

# Discrete Mathematics Set 4

**Math 435:** Complete 7 parts of the following exercises.

**Math 530:** Exercises 1, 2, and any two of the remaining exercises.

1. Let  $A_n$  be the set of paths in  $\mathbb{R}^2$  which start at  $(0, 0)$ , end at  $(n, n)$ , and only use steps of the form  $(1, 0)$  or  $(0, 1)$ .

a. Show that there are  $\binom{2n}{n}$  elements in  $A_n$ .

b. Denote the number of times the path  $p \in A_n$  touches the line  $y = x$  by  $\text{touch}(p)$ . Let

$$A(x, t) = \sum_{n=0}^{\infty} \left( \sum_{p \in A_n} t^{\text{touch}(p)} \right) x^n$$

and let  $c_n$  be the  $n^{\text{th}}$  Catalan number. Show that

$$\sum_{p \in A_{n+1}} t^{\text{touch}(p)} = 2t \sum_{k=0}^n c_k \left( \sum_{p \in A_{n-k}} t^{\text{touch}(p)} \right).$$

c. Show that  $A(x, t) = \frac{t}{1 - t + 2t\sqrt{\frac{1}{4} - x}}$ .

d. Find an asymptotic formula for the average number of times a path in  $C_n$  touches the line  $y = x$ .

2. Let  $M(x) = \sum_{n=0}^{\infty} m_n x^n$  be the generating function for  $m_n$ , the number of Motzkin paths of length  $n$ , as defined in Set 1 Exercise 6.

a. Find  $\lim_{x \rightarrow 1/3} (1/3 - x)^{1/2} x M'(x)$  and use this result to find an asymptotic formula for  $m_n$ . (As always, calculations can be done using a computer algebra system if desired.)

b. Let  $a_n$  be as defined in Set 1 Exercise 6b. Find an asymptotic formula for  $a_n$  and use it to find an asymptotic formula for the probability that a random path in the plane that from  $(0, 0)$  to  $(n, n)$  using steps of the form  $(1, 1)$ ,  $(1, -1)$  and  $(1, 0)$  is a Motzkin path.

3. Find an asymptotic formula for the average number of sets in an ordered set partition of  $n$  (see Set 2 Exercise 3, Set 3 Exercise 1b, and Set 3 Exercise 1d).

4. Find an asymptotic formula for the average number of cycles in a permutation of  $n$  with ordered cycles (see Set 3 Exercise 1c and Set 3 Exercise 1d.)

5. Find an asymptotic formula for the number of permutations of  $n$  that do not have any cycles of length 1, 2 or 3.

6. Define

1. the Chebyshev polynomial of the first kind  $T_n(y)$  by  $\sum_{n=0}^{\infty} T_n(y) x^n = \frac{1 - yx}{1 - 2yx + x^2}$ ,

2. the Chebyshev polynomial of the second kind  $U_n(y)$  by  $\sum_{n=0}^{\infty} U_n(y) x^n = \frac{1}{1 - 2yx + x^2}$ , and

3. the Legendre polynomial  $P_n(y)$  by  $\sum_{n=0}^{\infty} P_n(y)x^n = \frac{1}{\sqrt{1-2yx+x^2}}$ .

Find asymptotic formulas for  $T_n(5/4)$ ,  $U_n(5/4)$ , and  $P_n(5/4)$ .