Final Review Guide

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Topology

A **topology** τ is a collection of subsets of X such that:

1. $\emptyset, X \in \tau$
2. $\bigcup_{\alpha=1} U_\alpha$ (arbitrary union of elements of τ)
3. $\bigcap_{i=1}^{n} U_i$ (finite intersection of elements of τ)

(X, τ): a topological space consisting of space X and topology τ
a subset U of X is an **open set** of X if U belongs to τ

Basis

A **basis** β is a collection of subsets of X (i.e. basis elements) such that:

1. $\forall x \in X, \exists B \in \beta$ such that $x \in B$
2. $x \in B_1 \cap B_2 \Rightarrow \exists B_3$ such that $x \in B_3 \subset B_1 \cap B_2$

topology τ generated by basis β: a subset U of X is an element of τ if for each x $\in$ U, there is a basis element B $\in \beta$ such that x $\in B$ and $B \subset U$

Topology Examples:

- discrete topology: the collection of all subsets of X
- indiscrete (trivial) topology: the collection of only $\emptyset$, X
- finite complement topology (τ_f): the collection of all subsets U of X such that X-U is either finite or all of X
- countable complement topology (τ_c): the collection of all subsets U of X such that X-U is either countable (i.e. finite or countably infinite) or all of X
- standard topology ($\mathbb{R}$): the topology generated by the collection β of all open intervals in $\mathbb{R}$, $(a, b) = \{x | a < x < b\}$

- lower-limit topology ($\mathbb{R}_l$): the topology generated by the collection β' of all half-open intervals of the form: $[a, b) = \{x|a \leq x < b\}$
- K topology ($\mathbb{R}_k$): the topology generated by the collection β" of all open intervals (a,b) along with all sets of the form (a,b) - K such that $K = \{\frac{1}{n}|n \in \mathbb{N}\}$

Countability

A space X has a **countable basis at x** if there is a countable collection β of neighborhoods of x s.t. each neighborhood of x contains at least one of the elements of β

A space that has a countable basis at *each* of its points satisfies the **first countability axiom** (i.e. is "first-countable")

A space X that has a countable basis for its *topology* satisfies the **second countability axiom** (i.e. is "second-countable")

Every metrizable space is first-countable.
$\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^w$ have countable bases.

Theorem 30.1: Let X be a topological space.
(a) Let A be a subset of X. If there is a sequence of points of A converging to x, then x $\in \bar{A}$. (the converse holds if X is first-countable, ie X has a countable basis at each of its points ie there is a countable collection of neighborhoods of x s.t. each neighborhood contains at least one of the elements of the collection)
(b) Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence f(x_n) converges to f(x) in Y. (the converse holds if X is first-countable, ie X has a countable basis at each of its points)

Theorem 30.2:
(a) A subspace of a first-countable space is (also) first-countable. A countable product of a first-countable space is (also) first-countable.
(b) A subspace of a second-countable space is (also) second-countable. A countable product of a second-countable space is (also) second-countable.

Theorem 30.3: Suppose that X has a countable basis. Then,
(a) every open covering of X contains a countable subcollection covering X
(b) there exists a countable subset, A, of X that is dense (i.e. $\bar{A} = X$) in X

Continuity (Functions)

Let X, Y be topological spaces. $f : X \to Y$ is continuous if for each open subset V $\subset$ Y, the set $f^{-1}(V)$ is an open subset of X

Theorem 18.1: Let $f : X \to Y$. The following are equivalent:
(1) f is continuous
(2) for every subset A $\subset$ X, one has $f(\bar{A}) \subset f\bar{(a)}$

(3) for every closed set B of Y, the set $f^{-1}(B)$ is closed in X

(4) for each x $\in$ X and each neighborhood V of f(x), there is a neighborhood U of x such f(U) $\subset$ V

A function f from metric spaces $(X, d_x) \to (Y, d_y)$ is **uniformly continuous** if given $\epsilon > 0$, there is a $\delta > 0$ s.t. for every pair of points $x_0, x_1 \in X$, $d_x(x_0, x_1) < \delta \Rightarrow d_y(f(x_0), f(x_1)) < \epsilon$

<u>Theorem 18.2</u>: Let X, Y, Z be topological spaces The constant function, inclusion, composite, restricting domain, restring/expanding range, and local formulation are all continuous functions.

<u>Theorem 18.3</u> (Pasting Lemma): Let $X = A \cup B$, such A,B are closed sets in X. Let $f : A \to Y$, $g : B \to Y$ be continuous functions. If f(x) = g(x) for all x $\in A \cup B$, then f, g combine to form *continuous* $h : X \to Y$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases} \tag{1}$$

<u>Theorem 18.4</u> (Maps into Products): Let $f : A \to Y$ defined by f(a) = $(f_1(a), f_2(a))$. f is *continuous* iff $f_1 : A \to X$, $f_2 : A \to Y$ are continuous functions.

Let X, Y be topological spaces. Let $f : X \to Y$ be a bijection. If both f and $f^{-1} : Y \to X$ are *continuous*, then f is a **homeomorphism** (aka isomorphism- a bijective correspondence $f : X \to Y$ s.t. f(U) is open iff U is open).

Suppose $f : X \to Y$ is an injective, continuous map, where X, Y are topological spaces. Then $f' : X \to f(X)$, obtained by restricting the range of f (which we know is continuous by Theorem 18.2), is bijective. If f' is a homeomorphism of X with f(x), $f : X \to Y$ is an **imbedding** of X in Y.

Closed Sets

A subset A of X is **closed** if the set $A^c = X - A$ is open.

<u>Theorem 17.2</u>: Let Y $\subset$ X, a subspace. A set A is closed in Y iff it equals the intersection of a closed set of X with Y.

<u>Theorem 17.3</u>: Let Y $\subset$ X, a subspace. If A is closed in Y and Y is closed in X, then A is closed in Y.

interior (IntA): the union of all open sets contained in A

closure ($\bar{A}$): the intersection of all closed sets containing A

$$IntA \subset A \subset \bar{A}$$

Order, Product, and Subspace Topology

order topology: a topology on X with a basis β of the following types: all intervals (a,b) in X, all intervals of the form [a_0,b) where a_0 is the smallest element of X, all intervals of the form (a, b_0] where b_0 is the largest element of X

product topology: a topology on $X \times Y$ with a basis β of all sets of the form $U \times V$, where U $\subset X$ and V $\subset Y$, open subsets.

Theorem 15.1: If β, ζ are bases for the topologies X, Y respectively, then $\delta = \{B \times C | b \in \beta, C in \zeta\}$ is a basis for the topology of $X \times Y$

Let $\pi_1 : x \times Y \rightarrow X$ defined by $\pi_1(x,y) = x$ and $\pi_2 : x \times Y \rightarrow Y$ defined by $\pi_2(x,y) = y$ be **projections** (surjective maps) of $X \times Y$ onto X or Y

If U is an open subset of X, then the set $\pi_1^{-1}(U) = U \times Y$, which is open in $X \times Y$

If V is an open subset of Y, then the set $\pi_2^{-1}(V) = X \times V$, which is open in $X \times Y$

subspace topology: the collection $\tau_Y = \{Y \cup U | U \in \tau\}$ is a topology on Y, such that $Y \subset X$, a subspace (i.e. open sets of τ_Y consist of all intersections of open sets of X with Y

Lemma 16.2: Let $Y \subset X$, a subspace. If U is open in Y and Y is open in X, then U is open in X.

Lemma 16.3: Let $A \subset X$, $B \subset Y$, be subspaces. The product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$

Box vs Product Topology

(in the product space)

box topology: (on $\prod X_\alpha$) has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α, for all α

product topology: (on $\prod X_\alpha$) has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α, for all α AND $U_\alpha = X_\alpha$ except for finitely many values of α

$$\tau_{prod} \subset \tau_{box}$$

Theorem 19.3: Let $A_\alpha \subset X_\alpha$, a subspace, for each $\alpha \in J$. Then, $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box *and* product topology.

Theorem 19.4: If each X_α is a Hausdorff space, then $\prod X_\alpha$ is *also* a Hausdorff space in both the box *and* product topology.

Theorem 19.5: Let $\{X_\alpha\}$ be an indexed family of spaces. Let $A_\alpha \subset X_\alpha$ for each α. If $\prod X_\alpha$ is given either the box or product topology, then $\prod \bar{A}_\alpha = \overline{\prod A_\alpha}$.

Theorem 19.6: Let $f : A \to \prod_{\alpha\in J} X_\alpha$ defined by f(a) = $(f_\alpha(a))_{\alpha\in J}$ where $f_\alpha : A \to X_\alpha$, each α. Let $\prod X_\alpha$ have the product topology. Then, f is continuous iff each f_α is continuous.

Connected Spaces

Let X be a topological space.

A **separation** of X is a pair (U,V) of disjoint, nonempty, open subsets of X s.t. $U \cup V = X$. Otherwise, X is **connected** (i.e. the only subsets of X that are both open and closed in X are $\emptyset, X$.

Lemma 23.1: Let Y ⊂ X, a subspace. By definition, a separation of Y is a pair (A,B) of disjoint, nonempty, open subsets of X, s.t. $A \cup B = X$ AND neither of which contains a *limit point* of the other. The space Y is connected if there exists no separation of Y.

Lemma 23.2: If the sets C, D form a separation of X, and if Y ⊂ X, a *connected* subspace, then Y lies entirely in either C or D.

Proof. Suppose C, D form a separation of X. This implies C, D are disjoint, nonempty open subsets of X whose union is X.
Suppose Y is a connected subspace of X. Since C, D, and Y are open in Y, $C \cap Y$ and $D \cap Y$ are open in Y, by the subspace topology. $C \cap Y$ and $D \cap Y$ are disjoint, nonempty open subsets of Y, whose union is Y. This implies that they form a separation of Y, which would contradict our assumption that Y is connected.
This means that one of the sets must be empty. Thus, Y would lie enitrely in either $C \cap Y$ and $D \cap Y$, ie Y lies entirely in C or D.

□

Theorem 23.3: The union of a collection of connected subspaces of X that have a point in common is (also) connected.

Theorem 23.4: Let A ⊂ X, a *connected* subspace.

$$\text{A} \subset \text{X} \subset \bar{A} \Rightarrow \text{B is connected}$$

Theorem 23.5: The image f(X) of a connected space (X) under a continuous map (f) is connected.

Proof. Let $f : X \to Y$, continuous. Let X be a connected space. Since the map obtained from f by restricting its range to the space Z=f(X) is also continuous (by Theorem 18.2), consider the case of a continuous, surjective map of $g : X \to Z$. Suppose for contradiction, Z = g(X) is not connected. So, let B, C be nonempty, disjoint, open subsets of Z such $B \cup C = Z$ that form a separation of Z. Then, $g^{-1}(B), g^{-1}(C)$ are disjoint sets whose union is X. The sets are also open in X

because of the continuity of g and are nonementy because of the surjectivity of g. Thus, $g^{-1}(B), g^{-1}(C)$ are a pair of disjoint open subsets of X whose union is X, ie a separation of X.
Thus leads to a contradiction since X is assumed to be connected. □

Theorem 23.6: A finite cartesian product of connected spaces is (also) connected.

A **path** in X (from point x to y) is a continuous map $f : [a, b] \to Y$ of some closed interval in $\mathbb{R}$ into X, s.t. f(a) = x and f(b) = y.

A space X is **path connected** if every pair of points of X can be joined by a path in X.

- the punctured Euclidean space: $\mathbb{R}^n - \{0\}$
- the unit sphere
- S = $\{(x, sin(\frac{1}{x})|x \in (0, 1]$, which induces the topologist's sine curve $\bar{S} = S \cup (0 \times [-1, 1])$, which is only connected

path connected ⇒ connected

Compact Spaces

a collection A of subsets of a space X **covers** X if the union of the elements of A is equal to X, i.e. $\bigcup_{A \in A} A_\alpha = X$

A space X is **compact** if every open covering A contains a finite subcollection that also covers X.

Lemma 26.1: Let Y ⊂ X, a subspace. Y is compact iff every covering of Y by open sets in X contains a finite subcollection covering Y.

Theorem 26.2: Every closed subspace of a compact space is (also) compact.

Theorem 26.3: Every compact subspace of a Hausdorff space is *closed*.

Lemma 26.4: Let Y ⊂ X, a *compact* subspace of X, a Hausdorff space. Let x_0 not in Y. Then, there exist disjoint, open sets U, V of X which contain x_0, Y respectively.

Theorem 26.5: The image f(X) of a compact space (X) under a continuous map (f) is compact in Y.

Proof. Let $f : X \to Y$, continuous. Let X be compact space. Let A be a covering of set f(X) by sets open in Y. The collection $\{f^{-1}(A)|A \in A$ is a collection of open sets covering X; these sets are open in X because f is continuous. Hence, finitely many of them, $f^{-1}(A_1), ..., f^{-1}(A_n)$ cover f(X). So, the open covering of f(X) contains a finite subcollection that also covers f(X), ie f(X) is compact. □

Theorem 26.6: Let $f : X \to Y$, be a bijective and continuous map. If X is compact and Y is Hausdorff, then f is a **homeomorphism** (i.e. f and $f^{[1}$ are continuous).

Proof. Suppose X is compact and Y is Hausdorff space. Let A be a closed set in X. By theorem 26.2, every closed subspace of a compact space is (also) compact, so that A is compact. By theorem 26.5, the impact of a compact space under a continuous map is compact, so that f(A) is compact.
Further, f(A) is a compact subspace of Y, a Hausdorff space. So by theorem 26.3, every compact subspace of a Hausdorff space is *closed*, so that f(A) is closed in Y. Thus by definition of continuity, for every closed set A of X, the set $f^{-1}(A)$ is closed in Y, proves that f^{-1} is continuous. Since both f, f^{-1} are continuous, we deduce that f is a homeomorphism by definition.

□

Theorem 26.7: The product of finitely many compact spaces is (also) compact.

Lemma 26.8: Consider the product space X × Y, where Y is compact. If N is an open set of X × Y containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some **tube** $W \times Y$ about $x_0 \times Y$, where W is a neighborhood of x_0 in X

A collection $\{C_\alpha\}$ of subsets of X has the **finite intersection property** if for every finite subcollection $\{C_1, ..., C_n\}$ of $\{C_\alpha\}$, the intersection $\{C_1 \cap ... \cap C_n\}$ is nonempty.

Theorem 26.9: Let X be a topological space. X is compact iff for every collection $\{C_\alpha\}$ of closed sets in X having the finite intersection property, the intersection $\bigcap_C C$ of all the elements of $\{C_\alpha\}$ is nonempty

Given a collection A of subsets of X, let $C = \{X - A | A \in A\}$ be a collection of complements. So, the following statements hold:

(1) A is a collection of open sets iff C is a collection of closed sets, by definition.

(2) the collection A covers X iff the intersection $\bigcap C$ of all the elements of C is empty

(3) the finite subcollection $\{A_1, ..., A_n\}$ of A covers X iff the intersection of the corresponding elements $C_i = X - A_i$ of C is empty

Corollary 27.2: Every closed interval in $\mathbb{R}$ is compact.

Theorem 27.3: Let $A \subset \mathbb{R}^n$. A is compact iff it is *closed* and *bounded* in the euclidean metric d or square metric p.

Theorem 24.3 (Intermediate Value Theorem): Let $f : X \to Y$ a continuous function, where X is a connected space and Y is an ordered set in the order topology. If a,b $\in$ X and if r is a point of Y s.t. $f(a) \leq r \leq f(b)$, then there exists point c s.t. f(c) = r

Theorem 27.4 (Extreme Value Theorem): Let $f : X \rightarrow Y$ a continuous function, where Y is an ordered set in the order topology. If X is compact, then there exists points c,d in X s.t. $f(c) \leq f(x) \leq f(d) for \forall x \in X$

Lemma 27.5 (The Lebesgue Number): Let A be an open covering of the metric space (X, d). If X is compact, there exists $\delta > 0$ (called the Lebesgue Number for the covering) s.t for each subset A of X such diam(A) $< \delta$, there exists an element of A containing it.

Hausdorff, Regular, and Normal Spaces

A topological space X is a **Hausdorff space** if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1, U_2 of x_1, x_2 that are disjoint (i.e. $U_1 \cap U_2 = \emptyset$)

Suppose that one-point sets are closed in X.

X is **regular** if for each pair consisting of point x and closed set B disjoint from x, there exist disjoint open sets containing x, B respectively.

X is **normal** if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A, B respectively.

Normal $\Rightarrow$ Regular $\Rightarrow$ Hausdorff (any distinct points can be separated by two disjoint open sets) $\Rightarrow$ any one-point set is closed

Lemma 31.1: Let X be a topological space with one-points sets in it closed.

(a) X is regular iff given point x $\in$ X and neighborhood U of x, there is a neighborhood V of x s.t. $\bar{V} \subset U$

(b) X is normal iff given closed set A and open set U containing A, there is a neighborhood V containing A s.t. $\bar{V} \subset U$

Theorem 31.2:

(a) A subspace of a Hausdorff space is (also) Hausdorff. A product of Hausdorff spaces is (also) Hausdorff.

(b) A subspace of a regular space is (also) regular. A product of regular spaces is (also) regular.

Theorem 32.1: Every regular space with a countable basis is normal.

Theorem 32.2: Every metrizable space is normal.

Theorem 32.3: Every compact Hausdorff space is normal.

Proof. Let X be a compact Hausdorff space.
Claim: X is regular.
If x_0 is a point of X and B a closed set in X not containing x_0, then B is also a compact Hausdorff subspace. By lemma 26.4, there exist disjoint open sets U,V of X containing x_0 and B respectively. So, by definition, X is regular.
Claim: X is normal.
Given disjoint closed sets A, B in X, choose for each point $a \in A$, disjoint open sets U_a, V_a that contain a, B respectively (by definition of regularity). The collection $\{U_a\}$ covers A. Because A is compact, every open covering contains a finite subcollection that also covers A. So, A may be covered by finitely many sets $U_{a_1}, ..., U_{a_m}$. Then, let $U = \bigcup_{i=1}^{m} U_{a_i}$ and $V = \bigcup_{i=1}^{m} V_{a_i}$ be disjoint open sets containing A, B respectively. So, there exist disjoint open sets containing closed sets A, B of X i.e. X is normal.

□

Metric Topology

A **metric** is a function d: $X \times X \to \mathbb{R}$ having the following properties:
(1) d(x, y) ≥ 0 for $\forall x, y \in X$ (equality holds iff x=y)
(2) d(x, y) = d(y, x) for $\forall x, y \in X$
(3) *Triangle Inequality*: d(x,y) + d(y, x) $\geq$ d(x, z) for $\forall x, y, z \in X$

Given a metric d on X, the number d(x,y) is the *distance* between x and y in the metric.

Given $\epsilon > 0$, the set $B_d(x, \epsilon) = \{y | d(x, y) < \epsilon\}$ is the "ϵ-ball" centered at x whose all points y is less than ϵ. The collection of all "ϵ-balls" is a *basis* for the metric topology

A set U is open in the metric topology induced by d iff for each $y \in U$, there is a $\delta > 0$ s.t. $B_\delta(x, \delta) \subset U$

X is metrizable if there exists a metric d on X that induces the topology of X.

Let X be a metric space with d. A subset A of X is said to be bounded if there is some M s.t. $d(a_1, a_2) \leq M$ for $\forall a_1, a_2 \in A$. If A is bounded and nonempty, the diameter of A is defined as diam(A) = $sup\{d(a_1, a_2) | a_1, a_2 \in A\}$

Lemma 20.2: Let d, d' be two metrics on a set X. Let τ, τ' be their respective topologies. $\tau \subset \tau'$ iff for each $x \in X$ and $\epsilon > 0$, there exist $\delta > 0$ s.t. $B_{d'}(x, \delta) \subset B_d(x, \epsilon)$

- standard (Euclidean) metric (on $\mathbb{R}^n$) **d(x, y)** = $||x - y||$
- standard bounded metric (on $\mathbb{R}^1$) $\bar{d}$**(x,y)** = $min\{d(x, y), 1\}$ *(induces the same topology as d)*
- square metric (on $\mathbb{R}^n$) **p(x, y)** = $max\{|x_1 - y_1|, ..., |x_n - y_n|\}$

- uniform metric (on $\mathbb{R}^J$, an index set J)$\bar{p}$**(x,y)** $= sup\{\bar{d}(x_\alpha, y_\alpha)|\alpha \in J\}$

in $\mathbb{R}^n$, $\tau_\delta = \tau_p = \tau_{prod}$
in $\mathbb{R}^J$, $\tau_{prod} \subset \tau_{\bar{p}} \subset \tau_{box}$
(in $\mathbb{R}^w$, all the topologies are strictly related)

Urysohn Lemma

Let X be a normal space. Let A, B be disjoint closed subsets of X. Let [a,b] be a closed interval in $\mathbb{R}$. Then, there exists a continuous map $f : X \to [a, b]$, s.t.

$$f(x) = \begin{cases} a & \forall x \in A \\ b & \forall x \in B \end{cases} \tag{2}$$

Urysohn Metrization Theorem

Every regular space X with a countable basis is metrizable.