Ex. Let a_n = # ways to tile a $1 \times n$ chessboard with 1×2 and 1×3 tiles.

$a_7 = 3$

our Sequence: $1, 0, 1, 1, 1, 2, 2, 3, 4, \ldots$

Recursion: $a_n = a_{n-2} + a_{n-3}, \; n \geq 3, \; a_0 = 1, \; a_1 = 0, \; a_2 = 1$

Generating function: Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$

$$= 1 + 0x + 1x^2 + \sum_{n=3}^{\infty} (a_{n-2} + a_{n-3}) x^n$$

$$= 1 + x^2 + x^2 \sum_{n=3}^{\infty} a_{n-2} x^{n-2} + x^3 \sum_{n=3}^{\infty} a_{n-3} x^{n-3}$$

$$A(x) = 1 + x^2 + x^2 (A(x) - 1) + x^3 A(x)$$

$$(1 - x^2 - x^3) A(x) = 1 + x^2 - x^2$$

$$A(x) = \frac{1}{1 - x^2 - x^3}$$

Therefore $A(x) = \dfrac{1}{1 - x^2 - x^3}$

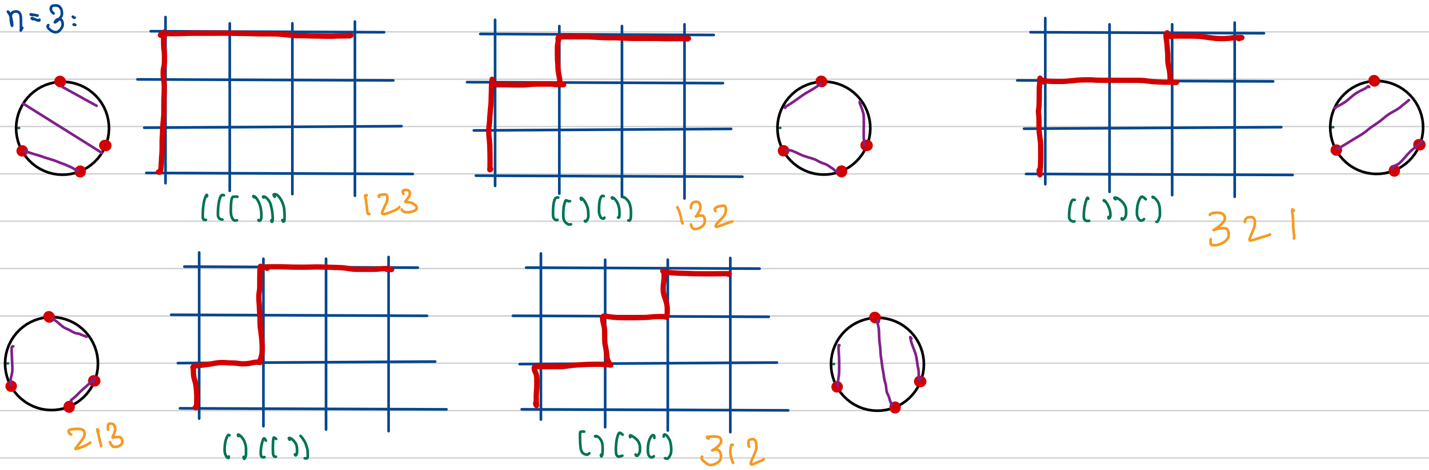
Theorem

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} a_k b_{n-k}\right] x^n$$

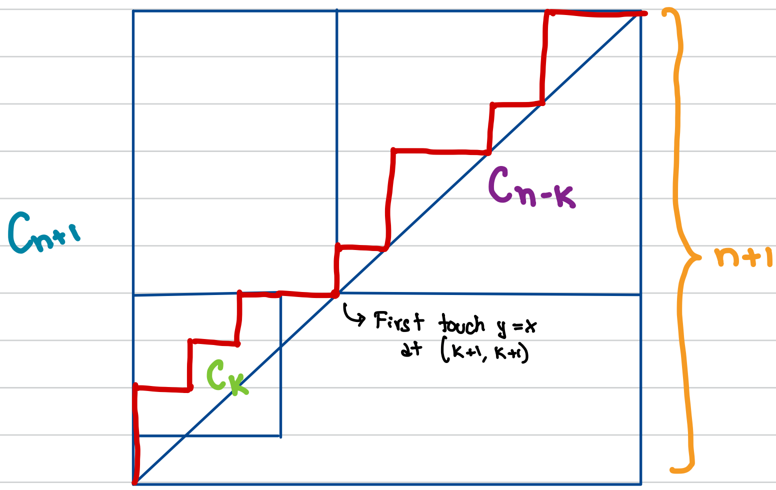
Ex. Catalan Sequence

Let C_n = # paths in plane from $(0,0)$ to (n,n) using $(1,0)$ or $(0,1)$ steps and does not travel below the line $y = x$

When $n = 3$:



Our sequence: $1, 1, 2, 5, 14$.



$(n-1)($

$\frac{(n-1)!}{(n-1)}$ $\quad (n-2)!$

$$C_{n+1} = \underbrace{C_0 C_{n-0}}_{k=0} + C_1 C_{n-1} + \dots + C_n C_0$$

$$= \sum_{k=0}^{n} C_k C_{n-k}$$

Let $C(x) = \sum_{n=0}^{\infty} C_n x^n$

Then $\sum_{n=0}^{\infty} C_{n+1} x^{n+1} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} C_k C_{n-k} \right) x^{n+1}$

$$C(x) - 1 = x^1 C(x)^2$$

$$x C(x)^2 - C(x) + 1 = 0$$

$$C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

$$\lim_{x \to 0} \frac{1-\sqrt{1-4x}}{2x} \quad \text{L'H} \quad \lim_{x\to 0} \frac{-\frac{1}{2}(1-4x)^{-1/2}(-4)}{2} = 1$$

$$C(x) = \frac{1-(1-4x)^{1/2}}{2x} = \frac{1 - \sum_{n=0}^{\infty} \binom{1/2}{n} (-4x)^n}{2x}$$ ← uses $(1+x)^a = \sum \binom{a}{n} x^n$

$$= -\sum_{n=1}^{\infty} \frac{1}{2} \binom{1/2}{n} (-4)^n x^{n-1}$$

So $C_n = \frac{-1}{2} \binom{1/2}{n+1} (-4)^{n+1}$

$$= \cancel{\left(\frac{-1}{2}\right)} \frac{\cancel{\left(\frac{1}{2}\right)}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\cdots\left(\frac{1}{2}-n\right)}{(n+1)!} (-1)^{n\cancel{+1}} 4^{n\cancel{+1}}$$

$$= \frac{(1-1/2)(2-1/2)(3-1/2)\cdots\cdots(n-1/2)}{(n+1)\cdot n!} 4^n$$

$$= \frac{1.3.5\cdots\cdots(2n-1)}{(n+1).n!} 2^n \cdot \frac{2.4.6.8\cdots\cdots(2n)}{2.4.6.8\cdots\cdots(2n)}$$

$$= \frac{(2n)!\ 2^n}{(n+1).n!\ 2^n\ n!} = \frac{(2n)!}{(n+1)(n!)^2} = \frac{1}{n+1}\binom{2n}{n}$$

Definition : A permutation of n is 231 - avoiding if no subsequence has the same relative order as 231.

Ex. 5 2 6 1 3 4 is a permutation of 6. It is not 231 avoiding.