Graph Theory Midterm 1 Solutions

1. State the definition of

a. a graph

b. a connected graph

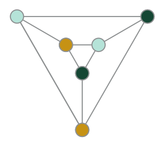
c. a spanning tree for a graph G

d. isomorphic graphs G_1 and G_2

Solution. *See the notes for the definitions.*

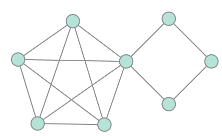
2. Find the chromatic number for the line graph of the complete bipartite graph $K_{2,3}$.

Solution. *The line graph for $K_{2,3}$ is shown below, colored with three colors:*



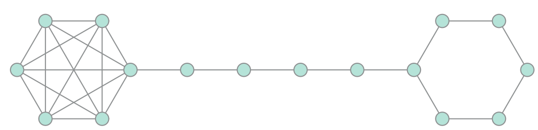
This is an optimal coloring since there is an odd cycle. So the chromatic number is 3.

3. A **coalescence** of the graphs G_1 and G_2 is a graph created by merging a vertex in G_1 with a vertex in G_2. For example, a coalescence of K_5 and C_4 is



a. Let G be a coalescence of G_1 and G_2. Explain why $P_G(x) = P_{G_1}(x)P_{G_2}(x)/x$.

b. Find the chromatic polynomial for the following graph.



Solution. *Select one of the $P_{G_1}(x)$ x-colorings of G_1. Color the vertex v in G_2 that is to be merged with a vertex in G_1 the same color as it appears in G_1. Color the remaining graph G_2 in $P_{G_2}(x)/x$ ways, with the division by x accounting for the already colored v.*

The graph shown is a coalescence of K_6, P_6, and C_6. The chromatic polynomial is

$$P_{K_6}(x)P_{P_6}(x)P_{C_6}(x)/x^2 = (x-1)^6(x-2)(x-3)(x-4)(x-5)((x-1)^6 + (x-1)).$$

4. Use the bijection in the (second) proof of Cayley's formula to show that there are $2n^{n-3}$ trees on n vertices with the property that vertex 1 and vertex 2 are adjacent.

Solution. *In order for vertex 1 and 2 to be adjacent, the function $f : \{2, \dots, n-1\} \to \{1, \dots, n\}$ in the second proof of Cayley's formula must have $f(2) = 1$ or $f(2) = 2$. There are 2 choices here and n choices for the remaining values of $f(i)$ for $i = 3, \dots, n-2$. So there are $2n^{n-3}$ total trees with 1 and 2 adjacent.*

5. Suppose T is a tree with n vertices without a vertex of degree $n-1$. Show that T^c is connected.

Solution. *The number of edges in the complement graph of a tree is $\binom{n}{2} - (n-1) = \binom{n-1}{2}$. Since T does not have a degree $n-1$ vertex, the complement graph cannot have a lone vertex as a disconnected component. By our theorem that says all graphs with more than $\binom{n-1}{2}$ edges is connected, This is the only way to have a disconnected graph with $\binom{n-1}{2}$ edges, and so T^c is connected.*