Faculté d'Économie et de Gestion
Aix Marseille Université

FINANCIAL ECONOMETRICS

Subject 1 : Log Normal Asset dynamic

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Subject 1: Log Normal asset dynamic

1/ Rectification on our database.

In this paper, we will assume log returns of a financial asset are Gaussian with mean $\mu = 0$ and an annual standard deviation $\sigma = 10\%$, and will simulate on a daily basis four years of data with a standard Gaussian realization as:

$$\epsilon_t = \Phi^{-1}(u_t) \sim \mathcal{N}(0,1)$$

To forecast the risk of our portfolio using the VaR, we must first standardize our database. In fact, we use log returns of the asset as:

$$r_t = \epsilon_t \times \frac{10\%}{\sqrt{252}}$$

Returns are calculated on a Monte Carlo basis with (ϵ_t) following a standard Gaussian distribution with mean $\mu = 0$ and an annual standard deviation $\sigma = 1$.

2/ Why are we multiplying the standard gaussian by $\frac{10\%}{\sqrt{252}}$?

On the one hand, as the asset price is described by a log normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 10\%$ (which allows the asset price to be non-negative), we must multiply by 10%. As a matter of fact, we have simulated a standard Gaussian realization ϵ_t with mean $\mu = 0$ and standard deviation $\sigma = 1$, yet log returns follow a Gaussian with mean $\mu = 0$ and standard deviation $\sigma = 10\%$, as we are interested in the daily asset return we have to normalize properly to take into account true variations.

On the other hand, we are multiplying by $\frac{1}{\sqrt{252}}$, because otherwise if we forgot to do it, we would only have annual log return r_t.

2/ How frequently do you breach the VaR ?

It's interesting to know how many times the Value at Risk is breached. To remind, the Value at Risk corresponds to the loss that we expect to have in a tail event, depending on a specific covering rate (here 5%).

To accomplish that, we use three backtesting procedures (unconditional, independence and conditional test) from the one day ahead 5% Historical Simulation VaR using for each day one year of returns.

We define the hit sequence of violations as: $I_{t+1} = \begin{cases} 1, if\ R_{PF,t+1} < -VaR_{t+1}^{p} \\ 0, if\ R_{PF,t+1} \geq -VaR_{t+1}^{p} \end{cases}$

Additionally, we consider the null hypothesis as: $H_0 : I_{t+1} \sim i.i.d.\ \text{Bernoulli}(p)$

Thus, with the unconditional coverage test, we want to test whether the estimated frequency of VaR violations is statistically equal to the appropriate coverage rate of 5% from the VaR calculation.

Therefore, we define the average proportion of VaR breaching by $\hat{\pi}$ from our sample and plug it into the likelihood function for an i.i.d Bernouilli random variable with parameter $\hat{\pi} = \frac{T_1}{T}$.

The optimized likelihood function is thus obtained from: $L(\hat{\pi}) = (1 - T_1/T)^{T_0} (T_1/T)^{T_1}$

where: T_1 = number of times where I_{t+1} = 1, T = sample size and T_0= $T - T_1$.

After that, from the alternative hypothesis (p = 5%, the coverage rate), we determine the appropriate likelihood function:

$$L(p) = \prod_{t=1}^{T} (1-p)^{1-I_{t+1}} p^{I_{t+1}} = (1-p)^{T_0} p^{T_1}$$

Ultimately, the unconditional test can be performed thanks to a likelihood ratio test, with the following likelihood ratio test statistic:

$$LR_{uc} = -2 \ln [L(p) / L(\hat{\pi})]$$

In this manner, we gather the following results for the test:

BACKTESTING VAR				
Unconditional Test			T_1	42
$\hat{\pi}$	5,56%		T_0	714
$L(\hat{\pi})$	3,59E-71		T	756
$L(p)$	2,83E-71			
LR_{uc}	0,474892737		$\chi_1^{2,1\%} = 6{,}63 > LR_{uc}$	

For the purpose of emphasizing our reasoning, we will take a particular case from the random results we obtain, which will be accordingly different each time we press the command enter (due to the ALEA function).

First, we must identify the frequency of the hit sequence, $\hat{\pi}$, that can be calculated straightforwardly from the maximum likelihood (ML) estimate as: $\hat{\pi} = \frac{T_1}{T} =$ 5,56%.

As we have collected a sample size of 756, which gives accurate statistical estimates (at least 30 to converge to the law of large numbers (LLN) even though we use a chi-squared distribution and not the standard normal distribution), for an econometric robustness, we will opt for a confident interval of 99% for all further tests (so we will consider a critical value at 1%).

According to our results, we can argue that: the unconditional test can be performed with a likelihood ratio test, that provides us a LR test statistic, LR_{uc}, equal to 0,47 (because the likelihood function from our ML estimate will be similar to the likelihood function from the alternative hypothesis that $\pi = p$, so the logarithm of the ratio will be close to 0 since $\log(1) = 0$). In addition, with a critical value at 1% from the chi-squared distribution with one degree of freedom ($\chi_1^{2,1\%} = 6{,}63$), the LR_{uc} test statistic is lower than the critical value ($LR_{uc}{}^{VaR} = 0{,}47 < \chi_1^{2,1\%} = 6{,}63$).

Consequently, from a statistical point of view, we cannot reject the hypothesis according to which the fraction of violations is significantly different at 1% from the corresponding coverage rate at 5%, in other words that our VaR model is correct.

Furthermore, we can test for the independence of the hit sequence by carrying out an independence test.

In this regard, we will test the null hypothesis under which the probability of a violation following a violation (π_{11}) is statistically equal to the probability of a violation following a non-violation (π_{01}), which should be equal, based on the unconditional test, to p.

To that end, we implement the maximum likelihood (ML) estimates from our corresponding VaR model using the following formulas:

$$\boxed{\begin{aligned} \hat{\pi}_{01} &= \frac{T_{01}}{T_{00}+T_{01}} & \hat{\pi}_{00} &= 1-\hat{\pi}_{01} \\ \hat{\pi}_{11} &= \frac{T_{11}}{T_{10}+T_{11}} & \hat{\pi}_{10} &= 1-\hat{\pi}_{11} \end{aligned}}$$

To highlight our thoughts, we can focus on T_{01} the number of violations following a non-violation and reiterate this reasoning for the others.

In such a case, we recognize an algorithmic structure by which, if the precedent value is 0 and the following value is 1, then count 1 or 0 otherwise.

As a matter of fact, it is an IF statement that we can translate into Excel using the IF function and the logical condition AND.

```
fx  = SI(ET(F256=1;F255=0);1;0)
```

Nevertheless, before we start to scroll the function to each column, we must fix the first observation to 0, because we need at each time the previous observation.

0	0	0	0	
0	1	0)	0	
0	1	0	0	

One more thing where we should pay attention to, is when we compute the likelihood function for our ML estimates. Indeed, as we have zero observations for T_{11}, we end up with $\widehat{\pi_{11}}^{T_{11}} = 0^0$ which is mathematically impossible.

This way, to remedy this issue, we must add an IF statement that will yield the value 1 if $\widehat{\pi_{11}} = 0$ or $\widehat{\pi_{11}}^{T_{11}}$ else.

```
fx  =(AD24^AG24)*(AD25^AG25)*(AD26^AG26)*SI(AD27= 0;1;AD27^AG27)
```

Subsequently, we glean the following findings for this test:

Independence Test				
$\widehat{\pi_{00}}$	94,39%		T_{00}	673
$\widehat{\pi_{01}}$	5,61%		T_{01}	40
$\widehat{\pi_{10}}$	95,24%		T_{10}	40
$\widehat{\pi_{11}}$	4,76%		T_{11}	2
$L(\hat{\Pi}_1)$	3,90564E-71			
$L(\hat{\Pi})$	2,97626E-71			
LR_{ind}	0,543510411		$\chi_1^{2,1\%} = 6{,}63 > LR_{ind}$	

The same arguments outlined formerly for the unconditional test can be advanced too.

Actually, the LR test statistic is lower than the corresponding critical value from the chi-squared distribution at 10% , 5% and 1% ($LR_{ind}{}^{VaR} = 0{,}54 < \chi_1^{2,1\%} = 6{,}63$; $LR_{ind}{}^{VaR} = 0{,}54 < \chi_1^{2,5\%} = 3{,}84$; $LR_{ind}{}^{VaR} = 0{,}54 < \chi_1^{2,10\%} = 2{,}71$), although as we mentioned earlier, we could focus only on the 1% critical value since we have accurate estimates.

As a result, statistically, we cannot reject at 10%, 5% or 1% the null hypothesis whereby there is no independence for the VaR violations. Symmetrically, we can reject the alternative hypothesis under which there the VaR violations are independent.

At last, we can combine the two previous tests all at once, if we want to test jointly that the proportion is statistically not different from our coverage rate and that the VaR violations are independent from one another.

Such a test is called a conditional test, and we can sum the two LRs computed beforehand and compare them with the critical value from the chi-squared distribution, but this time with two degrees of freedom (χ_2^2).

Last and not least, we raise the next outcomes:

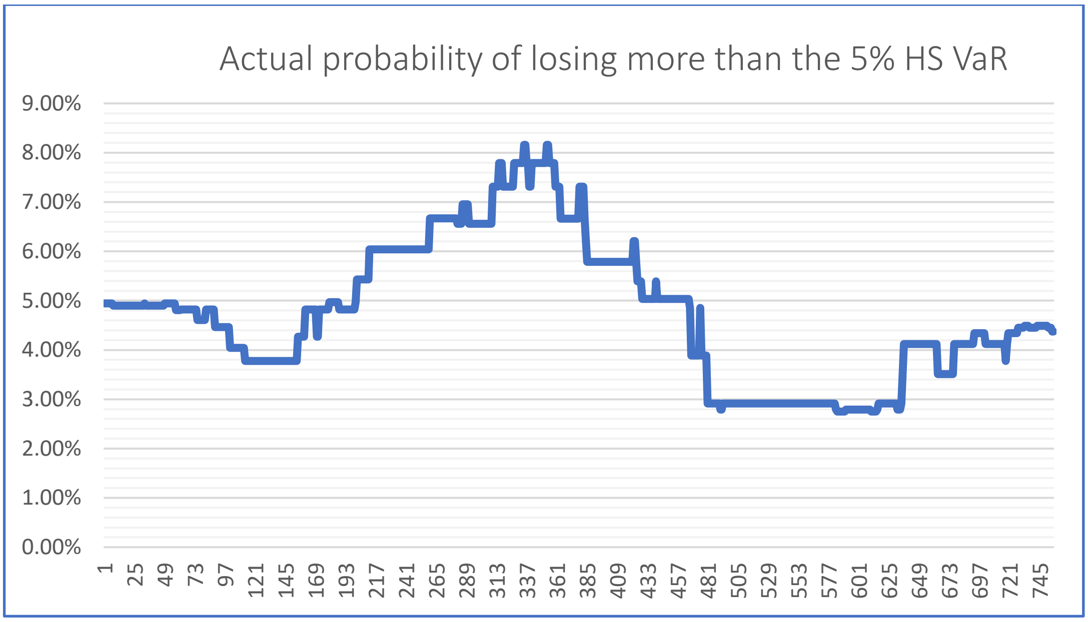
Conditional Coverage Test				
LR_{uc}	0,474892737			
LR_{ind}	0,543510411			
LR_{cc}	1,018403148		$\chi_2^{2,1\%} = 9{,}21 > LR_{cc}$	

In this case, we wind up with a LR test statistic, LR_{cc}, lower than the corresponding critical value from the chi-squared distribution with two degrees of freedom at 10%, 5% and 1% ($LR_{cc}{}^{VaR} = 1{,}02 < \chi_2^{2,1\%} = 9{,}21$; $LR_{cc}{}^{VaR} = 1{,}02 < \chi_2^{2,5\%} = 5{,}99$; $LR_{cc}{}^{VaR} = 1{,}02 < \chi_2^{2,10\%} = 4{,}61$).

Statistically, we cannot reject the null hypothesis at 10%, 5% and 1% under which the average proportion of violation is equal to p and that the violations are independent.

Finally, we can display the VaR breaching from our 756 predictions of the VaR one day ahead.

Not surprisingly, we observe an average breaching of 5%, with the part below 5% that compensate with that above 5%.



Back to the backtesting procedure, we obtained an average proportion of VaRviolations of approximately 5% which corroborate our scrutiny from the plot.

BACKTESTING VAR	
Unconditional Test	
$\hat{\pi}$	5,56%

3/ Backtest as well the Expected Shortfall. Show empirically that VaRt/ESt tends to one when p of the p% VaR decrease.

For the Expected Shortfall (ES) calculation, as it is not stated, we will presume the ES from HS the same way we did for the VaR so as to remain consistent.

Accordingly, we compute the ES one day ahead at 5% using one year of historical data as the average of the losses worse than the VaR computed from the same period.

We can achieve this operation in Excel using the AVERAGE.IF function which enable us to filter only the losses larger than the VaR.

```
fx  = -MOYENNE.SI(D3:D254;"<"&-E255)
```

However, for the backtesting procedure, the chapter 8 does not include the backtesting for the ES, despite the fact that it is quoted in the outline, hence, we just reiterate the same scheme we did for the VaR.

For the unconditional test we harvest the subsequent results:

BACKTESTING ES				
Unconditional Test			T_1	17
$\hat{\pi}$	2,25%		T_0	739
$L(\hat{\pi})$	4,82385E-36		T	756
$L(p)$	2,63169E-39			
LR_{uc}	15,02740648		$\chi_1^{2,1\%} = 6,63 < LR_{uc}$	

Unlike the VaR, the only thing that differs from the previous backtesting is that now we have a LR statistic test which is always larger than the critical value from the chi-squared distribution no matter which confident interval we choose ($LR_{uc}{}^{ES} = 15,03 > \chi_1^{2,10\%} = 2,71$; $LR_{uc}{}^{ES} = 15,03 > \chi_1^{2,5\%} = 5,99$; $LR_{uc}{}^{ES} = 15,03 > \chi_1^{2,1\%} = 6,63$).

As such, we can affirm that from a statistical point of view, we can accept the alternative hypothesis by which the average proportion of the ES violations is different from the coverage rate p of 5% (even though the test does not inform us if p should be lowered or increased, because we have realized a bilateral test).

Thereby, we reject our ES model and based on our empirical results (that the average proportion of violation $\hat{\pi} < p$), we can suggest that the coverage rate should be reduced, which would lead to a higher ES estimate.

Then, here is our results for the independence test:

Independence Test				
$\widehat{\pi_{00}}$	97,83%		T_{00}	722
$\widehat{\pi_{01}}$	2,17%		T_{01}	16
$\widehat{\pi_{10}}$	94,12%		T_{10}	16
$\widehat{\pi_{11}}$	5,88%		T_{11}	1
$L(\hat{\Pi}_1)$	7,11983E-36			
$L(\hat{\Pi})$	4,93482E-36			
LR_{ind}	0,73313299		$\chi_1^{2,1\%} = 6,63 > LR_{ind}$	

Moreover, for the independence test the same arguments for the VaR can be substantiated.

In fact, we end up with a LR test lower than the critical value from the chi-squared distribution ($LR_{ind}{}^{ES} = 0,73 < \chi_1^{2,10\%} = 2,71$; $LR_{ind}{}^{ES} = 0,73 < \chi_1^{2,5\%} = 5,99$; $LR_{ind}{}^{ES} = 0,73 < \chi_1^{2,1\%} = 6,63$).

Thus, we can allege that π_{01} is statistically equal to π_{11} and that the ES violations are clearly independent and unpredictable, which involve that our ES model is correct.

Lastly, by gathering the two previous tests we raise the conditional test, which is always superior to the critical value from the chi-squared distribution, regardless which significant level α we choose, so we reject both the HS model and the independence (whereas we rejected the dependence for the independence test).

Conditional Coverage Test				
LR_{uc}	15,02740648			
LR_{ind}	0,73313299			
LR_{cc}	15,76053947		$\chi_2^{2,1\%} = 9{,}21 < LR_{cc}$	

Now that we have computed both the VaR and the ES from the 5% HS one day ahead, we can express the ratio of these two risk measures as a function of p for a fixed t.

For instance, we can take the first year of historical data and compute for the 253-th day with different coverage rates the VaR and the ES from that same VaR.

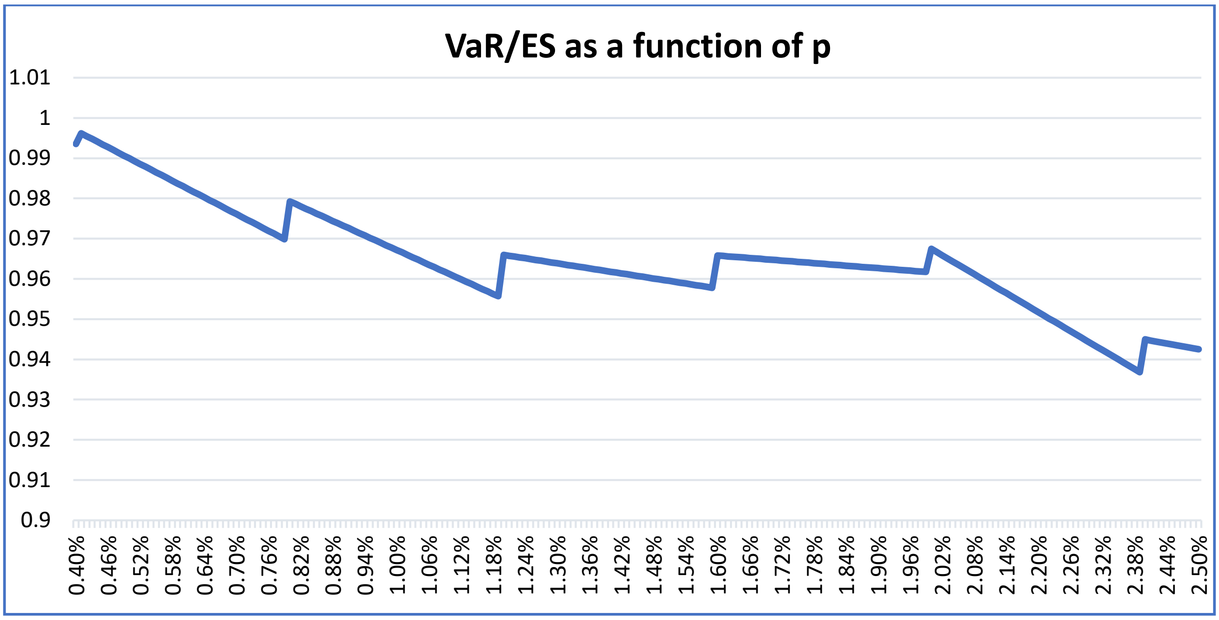
The coverage rates range from $\frac{1}{252} \cong 0{,}4\%$ (the lowest quantile we can compute from our sample) to 2,5% (as beyond that number the difference between the two will be significantly different).

The idea behind this procedure is that when the coverage rate drops, as the ES is the expected value of the losses larger than the VaR, it will detect less losses because the corresponding VaR has already captured those large losses.

As a result, we can expect these two measures to be closer when the p of the VaR decline, even though the ES will always be equal or larger than the VaR, so the ratio will always be less or equal to 1.

We can plot this ratio as function of p and expect the ratio to tend towards 1 when p shrinks.

Though, we may get unwarranted results because of the ALEA function, so we may press enter while waiting to see an interesting chart.



4/ The asset manager has a daily VaR of 100000EUR, what is the maximum amount, in EUR, he can invest every day to respect the VaR ? Plot this quantity for the three years of VaR.

As the VaR estimated is close to 1%, the $100{,}000€$ VaR limit is therefore the maximum loss that the asset manager cannot exceed, so he can invest in average no more that $10\ billion$ € which fits what we could expect.

To respect a Value at risk of $100{,}000€$ the asset manager can follow this plot for the next three years.

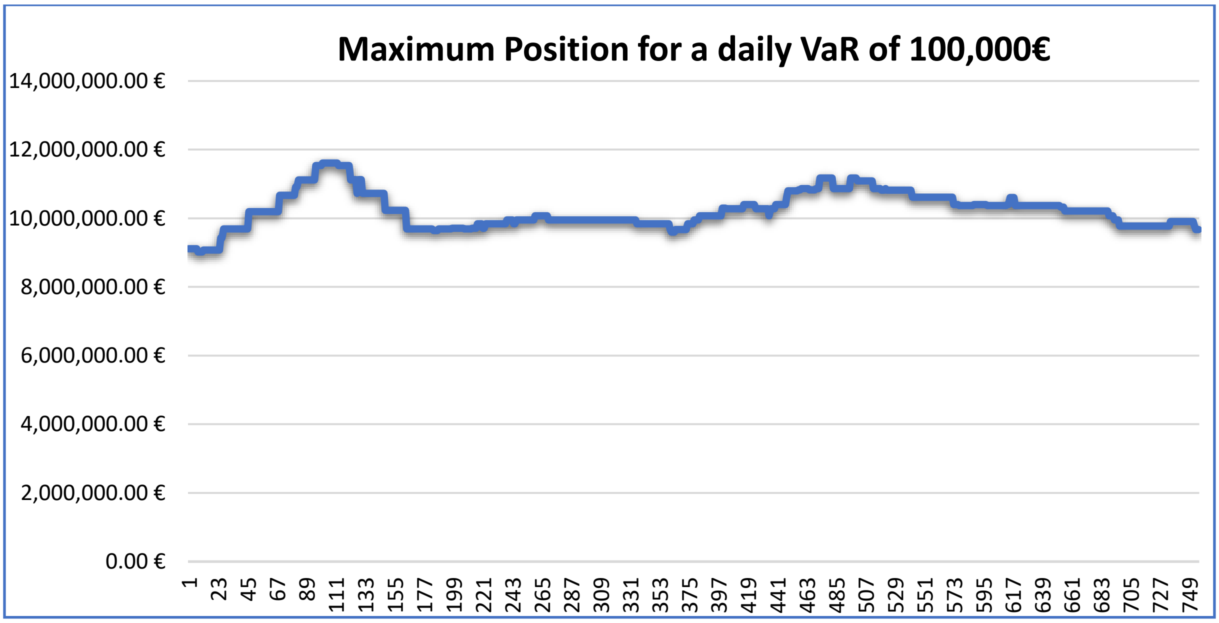
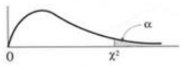


Table χ^2 : points de pourcentage supérieurs de la distribution χ^2



dl	.995	.990	.975	.950	.900	.750	.500	.250	.100	.050	.025	.010	.005
1	0.00	0.00	0.00	0.00	0.02	0.10	0.45	1.32	2.71	3.84	5.02	6.63	7.88
2	0.01	0.02	0.05	0.10	0.21	0.58	1.39	2.77	4.61	5.99	7.38	9.21	10.60
3	0.07	0.11	0.22	0.35	0.58	1.21	2.37	4.11	6.25	7.82	9.35	11.35	12.84
4	0.21	0.30	0.48	0.71	1.06	1.92	3.36	5.39	7.78	9.49	11.14	13.28	14.86
5	0.41	0.55	0.83	1.15	1.61	2.67	4.35	6.63	9.24	11.07	12.83	15.09	16.75
6	0.68	0.87	1.24	1.64	2.20	3.45	5.35	7.84	10.64	12.59	14.45	16.81	18.55
7	0.99	1.24	1.69	2.17	2.83	4.25	6.35	9.04	12.02	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	3.49	5.07	7.34	10.22	13.36	15.51	17.54	20.09	21.96
9	1.73	2.09	2.70	3.33	4.17	5.90	8.34	11.39	14.68	16.92	19.02	21.66	23.59
10	2.15	2.56	3.25	3.94	4.87	6.74	9.34	12.55	15.99	18.31	20.48	23.21	25.19
11	2.60	3.05	3.82	4.57	5.58	7.58	10.34	13.70	17.28	19.68	21.92	24.72	26.75
12	3.07	3.57	4.40	5.23	6.30	8.44	11.34	14.85	18.55	21.03	23.34	26.21	28.30
13	3.56	4.11	5.01	5.89	7.04	9.30	12.34	15.98	19.81	22.36	24.74	27.69	29.82
14	4.07	4.66	5.63	6.57	7.79	10.17	13.34	17.12	21.06	23.69	26.12	29.14	31.31
15	4.60	5.23	6.26	7.26	8.55	11.04	14.34	18.25	22.31	25.00	27.49	30.58	32.80
16	5.14	5.81	6.91	7.96	9.31	11.91	15.34	19.37	23.54	26.30	28.85	32.00	34.27
17	5.70	6.41	7.56	8.67	10.09	12.79	16.34	20.49	24.77	27.59	30.19	33.41	35.72
18	6.26	7.01	8.23	9.39	10.86	13.68	17.34	21.60	25.99	28.87	31.53	34.81	37.15
19	6.84	7.63	8.91	10.12	11.65	14.56	18.34	22.72	27.20	30.14	32.85	36.19	38.58
20	7.43	8.26	9.59	10.85	12.44	15.45	19.34	23.83	28.41	31.41	34.17	37.56	40.00
21	8.03	8.90	10.28	11.59	13.24	16.34	20.34	24.93	29.62	32.67	35.48	38.93	41.40
22	8.64	9.54	10.98	12.34	14.04	17.24	21.34	26.04	30.81	33.93	36.78	40.29	42.80
23	9.26	10.19	11.69	13.09	14.85	18.14	22.34	27.14	32.01	35.17	38.08	41.64	44.18
24	9.88	10.86	12.40	13.85	15.66	19.04	23.34	28.24	33.20	36.42	39.37	42.98	45.56
25	10.52	11.52	13.12	14.61	16.47	19.94	24.34	29.34	34.38	37.65	40.65	44.32	46.93
26	11.16	12.20	13.84	15.38	17.29	20.84	25.34	30.43	35.56	38.89	41.92	45.64	48.29
27	11.80	12.88	14.57	16.15	18.11	21.75	26.34	31.53	36.74	40.11	43.20	46.96	49.64
28	12.46	13.56	15.31	16.93	18.94	22.66	27.34	32.62	37.92	41.34	44.46	48.28	50.99
29	13.12	14.26	16.05	17.71	19.77	23.57	28.34	33.71	39.09	42.56	45.72	49.59	52.34
30	13.78	14.95	16.79	18.49	20.60	24.48	29.34	34.80	40.26	43.77	46.98	50.89	53.67
40	20.67	22.14	24.42	26.51	29.06	33.67	39.34	45.61	51.80	55.75	59.34	63.71	66.80
50	27.96	29.68	32.35	34.76	37.69	42.95	49.34	56.33	63.16	67.50	71.42	76.17	79.52
60	35.50	37.46	40.47	43.19	46.46	52.30	59.34	66.98	74.39	79.08	83.30	88.40	91.98
70	43.25	45.42	48.75	51.74	55.33	61.70	69.34	77.57	85.52	90.53	95.03	100.44	104.24
80	51.14	53.52	57.15	60.39	64.28	71.15	79.34	88.13	96.57	101.88	106.63	112.34	116.35
90	59.17	61.74	65.64	69.13	73.29	80.63	89.33	98.65	107.56	113.14	118.14	124.13	128.32
100	[illegible]	[illegible]	74.22	77.93	82.36	[illegible]	99.33	109.14	[illegible]	124.34	129.56	[illegible]	[illegible]