Discrete Mathematics Set 9

Math 435: Complete 7 parts of the following exercises.
Math 530: Complete exercises 1, 5, 6, 9, and one of the remaining exercises.

1. An alternating polynomial f in $x_1, \ldots, x_n$ is a polynomial such that for all $\sigma = \sigma_1 \cdots \sigma_n \in S_n$,

$$f(x_1, \ldots, x_n) = \text{sign}(\sigma) f(x_{\sigma_1}, \ldots, x_{\sigma_n}).$$

a. Show that an alternating polynomial is divisible by $\Delta = \prod_{i<j}(x_i - x_j)$.

b. Let $\mathcal{A}_k$ be the vector space of alternating polynomials with every term degree k. Show that division by Δ is a vector space isomorphism between $\mathcal{A}_{n+\binom{n}{2}}$ and Λ_n (the vector space of symmetric functions of degree n).

2. Prove that the coefficient of m_λ in h_μ is the number of matrices with nonnegative integer entries with row sum λ and column sum μ.

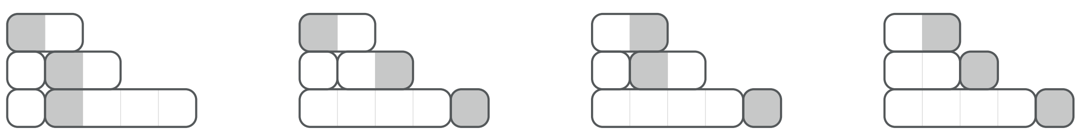
3. Let $\mu \vdash n$ and let $B_{\lambda,\mu}$ be the set of brick tabloids of shape Use a similar proof as used to prove

$$h_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,\mu}| e_\lambda$$

to prove

$$e_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,\mu}| h_\lambda.$$

4. A weighted brick tabloid of content λ and shape μ is the usual brick tabloid of content λ and shape μ but with one cell in the final brick in each row shaded. Let $WB_{\lambda,\mu}$ be the set of all weighted brick tabloids of content λ and shape μ. Here are 4 of the 30 possible examples of weighted brick tabloids of shape $(5, 3, 2)$ and content $(4, 2, 2, 1, 1)$:



a. Use a similar proof as used to prove $h_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,\mu}| e_\lambda$ to prove $p_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |WB_{\lambda,\mu}| e_\lambda$.

b. Prove $p_\mu = \sum_{\lambda \vdash n} (-1)^{\ell(\mu)-\ell(\lambda)} |WB_{\lambda,\mu}| h_\lambda$.

c. By counting weighted brick tabloids, find the 5×5 matrix with row and columns indexed by integer partitions of 4 and with row μ and column λ entry equal to $(-1)^{n-\ell(\lambda)} |WB_{\lambda,\mu}|$. Why does this matrix verify that $\{p_\lambda : \lambda \vdash 4\}$ is a basis for Λ_4? More generally, why is $\{p_\lambda : \lambda \vdash n\}$ a basis for Λ_n?

5. Let h_n and p_n be the homogeneous and power symmetric functions and let $H(t) = \sum_{n=0}^{\infty} h_n t^n$. Show that

$$\sum_{n=1}^{\infty} \frac{p_n}{n} t^n = \ln H(t) \qquad \text{and} \qquad \sum_{n=1}^{\infty} p_n t^n = \frac{tH'(t)}{H(t)}.$$

6. Define a ring homomorphism φ on Λ by $\varphi(e_n) = (-1)^{n-1}/n!$ for $n \geq 1$. Use $\varphi(h_n)$ to find the generating function for the number of ordered set partitions of n (which were first defined in Set 2 Exercise 3).

7. Define a ring homomorphism φ on Λ by $\varphi(e_n) = (-1)^{n-1}k(x-1)^{n-1}$ for $n \geq 1$. Use $\varphi(h_n)$ to find the generating function for

$$\sum_{w \in \{1,\ldots,k\}^n} x^{\text{equals}(w)}$$

where equals(w) denotes the number of times there are consecutive equal integers in a word $w \in \{1,\ldots,k\}^n$.

8. Define a ring homomorphism φ on Λ by $\varphi(e_n) = \begin{cases} 1 & \text{if } n = 0 \text{ or } n = 2, \\ 2x & \text{if } n = 1, \text{and} \\ 0 & \text{otherwise.} \end{cases}$

a. Recall from Set 4 exercise 6 the definitions of the Chebyshev polynomial of the first kind $T_n(x)$ and the Chebyshev polynomial of the second kind $U_n(x)$. Show that $\varphi(p_n) = 2T_n(x)$ for $n \geq 1$ and $\varphi(h_n) = U_n(x)$ for $n \geq 0$. It may help to use an identity found in Set 9 Exercise 5.

b. Use previously established relationships between e_n, h_n, and p_n (such as those in Set 9 Exercises 3,4, and 5 and this identity shown in class

$$\sum_{i=0}^{n-1} h_i p_{n-i} = nh_n$$

to show these identities hold for $n \geq 3$:

i. $U_n(x) = \displaystyle\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} (-1)^i (2x)^{n-2i}$

ii. $U_n(x) = \dfrac{2}{n} \displaystyle\sum_{i=0}^{n-1} U_i(x) T_{n-i}(x)$

iii. $U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0$

iv. $T_n(x) - 2xT_{n-1}(x) + T_{n-2}(x) = 0$

9. Define a ring homomorphism φ on Λ by $\varphi(e_n) = \begin{cases} (-1)^{k+k(3k-1)/2} & \text{if } n = k(3k-1)/2 \text{ for some } k \in \mathbb{Z}, \\ 0 & \text{if not.} \end{cases}$

a. Show that $\varphi(h_n) = p(n)$ where $p(n)$ is the number of integer partitions of n.

b. Apply φ to the generating function for p_n/n in Set 9 Exercise 5 to show that $\varphi(p_n) = \sigma(n)$ where $\sigma(n)$ is the sum of the positive integer divisors of n.

c. Use an identity found in the statement of Set 9 Exercise 9b to show that $p(n) = \dfrac{1}{n} \displaystyle\sum_{i=1}^{n} \sigma(i) p(n-i)$, thereby giving a recursion for the number of integer partitions of n. Calculate $p(7)$ using this recursion.