Last time $h_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,\mu}| e_\lambda$

$B_{\lambda,\mu}$: Bricks tabloids with fills λ and shapes μ

Ex: $n=4$

$$\begin{array}{cc|ccccc} & & & & \mu & & \\ & & (4) & (31) & (22) & (21^2) & (1^4) \\ \hline & (4) & -1 & 0 & 0 & 0 & 0 \\ & (31) & 2 & 1 & 0 & 0 & 0 \\ \lambda & (22) & 1 & 0 & 1 & 0 & 0 \\ & (21^2) & -3 & -2 & -2 & -1 & 0 \\ & (1^4) & 1 & 1 & 1 & 1 & 1 \end{array}$$

note as M

This matrix M is the h-to-e transition matrix.

Ex: $h_{(2,2)} = M \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$ this means $h_{22} = e_{22} - 2e_{21^2} + e_{1^4}$

Observation: $1, e_1, e_2, e_3 \cdots$ are linearly independent. They generate the ring of symmetric functions of any degrees.

Define a ring homomorphism φ by defining φ on $1, e_1, e_2, \cdots$ means

$\varphi(f+cg) = \varphi(f) + c\varphi(g)$ $\quad \varphi(fg) = \varphi(f)\varphi(g)$ $\quad f, g$ are symmetric functions, $c \in \mathbb{Q}$.

Ex: Define a ring homomorphism $\phi: \Lambda \longrightarrow \mathbb{Q}[x]$ → polynomial ring on $\mathbb{Q}$.

Λ: Ring of symmetric function

by $\varphi(e_n) = (-1)^{n-1} \frac{(x-1)^{n-1}}{n!}$ if $n \neq 0$ and $\varphi(e_0) = 1$

Ex: $\varphi(2!h_2) = 2!(-e_2 + e_1 \cdot e_1) = 2!(-(-1)^1 \frac{(x-1)^1}{2!} + 1^2) = x+1$

$\varphi(3!h_3) = 3!\,\varphi(e_3 - 2e_2e_1 + \cancel{e_2e_1}\, e_1 \cdot e_1 e_1) = 3!(\frac{(x-1)^2}{3!} - 2(\frac{-(x-1)}{2!}) + 1^3) = x^2 + 4x + 1$

which matches.

$n=2$	des(σ)
12	0
21	1

$n=3$	des(σ)
123	0
132	1
213	1
231	1
312	1
321	2

Theorem: Define homomorphism φ by $\varphi(1) = 1$ and $\varphi(e_n) = (-1)^{n-1} \frac{(x-1)^{n-1}}{n!}$ for $n \geq 1$

Then $\varphi(n!h_n) = \sum_{\sigma \in S_n} x^{des(\sigma)}$

pf: We have $n!\varphi(h_n) = n!\,\varphi(\sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,(n)}| e_\lambda)$

$= n! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,(n)}| \varphi(e_{\lambda_1}) \varphi(e_{\lambda_2}) \cdots$

$= n! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,(n)}| (-1)^{\lambda_1 - 1} \frac{(x-1)^{\lambda_1 - 1}}{\lambda_1!} (-1)^{\lambda_2 - 1} \frac{(x-1)^{\lambda_2 - 1}}{\lambda_2!} \cdots$

$= \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \lambda_2 \cdots} |B_{\lambda,(n)}| (x-1)^{n-\ell(\lambda)}$

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