Shared Notes for Friday

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Friday

Theorem 1 *Let R be the smallest singularity of our generating function (in complex magnitude),*

$$f(x) = \sum_{n=0}^{\infty} [a_n x^n]$$

and suppose

$$\lim_{x \to R} (R - x)^{\alpha} f(x) = c, \;\; c \neq 0, \infty$$

Then

$$a_n \sim \frac{c n^{\alpha - 1}}{R^{n+\alpha} \Gamma(\alpha)`}$$

Example 1 What is the probability that a permutation of n has no one cycles?
Answer:

$$\sum_{n=0}^{\infty} \left[\frac{\text{(number of cycles w/no one cycles)}}{n!} x^n \right] = e^{\sum_{n=2}^{\infty} \left[(n-1)! \frac{x^n}{n!}\right]}$$

Recall, we start at $n = 1$ because there are no cycles w/no one cycle for $n < 2$.

$$\begin{aligned} e^{\sum_{n=2}^{\infty} \left[(n-1)! \frac{x^n}{n!}\right]} &= e^{\sum_{n=1}^{\infty} \left[(n-1)! \frac{x^n}{n!}\right] - x} \\ &= e^{\log\left(\frac{1}{1-x}\right) - x} \\ &= \frac{e^{-x}}{1 - x} \end{aligned}$$

Which is perfect, since we have a singularity at $x = 1$. Now, we need

$$\lim_{x \to 1} (1 - x) \frac{e^{-x}}{1 - x} = e^{-1}$$

To be clear, $R = 1$, $\alpha = 1$, and $c = e^{-1}$. Therefore, we have

$$a_n \sim \frac{e^{-1} n^{\alpha - 1}}{1^{n+1} \Gamma(1)} = e^{-1}$$

Example 2 Let a_n be the number of 2-regular graphs on n vertices. We have seen that

$$\sum_{n=0}^{\infty}\left[\frac{a_n}{n!}x^n\right] = \frac{e^{-x/2-x^2/4}}{\sqrt{1-x}}$$

We see that

$$\lim_{x\to 1}(1-x)^{1/2}\frac{e^{-x.2-x^2/4}}{\sqrt{1-x}} = e^{-1/2-1/4} = e^{-3/4}$$

So here we have $c = e^{-3/4}$, $R = 1$, and $\alpha = \frac{1}{2}$. Therefore,

$$\begin{aligned}\frac{a_n}{n!} &\sim \frac{e^{-3/4}n^{-1/2-1}}{1^{n+1/2}\Gamma(1/2)} = \frac{e^{-3/4}n^{-1/2}}{\sqrt{\pi}}\\ a_n &\sim n!\frac{1}{e^{3/4}\sqrt{n\pi}}\\ &\sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\frac{1}{e^{3/4}\sqrt{n\pi}}\\ &= \frac{\sqrt{2}n^n}{e^{n+3/4}}\end{aligned}$$

Example 3 Let $F(x) = \frac{1}{1-x-x^2}$, the generating function for the fibbonaci sequence. This has singularities at $x = -\frac{1\pm\sqrt{5}}{2}$. Take $R = \frac{-1+\sqrt{5}}{2}$. Now we take the limit

$$\lim_{x\to R}\frac{(R-x)}{1-x-x^2} = \frac{1}{\sqrt{5}}$$

Therefore, $C = \frac{1}{\sqrt{5}}$, $R = \frac{-1+\sqrt{5}}{2}$, and $\alpha = 1$, and

$$a_n \sim \frac{1}{\sqrt{5}R^{n+1}}$$

Example 4 Let $C(x) = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty}[c_n x^n]$, the generating function for the Catalan numbers. Consider the following

$$xC'(x) = \frac{1-2x-\sqrt{1-4x}}{2x^2\sqrt{1-4x}} = \sum_{n=0}^{\infty}[nc_n x^n]$$

Then $\lim_{x\to 0} xC'(x) = 1$, so 0 is not really a problem. Thus, $x = \frac{1}{4}$ is the smallest singularity. Consider,

$$\lim_{x\to\frac{1}{4}}\left(\frac{1}{4}-x\right)^{1/2}xC'(x) = 2$$

In conclusion, we have $C = 2$, $\alpha = \frac{1}{2}$, and $R = \frac{1}{4}$, and

$$\begin{aligned}nc_n &\sim \frac{2n^{-1/2}}{\left(\frac{1}{4}\right)^{n+1/2}\sqrt{\pi}}\\ c_n &\sim \frac{4^{n+1}}{n^{3/2}\sqrt{\pi}}\end{aligned}$$