Calculus II Exercises!

Review Exercises

1. Evaluate these integrals:

a. $\displaystyle\int \sin x + 2\cos x + 3\sqrt{x} + 4\sec^2 x + \frac{5}{x^3}\,dx.$

b. $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx.$

c. $\displaystyle\int x^p\,dx$ where $p \neq -1$.

2. True or false:

a. $\displaystyle\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

b. $\displaystyle\int_a^b f(x)g(x)\,dx = \int_a^b f(x)\,dx \int_a^b g(x)\,dx$

c. $\displaystyle\int_a^t f'(x)\,dx = f(t) - f(a)$

d. $\displaystyle\frac{d}{dt}\left(\int_a^t f(x)\,dx\right) = f(t) - f(a).$

e. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

The substitution rule

3. Evaluate these integrals:

a. $\displaystyle\int x\cos(x^2)\,dx$

b. $\displaystyle\int \frac{3x}{(x^2+2)^2}\,dx$

c. $\displaystyle\int \sin(2\pi t)\,dt$

d. $\displaystyle\int \frac{1}{(3t-2)^{1.75}}\,dt$

e. $\displaystyle\int x^3\sqrt{x^2+1}\,dx$

f. $\displaystyle\int_0^a x\sqrt{a^2-x^2}\,dx$

g. $\displaystyle\int_1^2 x\sqrt{x-1}\,dx$

4. The function $f(t) = \sin(2\pi t/5)/2$ has been used to model the rate of air flow into human lungs. What is the volume of inhaled air at time t?

Areas between curves

5. Find the area of the region bounded by the graphs of $y = x^2 - 4x$ and $y = 0$.

6. Find the area of the region bounded by the graphs of $y = -3x(x-8)/8$, $y = 10 - x/2$, $x = 2$, and $x = 8$.

7. Find the area of the region bounded by the graphs of $y = |x|$ and $y = x^2 - 2$.

8. Find the area of the region bounded by the graphs of $x = y(2-y)$ and $x = -y$.

9. Find the number b such that the line $y = b$ divides the region bounded by the curves $y = x^2$ and $y = 4$ into two regions with equal areas.

10. Find the area of the region bounded by the parabola $y = x^2$, the tangent line to this parabola at $(2,4)$, and the x-axis.

11. Find the area of the region bounded by the graphs of $y = (x-1)^3$ and $y = x - 1$.

12. Find the area of the region bounded by the graphs of $y = \cos 2x$ and $y = \sin x$ for values of x between $-\pi/2$ and $\pi/6$.

Volumes

13. A sphere of radius r is cut by a plane h units above the equator. Find the volume of the part of the solid above the plane.

14. Find the volume of the solid generated by revolving the region bounded by the graphs of $y = 1/x$, $y = 0$, $x = 1$ and $x = 4$ about the x-axis.

15. Find the volume of the solids generated by revolving the regions bounded by the graphs of $y = \sqrt{x}$, $y = -x/2 + 4$, $x = 0$ and $x = 8$ about the x-axis.

16. Find the volume of the solid generated by revolving the circle $(x-a)^2 + (y-b)^2 = r^2$ around the x-axis.

17. A hole of radius r is bored through the center of a sphere of radius $R > r$. Find the volume of the remaining portion of the sphere.

18. Find the volume of the solid generated by revolving the region bounded by the graphs of $y = x^2$, $y = 1$ around the line $y = 2$.

19. The least spherical planet in our solar system is Saturn. Its shape, with units in kilometers, is found by rotating the graph of the portion of the ellipse

$$\frac{x^2}{60268^2} + \frac{y^2}{54364^2} = 1$$

lying to the right of the y-axis around the y-axis. Find the volume of Saturn.

20. Find the volume of the solid generating by revolving the region bounded by the graph of $y = 0$, $y = (\sin x)/x$, $x = 0$, and $x = \pi$ around the y-axis.

21. Find the volume of the solid generated by revolving the circle $(x - a)^2 + (y - b)^2 = r^2$ around the y-axis. Use a different method to find the volume than you did in exercise 16.

22. Find the volume of a right circular cone with height h and radius r.

23. Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ around the y-axis.

24. Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \cos(x^2)$, $y = 0$, $x = 0$, and $x = \sqrt{\pi/2}$ around the y-axis.

Work

25. A force of 800N stretches a spring 70cm. Find the work done in stretching the spring 70cm.

26. Newton's law of gravitation says that two bodies with masses m and M at a distance r away from one another attract each other with a force of

$$F = G\frac{mM}{r^2}$$

where G is a constant approximately equal to 6.67×10^{-11}. If one mass is fixed, find the work needed to move the other mass from $r = a$ to $r = b$.

27. How much work is required to launch a 1250kg satellite vertically into orbit 1000km high? The mass of the earth is approximately $5.98 \times 10^{24}\text{kg}$ and the radius of the earth is approximately 6367km. (See exercise 26).

28. A demolition crane has a 225kg ball suspended from a 12m cable that weighs $1\text{kg}/\text{m}$. How much work is needed to wind up the ball 6m?

29. A rectangular tank of width 1m, length 2m, and height 3m is full of water (the density of water is $1000\text{kg}/\text{m}^3$). How much work is needed to empty the tank by pumping the water out of the top? (Think about the center of mass.)

30. A cylindrical tank is 4m high with a radius of 2 meters. It is filled with gasoline (the density of gasoline is $720\text{kg}/\text{m}^3$). How much work is needed to empty the tank by pumping the water out of the top?

Averages

31. The speed of sound depends on its altitude. The speed of sound $s(x)$ in meters per second at an altitude of x kilometers is given by

$$s(x) = \begin{cases} -4x + 341 & \text{if } 0 \le x < 11.5 \\ 295 & \text{if } 11.5 \le x < 22 \\ 3x/4 + 278.5 & \text{if } 22 \le x < 32. \end{cases}$$

What is the average speed of sound for the first 32km of altitude?

32. The function $f(t) = \sin(2\pi t/5)/2$ has been used to model the rate of air flow into human lungs. What is the average volume of air in the lungs on the interval $[0, 5/2]$?

Inverse functions

33. Let $h(t)$ be your height after t years. Does $h(t)$ have an inverse? Why?

34. Let $P(t)$ be the human population living on Earth on year t. Does $P(t)$ have an inverse? Why?

35. Why does $f(x) = x^3 + 2x - 1$ have an inverse? Find $(f^{-1})'(2)$.

36. Why does $f(x) = \sqrt{x - 4}$ for $x \ge 4$ have an inverse? Find $(f^{-1})'(2)$.

37. Let $f(x)$ be the function $f(x) = \sin x$ for values of x between $-\pi/2$ and $\pi/2$. Why does have an inverse? Find $(f^{-1})'(1/2)$.

38. Give three different examples of functions f for which $f = f^{-1}$.

39. Let g be the inverse to f. Show that if g'' exists, then

$$g''(x) = -\frac{f''(g(x))}{f'(g(x))^3}.$$

40. Suppose f is increasing and concave up. Why must its inverse function be concave down?

The logarithm

41. Evaluate these limits:

a. $\lim_{x\to 2^-} \ln\left(x^2(3-x)\right)$

b. $\lim_{x\to\infty} \ln\left(\frac{x}{1+x}\right)$

c. $\lim_{x\to\infty} \ln\left(\ln\left(x+1/x\right)\right)$

42. Differentiate these functions:

a. $\ln\left(x+\sqrt{4+x^2}\right)$

b. $\ln\left|\frac{\cos x}{\cos x - 1}\right|$

c. $\ln\left(\ln x^2\right)$

d. $\int_1^{\ln x} (t^2+3)\,dt$

43. Show that the function $y = x\ln x - 4x$ satisfies the differential equation $x + y - xy' = 0$.

44. Find the line tangent to $\ln(1+x^3)$ at $x = 3$.

45. Which value of $x \geq 0$ maximizes $1/(1+\ln x)^2$?

46. Evaluate these integrals:

a. $\int \tan 5x\,dx$

b. $\int \frac{1}{3x+2}\,dx$

c. $\int \frac{x^2+2x+4}{x}\,dx$

d. $\int \frac{1}{1+\sqrt{2x}}\,dx$

e. $\int \frac{\cos x}{1+\sin x}\,dx$

The exponential function

47. Evaluate these limits:

a. $\lim_{x\to 2^-} e^{x^2(3-x)}$

b. $\lim_{x\to\infty} e^{1/(1+x)}$

48. Differentiate these functions:

a. $\ln\left(1+e^x\right)$

b. $\frac{2}{e^x+e^{-x}}$

49. Find the values of x for which the second derivative of $x^2e^{-x^2}$ is equal to 0.

50. Find the values of r which make the the function $y = e^{rx}$ satisfy the differential equation $y'' + 5y' + 6y = 0$.

51. Find the values of r which make the the function $y = e^{rx}$ satisfy the differential equation $y'' + 4y' + 4y = 0$.

52. Find the line tangent to the graph of $e^{2x}\cos x$ at $x = \pi$.

53. Evaluate these integrals:

a. $\int xe^{-x^2}\,dx$

b. $\int e^x\sqrt{1-e^x}\,dx$

c. $\int \frac{e^{2x}}{1+e^{2x}}\,dx$

d. $\int \frac{e^{2x}+e^x+1}{e^x}\,dx$

Other exponentiation

54. Differentiate these functions:

a. $4^t + t^4$

b. $x^{2/x}$

c. $\log_4(1+x^2)$

d. $(\ln x)e^{1+\ln x+\log_2 x^2}$

55. Evaluate these integrals

a. $\int_1^2 2^x + x2^{-x^2}\,dx$

b. $\int \ln x + \log_{10} x \, dx$

56. Find the value of $x \geq 0$ that minimizes $x^{\ln x}$.

57. Evaluate $\lim_{x\to\infty} \log_{10}(1 + \log_3 x)$

Growth and decay

58. A roast turkey is $185°$ when placed in a $75°$ room. The temperature is $145°$ after 30 minutes. When will the turkey be $100°$?

59. The half life of carbon 14 is 5730 years. When organic material dies, the amount of carbon 14 begins to decrease through radioactive decay. An old parchment was found to have about 70% as much carbon 14 as current plant material. How old is the parchment?

60. How long will it take an investment to double in value if the interest rate is 5% compounded continuously?

61. Suppose that an experimental population of fruit flies increases exponentially. There were 100 flies after the second day and 300 on the fourth day. How many flies were there to begin with?

62. The atmospheric pressure P decreases exponentially with increasing altitude. The pressure is 760 millimeters of mercury at sea level and 672.71 at an altitude of 1000m. What is the pressure at 3000m?

63. Four months after it stopped advertising, a company noticed that its sales dropped from 10000 units a month to 8000. If sales follow an exponential pattern of decline, what will the sales be after a year?

Inverse trig functions

64. Simplify each of these (without a calculator):

a. $\arctan \sqrt{3}$

b. $\cos(\arctan x)$

c. $\tan(\arcsin x)$

d. $\arcsin 1/\sqrt{2}$

65. Differentiate these functions:

a. $\arctan \sqrt{x}$

b. $\dfrac{\arcsin x}{\arctan x}$

66. Find the value of x between -1 and 1 which maximizes $\cos(1 + \arcsin x)$.

67. Evaluate these integrals:

a. $\int \dfrac{1}{\sqrt{2 - x^2}} \, dx$

b. $\int_0^{\pi/2} \dfrac{\cos x}{1 + \sin^2 x} \, dx$

c. $\int \dfrac{x}{x^4 + 16} \, dx$

d. $\int \dfrac{1 + x}{1 + x^2} \, dx.$

Integration by parts

68. Evaluate these integrals

a. $\int x^4 \cos 2x \, dx$

b. $\int_0^{1/2} \arcsin x \, dx$

c. $\int_0^{\pi} x^2 \sin nx \, dx$ where n is a positive integer

d. $\int x\sqrt{x - 1} \, dx$

e. $\int x \ln x \, dx$

f. $\int \dfrac{\ln x}{x^2} \, dx$

g. $\int \dfrac{x}{e^{2x}} \, dx$

h. $\int_0^2 e^{-x} \cos x \, dx$

Integrals with sines and cosines

69. Evaluate these integrals

a. $\int \sin^3 x \, \cos^2 x \, dx$

b. $\int \sin^4 x \, \cos^4 x \, dx$

c. $\int \sin^4 x \, dx$

d. $\int \tan^3 x\, dx$

70. Let m and n be positive integers. Show these three equations are true:

$$\int_{-\pi}^{\pi} \sin mx \cos nx\, dx = 0$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases}$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases}$$

This can be done with integration by parts or by using trig identities to simplify.

Integrals by trig substitution

71. Evaluate these integrals:

a. $\displaystyle\int \frac{1}{x^2\sqrt{16 - x^2}}\, dx$

b. $\displaystyle\int \left(\sqrt{1 + x^2}\right)^{-3}\, dx$

c. $\displaystyle\int \sqrt{1 - 4x^2}\, dx$

d. $\displaystyle\int x\sqrt{1 - x^4}\, dx$

Integrals of rational functions

72. Evaluate these integrals:

a. $\displaystyle\int \frac{1}{x^2 - 1}\, dx$

b. $\displaystyle\int \frac{3}{x^2 + x - 2}\, dx$

c. $\displaystyle\int \frac{1}{4x^2 - 9}\, dx$

d. $\displaystyle\int \frac{\sin x}{\cos x + \cos^2 x}\, dx$

e. $\displaystyle\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}\, dx$

f. $\displaystyle\int \frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)}\, dx$

g. $\displaystyle\int \frac{1}{x(a - x)}\, dx$

Approximating integrals

73. Approximate these integrals to within 0.001 of their true value:

a. $\displaystyle\int_0^1 \sqrt{1 + x^3}\, dx$

b. $\displaystyle\int_{\pi/2}^{\pi} \sqrt{x} \sin x\, dx$

c. $\displaystyle\int_0^1 e^{-x^2}\, dx$

d. $\displaystyle\int_{\pi/2}^{\pi} \sqrt{x} \sin x\, dx$

Random integrals

74. Evaluate these integrals:

a. $\displaystyle\int \cos x(1 + \sin^2 x)\, dx$

b. $\displaystyle\int_0^4 \frac{x - 1}{x^2 - 4x - 5}\, dx$

c. $\displaystyle\int x \sin^2 x\, dx$

d. $\displaystyle\int \frac{4^x + 10^x}{2^x}\, dx$

e. $\displaystyle\int_{-2}^{2} \left|x^2 - 4x\right|\, dx$

f. $\displaystyle\int \frac{e^x}{1 + e^x}\, dx$

g. $\displaystyle\int \ln x^2\, dx$

h. $\displaystyle\int_{-1}^{1} \frac{e^{\arctan x}}{1 + x^2}\, dx$

L'Hôpital's rule

75. Evaluate these limits

a. $\displaystyle\lim_{x\to\infty} \frac{x^{10}}{(x + e^x)}$

b. $\displaystyle\lim_{x\to\infty} (1 - x - x^2)^{1/x}$

c. $\displaystyle\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x$

d. $\displaystyle\lim_{x\to 0^+} (-x \ln x)$

76. We say that $f(x)$ grows faster than $g(x)$ provided $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$. List these functions in order from smallest to fastest growth:

$$x^x, e^{x^2}, \ln x, x^{10}, \arctan x, 2^x, (\ln x)^{100}, \frac{x}{\ln x}, \frac{x}{\sqrt{x}+1}$$

Improper integrals

77. Evaluate these integrals:

a. $\displaystyle\int_1^\infty \frac{1}{x^2+x}\,dx$

b. $\displaystyle\int_{-1}^0 \frac{1}{x^2+x}\,dx$

c. $\displaystyle\int_0^{3/2} \frac{1}{\sqrt{9-x^2}}\,dx$

d. $\displaystyle\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\,dx$

e. $\displaystyle\int_0^\infty (1-x)e^{-x}\,dx$

f. $\displaystyle\int_0^\infty \cos x\,dx$

g. $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$

Arclength

78. Find the exact arclength of the function $\cosh x$ on the interval $[0,2]$. (The function $\cosh x$ is shorthand notation for $(e^x+e^{-x})/2$).

79. The gateway arch in St. Louis, Missouri traces the graph of the function $693.85 - 68.76\cosh 0.01x$ on the interval $[-299.22, 299.22]$. Approximately what is the arclength of the arch?

80. Approximate the arclength of e^x on $[0,2]$.

81. Approximately what number a makes the arclength of x^3 on interval from $[0,a]$ equal to 1?

Surface areas

82. Find the surface area of the surface formed by revolving the graph of x^2 on the interval $[0,\sqrt{2}]$ around the y-axis.

83. Find the surface area of the part of the sphere formed by revolving the graph of $\sqrt{9-x^2}$ on the interval $[0,2]$ around the x-axis.

84. Approximately what number a makes the surface formed by revolving the graph of x^3 on the interval from $[0,a]$ around the x-axis equal to 1?

85. Find the volume of the solid formed by revolving the graph of $1/x$ on the interval $[0,\infty)$ around the x-axis. Then find the surface area of this solid.

Center of mass

86. Find the center of mass of a flat, uniformly dense object in the shape of the region bounded by the graph of $1-x^2$ and the x-axis?

87. Find the center of mass of a flat, uniformly dense object in the shape of the region bounded by the graph of 2^{-x} and the x-axis on the interval $[0,\infty)$.

88. Find the center of mass of a flat, uniformly dense object in the shape of the region bounded by the graphs of $\sin x$, $\cos x$, $x=0$, and $x=\pi/4$.

89. A cylindrical tank is 4m high with a radius of 2m. It is halfway filled with gasoline (the density of gasoline is $720\text{kg}/\text{m}^3$). How much work is needed to empty the tank by pumping the water out of the top?

Probability

90. A light bulb has an average lifetime of 1000 hours. Assuming an exponential density function, find the probability that a light bulb burns out in 500 hours. What is the probability that the light bulb burns longer than 2000 hours?

91. The height of adult males is normally distributed with mean 69 inches and standard deviation 2.8 inches. What is the probability an adult male is between 60 and 66 inches? What is the probability that an adult male is taller than 6 feet?

92. The standard deviation σ for a random variable with probability density function f with mean μ is

$$\sigma = \sqrt{\int_{-\infty}^\infty (x-\mu)^2 f(x)\,dx}.$$

Find the standard deviation for an exponential density function with mean μ.

93. The amount of paper thrown away by an American household is normally distributed with mean 9.4 pounds and standard deviation 4.2 pounds. What percentage of households throw out at least 12 pounds a week?

Separable differential equations

94. Solve these differential equations:

a. $y' = 4xy^2$

b. $y' = \sqrt{xy}$

c. $(x^2 + 4)y' = xy$

d. $\begin{cases} y' = y(1-y) \\ y(0) = .25 \end{cases}$

e. $\begin{cases} 2xy' = \ln x^2 \\ y(1) = 2 \end{cases}$

f. $y'' - 2xy' = 0$ (the answer may be written using an integral)

95. A lake was stocked with 500 fish. The lake can sustain a maximum population of 10000 fish. After the first year, the number of fish tripled. When will the population be 5000 fish?