

Problem 1.

(1)

Solution: The Langrangian is as follows:

$$L = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{i=0}^{\infty} \lambda_t (w_t - c_t - w_{t+1})$$

with the following FoCs

$$\begin{aligned} [c_t] \quad & \beta^t u'(c_t) = \lambda_t \\ [\lambda_t] \quad & w_t = c_t + w_{t+1} \\ [w_{t+1}] \quad & \lambda_{t+1} = \lambda_t \end{aligned}$$

We can see that we can derive the Euler equation,

$$\begin{aligned} \beta^{t+1} u'(c_t) &= \beta^t u'(c_t) \\ \beta u'(c_{t+1}) &= u'(c_t) \end{aligned}$$

□

(2)

Solution: We aim to extend the findings from the Finite Horizon model to the Infinite Horizon Model. Consider the following version of the finite horizon model.

$$L = \sum_{t=0}^T \{ \beta^t u(c_t) + \lambda_t (w_t - w_{t+1} - c_t) + \mu w_{t+1} \}$$

where we impose the condition $w_{t+1} \geq 0$. We have the following FOCs:

$$\begin{aligned} [c_t] \quad & \beta^t u'(c_t) - \lambda_t = 0 \\ [w_t] \quad & \lambda_t - \lambda_{t-1} = 0, \forall t \in \{0, 1, 2, \dots, T\} \\ [w_{T+1}] \quad & -\lambda_T + \mu = 0 \end{aligned}$$

with the complementary slackness condition where $\mu w_{t+1} = 0$. From the FOCs, we see that

$$\lambda_t \lambda_{t+1} \implies \lambda w_{t+1} = 0 \implies \beta^t u'(c_t) w_{t+1} = 0$$

and when we extend this idea to the infinite horizon, the following holds:

$$\lim_{t \rightarrow \infty} \beta^t u'(c_t) w_{t+1} = 0$$

Note that this is more of an assumption that we impose in the derivation. The intuition of this is that in the long run, we consume all our resources such that we maximize our utility in the long run. We can see this in the following manner, assume that:

$$\lim_{n \rightarrow \infty} \beta^n u'(c_n) < 0$$

the above does not hold from the assumptions on the utility function.

$$\lim_{n \rightarrow \infty} \beta^n u'(c_t) > 0$$

this implies that more room for consumption which implies that there can be more utility be consumed. Hence, by letting the limit equal 0, we imply there are no more room for optimization for the maximization of utility. \square

(3)

Solution: $u(c) = \ln(c)$ and thus $u'(c) = \frac{1}{c}$ Therefore, using the Euler Equation, we find that:

$$\begin{aligned} u'(c_t) &= \beta u'(c_{t+1}) \\ \frac{1}{c_t} &= \frac{\beta}{c_{t+1}} \\ c_{t+1} &= \beta c_t \end{aligned}$$

Note that if the consumer wants to maximize their utility, they should aim to consume all the goods. Thus,

$$\sum_{t=0}^{\infty} c_t = w_0$$

expanding the above, we can see that:

$$\sum_{t=0}^{\infty} \beta^t c_0 = w_t \iff c_0 = (1 - \beta)w_0$$

this implies that

$$c_t = \beta^t (1 - \beta)w_0$$

\square

Problem 2.

(1)

Solution: Since $K_{t+1} - (1 - \delta)K_t = I_t$, we find that:

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \beta^t \ln(c_t) \\ \text{s.t} \quad & C_t + K_{t+1} - (1 - \delta)K_t + G_t = A_t^{1-\alpha} K_t^\alpha \end{aligned}$$

with the following Langrangian:

$$L = \sum_{t=0}^{\infty} \beta^t \ln(c_t) + \sum_{t=0}^{\infty} \lambda_t (A_t^{1-\alpha} K_t^\alpha - C_t - K_{t+1} + (1 - \delta)K_t - G_t)$$

Thus, we derive the following FOCs:

$$\begin{aligned} [C_t] \quad & \frac{\beta^t}{C_t} = \lambda_t \\ [K_{t+1}] \quad & \lambda_{t+1}((1 - \delta) + \alpha A_{t+1}^{1-\alpha} K_{t+1}^{\alpha-1}) = \lambda_t \end{aligned}$$

□

(2)

Solution: Note using the $[C_t]$ and the $[K_{t+1}]$ conditions, we can derive the following:

$$\frac{1}{C_t} = \frac{\beta}{C_{t+1}} (\alpha A_{t+1}^{1-\alpha} K_{t+1}^{\alpha-1} + 1 - \delta)$$

Intuively, we can see that at optimum, consuming today must have have the same marginal benefit as consuming in the next time period, assuming we adjust for all discount factors and such. □

(3)

Solution: Define $k_t = \frac{K_t}{A_t}$, $c_t = \frac{C_t}{A_t}$, $y_t = \frac{Y_t}{A_t}$, $g_t = \frac{G_t}{A_t}$. Using the budget constraint, we can see:

$$\begin{aligned} A_t \left(\frac{K_t^\alpha}{A_t^\alpha} \right) &= C_t + K_{t+1} - K_t(1 - \delta) + G_t \\ k_t^\alpha &= c_t + K_{t+1} A_t^{-1} - k_t(1 - \delta) + g_t \\ k_t^\alpha &= c_t + \frac{K_{t+1} A_{t+1}}{A_{t+1} A_t} - k_t(1 - \delta) + g_t \\ k_t^\alpha &= c_t + k_{t+1}(1 + \gamma) - k_t(1 - \delta) + g_t \\ k_t^\alpha &= c_t + k_{t+1}(\gamma + \delta) + g_t \end{aligned}$$

Using the Euler's equation, we find that

$$\frac{1}{C_t} = \frac{\beta}{C_{t+1}} (\alpha A_{t+1}^{1-\alpha} K_{t+1}^{\alpha-1} + 1 - \delta)$$

$$C_{t+1} = C_t (\beta) (\alpha A_{t+1}^{1-\alpha} K_{t+1}^{\alpha-1} + 1 - \delta)$$

Since $k_t A_t = K_t$ and $c_t A_t = C_t$, we see that:

$$A_{t+1} C_{t+1} = A_t c_t \beta (\alpha A_{t+1}^{1-\alpha} K_{t+1}^{\alpha-1} + 1 - \delta)$$

$$c_{t+1} (1 + \gamma) = \beta c_t (\alpha A_{t+1}^{1-\alpha} K_{t+1}^{\alpha-1} + 1 - \delta)$$

$$\frac{c_{t+1}}{c_t} = \frac{(\alpha A_{t+1}^{1-\alpha} K_{t+1}^{\alpha-1} + 1 - \delta)}{(1 + p)(1 + \gamma)}$$

Thus, we can see that if we let the approximation hold, we get the following;

$$\frac{c_{t+1}}{c_t} = \alpha k_{t+1}^{\alpha-1} + 1 - \delta - \gamma - p$$

□

(4)

Solution: We can define the balanced steady state as the following, we want consumption and k to stop growing, which implies that $\frac{c_{t+1}}{c_t} = 1$. Therefore, we can solve for the following:

$$1 = \alpha k^{\alpha-1} + 1 - \delta - \gamma - \rho$$

$$\gamma + \delta + \rho = \alpha k^{\alpha-1}$$

$$k = \left(\frac{\alpha}{\gamma + \delta + \rho} \right)^{\frac{1}{1-\alpha}}$$

and thus using the fact that $k_t^\alpha = c_t + k_{t+1}(\gamma + \delta) + g_t$, we find that

$$c = \left(\frac{\alpha}{\gamma + \delta + \rho} \right)^{\frac{\alpha}{1-\alpha}} - \left(\frac{\alpha}{1\gamma + \delta + \rho} \right)^{\frac{1}{1-\alpha}} (\gamma + \delta) - g$$

□

(5)

Solution: We see that in steady state, or when $\delta c_t = 0$, we find that $k = \left(\frac{\alpha}{\gamma + \delta + \rho} \right)^{\frac{1}{1-\alpha}}$, which implies that we can draw the following graph.

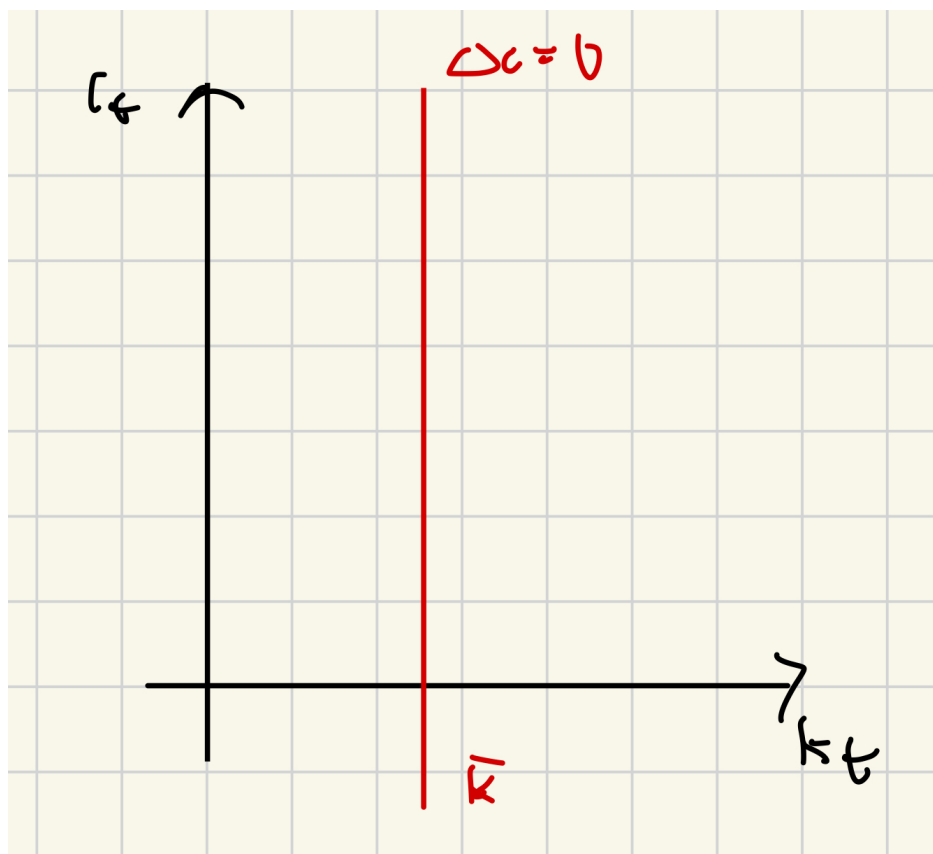


Figure 1: Graph for Question 2.5

□

(6)

Solution: If $\delta k = 0$, we find that:

$$\begin{aligned} k^\alpha - k(\gamma + \delta) &= c_t + g_t \\ k^\alpha - k(\gamma + \delta) - g &= c_t \\ \frac{\partial c_t}{\partial k} &= \alpha k^{\alpha-1} - (\gamma + \delta) \\ \frac{\partial^2 c_t}{\partial k^2} &= \alpha(\alpha - 1)k^{\alpha-2} < 0 \end{aligned}$$

Since $\frac{\partial^2 c_t}{\partial k^2}$, the shape is indeed hump shaped, and has a peak. which can be derived as follows:

$$\begin{aligned} 0 &= \alpha k^{\alpha-1} - (\gamma + \delta) \\ \gamma + \delta &= \alpha k^{\alpha-1} \\ \frac{\gamma + \delta}{\alpha} &= k^{\alpha-1} \\ k &= \left(\frac{\alpha}{\gamma + \delta} \right)^{\frac{1}{1-\alpha}} \end{aligned}$$

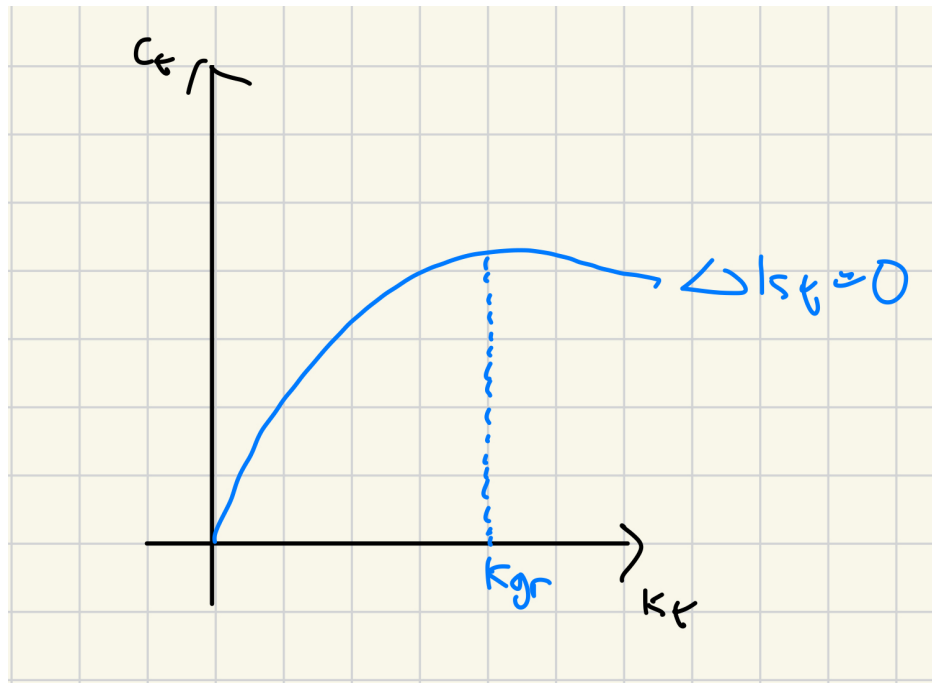


Figure 2: Question for Question 2.6

□

(7)

Solution: Note that:

$$k_{ss} = \left(\frac{\alpha}{\gamma + \delta + \rho} \right)^{\frac{1}{1-\alpha}} < \left(\frac{\alpha}{\gamma + \delta} \right)^{\frac{1}{1-\alpha}} = k_g$$

which implies that the steady state is indeed to the left. Intuitively, this makes sense, as we have to factor in the discount rate of future consumption, ρ . The left hand of the inequality has a ρ in it, indicating that we tend to consume more in the present because we discount consumption in future periods. The above is held with strict equality if $\rho = 0$, or where future and present consumption are the same to both individuals.

□

(8, 9, 10)

We begin with a general analysis. Recall this equation, we can see that:

$$\frac{c_{t+1}}{c_t} = \alpha k_{t+1}^{\alpha-1} + 1 - \delta - \gamma - \rho$$

If $\Delta k > 0$, this implies that Δc must increase and we are left to k_{ss} , which promotes a moving up of c . By symmetry, we can see that if $\Delta k < 0$, this implies that Δc must decrease and we are to the right of k_{ss} , which promotes a moving down of c . Similarly, if $\Delta c < 0$, we are in the bottom region of the hump, and thus, we see that Δk must increase. And, if $\Delta c > 0$, we are in the upper region of the hump, and thus, we see that Δk must decrease. We are left with the following diagram.

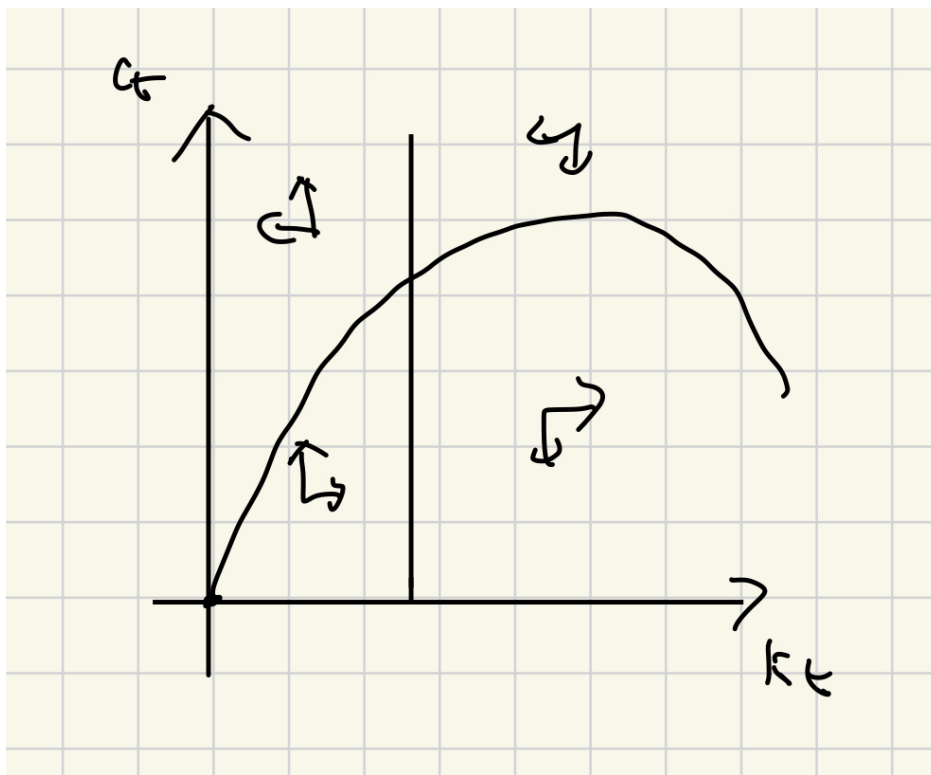


Figure 3: Graph for 8,9,10

for **8**, we see that $k_t \rightarrow 0$ and $c_t \rightarrow \infty$ and similarly, **9** we see that $k_t \rightarrow \infty$ and $c_t \rightarrow 0$. **10**, see above figure.

(11)

Solution: Note that in this graph, we can see that the saddle path describes the *optimal* path of consumption and k , which we know exists by the resource and feasibility constraint. Thus, we can see that:

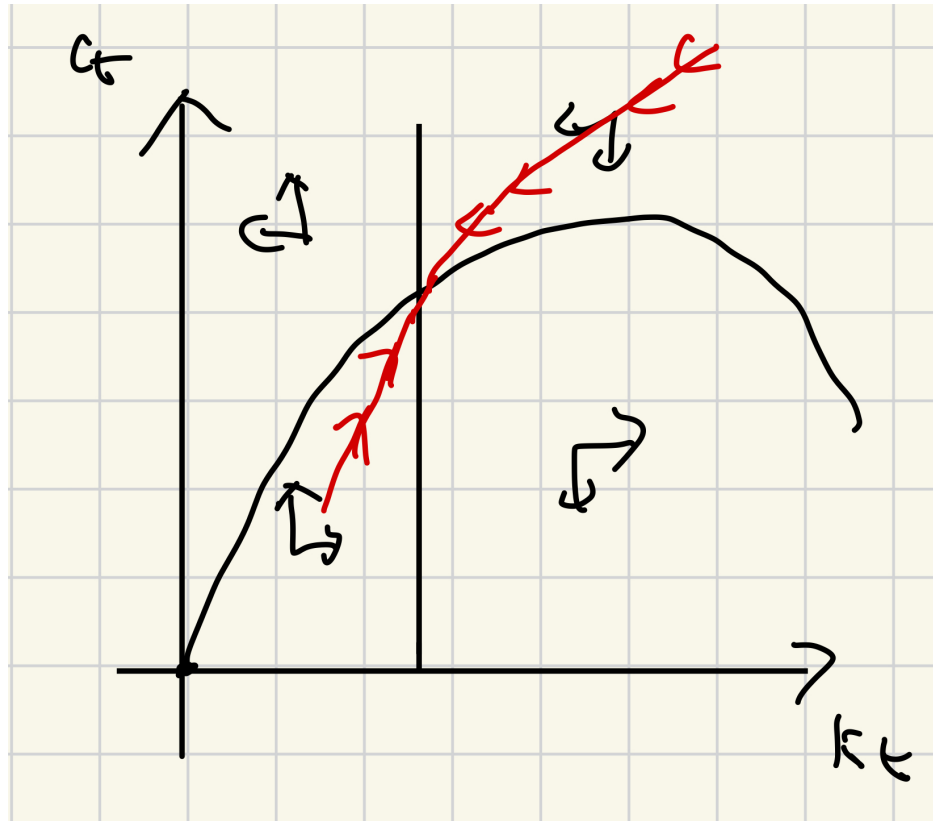


Figure 4: Graph for 11

Assume that we start in the SE region on the saddle path. We can see that I consuming not enough and also do not produce enough. Thus, we keep producing more and consuming more at rate such that we stay on the saddle path, until we hit steady state. Similarly, if we were to start in the NW saddle point region, we can see that we want to steadily decrease our production and consumption until we get to the steady state. \square

(12)

Solution: Note that:

$$k = \left(\frac{\alpha}{\gamma + \delta + \rho} \right)^{\frac{1}{1-\alpha}} \quad c_t = k_t^\alpha - k_t(\gamma + \delta) - g_t$$

note that nothing changes when $\Delta c_t = 0$, however when $\Delta k = 0$, we can see that g_t will decrease, which implies that the Δk curve will shift upwards as we see that g_t will decrease. We also see that when we increase the productivity, we want to shift down c and k at the time of the shock, as we can see that doing this will allow us to adjust to the new steady state path.

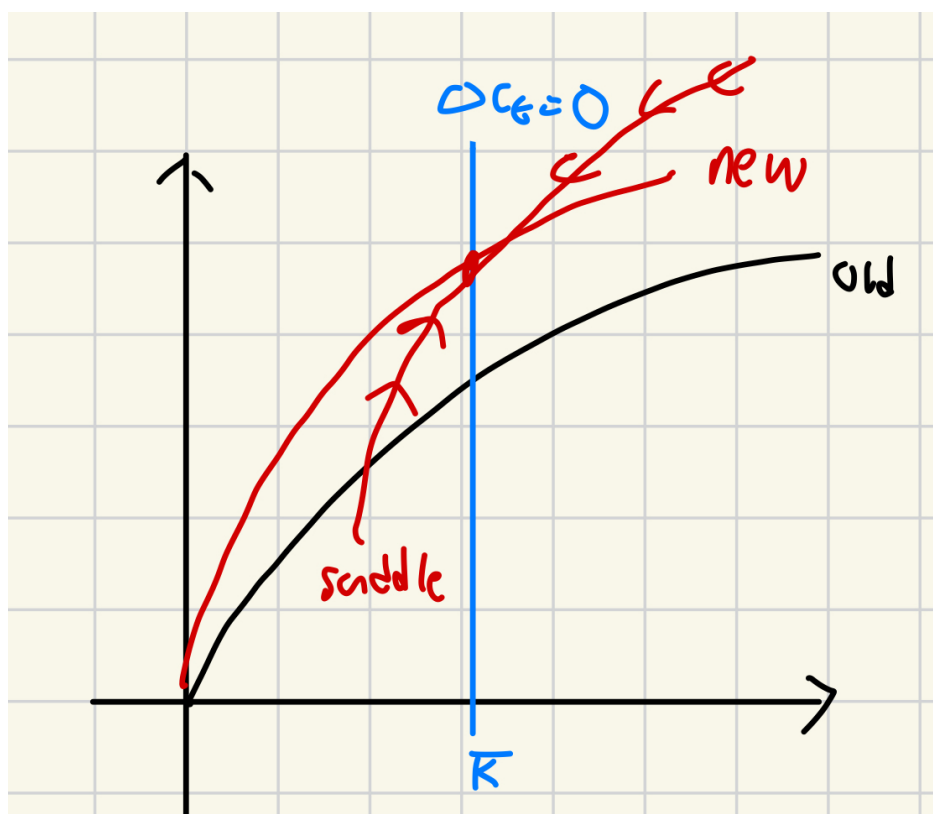


Figure 5: Change in Phase Diagram

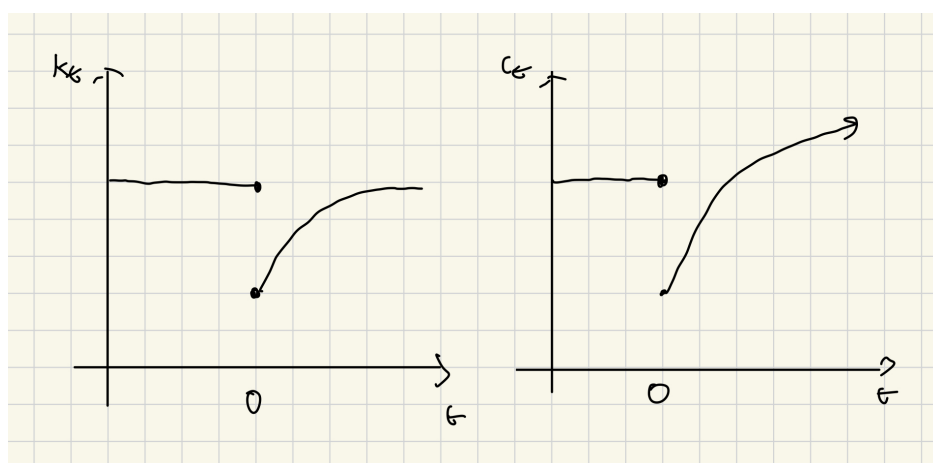


Figure 6: Change in Consumption and k

The intuition is that when we have a massive increase in productivity, but when we introduce this new level of productivity, to adopt this level of productivity, there will be a drop in production. This in turn, will incentivize consumers to adjust to consumption to be lower for this new increase of productivity, with the promise of increasing productivity yielding higher rewards in the long run. \square

(13)

Solution: Note that:

$$k = \left(\frac{\alpha}{\gamma + \delta + \rho} \right)^{\frac{1}{1-\alpha}} \quad c_t = k_t^\alpha - k_t(\gamma + \delta) - g_t$$

We see that Δc will shift to the left and the Δk will shift downwards. □

(14)

Solution:

Assume

$$Y_t = A_t K_t^\alpha.$$

The Lagrangian is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \ln C_t + \lambda_t (A_t K_t^\alpha - C_t - G_t - K_{t+1} + (1 - \delta)K_t).$$

From the first-order conditions:

For C_t :

$$\beta^t \frac{1}{C_t} - \lambda_t = 0$$

For K_{t+1} :

$$\lambda_{t+1} (\alpha A_{t+1} K_{t+1}^{\alpha-1} + (1 - \delta)) - \lambda_t = 0$$

From the Euler equation:

$$\beta^t \frac{1}{C_t} = \beta^{t+1} \frac{1}{C_{t+1}} (\alpha A_{t+1} K_{t+1}^{\alpha-1} + (1 - \delta))$$

Rewriting:

$$\frac{C_{t+1}}{C_t} = \beta (\alpha A_{t+1} K_{t+1}^{\alpha-1} + (1 - \delta))$$

Resource Constraint:

$$K_{t+1} = (1 - \delta)K_t + A_t K_t^\alpha - C_t - G_t$$

Notice:

$$\frac{C_{t+1}}{C_t} = \beta (\alpha A_{t+1} K_{t+1}^{\alpha-1} + (1 - \delta))$$

If $\Delta C_t = 0$, then:

$$\frac{1}{\beta} = \alpha A_{t+1} K_{t+1}^{\alpha-1} + (1 - \delta)$$

But since A_{t+1} is evolving, this doesn't make sense in a steady state. □

Problem 3.

(1)

Solution: We begin with the following Lagrangian.

$$L = \sum_{t=0}^{\infty} \beta^t \ln(c_t) + \lambda_t (K_t^\alpha - C_t - K_{t+1} + (1 - \delta)K_t - G_t)$$

We get the following FOCs:

$$\begin{aligned} [c_t] \quad & \frac{\beta^t}{C_t} = \lambda_t \\ [K_{t+1}] \quad & -\lambda_t + \lambda_{t+1}(\alpha K_{t+1}^{\alpha-1} + 1 - \delta) = 0 \end{aligned}$$

This implies that:

$$\frac{\beta^t}{C_t} = \lambda_{t+1}(\alpha k_{t+1}^{\alpha-1} + 1 - \delta) \iff \frac{C_{t+1}}{C_t} = \beta(\alpha k_{t+1}^{\alpha-1} + 1 - \delta)$$

Using this above condition, if $\Delta C_t = 0$, this means that:

$$k = \left(\frac{\alpha}{\beta^{-1} - 1 + \delta} \right)^{\frac{\alpha}{1-\alpha}}$$

Using the resource constraint we can derive that if $\Delta K_t = 0$:

$$C = K^\alpha - \delta K - G$$

Thus, if there is an increase in G , we must shift downwards the ΔK_t curve, hence yielding the following graph.

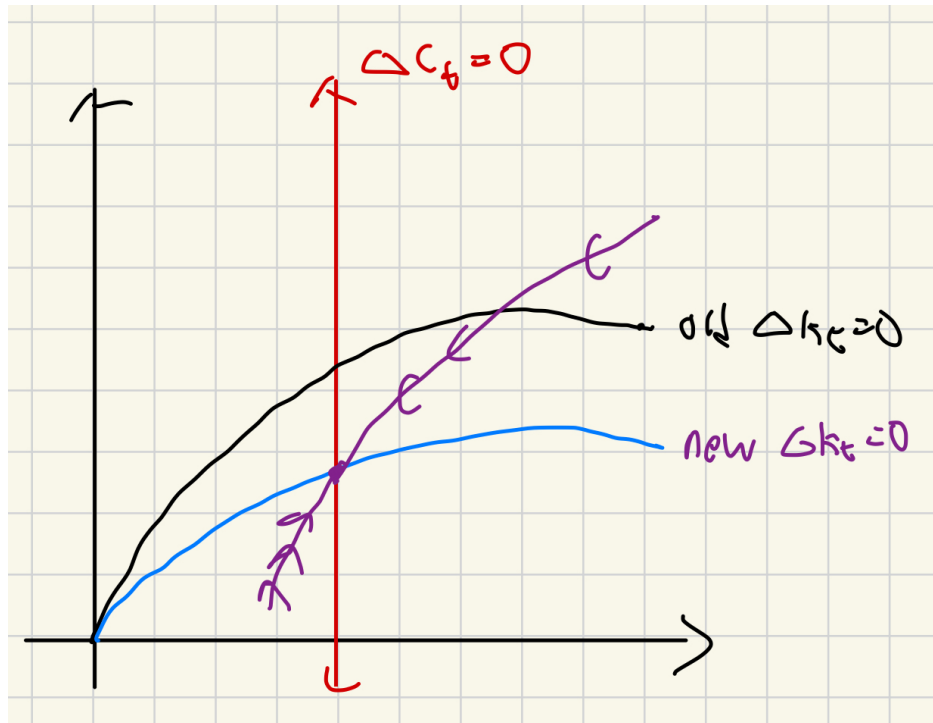


Figure 7: Graph for Question 3.1

□

(2)

Solution: We can note that consumption does indeed jump right away so that the individual can reach the steady state path, as the change in G is sudden and thus, the change in consumption must change immediately. We can see this at $t = 10$, the consumption will drop to A and then go to B then to C to encourage the economy to be on the steady state path.

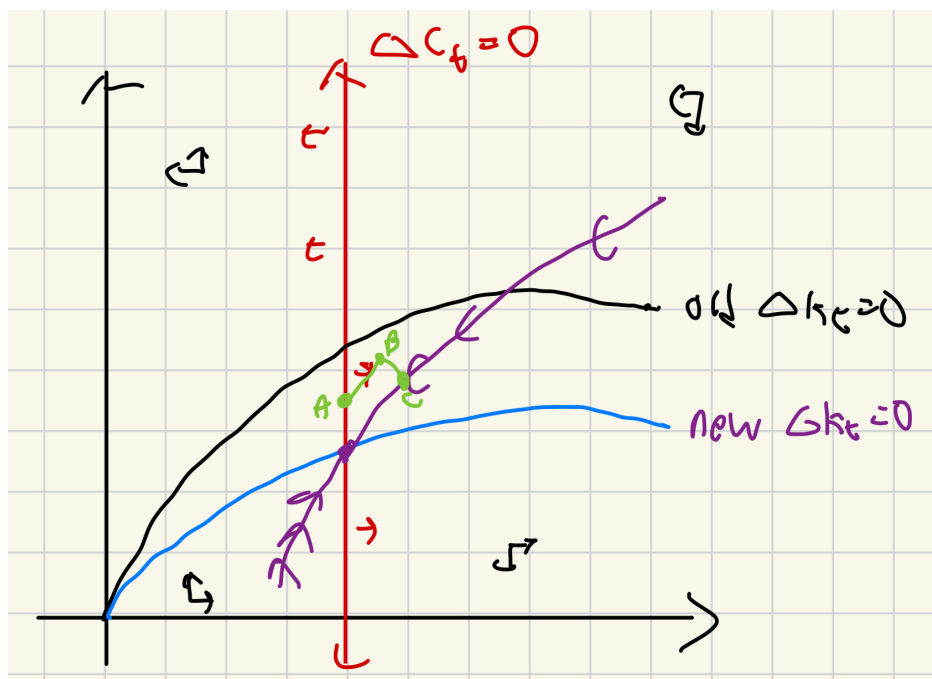


Figure 8: Graph for Question 3.2

and can analyze the short term and the long term effects of the move. Consider the following:

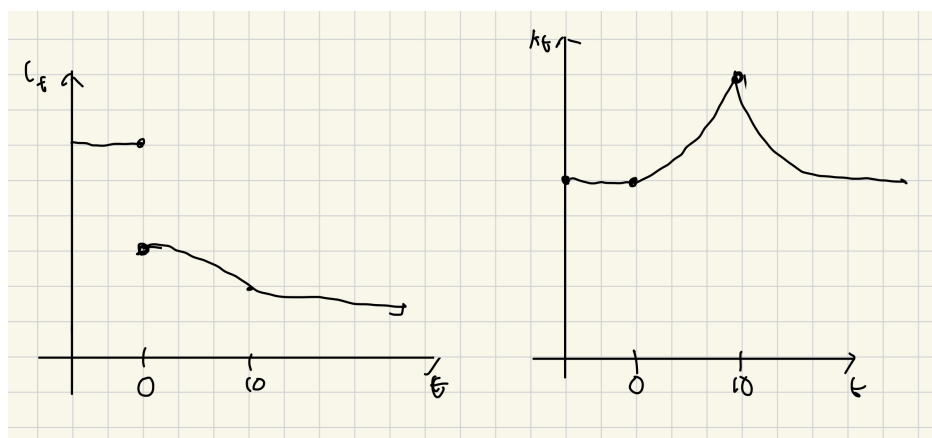


Figure 9: Short versus Long Term

The intuition for this is that we want to start saving for the day when the government starts increasing our taxes, that way we have enough padding to protect us from this shock.

□

Problem 4.

(1)

Solution: The transversality condition is the following:

$$\lim_{t \rightarrow \infty} \lambda_t K_t = \lim_{t \rightarrow \infty} \beta^t u'(c_t) K_t = 0$$

The intuition is that we want to maximize all of the resources given to us, with no room left to spend or to save. \square

(2)

Solution: At steady state, we can see that:

$$f(k) - c - \delta k = 0 \iff c = f(k) - \delta k$$

We want to maximize with respect to c , so we see that:

$$\frac{\partial c}{\partial k} = f'(k) - \delta = 0 \implies f'(k) = \delta$$

Thus, we see that:

$$\delta = f'(k_{gr}) > f'(k)$$

\square

(3)

Solution: Consider the Household's maximization problem. We see that:

$$L = \max_{c_t, k_t} \sum_{t=0}^{\infty} u(c_t) + \lambda_t (k_{t+1} - f(k_t) - c_t + (1 - \delta)k_t)$$

with the following FoCs:

$$\begin{aligned} [c_t] \quad \lambda_t &= \beta^t u'(c_t) \\ [k_t] \quad \lambda_{t-1} &= \lambda_t (f'(k_t) + 1 - \delta) \end{aligned}$$

From here, we can combine the above expressions to derive the Euler Equation,

$$u'(c_t) = \beta u'(c_{t+1}) (f'(k_{t+1}) + 1 - \delta)$$

Note the following:

$$\begin{aligned} u'(c_0) &= \beta u'(c_1) (f'(k_1) + 1 - \delta) \\ &= \beta (\beta u'(c_2) (f'(k_2) + 1 - \delta)) (f'(k_1) + 1 - \delta) \\ &\implies \beta^t u'(c_t) \Pi_{s=1}^t (f'(k_s) + 1 - \delta) \end{aligned}$$

Thus, we can see that

$$\beta^t u'(c_t) = u'(c_0) \Pi_{s=1}^t (f'(k_s) + 1 - \delta)^{-1}$$

Therefore,

$$\lim_{t \rightarrow \infty} u'(c_0) \Pi_{s=1}^t (1 + f'(k_s) - \delta)^{-1} k_{t+1} = 0$$

\square

(4)

Solution: Note from above, that following the given assumptions:

$$\begin{aligned} f'(k_{t+1}) &< \delta \\ f'(k_{t+1}) - \delta + 1 &< 1 \\ \frac{1}{f'(k_{t+1}) - \delta + 1} &> 1 \end{aligned}$$

However, from the $[k_t]$ condition in **3**, we know that this implies:

$$\lambda_t < \lambda_{t+1}$$

This implies that with the growth of capital that $\lim_{t \rightarrow \infty} \lambda_t K_t \neq 0$ □

(5)

Solution: As $\lim_{t \rightarrow \infty} = k_{ss}$ along the steady state path, and since $k_{ss} < k_g$, we know that by invoking a similar logic to **2** that:

$$\lim_{k \rightarrow k_{ss}} f'(k) > \delta \iff \frac{1}{1 + f'(k_t) - \delta} < 1$$

since in $\beta^t u'(c_t) \prod_{s=1}^t (f'(k_s) + 1 - \delta)$, everything is a constant but $\frac{1}{1 + f'(k_t) - \delta}$, which decreases, it implies that the TVC is satisfied. □