Matrix Multiplication

1. Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 0 & 3 \\ 5 & 1 & 1 \\ 4 & -4 & 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}.$$

Perform the following matrix operations if possible:

a. AB

b. BA

c. B^2

d. $B^\top B$

e. AC

f. DBC

g. CD

2. Let A be a $m \times n$ matrix and C an $r \times s$ matrix. What dimensions must B have so that ABC is defined?

3. Find A^2, A^3 and A^4 for

a. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ -1 & 0 & 2 \end{bmatrix}$

c. $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

4. Let A and B be $n \times n$ matrices. Show that

$$(A - B)^2 = A^2 - AB - BA + B^2.$$

5. Let $A = \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{bmatrix}$ show that that A satisfies

$$A^3 + A - 26I = 0$$

where I and 0 are the 3×3 identity and zero matrices.

6. Let $A = \begin{bmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Show that $A^4 = 0$.

7. A matrix A is symmetric if $A = A^\top$. Use properties of the transpose to show that

a. $AA^\top$ is symmetric for any matrix A

b. $A + A^\top$ is symmetric for any square matrix A

c. $(ABC)^\top = C^\top B^\top A^\top$.

Linear Systems

8. Write the linear system

$$\begin{cases} x_1 + x_2 - x_3 + 5x_4 = 0, \\ 2x_2 - x_3 + 7x_4 = 0, \\ 4x_1 + 2x_2 - 3x_3 + 13x_4 = 0, \end{cases}$$

as a matrix multiplication of the form $A\mathbf{x} = \mathbf{0}$ and then verify that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s \\ s - 2t \\ 2s + 3t \\ t \end{bmatrix}$$

is a solution to the system for any s and t.

9. Write the linear system

$$\begin{cases} 6x - 2y = 1, \\ -3x + y = 1. \end{cases}$$

as a matrix multiplication of the form $A\mathbf{x} = \mathbf{b}$ and then verify that there are no solutions to this system.

10. Let A be an $m \times n$ matrix.

a. If $A\mathbf{x} = \mathbf{0}$ for vectors $\mathbf{x}$ and $\mathbf{0}$, then what dimensions must $\mathbf{x}$ and $\mathbf{0}$ be?

b. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors that satisfy $A\mathbf{x} = \mathbf{0}$ and let c be a constant. Show that $\mathbf{x} + c\mathbf{y}$ satisfies $A\mathbf{x} = \mathbf{0}$.

Elementary Row Operations

11. Use elementary row operations to put these matrices into reduced row echelon form and then state the rank of each matrix.

a. $\begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$

c. $\begin{bmatrix} 2 - i & 2 \\ 1 & -i \end{bmatrix}$

d. $\begin{bmatrix} 0 & 1 & 5 \\ 0 & 2 & 4 \\ 0 & 5 & 1 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 1 & 5 & 1 & 2 \\ -1 & 2 & 4 & 2 & -1 \\ 3 & 5 & 1 & 3 & 0 \end{bmatrix}$

Solving linear systems

12. Solve the following linear systems using elementary row operations (Gaussian Elimination):

a. $\begin{cases} 4x_1 - x_2 = 8, \\ 2x_1 + x_2 = 1. \end{cases}$

b. $\begin{cases} 4x - y - z = 1, \\ x + y + z = 3. \end{cases}$

c. $\begin{cases} x - y - z = 0, \\ x + y + z = 0, \\ 2x - 2y = 0. \end{cases}$

d. $\begin{bmatrix} 4 & 2 \\ 4 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 0 & 5 \\ 3 & -2 & 11 \\ 2 & -2 & 6 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 0 & 3 & 6 & 9 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

g. $\begin{bmatrix} 1+i & 1-2i \\ -1+i & 2+i \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

h. $\begin{bmatrix} 2+i & i & 3-2i \\ i & 1-i & 4+3i \\ 3-i & 1+i & 1+5i \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Inverse matrices

13. Verify by matrix multiplication that these matrices are inverses, provided that $ad - bc \neq 0$:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

14. Find the inverse of the matrix if possible:

a. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 1+i \\ 1-i & 1 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & -2 \end{bmatrix}$

e. $\begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 4 & 1 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}$

15. Use the inverse matrix to solve the system:

a. $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 4 & 5 \\ 2 & 10 & 1 \\ 4 & 1 & 8 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

16. Let $A = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$. Show that $A^\top = A^{-1}$.

17. Suppose that A satisfies $A^n = 0$ for some positive integer n. Show that the inverse to $I - A$ is

$$I + A + A^2 + \cdots + A^{n-1}.$$

Determinants

18. Calculate the determinant:

a. $\begin{vmatrix} 3 & 5 & 7 \\ -1 & 2 & 4 \\ 6 & 3 & -2 \end{vmatrix}$

b. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

c. $\begin{vmatrix} 3 & 5 & -1 & 2 \\ 2 & 1 & 5 & 2 \\ 3 & 2 & 5 & 7 \\ 1 & -1 & 2 & 1 \end{vmatrix}$

19. Let A be invertible. Show that $\det(A^{-1}) = \dfrac{1}{\det A}$.

20. Let A and B be $n \times n$ with $\det A = 5$ and $\det B = -4$. Evaluate the determinant:

a. $\det(AB)$

b. $\det(A^\top BA)$

c. $\det(A^{-1}BA)$

d. $\det(3A)$

e. $\det C$ where C is A with its first two columns interchanged

f. $\det C$ where C is A with its first row multiplied by 2

21. Let A satisfy $A^\top A = I$. Show that $\det A = \pm 1$.

Subspaces

22. Either show that S is a subspace of the vector space V or give an example showing why it is not:

a. $V = \mathbb{R}^3$, S is the set of vectors of the form $\begin{bmatrix} x \\ 2x \\ 3x \end{bmatrix}$.

b. $V = \mathbb{R}^4$, S is the set of vectors of the form $\begin{bmatrix} x \\ y \\ x \\ 0 \end{bmatrix}$.

c. $V = \mathbb{R}^4$, S is the set of vectors of the form $\begin{bmatrix} x \\ 1 \\ 2x \\ 0 \end{bmatrix}$.

d. $V = \mathbb{R}^n$, S is the set of solutions to $A\mathbf{x} = \mathbf{0}$ where A is a fixed $m \times n$ matrix.

e. V is the vector space of 2×2 matrices with entries in $\mathbb{R}$, S is the set of matrices A with $\det A = 1$.

f. V is the vector space of 3×3 matrices with entries in $\mathbb{R}$, S is the set of upper triangular matrices.

g. V is the vector space of $n \times n$ matrices with entries in $\mathbb{R}$, S is the set of invertible matrices.

h. V is the vector space of real valued functions with domain $\mathbb{R}$, S is the set of functions $f(x)$ that satisfy $f(3) = 0$.

i. V is the vector space of real valued functions with domain $\mathbb{R}$, S is the set of functions of the form $ax^2 + bx + c$ where a, b, c are real numbers.

j. V is the vector space of real valued functions with domain $\mathbb{R}$, S is the set of solutions to the differential equation $y''(x) + y(x) = 0$.

Span

23. Determine if the set of vectors span $\mathbb{R}^3$:

a. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$

b. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right\}$

c. $\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \right\}$

24. Find a set of vectors that span the subspace S of the vector space V:

a. V is the space of 2×3 matrices with entries in $\mathbb{R}$, S is the set of matrices with entries that sum to 0.

b. V is the space of $n \times n$ matrices with entries in $\mathbb{R}$, S is the set of upper triangular matrices.

c. V is $\mathbb{R}^3$, S is the set of solutions to $x - 2y - z = 0$.

d. V is the space of polynomials of degree 5 or less with coefficients in $\mathbb{R}$, S is the set of polynomials p that satisfy $p'(x) = 0$.

Linear Independence

25. Determine if the following sets of vectors are linearly independent:

a. $\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \right\}$

b. $\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \right\}$

c. $\left\{ \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \right\}$

d. $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

e. $\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 1 \\ 0 \end{bmatrix} \right\}$

26. Determine if the given functions are linearly independent on the given interval I:

a. $1, x, x^2; I = \mathbb{R}$.

b. $\sin x, \cos x, \tan x; I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

c. $e^x, e^{-x}, \cosh x; I = \mathbb{R}$.

d. $e^x, x, \sin x; I = \mathbb{R}$.

e. $1 + x + x^2, 1 + x - x^2, 1 + x^2, 1 - x^2; I = \mathbb{R}$.

f. $x, \begin{cases} 1 & \text{if } x = 0, \\ x & \text{if } x \neq 0; \end{cases} I = \mathbb{R}$.

g. e^{ax}, e^{bx}, e^{cx} for a, b, c distinct; $I = \mathbb{R}$.

27. Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then so is $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}$.

Bases

28. Determine if the given set of vectors is a basis for the subspace S of the vector space V:

a. $V = \mathbb{R}^2, S = \mathbb{R}^2, \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.

b. $V = \mathbb{R}^3, S = \mathbb{R}^3, \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \right\}$.

c. V is space of 2×2 matrices with entries in $\mathbb{R}$, S is the subspace containing matrices with entries that sum to 0, $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}, \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix} \right\}$.

29. Find a basis for the nullspace of the matrix (a basis for the subspace of $\mathbb{R}^n$ containing solutions to $A\mathbf{x} = \mathbf{0}$):

a. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & -1 & 4 \\ 2 & 3 & -2 \\ 1 & 2 & -2 \end{bmatrix}$

30. Find a basis and the dimension of the subspace S of the vector space V:

a. V is the set of real valued functions on $\mathbb{R}$, S is the set of solutions to $f''(x) = 0$.

b. V is the set of polynomials of degree 3 or less with coefficients in $\mathbb{R}$, S is the set of polynomials p that satisfy $p(-1) = 0$.

c. V is $\mathbb{R}^3$, S is the span of the vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} \right\}$.

d. V is $\mathbb{R}^3$, S is the span of $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} \right\}$.

e. V is the space of 2×2 matrices over $\mathbb{R}$, S is the span of $\left\{ \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 5 & -6 \\ -5 & 2 \end{bmatrix} \right\}$.

f. V is the space of 4×4 matrices over $\mathbb{R}$, S is the set of matrices A that satisfy $A^\top = -A$.

Eigenvalues and Eigenvectors

31. Find the eigenvalues and eigenvectors:

a. $\begin{bmatrix} 5 & -4 \\ 8 & -7 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 6 \\ 2 & -3 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

d. $\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$

e. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

32. Show that if λ is an eigenvalue for an invertible matrix A, then λ^{-1} is an eigenvalue for A^{-1}.

33. Show that if A is square, then A and $A^\top$ have the same eigenvalues.

Diagonalization

34. Diagonalize the matrix A if possible: (provide a matrix S and D such that $A = S^{-1}DS$).

a. $\begin{bmatrix} -9 & 0 \\ 4 & -9 \end{bmatrix}$

b. $\begin{bmatrix} -7 & 4 \\ -4 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 7 \\ 1 & 1 & -3 \end{bmatrix}$

e. $\begin{bmatrix} -2 & 1 & 4 \\ -2 & 1 & 4 \\ -2 & 1 & 4 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

g. $\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Separable DEs

35. Verify that $y(t) = A\cos(\omega t - \phi)$ is a solution to $y'' + \omega^2 y = 0$ where A, ω, ϕ are constants. Determine constants A and ϕ that satisfy the initial conditions $y(0) = a, y'(0) = 0$.

36. When k is a positive integer, the Legendre equation

$$(1-x^2)y'' - 2xy' + k(k+1)y = 0$$

with $-1 < x < 1$ has a polynomial solution. Show that when $k = 3$ one such solution is $y(x) = x(5x^2 - 3)/2$.

37. Solve the differential equation:

a. $y' = 2xy$

b. $y' = y^2(x^2+1)$

c. $e^{x+y}y' = 1$

d. $y - xy' = 3 - 2x^2y'$

e. $(x^2+1)y' + xy = ax$ with $y(0) = 2a$ where a is a constant

f. $y' = y^3 \sin x$

38. An object of mass m falls from rest, starting near the earth's surface. Assuming air resistance varies as the square of the velocity of the object, the velocity $v(t)$ satisfies $mv' = mg - kv^2$ with $v(0) = 0$ where k, m, g are constants. Solve for $v(t)$.

First Order Linear DEs

39. Solve the differential equation:

a. $y' + y = 4e^x$

b. $y' + 2y/x = 5x^2, x > 0$

c. $y' + 2xy/(1+x^2) = 4/(1+x^2)^2$

d. $y' + 2xy/(1-x^2) = 4x, -1 < x < 1$

e. $y' + y/x = 2x^2 \ln x$

f. $y' + my/x = \ln x$ with m a constant

g. $y' + 2y/x = 4x$ with $y(1) = 2$

h. $y' + 2y/(4-x) = 5$ with $y(0) = 4$

Constant Coefficient Homogeneous DEs

40. Solve the differential equation:

a. $y'' - y' - 2y = 0$

b. $y'' - 6y' + 9y = 0$

c. $y'' + 8y' + 20y = 0$

d. $y'' - 14y' + 58y = 0$

e. $y''' - y'' + y' - y = 0$

f. $y'' - 8y' + 16y = 0$ with $y(0) = 2, y'(0) = 5$

g. $y'' - 2my' + (m^2+k^2)y = 0$ with $y(0) = 0$, $y'(0) = k$ where m, k are constants

Constant Coefficient Nonhomogeneous DEs

41. Solve the differential equation:

a. $y'' + y = 6e^x$

b. $y'' + 4y' + 4y = 5xe^{2x}$

c. $y'' + 2y' + 5y = 3\sin 2x$

d. $y''' + 2y'' - 5y' - 6y = 4x^2$

e. $y'' - 16y = 20\cos 4x$

f. $y'' + y = 3e^x \cos 2x$

g. $y'' + 9y = 5\cos 2x$ with $y(0) = 2, y'(0) = 3$

h. $y'' + y' - 2y = \sin x$ with $y(0) = 2, y'(0) = 1$

i. $y'' + \omega_0^2 y = F_0 \cos \omega t$ where ω, ω_0, F_0 are constants (treat the cases $\omega = \omega_0$ and $\omega \neq \omega_0$ separately)

Spring mass systems

42. At rest, a mass of 2 kilogram stretches a spring 1/4 meter. (Use $g = 10$ here.) Let $y(t)$ be the displacement of the spring at time t. Assuming friction is given by a term of the form $y'(t)/2$ and assuming with no external force, find $y(t)$ if $y(0) = 1$ and $y'(0) = 0$.

43. At rest, a mass of 2 kilogram stretches a spring 1/4 meter. (Use $g = 10$ here.) Assuming negligible friction, find an external force of the form $\cos(at)$ that produces resonance (the phenomenon of needing to "multiply the homogeneous solution" by t, creating spring oscillations of a greater and greater amplitude).

44. A spring mass system of the form $my''(t) + cy'(t) + ky(t) = 0$ is called critically damped if the characteristic equation $mr^2 + cr + k = 0$ has a repeated root. Assuming m and k are fixed, find the value of c in terms of m and k for which $my''(t) + cy'(t) + ky(t) = 0$ is critically damped.

Reduction of order

45. One solution to $y'' - 2ay' + a^2y = 0$ is $y = e^{ax}$ where a is a constant. Use reduction of order to find all solutions.

46. One solution to $y'' - xy' + y = 0$ is $y = x$. Find a second linearly independent solution. This second solution may involve an integral which cannot be evaluated, so the answer may involve an integral.

47. One solution to $x^2y'' + (1-2a)xy' + a^2y = 0$ for a a constant is $y_1(x) = x^a$. Use reduction of order to find all solutions to $xy'' + (1-2a)xy' + a^2y = x^b$ where b is another constant.

Cauchy-Euler

48. Solve the differential equation:

a. $x^2y'' + 5xy' + y = 0.$

b. $x^2y'' + 5xy' + 6y = 0.$

c. $4x^2y'' + y = 0.$

d. $x^2y'' - 5xy' + 9y = 0.$

e. $x^2y'' - 3xy' + 4y = x + 1.$

Linear Systems of DEs

49. Convert the differential equation into a first order linear system:

a. $y'' + 2ty' + y = \cos t$

b. $y''' + t^2y' - e^ty = t$

c. $y'' + ay' + by = F(t)$ where a, b are constants

50. Solve the system $\mathbf{x}' = A\mathbf{x}$ for the given matrix A:

a. $\begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$

b. $\begin{bmatrix} -2 & -7 \\ -1 & 4 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

d. $\begin{bmatrix} 2 & 0 & 3 \\ 0 & -4 & 0 \\ -3 & 0 & 2 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

f. $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

g. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

h. $\begin{bmatrix} -1 & 4 \\ 2 & -3 \end{bmatrix}$ with $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

i. $\begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix}$ with $\mathbf{x}(0) = \begin{bmatrix} -4 \\ 4 \\ 4 \end{bmatrix}$

Matrix Exponentials

51. Find the matrix exponential for the given matrix A and then state the solution to the system $\mathbf{x}' = A\mathbf{x}$:

a. $\begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}$ with $\mathbf{x}(0) = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$

b. $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ with $\mathbf{x}(0) = \begin{bmatrix} c \\ d \end{bmatrix}$ where a, b, c, d are constants

c. $\begin{bmatrix} 0 & 1 & 3 \\ 2 & 3 & -2 \\ 1 & 1 & 2 \end{bmatrix}$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$