Tuesday, October 3, 2023 1:00 PM

Named Series:

1) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$
2) $\ln \frac{1}{1-x} = \sum_{n=1}^{\infty} \frac{1}{n} x^n$
3) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
4) $(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$
5) $\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{2n!}$

Thm: Let H_n = # of exponential structures of size n

C_n = # of ways to arrange a single subset of size n into one component of an exponential structure.

Then $$\sum_{n=0}^{\infty} \sum_{h \in H_n} y^{\#\text{ cards in } h} \frac{x^n}{n!} = e^{y \sum_{n=1}^{\infty} c_n \frac{x^n}{n!}}$$

Proof:

$$e^{y \sum_{n=1}^{\infty} c_n \frac{x^n}{n!}} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(y \sum_{n=1}^{\infty} \frac{c_n}{n!} x^n \right)^k = \sum_{k=0}^{\infty} \frac{y^k}{k!} \sum_{n=0}^{\infty} \sum_{\substack{i_1, i_2, i_3, \dots, i_k \geq 1 \\ \text{w/ } i_1 + i_2 + \dots + i_k = n}} \left(\frac{C_{i_1} C_{i_2} \cdots C_{i_k}}{i_1! \, i_2! \cdots i_k!} \right) x^n$$

$$= \sum_{n=0}^{\infty} \Bigg(\sum_{k=0}^{\infty} \underbrace{\frac{y^k}{k!}}_{\text{unordering in exponential structure}} \underbrace{\sum_{\substack{i_1, \dots, i_k \geq 1 \\ i_1 + i_2 + \dots + i_k = n}} \binom{n}{i_1, i_2, \dots, i_k}}_{\text{assign the \#'s to be arranged on each card}} \overbrace{(C_{i_1} C_{i_2} \cdots C_{i_k})}^{\text{arranges \#'s in each card.}} \Bigg) \frac{x^n}{n!}$$

k = # of cards

Ex: Consider set partitions of n. $\underbrace{\sum_{n=0}^{\infty} \Bigg(\sum_{S \in \text{set partition}} y^{\#\text{ of sets in } S} \Bigg) \frac{x^n}{n!}}_{B(x,y)} = e^{y \left(\sum_{n=1}^{\infty} \frac{1}{n!} x^n \right)} = e^{y(e^x - 1)}$

Ex: Consider set partitions of n w/ sets of size 1, 2, 3 only

$$\sum_{n=0}^{\infty} \Bigg(\sum_{S \in \text{set part.}} y^{\#\text{ of sets in } S} \Bigg) \frac{x^n}{n!} = e^{y \left(\frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \right)}$$

Ex: Set Partitions w/ even size sets

$$\rightsquigarrow e^{y(\cosh x - 1)}$$

Ex: Consider set ptn's of n into an even total # of sets

Notice: $\frac{y^k + (-y)^k}{2} = \begin{cases} 0 & \text{if } k \text{ is odd} \\ y^k & k \text{ is even} \end{cases}$

Ans: $\frac{1}{2}\big(B(x,y) + B(x,-y)\big) = \frac{1}{2}\left(e^{y(e^x - 1)} + e^{-y(e^x - 1)}\right) = \cosh\left(y(e^x - 1)\right)$

Ex: $$\sum_{n=0}^{\infty} \Bigg(\sum_{\text{permutations of } n} y^{\#\text{ of cycles}} \Bigg) \frac{x^n}{n!} = e^{y \sum_{n=1}^{\infty} \frac{(n-1)!}{n!} x^n} = e^{y \ln\left(\frac{1}{1-x}\right)} = (1-x)^{-y} = \sum_{n=0}^{\infty} \binom{-y}{n} (-x)^n$$

Therefore $\left(\sum y^{\# \text{cycles}}\right) = y(y+1)(y+2)\cdots(y+n-1)$