CONGRUENCES BETWEEN RAMANUJAN'S TAU FUNCTION AND ELLIPTIC CURVES, AND MAZUR–TATE ELEMENTS AT ADDITIVE PRIMES

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Abstract. We show that if $E/\mathbb{Q}$ is an elliptic curve with a rational p-torsion for $p = 2$ or 3, then there is a congruence relation between Ramanujan's tau function and E modulo p. We make use of such congruences to compute the Iwasawa invariants of the 3-adic Mazur–Tate elements attached to Ramanujan's tau function. We also investigate numerically the Iwasawa invariants of the Mazur–Tate elements attached to an elliptic curve with additive reduction at a fixed prime number.

1. Introduction

Let $\Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ be the unique normalized cuspidal modular form of weight twelve and level one defined by Ramanujan's tau function $\tau : \mathbb{N} \to \mathbb{Z}$, given explicitly by the q-expansion

$$\Delta(z) = \sum_{n\geq 1} \tau(n)q^n = q \prod_{n\geq 1} (1-q^n)^{24},$$

where $q = e^{2\pi iz}$ with z being a variable in the upper half plane $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. This modular form encodes very rich arithmetic information and plays an important role in modern day Number Theory. Ramanujan's tau function satisfies interesting congruence relations, many of which can be explained by the theory of modular forms (see [SD73, SD77, SD88, Ser69, Ser73a] for detailed discussions). One important feature in the theory of p-adic families of modular forms is congruence relations between Fourier coefficients of modular forms (see [Hid86, Ser73b, Eme11]). It is therefore natural to study congruences between Δ and other modular forms. In [Suj20, §6], Sujatha discussed a congruence relation between Δ and the elliptic curve $X_0(11)$ modulo 11. We note in particular that $11 \nmid \tau(11) = 534612$, meaning that 11 is an ordinary prime for Δ. The congruence modulo 11 above originates from the fact that both Δ and the weight-two modular form attached to $X_0(11)$ lie inside a Hida family.

The starting point of the present article is to study congruence relations between Δ and other elliptic curves. Let f_1 and f_2 be two modular forms of weights k_1 and k_2 respectively. A necessary condition for the two modular forms to satisfy a congruence relation modulo a prime number p is that $k_1 \equiv k_2 \mod p-1$. The fact

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that Δ is of weight 12 means that if it satisfies a congruence relation with a weight two modular form, $p-1$ has to divide 10. The only possible values p can take are $2, 3$ and 11. We have $2 \mid \tau(2) = -24$ and $3 \mid \tau(3) = 252$. In particular, 2 and 3 are both non-ordinary primes for Δ. This means that any congruences between Δ and an elliptic curve modulo 2 or 3 cannot come from Hida Theory. Nonetheless, our first main result tells us that such congruences exist.

Theorem A (Theorem 2.2)**.** *Let $p \in \{2,3\}$. Let $E/\mathbb{Q}$ be an elliptic curve and denote its conductor by N_E. Suppose that E admits a p-torsion point defined over $\mathbb{Q}$. Then*

$$a_\ell(E) \equiv \tau(\ell) \mod p$$

for all primes $\ell \nmid pN_E$.

Curiously, even though Δ is non-ordinary at $p \in \{2,3\}$, an elliptic curve E admitting a rational p-torsion, as imposed by Theorem A can be ordinary at p. Indeed, if E has good reduction at p and $p \big| |E(\mathbb{F}_p)|$, then $a_p(E) \equiv 1 \mod p$, meaning that E is ordinary at p.

For a fixed prime p and a modular form f with good reduction at p, we write $\theta_{n,f}$ for the Mazur–Tate element attached to f over the sub-extension of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ of degree p^n. We shall write $\lambda(\theta_{n,f})$ for the λ-invariant of $\theta_{n,f}$ (see §3 for a summary of the definitions of these objects). When p is an ordinary prime for f, $\lambda(\theta_{n,f})$ is relatively well understood. We are interested in studying $\lambda(\theta_{n,\Delta})$ when p is a non-ordinary prime for Δ. Using algorithms of Pollack, we have been able to calculate these λ-invariants explicitly for small n. Our numerical data suggest that they are given by 3^n-2, 5^n-1 and 7^n-1 respectively; see Table 4 in §6 of the main body of the article.

Pollack and Weston [PW11] have proved several formulae for $\lambda(\theta_{n,f})$ when f is non-ordinary at p with Serre weight 2 under various hypotheses on the residual representation, the weight, the prime number p and the p-adic valuation of the p-th Fourier coefficient. One of the key ingredients in the work of Pollack–Weston is to compare f to a weight 2 modular form g via congruences modulo p. We may in fact describe the Iwasawa invariants of $\theta_{n,f}$ in terms of those of $\theta_{n,g}$, which can be described explicitly.

While the results of [PW11] do not apply to Δ at the prime 3, the congruences modulo 3 exhibited by Theorem A suggest that some of the techniques in loc. cit. may allow us to study the Iwasawa invariants of $\theta_{n,\Delta}$ when $p=3$, shedding light on the regular patterns exhibited by the numerical data given in Table 4. The following theorem where we compare the λ-invariants of $\theta_{n,\Delta}$ and $\theta_{n,E} := \theta_{n,f_E}$, where f_E is a weight-two modular form corresponding to an elliptic curve E defined over $\mathbb{Q}$ with conductor 27 (via the modularity theorem), is obtained along the lines of argument presented in [PW11]. We note that there exists a single isogeny class of such curves with four isomorphism classes. Three of the isomorphism classes admit a rational 3-torsion, so Theorem A applies for these curves. In fact, we obtain a full congruence between E and Δ in the sense that

$$a_\ell(E) \equiv \tau(\ell) \mod 3$$

for all primes ℓ (including $\ell = 3$). This allows us to establish:

Theorem B (Theorem 4.1)**.** *Let $p = 3$, $n \geq 0$ and E an elliptic curve defined over $\mathbb{Q}$ of conductor 27. If $\theta_{n,\Delta} \notin p\mathbb{Z}_p[\mathcal{G}_n]$, then*

$$\lambda(\theta_{n,\Delta}) = \lambda(\theta_{n,E}).$$

Note that an elliptic curve over $\mathbb{Q}$ with conductor 27 has additive reduction at 3. It led us to study the following question.

Question. Is there a general formula for $\lambda(\theta_{n,E})$ if $E/\mathbb{Q}$ is an elliptic curve with additive reduction at p?

We carry out numerical calculations of $\lambda(\theta_{n,E})$ at a prime $p \in \{3, 5, 7\}$, where E has additive reduction, using Pollack's algorithm. The codes we used are available on `https://github.com/anthonydoyon/Ramanujan-s-tau-and-MT-elts` and our results are presented in Tables 1-3 in §6. To our surprise, for a given E and a given p, there seems to always exist a very regular formula for $\lambda(\theta_{n,E})$ in terms of n when n is sufficiently large. While we are not able to fully explain the origins of these formulae, we are able to explain why $\lambda(\theta_{n,E})$ is always at least p^{n-1} (see Corollary 5.3). When E has potentially good ordinary or potentially multiplicative reduction, Delbourgo [Del98, Del02] has defined a p-adic L-function and formulated an Iwasawa main conjecture for E under the hypothesis that E is the twist of a modular form with good reduction at p by a Dirichlet character. We believe that our formulae on $\lambda(\theta_{n,E})$ should be related to Delbourgo's p-adic L-function. We intend to investigate this further in a future project. For certain potentially supersingular elliptic curves, the formulae we find depends on the parity of n, which has a great resemblance of the formulae for elliptic curves that have good supersingular reduction at p. We plan to develop the supersingular analogue of Delbourgo's theory in order to better understand these formulae from a theoretical view-point.

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2. Congruences between Ramanujan's tau function and elliptic curves with rational 2 or 3 torsions

We study a congruence relation between Δ and elliptic curves defined over $\mathbb{Q}$ which admit a rational 2-torsion or a 3-torsion. We begin with the following lemma on the values of τ modulo 2 and modulo 3.

Lemma 2.1. *Let $p \in \{2, 3\}$. For all primes $\ell \neq p$, we have*

$$\tau(\ell) \equiv 1 + \ell \mod p$$

Proof. Lehmer has proven that $\tau(\ell) \equiv 1 + \ell^{11} \mod 2^5$ if $\ell \neq 2$ and $\tau(\ell) \equiv \ell^2 + \ell^9 \mod 3^3$ if $\ell \neq 3$ (see for example [Ser69, §2.1]). But $\ell^{p-1} \equiv 1 \mod p$ for all $\ell \neq p$ by Fermat's little theorem. Hence the result follows. $\square$

We can now relate Δ to elliptic curves admitting a rational 2-torsion or a rational 3-torsion.

Theorem 2.2. *Let $p \in \{2,3\}$. Let $E/\mathbb{Q}$ be an elliptic curve and denote its conductor by N_E. Suppose that E admits a p-torsion point defined over $\mathbb{Q}$. Then*

$$a_\ell(E) \equiv \tau(\ell) \mod p$$

for all primes $\ell \nmid pN_E$.

Proof. Lemma 2.1 tells us that $\tau(\ell) \equiv 1+\ell \mod p$ for all $\ell \neq p$. Since

$$a_\ell(E) = 1+\ell-|E(\mathbb{F}_\ell)|,$$

it suffices to show that $|E(\mathbb{F}_\ell)| \equiv 0 \mod p$. Let $\alpha \in E(\mathbb{Q})[p]$ be a non-trivial p-torsion of E. Then, $\langle\alpha\rangle$ is a subgroup of $E(\mathbb{Q})[p]$ of order p.

Consider the natural group homomorphism $\pi_\ell : E(\mathbb{Q}) \to E(\mathbb{F}_\ell)$ given by reduction modulo ℓ. By [Sil86, Proposition VII.3.1], π_ℓ induces an injective group homomorphism

$$E(\mathbb{Q})[p] \hookrightarrow E(\mathbb{F}_\ell).$$

In particular, $\pi_\ell(\langle\alpha\rangle)$ is a subgroup of order p inside $E(\mathbb{F}_\ell)$. Thus, Lagrange's theorem tells us that $p\big||E(\mathbb{F}_p)|$ as required. □

Remark 2.3. An alternative approach to prove Theorem 2.2 is to consider the Galois representation $\rho_{E,p} : G_\mathbb{Q} \to \mathrm{GL}(E[p]) = \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$. Since E admits a rational p-torsion, $\rho_{E,p}$ admits a one-dimensional trivial $\mathbb{F}_p$-linear sub-representation. Since the determinant of $\rho_{E,p}$ is given by the mod p cyclotomic character $\chi_p : G_\mathbb{Q} \to (\mathbb{Z}/p\mathbb{Z})^\times$, we have

$$\rho_{E,p} \cong \begin{pmatrix} 1 & * \\ 0 & \chi_p \end{pmatrix}.$$

Therefore, for all $\ell \nmid pN_E$, we have

$$a_\ell(E) \equiv \mathrm{Tr}(\rho_{E,p}(\mathrm{Frob}_\ell)) = 1+\chi_p(\mathrm{Frob}_\ell) = 1+\ell \mod p,$$

where Frob_ℓ is the Frobenius at ℓ.

In the same vein as the results presented in [PW11], Theorem 2.2 suggests that there might be a link between Iwasawa-theoretic objects of Δ and elliptic curves admitting a rational 2-torsion or a 3-torsion. In the next section, we shall review the objects we are interested in, namely Mazur–Tate elements attached to modular forms. We will then relate the 3-adic Mazur–Tate elements attached to Δ to certain elliptic curves admitting a rational 3-torsion in §4.

3. Review of Mazur–Tate elements and Iwasawa invariants

3.1. **Definition and basic properties of Mazur–Tate elements.** In this section, we review the definition and some basic properties of the Mazur–Tate elements defined in [MT87]. We follow closely the exposition of Pollack–Weston in [PW11, §2.1 and §2.2]. Throughout this section, p is a fixed odd prime number and $f \in S_k(\Gamma_0(N))$ is a fixed normalized cuspidal eigenform. For simplicity, we assume throughout that the Fourier coefficients of f lie in $\mathbb{Z}$ (which is indeed the case when

$f = \Delta$ or when f corresponds to an elliptic curves defined over $\mathbb{Q}$, which are the cases of interest in the present article).

Let $G_n = \mathrm{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q})$. We identify an element $a \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ with the unique element $\sigma_a \in G_n$ satisfying $\sigma_a(\zeta) = \zeta^a$ for all $\zeta \in \mu_{p^n}$. We write $\mathcal{G}_n = \mathrm{Gal}\left(\mathbb{Q}(\mu_{p^{n+1}})/\mathbb{Q}(\mu_{p^n})\right)$, which we may identify with a quotient of G_{n+1} and we are equipped with a natural projection map $\pi_n : G_{n+1} \twoheadrightarrow \mathcal{G}_n$.

Definition 3.1. *Let $\mathcal{R}$ be a commutative ring. We denote by $V_k(\mathcal{R})$ the space of homogenous polynomials of degree k in two variables. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. As in* [PW11, §2.2]*, we define a modular symbol $\varphi_f \in H^1_c(\Gamma_0(N), V_{k-2}(\mathbb{C}))$ attached to f satisfying*

$$\varphi_f(\{r\} - \{s\}) = 2\pi i \int_s^r f(z)(zX+Y)^{k-2} dz$$

for $r, s \in \mathbb{P}^1(\mathbb{Q})$, where $\{r\}$ and $\{s\}$ are divisors associated to r and s respectively.

Definition 3.2. *Fix $n \in \mathbb{N}$. We define*

$$\Theta_{n,f} = \sum_{a \in (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times} \varphi_f(\{\infty\} - \{a/p^{n+1}\})\big|_{(X,Y)=(0,1)} \cdot \sigma_a \in \mathbb{C}[G_{n+1}]$$

and denote the image of $\Theta_{n,f}$ in $\mathbb{C}[\mathcal{G}_n]$ under the natural norm map induced by π_n by $\tilde{\Theta}_{n,f}$.

*The p-adic **Mazur–Tate element** of level n attached to f is defined to be*

$$\theta_{n,f} = \frac{\tilde{\Theta}_{n,f}}{\Omega_f^+},$$

where Ω_f^+ is the cohomological period for f given in [PW11, Definition 2.1].

Remark 3.3. We are only looking at the $+1$-eigenspace of the involution induced by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ on the space of modular forms since this is where our numerical calculations will be carried out. This is why we only make use of the period Ω_f^+ in the Definition 3.2. Furthermore, as explained in [PW11, Remark 2.2], the choice of Ω_f^+ ensures that $\theta_{n,f} \in \mathbb{Z}_p[\mathcal{G}_n]$.

By the modularity theorem, if $E/\mathbb{Q}$ is an elliptic curve of conductor N_E, then its L-function coincides with a unique normalized eigenform $f_E \in S_2(\Gamma_0(N_E))$. We let $\theta_{n,E}$ denote the Mazur–Tate element θ_{n,f_E}.

We now recall the definitions of Iwasawa μ and λ invariants attached to $\theta_{n,f}$. For further discussion on this topic, we invite the reader to consult [PW11, §3.1]. Given an element $F \in \mathbb{Z}_p[\mathcal{G}_n]$, we choose a generator γ_n of the Galois group $\mathcal{G}_n$. We may write F as a polynomial $\sum_{i=0}^{p^n-1} a_i X^i$, where $X = \gamma_n - 1$.

Definition 3.4. *For a non-zero element $F = \sum_{i=0}^{p^n-1} a_i X^i \in \mathbb{Z}_p[\mathcal{G}_n]$, we define the mu and lambda invariants of F by*

$$\mu(F) = \min_i \mathrm{ord}_p(a_i),$$
$$\lambda(F) = \min\{i : \mathrm{ord}_p(a_i) = \mu(L)\},$$

where ord_p *denotes the* p*-adic valuation on* $\mathbb{Z}$*. When* $F = 0$*, we set*

$$\mu(F) = \lambda(F) = \infty.$$

Remark 3.5. The definitions above are independent of the choice of the generator γ_n.

We explain the strategy we use to compute these Iwasawa invariants for $\theta_{n,f}$. Explicitly, given

$$\Theta_{n,f} = \sum_{a \in (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times} C_a \cdot \sigma_a \in \mathbb{C}[G_{n+1}],$$

we obtain its projection $\tilde{\Theta}_{n,f} \in \mathbb{C}[\mathcal{G}_n]$ via

$$\tilde{\Theta}_{n,f} = \sum_{a \in (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times} \frac{C_a}{\omega(a)} \cdot \pi_n(\sigma_a),$$

where $\omega : G_n \to \mathbb{Z}_p^\times$ is the Teichmüller character. Once we fix a generator γ_n of G_n, we may write $\pi_n(\sigma_n) = (1+X)^{a'}$ for some integer $a' \in \{0, 1, \ldots, p^n - 1\}$. This gives

$$\theta_{n,f} = \frac{1}{\Omega_f^+} \cdot \sum_{a \in (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times} \frac{C_a}{\omega(a)} \cdot (1+X)^{a'},$$

which is the formula we use in our numerical calculations below.

When $\theta_{n,E}$ is non-zero, we may multiply $\theta_{n,E}$ by an appropriate constant so that its coefficients as a polynomial in X are not divisible by p simultaneously. If we write $\tilde{\theta}_{n,E}$ for this scaled polynomial, we may calculate $\lambda(\theta_{n,f})$ by finding the degree of the polynomial $\mathbb{F}_p[X]$ obtained from $\tilde{\theta}_{n,E}$ modulo p.

For the elliptic curves we consider in our calculations, we may work with a particular choice of E' in the isogeny class containing E where $\tilde{\theta}_{n,E'} = \theta_{n,E'}$. Indeed, λ-invariants are constant in an isogeny class and it is conjectured that there always exists an E' in any given isogeny class satisfying $\mu(\theta_{n,E'}) = 0$ (see [Gre99, Conjecture 1.11] in the good ordinary case and [PR03, Conjecture 7.1] in the good supersingular case).

We are interested in p-adic Mazur-Tate elements mostly because they are closely related to the L-function. More precisely, p-adic Mazur-Tate elements satisfy the following interpolation property as pointed out in [PW11, §2].

Theorem 3.6. *Let* χ *be a Dirichlet character factoring through* $\mathcal{G}_n$*, but not* $\mathcal{G}_{n-1}$*, where* $n \geq 1$ *is an integer. Let* $\theta_{n,f}$ *be the* p*-adic Mazur-Tate element as defined in Definition 3.2. Then,*

$$\chi(\theta_{n,f}) = \tau(\chi) \frac{L(f, \chi^{-1}, 1)}{\Omega_f^+},$$

where $\tau(\chi)$ *is the Gauss sum of* χ*.*

Proof. See [PW11, §2] and [MT87, §8]. Note that χ is an even character, which is why we always have Ω_f^+ in the denominator. □

Remark 3.7. Let k denote the weight of f. When $k \geq 3$, $L(f, \chi^{-1}, 1) \neq 0$ for all χ (by the functional equation, it is a non-zero multiple of $L(f, \chi, k-1)$, which is non-zero since it can be expressed as an Euler product). In particular, Theorem 3.6 implies that $\theta_{n,f} \neq 0$ for all $n \geq 1$. When $k = 2$, the main result of [Roh88] tells us that $L(\overline{f}, \bar{\chi}, 1) \neq 0$ for all but finitely many χ. Thus, Theorem 3.6 implies that $\theta_{n,f} \neq 0$ for $n \gg 0$.

4. Mazur–Tate elements of Ramanujan's tau function at $p = 3$

We link the λ-invariants of the Mazur–Tate elements attached to Δ at $p = 3$ to those attached to an elliptic curve defined over $\mathbb{Q}$ of conductor 27.

According to the online database LMFDB, the complex vector space $S_2(\Gamma_0(27))$ is of dimension one. Let f denote the unique normalized cuspform in this space. Any elliptic curves of conductor 27 that are defined over $\mathbb{Q}$ have to correspond to this modular form under the modularity theorem.

There is one single isogeny class and four isomorphism classes of elliptic curves of conductor 27 that are defined over $\mathbb{Q}$. Three of the isomorphism classes admit non-trivial 3-torsions over $\mathbb{Q}$. Thus, Theorem 2.2 says that $a_\ell(E) \equiv \tau(\ell) \mod 3$ for all $\ell \neq 3$. Furthermore, since E has additive reduction at 3, we have $a_3(E) = 0$. Thus,

$$\tau_3(E) = 252 \equiv a_3(E) \mod 3.$$

Consequently, all Fourier coefficients of Δ and f are congruent modulo 3 and we have

$$f \equiv \Delta \mod 3 \tag{1}$$

as modular forms. This allows us to prove Theorem B:

Theorem 4.1. *Let $p = 3$, $n \geq 1$ and E an elliptic curve defined over $\mathbb{Q}$ of conductor 27. If $\mu(\theta_{n,\Delta}) = 0$, then*

$$\lambda(\theta_{n,\Delta}) = \lambda(\theta_{n,E}).$$

Proof. Let $p = 3$ in this proof. We define α' to be the composition

$$H_c^1(\Gamma_0(1), V_{10}(\mathbb{Z}_p)) \xrightarrow{\alpha} H_c^1(\Gamma_0(p^3), \mathbb{Z}_p/p^3\mathbb{Z}_p) \to H_c^1(\Gamma_0(p^3), \mathbb{F}_p),$$

where α is the Hecke equivariant map defined as in [PW11, §7] and the second arrow is given by the natural projection map.

Let us write $\overline{\varphi}_\Delta \in H_c^1(\Gamma_0(1), V_{10}(\mathbb{F}_p))$ (resp. $\overline{\varphi}_E \in H_c^1(\Gamma_0(1), V_{10}(\mathbb{F}_p))$) for the image of φ_Δ (resp. φ_{f_E}) modulo p. Here, $f_E \in S_2(\Gamma_0(27))$ denotes the modular form corresponding to E. After multiplying φ_Δ and φ_{f_E} by appropriate scalars if necessary, we may assume that $\overline{\varphi}_\Delta$ and $\overline{\varphi}_E$ are both non-zero.

Since α' is Hecke equivariant, (1) tells us that

$$\alpha'(\overline{\varphi}_\Delta) = \overline{\varphi}_E.$$

Hence, we deduce from [PW11, Lemma 4.6] that

$$\theta_{n,\Delta} \equiv \theta_{n,E} \mod p\mathbb{Z}_p[\mathcal{G}_n].$$

Therefore, we obtain the desired equality of λ-invariants under our hypothesis that $\theta_{n,\Delta} \notin p\mathbb{Z}_p[\mathcal{G}_n]$. $\square$

Note that E is of potentially good supersingular reduction at $p = 3$. Furthermore, our numerical investigations have led us to believe that the hypothesis $\theta_{n,\Delta} \notin p\mathbb{Z}_p[\mathcal{G}_n]$ is always true and that

$$\lambda(\theta_{n,\Delta}) = \lambda(\theta_{n,E}) = 3^n - 2$$

(see §.6).

5. Mazur–Tate elements at additive primes

Let $E/\mathbb{Q}$ be an elliptic curve having additive reduction at a fixed prime p. Other than the the settings where the works of Delbourgo [Del98, Del02] apply, it is not known how to define a p-adic L-function that would interpolate the complex L-values of E. Nonetheless, it is possible to compute p-adic Mazur–Tate elements of level n attached to E as given in definition 3.2. Interestingly, the calculations we made (see §6) show that the lambda invariants of such elements behave in a surprisingly regular manner, even though we do not know whether such patterns can be explained using Iwasawa-theoretic objects. In this section, we prove a theoretical lower bound on these lambda invariants (see Corollary 5.3 below). We note that this lower bound is attained by the curves $45a, 63a, 72a, 90c, 99a, 99b, 99d$ when $p = 3$, $150a$ when $p = 5$ and $147c, 294b$ when $p = 7$ (see Tables 1-3 in §6).

We recall the following norm relation satisfied by the Mazur–Tate elements.

Definition 5.1. *We denote by* $\mathrm{cor}_n^{n+1} : \mathbb{Z}[G_{n+1}] \to \mathbb{Z}[G_n]$ *the natural projection map.*

Lemma 5.2. *Let* $E/\mathbb{Q}$ *be an elliptic curve of conductor* N_E *and denote by* $\theta_{n,E}$ *its associated Mazur–Tate element as given in Definition 3.2. If* $p \big| N_E$ *and* $m \geq 1$*, then*

$$\mathrm{cor}_m^{m+1}(\theta_{m+1}) = a_p(E) \cdot \theta_m.$$

Proof. See [MT87, §1.3]. □

In the case of p being an additive prime, it has the following consequence on the λ-invariant of the Mazur–Tate elements.

Corollary 5.3. *Let* $E/\mathbb{Q}$ *be an elliptic curve with additive reduction at* p*. Then*

$$\lambda(\theta_{n,E}) \geq p^{n-1}$$

for all $n \geq 1$.

Proof. Lemma 5.2 tells us that that $\mathrm{cor}_n^{n+1}(\theta_n(f_E)) = 0$ for all $n \geq 1$ since $a_p(E) = 0$ when E has additive reduction at p. This implies that $\theta_{n+1} = g_n \cdot \omega_n$ for a certain polynomial $g_n \in \mathbb{Z}_p[x]$. So, $\lambda(\theta_{n+1}) = \lambda(g_n) + p^n \geq p^n$ as required. □

The same is true for the Mazur–Tate elements attached to Δ at $p = 3$:

Corollary 5.4. *At* $p = 3$*, if* $\mu(\theta_{n,\Delta}) = 0$*, we have*

$$\lambda(\theta_{n,\Delta}) \geq p^{n-1}$$

for all $n \geq 1$.

Proof. This follows from Theorem 4.1 and Corollary 5.3. □

6. Numerical data

In this section, we present a brief summary of the numerical results we have obtained. In Tables 1, 2 and 3, we give the λ-invariants of Mazur–Tate elements attached to elliptic curves having additive reduction at a fixed prime 3, 5 and 7 that we have computed respectively. We have found very uniform behaviour of these invariants, which suggest that after taking into account certain multiples of the lower bound exhibited by Corollary 5.3. It seems to suggest that the Mazur–Tate elements we computed might be related to certain bounded p-adic L-functions attached to these elliptic curves. We plan to study this in a future project.

The following tables contain our computations of the Iwasawa λ-invariants of p-adic Mazur–Tate elements of level n attached to elliptic curves having additive reduction at p. Since the Mazur–Tate elements are the same up to multiplication by a scalar for all elliptic curves in a given isogeny class, we organize our data by isogeny class using Cremona label. In the last column, we indicate our predictions for $\lambda(\theta_m)$ for m sufficiently large according to the values we computed.

The calculations we did were carried out on Sage modifying slightly Pollack's algorithm, available on `https://github.com/rpollack9974/OMS`. The codes we used for our computations can be found at `https://github.com/anthonydoyon/Ramanujan-s-tau-and-MT-elts`.

In what follows, we write

$$q_m = \begin{cases} p^{m-1} - p^{m-2} + \cdots + p - 1 & \text{if } m \text{ is even} \\ p^{m-1} - p^{m-2} + \cdots + p^2 - p & \text{if } m \text{ is odd.} \end{cases}$$

For some specific isogeny classes, for instance, $153a, 153c, 225a$ and $225b$, we could not find one single formula for $\lambda(\theta_m)$ in terms of m, but rather, two separate formulas depending on the parity of m, involving q_m. The term q_m appears naturally for elliptic curves with good supersingular reduction at p where the θ elements are related to Pollack's plus and minus p-adic L-functions defined in [Pol03] (see [PW11, §4.1]). As given in Tables 5 and 6, these curves all have potentially supersingular reduction at p. This suggests that the Mazur–Tate elements for these curves might be related to Pollack's plus and minus p-adic L-functions. Curiously, there are certain curves with potentially supersingular reduction whose Mazur–Tate elements do not exhibit such patterns. We will look for a theoretic explanation on how these two distinct cases arise in our follow-up project.

Table 1. λ-invariants for p-adic Mazur–Tate elements of level n of some elliptic curves with additive reduction at $p = 3$.

Isogeny class	$n = 1$	**2**	**3**	**4**	**5**	**6**	**7**	m
$27a, 54a$	1	7	25	79	241	727	2185	$3^m - 2$
$36a, 54b, 90a,$ $90b, 108a$	2	8	26	80	242	728	2186	$3^m - 1$
$45a, 63a, 72a,$ $90c, 99a, 99b,$ $99d$	1	3	9	27	81	243	729	3^{m-1}
$99c$	∞	6	18	54	162	486	1458	$2 \cdot 3^{m-1}$
$153a$	1	∞	11	39	101	309	911	$3^{m-1} + q_{m-1} + 6$ (m even) $3^{m-1} + q_{m-1}$ (m odd)
$153c$	∞	5	21	47	147	425	1281	$3^{m-1} + q_m$ (m even) $3^{m-1} + q_m + 6$ (m odd)
$153d$	2	6	20	60	182	546	1640	q_{m+1}

Table 2. λ-invariants for p-adic Mazur–Tate elements of level n of some elliptic curves with additive reduction at $p = 5$.

Isogeny class	$n = 1$	**2**	**3**	**4**	**5**	m
$50b, 75c$	4	24	124	624	3124	$5^m - 1$
$75b, 100a, 150c$	2	10	50	250	1250	$2 \cdot 5^{m-1}$
$50a, 75a, 150b, 175c$	3	15	75	375	1875	$3 \cdot 5^{m-1}$
$175b$	4	12	52	252	1252	$2 \cdot 5^{m-1} + 2$
$175a$	2	6	26	126	626	$5^{m-1} + 1$
$150a$	1	5	25	125	625	5^{m-1}
$225a$	1	8	37	188	937	$5^{m-1} + 3 \cdot q_{m-1} + 3$ (m even) $5^{m-1} + 3 \cdot q_{m-1}$ (m odd)
$225b$	4	17	88	437	2188	$3 \cdot 5^{m-1} + 3 \cdot q_{m-1} + 2$ (m even) $3 \cdot 5^{m-1} + 3 \cdot q_{m-1} + 1$ (m odd)

Table 3. λ-invariants for p-adic Mazur–Tate elements of level n of some elliptic curves with additive reduction at $p = 7$.

Isogeny class	$\boldsymbol{n = 1}$	**2**	**3**	**4**	$\boldsymbol{m}$
$49a, 245b, 294e, 294f, 392b, 441a$	5	35	245	1715	$5 \cdot 7^{m-1}$
$98a, 147a, 294c, 392d$	3	21	147	1029	$3 \cdot 7^{m-1}$
$147b, 196b, 294a, 392e, 441e$	4	28	196	1372	$4 \cdot 7^{m-1}$
$147c, 294b$	1	7	49	343	7^{m-1}
$245a, 294d, 294g, 441d$	2	14	98	686	$2 \cdot 7^{m-1}$
$196a, 392f$	2	8	50	344	$7^{m-1} + 1$
$245c, 392a, 441c$	4	22	148	1030	$3 \cdot 7^{m-1} + 1$
$392c, 441b$	3	15	99	687	$2 \cdot 7^{m-1} + 1$
$441f$	3	9	51	345	$7^{m-1} + 2$

In the following table, we give the values of λ-invariants of the Mazur–Tate elements attached to Δ at the non-ordinary primes $p = 3, 5, 7$ that we have been able to compute, We note that the values for p=3 agree with those for the isogeny classes 27a and 54a in Table 1, as predicted by Theorem 4.1.

Table 4. λ-invariants of p-adic Mazur–Tate elements of level n associated to Δ.

$\boldsymbol{p}$	$\boldsymbol{n = 1}$	**2**	**3**	**4**	$\boldsymbol{m}$
3	1	7	25	79	$3^m - 2$
5	4	24	124	624	$5^m - 1$
7	6	48	342	—	$7^m - 1$

If $E/\mathbb{Q}$ is an elliptic curve with additive reduction at p, we recall it has either potentially good or potentially multiplicative reduction at p. We have observed that the formulae of λ-invariants we have found in Tables 1 to 3 seem to be related to the potential reduction type of the curves. We give in Tables 5, 6 and 7 the potential reduction types that we have been able to work out for the curves we have studied with $p = 3, 5$ and 7 respectively. When $p = 3$ and $E = 54a$ or $54b$, we have found that these curves have potentially good reduction. But we have unable to find the number field where good reduction is attained. As a result, we do not know whether it has potentially good ordinary reduction or potentially good supersingular reduction.

TABLE 5. Potential reduction of elliptic curves having additive reduction at $p = 3$.

Isogeny class	$\boldsymbol{K = \mathbb{Q}(\cdot)}$	**Reduction of $\boldsymbol{E/K}$ at a prime ideal of $\boldsymbol{K}$ lying above $\boldsymbol{p}$**
$27a$	$54^{\frac{1}{12}}$	supersingular
$36a$	$3^{\frac{1}{4}}$	supersingular
$45a$	$3^{\frac{1}{2}}$	split multiplicative
$54a$	??	good
$54b$	??	good
$63a$	$3^{\frac{1}{2}}$	non-split multiplicative
$72a$	$3^{\frac{1}{2}}$	split multiplicative
$90a$	$3^{\frac{1}{4}}$	supersingular
$90b$	$3^{\frac{1}{4}}$	supersingular
$90c$	$3^{\frac{1}{2}}$	non-split multiplicative
$99a$	$3^{\frac{1}{4}}$	supersingular
$99b$	$3^{\frac{1}{2}}$	split multiplicative
$99c$	$3^{\frac{1}{4}}$	supersingular
$99d$	$3^{\frac{1}{2}}$	good ordinary
$108a$	$54^{\frac{1}{12}}$	supersingular
$153a$	$6^{\frac{1}{4}}$	supersingular
$153c$	$3^{\frac{1}{2}}$	supersingular
$153d$	$3^{\frac{1}{4}}$	supersingular

Table 6. Potential reduction of elliptic curves having additive reduction at $p = 5$.

Isogeny class	$K = \mathbb{Q}(\cdot)$	**Reduction of E/K at a prime ideal of K lying above p**
$50a$	$5^{\frac{1}{3}}$	supersingular
$50b$	$5^{\frac{1}{6}}$	supersingular
$75a$	$5^{\frac{1}{3}}$	supersingular
$75b$	$5^{\frac{1}{2}}$	split multiplicative
$75c$	$5^{\frac{1}{6}}$	supersingular
$100a$	$5^{\frac{1}{2}}$	non-split multiplicative
$150a$	$5^{\frac{1}{4}}$	good ordinary
$150b$	$5^{\frac{1}{4}}$	good ordinary
$150c$	$5^{\frac{1}{2}}$	non-split multiplicative
$175a$	$5^{\frac{1}{4}}$	good ordinary
$175b$	$5^{\frac{1}{2}}$	non-split multiplicative
$175c$	$5^{\frac{1}{4}}$	good ordinary
$225a$	$5^{\frac{1}{6}}$	supersingular
$225b$	$5^{\frac{1}{3}}$	supersingular

Table 7. Potential reduction of elliptic curves having additive reduction at $p = 7$.

Isogeny class	$\boldsymbol{K = \mathbb{Q}(\cdot)}$	**Reduction of $\boldsymbol{E/K}$ at a prime ideal of $\boldsymbol{K}$ lying above $\boldsymbol{p}$**
$49a$	$7^{\frac{1}{4}}$	supersingular
$98a$	$7^{\frac{1}{2}}$	non-split multiplicative
$147a$	$7^{\frac{1}{2}}$	split multiplicative
$147b$	$7^{\frac{1}{3}}$	good ordinary
$147c$	$7^{\frac{1}{6}}$	good ordinary
$196a$	$7^{\frac{1}{6}}$	good ordinary
$196b$	$7^{\frac{1}{3}}$	good ordinary
$245a$	$7^{\frac{1}{4}}$	supersingular
$245b$	$7^{\frac{1}{4}}$	supersingular
$245c$	$7^{\frac{1}{2}}$	non-split multiplicative
$294a$	$7^{\frac{1}{3}}$	good ordinary
$294b$	$7^{\frac{1}{6}}$	good ordinary
$294c$	$7^{\frac{1}{2}}$	split multiplicative
$294d$	$7^{\frac{1}{3}}$	good ordinary
$294e$	$7^{\frac{1}{6}}$	good ordinary
$294f$	$7^{\frac{1}{4}}$	supersingular
$294g$	$7^{\frac{1}{4}}$	supersingular
$392a$	$7^{\frac{1}{2}}$	split multiplicative
$392b$	$7^{\frac{1}{6}}$	good ordinary
$392c$	$7^{\frac{1}{3}}$	good ordinary
$392d$	$7^{\frac{1}{2}}$	non-split multiplicative
$392e$	$7^{\frac{1}{3}}$	good ordinary
$392f$	$7^{\frac{1}{6}}$	good ordinary
$441a$	$7^{\frac{1}{6}}$	good ordinary
$441b$	$7^{\frac{1}{3}}$	good ordinary
$441c$	$7^{\frac{1}{2}}$	split multiplicative
$441d$	$7^{\frac{1}{4}}$	supersingular
$441e$	$7^{\frac{1}{3}}$	good ordinary
$441f$	$7^{\frac{1}{6}}$	good ordinary

References

[Del98] Daniel Delbourgo, *Iwasawa theory for elliptic curves at unstable primes*, Compositio Math. **113** (1998), no. 2, 123–153.

[Del02] ———, *On the p-adic Birch, Swinnerton-Dyer conjecture for non-semistable reduction*, J. Number Theory **95** (2002), no. 1, 38–71.

[Eme11] Matthew Emerton, *p-adic families of modular forms (after Hida, Coleman, and Mazur)*, no. 339, 2011, Séminaire Bourbaki. Vol. 2009/2010. Exposés 1012–1026, pp. Exp. No. 1013, vii, 31–61.

[Gre99] Ralph Greenberg, *Iwasawa theory for elliptic curves*, Arithmetic theory of elliptic curves (Cetraro, 1997), Lecture Notes in Math., vol. 1716, Springer, Berlin, 1999, pp. 51–144.

[Hid86] Haruzo Hida, *Iwasawa modules attached to congruences of cusp forms*, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 2, 231–273.

[MT87] B. Mazur and J. Tate, *Refined conjectures of the "Birch and Swinnerton-Dyer type"*, Duke Math. J. **54** (1987), no. 2, 711–750.

[Pol03] Robert Pollack, *On the p-adic L-function of a modular form at a supersingular prime*, Duke Math. J. **118** (2003), no. 3, 523–558.

[PR03] Bernadette Perrin-Riou, *Arithmétique des courbes elliptiques à réduction supersingulière en p*, Experiment. Math. **12** (2003), no. 2, 155–186.

[PW11] Robert Pollack and Tom Weston, *Mazur-Tate elements of nonordinary modular forms*, Duke Math. J. **156** (2011), no. 3, 349–385.

[Roh88] David E. Rohrlich, *L-functions and division towers*, Math. Ann. **281** (1988), no. 4, 611–632.

[SD73] H. P. F. Swinnerton-Dyer, *On l-adic representations and congruences for coefficients of modular forms*, Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972), Springer, Berlin, 1973, pp. 1–55. Lecture Notes in Math., Vol. 350.

[SD77] ———, *On l-adic representations and congruences for coefficients of modular forms. II*, Modular functions of one variable, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), Springer, Berlin, 1977, pp. 63–90. Lecture Notes in Math., Vol. 601.

[SD88] ———, *Congruence properties of $\tau(n)$*, Ramanujan revisited (Urbana-Champaign, Ill., 1987), Academic Press, Boston, MA, 1988, pp. 289–311.

[Ser69] Jean-Pierre Serre, *Une interprétation des congruences relatives à la fonction τ de Ramanujan*, Séminaire Delange-Pisot-Poitou: 1967/68, Théorie des Nombres, Fasc. 1, Exp. 14, Secrétariat mathématique, Paris, 1969, p. 17.

[Ser73a] ———, *Congruences et formes modulaires [d'après H. P. F. Swinnerton-Dyer]*, Séminaire Bourbaki, 24e année (1971/1972), Exp. No. 416, Springer, Berlin, 1973, pp. 319–338. Lecture Notes in Math., Vol. 317.

[Ser73b] ———, *Formes modulaires et fonctions zêta p-adiques*, Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972), 1973, pp. 191–268. Lecture Notes in Math., Vol. 350.

[Sil86] Joseph H. Silverman, *The arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 106, Springer-Verlag, New York, 1986.

[Suj20] Ramdorai Sujatha, *Selmer groups in Iwasawa theory and congruences*, Philos. Trans. Roy. Soc. A **378** (2020), no. 2163, 20180442, 7.

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