Linear Analysis II Set 8

1. Find the vector in the span of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ that is closest to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Draw a clearly labeled picture in $\mathbb{R}^2$ depicting the situation.

2. Let L be the line in $\mathbb{R}^3$ containing the origin and parallel to the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Find the point on L closest to the point (x, y, z).

3. By inspection (not using our next topic, the Gram-Schmidt procedure), find vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that

1. $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal, and
2. the span of $\mathbf{v}_1$ and $\mathbf{v}_2$ is the same as the span of the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

Then use $\mathbf{v}_1$ and $\mathbf{v}_2$ to find the vector in this span that is closest to $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

4. Use orthogonal projections to find the distance from the point $(2, 3, 4)$ to the plane $2x + y + z = 0$.

5. Show that if $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are pairwise orthogonal vectors such that $\|\mathbf{u}_i\| = 1$ for all i, then

$$\|c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k\|^2 = c_1^2 + \cdots + c_k^2.$$

for constants $c_1, \ldots, c_k$.

6. Recall that the transpose of the $m \times n$ matrix A, denoted $A^\top$, is the $n \times m$ matrix found by interchanging the rows and columns of A. A column vector $\mathbf{x}$ in $\mathbb{R}^n$ is an $n \times 1$ matrix, and so $\mathbf{x}^\top$ is a $1 \times n$ matrix.

a. Let $\mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$. Find these four matrix products: $\mathbf{x}^\top\mathbf{y}$, $\mathbf{y}\mathbf{x}^\top$, $\mathbf{y}^\top\mathbf{x}\mathbf{y}$ and $\mathbf{y}\mathbf{y}^\top\mathbf{x}$.

b. Show that $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^\top\mathbf{y}$ holds for any vectors $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$.

c. Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$ and let

$$P = \frac{1}{\mathbf{v}^\top\mathbf{v}}\mathbf{v}\mathbf{v}^\top.$$

Show that the vector in the span of $\mathbf{v}$ that is closest to $\mathbf{x}$ is equal to $P\mathbf{x}$.