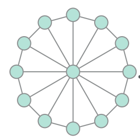
Graph Theory Set 3

11. The **join** of the graphs G and H, denoted $G \vee H$, is the graph created by drawing an edge between every vertex in G to every vertex of H.

a. Show that $\chi(G \vee H) = \chi(G) + \chi(H)$.

b. Show that $P_{K_n \vee G}(x) = x(x-1)\cdots(x-n+1)P_G(x-n)$.

c. The **wheel graph** W_n is $K_1 \vee C_{n-1}$. Here is W_{13}: . Find $P_{W_n}(x)$.

12. A graph G is k**-critical** if the chromatic number $\chi(G) = k$ and $\chi(G - v) = k - 1$ for any vertex v in G. The graph G is **critical** if it is k-critical for some k.

a. Why must every graph with $\chi(G) = k$ have a k-critical subgraph?

b. Show that the join $G \vee H$ is critical if and only if both G and H are critical.

c. Show that the even wheel graphs W_{2n} are 4-critical. (Open problem: Find a good characterization of all 4-critical graphs.)

d. Suppose G is k-critical. Why does every vertex have degree at least $k - 1$?

e. Suppose $\chi(G) = k$. Why must G have at least k vertices with degree $\geq k - 1$?

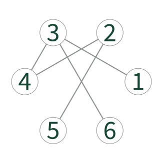
13. Let T be a tree. Properly color T green and gold such that the number of green vertices is at least the number of gold vertices. Prove that a green leaf exists.

14. Let T be a tree with n vertices and let G be a graph with minimum degree at least $n - 1$. Show that T can be found by removing vertices and edges from G.

15. This exercise gives another proof of Cayley's theorem. Starting with a labeled tree T with n vertices, create a list $(a_1, \ldots, a_{n-2})$ by implementing this algorithm:

1. Set $i = 1$.
2. Let u be the leaf in T with the minimum label. Let a_i be the label on the vertex adjacent to u.
3. If $i = n - 2$, stop. If not, increment i, change T to $T - u$, and go back to step 2.

For example, if T is the tree



then the list is $(3, 2, 4, 3)$.

Describe the inverse function; that is, give instructions on how to change the list $(a_1, \ldots, a_{n-2})$ with $1 \leq a_i \leq n$ into the corresponding labeled tree T. Why does this imply that there are n^{n-2} labeled trees?