Discrete Mathematics Set 3

Math 435: Complete 7 parts of the following exercises.
Math 530: Exercises 1, 2a, 3, and any one part of the remaining exercises.

1. Let L_n be the set of ordered lists of the form $(c_1, \ldots, c_m)$ where $c_1, \ldots, c_m$ are cards containing disjoint sets with union $\{1, \ldots, n\}$. This is similar to hands in the exponential formula with the difference being that hands are unordered and lists are ordered.

a. Let $C(x) = \sum_{n=1}^{\infty} C_n \frac{x^n}{n!}$ where C_n is the number of cards of size n, the same as in the exponential formula. Show that $\sum_{n=0}^{\infty} \left(\sum_{\ell \in L_n} y^{(\text{number of cards in } \ell)} \right) \frac{x^n}{n!} = \frac{1}{1 - yC(x)}$.

b. Use part a. of this exercise to find the result in Set 2 Exercise 3f.

c. A permutation of n with ordered cycles is a list $(\sigma_1, \ldots, \sigma_m)$ where $\sigma_1, \ldots, \sigma_m$ are the cycles in a permutation of n. Let $\mathcal{A}_n$ be the set of permutations of n with ordered cycles and find

$$\sum_{n=0}^{\infty} \left(\sum_{\ell \in \mathcal{A}_n} y^{(\text{number of cycles in } \ell)} \right) \frac{x^n}{n!}.$$

d. Let t_n be the total number of cards in all elements in L_n. Find a formula involving $C(x)$ for $\sum_{n=0}^{\infty} t_n \frac{x^n}{n!}$.

2. The generating function for the number of permutations of n with only even sized cycles is

$$\sqrt{\frac{1}{1 - x^2}}. \tag{1}$$

(The number of such permutations is $1^2 \cdot 3^2 \cdot 5^2 \cdots (n-1)^2$ if n is even and 0 if n is odd.)

a. Use the exponential formula to prove that

$$\sum_{n=0}^{\infty} (\text{the number of permutations of } n \text{ with only odd sized cycles}) \frac{x^n}{n!} = (1 + x)\sqrt{\frac{1}{1 - x^2}}. \tag{2}$$

b. The coefficients of x^2 in (1) and (2) are the same. Therefore the number of permutations of $2n$ with only even sized cycles is equal to the number of permutations of $2n$ with only odd sized cycles. Find a bijection between these two sets of permutations.

3. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a complex valued function with nonnegative real coefficients $a_n \geq 0$. Suppose that a singularity of f with smallest complex magnitude has magnitude R (meaning that R is the radius of convergence of $f(z)$ and the series $f(z_0)$ diverges for all z_0 with $|z_0| > R$). This exercise will show that R is a singularity of f (meaning that there is a singularity with smallest magnitude that is real).

a. Use the Taylor series of $f(x)$ centered at $z = R/2$ to show that

$$f(z) = \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} \binom{n}{k} a_n (R/2)^{n-k} \right) (z - R/2)^k$$

in some neighborhood of $R/2$.

b. Looking for a contradiction, assume that R is not a singularity of f. This means that there is an $\varepsilon > 0$ such that the above expression is valid for $R+\varepsilon$. Take $R+\varepsilon$ in the above expression and prove that

$$f(R+\varepsilon) = \sum_{n=0}^{\infty} a_n (R+\varepsilon)^n.$$

Why is this a contradiction? Hint: If all terms are positive, then Tonelli's theorem for non-negative measurable functions permits interchanging the order of summation in double sums.

4. Let a_n, b_n, c_n and d_n be sequences of real numbers. Does $\lim_{n\to\infty} |a_n - b_n| = 0$ imply $a_n \sim b_n$? Does $a_n \sim b_n$ and $c_n \sim d_n$ imply $a_n + c_n \sim b_n + d_n$?

5. Let $\alpha > 0$ and n be a nonnegative integer.

a. Use induction to show that $\displaystyle\int_0^1 x^{\alpha-1}(1-x)^n \, dx = \frac{n!}{\alpha(\alpha+1)\cdots(\alpha+n)}$.

b. Assuming that the limit and integral can be interchanged, use $\displaystyle\lim_{n\to\infty} \int_0^n x^{\alpha-1}\left(1-\frac{x}{n}\right)^n dx$ to show that

$$\Gamma(\alpha) = \lim_{n\to\infty} \frac{n!n^\alpha}{\alpha(\alpha+1)\cdots(\alpha+n)} = \lim_{n\to\infty} \frac{n^{\alpha-1}}{(-1)^n\binom{-\alpha}{n}}.$$

This means that $(-1)^n\dbinom{-\alpha}{n} \sim \dfrac{n^{\alpha-1}}{\Gamma(\alpha)}$, and so the coefficient of x^n in the series expansion of $(1-x)^{-\alpha}$ is asymptotic to $\dfrac{n^{\alpha-1}}{\Gamma(\alpha)}$.

c. Justify why the limit and integral can be interchanged in part b.

6. Let a_n be the number of ordered set partitions of n. Set 2 Exercise 3 gives that $a_n = \dfrac{1}{2}\displaystyle\sum_{k=0}^{\infty} k^n 2^{-k}$, which in turn can be approximated with $\dfrac{1}{2}\displaystyle\int_0^\infty x^n 2^{-x}\,dx$. Use a substitution in this integral together with Stirling's approximation to find $a_n \approx \dfrac{\sqrt{2\pi n}}{\ln 4}\left(\dfrac{n}{e\ln 2}\right)^n$.