MATH 27300 Notes

Anthony Yoon

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1 Lecture 1

1.1 Prereqs

This class assumes familiarity in linear algebra, calculus, and analysis. The concepts that are of note are as follows:

1.1.1 Linear Algebra Prereqs

We assume familiarity with

- $\mathbb{R}^n$ and abstract vectors spaces
- Determinants, traces. and the characteristic polynomial.
- Jordan Normal Form and Jordan Block Form

These have the following definitions/theorems:

Definition 1.1. *An abstract vector space V is a set over a field F equipped with vector addition and scalar multiplication. It has the following properties for all $u, v, w \in V$ and all $a, b \in F$:*

- *Closure of vector addition:* $u + v \in V$
- *Commutativity of vector addition:* $u + v = v + u$
- *Associativity of vector addition:* $u + (v + w) = (u + v) + w$
- *Additive Identity: There exists an element* $0 \in V$ *such that* $0 + v = v$
- *Additive Inverse: for all* u*, there exists a* $-u \in V$ *such that* $u + -u = 0$
- *Closure of scalar multiplication:* $av \in V$
- *Associativity of scalar multiplication:* $a(bu) = (ab)u$
- *Identity scalar: There exists a element* $1 \in F$ *such that* $1u = u$
- *Distributivity over scalar multiplication:* $(a + b)u = au + bu$
- *Distributivity over vector addition:* $a(u + v) = au + vu$

Definition 1.2. *Linear Transformation is a function between two vector spaces that preserves vector addition and scalar multiplication.*

Definition 1.3. *The trace is defined to be for a matrix as* $Tr(A) = \sum a_{ii} = \sum \lambda$

Definition 1.4. *Characteristic Polynomial of a matrix is* $\det(A - I\lambda)$

Definition 1.5. *Jordan Normal Form for a* $A \in \mathbb{R}^{n\times n}$ *is a matrix in the form*

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{pmatrix}$$

where J_k *is similar to As*

1.1.2 Calculus / Analysis

- Familiarity of the manipulation of complex numbers such as $z = x + iy = re^{i\theta}$ and $e^{i\pi} = -1$
- Evaluating intgrals and partial fraction decomposition
- Metric Spaces, $\epsilon - \delta$ contuinity
- Compact sets and complete metric space.
- Implicit and inverse function theorem.

Definition 1.6 (Metric Space)**.** *A metric space is a pair* (X, d) *where* $d : X \times X \to \mathbb{R}$ *satisfies for all* $x, y, z \in X$*:*

$$\begin{aligned} &d(x,y) \geq 0, \\ &d(x,y) = 0 \iff x = y, \\ &d(x,y) = d(y,x), \\ &d(x,z) \leq d(x,y) + d(y,z). \end{aligned}$$

Definition 1.7. *A function is continous if for all* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that* $|a - b| < \delta, |f(a) - f(b)| < \epsilon$

Definition 1.8. *Let* (X, d) *be a metric space. We say a set* A *is a compact set in* X *if for every open cover contains a finite subcover.*

Remark 1.1. *A subset of* $\mathbb{R}^n$ *is compact if and only if it is closed and bounded. (Heine Borel)*

Definition 1.9. *A subset* $K \subseteq X$ *is said to be* sequentially compact *if every sequence in* K *has a convergent subsequence whose limit lies in* K*.*

Theorem 1.1 (Inverse Function Theorem). *Let $U \subseteq \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}^n$ be continuously differentiable. Suppose $x_0 \in U$ and the derivative $Df(x_0)$ is invertible. Then there exist open neighborhoods V of x_0 and W of $f(x_0)$ such that $f : V \to W$ is a bijection with a continuously differentiable inverse $f^{-1} : W \to V$. Moreover,*

$$D(f^{-1})(f(x_0)) = [Df(x_0)]^{-1}.$$

Theorem 1.2 (Implicit Function Theorem). *Let $U \subseteq \mathbb{R}^{n+m}$ be open and let $F : U \to \mathbb{R}^m$ be continuously differentiable. Suppose there exists a $(x_0, y_0) \in U$ that satisfies*

$$F(x_0, y_0) = 0$$

and the partial derivative $D_y F(x_0, y_0)$ is invertible. Then there exist open neighborhoods $V \subseteq \mathbb{R}^n$ of x_0 and $W \subseteq \mathbb{R}^m$ of y_0 and a continuously differentiable function $g : V \to W$ such that

$$F(x, g(x)) = 0 \quad \textit{for all } x \in V,$$

and $g(x_0) = y_0$.

1.2 Introduction to ODEs

There are two types of differential equations: Ordinary and Partial. Ordinary differential equations have deriatives with one variable; partial has deriatives with resepect to many variables.

Typically with ODEs, we want to find a function φ subject to some constraints on its values and deriatives with respect to one variable.

Definition 1.10. *ϕ is a mapping from I to U where $I \subseteq \mathbb{R}$ and $U \subseteq \mathbb{R}^n$. We can think of I as a time axis.*

Typically, ODEs have a solution φ that solves the equation. This is often referred to as

$$F(t, \varphi(t), \varphi^{(1)}(t), \ldots, \varphi^{(n)}(t)) = 0$$

where n denotes the highest order deriative. This is the order of the ODE.

Remark 1.2. *There exists an maneuver that allows us to reduce equations of arbitary order to a system of ifrst order equations.* [1]. *This means that any theorems that are proven for first order ODEs implies that it holds for higher order definitions.*

These equations can be studied using the notion of vector fields.

Definition 1.11. *Let $U \subseteq \mathbb{R}^n$ be an open set. A vector field on U is a function $\vec{v} : U \to \mathbb{R}^n$. This is analogous to assigning a vector to every point in the set U*

[1] Will be covered later

Definition 1.12. *A time depedent vector field is a function $I \times U \to \mathbb{R}^n$ where $I \subseteq \mathbb{R}$ is an open interval. A vector field that does not have a dependence on time is an autonomous vector field.*

Definition 1.13. *The set $U \subseteq \mathbb{R}^n$ is called the phase space of the vector field. We can denote $I \times U$ as the extended phase space of an ODE. The extended phase space only applies to time depeendent vector fields.*

Definition 1.14. *The integral curve of $\vec{v}(t, \vec{x})$ on U is a differentiable function varphi : $I \subseteq \mathbb{R}^n \to U \supseteq \mathbb{R}^n$ such that $\varphi'(t) = \vec{v}(t, \varphi(t))$ Ommiting t here is equivalent for autonmous vector fields.* [2]

Remark 1.3. *The above means that for a vector field, the associated ODE is $\varphi'(t) = \vec{v}(t, \varphi(t))$. The integral curve itself is a solution to the ODE.*

Definition 1.15. *The phase curve is the trajectory of the particle through space.*

Remark 1.4. *The graph of an integral curve naturally sits in an extended phase space.*

The above holds because $T_\varphi = \{(t, \varphi(t) \in I \times U)\}$, which is exactly the graph of the integral curve.

1.3 Example of ODEs

Consider a 1D ODE. This means that $U \to \mathbb{R}$ and $V : \mathbb{R} \to \mathbb{R}$.

1.3.1 Constant Vector Field

Consider the vector field v where $v(x) = a$ where a is some constant. A diagram is provided below We thus have to find an integral curve $\varphi : I \to \mathbb{R}$ has to ssastify $\varphi(t)$ has to sastify $d\varphi(t)/dt = a$. This implies that $\varphi(t) = at + b$ determined by condition $\varphi(0) = b$

Definition 1.16. *Classicaly, we denote $\dot{x} = d\varphi/dt$ and $\ddot{x} = d^2\varphi/dt^2$*

1.3.2 Equation of normal growth

For some $k \neq 0$, consider the vector field $v(x) = kx$. We can see that the ODE here is $\dot{x} = kx$. Therefore, we can see that the solution to this ODE is $x(t) = ce^{kt}$ where $x(0) = c$

[2] The vector field here assigns a "velocity vector" to each point. So in a sense, the vector field dictates the form of the ODE.

1.4 Solving solution to ODEs

So far, we provided intuition based solutions. But, we have no introduced actual solutions to these problems. We have the following "Informal Method", where if we follow the follwing computations:

$$\begin{aligned}
\frac{dx}{dt} &= kx \\
\frac{dx}{x} &= kdt \\
\int \frac{dx}{x} &= k \int dt \\
\log(x) + c = k(t + c') &\iff x = Ce^{kt}
\end{aligned}$$

where C is a function of c and c'. Additionally, we can find solutions to ODEs using ones we already know. For example, if we know that $\dot{x} = kx$ and we let $y(t) = e^{-kt} \cdot x(t)$. Note that:

$$\dot{y} = -ke^{-kt}x + e^{-kt} \cdot \dot{x} \cdot kx = (k - k)e^{-kt}x = 0$$

which implies that $y(t) = c$ for some constant.