The Atiyah Class on Calabi–Yau Manifolds

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Contents

Chapter 1

Connections On Vector Bundles

1.1 Local Operators

In the following we consider vector bundles $E \to M$ and $F \to M$ over a smooth manifold M.

Definition 1.1.1. For vector bundles E and F over a smooth manifold M we call an $\mathbb{R}$-linear map $\Phi : \Gamma(E) \to \Gamma(F)$ a **local operator** if whenever $s \in \Gamma(E)$ vanishes on an open set $U \subset M$, then $\Phi(s) \in \Gamma(F)$ vanishes on U also.

There is also a notion of a **point operator** which replaces U with a single point $p \in M$. Before we look at the first example note that for a trivial bundle $M \times \mathbb{R} \to M$ a section s of $M \times \mathbb{R}$ is a map $s(p) = (p, f(p))$ for some function $f : M \to \mathbb{R}$. We thus have a one-to-one correspondence

$$\{\text{sections of } M \times \mathbb{R}\} \longleftrightarrow \{f \mid f : M \to \mathbb{R}\}.$$

In particular the space of smooth sections $\Gamma(M \times \mathbb{R})$ can be identified with $C^\infty(M)$.

Example 1.1.2. *The most basic example of a local operator is* $\frac{d}{dx} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$*. Here we are identifying* $C^\infty(\mathbb{R})$ *with* $\Gamma(\mathbb{R} \times \mathbb{R})$ *as noted above. If* $f(x) \equiv 0$ *on a neighborhood* $U \ni p$*, then* $f'(x) \equiv 0$ *on* U*.*

Another great example of such an operator is the exterior derivative

$$d : \Gamma\left(\Lambda^k T^* M\right) \to \Gamma\left(\Lambda^{k+1} T^* M\right).$$

Proposition 1.1.3. *If a map* $\Phi : \Gamma(E) \to \Gamma(F)$ *is* $C^\infty(M)$*-linear, then it is a local operator.*

Proof. Consider $s \in \Gamma(E)$ such that s vanishes on an open set $U \subset M$. Let $p \in U$ and consider a smooth bump function f such that $f(p) = 1$ and the support of f is contained in U. Now fs is a section of E and $fs \equiv 0$ on M. Thus $\Phi(fs) = 0$ and since Φ is $C^\infty(M)$-linear

$$\Phi(fs) = f\Phi(s) = 0.$$

At p we have $\Phi(s)(p) = 0$ so $\Phi(s) \equiv 0$ on U. □

Theorem 1.1.4. *If a map $\Phi : \Gamma(E) \to \Gamma(F)$ is a local operator, then for each open subset $U \subset M$ there exists a unique linear map*

$$\Phi|_U : \Gamma(U, E) \to \Gamma(U, F)$$

such that for any $s \in \Gamma(E)$ we have that

$$\Phi|_U(s|_U) = \Phi(s)|_U.$$

Proof. Let $s \in \Gamma(U, E)$ be a section and $p \in U$. Utilizing bump functions, there exists a global section $\overline{s}$ such that $\overline{s}$ agrees with s in some neighborhood W of p in U. Define

$$\Phi|_U(s)(p) := \Phi(\overline{s})(p).$$

If $\sigma \in \Gamma(E)$ is another global section with $\sigma = s$ in W, then $\sigma = \overline{s}$ in W. Now as Φ is a local operator we conclude that $\Phi(\sigma) = \Phi(\overline{s})$ in W i.e. $\Phi(\sigma)(p) = \Phi(\overline{s})(p)$ which shows that $\Phi|_U(s)(p)$ is independent of the choice of $\overline{s}$ and so $\Phi|_U$ is well-defined and unique. To show that $\Phi|_U$ is smooth we note that if $s \in \Gamma(U, E)$ and $\overline{s} \in \Gamma(M, E)$ agree on a neighborhood $W \ni p$, then $\Phi|_U(s) = \Phi(\overline{s})$ on W and the left-hand side is smooth. Lastly if $s \in \Gamma(M, E)$ is a global section, then it is a global section of its restriction $s|_U$ and so $\Phi|_U(s|_U)(p) = \Phi(s)(p)$ for all $p \in U$. Thus $\Phi|_U(s) = \Phi(\overline{s})$. □

Lemma 1.1.5. *A fiber-preserving map $\varphi : E \to F$ that is linear on each fiber is smooth if and only if the induced map $\Phi : \Gamma(E) \to \Gamma(F)$ takes smooth sections of E to smooth sections of F.*

Proof. Suppose that $\varphi : E \to F$ is smooth. Then for a smooth section $s \in \Gamma(E)$ the composition $\Phi(s) = \varphi \circ s$ is smooth as it's a composition of smooth maps. Conversely suppose that Φ takes smooth sections of E to smooth sections of F. As usual we proceed locally. Fix $p \in M$ and consider a chart $(U, x^1, \ldots, x^n)$ at p over which E and F are both trivial. Let $e_1, \ldots, e_r \in \Gamma(E)$ be a frame for E over U and $f_1, \ldots, f_m \in \Gamma(F)$ be a frame for F over U. Now a point in $E|_U$ can be written as a linear combination $\sum a^j e_j$. Suppose

$$\varphi \circ e_j = \sum_i b^i_j f_i.$$

Then

$$\varphi\left(\sum_j a^j e_j\right) = \sum_{i,j} a^j b_j^i f_i.$$

Taking local coordinates on $E|_U$ to be $(x^1, \ldots, x^n, a^1, \ldots, a^r)$ we have

$$\varphi(x^1, \ldots, x^n, a^1, \ldots, a^r) = \left(x^1, \ldots, x^n, \sum_j a^j b_j^1, \ldots, \sum_j a^j b_j^m\right)$$

which is smooth. □

Lemma 1.1.6. *A $C^\infty(M)$-linear map $\Phi : \Gamma(E) \to \Gamma(F)$ is a point operator.*

Proof. Suppose that $s \in \Gamma(E)$ such that s vanishes at $p \in M$. Let U be an open neighborhood of p over which E is trivial. Consider a frame $e_1, \ldots, e_r$ of E over U. We can write

$$s|_U = \sum_i a^i e_i$$

where the a^i's are smooth functions on U. Since

$$0 = s|_U(p) = \sum_i a^i(p) e_i(p)$$

we conclude that $a^i(p) = 0$ for every i. By Theorem 1.1.4 we know that Φ restricts to a unique map $\Phi|_U : \Gamma(U, E) \to \Gamma(U, F)$ and so

$$\begin{aligned}
\Phi(s)(p) &= \Phi|_U(s|_U)(p) \\
&= \Phi|_U(\sum_i a^i e_i)(p) \\
&= \left(\sum_i a^i \Phi|_U(e_i)\right)(p) \\
&= \sum_i a^i(p) \Phi|_U(e_i)(p) \\
&= 0.
\end{aligned}$$

□

Lemma 1.1.7. *If $\Phi : \Gamma(E) \to \Gamma(F)$ is $C^\infty(M)$-linear, then for each $p \in M$, there is a unique linear map $\Phi_p : E_p \to F_p$ such that for all $s \in \Gamma(E)$,*

$$\Phi_p(s(p)) = \Phi(s)(p).$$

Proof. Given $v \in E_p$ choose any $s \in \Gamma(E)$ such that $s(p) = v$ and set

$$\Phi_p(v) = \Phi(s)(p) \in F_p.$$

This definition is independent of the choice of s since as we saw previously Φ is a point operator. To show that Φ_p is linear suppose that $v_1, v_2 \in E_p$ and $a_1, a_2 \in \mathbb{R}$. Let s_1 and s_2 be global sections of E with $s_i(p) = v_i$. Now

$$\begin{aligned}\Phi_p(a_1v_1 + a_2v_2) &= \Phi(a_1s_1 + a_2s_2)(p) \\ &= a_1\Phi(s_1)(p) + a_2\Phi(s_2)(p) \\ &= a_1\Phi_p(v_1) + a_2\Phi_p(v_2).\end{aligned}$$

□

Theorem 1.1.8. *There is a one-to-one correspondence between bundle maps $\varphi : E \to F$ and $C^\infty(M)$-linear maps $\Phi : \Gamma(E) \to \Gamma(F)$ given by $\varphi \mapsto \Phi$.*

Proof. Suppose that $\alpha : \Gamma(E) \to \Gamma(F)$ is $C^\infty(M)$-linear. For each $p \in M$ there is a linear map $\Phi_p : E_p \to F_p$ such that for any $s \in \Gamma(E)$,

$$\Phi_p(s(p)) = \alpha(s)(p).$$

Define $\Phi : E \to F$ by setting $\Phi(v) = \Phi_p(v)$ if $v \in E_p$. Now for any $s \in \Gamma(E)$ and $p \in M$

$$\widetilde{\Phi}(s)(p) = \Phi(s(p)) = \alpha(s)(p),$$

and so $\alpha = \widetilde{\Phi}$ which proves surjectivity. Consider then bundle maps $\Phi, \Psi : E \to F$ with $\widetilde{\Phi} = \widetilde{\Psi}$. For any $v \in E_p$ pick $s \in \Gamma(E)$ with $s(p) = e$. Then

$$\Phi(v) = \Phi(s(p)) = \widetilde{\Phi}(s)(p) = \widetilde{\Psi}(s)(p) = \Psi(s(p)) = \Psi(v),$$

i.e. $\Phi = \Psi$. □

1.2 Connections

The study of connections on bundles is usually called **gauge theory**. From a mathematical standpoint, connections provide a way to differentiate sections of a vector bundle. Specifically, a connection allows one to compare values of sections at different points on the base manifold, giving rise to a covariant derivative. This derivative is crucial in defining how a vector bundle behaves under parallel transport, as well as in computing curvature, an intrinsic measure of the bundle's geometry.

Definition 1.2.1. Let $E \to M$ be a complex vector bundle. A **connection** on E is an $\mathbb{C}$-linear map

$$\nabla \colon \Gamma(E) \to \Gamma(T^*M \otimes E)$$

satisfying the Leibniz rule:

$$\nabla(fs) = df \otimes s + f\nabla s$$

for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$.

Proposition 1.2.2. *The set of all connections on a vector bundle $E \to M$ is an affine space modeled on the vector space of* $\mathrm{End}(E)$*-valued* 1*-forms.*

Proof. Let ∇_1, ∇_2 be two connections on E. Then for any $f \in C^\infty(M)$ and $s \in \Gamma(E)$,

$$\begin{aligned}(\nabla_1 - \nabla_2)(fs) &= \nabla_1(fs) - \nabla_2(fs)\\ &= (df \otimes s + f\nabla_1 s) - (df \otimes s + f\nabla_2 s)\\ &= f(\nabla_1 s - \nabla_2 s),\end{aligned}$$

that is, the difference $\nabla_1 - \nabla_2$ is $C^\infty(M)$-linear. Recall now that Lemma 1.1.7 states that such a map gives a unique map on fibers $E_p \to T_p^*M \otimes E_p$, for each $p \in M$. That is, we obtain a map $T_pM \to \mathrm{End}(E_p)$ which implies that $\nabla_1 - \nabla_2$ is an $\mathrm{End}(E)$-valued 1-form. Furthermore, if $A \in \mathcal{A}^1(M, \mathrm{End}(E))$, then

$$\begin{aligned}(\nabla + A)(fs) &= \nabla(fs) + A(fs)\\ &= df \otimes s + f\nabla s + fA(s)\\ &= df \otimes s + f(\nabla + A)(s).\end{aligned}$$

Therefore $\nabla + A$ satisfies the Leibniz rule and is a connection. □

Let $U \subset M$ be an open set over which E is trivial, and let $\{e_1, \ldots, e_r\}$ be a local frame of E over U. Any section $s \in \Gamma(U, E)$ can be written as $s = \sum_{j=1}^r s^j e_j$ for $s^j \in C^\infty(U)$. For $X \in \Gamma(U, TM)$, the connection is given by

$$\nabla_X s = \nabla_X \left(\sum_j s^j e_j \right) = \sum_j ds^j(X) e_j + \sum_j s^j \nabla_X e_j.$$

For each i, j, there exist unique 1-forms $\omega_j^i \in \Omega^1(U)$ such that

$$\nabla e_j = \sum_{i=1}^r \omega_j^i \otimes e_i.$$

The $r \times r$ matrix $\omega = [\omega_j^i]$ is called the *connection matrix* relative to the frame $\{e_i\}$. In this notation,

$$\begin{aligned}\nabla s &= ds^j \otimes e_j + s^j \nabla e_j\\ &= ds^j \otimes e_j + s^j \omega_j^i \otimes e_i\\ &= ds^j \otimes e_j + \omega_j^i s^j \otimes e_i\\ &= ds + \omega s,\end{aligned}$$

where $ds = (ds^1, \ldots, ds^r)^T$ and $s = (s^1, \ldots, s^r)^T$. Suppose now that $\{e_i\}$ and $\{\widetilde{e}_i\}$ are two local frames over U, related by $\widetilde{e}_j = \sum_i a^i_j e_i$, where $a = [a^i_j]$ is a smooth map $U \to \mathrm{GL}(r, \mathbb{R})$. If ω and $\widetilde{\omega}$ are the connection matrices relative to these frames, then

$$\widetilde{\omega} = a^{-1}\omega a + a^{-1}da.$$

Under a change of frame, the connection matrix transforms inhomogeneously. Given vector bundles $E_1, E_2 \to M$ with connections ∇^1 and ∇^2, there are natural ways to define connections on associated bundles:

(i) **Direct Sum:** The direct sum $E_1 \oplus E_2$ carries the connection

$$(\nabla_1 \oplus \nabla_2)(s_1 \oplus s_2) := \nabla_1 s_1 \oplus \nabla_2 s_2$$

for sections $s_1 \in \Gamma(E_1)$, $s_2 \in \Gamma(E_2)$.

(ii) **Tensor Product:** The tensor product $E_1 \otimes E_2$ has the connection

$$\nabla(s_1 \otimes s_2) := \nabla_1 s_1 \otimes s_2 + s_1 \otimes \nabla_2 s_2,$$

for $s_1 \in \Gamma(E_1)$, $s_2 \in \Gamma(E_2)$, extended by linearity.

(iii) **Dual Bundle:** If ∇ is a connection on E, the dual connection ∇^* on E^* is defined by

$$d\langle \lambda, s\rangle = \langle \nabla_* \lambda, s\rangle + \langle \lambda, \nabla s\rangle$$

for $\lambda \in \Gamma(E^*)$, $s \in \Gamma(E)$. Explicitly,

$$(\nabla^* \lambda)(s) := d(\lambda(s)) - \lambda(\nabla s).$$

(iv) **Hom Bundle:** Given ∇_1 on E_1 and ∇_2 on E_2, the induced connection ∇_{Hom} on $\mathrm{Hom}(E_1, E_2)$ is defined by

$$(\nabla_{\mathrm{Hom}}\varphi)(s) := \nabla_2(\varphi(s)) - \varphi(\nabla_1 s)$$

for $\varphi \in \Gamma(\mathrm{Hom}(E_1, E_2))$ and $s \in \Gamma(E_1)$.

Up to this point M has been a real manifold and E a smooth complex vector bundle. For now, we will consider a holomorphic vector bundle E over a complex manifold X.

Proposition 1.2.3. *Let E be a holomorphic vector bundle over a complex manifold X. Then there exists a natural $\mathbb{C}$-linear operator*

$$\bar{\partial}_E : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q+1}(E)$$

such that $\bar{\partial}_E^2 = 0$ and, for all $f \in \mathcal{A}^0(X)$ and $\alpha \in \mathcal{A}^{p,q}(E)$, the Leibniz rule holds:

$$\bar{\partial}_E(f\alpha) = \bar{\partial} f \wedge \alpha + f\,\bar{\partial}_E \alpha.$$

Proof. Let U be a framed open set with a frame $\{e_1, \dots, e_k\}$. A section $\alpha \in \mathcal{A}^{p,q}(U, E)$ can be written as $\alpha = \sum_i \alpha^i \otimes e_i$ for $\alpha^i \in \mathcal{A}^{p,q}(U)$. Set

$$\bar{\partial}_E \alpha := \sum_i \bar{\partial}\alpha^i \otimes e_i.$$

To see that this expression is independent of the choice of local frame, let $\{\widetilde{e}_1, \dots, \widetilde{e}_k\}$ be another frame over U, by the above construction, we obtain an operator $\bar{\partial}'_E$. Now $e_i = \sum_j g_i^j \widetilde{e}_j$ for a transition matrix of holomorphic functions $[g_i^j]$, and

$$\begin{aligned}
\bar{\partial}'_E \alpha &= \bar{\partial}'_E \left(\sum_{i,j} \alpha^i \otimes g_i^j \widetilde{e}_j \right) \\
&= \sum_{i,j} \bar{\partial}(\alpha^i g_i^j) \otimes \widetilde{e}_j \\
&= \sum_{i,j} \bar{\partial}\alpha^i \otimes g_i^j \widetilde{e}_j \\
&= \sum_i \bar{\partial}\alpha^i \otimes e_i \\
&= \bar{\partial}_E \alpha.
\end{aligned}$$

Hence we can extend this to all of $\mathcal{A}^{p,q}(E)$. By construction, this satisfies $\bar{\partial}_E^2 = 0$ and the Leibniz rule. □

If ∇ is now any connection on E, then since $\mathcal{A}^1(E) = \mathcal{A}^{1,0}(E) \oplus \mathcal{A}^{0,1}(E)$, we can decompose ∇ on E into its two components $\nabla = \nabla^{1,0} \oplus \nabla^{0,1}$. Note that

$$\nabla^{0,1}(fs) = \bar{\partial} f \otimes s + f \nabla^{0,1} s,$$

where $\bar{\partial} : \mathcal{A}^0(E) \to \mathcal{A}^{0,1}(E)$ is the operator we discussed above. Just as we have compatible connections with metrics in the case of Riemannian manifolds, there is an analog in the complex setting.

Definition 1.2.4. Let (E, h) be a Hermitian vector bundle. A connection ∇ on E is said to be *compatible with the metric* if, for all sections s, t of E, we have

$$d(h(s,t)) = h(\nabla s, t) + h(s, \nabla t).$$

Theorem 1.2.5. *Let E be a holomorphic vector bundle with a Hermitian metric h. Then there exists a unique connection that is compatible with the metric, and compatible with the holomorphic structure, in the sense that*

$$\nabla^{0,1} = \bar{\partial}.$$

Proof. We will first prove uniqueness. This is a local problem so consider a framed open set U with a holomorphic frame $e = (e_1, \ldots, e_k)$. Since $E|_U \cong U \times \mathbb{C}^k$, the connection is then determined completely by the connection 1-forms ω_i^j given by

$$\nabla e_j = \sum \omega_j^i \otimes e_i.$$

Since ∇ is compatible with h, we obtain

$$\begin{aligned} dh(e_i, e_j) &= h\left(\nabla e_i, e_j\right) + h\left(e_i, \nabla e_j\right) \\ &= h\left(\omega_i^k \otimes e_k, e_j\right) + h\left(e_i, \omega_j^k \otimes e_k\right) \\ &= \omega_i^k h(e_k, e_j) + h(e_i, e_k)\bar{\omega}_j^k. \end{aligned}$$

Now, note that ∇ being compatible with the holomorphic structure means that $\nabla s \in \mathcal{A}^{1,0}(U, E)$ for any holomorphic local section s. It follows that ω_i^k is of type $(1,0)$ so

$$\partial h(e_i, e_j) = \omega_i^k h(e_k, e_j) \quad \text{and} \quad \bar{\partial} h(e_i, e_j) = h(e_i, e_k)\bar{\omega}_j^k,$$

or as matrices $\partial h = \omega^T h$ and $\bar{\partial} h = h\bar{\omega}$. This yields that $\omega = \bar{h}^{-1}\partial\bar{h}$ is a unique solution to both equations. So ω is completely determined locally by the Hermitian metric, and thus so is ∇. The argument also constructs such a connection on each framed open set. Then by uniqueness, these local connections glue to a connection on all of E and so we are done. □

The unique connection described in Theorem 1.2.5 is called the **Chern connection**. To conclude this section, we will now define holomorphic connections which will play a key role in these notes.

Definition 1.2.6. Let E be a holomorphic vector bundle over a complex manifold X and denote by $\mathcal{E}$ the locally free sheaf associated to E. A holomorphic connection on E is a $\mathbb{C}$-linear map of sheaves

$$\nabla : \mathcal{E} \to \Omega_X^1 \otimes \mathcal{E},$$

such that $\nabla(fs) = \partial f \otimes s + f\nabla s$ for any local holomorphic function f and any local holomorphic section s.

1.3 Curvature

In the previous section, we introduced the notion of a connection as a generalization of the exterior differential to sections of vector bundles. While the exterior derivative d satisfies $d^2 = 0$, in general a connection ∇ does not satisfy $\nabla^2 = 0$. The curvature of a connection quantifies this failure and plays a key role in both differential and complex geometry.

Let $E \to M$ be a smooth vector bundle over a manifold M, and let $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ be a connection. There is a canonical way to extend ∇ to E-valued k-forms,

$$\nabla : \mathcal{A}^k(E) \to \mathcal{A}^{k+1}(E),$$

defined as follows: for a local k-form $\alpha \in \mathcal{A}^k(M)$ and a local section $s \in \Gamma(E)$,

$$\nabla(\alpha \otimes s) := d\alpha \otimes s + (-1)^k \alpha \wedge \nabla(s). \tag{1.1}$$

This formula ensures that for $k = 0$ we recover the usual connection, since $\nabla(s) = df \otimes s + f\nabla(s)$ for any smooth function f.

More generally, for $t \in \mathcal{A}^\ell(E)$ and $\beta \in \mathcal{A}^k(M)$, the extension satisfies the graded Leibniz rule:

$$\nabla(\beta \wedge t) = d\beta \wedge t + (-1)^k \beta \wedge \nabla(t). \tag{1.2}$$

Indeed, suppose $t = \alpha \otimes s$ with $\alpha \in \mathcal{A}^\ell(M)$ and $s \in \Gamma(E)$. Then,

$$\begin{aligned}
\nabla(\beta \wedge t) &= \nabla((\beta \wedge \alpha) \otimes s) \\
&= d(\beta \wedge \alpha) \otimes s + (-1)^{k+\ell}(\beta \wedge \alpha) \wedge \nabla(s) \\
&= (d\beta \wedge \alpha + (-1)^k \beta \wedge d\alpha) \otimes s + (-1)^k \beta \wedge \Big(d\alpha \otimes s + (-1)^\ell \alpha \wedge \nabla(s)\Big) \\
&= d\beta \wedge \alpha \otimes s + (-1)^k \beta \wedge d\alpha \otimes s + (-1)^k \beta \wedge d\alpha \otimes s + (-1)^{k+\ell} \beta \wedge \alpha \wedge \nabla(s) \\
&= d\beta \wedge \alpha \otimes s + (-1)^k \beta \wedge \nabla(\alpha \otimes s) \\
&= d\beta \wedge t + (-1)^k \beta \wedge \nabla(t).
\end{aligned}$$

Definition 1.3.1. The **curvature** of a connection ∇ on E is the composition

$$F_\nabla := \nabla \circ \nabla : \mathcal{A}^0(E) \to \mathcal{A}^2(E).$$

Explicitly, F_∇ is a map that takes a section $s \in \Gamma(E)$ and returns the E-valued 2-form $F_\nabla(s) = \nabla(\nabla(s))$. In fact, the curvature can be interpreted as a global section of $\mathcal{A}^2(\mathrm{End}(E))$:

$$F_\nabla \in \mathcal{A}^2(M, \mathrm{End}(E)).$$

This is justified by the following lemma:

Lemma 1.3.2. *The curvature $F_\nabla : \mathcal{A}^0(E) \to \mathcal{A}^2(E)$ is $\mathcal{A}^0(M)$-linear, i.e., for any $f \in C^\infty(M)$ and $s \in \Gamma(E)$,*

$$F_\nabla(fs) = f \cdot F_\nabla(s).$$

Proof.

$$\begin{aligned}
F_\nabla(fs) &= \nabla(\nabla(fs)) \\
&= \nabla(df \otimes s + f\nabla(s)) \\
&= d^2 f \otimes s - df \wedge \nabla(s) + df \wedge \nabla(s) + f\nabla(\nabla(s)) \\
&= f \cdot F_\nabla(s).
\end{aligned}$$

$\square$

Locally, if E is trivial over $U \subset M$ and $\{e_1, \ldots, e_r\}$ is a local frame, any connection ∇ can be written as

$$\nabla = d + A,$$

where $A = [A^i_j]$ is an $r \times r$ matrix of 1-forms. For a local section $s = \sum_j s^j e_j$ with $\nabla(s) = ds^j \otimes e_j + s^j A^i_j \otimes e_i$, we obtain:

$$\begin{aligned} F_\nabla(s) &= \nabla(\nabla(s)) = (d + A)(ds + As) \\ &= d^2 s + d(As) + A \wedge ds + A \wedge (As) \\ &= d(A)s + A \wedge As, \end{aligned}$$

so, as an $\mathrm{End}(E)$-valued 2-form,

$$F_\nabla = dA + A \wedge A. \tag{1.3}$$

If $A \in \mathcal{A}^1(M, \mathrm{End}(E))$ and $\widetilde{\nabla} = \nabla + A$ is another connection, then the curvatures are related by:

Lemma 1.3.3. *For any* $A \in \mathcal{A}^1(M, \mathrm{End}(E))$*,*

$$F_{\nabla + A} = F_\nabla + \nabla(A) + A \wedge A.$$

Proof. A calculation using the definition of the extended connection and the graded Leibniz rule shows:

$$\begin{aligned} F_{\nabla + A}(s) &= (\nabla + A) \circ (\nabla + A)(s) \\ &= \nabla(\nabla(s)) + \nabla(A(s)) + A(\nabla(s)) + A(A(s)) \\ &= F_\nabla(s) + \nabla(A(s)) + A(\nabla(s)) + A(A(s)) \\ &= F_\nabla(s) + \nabla(A)(s) + A \wedge A(s). \end{aligned}$$

□

Lemma 1.3.4 (Bianchi Identity)**.** *Let* $F_\nabla \in \mathcal{A}^2(M, \mathrm{End}(E))$ *be the curvature of a connection* ∇*. Then*

$$\nabla(F_\nabla) = 0 \in \mathcal{A}^3(M, \mathrm{End}(E)).$$

Proof. The action of the extended connection ∇ on F_∇ is

$$\nabla(F_\nabla)(s) = \nabla(F_\nabla(s)) - F_\nabla(\nabla(s)).$$

But, since $F_\nabla = \nabla^2$, this is

$$\nabla(\nabla^2(s)) - \nabla^2(\nabla(s)) = 0,$$

by associativity of composition and the definition of F_∇. □

As we did with connections, we will now describe how to define curvature on associated bundles: Let E_1, E_2 be vector bundles with connections ∇_1, ∇_2, respectively.

(i) The curvature of $E_1 \oplus E_2$ is

$$F_\nabla = F_{\nabla_1} \oplus F_{\nabla_2}.$$

(ii) The curvature of $E_1 \otimes E_2$ is

$$F_\nabla = F_{\nabla_1} \otimes 1 + 1 \otimes F_{\nabla_2}.$$

(iii) The curvature of the dual bundle E^* satisfies

$$F_{\nabla^*} = -(F_\nabla)^t.$$

(iv) For the pullback bundle f^*E under a smooth map $f : M' \to M$, the pullback connection $f^*\nabla$ satisfies

$$F_{f^*\nabla} = f^* F_\nabla.$$

When additional structure is present, such as a Hermitian metric or a holomorphic structure, the curvature obtains further properties:

Proposition 1.3.5.

(i) Let (E, h) be a Hermitian vector bundle and let ∇ be a connection compatible with h. Then the curvature F_∇ is skew-Hermitian in the sense that for all sections s_i, s_j,

$$h(F_\nabla s_i, s_j) + h(s_i, F_\nabla s_j) = 0.$$

In particular, F_∇ is a section of $\mathcal{A}^2(M, \mathrm{End}(E, h))$ consisting of skew-Hermitian endomorphisms.

(ii) Suppose ∇ is compatible with the holomorphic structure, i.e., $\nabla^{0,1} = \bar{\partial}$. Then the curvature F_∇ has no $(0, 2)$-component, that is,

$$F_\nabla \in \mathcal{A}^{2,0}(X, \mathrm{End}(E)) \oplus \mathcal{A}^{1,1}(X, \mathrm{End}(E)).$$

(iii) If ∇ is the Chern connection (i.e., compatible with both the Hermitian metric and the holomorphic structure), then F_∇ is a real, skew-Hermitian $(1, 1)$-form:

$$F_\nabla \in \mathcal{A}^{1,1}(X, \mathrm{End}(E, h)), \qquad \bar{F}_\nabla^T = -F_\nabla.$$

Proof. (i) We work locally in a holomorphic trivialization with a local orthonormal frame $\{e_1, \ldots, e_k\}$ for (E, h). For such a frame, the metric $h(e_i, e_j) = \delta_{ij}$ is constant. The connection ∇ can be written as

$$\nabla e_j = \sum_{i=1}^{k} \omega_j^i \otimes e_i,$$

where $\omega = [\omega_j^i]$ is a matrix of 1-forms. The compatibility of ∇ with h gives

$$dh(e_i, e_j) = h(\nabla e_i, e_j) + h(e_i, \nabla e_j).$$

Since $dh(e_i, e_j) = 0$, we compute

$$\begin{aligned} 0 &= h(\omega_i^k \otimes e_k, e_j) + h(e_i, \omega_j^k \otimes e_k) \\ &= \omega_i^j + \bar{\omega}_j^i, \end{aligned}$$

so the connection matrix ω is skew-Hermitian: $\omega_i^j = -\bar{\omega}_j^i$. The curvature $F_\nabla = d\omega + \omega \wedge \omega$ then also satisfies $\bar{F}_\nabla^T = -F_\nabla$. For sections s_i, s_j, this yields

$$h(F_\nabla s_i, s_j) + h(s_i, F_\nabla s_j) = 0.$$

(ii) If ∇ is compatible with the holomorphic structure, then $\nabla = \nabla^{1,0} + \bar{\partial}$, where $\bar{\partial}$ is the Dolbeault operator and $\nabla^{1,0}$ is of type $(1, 0)$. We compute

$$\begin{aligned} F_\nabla &= \nabla^2 \\ &= (\nabla^{1,0} + \bar{\partial})^2 \\ &= (\nabla^{1,0})^2 + \nabla^{1,0}\bar{\partial} + \bar{\partial}\nabla^{1,0} + (\bar{\partial})^2 \\ &= (\nabla^{1,0})^2 + \nabla^{1,0}\bar{\partial} + \bar{\partial}\nabla^{1,0}, \end{aligned}$$

where $(\bar{\partial})^2 = 0$ and $(\nabla^{1,0})^2$ is of type $(2, 0)$, while the mixed terms are of type $(1, 1)$. Thus F_∇ has no $(0, 2)$-component and

$$F_\nabla \in \mathcal{A}^{2,0}(X, \mathrm{End}(E)) \oplus \mathcal{A}^{1,1}(X, \mathrm{End}(E)).$$

(iii) If ∇ is the Chern connection, it is compatible with both the Hermitian metric and the holomorphic structure. Therefore, by (i), F_∇ is skew-Hermitian, and by (ii), it has no $(0, 2)$-component. To see that F_∇ is of type $(1, 1)$ and real, note that locally, if $A = \bar{h}^{-1}\partial\bar{h}$ is the $(1, 0)$ connection matrix, then

$$\begin{aligned} F_\nabla &= dA + A \wedge A \\ &= (\partial + \bar{\partial})A + A \wedge A \\ &= \bar{\partial}A + \partial A + A \wedge A \end{aligned}$$

but $\partial A + A \wedge A = 0$ by compatibility, so

$$F_\nabla = \bar{\partial}(\bar{h}^{-1}\partial\bar{h}) \in \mathcal{A}^{1,1}(X, \mathrm{End}(E)).$$

Thus F_∇ is a real, skew-Hermitian $(1,1)$-form as claimed.

$\square$

Chapter 2

Chern–Weil Theory

Classifying non-isomorphic vector bundles over a fixed base space is generally a challenging problem. Characteristic classes offer a way to approach this by providing topological invariants that can help differentiate between various vector bundles. These classes are elements in the cohomology groups of the base space. A notable example of such invariants are the Chern classes, which are specifically associated with complex vector bundles. These are maps c_k assigning to each complex vector bundle $E \to M$ over a complex manifold M a class $c_k(E) \in H^{2k}(M, \mathbb{R})$ that depends only on the isomorphism type of E. There are two ways to go about defining Chern classes, the axiomatic approach and the more geometric approach. We will adopt the latter one.

Recall that a connection ∇ on a vector bundle $E \to M$ of rank r can be expressed locally by the matrix $\omega = [\omega^i_j]$ of connection 1-forms. Similarly, the curvature can be represented locally by the matrix $\Omega = [\Omega^i_j]$ of curvature 2-forms.

Under a change of frame these transform as $\widetilde{\omega} = g^{-1}\omega g + g^{-1}dg$ and $\widetilde{\Omega} = g^{-1}\Omega g$. Now while both of these equations are nice, the one considering the curvature is quite remarkable. Essentially it's saying that a change of frame acts on Ω by conjugation with g.

The key thing to realize here is that since $\Omega \in \mathcal{A}^2(M, \operatorname{End}(E))$, we can utilize the framework we've developed for invariant polynomials. What this means is that if p is a homogeneous invariant polynomial of degree k in r^2 variables, then the form $2k$-form $p(\Omega)$ will be independent of the local frame, and hence define a global $2k$-form on M.

It turns out that $p(\Omega)$ is closed and independent of the chosen connection, hence $[p(\Omega)]$ yields a well-defined cohomology class in $H^*(M, \mathbb{C})$ depending only on the invariant polynomial p. This association gives rise to the **Chern-Weil homomorphism**

$$c_E : \operatorname{Inv}(\mathfrak{gl}(r, \mathbb{C})) \to H^*(M, \mathbb{C}).$$

2.1 Invariant Polynomials

We will begin this chapter with some algebraic preliminaries. Let $\mathbb{C}[x_1, \dots, x_n]$ denote the $\mathbb{C}$-algebra of polynomials in n indeterminates x_i.

Definition 2.1.1. A polynomial $p \in \mathbb{C}[x_1, \dots, x_n]$ is said to be homogeneous of degree k if

$$p(x_1, \dots, x_n) = \sum c_{i_1,\dots,i_k} x_{i_1} \cdots x_{i_k},$$

where the sum is taken over all n^k tuples $(i_1, \dots, i_k)$ such that $1 \leq i_j \leq n$ for each i_j.

Definition 2.1.2. Let V be a complex vector space of dimension n. A homogeneous polynomial of degree k on V is a map

$$f : V \to \mathbb{C}$$

such that for every basis (ε^i) of the dual V^*, there exists a unique homogeneous polynomial $p \in \mathbb{C}[x_1, \dots, x_n]$ such that

$$f(v) = p(\varepsilon^1, \dots, \varepsilon^n)(v) = \sum c_{i_1,\dots,i_k} \varepsilon^{i_1}(v) \cdots \varepsilon^{i_k}(v) \tag{2.1}$$

Note that Definition 2.1.2 is independent of the choice of a basis.

Definition 2.1.3. Let V be a vector space. Denote by $P^k(V)$ the set of all homogeneous polynomials of degree k on V, and

$$P(V) = \bigoplus_{k \geq 0} P^k(V).$$

Then $P(V)$ forms an algebra under the usual pointwise product of functions.

Definition 2.1.4. Let V be a vector space and suppose that $f \in P^k(V)$. The polarization of f is given by the tensor $P(f) \in (V^*)^{\otimes k}$ defined by

$$P(f) = \sum c_{i_1,\dots,i_k} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k},$$

where the coefficients are determined by 2.1 and (ε^i) is a basis for V^*.

Remark 2.1.5. To explicitly compute the coefficients $c_{i_1,\dots,i_k}$, we can use the partial derivatives of f in the following manner:

$$c_{i_1,\dots,i_k} = \frac{1}{k!} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}} \bigg|_{x=0}.$$

Example 2.1.6. *Let V be a complex vector space of dimension 2 with a basis $\{e_1, e_2\}$, and $f : V \to \mathbb{C}$ be the homogeneous polynomial of degree 2 given by*

$$f(v) = f(x_1e_1 + x_2e_2) = x_1^2 + 3x_1x_2 + 2x_2^2.$$

Then

$$P(f) = c_{1,1}\varepsilon^1 \otimes \varepsilon^1 + c_{1,2}\varepsilon^1 \otimes \varepsilon^2 + c_{2,1}\varepsilon^2 \otimes \varepsilon^1 + c_{2,2}\varepsilon^2 \otimes \varepsilon^2,$$

and for the coefficients we obtain

$$\begin{aligned}
c_{1,1} &= \frac{1}{2!}\frac{\partial^2 f}{\partial x_1 \partial x_1}\bigg|_{x=0} = 1,\\
c_{1,2} &= \frac{1}{2!}\frac{\partial^2 f}{\partial x_1 \partial x_2}\bigg|_{x=0} = \frac{3}{2},\\
c_{2,1} &= \frac{1}{2!}\frac{\partial^2 f}{\partial x_2 \partial x_1}\bigg|_{x=0} = \frac{3}{2},\\
c_{2,2} &= \frac{1}{2!}\frac{\partial^2 f}{\partial x_2 \partial x_2}\bigg|_{x=0} = 2.
\end{aligned}$$

Hence

$$P(f) = \varepsilon^1 \otimes \varepsilon^1 + \frac{3}{2}\left(\varepsilon^1 \otimes \varepsilon^2 + \varepsilon^2 \otimes \varepsilon^1\right) + 2\varepsilon^2 \otimes \varepsilon^2.$$

Definition 2.1.7. Let V be a vector space. A *symmetric k-linear function* (or *symmetric k-linear form*) on V is a k-linear map

$$f : V \times V \times \cdots \times V \to \mathbb{C}$$

that is invariant under any permutation of its arguments. More formally, for any permutation $\sigma \in S_k$ (the symmetric group on k elements) and for all $v_1, v_2, \ldots, v_k \in V$, the following holds:

$$f(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(k)}) = f(v_1, v_2, \ldots, v_k).$$

The set of all symmetric k-linear functions on V is denoted by $S^k(V^*)$.

Proposition 2.1.8. *Let $S^k(V^*)$ be the space of symmetric k-linear functions on a finite dimensional complex vector space V. Then there is an isomorphism from $S^k(V^*)$ to $P^k(V)$.*

Definition 2.1.9. Let G be a Lie group and denote by $\mathfrak{g}$ its Lie algebra. A homogeneous polynomial $p : \mathfrak{g} \to \mathbb{C}$ is called invariant if

$$p(\mathrm{Ad}_g(\xi)) = p(\xi),$$

for all $g \in G$ and $\xi \in \mathfrak{g}$.

We will be almost entirely focused on the case where $G = \mathrm{GL}(n, \mathbb{C})$ and the Lie algebra is given by $\mathfrak{gl}(n, \mathbb{C})$, the space of complex $(n \times n)$-matrices. In this case, the adjoint

$$\mathrm{Ad}_g : \mathfrak{gl}(n, \mathbb{C}) \to \mathfrak{gl}(n, \mathbb{C}),$$

is given by conjugation $B \mapsto ABA^{-1}$. Therefore, a polynomial $p : \mathfrak{gl}(n, \mathbb{C}) \to \mathbb{C}$ is invariant if

$$p(ABA^{-1}) = p(B),$$

for every $A \in \mathrm{GL}(n, \mathbb{C})$ and $B \in \mathfrak{gl}(n, \mathbb{C})$.

Lemma 2.1.10. *The determinant*

$$\det : \mathfrak{gl}(n, \mathbb{C}) \to \mathbb{C}$$

is an invariant polynoimal.

Proof. Let $A \in \mathrm{GL}(n, \mathbb{C})$ and $B \in \mathfrak{gl}(n, \mathbb{C})$. Then

$$\det(ABA^{-1}) = \det(A)^{-1} \det(B) \det(A) = \det(B).$$

$\square$

Proposition 2.1.11. *Let $A \in \mathfrak{gl}(n, \mathbb{C})$. Then the coefficients $f_k(A)$ in*

$$\det(\lambda I + A) = \sum_{k=0}^{n} f_k(A) \lambda^{n-k}$$

are invariant polynoimals

Proof. By Lemma 2.1.10, the determinant is an invariant polynomial. Hence for any $B \in \mathrm{GL}(n, \mathbb{C})$

$$\det(\lambda I + A) = \det\left(B(\lambda I + A) B^{-1}\right) = \det\left(\lambda I + BAB^{-1}\right).$$

That is,

$$\sum_{k=0}^{n} f_k(A) \lambda^{n-k} = \sum_{k=0}^{n} f_k(BAB^{-1}) \lambda^{n-k}.$$

Comparing the coefficients of λ^{n-k} gives

$$f_k(A) = f_k(BAB^{-1}),$$

which yields the result. $\square$

As a direct corollary of Proposition 2.1.11 we obtain that the trace

$$\mathrm{tr} : \mathfrak{gl}(n, \mathbb{C}) \to \mathbb{C} \tag{2.2}$$

is also an invariant polynomial.

Definition 2.1.12. Let $\mathfrak{gl}(n, \mathbb{C})$ denote the Lie algebra of $n \times n$ complex matrices. The algebra of invariant polynomials on $\mathfrak{gl}(n, \mathbb{C})$, denoted by $\mathrm{Inv}(\mathfrak{gl}(n, \mathbb{C}))$, consists of all complex-valued polynomials $p : \mathfrak{gl}(n, \mathbb{C}) \to \mathbb{C}$ that satisfy the invariance property:

$$p(ABA^{-1}) = p(B) \quad \text{for all } A \in \mathrm{GL}(n, \mathbb{C}) \text{ and } B \in \mathfrak{gl}(n, \mathbb{C}).$$

Theorem 2.1.13. *The ring of invariant complex-valued polynomials on $\mathfrak{gl}(n, \mathbb{C})$, denoted by $\mathrm{Inv}(\mathfrak{gl}(n, \mathbb{C}))$, is generated as a $\mathbb{C}$-algebra by the coefficients $f_k(A)$ of the characteristic polynomial of A. Specifically, for $A \in \mathfrak{gl}(n, \mathbb{C})$, let*

$$\det(\lambda I + A) = \sum_{k=0}^{n} f_k(A) \lambda^{n-k},$$

where $f_k(A)$ are the coefficients. Then,

$$\mathrm{Inv}(\mathfrak{gl}(n, \mathbb{C})) \cong \mathbb{C}[f_1(A), \ldots, f_n(A)].$$

For the proof of Theorem 2.1.13, see [17].

2.2 The Chern-Weil Homomorphism

Let $E \to M$ be a complex vector bundle over a complex manifold M equipped with a connection ∇. Let $\omega = [\omega^i_j]$ and $\Omega = [\Omega^i_j]$ denote the matrices of connection 1-forms and curvature 2-forms relative to a local frame $e = (e_1, \ldots, e_r)$ on an open subset $U \subset M$ respectively. Let p be a homogeneous invariant polynomial of degree k on $\mathfrak{gl}(n, \mathbb{C})$. If $\widetilde{e} = (\widetilde{e}_1, \ldots, \widetilde{e}_r)$ is another frame on U, then

$$\widetilde{e} = eg,$$

for a matrix-valued function $g : U \to \mathrm{GL}(r, \mathbb{C})$. Under this change of frame the curvature matrix transforms as

$$\Omega = g\widetilde{\Omega}g^{-1}.$$

Hence

$$p(\Omega) = p\left(g\widetilde{\Omega}g^{-1}\right) = p(\widetilde{\Omega}),$$

and so $p(\Omega)$ defines a global $2k$-form on M.

Lemma 2.2.1. *Let $A = [\alpha^i_j]$ and $\beta = [\beta^i_j]$ be matrices of forms of degree a and b respectively. In addition, suppose that $a + b \leq \dim(M)$. Then*

(i) $\mathrm{tr}(A \wedge B) = (-1)^{ab} \, \mathrm{tr}(B \wedge A)$.

(ii) If $A = [\alpha^i_j]$ is a square matrix of differential forms on M, then

$$d \, \mathrm{tr}(A) = \mathrm{tr}(dA).$$

Proof.

(i) The (i,j)'th entry of $A \wedge B$ is given by

$$(A \wedge B)^i_j = \sum_k \alpha^i_k \wedge \beta^k_j.$$

Therefore

$$\mathrm{tr}(A \wedge B) = \sum_k \alpha^i_k \wedge \beta^k_i = (-1)^{ab} \sum_k \beta^k_i \wedge \alpha^i_k = (-1)^{ab}\, \mathrm{tr}(B \wedge A).$$

(ii) We have

$$d\, \mathrm{tr}(A) = d \sum_k \alpha^k_k = \sum_k d\alpha^k_k = \mathrm{tr}(dA).$$

□

Proposition 2.2.2. *Let $E \to M$ be a complex vector bundle over a complex manifold M, ∇ a conncetion on E and p a homogeneous invariant polynomial of degree k. Then the global $2k$-form $p(\Omega)$ is closed.*

Proof. Let $\omega = [\omega^i_j]$ and $\Omega = [\Omega^i_j]$ be the connection 1-forms and curvature 2-forms on a framed open set U. Recall that $\mathrm{Inv}(\mathfrak{gl}(n, \mathbb{C}))$ is generated by the trace polynomials $\mathrm{tr}(\Omega^k)$, so it is sufficient to show that $d\,\mathrm{tr}(\Omega^k) = 0$. We have that:

$$\begin{aligned} d\,\mathrm{tr}(\Omega^k) &= \mathrm{tr}(d\Omega^k) \\ &= \mathrm{tr}(\Omega^k \wedge \omega - \omega \wedge \Omega^k) \\ &= \mathrm{tr}(\Omega^k \wedge \omega) - \mathrm{tr}(\omega \wedge \Omega^k) \\ &= (-1)^{2k}\, \mathrm{tr}(\omega \wedge \Omega^k) - \mathrm{tr}(\omega \wedge \Omega^k) \\ &= \mathrm{tr}(\omega \wedge \Omega^k) - \mathrm{tr}(\omega \wedge \Omega^k) \\ &= 0, \end{aligned}$$

That is, $p(\Omega)$ is closed. □

A subtlety we need to consider now is that defining Ω requires a choice of a connection ∇ on E. This appears to make the cohomology class $[p(\Omega)]$ reliant on ∇. However, any finite convex linear combination of connections yields a connection again, that is, if ∇^0 and ∇^1 are two connections on E, then for each $t \in [0, 1]$

$$\nabla^t = (1 - t)\nabla^0 + t\nabla^1$$

is again a connection on E. Also, for each $t \in [0, 1]$ the connection ∇^t yields the connection and curvature matrices ω_t and Ω_t. It's easy to see that $\omega_t = (1 - t)\omega_0 + t\omega_1$, which means that ω_t depends smoothly on t. By the second structural equation, the curvature matrix Ω_t also depends smoothly on t.

Our strategy is to show that, if $\frac{d}{dt}\operatorname{tr}(\Omega_t^k) = d\eta_t$ for some global form η_t, then integrating this with respect to t yields

$$\begin{aligned}\operatorname{tr}(\Omega_1^k) - \operatorname{tr}(\Omega_0^k) &= \int_0^1 \frac{d}{dt}\operatorname{tr}(\Omega_t^k)\ dt \\ &= \int_0^1 d\eta_t\ dt \\ &= d\int_0^1 \eta_t\ dt\end{aligned}$$

where $d\int_0^1 \eta_t\ dt$ is a global exact form. Passing to cohomology classes gives $[\operatorname{tr}(\Omega_1^k)] = [\operatorname{tr}(\Omega_0^k)]$ which would be enough to show that the class $[p(\Omega)]$ doesn't depend on the choice of the connection.

Lemma 2.2.3. *Let $E \to M$ be a complex vector bundle over a complex manifold M. Let $\omega_t \in \mathcal{A}^k(M, \operatorname{End}(E))$ and $\eta_t \in \mathcal{A}^l(M, \operatorname{End}(E))$ be families of* End(E)*-valued differential forms depending smoothly on the parameter $t \in \mathbb{R}$. Then:*

(i) If $\omega_t = [\omega_j^i(t)]$ is a square matrix, then

$$\frac{d}{dt}\operatorname{tr}(\omega_t) = \operatorname{tr}\left(\frac{d}{dt}\omega_t\right).$$

(ii) ω_t and η_t satisfy the product rule

$$\frac{d}{dt}(\omega_t \wedge \eta_t) = \dot{\omega}_t \wedge \eta_t + \omega_t \wedge \dot{\eta}_t.$$

Here $\dot{\omega}_t$ and $\dot{\eta}_t$ both refer to the derivative with respect to t.

(iii) The derivative $\frac{d}{dt}$ commutes with the exterior derivative d. That is,

$$\frac{d}{dt}(d\omega_t) = d\left(\frac{d}{dt}\omega_t\right).$$

(iv) For $[a, b] \subset \mathbb{R}$,

$$\int_a^b d\omega_t\, dt = d\left(\int_a^b \omega_t\, dt\right).$$

Proof. (i) A direct calculation gives

$$\frac{d}{dt}\operatorname{tr}(\omega_t) = \frac{d}{dt}\sum_i (\omega_t)_i^i = \sum_i \frac{d}{dt}(\omega_t)_i^i = \operatorname{tr}\left(\frac{d}{dt}\omega_t\right).$$

(ii) In local coordinates $(U, x = (x^1, \dots, x^n))$ we can write

$$\omega_t = \sum_I \omega_I(x,t)dx^I \quad \text{and} \quad \eta_t = \sum_J \eta_J(x,t)dx^J.$$

Then

$$\omega_t \wedge \eta_t = \sum_{I,J} \omega_I(x,t)\eta_J(x,t)dx^I \wedge dx^J,$$

which gives

$$\begin{aligned}
\frac{d}{dt}(\omega_t \wedge \eta_t) &= \sum_{I,J} \frac{d}{dt}\left(\omega_I(x,t)\eta_J(x,t)\right) dx^I \wedge dx^J \\
&= \sum_{I,J} \left(\frac{d}{dt}\omega_I(x,t)\eta_J(x,t) + \omega_I(x,t)\frac{d}{dt}\eta_J(x,t) \right) dx^I \wedge dx^J \\
&= \frac{d}{dt}\omega_t \wedge \eta_t + \omega_t \wedge \frac{d}{dt}\eta_t \\
&= \dot{\omega}_t \wedge \eta_t + \omega_t \wedge \dot{\eta}_t.
\end{aligned}$$

(iii) Locally $\omega_t = \sum_I \omega_I(x,t)dx^I$. This gives

$$d\omega_t = \sum_I \sum_j \frac{\partial \omega_I(x,t)}{\partial x^j} dx^j \wedge dx^I.$$

Now

$$\begin{aligned}
\frac{d}{dt}(d\omega_t) &= \frac{d}{dt}\left(\sum_I \sum_j \frac{\partial \omega_I(x,t)}{\partial x^j} dx^j \wedge dx^I \right) \\
&= \sum_I \sum_j \frac{\partial^2 \omega_I(x,t)}{\partial t \partial x^j} dx^j \wedge dx^I \\
&= \sum_I \sum_j \frac{\partial^2 \omega_I(x,t)}{\partial x^j \partial t} dx^j \wedge dx^I \\
&= d\left(\frac{d}{dt}\omega_t \right),
\end{aligned}$$

by the symmetry of the mixed partials.

(iv) Again, write $\omega_t = \sum_I \omega_I(x,t)dx^I$, so

$$d\omega_t = \sum_I \sum_j \frac{\partial \omega_I(x,t)}{\partial x^j} dx^j \wedge dx^I.$$

Integrating $d\omega_t$ over $t \in [a,b]$ gives

$$\int_a^b d\omega_t \, dt = \int_a^b \sum_I \sum_j \frac{\partial \omega_I(x,t)}{\partial x^j} dx^j \wedge dx^I \, dt.$$

By the Leibniz integral rule, we may interchange the order of integration and differentiation, yielding

$$\int_a^b d\omega_t \, dt = \sum_I \sum_j \frac{\partial}{\partial x^j} \int_a^b \omega_I(x,t) \, dt \, dx^j \wedge dx^I.$$

On the other hand, computing $d\int_a^b \omega_t \, dt$ gives

$$d\int_a^b \omega_t \, dt = \sum_I \frac{\partial}{\partial x^j} \left(\int_a^b \omega_I(x,t) \, dt \right) dx^j \wedge dx^I.$$

Since the expressions agree, we conclude

$$\int_a^b d\omega_t \, dt = d\left(\int_a^b \omega_t \, dt \right).$$

□

Proposition 2.2.4 (Generalized Second Bianchi Identity)**.** *Let ∇ be a connection on a vector bundle E. Suppose ω and Ω are the connection and curvature matrices of ∇ relative to a local frame on U. Then for any integer $k \geq 1$, we have*

$$d(\Omega^k) = \Omega^k \wedge \omega - \omega \wedge \Omega^k.$$

Proof. For $k = 1$, the second structural equation yields

$$\begin{aligned} d\Omega &= d(d\omega) + (d\omega) \wedge \omega - \omega \wedge d\omega \\ &= (\Omega - \omega \wedge \omega) \wedge \omega - \omega \wedge (\Omega - \omega \wedge \omega) \\ &= \Omega \wedge \omega - \omega \wedge \Omega. \end{aligned}$$

Now suppose the equation holds for $k = n$. We show it also holds for $k = n + 1$. Using the Leibniz rule, we have:

$$\begin{aligned} d\Omega^{n+1} &= d(\Omega^n \wedge \Omega) \\ &= d\Omega^n \wedge \Omega + \Omega^n \wedge d\Omega. \end{aligned}$$

Substituting the inductive hypothesis $d\Omega^n = \Omega^n \wedge \omega - \omega \wedge \Omega^n$ and $d\Omega = \Omega \wedge \omega - \omega \wedge \Omega$, we get

$$\begin{aligned} d\Omega^{n+1} &= (\Omega^n \wedge \omega - \omega \wedge \Omega^n) \wedge \Omega + \Omega^n \wedge (\Omega \wedge \omega - \omega \wedge \Omega) \\ &= \Omega^n \wedge \omega \wedge \Omega - \omega \wedge \Omega^n \wedge \Omega + \Omega^n \wedge \Omega \wedge \omega - \Omega^n \wedge \omega \wedge \Omega. \end{aligned}$$

Using the associativity of the wedge product and simplifying terms, we find

$$d\Omega^{n+1} = \Omega^{n+1} \wedge \omega - \omega \wedge \Omega^{n+1}.$$

Thus, the result holds for $k = n + 1$. By induction, the formula is true for all $k \geq 1$. □

Proposition 2.2.5. *If ∇^t is a family of connections on E with connection and curvature matrices ω_t and Ω_t relative to a local frame on an open set U depending smoothly on $t \in \mathbb{R}$, then*

$$\frac{d}{dt}\left(\operatorname{tr}\Omega_t^k\right) = d\left(k\operatorname{tr}(\Omega_t^{k-1} \wedge \dot{\omega}_t)\right).$$

Moreover, the $d(k\operatorname{tr}(\Omega_t^{k-1} \wedge \dot{\omega}_t))$ can be patched together to obtain a global form on M.

Proof. By Lemma 2.2.3, taking traces commutes with differentiation. Therefore, we have

$$\frac{d}{dt}\left(\operatorname{tr}(\Omega_t^k)\right) = \operatorname{tr}\left(\frac{d}{dt}\Omega_t^k\right).$$

Using the product rule for differentiation of wedge products, we expand $\frac{d}{dt}\Omega_t^k$ as follows:

$$\frac{d}{dt}\Omega_t^k = \sum_{m=0}^{k-1} \Omega_t^m \wedge \dot{\Omega}_t \wedge \Omega_t^{k-1-m},$$

where $\dot{\Omega}_t = \frac{d}{dt}\Omega_t$. Taking the trace, we obtain

$$\operatorname{tr}\left(\frac{d}{dt}\Omega_t^k\right) = \sum_{m=0}^{k-1} \operatorname{tr}\left(\Omega_t^m \wedge \dot{\Omega}_t \wedge \Omega_t^{k-1-m}\right).$$

The trace of a product of matrices (with wedge products) is cyclically invariant, and the degrees of the curvature 2-forms ensure that all exponents of -1 are even. Therefore, each term in the sum above is equal:

$$\operatorname{tr}\left(\Omega_t^m \wedge \dot{\Omega}_t \wedge \Omega_t^{k-1-m}\right) = \operatorname{tr}\left(\Omega_t^{k-1} \wedge \dot{\Omega}_t\right).$$

Since there are k identical terms in the sum, we obtain

$$\operatorname{tr}\left(\frac{d}{dt}\Omega_t^k\right) = k\operatorname{tr}\left(\Omega_t^{k-1} \wedge \dot{\Omega}_t\right).$$

Thus,

$$\frac{d}{dt}\left(\operatorname{tr}\Omega_t^k\right) = k\operatorname{tr}\left(\Omega_t^{k-1} \wedge \dot{\Omega}_t\right).$$

Now, the second structural equation gives

$$\Omega_t = d\omega_t + \omega_t \wedge \omega_t,$$

and differentiating both sides with respect to t, we obtain

$$\dot{\Omega}_t = d\dot{\omega}_t + \dot{\omega}_t \wedge \omega_t + \omega_t \wedge \dot{\omega}_t.$$

Substituting the expression for $\dot{\Omega}_t$ into the trace and using Proposition 2.2.4, we have

$$\begin{aligned}
\operatorname{tr}\left(\Omega_t^{k-1} \wedge \dot{\Omega}_t\right) &= \operatorname{tr}\left(\Omega_t^{k-1} \wedge d\dot{\omega}_t + \Omega_t^{k-1} \wedge \dot{\omega}_t \wedge \omega_t + \Omega_t^{k-1} \wedge \omega_t \wedge \dot{\omega}_t\right) \\
&= \operatorname{tr}\left(\Omega_t^{k-1} \wedge d\dot{\omega}_t - \omega_t \wedge \Omega_t^{k-1} \wedge \dot{\omega}_t + \Omega_t^{k-1} \wedge \omega_t \wedge \dot{\omega}_t\right) \\
&= \operatorname{tr}\left(\Omega_t^{k-1} \wedge d\dot{\omega}_t + \left(\Omega_t^{k-1} \wedge \omega_t - \omega_t \wedge \Omega_t^{k-1}\right) \wedge \dot{\omega}_t\right) \\
&= \operatorname{tr}\left(\Omega_t^{k-1} \wedge d\dot{\omega}_t + d\Omega_t^{k-1} \wedge \dot{\omega}_t\right) \\
&= \operatorname{tr}\left(d(\Omega_t^{k-1} \wedge \omega_t)\right) \\
&= d \operatorname{tr}(\Omega_t^{k-1} \wedge \omega_t).
\end{aligned}$$

We conclude that

$$\frac{d}{dt}\left(\operatorname{tr}\Omega_t^k\right) = k \operatorname{tr}\left(\Omega_t^{k-1} \wedge \dot{\Omega}_t\right) = d\left(k \operatorname{tr}(\Omega_t^{k-1} \wedge \omega_t)\right).$$

To complete the proof, we need to ensure that the form $k \operatorname{tr}(\Omega_t^{k-1} \wedge \dot{\omega}_t)$ patches together to define a global form on M. To achieve this, consider two overlapping local frames e and $\widetilde{e}$ on an open set $U \subset M$, related by a transition function $g : U \to \mathrm{GL}(n, \mathbb{C})$ such that $\widetilde{e} = eg$. Under this change of frame, the connection and curvature matrices transform as follows:

$$\widetilde{\omega}_t = g^{-1}\omega_t g + g^{-1}dg \quad \text{and} \quad \widetilde{\Omega}_t = g^{-1}\Omega_t g.$$

Differentiating the connection matrix with respect to t yields

$$\dot{\widetilde{\omega}}_t = g^{-1}\dot{\omega}_t g,$$

since $g^{-1}dg$ does not depend on t. Now, consider the form $\operatorname{tr}\left(\widetilde{\Omega}_t^{k-1} \wedge \dot{\widetilde{\omega}}_t\right)$:

$$\begin{aligned}
\operatorname{tr}\left(\widetilde{\Omega}_t^{k-1} \wedge \dot{\widetilde{\omega}}_t\right) &= \operatorname{tr}\left(g^{-1}\Omega_t^{k-1} g \wedge g^{-1}\dot{\omega}_t g\right) \\
&= \operatorname{tr}\left(g^{-1}\Omega_t^{k-1} g \wedge g^{-1}\dot{\omega}_t g\right).
\end{aligned}$$

Utilizing the cyclic property of the trace and the associativity of the wedge product, we can rewrite the above expression as

$$\operatorname{tr}\left(g^{-1}\Omega_t^{k-1} g \wedge g^{-1}\dot{\omega}_t g\right) = \operatorname{tr}\left(\Omega_t^{k-1} \wedge \dot{\omega}_t\right).$$

This equality shows that the local expressions $\operatorname{tr}\left(\Omega_t^{k-1} \wedge \dot{\omega}_t\right)$ computed with respect to different frames agree. Therefore, these local forms patch together to define a global form on M. □

Proposition 2.2.6. *Let $E \to M$ be a rank n complex vector bundle over a complex manifold M, ∇ a connection on E and p a homogeneous invariant polynomial of degree k on $\mathfrak{gl}(n, \mathbb{R})$. Then the cohomology class $[p(\Omega)] \in H^{2k}(M, \mathbb{C})$ is independent of the connection ∇.*

Proof. Suppose that ∇^0 and ∇^1 are two connections on E. Then the convex combination $\nabla^t = (1-t)\nabla^0 + t\nabla^1$ for $t \in \mathbb{R}$ is also a connection on E with connection and curvature matrices ω_t and Ω_t over a trivialized open set. Now

$$\frac{d}{dt}\left(\operatorname{tr}\Omega_t^k\right) = d(k \operatorname{tr}(\Omega_t^{k-1}\dot{\omega}_t))$$

and so

$$\int_0^1 \frac{d}{dt}\left(\operatorname{tr}\Omega_t^k\right)\, dt = \operatorname{tr}(\Omega_1^k) - \operatorname{tr}(\Omega_0^k).$$

By Lemma 2.2.3, we also have

$$\int_0^1 d(k \operatorname{tr}(\Omega_t^{k-1} \wedge \dot{\omega}_t))dt = d\int_0^1 k \operatorname{tr}(\Omega_t^{k-1} \wedge \dot{\omega}_t)dt.$$

Therefore

$$\operatorname{tr}(\Omega_1^k) - \operatorname{tr}(\Omega_0^k) = d\int_0^1 k \operatorname{tr}(\Omega_t^{k-1}\dot{\omega}_t)dt.$$

The right-hand side is now a global exact form so passing to cohomology yields that $[\operatorname{tr}(\Omega_1^k)] = [\operatorname{tr}(\Omega_0^k)]$ which proves that the cohomology class of $\operatorname{tr}(\Omega^k)$ is independent of the connection. $\square$

All in all we've have proved the following theorem:

Theorem 2.2.7. *Let $E \to M$ be a complex vector bundle over a complex manifold M, ∇ a connection on E and p an invariant homogeneous polynomial of degree k on $\mathfrak{gl}(r, \mathbb{C})$. Then*

(i) The global $2k$-form $p(\Omega)$ is closed.

(ii) The cohomology class $[p(\Omega)] \in H^{2k}(M, \mathbb{C})$ is independent of the choice of the connection.

2.3 Chern Classes

As demonstrated in Lemma 2.1.10, the determinant $\det : \mathfrak{gl}(n, \mathbb{C}) \to \mathbb{C}$ is an invariant polynomial. Proposition 2.1.11 showed that for $A \in \mathfrak{gl}(n, \mathbb{C})$, the coefficients $f_k(A)$ of the characteristic polynomial $\det(\lambda I + A)$ are also invariant polynomials.

Definition 2.3.1. Let $E \to X$ be a complex vector bundle over a complex manifold X of rank r. Let ∇ be a connection on E with curvature F_∇. The closed differential form

$$c_k(E, \nabla) := f_k\left(\frac{i}{2\pi}F_\nabla\right) \in \mathcal{A}^{2k}(X)$$

is called the k'th **Chern form**.

Definition 2.3.2. Let $E \to X$ be a complex vector bundle over a complex manifold X of rank r. Let ∇ be a connection on E with curvature F_∇. The k'th **Chern class** of E is defined to be the induced cohomology class

$$c_k(E) := [c_k(E, \nabla)] = \left[f_k\left(\frac{i}{2\pi}F_\nabla\right)\right] \in H^{2k}(X, \mathbb{C}).$$

In particular, $c_0(E) = 1$ and $c_k(E) = 0$ for $k > r$. It is worth noting that for $k = 1$, $f_1(F_\nabla)$ is given by the trace, so $c_1(E)$ is represented by

$$c_1(E, \nabla) = \frac{i}{2\pi}\Omega_k^k = \frac{i}{2\pi}R^k_{k\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^\beta = \frac{i}{2\pi}R_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^\beta. \tag{2.3}$$

Definition 2.3.3. The **total Chern class** of E is given by

$$c(E) := \left[\det\left(I + \frac{i}{2\pi}F_\nabla\right)\right] = 1 + c_1(E) + \cdots + c_r(E).$$

We now consider how the Chern classes and the total Chern class behave under the standard operations on vector bundles.

Proposition 2.3.4. *Let E and F be complex vector bundles of rank r over a complex manifold X. Let ∇_1 and ∇_2 be connections on E and F, with curvatures F_{∇_1} and F_{∇_2}. Then the total Chern class of the direct sum satisfies*

$$c(E \oplus F) = c(E) \cdot c(F).$$

Proof. The curvature of the direct sum connection $\nabla_1 \oplus \nabla_2$ is $F_{\nabla_1} \oplus F_{\nabla_2}$. Therefore,

$$\begin{aligned} c(E \oplus F, \nabla_1 \oplus \nabla_2) &= \det\left(I + \frac{i}{2\pi}(F_{\nabla_1} \oplus F_{\nabla_2})\right) \\ &= \det\left(I_E + \frac{i}{2\pi}F_{\nabla_1}\right) \cdot \det\left(I_F + \frac{i}{2\pi}F_{\nabla_2}\right) \\ &= c(E, \nabla_1) \cdot c(F, \nabla_2). \end{aligned}$$

Passing to cohomology classes yields $c(E \oplus F) = c(E) \cdot c(F)$. $\square$

Note that this actually gives a formula for $c_k(E \oplus F)$ also by comparing the degrees. We obtain

$$c_k(E \oplus F) = \sum_{i=0}^{r} c_i(E) \cdot c_{r-i}(F).$$

The total Chern class also satisfies the following naturality condition with respect to pullbacks.

Proposition 2.3.5. *Let $f : Y \to X$ be a smooth map between complex manifolds. Let $E \to X$ be a complex vector bundle over X equipped with a connection ∇. Then*

$$c(f^*E) = f^*c(E).$$

Proof. The curvature of the pullback connection $f^*\nabla$ is $F_{f^*\nabla} = f^*F_\nabla$. Thus,

$$\begin{aligned} c(f^*E, f^*\nabla) &= \det\left(I + \frac{i}{2\pi} f^*F_\nabla\right) \\ &= f^*\left(\det\left(I + \frac{i}{2\pi} F_\nabla\right)\right) \\ &= f^*c(E, \nabla). \end{aligned}$$

Passing to cohomology classes gives $c(f^*E) = f^*c(E)$. ☐

For the tensor product, the situation is a bit trickier. Using the so-called splitting principle, one obtains a formula for the total Chern class of the tensor product (see Bott and Tu [6] for details). For the first Chern class we do have the following.

Proposition 2.3.6. *Let E and F be complex vector bundles of rank r and s over a complex manifold X, with connections ∇_1 and ∇_2 of curvatures F_{∇_1} and F_{∇_2}. Then*

$$c_1(E \otimes F) = s\, c_1(E) + r\, c_1(F).$$

Proof. The curvature of the tensor product connection is

$$F_{\nabla_1 \otimes \nabla_2} = F_{\nabla_1} \otimes \mathrm{id}_F + \mathrm{id}_E \otimes F_{\nabla_2}.$$

Therefore,

$$\begin{aligned} c_1(E \otimes F, \nabla_1 \otimes \nabla_2) &= \frac{i}{2\pi}\, \mathrm{tr}(F_{\nabla_1 \otimes \nabla_2}) \\ &= \frac{i}{2\pi}\, (s\, \mathrm{tr}(F_{\nabla_1}) + r\, \mathrm{tr}(F_{\nabla_2})) \\ &= sc_1(E, \nabla_1) + rc_1(F, \nabla_2). \end{aligned}$$

Passing to cohomology yields $c_1(E \otimes F) = sc_1(E) + rc_1(F)$. ☐

The remedy for the Chern class not being very manageable with the tensor product is given by introducing something called the **Chern character**. Recall that the trace $\mathrm{tr} : \mathfrak{gl}(n, \mathbb{C}) \to \mathbb{C}$ is an invariant polynomial, this induces invariant homogeneous polynomials g_k of degree k defined by

$$\mathrm{tr}(\exp A) = g_0(A) + g_1(A) + \cdots$$

Definition 2.3.7. Let $E \to X$ be a complex vector bundle over a complex manifold X. Let ∇ be a connection on E with curvature F_∇. The k'th Chern character $\mathrm{ch}_k(E) \in H^{2k}(X, \mathbb{C})$ of E is defined as the cohomology class

$$\mathrm{ch}_k(E) := [\mathrm{ch}_k(E, \nabla)]\,,$$

where

$$\mathrm{ch}_k(E, \nabla) = g_k\left(\frac{i}{2\pi} F_\nabla\right) \in \mathcal{A}^{2k}(X)$$

is the Chern character form.

Definition 2.3.8. The **total Chern character** is defined by

$$\mathrm{ch}(E) := \mathrm{ch}_0(E) + \mathrm{ch}_1(E) + \cdots$$

Proposition 2.3.9. *Let E and F be complex vector bundles over a complex manifold X with connections ∇_1 and ∇_2. Then*

$$\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \cdot \mathrm{ch}(F).$$

Proof. The curvature of the tensor product bundle is

$$F_{\nabla_1 \otimes \nabla_2} = F_{\nabla_1} \otimes \mathrm{id}_F + \mathrm{id}_E \otimes F_{\nabla_2}.$$

The exponential map satisfies

$$\exp(A \otimes \mathrm{id} + \mathrm{id} \otimes B) = \exp(A) \otimes \exp(B),$$

so

$$\begin{aligned}
\mathrm{ch}(E \otimes F) &= \mathrm{tr}\left(\exp\left(\frac{i}{2\pi} F_{\nabla_1 \otimes \nabla_2}\right)\right) \\
&= \mathrm{tr}\left(\exp\left(\frac{i}{2\pi} F_{\nabla_1}\right) \otimes \exp\left(\frac{i}{2\pi} F_{\nabla_2}\right)\right) \\
&= \mathrm{tr}\left(\exp\left(\frac{i}{2\pi} F_{\nabla_1}\right)\right) \cdot \mathrm{tr}\left(\exp\left(\frac{i}{2\pi} F_{\nabla_2}\right)\right) \\
&= \mathrm{ch}(E) \cdot \mathrm{ch}(F).
\end{aligned}$$

□

To conclude this section we will specialize to the case where X is a compact Kähler manifold.

Proposition 2.3.10. *Let $E \to X$ be a complex vector bundle over a compact Kähler manifold X. If η is any real $(1,1)$-form representing $c_1(E)$, then there exists a Hermitian metric on E whose Chern connection has curvature F_∇ satisfying*

$$\frac{i}{2\pi}\operatorname{tr}(F_\nabla) = \eta.$$

Proof. Begin with an arbitrary Hermitian metric h on E, and let F_∇ be the curvature form of its Chern connection. Then $\frac{i}{2\pi}\operatorname{tr}(F_\nabla)$ is a real $(1,1)$-form representing $c_1(E)$. By assumption, the given real $(1,1)$-form η also represents $c_1(E)$, so their difference

$$\frac{i}{2\pi}\operatorname{tr}(F_\nabla) - \eta$$

is exact. By the global $\partial\bar{\partial}$-lemma, there exists a smooth real function f such that

$$\frac{i}{2\pi}\operatorname{tr}(F_\nabla) - \eta = \frac{i}{2\pi}\partial\bar{\partial}f.$$

Define a new Hermitian metric $\widetilde{h} = e^f h$. The curvature $\widetilde{F}$ of the Chern connection associated to $\widetilde{h}$ satisfies

$$\widetilde{F} = F_\nabla + \bar{\partial}\partial f,$$

and so

$$\frac{i}{2\pi}\operatorname{tr}(\widetilde{F}) = \frac{i}{2\pi}\operatorname{tr}(F_\nabla) - \frac{i}{2\pi}\partial\bar{\partial}f = \eta.$$

Thus, the new metric has the desired property. □

2.4 Chern Classes for Flat and Projectively Flat Bundles

In this section we will look at how Chern classes behave when the underlying vector bundle is equipped with a flat connection or a projectively flat connection. We will begin with the flat connections and state the following.

Theorem 2.4.1. *Let L be a complex line bundle over a complex manifold X. Then L admits a flat connection if and only if $c_1(L) = 0$.*

Proof. If L admits a flat connection, then clearly $c_1(L) = 0$. Conversely, suppose that $c_1(L) = 0$. Let ∇ be a connection on L and denote its curvature by F_∇. Then $F_\nabla = d\eta$ for some 1-form η. From 1.2.2 we know that the space of connections is an affine space, so consider the new connection

$$\nabla' = \nabla - \eta.$$

Denote the connection 1-forms for ∇ and ∇' by ω and ω' respectively. Then locally,

$$F_{\nabla'} = d\omega' = d(\omega - \eta) = d\omega - d\eta = F_{\nabla} - d\eta = 0.$$

Therefore, ∇' is a flat connection on L. $\square$

Definition 2.4.2. A *projectively flat bundle* is a complex vector bundle E over a complex manifold X equipped with a connection ∇ whose curvature F_{∇} satisfies

$$F_{\nabla} = \eta \otimes \mathrm{id}_E$$

for some closed 2-form $\eta \in \mathcal{A}^2(X)$. In other words, the curvature is proportional to the identity endomorphism of E. If $\eta = 0$, then the connection is flat, so every flat bundle is projectively flat, though the converse does not necessarily hold.

Proposition 2.4.3. *Let $E \to X$ be a projectively flat bundle of rank r over a complex manifold X. Then its total Chern class can be expressed in terms of the first Chern class and its rank as follows:*

$$c(E) = \left(1 + \frac{c_1(E)}{r}\right)^r.$$

Proof. Since E is projectively flat, it admits a connection ∇ with curvature

$$F_{\nabla} = \eta \otimes \mathrm{id}_E$$

for some 2-form η on X. The total Chern class is given by

$$c(E) = \left[\det\left(I + \frac{i}{2\pi} F_{\nabla}\right)\right].$$

Note that $\det\left(I + \frac{i}{2\pi} F_{\nabla}\right)$ is the determinant of a diagonal matrix with each diagonal entry $1 + \frac{i}{2\pi}\eta$, so

$$\det\left(I + \frac{i}{2\pi} F_{\nabla}\right) = \left(1 + \frac{i}{2\pi}\eta\right)^r.$$

Now, recall that $c_1(E)$ is represented by

$$\frac{i}{2\pi}\operatorname{tr}(F_{\nabla}) = \frac{i}{2\pi}\eta \operatorname{tr}(\mathrm{id}_E) = \frac{i}{2\pi} r\eta.$$

Therefore,

$$\left(1 + \frac{i}{2\pi}\eta\right)^r = \left(1 + \frac{c_1(E, \nabla)}{r}\right)^r,$$

and so

$$c(E) = \left(1 + \frac{c_1(E)}{r}\right)^r.$$

$\square$

Proposition 2.4.4. *Let $E \to X$ be a projectively flat bundle over a complex manifold X. Then the bundle* $\mathrm{End}(E)$ *admits a flat connection.*

Proof. The endomorphism bundle $\mathrm{End}(E)$ is naturally isomorphic to $E^* \otimes E$. The curvature of E^* is given by

$$F_{\nabla_{E^*}} = -\eta \otimes \mathrm{id}_{E^*},$$

so the curvature of $E^* \otimes E$ is

$$F_{\nabla_1 \otimes \nabla_2} = F_{\nabla_{E^*}} \otimes \mathrm{id}_E + \mathrm{id}_{E^*} \otimes F_{\nabla_E} = -(\eta \otimes \mathrm{id}_{E^*}) \otimes \mathrm{id}_E + \mathrm{id}_{E^*} \otimes (\eta \otimes \mathrm{id}_E) = 0.$$

Thus, $E^* \otimes E$ admits a flat connection, so $\mathrm{End}(E)$ does as well. □

Proposition 2.4.3 also yields formulas for the total Chern characters $\mathrm{ch}(E)$ and $\mathrm{ch}(E^*)$ for a projectively flat bundle and its duals in the following way:

$$\mathrm{ch}(E) = r \exp\left(\frac{1}{r} c_1(E)\right) \quad \text{and} \quad \mathrm{ch}(E^*) = r \exp\left(-\frac{1}{r} c_1(E)\right).$$

Chapter 3

Hermite–Einstein Vector Bundles

Hermite–Einstein vector bundles form a central object of study in the intersection of differential geometry, algebraic geometry, and gauge theory. These bundles provide a bridge between the notion of stability in algebraic geometry and the existence of canonical metrics on vector bundles in differential geometry. The study of these vector bundles has its origins in the famous Kobayashi–Hitchin correspondence, which proved that a holomorphic vector bundle admits a Hermite–Einstein metric if and only if it is slope polystable [18].

Moreover, Hermite–Einstein vector bundles are deeply intertwined with the theory of Yang–Mills connections, which arise in both mathematics and physics. In the physical setting, Hermitian Yang–Mills connections describe energy-minimizing configurations in gauge theory, particularly in the context of string theory and compactifications on Calabi–Yau manifolds.

This chapter introduces the Hermite–Einstein condition for vector bundles, explores the relationship between Hermite–Einstein metrics and stability, and discusses the Chern classes of these bundles, which capture essential topological information.

3.1 Hermite–Einstein Condition

To motivate the consideration for Hermite–Einstein vector bundles, let us begin by recalling what Einstein's field equations state. Suppose that (X, g) is a Riemannian manifold. The Einstein field equation is given by

$$\mathrm{Ric} = \frac{1}{2}S + T,$$

where S is the scalar curvature of g and T the so-called stress-energy tensor. Now in our case $T \equiv 0$ and so we obtain the following definition:

Definition 3.1.1. A Riemannian manifold (X, g) is called **Einstein** if the Ricci tensor Ric is proportional to the metric g. That is

$$\mathrm{Ric} = \lambda g,$$

for $\lambda \in \mathbb{R}$.

Consider now a compact Hermitian manifold (X, g) and the fundamental form $\omega = g(J(-), -)$. By definition, (X, g) is Kähler if and only if $d\omega = 0$ or equivalently if J is parallel with respect to the Levi–Civita connection of g. Now g can be viewed as a Hermitian metric on the tangent bundle of X and instead of asking if the Ricci curvature of the Levi–Civita connection on the tangent bundle of X is proportional to g, we can ask the same question about the curvature F_∇ of a Chern connection ∇ associated to a Hermitian metric on any holomorphic vector bundle (E, h) over (X, g).

Clearly, we are trying to generalize the definition above and since it depends on the Ricci curvature, we start with the curvature tensor F_∇ of E. This is an $\mathrm{End}(E)$-valued 2-form so locally on a framed open set we can write

$$\begin{aligned} F_\nabla &= \sum \Omega^i_j \otimes \varepsilon^j \otimes e_i \\ &= \sum \left(\sum R^i_{j\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta \right) \otimes \varepsilon^j \otimes e_i. \end{aligned}$$

Now, the Ricci tensor is given by tracing over the Levi–Civita connection on the tangent bundle of X. We will do something similar here and consider the trace/contraction of F_∇ with the fundamental form ω. However, tracing with the form ω is not as straightforward as it sounds. What we will do is consider the adjoint Λ of the Lefschetz operator to define this. Recall that the Lefschetz operator $L : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p+1,q+1}(X)$ is given by

$$\alpha \mapsto \alpha \wedge \omega.$$

Let $e_k : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p+1,q}(X)$ be the operator given by $\alpha \mapsto dz^k \wedge \alpha$ and define $\bar{e}_k : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X)$ similarly. Recall now also that the wedge product has the interior product as its adjoint. Denote these by ι_k and $\bar{\iota}_k$ respectively. Now

$$\begin{aligned} L(\alpha) &= \alpha \wedge \omega \\ &= i g_{j\bar{k}} \alpha \wedge dz^j \wedge d\bar{z}^k \\ &= \sum_{j,k} i g_{j\bar{k}} e_j \bar{e}_k(\alpha). \end{aligned}$$

The adjoint Λ is thus given by

$$\Lambda = -i \sum_{j,k} g^{j\bar{k}} \bar{\iota}_k \iota_j.$$

Applying this yields

$$\begin{aligned}\Lambda F_\nabla &= \Lambda\left(\sum \Omega^i_j \otimes \varepsilon^j \otimes e_i\right)\\ &= \Lambda\left(\sum\left(\sum R^i_{j\alpha\bar\beta} dz^\alpha \wedge d\bar z^\beta\right)\otimes\varepsilon^j\otimes e_i\right)\\ &= \sum\left(\sum R^i_{j\alpha\bar\beta}\Lambda(dz^\alpha\wedge d\bar z^\beta)\right)\otimes\varepsilon^j\otimes e_i\\ &= -i\sum\left(\sum g^{\alpha\bar\beta}R^i_{j\alpha\bar\beta}\right)\otimes\varepsilon^j\otimes e_i.\end{aligned}$$

Note that multiplying by i gives $i\Lambda F_\nabla = \sum\left(\sum g^{\alpha\bar\beta}R^i_{j\alpha\bar\beta}\right)\otimes\varepsilon^j\otimes e_i$. In literature you might see ΛF_∇ being denoted by K and called the **mean curvature**. This is defined by setting $K^i_j = g^{\alpha\bar\beta}R^i_{j\alpha\bar\beta}$ and $K(\xi) = K^i_j\xi^j e_i$, for a section $\xi = \xi^i e_i$. There is also the **mean curvature form** obtained by setting $K_{j\bar k} = h_{i\bar k}K^i_j$ and $\hat K(\xi,\eta) = K_{j\bar k}\xi^j\bar\eta^k$. Using this we will define the Hermite–Einstein condition as follows:

Definition 3.1.2. A Hermitian metric h on a holomorphic vector bundle E is called **weakly Hermite–Einstein** if there exists some real function λ for which

$$i\Lambda F_\nabla = \lambda\,\mathrm{id}_E\,.$$

If λ is a constant, we say that h is **Hermite–Einstein**.

Proposition 3.1.3. *(i) Every Hermitian line bundle (L,h) over a complex manifold X satisfies the weak Einstein condition (with respect to any Hermitian metric g on X).*

(ii) If (E,h) over (X,g) satisfies the (weak) Einstein condition with factor λ, then the dual bundle (E^*,h^*) satisfies the (weak) Einstein condition with factor $-\lambda$.

(iii) If (E_1,h_1) and (E_2,h_2) over (X,g) satisfy the (weak) Einstein condition with factors λ_1 and λ_2, respectively, then their tensor product $(E_1\otimes E_2, h_1\otimes h_2)$ satisfies the (weak) Einstein condition with factor $\lambda_1+\lambda_2$.

(iv) The Whitney sum $(E_1\oplus E_2, h_1\oplus h_2)$ satisfies the (weak) Einstein condition with factor λ if and only if both summands (E_1,h_1) and (E_2,h_2) satisfy the (weak) Einstein condition with the same factor λ.

Proof. (i) The curvature F_∇ of the Chern connection ∇ is an imaginary $(1,1)$-form. It follows that $i\Lambda F_\nabla = \Lambda i F_\nabla$ is a real-valued function on X and hence satisfies the weak Einstein condition.

(ii) Suppose that ∇ is the Chern connection on E and $i\Lambda F_\nabla = \lambda\,\mathrm{id}_E$. Then the induced connection ∇^* on E^* has curvature $F^*_\nabla = -F^T_\nabla$. Thus

$$i\Lambda F^*_\nabla = i\Lambda(-F^T_\nabla) = i\Lambda(F^T_\nabla) = -\lambda\,\mathrm{id}_E,$$

i.e. the dual bundle (E^*, h^*) satisfies the (weak) Einstein condition with factor $-\lambda$.

(iii) Recall that on $E_1 \otimes E_2$, the curvature is given by $F_{\nabla_1 \otimes \nabla_2} = F_{\nabla_1} \otimes \mathrm{id}_{E_2} + \mathrm{id}_{E_1} \otimes F_{\nabla_2}$. We obtain

$$\begin{aligned} i\Lambda F_{\nabla_1 \otimes \nabla_2} &= i\Lambda(F_{\nabla_1} \otimes \mathrm{id}_{E_2} + \mathrm{id}_{E_1} \otimes F_{\nabla_2}) \\ &= i\Lambda(F_{\nabla_1}) \otimes \mathrm{id}_{E_2} + \mathrm{id}_{E_1} \otimes i\Lambda(F_{\nabla_2}) \\ &= \lambda_1 \mathrm{id}_{E_1} \otimes \mathrm{id}_{E_2} + \mathrm{id}_{E_1} \otimes \lambda_2 \mathrm{id}_{E_2} \\ &= (\lambda_1 + \lambda_2) \mathrm{id}_{E_1 \otimes E_2} . \end{aligned}$$

(iv) If $(E_1 \oplus E_2, h_1 \oplus h_2)$ satisfies the (weak) Einstein condition with factor λ, then given the connection $\nabla = \nabla_1 + \nabla_2$ on $E_1 \oplus E_2$, the curvature is given by $F_\nabla = F_{\nabla_1} + F_{\nabla_2}$. This gives

$$\begin{aligned} \lambda \mathrm{id}_{E_1 \oplus E_2} &= i\Lambda F_\nabla \\ &= i\Lambda(F_{\nabla_1} + F_{\nabla_2}) \\ &= i\Lambda F_{\nabla_1} + i\Lambda F_{\nabla_2}. \end{aligned}$$

It follows that $i\Lambda F_{\nabla_j} = \lambda \mathrm{id}_{E_j}$. Conversely, if both summands (E_1, h_1) and (E_2, h_2) satisfy the (weak) Einstein condition with the same factor λ, then

$$\begin{aligned} i\Lambda F_\nabla &= i\Lambda(F_{\nabla_1} + F_{\nabla_2}) \\ &= i\Lambda F_{\nabla_1} + i\Lambda F_{\nabla_2} \\ &= \lambda \mathrm{id}_{E_1} + \lambda \mathrm{id}_{E_2} \\ &= \lambda \mathrm{id}_{E_1 \oplus E_2} . \end{aligned}$$

□

The Einstein condition imposes a strong restriction to possible sheaf morphisms. To see this we will state the following useful result which is mostly based on the maximum principle by E. Hopf.

Theorem 3.1.4. *Let (E, h) be a Hermitian vector bundle over a compact Hermitian manifold (X, g). Let ∇ be the Chern connection of E, F_∇ its curvature, and $\hat{K}$ the mean curvature.*

(i) If $\hat{K}$ is negative semi-definite everywhere on X, then $\nabla\xi = 0$ for any holomorphic section ξ of E and $\hat{K}(\xi, \xi) = 0$.

(ii) If $\hat{K}$ is negative semi-definite everywhere on X and negative definite at some point of X, then E admits no non-zero holomorphic sections.

Proposition 3.1.5. *Let (E_1, h_1) and (E_2, h_2) be Hermitian vector bundles over a compact Hermitian manifold (X, g) satisfying the (weak) Einstein condition with factors λ_1 and λ_2, respectively. If $\lambda_2 < \lambda_1$, then each sheaf homomorphism $f : E_1 \to E_2$ is zero.*

Proof. Let $f : E_1 \to E_2$ be a morphism of sheaves. The map f is a global holomorphic section of $E_1^* \otimes E_2$, which satisfies the weak Einstein condition with factor $\lambda_2 - \lambda_1$. Now Since $\lambda_2 - \lambda_1 < 0$ we have that

$$\begin{aligned} \Lambda F_{\nabla_1 \otimes \nabla_2} &= (\lambda_2 - \lambda_1) \operatorname{id}_{E_1 \otimes E_2} \\ &< 0, \end{aligned}$$

i.e. $\Lambda F_{\nabla_1 \otimes \nabla_2}$ is negative definite and hence $f = 0$. □

Generally speaking, these kinds of Hermite–Einstein metrics are not easy to describe, but they exist relatively frequently and the benefit is that those holomorphic bundles that admit such a metric can be described algebraically as we will see later on.

3.2 Vanishing Theorems

We will now consider some consequences of the Hermite–Einstein condition on the sections of a holomorphic Hermitian vector bundle (E, h) over a compact complex manifold X.

Lemma 3.2.1 (Bochner)**.** *If $f \in \Omega^0(X)$ is a real-valued function such that $\Delta f \geq 0$, then $\Delta f = 0$ and f is constant.*

The vanishing theorem we are aiming for follows from Bochner's result and the following lemma.

Lemma 3.2.2. *Let $U \subset X$ be an open subset. For a holomorphic section $\xi \in H^0(U, E)$ we have that*

$$\Delta h(\xi, \xi) = -i\Lambda h(F_\nabla \xi, \xi) + i\Lambda |\nabla \xi|^2,$$

where F_∇ denotes the curvature of the Chern connection on (E, h).

Proof. Let $\xi \in H^0(U, E)$ be a holomorphic section. Then as the Chern connection is compatible with h we have that

$$dh(\xi, \xi) = h(\nabla \xi, \xi) + h(\xi, \nabla \xi).$$

Also as $\nabla^{0,1} = \bar{\partial}$ for the Chern connection and ξ is holomorphic, $\nabla \xi$ is of type $(1, 0)$. A type comparison yields

$$\partial h(\xi, \xi) = h(\nabla \xi, \xi).$$

It follows that

$$\begin{aligned} \bar{\partial}\partial h(\xi, \xi) &= d\partial h(\xi, \xi) \\ &= dh(\nabla \xi, \xi) \\ &= h(F_\nabla \xi, \xi) - h(\nabla \xi, \nabla \xi), \end{aligned}$$

since generally for a p-form $\eta \in \mathcal{A}^p(U, E)$ and 1-form $\beta \in \Omega^1(U, E)$ we have

$$dh(\eta, \beta) = h(\nabla\eta, \beta) + (-1)^p h(\eta, \nabla\beta).$$

So all in all

$$\begin{aligned}\Delta h(\xi, \xi) &= i\Lambda(\partial\bar{\partial}h(\xi, \xi)) \\ &= -i\Lambda(\bar{\partial}\partial h(\xi, \xi)) \\ &= -i\Lambda h(F_\nabla \xi, \xi) + i\Lambda h(\nabla\xi, \nabla\xi) \\ &= -i\Lambda h(F_\nabla \xi, \xi) + i\Lambda|\nabla\xi|^2.\end{aligned}$$

□

Proposition 3.2.3. *If (E, h) is a Hermite–Einstein vector bundle over X with proportionality factor $\lambda < 0$, then E has no non-trivial global holomorphic sections. If $\lambda = 0$, then $\nabla\xi = 0$ for every $\xi \in H^0(X, E)$.*

Proof. Let ξ be a global holomorphic section. Then $\nabla\xi$ is of type $(1, 0)$ and locally

$$\nabla\xi = \sum \eta_j \otimes dz^j.$$

Now

$$\begin{aligned}i\Lambda h(\nabla\xi, \nabla\xi) &= i\Lambda \sum_{j,k} h(\eta_j, \eta_k) dz^j \wedge d\bar{z}^k \\ &= \sum_{j,k} h(\eta_j, \eta_k) g^{jk},\end{aligned}$$

and for each point $p \in X$ there are local coordinates with $g^{jk}(p) = \delta^{jk}$. It follows that

$$i\Lambda h(\nabla\xi, \nabla\xi) \geq 0,$$

and the equality holds when $\nabla\xi = 0$. Lemma 3.2.2 gives

$$\Delta h(\xi, \xi) = -i\Lambda h(F_\nabla \xi, \xi) + i\Lambda|\nabla\xi|^2,$$

and since $F_\nabla \in \mathcal{A}^2(X, \mathrm{End}(E))$ we can write it locally on a framed open set U as $F_\nabla = \sum \Omega^i_j \otimes \varepsilon^j \otimes e_i$. where $\Omega^i_j = R^i_{j\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta$ denote the curvature 2-forms. Writing $\xi = \xi^k e_k$ we obtain $F_\nabla \xi = \sum \Omega^i_j \xi^j \otimes e_i$ and so

$$\begin{aligned}h(F_\nabla \xi, \xi) &= h(\Omega^i_j \xi^j \otimes e_i, \xi^k e_k) \\ &= \Omega^i_j \xi^j \bar{\xi}^k h(e_i, e_k) \\ &= h_{ik} \Omega^i_j \xi^j \bar{\xi}^k \\ &= h_{ik} R^i_{j\alpha\beta} \xi^j \bar{\xi}^k dz^\alpha \wedge d\bar{z}^\beta.\end{aligned}$$

We obtain

$$\begin{aligned}\Lambda h(F_\nabla \xi, \xi) &= -i h_{ik} g^{\alpha\beta} R^i_{j\alpha\beta} \xi^j \bar{\xi}^k \\ &= -i g^{\alpha\beta} R^i_{j\alpha\beta} h(\xi, \xi),\end{aligned}$$

which yields $-i\Lambda h(F_\nabla \xi, \xi) = -g^{\alpha\beta} R^i_{j\alpha\beta} h(\xi, \xi)$ and the Hermite–Einstein condition gives

$$-i\Lambda h(F_\nabla \xi, \xi) = -\lambda h(\xi, \xi).$$

Thus

$$\Delta h(\xi, \xi) = -\lambda h(\xi, \xi) + i\Lambda |\nabla \xi|^2.$$

When $\lambda < 0$ we have $\Delta h(\xi, \xi) \geq 0$ and Bochner's lemma implies that $\Delta h(\xi, \xi) = 0$ and $h(\xi, \xi)$ is constant. Since both of the terms $-\lambda h(\xi, \xi)$ and $i\Lambda |\nabla \xi|^2$ are non-negative, they both vanish. In particular $\nabla \xi = 0$ and

$$h(\xi, \xi) = 0,$$

i.e. $\xi = 0$. □

Theorem 3.2.4 (Kobayashi). *Let (E, h) be a Hermite–Einstein vector bundle over X with proportionality factor λ. For the bundle $E^{\otimes r} \otimes E^{*\otimes s}$ the following holds:*

(i) If $\lambda < 0$, then $H^0(X, E^{\otimes r} \otimes E^{*\otimes s}) = 0$ for $r > s$, and for every

$$\xi \in H^0(X, E^{\otimes r} \otimes E^{*\otimes r})$$

we have that $\nabla \xi = 0$.

(ii) If $\lambda = 0$, then for any $\xi \in H^0(X, E^{\otimes r} \otimes E^{*\otimes s})$, $\nabla \xi = 0$ holds for all r and s.

(iii) If $\lambda > 0$, then $H^0(X, E^{\otimes r} \otimes E^{*\otimes s}) = 0$ for $r < s$, and for every

$$\xi \in H^0(X, E^{\otimes r} \otimes E^{*\otimes r})$$

we have that $\nabla \xi = 0$.

Proof. Recall that the tensor power $E^{\otimes r}$ of a Hermite–Einstein bundle E is Hermite–Einstein with proportionality factor $r\lambda$ and the dual bundle $E^{*\otimes s}$ is Hermite–Einstein with proportionality factor $-s\lambda$. It follows that $E^{\otimes r} \otimes E^{*\otimes s}$ is Hermite–Einstein with proportionality factor $(r - s)\lambda$. All of the statements now follow from Proposition 3.2.3 by analyzing the sign of $(r - s)\lambda$. □

Corollary 3.2.5. *If (E, h) is a Hermite–Einstein vector bundle over X with proportionality factor $\lambda = 0$ and $\xi \in H^0(X, E)$ has a zero, then $\xi = 0$.*

Proof. Let $\xi \in H^0(X, E)$, then the above theorem states that $\nabla\xi = 0$ and so

$$dh(\xi, \xi) = h(\nabla\xi, \xi) + h(\xi, \nabla\xi) = 0,$$

hence $h(\xi, \xi)$ is constant. If there exist $p \in X$ with $\xi(p) = 0$, then

$$h(\xi, \xi) = h(\xi(p), \xi(p)) = 0,$$

and so $\xi = 0$. $\square$

3.3 Chern Classes of Hermite–Einstein Vector Bundles

Throughout this section, let (E, h) denote a Hermitian vector bundle of rank r over a compact Hermitian manifold (X, g), where ω denotes the associated fundamental form of the metric g. Most of the considerations here follow Kobayashi's treatment [10] and Lübke's proof [11] for an inequality involving the first and second Chern classes.

We begin with a useful formula for the wedge product of a real $(1, 1)$-form with powers of the fundamental form. Let $(\varepsilon^1, \ldots, \varepsilon^n)$ be an unitary coframe on X such that

$$\omega = i \sum \varepsilon^\alpha \wedge \bar{\varepsilon}^\alpha.$$

If $\eta = i \sum \eta_{\alpha\bar{\beta}} \varepsilon^\alpha \wedge \bar{\varepsilon}^\beta$ is a real $(1, 1)$-form, then the wedge product $\eta \wedge \omega^{n-1}$ is given by

$$\eta \wedge \omega^{n-1} = \frac{1}{n} \left(\sum \eta_{\alpha\bar{\alpha}}\right) \omega^n.$$

The next lemma provides another formula that will be used in subsequent computations involving Chern classes.

Lemma 3.3.1. *Let $(\varepsilon^1, \ldots, \varepsilon^n)$ be an unitary coframe on the Hermitian manifold X. For any indices $\alpha, \beta, \gamma, \delta \in \{1, \ldots, n\}$, the following identity holds:*

$$n(n-1)\, \varepsilon^\alpha \wedge \bar{\varepsilon}^\beta \wedge \varepsilon^\gamma \wedge \bar{\varepsilon}^\delta \wedge \omega^{n-2} = \begin{cases} -\omega^n, & \text{if } \alpha = \beta \text{ and } \gamma = \delta,\ \alpha \neq \gamma, \\ \omega^n, & \text{if } \alpha = \delta \text{ and } \beta = \gamma,\ \alpha \neq \beta, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The fundamental form ω is given by

$$\omega = i \sum_{\lambda=1}^{n} \varepsilon^\lambda \wedge \bar{\varepsilon}^\lambda.$$

Raising ω to the $(n-2)$-th power, we obtain

$$\omega^{n-2} = i^{n-2} \sum_I \varepsilon^I \wedge \bar{\varepsilon}^I,$$

where the sum is over all multi-indices $I = (\lambda_1, \dots, \lambda_{n-2})$ with $1 \leq \lambda_1 < \cdots < \lambda_{n-2} \leq n$, and

$$\varepsilon^I \wedge \bar{\varepsilon}^I = \varepsilon^{\lambda_1} \wedge \bar{\varepsilon}^{\lambda_1} \wedge \cdots \wedge \varepsilon^{\lambda_{n-2}} \wedge \bar{\varepsilon}^{\lambda_{n-2}}.$$

Now consider the wedge product

$$\varepsilon^\alpha \wedge \bar{\varepsilon}^\beta \wedge \varepsilon^\gamma \wedge \bar{\varepsilon}^\delta \wedge \omega^{n-2}.$$

Expanding ω^{n-2}, the term $\varepsilon^\alpha \wedge \bar{\varepsilon}^\beta \wedge \varepsilon^\gamma \wedge \bar{\varepsilon}^\delta$ survives only if it pairs with terms from ω^{n-2} such that all n ε-type forms and $\bar{\varepsilon}$-type forms are accounted for exactly once. This follows from the antisymmetry of wedge products: repetition of any ε^λ or $\bar{\varepsilon}^\lambda$ results in the term vanishing. To compute explicitly, we analyze the different cases. First, if $\alpha = \beta$ and $\gamma = \delta$ with $\alpha \neq \gamma$, we obtain $\varepsilon^\alpha \wedge \bar{\varepsilon}^\beta = \varepsilon^\alpha \wedge \bar{\varepsilon}^\alpha$ and $\varepsilon^\gamma \wedge \bar{\varepsilon}^\delta = \varepsilon^\gamma \wedge \bar{\varepsilon}^\gamma$. These contribute negatively because the wedge product introduces a (-1)-sign when permuting $(\varepsilon^\gamma, \bar{\varepsilon}^\gamma)$ to align with the order in ω^{n-2}. Thus, the term contributes $-\omega^n$. For the second case we have $\alpha = \delta$ and $\beta = \gamma$ with $\alpha \neq \beta$. Here, $\varepsilon^\alpha \wedge \bar{\varepsilon}^\beta$ and $\varepsilon^\beta \wedge \bar{\varepsilon}^\alpha$ naturally align in ω^{n-2} without any sign issues. The contribution is positive ω^n. Finally, if $(\alpha, \beta, \gamma, \delta)$ do not satisfy the above cases, there are repeated indices or mismatched pairings, which result in the entire wedge product vanishing due to antisymmetry or incomplete pairing in ω^{n-2}. $\square$

Using Lemma 3.3.1, we obtain the following expressions for the first two Chern classes when we wedge them with the fundamental form.

Proposition 3.3.2. *Let (E, h) be a Hermitian vector bundle of rank r over a compact Hermitian manifold (X, g) of dimension n, with associated fundamental form ω. Then the following identities hold:*

$$\begin{aligned} c_1(E, \nabla)^2 \wedge \omega^{n-2} &= \frac{1}{4\pi^2 n(n-1)} \sum_{\alpha,\gamma} R_{\alpha\bar{\alpha}} R_{\gamma\bar{\gamma}} - R_{\alpha\bar{\gamma}} R_{\gamma\bar{\alpha}} \omega^n, \\ c_2(E, \nabla) \wedge \omega^{n-2} &= \frac{1}{8\pi^2 n(n-1)} \Big(\sum R_{\alpha\bar{\alpha}} R_{\gamma\bar{\gamma}} - R_{\alpha\bar{\gamma}} R_{\gamma\bar{\alpha}} \\ &\quad - \sum R_{ij\alpha\bar{\alpha}} R_{ji\gamma\bar{\gamma}} + \sum R_{ij\alpha\bar{\gamma}} R_{ji\gamma\bar{\alpha}} \Big) \omega^n. \end{aligned}$$

Proof. For the first identity, recall that

$$c_1(E, \nabla) = \frac{i}{2\pi} \sum \Omega_j^j = \frac{i}{2\pi} \sum R^j_{j\alpha\bar{\beta}} \varepsilon^\alpha \wedge \bar{\varepsilon}^\beta.$$

Therefore

$$c_1(E, \nabla)^2 = -\frac{1}{4\pi^2} \sum \Omega_j^j \wedge \Omega_k^k = -\frac{1}{4\pi^2} \sum R^j_{j\alpha\bar{\beta}} R^k_{k\gamma\bar{\delta}} \varepsilon^\alpha \wedge \bar{\varepsilon}^\beta \wedge \varepsilon^\gamma \wedge \bar{\varepsilon}^\delta.$$

Consider now

$$n(n-1)\sum \Omega^j_j \wedge \Omega^k_k \wedge \omega^{n-2} = n(n-1)\sum R^j_{j\alpha\bar\beta} R^k_{k\gamma\bar\delta} \varepsilon^\alpha \wedge \bar\varepsilon^\beta \wedge \varepsilon^\gamma \wedge \bar\varepsilon^\delta \wedge \omega^{n-2}.$$

By Lemma 3.3.1, the right-hand side simplifies to

$$-\sum (R_{\alpha\bar\alpha} R_{\gamma\bar\gamma} - R_{\alpha\bar\gamma} R_{\gamma\bar\alpha})\omega^n$$

and so

$$c_1(E,\nabla)^2 \wedge \omega^{n-2} = \frac{1}{4\pi^2 n(n-1)} \sum (R_{\alpha\bar\alpha} R_{\gamma\bar\gamma} - R_{\alpha\bar\gamma} R_{\gamma\bar\alpha})\, \omega^n.$$

We proceed similarly for the latter identity. Recall that

$$\begin{aligned} c_2(E,\nabla) &= -\frac{1}{8\pi} \sum (\Omega^j_j \wedge \Omega^k_k - \Omega^j_k \wedge \Omega^k_j) \\ &= -\frac{1}{8\pi} \left(\sum R^j_{j\alpha\bar\beta} R^k_{k\gamma\bar\delta} - R^j_{k\alpha\bar\beta} R^k_{j\gamma\bar\delta} \right) \varepsilon^\alpha \wedge \bar\varepsilon^\beta \wedge \varepsilon^\gamma \wedge \bar\varepsilon^\delta. \end{aligned}$$

Considering

$$n(n-1)\sum (\Omega^j_j \wedge \Omega^k_k - \Omega^j_k \wedge \Omega^k_j) \wedge \omega^{n-2} = n(n-1) \left(\sum R^j_{j\alpha\bar\beta} R^k_{k\gamma\bar\delta} - R^j_{k\alpha\bar\beta} R^k_{j\gamma\bar\delta} \right) \varepsilon^\alpha \wedge \bar\varepsilon^\beta \wedge \varepsilon^\gamma \wedge \bar\varepsilon^\delta \wedge \omega^{n-2}$$

and using Lemma 3.3.1 again, we obtain

$$n(n-1)\sum R^j_{j\alpha\bar\beta} R^k_{k\gamma\bar\delta} \varepsilon^\alpha \wedge \bar\varepsilon^\beta \wedge \varepsilon^\gamma \wedge \bar\varepsilon^\delta \wedge \omega^{n-2} = -\sum (R_{\alpha\bar\alpha} R_{\gamma\bar\gamma} - R_{\alpha\bar\gamma} R_{\gamma\bar\alpha})\omega^n$$

as before and for the latter term

$$-n(n-1)\sum R^j_{k\alpha\bar\beta} R^k_{j\gamma\bar\delta} \varepsilon^\alpha \wedge \bar\varepsilon^\beta \wedge \varepsilon^\gamma \wedge \bar\varepsilon^\delta \wedge \omega^{n-2},$$

observe that the indices j and k are being summed over. By relabeling the indices appropriately and applying the symmetry properties of the curvature tensor, we can rewrite the sum as:

$$\sum R^j_{k\alpha\bar\beta} R^k_{j\gamma\bar\delta} = \sum R_{jk\alpha\bar\beta} R_{kj\gamma\bar\delta}.$$

Applying Lemma 3.3.1 once more, the second term becomes:

$$-n(n-1)\sum R^j_{k\alpha\bar\beta} R^k_{j\gamma\bar\delta} \varepsilon^\alpha \wedge \bar\varepsilon^\beta \wedge \varepsilon^\gamma \wedge \bar\varepsilon^\delta \wedge \omega^{n-2} = -\sum (R_{jk\alpha\bar\alpha} R_{kj\gamma\bar\gamma} - R_{jk\alpha\bar\gamma} R_{kj\gamma\bar\alpha})\, \omega^n.$$

Combining both contributions, we have:

$$n(n-1)\sum (\Omega^j_j \wedge \Omega^k_k - \Omega^j_k \wedge \Omega^k_j) \wedge \omega^{n-2} =$$

$$-\sum (R_{\alpha\bar\alpha} R_{\gamma\bar\gamma} - R_{\alpha\bar\gamma} R_{\gamma\bar\alpha})\, \omega^n - \sum (R_{jk\alpha\bar\alpha} R_{kj\gamma\bar\gamma} - R_{jk\alpha\bar\gamma} R_{kj\gamma\bar\alpha})\, \omega^n.$$

Then, divide by $n(n-1)$ to get:

$$(\Omega^j_j \wedge \Omega^k_k - \Omega^j_k \wedge \Omega^k_j) \wedge \omega^{n-2} =$$

$$\frac{1}{n(n-1)}\Big(- \sum R_{\alpha\bar{\alpha}} R_{\gamma\bar{\gamma}} - R_{\alpha\bar{\gamma}} R_{\gamma\bar{\alpha}} - \sum R_{jk\alpha\bar{\alpha}} R_{kj\gamma\bar{\gamma}} - R_{jk\alpha\bar{\gamma}} R_{kj\gamma\bar{\alpha}} \Big)\omega^n.$$

Substituting this into $c_2(E, \nabla)$ we finally obtain:

$$c_2(E,\nabla) \wedge \omega^{n-2} = -\frac{1}{8\pi^2 n(n-1)}\Big(\sum R_{\alpha\bar{\alpha}} R_{\gamma\bar{\gamma}} - R_{\alpha\bar{\gamma}} R_{\gamma\bar{\alpha}}$$

$$+ \sum R_{jk\alpha\bar{\alpha}} R_{kj\gamma\bar{\gamma}} - R_{jk\alpha\bar{\gamma}} R_{kj\gamma\bar{\alpha}}\Big)\omega^n.$$

□

Proposition 3.3.3. *Let (E, h) be a Hermitian vector bundle of rank r over a compact Hermitian manifold (X, g). Then the following inequality holds:*

$$r \sum R_{i\bar{j}\alpha\bar{\gamma}} R_{j\bar{i}\gamma\bar{\alpha}} - \sum_{\alpha,\gamma} R_{\alpha\bar{\gamma}} R_{\gamma\bar{\alpha}} \geq 0.$$

Equality is achieved if and only if

$$r R^i_{j\alpha\bar{\beta}} = \delta^i_j R_{\alpha\bar{\beta}},$$

Proof. To establish the inequality, we consider the trace-free part of the curvature tensor. Define

$$T^i_{j\alpha\bar{\beta}} = R^i_{j\alpha\bar{\beta}} - \frac{1}{r}\delta^i_j R_{\alpha\bar{\beta}},$$

where $R_{\alpha\bar{\beta}} = \sum_j R^j_{j\alpha\bar{\beta}}$ is the Ricci curvature. Since $T^i_{j\alpha\bar{\beta}}$ is the trace-free part, its squared sum satisfies

$$0 \leq \sum T^i_{j\alpha\bar{\beta}} T^i_{j\alpha\bar{\beta}}.$$

Expanding this expression, we have

$$\begin{aligned} 0 &\leq \sum \left(R^i_{j\alpha\bar{\beta}} - \frac{1}{r}\delta^i_j R_{\alpha\bar{\beta}}\right)\left(R^i_{j\alpha\bar{\beta}} - \frac{1}{r}\delta^i_j R_{\alpha\bar{\beta}}\right) \\ &= \sum R^i_{j\alpha\bar{\beta}} R^i_{j\alpha\bar{\beta}} - \frac{2}{r}\sum \delta^i_j R^i_{j\alpha\bar{\beta}} R_{\alpha\bar{\beta}} + \frac{1}{r^2}\sum \delta^i_j \delta^i_j R_{\alpha\bar{\beta}} R_{\alpha\bar{\beta}}. \end{aligned}$$

Since $R^j_{j\alpha\bar{\beta}} = R_{\alpha\bar{\beta}}$, this simplifies to:

$$0 \leq \sum R^i_{j\alpha\bar{\beta}} R^i_{j\alpha\bar{\beta}} - \frac{2}{r}\sum R_{\alpha\bar{\beta}} R_{\alpha\bar{\beta}} + \frac{1}{r^2}\sum R_{\alpha\bar{\beta}} R_{\alpha\bar{\beta}}.$$

Reorganizing terms, we find:

$$\sum R^i_{j\alpha\bar{\beta}} R^i_{j\alpha\bar{\beta}} \geq \frac{1}{r} \sum R_{\alpha\bar{\beta}} R_{\alpha\bar{\beta}}.$$

Using the symmetry of the curvature tensor, the first term can be rewritten as:

$$\sum R_{i\bar{j}\alpha\bar{\gamma}} R_{j\bar{i}\gamma\bar{\alpha}}.$$

Similarly, the Ricci curvature term simplifies to:

$$\sum R_{\alpha\bar{\gamma}} R_{\gamma\bar{\alpha}}.$$

Thus, we obtain:

$$r \sum R_{i\bar{j}\alpha\bar{\gamma}} R_{j\bar{i}\gamma\bar{\alpha}} - \sum R_{\alpha\bar{\gamma}} R_{\gamma\bar{\alpha}} \geq 0,$$

as required. Equality holds if and only if $T^i_{j\alpha\bar{\beta}} = 0$, which implies:

$$R^i_{j\alpha\bar{\beta}} - \frac{1}{r}\delta^i_j R_{\alpha\bar{\beta}} = 0.$$

This gives:

$$r R^i_{j\alpha\bar{\beta}} = \delta^i_j R_{\alpha\bar{\beta}},$$

which completes the proof. □

We are now ready to prove the following theorem

Theorem 3.3.4 (Lübke)**.** *Let (E, h) be an Hermitian vector bundle of rank r over a compact Hermitian manifold (X, g) of dimension n with fundamental form ω. If (E, h) satisfies the weak Einstein condition, then*

$$\int_X \big((r-1)c_1(E,h)^2 - 2rc_2(E,h)\big) \wedge \omega^{n-2} \leq 0,$$

and the equality holds if and only if (E, h) is projectively flat.

Proof. Proposition 3.3.2 yields the following formulae:

$$\begin{aligned} c_1(E,\nabla)^2 \wedge \omega^{n-2} &= \frac{1}{4\pi^2 n(n-1)} \sum_{\alpha,\gamma} R_{\alpha\bar{\alpha}} R_{\gamma\bar{\gamma}} - R_{\alpha\bar{\gamma}} R_{\gamma\bar{\alpha}} \omega^n, \\ c_2(E,\nabla) \wedge \omega^{n-2} &= \frac{1}{8\pi^2 n(n-1)} \Big(\sum R_{\alpha\bar{\alpha}} R_{\gamma\bar{\gamma}} - R_{\alpha\bar{\gamma}} R_{\gamma\bar{\alpha}} \\ &\quad - \sum R_{ij\alpha\bar{\alpha}} R_{ji\gamma\bar{\gamma}} + \sum R_{ij\alpha\bar{\gamma}} R_{ji\gamma\bar{\alpha}} \Big) \omega^n. \end{aligned}$$

Now consider

$$\big((r-1)c_1(E,h)^2 - 2r\, c_2(E,h)\big).$$

Substituting the above expressions and wedging with ω^{n-2}, we obtain

$$\big((r-1)c_1(E,h)^2 - 2rc_2(E,h)\big)\wedge\omega^{n-2} = \frac{1}{4\pi^2 n(n-1)}\left(\sum_{\alpha,\gamma} R_{\alpha\bar{\gamma}}R_{\gamma\bar{\alpha}} - r\sum R_{i\bar{j}\alpha\bar{\gamma}}R_{j\bar{i}\gamma\bar{\alpha}}\right)\omega^n.$$

Proposition 3.3.3 asserts now that

$$\sum_{\alpha,\gamma} R_{\alpha\bar{\gamma}}R_{\gamma\bar{\alpha}} - r\sum R_{i\bar{j}\alpha\bar{\gamma}}R_{j\bar{i}\gamma\bar{\alpha}} \leq 0$$

which implies

$$\big((r-1)c_1(E,h)^2 - 2r\, c_2(E,h)\big)\wedge\omega^{n-2} \leq 0.$$

Finally, integrating over X yields the desired inequality:

$$\int_X \big((r-1)c_1(E,h)^2 - 2r\, c_2(E,h)\big)\wedge\omega^{n-2} \leq 0,$$

and the equality holds if and only if (E,h) is projectively flat. $\square$

Chapter 4

Stability

The motivation for slope stability has its origins in the work of Mumford in geometric invariant theory. We will follow closely the treatment of Friedman [8] and Okonek, Schneider and Spindler [12].

4.1 Slope Stability

We will begin this section by defining the degree.

Definition 4.1.1. Let E be holomorphic vector bundle over a compact Kähler manifold (X, ω). The **degree** of E is defined to be the integer

$$\deg(E) := \int_X c_1(E) \wedge \omega^{n-1},$$

where $c_1(E)$ is the first Chern class of E.

Definition 4.1.2. Let E be holomorphic vector bundle over a compact Kähler manifold (X, ω). The **slope** of E is defined to be the rational number

$$\mu(E) := \frac{\deg(E)}{\mathrm{rk}(E)}.$$

Definition 4.1.3. A holomorphic vector bundle E over a compact Kähler manifold (X, ω) is called **slope stable** (resp. **slope semistable**) if for all proper, non-zero coherent subsheaves $\mathcal{F} \subset E$ with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(E)$, the following inequality is satisfied:

$$\mu(\mathcal{F}) < \mu(E) \quad (\text{resp. } \leq).$$

Remark 4.1.4. An alternative definition for the slope without the Kähler condition would be to ask X to admit a polarization $L \to X$ and define $\deg(E) = \langle c_1(E) \cdot c_1(E)^{n-1}, [X] \rangle$.

In Definition 4.1.11 it is important to note that stability *must* be checked on coherent subsheaves and not only on subbundles. To illustrate why this is, consider the complex projective plane $\mathbb{P}^2$ and a point $p \in \mathbb{P}^2$. Then $\mathrm{Ext}^1(\mathcal{I}_p, \mathcal{O}_{\mathbb{P}^2}) \cong \mathbb{C}$, where $\mathcal{I}_p$ is the ideal sheaf and so there is a non-trivial extension

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow E \longrightarrow \mathcal{I}_p \longrightarrow 0.$$

As $H^0(\mathbb{P}^2, K_{\mathbb{P}^2}) = H^0(\mathbb{P}^2, \mathcal{O}(-3)) = 0$, the Cayley–Bacharach property is satisfied [9, Thm. 5.1.1] and so E is a vector bundle. Now the first Chern class is additive over short exact sequences so

$$c_1(E) = c_1(\mathcal{O}_{\mathbb{P}^2}) + c_1(\mathcal{I}_p) = 0,$$

which yields that $\mu(E) = 0$. However, $\mathcal{O}_{\mathbb{P}^2}$ is a subsheaf of E and $\mu(\mathcal{O}_{\mathbb{P}^2}) = 0$ so $\mu(\mathcal{O}_{\mathbb{P}^2}) \not< \mu(E)$. That is, E is not stable. On the other hand, as $\mathcal{O}_{\mathbb{P}^2}$ and $\mathcal{I}_p$ are stable of the same slope, E is semistable so any destabilizing subbundle would first and foremost be a line bundle and have slope zero. If $L \subset E$ is such a line bundle, then $L \cong \mathcal{O}_{\mathbb{P}^2}$ (The Picard variety of $\mathbb{P}^2$ is isomorphic to $\mathbb{Z}$). Note now that

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}, E) \cong \mathbb{C}.$$

Therefore, any morphism $\mathcal{O}_{\mathbb{P}^2} \to E$ is, up to scalar, the composition with the inclusion $\mathcal{O}_{\mathbb{P}^2} \hookrightarrow E$ from the short exact sequence. In particular, the image of $\mathcal{O}_{\mathbb{P}^2} \to E$ is precisely this subsheaf, which is not a subbundle at the point p, as the stalk of $\mathcal{I}_p$ fails to be locally free at p. Hence, although $\mathcal{O}_{\mathbb{P}^2}$ injects into E as a subsheaf, it is not a subbundle.

Note that for the determinant line bundle $\det(E)$, we have

$$c_1(E) = c_1(\det(E)),$$

and so $\deg(E) = \deg(\det(E))$. This observation allows us to generalize the notion of the degree to arbitary coherent sheaves.

Definition 4.1.5. A coherent sheaf $\mathcal{F}$ on a complex manifold X is called **reflexive** if $\mathcal{F} \cong \mathcal{F}^{**}$.

The kernel of the natural map $\mathcal{F} \to \mathcal{F}^{**}$ is exactly the torsion subsheaf of $\mathcal{F}$, and as the induced map $\mathcal{F}_x \to \mathcal{F}_x^{**}$ on stalks for any $x \in X$ is an isomorphism, it has a trivial kernel which in turn implies that $\mathcal{F}$ is torsion-free. Moreover, due to a result by Scheja [13], the singular set of a reflexive sheaf $\mathcal{F}$ has codimension at least three.

Definition 4.1.6. Let $\mathcal{F}$ be a coherent sheaf over a complex manifold X. The **determinant bundle** of $\mathcal{F}$ is defined by

$$\det(\mathcal{F}) := \bigwedge^r \mathcal{F}^{**}.$$

Due to [12], reflexive sheaves of rank 1 are line bundles. Since the dual $\bigwedge^r \mathcal{F}^*$ is reflexive of rank 1, $\det(\mathcal{F})$ is a line bundle and so we define the degree of $\mathcal{F}$ to be the degree of $\det(\mathcal{F})$.

Remark 4.1.7. As $E \cong E^{**}$ holds for vector bundles, the generalized definition does not affect Definition 4.1.1. In fact, this is the reason we are considering the double dual instead of the dual only.

The slope $\mu(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ over a complex manifold is then given analogously by

$$\mu(\mathcal{F}) = \frac{\det(\mathcal{F})}{\mathrm{rk}(\mathcal{F})}.$$

Definition 4.1.8. A torsion-free coherent sheaf $\mathcal{E}$ over a compact Kähler manifold (X, ω) is calle **slope stable** (resp. **slope semistable**) if for all proper, non-zero coherent subsheaves $\mathcal{F} \subset \mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{F})$, the following inequality is satisfied:

$$\mu(\mathcal{F}) < \mu(\mathcal{E}) \quad (\text{resp. } \leq).$$

Proposition 4.1.9. *If*

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0,$$

is an exact sequence of coherent sheaves over a compact Kähler manifold (X, ω)*, then*

$$\mu(\mathcal{F}) = \frac{\mathrm{rk}(\mathcal{E})}{\mathrm{rk}(\mathcal{F})}\mu(\mathcal{E}) + \frac{\mathrm{rk}(\mathcal{G})}{\mathrm{rk}(\mathcal{F})}\mu(\mathcal{G}).$$

Proof. Recall that the first Chern class is additive over short exact sequences. That is

$$c_1(\mathcal{F}) = c_1(\mathcal{E}) + c_1(\mathcal{G}),$$

and similarly for the rank. These give

$$\begin{aligned}(\mathrm{rk}(\mathcal{E}) + \mathrm{rk}(\mathcal{G}))\mu(\mathcal{F}) &= \deg(\mathcal{F}) \\ &= \deg(\mathcal{E}) + \deg(\mathcal{G}) \\ &= \mathrm{rk}(\mathcal{E})\mu(\mathcal{E}) + \mathrm{rk}(\mathcal{F})\mu(\mathcal{F}),\end{aligned}$$

which yields

$$\mu(\mathcal{F}) = \frac{\mathrm{rk}(\mathcal{E})}{\mathrm{rk}(\mathcal{F})}\mu(\mathcal{E}) + \frac{\mathrm{rk}(\mathcal{G})}{\mathrm{rk}(\mathcal{F})}\mu(\mathcal{G}).$$

□

Proposition 4.1.9 allows us to restate stability and semistability of $\mathcal{F}$ in terms of quotients instead of subsheaves.

Proposition 4.1.10. *Let $\mathcal{F}$ be a torsion-free coherent sheaf over a compact Kähler manifold (X, ω). Then $\mathcal{F}$ is stable (resp. semistable) if and only if*

$$\mu(\mathcal{F}) \leq \mu(\mathcal{Q}) \quad (\textit{resp. } \leq)$$

holds for any quotient sheaf $\mathcal{Q}$ with positive rank.

Proposition 4.1.11. *Let $\mathcal{E}$ and $\mathcal{F}$ be semistable coherent sheaves over a compact Kähler manifold (X, ω). Then:*

(i) If $\mu(\mathcal{E}) > \mu(\mathcal{F})$, we have $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) = 0$.

(ii) If $\mu(\mathcal{E}) = \mu(\mathcal{F})$ and if $\mathcal{E}$ is stable, we have

$$\mathrm{rk}(\mathcal{E}) = \mathrm{rk}(\mathrm{im}(f)),$$

and f is injective unless $f = 0$.

(iii) If $\mu(\mathcal{E}) = \mu(\mathcal{F})$ and if $\mathcal{F}$ is μ-stable, we have

$$\mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\mathrm{im}(f)),$$

and f is generically surjective unless $f = 0$.

Proof. (i) Suppose that f is non-zero and consider the sequence

$$\mathcal{E} \longrightarrow \mathrm{im}(f) \longrightarrow \mathcal{F}.$$

Since $\mathcal{E}$ and $\mathcal{F}$ are μ-semistable, we obtain a contradiction from

$$\mu(\mathcal{E}) \leq \mu(\mathrm{im}(f)) \leq \mu(\mathcal{F}) < \mu(\mathcal{E}).$$

(ii) Suppose that f is non-zero and that $\mathrm{rk}(\mathcal{E}) > \mathrm{rk}(\mathrm{im}(f))$. Then since $\mathcal{E}$ is stable, we obtain

$$\mu(\mathrm{im}(f)) \leq \mu(\mathcal{F}) = \mu(\mathcal{E}) < \mu(\mathrm{im}(f)),$$

which is impossible.

(iii) As $\mathcal{F}$ is stable, if $\mathrm{rk}(\mathcal{F}) > \mathrm{rk}(\mathrm{im}(f))$, then

$$\mu(\mathrm{im}(f)) < \mu(\mathcal{F}) = \mu(\mathcal{E}) \leq \mu(\mathrm{im}(f)),$$

which is again impossible.

□

Corollary 4.1.12. *Let E and F be semistable vector bundles over a compact Kähler manifold (X, ω) of the same rank and degree. Then if E or F is stable, any non-zero morphism $f : E \to F$ is an isomorphism.*

Proof. The morphism f is injective by Proposition 4.1.11. The induced morphism $\det(f) : \det(E) \to \det(F)$ is also non-zero. Consider $\det(f)$ as a holomorphic section of the line bundle $\mathrm{Hom}(\det(E), \det(F)) = \det(E)^* \otimes \det(F)$ and note that this has degree 0. Recall now that any holomorphic section of a degree 0 holomorphic line bundle over a compact Kähler manifold has no zeroes unless it vanishes identically. It follows that $\det(f)$ is an isomorphism and thus f is an isomorphism. □

Corollary 4.1.13. *If $\mathcal{F}$ is a semistable sheaf over a compact Kähler manifold (X, ω) such that $\deg(\mathcal{F}) < 0$, then $\mathcal{F}$ admits no non-zero holomorphic sections.*

Proof. Note that any non-zero holomorphic section $s \in H^0(X, \mathcal{F})$ yields a map $f : \mathcal{O}_X \to \mathcal{F}$. This gives maps

$$\mathcal{O}_X \longrightarrow \mathrm{im}(f) \longrightarrow \mathcal{F},$$

and $0 = \mu(\mathcal{O}_X) \leq \mu(\mathrm{im}(f)) \leq \mu(\mathcal{F}) < 0$ yielding a contradiction. □

Definition 4.1.14. A coherent sheaf $\mathcal{F}$ on a complex manifold X is called **simple** if

$$\mathrm{End}(\mathcal{F}) \cong \mathbb{C}.$$

Corollary 4.1.15. *If E is a stable vector bundle over a compact Kähler manifold (X, ω), then $\mathrm{End}(E) \cong \mathbb{C}$. That is, E is simple.*

Proof. Consider $f : E \to E$ and let λ be an eigenvalue of the map $f : E_x \to E_x$ on fibers for $x \in X$. Suppose that $f \neq \lambda \,\mathrm{id}_E$. Note that since E is stable, and $f - \lambda \,\mathrm{id}_E$ is a non-zero morphism from E to E, Corollary 4.1.12 implies that $f - \lambda \,\mathrm{id}_E$ is an isomorphism. However, an invertible linear map cannot have 0 eigenvalue so

$$f = \lambda \,\mathrm{id}_E \,.$$

□

Proposition 4.1.16. *Let $\mathcal{E}$ and $\mathcal{F}$ be torsion-free coherent sheaves over a compact Kähler manifold (X, ω). Then $\mathcal{E} \oplus \mathcal{F}$ is semistable if and only if $\mathcal{E}$ and $\mathcal{F}$ are both semistable with equal slopes.*

Proof. Suppose that $\mathcal{E}$ and $\mathcal{F}$ are semistable with slope λ. Then

$$\mu(\mathcal{E} \oplus \mathcal{F}) = \frac{\deg(\mathcal{E}) + \deg(\mathcal{F})}{\mathrm{rk}(\mathcal{E}) + \mathrm{rk}(\mathcal{F})} = \frac{\mathrm{rk}(\mathcal{E})\lambda + \mathrm{rk}(\mathcal{F})\lambda}{\mathrm{rk}(\mathcal{E}) + \mathrm{rk}(\mathcal{F})} = \lambda.$$

Let $\mathcal{G}$ be a subsheaf of $\mathcal{E} \oplus \mathcal{F}$, set $\mathcal{G}_1 = \mathcal{G} \cap (\mathcal{E} \oplus 0)$ and denote by $\mathcal{G}_2$ the image of $\mathcal{G}$ under $\mathcal{E} \oplus \mathcal{F} \to \mathcal{F}$. Since $\mathcal{E}$ and $\mathcal{F}$ are μ-semistable,

$$\deg(\mathcal{G}_i) \leq \lambda \,\mathrm{rk}(\mathcal{G}_i).$$

It follows that

$$\mu(\mathcal{G}) = \frac{\deg(\mathcal{G}_1) + \deg(\mathcal{G}_2)}{\mathrm{rk}(\mathcal{G}_1) + \mathrm{rk}(\mathcal{G}_2)} \leq \lambda = \mu(\mathcal{E} \oplus \mathcal{F}).$$

Conversely, if $\mathcal{E} \oplus \mathcal{F}$ is μ-semistable, If $\mathcal{G}$ is now a subsheaf of $\mathcal{E}$ or $\mathcal{F}$, it is a subsheaf of the direct sum $\mathcal{E} \oplus \mathcal{F}$ also. Now as $\mathcal{E}$ and $\mathcal{F}$ are both quotients and subsheaves of $\mathcal{E} \oplus \mathcal{F}$, we obtain equality for the slopes and

$$\mu(\mathcal{G}) \leq \mu(\mathcal{E} \oplus \mathcal{F}) = \mu(\mathcal{E}) = \mu(\mathcal{F}).$$

□

Proposition 4.1.17. *Let*

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

be a short exact sequence of coherent sheaves on (X, ω). Then the following equivalences hold:

$$\mu(\mathcal{E}) < \mu(\mathcal{F}) \iff \mu(\mathcal{F}) < \mu(\mathcal{G}),$$

and

$$\mu(\mathcal{E}) > \mu(\mathcal{F}) \iff \mu(\mathcal{F}) > \mu(\mathcal{G}).$$

4.2 The Harder–Narasimhan Filtration

In this section we establish the existence and uniqueness of the *Harder–Narasimhan filtration* for any torsion-free coherent sheaf on a compact Kähler manifold. This filtration decomposes an arbitrary sheaf into a tower of semistable quotients with strictly decreasing slopes. It provides a canonical way to measure and compare the degrees of instability of subsheaves, and underlies many deep results in the theory of moduli of sheaves and in the analysis of Hermite–Einstein metrics. We begin by showing that the slopes of all subsheaves are uniformly bounded above, which allows us to pick out a maximal slope component. Iterating this construction yields the full filtration, whose uniqueness follows from a simple slope-comparison argument.

Lemma 4.2.1 (Boundedness of slopes)**.** *There exists an integer m_0 (depending only on $\mathcal{F}$ and the Kähler form ω) such that*

$$\mu(\mathcal{G}) \leq m_0 \quad \textit{for every coherent subsheaf } \mathcal{G} \subset \mathcal{F}.$$

Proof. Recall that every coherent torsion-free sheaf is locally a subsheaf of a trivial bundle of the same rank so it suffices to bound the slopes of subsheaves of a holomorphic vector bundle E of rank r. Fix a Hermitian metric h on E. If $L \subset E$ is a holomorphic line subbundle, choose a local unitary frame $\{e_1, \ldots, e_r\}$ so that e_1 spans L. Writing the curvature of the Chern connection of L as

$$S_{\alpha\bar{\beta}} = R^1_{1\alpha\bar{\beta}} - \sum_{\lambda=2}^{r} A^\lambda_{1\alpha} \bar{A}^\lambda_{1\beta},$$

where A is the second fundamental form, we have

$$\sum S_{\alpha\bar{\alpha}} = \sum_{\alpha} R^1_{1\alpha\tilde{\alpha}} - \sum_{\alpha\cdot\lambda} \left|A^\lambda_{1\alpha}\right|^2 \leqq \sum_{\alpha} \left|R^1_{1\alpha\bar{x}}\right| \leqq \sum_{t,\alpha} \left|R^1_{1\alpha\tilde{\alpha}}\right| \leqq \left(nr \cdot \sum_{t,\alpha} \left|R^1_{t\alpha\bar{\alpha}}\right|^2\right)^{1/2}$$

which is independent of L. As

$$\deg(L) = \int_X \frac{i}{2\pi} \sum S_{\alpha\bar{\beta}} \varepsilon^\alpha \wedge \varepsilon^\beta \wedge \omega^{n-1} = \int_X \frac{1}{2n\pi} \left(\sum S_{\alpha\bar{\alpha}}\right) \omega^n,$$

we note that the degree is bounded below by a constant $q_0(E)$ only depending on E and ω. Considering the higher rank case, if $F \subset E$ has rank p, then $\det F \hookrightarrow \Lambda^p E$ is a line subbundle. Applying the above bound to $\Lambda^p E$ shows $\mu(F) = \frac{1}{p} \deg(\det F) \le \frac{1}{p}\, q_0(\Lambda^p E)$. Setting

$$m_0 = \max_{1 \le p \le r} \left\{ \tfrac{1}{p}\, q_0\left(\bigwedge^p E\right) \right\}$$

gives the desired uniform bound. $\square$

Lemma 4.2.2 (Maximal destabilising subsheaf)**.** *Let (X, ω) be a compact Kähler manifold and let $\mathcal{F}$ be a torsion-free coherent sheaf on X of positive rank. Then there exists a unique coherent subsheaf*

$$\mathcal{F}_1 \subset \mathcal{F},$$

often called the maximal destabilising subsheaf, which satisfies all of the following properties:

(i) The quotient $\mathcal{F}/\mathcal{F}_1$ is torsion-free.

(ii) For every coherent subsheaf $\mathcal{G} \subset \mathcal{F}$,

$$\mu(\mathcal{G}) \le \mu(\mathcal{F}_1).$$

(iii) If $\mathcal{G} \subset \mathcal{F}$ satisfies $\mu(\mathcal{G}) = \mu(\mathcal{F}_1)$, then

$$\operatorname{rk} \mathcal{G} \le \operatorname{rk} \mathcal{F}_1.$$

Consequently, $\mathcal{F}_1$ is semistable.

Proof. By Lemma 4.2.1 the set $\{\mu(\mathcal{G}) \mid \mathcal{G} \subset \mathcal{F}\}$ is bounded above, say by m_0. Choose a subsheaf

$$\mathcal{F}_1 \subset \mathcal{F} \quad \text{with} \quad \mu(\mathcal{F}_1) = \sup\{\mu(\mathcal{G})\},$$

and among those of slope $\mu(\mathcal{F}_1)$ pick one of maximal rank. Its saturation is still denoted $\mathcal{F}_1$, and after saturation, $\mathcal{F}/\mathcal{F}_1$ is torsion-free, so (i) holds by construction, and (ii) is built into the choice of maximal rank.

Suppose a proper subsheaf $\mathcal{H} \subset \mathcal{F}_1$ violates semistability. Then either $\mu(\mathcal{H}) > \mu(\mathcal{F}_1)$ or $\mu(\mathcal{H}) = \mu(\mathcal{F}_1)$ with $\mathrm{rk}(\mathcal{H}) > \mathrm{rk}(\mathcal{F}_1)$, both contradict property (ii) or the maximal rank property. Hence $\mathcal{F}_1$ is semistable.

Regarding uniqueness, assume another subsheaf $\mathcal{F}_1' \subset \mathcal{F}$ also satisfies (i) and (ii). If $\mathcal{F}_1 \subset \mathcal{F}_1'$ and $\mathcal{F}_1' \subset \mathcal{F}_1$ both failed, one of them is not contained in the other; without loss of generality, assume $\mathcal{F}_1 \not\subset \mathcal{F}_1'$. Let

$$\pi : \mathcal{F} \to \mathcal{F}/\mathcal{F}_1'$$

be the natural projection. Because $\mathcal{F}_1 \not\subset \mathcal{F}_1'$, its image $\pi(\mathcal{F}_1)$ is a nonzero subsheaf of $\mathcal{F}/\mathcal{F}_1'$. Consider the pullback $\pi^{-1}(\pi(\mathcal{F}_1))$; we have an exact sequence

$$0 \longrightarrow \mathcal{F}_1' \longrightarrow \pi^{-1}(\pi(\mathcal{F}_1)) \longrightarrow \pi(\mathcal{F}_1) \longrightarrow 0$$

and, inside $\mathcal{F}_1$,

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F}_1 \longrightarrow \pi(\mathcal{F}_1) \longrightarrow 0$$

where $\mathcal{G} := \ker(\mathcal{F}_1 \to \pi(\mathcal{F}_1))$. Because $\mathcal{F}_1$ is semistable, the quotient satisfies

$$\mu(\pi(\mathcal{F}_1)) \geq \mu(\mathcal{F}_1). \tag{1}$$

Next, apply (i) and (ii) to the subsheaf

$$\mathcal{K} := \pi^{-1}(\pi(\mathcal{F}_1)).$$

Since $\mathrm{rk}\,\mathcal{K} > \mathrm{rk}\,\mathcal{F}_1'$ (because $\pi(\mathcal{F}_1) \neq 0$), condition (ii) forces a strict inequality

$$\mu(\mathcal{K}) < \mu(\mathcal{F}_1').$$

Write $r' := \mathrm{rk}\,\mathcal{F}_1'$, $s := \mathrm{rk}\,\pi(\mathcal{F}_1)$, and degrees $\deg'$ and $\deg_\pi$ for $\mathcal{F}_1'$ and $\pi(\mathcal{F}_1)$,

$$\mu(\mathcal{K}) = \frac{\deg' + \deg_\pi}{r' + s} < \frac{\deg'}{r'} = \mu(\mathcal{F}_1').$$

Clearing denominators yields

$$\deg_\pi r' < \deg' s, \quad \text{i.e.} \quad \mu(\pi(\mathcal{F}_1)) < \mu(\mathcal{F}_1'). \tag{2}$$

Combining (1) and (2) gives

$$\mu(\mathcal{F}_1) \leq \mu(\pi(\mathcal{F}_1)) < \mu(\mathcal{F}_1') = \mu(\mathcal{F}_1),$$

a contradiction. Hence our assumption was impossible, so $\mathcal{F}_1 \subset \mathcal{F}_1'$. A symmetric argument shows $\mathcal{F}_1' \subset \mathcal{F}_1$, therefore $\mathcal{F}_1 = \mathcal{F}_1'$. □

Theorem 4.2.3 (Harder–Narasimhan)**.** *Let $\mathcal{F}$ be a torsion-free coherent sheaf over a compact Kähler manifold (X, ω). Then there exists a unique filtration by subsheaves*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{n-1} \subset \mathcal{F}_n = \mathcal{F},$$

such that, for $1 \leq i \leq n$, the quotients $\mathcal{F}_i/\mathcal{F}_{i-1}$ are semistable of slope μ_i satisfying

$$\mu_1 > \mu_2 > \cdots > \mu_n.$$

Proof. Set $\mathcal{F}_0 = 0$. Having constructed

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k$$

with each successive quotient $\mathcal{Q}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$ ω-semistable, we proceed as follows: The quotient $\mathcal{F}/\mathcal{F}_k$ is torsion-free, so by Lemma 4.2.2 it has a unique maximal destabilising subsheaf

$$\overline{\mathcal{F}}_{k+1} \subset \mathcal{F}/\mathcal{F}_k.$$

Pull this back under the natural projection $\pi \colon \mathcal{F} \to \mathcal{F}/\mathcal{F}_k$:

$$\mathcal{F}_{k+1} = \pi^{-1}(\overline{\mathcal{F}}_{k+1}) \subset \mathcal{F}.$$

By construction, $\mathcal{F}_k \subset \mathcal{F}_{k+1}$, the quotient $\mathcal{F}_{k+1}/\mathcal{F}_k \cong \overline{\mathcal{F}}_{k+1}$ is semistable of slope $\mu(\mathcal{F}_{k+1}/\mathcal{F}_k)$, and ω-saturation ensures $\mathcal{F}/\mathcal{F}_{k+1}$ is torsion-free.

Each step strictly decreases either the rank of the quotient $\mathcal{F}/\mathcal{F}_k$ or its slope, but rank cannot drop below zero, so after finitely many steps $\mathcal{F}_n = \mathcal{F}$. This yields the desired filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}.$$

By construction each $\mathcal{F}_i/\mathcal{F}_{i-1}$ is the maximal-slope subsheaf of $\mathcal{F}/\mathcal{F}_{i-1}$, so

$$\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) > \mu(\mathcal{F}_{i+1}/\mathcal{F}_i),$$

giving $\mu_1 > \mu_2 > \cdots > \mu_n$. Uniqueness of each successive $\mathcal{F}_i \subset \mathcal{F}/\mathcal{F}_{i-1}$ by Lemma 4.2.2 forces the entire filtration to be unique. This completes the proof of the Harder–Narasimhan filtration. □

4.3 Stability for Hermite–Einstein Vector Bundles

Our goal in this section is to prove that the Hermite–Einstein condition on an indecomposable holomorphic vector bundle (E, h) over a compact Kähler manifold (X, ω) is sufficient for E to be stable.

We will begin this section by introducing the second fundamental form and use it to present the fundamental principle that "curvature decreases in holomorphic subbundles".

First, we will recall how quotient bundles are defined.

Definition 4.3.1. Let $E \to X$ be a holomorphic vector bundle over a complex manifold X, and let $F \subset E$ be a holomorphic subbundle. Then the *quotient bundle* E/F is defined as the complex manifold obtained by taking the fiberwise quotient of E_x by the subspace F_x for each $x \in X$. More precisely, for each $x \in X$, consider the fiber E_x and its subspace F_x, and let $(E/F)_x := E_x/F_x$. The collection $\{(E/F)_x \mid x \in X\}$ forms a holomorphic vector bundle E/F over X whose dimension at each x is $\dim(E_x) - \dim(F_x)$.

Note that any Hermitian metric h on E restricts naturally to a Hermitian metric on the holomorphic subbundle F. Indeed, since $F_x \subset E_x$ is a subspace for each $x \in X$, and the metric h_x on E_x is a positive-definite inner product, its restriction to F_x remains positive-definite. Thus, $h_F := h|_F$ defines a Hermitian metric on F.

Similarly, given a Hermitian metric h on E, one would like to induce a Hermitian metric on the quotient bundle E/F. For this, one can choose a smooth splitting

$$E = F \oplus F^\perp,$$

where $F^\perp$ is the orthogonal complement of F with respect to h. Note that $F^\perp$ need not be a holomorphic subbundle in general. Despite this, $F^\perp$ provides a smooth isomorphism of complex vector bundles

$$E/F \cong F^\perp.$$

Via this identification, h induces a Hermitian metric on E/F. If, under additional conditions, $F^\perp$ happens to be a holomorphic subbundle (for example, if there is a holomorphic splitting), then E/F inherits a holomorphic structure compatible with this metric. Otherwise, E/F still inherits a Hermitian metric as a smooth complex vector bundle. This construction allows us to consider the Chern connections associated with these sub- and quotient bundles. If ∇ is the Chern connection associated to (E, h), then by restricting it to F we obtain a Chern connection on (F, h_F), and by projecting it onto $F^\perp$, we obtain a Chern connection on $(E/F, h_{E/F})$.

Lemma 4.3.2. *Let $(E, h) \to X$ be a holomorphic vector bundle equipped with a Hermitian metric h, and let $F \subset E$ be a holomorphic subbundle.*

Consider the direct sum decomposition induced by h:

$$E = F \oplus F^{\perp},$$

where $F^{\perp}$ is the h-orthogonal complement of F. Let ∇ be the Chern connection associated with (E, h). Define a connection ∇^F on F by

$$\nabla^F := \pi_F \circ \nabla,$$

where $\pi_F : E \to F$ is the projection onto F corresponding to the chosen decomposition. Then ∇^F is the Chern connection on (F, h_F).

Proof. Recall that the Chern connection ∇ on E is characterized by two properties:

(i) $(\nabla^{0,1}) = \bar{\partial}_E$, the $(0,1)$-component coincides with the holomorphic structure of E.

(ii) ∇ is compatible with the Hermitian metric h, i.e., $(\nabla h) = 0$.

Since F is a holomorphic subbundle, it inherits a holomorphic structure $\bar{\partial}_F$ defined by the restriction of $\bar{\partial}_E$. Let $s \in \Gamma(U, F)$ be a smooth local section of F (here $\Gamma(U, F)$ denotes the space of smooth sections on $U \subset X$ of the subbundle F). We have:

$$(\nabla^F)^{0,1} s = (\pi_F \circ \nabla)^{0,1} s = \pi_F(\nabla^{0,1} s) = \pi_F(\bar{\partial}_E s).$$

But since s is actually a section of F, we have $\bar{\partial}_E s \in \mathcal{A}^{0,1}(U, F)$, that is, the $(0,1)$-forms with values in E that come from the holomorphic structure will actually lie in F, because F is a holomorphic subbundle. Thus,

$$\bar{\partial}_E s = \bar{\partial}_F s.$$

Since $\bar{\partial}_F s \in \mathcal{A}^{0,1}(U, F)$, applying the projection π_F does nothing:

$$\pi_F(\bar{\partial}_E s) = \bar{\partial}_F s.$$

Therefore,

$$(\nabla^F)^{0,1} s = \bar{\partial}_F s.$$

This shows that the $(0,1)$-component of ∇^F coincides with the holomorphic structure of F. In other words, ∇^F is compatible with the complex structure on F. Next, we must show that ∇^F is compatible with the metric h_F. Since $h_F(s,t) = h(s,t)$ for all $s, t \in \Gamma(U, F)$, we need to verify that:

$$dh_F(s,t) = h_F\left(\nabla^F s, t\right) + h_F\left(s, \nabla^F t\right).$$

Since h_F is the restriction of h, we have $h_F(s,t) = h(s,t)$ for $s, t \in \Gamma(U, F)$. Thus:

$$dh_F(s,t) = dh(s,t).$$

Also, because ∇ is the Chern connection on E, it satisfies metric compatibility with h:

$$dh(s,t) = h(\nabla s, t) + h(s, \nabla t).$$

We obtain:

$$dh_F(s,t) = h(\nabla s, t) + h(s, \nabla t) = h_F\left(\nabla^F s, t\right) + h_F\left(s, \nabla^F t\right).$$

Thus, ∇^F is compatible with the metric. Together with the fact that $(\nabla^F)^{0,1} = \bar{\partial}_F$, we conclude that ∇^F is the unique Chern connection on (F, h_F). $\square$

Let $(E,h) \to X$ again be a holomorphic vector bundle over a complex manifold equipped with a Hermitian metric h. Let $S \subset E$ be a holomorphic subbundle and $Q = E/S$ the quotient bundle. Recall that Q is isomorphic (as a smooth vector bundle) to the orthogonal complement $S^\perp$ of S in E. We have the datum of three Hermitian bundles all of which therefore admit Chern connections ∇, ∇^S and ∇^Q. Consider now the short exact sequence

$$0 \longrightarrow S \longrightarrow E \longrightarrow Q \longrightarrow 0.$$

The Hermitian metric h on E defines a smooth splitting

$$E \cong S \oplus S^\perp,$$

and from Lemma 4.3.2, we know that

$$\nabla^S = \pi_S \circ \nabla.$$

Consider now a local section $s \in \Gamma(U,S) \subset \Gamma(U,E)$, then

$$\nabla s \in \Gamma(T^*X \otimes (S \oplus S^\perp)),$$

so we have a decomposition

$$\nabla s = \Phi(s) + A(s)$$

for $\Phi(s) \in \Gamma(U,S)$ and $A(s) \in \Gamma(U, S^\perp)$. Notice now that

$$\nabla^S s = \pi_S(\nabla s) = \pi_S(\Phi(s)) + \pi_S(A(s)) = \Phi(s)$$

so $\Phi = \nabla^S$. Moreover

$$A = \nabla - \nabla^S$$

maps $\Gamma(U,S)$ into $\Gamma(U,S^\perp)$, is a difference of two connections and is therefore C^∞-linear i.e. $A \in \mathcal{A}^1(U, \mathrm{Hom}(S, S^\perp))$. Finally, note that

$$\bar{\partial}_E(s) = \bar{\partial}_S(s) \in \mathcal{A}^{0,1}(U,S)$$

and thus

$$A^{0,1} = \nabla^{0,1} - (\nabla^S)^{0,1} = \bar{\partial}_E - \bar{\partial}_S = \bar{\partial}_S - \bar{\partial}_S = 0$$

so in fact, $A \in \mathcal{A}^{1,0}(U, \mathrm{Hom}(S, S^\perp))$. The operator A is called the *second fundamental form* of S in E.

We can now play the same game with the quotient Q, using the identification with $S^\perp$. We define operators $\Psi, B : \Gamma(U, S^\perp) \to \Gamma(U, T^*X \otimes S^\perp)$ by the decomposition

$$\nabla s = \Psi(s) + B(s),$$

for $s \in \Gamma(U, S^\perp)$. Ψ ends up as the connection ∇^Q, but B is this time in the space $\mathcal{A}^{0,1}(U, \mathrm{Hom}(S^\perp, S))$ and is the negative of the adjoint of A, i.e. $B = -A^*$. To see these, note that for $s \in \Gamma(U, S)$ and $t \in \Gamma(U, S^\perp)$ we have

$$\begin{aligned} 0 &= dh(s,t) \\ &= h(\nabla s, t) + h(s, \nabla t) \\ &= h\left(\nabla^S s + As, t\right) + h\left(s, \nabla^Q t + Bt\right) \\ &= h\left(\nabla^S s, t\right) + h\left(As, t\right) + h\left(s, \nabla^Q t\right) + h\left(s, Bt\right) \\ &= h(As, t) + h(s, Bt). \end{aligned}$$

So $h(As, t) = -h(s, Bt)$, i.e. $B = -A^* \in \mathcal{A}^{0,1}(U, \mathrm{Hom}(S^\perp, S))$.

Putting these together, the connection matrix ω_E for E takes the form

$$\omega_E = \begin{pmatrix} \omega_S & -A^* \\ A & \omega_Q \end{pmatrix}.$$

This gives us directly the curvature matrix from the second structural equation

$$\begin{aligned} \Omega_E &= d\omega_E + \omega_E \wedge \omega_E \\ &= \begin{pmatrix} d\omega_S & -dA^* \\ dA & d\omega_Q \end{pmatrix} + \begin{pmatrix} \omega_S & -A^* \\ A & \omega_Q \end{pmatrix} \wedge \begin{pmatrix} \omega_S & -A^* \\ A & \omega_Q \end{pmatrix} \\ &= \begin{pmatrix} d\omega_S & -dA^* \\ dA & d\omega_Q \end{pmatrix} + \begin{pmatrix} \omega_S \wedge \omega_S - A^* \wedge A & -\omega_S \wedge A^* - A^* \wedge \omega_Q \\ A \wedge \omega_S + \omega_Q \wedge A & -A \wedge A^* + \omega_Q \wedge \omega_Q \end{pmatrix} \\ &= \begin{pmatrix} d\omega_S + \omega_S \wedge \omega_S - A^* \wedge A & -dA^* - \omega_S \wedge A^* - A^* \wedge \omega_Q \\ dA + A \wedge \omega_S + \omega_Q \wedge A & d\omega_Q - A \wedge A^* + \omega_Q \wedge \omega_Q \end{pmatrix} \\ &= \begin{pmatrix} \Omega_S - A^* \wedge A & -dA^* - \omega_S \wedge A^* - A^* \wedge \omega_Q \\ dA + A \wedge \omega_S + \omega_Q \wedge A & \Omega_Q - A \wedge A^* \end{pmatrix}. \end{aligned}$$

This yields

$$\begin{aligned} \Omega_S &= \Omega_E|_S + A^* \wedge A, \\ \Omega_Q &= \Omega_E|_Q + A \wedge A^*. \end{aligned}$$

Definition 4.3.3 (Griffiths Positivity)**.** Let $(E, h) \to X$ be a holomorphic vector bundle over a complex manifold X, equipped with a Hermitian metric h. Let ∇ denote the Chern connection associated to (E, h), and let $\Omega \in$

$\mathcal{A}^{1,1}(\mathrm{End}(E))$ be its curvature tensor. The curvature operator Ω is said to be Griffiths positive if for any non-zero local holomorphic section s of E and $v \in T^{1,0}X$ one has

$$h(\Omega(s), s)(v, \bar{v}) > 0.$$

We write $\Omega > 0$ if Ω is positive and analogously in the other cases. Consider now the second fundamental form $A \in \mathcal{A}^{1,0}(\mathrm{Hom}(S, S^{\perp}))$. Since this is of type $(1,0)$ we have

$$A \wedge A^* \geq 0,$$

which yields

$$\Omega_S = \Omega_E|_S + A^* \wedge A \leq \Omega_E|_S,$$

i.e. the *curvature decreses in holomorphic subbundles.* Similarly

$$\Omega_Q = \Omega_E|_Q + A \wedge A^* \geq \Omega_E|_Q,$$

i.e. *curvature increases in holomorphic quotient bundles.* The main result we aim to prove is now a simple application of this principle.

Proposition 4.3.4. *Suppose $E \to (X, \omega)$ is an indecomposable holomorphic vector bundle over a compact Kähler manifold that admits a Hermite–Einstein metric h. Then E is slope stable with respect to holomorphic subbundles $S \subset E$.*

Proof. Let $S \subset E$ be a subbundle and consider the smooth direct sum decomposition $E \cong S \oplus Q$, where $Q = E/S$. Recall that in this case, the curvature Ω_E of E is given by

$$\Omega_E = \begin{pmatrix} \Omega_S - A^* \wedge A & -dA^* - \omega_S \wedge A^* - A^* \wedge \omega_Q \\ dA + A \wedge \omega_S + \omega_Q \wedge A & \Omega_Q - A \wedge A^* \end{pmatrix}.$$

The $\mathrm{End}(S)$-component of the Hermite–Einstein equation reads as

$$i\Lambda(\Omega_S - A^* \wedge A) = i\Lambda\Omega_S - i\Lambda(A^* \wedge A) = \lambda \,\mathrm{id}_S\,.$$

Tracing over this gives

$$i\,\mathrm{tr}(\Lambda\Omega_S) - i\,\mathrm{tr}(\Lambda(A^* \wedge A)) = \lambda\,\mathrm{rk}(S). \tag{4.1}$$

Now

$$-i\,\mathrm{tr}(\Lambda(A^* \wedge A)) = \mathrm{tr}(\Lambda(iA \wedge A^*)) = |A|^2,$$

and so integrating (1) yields

$$\begin{aligned}\int_X i\,\mathrm{tr}(\Lambda\Omega_S)\omega^n + \int_X |A|^2\omega^n &= i\int_X \Lambda\,\mathrm{tr}(\Omega_S)\omega^n + \|A\|^2\\ &= ni\int_X \mathrm{tr}(\Omega_S)\wedge\omega^{n-1} + \|A\|^2\\ &= 2\pi n\int_X c_1(S)\wedge\omega^{n-1} + \|A\|^2\\ &= 2\pi n\deg(S) + \|A\|^2\\ &= \lambda\,\mathrm{rk}(S)\,\mathrm{vol}(X).\end{aligned}$$

Also,

$$\lambda = \frac{2\pi n}{\mathrm{vol}(X)}\mu(E),$$

so we have

$$2\pi n\deg(S) + \|A\|^2 = 2\pi n\,\mathrm{rk}(S)\mu(E).$$

Note that the assumption that E is indecomposable forces $\|A\|^2 > 0$ as otheriwse we would have a holomorphic split $E = S \oplus Q$. This gives

$$2\pi n\,\mathrm{rk}(S)\mu(E) = 2\pi n\deg(S) + \|A\|^2 > 2\pi n\deg(S),$$

i.e.

$$\mu(E) > \mu(S).$$

□

4.4 The Kobayashi–Hitchin Correspondence

In the previous section we saw that any indecomposable holomorphic vector bundle admitting a Hermite–Einstein metric must be stable. We now turn to the converse implication, namely that polystability guarantees the existence of a Hermite–Einstein metric. This equivalence is known as the *Kobayashi–Hitchin correspondence*, and lies at the crossroads of complex differential geometry and algebraic geometry.

Originating in independent conjectures of Kobayashi and Hitchin in the late 1970s, and established in full generality by Donaldson and Uhlenbeck–Yau, the correspondence asserts that on a compact Kähler manifold (M, ω) a holomorphic vector bundle E admits a Hermite–Einstein metric if and only if E is polystable. In particular, when E is indecomposable, polystability is equivalent to stability, and so one obtains a precise bridge between an algebro–geometric criterion (slope stability) and an analytic one (the Hermite–Einstein PDE).

We have already established in Proposition 4.3.4 that an indecomposable Hermite–Einstein bundle must be stable. The converse requires the deep analytical machinery developed by Donaldson and Uhlenbeck–Yau. Let us state the key components:

Theorem 4.4.1. *Let E be a holomorphic vector bundle on a compact Kähler manifold (X, ω). If E is stable, then E admits a Hermite–Einstein metric, which is unique up to a constant multiple.*

Combining our previous results with Theorem 4.4.1, we obtain:

Corollary 4.4.2 (The Kobayashi–Hitchin Correspondence)**.** *An indecomposable holomorphic vector bundle E over a compact Kähler manifold (X, ω) admits a Hermite–Einstein metric if and only if it is stable.*

We present a brief overview of the proof strategy developed by Simpson [14], which combines the approaches of Donaldson and Uhlenbeck–Yau. The key idea is to use the heat flow method to deform an arbitrary Hermitian metric towards a Hermite–Einstein metric.

Let E be a stable holomorphic vector bundle on (X, ω). Fix a background Hermitian metric h_0 on E and consider the evolution equation

$$\frac{dh_t}{dt} h_t^{-1} = -\left(i\Lambda_\omega F_{h_t} - \lambda \,\mathrm{id}_E\right), \tag{4.2}$$

where h_t is a smooth family of Hermitian metrics on E, F_{h_t} is the curvature of the Chern connection associated to h_t, and $\lambda = \frac{2\pi n}{\mathrm{vol}(X)}\mu(E)$.

The right-hand side of (4.2) measures the failure of h_t to satisfy the Hermite–Einstein condition. Using the theory of parabolic partial differential equations, one can show:

(i) **Short-time existence**: There exists a solution to (4.2) on some interval $[0, T)$.

(ii) **Long-time existence**: Using a priori estimates, the solution can be extended to $[0, \infty)$.

(iii) **Convergence analysis**: Either the solution converges to a Hermite–Einstein metric as $t \to \infty$, or one can extract a destabilizing subsheaf, contradicting the stability of E.

The crucial analytical input is the following regularity result:

Definition 4.4.3. A **weakly holomorphic subbundle** of E is an L^2 section π of $\mathrm{End}(E)$ with L^2 first-order weak derivatives satisfying

$$\pi = \pi^* = \pi^2 \quad \text{and} \quad (\mathrm{id}_E - \pi) \circ \bar{\partial}\pi = 0,$$

where π^* denotes the adjoint with respect to h_0.

The conditions ensure that π is a projection operator compatible with the holomorphic structure. The technical achievement of Uhlenbeck–Yau is:

Theorem 4.4.4 (Uhlenbeck–Yau Regularity)**.** *Any weakly holomorphic subbundle of a holomorphic vector bundle over a compact Kähler manifold corresponds to a coherent subsheaf in the ordinary sense.*

This allows one to convert analytical obstructions to convergence into algebraic destabilizing subsheaves, completing the proof by contradiction when E is stable.

Chapter 5

The Atiyah Class

5.1 Atiyah Class via Čech Cocycles

Let X be a complex manifold (or more generally a paracompact topological space) and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on X.

Definition 5.1.1. An *open cover* of X is a family $\mathcal{U} = \{U_i\}_{i\in I}$ of open subsets of X such that

$$X = \bigcup_{i\in I} U_i.$$

For any finite ordered tuple $(i_0, \ldots, i_p)$ we set

$$U_{i_0\cdots i_p} = U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_p}.$$

Definition 5.1.2 (Čech p-Cochains)**.** The group of *Čech p-cochains* of $\mathcal{F}$ with respect to the cover $\mathcal{U}$ is

$$C^p(\mathcal{U}, \mathcal{F}) \;=\; \prod_{(i_0<\cdots<i_p)} \mathcal{F}\big(U_{i_0\cdots i_p}\big).$$

An element $\varphi \in C^p(\mathcal{U}, \mathcal{F})$ is a collection $\{\varphi_{\,i_0\cdots i_p}\}$ with $\varphi_{\,i_0\cdots i_p} \in \mathcal{F}(U_{i_0\cdots i_p})$.

Definition 5.1.3 (Čech Differential)**.** The Čech differential $\delta\colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ is defined by

$$(\delta\varphi)_{\,i_0\cdots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \, \varphi_{\,i_0\cdots\widehat{i_k}\cdots i_{p+1}}\Big|_{U_{i_0\cdots i_{p+1}}},$$

where the hat means omission of that index and the restriction is to the smaller intersection.

Definition 5.1.4 (Cocycles, Coboundaries)**.**

(i) A p-cochain $\varphi \in C^p(\mathcal{U}, \mathcal{F})$ is a *cocycle* if $\delta\varphi = 0$. The subgroup of such is denoted $Z^p(\mathcal{U}, \mathcal{F})$.

(ii) A p-cochain is a *coboundary* if there exists $\psi \in C^{p-1}(\mathcal{U}, \mathcal{F})$ with $\varphi = \delta\psi$. The subgroup of these is $B^p(\mathcal{U}, \mathcal{F})$.

Definition 5.1.5 (Čech Cohomology)**.** The *Čech cohomology* of the cover $\mathcal{U}$ is the quotient

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = \frac{Z^p(\mathcal{U}, \mathcal{F})}{B^p(\mathcal{U}, \mathcal{F})}.$$

Passing to the direct limit over all open covers of X gives the sheaf cohomology $\check{H}^p(X, \mathcal{F})$, which for good covers agrees with the derived functor cohomology $H^p(X, \mathcal{F})$.

Example 5.1.6. *A Čech 1-cocycle is a collection $\{\alpha_{ij}\} \in Z^1(\mathcal{U}, \mathcal{F})$ with $\alpha_{ij} \in \mathcal{F}(U_i \cap U_j)$ satisfying*

$$\alpha_{jk} - \alpha_{ik} + \alpha_{ij} = 0 \quad \text{on } U_i \cap U_j \cap U_k.$$

Two such cocycles differ by a coboundary $\alpha_{ij} = \beta_j - \beta_i$ for some $\{\beta_i\} \in C^0(\mathcal{U}, \mathcal{F})$.

Remark 5.1.7. In practice one often works with a *good cover* (each finite intersection is Stein or contractible), so that Čech cohomology computes the sheaf cohomology $\check{H}^p(\mathcal{U}, \mathcal{F}) \cong H^p(X, \mathcal{F})$.

We now specialize to the case $\mathcal{F} = \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E}nd(\mathcal{E})$ and explain how the Atiyah class of a holomorphic vector bundle $E \to X$ arises as a Čech 1-cocycle.

Now let $E \to X$ be a holomorphic vector bundle and denote by $\mathcal{E}$ its sheaf of holomorphic sections. By Definition 1.2.6, a holomorphic connection on E is a $\mathbb{C}$–linear map of sheaves

$$\nabla : \mathcal{E} \to \Omega^1_X \otimes \mathcal{E},$$

which satisfies the Leibniz rule. Although a global ∇ need not exist, locally on each member of a good cover $\mathcal{U} = \{U_i\}$ one can always choose

$$\nabla_i : \mathcal{E}\big|_{U_i} \to \Omega^1_X \otimes \mathcal{E}\big|_{U_i}$$

a holomorphic connection. On overlaps $U_{ij} = U_i \cap U_j$, the difference

$$\alpha_{ij} \; = \; \nabla_j - \nabla_i : \mathcal{E}|_{U_{ij}} \to \Omega^1_X \otimes \mathcal{E}|_{U_{ij}}$$

is $\mathcal{O}_X$–linear, hence defines

$$\alpha_{ij} \; \in \; \Gamma\big(U_{ij}, \, \Omega^1_X \otimes \mathcal{E}nd(\mathcal{E})\big).$$

One checks immediately that on triple overlaps

$$\alpha_{jk} - \alpha_{ik} + \alpha_{ij} = 0 \quad \text{in } \Gamma\big(U_i \cap U_j \cap U_k, \, \Omega^1_X \otimes \mathcal{E}nd(\mathcal{E})\big),$$

so $\{\alpha_{ij}\}$ is a Čech 1-cocycle in $Z^1(\mathcal{U}, \Omega^1_X \otimes \mathcal{E}nd(\mathcal{E}))$. Different choices of the local ∇_i change $\{\alpha_{ij}\}$ by a coboundary in B^1, and hence define a well-defined class

$$A(E) \;=\; \big[\{\alpha_{ij}\}\big] \;\in\; \check{H}^1\big(\mathcal{U},\, \Omega^1_X \otimes \mathcal{E}nd(\mathcal{E})\big) \;\cong\; H^1\big(X,\, \Omega^1_X \otimes \mathcal{E}nd(\mathcal{E})\big).$$

The essential thing to note here is that $A(E)$ acts as an obstruction for the existence of a holomorphic connection on E.

Definition 5.1.8. The *Atiyah class* of the holomorphic bundle E is the cohomology class

$$A(E) \in H^1\big(X, \Omega^1_X \otimes \mathcal{E}nd(\mathcal{E})\big)$$

represented by the Čech cocycle $\{\alpha_{ij}\}$.

Proposition 5.1.9. *A holomorphic vector bundle E admits a holomorphic connection if and only if its Atiyah class is trivial.*

Proof. If E admits a holomorphic connection ∇, then ∇ restricts to each ∇_i, hence $\nabla_j - \nabla_i = 0$ and the cocycle is trivial. Conversely, suppose that $A(E) = 0$. Pick local trivializations $\varphi_i : E|_{U_i} \to U_i \times \mathbb{C}^k$ and note that the local connections are of the form $\nabla_i = \partial + A_i$, for A_i a matrix of 1-forms. These glue to a global holomorphic connection if

$$\varphi_i^{-1} \circ (\partial + A_i) \circ \varphi_i = \varphi_j^{-1} \circ (\partial + A_j) \circ \varphi_j.$$

Equivalently if

$$\varphi_i^{-1} \circ \partial \circ \varphi_i - \varphi_j^{-1} \circ \partial \circ \varphi_j = \varphi_j^{-1} \circ A_j \circ \varphi_j - \varphi_i^{-1} \circ A_i \circ \varphi_i.$$

Note that

$$\varphi_i^{-1} \circ \partial \circ \varphi_i - \varphi_j^{-1} \circ \partial \circ \varphi_j = \varphi_j^{-1} \circ \Big(\varphi_{ij}^{-1}\partial(\varphi_{ij})\Big) \circ \varphi_j.$$

Thus the gluing condition becomes

$$\varphi_{ij}^{-1}\, \partial\varphi_{ij} = \varphi_j^{-1} A_j\, \varphi_j - \varphi_i^{-1} A_i\, \varphi_i.$$

But the left-hand side is exactly the Čech cocycle representing $A(E)$, and the right-hand side is the coboundary

$$(\varphi_j^{-1} A_j\, \varphi_j) - (\varphi_i^{-1} A_i\, \varphi_i) = \delta\, (\varphi_i^{-1} A_i\, \varphi_i)_{ij}.$$

Since this cocycle is a coboundary by assumption, it follows that we can choose the local A_i so that the local holomorphic connections $\nabla_i = \partial + A_i$ satisfy the gluing condition. Hence they define a global holomorphic connection on E, completing the proof. $\square$

The following theorem gives a geometric interpretation for the Atiyah class.

Theorem 5.1.10. *Let (E, h) be a holomorphic vector bundle equipped with a Hermitian metric h, and let ∇ denote its Chern connection. Then*

$$[F_\nabla] = A(E) \in H^1(X, \Omega^1_X \otimes \mathcal{E}nd(\mathcal{E})),$$

where $[F_\nabla]$ denotes the Dolbeault cohomology class of F_∇.

Proof. The key to relating the curvature F_∇ of the Chern connection with the Čech cocycle representing the Atiyah class $A(E)$ is the double complex which connects Čech and Dolbeault resolutions. This allows us to identify the Dolbeault cohomology class of F_∇ with the Čech class of the cocycle $\{\alpha_{ij}\}$ described above.

Let $\mathcal{U} = \{U_i\}$ be a good open cover of X trivializing E, with local holomorphic trivializations $\psi_i : E|_{U_i} \to U_i \times \mathbb{C}^r$. On each U_i, the Hermitian metric h is given by a positive-definite Hermitian matrix H_i, so that for sections s, t of E over U_i,

$$h(s, t) = \langle \psi_i(s), H_i \psi_i(t) \rangle_{\mathbb{C}^r}.$$

The Chern connection ∇ is the unique connection compatible with both the holomorphic structure and the Hermitian metric. On U_i, it is expressed as

$$\nabla = \partial + \bar{\partial} + A_i$$

where A_i is a matrix of 1-forms, and the $(1, 0)$-part of A_i is given by

$$A_i = H_i^{-1} \partial H_i$$

so that $\nabla' = \partial + A_i$ is compatible with h.

The curvature F_∇ of the Chern connection is a global $(1, 1)$-form with values in $\mathrm{End}(E)$, and locally on U_i is given by

$$F_\nabla|_{U_i} = \bar{\partial}(H_i^{-1} \partial H_i)$$

in the trivialization ψ_i. Consider now the Čech–Dolbeault double complex. Under the Dolbeault–Čech correspondence, the Dolbeault cohomology class $[F_\nabla]$ is mapped to the Čech cocycle

$$\left\{U_i,\, \psi_i^{-1} \circ \left(\bar{\partial}(H_i^{-1} \partial H_i)\right) \circ \psi_i\right\}$$

and its Čech coboundary δ_0 is given by

$$\delta_0 \left(F_\nabla\right) = \left\{U_{ij},\, \psi_j^{-1} \circ \left(\bar{\partial}(H_j^{-1} \partial H_j)\right) \circ \psi_j - \psi_i^{-1} \circ \left(\bar{\partial}(H_i^{-1} \partial H_i)\right) \circ \psi_i\right\}.$$

But since $\bar{\partial}$ commutes with restrictions and the transition functions $\psi_{ij} = \psi_j \circ \psi_i^{-1} : U_{ij} \to GL(r, \mathbb{C})$ are holomorphic, we have

$$\psi_j^{-1} \circ (H_j^{-1}\partial H_j) \circ \psi_j = \psi_{ij}^{-1}(H_i^{-1}\partial H_i)\psi_{ij} + \psi_{ij}^{-1}\partial\psi_{ij}.$$

Therefore,

$$\begin{aligned}
&\psi_j^{-1} \circ (H_j^{-1}\partial H_j) \circ \psi_j - \psi_i^{-1} \circ (H_i^{-1}\partial H_i) \circ \psi_i \\
&= \left(\psi_{ij}^{-1}(H_i^{-1}\partial H_i)\psi_{ij} + \psi_{ij}^{-1}\partial\psi_{ij}\right) - (H_i^{-1}\partial H_i) \\
&= \psi_{ij}^{-1}\partial\psi_{ij} + \psi_{ij}^{-1}(H_i^{-1}\partial H_i)\psi_{ij} - (H_i^{-1}\partial H_i).
\end{aligned}$$

But the terms involving $H_i^{-1}\partial H_i$ combine to a coboundary in the Čech complex, so the remaining cocycle is given by

$$\left\{U_{ij},\, \psi_{ij}^{-1}\partial\psi_{ij}\right\},$$

which is precisely the Čech 1-cocycle representing the Atiyah class $A(E)$, as constructed above. Thus, under the Dolbeault–Čech isomorphism,

$$[F_\nabla] = A(E) \in H^1(X, \Omega_X^1 \otimes \mathcal{E}nd(\mathcal{E})).$$

This completes the proof. □

Corollary 5.1.11. *If a holomorphic vector bundle E has a holomorphic connection, then all the rational Chern classes vanish.*

5.2 Atiyah Class via Extension Classes

In this section, we describe another important viewpoint on the Atiyah class through the language of extension classes. We begin by recalling general facts about extensions and Ext groups.

Definition 5.2.1. Let X be a topological space and consider sheaves of modules $\mathcal{F}, \mathcal{G}$ over a sheaf of rings $\mathcal{R}$ on X. An *extension of $\mathcal{G}$ by $\mathcal{F}$* is an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0$$

Two extensions

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0, \qquad 0 \to \mathcal{F} \to \mathcal{E}' \to \mathcal{G} \to 0$$

are said to be *equivalent* if there exists an isomorphism $\mathcal{E} \xrightarrow{\cong} \mathcal{E}'$ making the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{G} \arrow[r] & 0.
\end{tikzcd}$$

Definition 5.2.2. The set of equivalence classes of extensions of $\mathcal{G}$ by $\mathcal{F}$ naturally forms an abelian group under the Baer sum. This group is called the *extension group* and is denoted by

$$\operatorname{Ext}^1_{\mathcal{R}}(\mathcal{G}, \mathcal{F}).$$

More generally, the Ext groups can be computed as derived functors of the Hom functor:

$$\operatorname{Ext}^i_{\mathcal{R}}(\mathcal{G}, \mathcal{F}) = R^i \operatorname{Hom}_{\mathcal{R}}(\mathcal{G}, -)(\mathcal{F}),$$

where $\operatorname{Hom}_{\mathcal{R}}(\mathcal{G}, -)$ is viewed as a left-exact functor.

Remark 5.2.3. When $\mathcal{R}$ is the sheaf of holomorphic functions $\mathcal{O}_X$ on a complex manifold X, and $\mathcal{F}, \mathcal{G}$ are coherent sheaves, the groups $\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ can be computed using Čech cohomology and locally free resolutions.

We now turn to the specific situation relevant to the Atiyah class. Let E be a holomorphic vector bundle on a complex manifold X, and denote by $\mathcal{E}$ the corresponding locally free sheaf of holomorphic sections. Recall the tangent bundle T_X and its sheaf of holomorphic sections Ω_X.

Consider the exact sequence of vector bundles associated with the first *jet bundle* $J^1(E)$ of E:

$$0 \longrightarrow \Omega^1_X \otimes E \longrightarrow J^1(E) \longrightarrow E \longrightarrow 0.$$

Here, $J^1(E)$ is defined as the bundle whose fiber at a point $x \in X$ consists of equivalence classes of germs of holomorphic sections of E at x, modulo equivalence up to first-order.

This exact sequence gives rise to a short exact sequence of sheaves of $\mathcal{O}_X$-modules:

$$0 \longrightarrow \Omega^1_X \otimes \mathcal{E} \longrightarrow \mathcal{J}^1(\mathcal{E}) \longrightarrow \mathcal{E} \longrightarrow 0$$

Definition 5.2.4 (Atiyah Extension)**.** The short exact sequence of sheaves

$$0 \longrightarrow \Omega^1_X \otimes \mathcal{E} \longrightarrow \mathcal{J}^1(\mathcal{E}) \longrightarrow \mathcal{E} \longrightarrow 0$$

is called the *Atiyah extension* of $\mathcal{E}$.

By general theory, this short exact sequence defines an element in the extension group:

$$[\mathcal{J}^1(\mathcal{E})] \in \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, \Omega^1_X \otimes \mathcal{E}).$$

On the other hand, by using duality, we have a canonical isomorphism

$$\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, \Omega^1_X \otimes \mathcal{E}) \cong H^1(X, \Omega^1_X \otimes \mathcal{E}nd(\mathcal{E})).$$

Definition 5.2.5 (Atiyah Class via Extension)**.** The *Atiyah class* $A(E)$ is defined as the extension class of the Atiyah extension:

$$A(E) := [\mathcal{J}^1(\mathcal{E})] \in H^1(X, \Omega^1_X \otimes \mathcal{E}nd(\mathcal{E})).$$

This gives a purely algebraic characterization of the Atiyah class in terms of extensions of sheaves. Thus, we have two equivalent viewpoints of the Atiyah class: one via Čech cocycles arising from differences of local connections, and another via extension classes involving the first jet bundle of E.

We summarize this important fact in the following proposition:

Proposition 5.2.6. *Let $E \to X$ be a holomorphic vector bundle. Then the Atiyah class $A(E)$ coincides with the class represented by the Atiyah extension:*

$$A(E) = [\mathcal{J}^1(\mathcal{E})] \in H^1(X, \Omega^1_X \otimes \mathcal{E}nd(\mathcal{E})).$$

Moreover, $A(E)$ vanishes precisely when the Atiyah extension splits holomorphically.

Thus, the vanishing of the Atiyah class corresponds precisely to the existence of a holomorphic splitting of the Atiyah extension, or equivalently, to the existence of a holomorphic connection on E. This viewpoint provides another powerful algebraic tool to study the geometry of holomorphic vector bundles.

To illustrate the usefulness of this viewpoint we will show that any holomorphic vector bundle on a Stein manifold admits a holomorphic connection.

Definition 5.2.7. A complex manifold X is called a *Stein manifold* if it satisfies the following properties:

(i) For any two distinct points $x, y \in X$, there exists a holomorphic function $f \in \mathcal{O}(X)$ such that $f(x) \neq f(y)$.

(ii) For every compact subset $K \subset X$, the *holomorphically convex hull* of K,

$$\widehat{K} := \left\{ x \in X : |f(x)| \leq \sup_{y \in K} |f(y)| \text{ for all } f \in \mathcal{O}(X) \right\},$$

is compact in X.

Proposition 5.2.8. *Let X be a Stein manifold, and let E be a holomorphic vector bundle over X. Then E admits a holomorphic connection.*

Proof. The existence of a holomorphic connection on E is equivalent to the vanishing of the Atiyah class

$$A(E) \in H^1(X, \Omega^1_X \otimes \mathcal{E}nd(\mathcal{E})),$$

where $\mathcal{E}$ is the locally free sheaf of holomorphic sections of E. By Proposition 5.2.6, the vanishing of $A(E)$ is equivalent to the holomorphic splitting of the Atiyah extension

$$0 \longrightarrow \Omega^1_X \otimes \mathcal{E} \longrightarrow \mathcal{J}^1(\mathcal{E}) \longrightarrow \mathcal{E} \longrightarrow 0$$

where $\mathcal{J}^1(\mathcal{E})$ is the sheaf of holomorphic 1-jets of $\mathcal{E}$. On a Stein manifold, Cartan's Theorem B [7] asserts that the higher cohomology of any coherent analytic sheaf vanishes. In particular,

$$H^1(X, \Omega^1_X \otimes \mathcal{E}nd(\mathcal{E})) = 0,$$

since both Ω^1_X and $\mathcal{E}nd(\mathcal{E})$ are locally free, hence coherent, and the tensor product of coherent sheaves is coherent.

Therefore, the extension class defining the Atiyah extension is zero, so the sequence splits holomorphically. This splitting yields a holomorphic connection on E. $\square$

Chapter 6

Flat Bundles and Holomorphic Connections

This chapter looks at the relationship between flat holomorphic vector bundles, representations of the fundamental group, and the vanishing of the Atiyah class. The goal is to understand to what extent the existence of a holomorphic connection—or, equivalently, the vanishing of the Atiyah class—characterizes those bundles which arise from representations of the fundamental group.

Recall that the Atiyah class of a holomorphic vector bundle $E \to X$, denoted $A(E) \in H^1(X, \Omega^1_X \otimes \mathcal{E}nd(E))$, vanishes if and only if E admits a holomorphic connection. We aim to compare:

- The class of holomorphic vector bundles over X that admit a flat holomorphic connection.
- The class of holomorphic vector bundles whose Atiyah class vanishes (i.e., those that admit some holomorphic connection, not necessarily flat).

The key question is: When do these two classes coincide? In other words, when does the existence of a holomorphic connection automatically imply the existence of a flat one?

We begin by developing the general theory of flat holomorphic vector bundles and their correspondence with representations of the fundamental group. We then focus on the special case of curves, where every holomorphic connection is automatically flat, discuss the situation on projective spaces, and finally address the subtleties that arise in higher dimensions.

6.1 Flat Vector Bundles and Representations

Let X be a connected complex manifold. In this section, we focus on holomorphic vector bundles admitting a flat connection, and relate them to representations of the fundamental group of X.

Definition 6.1.1. A holomorphic vector bundle $E \to X$ of rank r is called *flat* if there exists an open cover $\{U_i\}_{i\in I}$ of X and holomorphic trivializations $\psi_i : E|_{U_i} \cong U_i \times \mathbb{C}^r$ such that the transition functions

$$\varphi_{ij} : U_i \cap U_j \to \mathrm{GL}(r, \mathbb{C})$$

are *locally constant*, i.e., constant on each connected component of $U_i \cap U_j$.

Alternatively, a holomorphic vector bundle E is flat if it admits a holomorphic connection ∇ with zero curvature, i.e., $F_\nabla = 0$. We now make precise the equivalence between these notions and their connection to representations of the fundamental group. Let $\tilde{X}$ denote the universal cover of X, with covering projection $p : \tilde{X} \to X$. Given a representation $\rho : \pi_1(X, x_0) \to \mathrm{GL}(r, \mathbb{C})$, there is a canonical way to associate a holomorphic vector bundle to ρ as follows:

Definition 6.1.2. The *bundle associated to* ρ is

$$E_\rho := (\tilde{X} \times \mathbb{C}^r)/\pi_1(X, x_0),$$

where $\pi_1(X, x_0)$ acts on $\tilde{X} \times \mathbb{C}^r$ by

$$\gamma \cdot (\tilde{x}, v) := (\gamma \cdot \tilde{x}, \rho(\gamma)v).$$

The projection $E_\rho \to X$ is holomorphic, and E_ρ is a holomorphic vector bundle of rank r.

Proposition 6.1.3. *Let $E \to X$ be a holomorphic vector bundle. The following are equivalent:*

- *(i) E admits a flat holomorphic structure, i.e., an open cover $\{U_i\}$ with locally constant holomorphic transition functions φ_{ij}.*
- *(ii) E admits a holomorphic connection ∇ such that $F_\nabla = 0$ (i.e., a flat holomorphic connection).*
- *(iii) E is isomorphic to E_ρ for some representation $\rho : \pi_1(X, x_0) \to \mathrm{GL}(r, \mathbb{C})$.*

Proof. (i) $\implies$ (ii): Suppose E is defined by locally constant transition functions φ_{ij}. On each U_i, the trivial bundle admits the standard holomorphic connection d. Since the transition functions are constant, these local connections patch to a global holomorphic connection on E, as for s a local section and on $U_i \cap U_j$,

$$d(\varphi_{ij}s) = \varphi_{ij}ds,$$

so the operators agree. Moreover, $d^2 = 0$ locally, so the global connection is flat: $F_\nabla = 0$. (ii) $\implies$ (i): Suppose ∇ is a flat holomorphic connection on E. By the existence and uniqueness theorem for ordinary differential equations, for each simply connected open subset $U \subset X$, parallel transport with respect to ∇ trivializes $E|_U$, giving a holomorphic frame $e_1, \ldots, e_r$ on U such that $\nabla e_j = 0$. On $U_i \cap U_j$, the change of basis matrix between two such frames is necessarily constant, as parallel sections are determined by their initial values. Thus, the transition functions are locally constant. (i) $\implies$ (iii): Let E be a holomorphic vector bundle with locally constant transition functions φ_{ij} with respect to some cover $\{U_i\}$. Choose a base point $x_0 \in X$ and a trivialization of E at x_0. Lifting to the universal cover $\tilde{X}$, the pullback p^*E is trivial, as the monodromy of E corresponds to the representation $\rho : \pi_1(X, x_0) \to \mathrm{GL}(r, \mathbb{C})$ defined by the transition matrices around loops based at x_0. Thus, E is isomorphic to E_ρ. (iii) $\implies$ (ii): On $\tilde{X} \times \mathbb{C}^r$, define the trivial holomorphic connection d. This connection is invariant under the diagonal $\pi_1(X, x_0)$-action:

$$d(\gamma \cdot (\tilde{x}, v)) = d(\gamma \cdot \tilde{x}, \rho(\gamma)v) = \rho(\gamma)dv,$$

and thus descends to a flat holomorphic connection on E_ρ. $\square$

Proposition 6.1.4. *Let $f : X \to Y$ be a holomorphic map, and let E be a flat holomorphic vector bundle on Y corresponding to a representation $\rho : \pi_1(Y, y_0) \to \mathrm{GL}(r, \mathbb{C})$. Then the pullback bundle f^*E is flat, and corresponds to the representation $\rho \circ f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0) \to \mathrm{GL}(r, \mathbb{C})$.*

Proof. The map f induces a group homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$. The pullback bundle f^*E is constructed so that, locally, the transition functions of f^*E are the pullbacks of the transition functions of E, and hence remain locally constant. On the universal cover, f lifts to a map $\tilde{f} : \widetilde{X} \to \widetilde{Y}$ equivariant for the fundamental group actions, so f^*E is isomorphic to the bundle associated to the representation $\rho \circ f_*$, and the flat connection on E pulls back to a flat connection on f^*E. $\square$

Proposition 6.1.5. *Let E_ρ and $E_{\rho'}$ be flat holomorphic vector bundles on X associated to representations $\rho, \rho' : \pi_1(X, x_0) \to \mathrm{GL}(r, \mathbb{C})$. Then $E_\rho \cong E_{\rho'}$ as holomorphic vector bundles if and only if there exists $A \in \mathrm{GL}(r, \mathbb{C})$ such that $\rho'(\gamma) = A\rho(\gamma)A^{-1}$ for all $\gamma \in \pi_1(X, x_0)$; that is, ρ and ρ' are conjugate representations.*

Proof. Suppose $E_\rho \cong E_{\rho'}$. Such an isomorphism lifts to a $\pi_1(X, x_0)$-equivariant holomorphic bundle map on $\tilde{X} \times \mathbb{C}^r$, i.e., a holomorphic map $F : \tilde{X} \times \mathbb{C}^r \to \tilde{X} \times \mathbb{C}^r$ of the form $F(\tilde{x}, v) = (\tilde{x}, Av)$ for some $A \in \mathrm{GL}(r, \mathbb{C})$, such that

$$F(\gamma \cdot (\tilde{x}, v)) = \gamma \cdot F(\tilde{x}, v)$$

for all $\gamma \in \pi_1(X, x_0)$. Expanding both sides gives

$$F(\gamma \cdot \tilde{x}, \rho(\gamma)v) = (\gamma \cdot \tilde{x}, A\rho(\gamma)v),$$

while

$$\gamma \cdot F(\tilde{x}, v) = \gamma \cdot (\tilde{x}, Av) = (\gamma \cdot \tilde{x}, \rho'(\gamma)Av).$$

Therefore, $A\rho(\gamma)v = \rho'(\gamma)Av$ for all v, i.e., $\rho'(\gamma) = A\rho(\gamma)A^{-1}$.

Conversely, given A with $\rho'(\gamma) = A\rho(\gamma)A^{-1}$, the map $(\tilde{x}, v) \mapsto (\tilde{x}, Av)$ descends to an isomorphism of the associated bundles. □

Proposition 6.1.6. *Let E and F be flat holomorphic vector bundles over X. Then the bundles $E \oplus F$, $E \otimes F$, E^*, $\Lambda^k E$, and $\mathrm{Sym}^k E$ are also flat.*

Proof. Suppose $E = E_\rho$ and $F = E_{\rho'}$, with $\rho : \pi_1(X, x_0) \to \mathrm{GL}(r, \mathbb{C})$ and $\rho' : \pi_1(X, x_0) \to \mathrm{GL}(s, \mathbb{C})$. Then:

- $E \oplus F$ is associated to the block-diagonal representation $\rho \oplus \rho'$;
- $E \otimes F$ is associated to the tensor product representation $\rho \otimes \rho'$;
- E^* is associated to the dual representation ρ^*;
- $\Lambda^k E$ is associated to the k'th exterior power of ρ;
- $\mathrm{Sym}^k E$ is associated to the k'th symmetric power of ρ.

In all cases, the resulting representation is a representation of $\pi_1(X, x_0)$, so the associated bundle is flat. □

Remark 6.1.7. Proposition 6.1.6 can be also proven using the fact that on the level of connections, the induced connection (on the sum, tensor, dual, etc.) of flat connections is again flat.

6.2 Flatness on Curves

We now explore the relationship between flat holomorphic vector bundles and bundles with vanishing Atiyah class, especially in the setting of complex curves. The primary result of this section is the equivalence between flatness and the vanishing of the Atiyah class for vector bundles over curves.

Recall from the previous chapter that the Atiyah class $A(E) \in H^1(X, \Omega^1_X \otimes \mathcal{E}nd(E))$ of a holomorphic vector bundle $E \to X$ measures the obstruction to the existence of a holomorphic connection. Explicitly, we have Proposition 5.1.9 stating that a holomorphic vector bundle $E \to X$ admits a holomorphic connection if and only if its Atiyah class vanishes, i.e., $A(E) = 0$. Thus, the vanishing of the Atiyah class corresponds exactly to the existence of a global holomorphic connection. However, the existence of a holomorphic connection alone does not guarantee that such a connection is flat. Flatness imposes the additional condition of zero curvature. Nonetheless, in the special case of complex curves, the situation simplifies significantly.

Theorem 6.2.1 (Flatness Criterion for Curves)**.** *Let X be a complex manifold of dimension one (a complex curve). A holomorphic vector bundle $E \to X$ admits a holomorphic connection if and only if it admits a flat holomorphic connection. In other words, on a complex curve, the vanishing of the Atiyah class characterizes flat vector bundles precisely:*

$$A(E) = 0 \iff E \text{ is flat.}$$

Proof. Let $E \to X$ be a holomorphic vector bundle over a one-dimensional complex manifold (curve) X, and suppose E admits a holomorphic connection ∇. By definition, the curvature of ∇ is given by:

$$F_\nabla \in H^0(X, \Omega_X^2 \otimes \mathrm{End}(E)).$$

However, since $\dim_{\mathbb{C}} X = 1$, the holomorphic 2-forms vanish:

$$\Omega_X^2 = 0.$$

Thus, necessarily:

$$F_\nabla = 0.$$

Therefore, every holomorphic connection on a curve is automatically flat. Conversely, any flat connection is clearly a holomorphic connection. Hence, the existence of a holomorphic connection implies flatness on curves, yielding the equivalence. □

Corollary 6.2.2. *Every holomorphic vector bundle on the Riemann sphere $\mathbb{P}^1$ with vanishing Atiyah class is trivial.*

Proof. Since $\pi_1(\mathbb{P}^1) = 0$, any flat vector bundle must be trivial. By Theorem 6.2.1, vanishing Atiyah class implies flatness, thus triviality. □

Corollary 6.2.3 (Flat bundles on projective spaces)**.** *Let $n \geq 1$. Every holomorphic vector bundle on $\mathbb{P}^n$ with vanishing Atiyah class is trivial:*

$$A(E) = 0 \implies E \cong \mathcal{O}_{\mathbb{P}^n}^{\oplus r}.$$

Proof. It is a classical fact [12, Thm. 2.1.1] that a vector bundle on $\mathbb{P}^n$ is trivial if and only if its restriction to every line is trivial. The previous corollary ensures triviality on each line (which is isomorphic to $\mathbb{P}^1$), hence global triviality. □

On compact curves, an even more refined description holds. We have Weil's celebrated theorem:

Theorem 6.2.4 (Weil)**.** *Let X be a compact Riemann surface (curve), and let E be an indecomposable holomorphic vector bundle on X. Suppose $\deg(E) = 0$. Then E admits a flat holomorphic structure.*

This theorem further implies that on compact curves, the existence of a holomorphic connection, flatness, and the vanishing of all rational Chern classes coincides with the vector bundles whose indecomposable components have degree zero.

Corollary 6.2.5 (Classification of flat bundles on compact curves)**.** *Let X be a compact Riemann surface and $E \to X$ a holomorphic vector bundle. Then E admits a flat structure (equivalently, vanishing Atiyah class) if and only if every indecomposable summand in its decomposition into indecomposable bundles has degree zero.*

Proof. Decompose E as a direct sum of indecomposable bundles:

$$E = E_1 \oplus \cdots \oplus E_k.$$

Each summand E_i must individually admit a flat connection if E does. By Weil's theorem, this occurs precisely when $\deg(E_i) = 0$. Conversely, a direct sum of flat bundles is clearly flat. □

6.3 Projective Manifolds

As we have seen, on complex curves these notions coincide: every holomorphic vector bundle with vanishing Atiyah class is flat. However, in higher dimensions, the relationship between the vanishing of the Atiyah class and flatness becomes significantly more subtle.

In this section, we will first consider results established by Atiyah [1] in 1957 and then follow the more recent results of Biswas [3].

A key result Atiyah proposed in his paper was that the existence of a holomorphic connection on a projective manifold X can be checked by analyzing a suitable surfaces inside X. More precisely:

Proposition 6.3.1 (Atiyah [1])**.** *Let X be a projective manifold with* $\dim X \geq 3$*, L an ample line bundle, and E a holomorphic vector bundle on X. Then there exists an n_0 such that for all larger n and all smooth hypersurfaces S with $\mathcal{O}_X(S) = L^n$, the Atiyah class of E vanishes if and only if the Atiyah class of $E|_S$ vanishes.*

Proof. The implication $A(E) = 0 \implies A(E|_S) = 0$ follows directly from functoriality of the Atiyah class. We prove the converse. Let F be any vector bundle on X, and write $F(n) := F \otimes L^n$. For a smooth hypersurface $S \subset X$ with $[S] = \mathcal{O}_X(n)$, there is a standard short exact sequence of sheaves:

$$0 \longrightarrow F(-n) \longrightarrow F \longrightarrow F|_S \longrightarrow 0$$

Set $F = \Omega^1_X \otimes \mathcal{E}nd(E)$, so the sequence becomes

$$0 \longrightarrow \Omega^1_X \otimes \mathcal{E}nd(E)(-n) \longrightarrow \Omega^1_X \otimes \mathcal{E}nd(E) \longrightarrow \Omega^1_X\big|_S \otimes \mathcal{E}nd(E|_S) \longrightarrow 0$$

Passing to cohomology, we obtain the long exact sequence:

$$\cdots \longrightarrow H^1(X, \Omega^1_X \otimes \mathcal{E}nd(E)(-n)) \longrightarrow H^1(X, \Omega^1_X \otimes \mathcal{E}nd(E)) \longrightarrow H^1(S, \Omega^1_X\big|_S \otimes \mathcal{E}nd(E|_S)) \longrightarrow H^2(X, \Omega^1_X \otimes \mathcal{E}nd(E)(-n)) \longrightarrow \cdots$$

By Kodaira vanishing, for large enough n and $i < 3 \leq \dim X$,

$$H^i(X, \Omega^1_X \otimes \mathcal{E}nd(E)(-n)) = 0,$$

so for $n \geq n_0$,

$$H^1(X, \Omega^1_X \otimes \mathcal{E}nd(E)) \cong H^1(S, \Omega^1_X\big|_S \otimes \mathcal{E}nd(E|_S)).$$

The inclusion $S \hookrightarrow X$ induces the short exact sequence of vector bundles on S:

$$0 \longrightarrow \mathcal{T}_S \longrightarrow \mathcal{T}_X|_S \longrightarrow \mathcal{N}_{X/S} \longrightarrow 0$$

where $\mathcal{N}_{X/S} \cong \mathcal{O}_S(S) \cong L^n|_S$. Dualizing and tensoring with $\mathcal{E}nd(E|_S)$ gives:

$$0 \longrightarrow \mathcal{E}nd(E|_S)(-n) \longrightarrow \Omega^1_X|_S \otimes \mathcal{E}nd(E|_S) \longrightarrow \Omega^1_S \otimes \mathcal{E}nd(E|_S) \longrightarrow 0$$

The associated long exact sequence in cohomology contains:

$$H^1(S, \mathcal{E}nd(E|_S)(-n)) \longrightarrow H^1(S, \Omega^1_X|_S \otimes \mathcal{E}nd(E|_S)) \xrightarrow{\phi} H^1(S, \Omega^1_S \otimes \mathcal{E}nd(E|_S))$$

Under the isomorphism above, the image of $A(E)$ in $H^1(S, \Omega^1_X|_S \otimes \mathcal{E}nd(E|_S))$ maps via ϕ to the restriction $A(E|_S)$. It remains to show ϕ is injective for large enough n. Consider again the exact sequence with $F = \mathcal{E}nd(E)(-n)$:

$$0 \longrightarrow \mathcal{E}nd(E)(-2n) \longrightarrow \mathcal{E}nd(E)(-n) \longrightarrow \mathcal{E}nd(E|_S)(-n) \longrightarrow 0$$

By the same vanishing, $H^1(S, \mathcal{E}nd(E|_S)(-n)) = 0$ for $n \gg 0$. Thus ϕ is injective for all sufficiently large n, and $A(E|_S) = 0$ if and only if $A(E) = 0$. □

Corollary 6.3.2. *Let X be a projective manifold with $\dim X \geq 3$ and E a holomorphic vector bundle on X. Then there is a complete intersection $S \subset X$ of dimension two such that*

$$A(E) = 0 \iff A\left(E|_S\right) = 0$$

Atiyah gives an explicit example showing that Theorem 6.3.1 does not hold if $\dim X = 2$ and neither that Weil's theorem (Theorem 6.2.4) can be generalized to higher dimensions. In particular, for vector bundles on higher-dimensional varieties, the vanishing of all Chern classes does not guarantee the existence of a holomorphic connection, nor can one always check the existence of such a connection by restricting to curves.

Let $Y = \mathbb{P}^1$, Z an elliptic curve, and $X = Y \times Z$. Fix a point $(y_0, z_0) \in X$ and consider the subvarieties

$$Y_0 := Y \times \{z_0\}, \quad Z_0 := \{y_0\} \times Z \subset X.$$

Let π_Y, π_Z denote the projections onto the factors. The canonical bundle of X is

$$K_X = \pi_Y^* K_Y \otimes \pi_Z^* K_Z = \pi_Y^* \mathcal{O}_{\mathbb{P}^1}(-2) = \mathcal{O}_X(-2Z_0),$$

where $\mathcal{O}_X(Z_0)$ denotes the line bundle associated to Z_0. Consider the short exact sequence for Z_0:

$$0 \longrightarrow \mathcal{O}_X(Z_0) \longrightarrow \mathcal{O}_X(2Z_0) \longrightarrow \mathcal{O}_{Z_0} \longrightarrow 0$$

Tensoring with $\mathcal{O}_X(2Z_0)$ gives

$$0 \longrightarrow \mathcal{O}_X(Z_0) \longrightarrow \mathcal{O}_X(2Z_0) \longrightarrow \mathcal{O}_{Z_0} \longrightarrow 0$$

since $\mathcal{O}_{Z_0}(Z_0) \cong \mathcal{O}_{Z_0}$. Taking cohomology and using Serre duality, we obtain

$$H^1(X, \mathcal{O}_X(2Z_0)) \xrightarrow{\phi} H^1(Z_0, \mathcal{O}_{Z_0}) \to H^2(X, \mathcal{O}_X(Z_0)) \cong H^0(X, \mathcal{O}_X(-3Z_0))^* = 0.$$

Since $Z_0 \cong Z$ is an elliptic curve, $h^1(\mathcal{O}_{Z_0}) = 1$. Thus, ϕ is surjective and we can pick $\xi \in H^1(X, \mathcal{O}_X(2Z_0))$ such that $\phi(\xi) = \eta \neq 0$. Extensions in $H^1(X, \mathcal{O}_X(2Z_0)) = H^1(X, \mathcal{H}om(\mathcal{O}_X(-Z_0), \mathcal{O}_X(Z_0)))$ correspond to short exact sequences

$$0 \longrightarrow \mathcal{O}_X(Z_0) \longrightarrow E \longrightarrow \mathcal{O}_X(-Z_0) \longrightarrow 0$$

representing the extension class ξ. The extension E now satisfies the following properties:

(i) The total Chern class of E is trivial. Indeed by the Whitney formula:

$$\begin{aligned} c(E) &= c(\mathcal{O}_X(Z_0)) \cdot c(\mathcal{O}_X(-Z_0)) \\ &= (1 + c_1(\mathcal{O}_X(Z_0)))(1 - c_1(\mathcal{O}_X(Z_0))) \\ &= 1 - c_1^2(\mathcal{O}_X(Z_0)) \end{aligned}$$

But $\mathcal{O}_X(Z_0)$ is pulled back from a point divisor on Y, so $c_1^2(\mathcal{O}_X(Z_0)) = 0$, hence $c(E) = 1$.

(ii) E is indecomposable. Restrict the extension sequence to Z_0:

$$0 \longrightarrow \mathcal{O}_{Z_0} \longrightarrow E|_{Z_0} \longrightarrow \mathcal{O}_{Z_0} \longrightarrow 0$$

This corresponds to the nonzero class η in $H^1(Z_0, \mathcal{O}_{Z_0})$, so $E|_{Z_0}$ is a nontrivial extension and thus indecomposable. If E were decomposable, $E|_{Z_0}$ would be as well, which is not the case.

(iii) E does not admit a holomorphic connection. If we restrict the sequence corresponding to ξ to Y_0, we obtain the exact sequence

$$0 \longrightarrow \mathcal{O}_{Y_0}(y_0) \longrightarrow E|_{Y_0} \longrightarrow \mathcal{O}_{Y_0}(-y_0) \longrightarrow 0$$

The extension class lies in $H^1(Y_0, \mathcal{O}_{Y_0}(2y_0)) \cong H^0(Y_0, \mathcal{O}_{Y_0}(-4y_0))^* = 0$, so the sequence splits and

$$E|_{Y_0} \cong \mathcal{O}_{Y_0}(y_0) \oplus \mathcal{O}_{Y_0}(-y_0).$$

Both summands are nontrivial line bundles of degree ± 1 and so $A(E|_{Y_0}) \neq 0$. If $A(E) = 0$, its restriction $A(E|_{Y_0})$ would vanish, contradiction. Thus E does not admit a holomorphic connection. Therefore

$$A\left(E|_Y\right) = A(\mathcal{O}_Y(y_0)) \oplus A(\mathcal{O}_Y(-y_0)) \neq 0,$$

so the Atiyah class of E cannot be zero yielding that E doesn't admit a holomorphic connection.

The above calculations imply that Weil's theorem is already false in dimension 2. That is, the vanishing of all Chern classes of an indecomposable bundle is not a sufficient condition for the existence of a holomorphic connection.

Furthermore, Atiyah [1] proves:

Proposition 6.3.3. *Let E be an indecomposable holomorphic vector bundle on X with* $\dim X \geq 2$. *Then for every hypersurface $S \subset X$ of sufficiently high degree, $E|_S$ remains indecomposable.*

Therefore, for a smooth curve $C \subset X$ of sufficiently high degree, $E|_C$ is indecomposable with $c(E|_C) = 1$, so by Weil's theorem and the results of Section 6.2, $A(E|_C) = 0$ and $E|_C$ is flat, while $A(E) \neq 0$ and E is not flat. Thus, the existence of a holomorphic connection *cannot* be checked on curves, and Theorem 6.3.1 does not generalize to codimension greater than one.

We will now turn to the results of Biswas. Let X be a projective manifold of complex dimension n, equipped with a fixed polarization, that is, an ample line bundle L. Let ω denote a Kähler form associated to L (for example, the curvature of a positive Hermitian metric on L).

Theorem 6.3.4 (Biswas [3]). *Let X be a smooth complex projective manifold, and fix a polarization as above. Let E be a holomorphic vector bundle on X with vanishing Atiyah class: $A(E) = 0$. Assume further:*

(i) The tangent bundle $\mathcal{T}_X$ satisfies $\deg(\mathcal{T}_X) \geq 0$, and its Harder–Narasimhan filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_l \subset \mathcal{F}_{l+1} = \mathcal{T}_X$$

satisfies $\deg(\mathcal{T}_X/\mathcal{F}_l) \geq 0$;

(ii) Or, more generally, suppose that $\mathcal{T}_X$ is not semistable, but the maximal destabilizing subbundle $\mathcal{F}_1$ is locally free and the Néron–Severi group of X has rank one:

$$\mathrm{NS}(X) \otimes \mathbb{Q} \cong \mathbb{Q}.$$

Then E admits a flat holomorphic connection, i.e., E is a flat bundle.

Remark 6.3.5. Condition (i) is always satisfied if $\mathcal{T}_X$ is semistable and has non-negative degree (e.g., on Calabi–Yau manifolds, or abelian varieties). The second case covers some exceptional geometries where the Néron–Severi group is small and the maximal destabilizing subbundle is regular.

Before proving the theorem, we note the following crucial result of Simpson.

Theorem 6.3.6 (Simpson [15, Cor. 3.10]). *Let X be a smooth projective manifold, and let E be a holomorphic vector bundle. If E is semistable with vanishing rational Chern classes, then E admits a flat holomorphic connection.*

Thus, to deduce flatness of E it suffices to show that E is semistable and has all $c_i(E) = 0$ (the latter follows from $A(E) = 0$ by the Atiyah results, see previous section).

Lemma 6.3.7. *Let*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{l+1} = \mathcal{T}_X$$

be the Harder–Narasimhan filtration of the tangent bundle. If $\mu(\mathcal{T}_X) \geq 0$ and $\mu(\mathcal{T}_X/\mathcal{F}_l) \geq 0$, then every *torsion–free quotient of $\mathcal{T}_X$ has nonnegative slope.*

Proof. First, recall that the Harder–Narasimhan filtration yields

$$0 \leq \mu(\mathcal{T}_X/\mathcal{F}_l) \leq \mu(\mathcal{F}_i/\mathcal{F}_{i-1}) \quad \text{for all } i.$$

We show by descending induction on i that $\mu(\mathcal{T}_X/\mathcal{F}_i) \geq 0$ for all $i = 0, \ldots, l$. For $i = l$, this is by assumption. Now consider the short exact sequence

$$0 \longrightarrow \mathcal{F}_i/\mathcal{F}_{i-1} \longrightarrow \mathcal{T}_X/\mathcal{F}_{i-1} \longrightarrow \mathcal{T}_X/\mathcal{F}_i \longrightarrow 0$$

Because the slope of a sheaf is at least the minimum slope among its semistable subquotients, and $\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) \geq 0$ by the filtration properties, induction yields $\mu(\mathcal{T}_X/\mathcal{F}_{i-1}) \geq 0$.

Next, let $\mathcal{G} \subset \mathcal{T}_X$ be any coherent subsheaf. We prove by induction on i that if $\mathcal{G} \subset \mathcal{F}_i$, then $\mu(\mathcal{F}_i/\mathcal{G}) \geq 0$.

Base case: $i = 1$. Here, $\mathcal{F}_1$ is semistable with $\mu(\mathcal{F}_1) \geq 0$. For any $\mathcal{G} \subset \mathcal{F}_1$, the quotient $\mathcal{F}_1/\mathcal{G}$ is torsion-free, so by semistability,

$$\mu(\mathcal{F}_1/\mathcal{G}) \geq \mu(\mathcal{F}_1) \geq 0.$$

Inductive step: Assume for $i-1$ that any $\mathcal{G}' \subset \mathcal{F}_{i-1}$ satisfies $\mu(\mathcal{F}_{i-1}/\mathcal{G}') \geq 0$. Now let $\mathcal{G} \subset \mathcal{F}_i$.

Consider the natural map $\varphi : \mathcal{G} \to \mathcal{F}_i/\mathcal{F}_{i-1}$, and let $\ker\varphi$ and $\operatorname{im}\varphi$ be the kernel and image sheaves. Then, we have the following commutative diagram of short exact sequences:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker\varphi \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & \operatorname{im}\varphi \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}_{i-1} \arrow[r] \arrow[d] & \mathcal{F}_i \arrow[r] \arrow[d] & \mathcal{F}_i/\mathcal{F}_{i-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}_{i-1}/\ker\varphi \arrow[r] \arrow[d] & \mathcal{F}_i/\mathcal{G} \arrow[r] \arrow[d] & (\mathcal{F}_i/\mathcal{F}_{i-1})/\operatorname{im}\varphi \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where each row and column is exact, and all quotients are torsion-free.

The middle column shows that $\mathcal{F}_i/\mathcal{G}$ fits into an exact sequence

$$0 \to \mathcal{F}_{i-1}/\ker\varphi \to \mathcal{F}_i/\mathcal{G} \to (\mathcal{F}_i/\mathcal{F}_{i-1})/\operatorname{im}\varphi \to 0.$$

By induction, $\mu(\mathcal{F}_{i-1}/\ker\varphi) \geq 0$, and because $\mathcal{F}_i/\mathcal{F}_{i-1}$ is semistable of slope at least 0, any quotient (in particular $(\mathcal{F}_i/\mathcal{F}_{i-1})/\operatorname{im}\varphi$) has slope at least 0. Since the slope of an extension is at least the minimum of the slopes of its torsion-free quotients, we have

$$\mu(\mathcal{F}_i/\mathcal{G}) \geq 0.$$

Finally, if $\mathcal{Q}$ is any torsion-free quotient of $\mathcal{T}_X$, it can be written as $\mathcal{T}_X/\mathcal{G}$ for some subsheaf $\mathcal{G}$, so the result follows. $\square$

Lemma 6.3.8. *Let X be a complex manifold. Then every coherent $\mathcal{O}_X$-module with a holomorphic connection is a vector bundle.*

Proof. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module and

$$\nabla : \mathcal{F} \to \Omega^1_X \otimes \mathcal{F}$$

a holomorphic connection. Let $x \in X$ be any point. Since $\mathcal{F}$ is coherent, its stalk $\mathcal{F}_x$ is a finitely generated $\mathcal{O}_{X,x}$-module. Let $s_1, \ldots, s_r$ be sections of $\mathcal{F}$ defined in a neighborhood of x whose images in the fiber $\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$ (where $\mathfrak{m}_x$ is the maximal ideal at x) form a basis. By Nakayama's lemma, the s_i generate $\mathcal{F}$ in a sufficiently small neighborhood of x. To prove that the s_i are linearly independent over $\mathcal{O}_X$ (hence form a local basis), suppose there is a nontrivial relation in some neighborhood of x:

$$\sum_{i=1}^{r} \varphi_i s_i = 0,$$

with holomorphic functions φ_i, not all zero. Let ν be the minimal vanishing order at x among the φ_i, i.e.,

$$\nu := \min_i \{\mathrm{ord}_x(\varphi_i) \mid \varphi_i \neq 0\} > 0,$$

since the $s_i(x)$ form a basis. Without loss of generality, assume $\nu = \mathrm{ord}_x(\varphi_1)$. Since φ_1 vanishes at x, there exists a local holomorphic vector field ξ such that the directional derivative $\partial_\xi \varphi_1$ has vanishing order at x strictly less than ν (this is a standard property of analytic functions: the derivative decreases the order of vanishing if the function does not vanish identically). Apply the connection ∇ in the direction ξ to the relation:

$$0 = \nabla_\xi \left(\sum_{i=1}^{r} \varphi_i s_i \right) = \sum_{i=1}^{r} \left((\partial_\xi \varphi_i) s_i + \varphi_i \nabla_\xi s_i \right).$$

This gives a new relation among the s_i:

$$\sum_{i=1}^{r} (\partial_\xi \varphi_i) s_i = -\sum_{i=1}^{r} \varphi_i \nabla_\xi s_i.$$

But for each i, the function φ_i vanishes at least to order ν, so the right-hand side is a linear combination of the s_i with coefficients vanishing to order at least ν at x. On the left-hand side, by construction, the coefficient $\partial_\xi \varphi_1$ vanishes at x with order strictly less than ν. Thus, this new relation is a nontrivial relation with strictly smaller minimal vanishing order at x. Repeating this process, we construct, after finitely many steps, a nontrivial relation among the s_i with at least one coefficient not vanishing at x (i.e., vanishing order zero). But this contradicts the fact that the $s_i(x)$ are linearly independent in the fiber. Therefore, there can be no nontrivial relation among the s_i, and they form a basis for $\mathcal{F}$ in a neighborhood of x. Thus, $\mathcal{F}$ is locally free. $\square$

Lemma 6.3.9. *Let X be a smooth projective manifold, and let*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{l+1} = \mathcal{T}_X$$

be the Harder–Narasimhan filtration of the tangent bundle. If $\mathcal{F}_1$ is locally free and maximal semistable with positive slope, then $\mathcal{F}_1$ is involutive and defines a holomorphic foliation.

Proof. Consider the Lie bracket of vector fields, inducing an $\mathcal{O}_X$-linear map

$$[\cdot,\cdot] : \mathcal{F}_1 \otimes \mathcal{F}_1 \to \mathcal{T}_X/\mathcal{F}_1.$$

Since $\mathcal{F}_1 \otimes \mathcal{F}_1$ is semistable with slope $2\mu(\mathcal{F}_1) > \mu(\mathcal{F}_1)$ and every subbundle of $\mathcal{T}_X/\mathcal{F}_1$ has strictly smaller slope than $\mathcal{F}_1$, this map must vanish. Hence, $\mathcal{F}_1$ is involutive. □

Lemma 6.3.10. *Under the conditions of Lemma 6.3.9, let E be a holomorphic bundle on X admitting a holomorphic connection. If $\mathcal{F} \subset \mathcal{E}$ is the maximal semistable subsheaf, then there is a partial holomorphic connection along $\mathcal{F}_1$, that is, a map*

$$\nabla' : \mathcal{F} \to \mathcal{F}_1^* \otimes \mathcal{F}$$

satisfying the Leibniz rule.

Proof. The given holomorphic connection induces a map

$$\nabla : \mathcal{F} \to \mathcal{F}_1^* \otimes (\mathcal{E}/\mathcal{F}).$$

Since $\mathcal{F}_1^*$ has negative slope and $\mathcal{E}/\mathcal{F}$ has slope strictly smaller than $\mathcal{F}$, no nonzero homomorphism can exist between these sheaves. Thus, $\nabla = 0$, giving the desired partial connection. □

Lemma 6.3.11. *Let $\mathcal{F}_1 \subset \mathcal{T}_X$ be an involutive subbundle defining a foliation, and let L be a holomorphic line bundle with a partial connection along $\mathcal{F}_1$. Then the $(1,1)$-part of the curvature of any extension of this partial connection lies in the differential ideal generated by* $\mathrm{Ann}(\mathcal{F}_1)$.

Proof of Theorem 6.3.4. Since $A(E) = 0$, by Proposition 5.1.9 the bundle E admits a holomorphic connection

$$\nabla : \mathcal{E} \to \Omega^1_X \otimes \mathcal{E},$$

and by Corollary 5.1.11 all rational Chern classes $c_i(E)$ vanish. Hence $\mu(E) = 0$. By Simpson's theorem 6.3.6 it suffices to show that E is semistable. Let

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_m = \mathcal{E}$$

be the Harder–Narasimhan filtration of $\mathcal{E}$. Consider the holomorphic connection

$$\nabla : \mathcal{E} \to \Omega^1_X \otimes \mathcal{E}$$

induced by the vanishing of the Atiyah class, i.e., $A(E) = 0$. Using duality between $\mathcal{T}_X$ and Ω^1_X, the connection ∇ induces a $\mathbb{C}$-linear homomorphism

$$\mathcal{T}_X \otimes_{\mathbb{C}} \mathcal{E}_1 \to \mathcal{E}/\mathcal{E}_1$$

by sending $\theta \otimes s$ to the projection of $\nabla_\theta s$ on $\mathcal{E}/\mathcal{E}_1$. By the Leibniz condition, this map is in fact $\mathcal{O}_X$-linear, thus giving a homomorphism of sheaves

$$\psi : \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{E}_1 \to \mathcal{E}/\mathcal{E}_1.$$

Now, under condition (i), by Lemma 6.3.7, every torsion-free quotient of $\mathcal{T}_X$ has nonnegative slope. Since $\mathcal{E}_1$ is semistable and has maximal slope in the filtration, we have

$$\mu(\mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{E}_1) = \mu(\mathcal{T}_X) + \mu(\mathcal{E}_1) \geq \mu(\mathcal{E}_1).$$

Thus, the slope of any quotient sheaf of $\mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{E}_1$ is at least $\mu(\mathcal{E}_1)$. On the other hand, $\mathcal{E}/\mathcal{E}_1$ has strictly smaller slope than $\mu(\mathcal{E}_1)$ by the definition of the Harder–Narasimhan filtration. Therefore, the homomorphism ψ must vanish.

Since $\psi = 0$, the connection ∇ restricts to a holomorphic connection on the subbundle $\mathcal{E}_1$. By Lemma 6.3.8, $\mathcal{E}_1$ is locally free. As a coherent sheaf with a holomorphic connection, it follows that $c_1(\mathcal{E}_1) = 0$, and thus $\mu(\mathcal{E}_1) = 0$. Since we have established $\mu(E) = 0$, we obtain $\mathcal{E} = \mathcal{E}_1$. Thus, E is semistable.

By Simpson's theorem (Theorem 6.3.6), since E is semistable with vanishing rational Chern classes, it follows that E admits a flat holomorphic connection.

Assume now that $\mathcal{T}_X$ is not semistable, but its maximal destabilizing subbundle $\mathcal{F}_1$ is locally free, and that

$$\mathrm{NS}(X) \otimes \mathbb{Q} \cong \mathbb{Q}.$$

Let $W \subset E$ be the maximal semistable subbundle. Suppose, by contradiction, $W \neq E$. Then $\mu(W) > \mu(E) = 0$.

By Lemma 6.3.9, $\mathcal{F}_1$ defines a holomorphic foliation. By Lemma 6.3.10, there is a partial holomorphic connection along $\mathcal{F}_1$:

$$\nabla' : W \to \mathcal{F}_1^* \otimes W.$$

Considering $\det(W)$, Lemma 6.3.11 implies that the $(1,1)$-part of the curvature of any extension of ∇' lies in the differential ideal generated by $\mathrm{Ann}(\mathcal{F}_1)$. Thus, for sufficiently large q, we obtain

$$c_1(W)^q = 0.$$

Since $\mathrm{NS}(X) \otimes \mathbb{Q} \cong \mathbb{Q}$, we must have $c_1(W) = m[\omega]$ for some integer m, where $[\omega]$ is the Kähler class. However, $[\omega]^q \neq 0$ for $q \leq \dim(X)$, forcing

$m = 0$, and thus $c_1(W) = 0$. Hence $\mu(W) = 0$, contradicting the assumption $\mu(W) > 0$. Therefore, $W = E$, and E is semistable.

Thus, in both cases, E is semistable with vanishing rational Chern classes, and by Simpson's theorem (Theorem 6.3.6), E admits a flat holomorphic connection. $\square$

Chapter 7

Holomorphic Connections on Calabi–Yau Manifolds

A recurring theme throughout these notes has been the interplay between the topology and geometry of complex manifolds, as measured by invariants such as the Atiyah class, and the existence of holomorphic or flat connections on vector bundles. In the case of Calabi–Yau manifolds, this interplay becomes especially interesting.

The main goal of this chapter is to present and analyze recent results due to Biswas and Dumitrescu, who showed that on compact Calabi–Yau manifolds, every holomorphic vector bundle admitting a holomorphic connection also admits a flat holomorphic connection.

7.1 Calabi–Yau Geometry

In this section we briefly recall the notions of Calabi–Yau manifolds that will be used throughout the chapter. We assume the reader is familiar with the basics of complex and Kähler geometry, but for completeness we summarize the key points and fix notation.

To start with, we will recall how tensor contraction is defined.

Definition 7.1.1. Let $V_1, \dots, V_k$ be finite-dimensional vector spaces over any field, and let W be another finite-dimensional vector space. Suppose $\alpha : V_i \times V_j \to W$ is a fixed bilinear map for some $1 \le i < j \le k$. The (i, j)-contraction with respect to α is the unique linear map

$$\mathrm{tr}_{\alpha,i,j} : V_1 \otimes \cdots \otimes V_k \to W \otimes V_1 \otimes \cdots \widehat{V_i} \otimes \cdots \otimes \widehat{V_j} \otimes \cdots \otimes V_k,$$

characterized by its action on pure tensors:

$$v_1 \otimes \cdots \otimes v_k \mapsto \alpha(v_i, v_j) \otimes v_1 \otimes \cdots \widehat{v_i} \otimes \cdots \otimes \widehat{v_j} \otimes \cdots \otimes v_k,$$

for all $v_\ell \in V_\ell$, $1 \leq \ell \leq k$. Here, the hat indicates omission of that factor in the tensor product and the resulting tensor. By linearity, this extends uniquely to all tensors in $V_1 \otimes \cdots \otimes V_k$.

Consider now a Riemannian manifold (X, g) of dimension n. Recall that the Ricci tensor is defined either as the contraction of the curvature tensor

$$R = R_{ijk}{}^{l} dx^i \otimes dx^j \otimes dx^k \otimes \partial_l$$

with respect to the bilinear pairing $T^*X \times TX \to \mathbb{R}$. More precisely,

$$\begin{aligned}
\mathrm{Ric} &= \mathrm{tr}_{\alpha,1,4}(R) \\
&= R_{ijk}{}^{l}\, \mathrm{tr}_{\alpha,1,4}(dx^i \otimes dx^j \otimes dx^k \otimes \partial_l) \\
&= R_{ijk}{}^{l} \alpha(dx^i, \partial_l) dx^j \otimes dx^k \\
&= R_{ijk}{}^{l} \delta^i_l dx^j \otimes dx^k \\
&= R_{ijk}{}^{i} dx^j \otimes dx^k \\
&= R_{jk} dx^j \otimes dx^k.
\end{aligned}$$

Note that this corresponds to taking the trace of the endomorphism $Z \mapsto R(Z, X)Y$, where X and Y are vector fields and R is viewed as a section of $\mathrm{End}(TX) \otimes T^*X \otimes T^*X$.

Definition 7.1.2. A Riemannian manifold (X, g) is called Ricci-flat if $\mathrm{Ric} \equiv 0$.

Definition 7.1.3. The Ricci curvature Ric of a Kähler manifold (X, g) is the real two-form given by

$$\mathrm{Ric}(X, Y) = r(J(X), Y),$$

where $r(X, Y) = \mathrm{tr}(Z \mapsto R(Z, X)Y)$ is the ordinary Ricci tensor.

Definition 7.1.4. A Calabi–Yau manifold is a compact n-dimensional Kähler manifold (X, g) such that g is a Ricci-flat metric.

Lemma 7.1.5. *Let (X, g) be a Kähler manifold, and let ∇ denote the Levi–Civita connection (which coincides with the Chern connection in the Kähler case). Then the Ricci curvature is related to the curvature of the Chern connection by*

$$\mathrm{Ric} = i\, \mathrm{tr}(F_\nabla),$$

where F_∇ is the curvature of the Chern connection viewed as an $\mathrm{End}(TX)$*-valued 2-form.*

Proof. Let J be the complex structure and let

$$e_1, \dots, e_n,\ \widetilde{e}_1 = J(e_1), \dots, \widetilde{e}_n = J(e_n)$$

be a local orthonormal frame. Recall that the Ricci curvature is defined by

$$\mathrm{Ric}(X,Y) = \mathrm{tr}\left(Z \mapsto R(Z,Y)X\right),$$

where R denotes the Riemannian curvature tensor. Since X is Kähler, we have $J\nabla = \nabla J$ and consequently $R(X,Y)J = JR(X,Y)$. Thus,

$$\begin{aligned}
\mathrm{Ric}(X,Y) &= \sum_{i=1}^{n} g(R(e_i,Y)X, e_i) + g(R(\widetilde{e}_i,Y)X, \widetilde{e}_i) \\
&= \sum_{i=1}^{n} -g(R(e_i,Y)e_i, X) - g(R(\widetilde{e}_i,Y)\widetilde{e}_i, X) \\
&= \sum_{i=1}^{n} g(X, R(e_i,Y)e_i) + g(X, R(\widetilde{e}_i,Y)\widetilde{e}_i) \\
&= \sum_{i=1}^{n} g(X, R(e_i,Y)\widetilde{e}_i) - g(X, R(\widetilde{e}_i,Y)e_i) \\
&= -\sum_{i=1}^{n} g(X, R(\widetilde{e}_i, e_i)Y) \\
&= -\sum_{i=1}^{n} g(R(e_i, \widetilde{e}_i)X, Y).
\end{aligned}$$

On the other hand, the curvature form F_∇ associated to the connection ∇ acts as

$$F_\nabla(X,Y)Z = R(X,Y)Z.$$

Thus, taking the trace,

$$\begin{aligned}
\mathrm{tr}(F_\nabla)(X,Y) &= \mathrm{tr}(Z \mapsto R(X,Y)Z) \\
&= \sum_{i=1}^{n} g(R(X,Y)e_i, e_i) + g(R(X,Y)\widetilde{e}_i, \widetilde{e}_i) \\
&= \sum_{i=1}^{n} g(R(e_i,e_i)X,Y) + g(R(\widetilde{e}_i,\widetilde{e}_i)X,Y) + g(R(e_i,\widetilde{e}_i)X,\widetilde{Y}) \\
&\quad + g(R(\widetilde{e}_i, e_i)X, \widetilde{Y}) \\
&= i\sum_{i=1}^{n} g(R(e_i,\widetilde{e}_i)X,Y)
\end{aligned}$$

Comparing this expression to our earlier result for the Ricci tensor, we have

$$\mathrm{Ric}(X,Y) = i\,\mathrm{tr}(F_\nabla)(X,Y).$$

□

Proposition 7.1.6. *Let (X, g) be a Calabi–Yau manifold. Then $c_1(X) = 0$.*

Proof. The first Chern class $c_1(X)$ of a complex manifold X is defined as the first Chern class of its holomorphic tangent bundle $T^{1,0}X$. If ∇ denotes the Chern connection on $T^{1,0}X$, the curvature F_∇ of ∇ is an $\mathrm{End}(T^{1,0}X)$-valued $(1,1)$-form. By Definition 2.3.2, the first Chern class is given by

$$c_1(X) = \left[\frac{i}{2\pi}\operatorname{tr}(F_\nabla)\right] \in H^2(X, \mathbb{C}).$$

By Lemma 7.1.5 above, we have the relation

$$\mathrm{Ric} = i\operatorname{tr}(F_\nabla).$$

Rearranging, this gives

$$\operatorname{tr}(F_\nabla) = -i\,\mathrm{Ric}\,.$$

Therefore, the first Chern class can be rewritten as

$$c_1(X) = \left[\frac{i}{2\pi}\operatorname{tr}(F_\nabla)\right] = \left[\frac{i}{2\pi}(-i\,\mathrm{Ric})\right] = \left[\frac{1}{2\pi}\,\mathrm{Ric}\right].$$

By definition, a Calabi–Yau manifold admits a Ricci-flat Kähler metric, so $\mathrm{Ric} \equiv 0$. Thus,

$$c_1(X) = \left[\frac{1}{2\pi}\cdot 0\right] = 0 \in H^2(X, \mathbb{C}).$$

□

Recall from Corollary 5.1.11 that if X is equipped with a holomorphic connection, then the rational Chern classes vanish. This immediately implies that any compact Kähler manifold that has a holomorphic tangent bundle equipped with a holomorphic connection is Calabi–Yau. We will now illustrate that if the tangent bundle of X admits a holomorphic connection, then it also admits a flat connection. We will first recall the notion of Kähler–Einstein metrics.

Definition 7.1.7. Let (X, g) be a Kähler manifold with Kähler form ω. The metric g (or the form ω) is called **Kähler–Einstein** if the Ricci curvature Ric is a constant multiple of the metric, that is, if there exists $\lambda \in \mathbb{R}$ such that

$$\mathrm{Ric} = \lambda\omega.$$

The following theorem characterizes the possible universal covers of Kähler–Einstein manifolds under a vanishing Chern class condition:

Theorem 7.1.8 ([16, Theorem 2.13])**.** *Let (X, g) be a Kähler–Einstein manifold. Then $\widetilde{X} \cong \mathbb{C}P^n, \mathbb{C}^n$ or B^n if and only if*

$$\big(2(n+1)c_2(X) - nc_1(X)^2\big) \wedge [\omega]^{n-2} = 0.$$

Here $\cong$ means isometric up to scaling.

Now, as a consequence of Yau's proof of the Calabi conjecture we obtain:

Proposition 7.1.9. *Let (X, g) be a compact Kähler manifold. Suppose that the tangent bundle TX admits a holomorphic connection, equivalently, that the Atiyah class of TX vanishes: $A(TX) = 0$. Then TX admits a holomorphic flat connection.*

Proof. Suppose the Atiyah class of the tangent bundle vanishes, i.e., $A(TX) = 0$. Therefore the tangent bundle TX admits a holomorphic connection. Since the existence of a holomorphic connection implies that all rational Chern classes vanish (Corollary 5.1.11), we conclude in particular that

$$c_1(X) = 0 \text{ and } c_2(X) = 0.$$

The vanishing of the first Chern class $c_1(X)$ implies, via Yau's resolution of the Calabi conjecture, that X admits a Ricci-flat Kähler metric. Thus, (X, g) is a Calabi–Yau manifold and consequently a special case of a Kähler–Einstein manifold with scalar $\lambda = 0$. Since $c_1(X) = 0$ and $c_2(X) = 0$, we have:

$$\big(2(n+1)c_2(X) - nc_1(X)^2\big) \wedge [\omega]^{n-2} = 0.$$

Thus, by Tian's uniformization result (Theorem 7.1.8), the universal cover of X must be holomorphically and isometrically equivalent (up to scaling) to one of the model spaces $\mathbb{CP}^n$, $\mathbb{C}^n$, or B^n. Since X admits a Ricci-flat metric (i.e., $\lambda = 0$), among these three possibilities only the flat complex Euclidean space $\mathbb{C}^n$ admits such a metric. Thus, we have

$$\widetilde{X} \cong \mathbb{C}^n.$$

Therefore, X itself is biholomorphic to a quotient of $\mathbb{C}^n$ by a discrete subgroup Γ acting freely and properly discontinuously:

$$X \cong \mathbb{C}^n/\Gamma.$$

Since $\mathbb{C}^n$ is simply connected and flat, it admits a trivial holomorphic tangent bundle with a canonical flat connection (the standard flat connection induced by the standard affine structure). The holomorphic tangent bundle TX of X then descends from this trivial bundle under the quotient by Γ, and thus inherits a flat holomorphic connection. □

7.2 Principal Bundles and Holomorphic Connections

In this section, we discuss the more general setting of holomorphic connections on holomorphic principal bundles, extending the discussion from vector bundles.

Definition 7.2.1. Let X be a complex manifold and G a complex Lie group. A **holomorphic principal G-bundle** $\pi : E_G \to X$ is a holomorphic fiber bundle whose fiber is G, with a free and holomorphic action of G on the right.

Let us consider the holomorphic tangent bundle TE_G of the total space E_G. The group G acts on TE_G via the differential of its action on E_G. In particular, the infinitesimal action of the Lie algebra $\mathfrak{g}$ of G gives rise to the vertical tangent subbundle.

Definition 7.2.2. The *relative tangent bundle* $T_{E_G/X}$ is the subbundle of TE_G defined as the kernel of the differential of the projection:

$$T_{E_G/X} := \ker(\pi_* : TE_G \to \pi^*TX).$$

The fibers of $T_{E_G/X}$ at a point $e \in E_G$ consist of vectors tangent to the fiber $\pi^{-1}(\pi(e)) \cong G$, and are naturally identified with $\mathfrak{g}$. This identification is equivariant with respect to the adjoint action of G on $\mathfrak{g}$.

Definition 7.2.3. The **adjoint bundle** $\mathrm{ad}(E_G)$ is the holomorphic vector bundle over X associated to the principal G-bundle E_G via the adjoint representation:

$$\mathrm{ad}(E_G) := E_G \times_{\mathrm{Ad}} \mathfrak{g},$$

where the fiber over $x \in X$ is the quotient $(E_G)_x \times \mathfrak{g}/\sim$, with the equivalence relation $(e \cdot g, X) \sim (e, \mathrm{Ad}(g)X)$.

Similarly, the full tangent bundle TE_G admits a quotient by the G-action.

Definition 7.2.4. The *Atiyah bundle* of the principal bundle E_G is defined as

$$\mathrm{At}(E_G) := (TE_G)/G \to E_G/G = X.$$

It is a holomorphic vector bundle over X, whose fiber at $x \in X$ consists of G-invariant holomorphic vector fields on the total space E_G projecting to vector fields on X.

The differential $\pi_* : TE_G \to \pi^*TX$ is G-equivariant, and therefore descends to a holomorphic vector bundle morphism

$$\widetilde{\pi}_* : \mathrm{At}(E_G) \to TX.$$

The short exact sequence of holomorphic vector bundles on E_G,

$$0 \longrightarrow T_{E_G/X} \longrightarrow TE_G \xrightarrow{\pi_*} \pi^*TX \longrightarrow 0,$$

is G-equivariant. Taking the quotient of each term by the action of G yields the sequence

$$0 \longrightarrow T_{E_G/X}/G \longrightarrow TE_G/G \xrightarrow{\widetilde{\pi}_*} \pi^*TX/G \longrightarrow 0$$

Since $T_{E_G/X}/G \cong \mathrm{ad}(E_G)$, $TE_G/G = \mathrm{At}(E_G)$, and $\pi^*TX/G \cong TX$, we obtain the Atiyah sequence.

$$0 \longrightarrow \mathrm{ad}(E_G) \longrightarrow \mathrm{At}(E_G) \xrightarrow{\widetilde{\pi}_*} TX \longrightarrow 0.$$

Definition 7.2.5. A *holomorphic connection* on E_G is a holomorphic splitting of the Atiyah sequence. That is, a holomorphic $\mathcal{O}_X$-linear map

$$\nabla : TX \to \mathrm{At}(E_G)$$

such that $\widetilde{\pi}_* \circ \nabla = \mathrm{id}_{TX}$.

Remark 7.2.6. The data of such a connection ∇ amounts to a choice of horizontal subspace in $\mathrm{At}(E_G)$ complementary to $\mathrm{ad}(E_G)$.

Given a holomorphic connection ∇, one defines the curvature as follows.

Definition 7.2.7. Let ∇ be a holomorphic connection on E_G. The **curvature** of ∇ is the $\mathrm{ad}(E_G)$-valued holomorphic 2-form defined by

$$R(\nabla)(X,Y) := [\nabla(X), \nabla(Y)] - \nabla([X,Y])$$

for all local holomorphic vector fields $X, Y \in TX$.

One checks that $R(\nabla) \in H^0(X, \Omega^2_X \otimes \mathrm{ad}(E_G))$. As before, a holomorphic connection ∇ is said to be flat if its curvature vanishes identically: $R(\nabla) = 0$. In the special case where E_G is the frame bundle associated to a holomorphic vector bundle E, a holomorphic connection ∇ on E_G induces a holomorphic connection on E, and vice versa.

Lemma 7.2.8. *Let E_G be a holomorphic principal G-bundle over a compact connected Kähler manifold X admitting a holomorphic connection. Then all characteristic classes of E_G of positive degree, with coefficients in $\mathbb{R}$, vanish.*

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be any finite-dimensional complex representation of G, with V a finite-dimensional complex vector space. The associated holomorphic vector bundle is

$$E_V := E_G \times_\rho V \to X.$$

A holomorphic connection ∇ on E_G induces a holomorphic connection on every associated bundle E_V. This follows from functoriality: the splitting $\nabla : TX \to \mathrm{At}(E_G)$ induces a splitting of the Atiyah sequence for E_V, hence a holomorphic connection on E_V. By Corollary 5.1.11

$$c_i(E_V) = 0 \in H^{2k}(X, \mathbb{R}) \quad \text{for all } k > 0.$$

Let P be any invariant polynomial on the Lie algebra $\mathfrak{g}$ of G. The characteristic class $P(E_G)$ of E_G (with real coefficients) is constructed as follows:

for each such p, there exists a representation ρ and an invariant polynomial $\widetilde{p}$ on $\mathfrak{gl}(V)$ such that

$$p = \widetilde{p} \circ \rho_*.$$

Then the associated characteristic class $p(E_G)$ coincides with $\widetilde{p}(E_V)$ under the natural identification. Since all such characteristic classes of the associated vector bundle E_V vanish in positive degrees, it follows that all characteristic classes of E_G of positive degree vanish in $H^{2k}(X, \mathbb{R})$. □

7.3 Pseudostability

In Section 6.3, we introduced the fundamental correspondence due to Simpson between semistable holomorphic vector bundles and flat bundles on projective manifolds. The aim of this section is to generalize this and introduce the concept of *pseudostability*. In 2007, Biswas and Gómez [5] proved a generalization of Simpson's theorem for compact connected Kähler manifolds, yielding an equivalence between pseudostable bundles with vanishing characteristic classes of degree one and two, and flat bundles.

Recall from Chapter 4 that a torsion-free coherent sheaf (or holomorphic vector bundle) E over a compact Kähler manifold (X, ω) is called:

- *Slope stable* if for all coherent subsheaves $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(E)$, we have $\mu(\mathcal{F}) < \mu(E)$.
- *Slope semistable* if for all such $\mathcal{F}$, $\mu(\mathcal{F}) \leq \mu(E)$.
- *Polystable* if it is a direct sum of stable sheaves, all of the same slope.

Semistability is a weakening of stability, polystability is a strengthening of semistability (but not necessarily of stability: indecomposable stable implies polystable, but the converse is not true in general).

Definition 7.3.1. Let E be a holomorphic vector bundle of rank r over a compact Kähler manifold (X, ω). We say that E is **pseudostable** if there exists a filtration of holomorphic subbundles

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{\ell-1} \subset F_\ell = E$$

such that for each $i = 1, \ldots, \ell$:

(i) The quotient bundle F_i/F_{i-1} is *stable*.

(ii) All successive quotients have equal slopes:

$$\mu(F_1) = \mu(F_2/F_1) = \cdots = \mu(F_\ell/F_{\ell-1}) = \mu(E).$$

In other words, E is pseudostable if it admits a filtration whose associated graded object is a direct sum of stable bundles, all with the same slope.

The relationship among these different notions of stability can be summarized as follows:

$$\text{Polystable} \implies \text{Pseudostable} \implies \text{Semistable.}$$

- If E is *polystable*, then it is a direct sum of stable bundles of the same slope, so the trivial filtration $0 \subset E$ already shows E is pseudostable.
- If E is *pseudostable* but not polystable, the associated graded need not split holomorphically. Only the filtration is required, not a direct sum decomposition.
- Every pseudostable bundle is semistable, since any destabilizing subbundle would yield a contradiction for one of the stable pieces in the filtration.
- The converse implications do not hold: a semistable bundle need not be pseudostable, and a pseudostable bundle need not be polystable.

To illustrate that a pseudostable bundle need not be polystable, consider an elliptic curve C and recall that for an elliptic curve, the Picard variety is given by $\mathrm{Pic}^0(C) \cong C$. Let L be a degree zero line bundle on C that is not trivial and consider the non-trivial extension

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow L \longrightarrow 0.$$

Both $\mathcal{O}_C$ and L are stable bundles of degree 0 and the extension E is semistable (since every line subbundle has degree ≤ 0). However, by construction E does not split as a direct sum so it is not polystable. Pseudostability follows as E has a filtration

$$0 \subset \mathcal{O}_C \subset E,$$

and both graded pieces are stable of equal slope. More generally, if E is an extension

$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0.$$

where E_1 and E_2 are stable of the same slope, but the extension is non-trivial and does not split, then E is pseudostable but not polystable.

The key result from [5] is described in the following theorem:

Theorem 7.3.2 (Biswas–Gómez [5, Theorem 1.1])**.** *Let X be a compact connected Kähler manifold. There is an equivalence of categories between the category of pseudostable holomorphic vector bundles E over X with vanishing Chern classes $c_1(E) = 0 = c_2(E)$, and the category of flat holomorphic vector bundles over X.*

7.4 The Main Theorem

The central result of this chapter addresses Atiyah's fundamental question: on compact Kähler manifolds, when does the existence of a holomorphic connection automatically imply the existence of a flat holomorphic connection? While we have seen that this question has a negative answer in general (as demonstrated by Atiyah's examples in Section 6.3), the situation becomes remarkably different for Calabi–Yau manifolds.

The main theorem, due to Biswas and Dumitrescu, provides a complete positive answer to Atiyah's question in the Calabi–Yau setting:

Theorem 7.4.1 (Biswas–Dumitrescu [4]). *Let X be a compact Calabi–Yau manifold and let E_G be a holomorphic principal G-bundle over X, where G is a complex affine algebraic group. If E_G admits a holomorphic connection, then E_G admits a flat holomorphic connection.*

In the special case of vector bundles (corresponding to $G = \mathrm{GL}(r, \mathbb{C})$), this gives:

Corollary 7.4.2. *Let X be a compact Calabi–Yau manifold and let E be a holomorphic vector bundle over X. Then E admits a holomorphic connection if and only if E admits a flat holomorphic connection. Equivalently:*

$$A(E) = 0 \iff E \text{ is flat.}$$

Recall that on curves (Theorem 6.2.1), every holomorphic connection is automatically flat due to the vanishing of Ω^2_X. On Calabi–Yau manifolds, despite the presence of non-trivial 2-forms, the same conclusion holds, but for much deeper reasons related to the Ricci-flat geometry.

For simply connected Calabi–Yau manifolds, we obtain an even stronger result:

Corollary 7.4.3. *Let X be a compact simply connected Calabi–Yau manifold and let E_G be a holomorphic principal G-bundle over X admitting a holomorphic connection. Then E_G is holomorphically trivial, and the holomorphic connection on it coincides with the trivialization.*

Proof. By Theorem 7.4.1, E_G admits a flat holomorphic connection. Since X is simply connected, any flat bundle is necessarily trivial. Moreover, any connection on the trivial bundle is determined by a holomorphic section of $\Omega^1_X \otimes \mathfrak{g}$. However, compact simply connected Kähler manifolds do not admit non-trivial holomorphic 1-forms (as $H^{1,0}(X) = 0$ by Hodge theory applied to simply connected compact Kähler manifolds). Therefore, the only holomorphic connection is the trivial one. $\square$

The proof of Theorem 7.4.1 relies on a result that connects the Ricci-flat geometry of Calabi–Yau manifolds to the stability properties of vector bundles with holomorphic connections:

Proposition 7.4.4. *Let X be a compact Calabi–Yau manifold equipped with a Ricci-flat Kähler form ω. Let E be a holomorphic vector bundle on X admitting a holomorphic connection. Then E is pseudostable.*

Before proving this proposition, we recall that the pseudostability property is crucial because of Theorem 7.3.2. Since any bundle with a holomorphic connection has vanishing Chern classes (Corollary 5.1.11), Proposition 7.4.4 combined with Theorem 7.3.2 immediately yields the flatness of E.

We now turn to the proof of Proposition 7.4.4, which requires a delicate analysis of the interplay between holomorphic connections and the Ricci-flat geometry.

Proof of Proposition 7.4.4. Since ω is a Ricci-flat Kähler form on X, the metric is Kähler–Einstein with Einstein constant zero. A fundamental result of Lübke states that any vector bundle on a compact Kähler manifold admitting a Hermitian–Einstein structure is polystable. Since the tangent bundle TX of a Ricci-flat Kähler manifold is polystable with $c_1(TX) = 0$, we can apply results of Biswas to conclude that any vector bundle E admitting a holomorphic connection is semistable.

The heart of the proof involves analyzing polystable subsheaves of E. Let $\mathcal{F} \subset E$ be a polystable subsheaf such that:

(i) $\deg(\mathcal{F}) = 0$

(ii) The quotient $E/\mathcal{F}$ is torsion-free

The second condition ensures that $\mathcal{F}$ is reflexive. Since $\deg(\mathcal{F}) = 0 = \deg(E)$ (the latter following from the existence of a holomorphic connection), we have $\deg(E/\mathcal{F}) = 0$. The semistability of both E and $\mathcal{F}$, combined with their equal degrees, implies that $E/\mathcal{F}$ is also semistable.

Let $d = \dim_{\mathbb{C}} X$, and denote $m = \mathrm{rk}(\mathcal{F})$, $n = \mathrm{rk}(E/\mathcal{F})$. By the Bogomolov inequality, both $\mathcal{F}$ and $E/\mathcal{F}$ satisfy:

$$\Big((2m \cdot c_2(\mathcal{F}) - (m-1)c_1(\mathcal{F})^2) \cup \omega^{d-2}\Big) \cap [X] \geq 0, \tag{1}$$

$$\Big((2n \cdot c_2(E/\mathcal{F}) - (n-1)c_1(E/\mathcal{F})^2) \cup \omega^{d-2}\Big) \cap [X] \geq 0. \tag{2}$$

We now show that the inequalities in (1) and (2) are in fact equalities. To see this, let $\mathcal{Q} := E/\mathcal{F}$. Using the Whitney formula for Chern classes and the additivity in short exact sequences, we have

$$c_1(E) = c_1(\mathcal{F}) + c_1(\mathcal{Q}), \qquad c_2(E) = c_2(\mathcal{F}) + c_2(\mathcal{Q}) + c_1(\mathcal{F}) \cup c_1(\mathcal{Q}).$$

Since E admits a holomorphic connection, $c_i(E) = 0$ for all $i > 0$. Thus,

$$c_1(\mathcal{F}) + c_1(\mathcal{Q}) = 0, \tag{3}$$

$$c_2(\mathcal{F}) + c_2(\mathcal{Q}) + c_1(\mathcal{F}) \cup c_1(\mathcal{Q}) = 0. \tag{4}$$

Now consider the following calculation:

$$2(m+n)\,c_2(\mathcal{F}\oplus\mathcal{Q})-(m+n-1)\,c_1(\mathcal{F}\oplus\mathcal{Q})^2=2(m+n)\big(c_2(\mathcal{F})+c_2(\mathcal{Q})+c_1(\mathcal{F})\,c_1(\mathcal{Q})\big)$$
$$-(m+n-1)\big(c_1(\mathcal{F})^2+c_1(\mathcal{Q})^2+2c_1(\mathcal{F})\,c_1(\mathcal{Q})\big)$$
$$=\frac{m+n}{m}\big(2m\,c_2(\mathcal{F})-(m-1)c_1(\mathcal{F})^2\big)+\frac{m+n}{n}\big(2n\,c_2(\mathcal{Q})-(n-1)c_1(\mathcal{Q})^2\big)$$
$$-\frac{1}{mn}\,(n\,c_1(\mathcal{F})-m\,c_1(\mathcal{Q}))^2$$

But as $c_1(E)=0$, $c_2(E)=0$, we also have

$$2(m+n)c_2(E)-(m+n-1)c_1(E)^2=0.$$

That is,

$$0=2(m+n)\,c_2(E)-(m+n-1)\,c_1(E)^2$$
$$=\frac{m+n}{m}\big(2m\,c_2(\mathcal{F})-(m-1)c_1(\mathcal{F})^2\big)+\frac{m+n}{n}\big(2n\,c_2(\mathcal{Q})-(n-1)c_1(\mathcal{Q})^2\big)$$
$$-\frac{1}{mn}\,(n\,c_1(\mathcal{F})-m\,c_1(\mathcal{Q}))^2$$

We now take the cup product with ω^{d-2} and integrate over $[X]$:

$$\begin{aligned}&\frac{m+n}{m}\Big((2m\,c_2(\mathcal{F})-(m-1)c_1(\mathcal{F})^2)\cup\omega^{d-2}\Big)\cap[X]\\&\quad+\frac{m+n}{n}\Big((2n\,c_2(\mathcal{Q})-(n-1)c_1(\mathcal{Q})^2)\cup\omega^{d-2}\Big)\cap[X]\\&\quad-\frac{1}{mn}\Big((n\,c_1(\mathcal{F})-m\,c_1(\mathcal{Q}))^2\cup\omega^{d-2}\Big)\cap[X]=0.\end{aligned}\tag{6}$$

Observe that since both $\mathcal{F}$ and $\mathcal{Q}$ have degree zero,

$$\Big(c_1(\mathcal{F})\cup\omega^{d-1}\Big)\cap[X]=0=\Big(c_1(\mathcal{Q})\cup\omega^{d-1}\Big)\cap[X],$$

so

$$\Big((n\,c_1(\mathcal{F})-m\,c_1(\mathcal{Q}))\cup\omega^{d-1}\Big)\cap[X]=0.$$

By the Hodge index theorem (see [19]), this implies

$$\Big((n\,c_1(\mathcal{F})-m\,c_1(\mathcal{Q}))^2\cup\omega^{d-2}\Big)\cap[X]\le 0.$$

Therefore, in (6), the third term is non-negative:

$$-\frac{1}{mn}\Big((n\,c_1(\mathcal{F})-m\,c_1(\mathcal{Q}))^2\cup\omega^{d-2}\Big)\cap[X]\ge 0.$$

But since both terms in (1) and (2) are non-negative by the Bogomolov inequality, and their sum (up to positive coefficients) is at most zero, we must have equality throughout. That is,

$$\Big((2m\,c_2(\mathcal{F})-(m-1)c_1(\mathcal{F})^2)\cup\omega^{d-2}\Big)\cap[X]=0,$$
$$\Big((2n\,c_2(\mathcal{Q})-(n-1)c_1(\mathcal{Q})^2)\cup\omega^{d-2}\Big)\cap[X]=0.$$

Thus, the Bogomolov inequalities for $\mathcal{F}$ and $E/\mathcal{F}$ are in fact equalities. Now as

$$\left(\left(2m \cdot c_2(\mathcal{F}) - (m-1)c_1(\mathcal{F})^2\right) \cup \omega^{d-2}\right) \cap [X] = 0,$$

Corollary 3 in [2] implies that $\mathcal{F}$ is a polystable vector bundle admitting a projectively flat unitary connection.

We now consider two cases based on whether the holomorphic connection ∇ preserves the subsheaf $\mathcal{F}$. **Case 1:** ∇ *preserves* $\mathcal{F}$*.* In this case, $\mathcal{F}$ is a subbundle of E which is preserved by ∇. The induced connection ∇_1 on $E/\mathcal{F}$ is also holomorphic. Thus, both $(\mathcal{F}, \nabla|_{\mathcal{F}})$ and $(E/\mathcal{F}, \nabla_1)$ are holomorphic vector bundles of strictly smaller rank, each equipped with a holomorphic connection. We may now apply the same argument recursively to $E/\mathcal{F}$, ultimately reducing to the case of bundles of rank one, which are necessarily flat, and thus polystable. Therefore, E is a successive extension of polystable flat bundles, and hence itself polystable of degree zero. **Case 2:** ∇ *does not preserve* $\mathcal{F}$*.* Consider the composition

$$\mathcal{F} \hookrightarrow E \xrightarrow{\nabla} E \otimes \Omega_X^1 \xrightarrow{q_{\mathcal{F}} \otimes \mathrm{id}_{\Omega_X^1}} (E/\mathcal{F}) \otimes \Omega_X^1,$$

where $q_{\mathcal{F}} : E \to E/\mathcal{F}$ is the quotient map. Since ∇ does not preserve $\mathcal{F}$, this composition is nonzero, and hence determines a nonzero section

$$s_{\mathcal{F}} \in H^0(X, \mathcal{H}om(\mathcal{F}, E/\mathcal{F}) \otimes \Omega_X^1).$$

This induces a homomorphism

$$\Phi : \mathcal{F} \otimes TX \to E/\mathcal{F}.$$

Both $\mathcal{F}$ and TX are polystable vector bundles of degree zero, so $\mathcal{F} \otimes TX$ is also polystable of degree zero. On the other hand, $E/\mathcal{F}$ is torsion-free and semistable of degree zero, so the image $\mathrm{im}(\Phi)$ is a polystable subsheaf of degree zero in $E/\mathcal{F}$, and the quotient

$$(E/\mathcal{F})/\,\mathrm{im}(\Phi)$$

is torsion-free and semistable of degree zero. If $\mathrm{rank}(\mathrm{im}(\Phi)) = \mathrm{rank}(E/\mathcal{F})$, then $\mathrm{im}(\Phi) = E/\mathcal{F}$. In this case, consider the preimage

$$\mathcal{F}_1 := q_{\mathcal{F}}^{-1}(\mathrm{im}(\Phi)) \subset E,$$

which gives a filtration $\mathcal{F} \subset \mathcal{F}_1 \subset E$. Over a dense open subset, the inclusion $\mathcal{F}_1 \hookrightarrow E$ is an isomorphism onto its image. The natural map

$$\iota : \mathcal{F}_1 \to E$$

gives rise to a section

$$\tilde{\iota} \in H^0\left(X, \det(E) \otimes \det(\mathcal{F}_1)^*\right),$$

whose divisor is effective. But since both E and $\mathcal{F}_1$ have degree zero, this divisor must vanish, so ι is an isomorphism everywhere. Consequently, $E/\mathcal{F}_1 = 0$, and we are done. If $\mathrm{rank}(\mathrm{im}(\Phi)) < \mathrm{rank}(E/\mathcal{F})$, then $E/\mathcal{F}_1$ is a torsion-free semistable sheaf of degree zero and strictly smaller rank. For $\mathcal{F}_1$ as above, consider the composition

$$\mathcal{F}_1 \hookrightarrow E \xrightarrow{\nabla} E \otimes \Omega^1_X,$$

which induces a map

$$(\mathcal{F}_1/\mathcal{F}) \to (E/\mathcal{F}_1) \otimes \Omega^1_X,$$

and thus a further homomorphism

$$\Phi_1 : (\mathcal{F}_1/\mathcal{F}) \otimes TX \to E/\mathcal{F}_1.$$

Again, $\mathcal{F}_1/\mathcal{F}$ and TX are polystable of degree zero, so $(\mathcal{F}_1/\mathcal{F}) \otimes TX$ is polystable of degree zero, and the image of Φ_1 is polystable of degree zero. If $\Phi_1 = 0$, then ∇ preserves $\mathcal{F}_1$, and we can repeat the earlier argument for $\mathcal{F}_1$. If $\Phi_1 \neq 0$, then the rank of the image increases, and we again construct a strictly increasing filtration. At each stage, the rank of the quotient decreases. Since the process strictly reduces the rank at each step, this process terminates after finitely many steps. To make this precise, let $q_1 : E \to E/\mathcal{F}_1$ be the quotient map. Consider the subsheaf

$$\mathcal{F}_2 := q_1^{-1}(\mathrm{im}(\Phi_1)) \subset E.$$

We then repeat the arguments above with $\mathcal{F}_1$ replaced by $\mathcal{F}_2$. Proceeding inductively in this way, we obtain a filtration of subbundles of E,

$$\mathcal{F}_0 = \mathcal{F} \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_{\ell-1} \subset \mathcal{F}_\ell = E,$$

such that

(i) the vector bundles $\mathcal{F}$ and $\mathcal{F}_i/\mathcal{F}_{i-1}$ for $1 \leq i \leq \ell$ are all stable, and

(ii) $\mu(\mathcal{F}) = \mu(\mathcal{F}_1/\mathcal{F}) = \mu(\mathcal{F}_2/\mathcal{F}_1) = \cdots = \mu(\mathcal{F}_\ell/\mathcal{F}_{\ell-1})$.

Hence E is pseudostable. □

With Proposition 7.4.4 established, we can now prove the main result.

Proof of Theorem 7.4.1. Let E be a holomorphic vector bundle on a compact Calabi–Yau manifold X admitting a holomorphic connection. By Proposition 7.4.4, E is pseudostable. By Corollary 5.1.11, all Chern classes of E vanish, so in particular $c_1(E) = 0 = c_2(E)$.

By the Biswas–Gómez correspondence (Theorem 7.3.2), there is an equivalence between pseudostable holomorphic vector bundles with vanishing first and second Chern classes and flat holomorphic vector bundles. Therefore, E admits a flat holomorphic connection.

The converse is immediate: any flat holomorphic connection is in particular a holomorphic connection. $\square$

Remark 7.4.5. The result extends to principal G-bundles for complex affine algebraic groups G using the structure theory of algebraic groups. The key idea is to reduce to the reductive case using the Levi decomposition $G = R_u(G) \ltimes L(G)$, where $R_u(G)$ is the unipotent radical, and then apply the vector bundle result to the adjoint bundle. However, since vector bundles (corresponding to $G = \mathrm{GL}(r, \mathbb{C})$) cover the most important geometric applications, we focus on this case for clarity.

This completes the proof of our main theorem, showing that on Calabi–Yau manifolds, the existence of holomorphic connections and flat holomorphic connections are equivalent conditions for principal bundles with complex affine algebraic structure groups.

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