Honors Econ PSET 6

Anthony Yoon and Chimin Liu

11/18/2024

19

1

Defining the Competitive Equilbirum

I am definig the competitive equilbrium as follows: Given a relative price p^E, and an allocation for each individual: $\{(x_1^E, y_1^E), (x_2^E, y_2^E)\}$, we sastify the following conditions:

- Each person will fufill their [UMP] problem given the p^E
- Markets clear for the given p^E

My guess is that the competitive equilbrium will be $(1, 1)$ as substituting away from these goods would cause these individuals to lose utilty, as indicated by the Quasilinear Utility Functions.

Computing the Competitive Equilbirum

These conditions imply the following: Given the Marshallian Demand Funtion for each individual, denoted as x_i^m and y_i^m respectively, we the following must hold:

$$x_1^m + x_2^m = \omega_x^1 + \omega_x^2$$
$$y_1^m + y_2^m = \omega_y^1 + \omega_y^2$$

Since it is given that both individuals have the exact same utility function, we can just solve one general case that applies to both individuals. We set up the Langrangian as follows:

$$L = \log(x) + \log(y) + \lambda(\omega_x p_x + \omega_y p_y - x p_x - y p_y)$$

With the following first order conditions:

$$[x]\ \frac{1}{x} = \lambda p_x$$
$$[y]\ \frac{1}{y} = \lambda p_y$$
$$[\lambda]\ \omega_x p_x + \omega_y p_y = x p_x + y p_y$$

Equating the λs together and simplifying, you get

$$p_x x = p_y y$$

Using the $[\lambda]$ first order condition, we see that we can do the following:

$$\begin{aligned} x p_x + y p_y &= \omega_x p_x + \omega_y p_y \\ 2 x p_x &= \omega_x p_x + \omega_y p_y \\ x &= \frac{\omega_x p_x + \omega_y p_y}{2 p_x} \\ x &= \frac{\omega_x}{2} + \frac{\omega_y}{2} \cdot \frac{p_y}{p_x} \end{aligned}$$

We cam do a similar process for y, where we see that

$$y = \frac{\omega_x}{2} \cdot \frac{p_x}{p_y} + \frac{\omega_y}{2}$$

Now, let $p = \frac{p_x}{p_y}$, we get the following expressions for the Marshallian demand for x and y:

$$x_m = \frac{\omega_x}{2} + \frac{\omega_y}{2}p \quad y_m = \frac{\omega_x}{2}p^{-1} + \frac{\omega_y}{2}$$

Putting these equations into the market clearning condition as defined earlier, we get the following system of equations:

$$\begin{aligned} x_m^1 + x_m^2 &= \omega_x^1 + \omega_x^2 \\ \frac{\omega_x^1}{2} + \frac{\omega_y^1}{2}p + \frac{\omega_x^2}{2} + \frac{\omega_y^2}{2}p &= \omega_x^1 + \omega_x^2 \end{aligned}$$

and

$$\begin{aligned} y_m^1 + y_m^2 &= \omega_y^1 + \omega_y^2 \\ \frac{\omega_x^1}{2}p^{-1} + \frac{\omega_y^1}{2} + \frac{\omega_x^2}{2}p^{-1} + \frac{\omega_y^2}{2} &= \omega_y^1 + \omega_y^2 \end{aligned}$$

However, note that we know that $\omega_x^i = \omega_y^i = 1$ When we insert these values into the above system equations, we would find that $p^E = x_m^i = y_m^i = 1$. Thus, the equilbrium conditions would be $x_m^1 = x_2^m = 1$ and $y_m^1 = y_m^2 = 1$ and the relative prices should be 1

2

The income distrubution at equilbrium price would be the same for both the individuals. Thus, we can see that the income distrubution for both individuals is:

$$m_pe = p\omega_x + \omega_y = 1(1) + 1 = 2$$

3

The implied weights in this scenario would be equal, as there is no preferential treatment towards another individual. So $\theta_1 = \theta_2 = 1$. We can set up the Langrangian as follows:

$$L = \theta_1(U(x,y)_1) + \theta_2(U(x,y)_2) + \phi_x(\omega_x^1 + \omega_x^2 - x_1 - x_2) + \phi_y(\omega_y^1 + \omega_y^2 - y_1 - y_2)$$

where we get the following first order conditions:

$$\begin{aligned} [x_1]\ \phi_x &= \frac{1}{x_1} \\ [x_2]\ \phi_x &= \frac{1}{x_2} \\ [y_1]\ \phi_y &= \frac{1}{y_1} \\ [y_2]\ \phi_y &= \frac{1}{y_2} \\ [\phi_x]\ \omega_x^1 + \omega_x^2 &= x_1 + x_2 \\ [\phi_y]\ \omega_y^1 + \omega_y^2 &= y_1 + y_2 \end{aligned}$$

Note the following first order conditions imply that $x_1 = x_2 = 1$ and $y_1 = y_2 = 1$. Thus, these implies that we can in fact show that competitive allocation is indeed Pareto optimal.

4

With these new weights, note the first order conditions change as follows:

$$[x_1] \;\; \phi_x = \frac{3}{x_1}$$
$$[x_2] \;\; \phi_x = \frac{1}{x_2}$$
$$[y_1] \;\; \phi_y = \frac{3}{y_1}$$
$$[y_2] \;\; \phi_y = \frac{1}{y_1}$$

Thus, we get that:

$$3x_2 = x_1 \quad 3y_2 = y_1$$

We know that:

$$x_1 + x_2 = \omega_x^1 + \omega_x^2$$
$$4x_2 = 2$$
$$x_2 = 1/2$$

A similar process can be done to x_2, y_1, y_2, which yields $\frac{3}{2}, \frac{1}{2}, \frac{1}{2}$ respectively. Thus, when we calculate the ϕs for each good, we see that $\phi_x = \phi_y = 2$. Note because of the optimality condition, we can know that:

$$\frac{p_x}{p_y} = \frac{U_x^i}{U_y^i} = \frac{\phi_x}{\phi_y} = 1$$

Additionally, we can calculate the langrange multipliers for good x and y using the 1st order conditions, which we can calculate as follows:

$$\phi_x = \phi_y = \frac{3}{\frac{3}{2}} = 2$$

5

The implied equilbrium price is 1, which is the same as original equilbrium. I think this is because the original endowment point already sastifies the conditions for equilbirum, in which the relative price would just stay constant as any consumer would not want to substitute away from the good as it would harm the individual.

6

The implied income distrubution can be calculated as follows:

$$m_{id}^1 = p\omega_x^1 + \omega_y^1 = \frac{3}{2} + \frac{3}{2} = 3$$

$$m_{id}^2 = p\omega_x^2 + \omega_y^2 = \frac{1}{2} + \frac{1}{2} = 1$$

The flow goes to person 1. It can be noted that the individual 1 can afford the equilbirum bundle and individual 2 cannot afford the equilbirum bundle.

7

I define the equilbrium as the following: Given an allocation and price p, the following should hold:

- Given the price p, each individual should maximize their utility
- The markets should clear given the allocations.

Consumer 1's [UMP] problem

We are given the following conditions:

$$\max \ \log(x) + \log(y)$$
$$s.t \ 3 = px + y$$

The Langrangian can be set up as the following:

$$L = \log(x) + \log(y) + \lambda(3 - px - y)$$

We derive the following first order conditions:

$$[x] \ \frac{1}{x} = \lambda p$$
$$[y] \ \frac{1}{y} = \lambda$$
$$[\lambda] \ 3 = px + y$$

Equating the lambdas to each other yields:

$$\frac{y}{x} = p$$

Thus, we can then solve the following:

$$2y = 3$$
$$y = \frac{3}{2}$$

Before we continue, I propose we analyze consumer 2's [UMP] problem.

Consumer 2's [UMP] problem

We are given the following conditions:

$$\max \ \log(x) + \log(y)$$
$$s.t \ 1 = px + y$$

Note that all the first order conditions remain the same, with the exception of lambda, which would be

$$[\lambda] \ 1 = px + y$$

So, the relation:

$$\frac{y}{x} = p$$

This implies that $y = \frac{1}{2}$. Note that the only possible p value that works is $p = 1$, as then all other conditions and constraints would be met. Threfore, we see that we get the bundles:

$$(x_1, y_1) = \left(\frac{3}{2}, \frac{3}{2}\right) \quad (x_2, y_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

Therefore, since $p = p^E = 1$, we have shown that quilibrium price that supports the allocation is identical to the planner's implied equilibrium price. and also that the allocation that emerges from your calculations is identical to the Pareto optimal allocation.

20

Note in this problem, I let $i = 1$ and $j = 2$.

1

We define the equilbirum conditions as given an allocation for each individual, denoted as (x_i, y_i), and a price p^E, the following must be true:

- Each consumer must maximize their utility given the p
- Markets must clear for the given allocations

2

Individual 1's UMP problem can be given as

$$\begin{aligned} \max\ & \alpha \log(x) + (1-\alpha)\log(y) \\ s.t\ & px + y = p\omega_x^1 + \omega_y^1 \end{aligned}$$

Individual 2's UMP problem can be given as

$$\begin{aligned} \max\ & \gamma \log(x) + (1-\gamma)\log(y) \\ s.t\ & px + y = p\omega_x^2 + \omega_y^2 \end{aligned}$$

3

Note that each individual has the same basic structure, but with the only exceptions being that each endowment bundle being different for each individual as well as α and γ being variables being representative of their respective individual. Hence, we can solve a general case, and then replace each respective parameter.

Solving the general [UMP] problem

We can set up the Langrangian as follows:

$$L = \alpha \log(x_i) + (1-\alpha)\log(y_i) + \lambda(p\omega_x^i + \omega_y^i - px_i - y_i)$$

with the following first order conditions:

$$\begin{aligned} [x]\ & \frac{\alpha}{x_i} = p\lambda \\ [y]\ & \frac{1-\alpha}{y_i} = \lambda \\ [\lambda]\ & p\omega_x^i + \omega_y^i = px_i + y_i \end{aligned}$$

equating the λs here yields the following equality:

$$\frac{y_i}{x_i} = \frac{1-\alpha}{\alpha}p$$

We can see that $y_i = \frac{1-\alpha}{\alpha}(p)(x_i)$. Thus, we can see that the following holds from the $[\lambda]$ first order condition:

$$\begin{aligned} p\omega_x^i + \omega_y^i &= px_i + y_i \\ p\omega_x^i + \omega_y^i &= px_i + \frac{1-\alpha}{\alpha}(p)(x_i) \\ p\omega_x^i + \omega_y^i &= px_i\left(\frac{1-\alpha}{\alpha} + 1\right) \\ p\omega_x^i + \omega_y^i &= px_i\left(\frac{1}{\alpha}\right) \\ x_i &= \frac{\alpha}{p}(p\omega_x^i + \omega_y^i) \end{aligned}$$

And using the derive relationship from earlier, we can see that

$$y_i = (1-\alpha)(p\omega_x + \omega_y)$$

Note by the envelope theorem, we know that

$$\lambda_i^* = \frac{\partial v(x_i, y_i)}{\partial m} = \frac{1-\alpha}{p\omega_x^i + \omega_y^i} + \frac{\alpha}{p\omega_x^i + \omega_y^i} = \frac{1}{p\omega_x^i + \omega_y^i}$$

Thus, we know each consumer's x^*, y^*, λ^* by substituting the proper values, which are as follows:

Consumer 1

- $x_1^* = \frac{\alpha}{p}(p\omega_x^1 + \omega_y^1)$
- $y_1^* = (1-\alpha)(p\omega_x^1 + \omega_y^1)$
- $\lambda_1^* = \frac{1}{p\omega_x^1 + \omega_y^1}$

Consumer 2

- $x_2^* = \frac{\gamma}{p}(p\omega_x^2 + \omega_y^2)$
- $y_2^* = (1-\gamma)(p\omega_x^2 + \omega_y^2)$
- $\lambda_2^* = \frac{1}{p\omega_x^2 + \omega_y^2}$

4

The market clearing conditions are:

$$\frac{\alpha}{p}(p\omega_x^1 + \omega_y^1) + \frac{\gamma}{p}(p\omega_x^2 + \omega_y^2) = \omega_x^1 + \omega_x^2$$
$$(1-\alpha)(p\omega_x^1 + \omega_y^1) + (1-\gamma)(p\omega_x^2 + \omega_y^2) = \omega_y^1 + \omega_y^2$$

5

We will use the 1st equation to do so. The work is as follows:

$$\begin{aligned}
\frac{\alpha}{p}(p\omega_x^1 + \omega_y^1) + \frac{\gamma}{p}(p\omega_x^2 + \omega_y^2) &= \omega_x^1 + \omega_x^2 \\
\alpha(p\omega_x^1 + \omega_y^1) + \gamma(p\omega_x^2 + \omega_y^2) &= p\omega_x^1 + p\omega_x^2 \\
\alpha p\omega_x^i + \alpha\omega_y^1 + \gamma p\omega_x^2 + \gamma\omega_y^2 &= p\omega_x^1 + p\omega_x^2 \\
\alpha p\omega_x^1 - p\omega_x^1 + \gamma p\omega_x^2 - p\omega_x^2 &= -\alpha\omega_y^1 - \gamma\omega_y^2 \\
p(\omega_x^1(1-\alpha) + \omega_x^2(1-\gamma)) &= \gamma\omega_y^2 + \alpha\omega_y^1 \\
p &= \frac{\gamma\omega_y^2 + \alpha\omega_y^1}{\omega_x^1(1-\alpha) + \omega_x^2(1-\gamma)}
\end{aligned}$$

This is the price of y in regards to goods x forgone. This is the calculations for x in regards to y to verify:

$$\begin{aligned}
(1-\alpha)(p\omega_x^1 + \omega_y^1) + (1-\gamma)(p\omega_x^2 + \omega_y^2) &= \omega_y^1 + \omega_y^2 \\
(1-\alpha)p\omega_x^1 + (1-\alpha)\omega_y^1 + (1-\gamma)p\omega_x^2 + (1-\gamma)\omega_y^2 &= \omega_y^1 + \omega_y^2 \\
\left[(1-\alpha)\omega_x^1 + (1-\gamma)\omega_x^2\right]p + \left[(1-\alpha)\omega_y^1 + (1-\gamma)\omega_y^2\right] &= \omega_y^1 + \omega_y^2 \\
\left[(1-\alpha)\omega_x^1 + (1-\gamma)\omega_x^2\right]p &= \omega_y^1 + \omega_y^2 - \left[(1-\alpha)\omega_y^1 + (1-\gamma)\omega_y^2\right] \\
\left[(1-\alpha)\omega_x^1 + (1-\gamma)\omega_x^2\right]p &= \alpha\omega_y^1 + \gamma\omega_y^2 \\
p &= \frac{\alpha\omega_y^1 + \gamma\omega_y^2}{(1-\alpha)\omega_x^1 + (1-\gamma)\omega_x^2}
\end{aligned}$$

6

a) Equal aggregate endowments does not imply that ples and oranges will trade at a rate of one-to-one in equilibrium as these values depend on α, γ to be certain values.

b) If consumers have identical preferences, this is still not enough for consumrs to trade at a 1-1 equilbirum as α, γ must be certain values for indivduals to trade at certain levels, combined with the fact that there must be equal aggregate endowments at play.

c) A set of conditions that would help this hold are

- $\omega_y^1 + \omega_y^2 = \omega_x^1 + \omega_x^2$
- $\alpha = \gamma = 0.5$

All these conditions must be true for trade to be one to one in equilbrium.

d) Something interesting about this question is that depending on how much each consumer values a good, which is determined by the α or γ values, we can see that the relative price also changes. This is interesting as the aggregate behavior of consumers will determine the relative price of the goods. Additionally, the relative price depends on the intial amount of endowments, which indicates that if a rich person had a really high endowment, then they would influence the relative price heavily. Another interesting thing to note is that you do not necessarily need equal endowments and preferences to reach one to one trading, as seen below:

Endowments: $\omega_x^1 = 8, \ \omega_y^1 = 2$
$\omega_x^2 = 2, \ \omega_y^2 = 8$

Aggregate Endowments: $\omega_x^1 + \omega_x^2 = 10 < \omega_y^1 + \omega_y^2 = 10$

1 likes x   2 likes y

Preferences: $\alpha = 0.7$, $\gamma = 0.3$ $\frac{1.4+2.4}{2.4+1.4}$

$$p = \frac{0.7(2) + 0.3(8)}{0.3(8) + 0.7(2)} = \frac{3.8}{3.8} = 1$$

and note the following: Assume that $\alpha = \gamma$ and $\omega_x = \omega_y$. Thus, we can see that we get the following:

$$p = \frac{\alpha}{1-\alpha} \frac{W_x}{W_y}$$

and when we rearrange these termns, we note that:

$$\alpha = \frac{W_x}{W_x + W_y}$$

and thus for there to be one to one trading, α should be the share of good x on the market.

7

Given this condition:

$$(1-\alpha)(p\omega_x^1 + \omega_y^1) + (1-\gamma)(p\omega_x^2 + \omega_y^2) = \omega_y^1 + \omega_y^2$$

We can rearrange terms on the right hand side to get:

$$p((1-\alpha)\omega_x^1 + (1-\gamma)\omega_x^2) + (1-\alpha)(\omega_y^1) + (1-\gamma)(\omega_y^2)$$

Substituting the known value of p yields the following:

$$\left(\frac{\alpha\omega_y^1 + \gamma\omega_y^2}{(1-\alpha)\omega_x^1 + (1-\gamma)\omega_x^2}\right)((1-\alpha)\omega_x^1 + (1-\gamma)\omega_x^2) + (1-\alpha)(\omega_y^1) + (1-\gamma)(\omega_y^2)$$

$$= \alpha\omega_y^1 + \gamma\omega_y^2 + (1-\alpha)(\omega_y^1) + (1-\gamma)(\omega_y^2)$$
$$= \omega_y^1 + \omega_y^2$$

which shows that markets clear for good y for the given market price. Note that we can do a similar thing when we determine how much each consumer consumes. For consumer one, we can see that consumer 1 will consume:

$$y_1^* = (1-\alpha)\left(\left(\frac{\alpha\omega_y^1 + \gamma\omega_y^2}{(1-\alpha)\omega_x^1 + (1-\gamma)\omega_x^2}\right)\omega_x^1 + \omega_y^1\right)$$

and similarly, we see that consumer 2 consumes

$$y_1^2 = (1-\gamma)\left(\left(\frac{\alpha\omega_y^1 + \gamma\omega_y^2}{(1-\alpha)\omega_x^1 + (1-\gamma)\omega_x^2}\right)\omega_x^2 + \omega_y^2\right)$$

8

Since we know that utility functions are quasilinear for both consumers, we know that

$$\frac{\partial v}{\partial m} = \lambda_i^* = \frac{1}{m}$$

Thus, we can do the following for each consumer. For consumer 1, we can see that:

$$\lambda_1^* = \frac{1}{m_1} = \frac{1}{p\omega_x^1 + \omega_y^1} = \frac{1}{\left(\left(\left(\frac{\alpha\omega_y^1+\gamma\omega_y^2}{(1-\alpha)\omega_x^1+(1-\gamma)\omega_x^2}\right)\omega_x^1 + \omega_y^1\right)\right)}$$

and similarly:

$$\lambda_2^* = \frac{1}{m_2} = \frac{1}{p\omega_x^2 + \omega_y^2} = \frac{1}{\left(\left(\left(\frac{\alpha\omega_y^1+\gamma\omega_y^2}{(1-\alpha)\omega_x^1+(1-\gamma)\omega_x^2}\right)\omega_x^2 + \omega_y^2\right)\right)}$$

The units of λ_i^* are $\frac{\text{utils}}{\text{good y}}$. Note that we cannot determine nominal value of λ_i^* as we are still working in relative prices and variables that have not been defined. Since we are working in relative prices, we can see that we will never have an unique solution. If we desired an unique answer for a given set of prices, we should be more concerned about the aboslute prices of goods.

9

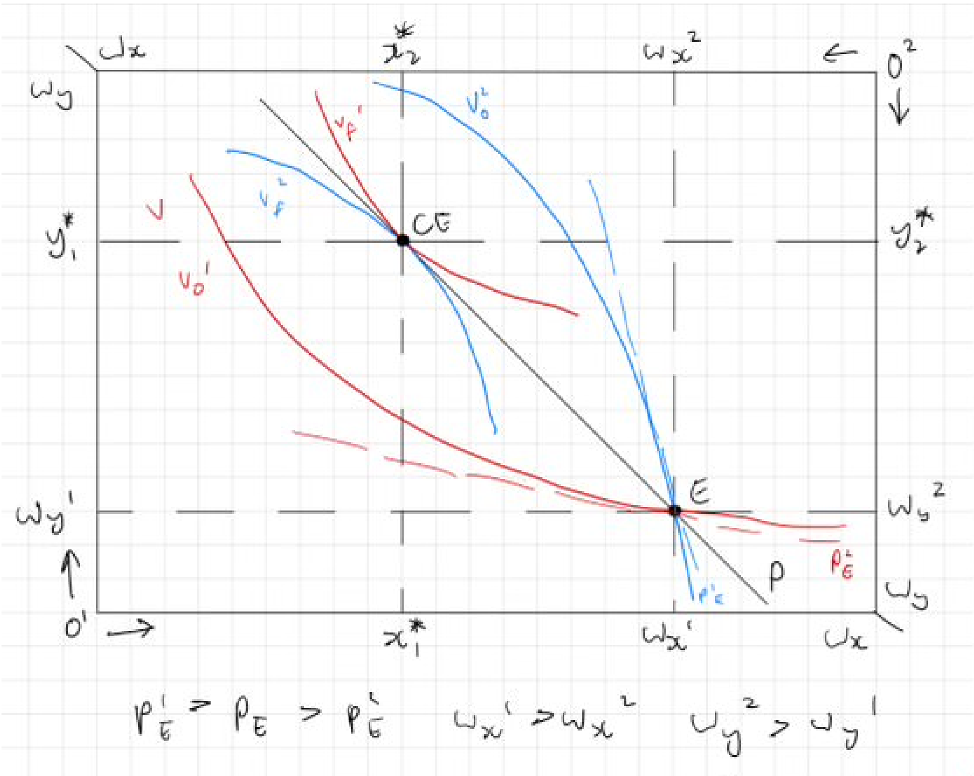

Figure 1: Edgeworth Graph

10

It is given that $\omega_x^1 > \omega_x^2$ and $\omega_y^1 > \omega_y^2$. In equibirum, each consumer desires to trade to the optimal bundle, as noted by the optimality conditions. So, for example, if consumer 1 had less goods of x than the optimal bundle $\omega_x^1 < x_1^*$, then consumer 1 would trade good y for good x. Similarly, if $\omega_x^1 > x_1^*$, consumer 1 would trade good x for good y. A similar logic applies for good y as well. However, if both consumers are at their optimal bundles, then consumers will not trade any further.

21

Now, we use notation as noted in the problem.

1

The social planner problem is as follows:

$$\begin{aligned} \max \ & (\alpha \ln(x_i) + (1-\alpha)\ln(y_i)) + \theta(\gamma \ln(x_j) + (1-\gamma)\ln(y_j)) \\ s.t \ & x_i + x_j = \omega_x^i + \omega_x^j \\ & y_i + y_j = \omega_y^i + \omega_y^j \end{aligned}$$

2

The Langrangian is:

$$L = (\alpha \ln(x_i) + (1-\alpha)\ln(y_i)) + \theta(\gamma \ln(x_j) + (1-\gamma)\ln(y_j)) + \phi_x(\omega_x^i + \omega_x^j - (x_i + x_j)) + \phi_y(\omega_y^i + \omega_y^j - (y_i + y_j))$$

The first order conditions are as follows:

$$
\begin{aligned}
&[x_i] \ \frac{\alpha}{x_i} = \phi_x \\
&[y_i] \ \frac{1-\alpha}{y_i} = \phi_y \\
&[x_j] \ \theta\frac{\gamma}{x_j} = \phi_x \\
&[y_j] \ \theta\frac{1-\gamma}{y_j} = \phi_y \\
&[\phi_x] \ x_i + x_j = \omega_x^i + \omega_x^j \\
&[\phi_y] \ y_i + y_j = \omega_y^i + \omega_y^j
\end{aligned}
$$

3

We know that the following is true given the first order conditions as aboves:

$$\frac{U_{x_i}}{U_{y_i}} = \frac{U_{x_j}}{U_{y_j}} = \frac{\phi_x^*}{\phi_y^*} = p^P$$

This is essential for the optimality arguements. Note that the slope of the indifference curves are the same, which indicates the same willingness to pay for x in terms of y. Additionally, this implies Pareto optimality because pushing the indifference curve of i up will necessarily mean pushing j onto a lower indifference curve, which means it's not possible to make i better off without making j worse off, due to the tangency of both utility functions with the equilibrium price We also wish to show that for any given allocation, we can find some θ weight that corresponds to each value. Note the following:

$$\theta = \left(\frac{\alpha}{\gamma}\right) \cdot \left(\frac{x_j}{x_i}\right) \quad \theta = \left(\frac{1-\alpha}{1-\gamma}\right) \cdot \left(\frac{y_j}{y_i}\right)$$

we observe that there are 2 θs, each dependent on one variable. Thus, we can see the following for each θ we can use the allocation found in the competitive equilbirum (or exercise 20):

$$
\begin{aligned}
\theta &= \frac{\alpha x_j}{\gamma x_i} \\
&= \frac{\alpha}{\gamma} \cdot \frac{p}{\alpha(p\omega_x^i + \omega_y^i)} \cdot \frac{\gamma(p\omega_x^j + \omega_y^j)}{p} \\
&= \frac{p\omega_x^j + \omega_y^j}{p\omega_x^i + \omega_y^i}
\end{aligned}
$$

and

$$
\begin{aligned}
\theta &= \left(\frac{1-\alpha}{1-\gamma}\right) \cdot \left(\frac{y_j}{y_i}\right) \\
&= \left(\frac{1-\alpha}{1-\gamma}\right) \cdot \frac{p}{(1-\alpha)(p\omega_x^i + \omega_y^i)} \cdot \frac{(1-\gamma)(p\omega_x^j + \omega_y^j)}{p} \\
&= \frac{p\omega_x^j + \omega_y^j}{p\omega_x^i + \omega_y^i}
\end{aligned}
$$

Thus, every allocation has a given θ that we can use, thus showing Pareto Optimality.

4

To find the solution, we can see that using the first order conditions $[x_i], [x_j]$ to see that

$$\frac{x_i}{x_j} = \frac{\theta\gamma}{\alpha}$$

Using this equality and the market clearing conditions, we can see that:

$$x_i + x_j = \omega_x^i + \omega_x^j$$
$$x_i \left(\theta\left(\frac{\gamma}{\alpha}\right) + 1\right) = \omega_x^i + \omega_x^j$$
$$x_i = \frac{\omega_x^i + \omega_x^j}{\left(\theta\left(\frac{\gamma}{\alpha}\right) + 1\right)}$$
$$x_i^P = \frac{\alpha(\omega_x^i + \omega_x^j)}{\theta\gamma + \alpha}$$

and a similar logic, we find that

$$x_j^P = \frac{\theta\gamma(\omega_x^i + \omega_x^j)}{\alpha + \theta\gamma}$$

We also derive the following equality:

$$\frac{y_j}{y_i} = \frac{\theta(1-\gamma)}{1-\alpha}$$

Thus, we can do the following to find y_i:

$$y_i + y_j = \omega_y^i + \omega_y^j$$
$$y_i + y_i\left(\frac{\theta(1-\gamma)}{1-\alpha}\right) = \omega_y^i + \omega_y^j$$
$$y_i\left(1 + \frac{\theta(1-\gamma)}{1-\alpha}\right) = \omega_y^i + \omega_y^j$$
$$y_i^P = \frac{(1-\alpha)(\omega_y^i + \omega_y^j)}{(1-\alpha) + \theta(1-\gamma)}$$

We can do a similar logic to find that:

$$y_j^P = \frac{\theta(1-\gamma)(\omega_y^i + \omega_y^j)}{(1-\alpha) + \theta(1-\gamma)}$$

Thus, the final bundles are:

$$(x_i^P, y_i^P) = \left(\frac{\alpha(\omega_x^i + \omega_x^j)}{\theta\gamma + \alpha}, \frac{(1-\alpha)(\omega_y^i + \omega_y^j)}{(1-\alpha) + \theta(1-\gamma)}\right)$$

and

$$(x_j^P, y_j^P) = \left(\frac{\theta\gamma(\omega_x^i + \omega_x^j)}{\alpha + \theta\gamma}, \frac{\theta(1-\gamma)(\omega_y^i + \omega_y^j)}{(1-\alpha) + \theta(1-\gamma)}\right)$$

5

In competitive equilbirum, note that we have already defined price to be:

$$p^E = \frac{\gamma\omega_y^2 + \alpha\omega_y^1}{\omega_x^1(1-\alpha) + \omega_x^2(1-\gamma)}$$

and in a Pareto problem, note that we have

$$p^P = \frac{\phi_x}{\phi_y} = \frac{\alpha}{1-\alpha} \cdot \frac{y_i^P}{x_i^P} = \frac{\omega_y^i + \omega_y^j}{(1-\alpha) + \theta(1-\gamma)} \cdot \frac{\alpha + \theta\gamma}{\omega_x^i + \omega_x^j}$$

Thus, the income distrubution in this scenario is

$$m_i^P = p^P x_i + y_i = \alpha\frac{\omega_x^i + \omega_x^j}{(1-\alpha) + \theta(1-\gamma)} + \frac{(1-\alpha)(\omega_y^i + \omega_y^j)}{(1-\alpha) + \theta(1-\gamma)} = \frac{\omega_y^i + \omega_y^j}{(1-\alpha) + \theta(1-\gamma)}$$

and

$$m_j^P = px_j + y_j = \frac{\theta\gamma(\omega_y^i + \omega_y^j)}{(1-\alpha)+\theta(1-\gamma)} + \frac{\theta(1-\gamma)(\omega_y^i + \omega_y^j)}{(1-\alpha)+\theta(1-\gamma)} = \frac{\theta(\omega_y^i + \omega_y^j)}{(1-\alpha)+\theta(1-\gamma)}$$

Thus, the transfer required for each individual can be caluclated as follows:

$$T_i = m_i^P - m_i^{CE} = \frac{\omega_y^i + \omega_y^j}{(1-\alpha)+\theta(1-\gamma)} - \left(\left(\frac{\alpha\omega_y^1 + \gamma\omega_y^2}{(1-\alpha)\omega_x^1 + (1-\gamma)\omega_x^2}\right)\omega_x^1 + \omega_y^1\right)$$

and

$$T_j = m_j^P - m_j^{CE} = \frac{\theta(\omega_y^i + \omega_y^j)}{(1-\alpha)+\theta(1-\gamma)} - \left(\left(\frac{\alpha\omega_y^1 + \gamma\omega_y^2}{(1-\alpha)\omega_x^1 + (1-\gamma)\omega_x^2}\right)\omega_x^2 + \omega_y^2\right)$$

6

The 1st welfare theorem states that if the equilibrium relative price pE and the allocation is a competitive equilibrium, then the allocation must also be pareto optimal. This is shown in 3. as the CE allocation was pareto optimal with the relative marginal utilities equivalent to the relative price.

The 2nd welfare theorem states that every pareto optimum can be supported as the competitive equilibrium of a decentralized economy with transfers. We showed this in 5. by finding the transfers Ti and Tj that will allow us to transform the PO from just a centralized allocation made by a social planner to a decentralized allocation arrived at by rational individuals who know only their own preferences and act according to their own preferences to maximise utility for both i and j, while also satisfying market clearing conditions. This PO is thus a CE made possible with transfers, which is just the difference in income needed to purchase the bundle at the socially optimal PO vs at the initial CE.

22

1

The descriptive problem is

$$\begin{aligned} \max\ & U(C,R) \\ s.t\ & pc = wL + V \\ & L + R = T \end{aligned}$$

2

The canonical form is:

$$\begin{aligned} \max\ & U(C,R) \\ s.t\ & pc + wR = wT + V \end{aligned}$$

3

The Langrangian is as follows:

$$L = \alpha\log(C) + (1-\alpha)\log(R) + \lambda(wT + V - pc - wR)$$

with the first order conditions:

$$\begin{aligned} [C]\ & \frac{\alpha}{C} + \lambda p \\ [R]\ & \frac{1-\alpha}{R} = \lambda w \\ [\lambda]\ & pc + wR = wT + V \end{aligned}$$

Using the first order conditins, we can derive the ratio:

$$\frac{\alpha}{pC} = \frac{1-\alpha}{Rw}$$

This implies that:

$$C^* = \frac{\alpha(wT+V)}{p} \quad R^* = \frac{(1-\alpha)(wT+V)}{w}$$

Using the $L + R = T$ constraint, we can see that;

$$L^* = \alpha T - \frac{(1-\alpha)V}{w}$$

4

So when a person works, a person has hours that they actually work and hours that they do not actually work. Let S denote the amount of time that an individual spends in work not working actively. Thus, we know that:

$$S = H - L^*$$

and thus:

$$S = H - \left(\alpha T - \frac{(1-\alpha)V}{w}\right)$$
$$S = H + \frac{(1-\alpha)V}{w} - \alpha T$$

5

Note that we know that $S = H - L^*$, thus we know that:

$$S = H - \left(\alpha T - \frac{(1-\alpha)V}{w}\right)$$
$$S = H + \frac{(1-\alpha)V}{w} - \alpha T$$

Note that S and V are directly proportional to each other. So it seems that wealthy inidviduals would be more likley to skip work, assuming that C, T, p, w are all held constant. However, this may change depedning on how much each individual values labor and leisure, as denoted by α and $1-\alpha$ respectively.

6

A lazy individual is going to work less than the allocated amount, so $L^* < H$. A diligant individual would work more, so $L^* > H$. We can then derive a mathematical inequality that shows how much of non-labor income an individual has. Note for the lazy individual,

$$L^* < H$$
$$H - L^* > 0$$
$$H + (1-\alpha)\frac{V_L}{w} - \alpha T > 0$$
$$V_L > \frac{w(\alpha T - H)}{1-\alpha}$$

and similarly for the diligent consumer,

$$L^* > H$$
$$H - L^* < 0$$
$$H + (1-\alpha)\frac{V_D}{w} - \alpha T < 0$$
$$V_D < \frac{w(\alpha T - H)}{1-\alpha}$$

Thus, the following inequality can be made:

$$V_L > \frac{w(\alpha T - H)}{1-\alpha} > V_D$$

so the lazy individual works:

$$L_l^* = \alpha T - (1-\alpha)\frac{V_L}{w}$$

and the diligent individual works:

$$L_D^* = \alpha T - (1-\alpha)\frac{V_D}{w}$$

such that the above inequality holds.

7

Note the goal of each policy is to make $L_l^* = H$. Since we are subsidizing the the income, we can do the following:

$$\begin{aligned} L_l^* &= H \\ \alpha T - (1-\alpha)\frac{V_L}{w+t} &= H \\ V_L\left(\frac{1-\alpha}{w+t}\right) &= -H + \alpha T \\ \frac{1-\alpha}{w+t} &= \frac{-H+\alpha T}{V_L} \\ w+t &= \frac{(1-\alpha)V_L}{-H+\alpha T} \\ t &= \frac{(1-\alpha)V_L}{-H+\alpha T} - w \end{aligned}$$

We calculate the cost as follows: This is the total price before this policy is

$$w(L_l^* + L_D^*) = 2\alpha wT - (1-\alpha)(V_L + V_D)$$

and the cost after the policy is:

$$(w+t)(L_l^* + L_D^*) = 2\alpha(w+t)T - (1-\alpha)(V_L + V_D)$$

and subtracting the two policies yields:

$$2\alpha tT = 2\alpha T\left(\frac{(1-\alpha)V_L}{H-\alpha T} - w\right)$$

The graph is as follows:

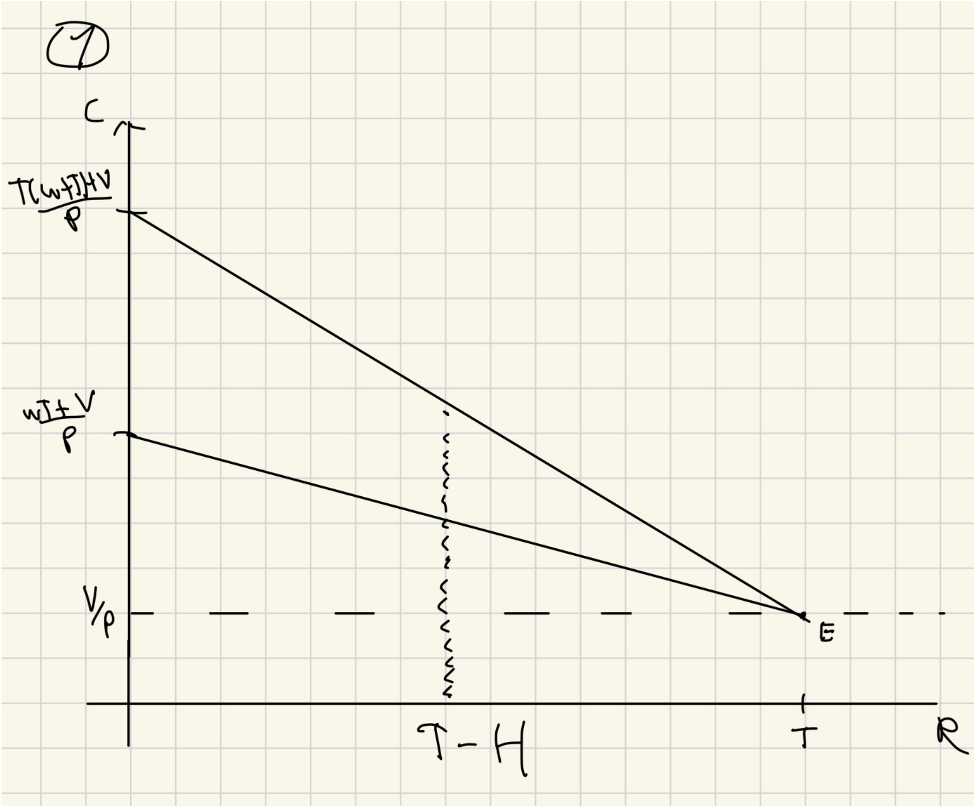

Figure 2: Graph for 7

8

We want $L_l^* = H$, and adding a bonus adds a modifer to the individual's non-labor income. Therefore, we now let $v_f = v_L + B$. Therefore, we can do the following:

$$\begin{aligned}
H &= \alpha T - (1-\alpha)\frac{V_L + B}{w} \\
\alpha T - H &= (1-\alpha)\left(\frac{V_L + B}{w}\right) \\
\frac{\alpha t - H}{1-\alpha} &= \frac{V_L + B}{w} \\
\frac{w(\alpha t - H)}{1-\alpha} &= V_L + B \\
B &= \frac{w(\alpha t - H)}{1-\alpha} - V_L
\end{aligned}$$

Note that both individuals qualify for the bonus, so the cost to the company would be:

$$2B + w(H - L_l^*) = 2\left(\frac{w(\alpha t - H)}{1-\alpha} - V_L\right) + w(H - \alpha T - (1-\alpha)(\frac{V_L + B}{w}))$$

and the graph is as follows:

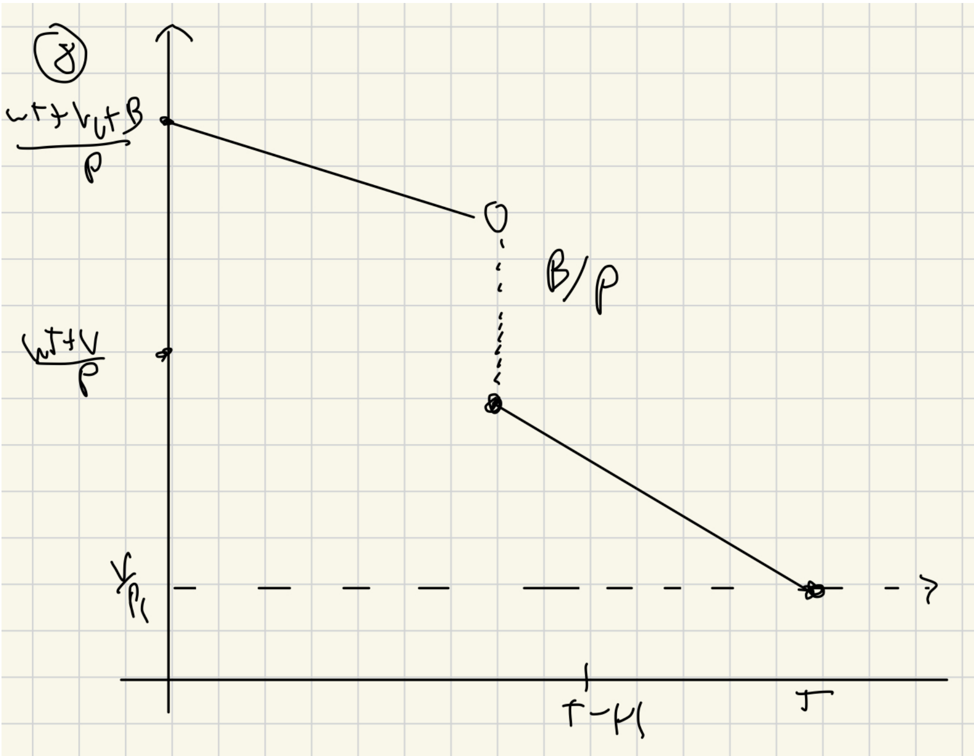

Figure 3: Graph for 8

9

We want $H = L_l^*$. Since this is a reduction in non-labor income, or $V - F$, we can do the following:

$$
\begin{aligned}
H &= \alpha T - (1-\alpha)\frac{V_L - F}{w} \\
H &= \alpha T - (1-\alpha)\frac{V_L - f(L_l^* - H)}{w} \\
\frac{-H + \alpha T}{1-\alpha} &= \frac{V - f(L_l^* - H)}{w} \\
w\left(\frac{-H + \alpha T}{1-\alpha}\right) &= V - f(L_l^* - H) \\
f &= \frac{V - w\left(\frac{-H+\alpha T}{1-\alpha}\right)}{L_l^* - H}
\end{aligned}
$$

Note there is actually a negative cost. Since the company does not actually give the workers any pay, rather they save money by taking away from their pay. Thus, the diligent worker does not cost the company any money, as this worker would not suffer the effects of this pay check decrease. However, the lazy worker would help the company save F dollars, as every hour that the worker does not meet the quoto saves the company f dollars. So the total cost would be:

$$F = f(H - L_l^*) = \left(\frac{v - w\left(\frac{-H+\alpha T}{1-\alpha}\right)}{L_l^* - H}\right) \cdot \left(H - \alpha T + \frac{V_L(1-\alpha)}{w - f}\right)$$

The graph is as follows:

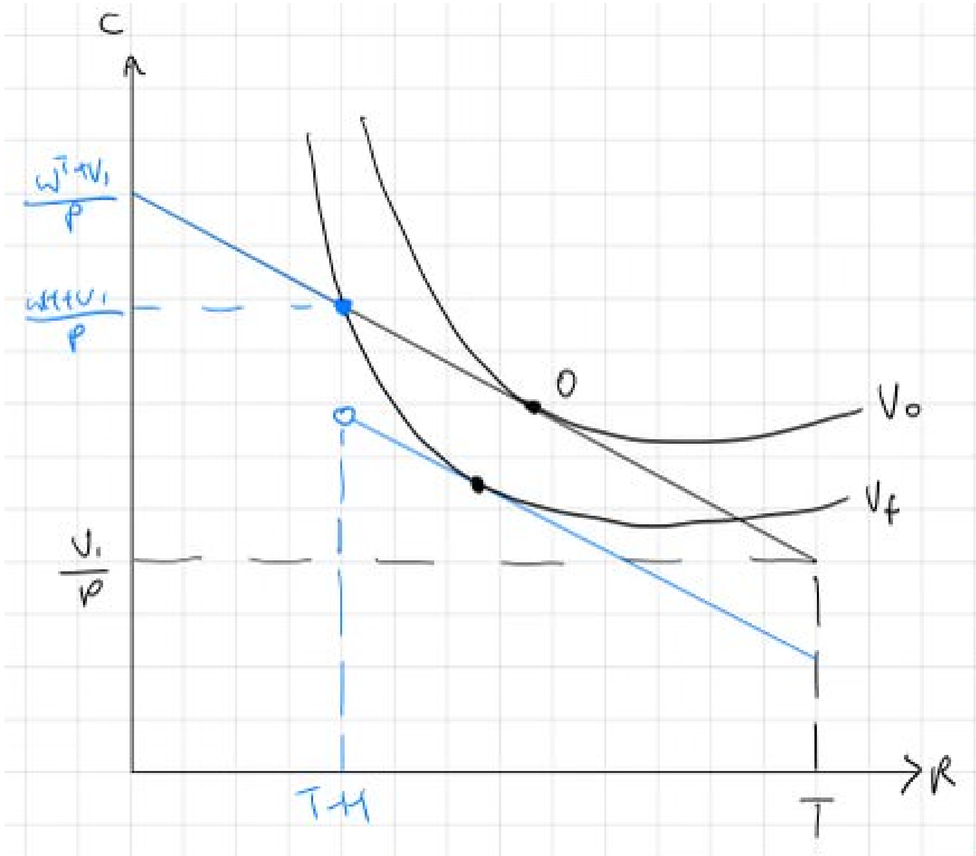

Figure 4: Graph for 9

10

All of the policies are effective at making the lazy worker work H hours. However, it regards to cost minimization, I would recommend the company implement Policy C, as this is the only policy that has a negative cost. We can see that Policy A and Policy B are bonuses and raises respectively and costs the company money, whearas Policy C actually saves the company money.