PSET 3

Anthony Yoon

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1

a

We are interested in the following optimization problem:

$$\max \quad p x_1^{\frac{1}{3}} x_2^{\frac{1}{3}} - \omega_1 x_1 - \omega_2 x_2$$

We see that the FOCs are

$$\begin{aligned} [x_1] \quad & \frac{1}{3} p x_1^{-\frac{2}{3}} x_2^{\frac{1}{3}} - \omega_1 \leq 0 \quad \text{for } x_1 \geq 0 \\ [x_2] \quad & \frac{1}{3} p x_1^{\frac{1}{3}} x_2^{-\frac{2}{3}} - \omega_2 \leq 0 \quad \text{for } x_1 \geq 0 \end{aligned}$$

We can see that $x_1, x_2, \neq 0$ as this would cause the FOCs to become undefined. From here, divide the FOCs to get the relation $\omega_1 x_1 = \omega_2 x_2$ Using this expression, we can substiute this into the FOCs to get that

$$x_1^* = \frac{p^3}{27\omega_1^2\omega_2} \quad x_2^* = \frac{p^3}{27\omega_1\omega_2^2}$$

Therefore, we see that:

$$y^* = \left(\frac{p^6}{3^6\omega_1^3\omega_2^3}\right)^{\frac{1}{3}} = \frac{p^2}{9\omega_1\omega_2}$$

Thus, we see that

$$PF = py^* - \omega_1 x_1^* - \omega_2 x_2^* = p\left(\frac{p^2}{9\omega_1\omega_2}\right) - \omega_1 \frac{p^3}{27\omega_1^2\omega_2} - \omega_2 \frac{p^3}{27\omega_1\omega_2^2} = \frac{p^3}{27\omega_1\omega_2}$$

b

We can see that for the IDFs

$$\frac{\partial x_1^*}{\partial \omega_1} = -2\left(\frac{p^3}{27\omega_2\omega_1^3}\right)$$

and

$$\frac{\partial x_2^*}{\partial \omega_2} = -2\left(\frac{p^3}{27\omega_1\omega_2^3}\right)$$

Note that both quantities are bounded above by 0, as p, ω are strictly positive. For the ODF, we see that

$$\frac{\partial y^*}{\partial p} = \frac{2p}{9\omega_1\omega_2}$$

which is always positive for the same reasons. For the PF, note that

$$\frac{\partial \pi(\omega, y)}{\partial p} = \frac{p^2}{9\omega_1\omega_2} > 0 \quad \frac{\partial \pi(\omega, p)}{\partial \omega_1} = \frac{-p^3}{27\omega_1^2\omega_2} < 0 \quad \frac{\partial \pi(\omega, p)}{\partial \omega_2} = \frac{-p^3}{27\omega_1\omega_2^2} < 0$$

c

Proof that IDF is homogenous in degree 0, let $t > 0$, we see that

$$x_1^*(t\omega, tp) = \frac{(tp)^3}{27(t\omega_1)^2 t\omega_2} = \frac{t^3p^3}{27t^3\omega_1^2\omega^2} = \frac{p^3}{27\omega_1^2\omega_2} = x_1^*(\omega, p)$$

and similarly

$$x_2^*(t\omega, tp) = \frac{(tp)^3}{27t\omega_1(t\omega_2)^2} = \frac{t^3p^3}{27t^3\omega_1\omega_2^2} = \frac{p^3}{27\omega_1\omega_2^2} = x_2^*(\omega, p)$$

Proof that OSF is homogenous in degree 0, let $t > 0$, we see that

$$y^*(t\omega, tp) = \frac{t^2p^2}{9t^2\omega_1\omega_2} = \frac{p^2}{9\omega_1\omega_2} = y^*(\omega, p)$$

Proof that PF is homogenous in degree 1, let $t > 0$, we see that

$$\pi(\mathbf{t}\omega, pt) = \frac{t^3p^3}{27t^2\omega_1\omega_2} = \frac{tp^3}{27\omega_1\omega_2} = t\pi(\omega, p)$$

d

To see if Hotelling's Lemma holds, note that

$$\frac{\partial \pi(\omega, p)}{\partial p} = \frac{3p^2}{27\omega_1\omega_2} = \frac{p^2}{9\omega_1\omega_2} = y^*$$

and

$$\frac{\partial \pi}{\partial \omega_1} = \frac{-p^3}{27\omega_1^2\omega_2} = -x_1^*$$

and

$$\frac{\partial \pi}{\partial \omega_1} = \frac{-p^3}{27\omega_1\omega_2^2} = -x_2^*$$

e

If $\alpha = \beta = 0.5$, this is a Cobb Douglas function. Thus, we can use the function derived from the notes to see that the FOCs yields:

$$p = \frac{\omega_1^{\frac{1}{2}}\omega_2^{\frac{1}{2}}}{0.25(0.25)} = 16$$

Thus, if price is less than 16, there has exists no solution as there is no profit to be found for any level of production.