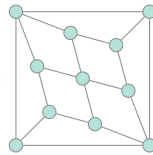


Graph Theory Midterm 2 Solutions

1. State the definition of

- a. A planar graph
- b. An Eulerian graph
- c. A u, v separating set
- d. A covering in a graph G

Solution. See the notes for the definitions.



2. Show the following graph is not Hamiltonian:

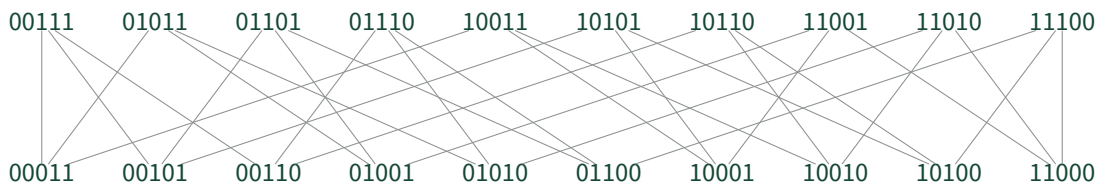
Solution. The planar graph has 9 faces with 4 edges. If a of these faces are inside the proposed Hamiltonian cycle, then $2(2a - 9) = 0$, giving $a = 9/2$, a contradiction.

3. Use the identities $M_G(x) = M_{G-e}(x) - M_{G-u-v}(x)$ and $M_G(x) = xM_{G-u}(x) - \sum_{v \text{ is adjacent to } u} M_{G-u-v}(x)$ to find a recursion for the matching polynomial for the wheel W_n in terms of matching polynomials for path graphs only.

Solution. Using the second recursion and taking u to be the middle vertex in the wheel and then using the first equation on the resulting $M_{C_n}(x)$ gives

$$\begin{aligned} M_{W_n}(x) &= xM_{C_{n-1}}(x) - (n-1)M_{P_{n-2}}(x) \\ &= x(M_{P_{n-1}}(x) - M_{P_{n-3}}(x)) - (n-1)M_{P_{n-2}}(x) \\ &= xM_{P_{n-1}}(x) - (n-1)M_{P_{n-2}}(x) - xM_{P_{n-3}}(x). \end{aligned}$$

4. Let G be the subgraph of the cube Q_{2n+1} containing those bit strings with either n 0's and $(n+1)$ 1's or with $(n+1)$ 0's and n 1's. For example, the graph G when $n = 2$ is below. Show that G has a perfect matching.



Solution. The graph G is a bipartite graph (since it is a subgraph of a bipartite graph) and every vertex is degree $n+1$. Let G have independent sets X and Y . If $S \subseteq X$, then there are $(n+1)|S|$ vertices leaving S . Since each vertex in Y has degree $n+1$, we have $|S| \leq |N(S)|$ and so Hall's theorem applies.

5. Suppose $\varepsilon(G) \neq 1$ and $e = \{u, v\}$ is an edge in G such that u and v are in a cycle in $G - e$. Show that $\varepsilon(G - e) \neq 1$.

Solution. There are two edge disjoint paths from x to y in G because $\varepsilon(G) \geq 2$. If there are not two edge disjoint paths from x to y in $G - e$, then the edge disjoint paths that would have used e can instead travel along one of the two ways around the cycle containing u and v , thus still finding two edge disjoint paths from x to y in $G - e$. Therefore $\varepsilon(G - e) \geq 2$.