Homotopy of Paths

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Abstract

We discuss and define the homotopy of paths. First, we define the path homotopy and its conditions. Then, we describe its characteristiscs and provide examples in the real numbers. Finally, we discuss and prove a theorem regarding a groupoid, namely the product operation, on the path homotopy.

Introduction

A **path homotopy** is an equivalence class between the paths on a fundamental group of a space X. If f is a path, we denote its path homotopy equivalence class by [f]. We will define a homotopy and the more specific case of the path homotopy. In addition, we define the product operation, a groupoid, on the path homotopy. A **groupoid** is a certain operation on the collection of the equivalence classes that make it into what is called in algebra.

Definitions

Homotopy

Let f, f': X → Y, continuous maps. f is said to be **homotopic** to f', denoted as $f \simeq f'$, if there exists a continuous map F: $X \times [0,1] \to Y$ s.t. $F(x,0) = f(x)$ and $F(x,1) = f'(x), \forall x$. We say that F is a **homotopy** between f and f'. In the case that $f \simeq f'$ s.t. f' is a constant map, we say that f is *nulhomotopic*. We describe a homotopy like a continuous family of maps from X to Y, with one parameter. F can be thought of as a continuous deformation of f to f' as t goes from 0 to 1.

Path Homotopy

Let f: $[0,1] \to X$, a continuous map s.t. f(0) is the initial point x_0 and f(1) is the final point x_1. Then, we say that f is a path in X. Two paths with common endpoints are homotopic if one path can be continuously deformed into the other path, leaving the end points fixed and remaining within its region.

Let f, f': $[0,1] \to$ X, be two paths (two continuous maps s.t. f(0) is the initial point x_0 and f(1) is the final point x_1 and similiary for f'). f is said to be **path homotopic** to f', denoted as $f \simeq_p f'$, if the two maps have the same initial and final point and there exists a continuous map F: $[0,1] \times [0,1] \to X$ s.t. $F(s,0) = f(s), F(s,1) = f'(s)$ and $F(0,t) = x_0, F(1,t) = x_1, \forall s,t \in [0,1]$. We say that F is a **path homotopy** between f and f'.

Applications and Examples

Lemma 51.1: The relations $\simeq, \simeq_p$ are equivalence relations.

Proof. So, we want to show that the relation $\simeq$ is (1) reflexive, (2) symmetric, and (3) transitive.

(1) Given a continuous map f, we can deduce that f $\simeq$ f where the map F(x,t) = f(x) is the required homotopy, since F(x,0) = F(x,1) = f(x). Thus, f is homotopic to itself i.e. f $\simeq$ f. In the case that f is a path, then F is a path homotopy. So, f $\simeq_p$ f holds as well. Thus, our relations are reflexive.

(2) Suppose f $\simeq$ f'. We want to show that f' $\simeq$ f. Let F be a homotopy between f and f', so that F(x,0) = f(x) and F(x,1) = f'(x). Then, G(x,t) = F(x, 1-t) is a homotopy between f' and f because G(x,0) = F(x,1) = f'(x) and G(x,1) = F(x,0) = f(x). Thus, f' is homotopic to f i.e. f' $\simeq$ f. In the case that F is a path homotopy so that F(s,0) = f(s), F(s,1) = f'(s) and F(0,t) = x_0, F(1,t) = x_1, we can deduce that G is also a path homotopy because G(s,0) = F(s,1) = f'(s), G(s,1) = F(s,0) = f(s) and G(0,t) = x_0, G(1,t) = x_1. So, f' $\simeq_p$ f holds as well. Thus, we conclude that $\simeq$, $\simeq_p$ are symmetric.

(3) Suppose f $\simeq$ f' and f' $\simeq$ f". We want to show that f $\simeq$ f", i.e. f is homotopic to f". Let F be a homotopy between f and f', so that F(x,0) = f(x) and F(x,1) = f'(x). Also, let F' be a homotopy between f' and f", so that F'(x,0) = f'(x) and F'(x,1) = f"(x). Define G: $X \times [0,1] \to Y$, s.t. G(x,t) = F(x,2t), $t \in [0, 0.5]$ and G(x,t) = F'(x,2t-1), $t \in [0.5, 1]$. G is well-defined since if t = 0.5, then F(x, 2t) = f"(x) = F'(x,2t-1). Because G is continuous on the closed subsets $X \times [0, 0.5]$ and $X \times [0.5, 1]$, or $X \times [0,1]$, we conclude that G is continuous on all of $X \times [0,1]$ by the pasting lemma. Thus, G is the homotopy between f and f" because G(x,0) = F(x,0) = f(x) and G(x,1) = F'(x,1) = f"(x). So, we conclude that f $\simeq$ f". In the case that F and F' are path homotopies, then G is also a path homotopy because G(s,0) = F(s,0) = f(s) and G(s,1) = F'(s,1) = f"(s) with the initial and endpoints staying constant. So, f $\simeq_p$ f" holds as well. We have now shown that the relations are transitive.

Therefore, $\simeq$, $\simeq_p$ are equivalence relations.

□

Example 1: Let f, g: $X \to \mathbb{R}^n$ be homotopic.

Then, F(x,t) = (1-t)f(x) +(t)g(x) is a homotopy called the *straight-line homotopy* between f and g, because it moves the point f(x) to the point g(x) along the straight line segment that connects them. If f and g are paths from initial to end points, then F is a path homotopy.

Example 2: Let X denote the punctured plane. The paths $f(s) = (cos(s\pi), sin(s\pi))$, $g(s) = (cos(s\pi), 2sin(s\pi))$ are path homotopic. However, f and $h(s) = (cos(s\pi), -sin(s\pi))$ are not path homotopic, for the image of h does not lie in X.

Theorem on Groupoid for Path Homotopy

Let f be a path in X from x_0 to x_1 and g be a path in X from x_1 to x_2. The product $f * g$ of f and g is the path h s.t. $h(s) = f(2s), \forall s \in [0, 0.5]$ and $h(s) = g(2s-1), \forall s \in [0.5, 1]$. By the pasting lemma, h is well-defined and continuous. Further, it is a path in X from x_0 to x_1, where the first half of the path is f and the second half is g. The **product operation** on paths induces a well-defined operation on path homotopy classes, defined by $[f] * [g] = [f * g]$.

Similarly, let F be a path homotopy between f and f' and G be a path homotopy between g and g'. Define $H(s) = F(2s, t), \forall s \in [0, 0.5]$ and $H(s) = G(2s-1, t), \forall s \in [0.5, 1]$. Since F(1,t) = x_1 = G(0,t), $\forall t$, we deduce H is well-defined. Furthermore, H is continuous by the pasting lemma. Note that $[f] * [g]$ is not defined for every pair of classes, but only for those pairs [f], [g] for which f(1) = g(0).

Theorem 51.2: The $*$ operation has the following properties:

(1) Associativity: If $[f]*([g]*[h])$ is defined, then $[f]*([g]*[h]) = ([f]*[g])*[h]$ and is also defined.

(2) Right and Left identities: Given x $\in$ X, let $e_x : [0, 1] \to X$ denote the constant path that carries all of [0,1] to the point x. If f is a path in X from x_0 to x_1, then $[f] * [e_{x_1}] = [f]$ and $[e_{x_0}] * [f] = [f]$.

(3) Inverse: Given the path f in X from x_0 to x_1, let the reverse of f, defined as f^- = f(1-s). Then, [f] $*$ $[f^-] = [e_{x_0}]$ and $[f^-]$ $*[f] = [e_{x_1}]$.

Proof. To begin our proof, we will first note the following two properties of composite functions:

Let k: $X \to Y$, continuous. Let F be the path homotopy in X between paths f and f'. Then, $k \circ F$ is a path homotopy in Y between paths $k \circ f$ and $k \circ f'$.

Also, if f, g are paths in X s.t. f(1) = g(0), the end point of f being equal to the initial point of g, then $k \circ (f * g) = (k \circ f) * (k \circ g)$, by definition of the $*$ operator.

To prove (2), the right and left identities, let e_0 denote the constant path in [0,1] at 0. Let i: $[0, 1] \to [0, 1]$ denote the identity map, which is a path in [0,1], since it is a continuous map with an initial and final point. Then, e_0 is also a path in [0,1] from 0 to 1. Since [0,1] is a closed interval in X, it is convex. Thus, there exists a path homotopy G in [0,1] between i and $e_0 * i$. Then, by the first property of composite functions above, $f \circ G$ is a path homotopy in X between the paths $f \circ i = f$ and $f \circ (e_0 * i) = (f \circ e_0) * (f \circ i) = e_{x_0} * f$, by the second property of composite functions as mentioned above. We have now shown that there is a path homotopy between the paths f and $e_{x_0} * f$. So, we conclude $[e_{x_0}] * [f] = [f]$.

Similarly, let e_1 denote the constant path in [0,1] at 1. We know e_1 is also a path in [0,1] from 0 to 1. Because [0,1] is convex, there is a path homotopy G in [0,1] between i and $i * e_1$. Then, $f \circ G$ is a path homotopy in X between the paths $f \circ i = f$ and $f \circ (i * e_1) = (f \circ i) * (f \circ e_1) = f * e_{x_1}$. We have now shown that there is a path homotopy between the paths $f * e_{x_1}$ and f. So, $[f] * [e_{x_1}] = [f]$. Thus, the right and left identities hold and (2) is true.

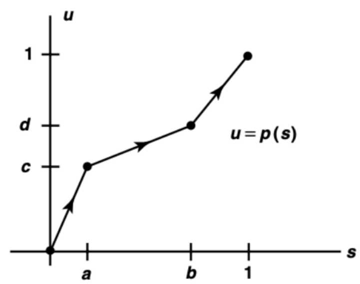
To prove (3), note that the reverse of the identity map i is $\bar{i}$(s) = 1-s. Then, $(i * \bar{i})$ and the constant path e_0 are both paths in the interval [0,1] starting and ending at point 0. Because [0,1] is convex, there exists a path homotopy H in [0,1] between e_0 and $i * \bar{i}$. Then, by the property of composite functions, $f \circ H$ is a path homotopy between $f \circ e_0 = e_{x_0}$ and $f \circ (i * \bar{i})$. By the second property, $f \circ (i * \bar{i}) = (f \circ i) * (f \circ \bar{i}) = f * \bar{f}$, by definition of the identity map. So, we conclude [f] $*$ $[f^{-}] = [e_{x_0}]$.

Similarly, observe that $\bar{i} * i$ and the constant path e_1 are both paths in the interval [0,1] starting and ending at 1. Because [0,1] is convex, there is a path homotopy H in [0,1] between e_1 and $\bar{i} * i$. Then, by the property of composite functions, $f \circ H$ is a path homotopy between $f \circ e_1 = e_{x_1}$ and $f \circ (\bar{i} * i)$. By the second property mentioned above, $f \circ (\bar{i} * i) = (f \circ \bar{i}) * (f \circ i) = \bar{f} * f$. So, we conclude $[f^{-}]$ $*[f] = [e_{x_1}]$. Hence, we have proven the inverse property (3) of the $*$ operation.

To prove (1), we will describe [a,b] and [c,d] as two intervals in $\mathbb{R}$. There exists a unique map p: $[a, b] \to [c, d]$ of the form p(x) = mx + k that sends point 'a' to 'c' and point 'b' to 'd'. This is a *positive linear map* of [a,b] to [c,d] because its graph is a straight line with a positive slope (m¿0). Note that the inverse of a positive linear map is *another* positive linear map, and so the composition of two positive linear maps would result in another positive linear map. Thus, the product $f * g$ can be described as: on [0, 0.5], $f * g$ equals the positive linear map of [0, 0.5] to [0,1], followed by f; on [0.5, 1], $f * g$ equals the positive linear map of [0.5, 1] to [0,1], followed by g.

Given paths f, g, h in X, the products $f * (g * h)$ and $(f * g) * h$ are defined precisely when f(1) = g(0) and g(1) = h(0), i.e. when the end point of f equals the initial point of g and the final point of g equals the initial point of h. Assuming these conditions, we define a *triple product* of the paths f, g, h as follows: Choose points a, b of [0,1] so that $0 < a < b < 1$. Define a path $k_{a,b}$ in X: On [0,a] $k_{a,b}$ equals the positive linear map of [0,a] to [0,1] followed by f; on [a,b] $k_{a,b}$ equals the positive linear map of [a,b] to [0,1] followed by g; and on [b,1] $k_{a,b}$ equals the positive linear map of [b,1] to [0,1] followed by h. The path $k_{a,b}$ depends on the choice of the paths a, b. However, $k_{a,b}$'s path-homotopy class does not. We show that if c, d are another pair of points of [0,1] with $0 < c < d < 1$, then $k_{c,d}$ is path homotopic to $k_{a,b}$.

Let $p : [0, 1] \to [0, 1]$ be a piecewise graph formed by connected line segments (refer to image below).



When restricted to the intervals [0,1], [a,b], and [b,1], the graph p equals the positive linear maps of these intervals onto [0,c], [c,d], and [d,1] respectively. Thus, it follows that $k_{c,d} \circ p = k_{a,b}$. However, p is a path in [0,1], and so the identity map $i : [0,1] \to [0,1]$ is a path also in [0,1]. Hence, there exists a path homotopy P in [0,1] between p and i. Then, $k_{c,d} \circ P$ is a path homotopy in X between $k_{a,b}$ and $k_{c,d}$. This shows that the product $f * (g * h) = k_{a,b}$ such $a = 1/2, b = 3/4$, a triple product. Similarly, $(f * g) * h = k_{c,d}$ such $c = 1/4, d = 1/2$, a triple product. Therefore, $k_{a,b}$ and $k_{c,d}$ are path homotopic and associativity holds (1).

□

Furthermore, this proof for associativity holds for any finite products of paths, which will be summarized in the following theorem:

Theorem 51.3: Let f be a path in X. Let $a_0, ..., a_n$ be numbers s.t. $0 = a_0 < a_1 < ... < a_n = 1$. Let $f_i : [0,1] \to X$ be the path that equals the positive linear map of [0,1] onto $[a_{i-1}, a_i]$ followed by f. Then, [f] = $[f_1] * ... * [f_n]$.

References

[1] James Munkres (2014) *Topology 2nd Edition*, Pearson Education.