PSET 3

Anthony Yoon

1/29/2025

1

a

We are interested in the following optimization problem:

$$\max \quad px_1^{\frac{1}{3}}x_2^{\frac{1}{3}} - \omega_1 x_1 - \omega_2 x_2$$

We see that the FOCs are

$$\begin{aligned}
[x_1] \quad & \frac{1}{3}px_1^{-\frac{2}{3}}x_2^{\frac{1}{3}} - \omega_1 \leq 0 \quad \text{for } x_1 \geq 0 \\
[x_2] \quad & \frac{1}{3}px_1^{\frac{1}{3}}x_2^{-\frac{2}{3}} - \omega_2 \leq 0 \quad \text{for } x_1 \geq 0
\end{aligned}$$

We can see that $x_1, x_2, \neq 0$ as this would cause the FOCs to become undefined. From here, divide the FOCs to get the relation $\omega_1 x_1 = \omega_2 x_2$ Using this expression, we can substiute this into the FOCs to get that

$$x_1^* = \frac{p^3}{27\omega_1^2\omega_2} \quad x_2^* = \frac{p^3}{27\omega_1\omega_2^2}$$

Therefore, we see that:

$$y^* = \left(\frac{p^6}{3^6\omega_1^3\omega_2^3}\right)^{\frac{1}{3}} = \frac{p^2}{9\omega_1\omega_2}$$

Thus, we see that

$$PF = py^* - \omega_1 x_1^* - \omega_2 x_2^* = p\left(\frac{p^2}{9\omega_1\omega_2}\right) - \omega_1\frac{p^3}{27\omega_1^2\omega_2} - \omega_2\frac{p^3}{27\omega_1\omega_2^2} = \frac{p^3}{27\omega_1\omega_2}$$

b

We can see that for the IDFs

$$\frac{\partial x_1^*}{\partial \omega_1} = -2\left(\frac{p^3}{27\omega_2\omega_1^3}\right)$$

and

$$\frac{\partial x_2^*}{\partial \omega_2} = -2\left(\frac{p^3}{27\omega_1\omega_2^3}\right)$$

Note that both quantities are bounded above by 0, as p, ω are strictly positive. For the ODF, we see that

$$\frac{\partial y^*}{\partial p} = \frac{2p}{9\omega_1\omega_2}$$

which is always positive for the same reasons. For the PF, note that

$$\frac{\partial \pi(\omega, y)}{\partial p} = \frac{p^2}{9\omega_1\omega_2} > 0 \quad \frac{\partial \pi(\omega, p)}{\partial \omega_1} = \frac{-p^3}{27\omega_1^2\omega_2} < 0 \quad \frac{\partial \pi(\omega, p)}{\partial \omega_2} = \frac{-p^3}{27\omega_1\omega_2^2} < 0$$

c

Proof that IDF is homogenous in degree 0, let $t > 0$, we see that

$$x_1^*(t\omega, tp) = \frac{(tp)^3}{27(t\omega_1)^2 t\omega_2} = \frac{t^3p^3}{27t^3\omega_1^2\omega^2} = \frac{p^3}{27\omega_1^2\omega_2} = x_1^*(\omega, p)$$

and similarly

$$x_2^*(t\omega, tp) = \frac{(tp)^3}{27t\omega_1(t\omega_2)^2} = \frac{t^3p^3}{27t^3\omega_1\omega_2^2} = \frac{p^3}{27\omega_1\omega_2^2} = x_2^*(\omega, p)$$

Proof that OSF is homogenous in degree 0, let $t > 0$, we see that

$$y^*(t\omega, tp) = \frac{t^2p^2}{9t^2\omega_1\omega_2} = \frac{p^2}{9\omega_1\omega_2} = y^*(\omega, p)$$

Proof that PF is homogenous in degree 1, let $t > 0$, we see that

$$\pi(\mathbf{t}\omega, pt) = \frac{t^3p^3}{27t^2\omega_1\omega_2} = \frac{tp^3}{27\omega_1\omega_2} = t\pi(\omega, p)$$

d

To see if Hotelling's Lemma holds, note that

$$\frac{\partial \pi(\omega, p)}{\partial p} = \frac{3p^2}{27\omega_1\omega_2} = \frac{p^2}{9\omega_1\omega_2} = y^*$$

and

$$\frac{\partial \pi}{\partial \omega_1} = \frac{-p^3}{27\omega_1^2\omega_2} = -x_1^*$$

and

$$\frac{\partial \pi}{\partial \omega_1} = \frac{-p^3}{27\omega_1\omega_2^2} = -x_2^*$$

e

If $\alpha = \beta = 0.5$, this is a Cobb Douglas function. Thus, we can use the function derived from the notes to see that the FOCs yields:

$$p = \frac{\omega_1^{\frac{1}{2}}\omega_2^{\frac{1}{2}}}{0.25(0.25)} = 16$$

Thus, if price is less than 16, there has exists no solution as there is no profit to be found for any level of production.

f

If $\alpha + \beta = 1$, we see that we are left with the Cobb Douglas function where the fucntion can be only be derived based on external given factors $(\omega, p, etc.)$. So if $\alpha + \beta \neq 1$, then we can derive a solution.

2

Note that cost is minized when $\alpha x_1 = \beta x_2 = y$. This is because we are working with a minimum function, a similar arguement of that to PSET 2 Q3b. This implies that

$$x_1^* = \frac{y}{\alpha} \quad x_2^* = \frac{y}{\beta}$$

Therefore, we can see that with the given assumptions that

$$c(\omega, y) = \frac{y}{\alpha} + \frac{y}{\beta}$$

Thus, we can see that we are interested in the following profit maxization problem:

$$\max_y py - y\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = \max_y y\left(p - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\right)$$

which implies that profit is dependent on exogenously given parameters, or rather we are in the form of *price times(output - input)*. Thus, we can see that for our OSF, which we can derive because we are soley focused on output versus input:

$$y^* = \begin{cases} \text{undefined} & p > \frac{1}{\alpha} + \frac{1}{\beta} \text{ as firms cannot have infinite output} \\ [0, \infty) & p = \frac{1}{\alpha} + \frac{1}{\beta} \text{ as there is 0 profit} \\ 0 & p < \frac{1}{\alpha} + \frac{1}{\beta} \text{ as cost} > \text{price} \end{cases}$$

Similarly, since IDF is dependent on the OSF, we can see that **similarly for the same reasons**

$$x_1^* = \begin{cases} \text{undefined} & p > \frac{1}{\alpha} + \frac{1}{\beta} \\ \frac{y}{\alpha} & p = \frac{1}{\alpha} + \frac{1}{\beta} \\ 0 & p < \frac{1}{\alpha} + \frac{1}{\beta} \end{cases}$$

$$x_2^* = \begin{cases} \text{undefined} & p > \frac{1}{\alpha} + \frac{1}{\beta} \\ \frac{y}{\beta} & p = \frac{1}{\alpha} + \frac{1}{\beta} \\ 0 & p < \frac{1}{\alpha} + \frac{1}{\beta} \end{cases}$$

Thus, this implies that the profit function is:

$$\pi^* = \begin{cases} \text{undefined} & p > \frac{1}{\alpha} + \frac{1}{\beta} \\ 0 & p \leq \frac{1}{\alpha} + \frac{1}{\beta} \end{cases}$$

3

Note that the following production function, or the perfect subsitute production function:

$$y = \alpha x_1 + \beta x_2$$

we can exchange between x_1 and x_2. From here, we can see that cost will be minimized if we purchase only of the cheaper of the 2 goods. To prove that, we see that we are interested in the following cost minimization problem

$$\begin{aligned} \min \quad & \omega_1 x_1 + \omega_2 x_2 \\ \text{s.t} \quad & y \leq \alpha x_1 + \beta x_2 \end{aligned}$$

However, plugging the constraint into the object function yields the following optimization problem:

$$\min_{x_1} \quad \omega_1 x_1 + \omega_2 \left(\frac{y - \alpha x_1}{\beta} \right)$$

From here, we differeniate with respect to x_1, we can se that we get

$$\omega_1 - \frac{\omega_2 \alpha}{\beta}$$

However, note that this quantity is dependent on parameters, so we can make the following deductions:

- If $\omega_1 - \frac{\omega_2 \alpha}{\beta} < 0$, we can see that increasing the input of x_1 will decrease cost, so we can see that in this case $x_1 = \frac{y}{\alpha}$ and $x_2 = 0$
- If $\omega_1 - \frac{\omega_2 \alpha}{\beta} > 0$, we can see that increasing the input of x_1 will increase cost, so we can see that in this case $x_2 = \frac{y}{\beta}$ and $x_1 = 0$
- If $\omega_1 = \frac{\omega_2 \alpha}{\beta}$, we can see that any input will give us the optimal amount. So this implies that $x_1 \in [0, \frac{y}{\alpha}]$ and $x_2 = \frac{y - \alpha x_1}{\beta}$

Thus, we can see that cost is minimized when we choose the minimum of the inputs, or rather

$$c(\omega, y) = \min \left\{ \frac{\omega_1 y}{\alpha}, \frac{\omega_2 y}{\beta} \right\}$$

For notational sake, let us call $c(\omega, y) = C$ Note that we are know interetsed in the following profit maximaization problem:

$$\max y(1 - C)$$

So we can see that our ODF (for same reasons as 2)

$$y^* = \begin{cases} \text{undefined} & C < 1 \\ [0, \infty] & C = 1 \\ 0 & C > 1 \end{cases}$$

and using the proof above and let $W = \omega_1 - \frac{\omega_2 \alpha}{\beta}$, we can see that

$$x_1^* = \begin{cases} 0 & W < 0 \text{ and } C = 1 \\ [0, \frac{y}{\alpha}] & W = 0 \text{ and } C = 1 \\ \frac{y}{\alpha} & W > 1 \text{ and } C = 1 \\ \text{undefined} & C < 1 \\ 0 & C > 1 \end{cases}$$

$$x_2^* = \begin{cases} 0 & W > 0 \text{ and } C = 1 \\ [0, \frac{y}{\beta}] & W = 0 \text{ and } C = 1 \\ \frac{y}{\beta} & W < 1 \text{ and } C = 1 \\ \text{undefined} & C < 1 \\ 0 & C > 1 \end{cases}$$

Thus, we can see our profit function is

$$\pi(\omega, p) = \begin{cases} \text{undefined} & C < 1 \\ 0 & C \geq 1 \end{cases}$$