Problem 1.

Solution: Note that f is continous at every point in $\mathbb{R}^3$. This implies that Jacobian exists. Let $f_1 : \mathbb{R}^3 \to \mathbb{R}$, $f_1(x_1, x_2, x_3) = x_1x_2 + \sin(x_3) + x_1^2$ and $f_2 : \mathbb{R}^3 \to \mathbb{R}^1$, $f_2(x_1, x_2, x_3) = 7 + e^{x_2}$. Therefore

$$\nabla f_1 = \begin{bmatrix} x_2 + 2x_1 & x_2 & \cos(x_3) \end{bmatrix} \quad \nabla f_2 = \begin{bmatrix} 0 & e^{x_2} & 0 \end{bmatrix}$$

This implies that

$$J_x = \begin{bmatrix} x_2 + 2x_1 & x_1 & \cos(x_3) \\ 0 & e^{x_2} & 0 \end{bmatrix}$$

We now aim to show what induced one norm on a matrix. For any $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m\times n}$, we can see that:

$$\begin{aligned} Ax &= \sum_{j=1}^{n} a_{ij}x_j \\ \|Ax\|_1 &= \sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_{ij}x_j \right| \\ &\leq \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}| \cdot |x_j| \\ &\leq \sum_{j=1}^{n} |x_j| \sum_{i=1}^{m} |a_{ij}| \\ &\leq \sum_{j=1}^{n} |x_j| \max_j |c_j| \\ &\leq \max_j |c_j| \end{aligned}$$

where c_j denotes the sum of the jth column. To prove the reverse direction, we can see that if we let $x = e_j$, where it is the maximum column sum, we can see that

$$\|Ax\|_1 = \sup_{\|x\|_1 = 1} \|Ax\|_1 \geq \max_j |c_j|$$

which implies that $|A\|_1 = \max_j |c_j|$. Therefore, we see that

$$k_{abs} = \max\{|x_2 + 2x_1|, |x_1 + e^{x_2}|, |\cos(x_3)|\}$$

Therefore, since $k_{rel} = k_{abs} \cdot \frac{\|x\|_1}{\|f(x)\|_1}$, we see that:

$$k_{abs} = \max\{|x_2 + 2x_1|, |x_1 + e^{x_2}|, |\cos(x_3)|\} \cdot \frac{|x_1| + |x_2| + |x_3|}{|x_1x_2 + \sin(x_3) + x_1^2 + 7 + e^{x_2}|}$$

$\square$

Problem 2.

Solution: Let $x, X, y, Y \in \mathbb{R}$, the following are derived from the statements given.

$$x\|\cdot\|_c \leq \|\cdot\|_a \leq X\|\cdot\|_c$$
$$y\|\cdot\|_b \leq \|\cdot\|_c \leq Y\|\cdot\|_b$$

We can combine these inequalities to find that:

$$xy\|\cdot\|_b \leq x\|\cdot\|_c \leq \|\cdot\|_a \leq X\|\cdot\|_c \leq XY\|\cdot\|_b$$

Thus, showing that $\|\cdot\|_a$ and $\|\cdot\|_b$ are indeed equivalent. $\square$

Problem 3.

Solution: If $p = q$, the statement is obvious. We aim to prove that the 1 norm is equivalent to the p norm and use transitivity of norms to prove the statement. First, consider $p, q \in \mathbb{R}, p > q \geq 1$. We can consider the following for any $x \in \mathbb{R}^n$ (or $\mathbb{C}^n$):

$$\|x\|_q^q = \sum_i^n |x_i|^q = \sum_{i=1}^n 1 \cdot |x_i|^q$$

By holder's inequality, we know that:

$$\sum_{i=1}^n 1 \cdot (x_i)^q \leq \left(\sum_{i=1}^n 1^a\right)^{\frac{1}{a}} \left(\sum_{i=1}^n (|x_i|^q)^b\right)^{\frac{1}{b}}$$

such that $\frac{1}{a} + \frac{1}{b} = 1$. Since the choice of a and b are arbitary, we can let b a value such that $b^{-1} = \frac{q}{p}$. From here, we can perform the following algebra to see that:

$$a = \frac{b}{b-1} = \frac{p}{p-q}$$

and thus:

$$\begin{aligned}
\|x\|_q^q &\leq \left(\sum_{i=1}^n 1^a\right)^{\frac{1}{a}} \left(\sum_{i=1}^n (|x_i|^q)^b\right)^{\frac{1}{b}} \\
\|x\|_q^q &\leq \left(\sum_{i=1}^n 1\right)^{\frac{p-q}{p}} \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{q}{p}} \\
\|x\|_q^q &\leq n^{\frac{p-q}{q}} \|x\|_p^q \\
\|x\|_q &\leq n^{\frac{p-q}{pq}} \|x\|_p
\end{aligned}$$

Let $q = 1$ and $p > 1$. we know aim to bound $\|x\|_1$ from below. Let $\{e_1, e_2, \ldots, e_n\}$ denote the standard basis vectors of $\mathbb{R}^n$. Let $x = (x_1, x_2, \ldots, x_n)$. and we can see that:

$$x = \sum_{i=1}^n e_i x_i$$

Thus, we can see that:

$$\|x\|_p = \left\|\sum_{i=1}^n x_i e_i\right\|_p$$

By the triangle inequality and the properties of the norm, we know that:

$$\left\|\sum_{i=1}^n x_i e_i\right\|_p \leq \sum_{i=1}^n \|x_i e_i\|_p = \sum_{i=1}^n |x_i| \cdot \|e_i\|_p$$

Let $M = \max_{i \in 1,2,\ldots,n} \|e_i\|_p$ Thus, it follows that:

$$\|x\|_p \leq \sum_{i=1}^{n} |x_i| \cdot \|e_i\|_p \leq M \sum_{i=1}^{n} |x_i| = M\|x\|_1$$

Thus, we can see that:

$$\frac{1}{M}\|x\|_p \leq \|x\|_1 \leq n^{\frac{p-1}{p}} \|x\|_p$$

Thus, we have proved that 1 norm and the p norm is equivalnet. Thus, by transitive of norms, we can see that 1 and q are equivalent, which implies that p and q are equivalent norms. $\square$

Problem 4.

Solution: Conisder the following: We know that by the definition of the induced norm that:

$$\|Ax\|_a \leq \|A\|_{a \leftarrow c} \cdot \|x\|_c$$

as

$$\|A\|_{a \leftarrow c} := \sup \frac{\|Ax\|_a}{\|x\|_c}, \forall x \in \mathbb{R}^n$$

Let $y = Bx$, we see that:

$$\|Ay\|_a \leq \|A\|_{a \leftarrow c} \cdot \|Bx\|_c$$

But, since we know that:

$$\|B\|_{c \leftarrow b} := \sup \frac{\|Bx\|_c}{\|x\|_b}, \forall x \in \mathbb{R}^n$$

we can see that:

$$\|Bx\|_c \leq \|B\|_{c \leftarrow b} \cdot \|x\|_b$$

Thus, we can see that, if we were to combine these two inequalities, we get that:

$$\|Ay\|_a \leq \|A\|_{a \leftarrow c} \cdot \| \cdot \|B\|_{c \leftarrow b} \cdot \|x\|_b$$

We can see that

$$\|ABx\|_a \leq \|A\|_{a \leftarrow c} \cdot \| \cdot \|B\|_{c \leftarrow b} \cdot \|x\|_b$$
$$\frac{\|ABx\|_a}{\|x\|_b} \leq \|A\|_{a \leftarrow c} \cdot \| \cdot \|B\|_{c \leftarrow b}$$

We can take the supremum of $\frac{\|ABx\|_a}{\|x\|_b}$, and we can see that:

$$\|AB\|_{a \leftarrow b} \leq \|A\|_{a \leftarrow c} \|B\|_{c \leftarrow b}$$

□

Problem 5.

Solution: Consider the subspace of V_1 and V_2 spanned by the basis $\{\cos(n\pi x), \sin(\sin(n\pi x))\}$. We can $f \in \{\cos(n\pi x), \sin(\sin(n\pi x))\}$ is a linear combination, denoted as follows:

$$f = a_n \sin(n\pi x) b_n + \cos(n\pi x)$$

Thus,

$$\frac{d}{dx} f = n\pi(a_n \cos(n\pi x) - b_n \sin(n\pi x))$$

Thus, we can see that:

$$T_n f = \begin{bmatrix} 0 & -n\pi \\ n\pi & 0 \end{bmatrix} \begin{bmatrix} a_n \sin(n\pi x) \\ b_n \cos(n\pi x) \end{bmatrix} = \begin{bmatrix} (a_n \cos(n\pi x) \\ -b_n \sin(n\pi x)) \end{bmatrix}$$

Which implies that

$$T_n = \begin{bmatrix} 0 & n\pi \\ -n\pi & 0 \end{bmatrix}$$

Which implies that the T is:

$$T = \begin{bmatrix} T_1 & 0 & 0 & \dots \\ 0 & T_2 & 0 & \dots \\ \vdots & \vdots & \ddots & \dots \\ 0 & 0 & 0 & T_n \end{bmatrix}$$

where T_n is defined as above. Note that any function in V_1 can be defined as follows:

$$f = \sum_{i=1}^{k} a_n \sin(n\pi x) + b_n \cos(n\pi x)$$

We now compute $\langle f, f \rangle$.

$$\langle f, f \rangle = \int_0^1 \left(\sum_{i=1}^{k} a_n \sin(n\pi x) + b_n \cos(n\pi x) \right)^2$$

$$= \int_0^1 \sum_{n,m}^{k} a_n a_m \sin(n\pi x) \sin(m\pi x) + b_n b_m \cos(n\pi x) \cos(m\pi x) + 2a_n b_m \sin(n\pi x) \cos(n\pi x)$$

Note that:

$$\int_0^1 \sin(n\pi x) \sin(m\pi x) = \begin{cases} 0.5 & n = m \\ 0 & n \neq m \end{cases}$$

$$\int_0^1 \cos(n\pi x) \cos(m\pi x) = \begin{cases} 0.5 & n = m \\ 0 & n \neq m \end{cases}$$

$$\int_0^1 \sin(n\pi x)\cos(n\pi x) = 0, \forall n, m$$

Thus, this implies that:

$$\langle f, f\rangle = \frac{1}{2}\sum_{i=1}^{k}(a_i^2 + b_i^2)$$

By a similar logic, we can find that:

$$\langle Tf, Tf\rangle = \frac{1}{2}\sum_{n=1}^{k}(n\pi)^2(a_n^2 + b_n^2)$$

Therefore, we can see that:

$$\left(\frac{\|Tf\|}{\|f\|}\right)^2 = \frac{\sum_{n=1}^{k}(n^2\pi^2)(a_n^2 + b_n^2)}{\sum_{n=1}^{k}(a_n^2 + b_n^2)}$$

Note that $\sup_f \left(\frac{\|Tf\|}{\|f\|}\right) = k\pi$, where this maximum is obtained when $a_n, b_n = 0$ when $n \neq k$, and $a_k, b_k = 1$, as this is the sum of all weighted averages, and the maximum is achived when we put all the weight when $n = k$. This implies that $\|T\| = k\pi$. We first also wish to prove the existance of a T^{-1}. Note that the deriative is a bjiective function, which implies that T^{-1} exists. In this case, we can see that T^{-1} can be defined as follows:

$$T_n^{-1} = \begin{bmatrix} 0 & \frac{-1}{n\pi} \\ \frac{1}{n\pi} & 0 \end{bmatrix}$$

where:

$$T^{-1} = \begin{bmatrix} T_1^{-1} & 0 & 0 & \dots \\ 0 & T_2^{-1} & 0 & \dots \\ \vdots & \vdots & \ddots & \dots \\ 0 & 0 & 0 & T_n^{-1} \end{bmatrix}$$

Let $Tf = g$, where $g, f \in V_1$. We can see that:

$$\sup_g \left(\frac{\|T^{-1}g\|}{\|g\|}\right)^2 = \sup_f \left(\frac{\|f\|}{\|Tf\|}\right)^2 = \sup_n \left(\frac{1}{(n\pi)^2}\right) = \frac{1}{\pi^2}$$

Thus, this implies that $\|T^{-1}\| = \frac{1}{\pi}$ Thus, we see that:

$$\max k_{rel} = \|T\|\|T^{-1}\| = k$$

Thius, this implies that $k_{rel} = O(k)$, as the bounded function as follows:

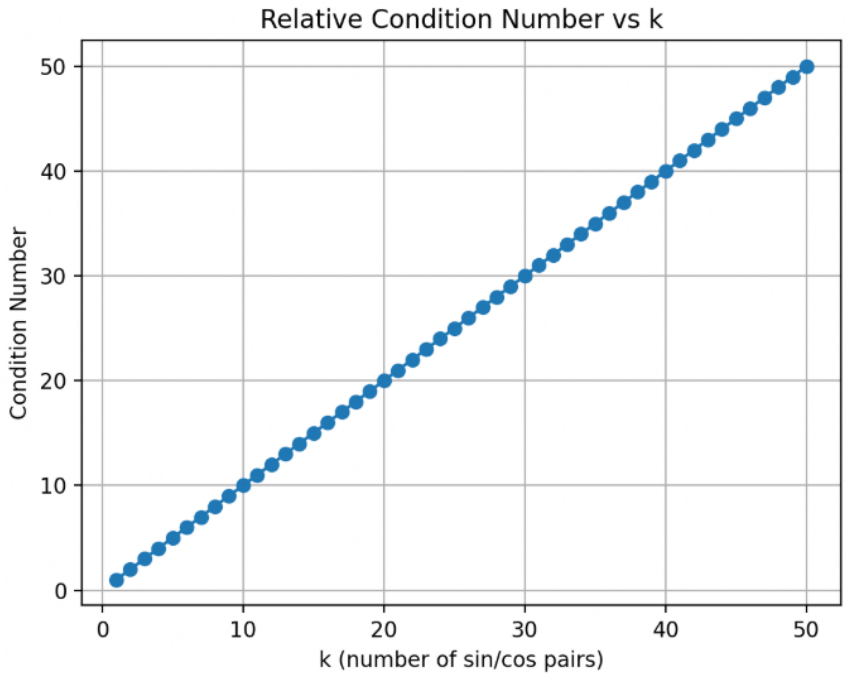

Figure 1: Showing that the relative condition is $O(k)$

$\square$

Problem 6.

Solution: Let $a_1 = 2\sin(\pi x) + \sin(\pi x)$ and $a_2 = -3\sin(\pi x) + \sin(2\pi x)$. Let $q_1 = \frac{a_1}{\|a_1\|}$. Thus, we can proceed with the folloiwng computation:

$$\begin{aligned}\langle a_1, a_1\rangle &= \langle 2\sin(\pi x) + \sin(\pi x), 2\sin(\pi x) + \sin(\pi x)\rangle \\ &= \int_0^1 (2\sin(2\pi x) + \sin(\pi x))^2 dx \\ &= \int_0^1 4\sin^2(2\pi x) + 4\sin(\pi x)\sin(2\pi x) + \sin^2(\pi x) dx\end{aligned}$$

Note that $\int_0^1 \sin(\pi x)\sin(2\pi x)dx = 0$. Thus, we can see that:

$$\begin{aligned}\langle a_1, a_1\rangle &= \int_0^1 4\sin^2(2\pi x) + 4\sin(\pi x)\sin(2\pi x) + \sin^2(\pi x) dx \\ &= \int_0^1 4\sin^2(2\pi x) + \sin^2(\pi x) dx\end{aligned}$$

Note $\int_0^1 \sin^2(n\pi x) = 0.5, \forall n \in \mathbb{N}$. Thus, we can see that:

$$\langle a_1, a_1\rangle = \int_0^1 4\sin^2(2\pi x) + \sin^2(\pi x) dx = 2 + 0.5 = \frac{5}{2}$$

This implies that:

$$q_1 = \sqrt{\frac{2}{5}}(2\sin(2\pi x) + \sin(\pi x))$$

How, we must consider solving the following $v_2 = a_2 - \text{proj}_{a_2} q_1 = a_2 - \langle a_2, q_1\rangle q_1$. We now the following the integral;

$$\begin{aligned}\sqrt{\frac{2}{5}}\int_0^1 (2\sin(2\pi x) + \sin(\pi x))(\sin(2\pi x) - 3\sin(\pi x))dx &= \sqrt{\frac{2}{5}}\int_0^1 2\sin^2(2\pi x) - 3\sin^2(\pi x) dx \\ &= \sqrt{\frac{2}{5}} * \frac{-1}{2} \\ &= -\sqrt{\frac{1}{10}}\end{aligned}$$

Thus, we see that $v_2 = a_2 - \text{proj}_{a_2} q_1 = a_2 - \langle a_2, q_1\rangle q_1 = a_2 + \left(\sqrt{\frac{1}{10}}\right)\left(\sqrt{\frac{2}{5}}\right) a_1 = a_2 + 0.2a_1$. Simplifying the vectors, we can see that we can simplify to:

$$-3\sin(\pi x) + \sin(2\pi x) + 0.2(2\sin(2\pi x) + \sin(\pi x)) = \frac{-14}{5}\sin(\pi x) + \frac{7}{5}\sin(2\pi x)$$

We now proceed with the following calcuation:

$$\int_0^1 \left(\frac{7}{5}(-2\sin(\pi x) + \sin(2\pi x))\right)^2 dx = \frac{49}{25}\int_0^1 4\sin^2(\pi x) + \sin^2(2\pi x) dx$$
$$= \frac{49}{25}\left(2 + \frac{1}{2}\right)$$
$$= \frac{49}{10}$$

We see that $\|v_2\| = \frac{7}{\sqrt{10}}$. Therefore, we can see that

$$q_1 = \sqrt{\frac{2}{5}}(2\sin(2\pi x) + \sin(\pi x)), q_2 = \frac{\sqrt{10}}{5}(\sin(2\pi x) - 2\sin(\pi x))$$

Note that $r_{11} = \sqrt{\frac{5}{2}}$, $r_{12} = \frac{\langle a_2, q_1\rangle}{\|q_1\|} = -\frac{1}{\sqrt{10}}, r_{22} = \frac{7}{\sqrt{10}} = \|v_2\|$. Thus, this implies that:

$$R = \begin{bmatrix} \sqrt{\frac{5}{2}} & -\sqrt{\frac{1}{10}} \\ 0 & \frac{7}{\sqrt{10}} \end{bmatrix}$$

$\square$