Honors Economics PSET 3

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Problem 10

Before we begin this problem, we will first derive the Hickensian Demand functions. We will note that we have already derived that the Marshallian Demand functions are

$$x^*(p_x, p_y, m) = \frac{\alpha m}{p_x}$$
$$y^*(p_x, p_y, m) = \frac{(1-\alpha)m}{p_y}$$
$$v(p_x, p_y, m) = \left(\frac{\alpha m}{p_x}\right)^{\alpha} \left(\frac{(1-\alpha)m}{p_y}\right)^{1-\alpha}$$

Now we will solve for the Hickensian Demand Functions. We can set up in the following:

$$\begin{array}{cc} \min_{x,y} & p_x x + p_y y \\ s.t & x^\alpha y^{1-\alpha} = \bar{U} \end{array}$$

and we can set up the Langrangian as:

$$L = p_x x + p_y y - \eta(\bar{U} - U(x, y))$$

we can note the following first order conditions

$$\begin{array}{ll} [\text{x}] & p_x = \eta U_x(x, y) \\ [\text{y}] & p_y = \eta U_y(x, y) \\ [\eta] & \bar{U} = U(x, y) \end{array}$$

And solving for η, we get

$$\frac{p_x}{U_x(x, y)} = \frac{p_y}{U_y(x, y)}$$

Noting that we can derive the following from the utility function:

$$U_x(x, y) = \alpha x^{\alpha-1} y^{1-\alpha} \qquad U_y(x, y) = (1-\alpha) x^\alpha y^{-\alpha}$$

Substituting values, we get:

$$\frac{p_x}{\alpha x^{\alpha-1} y^{1-\alpha}} = \frac{p_y}{(1-\alpha) x^\alpha y^{-\alpha}}$$
$$\frac{p_x}{\alpha x^{-1} y} = \frac{p_y}{(1-\alpha}$$
$$\frac{x p_x}{\alpha y} = \frac{p_y}{1-\alpha}$$
$$\frac{x}{y} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{p_y}{p_x}\right)$$

Note that we can solve for x and y respectively from this ratio. So if we were to solve for y_h^*, we would use the expression

$$x = y\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{p_y}{p_x}\right)$$

And solve as follows:

$$\begin{aligned}
x^\alpha y^{1-\alpha} &= \bar{U} \\
\left(y\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{p_y}{p_x}\right)\right)^\alpha y^{1-\alpha} &= \bar{U} \\
y\left(\frac{\alpha}{1-\alpha}\right)^\alpha \left(\frac{p_y}{p_x}\right)^\alpha &= \bar{U} \\
y = y^* &= \bar{U}\left(\frac{p_x(1-\alpha)}{\alpha p_y}\right)^\alpha
\end{aligned}$$

and using a similar line of logic, we would use the expression:

$$y = x\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{p_y}{p_x}\right)$$

and thus:

$$\begin{aligned}
x^\alpha y^{1-\alpha} &= \bar{U} \\
x^\alpha \left(x\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{p_x}{p_y}\right)\right)^{1-\alpha} &= \bar{U} \\
x\left(\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{p_x}{p_y}\right)\right)^{1-\alpha} &= \bar{U} \\
x = x^* &= \bar{U}\left(\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{p_y}{p_x}\right)\right)^{1-\alpha}
\end{aligned}$$

And thus we can then solve for the expenditure function as follows:

$$\begin{aligned}
p_x x^* + p_y y^* &= e \\
p_x \bar{U}\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{p_x}{p_y}\right)^{\alpha-1} + p_y \bar{U}\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{p_x}{p_y}\right)^\alpha &= e \\
\bar{U}\left(\frac{1-\alpha}{\alpha}\right)^\alpha \left(\frac{p_x}{p_y}\right)^\alpha \left(p_x\left(\frac{1-\alpha}{\alpha}\right)^{-1}\left(\frac{p_x}{p_y}\right)^{-1} + p_y\right) &= e \\
\bar{U}\left(\frac{1-\alpha}{\alpha}\right)^\alpha \left(\frac{p_x}{p_y}\right)^\alpha \left(p_y\left(\frac{\alpha}{1-\alpha}\right) + p_y\right) &= e \\
\bar{U}\cdot p_y \cdot \left(\frac{1-\alpha}{\alpha}\right)^\alpha \left(\frac{p_x}{p_y}\right)^\alpha \left(\frac{\alpha}{1-\alpha} + 1\right) &= e \\
\bar{U}\cdot p_y \cdot \left(\frac{1-\alpha}{\alpha}\right)^\alpha \left(\frac{p_x}{p_y}\right)^\alpha \left(\frac{1}{1-\alpha}\right) &= e \\
\bar{U}\left(\frac{p_x}{\alpha}\right)^\alpha \left(\frac{1-\alpha}{p_y}\right)^{\alpha-1} &= e
\end{aligned}$$

So we note that

x_m^*	$\frac{\alpha m}{p_x}$
y_m^*	$\frac{(1-\alpha)m}{p_y}$
$v^*(p_x, p_y, m)$	$m\left(\frac{a}{p_x}\right)^\alpha \left(\frac{1-\alpha}{p_y}\right)^{1-\alpha}$
x_h^*	$\bar{U}\left(\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{p_y}{p_x}\right)\right)^{1-\alpha}$
y_h^*	$\bar{U}\left(\frac{p_x(1-\alpha)}{\alpha p_y}\right)^\alpha$
$e^*(p_x, p_y, \bar{U})$	$\bar{U}\left(\frac{p_x}{\alpha}\right)^\alpha \left(\frac{1-\alpha}{p_y}\right)^{\alpha-1}$

1

Let $m = e(p_x, p_y, \bar{U})$. We can see that

$$\begin{aligned}
x_m^* = \frac{\alpha m}{p_x} &= \frac{\alpha}{p_x}\left(\left(\frac{p_x}{\alpha}\right)^{\alpha}\left(\frac{1-\alpha}{p_y}\right)^{1-\alpha}\right) \\
&= \bar{U}\left(\frac{p_x}{\alpha}\right)^{\alpha-1}\left(\frac{1-\alpha}{p_y}\right)^{\alpha-1} \\
&= \bar{U}\left(\frac{p_x}{p_y}\right)^{\alpha-1}\left(\frac{1-\alpha}{\alpha}\right)^{\alpha-1} \\
&= \bar{U}\left(\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{p_y}{p_x}\right)\right)^{1-\alpha} \\
&= x_h^*
\end{aligned}$$

and we can note for y^*:

$$\begin{aligned}
y_m^* = \frac{(1-\alpha)m}{p_y} &= \left(\frac{1-\alpha}{p_y}\right)\left(\frac{p_x}{\alpha}\right)^{\alpha}\left(\frac{1-\alpha}{p_y}\right)^{\alpha-1} \\
&= \left(\frac{p_x}{\alpha}\right)^{\alpha}\left(\frac{1-\alpha}{p_y}\right)^{\alpha}\bar{U} \\
&= \bar{U}\left(\frac{p_x(1-\alpha)}{\alpha p_y}\right)^{\alpha} \\
&= y_h^*
\end{aligned}$$

2

Let $\bar{U} = v(p_x, p_y, m)$ We can see that

$$\begin{aligned}
y_h^* = \bar{U}\left(\frac{p_x(1-\alpha)}{\alpha p_y}\right)^{\alpha} &= \left(\frac{\alpha m}{p_x}\right)^{\alpha}\left(\frac{(1-\alpha)m}{p_y}\right)^{1-\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\left(\frac{p_x}{p_y}\right)^{\alpha} \\
&= \frac{(1-\alpha)m}{p_y} \\
y_m^*
\end{aligned}$$

and also note for x^*

$$\begin{aligned}
x_h^* &= \bar{U}\left(\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{p_y}{p_x}\right)\right)^{1-\alpha} \\
&= \left(\frac{\alpha m}{x}\right)^{\alpha}\left(\frac{(1-\alpha)m}{p_y}\right)^{1-\alpha}\left(\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{p_y}{p_x}\right)\right)^{1-\alpha} \\
&= \frac{\alpha m}{p_x} \\
&= x_m^*
\end{aligned}$$

3

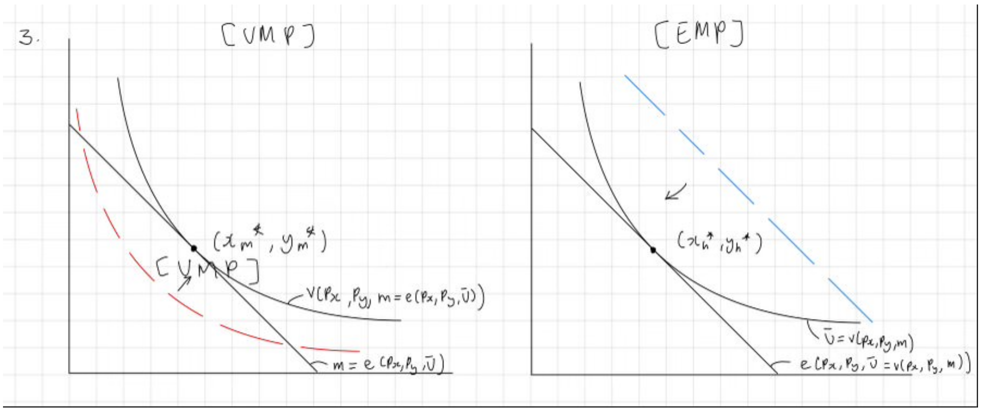

Figure 1: Graph for Figure 3

The intuition here relies on the fact that we first have fixed prices, p_x, p_y, as fixed. And thus, the only parameters that we are comparing are $\bar{U}$ and m. Note that when we plug in the Marshallian Demand Function into the Hicksian Demand Function, Hickensian Demand functions are designed to minimize the expenditure given some sort of utility level that needs to be fit. Given that we have already solved the EMP on the right graph, then we have an optimum expenditure level at $e(p_x, p_y, \bar{U})$. The Hickensian Demand Functions from this [EMP] indicate the consumption bundle to achieve the lowest expenditure possible given a target utility level $\bar{U}$. But, if we were to think about the expenditure function as not just the minimum expenditure for the [EMP] but also the budget for the [UMP], as noted on the left graph. Hence, solving the [UMP] by maximizing utility to this set budget (that was determined by the [EMP]), we will get Marshallian Demand functions that give us the same bundle as the Hicksian demand functions. A similar logic is applied to the opposite direction. When we set $\bar{U}$ as the target level of Utility, we can set this maximum utility we get for Marshallian demand. However, when we optimized the utility, we had to set the budget constant to a certain level. We can note that the bundle we get (the intersection) will be the same, as the Hicksian Demand Function will eventually minimize expenditure for the maximum utility we got from the Marshallian Demand, to get the budget determined in the Marshallian Demand. Thus, we can see that the demand functions are equal to each other when we set $m = e(p_x, p_y, \bar{U})$ and $\bar{U} = v(p_x, p_y, m)$ for each respective function.

4

Note that we can see that

$$\begin{aligned}
V &= v(p_x, p_y, m = e(p_x, p_y, \bar{U})) \\
&= m \left(\frac{a}{p_x}\right)^{\alpha} \left(\frac{1-\alpha}{p_y}\right)^{1-\alpha} \\
&= \bar{U} \left(\frac{p_x}{\alpha}\right)^{\alpha} \left(\frac{1-\alpha}{p_y}\right)^{\alpha-1} \left(\frac{a}{p_x}\right)^{\alpha} \left(\frac{1-\alpha}{p_y}\right)^{1-\alpha} \\
&= \bar{U}
\end{aligned}$$

Since we evaluated the indirect utility function at the expenditure function, we can see that the functions are inverses of each other as we return $\bar{U}$ as the result.

5

Note that:

$$
\begin{aligned}
E &= e(p_x, p_y, \bar{U} = v(p_x, p_y, m)) \\
&= m \left(\frac{a}{p_x}\right)^{\alpha} \left(\frac{1-\alpha}{p_y}\right)^{1-\alpha} \left(\frac{p_x}{\alpha}\right)^{\alpha} \left(\frac{1-\alpha}{p_y}\right)^{\alpha-1} \\
&= m
\end{aligned}
$$

A similar logic holds from the above question.

6

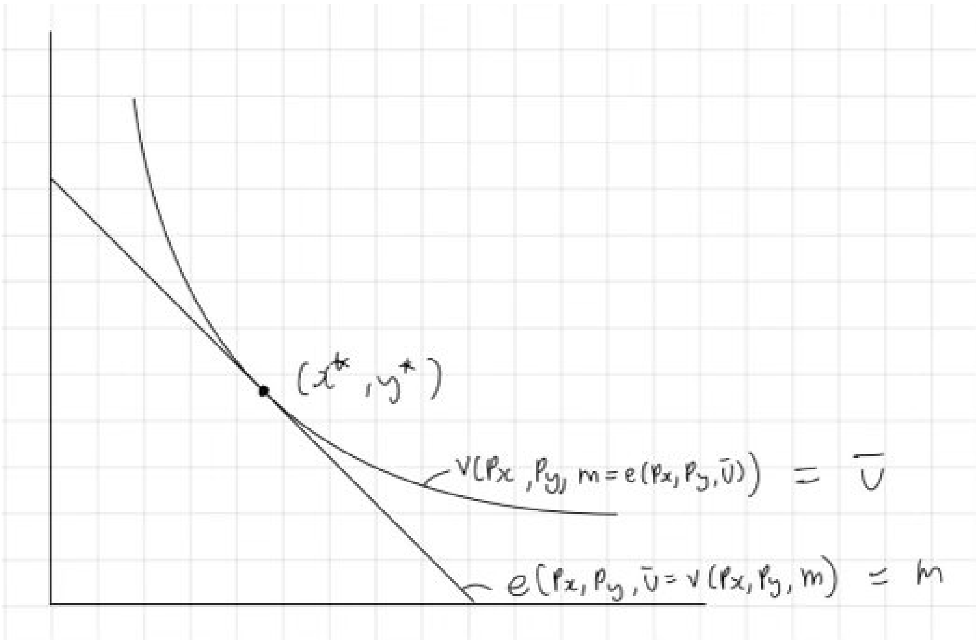

Figure 2: Graph for question 6

To explain the intuition, let us first examine what the indirect utility function and expenditure function is. The indirect utility function returns the highest utility possible for a given price of goods and budget. The expenditure function returns the lowest possible budget for a certain level of utility. So when we set $\bar{U} = v(p_x, p_y, m)$, when get the utility by minimizing m. Thus, graphically speaking, for a given $\bar{U}$, the minimum budget has already been set at m. So when we try to minimize the expenditure of the customer, there is only one possible budget m that enables tangency between the two possible points, which is m in the indirect utility function. The same logic applies for the opposite situation. Thus, the expenditure function (e) and the indirect utility functions (v) gives the same result as plugging v into e.

7

Note that

$$
\begin{aligned}
\lambda^* &= \frac{\alpha x^{*(\alpha-1)} y^{*(1-\alpha)}}{p_x} \\
&= \frac{1}{\frac{\alpha x^{*(\alpha-1)} y^{*(1-\alpha)}}{p_x}} \\
&= \frac{1}{\eta^*}
\end{aligned}
$$

λ^* is the marginal utility of income in $\frac{\text{utils}}{\$}$ whereas η is the marginal cost of utility $\frac{\$}{\text{utils}}$. Since this rate of exchange must be constant regardless of whether we are spending dollars to get utils or sacrificing utils to get dollars, therefore λ^* and η^* are reciprocals. ,

Problem 11

1

Note that the problem can be set as:

$$\begin{array}{ll} \max_{x_1,x_2} & (x_1^{-\rho} + \omega x_2)^{-\rho})^{\frac{-1}{\rho}} \\ s.t & p_1x_1 + p_2x_2 = m \end{array}$$

The Lagrangian is:

$$L = \left(\left(x_1^{-\rho} + \omega x_2^{-\rho}\right)^{\frac{-1}{\rho}}\right) - \lambda\left(m - p_1x_1 - p_2x_2\right)$$

The first-order conditions are:

$$\begin{array}{ll} [x_1] & \left(\frac{-1}{\rho}\right)\left(x_1^{-\rho} + \omega x_2^{-\rho}\right)^{\frac{-1}{\rho}-1} \cdot (-\rho)x_1^{-\rho-1} = \lambda p_1 \\ [x_2] & \left(\frac{-1}{\rho}\right)\left(x_1^{-\rho} + \omega x_2^{-\rho}\right)^{\frac{-1}{\rho}-1} \cdot (-\omega\rho)x_2^{-\rho-1} = \lambda p_2 \\ [\lambda] & p_1x_1 + p_2x_2 = m \end{array}$$

Equating lambdas yields

$$\begin{aligned} \frac{\left(\frac{-1}{\rho}\right)\left(x_1^{-\rho} + \omega x_2^{-\rho}\right)^{\frac{-1}{\rho}-1} \cdot (-\rho)x_1^{-\rho-1}}{p_1} &= \frac{\left(\frac{-1}{\rho}\right)\left(x_1^{-\rho} + \omega x_2^{-\rho}\right)^{\frac{-1}{\rho}-1} \cdot (-\omega\rho)x_2^{-\rho-1}}{p_2} \\ \frac{-\rho x_1^{-\rho-1}}{p_1} &= \frac{-\omega\rho x_2^{-\rho-1}}{p_2} \\ \frac{-x_1^{-\rho-1}}{p_1} &= \frac{-\omega x_2^{-\rho-1}}{\rho_2} \\ \left(\frac{x_1}{x_2}\right)^{-\rho-1} &= \frac{\omega p_1}{p_2} \\ \left(\frac{x_2}{x_1}\right)^{\rho+1} &= \frac{\omega p_1}{p_2} \\ x_2 &= x_1\left(\frac{\omega p_1}{p_2}\right)^{\frac{1}{\rho+1}} \end{aligned}$$

Therefore, the scalar is

$$\kappa = \left(\frac{\omega p_1}{p_2}\right)^{\frac{1}{\rho+1}}$$

2

We can see that:

$$\begin{aligned} p_2x_2^* + p_1x_1^* &= m \\ p_2x_2^* + p_1\left(x_2^*\left(\frac{p_2}{\omega p_1}\right)^{\frac{1}{\rho+1}}\right) &= m \\ x_2^*\left(p_2 + p_1\left(\frac{p_2}{\omega p_1}\right)^{\frac{1}{\rho+1}}\right) &= m \\ x_2^* &= \frac{m}{\left(p_2 + p_1\left(\frac{p_2}{\omega p_1}\right)^{\frac{1}{\rho+1}}\right)} \end{aligned}$$

We can see that indeed it is proportional to income. Therefore, we can note that Income Elasticity will be positive as the goods are directly proportional. Note that the income elasticity is determined as:

$$\epsilon = \frac{\partial x_2^*}{\partial m} \cdot \frac{m}{x_2^*}$$

And thus, when we go through the derivation, we can see that

$$\frac{\partial x_2^*}{\partial m} = \frac{1}{\left(p_2 + p_1 \left(\frac{p_2}{\omega p_1}\right)^{\frac{1}{\rho+1}}\right)}$$

and thus subsisting the proper values into the elasticity equation above, we get

$$\epsilon = 1$$

Thus, we know the elasticity for income demand is positive and 1.

3

We can note:

$$\lambda^* = \frac{U_{x_1}}{p_1}$$

and

$$v^*(p_x, p_y, m) = (x_1^{-\rho} + \omega x_2^{-\rho})^{-\frac{1}{\rho}}\Big|_*$$

Using the 1st equation, we can see that

$$\begin{aligned}
\lambda^* &= \frac{U_{x_1}}{p_1} \\
&= \frac{(x_1)^{-\rho-1}(x_1^{-\rho} + \omega x_2^{-\rho})^{\frac{-1}{\rho}-1}}{p_1} \\
&= \frac{(x_1^*)^{-\rho-1}}{(x_1^{-\rho} + \omega x_2^{-\rho})} \cdot \frac{(x_1^{-\rho} + \omega x_2^{-\rho})^{-\frac{1}{\rho}}}{p_1} \\
&= \frac{(x_1^*)^{-\rho-1}}{(x_1^{*-\rho} + \omega x_2^{*-\rho})} \cdot \frac{V(p_x, p_y, m)}{p_1}
\end{aligned}$$

Thus, we note that the indirect utility function is directly proportional to λ. To simplify λ even more, we can see that:

$$\begin{aligned}
\lambda^* &= \frac{U_{x_1}}{p_1} \\
&= \frac{(x_1)^{-\rho-1}(x_1^{-\rho} + \omega x_2^{-\rho})^{-\frac{1}{\rho}-1}}{p_1} \\
&= \frac{(x_1)^{-\rho-1}\left((x_1^{-\rho} + \omega\left(x_1\left(\frac{\omega p_1}{p_2}\right)^{\frac{1}{\rho+1}}\right)^{-\rho}\right)^{\frac{-1}{\rho}-1}}{p_1} \\
&= \frac{(x_1)^{-\rho-1}(x_1)^{\rho+1}\left(1 + \omega\left(\left(\frac{\omega p_1}{p_2}\right)^{\frac{1}{\rho+1}}\right)^{-\rho}\right)^{\frac{-1}{\rho}-1}}{p_1} \\
&= \frac{\left(1 + \omega\left(\left(\frac{\omega p_1}{p_2}\right)^{\frac{1}{\rho+1}}\right)^{-\rho}\right)^{\frac{-1}{\rho}-1}}{p_1}
\end{aligned}$$

4

Note we define elasticity as the following:

$$\sigma = \frac{d\log(\frac{x_1}{x_2})}{d\log(\frac{U_{x_2}}{U_{x_1}})}$$

We first find $\frac{x_1}{x_2}$, which is:

$$\frac{x_1}{x_2} = \frac{x_1}{x_1\left(\frac{\omega p_1}{p_2}\right)^{\frac{1}{\rho+1}}} = \left(\frac{\omega p_1}{p_2}\right)^{\frac{-1}{\rho+1}} = \left(\frac{p_2}{\omega p_1}\right)^{\frac{1}{\rho+1}}$$

and similarly we can do the same for $\frac{U_{x2}}{U_{x1}}$:

$$\frac{U_{x2}}{U_{x1}} = \omega\left(\frac{x_1}{x_2}\right)^{\rho+1}$$

and thus we can see that:

$$\begin{aligned}
\log\left(\frac{U_{x2}}{U_{x1}}\right) &= \log\left(\omega\left(\frac{x_1}{x_2}\right)^{\rho+1}\right) \\
&= \log(\omega) + (\rho+1)\log\left(\frac{x_1}{x_2}\right) \\
&= \log(\omega) + (\rho+1)\left(\frac{-1}{\rho+1}\right)\log\left(\frac{\omega p_1}{p_2}\right) \\
&= \log(\omega) - \log\left(\frac{\omega p_1}{p_2}\right) \\
&= \log(p_2) - \log(p_1) \\
&= \log\left(\frac{p_2}{p_1}\right)
\end{aligned}$$

Note that in this case we can note that:

$$\sigma = \frac{d\log(\frac{x_1}{x_2})}{d\log(\frac{U_{x_2}}{U_{x_1}})} = \frac{d\log(\frac{x_1}{x_2})}{d\log(\frac{p_2}{p_1})}$$

Thus, we note that:

$$\log\left(\frac{x_1}{x_2}\right) = \left(\frac{1}{\rho+1}\right)\left(\log\left(\frac{p_2}{p_1}\right) - \log(w)\right)$$

Taking this value and differentiating it with respect to yields:

$$\sigma = \frac{1}{\rho+1}$$

This significance of this value is that the elasticity of substitution is not reliant on the quantity or price of goods. It is also not reliant on ω , which means that regardless of the variables in the equation, the elasticity of substitution remains constant. This also means that the percent change in the ratio of goods consumed due to the percent change in the MRS is constant.

5

We can find the Hicksian demand functions by first setting the problem:

$$\begin{array}{ll}
\min_{x,y} & p_1x_1 + p_2x_2 \\
s.t & (x_1^{-\rho} + \omega x_2^{-\rho})^{\frac{-1}{\rho}} = \bar{U}
\end{array}$$

and thus set the Langrangian as:

$$L = p_1x_1 + p_2x_2 + \eta(\bar{U} - (x_1^{-\rho} + \omega x_2^{-\rho})^{\frac{-1}{\rho}})$$

And note that the following first order conditions can be set:

$$\begin{aligned}
&[x_1] \qquad && p_1 = \eta(x_1)^{-\rho-1}\left(x_1^{-\rho} + \omega x_2^{-\rho}\right)^{\frac{-1}{\rho}} \\
&[x_2] \qquad && p_2 = \eta\omega(x_2)^{-\rho-1}\left(x_1^{-\rho} + \omega x_2^{-\rho}\right)^{\frac{-1}{\rho}} \\
&[\lambda] \qquad && \bar{U} = (x_1^{-\rho} + \omega x_2^{-\rho})^{\frac{-1}{\rho}}
\end{aligned}$$

Letting the η equal to each other, we note that

$$\begin{aligned}
\frac{p_1}{(x_1)^{-\rho-1}} &= \frac{p_2}{\omega(x_2)^{-\rho-1}} \\
\frac{\omega p_1}{p_2} &= \left(\frac{x_2}{x_1}\right)^{\rho+1}
\end{aligned}$$

Rearranging the terms around yields:

$$x_1 = x_2\left(\frac{p_2}{\omega p_1}\right)^{\frac{1}{\rho+1}}$$

Substituting this value into the expression yields:

$$\begin{aligned}
(x_1^{-\rho} + \omega x_2^{-\rho})^{\frac{-1}{\rho}} &= \bar{U} \\
x_2\left(\left(\frac{p_2}{\omega p_1}\right)^{\frac{-\rho}{\rho+1}} + \omega\right)^{-\frac{1}{\rho}} &= \bar{U} \\
x_2 = x_2^H &= \frac{\bar{U}}{\left(\left(\frac{p_2}{\omega p_1}\right)^{\frac{-\rho}{\rho+1}} + \omega\right)^{-\frac{1}{\rho}}}
\end{aligned}$$

And similarly,

$$\begin{aligned}
\left(x_1^{-\rho} + \omega x_2^{-\rho}\right)^{\frac{-1}{\rho}} &= \bar{U} \\
\left(x_1^{-\rho} + \omega\left(x_1\left(\frac{\omega p_1}{p_2}\right)^{\frac{1}{\rho+1}}\right)^{-\rho}\right)^{-\frac{1}{\rho}} &= \bar{U} \\
x_1 = x_1^H &= \frac{\bar{U}}{\left(1 + \omega\left(\frac{\omega p_1}{p_2}\right)^{\frac{-\rho}{\rho+1}}\right)^{\frac{-1}{\rho}}}
\end{aligned}$$

And thus his expenditure function is:

$$p_1\left(\frac{\bar{U}}{\left(1 + \omega\left(\frac{\omega p_1}{p_2}\right)^{\frac{-\rho}{\rho+1}}\right)^{\frac{-1}{\rho}}}\right) + p_2\left(\frac{\bar{U}}{\left(\left(\frac{p_2}{\omega p_1}\right)^{\frac{-\rho}{\rho+1}} + \omega\right)^{-\frac{1}{\rho}}}\right) = e$$

Note that the demand functions are indeed proportional to $\bar{U}$

6

Note that in x_2^H, we have $\frac{p_2}{\omega p_1}$, and thus multiplying both p_1 and p_2 by the scalar t will yield the same thing as the t will cancel out. Thus, a similar logic applies to x_1^H. Thus, when we analyze the behavior of the expenditure function, we can be sure that multiplying p_1, p_2 by t will not change the value of x_2^H and x_1^H. Hence the following operation,

$$e(tp_1, tp_2, \bar{U}) = tp_1x_1 + tp_2x_2 = t(p_1x_1 + p_2x_2) = te(p_1, p_2.\bar{U})$$

hold, thus proving the idea that the function is homogeneous in degree 1.

7

Note that we can say that

$$\eta^* = \frac{p_1}{(x_1^{-\rho} + \omega x_2^{-\rho})^{\frac{1}{\rho}+1} \cdot (x_1)^{-\rho-1}}$$

Note that we can see that we can do the following:

$$\begin{aligned}
\eta^* &= \frac{p_1}{(x_1^{-\rho} + \omega x_2^{-\rho})^{\frac{1}{\rho}+1} \cdot (x_1)^{-\rho-1}} \\
&= p_1(x_1^{-\rho} + \omega(x_1^{-\rho}(\omega)\left(\frac{\omega p_1}{p_2}\right)^{\frac{-\rho}{\rho+1}})(x_1^{\rho+1}) \\
&= p_1(x_1)^{-\rho-1}(1 + \omega(\frac{\omega p_1}{p_2})^{\frac{-\rho}{\rho+1}})^{\frac{1}{\rho}+1}(x_1)^{\rho+1} \\
&= p_1(1 + \omega(\frac{\omega p_1}{p_2})^{\frac{-\rho}{\rho+1}})^{\frac{1}{\rho}+1} \\
&= \frac{1}{\lambda^*}
\end{aligned}$$

Problem 12

1

We are given

$$p_1x_1 + p_2x_2 + p_3x_3 = m$$

Differentiating in respect to m yields,

$$\begin{aligned}
\frac{\partial}{\partial m}(p_1x_1) + \frac{\partial}{\partial m}(p_2x_2) + \frac{\partial}{\partial m}(p_3x_3) &= \frac{\partial}{\partial m}m \\
p_1\frac{\partial x_1}{\partial m} + p_2\frac{\partial x_2}{\partial m} + p_3\frac{\partial x_3}{\partial m} &= 1
\end{aligned}$$

Note that

$$\eta_1 s_1 = \frac{p_1 x_1^m}{m} \cdot \frac{m}{x_1^m} \cdot \frac{\partial x_1^m}{\partial m}$$

Thus, we can divide everything $\frac{mx_1^m}{mx_1^m}$, where $i \in \{1, 2, 3\}$ for each respective index. Thus, we get the following equation:

$$\eta_1 s_1 + \eta_2 s_2 + \eta_3 s_3 = 1$$

2

a If all the η_i were to be negative, note that we can see that this implies that all the shares have to be negative, which is an inherent contradiction. Thus this is impossible.

b Note that if all η_i were all to be positive, this implies that all the shares have to be positive. This behavior follows from the idea that $\sum_{i=1}^{n} s_i = 1$. Since all the sums can be positive, this means that this is possible.

c If all $\eta > 1$, and we know that $s_1 + s_2 + s_3 = 1$, this means that $\eta_1 s_1 + \eta_2 s_2 + \eta_3 s_3 > 1$. Thus, not possible.

d Note:

$$\begin{aligned}
\eta_1 s_1 + \eta_2 s_2 + \eta_3 s_3 &= s_1 + s_2 + s_3 \\
\eta_1 s_1 - s_1 + \eta s_2 - s_2 + \eta_3 s_3 - s_3 &= 0 \\
s_1(\eta_1 - 1) + s_2(\eta_2 - 1) + s_3(\eta - 1) &= 0
\end{aligned}$$

Using the above calculation, we note that if $\eta < 1$, this implies that every $\eta_i - 1$ is negative. Thus, this means that at least one share of good must be negative, which is a contradiction with fact that all shares must be a positive value. Thus, this condition is false.

3

The budget constraint is given as

$$p_1 x_1 + p_2 x_2 + p_3 x_3 = m$$

Differentiating with respect to p_1 yields

$$\begin{aligned}
\frac{\partial}{\partial p_1}(p_1 x_1) + \frac{\partial}{\partial p_2}(p_2 x_2) + \frac{\partial}{\partial p_1}(p_3 x_3) &= \frac{\partial}{\partial p_1} m \\
x_1 + p_1 \frac{\partial x_1}{\partial p_1} + p_2 \frac{\partial x_2}{\partial p_1} + p_3 \frac{\partial x_3}{\partial p_1} &= 0 \\
x_1 + p_1 \frac{\partial x_1}{\partial p_1}\frac{x_1}{x_1} + p_2 \frac{\partial x_2}{\partial p_1}\frac{x_2}{x_2} + p_3 \frac{\partial x_3}{\partial p_1}\frac{x_3}{x_3} &= 0 \\
\frac{x_1 p_1}{m} + \frac{p_1}{m} p_1 \frac{\partial x_1}{\partial p_1}\frac{x_1}{x_1} + \frac{p_1}{m} p_2 \frac{\partial x_2}{\partial p_1}\frac{x_2}{x_2} + \frac{p_1}{m} p_3 \frac{\partial x_3}{\partial p_1}\frac{x_3}{x_3} &= 0 \\
s_1 + \epsilon_{1,1}^M s_1 + \epsilon_{2,1}^M s_2 + \epsilon_{3,1}^M s_3 &= 0 \\
s_1(1 + \epsilon_{1,1}^M) + \epsilon_{2,1}^M s_2 + \epsilon_{3,1}^M s_3 &= 0
\end{aligned}$$

4

If we were to assume that the good is to be price elastic, this means that

$$|\epsilon_{1,1}^M| > 1$$

And if we note if it is a good that follows the law of demand, then

$$\epsilon_{1,1} < -1$$

According to the Cournot Aggregation, we note that quantity

$$s_1(1 + \epsilon_{1,1}^M) < 0$$

This implies that

$$\epsilon_{2,1}^M s_2 + \epsilon_{3,1}^M s_3 > 0$$

which implies that at least one of $\epsilon_{2,1}$ or $\epsilon_{3,1}$ must be positive. However, to determine which of these cross price elasticities is positive, we need more information about the shares of these goods.

5

Setting up the equation and differentiating with respect with t yields:

$$\begin{aligned}
\frac{d}{dt}(x(tp_1, tp_2, tp_3, tm) &= x(p_1, p_2, p_3, m)) \\
\frac{\partial x^M}{\partial tp_1}p_1 + \frac{\partial x^M}{\partial tp_2}p_2 + \frac{\partial x^M}{\partial tp_3}p_3 + \frac{\partial x^M}{\partial tm}m &= 0 \\
\frac{\partial x^M}{\partial tp_1}p_1\frac{tx_1^M}{tx_1^M} + \frac{\partial x^M}{\partial tp_2}p_2\frac{tx_2^M}{tx_2^M} + \frac{\partial x^M}{\partial tp_3}p_3\frac{tx_3^M}{tx_3^M} + \frac{\partial x^M}{\partial tm}m\frac{tm}{tm} &= 0 \\
x_1^M(\epsilon_{11} + \epsilon_{21} + \epsilon_{31} + \eta_1) &= 0 \\
\epsilon_{11} + \epsilon_{21} + \epsilon_{31} + \eta_1 &= 0
\end{aligned}$$

6

According to the Cournot Aggregation, we can see that the quantity

$$s_1(1 + \epsilon_{1,1}^M) > 0$$

which implies that

$$\epsilon_{2,1}^M s_2 + \epsilon_{3,1}^M s_3 < 0$$

which also implies that at least one of $\epsilon_{2,1}$ and $\epsilon_{3,1}$ must be negative. Thus, this implies that the Geffen Good is strong complements with another good. Therefore, the statement is true.

7

If good 1 is an inferior good but not a Giffen Good, we note the conditions:

$$\epsilon_{1,1} < 0 \qquad \eta_1 < 0$$

Looking at the results of question 5, we see that

$$\epsilon_{11} + \epsilon_{21} + \epsilon_{31} + \eta_1 = 0$$

with the given conditions, note that the following conditions must hold:

$$\epsilon_{21} + \epsilon_{31} > 0$$

Thus implying at least one of $\epsilon_{21}, \epsilon_{31}$ must be positive. This means that at least one of the other goods must be a gross substitute.

Problem 13

1

Differentiating with respect to m yields:

$$\frac{\partial v}{\partial m} = U_x\left(\frac{\partial x^*}{\partial m}\right) + U_y\left(\frac{\partial y^*}{\partial m}\right)$$

For Marshallian, note the following first order conditions are derived.

$$\begin{aligned}
&[x] \qquad U_x = p_x\lambda^* \\
&[y] \qquad U_y = p_y\lambda^* \\
&[\lambda] \qquad p_x x + p_y y = m
\end{aligned}$$

and thus we can use these conditions to solve for $\frac{\partial v}{\partial m}$, which is as shown:

$$\begin{aligned}
\frac{\partial v}{\partial m} &= U_x\left(\frac{\partial x^*}{\partial m}\right) + U_y\left(\frac{\partial y^*}{\partial m}\right)] \\
&= p_x\lambda^*\left(\frac{\partial x^*}{\partial m}\right) + p_y\lambda^*\left(\frac{\partial y^*}{\partial m}\right) \\
&= \lambda^*\left(p_x\frac{\partial x^*}{\partial m} + p_y\frac{\partial y^*}{\partial m}\right) \\
&= \lambda^*\left(p_x\frac{\partial x^*}{\partial m}\frac{x^m m}{x^m m} + p_y\frac{\partial y^*}{\partial m}\frac{y^m m}{y^m m}\right) \\
&= \lambda^*(s_x\eta_x + s_y\eta_y)
\end{aligned}$$

Since we have already proven in Exercise 12 that

$$s_x\eta_x + s_y\eta_y = 1$$

which implies that

$$\frac{\partial v}{\partial m} = \lambda^*(s_x\eta_x + s_y\eta_y) = \lambda^*$$

The reason why this works is that when we substitute the first order conditions and note that we could use the Engel Aggregation result to simplify the equation down to λ^*

2

Applying a similar process to earlier question, we see that

$$E = e(p_x, p_y, \bar{U})$$

and note that

$$e = p_x x + p_y$$

We can calculate the following:

$$e_x = p_x \qquad e_y = p_y$$

And the first order of conditions of we can find that:

$$\begin{aligned}
&[x] \qquad p_x = \eta U_x \\
&[y] \qquad p_y = \eta U_y \\
&[\eta] \qquad U(x,y) = \bar{U}
\end{aligned}$$

We can use this conditions to note that differentiating with respect to U means that:

$$\begin{aligned}
e &= p_x x + p_y y \\
\frac{\partial e}{\partial U} &= p_x\frac{\partial x^*}{\partial U} + p_y\frac{\partial y^*}{\partial U} \\
\frac{\partial e}{\partial U} &= \eta\left(\frac{\partial x^*}{\partial U}U_x + \frac{\partial y^*}{\partial U}U_y\right)
\end{aligned}$$

Note that

$$\frac{\partial}{\partial U}U(x,y) = \frac{\partial x^*}{\partial U}U_x + \frac{\partial y^*}{\partial U}U_y = 1$$

Thus, we can conclude that

$$\frac{\partial e}{\partial U} = \eta\left(\frac{\partial x^*}{\partial U}U_x + \frac{\partial y^*}{\partial U}U_y\right) = \eta^*$$

The reason why all of this works is that when we do some algebraic manipulation and substitution, we note that we end up something that resembles the total differential of $U(x,y)$, which allows us to simplify the equation further into the desired result.

3

$$
\begin{aligned}
V &= U(x, y) \\
\frac{\partial V}{\partial p_x} &= \frac{\partial U}{\partial x^*} \cdot \frac{\partial x^*}{\partial p_x} + \frac{\partial U}{\partial y^*} \cdot \frac{\partial y^*}{\partial p_x}
\end{aligned}
$$

Using the first order conditions that were derived earlier,

$$
\begin{aligned}
\frac{\partial V}{\partial p_x} &= p_x \lambda^* \frac{\partial x^*}{\partial p_x} + p_y \lambda^* \frac{\partial y^*}{\partial p_x} \\
\frac{\partial V}{\partial p_x} &= \lambda^* \left(p_x \frac{\partial x^*}{\partial p_x} + p_Y \frac{\partial y^*}{\partial p_x} \right)
\end{aligned}
$$

Noting the aggregation results from 12.3, we see that

$$
x_1 + p_1 \frac{\partial x_1}{\partial p_1} \frac{x_1}{x_1} + p_2 \frac{\partial x_2}{\partial p_1} \frac{x_2}{x_2} + p_3 \frac{\partial x_3}{\partial p_1} \frac{x_3}{x_3} = 0
$$

Thus, we can use this result in conjunction with the derivation from 13.1 to show that

$$
\begin{aligned}
\frac{\partial V}{\partial p_x} &= \lambda^* (-x^*) \\
x^* &= -\frac{\frac{\partial V}{\partial p_x}}{\lambda^*} \\
x^* &= -\frac{\frac{\partial V}{\partial p_x}}{\frac{\partial V}{\partial m}}
\end{aligned}
$$

The logic here is that upon some algebraic manipulation, we can see that the partial derivative of the indirect utility with respect to p_x yields λ^* multiplied by the equation derived in 12.3. Thus, this allows us to reduce the form down to $\lambda^* x^*$, which using the result from 13.1 allows us to prove Roy's Identity.