PSET5

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1

a

We can first begin calulating MRS for each individual. Without a loss of generality, we can begin to note the following:

$$MRS_i = \frac{\alpha(x_1^i)^{\alpha-1}(x_2^i)^{\beta}}{\beta(x_2^i)^{\beta-1}(x_1^i)^{\alpha}}$$

This above quantity implies that $\frac{x_2^1}{x_1^1} = \frac{x_2^2}{x_1^2}$. Let $e_1 = x_1^1 + x_1^2$ and $e_2 = x_2^1 + x_2^2$. Using this equalities as well as the implication derived from the MRSes, we can find that we get:

$$x_2^1 = \frac{e_2}{e_1}x_1^1 \quad x_2^2 = \frac{e_2}{e_1}x_1^2$$

Since both the above quantities are linear in nature with no intercept, this implies that indeed the contact curve is that of connecting endpoints.

b

Since we we are working with different utility functions, we can find that after similar caluclations to above that:

$$MRS_1 = \frac{\alpha(x_1^1)^{\alpha-1}(x_2^1)^{1-\alpha}}{(1-\alpha)(x_1^1)^{\alpha}(x_2^1)^{-\alpha}} = \frac{\alpha x_2^1}{(1-\alpha)x_1^1}$$

and using similar calculations, we find that:

$$MRS_2 = \frac{\beta x_2^2}{(1-\beta)x_1^2}$$

Since we know that $1 > \alpha > \beta > 0$, we find that:

$$\frac{\alpha}{1-\alpha} > \frac{\beta}{1-\beta}$$

Thus, we can see that for MRS to equal to each other, we know that:

$$\frac{x_2^1}{x_1^1} < \frac{x_2^2}{x_1^2}$$

Using the equations derived above, we can find that:

$$x_2^1 < \frac{e_2}{e_1} x_1^1$$

this implies that the graph still intersects the origins, but now $x_2^1 < x_1^1$, where we have all a curve that will be strictly below that of the original line derived in **a**. The contract curve is seen as below.

c

For the contract curve to exist, we want $MRS_1 = MRS_2$. Let $e_1 = x_1^1 + x_1^2$. We can see that

$$MRS_1 = MRS_2 \implies \alpha(x_1^1)^{\alpha-1} = \beta(x_1^2)^{\beta-1}$$

Thus, substuting the endowment, we find that:

$$\alpha(x_1^1)^{\alpha-1} = \beta(e_1 - x_1^1)^{\beta-1}$$

So we see that as $x_1^1 \to e_1^1$, we find that the consumers will not consume any x_2, and consume only x_1 However, since we know that $\alpha > \beta$, this implies that consumer x_1^1 has greater value on x_1^1, which implies that $x_1^1 > x_1^2$. Note that any level of x_2 sastifies the MRS equality arguement, thus we are only concerned about when $e_1 = x_1^1 + x_1^2$ and one of these inputs are the utility maximizing solution. The contract curve will like the one below:

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a

See Graph

b

See Graph

c

See Graph

d

The core would just be the point (e_1^1, e_2^1) as any trade would make this individual worse off, hence no trade would not be blocked by any coalition.

e

We first derive the general Marshallian Demand function for each individual. We are interested in the following optimization problem:

$$\begin{aligned} \max \quad & x_1 x_2 \\ \text{s.t} \quad & p_1 e_1 + p_2 e_2 \geq p_1 x_1 + p_2 x_2 \end{aligned}$$

Where the langrangian is:

$$L = x_1 x_2 - \lambda(p_1 e_1 + p_2 e_2 - p_1 x_1 - p_2 x_2)$$

wherw we see that the FOCs are:

$$\begin{aligned} [x_1] \quad & x_2 + \lambda p_1 \leq 0 \text{ and } x_1 \geq 0 \\ [x_2] \quad & x_1 + \lambda p_2 \leq 0 \text{ and } x_2 \geq 0 \\ [\lambda] \quad & p_1 e_1 + p_2 e_2 \leq p_1 x_1 + p_2 x_2 \end{aligned}$$

We can see all FOCs must be strict equality, as if that is not the case, then markets will fail to clear $([x_1], [x_2])$ and by the AU assumption that we want to use all of our endowment to maximize utility. Using these FOCs, we can find that:

$$p_1 x_1 = p_2 x_2$$

Thus, using this equation and the constraint, we find that:

$$x_1^m = \frac{e_1}{2} + \frac{p_2}{2p_1} e_2 \quad x_2^m = \frac{p_1}{2p_2} e_1 + \frac{e_2}{2}$$

Thus, we find that:

$$x_1^1 = \frac{p_2}{2p_1} + 1 \quad x_2^1 = \frac{p_1}{p_2} + \frac{1}{2}$$

$$x_1^2 = \frac{3p_2}{2p_1} + 1 \quad x_2^2 = \frac{p_1}{p_2} + \frac{3}{2}$$

Thus we can find that where we let $\mathbf{p} = (p_1, p_2)$:

$$z_1(\mathbf{p}) = 2 + 2\left(\frac{p_2}{p_1}\right) - 4 = -2 + 2\left(\frac{p_2}{p_1}\right)$$

$$z_2(\mathbf{p}) = 2 + 2\left(\frac{p_1}{p_2}\right) - 4 = -2 + 2\left(\frac{p_1}{p_2}\right)$$

and we can verify that:

$$\mathbf{p} \cdot \mathbf{z} = \begin{bmatrix} p_1 & p_2 \end{bmatrix} \cdot \begin{bmatrix} 2\left(\frac{p_2}{p_1}\right) - 2 \\ 2\left(\frac{p_1}{p_2}\right) - 2 \end{bmatrix} = 2p_2 - 2p_1 + 2p_1 - 2p_2 = 0$$

f

We can see if $p_2 = p_1$, we find that obviously, $\mathbf{z}$ goes to 0. Thus, we find that the set of Walrasian equlibria is $p^*(\mathcal{E}) = \{p_1, p_2)|p_1 = p_2\}$ We find that a Walrasian equilbrium allocation is

$$x^W = \{(x_1, x_2)\} = \{(1.5, 1.5), (2.5, 2.5)\}$$

since there is one relative price, we can find that the set of Walrasian Equilbirum Allocations is:

$$W(\mathcal{E}) = \bigcup_{p^*} x^W(p^*(\mathcal{E}), \mathcal{E}) = \{(1.5, 1.5), (2.5, 2.5)\}$$

g

Note that $e_1 = e_2$, this implies that the slope of the line is 1, which means that all allocations that have $x_1 = x_2$ will be in the core. Thus, we can see that the above set is a subset of the core.

h

Switching the utility function, we find that we are interested in the following optimization problem:

$$\begin{aligned} \max \quad & x_1^{\frac{2}{3}} x_2^{\frac{1}{3}} \\ \text{s.t} \quad & p_1 e_1 + p_2 e_2 \leq p_1 x_1 + p_2 x_2 \end{aligned}$$

the constraint remains the same, but with the following FOCs.

$$\begin{aligned} [x_1] \quad & \left(\frac{2}{3} x_1^{-\frac{1}{3}}\right) x_2^{\frac{1}{3}} = p_1 \lambda \\ [x_2] \quad & \left(\frac{1}{3} x_1^{-\frac{1}{3}}\right) x_1^{\frac{2}{3}} = p_2 \lambda \end{aligned}$$

We know that these FOCs must have strict equality due to the same reasons as stated above. Using these FOCs, the following can be derived:

$$2x_2 p_2 = x_1 p_1$$

which implies that:

$$x_1^2 = \frac{2}{3p_1}(p_1e_1 + p_2e_2) \quad x_2^2 = \frac{1}{3p_1}(p_1e_1 + p_2e_2)$$

Using, previous results, we can find that:

$$z_1(\mathbf{p}) = 1 + \frac{p_2}{2p_1} + \frac{4}{3} + \frac{2p_2}{p_1} - 4 = \frac{5p_2}{2p_1} - \frac{5}{3}$$

$$z_2(\mathbf{p}) = \frac{1}{2} + \frac{p_1}{p_2} + \frac{2p_1}{3p_2} + 1 - 4 = \frac{5p_1}{3p_2} - \frac{5}{2}$$

Note that this implies that $3p_2 = 2p_1$, as this is the only relative price that makes Walras' law hold. Thus, we see that:

$$x^W = \{(x_1, x_2)\} = \left\{ \left(\frac{4}{3}, 2\right), \left(\frac{8}{3}, 2\right) \right\}$$

and since we have only one relative price, we can find that the set of Walrasian equilbirum is

$$W(\mathcal{E}) = \bigcup_{p^*} x^W(p^*(\mathcal{E}), \mathcal{E}) = \left\{ \left(\frac{4}{3}, 2\right), \left(\frac{8}{3}, 2\right) \right\}$$

i

Consider the general form of the equation,

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4

a

We can note that as we are all using the minimum function, we can see that $x_1^1 = x_2^1 = x_2^2 = x_3^2 = x_1^3 = x_3^3$, which implies that we are interested in the allocation:

$$x_1^1 = x_2^1 = x_2^2 = x_3^2 = x_1^3 = x_3^3 = 0.5$$

which is optimal as moving away would cause utilty lose and maximizes utlity for all and markets clear. Thus, this allocation is in the core, and let this allocation equal $\overline{x}$

b

Consider the UMP for consumer 1. We are interested in the following optimization problem:

$$\begin{aligned} \max \quad & \min\{x_1^1, x_2^1\} \\ \text{s.t} \quad & x_1^1 p_1 + p_1 x_2^1 \leq pe_1 \end{aligned}$$

note that we want $x_1^1 = x_2^1$, which implies using the strict equality of the constraint (which we know is true by the AU assumption) that

$$x_1^1 = x_2^1 = \frac{p_1}{p_1 + p_2}$$

using a similar logic, we can see that:

$$x_2^2 = x_3^2 = \frac{p_2}{p_2 + p_3}$$

and

$$x_1^3 = x_3^3 = \frac{p_3}{p_1 + p_3}$$

Thus, we can now analyze the aggegate demand functions. We find that $z_i = x_i^1 + x_i^2 + x_i^3 - 1$ where $i \in \{1, 2, 3\}$. After some algebra and using the Marshallian demand functions taht we derived, we find that $p_1 = p_2 = p_3$. This implies that:

$$x_1^1 = x_2^1 = x_2^2 = x_3^2 = x_1^3 = x_3^3 = 0.5$$

Thus, we can see that $p^*(\mathcal{E}) = \{(p_1, p_2, p_3) \in \mathbb{R}^3 | p_1 = p_2 = p_3\}$. Thus, we can see tha $x^W(p^*(\mathcal{E}), \mathcal{E}) = \{(0.5, 0.5, 0), (0, 0.5, 0.5), (0.5, 0, 0.5)\}$ Since we are working with only one relative price, we can note that

$$W(\mathcal{E}) = \bigcup_{p^*} x^W(p^*(\mathcal{E}), \mathcal{E}) = \{(0.5, 0.5, 0), (0, 0.5, 0.5), (0.5, 0, 0.5)\}$$

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