PSET 3

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1/29/2025

1

a

We are interested in the following optimization problem:

$$\max \quad p x_1^{\frac{1}{3}} x_2^{\frac{1}{3}} - \omega_1 x_1 - \omega_2 x_2$$

We see that the FOCs are

$$\begin{aligned}
&[x_1] \quad \frac{1}{3} p x_1^{-\frac{2}{3}} x_2^{\frac{1}{3}} - \omega_1 \leq 0 \quad \text{for } x_1 \geq 0 \\
&[x_2] \quad \frac{1}{3} p x_1^{\frac{1}{3}} x_2^{-\frac{2}{3}} - \omega_2 \leq 0 \quad \text{for } x_1 \geq 0
\end{aligned}$$

We can see that $x_1, x_2, \neq 0$ as this would cause the FOCs to become undefined. From here, divide the FOCs to get the relation $\omega_1 x_1 = \omega_2 x_2$ Using this expression, we can substiute this into the FOCs to get that

$$x_1^* = \frac{p^3}{27\omega_1^2\omega_2} \quad x_2^* = \frac{p^3}{27\omega_1\omega_2^2}$$

Therefore, we see that:

$$y^* = \left(\frac{p^6}{3^6\omega_1^3\omega_2^3}\right)^{\frac{1}{3}} = \frac{p^2}{9\omega_1\omega_2}$$

Thus, we see that

$$PF = py^* - \omega_1 x_1^* - \omega_2 x_2^* = p\left(\frac{p^2}{9\omega_1\omega_2}\right) - \omega_1 \frac{p^3}{27\omega_1^2\omega_2} - \omega_2 \frac{p^3}{27\omega_1\omega_2^2} = \frac{p^3}{27\omega_1\omega_2}$$

b

We can see that for the IDFs

$$\frac{\partial x_1^*}{\partial \omega_1} = -2\left(\frac{p^3}{27\omega_2\omega_1^3}\right)$$

and

$$\frac{\partial x_2^*}{\partial \omega_2} = -2\left(\frac{p^3}{27\omega_1\omega_2^3}\right)$$

Note that both quantities are bounded above by 0, as p, ω are strictly positive. For the ODF, we see that

$$\frac{\partial y^*}{\partial p} = \frac{2p}{9\omega_1\omega_2}$$

which is always positive for the same reasons. For the PF, note that

$$\frac{\partial \pi(\omega, y)}{\partial p} = \frac{p^2}{9\omega_1\omega_2} > 0 \quad \frac{\partial \pi(\omega, p)}{\partial \omega_1} = \frac{-p^3}{27\omega_1^2\omega_2} < 0 \quad \frac{\partial \pi(\omega, p)}{\partial \omega_2} = \frac{-p^3}{27\omega_1\omega_2^2} < 0$$

c

Proof that IDF is homogenous in degree 0, let $t > 0$, we see that

$$x_1^*(t\omega, tp) = \frac{(tp)^3}{27(t\omega_1)^2 t\omega_2} = \frac{t^3p^3}{27t^3\omega_1^2\omega^2} = \frac{p^3}{27\omega_1^2\omega_2} = x_1^*(\omega, p)$$

and similarly

$$x_2^*(t\omega, tp) = \frac{(tp)^3}{27t\omega_1(t\omega_2)^2} = \frac{t^3p^3}{27t^3\omega_1\omega_2^2} = \frac{p^3}{27\omega_1\omega_2^2} = x_2^*(\omega, p)$$

Proof that OSF is homogenous in degree 0, let $t > 0$, we see that

$$y^*(t\omega, tp) = \frac{t^2p^2}{9t^2\omega_1\omega_2} = \frac{p^2}{9\omega_1\omega_2} = y^*(\omega, p)$$

Proof that PF is homogenous in degree 1, let $t > 0$, we see that

$$\pi(\mathbf{t}\omega, pt) = \frac{t^3p^3}{27t^2\omega_1\omega_2} = \frac{tp^3}{27\omega_1\omega_2} = t\pi(\omega, p)$$

d

To see if Hotelling's Lemma holds, note that

$$\frac{\partial \pi(\omega, p)}{\partial p} = \frac{3p^2}{27\omega_1\omega_2} = \frac{p^2}{9\omega_1\omega_2} = y^*$$

and

$$\frac{\partial \pi}{\partial \omega_1} = \frac{-p^3}{27\omega_1^2\omega_2} = -x_1^*$$

and

$$\frac{\partial \pi}{\partial \omega_1} = \frac{-p^3}{27\omega_1\omega_2^2} = -x_2^*$$

e

REDO If $\alpha = \beta = 0.5$, this is a Cobb Douglas function. Thus, we can use the function derived from the notes to see that the FOCs yields:

$$p = \frac{\omega_1^{\frac{1}{2}}\omega_2^{\frac{1}{2}}}{0.25(0.25)} = 16$$

Thus, if price is less than 16, there has exists no solution as there is no profit to be found for any level of production.

f

If $\alpha + \beta = 1$, we see that we are left with the Cobb Douglas function where the fucntion can be only be derived based on external given factors $(\omega, p, etc.)$. So if $\alpha + \beta \neq 1$, then we can derive a solution.

2

Note that cost is minized when $\alpha x_1 = \beta x_2 = y$. This is because we are working with a minimum function, a similar arguement of that to PSET 2 Q3b. This implies that

$$x_1^* = \frac{y}{\alpha} \quad x_2^* = \frac{y}{\beta}$$

Therefore, we can see that with the given assumptions that

$$c(\omega, y) = \frac{y}{\alpha} + \frac{y}{\beta}$$

Thus, we can see that we are interested in the following profit maxization problem:

$$\max_y py - y\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = \max_y y\left(p - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\right)$$

which implies that profit is dependent on exogenously given parameters, or rather we are in the form of *price times(output - input)*. Thus, we can see that for our OSF, which we can derive because we are soley focused on output versus input:

$$y^* = \begin{cases} \text{undefined} & p > \frac{1}{\alpha} + \frac{1}{\beta} \text{ as firms cannot have infinite output} \\ [0, \infty) & p = \frac{1}{\alpha} + \frac{1}{\beta} \text{ as there is 0 profit} \\ 0 & p < \frac{1}{\alpha} + \frac{1}{\beta} \text{ as cost} > \text{price} \end{cases}$$

Similarly, since IDF is dependent on the OSF, we can see that **similarly for the same reasons**

$$x_1^* = \begin{cases} \text{undefined} & p > \frac{1}{\alpha} + \frac{1}{\beta} \\ \frac{y}{\alpha} & p = \frac{1}{\alpha} + \frac{1}{\beta} \\ 0 & p < \frac{1}{\alpha} + \frac{1}{\beta} \end{cases}$$

$$x_2^* = \begin{cases} \text{undefined} & p > \frac{1}{\alpha} + \frac{1}{\beta} \\ \frac{y}{\beta} & p = \frac{1}{\alpha} + \frac{1}{\beta} \\ 0 & p < \frac{1}{\alpha} + \frac{1}{\beta} \end{cases}$$

Thus, this implies that the profit function is:

$$\pi^* = \begin{cases} \text{undefined} & p > \frac{1}{\alpha} + \frac{1}{\beta} \\ 0 & p \leq \frac{1}{\alpha} + \frac{1}{\beta} \end{cases}$$

3

Note that the following production function, or the perfect subsitute production function:

$$y = \alpha x_1 + \beta x_2$$

we can exchange between x_1 and x_2. From here, we can see that cost will be minimized if we purchase only of the cheaper of the 2 goods. To prove that, we see that we are interested in the following cost minimization problem

$$\begin{aligned} \min \quad & \omega_1 x_1 + \omega_2 x_2 \\ \text{s.t} \quad & y \leq \alpha x_1 + \beta x_2 \end{aligned}$$

However, plugging the constraint into the object function yields the following optimization problem:

$$\min_{x_1} \quad \omega_1 x_1 + \omega_2 \left(\frac{y - \alpha x_1}{\beta} \right)$$

From here, we differeniate with respect to x_1, we can se that we get

$$\omega_1 - \frac{\omega_2 \alpha}{\beta}$$

However, note that this quantity is dependent on parameters, so we can make the following deductions:

- If $\omega_1 - \frac{\omega_2\alpha}{\beta} < 0$, we can see that increasing the input of x_1 will decrease cost, so we can see that in this case $x_1 = \frac{y}{\alpha}$ and $x_2 = 0$
- If $\omega_1 - \frac{\omega_2\alpha}{\beta} > 0$, we can see that increasing the input of x_1 will increase cost, so we can see that in this case $x_2 = \frac{y}{\beta}$ and $x_1 = 0$
- If $\omega_1 = \frac{\omega_2\alpha}{\beta}$, we can see that any input will give us the optimal amount. So this implies that $x_1 \in [0, \frac{y}{\alpha}]$ and $x_2 = \frac{y-\alpha x_1}{\beta}$

Thus, we can see that cost is minimized when we choose the minimum of the inputs, or rather

$$c(\omega, y) = \min\left\{\frac{\omega_1 y}{\alpha}, \frac{\omega_2 y}{\beta}\right\}$$

For notational sake, let us call $c(\omega, y) = C$ Note that we are know interetsed in the following profit maximaization problem:

$$\max y(1 - C)$$

So we can see that our ODF (for same reasons as 2)

$$y^* = \begin{cases} \text{undefined} & C < 1 \\ [0, \infty] & C = 1 \\ 0 & C > 1 \end{cases}$$

and using the proof above and let $W = \omega_1 - \frac{\omega_2\alpha}{\beta}$, we can see that

$$x_1^* = \begin{cases} 0 & W < 0 \text{ and } C = 1 \\ [0, \frac{y}{\alpha}] & W = 0 \text{ and } C = 1 \\ \frac{y}{\alpha} & W > 1 \text{ and } C = 1 \\ \text{undefined} & C < 1 \\ 0 & C > 1 \end{cases}$$

$$x_2^* = \begin{cases} 0 & W > 0 \text{ and } C = 1 \\ [0, \frac{y}{\beta}] & W = 0 \text{ and } C = 1 \\ \frac{y}{\beta} & W < 1 \text{ and } C = 1 \\ \text{undefined} & C < 1 \\ 0 & C > 1 \end{cases}$$

Thus, we can see our profit function is

$$\pi(\omega, p) = \begin{cases} \text{undefined} & C < 1 \\ 0 & C \geq 1 \end{cases}$$

4

a

If we fix $x_2 = 1$, we can see that

$$y = x_1^\alpha \iff x_1^* = y^{\frac{1}{\alpha}}$$

This implies that

$$sc(\omega, y) = \omega_2 + \omega_1 y^{\frac{1}{\alpha}}$$

We can see that if $\alpha = 1$, we get constant return to scale, and if $\alpha > 1$ we can see we get decreasing return to scale and $\alpha < 1$ we get increasing return to scale. Now we solve the profit maximaization case. where we want to

$$py - \omega_2 - \omega_1 y^{\frac{1}{\alpha}}$$

Now, we are interested in the profit maximizing short run supply function. If we let $\alpha = 1$, we get the equation:

$$py - \omega_2 - \omega_1 y \iff y(p - \omega_1) - \omega_2$$

which implies that ω_2 is a fixed cost that the company must pay for. Thus, we can see the following production functions.

$$y^* = \begin{cases} \text{undefined} & p > \omega_1 \\ [0, \infty) & p = \omega_1 \\ 0 & p < \omega_1 \end{cases}$$

If $\alpha > 1$, note the following. Let us look at the marginal cost.

$$\frac{\partial sc}{\partial y} = -\frac{\omega_2}{y^2} - \left(\frac{1-\alpha}{\alpha}\right) \omega_1 y^{\frac{1-2\alpha}{\alpha}}$$

Note that with $\alpha > 1$, we can see that the above quantity is always negative, which implies that average cost is strictly less than that of marginal cost, which means the firm would want to produce infinite amounts. If $\alpha < 1$, we know that