Lecture 28 Week 7 Thurs 11/8/23

Symmetric Functions

Def. A symmetric function in the variables $x_1, \dots, x_N$ is a polynomial f in $x_1, \dots, x_N$ such that $f(x_1, \dots, x_N) = f(x_{\sigma(1)}, \dots, x_{\sigma(N)})$ for all $\sigma \in S_N$.

Ex. $2x_1^3 + 2x_2^3 + 2x_3^3 + x_1x_2x_3 + x_1 + x_2 + x_3 + 1$ is symmetric

Def. Let $\Lambda_n(x_1, \dots, x_N)$ denoted the vector space of symmetric polynomials in $x_1, \dots, x_N$ of degree n. over $\mathbb{Q}$. (every term has degree n)

Ex. $\Lambda_4(x_1, x_2, x_3, x_4)$ has the following basis:

$\{ m_{(4)} := x_1^4 + x_2^4 + x_3^4 + x_4^4,$

$m_{(3,1)} := x_1^3 x_2 + x_3^3 x_4 + \cdots,$

$m_{(2,2)} := x_1^2 x_2^2 + x_1^2 x_3^2 + \cdots,$

$m_{(2,1,1)} := x_1^2 x_2 x_3 + x_1^2 x_2 x_4 + \cdots,$

$m_{(1,1,1,1)} := x_1 x_2 x_3 x_4. \}$

Pretty clearly both an independent and a spanning list.

Def. The monomial symmetric function m_λ is the symmetric function with fewest terms containing $x_1^{\lambda_1} x_2^{\lambda_2} \cdots$ if $\lambda = (\lambda_1, \lambda_2, \dots)$

Thm. $\{m_\lambda : \lambda \vdash n\}$ is a basis for Λ_n.

Ex. $8x_1^4 + \cdots + 30 x_1 x_2 x_3 x_4 - x_1^3 x_2 - \cdots = 8 m_{(4)} + 30 m_{(1^4)} - m_{(3,1)}$.

Coro. The dimension of Λ_n is the # of $\lambda \vdash n$.

Def. The elementary symmetric function $e_n(x_1, x_2, \dots)$ is defined by

$$\sum_{n=0}^{\infty} e_n(x_1, x_2, \dots) z^n = \prod_i (1 + x_i z)$$

Ex. $e_3(x_1, x_2, x_3, x_4)$ is the coefficient of z^3 in $(1+x_1 z)(1+x_2 z)(1+x_3 z)(1+x_4 z)$, which is $x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4$.

Thm. $e_n = m_{(1^n)}$.

Note: Elementary symmetric functions are "square free"

Def. The homogeneous symmetric function $h_n(x_1, x_2, \dots)$ is defined by $\displaystyle\sum_{n=0}^{\infty} h_n(x_1, x_2, \dots) z^n = \prod_i \frac{1}{1 - x_i z}$

Ex. $h_3(x_1, x_2, x_3)$ is the coefficient of z^3 in $\dfrac{1}{(1-x_1 z)(1-x_2 z)(1-x_3 z)}$

$(1 + x_1 z + (x_1 z)^2 + \cdots)(1 + x_2 z + (x_2 z)^2 + \cdots)(1 + x_3 z + (x_3 z)^2 + \cdots)$.

This coeff. is $x_1 x_2 x_3 + x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2$

$+ x_1^3 + x_2^3 + x_3^3 = m_{(1,1,1)} + m_{(2,1)} + m_{(3)}$