Proactive and Adaptive Energy-Aware Programming with Hybrid Typing — Proofs

⟨⟨⟨ init: Don't think it works properly right now (we have to subst over supertype expressions). -Anthony ⟩⟩⟩
⟨⟨⟨ cast: Do we need to strengthen casting for preservation?). -Anthony ⟩⟩⟩
⟨⟨⟨ bad cast and bad check: We get stuck during progress right now. How to handle. -Anthony ⟩⟩⟩
⟨⟨⟨ Need a way to translate from $\mathtt{m} : \mathtt{modev}$ to the actual mode. I used emode – poorly – for now. -Anthony ⟩⟩⟩
⟨⟨⟨ We may still have some trouble with $\mathtt{c}\langle\iota\rangle$, T, and τ. -Anthony ⟩⟩⟩
⟨⟨⟨ We would get stuck with the old Snapshot2 rule (optimization). -Anthony ⟩⟩⟩
⟨⟨⟨ Check adjustment to reduction context and lemma relating static and dynamic this mode. -Anthony ⟩⟩⟩
⟨⟨⟨ I think this needs a type rule.... -Anthony ⟩⟩⟩
⟨⟨⟨ Need to add weaking for subtype... -Anthony ⟩⟩⟩

1. Proofs

Lemma 1 (Weakening)**.**

(1) If $\mathtt{K} \vdash_{\mathtt{wft}} \tau$ *and* $\mathtt{K} \models \{\eta \leq \eta\}$ *then* $\mathtt{K}, \eta \leq \eta' \vdash_{\mathtt{wft}} \tau$.
(2) If $\mathtt{K} \vdash \tau <: \tau'$ *and* $\mathtt{K} \models \{\eta \leq \eta\}$ *then* $\Gamma; \mathtt{K}, \eta \leq \eta' \vdash \tau <: \tau'$.
(3) If $\Gamma; \mathtt{K} \vdash e : \tau$*, and* $\mathtt{K} \models \{\eta \leq \eta'\}$*, then* $\Gamma; \mathtt{K}, \eta \leq \eta' \vdash e : \tau$.
(4) If $\Gamma; \mathtt{K} \vdash e : \tau$*, and* $\Gamma \vdash \mathtt{y} : \tau'$*, then* $\Gamma, \mathtt{y} : \tau'; \mathtt{K} \vdash e : \tau$.

Proof. Each is proved by straightforward induction on the derivations of $\mathtt{K} \vdash_{\mathtt{wft}} \tau$, $\mathtt{K} \vdash \tau <: \tau'$, and $\Gamma; \mathtt{K} \vdash e : \tau$. □

Lemma 2 (Mode Substitution Perserves Submoding)**.** *If* $\Omega_1, \eta \leq \mathtt{mt} \leq \eta', \Omega_2 \vdash \{\eta_1 \leq \mathtt{mt}_1, \mathtt{mt}_1 \leq \eta_1', \eta_1 \leq \eta_1'\}$, $\Omega_1 \vdash \{\eta \leq \eta'', \eta'' \leq \eta'\}$*, and* $\mathtt{mt}_1 \notin \Omega_1$ *if* $\eta'' = \mathtt{mt}_1$*, then* $\Omega_1, \Omega_2\{\eta''/\mathtt{mt}\} \vdash \{\eta_1 \leq \mathtt{mt}_1, \mathtt{mt}_1 \leq \eta_1', \eta_1 \leq \eta_1'\}\{\eta''/\mathtt{mt}\}$.

Proof. Case analysis on the derivation $\Omega_1, \eta \leq \mathtt{mt} \leq \eta', \Omega_2 \vdash \{\eta_1 \leq \mathtt{mt}_1, \mathtt{mt}_1 \leq \eta_1', \eta_1 \leq \eta_1'\}$.

Case $\quad \eta_1, \mathtt{mt}_1, and\ \eta_1' \neq \mathtt{mt} \quad \eta_1 \leq \mathtt{mt}_1 \leq \eta_1' \in \Omega_1, \eta \leq \mathtt{mt} \leq \eta', \Omega_2$
$\eta_1 \leq \mathtt{mt}_1 \leq \eta_1' \in \Omega_1, \Omega_2\{\eta''/\mathtt{mt}\}$ is immediately apparent, since $\eta'' \neq \eta_1, \mathtt{mt}_1$ and η_1'. Then, by M-Sub, $\Omega_1, \Omega_2\{\eta''/\mathtt{mt}\} \vdash \{\eta_1 \leq \mathtt{mt}_1, \mathtt{mt}_1 \leq \eta_1', \eta_1 \leq \eta_1'\}$.

Case $\quad \eta_1 = \mathtt{mt} \quad \mathtt{mt}_1 and\ \eta_1' \neq \mathtt{mt} \quad \eta_1 \leq \mathtt{mt}_1 \leq \eta_1' \in \Omega_1, \eta \leq \mathtt{mt} \leq \eta', \Omega_2$
$\eta'' \leq \mathtt{mt}_1 \leq \eta_1' \in \Omega_1, \Omega_2\{\eta''/\mathtt{mt}\}$ is immediately apparent. Then, by M-Sub, $\Omega_1, \Omega_2\{\eta''/\mathtt{mt}\} \vdash \{\eta'' \leq \mathtt{mt}_1, \mathtt{mt}_1 \leq \eta_1', \eta'' \leq \eta_1'\}$ by M-Sub.

The remaining cases are similar. □

Corollary 1. *If* $\mathtt{K}_1, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta', \mathtt{K}_2 \models \{\eta_1 \leq \eta_1'\}$ *and* $\mathtt{K}_1 \models \{\eta \leq \eta'', \eta'' \leq \eta'\}$*, then* $\mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \models \{\eta_1\{\eta''/\mathtt{mt}\} \leq \eta_1'\{\eta''/\mathtt{mt}\}\}$.

Proof. **⟨⟨⟨ Come back to prove. -Anthony ⟩⟩⟩** □

Lemma 3. *If* $\mathtt{mode}(T) = \mu$ *and* $\vdash_{\mathtt{wft}} T\{\eta/\eta'\}$*, then* $\mathtt{mode}(T\{\eta/\eta'\}) = \mu\{\eta/\eta'\}$.

Proof. **⟨⟨⟨ Come back to prove. -Anthony ⟩⟩⟩** □

Lemma 4. *If* $\texttt{eparam}(\Omega_1, \eta \leq \texttt{mt} \leq \eta', \Omega_2) = \iota$, $\Omega_1 \vdash \{\eta \leq \eta'', \eta'' \leq \eta'\}$, *and* $\texttt{mt}_1 \notin \Omega_1$ *if* $\eta'' = \texttt{mt}_1$, *then* $\texttt{eparam}(\Omega_1, \{\eta''/\texttt{mt}\}\Omega_2) = \iota\{\eta''/\texttt{mt}\}$.

Proof. Trivial. □

Lemma 5. *If* $\Omega\{\overline{\eta}/\iota\} \subseteq \Omega_1, \eta \leq \texttt{mt} \leq \eta', \Omega_2$, $\Omega_1 \vdash \{\eta \leq \eta'', \eta'' \leq \eta'\}$, *and* $\texttt{mt}_1 \notin \Omega_1$ *if* $\eta'' = \texttt{mt}_1$, *then* $\Omega\{\overline{\eta}/\iota\}\{\eta''/\texttt{mt}\} \subseteq \Omega_1, \Omega_2\{\eta''/\texttt{mt}\}$.

Proof. ⟨⟨⟨ **Come back to prove. -Anthony** ⟩⟩⟩ □

Lemma 6 (Mode Substitution Perserves Type Well-Formedness)**.** *If* $\Omega_1, \eta \leq \texttt{mt} \leq \eta', \Omega_2 \vdash_{\texttt{wft}} T$, $\Omega_1 \vdash \{\eta \leq \eta'', \eta'' \leq \eta'\}$, *and* $\texttt{mt}_1 \notin \Omega_1$ *if* $\eta'' = \texttt{mt}_1$, *then* $\Omega_1, \Omega_2\{\eta''/\texttt{mt}\} \vdash T\{\eta''/\texttt{mt}\}$.

Proof. By induction on the derivation of $\Omega_1, \eta \leq \texttt{mt} \leq \eta', \Omega_2 \vdash_{\texttt{wft}} T$.

Case WF-Top $\quad T = \texttt{Object}\langle\eta\rangle$
Trivial.

Case WF-MCase $\quad T = \textbf{mcase}\langle \texttt{c}\langle\iota\rangle\rangle$
$\Omega_1, \eta \leq \texttt{mt} \leq \eta', \Omega_2 \vdash_{\texttt{wft}} \texttt{c}\langle\iota\rangle$
By the induction hypothesis, $\Omega_1, \Omega_2\{\eta''/\texttt{mt}\} \vdash_{\texttt{wft}} \texttt{c}\langle\iota\rangle\{\eta''/\texttt{mt}\}$. Then, by WF-MCase, $\Omega_1, \Omega_2\{\eta''/\texttt{mt}\} \vdash_{\texttt{wft}}$ $\textbf{mcase}\langle \texttt{c}\langle\iota\rangle\{\eta''/\texttt{mt}\}\rangle$.

Case WF-Class $\quad T = \texttt{c}\langle\overline{\eta}\rangle$
$\textbf{class}\ \texttt{c}\ \Omega\ \ldots \in P \quad \texttt{eparam}(\Omega) = \iota \quad \Omega\{\overline{\eta}/\iota\} \subseteq \Omega_1, \eta \leq \texttt{mt} \leq \eta', \Omega_2$
Trivial by Lemma 5.

Case WF-ClassDyn $\quad T = \texttt{c}\langle ?, \overline{\eta}\rangle$
$\textbf{class}\ \texttt{c}\ ? \to \omega, \Omega\ \ldots \in P \quad \texttt{eparam}(? \to \omega, \Omega) = \iota \quad \Omega\{\overline{\eta}/\iota\} \subseteq \Omega_1, \eta \leq \texttt{mt} \leq \eta', \Omega_2$
Trivial by Lemma 5.

□

Lemma 7. *If* $\texttt{K}_1, \eta \leq \texttt{mt}, \texttt{mt} \leq \eta', \texttt{K}_2 = \texttt{cons}(\Delta_1, \eta \leq \texttt{mt} \leq \eta', \Omega_2)$, $\texttt{K}_1 \models \{\eta \leq \eta'', \eta'' \leq \eta'\}$, *and* $\texttt{mt}' \notin \texttt{K}_1$ *and* $\texttt{mt}' \notin \Delta_1$ *if* $\eta'' = \texttt{mt}'$, *then* $\texttt{K}_1, \texttt{K}_2\{\eta''/\texttt{mt}\} = \texttt{cons}(\Delta_1, \Omega_2\{\eta''/\texttt{mt}\})$.

Proof. ⟨⟨⟨ **Come back to prove. -Anthony** ⟩⟩⟩ □

Lemma 8. *If* $\texttt{K}_1, \eta \leq \texttt{mt}, \texttt{mt} \leq \eta', \texttt{K}_2 = \{\eta_1 \leq \texttt{mt}_1, \texttt{mt}_1 \leq \eta_2\} \cup \texttt{K}'$ *and* $\texttt{mt}_1 \notin \texttt{K}'$, *with the constraints that* $\texttt{K}_1 \models \{\eta \leq \eta'', \eta'' \leq \eta'\}$, $\texttt{mt}' \notin \texttt{K}_1$ *if* $\eta'' = \texttt{mt}'$, *and* $\texttt{mt}_1$ *does not appear in* $\texttt{K}_1, \eta \leq \texttt{mt}, \texttt{mt} \leq \eta', \texttt{K}_2$, *then* $\texttt{K}_1, \texttt{K}_2\{\eta''/\texttt{mt}\} = \{\eta_1\{\eta''/\texttt{mt}\} \leq \texttt{mt}_1, \texttt{mt}_1 \leq \eta_2\{\eta''/\texttt{mt}\}\} \cup \texttt{K}'\{\eta''/\texttt{mt}\}$.

Proof. ⟨⟨⟨ **Come back to prove. -Anthony** ⟩⟩⟩ □

Lemma 9 (Mode Substitution Perserves Subtyping)**.** *If* $\texttt{K}_1, \eta \leq \texttt{mt}, \texttt{mt} \leq \eta', \texttt{K}_2 \vdash \tau <: \tau'$, $\texttt{K}_1 \models \{\eta \leq \eta'', \eta'' \leq \eta'\}$, *and* $\texttt{mt}' \notin \texttt{K}_1$ *if* $\eta'' = \texttt{mt}'$, *then* $\texttt{K}_1, \texttt{K}_2\{\eta''/\texttt{mt}\} \vdash \tau\{\eta''/\texttt{mt}\} <: \tau'\{\eta''/\texttt{mt}\}$.

Proof. Induction on the derivation of $\texttt{K}_1, \eta \leq \texttt{mt}, \texttt{mt} \leq \eta', \texttt{K}_2 \vdash \tau <: \tau'$.

Case (S-Dynamic) $\quad \tau = \texttt{c}\langle\mu, \overline{\eta}\rangle \quad \tau' = \texttt{c}\langle ?, \overline{\eta}\rangle$
If $\mu = \texttt{mt}$, then we have $\texttt{K}_1, \texttt{K}_2\{\eta''/\texttt{mt}\} \vdash \texttt{c}\langle\eta'', \overline{\eta}\{\eta''/\texttt{mt}\}\rangle <: \texttt{c}\langle ?, \overline{\eta}\{\eta''/\texttt{mt}\}\rangle$. If $\mu \neq \texttt{mt}$, then we have $\texttt{K}_1, \texttt{K}_2\{\eta''/\texttt{mt}\} \vdash \texttt{c}\langle\mu, \overline{\eta}\{\eta''/\texttt{mt}\}\rangle <: \texttt{c}\langle ?, \overline{\eta}\{\eta''/\texttt{mt}\}\rangle$. Both cases are exactly what is needed.

Case (S-Mcase) $\quad \tau = \textbf{mcase}\langle\tau_1\rangle \qquad \tau' = \textbf{mcase}\langle\tau_1\rangle$
$\texttt{K}_1, \eta \leq \texttt{mt}, \texttt{mt} \leq \eta', \texttt{K}_2 \vdash \tau_1 <: \tau_1'$

By the induction hypothesis, $\texttt{K}_1, \texttt{K}_2\{\eta''/\texttt{mt}\} \vdash \tau_1\{\eta''/\texttt{mt}\} <: \tau_1'\{\eta''/\texttt{mt}\}$ Then, by S-MCase, $\texttt{K}_1, \texttt{K}_2\{\eta''/\texttt{mt}\} \vdash$ $\textbf{mcase}\langle\tau_1\{\eta''/\texttt{mt}\}\rangle <: \textbf{mcase}\langle\tau_1'\{\eta''/\texttt{mt}\}\rangle$.

Case (S-Exists) $\tau = \exists\omega.\tau_1$ $\quad \tau' = \tau_1$
$\omega = \eta_1 \leq \mathtt{mt}_1 \leq \eta_2 \quad \mathtt{K}_1, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta', \mathtt{K}_2 \models \{\eta_1 \leq \mathtt{mt}_1, \mathtt{mt}_1 \leq \eta_2\} \cup \mathtt{K}' \quad \mathtt{mt}_1 \notin \mathtt{K}'$

Since $\mathtt{mt}_1$ cannot appear in $\mathtt{K}_1\eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta', \mathtt{K}_2$, Lemma 8 applies, giving us $\mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} = \{\eta_1\{\eta''/\mathtt{mt}\} \leq \mathtt{mt}_1, \mathtt{mt}_1 \leq \eta_2\{\eta''/\mathtt{mt}\}\} \cup \mathtt{K}'\{\eta''/\mathtt{mt}\}$. We may then take $\eta_1\{\eta''/\mathtt{mt}\} \leq \mathtt{mt}_1 \leq \eta_2\{\eta''/\mathtt{mt}\}$ as our ω, and $\vdash_{\mathtt{wft}} \tau_1\{\eta''/\mathtt{mt}\}$ by Lemma 6.

We may now apply S-Exist to get:

$$\mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash \exists\eta_1\{\eta''/\mathtt{mt}\} \leq \mathtt{mt}_1 \leq \eta_2\{\eta''/\mathtt{mt}\}.\tau_1\{\eta''/\mathtt{mt}\} <: \tau_1\{\eta''/\mathtt{mt}\}.$$

Which is exactly what we need.

$\langle\langle\langle$ **Come back to double check my treatment of substitution. I may also have to subst over $\mathtt{c}$ in the well-formed type substitution. -Anthony** $\rangle\rangle\rangle$

Case (S-Class) $\tau = \mathtt{c}\langle\iota\rangle \quad \tau' = T\{\iota/\iota'\}$
class $\mathtt{c}\ \Delta_1, \eta \leq \mathtt{mt} \leq \eta', \Omega_2$ **extends** $T\ \ldots \in P \quad \mathtt{eparam}(\Delta_1, \eta \leq \mathtt{mt} \leq \eta', \Omega_2) = \iota'$
$\mathtt{K}_1, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta', \mathtt{K}_2 = \mathtt{cons}(\Delta_1, \eta \leq \mathtt{mt} \leq \eta', \Omega_2)$
$\Delta_1, \eta \leq \mathtt{mt} \leq \eta', \Omega_2 \vdash_{\mathtt{wft}} \mathtt{c}\langle\iota\rangle \quad \Delta_1, \eta \leq \mathtt{mt} \leq \eta', \Omega_2 \vdash_{\mathtt{wft}} T\{\iota/\iota'\}$

By Lemma 6,

$$\Delta_1, \Omega_2\{\eta''/\mathtt{mt}\} \vdash_{\mathtt{wft}} \mathtt{c}\langle\iota\rangle\{\eta''/\mathtt{mt}\}$$
$$\Delta_1, \Omega_2\{\eta''/\mathtt{mt}\} \vdash_{\mathtt{wft}} T\{\iota/\iota'\}\{\eta''/\mathtt{mt}\},$$

which, by Lemma **??** are,

$$\Delta_1, \Omega_2\{\eta''/\mathtt{mt}\} \vdash_{\mathtt{wft}} \mathtt{c}\langle\iota\{\eta''/\mathtt{mt}\}\rangle$$
$$\Delta_1, \Omega_2\{\eta''/\mathtt{mt}\} \vdash_{\mathtt{wft}} T\{\iota\{\eta''/\mathtt{mt}\}/\iota'\{\eta''/\mathtt{mt}\}\}.$$

Lemma 4 gives $\mathtt{eparam}(\Delta_1, \Omega_2\{\eta''/\mathtt{mt}\}) = \iota'\{\eta''/\mathtt{mt}\}$, and Lemma 7 gives $\mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} = \mathtt{cons}(\Delta_1, \Omega_2\{\eta''/\mathtt{mt}\})$. Lastly, **class** $\mathtt{c}\ \Delta_1, \Omega_2\{\eta''/\mathtt{mt}\}$ **extends** $T\ \ldots \in P$ by Lemma **??**.

We may now apply S-Class to get:

$$\mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash \mathtt{c}\langle\iota\{\eta''/\mathtt{mt}\}\rangle <: T\{\iota\{\eta''/\mathtt{mt}\}/\iota'\{\eta''/\mathtt{mt}\}\}.$$

Which is exactly what we need.

$\langle\langle\langle$ **Come back to double check my treatment of substitution. I may also have to subst over $\mathtt{c}$ in the well-formed type substitution. -Anthony** $\rangle\rangle\rangle$

□

Lemma 10. *If* $\mathtt{mtype}(\mathtt{md}, T) = \overline{T} \to T'$ *and* $\vdash_{\mathtt{wft}} T\{\eta''/\mathtt{mt}\}$, *then* $\mathtt{mtype}(\mathtt{md}, T\{\eta''/\mathtt{mt}\}) = \overline{T\{\eta''/\mathtt{mt}\}} \to T'\{\eta''/\mathtt{mt}\}$.

Proof. Come back to prove. □

Lemma 11. *If* $\mathtt{fields}(T) = \overline{T}\ \overline{\mathtt{fd}}$ *and* $\vdash_{\mathtt{wft}} T\{\eta''/\mathtt{mt}\}$, *then* $\mathtt{fields}(T\{\eta''/\mathtt{mt}\}) = \overline{T\{\eta''/\mathtt{mt}\}}\ \overline{\mathtt{fd}}$.

Proof. Come back to prove. □

Lemma 12. *If* $\mathtt{mtype}(\mathtt{md}, T) = \overline{T} \to T'$ *and* $\mathtt{K} \vdash \tau' <: T$, *then* $\mathtt{mtype}(\mathtt{md}, \tau') = \overline{T} \to T'$.

Proof. By induction on the derivation of $\mathtt{K} \vdash \tau <: T$.

Case (S-Dynamic) $\tau = \mathtt{c}\langle\mu;\overline{\eta}\rangle \quad T = \mathtt{c}\langle ?;\overline{\eta}\rangle$

$\langle\langle\langle$ **Come back. -Anthony** $\rangle\rangle\rangle$

Case (S-Mcase) $\quad \tau = \textbf{mcase}\langle \tau_1 \rangle$
Cannot occur; T cannot be $\textbf{mcase}\langle \tau_1' \rangle$ by the subtype relation.

Case (S-Exists) $\quad \tau = \exists\omega.\tau_1 \quad T = \tau_1$
⟨⟨⟨ **Come back. -Anthony** ⟩⟩⟩

Case (S-Class) $\quad \tau = \mathtt{c}\langle \iota \rangle \qquad T = T_1\{\iota/\iota'\}$
$\textbf{class}\ \mathtt{c}\ \Delta\ \textbf{extends}\ T_1 \cdots \in P \quad \mathtt{eparam}(\Delta) = \iota' \quad \mathtt{K} = \mathtt{cons}(\Delta)$
⟨⟨⟨ **Come back. -Anthony** ⟩⟩⟩

□

Lemma 13. *If* $\mathtt{fields}(T) = \overline{T}\ \overline{\mathtt{fd}}$ *and* $\mathtt{K} \vdash \tau' <: T$*, then* $\mathtt{fields}(\tau') = \overline{\tau'}\ \overline{\mathtt{fd}}$ *with* $\mathtt{K} \vdash \overline{\tau'} <: \overline{T}$.

Proof. ⟨⟨⟨ **Come back to prove. -Anthony** ⟩⟩⟩ □

Lemma 14. *If* $\mathtt{mode}(T) = \mu$ *and* $\mathtt{K} \vdash \tau' <: T$*, then* $\mathtt{mode}(\tau') = \mu'$ *with* $\mu' \leq \mu$.

Proof. ⟨⟨⟨ **Come back to prove. -Anthony** ⟩⟩⟩ □

Lemma 15. *If* $\Gamma, \mathtt{y} : \tau; \mathtt{K} \vdash \textbf{this} : T$ *and* $\Gamma; \mathtt{K} \vdash \mathtt{s} : \tau'$ *with* $\mathtt{K} \vdash \tau' <: \tau$*, then* $\Gamma\{\mathtt{s}/\mathtt{y}\}; \mathtt{K} \vdash \textbf{this} : T$.

Proof. ⟨⟨⟨ **Is this true? -Anthony** ⟩⟩⟩ □

Lemma 16 (Mode Substitution Perserves Typing)**.** *If* $\Gamma; \mathtt{K}_1, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta', \mathtt{K}_2; \vdash e : \tau$, $\mathtt{K}_1 \models \{\eta \leq \eta'', \eta'' \leq \eta'\}$*, and* $\mathtt{mt}_1 \notin \mathtt{K}_1$ *if* $\eta'' = \mathtt{mt}_1$*, then* $\Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash e\{\eta''/\mathtt{mt}\} : \tau'$*, with* $\mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash \tau' <: \tau\{\eta''/\mathtt{mt}\}$.

Proof. Induction on the derivation of $\Gamma; \mathtt{K}_1, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta', \mathtt{K}_2 \vdash e : \tau$.

Case T-Var $\quad e = \mathtt{x} \quad \tau = \Gamma(\mathtt{x})$
⟨⟨⟨ **Our substitution does not effect types directly; it acts on thier parameteres. I think I need a subcase analysis here. -Anthony** ⟩⟩⟩

Case T-New $\quad e = \textbf{new}\ \mathtt{c}\langle \iota \rangle \qquad \tau = \mathtt{c}\langle \iota \rangle$
$\iota = ?, \iota'$ iff $\textbf{class}\ \mathtt{c}\ \Delta_1, \eta \leq \mathtt{mt} \leq \eta', \Omega_2 \cdots \in P$ and $\mathtt{ethis}(\Delta') = ?$
$\iota \neq ?, \iota'$ iff $\textbf{class}\ \mathtt{c}\ \Delta_1, \eta \leq \mathtt{mt} \leq \eta', \Omega_2 \cdots \in P$ and $\mathtt{ethis}(\Delta') \neq ?$
$\mathtt{K}_1, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta', \mathtt{K}_2 \models \mathtt{cons}(\Delta_1, \eta \leq \mathtt{mt} \leq \eta', \Omega_2)$
By Lemma **??**, we have $\textbf{class}\ \mathtt{c}\ \Delta_1, \Omega_2\{\eta''/\mathtt{mt}\} \cdots \in P$ with $\Delta_1, \Omega_2\{\eta''/\mathtt{mt}\} \vdash_{\mathtt{wft}} \mathtt{c}\langle \iota \rangle\{\eta''/\mathtt{mt}\}$ by Lemma 6. By Lemma 7, $\mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} = \mathtt{cons}(\Delta_1, \Omega_2\{\eta''/\mathtt{mt}\})$.

Then, by T-New, $\mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash \textbf{new}\ \mathtt{c}\langle \iota \rangle\{\eta''/\mathtt{mt}\} : \mathtt{c}\langle \iota \rangle\{\eta''/\mathtt{mt}\}$. Letting τ' be $\mathtt{c}\langle \iota \rangle\{\eta''/\mathtt{mt}\}$ finishes the case.

⟨⟨⟨ **Double check -Anthony** ⟩⟩⟩

Case T-Cast $\quad e = (T)e_1 \qquad \tau = T$
$\Gamma; \mathtt{K}_1, \eta \leq \mathtt{mt}, \mathtt{mt} \leq \eta', \mathtt{K}_2 \vdash e_1 : T'$

By the induction hypothesis,

$$\Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash e_1\{\eta''/\mathtt{mt}\} : \tau_1'$$

with

$$\mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash \tau_1' <: T'\{\eta''/\mathtt{mt}\}.$$

T-Sub gives us $\Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash e_1\{\eta''/\mathtt{mt}\} : T'\{\eta''/\mathtt{mt}\}$. Then, by T-Cast, $\Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash (T\{\eta''/\mathtt{mt}\})e_1\{\eta''/\mathtt{mt}\} : T\{\eta''/\mathtt{mt}\}$. Letting τ' be $T\{\eta''/\mathtt{mt}\}$ finishes the case.

⟨⟨⟨ **Double check -Anthony** ⟩⟩⟩

Case T-Msg $\quad e = e_1.\mathtt{md}(\overline{e_1}) \qquad \tau = T'$

$\Gamma; \mathtt{K}_1, \eta \le \mathtt{mt} \le \eta', \mathtt{K}_2 \vdash e_1 : T \qquad \Gamma; \mathtt{K}_1, \eta \le \mathtt{mt} \le \eta', \mathtt{K}_2 \vdash \overline{e_1} : \overline{T}$

$\mathtt{mtype}(\mathtt{md}, T) = \overline{T} \to T' \qquad \Gamma; \mathtt{K}_1, \eta \le \mathtt{mt} \le \eta', \mathtt{K}_2 \vdash \textbf{this} : T_{this}$

$\mathtt{K}_1, \eta \le \mathtt{mt} \le \eta', \mathtt{K}_2 \models \{\mathtt{mode}(T) \le \mathtt{mode}(T_{this})\} \qquad \mathtt{mode}(T) \ne ?$

By the induction hypothesis,

$$\begin{aligned}
&\Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash e_1\{\eta''/\mathtt{mt}\} : \tau_1' \\
&\Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash \overline{e_1\{\eta''/\mathtt{mt}\}} : \overline{\tau_1'} \\
&\Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash \textbf{this}\{\eta''/\mathtt{mt}\} : \tau_{this}'
\end{aligned}$$

with

$$\begin{aligned}
&\mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash \tau_1' <: T \\
&\mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash \overline{\tau_1'} <: \overline{T\{\eta''/\mathtt{mt}\}} \\
&\mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash \tau_{this}' <: T_{this}\{\eta''/\mathtt{mt}\}.
\end{aligned}$$

Now, by Lemma 10 and Lemma 12 we have $\mathtt{mtype}(\mathtt{md}, \tau_1') = \overline{T\{\eta''/\mathtt{mt}\}} \to T'\{\eta''/\mathtt{mt}\}$. By T-Sub we have $\Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash \overline{e_1\{\eta''/\mathtt{mt}\}} : \overline{T\{\eta''/\mathtt{mt}\}}$; hence, $\mathtt{mtype}$ is taken care of.

We must now show that $\mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \models \{\mathtt{mode}(T\{\eta''/\mathtt{mt}\}) \le \mathtt{mode}(T\{\eta''/\mathtt{mt}\})\}$. Corollary 1 gives us $\mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \models \{\mathtt{mode}(T)\{\eta''/\mathtt{mt}\} \le \mathtt{mode}(T_{this})\{\eta''/\mathtt{mt}\}\}$, but $\mathtt{mode}(T)\{\eta''/\mathtt{mt}\} \le \mathtt{mode}(T_{this})\{\eta''/\mathtt{mt}\} = \mathtt{mode}(T\{\eta''/\mathtt{mt}\}) \le \mathtt{mode}(T_{this}\{\eta''/\mathtt{mt}\})$ by Lemma 3; hence, our constraint is handled.

Thus we may now apply T-Msg to get $\Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash e_1\{\eta''/\mathtt{mt}\}.\mathtt{md}(\overline{e_1\{\eta''/\mathtt{mt}\}}) : T'\{\eta''/\mathtt{mt}\}$. Letting τ' be $T'\{\eta''/\mathtt{mt}\}$ finishes the case, since $e_1\{\eta''/\mathtt{mt}\}.\mathtt{md}(\overline{e_1\{\eta''/\mathtt{mt}\}}) : T'\{\eta''/\mathtt{mt}\}$ is $e\{\eta''/\mathtt{mt}\} : \tau'$.

Case T-Field $\quad e = e_1.\mathtt{fd}_i \qquad \tau = T_i$

$\Gamma; \mathtt{K}_1, \eta \le \mathtt{mt} \le \eta', \mathtt{K}_2 \vdash e_1 : T \qquad \mathtt{fields}(T) = \overline{T}\ \overline{\mathtt{fd}}$

$\Gamma; \mathtt{K}_1, \eta \le \mathtt{mt} \le \eta', \mathtt{K}_2 \vdash \textbf{this} : T_{this} \qquad \mathtt{K}_1, \eta \le \mathtt{mt} \le \eta', \mathtt{K}_2 \models \{\mathtt{mode}(T) \le \mathtt{mode}(T_{this})\} \quad \mathtt{mode}(T) \ne ?$

By the induction hypothesis,

$$\begin{aligned}
&\Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash e_1\{\eta/\mathtt{mt}\} : \tau_1' \\
&\Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash \textbf{this} : \tau_{this}'
\end{aligned}$$

with

$$\begin{aligned}
&\mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash \tau_1' <: T\{\eta''/\mathtt{mt}\} \\
&\mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash \tau_{this}' <: T_{this}\{\eta''/\mathtt{mt}\}.
\end{aligned}$$

Now by Lemma 11 and Lemma 13 we have $\mathtt{fields}(\tau_1') = \overline{\tau_1'}\ \overline{\mathtt{fd}}$ with $\mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash \overline{\tau_1'} <: \overline{T\{\eta''/\mathtt{mt}\}}$.

$\langle\langle\langle$ **T-MSG and T-Fields are wrong. The "this" subtype issue still remains. -Anthony** $\rangle\rangle\rangle$

Case T-Snapshot $\quad e = \textbf{snapshot}\ e_1\ [\eta_1, \eta_2] \qquad \tau = \exists\omega.\mathtt{c}\langle \mathtt{mt}_1, \iota\rangle$

$\Gamma; \mathtt{K}_1, \eta \le \mathtt{mt} \le \eta', \mathtt{K}_2 \vdash e_1 : \mathtt{c}\langle ?, \iota\rangle \qquad \omega = \eta_1 \le \mathtt{mt}_1 \le \eta_2$

By the induction hypothesis,

$$\Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash e_1\{\eta''/\mathtt{mt}\} : \tau_1'$$

with

$$\Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash \tau_1' <: \mathtt{c}\langle ?, \iota\rangle\{\eta''/\mathtt{mt}\}.$$

Since $\mathtt{c}\langle ?, \iota\rangle\{\eta''/\mathtt{mt}\}$ is $\mathtt{c}\langle ?, \iota\{\eta''/\mathtt{mt}\}\rangle$, by T-Sub we have $\Gamma\{\eta''/\mathtt{mt}\}; \mathtt{K}_1, \mathtt{K}_2\{\eta''/\mathtt{mt}\} \vdash e_1 : \mathtt{c}\langle ?, \iota\{\eta''/\mathtt{mt}\}\rangle$. Now, consider $\omega = \eta_1 \le \mathtt{mt}_1 \le \eta_2$: $\mathtt{mt}_1$ must be unique; hence, $(\eta_1 \le \mathtt{mt}_1 \le \eta_2\{\eta''/\mathtt{mt}\})$ is $\eta_1\{\eta''/\mathtt{mt}\} \le \mathtt{mt}_1 \le \eta_2\{\eta''/\mathtt{mt}\}$ by Lemma 2.

Thus we may now apply T-Snapshot to get

$$\Gamma\{\eta''/\texttt{mt}\};\texttt{K}_1,\texttt{K}_2\{\eta''/\texttt{mt}\} \vdash \textbf{snapshot}\ e_1\{\eta''/\texttt{mt}\}\ [\eta_1\{\eta''/\texttt{mt}\},\eta_2\{\eta''/\texttt{mt}\}] : \exists\omega\{\eta''/\texttt{mt}\}.\texttt{c}\langle\texttt{mt}_1,\iota\{\eta''/\texttt{mt}\}\rangle$$

Letting τ' be $\exists\omega\{\eta''/\texttt{mt}\}.\texttt{c}\langle\texttt{mt}_1,\iota\{\eta''/\texttt{mt}\}\rangle$ finishes the case.

Case T-MCase $\quad e = \{\overline{\texttt{m} : e_1}\}^T \qquad \tau = \textbf{mcase}\langle T\rangle$
$\quad \Gamma;\texttt{K}_1,\eta \le \texttt{mt} \le \eta',\texttt{K}_2 \vdash e_{1_i} : T$ for all $i \qquad \overline{\texttt{m}} = \texttt{modes}(P)$

By the induction hypothesis,

$$\Gamma\{\eta''/\texttt{mt}\};\texttt{K}_1,\texttt{K}_2\{\eta''/\texttt{mt}\} \vdash e_{1_i}\{\eta''/\texttt{mt}\} : \tau_1' \text{ for all } i$$

with

$$\texttt{K}_1,\texttt{K}_2\{\eta''/\texttt{mt}\} \vdash \tau_1' <: T\{\eta''/\texttt{mt}\}.$$

Then, by T-MCase, $\Gamma\{\eta''/\texttt{mt}\};\texttt{K}_1,\texttt{K}_2\{\eta''/\texttt{mt}\} \vdash \{\overline{\texttt{m} : e_1\{\eta''/\texttt{mt}\}}\}^{\tau_1'} : \textbf{mcase}\langle\tau_1'\rangle$ with $\texttt{K}_1,\texttt{K}_2\{\eta''/\texttt{mt}\} \vdash \textbf{mcase}\langle\tau_1'\rangle <: \textbf{mcase}\langle T\{\eta''/\texttt{mt}\}\rangle$.

Case T-ElimCase $\quad e = e_1 \triangleright \eta_1 \qquad \tau = T$
$\quad \Gamma;\texttt{K}_1,\eta \le \texttt{mt} \le \eta',\texttt{K}_2 \vdash e_1 : \textbf{mcase}\langle T\rangle \qquad \eta_1 \in \texttt{modes}(P)$ or η_1 appears in $\texttt{K}_1,\eta \le \texttt{mt} \le \eta',\texttt{K}_2$

By the induction hypothesis,

$$\Gamma\{\eta''/\texttt{mt}\};\texttt{K}_1,\texttt{K}_2\{\eta''/\texttt{mt}\} \vdash e_1\{\eta''/\texttt{mt}\} : \tau_1'$$

with

$$\texttt{K}_1,\texttt{K}_2\{\eta''/\texttt{mt}\} \vdash \tau_1' <: \textbf{mcase}\langle T\rangle\{\eta''/\texttt{mt}\}.$$

$\textbf{mcase}\langle T\rangle\{\eta''/\texttt{mt}\}$ is $\textbf{mcase}\langle T\{\eta''/\texttt{mt}\}\rangle$; hence, by T-Sub we have $\Gamma\{\eta''/\texttt{mt}\};\texttt{K}_1,\texttt{K}_2\{\eta''/\texttt{mt}\} \vdash e_1\{\eta''/\texttt{mt}\} : \textbf{mcase}\langle T\{\eta''/\texttt{mt}\}\rangle$. Both cases of the remaining constraint are trivial: If $\eta_1 \in \texttt{modes}(P)$, then $\eta_1\{\eta''/\texttt{mt}\} \in \texttt{modes}(P)$, and if $\eta_1 \in \texttt{K}_1,\eta \le \texttt{mt} \le \eta',\texttt{K}_2$, then $\eta_1\{\eta''/\texttt{mt}\} \in \texttt{K}_1,\texttt{K}_2\{\eta''/\texttt{mt}\}$.

Then, by T-ElimCase, $\Gamma\{\eta''/\texttt{mt}\};\texttt{K}_1,\texttt{K}_2 \vdash e_1\{\eta''/\texttt{mt}\} \triangleright \eta_1\{\eta''/\texttt{mt}\} : T\{\eta''/\texttt{mt}\}$ Letting τ' be $T\{\eta''/\texttt{mt}\}$ finishes the case.

Case T-Mode $\quad e = \texttt{m} \quad \tau = \texttt{modev}$
Trivial.

Case T-Sub $\quad e = e_1 \qquad \tau = \tau_1'$
$\quad \Gamma;\texttt{K}_1,\eta \le \texttt{mt} \le \eta',\texttt{K}_2 \vdash e_1 : \tau_1 \qquad \texttt{K}_1,\eta \le \texttt{mt} \le \eta',\texttt{K}_2 \vdash \tau_1 <: \tau_1'$

By the induction hypothesis,

$$\Gamma\{\eta''/\texttt{mt}\};\texttt{K}_1,\texttt{K}_2\{\eta''/\texttt{mt}\} \vdash e_1\{\eta''/\texttt{mt}\} : \tau_1''$$

with

$$\texttt{K}_1,\texttt{K}_2\{\eta''/\texttt{mt}\} \vdash \tau_1'' <: \tau_1\{\eta''/\texttt{mt}\}.$$

Lemma 9 gives $\texttt{K}_1,\texttt{K}_2\{\eta''/\texttt{mt}\} \vdash \tau_1\{\eta''/\texttt{mt}\} <: \tau_1'\{\eta''/\texttt{mt}\}$; therefore, by S-Trans, $\texttt{K}_1,\texttt{K}_2\{\eta''/\texttt{mt}\} \vdash \tau_1'' <: \tau_1'\{\eta''/\texttt{mt}\}$. Then, by T-Sub we have $\Gamma\{\eta''/\texttt{mt}\};\texttt{K}_1,\texttt{K}_2\{\eta''/\texttt{mt}\} \vdash e_1\{\eta''/\texttt{mt}\} : \tau_1'\{\eta''/\texttt{mt}\}$. Letting τ' be $\tau_1'\{\eta''/\texttt{mt}\}$ finishes the case.

$\langle\langle\langle$ **Finish the proof. -Anthony** $\rangle\rangle\rangle$

□

Lemma 17 (Term Substitution Perserves Typing)**.** *If* $\Gamma,\texttt{y} : \tau_0;\texttt{K} \vdash e : \tau$ *and* $\Gamma;\texttt{K} \vdash \texttt{s} : \tau_0'$ *with* $\texttt{K} \vdash \tau_0' <: \tau_0$*, then* $\Gamma\{\texttt{s}/\texttt{y}\};\texttt{K} \vdash e : \tau'$ *with* $\texttt{K} \vdash \tau' <: \tau$.

Proof. By induction on the derivation of $\Gamma,\texttt{y} : \tau_0;\texttt{K} \vdash e : \tau$.

Case T-Var $\quad e = \mathtt{x} \quad \tau = \Gamma(\mathtt{x})$
If $\mathtt{x} \neq \mathtt{y}$ then we have $\Gamma\{\mathtt{s}/\mathtt{y}\}; \mathtt{K} \vdash \mathtt{x} : \Gamma(\mathtt{x})$, with $\Gamma(\mathtt{x}) <: \Gamma(\mathtt{x})$ which is exactly what we need, since $\mathtt{x} : \Gamma(\mathtt{x})$ is $e\{\mathtt{s}/\mathtt{y}\} : \tau$. If $\mathtt{x} = \mathtt{y}$, then we have $\Gamma\{\mathtt{s}/\mathtt{y}\}; \mathtt{K} \vdash \mathtt{s} : \Gamma(\mathtt{s})$ with $\Gamma(\mathtt{s}) <: \Gamma(\mathtt{s})$ which is exactly what we need, since $\mathtt{s} : \Gamma(\mathtt{s})$ is $e\{\mathtt{s}/\mathtt{y}\} : \tau$.

⟨⟨⟨ **Come back to check $\Gamma(\mathrm{x})$ definition -Anthony** ⟩⟩⟩ .

Case T-New $\quad e = \textbf{new}\ \mathtt{c}\langle\iota\rangle \quad \tau = \mathtt{c}\langle\iota\rangle$
Trivial.

Case T-Cast $\quad e = (T)e_1 \qquad \tau = T$
$\qquad \Gamma, \mathtt{y} : \tau_0; \mathtt{K} \vdash e_1 : T'$

By the induction hypothesis,

$$\Gamma\{\mathtt{s}/\mathtt{y}\}; \mathtt{K} \vdash e_1\{\mathtt{s}/\mathtt{y}\} : \tau_1$$

with

$$\mathtt{K} \vdash \tau_1 <: T'.$$

T-Sub gives $\Gamma\{\mathtt{s}/\mathtt{y}\}; \mathtt{K} \vdash e_1\{\mathtt{s}/\mathtt{y}\} : T'$. Then, by T-Cast, $\Gamma\{\mathtt{s}/\mathtt{y}\}; \mathtt{K} \vdash (T)e_1\{\mathtt{s}/\mathtt{y}\} : T$. Now, letting τ' be T.

Case T-Msg $\quad e = e_1.\mathtt{md}(\overline{e_1}) \qquad \tau = T$
$\qquad \Gamma, \mathtt{y} : \tau_0; \mathtt{K} \vdash e_1 : T \qquad \Gamma, \mathtt{y} : \tau_0; \mathtt{K} \vdash \overline{e_1} : \overline{T} \qquad \mathtt{mtype}(\mathtt{md}, T) = \overline{T} \rightarrow T'$
$\qquad \Gamma, \mathtt{y} : \tau_0; \mathtt{K} \vdash \textbf{this} : T_{this} \qquad \mathtt{K} \models \{\mathtt{mode}(T) \leq \mathtt{mode}(T_{this})\} \qquad \mathtt{mode}(T) \neq ?$

By the induction hypothesis,

$$\Gamma\{\mathtt{s}/\mathtt{y}\}; \mathtt{K} \vdash e_1\{\mathtt{s}/\mathtt{y}\} : \tau_1$$
$$\Gamma\{\mathtt{s}/\mathtt{y}\}; \mathtt{K} \vdash \overline{e_1}\{\mathtt{s}/\mathtt{y}\} : \overline{\tau_1}$$

with

$$\mathtt{K} \vdash \tau_1 <: T$$
$$\mathtt{K} \vdash \overline{\tau_1} <: \overline{T}.$$

Lemma 15 gives $\Gamma\{\mathtt{s}/\mathtt{y}\}; \mathtt{K} \vdash \textbf{this} : T_{this}$. Now, $\mathtt{mtype}(\mathtt{md}, \tau_1) = \overline{T} \rightarrow T'$ by Lemma 12, but $\mathtt{K} \vdash \overline{\tau_1} <: \overline{T}$; therefore, our method types and arguments are still satisfied.

Now, by Lemma 14, $\mathtt{mode}(\tau_1) \leq \mathtt{mode}(T)$; hence, $\mathtt{K} \models \{\mathtt{mode}(\tau_1) \leq \mathtt{mode}(T_{this})\}$ and $\mathtt{mode}(\tau_1) \neq ?$. Then, by T-Msg, $\Gamma\{\mathtt{s}/\mathtt{y}\}; \mathtt{K} \vdash e_1\{\mathtt{s}/\mathtt{y}\}.\mathtt{md}(\overline{e_1}\{\mathtt{s}/\mathtt{y}\}) : \tau_1$ with $\mathtt{K} \vdash \tau_1 <: T$.

⟨⟨⟨ **Use mode substitution preseves typing on mtype. -Anthony** ⟩⟩⟩

Case T-Field $\quad e = e_1.\mathtt{fd}_i \qquad \tau = T$
$\qquad \Gamma, \mathtt{y} : \tau_0; \mathtt{K} \vdash e_1 : T \qquad \mathtt{fields}(T) = \overline{T}\ \overline{\mathtt{fd}}$
$\qquad \Gamma, \mathtt{y} : \tau_0; \mathtt{K} \vdash \textbf{this} : T_{this} \qquad \mathtt{K} \models \{\mathtt{mode}(T) \leq \mathtt{mode}(T_{this})\} \qquad \mathtt{mode}(T) \neq ?$

By the induction hypothesis,

$$\Gamma\{\mathtt{s}/\mathtt{y}\}; \mathtt{K} \vdash e_1\{\mathtt{s}/\mathtt{y}\} : \tau_1$$

with

$$\mathtt{K} \vdash \tau_1 <: T.$$

Lemma 15 gives $\Gamma\{\mathtt{s}/\mathtt{y}\}; \mathtt{K} \vdash \textbf{this} : T_{this}$. Lemma 13 gives $\mathtt{fields}(\tau_1) = \overline{\tau_1}\ \overline{\mathtt{fd}}$ with $\overline{\tau_1} <: \overline{T}$.

Now, by Lemma 14, $\mathtt{mode}(\tau_1) \leq \mathtt{mode}(T)$; hence, $\mathtt{K} \models \{\mathtt{mode}(\tau_1) \leq \mathtt{mode}(T_{this})\}$ and $\mathtt{mode}(\tau_1) \neq ?$. Then, by T-Field, $\Gamma\{\mathtt{s}/\mathtt{y}\}; \mathtt{K} \vdash e_1\{\mathtt{s}/\mathtt{y}\}.\mathtt{fd}_i : \tau_{1_i}$ with $\tau_{1_i} <: T_i$.

⟨⟨⟨ **Use mode substitution preseves typing on fields. -Anthony** ⟩⟩⟩

Case T-Snapshot $\quad e = \textbf{snapshot}\; e_1\; [\eta_1, \eta_2] \quad \tau = \exists\omega.\mathtt{c}\langle \mathtt{mt}, \iota\rangle$
$\qquad \Gamma, \mathtt{y} : \tau_0; \mathtt{K} \vdash e_1 : \mathtt{c}\langle ?, \iota\rangle \quad \omega = \eta_1 \leq \mathtt{mt} \leq \eta_2$

By the induction hypothesis,

$$\Gamma\{\mathtt{s}/\mathtt{y}\} \vdash e_1\{\mathtt{s}/\mathtt{y}\} : \tau_1'$$

with

$$\tau_1' <: \mathtt{c}\langle ?, \iota\rangle.$$

We may now use T-Sub to get $\Gamma\{\mathtt{s}/\mathtt{y}\} \vdash e_1\{\mathtt{s}/\mathtt{y}\} : \mathtt{c}\langle ?, \iota\rangle$. Then, by T-Snapshot, $\Gamma\{\mathtt{s}/\mathtt{y}\} \vdash \textbf{snapshot}\; e_1\{\mathtt{s}/\mathtt{y}\}\; [\eta_1, \eta_2] : \exists\omega.\mathtt{c}\langle \mathtt{mt}, \iota\rangle$. Letting $\tau' = \exists\omega.\mathtt{c}\langle \mathtt{mt}, \iota\rangle$ finishes the case.

Case T-MCase $\quad e = \{\overline{\mathtt{m} : e_1}\}^T \quad \tau = \textbf{mcase}\langle T\rangle$
$\qquad \Gamma, \mathtt{y} : \tau_0; \mathtt{K} \vdash e_{1_i} : T$ for all $i \quad \overline{\mathtt{m}} = \mathtt{modes}(P)$

By the induction hypothesis,

$$\Gamma\{\mathtt{s}/\mathtt{y}\} \vdash e_{1_i}\{\mathtt{s}/\mathtt{y}\} : \tau_1'$$

with

$$\mathtt{K} \vdash \tau_1 <: T.$$

By the inversion of the subtype relation, $\tau_1 = T'$ with $\mathtt{K} \vdash T' <: T$. Then, by T-Mcase, $\Gamma\{\mathtt{s}/\mathtt{y}\} \vdash \{\overline{\mathtt{m} : e_1\{\mathtt{s}/\mathtt{y}\}}\}^{T'} : \textbf{mcase}\langle T'\rangle$ with $\mathtt{K} \vdash \textbf{mcase}\langle T'\rangle <: \textbf{mcase}\langle T\rangle$ by S-MCase.

⟨⟨⟨ **Check subtype relation -Anthony** ⟩⟩⟩

Case T-ElimCase $\quad e = e_1 \triangleright \eta \quad \tau = T$
$\qquad \Gamma, \mathtt{y} : \tau_0; \mathtt{K} \vdash e_1 : \textbf{mcase}\langle T\rangle \quad \eta \in \mathtt{modes}(P)$ or η appears in $\mathtt{K}$

By the induction hypothesis,

$$\Gamma\{\mathtt{s}/\mathtt{y}\}\mathtt{K} \vdash e_1\{\mathtt{s}/\mathtt{y}\} : \tau_1$$

with

$$\mathtt{K} \vdash \tau_1 <: \textbf{mcase}\langle T\rangle.$$

By the inversion of the subtype relation, $\tau_1 = \textbf{mcase}\langle T'\rangle$ with $\mathtt{K} \vdash T' <: T$. Then, by T-ElimCase, $\Gamma\{\mathtt{s}/\mathtt{y}\}; \mathtt{K} \vdash e_1\{\mathtt{s}/\mathtt{y}\} \triangleright \eta' : T$, with $\mathtt{K} \vdash T' <: T$.

⟨⟨⟨ **Check subtype relation -Anthony** ⟩⟩⟩

Case T-Mode $\quad e = \mathtt{m} \quad \tau = \mathtt{modev}$

Trivial.

Case T-Sub $\quad e = e_1 \quad \tau = \tau_1'$
$\qquad \Gamma, \mathtt{y} : \tau_0; \mathtt{K} \vdash e_1 : \tau_1 \quad \mathtt{K} \vdash \tau_1 <: \tau_1'$

By the induction hypothesis,

$$\Gamma\{\mathtt{s}/\mathtt{y}\}; \mathtt{K} \vdash e_1\{\mathtt{s}/\mathtt{y}\} : \tau_2$$

with

$$\mathtt{K} \vdash \tau_2 <: \tau_1.$$

By S-Trans, we have $\mathtt{K} \vdash \tau_2 <: \tau_1'$. Then, by T-Sub, $\Gamma\{\mathtt{s}/\mathtt{y}\}; \mathtt{K} \vdash e_1\{\mathtt{s}/\mathtt{y}\} : \tau_1'$. Letting $\tau' = \tau_1'$ finishes the case.

$\square$

Lemma 18. *If* $\mathtt{K} \vdash_{\mathtt{wft}} \mathtt{c}\langle\iota\rangle$, $\mathtt{mtype}(\mathtt{md}, \mathtt{c}\langle\iota\rangle) = \overline{T} \to T$ *and* $\mathtt{mbody}(\mathtt{md}, \mathtt{c}\langle\iota\rangle)$ *then* $\overline{\mathtt{x}} : \overline{T}; \mathbf{this} : T_0; \mathtt{K} \vdash e : T'$ *with* $\mathtt{K} \vdash_{\mathtt{wft}} T_0$, $\mathtt{K} \vdash \mathtt{c}\langle\iota\rangle <: T_0$, *and* $\mathtt{K} \vdash T' <: T$.

Proof. Induction on the derivation of $\mathtt{mtype}(\mathtt{md}, \mathtt{c}\langle\iota\rangle) = \overline{\mathtt{x}}.e$ using Lemmas 9 and 16.

Case MB-Class $\quad$ $\mathbf{class}\ \mathtt{c}\ \Delta\ \mathbf{extends}\ T_0\{\overline{F}\ \overline{M}\ A\} \quad \mathtt{eparam}(\Delta) = \iota'$
$\tau\ \mathtt{md}(\overline{\tau}\ \overline{\mathtt{x}})\{e\} \in \overline{M}$

Let $\Gamma = \overline{\mathtt{x}} : \overline{T}; \mathbf{this} : \mathtt{c}\langle\iota\rangle$. From T-Class and T-Method we have $\Gamma; \mathtt{K}' \vdash e : \tau'$ with $\mathtt{K}' \vdash \tau' <: \tau$. Since $\mathtt{K} \vdash_{\mathtt{wft}} \mathtt{c}\langle\iota\rangle$ we have $\mathtt{K} \models \mathtt{K}'\{\iota/\iota'\}$ and $\mathtt{K}' = \mathtt{cons}(\Delta)$ from WF-Class. Using Lemmas 1, 9 and 16 we have

$$\mathtt{K} \vdash \tau'\{\iota/\iota'\} <: \tau\{\iota/\iota'\}$$

and

$$\overline{\mathtt{x}} : \overline{\tau}\{\iota/\iota'\}; \mathbf{this} : \mathtt{c}\langle\iota\rangle; \mathtt{K} \vdash e\{\iota/\iota'\} : \tau''$$

with

$$\mathtt{K} \vdash \tau'' <: \tau'\{\iota/\iota'\}.$$

From MT-Class we have

$$\overline{\tau}\{\iota/\iota'\} = \overline{T} \quad \tau\{\iota/\iota'\} = T$$

S-Trans gives us $\mathtt{K} \vdash \tau'' <: \tau\{\iota/\iota'\}$ from which we have $\overline{\mathtt{x}} : \overline{T}, \mathbf{this} : \mathtt{c}\langle\iota\rangle; \mathtt{K} \vdash e : \tau''$ with $\mathtt{K} \vdash \tau'' <: T$. Letting T_0 be $\mathtt{c}\langle\iota\rangle$ finishes the case.

Case MB-Super $\quad$ $\mathbf{class}\ \mathtt{c}\ \Delta\ \mathbf{extends}\ T_0\{\overline{F}\ \overline{M}\ A\} \quad \mathtt{eparam}(\Delta) = \iota'$
$\mathtt{md} \notin \overline{M}$

Immediate from the inductive hypothesis and the fact that $\mathtt{K} \vdash \mathtt{c}\langle\iota\rangle <: T_0\{\iota/\iota'\}$.

$\square$

Lemma 19. *If* $\Gamma; \mathtt{K} \vdash \mathtt{obj}(\alpha, \mathtt{c}\langle\iota\rangle, \overline{v})$ *and* $\mathtt{fields}(\mathtt{c}\langle\iota\rangle) = \overline{\tau}\ \overline{\mathtt{fd}} = \overline{e}$ *then* $\Gamma; \mathtt{K} \vdash v_i : \tau_i'$ *with* $\mathtt{K} \vdash \tau_i' <: \tau_i$.

Proof. Induction on the derivation of $\mathtt{fields}(\mathtt{c}\langle\iota\rangle) = \overline{\tau}\ \overline{\mathtt{fd}}$.

Case FD-Class $\quad$ $\mathbf{class}\ \mathtt{c}\ \Delta\ \mathbf{extends}\ T_0\{\overline{\tau}\ \overline{\mathtt{fd}} = \overline{e}\ \ldots\} \quad \mathtt{eparam}(\Delta) = \iota'$
$\mathtt{fields}(T_0\{\iota/\iota'\}) = \overline{\tau_0}\ \overline{\mathtt{fd}_0} = \overline{e_0}$

From T-Class we have $\varnothing; \mathtt{K}' \vdash \overline{e} : \overline{\tau'}$ with $\mathtt{K}' \vdash \overline{\tau'} <: \overline{\tau}$. Since $\mathtt{K} \vdash_{\mathtt{wft}} \mathtt{c}\langle\iota\rangle$ we have $\mathtt{K} \models \mathtt{K}'\{\iota/\iota'\}$ and $\mathtt{K}' = \mathtt{cons}(\Delta)$ from WF-Class.

Using Lemmas 1, 9, and 16 we have

$$\mathtt{K} \vdash \overline{\tau'}\{\iota/\iota'\} <: \overline{\tau}\{\iota/\iota'\}$$

and

$$\Gamma; \mathtt{K} \vdash \overline{e}\{\iota/\iota'\} : \overline{\tau''}$$

with

$$\mathtt{K} \vdash \overline{\tau''} <: \overline{\tau'}\{\iota/\iota'\}.$$

Now, from T-Object we have $\overline{\tau'}\{\iota/\iota'\} = \overline{\tau}$ and $\overline{e}\{\iota/\iota'\} = \overline{v}$. Then, by S-Trans we have $\Gamma; \mathtt{K} \vdash \overline{v} : \overline{\tau''}$ with $\mathtt{K} \vdash \overline{\tau''} <: \overline{\tau}$. Choosing $\Gamma; \mathtt{K} \vdash v_i : \tau''_i$ with $\mathtt{K} \vdash \tau''_i <: \tau_i$ and letting τ'_i be τ''_i finishes the case.

Case FD-Object Trivial.

□

Lemma 20 (Preservation)**.** *If* $\Gamma; \mathtt{K} \vdash e : \tau, e \overset{\mathtt{m}}{\Longrightarrow} e'$*, then* $\Gamma; \mathtt{K} \vdash e' : \tau'$ *with* $\mathtt{K} \vdash \tau' <: \tau$.

Proof. By induction on the derivation of $\Gamma, \mathtt{K} \vdash e : \tau$, with a case analysis on the last rule used.

Case T-Var $\quad e = \mathtt{x} \quad \tau = \Gamma(\mathtt{x})$
Trival: Cannot occur.

Case T-New $\quad e = \textbf{new}\ \mathtt{c}\langle\iota\rangle \qquad \tau = \mathtt{c}\langle\iota\rangle$
$\iota = ?, \iota'$ iff **class** $\mathtt{c}\ \Delta \cdots \in P$ and $\mathtt{ethis}(\Delta) = ?$
$\iota \neq ?, \iota'$ iff **class** $\mathtt{c}\ \Delta \cdots \in P$ and $\mathtt{ethis}(\Delta) \neq ?$
$\mathtt{K} \models \mathtt{cons}(\Delta)$
Trivial.

Case T-Cast $\quad e = (T)e_1 \qquad \tau = T$
$\Gamma; \mathtt{K} \vdash e_1 : T_1$

Subcase $\quad e_1 \overset{\mathtt{m}}{\Longrightarrow} e'_1 \quad e' = (T)e'_1$

By the induction hypothesis, $\Gamma; \mathtt{K} \vdash e'_1 : T'_1$ with $\mathtt{K} \vdash T'_1 <: T_1$. T-Sub gives $\Gamma; \mathtt{K} \vdash e'_1 : T_1$. Then, by T-Cast, $\Gamma; \mathtt{K} \vdash (T)e'_1 : T$. Letting τ' be T finishes the subcase.

Subcase $\quad e_1 = \mathtt{obj}(\alpha, T'_1, \overline{v})$
$(T)\mathtt{obj}(\alpha, T'_1, \overline{v}) \overset{\mathtt{m}}{\Longrightarrow} \mathtt{obj}(\alpha, T'_1, \overline{v}) \quad T'_1 <: T'$
$e' = \mathtt{obj}(\alpha, T'_1, \overline{v})$
Trivial. We have $\Gamma; \mathtt{K} \vdash \mathtt{obj}(\alpha, T'_1, \overline{v}) : T'_1$ from T-Cast and T-Obj, and $T'_1 <: T$ from R-Cast.

Case T-Msg $\quad e = e_1.\mathtt{md}(\overline{e_1}) \qquad \tau = T'$
$\Gamma; \mathtt{K} \vdash e_1 : T \qquad \Gamma; \mathtt{K} \vdash \overline{e_1} : \overline{T} \qquad \mathtt{mtype}(\mathtt{md}, T) = \overline{T} \rightarrow T'$
$\Gamma; \mathtt{K} \vdash \textbf{this} : T_{this} \qquad \mathtt{K} \models \{\mathtt{mode}(T) \leq \mathtt{mode}(T_{this})\} \qquad \mathtt{mode}(T) \neq ?$

Subcase $\quad e_1 \overset{\mathtt{m}}{\Longrightarrow} e'_1 \quad e' = e'_1.(\overline{e_1})$
Easy.

Subcase $\quad e_1 = o \quad e_{1_i} \overset{\mathtt{m}}{\Longrightarrow} e'_{1_i} \quad e' = o.(v_{1_i}, \ldots, e'_{1_i}, \ldots, e_n)$
Easy.

Subcase R-Msg $\quad e_1 = o \qquad o = \mathtt{obj}(\alpha, \mathtt{c}\langle\mu, \iota\rangle, \overline{v})$
$o.\mathtt{md}(\overline{v'}) \overset{\mathtt{m}}{\Longrightarrow} \mathbf{E}_{\mathtt{m}'}[\, e_b\{\overline{v}'/\overline{\mathtt{x}}\}\{o/\textbf{this}\}\,] \qquad \mathtt{mbody}(\mathtt{md}, \mathtt{c}\langle\mu, \iota\rangle) = \overline{\mathtt{x}}.e_b \quad \mu \leq \mathtt{m}$
$\mathtt{m}' = \mathtt{emode}(o)$
$e' = \mathbf{E}_{\mathtt{m}'}[\, e_b\{\overline{v}'/\overline{\mathtt{x}}\}\{o/\textbf{this}\}\,]$

From Lemma 18 we have $\overline{\mathtt{x}} : \overline{\tau}, \textbf{this} : \mathtt{c}\langle\mu, \iota\rangle; \mathtt{K} \vdash \overline{\mathtt{x}}.e_b : \tau_b$ with $\mathtt{K} \vdash \tau_b <: T'$. Using Lemma 17 twice and S-Trans we get $\varnothing; \mathtt{K}' \vdash e_b\{\overline{v}'/\overline{\mathtt{x}}\}\{o/\textbf{this}\} : \tau_b$.

Now, we may weaken $\varnothing$ to Γ by Lemma 1 which gives us $\Gamma; \mathtt{K} \vdash e_b\{\overline{v}'/\overline{\mathtt{x}}\}\{o/\textbf{this}\} : \tau_b$. Letting τ' be τ_b finishes the case.

Case T-Field $\quad e = e_1.\mathtt{fd}_i \qquad \tau = T_i$
$\Gamma; \mathtt{K} \vdash e_1 : T \qquad \mathtt{fields}(T) = \overline{T}\ \overline{\mathtt{fd}}$
$\Gamma; \mathtt{K} \vdash \textbf{this} : T_{this} \qquad \mathtt{K} \models \{\mathtt{mode}(T) \leq \mathtt{mode}(T_{this})\} \quad \mathtt{mode}(T) \neq ?$

Subcase $\quad e_1 \stackrel{\mathtt{m}}{\Longrightarrow} e_1' \quad e' = e_1'.\mathtt{fd}_i$
Easy.

Subcase R-Field $\quad e_1 = \mathtt{obj}(\alpha, \mathtt{c}\langle \mu, \iota \rangle, \overline{v})$
$\mathtt{obj}(\alpha, \mathtt{c}\langle \mu, \iota \rangle, \overline{v}).\mathtt{fd}_i \stackrel{\mathtt{m}}{\Longrightarrow} v_i \quad \mu \leq \mathtt{m}$
$e' = v_i$
Lemma 19 gives $\Gamma; \mathtt{K} \vdash v_i : \tau_i$ with $\mathtt{K} \vdash \tau_i <: T_i$ which is exactly what we need.

Case T-Snapshot $\quad e = \textbf{snapshot}\ e_1\ [\eta_1, \eta_2] \qquad \tau = \exists \omega.\mathtt{c}\langle \mathtt{mt}, \iota \rangle$
$\Gamma; \mathtt{K} \vdash e_1 : \mathtt{c}\langle ?, \iota \rangle \qquad \omega = \eta_1 \leq \mathtt{mt} \leq \eta_2$

Subcase $\quad e_1 \stackrel{\mathtt{m}}{\Longrightarrow} e_1' \quad e' = \textbf{snapshot}\ e_1'\ [\eta_1, \eta_2]$
Easy.

Subcase R-Snapshot1 $\quad \textbf{snapshot}\ o\ [\eta_1, \eta_2] \stackrel{\mathtt{m}}{\Longrightarrow} \textbf{let}\ \mathtt{x} = \textbf{check}(e_a\{o/\textbf{this}\}, \mathtt{m}', \mathtt{m}'')\ \textbf{in}\ \mathtt{obj}(\alpha', \mathtt{c}\langle \mathtt{x}, \iota \rangle, \overline{v})$
$o = \mathtt{obj}(\alpha, \mathtt{c}\langle ?, \iota \rangle, \overline{v}) \quad \textbf{class}\ \mathtt{c}\ \cdots\ \{\ \cdots\ A\ \} \in P \quad e_a = \mathtt{abody}(\mathtt{c}\langle \iota \rangle)$
By T-Attr, $e_a : \mathtt{modev}$ which is preserved by Lemma 16 From T-Attr, we have $e_a : \mathtt{modev}$

Subcase R-Snapshot2 $\quad \textbf{snapshot}\ o\ [\eta_1, \eta_2] \stackrel{\mathtt{m}}{\Longrightarrow} \textbf{let}\ \mathtt{x} = \textbf{check}(\mathtt{m}, \mathtt{m}', \mathtt{m}'')\ \textbf{in}\ \mathtt{obj}(\alpha', \mathtt{c}\langle \mathtt{x}, \iota \rangle, \overline{v})$
$o = \mathtt{obj}(\alpha, \mathtt{c}\langle \mathtt{m}, \iota \rangle, \overline{v})$
Easy.

Case T-MCase $\quad e = \{\overline{\mathtt{m} : e_1}\}^T \qquad \tau = \textbf{mcase}\langle T \rangle$
$\Gamma; \mathtt{K} \vdash e_{1_i} : T$ for all $i \qquad \overline{\mathtt{m}} = \mathtt{modes}(P)$

Subcase $\quad e_{1_i} \stackrel{\mathtt{m}}{\Longrightarrow} e_{1_i}' \quad e' = \{\mathtt{m} : v_{1_i}; \ldots; \mathtt{m} : e_{1_i}'; \ldots; \mathtt{m} : e_{1_n}\}$
Easy.

Case T-ElimCase $\quad e = e_1 \triangleright \eta \qquad \tau = T$
$\Gamma; \mathtt{K} \vdash e_1 : \textbf{mcase}\langle T \rangle \qquad \eta \in \mathtt{modes}(P)$ or η appears in $\mathtt{K}$

Subcase $\quad e_1 \stackrel{\mathtt{m}}{\Longrightarrow} e_1' \quad e' = e_1' \triangleright \eta$
Easy.

Subcase R-McaseProj $\quad e_1 = \{\overline{\mathtt{m} : v}\}^T \qquad \eta = \mathtt{m}_j$
$\{\overline{\mathtt{m} : v}\}^T \triangleright \mathtt{m}_j \stackrel{\mathtt{m}}{\Longrightarrow} v_j$
$e' = v_j$
From T-Mcase,

$\overline{\mathtt{m}} = \mathtt{modes}(P)$
$\Gamma; \mathtt{K} \vdash v_i : T$ for all i.

$\Gamma; \mathtt{K} \vdash v_j : T$ gives us what we need. Letting τ' be T finishes the case.

Case T-Mode $\quad e = \mathtt{m} \quad \tau = \mathtt{modev}$
Trival: Cannot occur.

Case T-Sub $\quad e = e_1 \qquad \tau = \tau_1'$
$\Gamma; \mathtt{K} \vdash e_1 : \tau_1 \quad \mathtt{K} \vdash \tau_1 <: \tau_1'$

Trivial.

Case T-Object $\quad e = \mathtt{obj}(\alpha, \mathtt{c}\langle\iota\rangle, \overline{e}) \quad \tau = \mathtt{c}\langle\iota\rangle$
$\Gamma; \mathtt{K} \vdash \overline{e} : \overline{\tau} \qquad \mathtt{fields}(\mathtt{c}\langle\iota\rangle) = \overline{\tau}\ \overline{\mathtt{fd}} = \overline{e}$

Trival: Cannot occur.

Case T-Check $\quad e = \mathbf{check}(e_1, \mathtt{m}_1, \mathtt{m}_2) \quad \tau = \mathtt{modev}$
$\Gamma; \mathtt{K} \vdash e_1 : \mathtt{modev}$

Subcase $\quad e_1 \xRightarrow{\mathtt{m}} e_1' \quad e' = \mathbf{check}(e_1', \mathtt{m}_1, \mathtt{m}_2)$
Easy.

Subcase R-Check $\quad e_1 = \mathtt{m}'$
$\mathbf{check}(\mathtt{m}', \mathtt{m}_1, \mathtt{m}_2) \xRightarrow{\mathtt{m}} \mathtt{m}' \quad \mathtt{m}_1 \leq \mathtt{m}' \leq \mathtt{m}_2$
$e' = \mathtt{m}'$
Trivial: From T-Check we have $\Gamma; \mathtt{K} \vdash \mathtt{m}' : \mathtt{modev}$ which is exactly what we need. Letting τ' be $\mathtt{modev}$ finishes the subcase.

Case T-Let $\quad e = \mathbf{let}\ \mathtt{x} = e_1\ \mathbf{in}\ e_2 \quad \tau = T$
$\Gamma; \mathtt{K} \vdash e_1 : \tau_1 \qquad \Gamma, \mathtt{x} : \tau_1; \mathtt{K} \vdash e_2 : T$

Trivial.

⟨⟨⟨ **Finish the proof. -Anthony** ⟩⟩⟩

□

Lemma 21.

(1) If $\Gamma; \mathtt{K} \vdash v : \tau$ *and* $\mathtt{K} \vdash \tau <: \mathtt{c}\langle\mu, \overline{\eta}\rangle$*, then* $\tau = \mathtt{c}'\langle\mu', \overline{\eta}\rangle$ *with* $\mathtt{K} \vdash \mathtt{c}'\langle\mu', \overline{\eta}\rangle <: \mathtt{c}\langle\mu, \overline{\eta}\rangle$*.*
(2) If $\Gamma; \mathtt{K} \vdash v : \tau$ *and* $\mathtt{K} \vdash \tau <: \mathbf{mcase}\langle T\rangle$*, then* $\tau = \mathbf{mcase}\langle T'\rangle$ *with* $\mathtt{K} \vdash T' <: T$*.*

Proof.

(1) Case analysis on the induction of the derivation of $\mathtt{K} \vdash \tau <: \mathtt{c}\langle\mu, \overline{\eta}\rangle$: Only S-Dynamic and S-Class apply, we present S-Exists to clarify.

Case (S-Dynamic) $\quad \tau = \mathtt{c}\langle\mu', \overline{\eta}\rangle$
Letting $\mathtt{c}'$ be $\mathtt{c}$ and μ be ? finishes the case.

Case (S-Class) $\quad \tau = \mathtt{c}'\langle\iota\rangle$
Trivial. Exactly what we need.

Case (S-Exists) $\quad \tau = \exists\omega.\mathtt{c}\langle\mu, \overline{\eta}\rangle$
If $\tau = \exists\omega.\mathtt{c}\langle\mu, \overline{\eta}\rangle$ then we need to have a value with type $\exists\omega.\mathtt{c}\langle\mu, \overline{\eta}\rangle$, but by the structure of our terms and typing rules this cannot occur; hence, S-Exists contradicts our hypothesis and cannot occur.

(2) Induction on the derivation of $\mathtt{K} \vdash \tau <: \mathbf{mcase}\langle T\rangle$: Only S-Mcase applies.

Case (S-Mcase) $\quad \tau = \mathbf{mcase}\langle T'\rangle$
$\mathtt{K} \vdash T' <: T$
Trivial. Exactly what we need.

□

Lemma 22 (Canonical Forms)**.** *Given* $\Gamma; \mathtt{K} \vdash v : \tau$*,*

(1) If $\tau = \mathtt{c}\langle\iota\rangle$ *then* v *has the shape* $\mathtt{obj}(\alpha, \tau', \overline{v})$ *with* $\mathtt{K} \vdash \tau' <: \mathtt{c}\langle\iota\rangle$*.*
(2) If $\tau = \textbf{mcase}\langle T\rangle$ *then* v *has the shape* $\{\overline{\mathtt{m} : v}\}^{T'}$ *with* $\mathtt{K} \vdash T' <: T$*.*
(3) If $\tau = \mathtt{modev}$ *then* v *has the shape* $\mathtt{m}$ *with* $\mathtt{m} \in \mathtt{modes}(P)$*.*

Proof.

(1) Induction on the derivation $\Gamma; \mathtt{K} \vdash v : \mathtt{c}\langle ?, \iota\rangle$. Two rules may apply: T-Obj and T-Sub.

Case T-Obj $\quad v = \mathtt{obj}(\alpha, \mathtt{c}\langle\iota\rangle, \overline{v})$
Letting τ' be $\mathtt{c}\langle\iota\rangle$ finishes the case.

Case T-Sub $\quad v = v_1$
$\Gamma; \mathtt{K} \vdash v_1 : \tau_1 \quad \mathtt{K} \vdash \tau_1 <: \mathtt{c}\langle\iota\rangle$
By Lemma 21 $\tau_1 = \mathtt{c}'\langle\iota\rangle$. Then, by the induction hypothesis, $v_1 = \mathtt{obj}(\alpha, \tau_1', \overline{v})$ with $\mathtt{K} \vdash \tau_1' <: \mathtt{c}'\langle\iota\rangle$. By S-Trans, $\mathtt{K} \vdash \tau_1' <: \mathtt{c}\langle\iota\rangle$. We may now apply T-Sub to get $\Gamma; \mathtt{K} \vdash \mathtt{obj}(\alpha, \tau_1', \overline{v}) : \mathtt{c}\langle\iota\rangle$.

(2) Induction on the derivation $\Gamma; \mathtt{K} \vdash v : \textbf{mcase}\langle T\rangle$. Two rules may apply: T-Mcase and T-Sub.

Case T-Mcase $\quad v = \{\overline{\mathtt{m} : v}\}^{T}$
Letting T' be T finishes the case.

Case T-Sub $\quad v = v_1$
$\Gamma; \mathtt{K} \vdash v_1 : \tau_1 \quad \mathtt{K} \vdash \tau_1 <: \textbf{mcase}\langle T\rangle$

By Lemma 21 $\tau_1 = \textbf{mcase}\langle T_1\rangle$ with $\mathtt{K} \vdash T_1 <: T$. Then, by the induction hypothesis, $v_1 = \{\overline{\mathtt{m} : v}\}^{T_1'}$ with $\mathtt{K} \vdash T_1' <: T_1$. By S-Trans, $\mathtt{K} \vdash T_1' <: T$. We may now apply T-Sub to get $\Gamma; \mathtt{K} \vdash \{\overline{\mathtt{m} : v}\}^{T_1} : \textbf{mcase}\langle T\rangle$.

(3) Only T-ModeValue may apply from which $\mathtt{m} \in \mathtt{modes}(P)$ is immediate.

□

Definition 1 (Bad Cast)**.** *Expression* $(T')\mathtt{obj}(\alpha, T, \overline{v})$ *is a bad cast iff* $\emptyset \vdash T <: T'$ *does not hold.*

Definition 2 (Bad Check)**.** *Expression* $\textbf{check}(\mathtt{m}, \mathtt{m}', \mathtt{m}'')$ *is a bad check iff* $\mathtt{m}' \leq \mathtt{m} \leq \mathtt{m}''$ *does not hold.*

Lemma 23. *If* $\mathbf{E}_{\mathtt{m}}[\,e\,]$, $\Gamma; \mathtt{K} \vdash e : \tau$ *with a premise containing* $\Gamma; \mathtt{K} \vdash \textbf{this} : T_{this}$*, then* $\mathtt{mode}(T_{this}) = \mathtt{m}$*.*

Proof. ⟨⟨⟨ **Come back to prove. -Anthony** ⟩⟩⟩ □

Lemma 24 (Progress)**.** *If* $\Gamma; \mathtt{K} \vdash e : \tau$*, then either* e *is a value,* e *is a bad cast,* e *is a bad check, or there exists* e' *such that* $e \overset{\mathtt{m}}{\Longrightarrow} e'$*.*

Proof. By induction on the derivation of $\Gamma, \mathtt{K} \vdash e : \tau$.

Case T-Var $\quad e = \mathtt{x} \quad \tau = \Gamma(\mathtt{x})$
Trivial.

Case T-New $\quad e = \textbf{new}\ \mathtt{c}\langle\iota\rangle \quad \tau = \mathtt{c}\langle\iota\rangle$
Trivial by R-New, with $e' = \mathtt{obj}(\alpha, \mathtt{c}\langle\iota\rangle, \mathtt{init}(P, \mathtt{c}))$.

Case T-Cast $\quad e = (T')e_1 \qquad \tau = T'$
$\Gamma; \mathtt{K} \vdash e_1 : \mathtt{c}\langle\iota\rangle$

By the induction hypothesis, e_1 is a value, bad cast, bad check, or there exists e_1' such that $e_1 \overset{\mathtt{m}}{\Longrightarrow} e_1'$. If e_1 is a value, then by Lemma 22, $e_1 = \mathtt{obj}(\alpha, T, \overline{v})$ with $\mathtt{K} \vdash T <: \mathtt{c}\langle\iota\rangle$. Now, if $\mathtt{K} \vdash \mathtt{c}\langle\iota\rangle <: T'$ then by S-Trans we have $\mathtt{K} \vdash T <: T'$, from which R-Cast applies, giving $e' = \mathtt{obj}(\alpha, T, \overline{v})$. If $\mathtt{K} \vdash \mathtt{c}\langle\iota\rangle <: T'$ does not hold, then by S-Trans $\mathtt{K} \vdash T <: T'$ does not hold; hence, we have a bad cast. If $e_1 \overset{\mathtt{m}}{\Longrightarrow} e_1'$ then by the reduction context we may replace e_1 with e_1', giving $e' = (T')e_1'$.

Case T-Msg $\quad e = e_1.(\overline{e_1}) \qquad \tau = T'$

$\Gamma; \mathtt{K} \vdash e_1 : T \qquad \Gamma; \mathtt{K} \vdash \overline{e_1} : \overline{T} \qquad \mathtt{mtype}(\mathtt{md}, T) = \overline{T} \to T'$

$\Gamma; \mathtt{K} \vdash \textbf{this} : T_{this} \qquad \mathtt{K} \models \{\mathtt{mode}(T) \leq \mathtt{mode}(T_{this})\} \qquad \mathtt{mode}(T) \neq ?$

By the induction hypothesis,

e_1 is a value, bad cast, bad check, or there exists e'_1 such that $e_1 \stackrel{\mathtt{m}}{\Longrightarrow} e'_1$
e_{1_i} is a value, bad cast, bad check, or there exists e'_{1_i} for each i such that $e_{1_i} \stackrel{\mathtt{m}}{\Longrightarrow} e'_{1_i}$.

If $e_1 \stackrel{\mathtt{m}}{\Longrightarrow} e'_1$ then we may replace e_1 with e'_1 giving us $e' = e'_1.(\overline{e_1})$.

If e_1 is a value then by Lemma 22, $e_1 = \mathtt{obj}(\alpha, T, \overline{v})$ with $\mathtt{K} \vdash T <: \mathtt{c}\langle\iota\rangle$. We consider the case that all e_{1_i} are values first. By Lemma 23 we have $\mathtt{K} \models \{\mathtt{mode}(T) \leq \mathtt{m}\}$. R-Msg now applies, giving us $e' = \mathbf{E}_{\mathtt{m}'}[\, e\{\overline{v}'/\overline{\mathtt{x}}\}\{\mathtt{obj}(\alpha, T, \overline{v})/\textbf{this}\}\,]$. Otherwise we may replace the first e_{1_i} with e'_{1_i} giving us $e' = \mathtt{obj}(\alpha, T, v_{1_1}, \ldots, e'_{1_i}, \ldots, e_{1_n})$.

Case T-Field $\quad e = e_1.\mathtt{fd}_i \qquad \tau = T_i$

$\Gamma; \mathtt{K} \vdash e_1 : T \qquad \mathtt{fields}(T) = \overline{T}\;\overline{\mathtt{fd}}$

$\Gamma; \mathtt{K} \vdash \textbf{this} : T_{this} \qquad \mathtt{K} \models \{\mathtt{mode}(T) \leq \mathtt{mode}(T_{this})\} \qquad \mathtt{mode}(T) \neq ?$

By the induction hypothesis, e_1 is a value, bad cast, bad check, or there exists e'_1 such that $e_1 \stackrel{\mathtt{m}}{\Longrightarrow} e'_1$.

If $e_1 \stackrel{\mathtt{m}}{\Longrightarrow} e'_1$ then we may replace e_1 with e'_1 giving us $e' = e'_1.\mathtt{fd}_i$. If e_1 is a value then by Lemma 22, $e_1 = \mathtt{obj}(\alpha, T, \overline{v})$ with $\mathtt{K} \vdash T <: \mathtt{c}\langle\iota\rangle$. By Lemma 23 we have $\mathtt{K} \models \{\mathtt{mode}(T) \leq \mathtt{m}\}$. R-Field now applies, giving us $e' = v_i$.

Case T-Snapshot $\quad e = \textbf{snapshot}\ e_1\ [\eta_1, \eta_2] \qquad \tau = \exists\omega.\mathtt{c}\langle\mathtt{mt}, \iota\rangle$

$\Gamma; \mathtt{K} \vdash e_1 : \mathtt{c}\langle ?, \iota\rangle \qquad \omega = \eta_1 \leq \mathtt{mt} \leq \eta_2$

By the induction hypothesis, e_1 is a value, bad cast, bad check, or there exists e'_1 such that $e_1 \stackrel{\mathtt{m}}{\Longrightarrow} e'_1$.

If e_1 is a value then by Lemma 22 $e_1 = \mathtt{obj}(\alpha, T, \overline{v})$ with $\mathtt{K} \vdash T <: \mathtt{c}\langle ?, \iota\rangle$. Now, if $\mathtt{eargs}(T) = ?, \iota$, then R-Snapshot1 applies, with $e' = \textbf{let}\ \mathtt{x} = \textbf{check}(A\{\mathtt{obj}(\alpha, T, \overline{v})/\textbf{this}\}, \mathtt{m}', \mathtt{m}'')\ \textbf{in}\ \mathtt{obj}(\alpha', T\{\mathtt{x}, \iota/\mathtt{eargs}(T)\}, \overline{v})$. Otherwise $\mathtt{eargs}(T) = \mathtt{m}, \iota$ from which we may apply R-Snapshot2 to get $e' = \textbf{let}\ \mathtt{x} = \textbf{check}(\mathtt{m}, \mathtt{m}', \mathtt{m}'')\ \textbf{in}\ \mathtt{obj}(\alpha, T, \overline{v})$.

If $e_1 \stackrel{\mathtt{m}}{\Longrightarrow} e'_1$ then by the reduction context we may replace e_1 with e'_1 to get $e' = \textbf{snapshot}\ e'_1\ [\eta_1, \eta_2]$.

Case T-MCase $\quad e = \{\overline{\mathtt{m} : e_1}\}^T \qquad \tau = \textbf{mcase}\langle T\rangle$

$\Gamma; \mathtt{K} \vdash e_{1_i} : T$ for all $i \qquad \overline{\mathtt{m}} = \mathtt{modes}(P)$

By the induction hypothesis, e_{1_i} is a value, bad cast, bad check, or there exists e'_{1_i} such that $e_{1_i} \stackrel{\mathtt{m}}{\Longrightarrow} e'_{1_i}$.

If all e_{1_i} are values, then e is a value. Otherwise by the reduction context we may replace e_{1_i} with e'_{1_i}, giving us e'.

Case T-ElimCase $\quad e = e_1 \triangleright \eta \qquad \tau = T$

$\Gamma; \mathtt{K} \vdash e_1 : \textbf{mcase}\langle T\rangle \qquad \eta \in \mathtt{modes}(P)$ or η appears in $\mathtt{K}$

By the induction hypothesis, e_1 is a value, bad cast, bad check, or there exists e'_1 such that $e_1 \stackrel{\mathtt{m}}{\Longrightarrow} e'_1$.

If e_1 is a value then by Lemma 22, e_1 has the shape $\{\overline{\mathtt{m} : v}\}^T$, from which R-McaseProj applies, giving us $e' = v_j$. If $e_1 \stackrel{\mathtt{m}}{\Longrightarrow} e'_1$ then by the reduction context we may replace e_1 with e'_1, giving us $e' = e'_1 \triangleright \eta$.

Case T-Mode $\quad e = \mathtt{m} \qquad \tau = \mathtt{modev}$

Trivial.

Case T-Sub $\quad e = e_1 \qquad \tau = \tau'_1$

$\Gamma; \mathtt{K} \vdash e_1 : \tau_1 \qquad \mathtt{K} \vdash \tau_1 <: \tau'_1$

By the induction hypothesis, e_1 is a value, bad cast, bad check, or there exists e'_1 such that $e_1 \stackrel{\mathtt{m}}{\Longrightarrow} e'_1$.

If e_1 is a value, we are done. If $e_1 \stackrel{\mathtt{m}}{\Longrightarrow} e'_1$ then we may replace e_1 with e'_1 giving us $e' = e'_1$.

Case T-Object $\quad e = \texttt{obj}(\alpha, \texttt{c}\langle\iota\rangle, \overline{e}) \quad \tau = \texttt{c}\langle\iota\rangle$
$\Gamma; \texttt{K} \vdash \overline{e} : \overline{\tau} \quad \texttt{fields}(\texttt{c}\langle\iota\rangle) = \overline{\tau}\ \overline{\texttt{fd}} = \overline{e}$

By the induction hypothesis, e_i is a value, bad cast, bad check, or there exists e'_i such that $e_i \overset{\texttt{m}}{\Longrightarrow} e'_i$ for each i.

If all e_i are values, then e is a value and we are done. Otherwise, by the reduction context, we may replace an e_i with e'_i, giving us $e' = \texttt{obj}(\alpha, \texttt{c}\langle\iota\rangle, v_1, \ldots, e'_i, \ldots, e_n)$.

Case T-Check $\quad e = \textbf{check}(e_1, \texttt{m}_1, \texttt{m}_2) \quad \tau = \texttt{modev}$
$\Gamma; \texttt{K} \vdash e_1 : \texttt{modev}$

By the induction hypothesis, e_1 is a value, bad cast, bad check, or there exists e'_1 such that $e_1 \overset{\texttt{m}}{\Longrightarrow} e'_1$.

If e_1 is a value, then by Lemma 22, e_1 has the shape $\texttt{m}$. Now, we have two cases: If $\texttt{m}_1 \le \texttt{m} \le \texttt{m}_2$ then R-Check applies, giving us $e' = \texttt{m}$. If $\texttt{m}_1 \le \texttt{m} \le \texttt{m}_2$ *does not hold* then by definition we have a bad check.

If $e_1 \overset{\texttt{m}}{\Longrightarrow} e'_1$ then we may replace e_1 with e'_1 by the reduction context, giving us $e' = \textbf{check}(e'_1, \texttt{m}_1, \texttt{m}_2)$.

Case T-Let $\quad e = \textbf{let}\ \texttt{x} = e_1\ \textbf{in}\ e_2 \quad \tau = T$
$\Gamma; \texttt{K} \vdash e_1 : \tau_1 \quad \Gamma, \texttt{x} : \tau_1; \texttt{K} \vdash e_2 : T$

By the induction hypothesis, e_1 is a value, bad cast, bad check, or there exists e'_1 such that $e_1 \overset{\texttt{m}}{\Longrightarrow} e'_1$.

If e_1 is a value, then T-Let applies, giving us $e' = e_2\{e_1/\texttt{x}\}$. If $e_1 \overset{\texttt{m}}{\Longrightarrow} e'_1$ then we may replace e_1 with e'_1 by the reduction context, giving us $e' = \textbf{let}\ \texttt{x} = e'_1\ \textbf{in}\ e_2$.

□

Theorem 1 (Type Soundness). *If P is well-typed and $\texttt{boot}(P) = \langle\top, e\rangle$, then either $e \overset{\top}{\Longrightarrow}_* v$, $\langle\top, e\rangle \Uparrow$, or $e \overset{\top}{\Longrightarrow}_* e'$ and e' is a bad cast or a bad check.*

Let us say $\langle\texttt{m}_0; e_0\rangle$ is a *sub-redex* of reduction $e \overset{\texttt{m}}{\Longrightarrow} e'$ iff $e_0 \overset{\texttt{m}_0}{\Longrightarrow} e'_0$ is a sub-derivation of $e \overset{\texttt{m}}{\Longrightarrow} e'$. We next state two important properties of ENT.

Theorem 2 (Type Decidability). *For any program P, it is decidable whether $\vdash P$ holds.*

Theorem 3 (Monotone Snapshotting). *If P is well-typed, $\texttt{boot}(P) = \langle\top, e\rangle$, $e \overset{\top}{\Longrightarrow} \ldots e_1 \overset{\top}{\Longrightarrow} e_2 \ldots \overset{\top}{\Longrightarrow} e_3 \overset{\top}{\Longrightarrow} e_4$, $\langle\texttt{m}; \texttt{obj}(\alpha, T, \overline{v},)\rangle$ is a sub-redex of $e_1 \overset{\top}{\Longrightarrow} e_2$ and $\langle\texttt{m}'; \texttt{obj}(\alpha, T', \overline{v}',)\rangle$ is a sub-redex of $e_3 \overset{\top}{\Longrightarrow} e_4$, then if $\texttt{mode}(T) \neq ?$, $T = T'$.*

In other words, once the type of an object becomes static, it can never be changed any more. This theorem reveals the *monotone* nature of object type change throughout the object lifetime, a crucial property to guarantee type soundness.

Theorem 4 (Waterfall Invariant with Hybrid Typing). *If P is well-typed, $\texttt{boot}(P) = \langle\top, e\rangle$, $e \overset{\top}{\Longrightarrow} \ldots e_1 \overset{\top}{\Longrightarrow} e_2$, and $\langle\texttt{m}, \texttt{obj}(\alpha, T, \overline{v}, .)\texttt{md}(\overline{v'})\rangle$ or $\langle\texttt{m}, \texttt{obj}(\alpha, T, \overline{v}, .)\texttt{fd}(\overline{v'})\rangle$ is a sub-redex of $e_1 \overset{\top}{\Longrightarrow} e_2$, then $R \models \texttt{mode}(T) <: \texttt{m}$ where $P = R\ \overline{C}\ e$.*

This theorem says even in the presence of hybrid typing, waterfall invariant — a key principle to regulate mode-based energy management — is still preserved. Observe that this theorem says run-time errors are never delayed to messaging or field access time. If any potential violation may happen due to dynamic typing, a run-time error would result from a bad check, *i.e.*, at snapshotting time.