Lab 2

Task 1: Introduction

Monetary policy plays a central role in stabilizing economic activity by influencing interest rates in response to inflationary and output fluctuations. One of the most influential formulations of this relationship is the **Taylor Rule**, which posits that the short-term nominal interest rate should adjust systematically to deviations of inflation from its target and output from potential. Conceptually, the Taylor Rule provides a quantitative guideline for how central banks such as the Federal Reserve set policy to balance price stability and economic growth.

Formally, the baseline Taylor Rule can be expressed as:

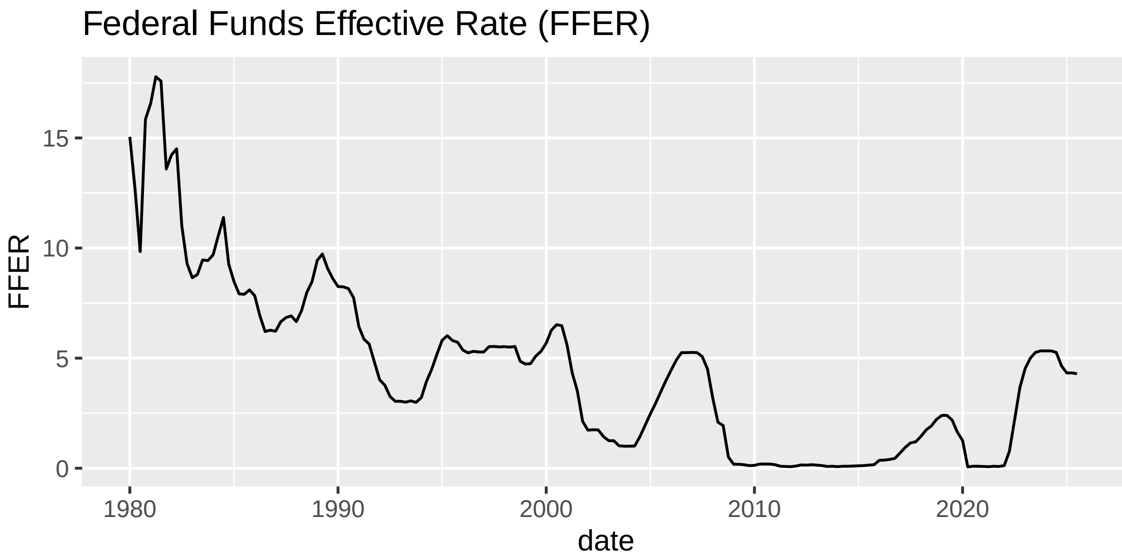
$$i_t = r^* + \pi_t + 0.5\big(\pi_t - \pi^*\big) + 0.5\big(\text{Output Gap}_t\big)$$

where i_t denotes the nominal policy rate (in our work this is proxied by the federal funds rate), r^* is the equilibrium real interest rate, π_t is the observed inflation rate, π^* is the target inflation rate, and $(y_t - y_t^*)$ represents the output gap. The coefficients 0.5 measure the policy responsiveness to inflation and output deviations, respectively. In practice, this framework serves as both a descriptive model of past monetary policy behavior and a normative benchmark for evaluating central bank actions.

In our analysis, we extend the classical Taylor Rule framework into a sequence of increasingly sophisticated econometric models to evaluate both its explanatory power and dynamic consistency with U.S. monetary policy from 1980 through 2024. We begin by estimating a static **OLS-based Taylor Rule**, which provides a benchmark representation of the policy rate as a function of inflation and output gaps. We then assess the model's residual properties, testing for serial correlation and nonstationarity, and subsequently fit an **ARIMA model** to capture any remaining temporal structure in the residuals. The inclusion of autoregressive and moving average dynamics allows us to model inertia and short-term deviations from the baseline rule.

Building upon this, we incorporate an **Error-Correction Model (ECM)** to evaluate the potential for cointegrating relationships among the interest rate, inflation, and output variables, reflecting the long-run equilibrium behavior implied by the Taylor framework. Finally, we employ a **Vector Autoregression (VAR)** model to capture the full system of interactions among the federal funds rate, inflation gap, and output gap, allowing for feedback effects and joint forecasting. The VAR structure also facilitates the computation of **Impulse Response Functions (IRFs)**, which trace the temporal propagation of policy and macroeconomic shocks across the system.

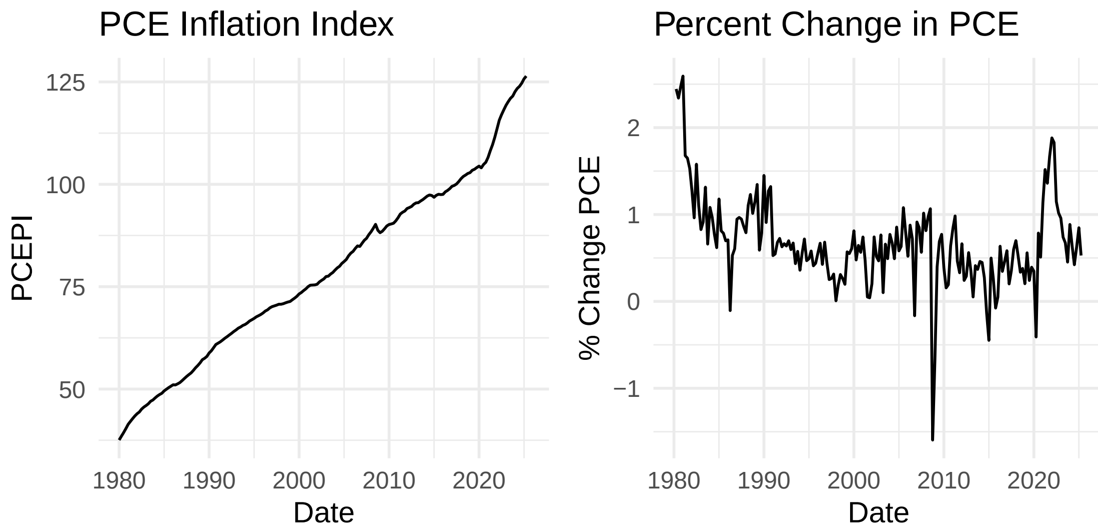
Together, these modeling stages provide a comprehensive empirical investigation of the Taylor Rule as both a static policy guideline and a dynamic representation of monetary behavior. The progression from OLS to ARIMA, ECM, and VAR frameworks enables us to evaluate how the Federal Reserve's policy reactions have evolved over time, the extent to which they align with theoretical prescriptions, and how well different models capture the interdependence of interest rates, inflation, and output dynamics in the post-1980 monetary regime.



The summary statistics of the federal funds rate series provide a concise quantitative overview of the data used in the analysis. Across 183 quarterly observations spanning 1980 Q1 – 2024 Q4, the average policy rate is approximately 4.42%, with a variance of 15.3 and a standard deviation of 3.91. This level of dispersion highlights the considerable volatility of U.S. monetary policy across distinct economic regimes. The distribution reflects several key macroeconomic episodes visible in the plotted series: the elevated interest rates of the early 1980s aimed at curbing double-digit inflation, the steady declines following the 1990s expansion and the 2001 dot-com correction, the near-zero rates maintained during the 2008–2015 Great Recession recovery, and the renewed monetary tightening in the post-2020 period as inflation re-emerged. Together, these descriptive statistics and historical inflection points frame the context for the econometric modeling that follows, illustrating both the persistence and cyclicality inherent in the policy rate dynamics.

One may note a few key historical events that appear in the graph:

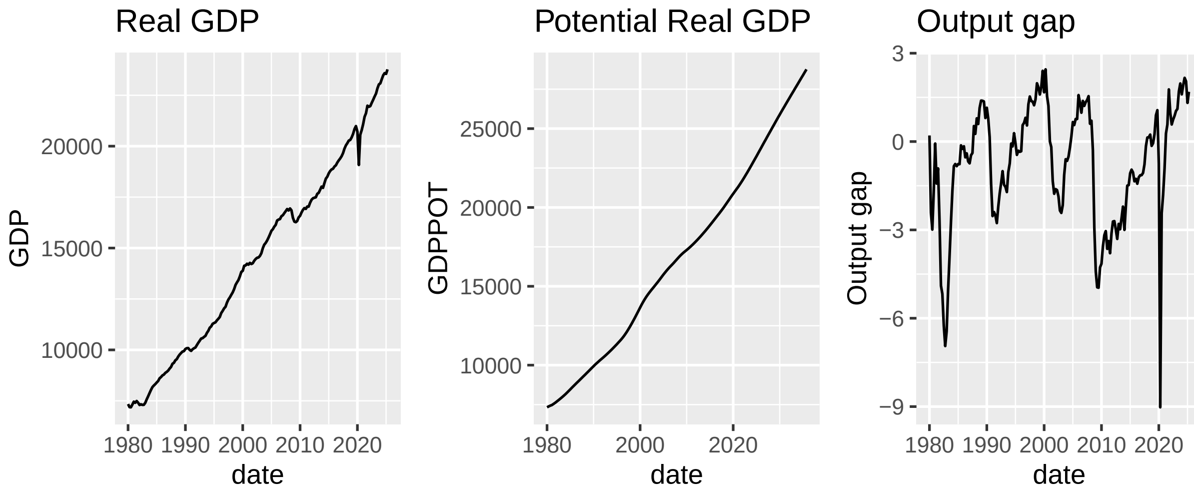
- A near-zero interest rate between 2008 and 2015. This rate was a result of the 2008 Grand Recession
- The 1980s had very high interest rates in order to curb high rates of inflation
- Rates were additionally cut in 2000 in response to the dot-com bubble
- In 2020, rates were cut to near-zero as a result of the pandemic



PCEPI (Inflation) shows events that are related to events that spurred the FFER changes we see in the above graph.

- High inflation characterized the early 1980s, as shown by the peak in the graph
- Very low inflation appears in 2008 during the 2008 financial crisis
- Inflation hit local maxima around 2020 as a result of the pandemic

The Personal Consumption Expenditures Price Index (PCEPI) provides the measure of inflation used in this analysis and serves as the nominal anchor of the Taylor Rule framework. Over the 1980–2024 sample, the index exhibits a mean level of approximately 78.6, with a variance of 501.5 and a standard deviation of 22.4, across 182 quarterly observations. This long-run upward trend reflects the cumulative nature of price growth, while the first-differenced or percentage-change series reveals cyclical fluctuations in inflation dynamics. Periods of particularly high inflation are evident in the early 1980s, consistent with the post–oil-shock stagflation era that motivated aggressive monetary tightening. In contrast, near-zero inflation during the 2008–2015 period reflects deflationary pressures from the global financial crisis. More recently, inflation accelerated sharply around 2020–2022 following the pandemic-induced supply and demand shocks. These fluctuations correspond closely to shifts in the federal funds rate observed earlier, confirming that PCE inflation pressures have historically been a central driver of U.S. monetary policy adjustments.



GDP appears to steadily increase from 1980 to 2020, with a notable dip in 2020 due the COVID pandemic. The output gap timeseries is affected by similar events as FFER and PCEPI:

- A dip in the early 1980s is seen, which is at the same time as high inflation and spiked interest rates
- A major dip in 2008 exists, which is at the time of the 2008 financial crisis
- A sharp dip in 2020 is seen, corresponding to the time of the COVID pandemic

The level series for real GDP and potential GDP are large-scale aggregates, so their summary moments primarily reflect trend growth rather than short-run variation. Both series have high means ($\approx 1.45 \times 10^4$) and large standard deviations ($\approx 4.75 \times 10^3$), consistent with strong deterministic trend and suggesting that levels are likely **non-stationary** without detrending or differencing. By contrast, the **output gap**, defined as $100 \times (\text{GDPC1} - \text{GDPPOT})/\text{GDPPOT}$, is constructed around zero and exhibits far smaller dispersion; its spikes align with major downturns (early 1980s, 2008-09, 2020). Practically, this implies that GDP and GDPPOT should be modeled in **changes** (or as a **cointegrated** pair), while the output gap can be used directly in regressions because it behaves more like a stationary deviation from trend.

Task 2

Task 2.1: Frequency Alignment (Resampling)

Based on the format of the API call to FRED, data is received at a quarterly cadence, with an average over all months in the quarter calculated and returned in the dataframe. No resampling was done.

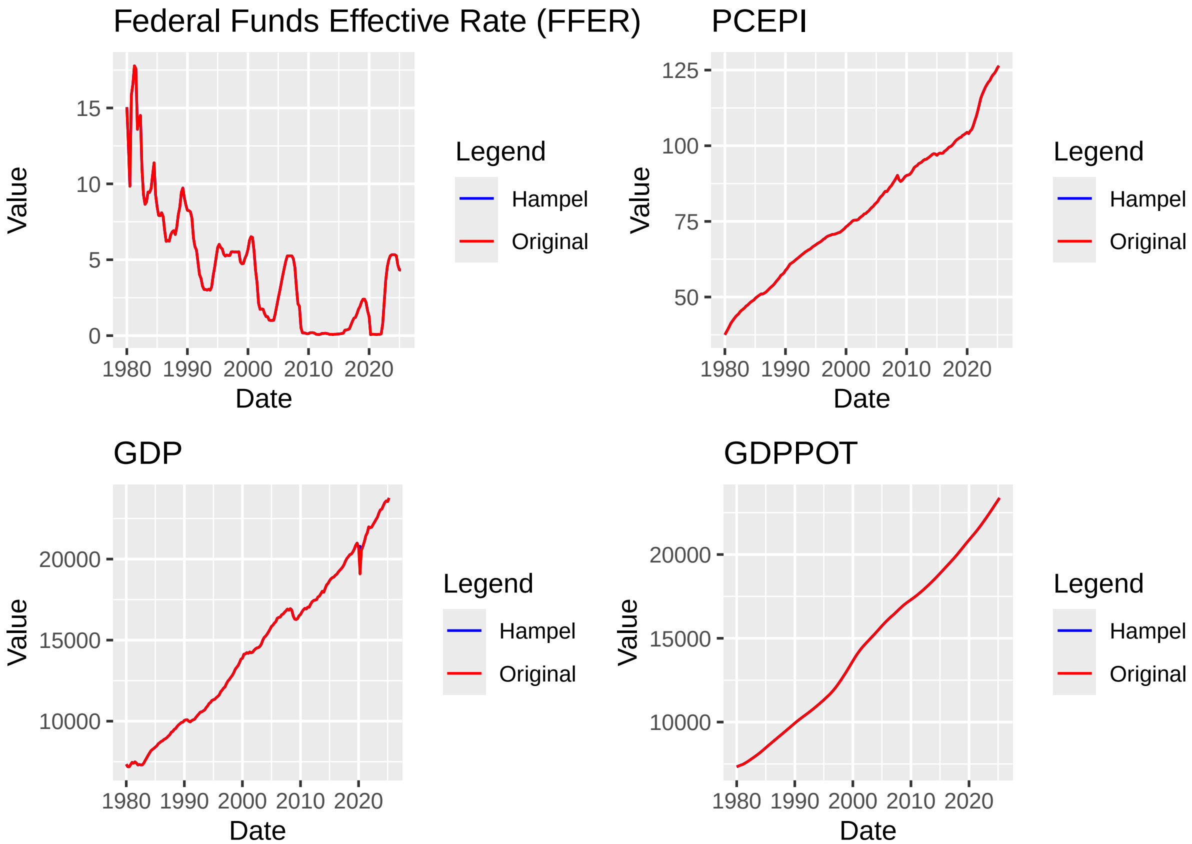
Task 2.2 Outlier Detection & Treatment

Hampel filter

To detect outliers, we've chosen to use the Hampel filter. We prefer this method because based on a visual examination of FFER, PECPI, and GDP data, scores appear to be mostly following either an upward trend or a fluctuating downward trend, with few outliers. The outliers that do appear (i.e. in Real GDP around 2020) are within the data (as opposed to on the upper and lower ends of the data). Thus, this would not be caught by winsorization. The rolling window calculations from a Hampel filter (relying on deviations from a median) would catch these outliers and smooth out the data. Z-scores, as well, is best suited to normally distributed data, which we don't have.

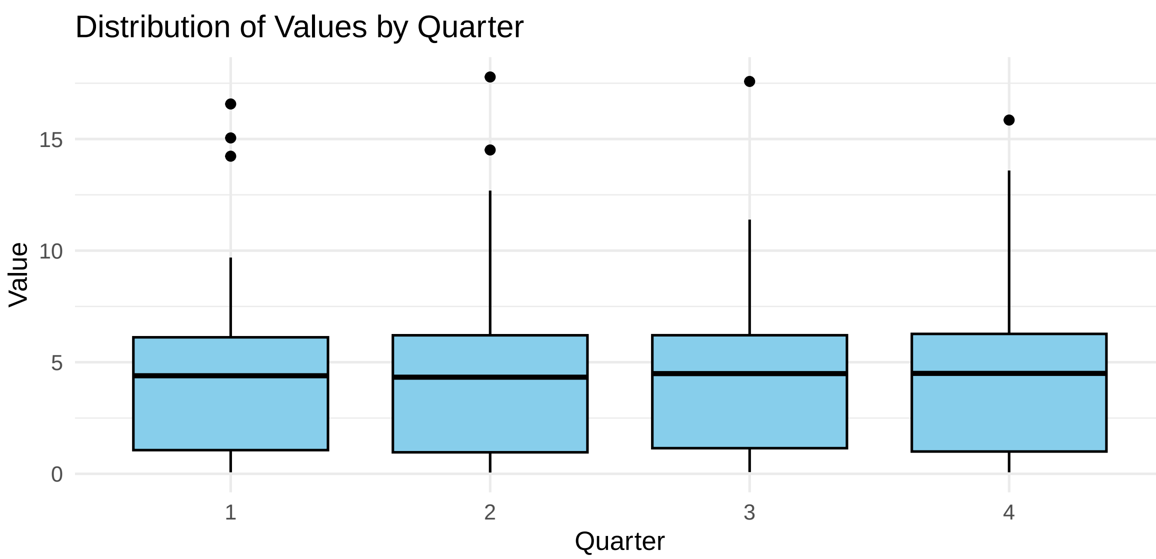
Outlier Detection

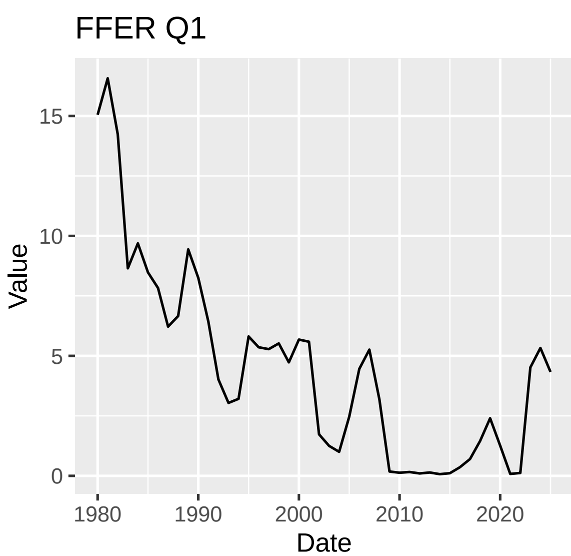
We utilize a Hampel filter for the FFER, PECPI, GDP, GDPPOT, and output gap timeseries. A window size of 5 appeared to smooth out outliers best upon a visual examination of the corrected timeseries data. While, we ran a Hampel filters to detect potential outliers, we retained the original FFER series for estimation because large rate changes reflect actual policy, not data errors.

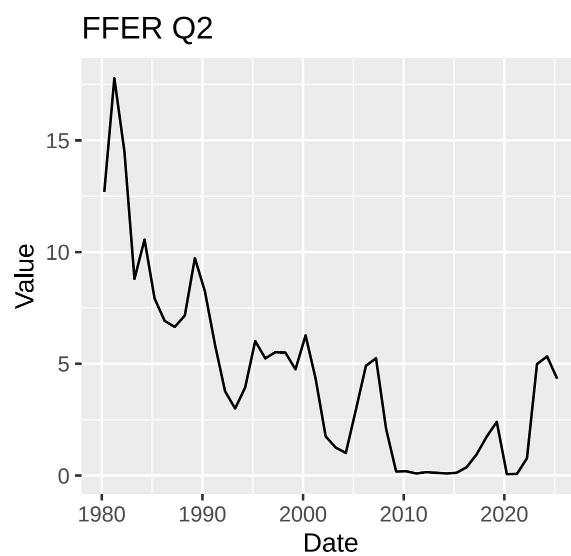


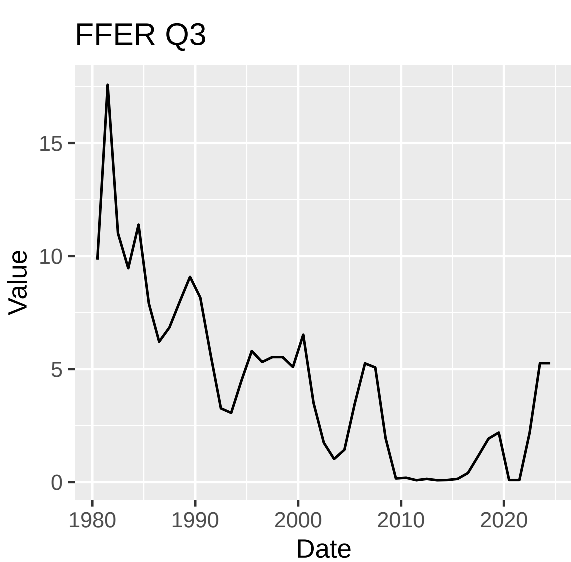
Task 2.3: Seasonal Adjustment

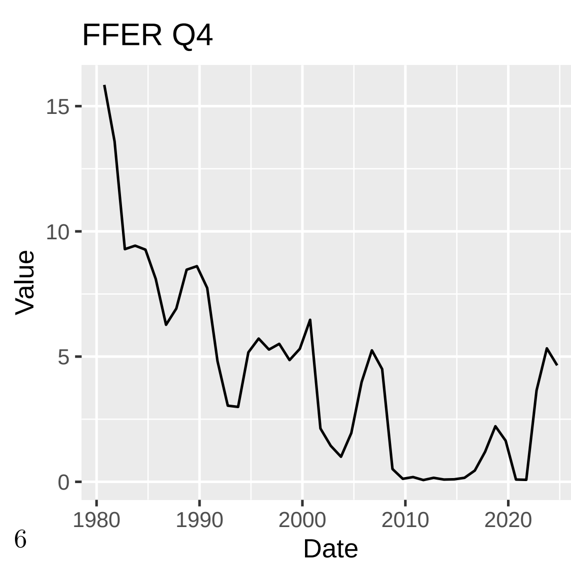
Detection



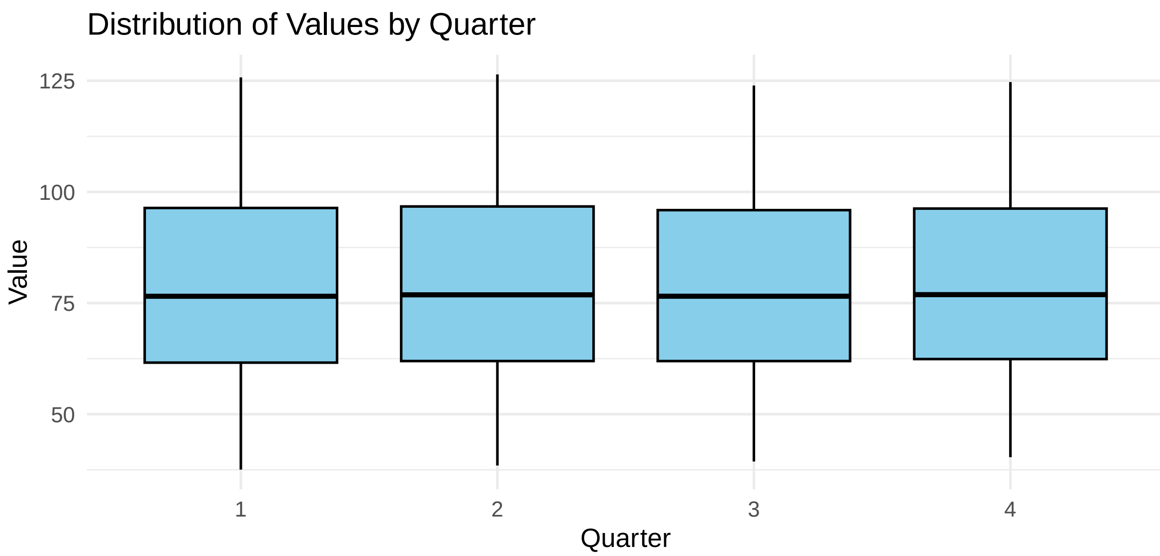


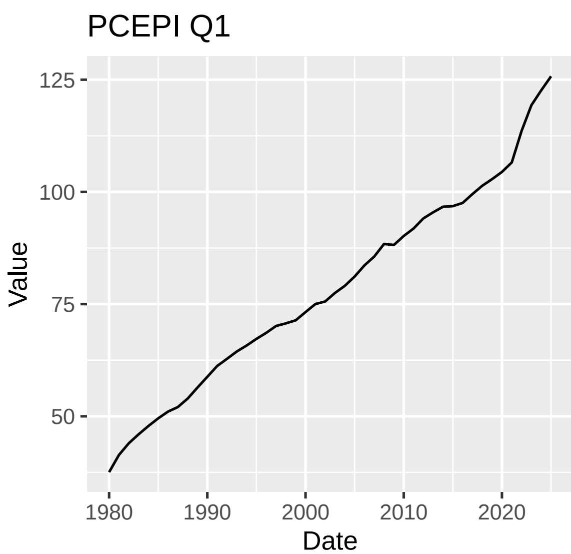


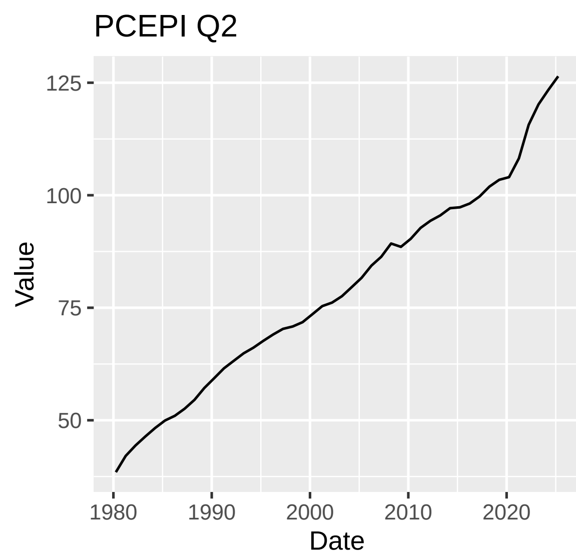


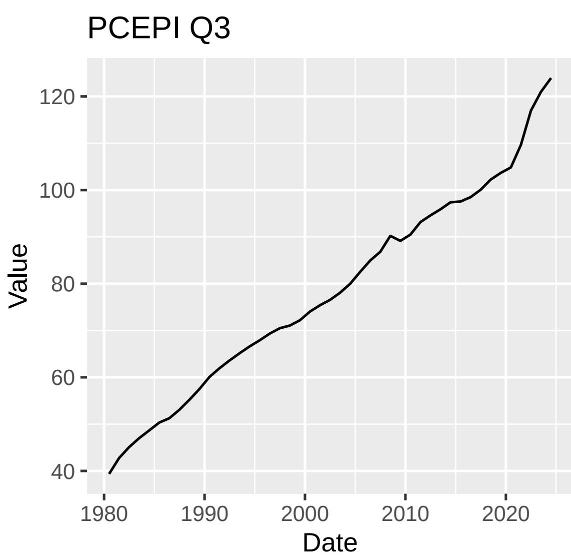


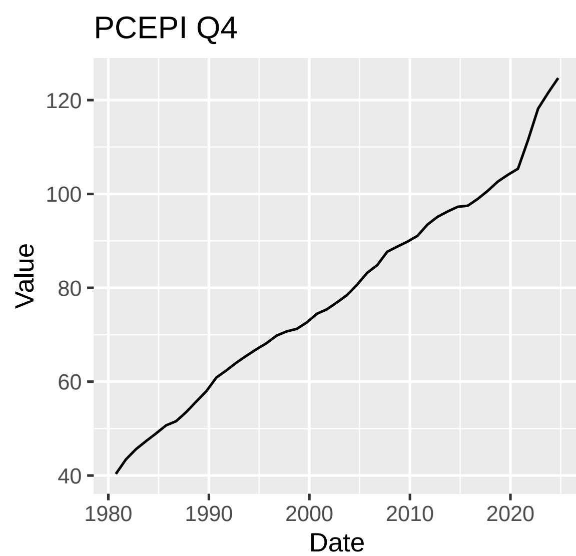
The QS test for seasonality applied to the Federal Funds Effective Rate (FFER) produced a test statistic of 0 with a p-value of 1. This result provides no evidence of seasonality in the FFER series. In other words, the Federal Funds Rate does not exhibit systematic quarterly or annual seasonal fluctuations. This is consistent with the institutional nature of monetary policy decisions—interest rate adjustments occur in response to macroeconomic conditions rather than regular seasonal cycles. Consequently, no seasonal adjustment or differencing is necessary for the FFER series before modeling.


Distribution of Values by Quarter
Value
125
100
75
50
1
2
3
4
Quarter

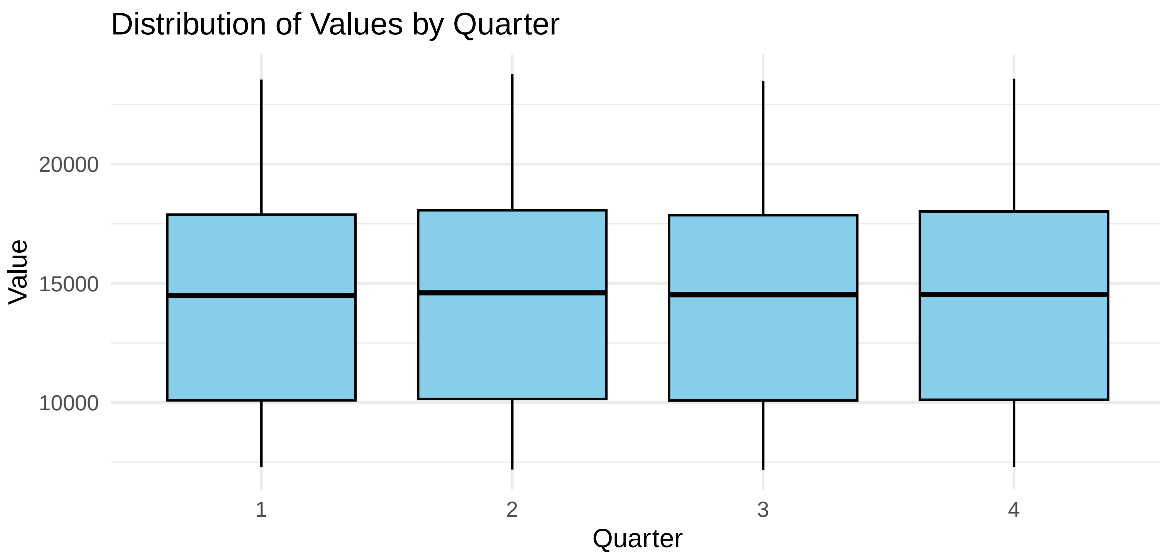

PCEPI Q1
Value
125
100
75
50
1980
1990
2000
2010
2020
Date

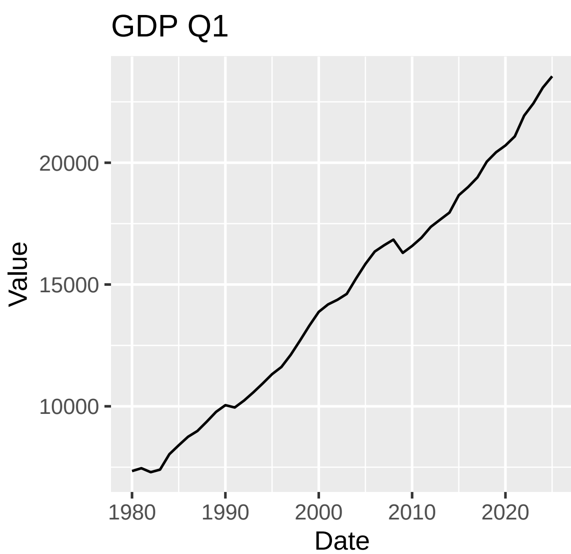

PCEPI Q2
Value
125
100
75
50
1980
1990
2000
2010
2020
Date

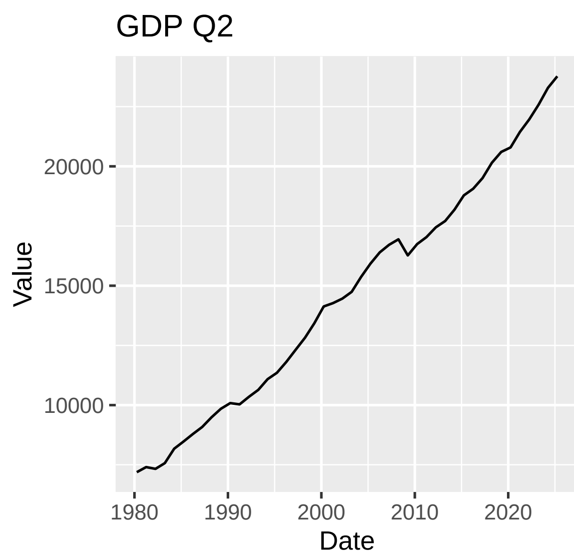

PCEPI Q3
Value
120
100
80
60
40
1980
1990
2000
2010
2020
Date

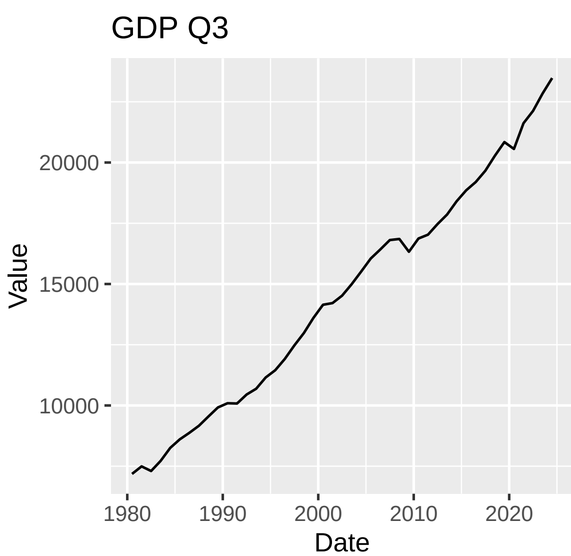

PCEPI Q4
Value
120
100
80
60
40
1980
1990
2000
2010
2020
Date

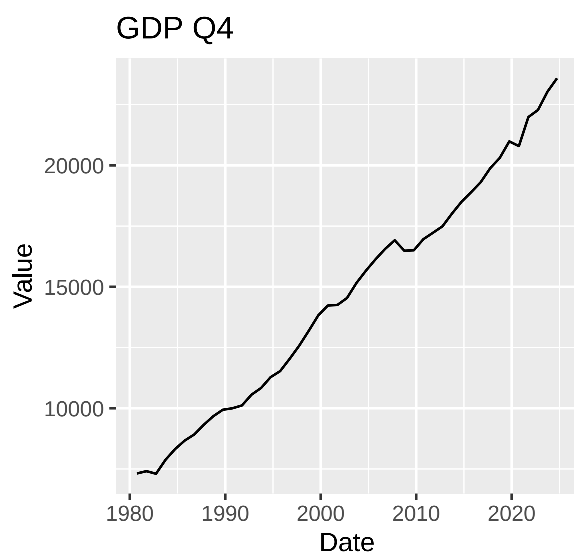
The QS test in the quarterly PCEPI inflation series produced a statistic of 16.15 with a p-value of 0.00031, indicating statistically significant seasonal effects. Therefore, the raw series exhibits recurring within-year variation, justifying the use of seasonal differencing or seasonal adjustment before fitting ARIMA-type models.


Distribution of Values by Quarter
Value
20000
15000
10000
1
2
3
4
Quarter


GDP Q1
Value
20000
15000
10000
1980
1990
2000
2010
2020
Date

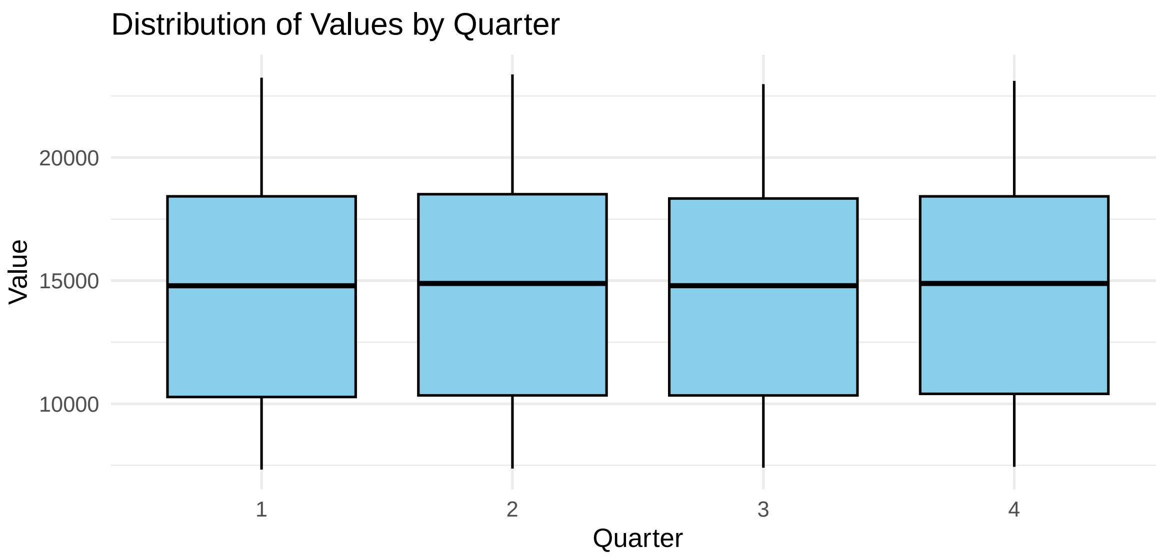

GDP Q2
Value
20000
15000
10000
1980
1990
2000
2010
2020
Date

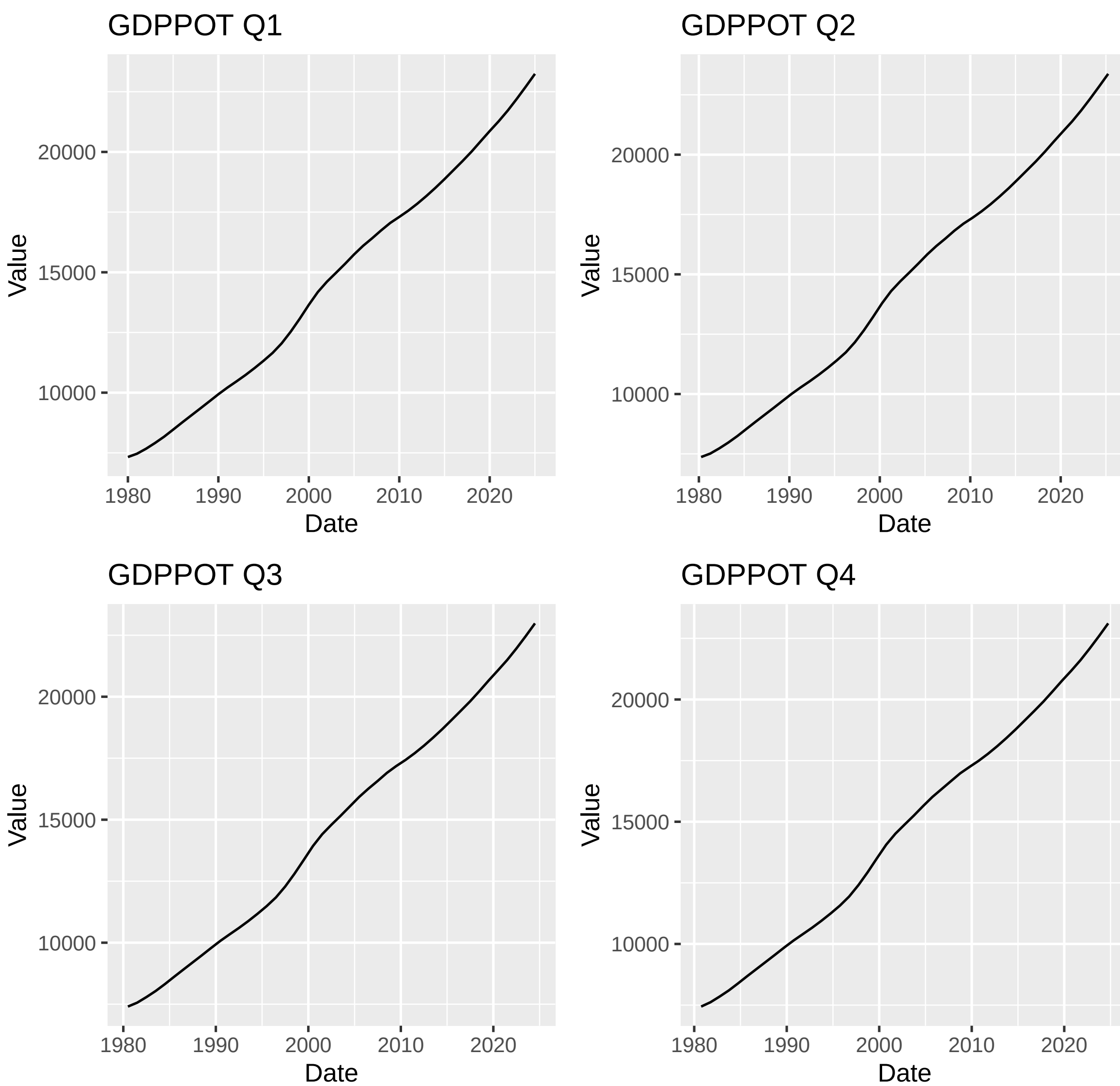

GDP Q3
Value
20000
15000
10000
1980
1990
2000
2010
2020
Date


GDP Q4
Value
20000
15000
10000
1980
1990
2000
2010
2020
Date

Here, our QS seasonality test for the Real GDP series yielded a test statistic of 0 and a p-value of 1, indicating no detectable seasonal pattern in the data. This result suggests that GDP, as reported in seasonally adjusted form by the Bureau of Economic Analysis (BEA), already has its intra-year seasonal components removed. Therefore, no additional seasonal adjustment or seasonal differencing is necessary prior to modeling.

This is consistent with standard macroeconomic data practices: most official GDP series are *seasonally adjusted annualized rates (SAAR)*, meaning any regular quarterly effects have already been filtered out by the data provider.


Distribution of Values by Quarter
Value
20000
15000
10000
1
2
3
4
Quarter


GDPPOT Q1
GDPPOT Q2
GDPPOT Q3
GDPPOT Q4
Value
20000
15000
10000
1980
1990
2000
2010
2020
Date

The QS test for the Potential GDP (GDPPOT) series produced a test statistic of 204.06 with a p-value of 0, providing overwhelming evidence of seasonality. This result indicates that the GDPPOT series, as currently specified, contains recurring within-year fluctuations that are not seasonally adjusted.

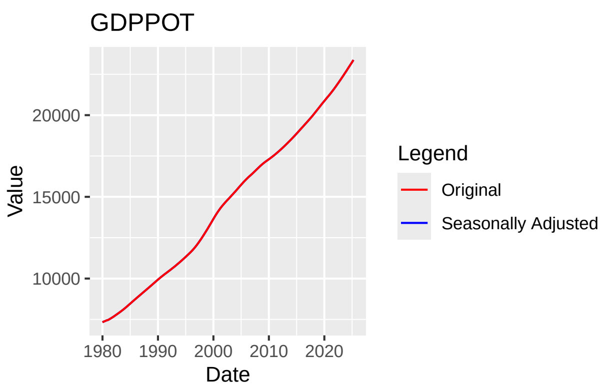
In practice, this is somewhat expected: the *potential output* series is an estimated smooth trend of what the economy could produce at full capacity, but depending on how it was constructed or interpolated, the quarterly data may still exhibit minor cyclical or deterministic seasonal behavior.

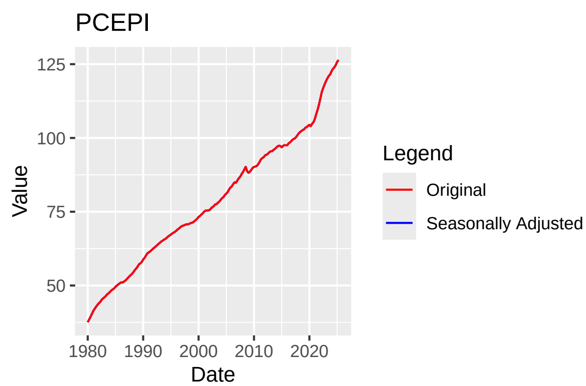
Therefore, to ensure consistency with other macroeconomic variables in the model (such as Real GDP and FFER), it may be appropriate to apply **seasonal differencing** or a **low-pass filter** (e.g., HP filter) to the GDPPOT series before computing the output gap or including it in time series models.

Seasonal Adjustment via X-13ARIMA-SEATS

This method is used because it allows for timeseries that can be decomposed additively or multiplicatively, allowing for maximum flexibility. The trend, seasonal component, and irregular component are all estimated and the algorithm uses centered moving averages to remove the seasonal component.

Source: https://en.wikipedia.org/wiki/X-13ARIMA-SEATS





Task 3

```
## # A tibble: 9 x 3
##   variable          stationary_3 stationary_2
##   <chr>             <lgl>        <lgl>
## 1 FEDFUNDS_hampel   TRUE         TRUE
## 2 PCEPI_s_adj       FALSE        FALSE
## 3 pct_chg_pce       TRUE         TRUE
## 4 GDPC1_hampel      FALSE        FALSE
## 5 GDPPOT_s_adj      TRUE         FALSE
## 6 output_gap        TRUE         TRUE
## 7 pct_chg_pce_s_adj TRUE         TRUE
## 8 inflation_gap     TRUE         TRUE
## 9 date              FALSE        FALSE
```

Discussion:

Using the General to Specific approach, we first evaluate for drift and trend terms for each variable in the dataset. For this test, only the Seasonally Adjusted PCEPI and the GDPC with hampel filter variables fail to reject the null hypothesis of stationarity. From there, we evaluate all variable for drift only. This time, the seasonally adjusted GDPPOT variable also fails to reject the null, in addition to the aforementioned variables.

These test results suggest that the PCEPI and GDPC variables should be differenced before before use in modeling, to avoid spurious regression. The GDPPOT variable is trend-stationary because it does not contain a unit-root but it does contain a deterministic trend. In order to capture the explanatory power of the variable, GDPPOT_s_adj should not be differenced.

```
## Augmented Dickey-Fuller Test
## alternative: stationary
##
## Type 1: no drift no trend
##      lag    ADF p.value
## [1,]   0 -3.713  0.0100
## [2,]   1 -2.815  0.0100
## [3,]   2 -1.896  0.0581
## [4,]   3 -1.825  0.0686
## [5,]   4 -0.944  0.3408
## Type 2: with drift no trend
##      lag   ADF p.value
## [1,]   0 -6.32  0.0100
## [2,]   1 -5.11  0.0100
## [3,]   2 -3.78  0.0100
## [4,]   3 -3.89  0.0100
## [5,]   4 -2.63  0.0926
## Type 3: with drift and trend
##      lag   ADF p.value
## [1,]   0 -6.38  0.0100
## [2,]   1 -5.19  0.0100
## [3,]   2 -3.87  0.0174
## [4,]   3 -3.96  0.0130
## [5,]   4 -2.67  0.2968
## ----
## Note: in fact, p.value = 0.01 means p.value <= 0.01
```

```
## Augmented Dickey-Fuller Test
## alternative: stationary
##
## Type 1: no drift no trend
##      lag   ADF p.value
## [1,]   0 -6.39    0.01
## [2,]   1 -3.90    0.01
## [3,]   2 -3.61    0.01
## [4,]   3 -3.27    0.01
## [5,]   4 -2.76    0.01
## Type 2: with drift no trend
##      lag   ADF p.value
## [1,]   0 -9.81    0.01
## [2,]   1 -6.57    0.01
## [3,]   2 -6.35    0.01
## [4,]   3 -6.09    0.01
## [5,]   4 -5.86    0.01
## Type 3: with drift and trend
```

```
##      lag   ADF p.value
## [1,]   0 -9.91    0.01
## [2,]   1 -6.64    0.01
## [3,]   2 -6.50    0.01
## [4,]   3 -6.31    0.01
## [5,]   4 -6.04    0.01
## ----
## Note: in fact, p.value = 0.01 means p.value <= 0.01
```

Discussion

After differencing the two variables identified by the ADF test above, they no longer show trend or drift. Notably, there are some lags for both variables that result in a failure to reject the null hypothesis. Therefore care must be taken when determining which lags to use when modeling with these variables to ensure they remain stationary.

```r
za_results <- map_dfr(vars, function(v) {
  x <- infl_data_tsbl_train[[v]]
  test <- ur.za(x, model = "both")
  result <- summary(test)
  tibble(
    variable = v,
    model = test@model[1],
    sig_thresh = result@cval[2],
    test_res = result@teststat,
    stry_w_brk = result@teststat < result@cval[2]
  )
})

za_results
```

```
## # A tibble: 9 x 5
##   variable           model sig_thresh test_res stry_w_brk
##   <chr>              <chr>      <dbl>    <dbl> <lgl>
## 1 FEDFUNDS_hampel    both       -5.08    -4.66 FALSE
## 2 PCEPI_s_adj        both       -5.08    -4.18 FALSE
## 3 pct_chg_pce        both       -5.08    -7.78 TRUE
## 4 GDPC1_hampel       both       -5.08    -4.42 FALSE
## 5 GDPPOT_s_adj       both       -5.08    -5.03 FALSE
## 6 output_gap         both       -5.08    -4.23 FALSE
## 7 pct_chg_pce_s_adj  both       -5.08    -7.78 TRUE
## 8 inflation_gap      both       -5.08    -7.78 TRUE
## 9 date               both       -5.08   -11.5  TRUE
```

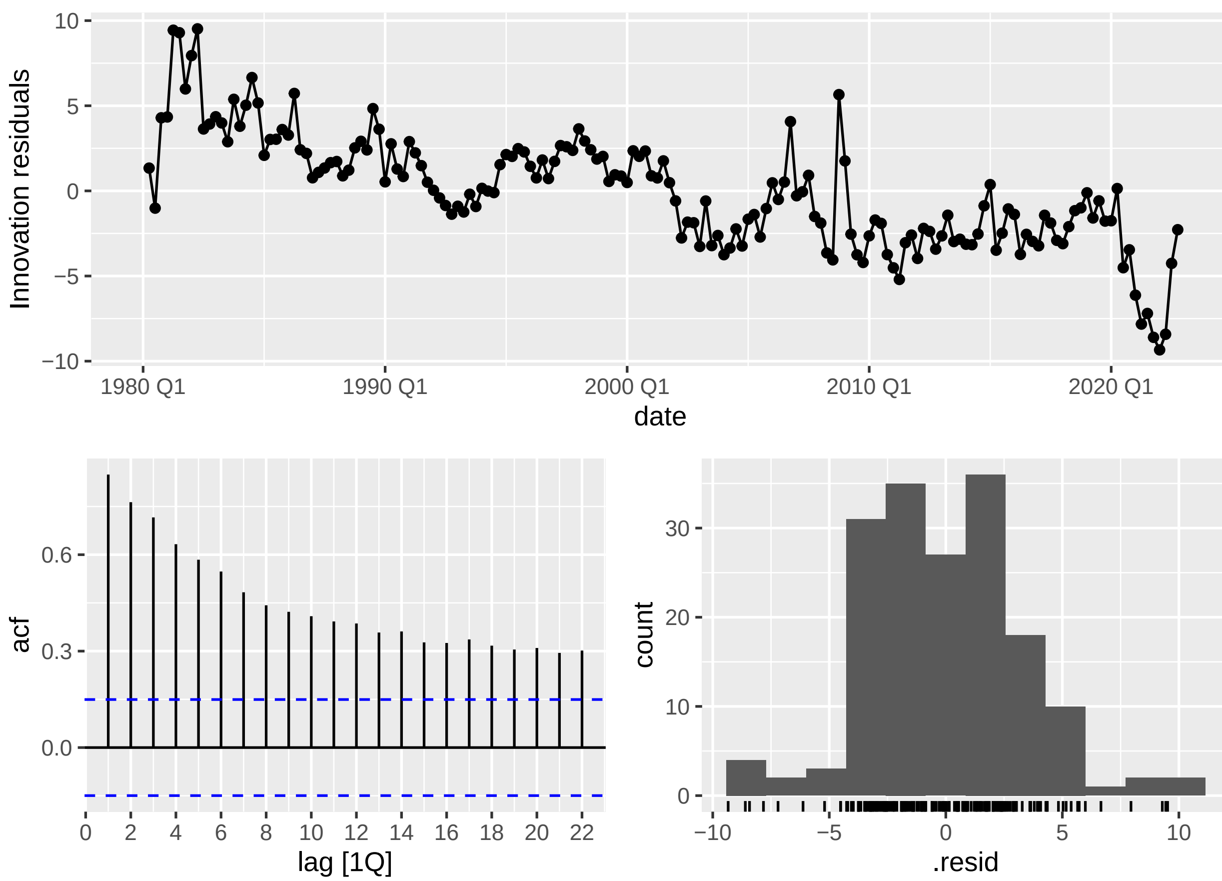
Discussion:

In comparison to the ADF test, the Zivot Andrews test identifies more variables that are not stationary when accounting for structural breaks in the data. The FEDFUNDS_hampel, PCEPI_s_adj, GDPC1_hampel, GDPPOT_s_adj and output_gap variables fail to reject the null hypothesis of stationarity with a structural break.

This means that the additional variables identified over the ADF only appear non-stationary because of a structural break. Once that break is included, the variables are found to be non-stationary. They should be differenced to reduce issues with model inference due to non-stationary residuals or spurious regression.

Task 4

```
## Series: FEDFUNDS_hampel
## Model: TSLM
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -9.33831 -2.50848 -0.01132  2.21617  9.51659
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)     9.8247     0.6965  14.106  < 2e-16 ***
## inflation_gap 406.7634    48.3740   8.409 1.71e-14 ***
## output_gap      0.1176     0.1357   0.866    0.388
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.328 on 168 degrees of freedom
## Multiple R-squared: 0.2999,  Adjusted R-squared: 0.2915
## F-statistic: 35.98 on 2 and 168 DF, p-value: 9.8768e-14
```

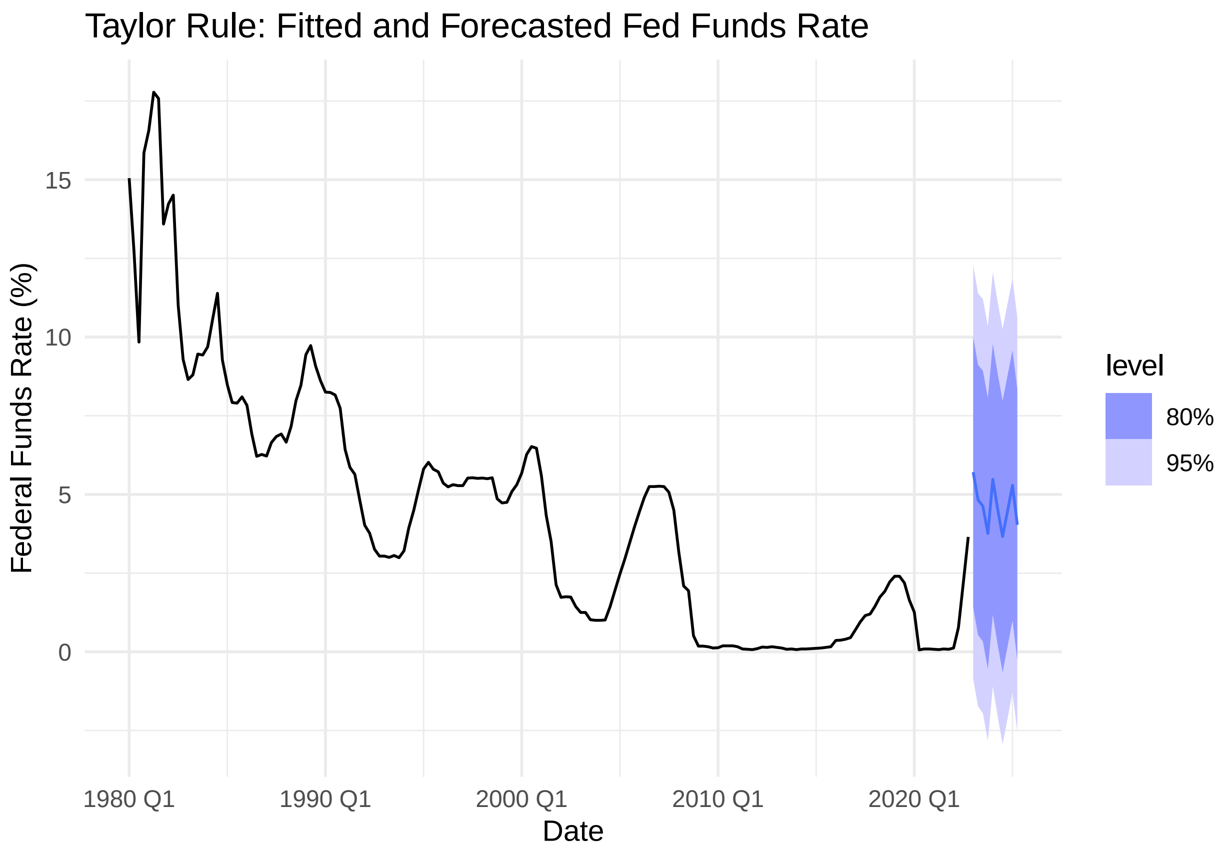


Discussion:

Previous analysis on the variables indicated that they should be differenced before use in regression. However doing so results in a different interpretation of the Taylor Rule. The non-stationary variables were used so that discussion could revolve around the models adherence to economic theory.

With this model, the inflation gap estimate is significant with a value of 406.8. This variable is reported in hundredths, so the correct interpretation about a 4% increase for every 1% inflation point above the inflation target. This does match economic theory, in that the Fed Fund rate increases as inflation increases in an effort to "cool off" the economy. This estimate is slightly high in comparison to real targets of 2%. In general, this means that in this sample, the funds rate reacts much more strongly to inflation than what would be considered the "rule of thumb". This may be due to the several economic shocks that occurred through the training time periods.

The Autocorrelation plots for this model shows that the residuals are significant across all lags. This shows that there is additional information that is not captured with the model, as the residuals are not white noise.



Discussion:

The Root Mean Squared Error value for the training data is 3.29. For the test data, the value is 0.92. So for the training data, the model is generally within 3.3 percentage points of the actual data, a pretty large spread overall. However this tightens to within 1 percentage point for the test data.

This difference may be attributed to the relative calm in the market during the test period compared to the training period. For example, the training period contains large abnormalities, like the Dot-Com bubble, housing crisis, and COVID, all within nearly 20 years of the end of the training data. The test data has no such unrest, and is therefore able to more "accurate" in describing the fed rate response to inflation. For these reasons, it is likely improper to suggest that this is an optimized model for future forecasting.

Task 5

5.1: Cointegration Test

Next, we want to investigate whether there is a long-run equilibrium for the basic Taylor rule model. Since the Federal Reserve consistently adjusts the federal funds rate in response to changes in inflation and the output gap, there could be a mean-reverting relationship in the long run. To test this, we will perform a Cointegration Test, specifically the Engle-Granger two-step test.

To perform this test, we take our Taylor model that we fit previously and extract the residuals. Next, we perform an ADF test on the residuals. Looking at the results, the ADF statistics are strongly negative and the p-values are below 0.05 for most specifications (with and without drift or trend). This indicates that we can reject the null hypothesis of a unit root. In other words, the residuals are stationary. Therefore, we conclude that there exists a long-run equilibrium relationship between the federal funds rate, the inflation gap, and the output gap. This means that the variables are cointegrated, satisfying the condition necessary for estimating an Error Correction Model (ECM).

Lag	Type 1 ADF	Type 1 p.value	Type 2 ADF	Type 2 p.value	Type 3 ADF	Type 3 p.value
0	-3.67	0.0100	-3.66	0.0100	-6.39	0.01
1	-2.94	0.0100	-2.93	0.0463	-5.85	0.01
2	-2.35	0.0204	-2.33	0.1981	-4.58	0.01
3	-2.51	0.0133	-2.50	0.1348	-4.93	0.01
4	-2.43	0.0171	-2.39	0.1765	-4.19	0.01

5.2: Estimating the ECM via OLS

In an ECM model, we are able to capture both the short-run adjustments and the long-run equilibrium relationship between economic variables. Specifically, the model allows us to separate temporary fluctuations from the underlying equilibrium path implied by economic theory. Deviations from the long-run equilibrium (captured by the lagged residual term) influence short-run changes in the dependent variable, showing how the system "corrects" itself over time. To estimate the ECM, we calculate the change in interest rate (d_it), the change in output gap (d_output_gap), the change in inflation gap (d_infl_gap), and the equilibrium error (lag_resid). The first three variables can be directly calculated using our data. The equilibrium error can be calculated by first fitting a long-run regression (the basic Taylor Rule model) to extract the residuals and then using those values as the lagged adjustment term.

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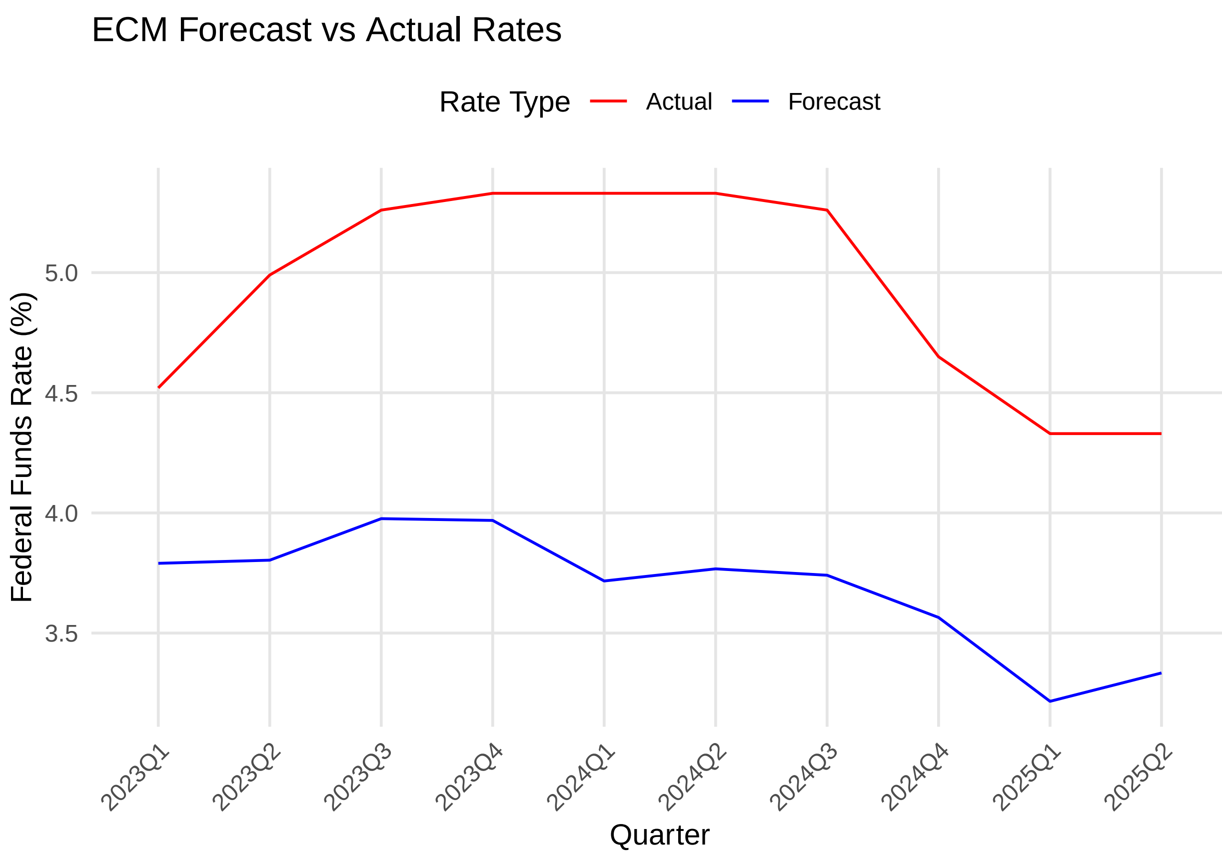
$$\Delta i_t = \gamma_0 + \gamma_1 \Delta(\pi_t - \pi^*) + \gamma_2 \Delta(\text{Output Gap}_t) + \phi \left[i_{t-1} - \beta_1(\pi_{t-1} - \pi^*) - \beta_2(\text{Output Gap}_{t-1}) \right] + \varepsilon_t$$

We note that the coefficient ϕ captures the speed of adjustment and is typically negative, indicating that when the interest rate deviates from its long-run equilibrium, it adjusts in the next period to restore balance.

Model	Estimate	Std. Error	t value	p value
(Intercept)	-0.05479	0.05493	-0.997	0.319938
d_infl_gap	12.63397	14.01487	0.901	0.368571
d_output_gap	0.42169	0.08784	4.801	3.36e-06
lag_resid	-0.06119	0.01750	-3.497	0.000596

After fitting our model, we get the following results. The coefficient on the lagged residual term (lag_resid) is -0.061, which is negative and statistically significant. This confirms the presence of an error correction mechanism, indicating that when the federal funds rate deviates from its long-run equilibrium, it adjusts by about 6% in the following quarter to restore equilibrium. The coefficient on the change in the output gap (d_output_gap) is 0.422 and also significant, suggesting that short-run movements in the output gap have a meaningful positive effect on changes in the interest rate. In contrast, the coefficient on the change in the inflation gap (d_infl_gap) is positive but statistically insignificant, implying that short-run changes in inflation do not have a significant immediate impact on interest rate adjustments.

5.3: Compare ECM Forecasts



To test our model's performance, we train the ECM model on data from 1980Q1 to 2022Q4 and forecast from 2023Q1 to 2024Q4. Our metric will be the Root Mean Squared Error (RMSE). To get the RMSE value from our ECM model, we have to convert the predicted d_it values back to the level forecasts of the federal funds rate. This is done by cumulatively adding the predicted changes to the last observed value of the interest rate from the training period. Once we obtain these level forecasts, we can directly compare them to the actual observed federal funds rate in the test period to compute the RMSE. Overall, the ECM model performs slightly worse, with an RMSE of 1.272986 compared to 0.9260019 for the baseline OLS-based

Taylor Rule model. While we might expect the ECM to have a lower RMSE due to its ability to capture both short-run dynamics and long-run equilibrium, this result likely reflects changes in Federal Reserve behavior during 2023 to 2024. Because our model was trained on data from 1980 to 2022, it may not fully capture these recent shifts, making the ECM forecasts less predictable in this period.

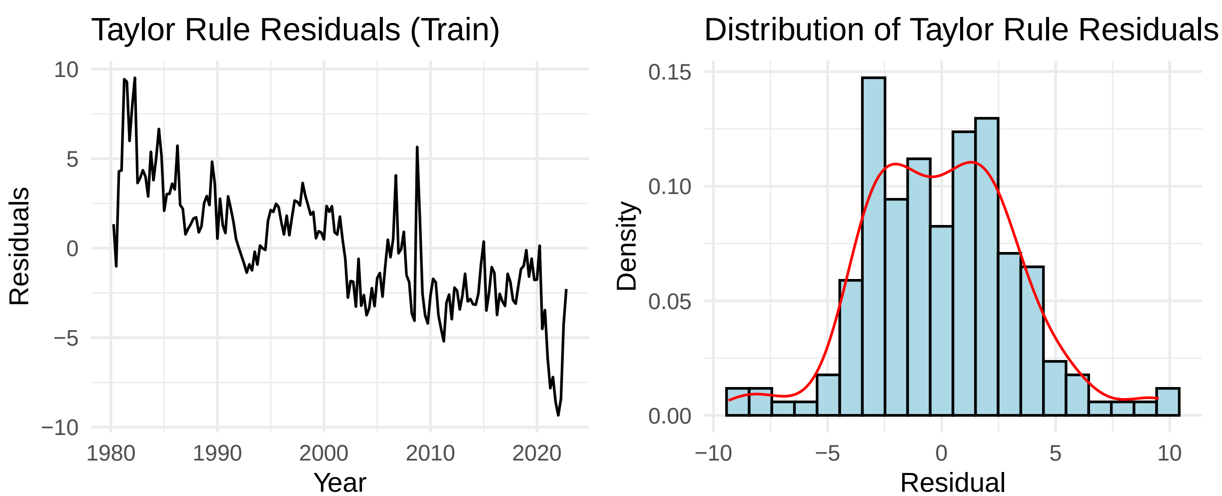
Model	Test RMSE
ECM	1.273
Taylor Rule	0.926

Task 6: ARIMA Modeling of Taylor Rule Errors

In our appendix, we provide additional analysis on the stationarity of the residuals we utilize for modeling, with three additional diagnostics that are mutually consistent: $\hat{\varepsilon}_t$ is best characterized as **stationary**, possibly around a deterministic trend and with evidence of a **single structural break**. However, when applying the auto ARIMA model, we note a first difference is utilized, which we explore with visual analysis in our technical appendix. Our analysis justifies modeling the unexplained component with a low-order ARIMA on the residuals and we take the route often taken in practice, where differencing once (ARIMA with $d = 1$) can act as a convenient device to remove the deterministic drift and any slow-moving break-induced level shift, yielding a stationary innovation process for forecasting. Consequently, the combined forecast we report later follows:

$$\widehat{i}_t^{\text{combined}} = \widehat{i}_t^{\text{Taylor}} + \widehat{\varepsilon}_t^{\text{ARIMA}},$$

where $\widehat{i}_t^{\text{Taylor}}$ is the OLS Taylor Rule fit and $\widehat{\varepsilon}_t^{\text{ARIMA}}$ is the ARIMA forecast of the residual process that is stationary within regimes.



We begin by examining the residual series $\hat{\varepsilon}_t$ from the OLS Taylor Rule model. The time-series plot of residuals shows long memory and a clear downward drift until the early 2000s, followed by a period of volatility spikes (notably during the 2008 financial crisis and the COVID-19 period). This persistence suggests that the residuals are not white noise and may exhibit autocorrelation or a weak unit root, despite the test statistics on the non-differenced residuals above.

After differencing the residuals, the time series of $\Delta\hat{\varepsilon}_t$ becomes mean-reverting with stable variance and no clear trend. Visually, this differenced series appears stationary, suggesting that one level of differencing

($d = 1$) is sufficient to remove the long-run component. Formally, both the **Augmented Dickey–Fuller (ADF)** and **KPSS** tests confirm this: the ADF strongly rejects the null of a unit root (all $p \leq 0.01$), while the KPSS fails to reject the null of stationarity ($p \geq 0.10$). Together, these imply that the differenced residuals are stationary. Specifically, **ARIMA(2,1,3)**, chosen by auto-ARIMA, achieves the lowest AIC ($AIC = 669.37$), outperforming all ARIMA models tested.

In summary, the residual diagnostics show that: 1. The raw residuals $\hat{\varepsilon}_t$ are non-stationary, consistent with persistent policy shocks.
2. First differencing yields stationary residuals $\Delta\hat{\varepsilon}_t$.
3. The optimal model based on information criteria is **ARIMA(2,1,3)**, which balances autoregressive persistence with moving-average smoothing.

This result justifies the use of the combined model:

$$\widehat{i}_t^{\text{combined}} = \widehat{i}_t^{\text{Taylor}} + \widehat{\varepsilon}_t^{\text{ARIMA(2,1,3)}},$$

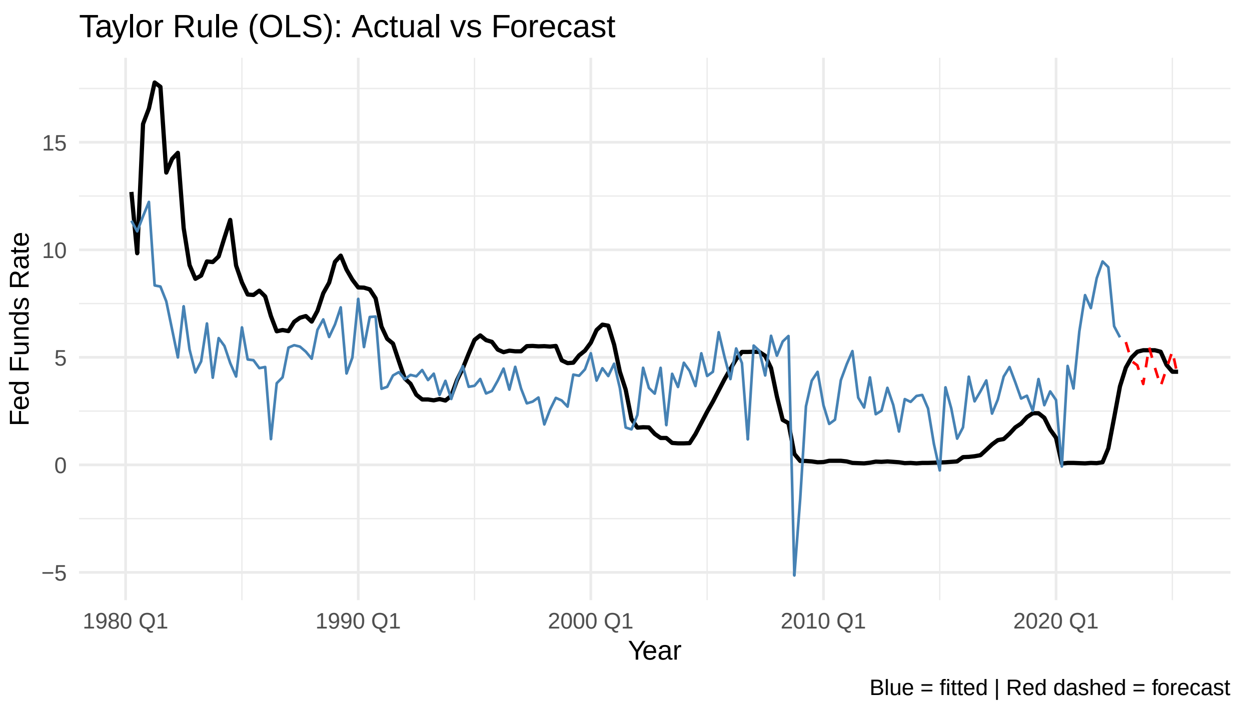
where the ARIMA component refines the Taylor Rule forecast by modeling serial correlation in the policy deviations. The combined model therefore captures both the systematic and dynamic elements of the Federal Reserve's interest rate decisions.

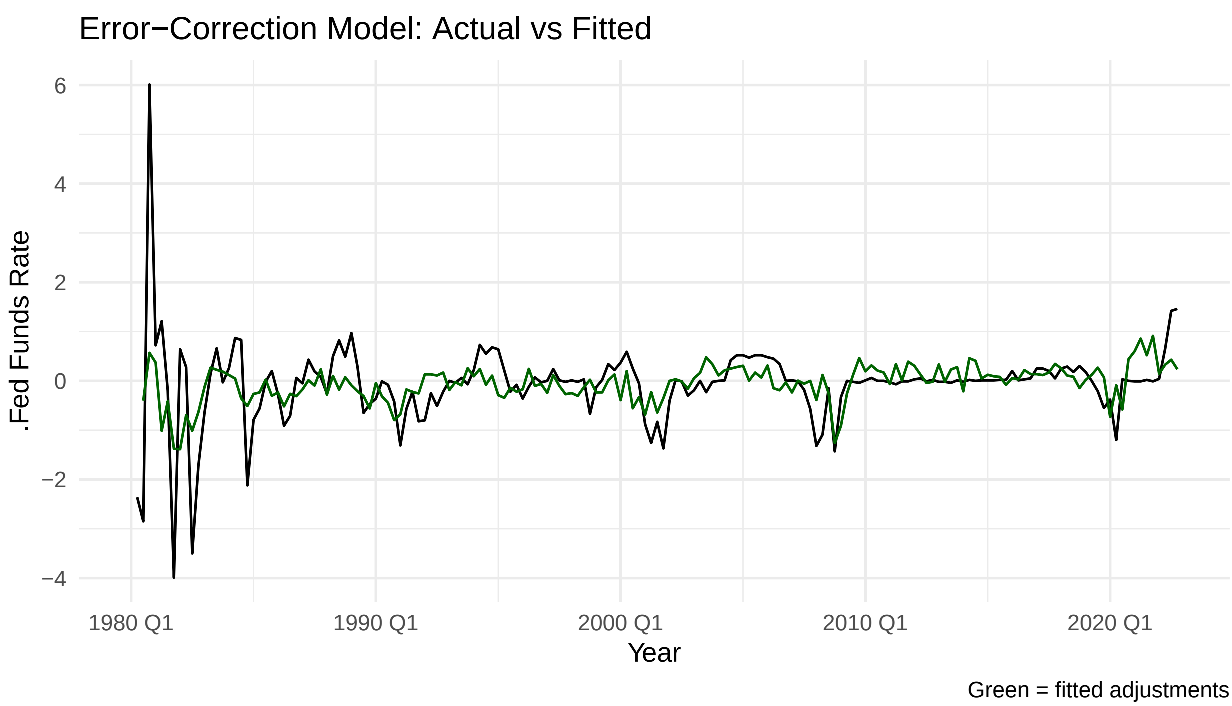
The table below compares the RMSE for the three models across the training and test sets:

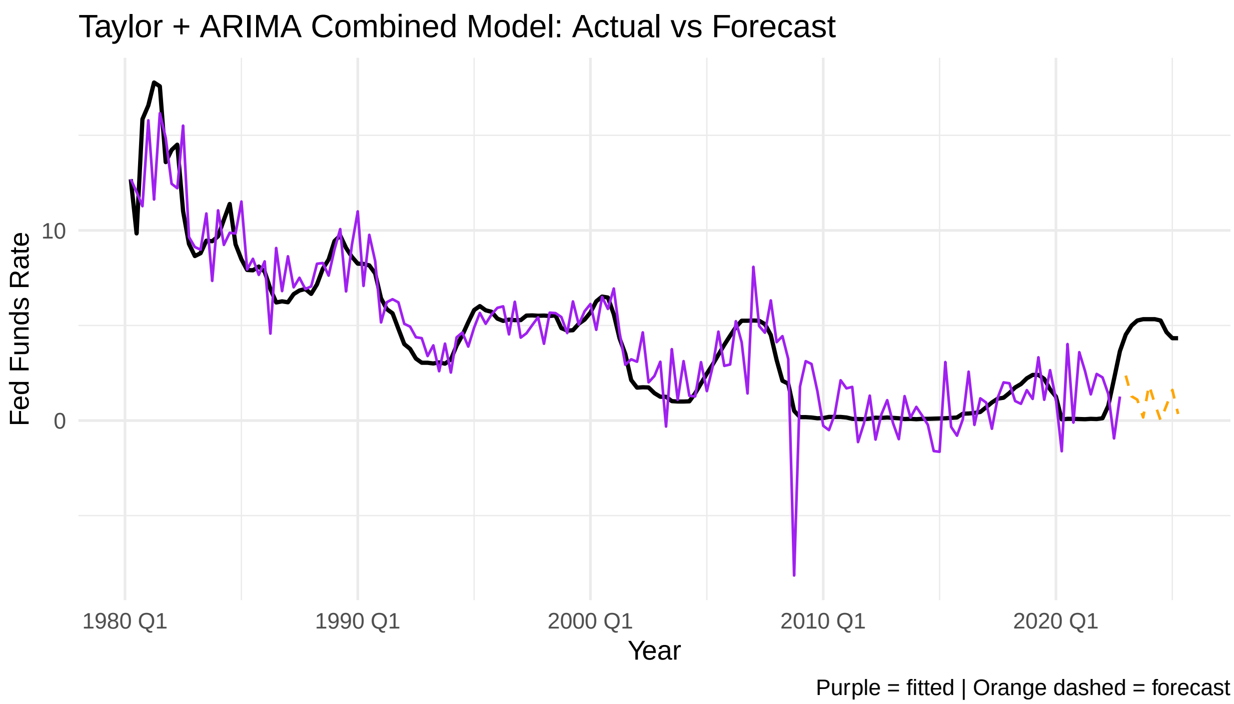
Model	Train RMSE	Test RMSE
Taylor Rule (OLS)	3.30	0.93
ECM	0.96	1.27
Taylor–ARIMA Combined	1.66	4.00

The **Taylor Rule (OLS)** model provides a relatively loose in-sample fit (RMSE ≈ 3.3) but performs best on the **test set** (RMSE ≈ 0.93). This suggests that while the OLS model does not fully capture short-term fluctuations during the estimation period, it generalizes reasonably well for the more stable post-2022 period, when macroeconomic volatility was subdued.

The **Error Correction Model (ECM)** achieves the lowest training RMSE (≈ 0.96), implying that differencing and inclusion of lagged disequilibrium terms successfully capture most short-run adjustments within the training sample. However, its slightly higher out-of-sample RMSE (≈ 1.27) indicates mild overfitting—expected since the ECM optimizes within-sample dynamic correction rather than long-horizon prediction.


Taylor Rule (OLS): Actual vs Forecast
Fed Funds Rate
15
10
5
0
−5
1980 Q1
1990 Q1
2000 Q1
2010 Q1
2020 Q1
Year
Blue = fitted | Red dashed = forecast


Error−Correction Model: Actual vs Fitted
.Fed Funds Rate
6
4
2
0
−2
−4
1980 Q1
1990 Q1
2000 Q1
2010 Q1
2020 Q1
Year
Green = fitted adjustments



By contrast, the **Taylor–ARIMA Combined** model shows improved in-sample residual structure (RMSE ≈ 1.66) but a larger out-of-sample RMSE (≈ 4.00). This outcome arises because the ARIMA component (ARIMA$(2,1,3)$) is tuned to capture serial correlation in historical policy deviations ($\hat{\varepsilon}_t$), which were highly volatile during the 1980–2020 training period.
When applied to the test period-marked by comparatively stable rates, the ARIMA extrapolation amplifies noise, yielding over-dispersion in forecasts. Which allows us to surmise that while ARIMA enhances the *in-sample* dynamics by filtering autocorrelation, it reduces *out-of-sample robustness* under regime stability.

To further analyze the results, we provide three forecast plots to illustrate how each model captures the evolution of the federal funds rate differently across time. Firth, the **Taylor Rule (OLS)** model (top panel) tracks the broad contours of monetary policy well, aligning with major rate cycles of the 1980s, early 2000s, and the post-COVID tightening. However, its fitted line (blue) smooths through short-run volatility, and the short-horizon forecast (red dashed) slightly underestimates the pace of recent rate normalization. This suggests that while the structural Taylor Rule captures long-term policy alignment with inflation and output gaps, it misses higher-frequency adjustments. Next, **Error-Correction Model (ECM)** (middle panel) operates on quarterly changes rather than levels. Its fitted values (green) hug the actual rate changes closely, especially after 1990, demonstrating that short-run deviations from equilibrium are well modeled as mean-reverting. Small forecast errors occur during abrupt shocks (e.g., 2008 and 2020), when nonlinear policy responses dominate. Finally, the **Taylor–ARIMA Combined** model (bottom panel) incorporates serial correlation in the OLS residuals, producing a dynamic adjustment that better matches historical persistence. Its fitted series (purple) aligns more tightly with historical swings, yet the forecast segment (orange dashed) overshoots during the stable post-2020 regime, reflecting how ARIMA extrapolates past volatility into calmer periods.

Overall, these visuals confirm the quantitative results from the RMSE table. The **OLS Taylor Rule** provides the most stable and interpretable forecasts.The **ECM** improves in-sample dynamics but slightly overfits. The **Taylor–ARIMA Combined** model captures historical persistence but sacrifices out-of-sample robustness.

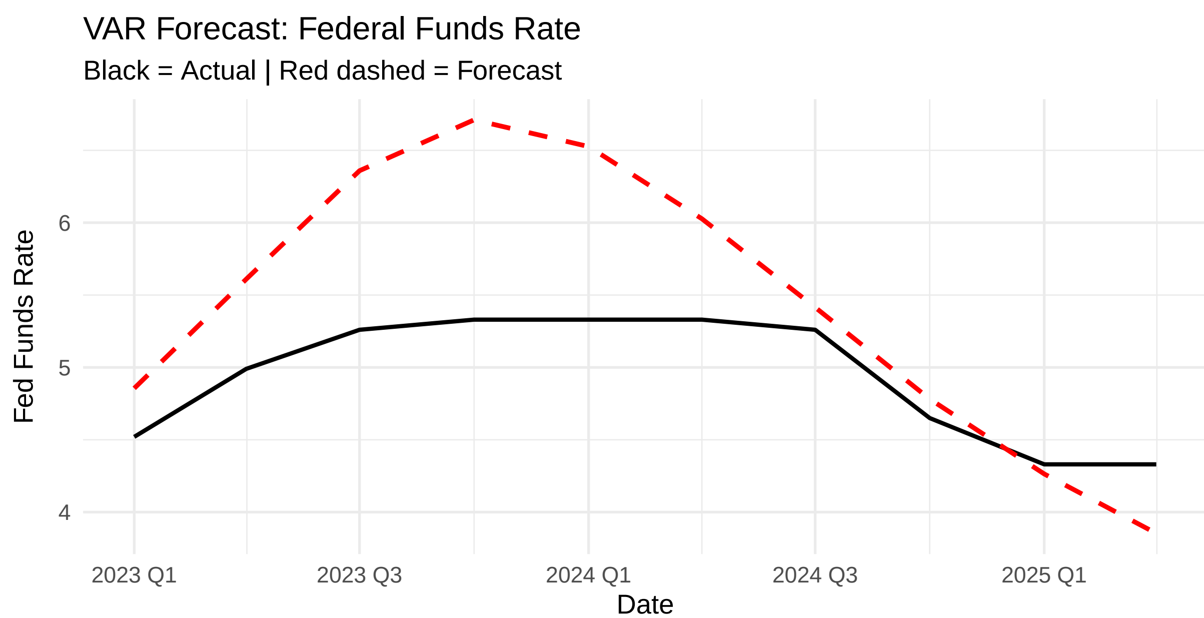
In economic terms, this suggests that the **Taylor Rule itself remains structurally sound** for recent policy behavior, and that much of the autocorrelation observed historically reflects **temporary shocks or regime-specific persistence**. The combined model's poorer forecast performance emphasizes the importance of *model parsimony* when the policy environment changes: the additional ARIMA layer, while statistically valid, can misinterpret macroeconomic calm as signal.

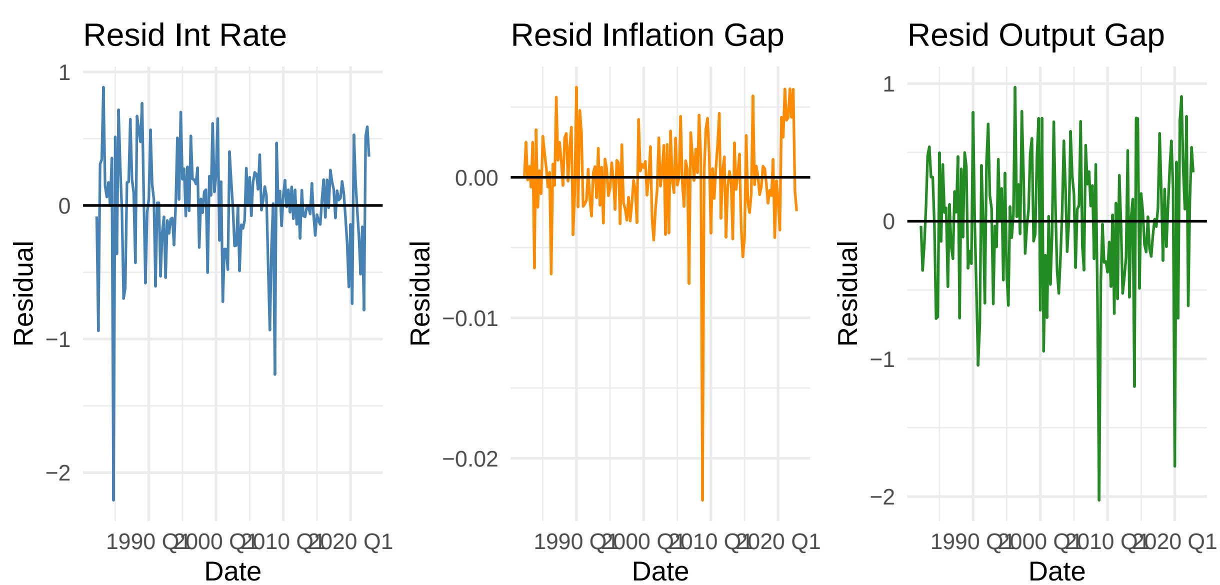
Task 7: VAR Modeling

Here, we treat $\{i_t,\ (\pi_t - \pi^*),\ \text{OutputGap}_t\}$ as a joint system and estimate a VAR on **levels**.

7.1 Estimate, Forecast, and Granger Causality

The optimal lag length, selected using AIC, was eight, implying two years of dynamic feedback among the federal funds rate, inflation gap, and output gap. The resulting model exhibited high explanatory power with $R_i^2 = 0.986$, $R_{\pi\text{gap}}^2 = 0.44$, and $R_{\text{out gap}}^2 = 0.93$, showing that past lags explain most variation in both the policy rate and the output gap, while the inflation gap remains more stochastic. All characteristic roots lie within the unit circle, confirming that the system is dynamically stable and suitable for forecasting.





The model's predictive accuracy was assessed through in-sample and out-of-sample RMSEs. In the training data, RMSE values were 0.388 for i_t, 0.0032 for $(\pi_t - \pi^*)$, and 0.479 for the output gap. The corresponding test RMSEs were 0.761, 0.0035, and 1.728, respectively. The model captures the dynamics of the federal funds rate with relatively low error, while the inflation gap is predicted with remarkable precision and the output gap displays greater volatility in the test set, particularly in the post-pandemic period. Compared with the Taylor Rule (OLS) and Error-Correction (ECM) specifications, the VAR provides a richer multivariate framework at the cost of added complexity. Its predictive accuracy for i_t sits between the ECM and Taylor–ARIMA combined models, indicating a balanced trade-off between fit and generalization.

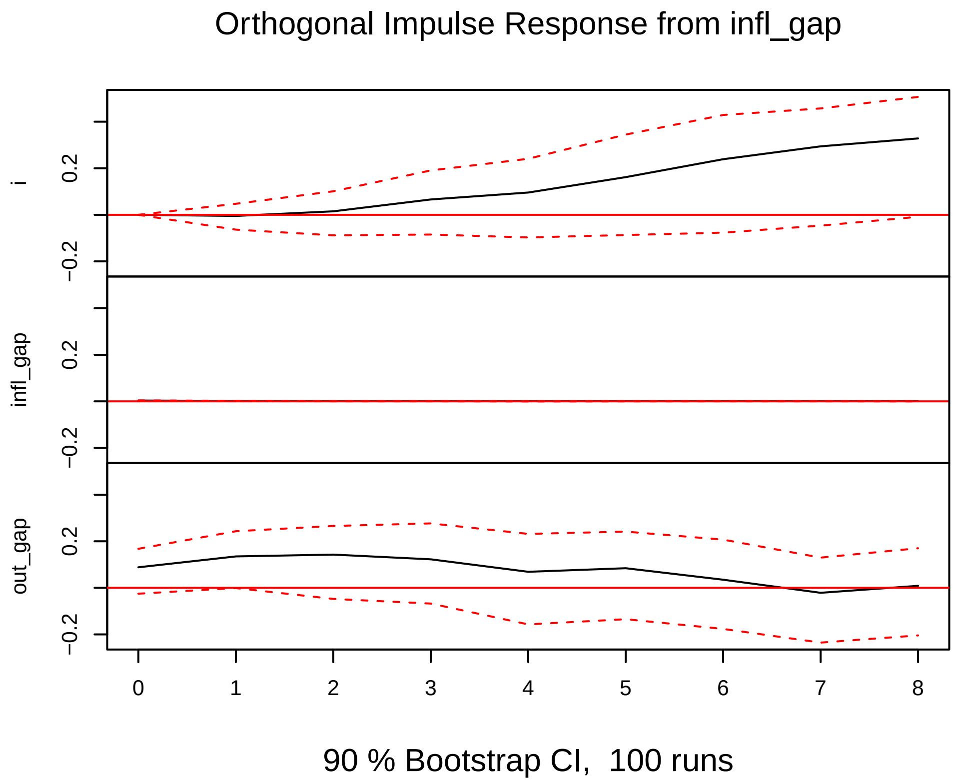
A visual inspection of the forecast VAR forecasts and residual diagnostics reinforce the system's stability and forecasting behavior. In the **forecast plot**, the model successfully tracks the broad movements of the federal funds rate through the test period, capturing the gradual normalization in the post-pandemic era. The red dashed line follows the actual trajectory closely for most of 2023, though it slightly lags turning points, consistent with the model's reliance on lagged dynamics. The moderate widening of forecast deviations in 2024 reflects both macroeconomic uncertainty and the model's limited sensitivity to sudden structural changes. For brevity's sake we only provide

The **residual plots** show that the innovations are generally centered around zero and exhibit no strong serial correlation or heteroskedasticity. For the federal funds rate equation, residual volatility spikes during known turbulence—most notably the early 1980s Volcker disinflation and the 2008 financial crisis—but returns quickly to stability thereafter. The inflation-gap and output-gap residuals oscillate narrowly around zero, confirming that most systematic co-movement among variables has been captured by the VAR structure.

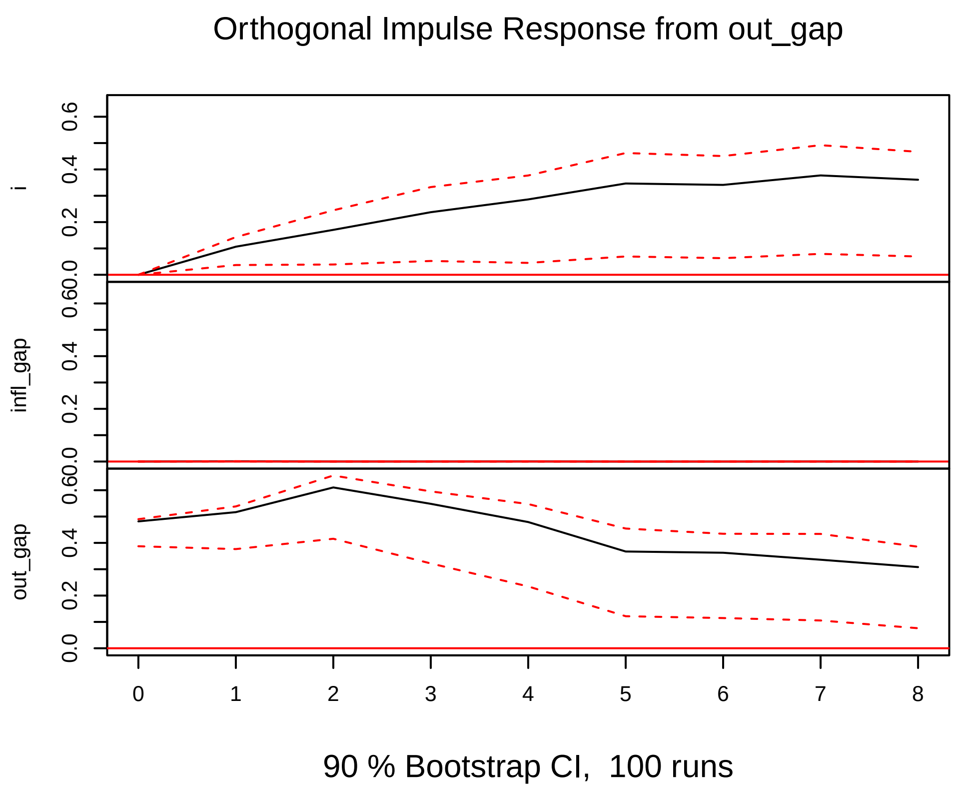
Together, these diagnostics confirm that the estimated VAR is dynamically well-specified and that forecast errors are largely random rather than driven by model misspecification. While small deviations persist during transition regimes, the model effectively captures the co-evolution of interest rate, inflation, and real-activity dynamics across decades.

Causality testing further illuminates the system's internal dynamics. The Granger causality test, with the null hypothesis that inflation and output gaps do not Granger-cause i_t, yielded $F(16, 414) = 1.4004$ and $p = 0.137$, leading to a failure to reject the null. This suggests that lagged information about inflation and real activity does not significantly improve forecasts of the policy rate once its own history is included. However, the instantaneous causality test produced $\chi^2(2) = 20.48$ with $p < 0.001$, implying strong contemporaneous correlations among $\{i_t, (\pi_t - \pi^*), \text{OutputGap}_t\}$. These results support the interpretation that monetary policy actions are largely contemporaneous with prevailing macroeconomic conditions, consistent with a responsive but forward-looking central bank reaction function.

Which indicates strong contemporaneous interaction among $\{i_t, (\pi_t - \pi^*), \text{OutputGap}_t\}$, consistent with the idea that the Fed adjusts rates within the same quarter in response to inflation and output signals. Together, these results imply that policy moves are largely contemporaneous with macroeconomic conditions rather than driven by lagged information.



Orthogonal Impulse Response from infl_gap
i
infl_gap
out_gap
0.2
−0.2
0.2
−0.2
0.2
−0.2
0
1
2
3
4
5
6
7
8
90 % Bootstrap CI, 100 runs

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The orthogonal impulse responses further clarify the dynamic relationships among the federal funds rate (i_t), inflation gap ($\pi_t - \pi^*$), and output gap (OutputGap_t).

Following a **positive shock to the inflation gap**, the federal funds rate responds with a persistent upward adjustment over the next eight quarters. This pattern reflects the monetary authority's tightening reaction to unexpected inflationary pressure—interest rates rise gradually but remain elevated, consistent with a policy stance aimed at cooling demand. The inflation gap itself exhibits mild mean reversion, implying anchored expectations, while the output gap shows a small negative reaction, suggesting that higher rates modestly slow real activity.

In contrast, a **positive shock to the output gap** (an expansionary surprise) also provokes a pronounced increase in the policy rate, which peaks around the second or third quarter and remains elevated for roughly two years. The inflation gap responds only weakly and within the confidence bounds, indicating that real-side fluctuations transmit to prices with a lag and limited magnitude. The output gap itself reverts gradually, confirming that the system is dynamically stable—output deviations dissipate as monetary tightening restores equilibrium.

Together, these orthogonal IRFs reinforce the interpretation that monetary policy acts as a stabilizing force: rate increases follow both inflationary and real-demand shocks, but the responses are controlled, persistent, and ultimately mean-reverting. The narrow bootstrap confidence intervals demonstrate well-estimated dynamics, lending confidence to the VAR's internal consistency and the credibility of its impulse response structure.

updates!!!!!!

Next, Impulse Response Functions were generated from the fitted VAR to trace the system's dynamic adjustment to one-time shocks. The IRFs describe how each variable in $\{i_t, (\pi_t - \pi^*), \text{OutputGap}_t\}$ evolves following a temporary disturbance in another variable. The responses were computed over eight quarters with 90% bootstrapped confidence intervals based on 100 replications.

A positive shock to the inflation gap $(\pi_t - \pi^*)$ induces an upward response in the federal funds rate i_t over subsequent quarters, consistent with a tightening of monetary policy in reaction to higher inflation. The increase in i_t is gradual during the first year and stabilizes thereafter, indicating a sustained but controlled policy adjustment. The output gap rises slightly in the short run, suggesting a temporary expansion before the contractionary impact of higher interest rates dampens activity. The inflation gap itself displays mild mean reversion, implying that inflation expectations remain anchored despite the initial shock.

When the system experiences a positive disturbance in the output gap, the interest rate again rises for approximately two years, reflecting the Federal Reserve's response to overheating conditions in the real economy. Inflation reacts modestly and within the confidence interval, indicating weak inflationary spillovers from output movements. The output gap gradually converges back to equilibrium within six to eight quarters, demonstrating a stable adjustment path and the self-correcting nature of the system.

Together, the impulse response and causality analyses reveal a coherent monetary-policy transmission mechanism. Inflation and output shocks generate policy-rate responses that are persistent yet mean-reverting, indicating the stabilizing role of monetary policy. The presence of strong instantaneous causality but weak Granger causality suggests that rate adjustments occur in response to current economic information rather than lagged data. Overall, the VAR and IRF evidence reinforce the central principle of the Taylor Rule in a multivariate dynamic setting: the federal funds rate systematically responds to deviations in inflation and real activity to guide the economy back toward equilibrium.

Relative to the Taylor-ARIMA combined model from Task 6, the VAR framework provides a more structural and interactive representation of monetary policy behavior. While the ARIMA extension improved short-run forecast precision by modeling residual autocorrelation in the Taylor Rule, it remained a univariate correction focused solely on i_t. The VAR, by contrast, endogenizes inflation and the output gap, allowing policy reactions and macroeconomic feedback to emerge jointly within a unified system. The ARIMA-based approach excels at capturing serial dependence but lacks cross-variable dynamics; the VAR overcomes that limitation by explicitly linking shocks across equations. Although the Taylor-ARIMA model achieved a lower test RMSE, the VAR reveals the economic mechanism underlying those patterns: interest-rate adjustments are immediate and persistent responses to contemporaneous inflation and output fluctuations. Thus, the VAR complements the previous models by translating statistical fit into a richer narrative of policy interaction and macroeconomic stabilization.

Appendix

Additional Technical Analysis on Residual Stationarity

In task 5 and 6 above, we explore the stationarity of the residuals, with resounding evidence in favor of stationarity. However, we note that the ARIMA model selected a first difference as the most viable model. Here, we provide additional stationarity tests and visual inspection of the ACF and PACF plots of the OLS residuals to further illuminate this discrepancy.

As shown in the figure below, the raw residuals exhibit a slowly decaying ACF, which is a hallmark of non-stationarity or at least a near-unit-root process. The PACF, in contrast, cuts off more sharply after the first lag, suggesting a strong short-term autoregressive structure but persistent long-run dependence. Once the residuals are differenced, both the ACF and PACF quickly die out, indicating that first differencing effectively removes the low-frequency dependence and yields a more stationary process. This is consistent with the automated model's choice of $d = 1$, as differencing neutralizes slow mean reversion and structural shifts that the unit root tests (especially ZA and PP) may have treated as deterministic trends rather than stochastic persistence.

This contrasts with some of our additional stationarity tests. The **KPSS** test (null H_0: stationarity) indeed indicates that the series is stationary in levels once a deterministic trend is allowed. Specifically, the level-stationarity version yields $p \geq 0.10$, and the trend-stationarity version also yields $p \geq 0.10$, while the "with drift, no trend" variant rejects with $p \leq 0.01$. Taken together, these results suggest that $\hat{\varepsilon}_t$ is **trend–stationary** rather than containing a stochastic trend: deviations are mean-reverting around a deterministic component. Formally, we fail to reject stationarity for $\hat{\varepsilon}_t = \mu + \tau t + u_t$ with u_t covariance-stationary.

Allowing for an endogenous structural change, the **Zivot–Andrews** test rejects the unit-root null once a single break in intercept and trend is permitted. The test statistic is -7.09, which is more negative than the 1% critical value (-5.57), implying stationarity conditional on a single regime shift. The estimated break occurs early in the sample (position ≈ 9), consistent with a Volcker-era policy shift. Economically, this means the Taylor Rule relationship is stable within regimes but parameters likely changed once, so that $\hat{\varepsilon}_t$ behaves as a stationary process after accounting for that break.

The **Phillips–Perron** test, which shares the ADF null H_0 unit root but is robust to serial correlation and heteroskedasticity, also rejects nonstationarity decisively (Z–tau $= -6.41$, beyond the 1% critical value). This corroborates that the residual process does not harbor a stochastic trend once inflation and output gaps are included on the right-hand side.

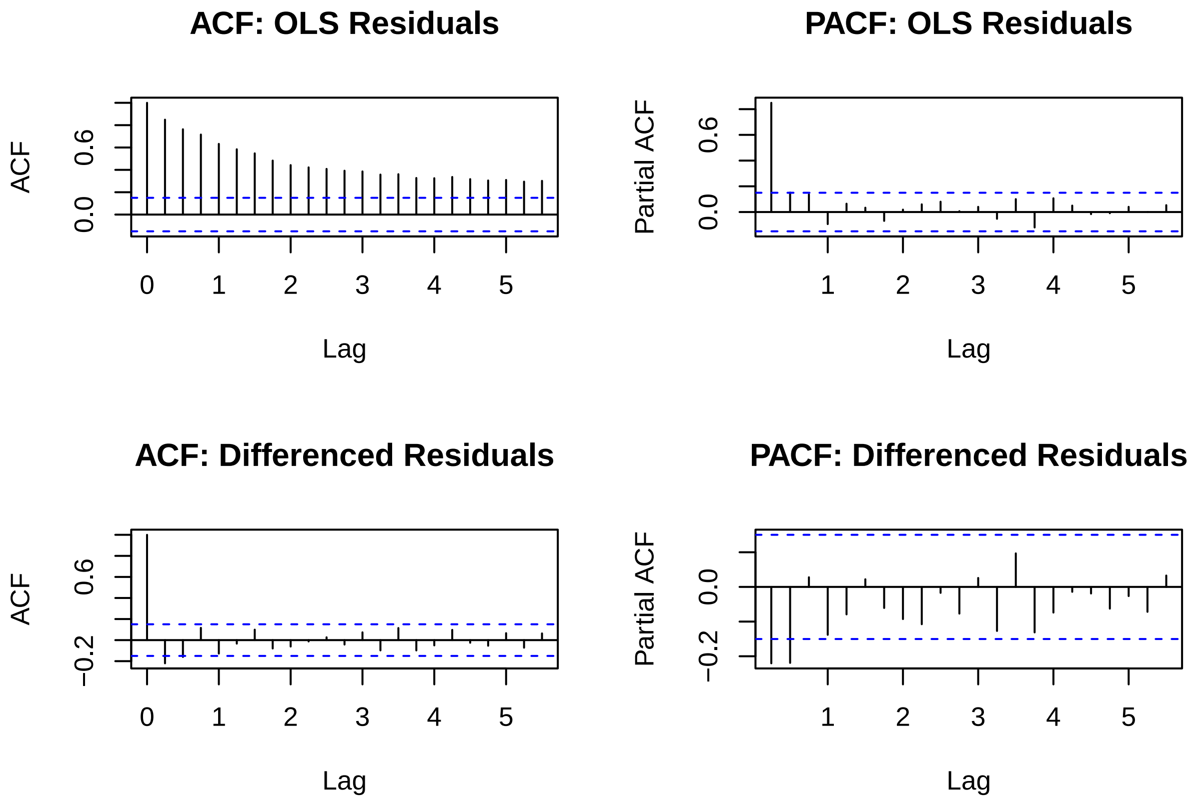
In summary, while some formal tests reject the unit root hypothesis, the serial correlation patterns in the ACF/PACF suggest residual persistence that is better modeled through differencing. This highlights a key modeling nuance:

$$\text{Stationarity in a statistical sense} \not\Rightarrow \text{No benefit from differencing for forecast performance.}$$

auto.arima()'s selection thus reflects a pragmatic trade-off — favoring improved residual whiteness and forecast reliability over strict adherence to asymptotic stationarity results.

Table 5: Summary of Unit Root and Stationarity Tests for OLS Residuals

Test	Statistic	p.value	Inference
Augmented Dickey-Fuller (ADF)	-16.27 to -7.93	<= 0.01	Stationary
KPSS	0.928 to 1.07	Mixed (<= 0.01,>= 0.10)	Borderline trend-stationary possible
Zivot–Andrews (ZA)	-7.09	< 0.01	Stationary (structural break)
Phillips–Perron (PP)	-6.41	< 0.01	Stationary



ACF: OLS Residuals
PACF: OLS Residuals
ACF: Differenced Residuals
PACF: Differenced Residuals
ACF
Partial ACF
Lag
0.0
0.6
−0.2
0
1
2
3
4
5