Math 344 Midterm 2 Solutions

1. Find the vector in the span of $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}1\\0\\1\end{bmatrix}$, and $\begin{bmatrix}2\\-4\\2\end{bmatrix}$ closest to $\begin{bmatrix}5\\1\\0\end{bmatrix}$.

Solution: Gram-Schmidt gives $\mathbf{u}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}1\\-2\\1\end{bmatrix}$. There is no $\mathbf{u}_3$ because this turns out to be the zero vector when applying Gram-Schmidt. The projection vector is $\dfrac{\mathbf{u}_1 \cdot \begin{bmatrix}5\\1\\0\end{bmatrix}}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \dfrac{\mathbf{u}_2 \cdot \begin{bmatrix}5\\1\\0\end{bmatrix}}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = \begin{bmatrix}5/2\\1\\5/2\end{bmatrix}$.

2. Which function of the form $f(x) = ax + bx^2$ best fits the data $\{(-1,0),(0,1),(1,2)\}$?

Solution: If $V = \begin{bmatrix}-1 & 1\\0 & 0\\1 & 1\end{bmatrix}$, then $\begin{bmatrix}a\\b\end{bmatrix} = (V^\top V)^{-1}V^\top \begin{bmatrix}0\\1\\2\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$.

3. Find a pairwise orthogonal basis for span $\{e^{-x}, e^x\}$ in $PS[0, \ln 2]$.

Solution: Applying Gram-Schmidt, $u_1(x) = e^{-x}$ and $u_2(x) = e^x - \dfrac{\langle e^{-x}, e^x\rangle}{\langle e^{-x}, e^{-x}\rangle}e^{-x} = e^x - \dfrac{\ln 2}{3/8}e^{-x}$.

4. Find the first two nonzero terms in one of the series solutions to $x^2y'' + x(x+3)y' + (1+3x)y = 0$.

Solution: See example 11.4.6 on page 756 of the textbook.

5. Let P be the projection matrix onto the span of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ and let $\mathbf{x}$ be any vector in $\mathbb{R}^n$. Show that $\mathbf{x} - P\mathbf{x}$ and $P\mathbf{x}$ are orthogonal.

Solution: We have

$$(\mathbf{x} - P\mathbf{x})^\top P\mathbf{x} = \mathbf{x}^\top P\mathbf{x} - \mathbf{x}^\top P^\top P\mathbf{x} = \mathbf{x}^\top P\mathbf{x} - \mathbf{x}^\top P^2\mathbf{x} = 0$$

because $P^\top = P$ and $P^2 = P$.