Def: a permutation of $n$ is a rearrangement of $1, \dots, n$

The set of permutations of $n$ is denoted $S_n$, the symmetric group

If $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n$, then

[$\sigma_i = k \in \{1, \dots, n\}$. ex: $\sigma = 25134$, so $\sigma_1 = 2$]

① $des(\sigma) = (\#\ i$ w/ $\sigma_i > \sigma_{i+1}$ for $1 \le i \le n-1)$

Descents

② $exc(\sigma) = (\#\ i$ w/ $i > \sigma_i)$

Exceedances

③ $inv(\sigma) = (\#$ pairs $i, j$ w/ $i < j$ & $\sigma_i > \sigma_j)$

Inversions

④ $maj(\sigma) = ($the sum of all $i$ w/ $\sigma_i > \sigma_{i+1})$

Major Index

All examples of permutation statistics, which takes a permutation and outputs a number

Ex:

| $S_3$ | $des(\sigma)$ | $exc(\sigma)$ | $inv(\sigma)$ | $maj(\sigma)$ |
|---|---|---|---|---|
| 123 | 0 | 0 | 0 | 0 |
| 132 | 1 ($i=2$, $3>2$) | 1 | 1 (3,2) | 2 |
| 213 | 1 ($i=1$, $2>1$) | 1 | 1 (2,1) | 1 |
| 231 | 1 ($i=2$, $3>1$) | 2 | 2 (2,1) (3,1) | 2 |
| 312 | 1 ($i=1$, $3>1$) | 1 | 2 (3,1) (3,2) | 1 |
| 321 | 2 ($i=1$, $3>2$; $i=2$, $2>1$) | 1 | 3 (3,2) (3,2) (2,1) | 3 (1+2) |

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Questions:

① What is the max value for each perm stat in $S_n$?

$des(\sigma) \to n-1$

$exc(\sigma) \to n-1$ cannot put exceedance in final spot

$inv(\sigma) \to \binom{n}{2}$ max is decreasing perm, every pair

decreasing perm: $n \; (n-1) \; (n-2) \cdots 2 \; 1$

$(n-1)+(n-2)+(n-3)+\cdots+1+0 = \binom{n}{2}$

(can prove by induction)

$maj(\sigma) \to \binom{n}{2}$

② What is $\sum_{\sigma \in S_n} des(\sigma)$?

observe: every permutation in $S_n$ has a reverse

(123 → 321, 231 → 132, etc.)

a pair of permutation + its reverse will have $n-1$ $des(\sigma)$

[if you go up in one, you go down in another]

there are $\frac{n!}{2}$ total pairs

so the answer is: $\frac{n!}{2}(n-1)$

③ What is $\sum_{\sigma \in S_n} inv(\sigma)$?

Same argument as question #2

so the answer is: $\frac{n!}{2}\binom{n}{2}$

Theorem: des($\sigma$) and exc($\sigma$) are equidistributed over $S_n$

[This means there is a bijection $\phi: S_n \to S_n$ such that des($\sigma$) = exc($\phi(\sigma)$) $\forall \sigma \in S_n$]

Proof:

Take a permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n$

Suppose $\sigma_j = 1$

Create a cycle: $(\sigma_j \ \sigma_{j-1} \ \sigma_{j-2} \ \cdots \ \sigma_2 \ \sigma_1)$

Then find the next smallest remaining integer not in a cycle and repeat, creating a new cycle

ex: $\boxed{1}$ 3 5 $\boxed{2}$ 6 $\boxed{4}$ $\boxed{7}$ $\boxed{8}$ $\boxed{9}$

new permutation: (1)(2 5 3)(4 6)(7)(8)(9)

ex: 6 3 8 2 $\boxed{1}$ $\boxed{4}$ 9 $\boxed{5}$ $\boxed{7}$

new permutation: (1 2 8 3 6)(4)(5 9)(7)

[cycle notation to one-line notation:

(1)(2 5 3)(4 6)(7)(8)(9)

| 1 | 3 | 5 | 6 | 2 | 4 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

(e.g. 2 in 5th spot, 5 in 3rd spot, etc)]

To invert this function:

put smallest # in each cycle first

sort cycles by smallest number

report each cycle backwards

Thus, the function is a bijection

Proof To Be continued...