Math 344 Final Sample Questions

The final exam is cumulative. Here are two questions (with solutions!) involving the new material, one question on solving a heat equation problem and one question involving the Fourier transform. Both types of questions (along with other types of problems) will appear on the final.

1. Give a physical interpretation and solve the partial differential equation $\begin{cases} u_t = ku_{xx}, \\ u(0,t) = 0, u_x(L,t) = 0, \\ u(x,0) = x. \end{cases}$

Solution: The physical interpretation could be the description of how heat diffuses through a metal rod of length L. The first equation is the heat equation in one spacial dimension. On the left end of the rod the temperature is held at 0, the right end of the rod is insulated, and the initial temperature distribution is given by x. Following through with the separation of variables technique, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{(-1)^n 8L}{(2n+1)^2\pi^2} e^{-k\frac{(2n+1)\pi}{2L}t} \sin\left(\frac{(2n+1)\pi}{2L}x\right)$$

The constants in this sum were found by taking inner products.

2. The Fourier transform of the function $f(t)$ that is equal to $\pi^2 - t^2$ on $[-\pi, \pi]$ and 0 otherwise is

$$F(\omega) = \frac{2\sin(\pi\omega) - 2\pi\omega\cos(\pi\omega)}{\pi\omega^3}.$$

Evaluate $\int_{-\infty}^{\infty} F(\omega)\cos(\omega t)\, d\omega$ and $\int_{-\infty}^{\infty} F(\omega)\sin(\omega t)\, d\omega$.

Solution: The answer can be found using the inverse Fourier transform, which gives

$$f(t) = \int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\, d\omega = \int_{-\infty}^{\infty} F(\omega)\cos(\omega t)\, d\omega + i\int_{-\infty}^{\infty} F(\omega)\sin(\omega t)\, d\omega$$

where we used $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$. Looking at the real and imaginary parts separately, we see the answers are $f(t)$ (and not $\pi^2 - t^2$ as this is only valid on $[-\pi, \pi]$) and 0.