Thm $[n]_q! = \sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma)}$ ← sum of indices with descents

EX $\underset{4}{\_}\ 3\ \underset{5}{\_}\ \overset{*}{8}\ \underset{3}{\_}\ 1\ \underset{6}{\_}\ \overset{*}{7}\ \underset{2}{\_}\ 2\ \underset{7}{\_}\ \overset{*}{6}\ \underset{1}{\_}\ 4\ \underset{8}{\_}\ 5\ \underset{0}{\_}$

$*$ = descent

9 places to put a 9

↑ shifting two descents over by 1

↑ descent position is shifted by 1

↖ net increase to $\mathrm{maj}(\sigma)$ when "9" is inserted

Pf By induction, we consider $[n+1]! = [n+1][n]!$

$$= (1+q+\dots+q^n)\sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma)}$$

Multiplying gives terms of the form $q^j q^{\mathrm{maj}(\sigma)}$ for $0 \le j \le n$, $\sigma \in S_n$.

Label positions to insert "n+1" into σ:

① Label last position with 0.

② Label positions after descents w/ $1, 2, \dots, \mathrm{des}(\sigma)$ from right to left.

③ Label remaining positions w/ $\mathrm{des}(\sigma)+1, \dots, n$ from left to right

Use j and labeled position to create a permutation of $n+1$ w/ major index $j + \mathrm{maj}(\sigma)$. □

Ex Given $\sigma \in S_n$ w/ $\mathrm{inv}(\sigma) = k$, can we find a $\varphi(\sigma)$ w/ $\mathrm{maj}(\varphi(\sigma)) = k$?

σ	Change in inv	$\varphi(\sigma)$
234651		623451
23451	2	23451 2 3 4 5 1 0
2341	1	2341
231	1	231
21	1	21
1	1	1

labels based off of proof ↗

Def Let $R(0^k, 1^{n-k})$ denote the set of rearrangements of k zeros and $n-k$ ones.

Ex

$R(0^2,1^2)$	$\mathrm{inv}(\sigma)$	$\mathrm{maj}(\sigma)$
0011	0	0
0101	1	2
0110	2	3
1001	2	1
1010	3	4
1100	4	2

$$\begin{bmatrix}4\\2\end{bmatrix}_q = q^0+q^1+2q^2+q^3+q^4$$

Thm $$\begin{bmatrix}n\\k\end{bmatrix}_q = \sum_{r\in R(0^k,1^{n-k})} q^{\mathrm{inv}(r)}$$

Pf We show $[n]! = [k]!\,[n-k]! \cdot \sum_{r\in R(0^k,1^{n-k})} q^{\mathrm{inv}(r)}$

$$= \left(\sum_{\sigma\in S_k} q^{\mathrm{inv}(\sigma)}\right)\left(\sum_{\tau\in S_{n-k}} q^{\mathrm{inv}(\tau)}\right)\left(\sum_{r\in R(0^k,1^{n-k})} q^{\mathrm{inv}(r)}\right)$$

→ Choose $r\in R(0^k,1^{n-k})$, $\sigma\in S_k$, $\tau\in S_{n-k}$ and create an element in S_n.

Ex

$$\text{inv's here sum to}\left\{\begin{array}{cccccccccccccc} 0&1&1&1&0&0&1&0&0&0&1&1&0&1\\ 3&&&&1&7&&2&5&6&&&4&\\ &11&8&14&&&10&&&&9&12&&13 \\ \hline \end{array}\right.$$

$$\hookrightarrow \mathrm{inv}(3\ 11\ 8\ 14\ 1\ 7\ 10\ 2\ 5\ 6\ 9\ 12\ 4\ 13)$$

r

$\sigma\in S_7$

"τ"$\in S_{(14-7)} = S_7$

$(k=7)$