Ex: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Let $u = \sqrt{x}$

$\Gamma(\frac{1}{2}) = \int_0^\infty \frac{1}{\sqrt{x}} e^{-x}\,dx = \int_0^\infty 2e^{-u^2}\,du$

Theorem: Sterling's Approximation – $\Gamma(\alpha+1) \sim \sqrt{2\pi\alpha}\left(\frac{\alpha}{e}\right)^\alpha$

Corr. $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

Ex: $\frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\frac{(2n)!}{(n!)^2} \sim \frac{1}{n+1}\cdot\frac{\sqrt{2\pi(2n)}\left(\frac{2n}{e}\right)^{2n}}{2\pi n\left(\frac{n}{e}\right)^{2n}} = \frac{1}{n+1}\cdot\frac{4^n}{\sqrt{\pi n}}$

$\sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$

Pf. Consider $f_\alpha(x) = \begin{cases}\left(1+\frac{x}{\sqrt{\alpha}}\right)^\alpha e^{-\sqrt{\alpha}x} & , x > -\sqrt{\alpha} \\ 0 & , x \leq -\sqrt{\alpha}\end{cases}$

Then $\ln(f_\alpha(x)) = \begin{cases}\alpha\ln\left(1+\frac{x}{\sqrt{\alpha}}\right) - \sqrt{\alpha}x \\ x > -\sqrt{\alpha}\end{cases}$

$= -\alpha\ln\left(\frac{1}{1+\frac{x}{\sqrt{\alpha}}}\right) - \sqrt{\alpha}x$

$= -\alpha\ln\left(\frac{1}{1-\frac{-x}{\sqrt{\alpha}}}\right) - \sqrt{\alpha}x$

$= -\alpha\sum_{n=1}^{\infty}\frac{\left(\frac{-x}{\sqrt{\alpha}}\right)^n}{n} - \sqrt{\alpha}x$

$= -\alpha\left(\frac{-x}{\sqrt{\alpha}} + \frac{x^2}{2\alpha} - \frac{x^3}{3\alpha^{3/2}} + \frac{x^4}{4\alpha^2} \mp \cdots\right) - \sqrt{\alpha}x$

$= \left(x\sqrt{\alpha} - \frac{x^2}{2} + \frac{x^3}{3\alpha^{\frac{1}{2}}} \mp \cdots\right) - \sqrt{\alpha}x$

$= \left(-\frac{x^2}{2} + \frac{x^3}{3\alpha^{\frac{1}{2}}} \mp \cdots\right) = \sum_{n=2}^{\infty}\frac{(-1)^{n+1}x^n}{n\alpha^{\frac{n-2}{2}}}$

$\lim_{\alpha\to\infty}\ln(f_\alpha(x)) = \frac{-x^2}{2}$, so $\lim_{\alpha\to\infty} f_\alpha(x) = e^{\frac{-x^2}{2}}$

Thus $\sqrt{2\pi} = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \int_{-\infty}^{\infty} \lim_{\alpha\to\infty} f_\alpha(x)\,dx$

$= \lim_{\alpha\to\infty} \int_{-\infty}^{\infty} f_\alpha(x)\,dx = \lim_{\alpha\to\infty} \int_{-\sqrt{\alpha}}^{\infty} (1+\frac{x}{\sqrt{\alpha}})^{\alpha} e^{-\sqrt{\alpha}x}$

Let $u = x\sqrt{\alpha} + \alpha$

$du = \sqrt{\alpha}\,dx$ $\quad \lim_{\alpha\to\infty} \int_0^{\infty} \frac{1}{\sqrt{\alpha}} (\frac{u}{\alpha})^{\alpha} e^{\alpha - u} du$

$= \lim_{\alpha\to\infty} \frac{e^{\alpha}}{\alpha^{\alpha+\frac{1}{2}}} \int_0^{\infty} (u)^{\alpha} e^{-u} du = \lim_{\alpha\to\infty} \frac{1}{\sqrt{\alpha}(\frac{\alpha}{e})^{\alpha}} \int_0^{\infty} u^{\alpha} e^{-u} du$

$= \lim_{\alpha\to\infty} \frac{\Gamma(\alpha+1)}{\sqrt{\alpha}(\frac{\alpha}{e})^{\alpha}} \Rightarrow \lim_{\alpha\to\infty} \frac{\Gamma(\alpha+1)}{\sqrt{2\pi\alpha}(\frac{\alpha}{e})^{\alpha}} = 1$

Let $R>0$, $\alpha>0$, $c>0$. Then if $\frac{c}{(R-x)^{\alpha}} = \sum_{n=0}^{\infty} a_n x^n$

then $a_n \sim \frac{c n^{\alpha-1}}{R^{n+\alpha}\Gamma(\alpha)}$

Ex: $\frac{3}{(1-x)^{1/2}} \Rightarrow a_n \sim \frac{3n^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} = \frac{3n^{-\frac{1}{2}}}{\sqrt{\pi n}}$