Homework for Chapter 5.5-5.6

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Chapter 5.5

Problem 4

We are given the recursive formula $d_k = k(d_{k-1})^2$ for all $k \in \mathbb{Z}$, $k \geq 1$. The first term is given as $d_0 = 3$, so the second term is given by $d_1 = 1 \cdot (d_0)^2 = 9$. Then, the third term d_2 is given by $d_2 = 2 \cdot (d_1)^2 = 162$. The fourth term d_3 is given by $d_3 = 3 \cdot (d_2)^2 = 78,732$. So, the first four terms of the series are $d_0 = 3$, $d_1 = 9$, $d_2 = 162$, and $d_3 = 78,732$.

Problem 14

We are given that d_0, d_1, d_2, ... are defined by the formula $d_n = 3^n - 2^n$ for $\forall n \in \mathbb{Z}$, $n \geq 0$. Then, the recurrence relation is given by $d_k = 5d_{k-1} - 6d_{k-2}$. So, we can use the given formula to find d_k, d_{k-1}, and d_{k-2}, which are given by $d_k = 3^k - 2^k$, $d_{k-1} = 3^{k-1} - 2^{k-1}$, and $d_{k-2} = 3^{k-2} - 2^{k-2}$. Next, we can substitute these values into the recurrence relation to get $d_k = 5(3^{k-1} - 2^{k-1}) - 6(3^{k-2} - 2^{k-2})$. Now, we can rearrange this to get $d_k = 5(3^k \cdot 3^{-1} - 2^k \cdot 2^{-1}) - 6(3^k \cdot 3^{-2} - 2^k \cdot 2^{-2})$ and write it more suggestively as $d_k = 5(\frac{3^k}{3} - \frac{2^k}{2}) - 6(\frac{3^k}{3^2} - \frac{2^k}{2^2})$. Then, by basic algebra, we can factor out a 3^k from the first term and a 2^k from the second term after distributing to get $3^k(\frac{5}{3} - \frac{6}{9}) - 2^k(\frac{5}{2} - \frac{6}{4})$, which simplifies to $3^k(\frac{15-6}{9}) - 2^k(\frac{10-6}{4})$ after giving terms common denominators. Finally, we can see that this simplifies to $3^k - 2^k$, and $d_k = 3^k - 2^k$, which is what we were trying to show. Therefore, this sequence must satisfy the given recurrence relation.

Problem 19c

We are asked to show that $s_k \leq 2s_{k-2} + 3$ for $\forall k \in \mathbb{Z}$, $k \geq 3$ for the Tower of Hanoi problem with four poles, where s_k is the minimum number of moves needed to transfer the entire tower of k disks from the left-most to the right-most pole. First, we must find the recurrence relation for the four-pole Tower of Hanoi. To do so, we can begin by iteration and looking for a pattern: $s_1 = 1$ since we can move the single disk from the first pole to the last in a single move. Next, s_2 can be achieved by moving the top disk to the second pole, the bottom disk to the fourth pole, and then the top disk to the fourth pole, giving a total of $s_2 = 3$. Next, for s_3, we can move the smallest disk to the third pole, the medium disk to the second pole, the biggest disk to the fourth pole, then the medium disk to the fourth pole and finally the small disk to the fourth pole, giving a total of $s_3 = 5$. Now, we can start to see the pattern, so we will generalize this for s_n: first, we must move $n-2$ disks to the second pole which takes s_{n-2} moves, and then move the second to last disk to the unoccupied third pole which is one move, and then the final disk to the fourth pole which is one move, and then the second to last disk to the fourth pole which is again one move, and then the stack of $n-2$ onto the fourth pole which takes another s_{n-2} moves. Therefore, we have generated a recurrence relation that says $s_n = 2s_{n-2} + 3$. So, since this is an algorithm for the least number of moves, we can rule out all $s_k > 2s_{k-2} + 3$ since they will be less efficient. However, we cannot prove that this algorithm is the most efficient, and we don't need to prove that it is for the context of this problem. What we have shown is sufficient to say that $s_k \leq 2s_{k-2} + 3$ for $\forall k \in \mathbb{Z}$, $k \geq 3$ since we know that s_k cannot be greater than $2s_{k-2} + 3$ because we have proven a more efficient algorithm, but that does not rule out the situation of $s_k < 2s_{k-2} + 3$.

Problem 28

We are asked to prove that $F_{k+1}^2 - F_k^2 - F_{k-1}^2 = 2F_kF_{k-1}$ for $\forall k \in \mathbb{Z}$, $k \geq 1$. First, we can regroup the left side of the equation as $(F_{k+1}^2 - F_{k-1}^2) - F_k^2$. Then, by basic algebra we know that $a^2 - b^2 = (a+b)(a-b)$, so we can use this identity to write the left side as $(F_{k+1} + F_{k-1})(F_{k+1} - F_{k-1}) - F_k^2$. Then, by the definition of a Fibonacci sequence, we know that $F_{k+1} = F_k + F_{k-1}$ and rearranging, $F_k = F_{k+1} - F_{k-1}$. Then, we can use this result to substitute into our previous equation to get $(F_{k+1} + F_{k-1})(F_k) - F_k^2 = 2F_kF_{k-1}$ and factor out a F_k term to get the left side of the equation to be $F_k[(F_{k+1} + F_{k-1}) - F_k]$. Next, we can rearrange the internal terms to get $F_k[(F_{k+1} - F_{k-1}) + F_k]$ Then, we can again use the definition of the Fibonacci sequence, $F_{k+1} = F_k + F_{k-1}$, and then rearrange to get $F_{k-1} = F_{k+1} - F_k$ and substitute this into the left side of our equation to get $F_k(F_{k-1} + F_{k-1})$. Finally, adding the two F_{k-1} terms inside the parentheses gives $2F_kF_{k-1}$, which is what we were trying to prove. Therefore, $F_{k+1}^2 - F_k^2 - F_{k-1}^2 = 2F_kF_{k-1}$. Q.E.D.

Problem 31

We are asked to use strong mathematical induction to prove that $F_n < 2^n$ for $\forall n \in \mathbb{Z}$, $n \geq 1$.

PROOF: First, we must show that the base cases are true. That is, $F_1 = 1 < 2^1$, which is true since $1 < 2$, and $F_2 = 2 < 2^2$, which is also true since $2 < 4$. Then, suppose $F_k < 2^k$ is true for all $k \leq n$. This is our inductive hypothesis. Then, we must show that $F_{k+1} < 2^{k+1}$. We can rewrite the right side of the inequality as $2 \cdot 2^k$ by the properties of exponents, which we can then rewrite again as $2^k + 2^k$ by basic algebra. Now, we must show $F_{k+1} < 2^k + 2^k$. From the definition of Fibonacci numbers, we know that $F_{k+1} = F_k + F_{k-1}$, and from the inductive hypothesis, we know that $F_k < 2^k$, and $F_{k-1} < 2^{k-1}$, so $F_k + F_{k-1} < 2^k + 2^{k-1}$, and therefore $F_{k+1} < 2^k + 2^{k-1}$. Now, we can see that $F_{k+1} < 2^k + 2^{k-1}$, and from what we are trying to show, $F_{k+1} < 2^k + 2^k$, $2^k + 2^{k-1} < 2^k + 2^k$ by basic algebra since $2^{k-1} < 2^k$ for $\forall k \in \mathbb{Z}$, $k \geq 1$. Therefore, since $2^k + 2^{k-1} < 2^k + 2^k$, we know that $F_{k+1} < 2^k + 2^k$, which is what we were trying to prove. Thus, by strong mathematical induction, $F_n < 2^n$ must be true for $\forall n \in \mathbb{Z}$, $n \geq 1$. Q.E.D.

Problem 37a

We are given that the account pays 3% in annual interest compounded monthly, and that S_n is the amount in the account at the end of the n^{th} month. First, we know that the monthly interest rate is given by $\frac{3\%}{12} = 0.25\%$ per month. Then, we know that the recurrence relation for any particular month $k \geq 1$ is simply the amount at the end of the month before it plus that same amount multiplied by the monthly interest. That is, $S_k = S_{k-1} + 0.0025S_{k-1}$, which we can finally rewrite as $S_k = 1.0025S_{k-1}$.

Problem 39

We can start by solving for each value of t_n explicitly and trying to find a pattern. First, t_1 can only be obtained using one 1-cm block, so $t_1 = 1$. Then, t_2 can be obtained in two ways: one 2-cm block or two 1-cm blocks, so $t_2 = 2$. Next, t_3 can be obtained in three ways: three 1-cm blocks, one 1-cm block on top of a 2-cm block, and one 2-cm block on top of a 1-cm block. Then, t_4 can be obtained in six ways: one 4-cm block, two 2-cm blocks, four 1-cm blocks, one 2-cm block on top of two 1-cm blocks, one 2-cm block in between two 1-cm blocks, and one 2-cm block beneath two 1-cm blocks. Therefore, $t_4 = 6$. Finally, t_5 can be obtained in 10 ways: using five 1-cm blocks, one 4-cm block on top of a 1-cm block, one 1-cm block on top of a 4-cm block, three different combinations using two 2-cm blocks and one 1-cm block, and four different combinations using three 1-cm blocks and one 2-cm block. Therefore, $t_5 = 10$. Now, we can see the formula that is emerging: the final block of a tower of height n can be either a 1-cm, 2-cm, or 4-cm. Therefore, we can write $t_n = t_{n-1} + t_{n-2} + t_{n-4}$, which fits our pattern thus far and makes intuitive sense since it recursively defines the pattern. In other words, since the bottom block can only be one of our three choices, we can use the results from our previously calculated values for heights 1, 2, and 4 centimeters less and add them together to get the new value.

Problem 43

We are asked to show through induction that $|\sum_{i=1}^{n} a_i| \leq \sum_{i=1}^{n} |a_i|$. By the definition of summation, we can rewrite this in expanded form as $|a_1 + a_2 + a_3 + ... + a_n| \leq |a_1| + |a_2| + |a_3| + ... + |a_n|$.

PROOF: First, we must show that the base case is true. For the base case, we consider the case $n = 1$: $|a_1| \leq |a_1|$ is clearly true. Then, assume that $|a_1 + a_2 + a_3 + ... + a_k| \leq |a_1| + |a_2| + |a_3| + ... + |a_k|$ is true for some integer $k \geq 1$. This is our inductive hypothesis. We then must show that this inequality also holds for the case $k+1$; that is, $|a_1 + a_2 + a_3 + ... + a_k + a_{k+1}| \leq |a_1| + |a_2| + |a_3| + ... + |a_k| + |a_{k+1}|$. Then, we know that $\left|\sum_{i=1}^{k+1} a_i\right| = \left|(\sum_{i=1}^{k} a_i) + a_{k+1}\right| \leq \left|\sum_{i=1}^{k} a_i\right| + |a_{k+1}|$, which must be true because of the triangular inequality. Then, since $|a_1 + a_2 + a_3 + ... + a_k| \leq |a_1| + |a_2| + |a_3| + ... + |a_k|$ by our inductive hypothesis, we can add a $|x_{k+1}|$ term to both sides to get $|a_1 + a_2 + a_3 + ... + a_k| + |x_{k+1}| \leq |a_1| + |a_2| + |a_3| + ... + |a_k| + |x_{k+1}|$. Now, we can see that the left side of the previous inequality, $|a_1 + a_2 + a_3 + ... + a_k| + |x_{k+1}|$, is the same as the right side of our previously derived inequality $\left|\sum_{i=1}^{k+1} a_i\right| = \left|(\sum_{i=1}^{k} a_i) + a_{k+1}\right| \leq |\sum_{i=1}^{n} a_i| + |a_{k+1}|$, and that $\left|\sum_{i=1}^{k+1} a_i\right| = \left|(\sum_{i=1}^{k} a_i) + a_{k+1}\right| \leq \left|\sum_{i=1}^{k} a_i\right| + |a_{k+1}| \leq \sum_{i=1}^{k} |a_i| + |a_{k+1}| = \sum_{i=1}^{k+1} |a_i|$. Therefore, by mathematical induction, it must be true that $|\sum_{i=1}^{n} a_i| \leq \sum_{i=1}^{n} |a_i|$. Q.E.D.

Chapter 5.6

Problem 1c

We are given the series $3 + 3 \cdot 2 + 3 \cdot 3 + ... + 3 \cdot n + n$, where $n \in \mathbb{Z}$, and $n \geq 1$. It is easy to see that we can factor a 3 out of this expression to get $3(1 + 2 + 3 + ... + n) + n$, and we know that the series $1+2+3+...+n=\frac{n(n+1)}{2}$ as given by the initial formula. So, we can simply substitute this formula into our expression to get $3+3\cdot 2+3\cdot 3+...+3\cdot n+n=3(\frac{n(n+1)}{2})+n$. Simplifying the right hand side by distributing and getting common denominators, we end up with the expression $3+3\cdot 2+3\cdot 3+...+3\cdot n+n=\frac{3n^2+5n}{2}$.

Problem 2d

We are given the expression $2^n-2^{n-1}+2^{n-2}-2^{n-3}+...+(-1)^{n-1}\cdot 2+(-1)^n$ where $n\in\mathbb{Z}$, and $n\geq 1$. Then, we can rewrite this as $2^n\cdot 2^0-2^n\cdot 2^{-1}+2^n\cdot 2^{-2}-2^n\cdot 2^{-3}+...+(-1)^{n-1}\cdot\frac{2^n}{2^{n-1}}+(-1)^n\cdot\frac{2^n}{2^n}$ after recognizing the pattern. From this, we can factor out a 2^n from each term to get $2^n(1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+...+(-1)^{n-1}\cdot\frac{1}{2^{n-1}}+(-1)^n\cdot\frac{1}{2^n})$. Now, given the formula $1+r+r^2+...+r^n=\frac{r^{n+1}-1}{r-1}$, we can see that in our case, $r=-\frac{1}{2}$. Substituting this in on the right side of the equation, we get $2^n(\frac{(-\frac{1}{2})^{n+1}-1}{-\frac{1}{2}-1})$, which then simplifies to $2^n(\frac{(-1)^{n+1}(\frac{1}{2})^{n+1}-1}{-\frac{3}{2}})$. We can reduce this farther to $2^n(\frac{-(-1)^n\frac{1}{2^{n+1}}-1}{-\frac{3}{2}})$, and then to $2^n\cdot\frac{2}{3}((-1)^n\frac{1}{2^{n+1}}+1)$. Doing more rearranging, we get $\frac{2^{n+1}}{3}((-1)^n\frac{1}{2^{n+1}}+1)$, and then $\frac{2^{n+1}}{3}\cdot(-1)^n\cdot\frac{1}{2^{n+1}}+\frac{2^{n+1}}{3}$. Finally, we can simplify this to $\frac{(-1)^n}{3}+\frac{2^{n+1}}{3}$. Therefore, the formula is $2^n-2^{n-1}+2^{n-2}-2^{n-3}+...+(-1)^{n-1}\cdot 2+(-1)^n=\frac{(-1)^n}{3}+\frac{2^{n+1}}{3}$.

Problem 8

We are given the recurrence relation $f_k=f_{k-1}+2^k$, for $\forall k\in\mathbb{Z}$, $k\geq 2$. We are also given that $f_1=1$, so we can start to iterate this relation for f_2, f_3, and f_4. We find that $f_2=f_1+2^2=1+2^2$, and $f_3=f_2+2^3=1+2^2+2^3$, and $f_4=f_3+2^4=1+2^2+2^3+2^4$. Thus, we can guess that the pattern will be $f_n=1+2^2+2^3+...+2^n$. Then, if we add and subtract 2 from the equation, we can insert the missing 2^1 term to get $f_n=[1+2+2^2+2^3+...+2^n]-2$. We can see that the bracketed series is of the form $1+r+r^2+...+r^n=\frac{r^{n+1}-1}{r-1}$, where $r=2$, so we can substitute in for the formula on the right side to get $1+2+2^2+...+2^n=\frac{2^{n+1}-1}{2-1}$, which simplifies down to $1+2+2^2+...+2^n=2^{n+1}-1$. We can substitute this into our previous equation for f_n, $f_n=[1+2+2^2+2^3+...+2^n]-2$, to get $f_n=2^{n+1}-1-2$ and therefore we can see that the explicit formula for the given sequence is $f_n=2^{n+1}-3$.

Problem 23

Let the initial population be $P_0=50,000,000$. Then, we know that the population in future years can be calculated by $P_k=P_{k-1}+r\cdot P_{k-1}$, where P_{k-1} is the population from the previous year and r is the rate of growth. Therefore, we can substitute in $r=0.03$ and factor out a P_{k-1} from each term to get $P_k=P_{k-1}(1+0.03)=1.03\cdot P_{k-1}$. Then, by iterating this formula for $k=1,2,3$, we get $P_1=1.03\cdot P_0$, $P_2=1.03\cdot P_1=(1.03)^2\cdot P_0$, and $P_3=1.03\cdot P_2=(1.03)^3\cdot P_0$. So, we can guess that the general formula is $P_n=(1.03)^n\cdot P_0$. Therefore, 25 years later, the population will be $P_{25}=(1.03)^{25}\cdot P_0=(1.03)^{25}\cdot 50,000,000=2.09378\cdot 50,000,000=104,000,000$.

Problem 35

PROOF: From the hint, we are given that the formula obtained from Exercise 10 is $h_n=\frac{2^{n+1}-(-1)^{n+1}}{3}$, and the recurrence relation is defined as $h_k=2^k-h_{k-1}$, for $\forall k\in\mathbb{Z}$, $k\geq 1$. Then, we must prove our base case is true for n = 1: $h_1 = \frac{2^{1+1}-(-1)^{1+1}}{3} = \frac{3}{3} = 1$, and $h_1 = 2^1 - h_0 = 2-1 = 1$. Therefore, the base case is true. Next, assume that the formula is true for $n = k$; that is, $h_k = \frac{2^{k+1}-(-1)^{k+1}}{3}$. This is our inductive hypothesis. Then, we must show that the formula is true for $k+1$; that is, $h_{k+1} = \frac{2^{(k+1)+1}-(-1)^{(k+1)+1}}{3}$. Then, by the recursion relation, we know that $h_{k+1} = 2^{k+1} - h_k$, where $h_k = \frac{2^{k+1}-(-1)^{k+1}}{3}$ by the inductive hypothesis. Now, we have $h_{k+1} = 2^{k+1} - \frac{2^{k+1}-(-1)^{k+1}}{3}$, which we can rearrange to get $\frac{3\cdot 2^{k+1}-2^{k+1}+(-1)^{k+1}}{3}$. Factoring out a 2^{k+1} from the first two terms, we get $\frac{2^{k+1}(3-1)+(-1)^{k+1}}{3}$, which we can then add a multiplicative factor of -(-1) to the second term in the numerator to get $\frac{2\cdot 2^{k+1}-(-1)(-1)^{k+1}}{3}$. We can rewrite this as $\frac{2^{k+2}-(-1)^{k+2}}{3}$ by the properties of exponents. Finally, we can rewrite this one last time as $\frac{2^{(k+1)+1}-(-1)^{(k+1)+1}}{3}$. Thus, $\frac{2^{(k+1)+1}-(-1)^{(k+1)+1}}{3} = \frac{2^{(k+1)+1}-(-1)^{(k+1)+1}}{3}$, which is what we were trying to show. Therefore, by mathematical induction, the explicit formula must be true for this recurrence relation. Q.E.D.

Problem 49

(a) We are given the recurrence relation $u_k = u_{k-2} \cdot u_{k-1}$, for $\forall k \in \mathbb{Z}$, $k \geq 2$. First, we can use iteration to solve for $k = 2, 3, 4, 5$ since $k = 0, 1$ are given: $u_2 = u_0 \cdot u_1 = 2 \cdot 2$, $u_3 = u_1 \cdot u_2 = 2 \cdot (2 \cdot 2)$, $u_4 = u_2 \cdot u_3 = (2 \cdot 2) \cdot (2 \cdot 2 \cdot 2)$, and $u_5 = u_3 \cdot u_4 = (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2)$. Then, we can guess the explicit formula to be $u_n = 2^{n-2} \cdot 2^{n-1} = 2^{(n-2)+(n-1)} = 2^{2n-3}$.

(b) PROOF: Let $P(n)$ be the explicit formula; that is, $P(n) = u_n = 2^{2n-3}$. Then, our base case is to show that $P(2)$ is true; $P(2) = 2^{4-3} = 2$, and we are given that $u_2 = 2$, so our base case must be true. Now, we must show that for $\forall k \in \mathbb{Z}$ where $k \geq 1$, if $P(i)$ is true for $\forall i \in \mathbb{Z}$ where $1 \leq i \leq k$, then $P(k+1)$ is also true. That is, we must show that $u_{k+1} = 2^{2(k+1)-3} = 2^{2k+2-3} = 2^{2k-1}$. From the recurrence relation, we know that $u_{k+1} = u_{(k+1)-2} \cdot u_{(k+1)-1} = u_{k-1} \cdot u_k$, which we can then rewrite using the inductive hypothesis as $u_{k+1} = 2^{2(k-1)-3} \cdot 2^{2k-3} = 2^{2k-2-3+2k-3} = 2^{4k-8} = 2^{2(k+1)-3} = 2^{2k-1}$, which is what we were trying to show. Therefore, by strong mathematical induction, we know that the explicit formula we found must be true for this recurrence relation. Q.E.D.

Problem 52

(a) We are asked to derive a recurrence relation for P_k in terms of P_{k-1}, for $\forall k \in \mathbb{Z}$, $k \geq 2$. First, we know that $P_1 = 2$, since a single line divides a plane into two regions. Then, we know that $P_2 = 4 = 2 + 2$ since two lines divide a plane into four regions. Then, $P_3 = 7 = 4 + 3$, and similarly $P_4 = 11 = 4 + 7$. Therefore, the pattern that is emerging must be the recurrence relation, which we can write as $P_k = P_{k-1} + k$ since we can see that each successive P takes the value of P before it and adds the total number of lines (k).

(b) We have:
$P_1 = 1 + 1$
$P_2 = 1 + 1 + 2$
$P_3 = 1 + 1 + 2 + 3$
$P_4 = 1 + 1 + 2 + 3 + 4$

$P_5 = 1 + 1 + 2 + 3 + 4 + 5$

Therefore, it looks like we can write the series as $P_n = 1 + (1 + 2 + 3 + ... + n)$, where we know that the formula for the series $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$. Therefore, we can write the explicit formula for P_n as $P_n = 1 + \frac{n(n+1)}{2}$ for $\forall n \in \mathbb{Z}$, $n \geq 1$.