Recall

$\boxed{\text{Ex}}$ $\dfrac{1}{1-\sin x}$ has singularities at

$$x = \frac{\pi}{2} + 2\pi k \qquad k \in \mathbb{Z}$$

Can we "remove" singularity at $\pi/2$ by multiplying by $(\pi/2 - x)^\alpha$

$$\lim_{x\to\pi/2} \left(\frac{\pi}{2}-x\right)^{1} \frac{1}{1-\sin x} \overset{L'H}{=} \lim_{x\to\pi/2} \frac{-1}{-\cos x}$$

not finite number since $\cos\left(\frac{\pi}{2}\right)=0$

Not high enough, try

$$\lim_{x\to\pi/2} \left(\frac{\pi}{2}-x\right)^2 \frac{1}{1-\sin x} \overset{LH}{=} \lim_{x\to\pi/2} \frac{-2\left(\frac{\pi}{2}-x\right)}{-\cos(x)}$$

$$\frac{-2(\pi/2-\pi/2)}{-\cos(\pi/2)} = 0$$

$$\overset{LH}{=} \lim_{x\to\pi/2} \frac{2}{\sin x} = \frac{2}{\sin \pi/2} = \frac{2}{1}$$

$$= 2$$

Might need to increase the power's to remove singularities.

★ Remove singularity closest to 0 for convience & mystery reason....

↳ because we want R to be the smallest singularity.

Theorem: (Main Asymptotic Result):

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. If $\exists\ \alpha > 0, R > 0$ such that:

① R is the smallest singularity in complex magnitude.

"singularity closest to 0."

② $(R-x)^{\alpha} f(x)$ is analytic at $x = R$.

③ $\lim_{x \to R} (R-x)^{\alpha} f(x) = C$ with $C \neq 0, \infty$

"Don't overdo your exponet"

Then

$$a_n \sim \frac{C n^{\alpha - 1}}{R^{n+\alpha} \Gamma(\alpha)}$$

Proof: We have $(R-x)^{\alpha} f(x)$ does not have singularity at R so...

$$(R-x)^{\alpha} f(x) = \sum_{n=0}^{\infty} c_n (R-x)^n \quad \text{for constants } c_0, c_1, \ldots$$

Note $c_0 = c$ since took lim $x \to R$

Divide by $(R-x)^{\alpha}$

$$f(x) = \boxed{\frac{c}{(R-x)^{\alpha}} + \frac{c_1}{(R-x)^{\alpha-1}} + \cdots + \frac{c_K}{(R-x)^{\alpha-K}}} + \sum_{n=K+1}^{\infty} c_n (R-x)^{n-\alpha}$$

call this $g(x) = \sum_{n=0}^{\infty} b_n x^n$ "corollary"

where K is max integer s.t $\alpha - K > 0$

Then $f(x) - g(x) = \sum_{n=0}^{\infty} (a_n - b_n) x^n$

Notice that $\sum_{n=0}^{\infty} (a_n - b_n) x^n$ does not have a singularity R or any c w/ $|c| < R$

Thus, $\sum_{n=0}^{\infty} (a_n - b_n)(R+\varepsilon)^n$ is convergent for some $\varepsilon > 0$.

So
$$0 = \lim_{n\to\infty} |a_n - b_n| (R+\varepsilon)^n$$
$$= \lim_{n\to\infty} \left| \frac{a_n}{b_n} - 1 \right| |b_n| (R+\varepsilon)^n$$
$$= \lim_{n\to\infty} \left| \frac{a_n}{b_n} - 1 \right| \frac{c n^{\alpha-1}}{R^{n+\alpha}\Gamma(\alpha)} (R+\varepsilon)^n$$
by corollary
$$= \lim_{n\to\infty} \left| \frac{a_n}{b_n} - 1 \right| \frac{c n^{\alpha-1}}{R^{\alpha}\Gamma(\alpha)} \left(\frac{R+\varepsilon}{R}\right)^n$$

$$0 = \lim_{n\to\infty} \left| \frac{a_n}{b_n} - 1 \right| \quad \frac{c\,n^{\alpha-1}}{R^{\alpha}\Gamma(\alpha)} \quad \boxed{\left(1+\varepsilon/_{12}\right)^n} \longrightarrow \text{goes to } \infty$$

is power of n

asymptotically $(\ )^n > n$.

$\to \left|\frac{a_n}{b_n} - 1\right| = 0$ since the other stuff goes to ∞.

Conclusion $\lim_{n\to\infty} \left| \frac{a_n}{b_n} - 1 \right| = 0$

therefore $a_n \sim b_n$

$\boxed{Ex}$ Let $a_n = 3^n - 2a_{n-1}$ for $n \geq 1$ with $a_0 = 1$

* first gf. function.. *

Then $A(x) = \dfrac{1}{(1-3x)(1+2x)}$ and $a_n = \frac{1}{5}3^{n+1} + (-1)^n \frac{1}{5} \cdot 2^{n+1}$

Note that $x \neq 1/3, -1/2$. $1/3 < 1/2$.

we have $\left(\frac{1}{3} - x\right) A(x) = \dfrac{3\left(1/3 - x\right)}{3(1-3x)(1+2x)}$

$$= \frac{1}{3(1+2x)}$$

$$C = \lim_{x\to 1/3} \left(\tfrac{1}{3} - x\right)^{1} A(x) = \frac{1}{5}$$

α R C

Thus $a_n \sim \dfrac{1/5\, n^{1-1}}{\left(\frac{1}{3}\right)^{n+1}\Gamma(1)} = \dfrac{1}{5} \cdot 3^{n+1}$