Linear Analysis II Exercise Set 8

1. a. Let k be a nonnegative integer. Show that the equation

$$\frac{d}{dx}\left(\sqrt{1-x^2}y'\right) = -k^2\frac{y}{\sqrt{1-x^2}}.$$

is the same as the differential equation

$$(1-x^2)y'' - xy' + k^2y = 0.$$

b. Define t_k to be the polynomial solution to the differential equation in a. that satisfies $t_k(1) = 1$. Find t_0, t_1, t_2, and t_3.

c. Let $t_k(x)$ and $t_m(x)$ be two different polynomials as defined in part b.. This means that t_k and t_m satisfy the equations

$$\frac{d}{dx}\left(\sqrt{1-x^2}t_k'\right) = -k^2\frac{t_k}{\sqrt{1-x^2}},$$
$$\frac{d}{dx}\left(\sqrt{1-x^2}t_m'\right) = -m^2\frac{t_m}{\sqrt{1-x^2}}.$$

Multiply the first of these equations by t_m, multiply the second equation by t_k, and then subtract the two equations. After integrating both sides of the result from -1 to 1, use integration by parts to show that

$$\int_{-1}^{1}\frac{t_k(x)t_m(x)}{\sqrt{1-x^2}}\,dx = 0.$$

d. Let $T_k(x) = \cos(k \arccos x)$. Show that $T_k(1) = 1$ and that

$$\frac{d}{dx}\left(\sqrt{1-x^2}T_k'\right) = -k^2\frac{T_k}{\sqrt{1-x^2}}.$$

This means that $T_k(x) = t_k(x)$.

2. Let $p_k(x)$ be the k^{th} Legendre polynomial. Find the constants $a_0, \ldots, a_3$ so that the approximation

$$u_0(x)x \approx a_0p_0(x) + a_1p_1(x) + a_2p_2(x) + a_3p_3(x)$$

on $[-1, 1]$ is as accurate as possible. (Here, $u_0(x)$ is the unit step function.) To do this, recall that

$$a_k = \frac{2k+1}{2}\int_{-1}^{1} f(x)p_k(x)\,dx.$$