \mathtt{m}_j \overset{\mathtt{m}}{\Longrightarrow} v_j$
$e' = v_j$
From T-Mcase we have $\overline{\mathtt{m}} = \mathtt{modes}(P)$ and $\Gamma; \mathtt{K} \vdash v_i : T$ for all $i$. $\Gamma; \mathtt{K} \vdash v_j : T$ gives us what we need.

*Case* T-ModeValue $\quad e = \mathtt{m} \quad \tau = \mathtt{modev}$
Trival: Cannot occur.

*Case* T-Sub $\quad e = e_1 \qquad \tau = \tau_1'$
$\Gamma; \mathtt{K} \vdash e_1 : \tau_1 \quad \mathtt{K} \vdash \tau_1 <: \tau_1'$

Trivial.

*Case* T-Object $\quad e = \mathtt{obj}(\alpha, \mathtt{c}\langle \iota \rangle, \overline{e}) \quad \tau = \mathtt{c}\langle \iota \rangle$
$\Gamma; \mathtt{K} \vdash \overline{e} : \overline{\tau} \qquad \mathtt{fields}(\mathtt{c}\langle \iota \rangle) = \overline{\tau}\ \overline{\mathtt{fd}} = \overline{e}$

Trival: Cannot occur.

*Case* T-Check $\quad e = \mathbf{check}(e_1, \mathtt{m}_1, \mathtt{m}_2) \quad \tau = \mathtt{modev}$
$\Gamma; \mathtt{K} \vdash e_1 : \mathtt{modev}$

*Subcase* $\quad e_1 \overset{\mathtt{m}}{\Longrightarrow} e_1' \quad e' = \mathbf{check}(e_1', \mathtt{m}_1, \mathtt{m}_2)$
Easy.

*Subcase* R-Check $\quad e_1 = \mathbf{check}(\mathtt{m}', \mathtt{m}_1, \mathtt{m}_2) \quad \mathbf{check}(\mathtt{m}', \mathtt{m}_1, \mathtt{m}_2) \overset{\mathtt{m}}{\Longrightarrow} \mathtt{m}' \quad e' = \mathtt{modev}$
Easy.

*Case* T-Closure $\quad e = \mathtt{cl}(\mathtt{m}', e_1) \quad \tau = \tau_1$
$\Gamma; \mathtt{K} \vdash e_1 : \tau_1$

*Subcase* R-Closure1 $\quad e_1 \overset{\mathtt{m}'}{\Longrightarrow} e_1' \quad e' = \mathtt{cl}(\mathtt{m}', e_1')$
Trivial.

*Subcase* R-Closure1 $\quad \mathtt{cl}(\mathtt{m}', v) \overset{\mathtt{m}'}{\Longrightarrow} v \quad e' = v$
Trivial.

$\langle\langle\langle$ **Finish the proof. -Anthony** $\rangle\rangle\rangle$

□

**Lemma 12.**

*(1) If* $\Gamma; \mathtt{K} \vdash v : \tau$ *and* $\mathtt{K} \vdash \tau <: \mathtt{c}\langle \mu, \overline{\eta} \rangle$*, then* $\tau = \mathtt{c}'\langle \mu', \overline{\eta} \rangle$ *with* $\mathtt{K} \vdash \mathtt{c}'\langle \mu', \overline{\eta} \rangle <: \mathtt{c}\langle \mu, \overline{\eta} \rangle$.

*(2) If* $\Gamma; \mathtt{K} \vdash v : \tau$ *and* $\mathtt{K} \vdash \tau <: \mathbf{mcase}\langle T \rangle$*, then* $\tau = \mathbf{mcase}\langle T' \rangle$ *with* $\mathtt{K} \vdash T' <: T$.

*Proof.*

(1) Case analysis on the induction of the derivation of $\mathtt{K} \vdash \tau <: \mathtt{c}\langle \mu, \overline{\eta} \rangle$: Only S-Dynamic and S-Class apply, we present S-Exists to clarify.

*Case* (S-Dynamic) $\quad \tau = \mathtt{c}\langle \mu', \overline{\eta} \rangle$
Letting $\mathtt{c}'$ be $\mathtt{c}$ and $\mu$ be $?$ finishes the case.

*Case* (S-Class) $\quad \tau = \mathtt{c}'\langle \iota \rangle$
Trivial. Exactly what we need.

*Case* (S-Exists) $\quad \tau = \exists \omega.\mathtt{c}\langle \mu, \overline{\eta} \rangle$
If $\tau = \exists \omega.\mathtt{c}\langle \mu, \overline{\eta} \rangle$ then we need to have a value with type $\exists \omega.\mathtt{c}\langle \mu, \overline{\eta} \rangle$, but by the structure of our terms and typing rules this cannot occur; hence, S-Exists contradicts our hypothesis and cannot occur.

(2) Induction on the derivation of $\mathtt{K} \vdash \tau <: \mathbf{mcase}\langle T \rangle$: Only S-Mcase applies.

*Case* (S-Mcase) $\quad \tau = \mathbf{mcase}\langle T' \rangle$
$\mathtt{K} \vdash T' <: T$
Trivial. Exactly what we need.

$\square$

**Lemma 13** (Canonical Forms)**.** *Given* $\Gamma; \mathtt{K} \vdash v : \tau$*,*

*(1) If* $\tau = \mathtt{c}\langle \iota \rangle$ *then* $v$ *has the shape* $\mathtt{obj}(\alpha, \tau', \overline{v})$ *with* $\mathtt{K} \vdash \tau' <: \mathtt{c}\langle \iota \rangle$*.*

*(2) If* $\tau = \mathbf{mcase}\langle T \rangle$ *then* $v$ *has the shape* $\{\overline{\mathtt{m} : v}\}^{T'}$ *with* $\mathtt{K} \vdash T' <: T$*.*

*(3) If* $\tau = \mathtt{modev}$ *then* $v$ *has the shape* $\mathtt{m}$ *with* $\mathtt{m} \in \mathtt{modes}(P)$*.*

*Proof.*

(1) Induction on the derivation $\Gamma; \mathtt{K} \vdash v : \mathtt{c}\langle ?, \iota \rangle$. Two rules may apply: T-Obj and T-Sub.

*Case* T-Obj $\quad v = \mathtt{obj}(\alpha, \mathtt{c}\langle \iota \rangle, \overline{v})$
Letting $\tau'$ be $\mathtt{c}\langle \iota \rangle$ finishes the case.

*Case* T-Sub $\quad v = v_1$
$\Gamma; \mathtt{K} \vdash v_1 : \tau_1 \quad \mathtt{K} \vdash \tau_1 <: \mathtt{c}\langle \iota \rangle$
By Lemma 12 $\tau_1 = \mathtt{c}'\langle \iota \rangle$. Then, by the induction hypothesis, $v_1 = \mathtt{obj}(\alpha, \tau_1', \overline{v})$ with $\mathtt{K} \vdash \tau_1' <: \mathtt{c}'\langle \iota \rangle$. By S-Trans, $\mathtt{K} \vdash \tau_1' <: \mathtt{c}\langle \iota \rangle$. We may now apply T-Sub to get $\Gamma; \mathtt{K} \vdash \mathtt{obj}(\alpha, \tau_1', \overline{v}) : \mathtt{c}\langle \iota \rangle$.

(2) Induction on the derivation $\Gamma; \mathtt{K} \vdash v : \mathbf{mcase}\langle T \rangle$. Two rules may apply: T-Mcase and T-Sub.

*Case* T-Mcase $\quad v = \{\overline{\mathtt{m} : v}\}^{T}$
Letting $T'$ be $T$ finishes the case.

*Case* T-Sub $\quad v = v_1$
$\Gamma; \mathtt{K} \vdash v_1 : \tau_1 \quad \mathtt{K} \vdash \tau_1 <: \mathbf{mcase}\langle T \rangle$

By Lemma 12 $\tau_1 = \mathbf{mcase}\langle T_1 \rangle$ with $\mathtt{K} \vdash T_1 <: T$. Then, by the induction hypothesis, $v_1 = \{\overline{\mathtt{m} : v}\}^{T_1'}$ with $\mathtt{K} \vdash T_1' <: T_1$. By S-Trans, $\mathtt{K} \vdash T_1' <: T$. We may now apply T-Sub to get $\Gamma; \mathtt{K} \vdash \{\overline{\mathtt{m} : v}\}^{T_1} : \mathbf{mcase}\langle T \rangle$.

(3) Only T-ModeValue may apply from which $\mathtt{m} \in \mathtt{modes}(P)$ is immediate.

$\square$

**Definition 1** (Bad Cast)**.** *Expression* $(T')\mathtt{obj}(\alpha, T, \overline{v})$ *is a bad cast iff* $\emptyset \vdash T <: T'$ *does not hold.*
**Definition 2** (Bad Check)**.** *Expression* $\mathbf{check}(\mathtt{m}, \mathtt{m}', \mathtt{m}'')$ *is a bad check iff* $\mathtt{m}' \leq \mathtt{m} \leq \mathtt{m}''$ *does not hold.*
**Lemma 14.** *If* $\mathbf{E}_{\mathtt{m}}[\,e\,]$*,* $\Gamma; \mathtt{K} \vdash e : \tau$ *with a premise containing* $\Gamma; \mathtt{K} \vdash \mathbf{this} : T_{this}$*, then* $\mathtt{mode}(T_{this}) = \mathtt{m}$*.*

*Proof.* $\langle\langle\langle$ **Come back to prove. -Anthony** $\rangle\rangle\rangle$ $\square$

**Lemma 15** (Progress)**.** *Suppose* $\Gamma; \mathtt{K} \vdash e : \tau$*, then either*

*(1) $e \overset{\mathtt{m}}{\Longrightarrow} e'$ for some $e'$.*

*(2) $e$ is a value.*

*(3) $e = \mathbf{E}[\,\mathbf{check}(\mathtt{m}', \mathtt{m}_1, \mathtt{m}_2)\,]$ where $\mathtt{m}_1 \le \mathtt{m} \le \mathtt{m}_2$ does not hold.*

*(4) $e = \mathbf{E}[\,(T')\mathtt{obj}(\alpha, T, \overline{v})\,]$ where $\mathtt{K} \vdash T <: T'$ does not hold.*

*Proof.* By induction on the derivaiton of $\Gamma; \mathtt{K} \vdash e : \tau$.

*Case* T-Var $\quad e = \mathtt{x} \quad \tau = \Gamma(\mathtt{x})$
Trivial.

*Case* T-New $\quad e = \mathbf{new}\ \mathtt{c}\langle\iota\rangle \quad \tau = \mathtt{c}\langle\iota\rangle$
Trivial by R-New, with $e' = \mathtt{obj}(\alpha, \mathtt{c}\langle\iota\rangle, \mathtt{init}(P, \mathtt{c}))$.

*Case* T-Cast $\quad e = (T')e_1 \qquad \tau = T'$
$\Gamma; \mathtt{K} \vdash e_1 : \mathtt{c}\langle\iota\rangle$

By the induction hypothesis the following may occur: (1) $e_1 \overset{e}{\Longrightarrow}{}'_1$ from which we have $e' = (T')e'_1$ by R-?. (2) $e_1$ is a value from which Lemma **??** gives $e_1 = \mathtt{obj}(\alpha, T, \overline{v})$. If $\mathtt{K} \vdash T <: T'$ then R-Cast applies, giving $e' = \mathtt{obj}(\alpha, T, \overline{v})$. If $\mathtt{K} \vdash T <: T'$ does not hold, then $e = \mathbf{E}[\,(T')\mathtt{obj}(\alpha, T, \overline{v})\,]$. (3) $e_1 = \mathbf{E}_1[\,\mathbf{check}(\mathtt{m}', \mathtt{m}_1, \mathtt{m}_2)\,]$ where $\mathtt{m}_1 \le \mathtt{m} \le \mathtt{m}_2$ does not hold, which gives $e = \mathbf{E}[\,\mathbf{check}(\mathtt{m}', \mathtt{m}_1, \mathtt{m}_2)\,]$ for $\mathbf{E} = (T')\mathbf{E}_1$. (4) $e_1 = \mathbf{E}_1[\,(T')\mathtt{obj}(\alpha, T, \overline{v})\,]$ where $\mathtt{K} \vdash T <: T'$ does not hold, which gives $e = \mathbf{E}[\,(T')\mathtt{obj}(\alpha, T, \overline{v})\,]$ for $\mathbf{E} = (T')\mathbf{E}_1$.

*Case* T-Msg $\quad e = e_1.(\overline{e_1}) \qquad \tau = T'$
$\Gamma; \mathtt{K} \vdash e_1 : T \qquad \Gamma; \mathtt{K} \vdash \overline{e_1} : \overline{T} \qquad \mathtt{mtype}(\mathtt{md}, T) = \overline{T} \to T'$
$\Gamma; \mathtt{K} \vdash \mathbf{this} : T_{this} \qquad \mathtt{K} \models \{\mathtt{mode}(T) \le \mathtt{mode}(T_{this})\} \quad \mathtt{mode}(T) \ne ?$

By the induction hypothesis,

$e_1 \overset{\mathtt{m}}{\Longrightarrow} e'_1$, $e'_1$ is a value, $e_1 = \mathbf{E}_1[\,\mathbf{check}(\mathtt{m}', \mathtt{m}_1, \mathtt{m}_2)\,]$ where $\mathtt{m}_1 \le \mathtt{m}' \le \mathtt{m}_2$ does not hold, or
$e_1 = \mathbf{E}_1[\,(T')\mathtt{obj}(\alpha, T, \overline{v})\,]$ where $\mathtt{K} \vdash T <: T'$ does not hold.
$e_{1_i} \overset{\mathtt{m}}{\Longrightarrow} e'_{1_i}$, $e'_{1_i}$ is a value, $e_{1_i} = \mathbf{E}_{1_i}[\,\mathbf{check}(\mathtt{m}', \mathtt{m}_1, \mathtt{m}_2)\,]$ where $\mathtt{m}_1 \le \mathtt{m}' \le \mathtt{m}_2$ does not hold, or
$e_{1_i} = \mathbf{E}_{1_i}[\,(T')\mathtt{obj}(\alpha, T, \overline{v})\,]$ where $\mathtt{K} \vdash T <: T'$ does not hold.

If $e_1 \overset{\mathtt{m}}{\Longrightarrow} e'_1$ then we have $e' = e'_1.\overline{e_1}$ by R-?. If $e_1 = \mathbf{E}_1[\,\mathbf{check}(\mathtt{m}', \mathtt{m}_1, \mathtt{m}_2)\,]$ or $e_1 = \mathbf{E}_1[\,(T')\mathtt{obj}(\alpha, T, \overline{v})\,]$ then $e = \mathbf{E}[\,\mathbf{check}(\mathtt{m}', \mathtt{m}_1, \mathtt{m}_2)\,]$ or $e = \mathbf{E}[\,(T')\mathtt{obj}(\alpha, T, \overline{v})\,]$ for $\mathbf{E} = \mathbf{E}_1.\mathtt{md}(\overline{e_1})$.

If $e_1$ is a value then by Lemma 13, $e_1 = \mathtt{obj}(\alpha, T, \overline{v})$. If $e_{1_i} \overset{\mathtt{m}}{\Longrightarrow} e'_{1_i}$, then we have $e' = \mathtt{obj}(\alpha, T, \overline{v}).\mathtt{md}(\ldots, e'_{1_i} \ldots)$ by R-?. If $e_{1_i} = \mathbf{E}_{1_i}[\,\mathbf{check}(\mathtt{m}', \mathtt{m}_1, \mathtt{m}_2)\,]$ or $e_{1_i} = \mathbf{E}_{1_i}[\,(T')\mathtt{obj}(\alpha, T, \overline{v})\,]$ then $e = \mathbf{E}[\,\mathbf{check}(\mathtt{m}', \mathtt{m}_1, \mathtt{m}_2)\,]$ or $e = \mathbf{E}[\,(T')\mathtt{obj}(\alpha, T, \overline{v})\,]$ for $\mathbf{E} = \mathtt{obj}(\alpha, T, \overline{v}).\mathtt{md}(\ldots, \mathbf{E}_{1_i}, \ldots)$.

We now consider the case that all $e_{1_i}$ are values . By Lemma 14 we have $\mathtt{K} \models \{\mathtt{mode}(T) \le \mathtt{m}\}$. R-Msg now applies, giving $e' = \mathtt{cl}(\mathtt{m}', e_b\{\overline{v}'/\overline{\mathtt{x}}\}\{o/\mathbf{this}\})$ with $\mathtt{mbody}(\mathtt{md}, T) = \overline{\mathtt{x}}.e_b$.

*Case* T-Field $\quad e = e_1.\mathtt{fd}_i \qquad \tau = T_i$
$\Gamma; \mathtt{K} \vdash e_1 : T \qquad \mathtt{fields}(T) = \overline{T}\ \overline{\mathtt{fd}}$
$\Gamma; \mathtt{K} \vdash \mathbf{this} : T_{this} \qquad \mathtt{K} \models \{\mathtt{mode}(T) \le \mathtt{mode}(T_{this})\} \quad \mathtt{mode}(T) \ne ?$

Similar.

*Case* T-Snapshot $\quad e = \mathbf{snapshot}\ e_1\ [\eta_1, \eta_2] \quad \tau = \exists\omega.\mathtt{c}\langle\mathtt{mt}, \iota\rangle$
$\Gamma; \mathtt{K} \vdash e_1 : \mathtt{c}\langle ?, \iota\rangle \qquad \omega = \eta_1 \le \mathtt{mt} \le \eta_2$

Easy.

*Case* T-MCase $\quad e = \{\overline{\mathtt{m} : e_1}\}^T \qquad \tau = \mathbf{mcase}\langle T\rangle$
$\qquad \Gamma; \mathtt{K} \vdash e_{1_i} : T$ for all $i \quad \overline{\mathtt{m}} = \mathtt{modes}(P)$

Easy.

*Case* T-ElimCase $\quad e = e_1 \triangleright \eta \qquad \tau = T$
$\qquad \Gamma; \mathtt{K} \vdash e_1 : \mathbf{mcase}\langle T\rangle \quad \eta \in \mathtt{modes}(P)$ or $\eta$ appears in $\mathtt{K}$

Similar, except for the case that $e_1$ is a value. by Lemma 13, $e_1$ has the shape $\{\overline{\mathtt{m} : v}\}^T$, from which (R-McaseProj) applies, giving us $e' = v_j$.

*Case* T-Mode $\quad e = \mathtt{m} \quad \tau = \mathtt{modev}$
Trivial.

*Case* T-Sub $\quad e = e_1 \qquad \tau = \tau_1'$
$\qquad \Gamma; \mathtt{K} \vdash e_1 : \tau_1 \quad \mathtt{K} \vdash \tau_1 <: \tau_1'$

Easy.

*Case* T-Object $\quad e = \mathtt{obj}(\alpha, \mathtt{c}\langle\iota\rangle, \overline{e}) \quad \tau = \mathtt{c}\langle\iota\rangle$
$\qquad \Gamma; \mathtt{K} \vdash \overline{e} : \overline{\tau} \qquad \mathtt{fields}(\mathtt{c}\langle\iota\rangle) = \overline{\tau}\ \overline{\mathtt{fd}} = \overline{e}$

Easy.

*Case* T-Check $\quad e = \mathbf{check}(e_1, \mathtt{m}_1, \mathtt{m}_2) \quad \tau = \mathtt{modev}$
$\qquad \Gamma; \mathtt{K} \vdash e_1 : \mathtt{modev}$

By the induction hypothesis, $e_1$ is a value, bad cast, bad check, or there exists $e_1'$ such that $e_1 \overset{\mathtt{m}}{\Longrightarrow} e_1'$.

If $e_1$ is a value, then by Lemma 13, $e_1$ has the shape $\mathtt{m}$. Now, we have two cases: If $\mathtt{m}_1 \le \mathtt{m} \le \mathtt{m}_2$ then R-Check applies, giving us $e' = \mathtt{m}$. If $\mathtt{m}_1 \le \mathtt{m} \le \mathtt{m}_2$ *does not hold* then by definition we have a bad check.

If $e_1 \overset{\mathtt{m}}{\Longrightarrow} e_1'$ then we may replace $e_1$ with $e_1'$ by the reduction context, giving us $e' = \mathbf{check}(e_1', \mathtt{m}_1, \mathtt{m}_2)$.

*Case* T-Cl $\quad e = \mathtt{cl}(\mathtt{m}', e_1) \quad \tau = \tau_1$
$\qquad \Gamma; \mathtt{K} \vdash e_1 : \tau_1$

Easy. □

**Theorem 1** (Type Soundness)**.** *If $P$ is well-typed and $\mathtt{boot}(P) = \langle\top, e\rangle$, then either $e \overset{\top}{\Longrightarrow}_* v$, $\langle\top, e\rangle \Uparrow$, or $e \overset{\top}{\Longrightarrow}_* e'$ and $e'$ is a bad cast or a bad check.*

Let us say $\langle\mathtt{m}_0; e_0\rangle$ is a *sub-redex* of reduction $e \overset{\mathtt{m}}{\Longrightarrow} e'$ iff $e_0 \overset{\mathtt{m}_0}{\Longrightarrow} e_0'$ is a sub-derivation of $e \overset{\mathtt{m}}{\Longrightarrow} e'$. We next state two important properties of Ent.

**Theorem 2** (Type Decidability)**.** *For any program $P$, it is decidable whether $\vdash P$ holds.*

**Theorem 3** (Monotone Snapshotting)**.** *If $P$ is well-typed, $\mathtt{boot}(P) = \langle\top, e\rangle$, $e \overset{\top}{\Longrightarrow} \ldots e_1 \overset{\top}{\Longrightarrow} e_2 \ldots \overset{\top}{\Longrightarrow} e_3 \overset{\top}{\Longrightarrow} e_4$, $\langle\mathtt{m}; \mathtt{obj}(\alpha, T, \overline{v},)\rangle$ is a sub-redex of $e_1 \overset{\top}{\Longrightarrow} e_2$ and $\langle\mathtt{m}'; \mathtt{obj}(\alpha, T', \overline{v}',)\rangle$ is a sub-redex of $e_3 \overset{\top}{\Longrightarrow} e_4$, then if $\mathtt{mode}(T) \neq ?$, $T = T'$.*

In other words, once the type of an object becomes static, it can never be changed any more. This theorem reveals the *monotone* nature of object type change throughout the object lifetime, a crucial property to guarantee type soundness.

**Theorem 4** (Waterfall Invariant with Hybrid Typing)**.** *If $P$ is well-typed, $\mathtt{boot}(P) = \langle\top, e\rangle$, $e \overset{\top}{\Longrightarrow} \ldots e_1 \overset{\top}{\Longrightarrow} e_2$, and $\langle\mathtt{m}, \mathtt{obj}(\alpha, T, \overline{v}, .)\mathtt{md}(\overline{v'})\rangle$ or $\langle\mathtt{m}, \mathtt{obj}(\alpha, T, \overline{v}, .)\mathtt{fd}(\overline{v'})\rangle$ is a sub-redex of $e_1 \overset{\top}{\Longrightarrow} e_2$, then $R \models \mathtt{mode}(T) <: \mathtt{m}$ where $P = R\ \overline{C}\ e$.*

This theorem says even in the presence of hybrid typing, waterfall invariant — a key principle to regulate mode-based energy management — is still preserved. Observe that this theorem says run-time errors are never delayed to messaging or field access time. If any potential violation may happen due to dynamic typing, a run-time error would result from a bad check, *i.e.*, at snapshotting time.