Nested AMMs

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Suppose we want to design a three-product AMM: an AMM that can trade cash (that is, say, some stablecoin), which we'll call y, with two different kinds of tokens, a and b. Suppose also that a and b are "similar" products in some sense. Here are some hypothetical examples.

- Suppose a and b are both ETH perps, with slighly different contract specs. However, a and b generally have very similar market prices.
- a and b are options on ETH with similar strikes
- a and b are both claims on yields, so their prices tend to move closely together.
- Many other case where tokens a and b's prices tend to have some correlation.

In such settings, we might want our AMM to behave in a way that recognizes that a and b are very similar to each other. What does this mean? If a and b are economically similar, the AMM's portfolio has roughly the same risk exposures if it holds a lot of a and a little b, or vice versa. Thus, the AMM should be willing to trade a for b, allowing relatively low price slippage. On the other hand, the AMM should allow higher price slippage when trading a or b for cash, since this changes the AMM's total risk exposure.

In this note, I describe a simple AMM design which accomplishes this, which I call the *nested AMM.*

1 Nested AMMs

Consider an AMM which uses the following class of functions as a "constant curve":

$$U(a, b, y) = \left(\left(a^{1-\sigma} + b^{1-\sigma}\right)^{\frac{1}{1-\sigma}}\right)^{1-\eta} + y^{1-\eta} \tag{1}$$

Where $\eta > y$. This defines an AMM, since, suppose the stablecoin begins with a, b, y. Suppose an agent wants to buy y by paying a. The AMM is willing to make any trade of Δ_y of y for Δ_a of a, as long as U

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stays fixed:

$$U(a,b,y) = U(a+\Delta_a, b, y-\Delta_y) \tag{2}$$

This is a simple example of a constant function market maker (Angeris and Chitra (2020)). I claim that it accomplishes the desired goal, of allowing the a,b slippage to be low while the a,y and b,y trade price slippage is high. Let's use a simple example to illustrate this, setting $\sigma = 0.2, \eta = 0.8$. Hence, we have:

$$U(a,b,y) = \left(\left(a^{1-0.2} + b^{1-0.2}\right)^{\frac{1}{1-0.2}}\right)^{1-0.8} + y^{1-0.8}$$

Now let's do some numerical experiments. Suppose we start with 10 units each of a,b,y. This gives the constant approximately 3.58. I calculate prices for different trades in accompanying Github code available here.

To calculate price slippage, we'll ask a simple question: let's try to spend 1, 2, and 3 units of a to buy y from the AMM, and see how much the price of y in terms of a changes. Then, let's try to spend 1, 2, and 3 units of a to buy b, and see how the price changes.

- If you use 1, 2, or 3 units of a to buy y, you can buy respectively 0.597, 1.136, 1.624 units of y. The prices are 1.76, 1.84, 1.92. To calculate slippage, note that spending 3 units of a instead of 1 unit increases the price by $\frac{1.92}{1.76} - 1 = 9.2\%$.
- If you use 1, 2, or 3 units of a to buy b, you can buy respectively 0.96, 1.85, 2.68 units of b. The prices are 1.02, 1.04, 1.06. Buying 3 units of 1 unit increases the price of b in terms of a, by $\frac{1.06}{1.02} - 1 = 4.0\%$.

So, this accomplishes the desired behavior: a,b price slippage is lower than a,y price slippage.

To see mathematically what is going on, note that in (1), a and b live in a "nest", with the curvature/substitution elasticity between them governed by the parameter σ. They effectively create a "composite product":

$$\left(a^{1-\sigma} + b^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

This "composite" then substitutes with y, with the curvature/substitution elasticity between them governed by the parameter η. When σ is low, a and b are "good substitutes" and can be traded with low slippage. When η is low, the a,b "composite product" can be traded with y at low slippage. So when σ is low but η is high, a,b can be traded with low slippage, but a,y and b,y trade at high slippage.

2 General formula

More than three tokens. The three-product example easily generalizes. We can in general do:

$$U(x_1 \dots x_n, y) = \left(\left(\sum_{i=1}^{n} w_i x_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \right)^{1-\eta} + w_y y^{1-\eta} \tag{3}$$

Expression (3) allows multiple kinds of tokens in the "nest". Tokens can be weighted with weights w_i: this could be useful, for example, if the tokens have different risk exposures: for example if the AMM is trading 2x and 4x leveraged perps, the AMM may want to put a different weight on the tokens, to reflect their risk exposures.

Multiple nests. We could include multiple nests:

$$U(x_1 \dots x_n, z_1 \dots z_m, y) = \left(\left(\sum_{i=1}^{n} w_{xi} x_i^{1-\sigma_x} \right)^{\frac{1}{1-\sigma_x}} \right)^{1-\eta} + \left(\left(\sum_{j=1}^{m} w_{zj} z_j^{1-\sigma_z} \right)^{\frac{1}{1-\sigma_z}} \right)^{1-\eta} + w_y y^{1-\eta} \tag{4}$$

Things like (4) might be useful if we want one AMM to make markets for multiple nests of products. For example, the x nest could be BTH futures or options, and the z nest could be ETH futures and options. There's correlations within the buckets, so we want to make it easy (low price slippage) to trade tokens within the nests for each other, and harder – but still possible, though with higher price slippage – to trade tokens across nests. Note that the substitution elasticity parameters, σ_x and σ_z in the example, can vary across nests.

If an AMM wants to allow multiple kinds of stablecoins, one option would be to expand y into a "cash nest", with a substitution parameter σ_y very close to 0, so that there's very little slippage (i.e. the AMM is happy to hold many different mixes of stablecoins, but maintains a certain ratio of total stablecoins relative to non-stablecoin tokens)

Multi-layered nests. Another trick is that we can put *nests within nests*. The simplest example is:

$$U(x_1 \dots x_n, z_1 \dots z_m, y) = \left(\left(\left(\left(x_1^{1-\theta} + x_2^{1-\theta} \right)^{\frac{1}{1-\theta}} \right)^{1-\sigma} + x_3^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \right)^{1-\eta} + y^{1-\eta}$$

Where we have $\eta > \sigma > \theta$. The intuition behind the previous cases extends: the closer two tokens are in terms of nests, the lower the price slippage. x_1 and x_2 can be traded with low slippage; x_1 and x_3 with slightly higher slippage; and x_1 and y with even higher price slippage. Once again, this tends to encourage trading products within nests.

3 Literature

This builds off a trick from the economics literature of *nested CES utility functions*. These are used in economics to model settings in which products are more substitutable within nests than across them. It's a common trick in economics: I don't know the original reference, but see for example Engel and Herlihy (2021), Perroni and Rutherford (1995), Keller (1976). The mapping to AMMs exploits a known result in the AMMs literature that there is a duality between quasi-concave utility functions and AMMs: see for example Angeris and Chitra (2020), Angeris et al. (2021), Engel and Herlihy (2021).

4 Detailed math

This section is aimed at those who are familiar with some AMM theory. To characterize the behavior of a given AMM, we basically want to characterize the behavior of marginal rates of substitution. Consider the function:

$$U(a,b,y) = \left(\left(a^{1-\sigma}+b^{1-\sigma}\right)^{\frac{1}{1-\sigma}}\right)^{1-\eta} + y^{1-\eta} \tag{5}$$

We want to solve for the marginal rates of substitution, which are the derivatives:

$$\frac{db}{da} = \frac{\frac{\partial U}{\partial a}}{\frac{\partial U}{\partial b}}, \frac{dy}{da} = \frac{\frac{\partial U}{\partial a}}{\frac{\partial U}{\partial y}}$$

(Note that by symmetry we'll have $\frac{\frac{\partial U}{\partial a}}{\frac{\partial U}{\partial y}} = \frac{\frac{\partial U}{\partial b}}{\frac{\partial U}{\partial y}}$ so we skip the last one). In AMM terms, the marginal rates of substitution are the rates at which the AMM will make infinitisemally small trades, if its current holdings are a, b, y.

To calculate these, note that we can alternatively write $U(a,b,y)$ as:

$$U(a,b,y) = x^{1-\eta} + y^{1-\eta}$$

$$x^{1-\sigma} = a^{1-\sigma} + b^{1-\sigma} \tag{6}$$

where x is basically a "composite good" made from a and b. Now, totally differentiating (6), we have:

$$(1-\sigma)\,x^{-\sigma}dx = (1-\sigma)\,a^{-\sigma}da + (1-\sigma)\,b^{-\sigma}db$$

This gives us:

$$\frac{\partial x}{\partial a} = \left(\frac{a}{x}\right)^{-\sigma} \tag{7}$$

$$\frac{\partial x}{\partial b} = \left(\frac{b}{x}\right)^{-\sigma}$$

Now, using (7) and (5), we have:

$$\frac{\partial U}{\partial y} = (1-\eta)\, y^{-\eta}$$

$$\frac{\partial U}{\partial x} = (1-\eta)\, x^{-\eta}$$

$$\frac{\partial x}{\partial a} = \left(\frac{a}{x}\right)^{-\sigma}$$

$$\frac{\partial U}{\partial a} = \frac{\partial U}{\partial x}\frac{\partial x}{\partial a} = (1-\eta)\, x^{-\eta}\left(\frac{a}{x}\right)^{-\sigma}$$

$$\frac{\partial U}{\partial b} = \frac{\partial U}{\partial x}\frac{\partial x}{\partial b} = (1-\eta)\, x^{-\eta}\left(\frac{b}{x}\right)^{-\sigma}$$

Hence we can get the MRSs:

$$\frac{dy}{da} = \frac{\frac{\partial U}{\partial a}}{\frac{\partial U}{\partial y}} = \frac{(1-\eta)\, x^{-\eta}\left(\frac{a}{x}\right)^{-\sigma}}{(1-\eta)\, y^{-\eta}} = \left(\frac{x}{y}\right)^{-\eta}\left(\frac{a}{x}\right)^{-\sigma} = \left(\frac{(a^{1-\sigma}+b^{1-\sigma})^{\frac{1}{1-\sigma}}}{y}\right)^{-\eta}\left(\frac{a}{(a^{1-\sigma}+b^{1-\sigma})^{\frac{1}{1-\sigma}}}\right)^{-\sigma} \tag{8}$$

$$\frac{da}{db} = \frac{\frac{\partial U}{\partial b}}{\frac{\partial U}{\partial a}} = \left(\frac{b}{a}\right)^{-\sigma} \tag{9}$$

Expressions (8) and (9) characterize MRSs. The MRS between a and b only depends on the relative amounts of b and a the AMM holds. In particular, it's independent of y, how much cash the AMM holds. On the other hand, the MRS between y and a depends both on how much of a there is relative to b (which determines the ratio $\frac{a}{x}$), as well as how much of a and b together the AMM holds relative to cash y. Moreover, since $\eta > \sigma$, we have that a, y prices respond more overall to movements in the ratio $\frac{x}{y}$, than a, b prices respond to movements in $\frac{b}{a}$.

5 References

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