

1 Introduction to Probability

Random Experiment: any well defined procedure that produces an observable outcome that cannot be perfectly predicted.

- (a) an outcome is called a sample point, it cannot be further decomposed
- (b) outcomes are disjoint; i.e. **mutually exclusive**

Ω = Sample Space

Event: subset of the sample space

Given the set: $\Omega = \{X_1, X_2, \dots, X_n\}$, there are a total of 2^n possible outcomes

1.1 Operation on Sets

- 1. Union: $A \cup B$
- 2. Intersection: $A \cap B$
- 3. Compliment: A^C
- (a) Two events are said to be mutually exclusive if $A \cap B = \emptyset$ can be extended to N events
- (b) Events A_1, A_2, \dots, A_n are said to be mutually exclusive (span the entire sample space) if $A_i \cap A_j = \emptyset$ for all $i \neq j$

1.2 The 3 Axioms of Probability

- I $P(A) > 0$
- II $P(\Omega) = 1$
- III if two events are mutually exclusive, $P(A + B) = P(A) + P(B)$

$$P\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i)$$

Also

$$P(A^C) = 1 - P(A)$$

1.3 Subsets

$A \subset B$: A is a subset of B, $P(A) \leq P(B)$

$$P(A + B) = P(A) + P(B) - P(AB) \quad (1)$$

$$P(A + B + C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC) \quad (2)$$

2 Conditional Probability

$P(A|B)$: read as "A given B" (can be independent, i.e. no relationship)

$$P(A|B) = \frac{P(AB)}{P(B)}$$

Or

$$P(B|A) = \frac{P(AB)}{P(A)}$$

If A and B are *independent*, then

$$P(AB) = P(A) \cdot P(B)$$

Bayes Rule

$$P(A|B)P(B) = P(B|A)P(A)$$

3 Total Probability

Suppose events A_1, A_2, \dots, A_n are mutually exclusive and exhaustive

$$\begin{aligned} P(B) &= P(A_1B) + P(A_2B) + \dots + P(A_nB) \\ &= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \dots + P(B|A_n)P(A_n) \\ &= P(B) = \sum_{i=1}^n P(B|A_i)P(A_i) \end{aligned}$$

4 Combinatorics

Deals with counting, sampling, ordering, selection,...

4.1 Basic Concept of Counting

Combined number of outcomes: $m \times n \times k \dots$

Example: How many passwords can you create such that the first two are small letters, next one is a capital letter, and the last four are numbers?

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 = N$$

Example: Given a hacker generates 10^8 passwords (can be repeated) what is the probability that at least one password will match yours?

$$P(\text{passwordMatches}) = \frac{1}{N}$$

$$P(\text{passwordDoesNotMatch}) = 1 - \frac{1}{N}$$

$$P(\text{allPasswordsDoNotMatch}) = \left(1 - \frac{1}{N}\right)^{10^8}$$

$$P(\text{atLeastOneMatch}) = 1 - \left(1 - \frac{1}{N}\right)^{10^8}$$

4.2 Permutations

Example: Suppose we have n distinguishable objects. How many ways can you arrange the objects? (without replacement)

$$n!$$

with replacement

$$n^n$$

Now suppose we are interested in choosing k objects out of n ($k \leq n$), **without replacement**; order is **important**

$$n(n-1)(n-2)\dots(n-k+1)$$

$$= \frac{n!}{(n-k)!}$$

with replacement (given k terms)

$$n(n)(n)\dots(n) = n^k$$

Example: You have a bookshelf with 4 math books, 3 physics, 2 economics, and 1 history book. How many ways can you arrange the bookshelf such that similar books are bundled/adjacent?

Ways of arranging the books:

$$(4!)(3!)(2!)(1!)$$

Now account for the ways you can arrange the bundles:

$$(4!)(3!)(2!)(1!)(4!)$$

Example: Ways of creating a password of 5 numbers without replacement:

$$\frac{10!}{(10-5)!}$$

Combination: How many ways can you select k objects out of n distinguishable objects without replacement (n choose k); order does **not** matter

$$\frac{n!}{(n-k)!k!}$$

represented as:

$$\binom{n}{k}$$

note that $\binom{n}{n-k}$ is the same thing

Example: Given a deck of cards, you choose 5 cards. What is the probability you get two aces?

$$\frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}$$

Example: A team has 25 players, 15 are position players, 10 are pitchers. The lineup consists of 9 players.

(a) How many lineups can the manager create?

$$\binom{10}{1} \binom{15}{8} = \frac{(10)(15!)}{(7!)(8!)} = 64350$$

(b) How many batting orders can he create?

$$64350(9!)$$

Example: Suppose you pick 3 cards from a deck. What is the probability of getting at least an ace?

$$\begin{aligned} &= 1 - P(\text{no Ace}) \\ &= 1 - \frac{\binom{48}{3}}{\binom{52}{3}} \end{aligned}$$

Example: Suppose you have 2 letters, A and B. How many sequences can you generate having 3 As and 5 Bs.

$$= \binom{8}{3} \times \binom{5}{5}$$

Example: You have 4 mailboxes and 8 indistinguishable balls. How many ways can you distribute the balls so that no box is empty. Let x_1 be the number of balls in box 1. We know $x_1 + x_2 + x_3 + x_4 = 8$. x_i is ≥ 0 for all i .

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Since there are 7 spaces, we choose three to split them up in any box.

Essentially

$$\binom{n-1}{k-1}$$

How many ways can you distribute the balls into the boxes. A box can be empty.

let $y_i = x_i + 1$ so

$$y_1 + y_2 + y_3 + y_4 = 12$$

$$= \binom{11}{3}$$

$$\sum_{k=0}^n \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$$

4.3 Multinomial Coefficients

Suppose we have n distinguishable objects. We want to break them into k groups. G_1 has n_1 objects, G_2 has n_2 objects, G_k has n_k objects. $n_1 + n_2 + \dots + n_k = n$. How many ways can we do it?

$$\begin{aligned} &= \binom{n}{n_1} \binom{n-n_1}{n_2} \binom{n-n_1-n_2}{n_3} \dots \binom{n_1-\dots-n_{k-1}}{n_k} \\ &= \frac{n!}{n_1!n_2!\dots n_k!} \end{aligned}$$

5 Random Variables

A random variable (X) is a function mapping the outcomes of a random experiment into the real line. Mapping must be 1 to 1 *or* many to 1. Mapping 1 to many is *unacceptable*.

Example: Suppose you toss a coin twice.

$$\Omega = HH, HT, TH, TT$$

X is a random variable as the number of heads. X can take the value 0, 1, 2

$$P(X = 0) = \frac{1}{4}$$

5.1 Classification of Random Variables

Discrete, continuous, or mixed

a R.V. is said to be discrete if it can take a finite or infinitely countable (i.e. integer values) number of values

a R.V. is continuous if it can take infinitely uncountable number of values (i.e. real numbers) number of values

How do we describe a discrete RV

PMF - probability mass function $P(X = k)$ for all possible values of k

$$\sum_k^{max} P(X = K) = 1$$

CDF - Cumulative Distributive Function

$$F_X(x) = P(X \leq x)$$

Example: Suppose you toss a coin twice $\Omega = HH, HT, TH, TT$. Let $X =$ number of heads. HH to 2, HT to 1, TH to 1, TT to 0. X is a discrete RV taking on values of 0, 1, 2. Sketch the PMF and CDF.