Graph Theory Midterm 2 Review

Definitions: bridge, covering, crossing number, the cube graph Q_n, directed graph, (u, v) disconnecting set, dual graph, edge connectivity $\varepsilon(G)$, Eulerian graph, face, flow value, flow, Hamiltonian graph, matching polynomial, matching, network, perfect matching, planar graph, Platonic solid, a matching saturates a set, (u, v) separating set, trail, vertex connectivity $\kappa(G)$, walk.

Theorems:

- *Euler*: If G is connected and planar with V vertices, E edges, and F faces, then $V - E + F = 2$.
- If G is planar, then $E \leq 3V - 6$. If bipartite, then $E \leq 2V - 4$.
- *Kuratowski*: A graph is not planar if and only if K_5 or $K_{3,3}$ can be found by contracting edges or removing vertices and edges.
- A planar graph has a vertex of degree at most 5.
- *The four color theorem*: If G is planar, then $\chi(G) \leq 4$.
- There are exactly 5 Platonic solids.
- A connected graph is Eulerian if and only if every vertex degree is even.
- *Bondy-Chvátal*: Let u and v be non-adjacent vertices such that the sum of the degrees of u and v is at least n, then total number of vertices. Then G is Hamiltonian if an only if $G + \{u, v\}$ is Hamiltonian.
- Let C be a Hamiltonian cycle in a planar graph, let inside(i) be the number of i-edged faces inside C and outside(i) be the number of i-edged faces outside C. Then $\sum_i (i-2)\left(\text{inside}(i) - \text{outside}(i)\right) = 0$.
- *Menger*: The minimum size of a u, v disconnecting (separating) set is the maximum number of edge (vertex) disjoint u, v paths.
- *Max flow min cut*: Let N be a network with vertices u, v. The maximum value for a flow from u to v is equal to the minimum weight u, v disconnecting set.
- The Ford-Fulkerson algorithm produces a network flow with maximum flow value.
- If the maximum degree in a graph is 3, then $\kappa(G) = \varepsilon(G)$.
- *Hall*: Let G be a bipartite graph with independent sets X and Y. Then there is a matching for G that saturates X if and only if $|S| \leq |N(S)|$ for all $S \subseteq X$.
- *Kőnig*: The maximum number of edges in a matching in a bipartite graph is equal to the minimum number of vertices in a covering.
- *Tutte*: For any subset S of vertices in a graph G, let $\text{odd}_G(S)$ denote the number of components of $G - S$ that have an odd number of vertices. Then G has a perfect matching if and only if $\text{odd}_G(S) \leq |S|$ for all subsets S of vertices.
- The matching polynomial satisfies $M_G(x) = M_{G-e}(x) - M_{G-u-v}(x)$ and $M_G(x) = xM_{G-u}(x) - \sum_{v \text{ is adjacent to } u} M_{G-u-v}(x)$.

Extra exercises:

1. Let u, v, w be vertices in a graph G with $\kappa(G) \geq 3$. Show that there is a cycle in G that contains u, v and w.

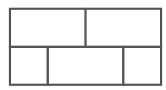
2. Find a recursion for the matching polynomial for $K_{m,n}$.

3. Show that if G is planar and has at least 11 vertices, then G^c is not planar.

4. Show that if G is Hamiltonian, then $\kappa(G) \geq 2$.

5. Show that the line graph of an Eulerian graph is Eulerian and Hamiltonian.

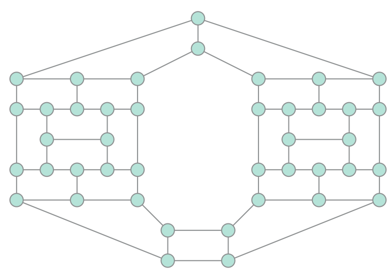
6. The "five room puzzle" is a brainteaser that asks to find a continuous path in $\mathbb{R}^2$ that passes through exactly once each wall (without going through a corner) in each of the five rooms depicted below:



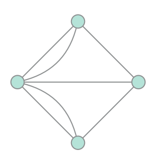
Show that the five room puzzle is impossible.

7. Show that $K_{m,n}$ is Hamiltonian if and only if $m = n$.

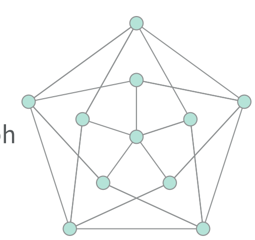
8. Show the following graph is not Hamiltonian:



9. The seven bridges of Königsburg is a famous problem that is equivalent to asking for an Eulerian trail that need not start and end at the same vertex in the this multigraph:



Show that finding such a trail is not possible.

10. Show that the Grötzsch graph  is Hamiltonian.

11. Let E be a minimal disconnecting set of edges. Why does E share an even number of edges with every cycle?

12. Show that a Hamiltonian graph with $2n$ vertices has a perfect matching.