Lecture 9 - Week 2 Thurs 10/5/23

The exponential formula: $\sum_{n=0}^{\infty}\left(\sum_{h\in H_n} y^{\#\text{ of cards in } h}\right)\frac{x^n}{n!} = e^{y\sum_{n=1}^{\infty} c_n \frac{x^n}{n!}}$

Ex: Last time: perms in cycle notation. $\sum_{n=0}^{\infty}\left(\sum_{\sigma\in S_n} y^{\#\text{ of cycles in }\sigma}\right)\frac{x^n}{n!} = e^{y\ln\left(\frac{1}{1-x}\right)}$
$= (1-x)^{-y}$

Q. What is the expected # of cycles in a randomly selected permutation of n?
Taking $\frac{\partial}{\partial y}$ and taking $y=1$, we get

$n! = $ # of perms of S_n. So take coeff of $x_n!$

$$\sum_{n=0}^{\infty}\left(\sum_{\sigma\in S_n}(\#\text{ of cycles in }\sigma)\right)\frac{x^n}{n!} = e^{\ln\left(\frac{1}{1-x}\right)}\ln\left(\frac{1}{1-x}\right)$$

$$=\left(\frac{1}{1-x}\right)\ln\left(\frac{1}{1-x}\right) = \left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=1}^{\infty}\frac{x^n}{n}\right) = \sum_{n=0}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{k}\cdot 1\right)x^n$$

A: $\sum_{k=1}^{n}\frac{1}{k}$, the n^{th} harmonic number.

Ex: How many $\sigma\in S_n$ do not have any cycles of length 1?

$$\sum_{n=0}^{\infty}\left(\sum_{\sigma\in S_n} y^{\#\text{ of cycles in }\sigma}\right)\frac{x^n}{n!} = e^{y\sum_{n=2}^{\infty}(n-1)!/n!\, x^n} = e^{y\sum_{n=2}^{\infty}\frac{x^n}{n}} = e^{y\left(\ln\left(\frac{1}{1-x}\right)-x\right)}$$

$$= \frac{e^{-yx}}{(1-x)^y}.$$

Woah!

Q. What is the probability that a randomly constructed permutation in S_n has no 1-cycle?

A. Plug in $y=1$! (You're already dividing by $n!$).
It's the coefficient of x^n in

$$\frac{e^{-x}}{1-x} = \left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n\right)\left(\sum_{n=0}^{\infty}x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^k}{k!}\right)x^n$$

So the answer is $\sum_{k=0}^{n}\frac{(-1)^k}{k!} \sim \frac{1}{e}$

Interesting that the probability approximately is constant.
On midterm!

On midterm: Find a generating function for the # of permutations of n that have
- no cycles of length 5?
- cycles of length 1 or 2 only?
- only cycles of even length?

$c_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (n-1)! & \text{if } n \text{ is even} \end{cases}$

Last one: $\displaystyle\sum_{n=0}^{\infty} \Bigg(\sum_{\substack{\sigma \in S_n \\ \text{tot even}}} y^{\#\text{ of cycles in } \sigma} \Bigg) \frac{x^n}{n!} = e^{y \sum_{n=1}^{\infty} \frac{x^{2n}}{2n}} = e^{y \frac{1}{2} \sum_{n=1}^{\infty} \frac{(x^2)^n}{n}} = e^{y \frac{1}{2} \ln\left(\frac{1}{1-x^2}\right)}$

Taking $y=1$, $\displaystyle\sum_{n=0}^{\infty} (\#\, \sigma \in S_n \text{ w/ only even cycles}) \frac{x^n}{n!} = (1-x^2)^{-1/2}$

$$= \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n x^{2n}$$

The # of permutations of $2n$ w/ only even cycles is $(-1)^n \binom{-1/2}{n} (2n)! = (*)$
(If n is odd, there are no such permutations!)

$$(*) = (-1)^n \frac{(-1/2)(-1/2-1)\cdots(-1/2-n+1)}{1\cdot 2\cdot 3 \cdot \cdots \cdot n} \big(1\cdot 2\cdot 3 \cdot \cdots \cdot (2n-1)(2n)\big) ✓$$

$$= 1^2 \cdot 3^2 \cdot 5^2 \cdots (2n-1)^2$$

(The even terms in the $(2n)!$ are "hiding" an $n!$. Cancel w/ $n!$ in denominator. Left with 2^n. Distribute through fractions in numerator.)

Apparently the # of $\sigma \in S_{2n}$ w/ only even cycles is the same as the # of $\sigma \in S_{2n}$ w/ only odd cycles. On next wk's HW.