Discrete Mathematics Set 7

Math 435: Complete 4 parts of the following exercises.
Math 530: Exercises 5, 6 and one of the remaining exercises.

1. Prove $\sum_{n=1}^{\infty} x^n y^n z(1+zx^1)\cdots(1+zx^{n-1}) = \sum_{n=1}^{\infty} \frac{x^{\binom{n+1}{2}} y^n z^n}{(1-yx^1)\cdots(1-yx^n)}$.

2. Recover Euler's pentagonal number theorem by taking $y=-q$ and $x=q^3$ in Jacobi's triple product

$$(1+y)\prod_{n=1}^{\infty}(1-x^n)(1+yx^n)(1+y^{-1}x^n) = \sum_{k\in\mathbb{Z}} y^k x^{k(k-1)/2}.$$

3. Use the function $x^2 \prod_{n=1}^{\infty}(1-x^n)^6$ to show that the number of integer partitions of $7n+5$ is divisible by 7.

4. This exercise proves a finite version of Jacobi's triple product identity.

a. Prove $\prod_{i=1}^{n}(1+yq^{i-1})(1+y^{-1}q^i) = \sum_{k=-n}^{n} y^k q^{k(k-1)/2} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q$.

b. Take $\lim_{n\to\infty}$ of the above expression to find the full Jacobi triple product.

5. Suppose $F(x,y)$ is an infinite product in two indeterminates y and x. Euler's device refers to this process of turning $F(x,y)$ into a sum:

1. Find an equation providing a relationship between $F(x,xy)$ and $F(x,y)$.
2. Assume that $F(x,y) = \sum_{n=0}^{\infty} a_n(x)y^n$ and plug this into the equation found in step 1.
3. Compare coefficients of y^n to find a recursion for $a_n(x)$.
4. Iterate this recursion to find a formula for $a_n(x)$.

Show $F(x,y) = \prod_{i=0}^{\infty} \frac{1}{1-yx^i}$ satisfies $F(x,xy) = (1-y)F(x,y)$, thereby completing step 1 of the above process. Continue following the next three steps outlined above to express $F(x,y)$ as a sum using Euler's device.

6. Let $F(x,y)$ be the function that satisfies the recursion $F(x,y) = F(x,xy) + xyF(x,x^2y)$ and $F(x,0)=1$. An explicit formula for $F(x,y)$ is given on the next page.

a. Use Euler's device to show $F(x,y) = \sum_{n=0}^{\infty} \frac{y^n x^{n^2}}{(1-x)\cdots(1-x^n)}$.

b. Use the result in the theorem on the next page to show $F(q,1) = \left(\sum_{k\in\mathbb{Z}} (-1)^n q^{n(5n-1)/2}\right) \prod_{i=1}^{\infty} \frac{1}{1-q^i}$.

c. Take $y=-q^2$ and $x=q^5$ in Jacobi's triple product to show $F(q,1) = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$.

d. Show that (the number of integer partitions of n with parts differing by at least 2) is equal to (the number of integer partitions of n with parts congruent to $\pm 1 \bmod 5$). (Hint: $1+3+5+\cdots+(2n-1) = n^2$.)

Theorem. *The function*

$$F(x,y)=\left(1+\sum_{n=1}^{\infty}(-1)^n y^{2n}x^{n(5n-1)/2}\left(1-yx^{2n}\right)\frac{(1-yx)\cdots(1-yx^{n-1})}{(1-x)\cdots(1-x^n)}\right)\prod_{i=1}^{\infty}\frac{1}{1-yx^i} \qquad (1)$$

satisfies $F(x,y)=F(x,xy)+xyF(x,x^2y)$ *and* $F(x,0)=1$.

Proof. Using $\left(1-yx^{2n}\right)=(1-x^n)+x^n\left(1-yx^n\right)$, we have that

$$\begin{aligned}F(x,y)\prod_{i=1}^{\infty}\left(1-yx^i\right)&=1+\sum_{n=1}^{\infty}(-1)^n y^{2n}x^{n(5n-1)/2}\frac{(1-yx)\cdots(1-yx^{n-1})}{(1-x)\cdots(1-x^{n-1})}\\&\quad+\sum_{n=1}^{\infty}(-1)^n y^{2n}x^{n(5n+1)/2+n}\frac{(1-yx)\cdots(1-yx^{n})}{(1-x)\cdots(1-x^{n})}\\&=\sum_{n=0}^{\infty}(-1)^n y^{2n}x^{n(5n+1)/2}\left(1-y^2x^{4n+2}\right)\frac{(1-yx)\cdots(1-yx^{n})}{(1-x)\cdots(1-x^{n})}.\end{aligned} \qquad (2)$$

where we changed the first line into the second by reindexing the first infinite sum. Therefore, using (2) to simplify the first product and (1) to simplify the second product,

$$(F(x,y)-F(x,xy))\prod_{i=1}^{\infty}\left(1-yx^i\right)=\sum_{n=0}^{\infty}(-1)^n y^{2n}x^{n(5n+1)/2}\frac{(1-yx)\cdots(1-yx^{n})}{(1-x)\cdots(1-x^{n})}\left(1-y^2x^{4n+2}-x^n\left(1-yx^{2n+1}\right)\right).$$

Using $\left(1-y^2x^{4n+2}-x^n\left(1-yx^{2n+1}\right)\right)=(1-x^n)+yx^{3n+1}\left(1-yx^{n+1}\right)$, the above expression is equal to

$$\sum_{n=1}^{\infty}(-1)^n y^{2n}x^{(5n+1)/2}\frac{(1-yx)\cdots(1-yx^{n})}{(1-x)\cdots(1-x^{n-1})}+yx\sum_{n=0}^{\infty}(-1)^n y^{2n}x^{n(5n+7)/2}\frac{(1-yx)\cdots(1-yx^{n+1})}{(1-x)\cdots(1-x^{n})},$$

which in turn, by reindeying the first sum, is equal to

$$yx\sum_{n=0}(-1)^n y^{2n}x^{n(5n+7)/2}\left(1-yx^{2n+2}\right)\frac{(1-yx)\cdots(1-yx^{n+1})}{(1-x)\cdots(1-x^{n})}=yx\left(\prod_{i=1}^{\infty}\left(1-yx^i\right)\right)F(x,x^2y).$$

The last step used (1) again. Thus we have proved

$$(F(x,y)-F(x,xy))\prod_{i=1}^{\infty}\left(1-yx^i\right)=xyF(x,x^2y)\prod_{i=1}^{\infty}\left(1-yx^i\right),$$

which implies the desired result. □