

Discrete Mathematics Set 1

Math 435: Complete 6 parts of the following exercises. (For example, one option is {1, 2, 3a, 4, 6a, 6c}.)

Math 530: Complete exercises 1, 3, 5 and 6.

1. Using the Taylor series centered at $x = 0$, show that $(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n$ where $\binom{a}{n} = \frac{a(a-1) \cdots (a-n+1)}{n!}$.

2. Prove that $\frac{1}{(1-x)^a} = \sum_{n=0}^{\infty} \binom{a+n-1}{n} x^n$.

3. Verify the following identities involving the products of series:

a. $\left(\sum_{n=0}^{\infty} a_n x^n \right) \left(\sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k b_{n-k} \right) x^n$

b. $\left(\sum_{n=0}^{\infty} a_n x^n \right)^k = \sum_{n=0}^{\infty} \left(\sum_{\substack{i_1, \dots, i_k \geq 0 \\ i_1 + \dots + i_k = n}} a_{i_1} \cdots a_{i_k} \right) x^n$.

4. By multiplying $(1+x)^a$ and $(1+x)^b$, prove that $\binom{a+b}{n} = \sum_{k=0}^n \binom{a}{k} \binom{b}{n-k}$ holds for all $a, b \in \mathbb{C}$.

5. Let a_n be the number of ways to tile a $2 \times n$ chessboard with dominoes of sizes 2×1 and 2×2 . For example, there are 11 such tilings when $n = 4$:



Find a recurrence for a_n , the generating function for a_n , and a formula for a_n .

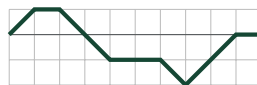
6. A Motzkin path of length n is a path in the plane which starts at $(0,0)$, ends at $(n,0)$, uses steps of the form $(1,1)$, $(1,-1)$, and $(1,0)$, and never travels below (but may touch) the x -axis. For example,



are the 9 Motzkin paths of length 4. Let m_n be the number of Motzkin paths of length n and let $M(x) = \sum_{n=0}^{\infty} m_n x^n$.

a. Show that $(M(x) - 1)/x = M(x) + xM(x)^2$ and then find an explicit formula for $M(x)$.

b. Let a_n be the number of paths in the plane which start at $(0,0)$, end at $(n,0)$, and use steps of the form $(1,1)$, $(1,-1)$, and $(1,0)$. For example, one path when $n = 11$ is



By looking at the first time a path touches the x axis, show that $a_{n+2} = a_{n+1} + 2 \sum_{k=0}^n m_k a_{n-k}$ for $n \geq 0$.

c. Show that $A(x) = 1/\sqrt{1-2x-3x^2}$.

Discrete Mathematics Set 2

Math 435: Complete 8 parts of the following exercises.

Math 530: Complete exercises 8 and 9.

7. Let $b_{n,k}$ be the coefficient of $e^{e^x} e^{kx}$ in $\frac{d^n}{dx^n} (e^{e^x})$. Prove that $b_{n,k}$ is the number of set partitions of n into k sets.

8. Let $b_{n,k}$ be the number of set partitions of n into k sets and $T_n(y) = \sum_{k=0}^n b_{n,k} y^k$. We have $\sum_{n=0}^{\infty} T_n(y) \frac{x^n}{n!} = e^{y(e^x-1)}$.

a. Show that $T_n(a+b) = \sum_{k=0}^n \binom{n}{k} T_k(a) T_{n-k}(b)$.

b. Show that $\sum_{n=0}^{\infty} T_{n+1}(y) \frac{x^n}{n!} = e^{y(e^x-1)} y e^x$ and use this to show $T_{n+1}(y) = y \sum_{k=0}^n \binom{n}{k} T_k(y)$.

9. An ordered set partition of n is an ordered list of disjoint nonempty sets with union $\{1, \dots, n\}$. For example, there are 13 ordered set partitions of 3:

$(\{1, 2, 3\})$,
 $(\{1\}, \{2, 3\})$, $(\{2, 3\}, \{1\})$, $(\{2\}, \{1, 3\})$, $(\{1, 3\}, \{2\})$, $(\{3\}, \{1, 2\})$, $(\{1, 2\}, \{3\})$,
 $(\{1\}, \{2\}, \{3\})$, $(\{1\}, \{2\}, \{3\})$, $(\{2\}, \{1\}, \{3\})$, $(\{2\}, \{3\}, \{1\})$, $(\{3\}, \{1\}, \{2\})$, $(\{3\}, \{2\}, \{1\})$.

Let a_n be the number of ordered set partitions of n and let $A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$.

a. Show that $a_0 = 1$ and $a_n = \sum_{k=1}^n \binom{n}{k} a_{n-k}$ for $n \geq 1$.

b. Show that $A(x) = 1/(2 - e^x)$.

c. Expand $A(x)$ as a geometric series to show that $a_n = \frac{1}{2} \sum_{k=0}^{\infty} \frac{k^n}{2^k}$.

d. Let $a_{n,k}$ be the number of ordered set partitions of n into exactly k sets. Show that $a_{n+1} = k a_{n,k} + k a_{n,k-1}$.

e. Let $A(x, y) = \sum_{n=0}^{\infty} \left(\sum_{k=1}^n a_{n,k} y^k \right) \frac{x^n}{n!}$. Show that $A(x, y)$ satisfies $A_x = yA + (y + y^2)A_y$ and $A(x, 1) = 1/(2 - e^x)$.

f. Verify that $A(x, y) = 1/(1 - y(e^x - 1))$.

g. Let t_n be the total number of sets in all ordered set partitions of n . Find a generating function for $\sum_{n=0}^{\infty} t_n \frac{x^n}{n!}$.

h. (Optional!) Show the change of variables $w(x, y) = (1 + 1/y)e^{-x}$ and $z(x, y) = y$ can solve the PDE in part e.

Discrete Mathematics Set 3

Math 435: Complete 6 parts of the following exercises.

Math 530: Complete exercises 12 and 13.

10. Use the exponential formula to find a generating function for the number of permutations of n that do not have any cycles of size 1 (such a permutation is called a derangement). Use this generating function to find an explicit formula for the number of such permutations of n .

11. Find a generating function for the number of set partitions of n which have an even total number of sets, all of which are an even size. Write the answer in terms of the function $\cosh x = \sum_{n=0}^{\infty} x^{2n} / (2n)!$.

12. The generating function for the number of permutations of n with only even sized cycles is

$$\sqrt{\frac{1}{1-x^2}}. \quad (1)$$

(The number of such permutations is $1^2 \cdot 3^2 \cdot 5^2 \cdots (n-1)^2$ if n is even and 0 if n is odd.)

a. Use the exponential formula to prove that

$$\sum_{n=0}^{\infty} (\text{the number of permutations of } n \text{ with only odd sized cycles}) \frac{x^n}{n!} = (1+x) \sqrt{\frac{1}{1-x^2}}. \quad (2)$$

b. The coefficients of x^2 in (1) and (2) are the same. Therefore the number of permutations of $2n$ with only even sized cycles is equal to the number of permutations of $2n$ with only odd sized cycles. Find a bijection between these two sets of permutations.

13. Let L_n be the set of ordered lists of the form (C_1, \dots, C_m) where C_1, \dots, C_m are cards containing disjoint sets with union $\{1, \dots, n\}$. This is similar to hands in the exponential formula with the difference being that hands are unordered and lists are ordered.

a. Let $C(x) = \sum_{n=1}^{\infty} |C_n| \frac{x^n}{n!}$. Show that $\sum_{n=0}^{\infty} \left(\sum_{\ell \in L_n} y^{(\text{number of cards in } \ell)} \right) \frac{x^n}{n!} = \frac{1}{1-yC(x)}$.

b. Use part a. of this exercise to find the result in part f. of Exercise 9 in Set 2.

c. A permutation of n with ordered cycles is a list $(\sigma_1, \dots, \sigma_m)$ where $\sigma_1, \dots, \sigma_m$ are the cycles in a permutation of n . Let \mathcal{A}_n be the set of permutations of n with ordered cycles and find

$$\sum_{n=0}^{\infty} \left(\sum_{\ell \in \mathcal{A}_n} y^{(\text{number of cycles in } \ell)} \right) \frac{x^n}{n!}.$$

d. Let t_n be the total number of cards in all elements in L_n . Find a generating function involving $C(x)$ for $\sum_{n=0}^{\infty} t_n \frac{x^n}{n!}$.

Discrete Mathematics Set 4

Math 435: Complete 7 parts of the following exercises.

Math 530: Complete exercises 16, 17 and 18.

14. Show $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$. (This was used when introducing the Gamma function and Stirling's approximation.)

15. Let a_n, b_n, c_n and d_n be sequences of real numbers.

- Show that $\lim_{n \rightarrow \infty} |a_n - b_n| = 0$ does not imply $a_n \sim b_n$.
- Show that $a_n \sim b_n$ and $c_n \sim d_n$ does not imply $a_n + c_n \sim b_n + d_n$.

16. Let $\alpha > 0$ and n be a nonnegative integer.

- Use induction to show that $\int_0^1 x^{\alpha-1} (1-x)^n dx = \frac{n!}{\alpha(\alpha+1) \cdots (\alpha+n)}$.
- Assuming that the limit and integral can be interchanged, use $\lim_{n \rightarrow \infty} \int_0^1 x^{\alpha-1} \left(1 - \frac{x}{n}\right)^n dx$ to show that

$$\Gamma(\alpha) = \lim_{n \rightarrow \infty} \frac{n! n^\alpha}{\alpha(\alpha+1) \cdots (\alpha+n)}.$$

- (Optional!) Justify why the limit and integral can be interchanged in part b.

17. Let a_n be the number of ordered set partitions of n . Set 2 Exercise 9c gives $a_n = \frac{1}{2} \sum_{k=0}^\infty k^n 2^{-k} \approx \frac{1}{2} \int_0^\infty x^n 2^{-x} dx$.

Use a substitution in the above integral and then use Stirling's approximation to find $a_n \approx \frac{\sqrt{2\pi n}}{\ln 4} \left(\frac{n}{e \ln 2}\right)^n$.

18. Let A_n be the set of paths in \mathbb{R}^2 which start at $(0,0)$, end at (n,n) , and only use steps of the form $(1,0)$ or $(0,1)$. Denote the number of times the path $p \in A_n$ touches the line $y = x$ by $\text{touch}(p)$. Let

$$A(x, t) = \sum_{n=0}^\infty \left(\sum_{p \in A_n} t^{\text{touch}(p)} \right) x^n.$$

- Let c_n be the n^{th} Catalan number. Show that

$$\sum_{p \in A_{n+1}} t^{\text{touch}(p)} = 2t \sum_{k=0}^n c_k \left(\sum_{p \in A_{n-k}} t^{\text{touch}(p)} \right).$$

- Show that $A(x, t) = \frac{t}{1 - t + 2t\sqrt{\frac{1}{4} - x}}$.

- With the help of the corollary to the first asymptotic result in video 17, find an asymptotic formula for the average number of times a path in A_n touches the line $y = x$.

Discrete Mathematics Set 5

Math 435: Complete 6 parts of the following exercises.

Math 530: Complete exercises 21–24.

19. Let a_n be as defined in Set 1 Exercise 6b. Find an asymptotic formula for a_n using the result in Set 1 Exercise 6c.

20. Let a_n be the number of ordered set partitions of n .

- Use the generating function in Set 2 Exercise 9 to find an asymptotic formula for a_n given in Set 4 Exercise 17.
- Using the ideas in Set 3 Exercise 13d, show that the average number of sets in an ordered set partition of n is approximately $n / \ln 4$.

21. Let a_n be the number of permutations of n with ordered cycles.

- Use the generating function in Set 3 Exercise 13c to find an asymptotic formula for a_n .
- Find an asymptotic formula for the average number of cycles in a permutation of n with ordered cycles.

22. Find an asymptotic formula for the probability that a permutation of n does not have a cycle of length 1, 2 or 3.

23. We define

- the Chebyshev polynomial of the first kind $T_n(y)$ by $\sum_{n=0}^{\infty} T_n(y)x^n = \frac{1-yx}{1-2yx+x^2}$,
- the Chebyshev polynomial of the second kind $U_n(y)$ by $\sum_{n=0}^{\infty} U_n(y)x^n = \frac{1}{1-2yx+x^2}$, and
- the Legendre polynomial $P_n(y)$ by $\sum_{n=0}^{\infty} P_n(y)x^n = \frac{1}{\sqrt{1-2yx+x^2}}$.

Find asymptotic formulas for $T_n(5/4)$, $U_n(5/4)$, and $P_n(5/4)$.

24. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a complex valued function with nonnegative real coefficients $a_n \geq 0$. Suppose that a singularity of f with smallest complex magnitude has magnitude R (this means that R is the radius of convergence of $f(z)$ and that the series $f(z_0)$ diverges for all z_0 with $|z_0| > R$). This exercise will show that R is a singularity of f .

- Show that $f(z) = \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} \binom{n}{k} a_n (R/2)^{n-k} \right) (z - R/2)^k$ in some neighborhood of $R/2$.
- Looking for a contradiction, assume that R is not a singularity of f . This means that there is an $\varepsilon > 0$ such that the above expression is valid for $R + \varepsilon$. Take $R + \varepsilon$ in the above expression and prove that

$$f(R + \varepsilon) = \sum_{n=0}^{\infty} a_n (R + \varepsilon)^n.$$

Why is this a contradiction? Where was the hypothesis that $a_n \geq 0$ used?

Discrete Mathematics Set 6

Math 435: Complete 6 of the following exercises.

Math 530: Complete exercises 32 and 33 and 4 of the remaining exercises.

25. Find the average value of the descent statistic over all permutations of n .

26. Find the average value of the inversion statistic over all permutations of n .

27. Find $\lim_{q \rightarrow 1} \frac{d}{dq} [n]_q!$.

28. Find $\lim_{q \rightarrow -1} \frac{d}{dq} [n]_q!$.

29. Let $a_{n,k}$ be the number of permutations in S_n with exactly k descents. By inserting " $n+1$ " into a permutation of n , show $a_{n+1,k} = (k+1)a_{n,k} + (n+1-k)a_{n,k-1}$.

30. Let $\varphi : S_n \rightarrow S_n$ be a bijection such that $\text{des}(\sigma) = \text{exc}(\varphi(\sigma))$ for all $\sigma \in S_n$. Write a computer program in Python or Mathematica (if you would like to use another programming language for any of the programming exercises, just ask me!) that inputs σ in one line notation and outputs $\varphi(\sigma)$ in one line notation.

31. Let $\varphi : S_n \rightarrow S_n$ be a bijection such that $\text{inv}(\sigma) = \text{maj}(\varphi(\sigma))$ for all $\sigma \in S_n$. Write a computer program in Python or Mathematica that inputs σ in one line notation and outputs $\varphi(\sigma)$ in one line notation.

32. Suppose that in one line notation, the permutation $\sigma \in S_n$ has σ_i in position i . Then the inverse permutation σ^{-1} written in one line notation has i in position σ_i . Show that $\text{inv}(\sigma) = \text{inv}(\sigma^{-1})$ for all $\sigma \in S_n$.

33. Using terminology from Abstract Algebra, prove that the sign of σ (as found by writing σ as a product of transpositions) is equal to $(-1)^{\text{inv}(\sigma)}$.

Discrete Mathematics Set 7

Math 435: Complete 6 parts of the following exercises.

Math 530: Complete exercise 38 and 3 parts of the remaining exercises.

34. Find the average number of inversions in $R(0^k, 1^{n-k})$ and use the result to simplify $\lim_{q \rightarrow 1} \frac{d}{dq} \begin{bmatrix} n \\ k \end{bmatrix}_q$.

35. The q -multinomial coefficient $\begin{bmatrix} n \\ k_1, \dots, k_\ell \end{bmatrix}_q$ is defined to be $\frac{[n]_q!}{[k_1]_q! \cdots [k_\ell]_q!}$ for $n = k_1 + \cdots + k_\ell$. Show that

$$\begin{bmatrix} n \\ k_1, \dots, k_\ell \end{bmatrix}_q = \sum_{r \in R(1^{k_1}, \dots, \ell^{k_\ell})} q^{\text{inv}(r)}$$

where $R(1^{k_1}, \dots, \ell^{k_\ell})$ denotes the set of rearrangements of k_1 1's, k_2 2's, etc.

36. Let φ be the bijection in the proof of $\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{r \in R(0^k, 1^{n-k})} q^{\text{maj}(r)}$ found on the next page.

a. Find both $\varphi(110111011001)$ and $\varphi^{-1}(110111011001)$.

b. Write a computer program in Python or Mathematica that inputs r and outputs $\varphi(r)$.

37. Prove these identities without writing $\begin{bmatrix} n \\ k \end{bmatrix}_q$ as a fraction and manipulating powers of q . Instead, interpret both sides of the identity as rearrangements or integer partitions and show the result by double counting or a bijection.

a. (The q -Pascal identity) $\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$.

b. (The q -Vandermonde identity) $\begin{bmatrix} a+b \\ n \end{bmatrix}_q = \sum_{k=0}^n q^{(a-k)(n-k)} \begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} b \\ n-k \end{bmatrix}_q$.

c. (The q -binomial theorem) $(1+xq^0) \cdots (1+xq^{n-1}) = \sum_{k=0}^n q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k$.

38. Let q a prime power, \mathbb{F}_q be the finite field with q elements, and \mathbb{F}_q^n be the n -dimensional vector space over \mathbb{F}_q .

a. Prove that the number of k dimensional subspaces in \mathbb{F}_q^n is equal to $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

b. Let X be a vector space with a finite number of elements x . Show that there are

$$\begin{bmatrix} n \\ n-k \end{bmatrix}_q (x - q^0) \cdots (x - q^{k-1})$$

linear maps $L : \mathbb{F}_q^n \rightarrow X$ which have a null space of dimension $n - k$.

c. By counting linear maps $L : \mathbb{F}_q^n \rightarrow X$, prove the q -Cauchy identity:

$$x^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q (x - q^0) \cdots (x - q^{k-1}).$$

d. The identity in part c. has been shown true for prime powers q . How can we conclude that this identity is true for any complex number q ?

Theorem 1. If $0 \leq k \leq n$, then $\left[\begin{matrix} n \\ k \end{matrix} \right]_q = \sum_{r \in R(0^k, 1^{n-k})} q^{\text{maj}(r)}$.

Proof. We prove this result by defining a bijection $\varphi : R(0^k, 1^{n-k}) \rightarrow R(0^k, 1^{n-k})$ such that $\text{maj}(r) = \text{inv}(\varphi(r))$ for all $r \in R(0^k, 1^{n-k})$. This is enough because we have already shown that

$$\left[\begin{matrix} n \\ k \end{matrix} \right]_q = \sum_{r \in R(0^k, 1^{n-k})} q^{\text{inv}(r)}.$$

We first define an auxiliary bijection $\Gamma : R(0^k, 1^{n-k}) \rightarrow R(0^k, 1^{n-k})$. If r ends with a 0, define $\Gamma(r)$ to be r with every consecutive substring of the form $11 \cdots 110$ changed to $011 \cdots 11$. If r ends with a 1, define $\Gamma(r)$ to be r with every consecutive substring of the form $00 \cdots 001$ changed to $100 \cdots 00$. For example,

$$\Gamma(1100011110100) = 0110001111010.$$

If r ends with a 0, then $\text{inv}(\Gamma(r)) = \text{inv}(r) - (n - k)$ because changing $11 \cdots 110$ into $011 \cdots 11$ for all 1's in r decreases the number of inversions in r by 1 for each of the $n - k$ 1's in r . Similarly, if r ends with a 1, then $\text{inv}(\Gamma(r)) = \text{inv}(r) + k$.

We can now define our main bijection φ . If r contains no 0's, then we define $\varphi(r) = r$. Otherwise, let w be r with the last 0 and all trailing 1's deleted so that r can be written as $w011 \cdots 11$. For any rearrangement $r \in R(0^k, 1^{n-k})$, define $\varphi(r)$ recursively by $\varphi(r) = \Gamma(\varphi(w))011 \cdots 11$. For example, it can be checked that

$$\varphi(10110100011) = 00111010011.$$

By definition, $\varphi(r)$ ends with a 0 if and only if r ends with a 0.

The fact that φ is a bijection follows from the fact that Γ is a bijection. To complete the proof, we will show that $\text{maj}(r) = \text{inv}(\varphi(r))$ by induction on the length of r . Suppose we add a 0 to the end of $r \in R(0^k, 1^{n-k})$. Then we have

$$\begin{aligned} \text{inv}(\varphi(r0)) &= \text{inv}(\Gamma(\varphi(r))0) \\ &= \text{inv}(\Gamma(\varphi(r))) + (n - k) \\ &= \begin{cases} \text{inv}(\varphi(r)) - (n - k) + (n - k) & \text{if } \varphi(r) \text{ ends in 0,} \\ \text{inv}(\varphi(r)) + k + (n - k) & \text{if } \varphi(r) \text{ ends in 1.} \end{cases} \end{aligned}$$

Using the induction hypothesis and the fact that $\varphi(r)$ ends in a 0 if and only if r does, this is equal to

$$\begin{cases} \text{maj}(r) & \text{if } r \text{ ends in 0,} \\ \text{maj}(r) + n & \text{if } r \text{ ends in 1.} \end{cases}$$

In both cases, this is equal to $\text{maj}(r0)$. We have shown that $\text{inv}(\varphi(r0)) = \text{maj}(r0)$.

Now suppose we add a 1 onto the end of r . Since $\varphi(r1) = \Gamma(\varphi(w))01 \cdots 11 = \varphi(r)1$, we have

$$\text{inv}(\varphi(r1)) = \text{inv}(\varphi(r)1) = \text{inv}(\varphi(r)) = \text{maj}(r) = \text{maj}(r1).$$

This completes the proof. □

Discrete Mathematics Set 8

Math 435: Complete 6 parts of the following exercises.

Math 530: Complete exercise 41 and 3 parts of the remaining exercises.

39. Prove the following identities by either proving that they have the same generating functions or by proving them with a bijection.

- Show that the number of integer partitions of n with no part divisible by d is equal to the number of integer partitions of n with no part repeated d or more times.
- Prove that the number of integer integer partitions of n with both odd and distinct parts is equal to the number of integer integer partitions of n that are equal to their conjugate.
- Prove that the number of integer partitions of n in which each part appears exactly 2, 3, or 5 times equals the number integer partitions of n into parts which are congruent to 2, 3, 6, 9, or 10 modulo 12.
- Show that the number of integer partitions of n in which no part appears exactly once is equal to the number of integer partitions of n with no part equal to 1 and where consecutive integers do not both appear as parts.
- Show that the number of integer partitions of n in which no part appears exactly once is equal to the number of integer partitions of n where no part is congruent to 1 or 5 modulo 6.

40. Prove that
$$\sum_{n=1}^{\infty} x^n y^n z (1 + zx^1) \cdots (1 + zx^{n-1}) = \sum_{n=1}^{\infty} \frac{x^{\binom{n+1}{2}} y^n z^n}{(1 - yx^1) \cdots (1 - yx^n)}.$$

41. Let $p_k(n)$ be the number of integer partitions with $\ell(\lambda) = k$.

- Show there are $\binom{n-1}{k-1}$ solutions to $x_1 + \cdots + x_k = n$ where x_1, \dots, x_k are positive integers. (One way is to use a “balls and bars” or “stars and bars” argument from an introductory combinatorics course.)
- By considering rearrangements of the parts of partitions, show that $\binom{n-1}{k-1} \leq k! p_k(n)$.
- By making the parts of a partition distinct, show that $k! p_k(n) \leq \binom{n + \binom{k}{2} - 1}{k-1}$.
- Show that $\binom{n+a-1}{k-1} \sim \frac{n^{k-1}}{(k-1)!}$ for any nonnegative integer a and then show that $p_k(n) \sim \frac{n^{k-1}}{k!(k-1)!}$.

Discrete Mathematics Set 9

Math 435: Complete 6 parts of the following exercises.

Math 530: Complete exercises 43, 44, 46a and ((46b, 46c, and 46d) or (46e, 46f, and 46g)).

42. Recover Euler's pentagonal number theorem by taking $y = -q$ and $x = q^3$ in Jacobi's triple product

$$(1+y) \prod_{n=1}^{\infty} (1-x^n)(1+yx^n)(1+y^{-1}x^n) = \sum_{k \in \mathbb{Z}} y^k x^{k(k-1)/2}.$$

43. Use the function $x^2 \prod_{n=1}^{\infty} (1-x^n)^6$ to show that the number of integer partitions of $7n+5$ is divisible by 7.

44. This exercise proves a finite version of Jacobi's triple product identity.

a. Prove $\prod_{i=1}^n (1+yq^{i-1})(1+y^{-1}q^i) = \sum_{k=-n}^n y^k q^{k(k-1)/2} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q.$

b. Take $\lim_{n \rightarrow \infty}$ of the above expression to find the full Jacobi triple product.

45. Suppose $F(x, y)$ is an infinite product in two indeterminates y and x . Euler's device refers to this process of turning $F(x, y)$ into a sum:

1. Find an equation providing a relationship between $F(x, xy)$ and $F(x, y)$.
2. Assume that $F(x, y) = \sum_{n=0}^{\infty} a_n(x) y^n$ and plug this into the equation found in step 1.
3. Compare coefficients of y^n to find a recursion for $a_n(x)$.
4. Iterate this recursion to find a formula for $a_n(x)$.

Show $F(x, y) = \prod_{i=0}^{\infty} \frac{1}{1-yx^i}$ satisfies $F(x, xy) = (1-y)F(x, y)$ and then express $F(x, y)$ as a sum using Euler's device.

46. Let $F(x, y)$ be the function that satisfies the recursion $F(x, y) = F(x, xy) + xyF(x, x^2y)$ and $F(x, 0) = 1$.

a. Use Euler's device to show $F(x, y) = \sum_{n=0}^{\infty} \frac{y^n x^{n^2}}{(1-x) \cdots (1-x^n)}.$

b. Use the result in the theorem on the next page to show $F(q, 1) = \left(\sum_{k \in \mathbb{Z}} (-1)^k q^{n(5n-1)/2} \right) \prod_{i=1}^{\infty} \frac{1}{1-q^i}.$

c. Take $y = -q^2$ and $x = q^5$ in Jacobi's triple product to show $F(q, 1) = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}.$

d. Show that (the number of integer partitions of n with parts differing by at least 2) is equal to (the number of integer partitions of n with parts congruent to $\pm 1 \pmod{5}$). (Hint: $1 + 3 + 5 + \cdots + (2n-1) = n^2$.)

e. Use the result in the theorem on the next page to show $F(q, q) = \left(\sum_{k \in \mathbb{Z}} (-1)^k q^{n(5n+3)/2} \right) \prod_{i=1}^{\infty} \frac{1}{1-q^i}.$

f. Take $y = -q^4$ and $x = q^5$ in Jacobi's triple product to show $F(q, q) = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$

g. Show that (the number of integer partitions of n with parts differing by at least 2 and no part of size 1) is equal to (the number of integer partitions of n with parts congruent to $\pm 2 \pmod{5}$). (Hint: $2 + \cdots + 2n = n^2 + n$.)

Theorem. The function

$$F(x, y) = \left(1 + \sum_{n=1}^{\infty} (-1)^n y^{2n} x^{n(5n-1)/2} (1 - yx^{2n}) \frac{(1 - yx) \cdots (1 - yx^{n-1})}{(1 - x) \cdots (1 - x^n)} \right) \prod_{i=1}^{\infty} \frac{1}{1 - yx^i} \quad (3)$$

satisfies $F(x, y) = F(x, xy) + xyF(x, x^2y)$ and $F(x, 0) = 1$.

Proof. Using $(1 - yx^{2n}) = (1 - x^n) + x^n (1 - yx^n)$, we have that

$$\begin{aligned} F(x, y) \prod_{i=1}^{\infty} (1 - yx^i) &= 1 + \sum_{n=1}^{\infty} (-1)^n y^{2n} x^{n(5n-1)/2} \frac{(1 - yx) \cdots (1 - yx^{n-1})}{(1 - x) \cdots (1 - x^{n-1})} \\ &\quad + \sum_{n=1}^{\infty} (-1)^n y^{2n} x^{n(5n+1)/2+n} \frac{(1 - yx) \cdots (1 - yx^n)}{(1 - x) \cdots (1 - x^n)} \\ &= \sum_{n=0}^{\infty} (-1)^n y^{2n} x^{n(5n+1)/2} (1 - y^2 x^{4n+2}) \frac{(1 - yx) \cdots (1 - yx^n)}{(1 - x) \cdots (1 - x^n)}. \end{aligned} \quad (4)$$

where we changed the first line into the second by reindexing the first infinite sum. Therefore, using (4) to simplify the first product and (3) to simplify the second product,

$$(F(x, y) - F(x, xy)) \prod_{i=1}^{\infty} (1 - yx^i) = \sum_{n=0}^{\infty} (-1)^n y^{2n} x^{n(5n+1)/2} \frac{(1 - yx) \cdots (1 - yx^n)}{(1 - x) \cdots (1 - x^n)} (1 - y^2 x^{4n+2} - x^n (1 - yx^{2n+1})).$$

Using $(1 - y^2 x^{4n+2} - x^n (1 - yx^{2n+1})) = (1 - x^n) + yx^{3n+1} (1 - yx^{n+1})$, the above expression is equal to

$$\sum_{n=1}^{\infty} (-1)^n y^{2n} x^{n(5n+1)/2} \frac{(1 - yx) \cdots (1 - yx^n)}{(1 - x) \cdots (1 - x^{n-1})} + yx \sum_{n=0}^{\infty} (-1)^n y^{2n} x^{n(5n+7)/2} \frac{(1 - yx) \cdots (1 - yx^{n+1})}{(1 - x) \cdots (1 - x^n)},$$

which in turn, by reindexing the first sum, is equal to

$$yx \sum_{n=0}^{\infty} (-1)^n y^{2n} x^{n(5n+7)/2} (1 - yx^{2n+2}) \frac{(1 - yx) \cdots (1 - yx^{n+1})}{(1 - x) \cdots (1 - x^n)} = yx \left(\prod_{i=1}^{\infty} (1 - yx^i) \right) F(x, x^2y).$$

The last step used (3) again. Thus we have proved

$$(F(x, y) - F(x, xy)) \prod_{i=1}^{\infty} (1 - yx^i) = xyF(x, x^2y) \prod_{i=1}^{\infty} (1 - yx^i),$$

which implies the desired result. □

Discrete Mathematics Set 10

Math 435: Complete either exercises 47–50 or exercise 51.

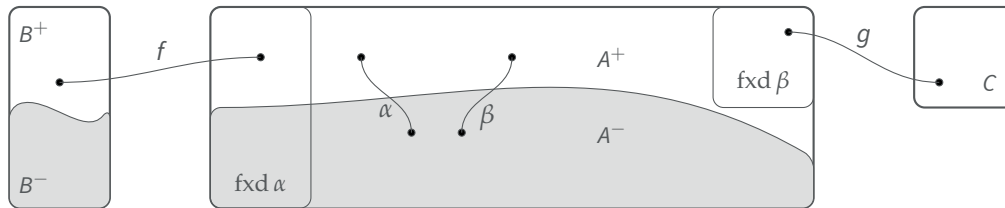
Math 530: Complete either exercises 47–50 or exercise 51.

47. Provide a bijection to prove that (the number of partitions of n in which only odd parts may be repeated) is equal to (the number of partitions of n in which no part appears more than 3 times). Give an explicit, nontrivial example of your bijection.

48. Show that the number of partitions of n in which 3 consecutive parts may not repeated equals the number of partitions of n in which 3 consecutive even parts do not appear.

49. Suppose A, B and C are finite sets such that

1. A is the disjoint union of two sets A^+ and A^- ,
2. B is the disjoint union of two sets B^+ and B^- ,
3. there is an involution $\alpha : A \rightarrow A$ such that $\alpha(A^+ \setminus \text{fxd } \alpha) \subseteq A^-$,
4. there is a bijection $f : \text{fxd } \alpha \rightarrow B$ such that $f(\text{fxd } \alpha \cap A^+) = B^+$ and $f(\text{fxd } \alpha \cap A^-) = B^-$,
5. there is an involution $\beta : A \rightarrow A$ such that $\text{fxd } \beta \subseteq A^+$, and
6. there is a bijection $g : \text{fxd } \beta \rightarrow C$.



Prove that there is an involution $\gamma : B \rightarrow B$ such that $\text{fxd } \gamma \subseteq B^+$ and a bijection $h : \text{fxd } (\gamma) \rightarrow C$.

50. Let A_1, \dots, A_n be finite sets. Prove the principle of inclusion/exclusion:

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{S \subseteq \{1, \dots, n\}} (-1)^{|S|-1} \left| \bigcap_{i \in S} A_i \right|$$

by way of a sign reversing involution.

51. Write Python or Mathematica code defining a function `bijection_machine`. The input is (λ, A, B) where

1. $A = (A_1, \dots, A_k)$ and $B = (B_1, \dots, B_k)$ are lists of pairwise disjoint lists such that the sum of the elements in A_i and B_i are the same for all i (these are “diseases”), and
2. λ is an integer partition without any diseases in A .

The output is the integer partition without any diseases in B as produced by Remmel’s bijection machine.

Discrete Mathematics Set 11

Math 435: Complete 4 parts of the following exercises.

Math 530: Complete exercise 53, 54, and 55.

52. Let $K_{\lambda, \mu}$ be the number of column strict tableau T of shape $\lambda \vdash n$ and content $\mu = (\mu_1, \mu_2, \dots) \vdash n$ (the content means that there are μ_1 copies of 1 in T , μ_2 copies of 2 in T , and so on). The Kostka matrix K_n is matrix with rows and columns indexed by integer partitions of n with the row λ and column μ entry equal to $K_{\mu, \lambda}$. For example, when $n = 4$, this matrix is

$$\begin{array}{c} (4) \quad (3,1) \quad (2^2) \quad (2,1^2) \quad (1^4) \\ \begin{array}{c} (4) \\ (3,1) \\ (2^2) \\ (2,1^2) \\ (1^4) \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & 3 & 2 & 3 & 1 \end{bmatrix} \end{array}.$$

a. Find K_5 and (using a machine) find K_5^{-1} .

b. Let a_λ be the vector in $\mathbb{R}^{p(n)}$ with a 1 in the λ entry and 0 elsewhere. What do the matrix multiplications $K_n a_\lambda$ and $K_n^{-1} a_\lambda$ mean in terms of the monomial and Schur symmetric functions?

53. Let RCS_λ denote the set of reverse column strict tableaux; that is, all tableaux where the integer labeling weakly decreases in rows and strictly decreases up columns. Show that $s_\lambda = \sum_{RCS_\lambda} w(T)$ for any $\lambda \vdash n$. For example, here are all elements in $RCS_{(2,1)}$ that are filled with integers ≤ 3 :

$$\begin{array}{|c|c|} \hline 2 & \\ \hline 3 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & \\ \hline 3 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & \\ \hline 2 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & \\ \hline 3 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & \\ \hline 2 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & \\ \hline 3 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 2 & \\ \hline 3 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 2 & \\ \hline 3 & 3 \\ \hline \end{array}.$$

54. Prove that the power symmetric polynomial $p_\lambda(x_1, \dots, x_N)$, the homogeneous symmetric polynomial $h_\lambda(x_1, \dots, x_N)$, and the elementary symmetric polynomial $e_\lambda(x_1, \dots, x_N)$ are indeed symmetric polynomials.

55. An alternating polynomial f in x_1, \dots, x_n is a polynomial such that for all $\sigma = \sigma_1 \cdots \sigma_n \in S_n$,

$$f(x_1, \dots, x_n) = \text{sign}(\sigma) f(x_{\sigma_1}, \dots, x_{\sigma_n}).$$

a. Show that an alternating polynomial is divisible by $\Delta = \prod_{i < j} (x_i - x_j)$.

b. Let \mathcal{A}_k be the vector space of alternating polynomials with every term degree k . Show that division by Δ is a vector space isomorphism between $\mathcal{A}_{n+\binom{n}{2}}$ and Λ_n . (Therefore understanding Λ_n is the same as understanding $\mathcal{A}_{n+\binom{n}{2}}$.)

Discrete Mathematics Set 12

Math 435: Complete 7 parts of the following exercises.

Math 530: Complete 7 parts of the following exercises.

56. Prove that the coefficient of m_λ in h_μ is the number of matrices with nonnegative integer entries with row sum λ and column sum μ .

57. Let $\mu \vdash n$.

- Use a similar proof as used to prove $h_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,\mu}| e_\lambda$ to prove $e_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,\mu}| h_\lambda$.
- Let $p(n)$ be the number of integer partitions of n and let B_n be the $p(n) \times p(n)$ matrix with row μ , column λ entry equal to $(-1)^{n-\ell(\lambda)} |B_{\lambda,\mu}|$. Why does part a. imply $B_n^{-1} = B_n$?

58. A weighted brick tabloid of content λ and shape μ is the usual brick tabloid of content λ and shape μ but with one cell in the final brick in each row shaded. Let $WB_{\lambda,\mu}$ be the set of all weighted brick tabloids of content λ and shape μ . Here are 4 of the 30 examples of weighted brick tabloids found on the next page:



- Use a similar proof as used to prove $h_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,\mu}| e_\lambda$ to prove $p_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |WB_{\lambda,\mu}| e_\lambda$.
- Prove $p_\mu = \sum_{\lambda \vdash n} (-1)^{\ell(\mu)-\ell(\lambda)} |WB_{\lambda,\mu}| h_\lambda$.
- By counting weighted brick tabloids, find the 5×5 matrix with row and columns indexed by integer partitions of 4 and with row μ and column λ entry equal to $(-1)^{n-\ell(\lambda)} |WB_{\lambda,\mu}|$. Why does this matrix verify that $\{p_\lambda : \lambda \vdash 4\}$ is a basis for Λ_4 ? More generally, why is $\{p_\lambda : \lambda \vdash n\}$ a basis for Λ_n ?

59. Prove these identities are true for $n \geq 1$ using bijections or sign reversing involutions:

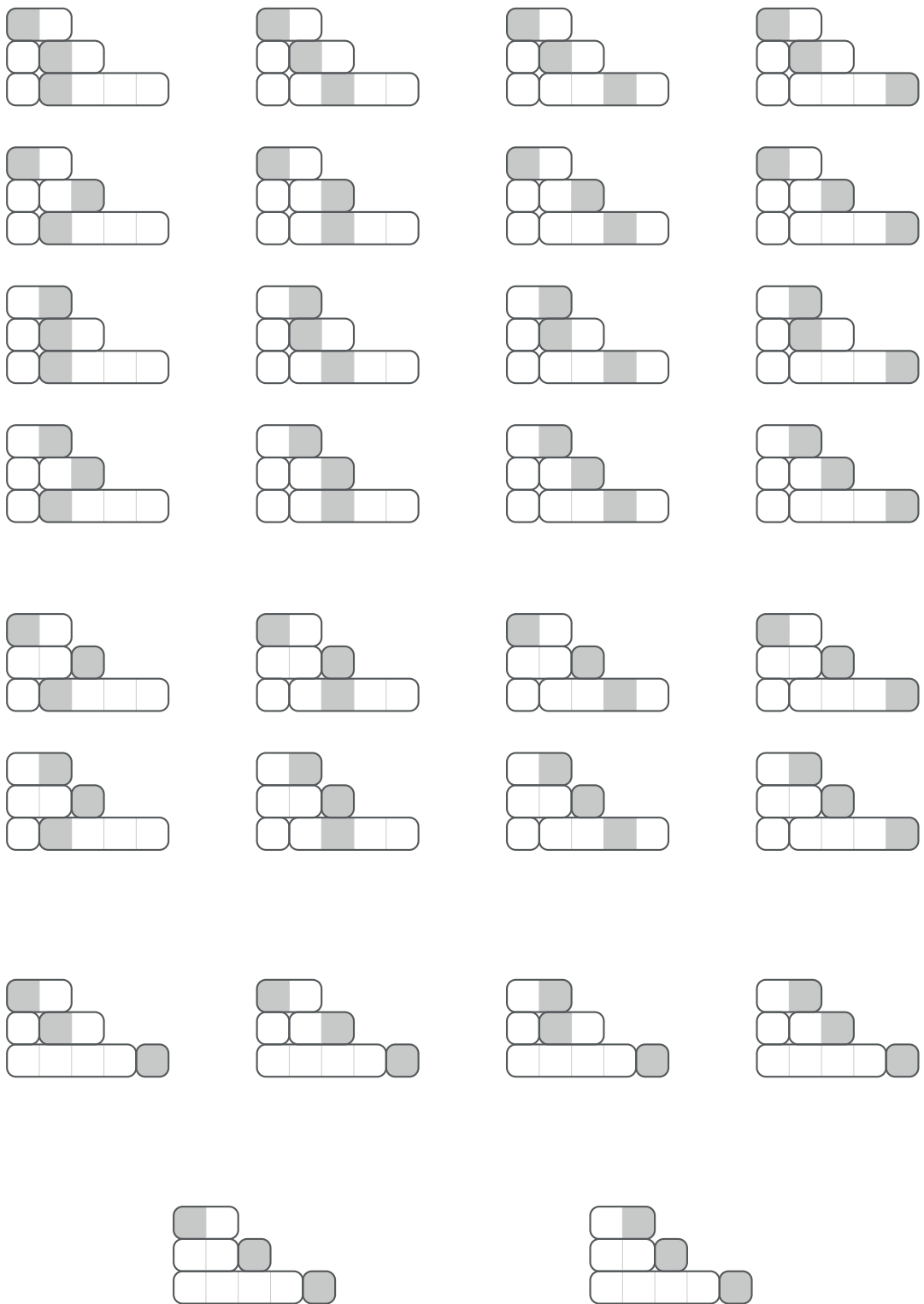
- $p_n = \sum_{i=0}^{n-1} (-1)^i s_{(n-i,1^i)}.$
- $\sum_{i=0}^{n-1} h_i p_{n-i} = n h_n.$
- $\sum_{i=0}^{n-1} (-1)^i e_i p_{n-i} = (-1)^{n-1} n e_n.$

60. Let $p_n = p_{(n)}(x_1, \dots, x_N)$ be the power symmetric polynomial in x_1, \dots, x_N , let $h_n = h_{(n)}(x_1, \dots, x_N)$ be the homogeneous symmetric polynomial, and let $H(t) = \sum_{n=0}^{\infty} h_n t^n$. Show $\sum_{n=1}^{\infty} \frac{p_n}{n} t^n = \ln H(t)$ and $\sum_{n=1}^{\infty} p_n t^n = \frac{t H'(t)}{H(t)}.$

61. Show that $\sum_{n=0}^{\infty} \sum_{\lambda \vdash n} h_\lambda(x_1, \dots, x_N) m_\lambda(y_1, \dots, y_M) = \prod_{j=1}^M \prod_{i=1}^N \frac{1}{1 - x_i y_j}.$

Hint: Multiply sums of the form $\prod_{i=1}^N \frac{1}{1 - x_i y_j} = \sum_{n=0}^{\infty} h_n(x_1, \dots, x_N) y_j^n.$

All 30 weighted brick tabloids of shape $(5, 3, 2)$ and content $(4, 2, 2, 1, 1)$.



Discrete Mathematics Set 13

Math 435: Complete 4 parts of the following exercises.

Math 530: Complete exercise 65 and two parts of the remaining exercises.

62. Define a ring homomorphism φ on Λ by $\varphi(e_n) = (-1)^{n-1}/n!$ for $n \geq 1$. Use $\varphi(h_n)$ to find the generating function for the number of ordered set partitions of n first found in Exercise 9 in Set 2.

63. Define a ring homomorphism φ on Λ by $\varphi(e_n) = (-1)^{n-1}k(x-1)^{n-1}$ for $n \geq 1$. Use $\varphi(h_n)$ to find the generating function for

$$\sum_{w \in \{1, \dots, k\}^n} x^{\text{equals}(w)}$$

where $\text{equals}(w)$ denotes the number of times there are consecutive equal integers in a word $w \in \{1, \dots, k\}^n$.

64. Define a ring homomorphism φ on Λ by $\varphi(e_n) = \begin{cases} 1 & \text{if } n = 0 \text{ or } n = 2, \\ 2x & \text{if } n = 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$

a. Recall from exercise 23 the definitions of the Chebyshev polynomial of the first kind $T_n(x)$ and the Chebyshev polynomial of the second kind $U_n(x)$. Show that $\varphi(p_n) = 2T_n(x)$ for $n \geq 1$ and $\varphi(h_n) = U_n(x)$ for $n \geq 0$. It may help to use an identity found in Exercise 60.

b. Use previously established relationships between e_n , h_n , and p_n (such as those in Exercise 59) to show these identities hold for $n \geq 3$:

$$\text{i. } U_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} (-1)^i (2x)^{n-2i}$$

$$\text{ii. } U_n(x) = \frac{2}{n} \sum_{i=0}^{n-1} U_i(x) T_{n-i}(x)$$

$$\text{iii. } U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0$$

$$\text{iv. } T_n(x) - 2xT_{n-1}(x) + T_{n-2}(x) = 0$$

65. Define a ring homomorphism φ on Λ by $\varphi(e_n) = \begin{cases} (-1)^{k+k(3k-1)/2} & \text{if } n = k(3k-1)/2 \text{ for some } k \in \mathbb{Z}, \\ 0 & \text{if not.} \end{cases}$

a. Show that $\varphi(h_n) = p(n)$ where $p(n)$ is the number of integer partitions of n .

b. Apply φ to the generating function for p_n/n in Exercise 60 to show that $\varphi(p_n) = \sigma(n)$ where $\sigma(n)$ is the sum of the positive integer divisors of n .

c. Use an identity found in Exercise 59 to show that $p(n) = \frac{1}{n} \sum_{i=1}^n \sigma(i)p(n-i)$, thereby giving a recursion for the number of integer partitions of n . Calculate $p(7)$ using this recursion.

Discrete Mathematics Set 14

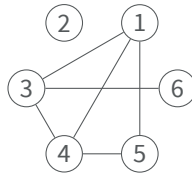
Math 435: Complete 3 of the following exercises.

Math 530: Complete exercise 70 and 2 exercises in {67, 68, 71, 72}.

66. How many ways are there to color the vertices of a cube using N colors (two colorings are the same if the cube can be rotated to turn one coloring into another)? How many ways are there to color the vertices if 3 vertices must be red and 5 must be black?

67. How many ways are there to color the edges of a cube using N colors (two colorings are the same if the cube can be rotated to turn one coloring into another)? How many ways are there to color the vertices if 6 edges must be red and 6 must be black?

68. Let E be the set of two element subsets of $\{1, \dots, n\}$. A simple graph on n vertices corresponds to a coloring of E which uses two different colors: a set $\{i, j\}$ is colored q if the edge between i and j appears in a simple graph and 1 if not. For example, the graph



corresponds to coloring each of $\{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$, $\{3, 4\}$, $\{3, 6\}$, and $\{4, 5\}$ with q and all other elements of E with 1. In this way, the number of edges in the graph is the number of times q is used in the coloring.

By defining $\sigma\{i, j\} = \{\sigma(i), \sigma(j)\}$ for all $\sigma \in S_n$, the symmetric group S_n acts on elements of E . Find

$$\sum_{\text{inequivalent 2 colorings } f \text{ of } E} q^{\text{the number of times color } q \text{ is used in } f}$$

when $n = 4$. Using the language of graph theory, we are finding

$$\sum_{\text{nonisomorphic simple graphs } g \text{ on 4 vertices}} q^{\text{the number edges in } g}.$$

69. Show that the cycle index polynomial satisfies $Z_{G \times H} = Z_G Z_H$.

70. Let C_n be the cyclic group of order n (the group generated by rotations of an n -sided regular polygon) and let D_n be the dihedral group of order $2n$ (the group generated by rotations and reflections of an n -sided regular polygon). Show that the cycle index polynomials for these groups are

$$Z_{C_n} = \frac{1}{n} \sum_{i=1}^n (p_{n/\gcd(i,n)})^{\gcd(i,n)} \quad \text{and} \quad Z_{D_n} = \frac{1}{2} Z_{C_n} + \begin{cases} p_1 p_2^{(n-1)/2} / 2 & \text{if } n \text{ is odd,} \\ (p_2^{n/2} + p_1^2 p_2^{(n-2)/2}) / 4 & \text{if } n \text{ is even.} \end{cases}$$

where $\gcd(i, n)$ is the greatest common divisor of i and n .

71. Show the cycle index polynomial for the symmetric group S_n is the homogeneous symmetric function h_n .

72. Let A_n be the alternating group, the subgroup of S_n containing all permutations σ with an even number of even sized cycles. Show that the cycle index polynomial for the alternating group A_n is $h_n + e_n$.