Math 344 Sample Final

The final exam is cumulative, with the majority of the exam consisting of the material found on the first two exams. Please see those review problems and the blue book problems for sample questions on that content.

1. Give a physical interpretation and solve the partial differential equation $\begin{cases} u_t = ku_{xx}, \\ u(0,t) = 0, u_x(L,t) = 0, \\ u(x,0) = x. \end{cases}$

Solution: The physical interpretation could be the description of how heat diffuses through a metal rod of length L. The first equation is the heat equation in one spacial dimension. On the left end of the rod the temperature is held at 0, the right end of the rod is insulated, and the initial temperature distribution is given by x. Following through with the separation of variables technique, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{(-1)^n 8L}{(2n+1)^2\pi^2} e^{-k\frac{(2n+1)\pi}{2L}t} \sin\left(\frac{(2n+1)\pi}{2L}x\right)$$

The constants in this sum were found by taking inner products.

2. The Fourier transform of the function $f(t)$ that is equal to $\pi^2 - t^2$ on $[-\pi, \pi]$ and 0 otherwise is

$$F(\omega) = \frac{2\sin(\pi\omega) - 2\pi\omega\cos(\pi\omega)}{\pi\omega^3}.$$

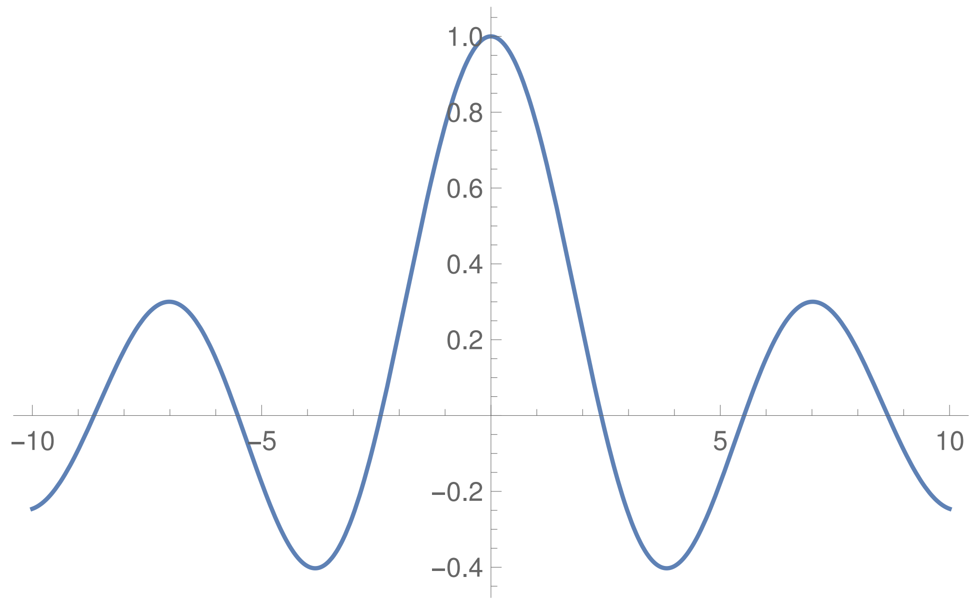
Evaluate $\int_{-\infty}^{\infty} F(\omega)\cos(\omega t)\,d\omega$ and $\int_{-\infty}^{\infty} F(\omega)\sin(\omega t)\,d\omega$.

Solution: The answer can be found using the inverse Fourier transform (this is labeled as the "Fourier relations", the second item on the table of Fourier transforms), which gives

$$f(t) = \int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega = \int_{-\infty}^{\infty} F(\omega)\cos(\omega t)\,d\omega + i\int_{-\infty}^{\infty} F(\omega)\sin(\omega t)\,d\omega$$

where we used $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$. Looking at the real and imaginary parts separately, we see the answers are $f(t)$ (and not $\pi^2 - t^2$ as this is only valid on $[-\pi, \pi]$) and 0.

3. The Fourier transform of $f(t) = \begin{cases} \dfrac{2}{\sqrt{1-t^2}} & \text{if } t \in [-1,1], \\ 0 & \text{if not.} \end{cases}$ is the real valued $F(\omega)$ plotted below:



Approximately what is the coefficient c_{-7} in $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$, the complex Fourier series?

4. Find the Fourier series for the function e^x on $PS[-L, L]$. You can leave the solution in terms of integrals.