Systems of equations

1. Use elementary row operations to put these matrices into row echelon form.

a. $\begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$

c. $\begin{bmatrix} 2-i & 2 \\ 1 & -i \end{bmatrix}$

d. $\begin{bmatrix} 0 & 1 & 5 \\ 0 & 2 & 4 \\ 0 & 5 & 1 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 1 & 5 & 1 & 2 \\ -1 & 2 & 4 & 2 & -1 \\ 3 & 5 & 1 & 3 & 0 \end{bmatrix}$

2. Find the general solutions to the systems with these augmented matrices by putting the matrix into Row Echelon Form:

a. $\begin{bmatrix} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

b. $\begin{bmatrix} 1 & -5 & 4 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -1 & 0 & 0 & 5 \\ 0 & 1 & -2 & 0 & 7 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix}$

e. $\begin{bmatrix} 1 & -3 & 0 & -5 \\ -3 & 7 & 0 & 9 \end{bmatrix}$

f. $\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

3. Solve the system of equations:

a.

$$\begin{aligned} x_2 + 5x_3 &= -4 \\ x_1 + 4x_2 + 3x_3 &= -2 \\ 2x_1 + 7x_2 + x_3 &= -2 \end{aligned}$$

b.

$$\begin{aligned} x_1 - 5x_2 + 4x_3 &= -3 \\ 2x_1 - 7x_2 + 3x_3 &= -2 \\ -2x_1 + x_2 + 7x_3 &= -1 \end{aligned}$$

c.

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ 2x_1 - 7x_2 &= -5 \end{aligned}$$

4. Determine if the following system is consistent. Do not completely solve the system.

$$\begin{aligned} x_1 - 6x_2 &= 5 \\ x_2 - 4x_3 + x_4 &= 0 \\ -x_1 + 6x_2 + x_3 + 5x_4 &= 3 \\ -x_2 + 5x_3 + 4x_4 &= 0 \end{aligned}$$

5. Do these three planes have at least one point in common? Why?

$$\begin{aligned} 2x_1 + 4x_2 + 4x_3 &= 4 \\ x_2 - 2x_3 &= -2 \\ 2x_1 + 3x_2 &= 0 \end{aligned}$$

6. True or false:

a. Every elementary row operation is reversible.

b. A 5×6 matrix has six rows.

c. Elementary row operations on an augmented matrix never change the solution set of the associated linear system.

7. Give an example of an inconsistent system (a system with no solution) of two equations in three unknowns.

Vectors and $A\mathbf{x} = \mathbf{b}$

8. Write the linear system

$$\begin{cases} 6x - 2y = 1, \\ -3x + y = 1. \end{cases}$$

as a multiplication of the form $A\mathbf{x} = \mathbf{b}$ and then verify that there are no solutions to this system.

9. Solve the following linear systems using elementary row operations:

a. $\begin{cases} 4x_1 - x_2 = 8, \\ 2x_1 + x_2 = 1. \end{cases}$

b. $\begin{bmatrix} 1 & 0 & 5 \\ 3 & -2 & 11 \\ 2 & -2 & 6 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$

c. $\begin{bmatrix} 1+i & 1-2i \\ -1+i & 2+i \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

10. Determine if $\mathbf{b}$ is a linear combination of the columns of A when:

a. $A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$.

b. $A = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$.

11. If A is an $m \times n$ matrix and $A\mathbf{x} = \mathbf{0}$ for vectors $\mathbf{x}$ and $\mathbf{0}$, then what dimensions must $\mathbf{x}$ and $\mathbf{0}$ be?

12. List 4 vectors in the span of $\mathbf{v}_1, \mathbf{v}_2$ in the cases below. For each example, show the weights on $\mathbf{v}_1$ and $\mathbf{v}_2$ used to generate the example vectors.

a. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$.

b. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -6 \\ -2 \\ 4 \end{bmatrix}$.

13. True or false:

a. Another notation for $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 2 \end{bmatrix}$.

b. An example of a linear combination of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is $\frac{1}{2}\mathbf{v}_1$.

c. Asking whether the linear system corresponding to the augmented matrix $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{bmatrix}$ has a solution is equivalent to asking if $\mathbf{b}$ is in the span of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

d. The coefficients $c_1, \ldots, c_n$ in a linear combination $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ cannot be all 0.

14. Write the system as a matrix equation $A\mathbf{x} = \mathbf{b}$:

a.

$$\begin{aligned} 5x_1 + x_2 - 3x_3 &= -2 \\ 7x_2 + x_3 &= 0 \end{aligned}$$

b.

$$\begin{aligned} 4x_1 - x_2 &= 9 \\ 7x_1 + x_2 &= 0 \\ 7x_1 + 3x_2 &= 1 \end{aligned}$$

15. Given the following examples of A and $\mathbf{b}$, solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$. Write the solutions as a vector.

a. $A = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 5 & 2 \\ -3 & -7 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -2 \\ 4 \\ 12 \end{bmatrix}$.

b. $A = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -4 & 2 \\ 5 & 2 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$.

16. Let $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 5 & -1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}$. Is $\mathbf{u}$ in the subset of $\mathbb{R}^3$ spanned by the columns of A? Why?

17. Can every vector in $\mathbb{R}^4$ be written as a linear combination of the columns in $\begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$? Do these columns span $\mathbb{R}^4$?

18. True or false:

a. A vector $\mathbf{b}$ is a linear combination of the columns of a matrix A if and only if the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution.

b. Any linear combination of vectors can always be written as $A\mathbf{x}$ for some matrix A and vector $\mathbf{x}$.

c. If $\mathbf{x}$ is a nontrivial solution to $A\mathbf{x} = \mathbf{0}$, then every entry in $\mathbf{x}$ is nonzero.

Linear Independence

19. Write the linear system

$$\begin{cases} x_1 + x_2 - x_3 + 5x_4 = 0, \\ 2x_2 - x_3 + 7x_4 = 0, \\ 4x_1 + 2x_2 - 3x_3 + 13x_4 = 0, \end{cases}$$

as a multiplication of the form $A\mathbf{x} = \mathbf{0}$ and then verify that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s \\ s-2t \\ 2s+3t \\ t \end{bmatrix}$$

is a solution to the system for any s and t.

20. Describe all solutions to $A\mathbf{x} = \mathbf{0}$ using parameters and vectors where A is each one of these matrices:

a. $\begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix}$

b. $\begin{bmatrix} 3 & -6 & 6 \\ -2 & 4 & -2 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -2 & 3 & -6 & 5 & 0 \\ 0 & 0 & 0 & 1 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

21. Let A be an $m \times n$ matrix and suppose $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$ such that $A\mathbf{v} = \mathbf{0}$ and $A\mathbf{w} = \mathbf{0}$; in other words, $\mathbf{v}$ and $\mathbf{w}$ are solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$. Show that $c\mathbf{v} + d\mathbf{w}$ is also a solution to $A\mathbf{x} = \mathbf{0}$.

22. Determine if the following vectors are linearly independent:

a. $\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$

b. $\begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -7 \\ 2 \\ 4 \end{bmatrix}.$

c. $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}.$

23. True or false:

a. The columns of A are linearly independent if the equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution.

b. If S is a linearly dependent set, then each vector in S is a linear combination of the other vectors in S.

c. The columns of any 4×5 matrix are linearly dependent.

d. If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent and if $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent, then $\mathbf{z}$ is in the span of $\mathbf{x}$ and $\mathbf{y}$.

e. If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent and if $\mathbf{z}$ is in the span of $\mathbf{x}$ and $\mathbf{y}$, then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent.

f. If a set in $\mathbb{R}^n$ is linearly dependent, then the set contains more than n vectors.

24. The following statements are either True (in all cases) or False. If the statement is False, give an example illustrating that it is false. If true, explain why.

a. If $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are linearly independent and if $\mathbf{x} = \mathbf{y} + 2\mathbf{z}$, then the set $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent.

b. If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{R}^5$ and $\mathbf{x}$ is not a scale multiple of $\mathbf{y}$, then $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent.

c. If $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are in $\mathbb{R}^3$ and $\mathbf{z}$ is not a linear combination of $\mathbf{x}$ and $\mathbf{y}$, then the set $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly independent.

d. If $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly independent, then so is $\{\mathbf{x}, \mathbf{y}\}$.

25. Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then so is $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}$.

Linear Maps

26. Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{e}_1$ to $\mathbf{y}_1$ and $\mathbf{e}_2$ to $\mathbf{y}_2$. Find the images of $\mathbf{e}_2 = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ under T.

27. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{x}$ to $x_1\mathbf{v}_1 + x_2\mathbf{v}_2$. Find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$.

28. True or false:

a. A linear transformation is a special type of function.

b. If A is a 3×5 matrix and T is a linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, then the domain of T is $\mathbb{R}^3$.

c. If A is a $m \times n$ matrix and T is a linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, then the range of T is $\mathbb{R}^m$.

d. Every linear transformation is a matrix transformation.

e. A linear transformation always sends the zero vector to the zero vector.

f. A linear transformation preserves the operations of vector addition and scalar multiplication.

29. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the function that sends $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to $\begin{bmatrix} x_1 \\ 0 \\ x_3 \end{bmatrix}$ for all real numbers x_1, x_2, x_3. Show that T is a linear transformation.

Matrix operations and Inverses

30. Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 0 & 3 \\ 5 & 1 & 1 \\ 4 & -4 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}.$$

Perform the following matrix operations if possible:

a. AB

b. BA

c. B^2

d. $B^\top B$

e. AC

f. DBC

g. CD

31. Let A be a $m \times n$ matrix and C an $r \times s$ matrix. What dimensions must B have so that ABC is defined?

32. Find A^2, A^3 and A^4 for

a. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ -1 & 0 & 2 \end{bmatrix}$

c. $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

33. Let A and B be $n \times n$ matrices. Show that

$$(A-B)^2 = A^2 - AB - BA + B^2.$$

34. Let $A = \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{bmatrix}$ show that that A satisfies

$$A^3 + A - 26I = 0$$

where I and 0 are the 3×3 identity and zero matrices.

35. Let $A = \begin{bmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Show that $A^4 = 0$.

36. A matrix A is symmetric if $A = A^\top$. Use properties of the transpose to show that

a. $AA^\top$ is symmetric for any matrix A

b. $A + A^\top$ is symmetric for any square matrix A

c. $(ABC)^\top = C^\top B^\top A^\top$.

37. Verify by matrix multiplication that these matrices are inverses, provided that $ad - bc \neq 0$:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

38. Find the inverse of the matrix if possible:

a. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 1+i \\ 1-i & 1 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & -2 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}$

39. Use the inverse matrix to solve the system:

a. $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 4 & 5 \\ 2 & 10 & 1 \\ 4 & 1 & 8 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

40. Let $A = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$. Show that $A^\top = A^{-1}$.

41. Suppose that A satisfies $A^n = 0$ for some positive integer n. Show that the inverse to $I - A$ is

$$I + A + A^2 + \cdots + A^{n-1}.$$