

Proactive and Adaptive Energy-Aware Programming with Hybrid Typing — Proofs

1. Proofs

⟨⟨⟨ Internal vs External issue regarding : wft, abody, preservation. -Anthony ⟩⟩⟩

Lemma 1 (Weakening).

- (1) If $K \vdash_{\text{wft}} \tau$ and $K \models \{\eta \leq \eta'\}$ then $K, \eta \leq \eta' \vdash_{\text{wft}} \tau$.
- (2) If $K \vdash \tau <: \tau'$ and $K \models \{\eta \leq \eta'\}$ then $\Gamma; K, \eta \leq \eta' \vdash \tau <: \tau'$.
- (3) If $\Gamma; K \vdash e : \tau$, and $K \models \{\eta \leq \eta'\}$, then $\Gamma; K, \eta \leq \eta' \vdash e : \tau$.
- (4) If $\Gamma; K \vdash e : \tau$, and $\Gamma \vdash y : \tau'$, then $\Gamma, y : \tau'; K \vdash e : \tau$.

Proof. Each is proved by straightforward induction on the derivations of $K \vdash_{\text{wft}} \tau$, $K \vdash \tau <: \tau'$, and $\Gamma; K \vdash e : \tau$. □

Lemma 2. If $K, \eta \leq \text{mt}, \text{mt} \leq \eta' \models \{\eta_2 \leq \eta'_2\}$, $K \models \{\eta \leq \eta'', \eta'' \leq \eta'\}$, and $\text{mt} \notin K$, then $K\{\eta''/\text{mt}\} \models \{\eta_2\{\eta''/\text{mt}\} \leq \eta'_2\{\eta''/\text{mt}\}\}$.

Proof. Trivial. □

Lemma 3 (Mode Substitution Perserves Type Well-Formedness). If $K, \eta \leq \text{mt}, \text{mt} \leq \eta' \vdash_{\text{wft}} \tau$, $K \models \{\eta \leq \eta'', \eta'' \leq \eta'\}$, and $\text{mt} \notin K$, then $K\{\eta''/\text{mt}\} \vdash_{\text{wft}} \tau\{\eta''/\text{mt}\}$.

Proof. By induction on the derivation of $K, \eta \leq \text{mt}, \text{mt} \leq \eta' \vdash_{\text{wft}} \tau$.

Case WF-Top $T = \text{Object}\langle\eta\rangle$
Trivial.

Case WF-MCase $T = \mathbf{mcase}\langle T_1 \rangle$

$$K, \eta \leq \text{mt}, \text{mt} \leq \eta' \vdash_{\text{wft}} T_1$$

By the induction hypothesis, $K\{\eta''/\text{mt}\} \vdash_{\text{wft}} T_1\{\eta''/\text{mt}\}$. Then, by WF-MCase, $K\{\eta''/\text{mt}\} \vdash_{\text{wft}} \mathbf{mcase}\langle T_1\{\eta''/\text{mt}\} \rangle$.

Case WF-Class $T = c\langle\bar{\eta}\rangle$

$$\mathbf{class} \ c \ \Delta \ \mathbf{extends} \ c' \ \dots \in P \quad \text{eparam}(\Delta) = \iota' \quad \text{cons}(\Delta) = K'$$

$$K, \eta \leq \text{mt}, \text{mt} \leq \eta' \models K'\{\bar{\eta}/\iota'\} \quad K, \eta \leq \text{mt}, \text{mt} \leq \eta' \vdash_{\text{wft}} c'\langle\bar{\eta}\rangle$$

By the induction hypothesis, $K\{\eta''/\text{mt}\} \vdash_{\text{wft}} c'\langle\bar{\eta}\rangle\{\eta''/\text{mt}\}$. Lemma 2 gives us $K_1, K_2\{\eta''/\text{mt}\} \models K'\{\iota/\iota'\}\{\eta''/\text{mt}\}$.

Then, by WF-Class, $K\{\eta''/\text{mt}\} \vdash_{\text{wft}} c\langle\bar{\eta}\rangle\{\eta''/\text{mt}\}$.

Case WF-ClassDyn $T = c\langle?, \bar{\eta}\rangle$

$$\mathbf{class} \ c \ ? \rightarrow \omega, \Omega \ \mathbf{extends} \ c' \ \dots \in P \quad \text{eparam}(? \rightarrow \omega, \Omega) = \iota' \quad \text{cons}(? \rightarrow \omega, \Omega) = K'$$

$$K, \eta \leq \text{mt}, \text{mt} \leq \eta' \models K'\{\bar{\eta}/\iota'\} \quad K, \eta \leq \text{mt}, \text{mt} \leq \eta' \vdash_{\text{wft}} c'\langle?, \bar{\eta}\rangle$$

Similar.

□

Lemma 4 (Mode Substitution Perserves Subtyping). *If $K, \eta \leq \text{mt}, \text{mt} \leq \eta' \vdash \tau <: \tau', K \models \{\eta \leq \eta'', \eta'' \leq \eta'\}$, and $\text{mt} \notin K$, then $K\{\eta''/\text{mt}\} \vdash \tau\{\eta''/\text{mt}\} <: \tau'\{\eta''/\text{mt}\}$.*

Proof. Induction on the derivation of $K, \eta \leq \text{mt}, \text{mt} \leq \eta' \vdash \tau <: \tau'$.

Case (S-Dynamic) $\tau = c\langle\mu, \bar{\eta}\rangle \quad \tau' = c\langle?, \bar{\eta}\rangle$

If $\mu = \text{mt}$, then we have $K\{\eta''/\text{mt}\} \vdash c\langle\eta'', \bar{\eta}\{\eta''/\text{mt}\}\rangle <: c\langle?, \bar{\eta}\{\eta''/\text{mt}\}\rangle$. If $\mu \neq \text{mt}$, then we have $K\{\eta''/\text{mt}\} \vdash c\langle\mu, \bar{\eta}\{\eta''/\text{mt}\}\rangle <: c\langle?, \bar{\eta}\{\eta''/\text{mt}\}\rangle$. Both cases are exactly what is needed.

Case (S-Mcase) $\tau = \mathbf{mcase}\langle T_1 \rangle \quad \tau' = \mathbf{mcase}\langle T'_1 \rangle$
 $K, \eta \leq \text{mt}, \text{mt} \leq \eta' \vdash T_1 <: T'_1$

By the induction hypothesis, $K\{\eta''/\text{mt}\} \vdash T_1\{\eta''/\text{mt}\} <: T'_1\{\eta''/\text{mt}\}$. Then, by S-MCase, $K\{\eta''/\text{mt}\} \vdash \mathbf{mcase}\langle T_1\{\eta''/\text{mt}\} \rangle <: \mathbf{mcase}\langle T'_1\{\eta''/\text{mt}\} \rangle$.

Case (S-Exists) $\tau = \exists\omega. \tau_1 \quad \tau' = \tau_1$
 $\omega = \eta_1 \leq \text{mt}_1 \leq \eta_2 \quad K_1, \eta \leq \text{mt}, \text{mt} \leq \eta', K_2 \models \{\eta_1 \leq \text{mt}_1, \text{mt}_1 \leq \eta_2\} \cup K' \quad \text{mt}_1 \notin K'$

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Case (S-Class) $\tau = c\langle\iota\rangle \quad \tau' = c'\langle\iota\rangle$
 $\mathbf{class} \ c \ \Delta \ \mathbf{extends} \ c' \ \dots \in P \quad \mathbf{eparam}(\Delta) = \iota'$
 $K, \eta \leq \text{mt}, \text{mt} \leq \eta' = \mathbf{cons}(\Delta)$
 $K, \eta \leq \text{mt}, \text{mt} \leq \eta' \vdash_{\mathbf{wft}} c'\langle\iota\rangle$

By Lemma 3 we have $K\{\eta''/\text{mt}\} \vdash_{\mathbf{wft}} c\langle\iota\rangle\{\eta''/\text{mt}\}$ and $K\{\eta''/\text{mt}\} \vdash_{\mathbf{wft}} c'\langle\iota\rangle\{\eta''/\text{mt}\}$. Lemma 2 we have $K\{\eta''/\text{mt}\} \models \mathbf{cons}(\Delta)\{\eta''/\text{mt}\}$.

Then, by S-Class, $K\{\eta''/\text{mt}\} \vdash c\langle\iota\{\eta''/\text{mt}\}\rangle <: c'\langle\iota\{\eta''/\text{mt}\}\rangle$.

□

Lemma 5. *If $K, \eta \leq \text{mt}, \text{mt} \leq \eta' \vdash_{\mathbf{wft}} c\langle\iota\rangle$, $K \models \{\eta \leq \eta'', \eta'' \leq \eta'\}$, $\text{mt} \notin K$, and $\mathbf{mtype}(\mathbf{md}, c\langle\iota\rangle) = \overline{T} \rightarrow T$ then $\mathbf{mtype}(\mathbf{md}, c\langle\iota\rangle\{\eta''/\text{mt}\}) = \overline{T\{\eta''/\text{mt}\}} \rightarrow T'\{\eta''/\text{mt}\}$.*

Proof. Induction on the derivation of $\mathbf{mtype}(\mathbf{md}, c\langle\iota\rangle) = \overline{T} \rightarrow T$.

Case MT-Class

Case MT-Super

□

Lemma 6. *If $K, \eta \leq \text{mt}, \text{mt} \leq \eta' \vdash_{\mathbf{wft}} c\langle\iota\rangle$, $K \models \{\eta \leq \eta'', \eta'' \leq \eta'\}$, $\text{mt} \notin K$ and $\mathbf{fields}(T) = \overline{T} \ \mathbf{fd}$ then $\mathbf{fields}(c\langle\iota\rangle\{\eta''/\text{mt}\}) = \overline{T\{\eta''/\text{mt}\}} \ \mathbf{fd}$.*

Proof. Induction on the derivation of $\mathbf{fields}(\mathbf{md}, c\langle\iota\rangle) = \overline{T} \ \mathbf{fd}$.

Case FD-Class

Case FD-Object

□

Lemma 7 (Mode Substitution Preserves Typing). *If $\Gamma; K, \eta \leq \text{mt}, \text{mt} \leq \eta' \vdash e : \tau$, $K \models \{\eta \leq \eta'', \eta'' \leq \eta'\}$, and $\text{mt} \notin K$, then $\Gamma\{\eta''/\text{mt}\}; K\{\eta''/\text{mt}\} \vdash e\{\eta''/\text{mt}\} : \tau\{\eta''/\text{mt}\}$.*

Proof. Induction on the derivation of $\Gamma; K, \eta \leq \mathbf{mt}, \mathbf{mt} \leq \eta' \vdash e : \tau$.

Case T-Var $e = x \quad \tau = \Gamma(x)$

⟨⟨⟨ **Our substitution does not effect types directly; it acts on thier parameteres. I think I need a subcase analysis here. -Anthony** ⟩⟩⟩

Case T-New $e = \mathbf{new} \ c\langle\iota\rangle \quad \tau = c\langle\iota\rangle$
 $\iota = ?, \iota' \text{ iff } \mathbf{class} \ c \ \Delta \cdots \in P \text{ and } \mathbf{ethis}(\Delta) = ?$
 $\iota \neq ?, \iota' \text{ iff } \mathbf{class} \ c \ \Delta \cdots \in P \text{ and } \mathbf{ethis}(\Delta) \neq ?$
 $K, \eta \leq \mathbf{mt}, \mathbf{mt} \leq \eta' \models \mathbf{cons}(\Delta) \quad K, \eta \leq \mathbf{mt}, \mathbf{mt} \leq \eta' \vdash_{\mathbf{wft}} c\langle\iota\rangle$

Using Lemmas 2 and 3 gives us $K\{\eta''/\mathbf{mt}\} \models \mathbf{cons}(\Delta)\{\eta''/\mathbf{mt}\}$ and $K\{\eta''/\mathbf{mt}\} \vdash c\langle\iota\rangle\{\eta''/\mathbf{mt}\}$. Then, by T-New, we have $\Gamma\{\eta''/\mathbf{mt}\}; K\{\eta''/\mathbf{mt}\} \vdash \mathbf{new} \ c\langle\iota\rangle\{\eta''/\mathbf{mt}\} : c\langle\iota\rangle\{\eta''/\mathbf{mt}\}$.

Case T-Cast $e = (T)e_1 \quad \tau = T$
 $\Gamma; K, \eta \leq \mathbf{mt}, \mathbf{mt} \leq \eta' \vdash e_1 : T'$

Easy.

Case T-Msg $e = e_1.\mathbf{md}(\overline{e_1}) \quad \tau = T$
 $\Gamma; K, \eta \leq \mathbf{mt}, \mathbf{mt} \leq \eta' \vdash e_1 : c\langle\iota\rangle \quad \Gamma; K, \eta \leq \mathbf{mt}, \mathbf{mt} \leq \eta' \vdash \overline{e_1} : \overline{T}$
 $\mathbf{mtype}(\mathbf{md}, c\langle\iota\rangle) = \overline{T} \rightarrow T \quad \Gamma; K, \eta \leq \mathbf{mt}, \mathbf{mt} \leq \eta' \vdash \mathbf{this} : T_{this}$
 $K, \eta \leq \mathbf{mt}, \mathbf{mt} \leq \eta' \models \{\mathbf{mode}(c\langle\iota\rangle) \leq \mathbf{mode}(T_{this})\} \quad \mathbf{mode}(c\langle\iota\rangle) \neq ?$

By the induction hypothesis we have,

$$\begin{aligned} \Gamma\{\eta''/\mathbf{mt}\}; K\{\eta''/\mathbf{mt}\} &\vdash \overline{e_1\{\eta''/\mathbf{mt}\}} : c\langle\iota\rangle\{\eta''/\mathbf{mt}\} \\ \Gamma\{\eta''/\mathbf{mt}\}; K\{\eta''/\mathbf{mt}\} &\vdash e_1\{\eta''/\mathbf{mt}\} : T\{\eta''/\mathbf{mt}\} \\ \Gamma\{\eta''/\mathbf{mt}\}; K\{\eta''/\mathbf{mt}\} &\vdash \mathbf{this}\{\eta''/\mathbf{mt}\} : T_{this}\{\eta''/\mathbf{mt}\}. \end{aligned}$$

Now, by Lemma 5 we have $\mathbf{mtype}(\mathbf{md}, c\langle\iota\rangle\{\eta''/\mathbf{mt}\}) = \overline{T\{\eta''/\mathbf{mt}\}} \rightarrow T\{\eta''/\mathbf{mt}\}$.

Using Lemma 2 gives us $K\{\eta''/\mathbf{mt}\} \models \{\mathbf{mode}(c\langle\iota\rangle\{\eta''/\mathbf{mt}\}) \leq \mathbf{mode}(T_{this}\{\eta''/\mathbf{mt}\})\}$.

Then, by T-Msg, we have $\Gamma\{\eta''/\mathbf{mt}\}; K\{\eta''/\mathbf{mt}\} \vdash e_1\{\eta''/\mathbf{mt}\}.\mathbf{md}(\overline{e_1\{\eta''/\mathbf{mt}\}}) : T\{\eta''/\mathbf{mt}\}$.

Case T-Field $e = e_1.\mathbf{fd}_i \quad \tau = T_i$
 $\Gamma; K, \eta \leq \mathbf{mt}, \mathbf{mt} \leq \eta' \vdash e_1 : c\langle\iota\rangle \quad \mathbf{fields}(c\langle\iota\rangle) = \overline{T} \ \overline{\mathbf{fd}}$
 $\Gamma; K, \eta \leq \mathbf{mt}, \mathbf{mt} \leq \eta' \vdash \mathbf{this} : T_{this} \quad K\eta \leq \mathbf{mt}, \mathbf{mt} \leq \eta' \models \{\mathbf{mode}(c\langle\iota\rangle) \leq \mathbf{mode}(T_{this})\}$
 $\mathbf{mode}(c\langle\iota\rangle) \neq ?$

By the induction hypothesis we have,

$$\begin{aligned} \Gamma\{\eta''/\mathbf{mt}\}; K\{\eta''/\mathbf{mt}\} &\vdash e_1\{\eta''/\mathbf{mt}\} : c\langle\iota\rangle\{\eta''/\mathbf{mt}\} \\ \Gamma\{\eta''/\mathbf{mt}\}; K\{\eta''/\mathbf{mt}\} &\vdash \mathbf{this} : T_{this}\{\eta''/\mathbf{mt}\}. \end{aligned}$$

Now by Lemma 6 we have $\mathbf{fields}(c\langle\iota\rangle\{\eta''/\mathbf{mt}\}) = \overline{T\{\eta''/\mathbf{mt}\}} \ \overline{\mathbf{fd}}$.

Using Lemma 2 gives us $K\{\eta''/\mathbf{mt}\} \models \{\mathbf{mode}(c\langle\iota\rangle\{\eta''/\mathbf{mt}\}) \leq \mathbf{mode}(T_{this}\{\eta''/\mathbf{mt}\})\}$.

Then, by T-Field, we have $\Gamma\{\eta''/\mathbf{mt}\}; K\{\eta''/\mathbf{mt}\} \vdash e_1\{\eta''/\mathbf{mt}\}.\mathbf{fd}_i : T_i\{\eta''/\mathbf{mt}\}$.

Case T-Snapshot $e = \mathbf{snapshot} \ e_1 \ [\eta_1, \eta_2] \quad \tau = \exists \omega. c\langle \mathbf{mt}_1, \iota \rangle$
 $\Gamma; K, \eta \leq \mathbf{mt}, \mathbf{mt} \leq \eta' \vdash e_1 : c\langle ?, \iota \rangle \quad \omega = \eta_1 \leq \mathbf{mt}_1 \leq \eta_2$

By the induction hypothesis, $\Gamma\{\eta''/\mathbf{mt}\}; K\{\eta''/\mathbf{mt}\} \vdash e_1\{\eta''/\mathbf{mt}\} : c\langle ?, \iota \rangle\{\eta''/\mathbf{mt}\}$.

Now, consider $\omega = \eta_1 \leq \text{mt}_1 \leq \eta_2$: mt_1 must be unique; hence, $(\eta_1 \leq \text{mt}_1 \leq \eta_2\{\eta''/\text{mt}\})$ is $\eta_1\{\eta''/\text{mt}\} \leq \text{mt}_1 \leq \eta_2\{\eta''/\text{mt}\}$ by Lemma ??.

Then, by T-Snapshot,

$$\Gamma\{\eta''/\text{mt}\}; K\{\eta''/\text{mt}\} \vdash \mathbf{snapshot} \ e_1\{\eta''/\text{mt}\} \ [\eta_1\{\eta''/\text{mt}\}, \eta_2\{\eta''/\text{mt}\}] : \exists \omega\{\eta''/\text{mt}\}. c\langle \text{mt}_1, \iota\{\eta''/\text{mt}\} \rangle$$

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$$\begin{array}{ll} \text{Case T-MCase} & e = \{\bar{m} : e_1\}^T \quad \tau = \mathbf{mcase}\langle T \rangle \\ & \Gamma; K, \eta \leq \text{mt}, \text{mt} \leq \eta' \vdash e_{1_i} : T \text{ for all } i \quad \bar{m} = \mathbf{modes}(P) \end{array}$$

Easy.

$$\begin{array}{ll} \text{Case T-ElimCase} & e = e_1 \triangleright \eta_1 \quad \tau = T \\ & \Gamma; K, \eta \leq \text{mt}, \text{mt} \leq \eta' \vdash e_1 : \mathbf{mcase}\langle T \rangle \quad \eta_1 \in \mathbf{modes}(P) \text{ or } \eta_1 \text{ appears in } K, \eta \leq \text{mt}, \text{mt} \leq \eta' \end{array}$$

Easy.

$$\text{Case T-Mode} \quad e = m \quad \tau = \mathbf{modev}$$

Trivial.

$$\begin{array}{ll} \text{Case T-Sub} & e = e_1 \quad \tau = \tau'_1 \\ & \Gamma; K, \eta \leq \text{mt}, \text{mt} \leq \eta' \vdash e_1 : \tau_1 \quad K\eta \leq \text{mt}, \text{mt} \leq \eta' \vdash \tau_1 <: \tau'_1 \end{array}$$

By the induction hypothesis, $\Gamma\{\eta''/\text{mt}\}; K\{\eta''/\text{mt}\} \vdash e_1\{\eta''/\text{mt}\} : \tau_1\{\eta''/\text{mt}\}$. Using Lemma 4 gives us $K\{\eta''/\text{mt}\} \vdash \tau_1\{\eta''/\text{mt}\} <: \tau'_1\{\eta''/\text{mt}\}$

Then, by T-Sub, we have $\Gamma\{\eta''/\text{mt}\}; K\{\eta''/\text{mt}\} \vdash e_1\{\eta''/\text{mt}\} : \tau'_1\{\eta''/\text{mt}\}$.

⟨⟨⟨ **Finish the proof. -Anthony** ⟩⟩⟩

□

Lemma 8 (Term Substitution Perserves Typing). *If $\Gamma, y : \tau_0; K \vdash e : \tau$ and $\Gamma; K \vdash s : \tau_0$ then $\Gamma\{s/y\}; K \vdash e\{s/y\} : \tau$.*

Proof. Easy induction on the derivation of $\Gamma, y : \tau_0; K \vdash e : \tau$.

□

Lemma 9. *If $K \vdash_{\text{wft}} c\langle \iota \rangle$, $\text{mtype}(\text{md}, c\langle \iota \rangle) = \bar{T} \rightarrow T$ and $\text{mbody}(\text{md}, c\langle \iota \rangle) = \bar{x}.e$ then $\bar{x} : \bar{T}; \mathbf{this} : T; K \vdash e : T$.*

Proof. Induction on the derivation of $\text{mbody}(\text{md}, c\langle \iota \rangle) = \bar{x}.e$ using Lemmas 4 and 7.

$$\begin{array}{ll} \text{Case MB-Class} & \bar{x}.e = \bar{y}.e_0\{\iota/\iota'\} \\ & \mathbf{class} \ c \ \Delta \ \mathbf{extends} \ c'\{\bar{F} \ \bar{M} \ A\} \quad \text{eparam}(\Delta) = \iota' \\ & T_0 \ \text{md}(\bar{T}_0 \ \bar{y})\{e_0\} \in \bar{M} \end{array}$$

From T-Class and T-Method we have $\bar{y} : \bar{T}_0; \mathbf{this} : c\langle \iota \rangle; K' \vdash e_0 : T_0$. Since $K \vdash_{\text{wft}} c\langle \iota \rangle$ we have $K \models K'\{\iota/\iota'\}$ and $K' = \text{cons}(\Delta)$ from WF-Class. Using Lemmas 1 and 7 we have $\bar{y} : \bar{T}_0\{\iota/\iota'\}; \mathbf{this} : c\langle \iota \rangle; K \vdash e_0\{\iota/\iota'\} : T_0\{\iota/\iota'\}$.

Then, by MT-Class we have $\bar{T}_0\{\iota/\iota'\} = \bar{T}$ and $T_0\{\iota/\iota'\} = T$, from which $\bar{x} : \bar{T}, \mathbf{this} : c\langle \iota \rangle; K \vdash e : T$ is immediate.

Case MB-Super $\bar{x}.e = \text{mbody}(\text{md}, c'\langle\iota\rangle)$
 $\text{class } c \Delta \text{ extends } c'\{\bar{F} \ \bar{M} \ A\} \quad \text{md} \notin \bar{M}$

Immediate from the inductive hypothesis and the fact that $K \vdash c\langle\iota\rangle <: c'\langle\iota\rangle$.

□

Lemma 10. *If $\Gamma; K \vdash \text{obj}(\alpha, c\langle\iota\rangle, \bar{v})$ and $\text{fields}(c\langle\iota\rangle) = \bar{T} \ \bar{\text{fd}} = \bar{e}$ then $\Gamma; K \vdash v_i : T'_i$.*

Proof. Induction on the derivation of $\text{fields}(c\langle\iota\rangle) = \bar{\tau} \ \bar{\text{fd}}$.

Case FD-Class $\text{class } c \Delta \text{ extends } c'\{\bar{T}_0 \ \bar{\text{fd}} = \bar{e}_0 \ \dots\} \quad \text{eparam}(\Delta) = \iota'$
 $\text{fields}(c'\langle\iota\rangle) = \bar{T}_1 \ \bar{\text{fd}}_1 = \bar{e}_1$

From T-Class we have $\emptyset; K' \vdash \bar{e}_0 : \bar{T}_0$. Since $K \vdash_{\text{wft}} c\langle\iota\rangle$ we have $K \models K'\{\iota/\iota'\}$ and $K' = \text{cons}(\Delta)$ from WF-Class. Using Lemmas 1 and 7 we have $\Gamma; K \vdash \bar{e}_0\{\iota/\iota'\} : \bar{T}_0\{\iota/\iota'\}$

Now, from T-Object we have $\bar{T}_1, \bar{T}_0\{\iota/\iota'\} = \bar{T}$ and $\bar{e}_1, \bar{e}_0\{\iota/\iota'\} = \bar{v}$. Choosing $\Gamma; K \vdash v_i : T_i$ finishes the case.

Case FD-Object Trivial.

□

Lemma 11. *If $\Gamma; K \vdash \text{obj}(\alpha, c\langle?, \iota\rangle, \bar{v}) : c\langle?, \iota\rangle$ and $K \vdash_{\text{wft}} c\langle\mathfrak{m}, \iota\rangle$ then $\Gamma; K \vdash \text{obj}(\alpha', c\langle\mathfrak{m}, \iota\rangle, \bar{v})$.*

Proof.

□

Lemma 12 (Preservation). *If $\Gamma; K \vdash e : \tau$, $e \xRightarrow{\text{m}} e'$, then $\Gamma; K \vdash e' : \tau$.*

Proof. By induction on the derivation of $\Gamma, K \vdash e : \tau$, with a case analysis on the last rule used.

Case T-Var $e = x \quad \tau = \Gamma(x)$

Trivial: Cannot occur.

Case T-New $e = \text{new } c\langle\iota\rangle \quad \tau = c\langle\iota\rangle$
 $\iota = ?, \iota' \text{ iff } \text{class } c \Delta \dots \in P \text{ and } \text{ethis}(\Delta) = ?$
 $\iota \neq ?, \iota' \text{ iff } \text{class } c \Delta \dots \in P \text{ and } \text{ethis}(\Delta) \neq ?$
 $K \models \text{cons}(\Delta)$

Trivial.

Case T-Cast $e = (T)e_1 \quad \tau = T$
 $\Gamma; K \vdash e_1 : T_1$

Subcase $e_1 \xRightarrow{\text{m}} e'_1 \quad e' = (T)e'_1$

Easy.

Subcase $e_1 = \text{obj}(\alpha, T_1, \bar{v})$
 $(T)\text{obj}(\alpha, T_1, \bar{v}) \xRightarrow{\text{m}} \text{obj}(\alpha, T_1, \bar{v}) \quad T_1 <: T$
 $e' = \text{obj}(\alpha, T_1, \bar{v})$

Trivial. We have $\Gamma; K \vdash \text{obj}(\alpha, T_1, \bar{v}) : T_1$ from T-Cast and T-Obj. Then, by T-Sub we have $\Gamma; K \vdash \text{obj}(\alpha, T_1, \bar{v})$.

Case T-Msg $e = e_1.\text{md}(\bar{e}_1) \quad \tau = T'$
 $\Gamma; K \vdash e_1 : T \quad \Gamma; K \vdash \bar{e}_1 : \bar{T} \quad \text{mtype}(\text{md}, T) = \bar{T} \rightarrow T'$
 $\Gamma; K \vdash \text{this} : T_{\text{this}} \quad K \models \{\text{mode}(T) \leq \text{mode}(T_{\text{this}})\} \quad \text{mode}(T) \neq ?$

Subcase $e_1 \xRightarrow{m} e'_1 \quad e' = e'_1.(\overline{e_1})$

Easy.

Subcase $e_1 = o \quad e_{1_i} \xRightarrow{m} e'_{1_i} \quad e' = o.(v_{1_i}, \dots, e'_{1_i}, \dots, e_n)$

Easy.

Subcase R-Msg $e_1 = o \quad o = \text{obj}(\alpha, c\langle\mu, \iota\rangle, \bar{v})$
 $o.\text{md}(\bar{v}') \xRightarrow{m} \mathbf{E}_{m'}[e_b\{\bar{v}'/\bar{x}\}\{o/\mathbf{this}\}] \quad \text{mbody}(\text{md}, c\langle\mu, \iota\rangle) = \bar{x}.e_b \quad \mu \leq m$
 $m' = \text{emode}(o)$
 $e' = \mathbf{E}_{m'}[e_b\{\bar{v}'/\bar{x}\}\{o/\mathbf{this}\}]$

From Lemma 9 we have $\bar{x} : \bar{T}, \mathbf{this} : c\langle\mu, \iota\rangle; K \vdash \bar{x}.e_b : T'$. Using Lemma 8 twice gives us $\emptyset; K \vdash e_b\{\bar{v}'/\bar{x}\}\{o/\mathbf{this}\} : T'$.

Now, we may weaken \emptyset to Γ by Lemma 1 which gives us $\Gamma; K \vdash e_b\{\bar{v}'/\bar{x}\}\{o/\mathbf{this}\} : T'$.

Case T-Field $e = e_1.\text{fd}_i \quad \tau = T_i$
 $\Gamma; K \vdash e_1 : T \quad \text{fields}(T) = \bar{T} \bar{\text{fd}}$
 $\Gamma; K \vdash \mathbf{this} : T_{\text{this}} \quad K \models \{\text{mode}(T) \leq \text{mode}(T_{\text{this}})\} \quad \text{mode}(T) \neq ?$

Subcase $e_1 \xRightarrow{m} e'_1 \quad e' = e'_1.\text{fd}_i$

Easy.

Subcase R-Field $e_1 = \text{obj}(\alpha, c\langle\mu, \iota\rangle, \bar{v})$
 $\text{obj}(\alpha, c\langle\mu, \iota\rangle, \bar{v}).\text{fd}_i \xRightarrow{m} v_i \quad \mu \leq m$
 $e' = v_i$

Lemma 10 gives $\Gamma; K \vdash v_i : T_i$ which is exactly what we need.

Case T-Snapshot $e = \mathbf{snapshot} e_1 [\eta_1, \eta_2] \quad \tau = c\langle\text{mt}, \iota\rangle$
 $\Gamma; K' \vdash e_1 : c\langle?, \iota\rangle \quad \text{mt fresh}$
 $K = K' \cup \{\eta_1 \leq \text{mt}, \text{mt} \leq \eta_2\} \quad K \vdash_{\text{wft}} c\langle\text{mt}, \iota\rangle$

Subcase $e_1 \xRightarrow{m} e'_1 \quad e' = \mathbf{snapshot} e'_1 [\eta_1, \eta_2]$

Easy.

Subcase R-Snapshot1 $\mathbf{snapshot} o [\mathbf{m}_1, \mathbf{m}_2] \xRightarrow{m} \mathbf{check}(e_a\{o/\mathbf{this}\}, \mathbf{m}_1, \mathbf{m}_2, o) \quad o = \text{obj}(\alpha, c\langle?, \iota\rangle, \bar{v})$
 $\mathbf{class} c \dots \{ \dots A \} \in P \quad e_a = \text{abody}(c\langle?, \iota\rangle)$

From Lemma ?? we have $\mathbf{this} : c\langle?, \iota\rangle; K' \vdash e_a : \text{modev}$. Then, by Lemma 8 we have $\emptyset; K' \vdash e_a\{o/\mathbf{this}\} : \text{modev}$. Using Lemma 1 gives us $\Gamma; K \vdash e_a\{o/\mathbf{this}\} : \text{modev}$.

Now, we have $K = K' \cup \{\mathbf{m}_1 \leq \text{mt}, \text{mt} \leq \mathbf{m}_2\}$ from T-Snapshot, from which $K \models \{\mathbf{m}_1 \leq \text{mt}, \text{mt} \leq \mathbf{m}_2\}$ is immediate. Then, by T-Check, we have $\Gamma; K \vdash \mathbf{check}(\{o/\mathbf{this}\}, \mathbf{m}_1, \mathbf{m}_2, o)$.

Subcase R-Snapshot2 $\mathbf{snapshot} o [\eta_1, \eta_2] \xRightarrow{m} o \quad o = \text{obj}(\alpha, c\langle\mathbf{m}', \iota\rangle, \bar{v})$

Trivial.

Case T-MCase $e = \{\bar{m} : \bar{e}_1\}^T \quad \tau = \mathbf{mcase}\langle T \rangle$
 $\Gamma; K \vdash e_{1_i} : T \text{ for all } i \quad \bar{m} = \text{modes}(P)$

Subcase $e_{1_i} \xRightarrow{m} e'_{1_i} \quad e' = \{\bar{m} : v_{1_i}; \dots; \bar{m} : e'_{1_i}; \dots; \bar{m} : e_{1_n}\}$

Easy.

Case T-ElimCase $e = e_1 \triangleright \eta \quad \tau = T$
 $\Gamma; K \vdash e_1 : \mathbf{mcase}\langle T \rangle \quad \eta \in \text{modes}(P) \text{ or } \eta \text{ appears in } K$

Subcase $e_1 \xRightarrow{m} e'_1 \quad e' = e'_1 \triangleright \eta$

Easy.

$$\begin{array}{lcl}
\text{Subcase R-McaseProj} & e_1 = \{\overline{m} : v\}^T & \eta = m_j \\
& \{\overline{m} : v\}^T \triangleright m_j \xRightarrow{m} v_j & \\
& e' = v_j &
\end{array}$$

From T-Mcase we have $\overline{m} = \text{modes}(P)$ and $\Gamma; K \vdash v_i : T$ for all i . $\Gamma; K \vdash v_j : T$ gives us what we need.

$$\text{Case T-ModeValue} \quad e = m \quad \tau = \text{modev}$$

Trivial: Cannot occur.

$$\begin{array}{lcl}
\text{Case T-Sub} & e = e_1 & \tau = \tau'_1 \\
& \Gamma; K \vdash e_1 : \tau_1 & K \vdash \tau_1 <: \tau'_1
\end{array}$$

Trivial.

$$\begin{array}{lcl}
\text{Case T-Object} & e = \text{obj}(\alpha, c\langle \iota \rangle, \bar{e}) & \tau = c\langle \iota \rangle \\
& \Gamma; K \vdash \bar{e} : \bar{\tau} & \text{fields}(c\langle \iota \rangle) = \bar{\tau} \bar{\text{fd}} = \bar{e}
\end{array}$$

Trivial: Cannot occur.

$$\begin{array}{lcl}
\text{Case T-Check} & e = \mathbf{check}(e_1, m_1, m_2, e_2) & \tau = c\langle \text{mt}, \iota \rangle \\
& \Gamma; K \vdash e_1 : \text{modev} & \Gamma; K \vdash e_2 : c\langle ?, \iota \rangle \\
& K \models \{\eta_1 \leq \text{mt}, \text{mt} \leq \eta_2\} & K \models \{\eta_1 \leq m_1, m_2 \leq \eta_2\}
\end{array}$$

$$\text{Subcase} \quad e_1 \xRightarrow{m} e'_1 \quad e' = \mathbf{check}(e'_1, m_1, m_2, e_2)$$

Easy.

$$\text{Subcase} \quad e_2 \xRightarrow{m} e'_2 \quad e' = \mathbf{check}(m', m_1, m_2, e'_2)$$

Easy.

$$\begin{array}{lcl}
\text{Subcase R-Check} & e_1 = \mathbf{check}(m', m_1, m_2, o) & o = \text{obj}(\alpha, c\langle ?, \iota \rangle, \bar{v}) \\
& \mathbf{check}(m', m_1, m_2, o) \xRightarrow{m} \text{obj}(\alpha', c\langle m', \iota \rangle, \bar{v}) & m_1 \leq m' \leq m_2 \\
& e' = \text{obj}(\alpha', c\langle m', \iota \rangle, \bar{v}) &
\end{array}$$

We have $K \vdash_{\text{wft}} c\langle \text{mt}, \iota \rangle$. Since $K \models \{\eta_1 \leq m_1, m_2 \leq \eta_2\}$ and $K \models \{m_1 \leq m', m' \leq \eta_2\}$ we have $K \models \{\eta_1 \leq m', m' \leq \eta_2\}$; hence, $K \vdash_{\text{wft}} c\langle m', \iota \rangle$.

From Lemma 11 we have $\Gamma; K \vdash \text{obj}(\alpha', c\langle m', \iota \rangle, \bar{v}) : c\langle m', \iota \rangle$. Then, by T-Sub, $\Gamma; K \vdash \text{obj}(\alpha', c\langle m', \iota \rangle, \bar{v}) : c\langle \text{mt}, \iota \rangle$, which is exactly what we need.

$$\begin{array}{lcl}
\text{Case T-Closure} & e = \text{cl}(m', e_1) & \tau = \tau_1 \\
& \Gamma; K \vdash e_1 : \tau_1 &
\end{array}$$

$$\text{Subcase R-Closure1} \quad e_1 \xRightarrow{m'} e'_1 \quad e' = \text{cl}(m', e'_1)$$

Trivial.

$$\text{Subcase R-Closure1} \quad \text{cl}(m', v) \xRightarrow{m'} v \quad e' = v$$

Trivial.

⟨⟨⟨ **Finish the proof. -Anthony** ⟩⟩⟩

□

Lemma 13.

(1) If $\Gamma; K \vdash v : \tau$ and $K \vdash \tau <: c\langle \mu, \bar{\eta} \rangle$, then $\tau = c'\langle \mu', \bar{\eta} \rangle$ with $K \vdash c'\langle \mu', \bar{\eta} \rangle <: c\langle \mu, \bar{\eta} \rangle$.

(2) If $\Gamma; K \vdash v : \tau$ and $K \vdash \tau <: \mathbf{mcase}\langle T' \rangle$, then $\tau = \mathbf{mcase}\langle T' \rangle$ with $K \vdash T' <: T$.

Proof.

- (1) Case analysis on the induction of the derivation of $K \vdash \tau <: c\langle\mu, \bar{\eta}\rangle$: Only S-Dynamic and S-Class apply, we present S-Exists to clarify.

Case (S-Dynamic) $\tau = c\langle\mu', \bar{\eta}\rangle$

Letting c' be c and μ be $?$ finishes the case.

Case (S-Class) $\tau = c'\langle\iota\rangle$

Trivial. Exactly what we need.

Case (S-Exists) $\tau = \exists\omega.c\langle\mu, \bar{\eta}\rangle$

If $\tau = \exists\omega.c\langle\mu, \bar{\eta}\rangle$ then we need to have a value with type $\exists\omega.c\langle\mu, \bar{\eta}\rangle$, but by the structure of our terms and typing rules this cannot occur; hence, S-Exists contradicts our hypothesis and cannot occur.

- (2) Induction on the derivation of $K \vdash \tau <: \mathbf{mcase}\langle T \rangle$: Only S-Mcase applies.

Case (S-Mcase) $\tau = \mathbf{mcase}\langle T' \rangle$

$K \vdash T' <: T$

Trivial. Exactly what we need.

□

Lemma 14 (Canonical Forms). *Given $\Gamma; K \vdash v : \tau$,*

- (1) *If $\tau = c\langle\iota\rangle$ then v has the shape $\text{obj}(\alpha, \tau', \bar{v})$ with $K \vdash \tau' <: c\langle\iota\rangle$.*
- (2) *If $\tau = \mathbf{mcase}\langle T \rangle$ then v has the shape $\{\bar{m} : \bar{v}\}^{T'}$ with $K \vdash T' <: T$.*
- (3) *If $\tau = \text{modev}$ then v has the shape m with $m \in \text{modes}(P)$.*

Proof.

- (1) Induction on the derivation $\Gamma; K \vdash v : c\langle\iota\rangle$. Two rules may apply: T-Obj and T-Sub.

Case T-Obj $v = \text{obj}(\alpha, c\langle\iota\rangle, \bar{v})$

Letting τ' be $c\langle\iota\rangle$ finishes the case.

Case T-Sub $v = v_1$

$\Gamma; K \vdash v_1 : \tau_1 \quad K \vdash \tau_1 <: c\langle\iota\rangle$

By Lemma 13 $\tau_1 = c'\langle\iota\rangle$. Then, by the induction hypothesis, $v_1 = \text{obj}(\alpha, \tau'_1, \bar{v})$ with $K \vdash \tau'_1 <: c'\langle\iota\rangle$. By S-Trans, $K \vdash \tau'_1 <: c\langle\iota\rangle$. We may now apply T-Sub to get $\Gamma; K \vdash \text{obj}(\alpha, \tau'_1, \bar{v}) : c\langle\iota\rangle$.

- (2) Induction on the derivation $\Gamma; K \vdash v : \mathbf{mcase}\langle T \rangle$. Two rules may apply: T-Mcase and T-Sub.

Case T-Mcase $v = \{\bar{m} : \bar{v}\}^T$

Letting T' be T finishes the case.

Case T-Sub $v = v_1$

$\Gamma; K \vdash v_1 : \tau_1 \quad K \vdash \tau_1 <: \mathbf{mcase}\langle T \rangle$

By Lemma 13 $\tau_1 = \mathbf{mcase}\langle T_1 \rangle$ with $K \vdash T_1 <: T$. Then, by the induction hypothesis, $v_1 = \{\bar{m} : \bar{v}\}^{T'_1}$ with $K \vdash T'_1 <: T_1$. By S-Trans, $K \vdash T'_1 <: T$. We may now apply T-Sub to get $\Gamma; K \vdash \{\bar{m} : \bar{v}\}^{T'_1} : \mathbf{mcase}\langle T \rangle$.

- (3) Only T-ModeValue may apply from which $m \in \text{modes}(P)$ is immediate.

□

Definition 1 (Bad Cast). *Expression $(T')\text{obj}(\alpha, T, \bar{v})$ is a bad cast iff $\emptyset \vdash T <: T'$ does not hold.*

Definition 2 (Bad Check). *Expression $\mathbf{check}(m, m', m'')$ is a bad check iff $m' \leq m \leq m''$ does not hold.*

Lemma 15. If $\mathbf{E}_m[e]$, $\Gamma; K \vdash e : \tau$ with a premise containing $\Gamma; K \vdash \mathbf{this} : T_{this}$, then $\text{mode}(T_{this}) = m$.

Proof. $\langle\langle\langle \text{Come back to prove. -Anthony} \rangle\rangle\rangle$ □

Lemma 16 (Progress). If $\Gamma; K \vdash e : \tau$, then either e is a value, e is a bad cast, e is a bad check, or there exists e' such that $e \xRightarrow{m} e'$.

Proof. By induction on the derivation of $\Gamma, K \vdash e : \tau$.

Case T-Var $e = x \quad \tau = \Gamma(x)$

Trivial.

Case T-New $e = \mathbf{new} \ c\langle\iota\rangle \quad \tau = c\langle\iota\rangle$

Trivial by R-New, with $e' = \text{obj}(\alpha, c\langle\iota\rangle, \text{init}(P, c))$.

Case T-Cast $e = (T')e_1 \quad \tau = T'$
 $\Gamma; K \vdash e_1 : c\langle\iota\rangle$

By the induction hypothesis, e_1 is a value, bad cast, bad check, or there exists e'_1 such that $e_1 \xRightarrow{m} e'_1$. If e_1 is a value, then by Lemma 14, $e_1 = \text{obj}(\alpha, T, \bar{v})$ with $K \vdash T <: c\langle\iota\rangle$. Now, if $K \vdash c\langle\iota\rangle <: T'$ then by S-Trans we have $K \vdash T <: T'$, from which R-Cast applies, giving $e' = \text{obj}(\alpha, T, \bar{v})$. If $K \vdash c\langle\iota\rangle <: T'$ does not hold, then by S-Trans $K \vdash T <: T'$ does not hold; hence, we have a bad cast. If $e_1 \xRightarrow{m} e'_1$ then by the reduction context we may replace e_1 with e'_1 , giving $e' = (T')e'_1$.

Case T-Msg $e = e_1.(\bar{e}_1) \quad \tau = T'$
 $\Gamma; K \vdash e_1 : T \quad \Gamma; K \vdash \bar{e}_1 : \bar{T} \quad \text{mtype}(\text{md}, T) = \bar{T} \rightarrow T'$
 $\Gamma; K \vdash \mathbf{this} : T_{this} \quad K \models \{\text{mode}(T) \leq \text{mode}(T_{this})\} \quad \text{mode}(T) \neq ?$

By the induction hypothesis,

e_1 is a value, bad cast, bad check, or there exists e'_1 such that $e_1 \xRightarrow{m} e'_1$
 e_{1_i} is a value, bad cast, bad check, or there exists e'_{1_i} for each i such that $e_{1_i} \xRightarrow{m} e'_{1_i}$.

If $e_1 \xRightarrow{m} e'_1$ then we may replace e_1 with e'_1 giving us $e' = e'_1.(\bar{e}_1)$.

If e_1 is a value then by Lemma 14, $e_1 = \text{obj}(\alpha, T, \bar{v})$ with $K \vdash T <: c\langle\iota\rangle$. We consider the case that all e_{1_i} are values first. By Lemma 15 we have $K \models \{\text{mode}(T) \leq m\}$. R-Msg now applies, giving us $e' = \mathbf{E}_m[e\{\bar{v}/\bar{x}\}\{\text{obj}(\alpha, T, \bar{v})/\mathbf{this}\}]$. Otherwise we may replace the first e_{1_i} with e'_{1_i} giving us $e' = \text{obj}(\alpha, T, v_{1_1}, \dots, e'_{1_i}, \dots, e_{1_n})$.

Case T-Field $e = e_1.\text{fd}_i \quad \tau = T_i$
 $\Gamma; K \vdash e_1 : T \quad \text{fields}(T) = \bar{T} \ \bar{\text{fd}} \quad \text{mode}(T) \neq ?$
 $\Gamma; K \vdash \mathbf{this} : T_{this} \quad K \models \{\text{mode}(T) \leq \text{mode}(T_{this})\}$

By the induction hypothesis, e_1 is a value, bad cast, bad check, or there exists e'_1 such that $e_1 \xRightarrow{m} e'_1$.

If $e_1 \xRightarrow{m} e'_1$ then we may replace e_1 with e'_1 giving us $e' = e'_1.\text{fd}_i$. If e_1 is a value then by Lemma 14, $e_1 = \text{obj}(\alpha, T, \bar{v})$ with $K \vdash T <: c\langle\iota\rangle$. By Lemma 15 we have $K \models \{\text{mode}(T) \leq m\}$. R-Field now applies, giving us $e' = v_i$.

Case T-Snapshot $e = \mathbf{snapshot} \ e_1 \ [\eta_1, \eta_2] \quad \tau = \exists \omega. c\langle \text{mt}, \iota \rangle$
 $\Gamma; K \vdash e_1 : c\langle ?, \iota \rangle \quad \omega = \eta_1 \leq \text{mt} \leq \eta_2$

By the induction hypothesis, e_1 is a value, bad cast, bad check, or there exists e'_1 such that $e_1 \xRightarrow{m} e'_1$.

If e_1 is a value then by Lemma 14 $e_1 = \text{obj}(\alpha, T, \bar{v})$ with $K \vdash T <: c\langle?, \iota\rangle$. Now, if $\text{eargs}(T) = ?, \iota$, then R-Snapshot1 applies, with $e' = \text{let } x = \text{check}(A\{\text{obj}(\alpha, T, \bar{v})/\text{this}\}, m', m'') \text{ in } \text{obj}(\alpha', T\{x, \iota/\text{eargs}(T)\}, \bar{v})$. Otherwise $\text{eargs}(T) = m, \iota$ from which we may apply R-Snapshot2 to get $e' = \text{let } x = \text{check}(m, m', m'') \text{ in } \text{obj}(\alpha, T, \bar{v})$.

If $e_1 \xRightarrow{m} e'_1$ then by the reduction context we may replace e_1 with e'_1 to get $e' = \text{snapshot } e'_1 [\eta_1, \eta_2]$.

Case T-MCase $e = \{\bar{m} : e_1\}^T \quad \tau = \text{mcase}\langle T \rangle$
 $\Gamma; K \vdash e_{1_i} : T \text{ for all } i \quad \bar{m} = \text{modes}(P)$

By the induction hypothesis, e_{1_i} is a value, bad cast, bad check, or there exists e'_{1_i} such that $e_{1_i} \xRightarrow{m} e'_{1_i}$.

If all e_{1_i} are values, then e is a value. Otherwise by the reduction context we may replace e_{1_i} with e'_{1_i} , giving us e' .

Case T-ElimCase $e = e_1 \triangleright \eta \quad \tau = T$
 $\Gamma; K \vdash e_1 : \text{mcase}\langle T \rangle \quad \eta \in \text{modes}(P) \text{ or } \eta \text{ appears in } K$

By the induction hypothesis, e_1 is a value, bad cast, bad check, or there exists e'_1 such that $e_1 \xRightarrow{m} e'_1$.

If e_1 is a value then by Lemma 14, e_1 has the shape $\{\bar{m} : \bar{v}\}^T$, from which R-McaseProj applies, giving us $e' = v_j$. If $e_1 \xRightarrow{m} e'_1$ then by the reduction context we may replace e_1 with e'_1 , giving us $e' = e'_1 \triangleright \eta$.

Case T-Mode $e = m \quad \tau = \text{modev}$
Trivial.

Case T-Sub $e = e_1 \quad \tau = \tau'_1$
 $\Gamma; K \vdash e_1 : \tau_1 \quad K \vdash \tau_1 <: \tau'_1$

By the induction hypothesis, e_1 is a value, bad cast, bad check, or there exists e'_1 such that $e_1 \xRightarrow{m} e'_1$.

If e_1 is a value, we are done. If $e_1 \xRightarrow{m} e'_1$ then we may replace e_1 with e'_1 giving us $e' = e'_1$.

Case T-Object $e = \text{obj}(\alpha, c\langle\iota\rangle, \bar{e}) \quad \tau = c\langle\iota\rangle$
 $\Gamma; K \vdash \bar{e} : \bar{\tau} \quad \text{fields}(c\langle\iota\rangle) = \bar{\tau} \bar{\text{fd}} = \bar{e}$

By the induction hypothesis, e_i is a value, bad cast, bad check, or there exists e'_i such that $e_i \xRightarrow{m} e'_i$ for each i .

If all e_i are values, then e is a value and we are done. Otherwise, by the reduction context, we may replace an e_i with e'_i , giving us $e' = \text{obj}(\alpha, c\langle\iota\rangle, v_1, \dots, e'_i, \dots, e_n)$.

Case T-Check $e = \text{check}(e_1, m_1, m_2) \quad \tau = \text{modev}$
 $\Gamma; K \vdash e_1 : \text{modev}$

By the induction hypothesis, e_1 is a value, bad cast, bad check, or there exists e'_1 such that $e_1 \xRightarrow{m} e'_1$.

If e_1 is a value, then by Lemma 14, e_1 has the shape m . Now, we have two cases: If $m_1 \leq m \leq m_2$ then R-Check applies, giving us $e' = m$. If $m_1 \leq m \leq m_2$ *does not hold* then by definition we have a bad check.

If $e_1 \xRightarrow{m} e'_1$ then we may replace e_1 with e'_1 by the reduction context, giving us $e' = \text{check}(e'_1, m_1, m_2)$.

□

Theorem 1 (Type Soundness). *If P is well-typed and $\text{boot}(P) = \langle \top, e \rangle$, then either $e \xRightarrow{\top}_* v$, $\langle \top, e \rangle \uparrow$, or $e \xRightarrow{\top}_* e'$ and e' is a bad cast or a bad check.*

Let us say $\langle m_0; e_0 \rangle$ is a *sub-redex* of reduction $e \xRightarrow{m} e'$ iff $e_0 \xRightarrow{m_0} e'_0$ is a sub-derivation of $e \xRightarrow{m} e'$. We next state two important properties of ENT.

Theorem 2 (Type Decidability). *For any program P , it is decidable whether $\vdash P$ holds.*

Theorem 3 (Monotone Snapshotting). *If P is well-typed, $\text{boot}(P) = \langle \top, e \rangle$, $e \xRightarrow{\top} \dots e_1 \xRightarrow{\top} e_2 \dots \xRightarrow{\top} e_3 \xRightarrow{\top} e_4$, $\langle \mathfrak{m}; \text{obj}(\alpha, T, \bar{v}, \cdot) \rangle$ is a sub-redex of $e_1 \xRightarrow{\top} e_2$ and $\langle \mathfrak{m}'; \text{obj}(\alpha, T', \bar{v}', \cdot) \rangle$ is a sub-redex of $e_3 \xRightarrow{\top} e_4$, then if $\text{mode}(T) \neq ?$, $T = T'$.*

In other words, once the type of an object becomes static, it can never be changed any more. This theorem reveals the *monotone* nature of object type change throughout the object lifetime, a crucial property to guarantee type soundness.

Theorem 4 (Waterfall Invariant with Hybrid Typing). *If P is well-typed, $\text{boot}(P) = \langle \top, e \rangle$, $e \xRightarrow{\top} \dots e_1 \xRightarrow{\top} e_2$, and $\langle \mathfrak{m}, \text{obj}(\alpha, T, \bar{v}, \cdot) \text{md}(\bar{v}') \rangle$ or $\langle \mathfrak{m}, \text{obj}(\alpha, T, \bar{v}, \cdot) \text{fd}(\bar{v}') \rangle$ is a sub-redex of $e_1 \xRightarrow{\top} e_2$, then $R \models \text{mode}(T) <: \mathfrak{m}$ where $P = R \ \bar{C} \ e$.*

This theorem says even in the presence of hybrid typing, waterfall invariant — a key principle to regulate mode-based energy management — is still preserved. Observe that this theorem says run-time errors are never delayed to messaging or field access time. If any potential violation may happen due to dynamic typing, a run-time error would result from a bad check, *i.e.*, at snapshotting time.