Linear Analysis II Exercise Set 4

1. Solve these differential equations, systems, or integral equations:

a. Solve $y'' + y = f(t)$ with $y(0) = 0, y'(0) = 0$, and $f(t) = \begin{cases} 1 & \text{if } 0 \le t < 1, \\ 0 & \text{otherwise.} \end{cases}$

b. Solve $y'' + 4y' + 3y = \delta(t-2)$ with the conditions $y(0) = 0, y'(0) = -1$.

c. Solve $\begin{cases} x' = y + \delta(t-\pi), \\ y' = -4x \end{cases}$ with $x(0) = 1$ and $y(0) = 0$.

d. $f(t) = e^{-t} - \int_0^t f(x) e^{t-x}\, dx$.

2. Use the definition of convolution to find $f * g$ where $f(t) = \cos t$ and $g(t) = t$.

3. Find $\mathcal{L}^{-1}\left[\frac{1}{s} \cdot \frac{1}{s-2}\right]$ using convolution.

4. Use the Laplace transform to explain why $f * g = g * f$.

5. Find $\mathcal{L}\left[(e^{2t}) * 1 + (t^2) * \left(e^{2t} \sin 2t\right)\right]$.