

Theorem 1. We have

$$\prod_{n=1}^{\infty} (1 - z^n)^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) z^{n(n+1)/2}.$$

Proof. Using Jacobi's triple product identity, we have

$$\begin{aligned} \prod_{n=1}^{\infty} (1 - z^n)^3 &= \lim_{x \rightarrow -1} \prod_{n=1}^{\infty} (1 - z^n)(1 + xz^n)(1 + x^{-1}z^n) \\ &= \lim_{x \rightarrow -1} \frac{\sum_{k \in \mathbb{Z}} x^k z^{k(k-1)/2}}{1 + x} \\ &= \lim_{x \rightarrow -1} \frac{\sum_{k=1}^{\infty} x^k z^{k(k-1)/2} + \sum_{n=0}^{\infty} x^{-n} z^{-n(-n-1)/2}}{1 + x} \\ &= \lim_{x \rightarrow -1} \frac{\sum_{n=0}^{\infty} (x^{n+1} + x^{-n}) z^{-n(-n-1)/2}}{1 + x}. \end{aligned}$$

Now, since $((-1)^{n+1} + (-1)^{-n})$ is zero for all integers n , we have a limit that is an indeterminate form, ripe for the application of L'Hôpital's rule. Doing this gives

$$\lim_{x \rightarrow -1} \frac{\sum_{n=0}^{\infty} ((n+1)(x)^{n+1} - n(x)^{-n}) z^{-n(-n-1)/2}}{1} = \sum_{n=0}^{\infty} (-1)^n (2n+1) z^{n(n+1)/2},$$

as needed. □