Review. (Exam 1)

① Generating Functions from recursions.

ⓐ Catalan/Motzkin paths:

Required multiplication of power series

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)x^n$$

ⓑ Solve using ODEs/PDEs.

Memorize

② Generating Functions from Exponential Formula. (Unordered, Ordered)

ⓐ Best examples: restricted set partitions,

permutations w/ cycle structure,

graphs.

③ Extracting information from Generating Functions.

ⓐ Exact formulas using: partial fractions,

known series.

Memorize

↳ $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

↳ $e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!}$

↳ $\ln\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} \frac{x^n}{n}$

↳ $(1+x)^{\alpha} = \sum_{n=0}^{} \binom{\alpha}{n} x^n$

ⓑ Generate examples (small coefficients of x^n) w/ software.

ⓒ Used bivariate Generating Functions : $A(x,y)$

↳ averages (expected number) ; partial derivatives.

↳ probability

↳ Pick out even/odd terms : $\frac{1^n + (-1)^n}{2} = \begin{cases} 0, & \text{if } n \text{ odd.} \\ 1, & \text{if } n \text{ even.} \end{cases}$

ⓓ Asymptotics:

Memorize

ⓘ $\Gamma(\alpha+1) \sim \sqrt{2\pi\alpha}\left(\frac{\alpha}{e}\right)^{\alpha}$; $\Gamma(n+1) = n!$ for $n \in \mathbb{Z}$.

ⓘⓘ $\lim_{x \to R} (R-x)^{\alpha} \cdot f(x) = c \neq 0, \infty. \Rightarrow a_n \sim \frac{c n^{\alpha-1}}{R^{n+\alpha}\Gamma(\alpha)}$

(Recall assumptions)

④ Basic Combinatorial Structures.

ⓐ Set Partitions

ⓑ Permutations

↳ Notation: Cycle, one-line

ⓒ Lattice paths (in xy-plane)

↳ e.g. Catalan / Motzkin

⑤ Permutation Statistics.

ⓐ $\mathrm{des}(\sigma)$, $\mathrm{exc}(\sigma)$, $\mathrm{inv}(\sigma)$, $\mathrm{maj}(\sigma)$

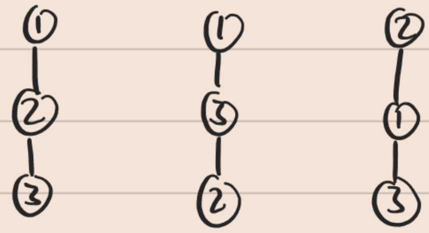
ⓑ $[n]$, $[n]!$, $\begin{bmatrix} n \\ k \end{bmatrix}$

ⓒ $$[n]! = \sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)}$$

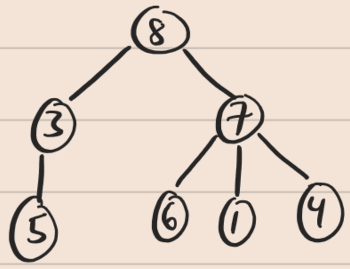
THM. There are n^{n-2} labeled trees.

- connected
- exactly one path between any 2 vertices.

Ex. $n=3$:



$n=8$:

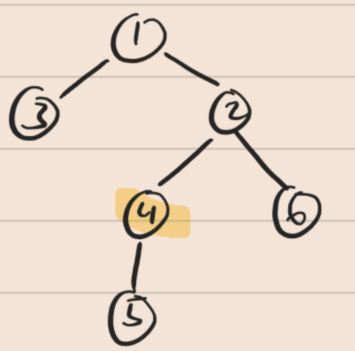


Cards = Labeled Trees ⟹ Hands = Forest.

(Add a condition to the cards.)

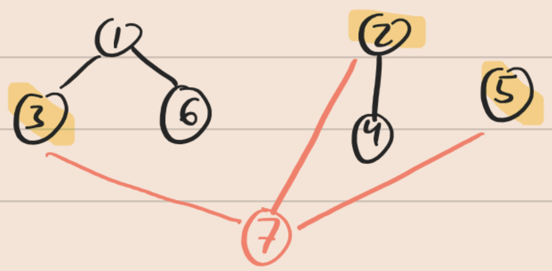
In exponential formula, have cards which are labeled trees with one distinguished vertex.

Ex. A card when $n=6$:



A hand of size n corresponds to a labeled tree with $n+1$ vertices.

Ex.



Becomes a card of size $n+1$.

Assume there are n^{n-2} labeled trees.

The exponential formula (taking $y=1$) says:

$$\sum_{n=0}^{\infty} \begin{pmatrix} \text{\# hands} \\ \text{of size } n \end{pmatrix} \frac{x^n}{n!} = e^{\sum_{n=1}^{\infty} (\text{\# cards of size } n) \frac{x^n}{n!}}$$

↳ Hand on n vertices = Card on $n+1$ vertices

$\Rightarrow (n+1)^{n+1-2} = (n+1)^{n-1}$

↳ Cards: For each labeled tree, pick one vertex to be yellow

$\Rightarrow n \cdot n^{n-2} = n^{n-1}$

$$\sum_{n=0}^{\infty} \left((n+1)^{n+2-1} \right) \frac{x^n}{n!} = e^{\sum_{n=1}^{\infty} n^{n-1} \cdot \frac{x^n}{n!}}$$

Ex. What is the average # of cycles in a permutation of n with no cycles of length 4 or more?

By exponential formula:

$$\sum_{n=0}^{\infty} \Bigg(\sum_{\substack{\sigma \in S_n \\ \text{cycles} \le 3}} y^{\text{\# cycles in } \sigma} \Bigg) \frac{x^n}{n!} = e^{y\left(\frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{2}{3!}x^3\right)}$$

$$= e^{y\left(x + \frac{x^2}{2} + \frac{x^3}{3}\right)}$$

↳ The number of permutations (Hands) $\sigma \in S_n$ with needed property is the coefficient of $\frac{x^n}{n!}$ in $e^{x + x^2/2 + x^3/3}$ (plug $y=1$)

↳ The total number of cycles (Cards) in such $\sigma \in S_n$ is the coefficient of $\frac{x^n}{n!}$ in $\frac{\partial}{\partial y}\left(e^{y(x + x^2/2 + x^3/3)}\right)\Big|_{y=1}$

Ex. Find $\mathrm{maj}(3 > 1 \; 4 > 2 \; 6 > 5)$

position: $1 + 3 + 5 = 9$