Math 244 Sample Final

Main topics

1. Solving linear systems using row reductions
2. Matrix inverses and the determinant
3. Vector spaces, subspaces, linear independence, span, and basis
4. Eigenvalues and eigenvectors
5. Diagonalization
6. Separable and first order linear differential equations
7. Linear differential equations with constant coefficients
8. Undetermined coefficients for nonhomogeneous differential equations
9. Cauchy-Euler differential equations
10. Reduction of order
11. Solving linear systems with eigenvalues/vectors
12. The matrix exponential

Sample problems

1. Are the following statements **True** or **False**? (Warning: some of these are a bit tricky!)

a. It is possible for a 3×3 matrix A to have only one eigenvector.

b. Solutions to a nonhomogeneous linear system of differential equations form a subspace.

c. Let A be an $n \times m$ matrix. Then

$$(\text{the dimension of the subspace } \{\mathbf{x} \text{ in } \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}\ ) + \text{rank}(A) = m.$$

d. If 0 is not an eigenvalue of a square matrix A, then A^{-1} exists.

e. If A is diagonalizable, then A^{-1} exists.

f. If the columns of A are linearly independent, then A^{-1} exists.

g. If the system $A\mathbf{x} = \mathbf{b}$ has a unique solution for $\mathbf{x}$, then A^{-1} exists.

h. Any collection of $n+1$ vectors in $\mathbb{R}^n$ must be linearly dependent.

i. If a collection of n vectors span $\mathbb{R}^n$, then those n vectors must form a basis for $\mathbb{R}^n$.

j. The determinant of a diagonalizable matrix is the product of its eigenvalues.

2. Consider the differential equation $y'' - 6y' + 9y = 0$.

a. Solve in the usual way.

b. Write as a linear system and solve using eigenvalues and eigenvectors.

3. Consider the differential equation $y'' + 9y = 0$.

a. Solve in the usual way.

b. Write as a linear system and solve using the matrix exponential.

4. Solve $\begin{cases} x' = y, \\ y' = -5x + 2y. \end{cases}$

5. Find the matrix exponential e^{At} where $A = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}$ and use it to solve $\mathbf{x}' = A\mathbf{x}$.

6. Show that solutions to $\mathbf{x}' = A\mathbf{x}$ form a subspace.

7. Solve $x^2y'' + xy' - y = x$.

8. Solve $y'' - y = e^x + e^{2x}$.

9. One solution to $xy'' - (1 + 2x)y' + 2y = 0$ is e^{2x}. Find a second linearly independent solution.

10. Are the matrices $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, and $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ linearly independent?

11. Solve $\begin{cases} x - y + z - w = 1, \\ 2x - y + 2z - w = 1, \\ x + 3z + z + 3w = 1. \end{cases}$

12. Solve $\begin{cases} y' = ye^x - e^x, \\ y(0) = 1. \end{cases}$

13. Let λ be an eigenvalue for A and show that $S = \{\mathbf{x} \text{ in } \mathbb{R}^n : A\mathbf{x} = \lambda\mathbf{x}\}$ is a subspace of $\mathbb{R}^n$.

14. Give an example of a 3 dimensional subspace of $\mathbb{R}^5$ which does not contain the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$.

15. Let $S = \{n \times n \text{ matrices } A : \det(A) \neq 0\}$. Either show S is a subspace or show it is not a subspace.

16. Let $S = \text{span}\{1-2x, x^2-2x, 2x, x^3-3x\}$. Find a basis for S and the dimension of S.

17. Find the rank of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 \end{bmatrix}$. What is the dimension of the subspace of solutions to $A\mathbf{x} = \mathbf{0}$?

18. Find a basis for the subspace of 2×2 matrices A with diagonal entries which sum to 0.

19. Solve $\begin{cases} x' = y + z, \\ y' = x + z, \\ z' = y + z \end{cases}$ where x, y, and z are functions of t.

20. Solve $y'' - 2y' + 4y = 1 + 13\cos x$.

21. Find the inverse matrix to $\begin{bmatrix} -2 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$ and use it to solve the linear system $\begin{cases} -2x + y + z = a, \\ y + 2z = b, \\ x = c. \end{cases}$

22. Find the matrix exponential e^{At} where $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and use it to solve $\mathbf{x}' = A\mathbf{x}$.

23. Are $\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ linearly independent? Why or why not?

24. Let $A = \begin{bmatrix} 2 & 2 & 1 & 1 \\ -2 & 1 & -3 & 4 \end{bmatrix}$ and let $S = \{\mathbf{x} \text{ in } \mathbb{R}^4 : A\mathbf{x} = \mathbf{0}\}$. Find a basis the dimension of S.

25. Solve $x^2y'' - xy' + y = x \ln x$ if one solution to the homogeneous problem is $y = x^r$ for some r.

26. Solve $xy' + y = xy + 2xe^x$.

27. Let A be an $n \times n$ matrix and consider $S = \{n \times n \text{ matrices } B : AB = BA\}$. Either show S is a subspace or show it is not a subspace.

28. Solve, by any means, the linear system $\begin{cases} x - y + z - w = 2, \\ -x + y - z + w = -2, \\ x + y + 2w = 0, \\ x - y + 2z = 1. \end{cases}$