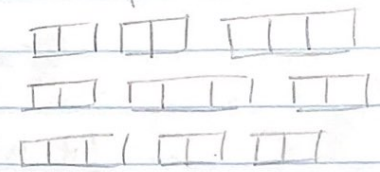
Generating Functions (cont.)

ex/ Let a_n = # of ways to tile a $1 \times n$ chessboard w/ 1×2 and 1×3 tiles.

When $n=7$:  $a_7 = 3$

Our Sequence: $1, 0, 1, 1, 1, \ldots$

Recursion $a_n = a_{n-2} + a_{n-3}$, $n \geq 3$, $a_0 = 1$, $a_1 = 0$, $a_2 = 1$

Generating function: Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$

$$= 1 + 0x + 1x^2 + \sum_{n=3}^{\infty} (a_{n-2} + a_{n-3}) x^n$$

$$= 1 + x^2 + x^2 \sum_{n=3}^{\infty} a_{n-2} x^{n-2} + x^3 \sum_{n=3}^{\infty} a_{n-3} x^{n-3}$$

$$= 1 + x^2 + x^2 \left(A(x) - 1\right) + x^3 A(x)$$

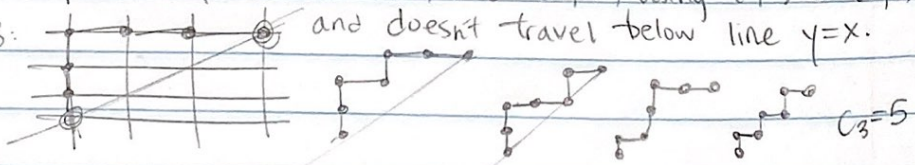
Therefore $A(x) = \dfrac{1}{1 - x^2 - x^3}$

Theorem $\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} a_k b_{n-k}\right] x^n$

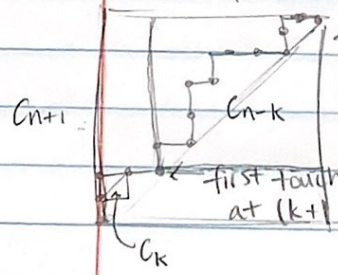
$(a_0 + a_1 x + a_2 x^2 + \ldots)(b_0 + b_1 x + b_2 x^2 + \ldots)$

ex/ Catalan Sequence

Let C_n = # paths in plane from $(0,0)$ to (n,n) using $(1,0)$ or $(0,1)$ steps and doesn't travel below line $y=x$.

When $n=3$:  $C_3 = 5$

Our Sequence: $1, 1, 2, 5, 14, \ldots$ $\quad n \geq 0$



$$C_{n+1} = \underbrace{C_0 C_{n-0}}_{k=0} + \underbrace{C_1 C_{n-1}}_{k=1} + \ldots + C_n C_0$$

$$= \sum_{k=0}^{n} C_k C_{n-k}$$

Let $C(x) = \sum_{n=0}^{\infty} C_n x^n$

Then $\sum_{n=0}^{\infty} C_{n+1} x^{n+1} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} C_k C_{n-k}\right) x^{n+1}$

$C(x) - 1 = x\, C(x)^2 \Rightarrow x\, C(x)^2 - C(x) + 1 = 0$

$\Rightarrow C(x) = \dfrac{1 \pm \sqrt{1-4x}}{2x}$