Graph Theory Final Review

This review only covers material introduced after midterm 2, but **the final exam is cumulative**! Combine this review with the midterm 1 and midterm 2 reviews for all topics that will appear on the final.

Definitions: adjacency matrix, eigenvalue and eigenvector for a graph, strongly connected network, probability vector, Perron value, Perron vector, random walk, distance, diameter, incidence matrix, Laplacian matrix, algebraic connectivity, Tutte layout.

Theorems:

Let $\lambda_{\text{max}} \geq \cdots \geq \lambda_n$ be the eigenvalues for the adjacency matrix $A = A(G)$ and let $0 \leq \mu_2 \leq \cdots \leq \mu_n$ be the eigenvalues for the Laplacian matrix $L(G)$.

- The number of walks of length k that start and end at the same vertex is $\lambda_{\text{max}}^k + \cdots + \lambda_n^k$. As a corollary, the graph has is $(\lambda_{\text{max}}^3 + \cdots + \lambda_n^3)/6$ triangles.
- If $\mathbf{v}$ is the Perron vector for A, then $\mathbf{v}$ gives the limiting distribution of landing on a given vertex after a long random walk.
- The graph is bipartite if and only if $-\lambda_{\text{max}}$ is an eigenvalue. In this case the positive and negative components of the corresponding eigenvector gives the independent sets.
- (average degree) $\leq \lambda_{\text{max}} \leq$ (maximum degree).
- We have $1 - \lambda_{\text{max}}/\lambda_n \leq \chi(G) \leq \lambda_{\text{max}} + 1$.
- The graph G has no cycles if and only if $M_G(-x)$ is the characteristic polynomial for $A(G)$.
- The diameter of G is less than the number of distinct eigenvalues of $A(G)$.
- If G is d-regular, then $\mu_i = d - \lambda_i$.
- $\mu_2 \leq \sum_{\{i,j\} \text{ is an edge}} (x_i - x_j)^2/(x_1^2 + \cdots + x_n^2)$ for any $\mathbf{x}$ with $1^\top \mathbf{x} = 0$.
- $\mu_2 \leq \kappa(G)$
- The multiplicity of 0 as an eigenvalue for $L(G)$ is the number of components of G.
- The number of spanning trees in G is $\mu_2 \cdots \mu_n/n$.
- The eigenvectors corresponding to μ_2, μ_3, μ_4 give good coordinates for a graph layout.
- The Tutte layout can give planar embeddings.

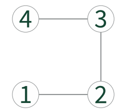
Sample questions:

1. Find the Perron value for the adjacency matrix for $K_{m,n}$.

2. Find a reasonable upper bound for the algebraic multiplicity μ_2 for the Petersen graph (or your favorite graph) using the Courant-Fischer theorem.

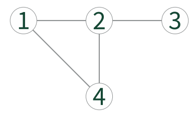
3. Find the eigenvalues for the star graph with n vertices. This is the star graph with 10 vertices:

4. Place vertices 1, 2, 3, 4 in C_6 at $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$, shown below:



Where should vertices 5 and 6 be placed in a Tutte layout?

5. How would you find the limiting distribution of landing on a given vertex after a long random walk in this graph?



6. How many spanning trees are there for the cube Q_n?

7. Find a formula for the number of cycles in length 4 in a d regular graph with eigenvalues $\lambda_1, \ldots, \lambda_n$.