ECE 450 Notes

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1 Introduction to Probability

Random Experiment: any well defined procedure that produces an observable outcome that cannot be perfectly predicted:

(a) an outcome is called a sample point, it cannot be further decomposed

(b) outcomes are disjoint; i.e. **mutually exclusive**

Ω = Sample Space
Event: subset of the sample space

Given the set: $\Omega = \{X_1, X_2, ..., X_n\}$, there are a total of 2^n possible outcomes

1.1 Operation on Sets

1. Union: $A \cup B$
2. Intersection: $A \cap B$
3. Compliment: A^C

(a) Two events are said to be mutually exclusive if $A \cap B = \emptyset$ can be extended to N events

(b) Events $A_1, A_2, ..., A_n$ are said to be mutually exclusive (span the entire sample space) if $A_i \cap A_j = \emptyset$ for all $i \neq j$

1.2 The 3 Axioms of Probability

I $P(A) > 0$

II $P(\Omega) = 1$

III If two events are mutually exclusive, then $P(A + B) = P(A) + P(B)$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) \tag{1}$$

$$P(A^C) = 1 - P(A) \tag{2}$$

1.3 Subsets

$A \subset B$: A is a subset of B, $P(A) \leq P(B)$

$$P(A+B) = P(A) + P(B) - P(AB) \tag{3}$$

$$P(A+B+C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC) \tag{4}$$

2 Conditional Probability

$P(A|B)$: read as "A given B" (can be independent, i.e. no relationship)

$$P(A|B) = \frac{P(AB)}{P(B)} \tag{5}$$

$$P(B|A) = \frac{P(AB)}{P(A)} \tag{6}$$

If A and B are *independent*, then

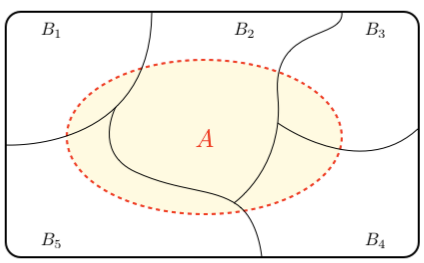
$$P(AB) = P(A) \cdot P(B) \tag{7}$$

Bayes Rule

$$P(A|B)P(B) = P(B|A)P(A) \tag{8}$$

3 Total Probability

Suppose events $A_1, A_2, \ldots, A_n$ are mutually exclusive and exhaustive



$$\begin{aligned} P(B) &= P(A_1B) + P(A_2B) + ... + P(A_nB) \\ &= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + ...P(B|A_n)P(A_n) \\ &= P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i) \end{aligned}$$

4 Combinatorics

Deals with counting, sampling, ordering, selection,...

4.1 Basic Concept of Counting

Combined number of outcomes: $m \times n \times k \ldots$
Example: How many passwords can you create such that the first two are small letters, next one is a capital letter, and the last four are numbers?

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 = n$$

Example: Given a hacker generates 10^8 passwords (can be repeated) what is the probability that at least one password will match yours?

$$P(passwordMatches) = \frac{1}{N}$$
$$P(passwordDoesNotMatch) = 1 - \frac{1}{N}$$
$$P(allPasswordsDoNotMatch) = \left(1 - \frac{1}{N}\right)^{10^8}$$
$$P(atLeastOneMatch) = 1 - \left(1 - \frac{1}{N}\right)^{10^8}$$

4.2 Permutations

Example: Suppose we have n distinguishable objects. How many ways can you arrange the objects? *(without replacement)*

$$n!$$

with replacement

$$n^n$$

Now suppose we are interested in choosing k objects out of n ($k \leq n$), *without replacement*; order is *important*

$$n(n-1)(n-2)...(n-k+1)$$

$$= \frac{n!}{(n-k)!} \tag{9}$$

with replacement (given k terms)

$$n(n)(n)...(n) = n^k \tag{10}$$

Example: You have a bookshelf with 4 math books, 3 physics, 2 economics, and 1 history book. How many ways can you arrange the bookshelf such that similar books are bundled/adjancent?
Ways of arranging the books:

$$(4!)(3!)(2!)(1!)$$

Now account for the ways you can arrange the bundles:

$$(4!)(3!)(2!)(1!)(4!)$$

Example: Ways of creating a password of 5 numbers without replacement:

$$\frac{10!}{(10-5)!}$$

Combination: How many ways can you select k objects out of n distinguishable objects without replacement (n choose k); order does **not** matter

$$\frac{n!}{(n-k)!k!} = \binom{n}{k} \tag{11}$$

*note that $\binom{n}{n-k}$ is the same thing

Example: Given a deck of cards, you choose 5 cards. What is the probability you get two aces?

$$\frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}$$

Example: A team has 25 players, 15 are position players, 10 are pitchers. The lineup consists of 9 players.

(a) How many lineups can the manager create?
$\binom{10}{1}\binom{15}{8} = \frac{(10)(15!)}{(7!)(8!)} = 64350$

(b) How many batting orders can he create?
64350(9!)

Example: Suppose you pick 3 cardsfrom a deck. What is the probability of getting atleast an ace?

$$= 1 - P(noAce)$$
$$= 1 - \frac{\binom{48}{3}}{\binom{52}{3}}$$

Example: Suppose you have 2 letters, A and B. How many sequences can you generate having 3 As and 5 Bs.

$$= \binom{8}{3} \times \binom{5}{5}$$

Example: You have 4 mailboxes and 8 indistinguishable balls. (a) How many ways can you distribute the balls so that no box is empty. Let x_1 be the number of balls in box 1. We know $x_1 + x_2 + x_3 + x_4 = 8$. x_i is > 0 for all i.

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Since there are 7 spaces, we choose three to split them up in any box.

$$\binom{n-1}{k-1}$$

(b) How many ways can you distribute the balls into the boxes? A box can be empty.

$$\text{let } y_i = x_i + 1 \text{ so}$$
$$y_1 + y_2 + y_3 + y_4 = 12$$

$$\binom{11}{3}$$

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + ... + \binom{n}{n} = 2^n$$

4.3 Multinomial Coefficients

Suppose we have n distinguishable objects. We want to break them into k groups. G_1 has n_1 objects, G_2 has n_2 objects, G_k has n_k objects. $n_1 + n_2 + ... + n_k = n$. How many ways can we do it?

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}...\binom{n_1-...-n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1!n_2!...n_k!}$$

5 Random Variables

A random variable (X) is a function mapping the outcomes of a random experiment into the real line. Mapping must be 1 to 1 *or* many to 1.Mapping 1 to many is *unacceptable.*
Example: Suppose you toss a coin twice.

$$\Omega = HH, HT, TH, TT$$

X is a random variable as the number of heads. X can take the value 0, 1, 2

$$P(X = 0) = \frac{1}{4}$$

5.1 Classification of Random Variables

(a) ***Discrete*** - said to be discrete if it can take a finite or infinitely countable (i.e. integer values) number of values

(b) ***Continuous*** - continuous if it can take infinitely uncountable number of values (i.e. real numbers) number of values

(c) ***Mixed***

5.2 How do we describe a discrete RV

PMF - probability mass function $P(X = k)$ for all possible values of k

$$\sum_{k}^{max} P(X = K) = 1$$

CDF - Cumulative Distributive Function

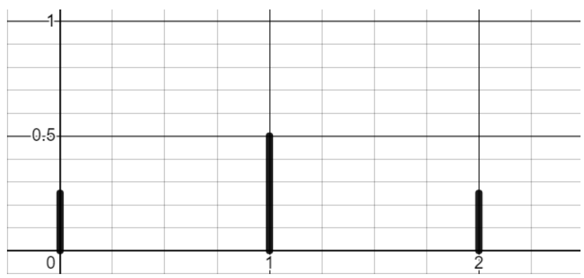
$$F_X(x) = P(X \leq x)$$

I Cannot be negative

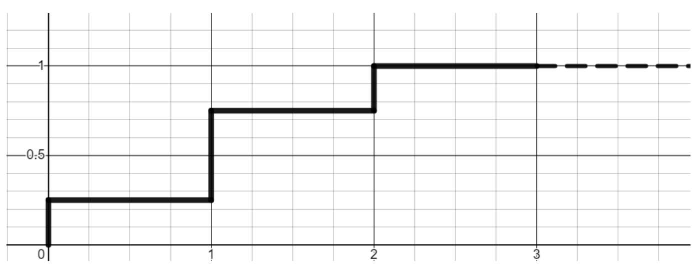
II Cannot slope downwards

III Must be a stair function and the number of stairs equals the number of values the random variable takes on

Example: Suppose you toss a coin twice $\Omega = HH, HT, TH, TT$. Let $X =$ number of heads. HH = 2, HT = 1, TH = 1, TT = 0. X is a disccrete RV taking on values of 0, 1, 2. Sketch the PMF and CDF.
PDF:



CDF:



$$F_X(1) = 3/4$$
$$F_X(0.5) = 1/4$$
$$P(X = 0.5) = 0$$

*Note that $<$ is denoted by a '*parenthises*' on a number line and $\leq$ is denoted by '*brackets*'
Example Cases:

1. $P(x_1 < x \leq x_2) = F_X(x_2) - F_X(x_1)$
2. $P(x_1 \leq x \leq x_2) = F_X(x_2) - F_X(x_1) + P(X = x_1)$
3. $P(x_1 \leq x < x_2) = F_X(x_2) - F_X(x_1) + P(X = x_1) - P(X = x_2)$
4. $P(x_1 < x < x_2) = F_X(x_2) - F_X(x_1) - P(X = x_2)$

5.3 Famous Discrete Random Variables

I ***Bernoulli*** -
Can take 1 of 2 possible values

II ***Binomial*** -
Suppose we have a bernoulli experiment repeated n times (n is not random). Let $X =$ number of successes in n trials

$$PMF = P(X = k) \text{ where } k = 0, 1, \ldots, n$$

Example: Suppose we toss a fair coin 3 times, let X be the number of heads. Find the PMF of X

X is a discrete random variable that can take the values: 0, 1, 2, & 3
$P(X = 3) = P(HHH) = 1/8$

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \tag{12}$$

$k = 0, 1, \ldots, n$

where p = probability of success in a single trial
k = number of successes in n trials

III ***Geometric*** -
Number of trials until the first success; will be represented as a non-increasing PDF

$$P(X = k) = (1-p)^{k-1} p \tag{13}$$

where $k = 0, 1, \ldots$

IV ***Pascal*** -
The number of trials until getting m successes

$$P(X = k) = \binom{k-1}{m-1} p^{m-1} (1-p)^{k-1-(m-1)}$$

$$\binom{k-1}{m-1} p^m (1-p)^{k-m} \tag{14}$$

where $k = m, m+1, m+2, \ldots$

V ***Poisson*** -
X is said to be a poisson with parameter λ (average)

$$P(X = k) = \frac{e^{-\lambda}(\lambda^k)}{k!} \tag{15}$$

where $\lambda = np$

VI ***Hyper-Geometric*** **Example**: Given a bowl, we have M red balls and N blue balls.
Pick L balls without replacement
Let X be the number of red balls drawn:

$$P(X = k) = \frac{\binom{M}{K}\binom{N}{L-K}}{\binom{M+N}{L}} \tag{16}$$