PSET5

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1

a

We can first begin calulating MRS for each individual. Without a loss of generality, we can begin to note the following:

$$MRS_i = \frac{\alpha(x_1^i)^{\alpha-1}(x_2^i)^{\beta}}{\beta(x_2^i)^{\beta-1}(x_1^i)^{\alpha}}$$

This above quantity implies that $\frac{x_2^1}{x_1^1} = \frac{x_2^2}{x_1^2}$. Let $e_1 = x_1^1 + x_1^2$ and $e_2 = x_2^1 + x_2^2$. Using this equalities as well as the implication derived from the MRSes, we can find that we get:

$$x_2^1 = \frac{e_2}{e_1}x_1^1 \quad x_2^2 = \frac{e_2}{e_1}x_1^2$$

Since both the above quantities are linear in nature with no intercept, this implies that indeed the contact curve is that of connecting endpoints.

b

Since we we are working with different utility functions, we can find that after similar caluclations to above that:

$$MRS_1 = \frac{\alpha(x_1^1)^{\alpha-1}(x_2^1)^{1-\alpha}}{(1-\alpha)(x_1^1)^{\alpha}(x_2^1)^{-\alpha}} = \frac{\alpha x_2^1}{(1-\alpha)x_1^1}$$

and using similar calculations, we find that:

$$MRS_2 = \frac{\beta x_2^2}{(1-\beta)x_1^2}$$

Since we know that $1 > \alpha > \beta > 0$, we find that:

$$\frac{\alpha}{1-\alpha} > \frac{\beta}{1-\beta}$$

Thus, we can see that for MRS to equal to each other, we know that:

$$\frac{x_2^1}{x_1^1} < \frac{x_2^2}{x_1^2}$$

Using the equations derived above, we can find that:

$$x_2^1 < \frac{e_2}{e_1} x_1^1$$

this implies that the graph still intersects the origins, but now $x_2^1 < x_1^1$, where we have all a curve that will be strictly below that of the original line derived in **a**. The contract curve is seen as below.

c

For the contract curve to exist, we want $MRS_1 = MRS_2$. Let $e_1 = x_1^1 + x_1^2$. We can see that

$$MRS_1 = MRS_2 \implies \alpha(x_1^1)^{\alpha-1} = \beta(x_1^2)^{\beta-1}$$

Thus, substuting the endowment, we find that:

$$\alpha(x_1^1)^{\alpha-1} = \beta(e_1 - x_1^1)^{\beta-1}$$

So we see that as $x_1^1 \to e_1^1$, we find that the consumers will not consume any x_2, and consume only x_1 However, since we know that $\alpha > \beta$, this implies that consumer x_1^1 has greater value on x_1^1, which implies that $x_1^1 > x_1^2$. Note that any level of x_2 sastifies the MRS equality arguement, thus we are only concerned about when $e_1 = x_1^1 + x_1^2$ and one of these inputs are the utility maximizing solution. The contract curve will like the one below:

2

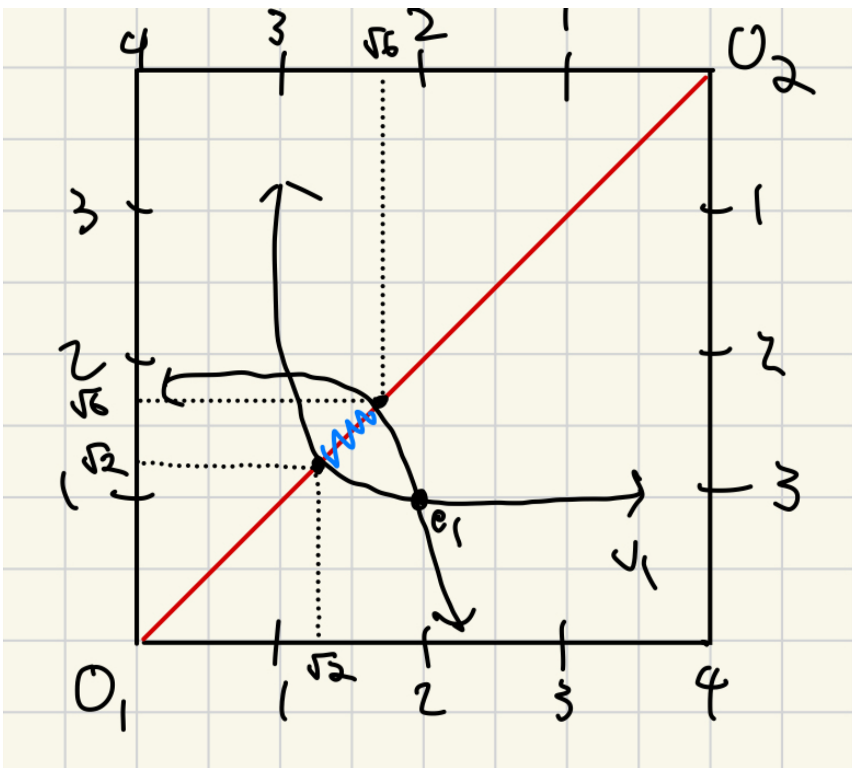

Figure 1: Graph for Question 2

a

See Graph

b

Given in Question **1**

c

By definition, it is the contract curve between the curves.

d

The core would just be the point (e^1_1, e^1_2) as any trade would make this individual worse off, hence no trade would not be blocked by any coalition.

e

We first derive the general Marshallian Demand function for each individual. We are interested in the following optimization problem:

$$\begin{aligned} \max \quad & x_1 x_2 \\ \text{s.t} \quad & p_1 e_1 + p_2 e_2 \geq p_1 x_1 + p_2 x_2 \end{aligned}$$

Where the langrangian is:

$$L = x_1 x_2 - \lambda(p_1 e_1 + p_2 e_2 - p_1 x_1 - p_2 x_2)$$

wherw we see that the FOCs are:

$$\begin{aligned} [x_1] \quad & x_2 + \lambda p_1 \leq 0 \text{ and } x_1 \geq 0 \\ [x_2] \quad & x_1 + \lambda p_2 \leq 0 \text{ and } x_2 \geq 0 \\ [\lambda] \quad & p_1 e_1 + p_2 e_2 \leq p_1 x_1 + p_2 x_2 \end{aligned}$$

We can see all FOCs must be strict equality, as if that is not the case, then markets will fail to clear $([x_1], [x_2])$ and by the AU assumption that we want to use all of our endowment to maximize utility. Using these FOCs, we can find that:

$$p_1 x_1 = p_2 x_2$$

Thus, using this equation and the constraint, we find that:

$$x^m_1 = \frac{e_1}{2} + \frac{p_2}{2p_1} e_2 \quad x^m_2 = \frac{p_1}{2p_2} e_1 + \frac{e_2}{2}$$

Thus, we find that:

$$\begin{aligned} x^1_1 = \frac{p_2}{2p_1} + 1 \quad x^1_2 = \frac{p_1}{p_2} + \frac{1}{2} \\ x^2_1 = \frac{3p_2}{2p_1} + 1 \quad x^2_2 = \frac{p_1}{p_2} + \frac{3}{2} \end{aligned}$$

Thus we can find that where we let $\mathbf{p} = (p_1, p_2)$:

$$z_1(\mathbf{p}) = 2 + 2\left(\frac{p_2}{p_1}\right) - 4 = -2 + 2\left(\frac{p_2}{p_1}\right)$$

$$z_2(\mathbf{p}) = 2 + 2\left(\frac{p_1}{p_2}\right) - 4 = -2 + 2\left(\frac{p_1}{p_2}\right)$$

and we can verify that:

$$\mathbf{p} \cdot \mathbf{z} = \begin{bmatrix} p_1 & p_2 \end{bmatrix} \cdot \begin{bmatrix} 2\left(\frac{p_2}{p_1}\right) - 2 \\ 2\left(\frac{p_1}{p_2}\right) - 2 \end{bmatrix} = 2p_2 - 2p_1 + 2p_1 - 2p_2 = 0$$

f

We can see if $p_2 = p_1$, we find that obviously, $\mathbf{z}$ goes to 0. Thus, we find that the set of Walrasian equlibria is $p^*(\mathcal{E}) = \{p_1, p_2)|p_1 = p_2\}$ We find that a Walrasian equilbrium allocation is

$$x^W = \{(x_1, x_2)\} = \{(1.5, 1.5), (2.5, 2.5)\}$$

since there is one relative price, we can find that the set of Walrasian Equilbirum Allocations is:

$$W(\mathcal{E}) = \bigcup_{p^*} x^W(p^*(\mathcal{E}), \mathcal{E}) = \{(1.5, 1.5), (2.5, 2.5)\}$$

g

Note that $e_1 = e_2$, this implies that the slope of the line is 1, which means that all allocations that have $x_1 = x_2$ will be in the core. Thus, we can see that the above set is a subset of the core.

h

Switching the utility function, we find that we are interested in the following optimization problem:

$$\begin{aligned} \max \quad & x_1^{\frac{2}{3}} x_2^{\frac{1}{3}} \\ \text{s.t} \quad & p_1 e_1 + p_2 e_2 \leq p_1 x_1 + p_2 x_2 \end{aligned}$$

the constraint remains the same, but with the following FOCs.

$$\begin{aligned} [x_1] \quad & \left(\frac{2}{3} x_1^{-\frac{1}{3}}\right) x_2^{\frac{1}{3}} = p_1 \lambda \\ [x_2] \quad & \left(\frac{1}{3} x_1^{-\frac{1}{3}}\right) x_1^{\frac{2}{3}} = p_2 \lambda \end{aligned}$$

We know that these FOCs must have strict equality due to the same reasons as stated above. Using these FOCs, the following can be derived:

$$2x_2 p_2 = x_1 p_1$$

which implies that:

$$x_1^2 = \frac{2}{3p_1}(p_1e_1 + p_2e_2) \quad x_2^2 = \frac{1}{3p_1}(p_1e_1 + p_2e_2)$$

Using, previous results, we can find that:

$$z_1(\mathbf{p}) = 1 + \frac{p_2}{2p_1} + \frac{4}{3} + \frac{2p_2}{p_1} - 4 = \frac{5p_2}{2p_1} - \frac{5}{3}$$

$$z_2(\mathbf{p}) = \frac{1}{2} + \frac{p_1}{p_2} + \frac{2p_1}{3p_2} + 1 - 4 = \frac{5p_1}{3p_2} - \frac{5}{2}$$

Note that this implies that $3p_2 = 2p_1$, as this is the only relative price that makes Walras' law hold. Thus, we see that:

$$x^W = \{(x_1, x_2)\} = \left\{ \left(\frac{4}{3}, 2\right), \left(\frac{8}{3}, 2\right) \right\}$$

and since we have only one relative price, we can find that the set of Walrasian equilbirum is

$$W(\mathcal{E}) = \bigcup_{p^*} x^W(p^*(\mathcal{E}), \mathcal{E}) = \left\{ \left(\frac{4}{3}, 2\right), \left(\frac{8}{3}, 2\right) \right\}$$

i

Note that the optimal values of x_1 and x_2 are only dependent on prices, we are interested in solving the cases purely dependent on making the value of the endowments equal to each other.

First Economy

since we know that $p_1 = p_2$, and we are interested in equating market values to each other, we should know that:

$$p_1(e_1^1 + e_2^1) = p_2(e_1^2 + p_2^2) \implies (e_1^1 + e_2^1) = (e_1^2 + p_2^2)$$

Let $T = (T_1, T_2)$ where each T_i represents the transfer from each individual 1 to individual 2 in respect to each good, we can see that:

$$2 - T_1 + 1 - T_2 = 2 + T_1 + 3 + T_2$$

However, since prices are equal, we can solve with respect with each good. Thus, we can see that:

$$2 - T_1 = 2 + T_1 \iff T_1 = 0 \quad 1 - T_2 = 3 + T_2 \iff T_2 = -1$$

Thus, $T = (0, -1)$, which means that no transfer of good is done, but one x_1 is transfered from person 2 to person 1, with leads to:

$$\mathbf{e^1} = \mathbf{e^2} = (2, 2)$$

Second Economy

Now. we see that $3p_2 = 2p_1$, using a similar logic to that above and same definition of T, we see that:

$$p_1(2 - T_1) + p_2(1 - T_2) = p_1(2 + T_1) + p_2(3 + T_1)$$
$$\frac{3p_2}{2}(2 - T_1) + p_2(1 - T_2) = \frac{3p_2}{2}(2 + T_1) + p_2(3 + T_1)$$

Using a similar arguement, we find that $T = (0, -1)$, which means that we obtain the same individual endowment as before.

3

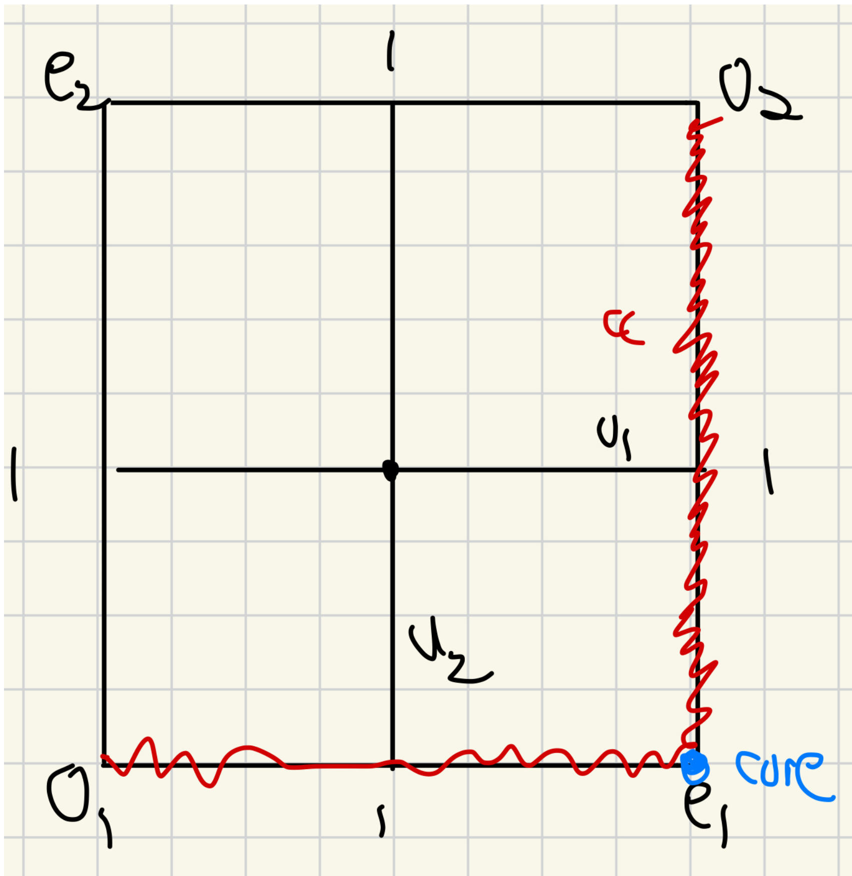

Figure 2: Graph for 3a - 3d

a

See Graph

b

Note that equating the MRSes here is not ideal due to the given utility functions, so a more logical arguement must be used. We can see that in order for each consumer to maximize their utility, consumer 1 and 2 should have only have good 1 and 2 respectively. Thus, the only pareto optimal points would be the bottom right corners of the edgeworth box, where we would draw the indfference curves as seen as below, making sure that consumer's

1 indifference curve goes through the bottom right corner of the indifference point and consumer's 2 indifference curve goes thorugh the top left corner of the Edgeworth Box. A graph can be seen above, where the red line is the contract curve.

c

The core would be characterized as $\{e^1, e^2\} = \{(2,0),(0,2)\}$, as moving away from this allocation will cause a coalition to reject it.

d

Since the Walrasian Equilbrium is a subset of the core, we find that that the only allocation in Walrasian Equilbirum is:

$$x^W(p^*(\mathcal{E}), \mathcal{E}) = \{(2,0),(0,2)\}$$

Thus, we see that:

$$W(\mathcal{E}) = \bigcup_{p^*}(p^*(\mathcal{E}), \mathcal{E}) = \{(2,0),(0,2)\}$$

e

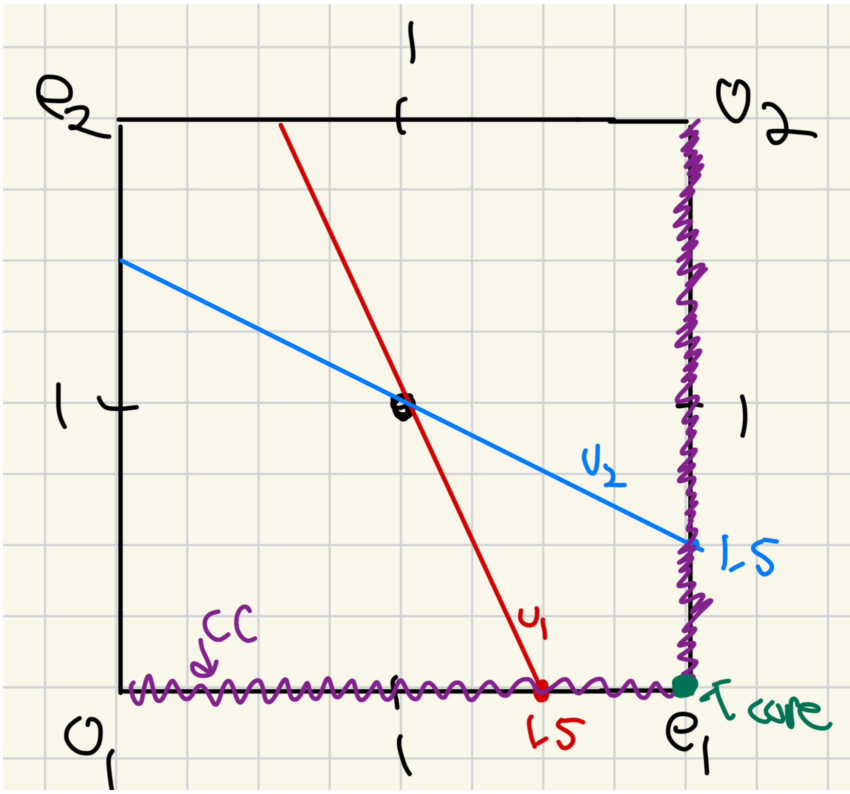

Figure 3: Graph for 3e

a

See above

b

Note that each consumer maximize utilty if they consume only good x_1 and x_2 respectively. Thus, we see that the contract curve follows.

c

Same logic in the previous exchange economy, which yields the same answer as above.

d

Same logic in the previous exchange economy, which yields the same answer as above.

f

We can consider the 3 cases, $\alpha = 1, \alpha < 1, \alpha > 1$.

- If $\alpha > 1$ we are left with the case above, as consumer 1 and 2 value x_1 and x_2 more heavily as we see respectively.
- If $\alpha < 1$ we see that the order of preference between x_1, x_2 is changed between consumer 1 and 2. Thus, we would get a very similar result, but we would have to swap the Walrsian equilbrium, which yields the result:

  $$W(\mathcal{E}) = \bigcup_{p^*}(p^*(\mathcal{E}), \mathcal{E}) = \{(0,2),(2,0)\}$$

  and we would simply reflect our core and contract curve over the line that connects the bottom left and top right corners of the edgeworth box.
- If $\alpha = 1$, we see that both consumers value x_1 and x_2 equally. Thus, we are left the graph seen in **2**

4

a

We can note that as we are all using the minimum function, we can see that $x_1^1 = x_2^1 = x_2^2 = x_3^2 = x_1^3 = x_3^3$, which implies that we are interested in the allocation:

$$x_1^1 = x_2^1 = x_2^2 = x_3^2 = x_1^3 = x_3^3 = 0.5$$

which is optimal as moving away would cause utilty lose and maximizes utlity for all and markets clear. Thus, this allocation is in the core, and let this allocation equal $\overline{x}$

b

Consider the UMP for consumer 1. We are interested in the following optimization problem:

$$\begin{aligned} \max \quad & \min\{x_1^1, x_2^1\} \\ \text{s.t} \quad & x_1^1 p_1 + p_1 x_2^1 \leq p e_1 \end{aligned}$$

note that we want $x_1^1 = x_2^1$, which implies using the strict equality of the constraint (which we know is true by the AU assumption) that

$$x_1^1 = x_2^1 = \frac{p_1}{p_1 + p_2}$$

using a similar logic, we can see that:

$$x_2^2 = x_3^2 = \frac{p_2}{p_2 + p_3}$$

and

$$x_1^3 = x_3^3 = \frac{p_3}{p_1 + p_3}$$

Thus, we can now analyze the aggegate demand functions. We find that $z_i = x_i^1 + x_i^2 + x_i^3 - 1$ where $i \in \{1, 2, 3\}$. After some algebra and using the Marshallian demand functions taht we derived, we find that $p_1 = p_2 = p_3$. This implies that:

$$x_1^1 = x_2^1 = x_2^2 = x_3^2 = x_1^3 = x_3^3 = 0.5$$

Thus, we can see that $p^*(\mathcal{E}) = \{(p_1, p_2, p_3) \in \mathbb{R}^3 | p_1 = p_2 = p_3\}$. Thus, we can see tha $x^W(p^*(\mathcal{E}), \mathcal{E}) = \{(0.5, 0.5, 0), (0, 0.5, 0.5), (0.5, 0, 0.5)\}$ Since we are working with only one relative price, we can note that

$$W(\mathcal{E}) = \bigcup_{p^*} x^W(p^*(\mathcal{E}), \mathcal{E}) = \{(0.5, 0.5, 0), (0, 0.5, 0.5), (0.5, 0, 0.5)\}$$

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