11/2/23
Samuel Tiscareno

We have ① $\sum_{n=0}^{\infty} p(n) z^n = \prod_{i=1}^{\infty} \frac{1}{1-z^i}$

② $\prod_{i=1}^{\infty} (1-z^i) = \sum_{k\in\mathbb{Z}} (-1)^k z^{k(3k-1)/2}$

Therefore $(p(0) + p(1)z + p(2)z^2 + \dots)(1 - z^1 - z^2 + z^5 + z^7 - \dots)$

Now compare coef. of z^n on both sides

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) + \dots = 0$$
$$p(0) = 1$$

$$\boxed{\begin{aligned} p(n) &= p(n-1) + p(n-2) - p(n-5) - p(n-7) + \dots \quad \text{if } n \geq 1 \\ p(0) &= 1 \end{aligned}}$$

$$p(11) = p(10) + p(9) - p(6) - p(4) + \underset{0}{p(-1)} + \underset{0}{\dots} \dots$$

$$= (p(9) + p(8) - p(5) - p(3)) + (p(8) + p(7) - p(4) - p(2)) - (p(5) + p(4) - p(1))$$
$$- p(4)$$

$$= p(9) + 2p(8) - 7 - 3 + p(7) - 5 - 2 - 11 - 5$$

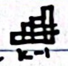
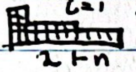
$$= p(9) + 2p(8) + p(7) - 33$$
$$= p(8) + p(7) - p(3) - p(1) + 2p(8) + p(7) - 33$$
$$= 3p(8) + 2p(7) - 37$$
$$= 3(p(7) + p(6) - p(3) - p(1)) + 2p(7) - 37$$
$$= 5p(7) + 3p(6) - 49$$
$$= 5(p(6) + p(5) - p(2) - p(0)) + 3p(6) - 49$$
$$= 8p(6) + 35 - 15 - 49 = 8(p(5) + p(4) - p(1)) - 29$$
$$= 88 - 29 = 59$$

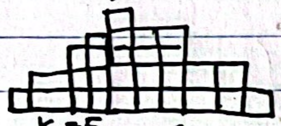
Made some mistake since $p(10) = 56$

11/2/23

Thm: The coefficient of x^k in $\prod_{n=0}^{\infty}(1+xz^n)\cdot\prod_{n=1}^{\infty}(1+x^{-1}z^n)$

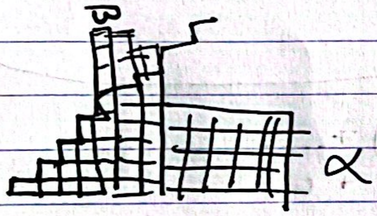
is $z^{k(k-1)/2}\prod_{i=1}^{\infty}\frac{1}{1-z^i}$

Proof: The $z^{\binom{k}{2}}\prod_{i=1}^{\infty}\frac{1}{1-z^i}$ term gives the number of boxes in a pair ( $k-1$,  $\lambda\vdash n$)



$k=5$ λ

Note $\prod_{n=0}^{\infty}(1+xz^n)$ counts partitions, possible w/ ~~size~~ a part of size 0, with distinct parts where x counts the length



α has distinct parts since added staircase

β is on its side and β also has distinct parts

The power ~~coef.~~ of x^k with this picture is $x^{\ell(\alpha)-\ell(\beta)}$ in this case that is 4.

It depends if the last step is up or over. If up, $k-1$, if over k.
If you get $k-1$ select the 0 part

Rewritten, $(1+x)\prod_{n=1}^{\infty}(1-z^n)(1+xz^n)(1+x^{-1}z^n)=\sum_{k\in\mathbb{Z}}x^k z^{k(k-1)/2}$

Jacobi's triple product identity