

## Matrix Multiplication

1. Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 3 \\ 5 & 1 & 1 \\ 4 & -4 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}.$$

Perform the following matrix operations if possible:

- $AB$
- $BA$
- $B^2$
- $B^T B$
- $AC$
- $DBC$
- $CD$

2. Let  $A$  be a  $m \times n$  matrix and  $C$  an  $r \times s$  matrix. What dimensions must  $B$  have so that  $ABC$  is defined?

3. Find  $A^2$ ,  $A^3$  and  $A^4$  for

- $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
- $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ -1 & 0 & 2 \end{bmatrix}$
- $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

4. Let  $A$  and  $B$  be  $n \times n$  matrices. Show that

$$(A - B)^2 = A^2 - AB - BA + B^2.$$

5. Let  $A = \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{bmatrix}$  show that that  $A$  satisfies

$$A^3 + A - 26I = 0$$

where  $I$  and  $0$  are the  $3 \times 3$  identity and zero matrices.

6. Let  $A = \begin{bmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix}$ . Show that  $A^4 = 0$ .

7. A matrix  $A$  is symmetric if  $A = A^T$ . Use properties of the transpose to show that

- $AA^T$  is symmetric for any matrix  $A$
- $A + A^T$  is symmetric for any square matrix  $A$
- $(ABC)^T = C^T B^T A^T$ .

## Linear Systems

8. Write the linear system

$$\begin{cases} x_1 + x_2 - x_3 + 5x_4 = 0, \\ 2x_2 - x_3 + 7x_4 = 0, \\ 4x_1 + 2x_2 - 3x_3 + 13x_4 = 0, \end{cases}$$

as a matrix multiplication of the form  $A\mathbf{x} = \mathbf{0}$  and then verify that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s \\ s - 2t \\ 2s + 3t \\ t \end{bmatrix}$$

is a solution to the system for any  $s$  and  $t$ .

9. Write the linear system

$$\begin{cases} 6x - 2y = 1, \\ -3x + y = 1. \end{cases}$$

as a matrix multiplication of the form  $A\mathbf{x} = \mathbf{b}$  and then verify that there are no solutions to this system.

10. Let  $A$  be an  $m \times n$  matrix.

- If  $A\mathbf{x} = \mathbf{0}$  for vectors  $\mathbf{x}$  and  $\mathbf{0}$ , then what dimensions must  $\mathbf{x}$  and  $\mathbf{0}$  be?
- Let  $\mathbf{x}$  and  $\mathbf{y}$  be vectors that satisfy  $A\mathbf{x} = \mathbf{0}$  and let  $c$  be a constant. Show that  $\mathbf{x} + c\mathbf{y}$  satisfies  $A\mathbf{x} = \mathbf{0}$ .

## Elementary Row Operations

11. Use elementary row operations to put these matrices into reduced row echelon form and then state the rank of each matrix.

- $\begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix}$

- $\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$

- $\begin{bmatrix} 2-i & 2 \\ 1 & -i \end{bmatrix}$

$$\text{d. } \begin{bmatrix} 0 & 1 & 5 \\ 0 & 2 & 4 \\ 0 & 5 & 1 \end{bmatrix}$$

$$\text{e. } \begin{bmatrix} 1 & 1 & 5 & 1 & 2 \\ -1 & 2 & 4 & 2 & -1 \\ 3 & 5 & 1 & 3 & 0 \end{bmatrix}$$

## Solving linear systems

**12.** Solve the following linear systems using elementary row operations (Gaussian Elimination):

$$\text{a. } \begin{cases} 4x_1 - x_2 = 8, \\ 2x_1 + x_2 = 1. \end{cases}$$

$$\text{b. } \begin{cases} 4x - y - z = 1, \\ x + y + z = 3. \end{cases}$$

$$\text{c. } \begin{cases} x - y - z = 0, \\ x + y + z = 0, \\ 2x - 2y = 0. \end{cases}$$

$$\text{d. } \begin{bmatrix} 4 & 2 \\ 4 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{e. } \begin{bmatrix} 1 & 0 & 5 \\ 3 & -2 & 11 \\ 2 & -2 & 6 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

$$\text{f. } \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 0 & 3 & 6 & 9 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{g. } \begin{bmatrix} 1+i & 1-2i \\ -1+i & 2+i \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{h. } \begin{bmatrix} 2+i & i & 3-2i \\ i & 1-i & 4+3i \\ 3-i & 1+i & 1+5i \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

## Inverse matrices

**13.** Verify by matrix multiplication that these matrices are inverses, provided that  $ad - bc \neq 0$ :

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

**14.** Find the inverse of the matrix if possible:

$$\text{a. } \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{b. } \begin{bmatrix} 1 & 1+i \\ 1-i & 1 \end{bmatrix}$$

$$\text{c. } \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\text{d. } \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & -2 \end{bmatrix}$$

$$\text{e. } \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 4 & 1 \end{bmatrix}$$

$$\text{f. } \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

**15.** Use the inverse matrix to solve the system:

$$\text{a. } \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\text{b. } \begin{bmatrix} 3 & 4 & 5 \\ 2 & 10 & 1 \\ 4 & 1 & 8 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

**16.** Let  $A = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$ . Show that  $A^T = A^{-1}$ .

**17.** Suppose that  $A$  satisfies  $A^n = 0$  for some positive integer  $n$ . Show that the inverse to  $I - A$  is

$$I + A + A^2 + \cdots + A^{n-1}.$$

## Determinants

**18.** Calculate the determinant:

$$\text{a. } \begin{vmatrix} 3 & 5 & 7 \\ -1 & 2 & 4 \\ 6 & 3 & -2 \end{vmatrix}$$

$$\text{b. } \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$\text{c. } \begin{vmatrix} 3 & 5 & -1 & 2 \\ 2 & 1 & 5 & 2 \\ 3 & 2 & 5 & 7 \\ 1 & -1 & 2 & 1 \end{vmatrix}$$

**19.** Let  $A$  be invertible. Show that  $\det(A^{-1}) = \frac{1}{\det A}$ .

20. Let  $A$  and  $B$  be  $n \times n$  with  $\det A = 5$  and  $\det B = -4$ . Evaluate the determinant:

- $\det(AB)$
- $\det(A^T BA)$
- $\det(A^{-1}BA)$
- $\det(3A)$
- $\det C$  where  $C$  is  $A$  with its first two columns interchanged
- $\det C$  where  $C$  is  $A$  with its first row multiplied by 2

21. Let  $A$  satisfy  $A^T A = I$ . Show that  $\det A = \pm 1$ .

## Subspaces

22. Either show that  $S$  is a subspace of the vector space  $V$  or give an example showing why it is not:

- $V = \mathbb{R}^3$ ,  $S$  is the set of vectors of the form  $\begin{bmatrix} x \\ 2x \\ 3x \end{bmatrix}$ .
- $V = \mathbb{R}^4$ ,  $S$  is the set of vectors of the form  $\begin{bmatrix} x \\ y \\ x \\ 0 \end{bmatrix}$ .
- $V = \mathbb{R}^4$ ,  $S$  is the set of vectors of the form  $\begin{bmatrix} x \\ 1 \\ 2x \\ 0 \end{bmatrix}$ .
- $V = \mathbb{R}^n$ ,  $S$  is the set of solutions to  $Ax = 0$  where  $A$  is a fixed  $m \times n$  matrix.
- $V$  is the vector space of  $2 \times 2$  matrices with entries in  $\mathbb{R}$ ,  $S$  is the set of matrices  $A$  with  $\det A = 1$ .
- $V$  is the vector space of  $3 \times 3$  matrices with entries in  $\mathbb{R}$ ,  $S$  is the set of upper triangular matrices.
- $V$  is the vector space of  $n \times n$  matrices with entries in  $\mathbb{R}$ ,  $S$  is the set of invertible matrices.
- $V$  is the vector space of real valued functions with domain  $\mathbb{R}$ ,  $S$  is the set of functions  $f(x)$  that satisfy  $f(3) = 0$ .
- $V$  is the vector space of real valued functions with domain  $\mathbb{R}$ ,  $S$  is the set of functions of the form  $ax^2 + bx + c$  where  $a, b, c$  are real numbers.
- $V$  is the vector space of real valued functions with domain  $\mathbb{R}$ ,  $S$  is the set of solutions to the differential equation  $y''(x) + y(x) = 0$ .

## Span

23. Determine if the set of vectors span  $\mathbb{R}^3$ :

- $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$
- $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right\}$
- $\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \right\}$

24. Find a set of vectors that span the subspace  $S$  of the vector space  $V$ :

- $V$  is the space of  $2 \times 3$  matrices with entries in  $\mathbb{R}$ ,  $S$  is the set of matrices with entries that sum to 0.
- $V$  is the space of  $n \times n$  matrices with entries in  $\mathbb{R}$ ,  $S$  is the set of upper triangular matrices.
- $V$  is  $\mathbb{R}^3$ ,  $S$  is the set of solutions to  $x - 2y - z = 0$ .
- $V$  is the space of polynomials of degree 5 or less with coefficients in  $\mathbb{R}$ ,  $S$  is the set of polynomials  $p$  that satisfy  $p'(x) = 0$ .

## Linear Independence

25. Determine if the following sets of vectors are linearly independent:

- $\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \right\}$
- $\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \right\}$
- $\left\{ \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \right\}$
- $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$
- $\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 1 \\ 0 \end{bmatrix} \right\}$

**26.** Determine if the given functions are linearly independent on the given interval  $I$ :

- $1, x, x^2; I = \mathbb{R}.$
- $\sin x, \cos x, \tan x; I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$
- $e^x, e^{-x}, \cosh x; I = \mathbb{R}.$
- $e^x, x, \sin x; I = \mathbb{R}.$
- $1 + x + x^2, 1 + x - x^2, 1 + x^2, 1 - x^2; I = \mathbb{R}.$
- $x, \begin{cases} 1 & \text{if } x = 0, \\ x & \text{if } x \neq 0; \end{cases} I = \mathbb{R}.$
- $e^{ax}, e^{bx}, e^{cx}$  for  $a, b, c$  distinct;  $I = \mathbb{R}.$

**27.** Show that if  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly independent, then so is  $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}.$

## Bases

**28.** Determine if the given set of vectors is a basis for the subspace  $S$  of the vector space  $V$ :

- $V = \mathbb{R}^2, S = \mathbb{R}^2, \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$
- $V = \mathbb{R}^3, S = \mathbb{R}^3, \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \right\}.$
- $V$  is space of  $2 \times 2$  matrices with entries in  $\mathbb{R}$ ,  $S$  is the subspace containing matrices with entries that sum to 0,  $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}, \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$

**29.** Find a basis for the nullspace of the matrix (a basis for the subspace of  $\mathbb{R}^n$  containing solutions to  $A\mathbf{x} = \mathbf{0}$ ):

- $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
- $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$
- $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$
- $\begin{bmatrix} 1 & -1 & 4 \\ 2 & 3 & -2 \\ 1 & 2 & -2 \end{bmatrix}$

**30.** Find a basis and the dimension of the subspace  $S$  of the vector space  $V$ :

a.  $V$  is the set of real valued functions on  $\mathbb{R}$ ,  $S$  is the set of solutions to  $f''(x) = 0.$

b.  $V$  is the set of polynomials of degree 3 or less with coefficients in  $\mathbb{R}$ ,  $S$  is the set of polynomials  $p$  that satisfy  $p(-1) = 0.$

c.  $V$  is  $\mathbb{R}^3$ ,  $S$  is the span of the vectors  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} \right\}.$

d.  $V$  is  $\mathbb{R}^3$ ,  $S$  is the span of  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} \right\}.$

e.  $V$  is the space of  $2 \times 2$  matrices over  $\mathbb{R}$ ,  $S$  is the span of  $\left\{ \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 5 & -6 \\ -5 & 2 \end{bmatrix} \right\}.$

f.  $V$  is the space of  $4 \times 4$  matrices over  $\mathbb{R}$ ,  $S$  is the set of matrices  $A$  that satisfy  $A^\top = -A.$

## Eigenvalues and Eigenvectors

**31.** Find the eigenvalues and eigenvectors:

- $\begin{bmatrix} 5 & -4 \\ 8 & -7 \end{bmatrix}$
- $\begin{bmatrix} 1 & 6 \\ 2 & -3 \end{bmatrix}$
- $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
- $\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$
- $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$
- $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

**32.** Show that if  $\lambda$  is an eigenvalue for an invertible matrix  $A$ , then  $\lambda^{-1}$  is an eigenvalue for  $A^{-1}.$

**33.** Show that if  $A$  is square, then  $A$  and  $A^\top$  have the same eigenvalues.

## Diagonalization

**34.** Diagonalize the matrix  $A$  if possible: (provide a matrix  $S$  and  $D$  such that  $A = S^{-1}DS$ ).

a.  $\begin{bmatrix} -9 & 0 \\ 4 & -9 \end{bmatrix}$

b.  $\begin{bmatrix} -7 & 4 \\ -4 & 1 \end{bmatrix}$

c.  $\begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$

d.  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 7 \\ 1 & 1 & -3 \end{bmatrix}$

e.  $\begin{bmatrix} -2 & 1 & 4 \\ -2 & 1 & 4 \\ -2 & 1 & 4 \end{bmatrix}$

f.  $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

g.  $\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

## Separable DEs

**35.** Verify that  $y(t) = A \cos(\omega t - \phi)$  is a solution to  $y'' + \omega^2 y = 0$  where  $A, \omega, \phi$  are constants. Determine constants  $A$  and  $\phi$  that satisfy the initial conditions  $y(0) = a, y'(0) = 0$ .

**36.** When  $k$  is a positive integer, the Legendre equation

$$(1 - x^2)y'' - 2xy' + k(k+1)y = 0$$

with  $-1 < x < 1$  has a polynomial solution. Show that when  $k = 3$  one such solution is  $y(x) = x(5x^2 - 3)/2$ .

**37.** Solve the differential equation:

a.  $y' = 2xy$

b.  $y' = y^2(x^2 + 1)$

c.  $e^{x+y}y' = 1$

d.  $y - xy' = 3 - 2x^2y'$

e.  $(x^2 + 1)y' + xy = ax$  with  $y(0) = 2a$  where  $a$  is a constant

f.  $y' = y^3 \sin x$

**38.** An object of mass  $m$  falls from rest, starting near the earth's surface. Assuming air resistance varies as the square of the velocity of the object, the velocity  $v(t)$  satisfies  $mv' = mg - kv^2$  with  $v(0) = 0$  where  $k, m, g$  are constants. Solve for  $v(t)$ .

## First Order Linear DEs

**39.** Solve the differential equation:

a.  $y' + y = 4e^x$

b.  $y' + 2y/x = 5x^2, x > 0$

c.  $y' + 2xy/(1 + x^2) = 4/(1 + x^2)^2$

d.  $y' + 2xy/(1 - x^2) = 4x, -1 < x < 1$

e.  $y' + y/x = 2x^2 \ln x$

f.  $y' + my/x = \ln x$  with  $m$  a constant

g.  $y' + 2y/x = 4x$  with  $y(1) = 2$

h.  $y' + 2y/(4 - x) = 5$  with  $y(0) = 4$

## Constant Coefficient Homogeneous DEs

**40.** Solve the differential equation:

a.  $y'' - y' - 2y = 0$

b.  $y'' - 6y' + 9y = 0$

c.  $y'' + 8y' + 20y = 0$

d.  $y'' - 14y' + 58y = 0$

e.  $y''' - y'' + y' - y = 0$

f.  $y'' - 8y' + 16y = 0$  with  $y(0) = 2, y'(0) = 5$

g.  $y'' - 2my' + (m^2 + k^2)y = 0$  with  $y(0) = 0, y'(0) = k$  where  $m, k$  are constants

## Constant Coefficient Nonhomogeneous DEs

**41.** Solve the differential equation:

a.  $y'' + y = 6e^x$

b.  $y'' + 4y' + 4y = 5xe^{2x}$

c.  $y'' + 2y' + 5y = 3 \sin 2x$

d.  $y''' + 2y'' - 5y' - 6y = 4x^2$

e.  $y'' - 16y = 20 \cos 4x$

- f.  $y'' + y = 3e^x \cos 2x$
- g.  $y'' + 9y = 5 \cos 2x$  with  $y(0) = 2, y'(0) = 3$
- h.  $y'' + y' - 2y = \sin x$  with  $y(0) = 2, y'(0) = 1$
- i.  $y'' + \omega_0^2 y = F_0 \cos \omega t$  where  $\omega, \omega_0, F_0$  are constants (treat the cases  $\omega = \omega_0$  and  $\omega \neq \omega_0$  separately)
- b.  $x^2 y'' + 5xy' + 6y = 0.$
- c.  $4x^2 y'' + y = 0.$
- d.  $x^2 y'' - 5xy' + 9y = 0.$
- e.  $x^2 y'' - 3xy' + 4y = x + 1.$

## Spring mass systems

**42.** At rest, a mass of 2 kilogram stretches a spring 1/4 meter. (Use  $g = 10$  here.) Let  $y(t)$  be the displacement of the spring at time  $t$ . Assuming friction is given by a term of the form  $y'(t)/2$  and assuming with no external force, find  $y(t)$  if  $y(0) = 1$  and  $y'(0) = 0$ .

**43.** At rest, a mass of 2 kilogram stretches a spring 1/4 meter. (Use  $g = 10$  here.) Assuming negligible friction, find an external force of the form  $\cos(at)$  that produces resonance (the phenomenon of needing to “multiply the homogeneous solution” by  $t$ , creating spring oscillations of a greater and greater amplitude).

**44.** A spring mass system of the form  $my''(t) + cy'(t) + ky(t) = 0$  is called critically damped if the characteristic equation  $mr^2 + cr + k = 0$  has a repeated root. Assuming  $m$  and  $k$  are fixed, find the value of  $c$  in terms of  $m$  and  $k$  for which  $my''(t) + cy'(t) + ky(t) = 0$  is critically damped.

## Reduction of order

**45.** One solution to  $y'' - 2ay' + a^2y = 0$  is  $y = e^{ax}$  where  $a$  is a constant. Use reduction of order to find all solutions.

**46.** One solution to  $y'' - xy' + y = 0$  is  $y = x$ . Find a second linearly independent solution. This second solution may involve an integral which cannot be evaluated, so the answer may involve an integral.

**47.** One solution to  $x^2 y'' + (1 - 2a)xy' + a^2 y = 0$  for  $a$  a constant is  $y_1(x) = x^a$ . Use reduction of order to find all solutions to  $x^2 y'' + (1 - 2a)xy' + a^2 y = x^b$  where  $b$  is another constant.

## Cauchy-Euler

**48.** Solve the differential equation:

a.  $x^2 y'' + 5xy' + y = 0.$

## Linear Systems of DEs

**49.** Convert the differential equation into a first order linear system:

- a.  $y'' + 2ty' + y = \cos t$
- b.  $y''' + t^2 y' - e^t y = t$
- c.  $y'' + ay' + by = F(t)$  where  $a, b$  are constants

**50.** Solve the system  $\mathbf{x}' = A\mathbf{x}$  for the given matrix  $A$ :

a.  $\begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$

b.  $\begin{bmatrix} -2 & -7 \\ -1 & 4 \end{bmatrix}$

c.  $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

d.  $\begin{bmatrix} 2 & 0 & 3 \\ 0 & -4 & 0 \\ -3 & 0 & 2 \end{bmatrix}$

e.  $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

f.  $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

g.  $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

h.  $\begin{bmatrix} -1 & 4 \\ 2 & -3 \end{bmatrix}$  with  $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

i.  $\begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix}$  with  $\mathbf{x}(0) = \begin{bmatrix} -4 \\ 4 \\ 4 \end{bmatrix}$

## Matrix Exponentials

**51.** Find the matrix exponential for the given matrix  $A$  and then state the solution to the system  $\mathbf{x}' = A\mathbf{x}$ :

a.  $\begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}$  with  $\mathbf{x}(0) = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$

b.  $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$  with  $\mathbf{x}(0) = \begin{bmatrix} c \\ d \end{bmatrix}$  where  $a, b, c, d$  are constants

c.  $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$  with  $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$