Linear Analysis II Set 11

1. Find the g in the span of $\{1, x\}$ closest to x^2 on $PS[-1, 1]$.

2. Find the function g in the span of 1, $\cos x$ and $\sin x$ in $PS[-\pi, \pi]$ that is closest to x.

3. Find the function of the form $ax + bx^3$ in $PS[0, 1]$ closest to x^2.

4. For any two functions f and g in $PS[a, b]$, verify that

$$0 \leq \frac{1}{2} \int_a^b \int_a^b [f(x)g(y) - g(x)f(y)]^2 \, dx \, dy = \|f\|^2 \|g\|^2 - \langle f, g \rangle^2.$$

Use this fact to establish the inequality $|\langle f, g \rangle| \leq \|f\| \|g\|$.

5. For any two functions f and g in $PS[a, b]$, show that $\|f + g\| \leq \|f\| + \|g\|$. (Hint: Verify that $\|f + g\|^2 = \|f\|^2 + 2\langle f, g \rangle + \|g\|^2$ and then use the above exercise)