Lemma: $\begin{bmatrix} n \\ \lambda_1, \lambda_2, \dots \end{bmatrix}_q \left(q^{\binom{\lambda_1}{2} + \binom{\lambda_2}{2} + \dots} \right) = \sum_{\sigma \in S_n} q^{inv(\sigma)}$ w/ decreasing runs of length $\lambda_1, \lambda_2, \dots$

where $\sum \lambda_i = n$

Ex: $\underbrace{8\,7\,2\,1}_{\substack{\text{decreasing run} \\ \lambda_1 = 4}} \mid \underbrace{4\,3}_{\substack{\text{dec. run} \\ \lambda_2 = 2}} \quad \underbrace{6\,5}_{\substack{\text{dec. run} \\ \lambda_3 = 2}}$

Now define $\varphi(e_n) = \frac{(-1)^{n-1}}{[n]_q!} q^{\binom{n}{2}} (x-1)^{n-1}$ to be a homomorphism

Then $[n]!\,\varphi(h_n) = [n]! \sum_{\lambda \vdash n} (-1)^{n - \ell(\lambda)} \left| B_{\lambda, (n)} \right| \varphi(e_\lambda)$

$$= \underbrace{\sum_{\lambda \vdash n}}_{\substack{\text{What type of} \\ \text{content in } (n)}} \underbrace{\begin{bmatrix} n \\ \lambda_1, \lambda_2, \dots \end{bmatrix}}_{\substack{\text{Chooses} \\ \text{all \#'s} \\ \text{in content}}} \underbrace{\left| B_{\lambda, (n)} \right|}_{\substack{\text{Order of} \\ \text{content in } (n)}} \underbrace{\left(q^{\binom{\lambda_1}{2} + \binom{\lambda_2}{2} + \dots} \right)}_{\substack{\text{Counts} \\ \text{inversions}}} \underbrace{\left((x-1)^{\lambda_1 - 1} (x-1)^{\lambda_2 - 1} \dots \right)}_{\substack{\text{Place } x \text{ everywhere} \\ \text{except last cell.}}}$$

goes with a q^{24}

Apply similar involution as before to get $(\dots) = \sum_{\sigma \in S_n} q^{inv(\sigma)} x^{dec(\sigma)}$

Apply φ to $\sum h_n z^n = \frac{1}{\sum_{n=0}^{\infty} (-1)^n e_n z^n}$ to get:

$$\sum_{n=0}^{\infty} \left(\frac{1}{[n]!} \sum_{\sigma \in S_n} q^{inv(\sigma)} x^{des(\sigma)} \right) z^n = \frac{1}{1 - \sum_{n \geq 1} \frac{1}{[n]!} q^{\binom{n}{2}} (x-1)^{n-1} z^n} \cdot \frac{(x-1)}{(x-1)}$$

$$= \frac{x-1}{x - e_q^{(x-1)z}} \quad \text{where} \quad e_q^z = \sum_{n=0}^{\infty} \frac{1}{[n]!} q^{\binom{n}{2}} z^n$$

Lemma: from earlier $\begin{bmatrix} n \\ \lambda_1, \dots, \dots \end{bmatrix}_q q^{\binom{\lambda_1}{2} + \dots} = \sum_{\substack{\sigma \in S_n \\ \text{w/ dec runs } \lambda_1, \dots}} q^{inv(\sigma)}$

Pf. Example: $r = 1\ 3\ 1\ 2\ 3\ 3\ 1\ 1\ 2\ 1\ 2$

$\sigma^{-1} : 5\ 11\ 4\ 8\ 10\ 9\ 3\ 2\ 7\ 1\ 6$

$\sigma = \underbrace{10\ 8\ 7\ 3\ 1}_{q^{\binom{5}{2}}}\ \underbrace{11\ 9\ 4}_{q^{\binom{3}{2}}}\ \underbrace{6\ 5\ 2}_{q^{\binom{3}{2}}}$

dec runs

"21 inversions, thus q^{21}"

make σ^{-1} by going R to L for each number, starting with 1 and going up, and labelling up starting with 1.

The values of the dec. runs are the indices of the #'s in r

"10, 8, 7, 3, 1 are the positions of 1 in r"

Each dec run represents $q^{\binom{\lambda_i}{2}}$, which is the # of inversions

QUESTION ON FINAL: State and Prove Euler Pentagonal # Thm.

- 1' on ring hom.
- 1 on Remmel bij
- 1' on g.f w/ exp. formula
- 1 on q analogues
- 1 on asymptotics, given a g.f
- No coming up w/ recursions on the spot

$\boxed{\text{8 questions!}}$