def: An alternating permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n$ sastifies $\sigma_1 > \sigma_2$, $\sigma_2 < \sigma_3$, $\sigma_3 > \sigma_4$, $\sigma_4 < \sigma_5$ ...

$\boxed{\text{Ex}}$ what ~~σ~~ $\in S_4$ are alternating

2143, 4231, 3241

4132, 3142 $\boxed{5 \text{ ways}}$

These are in one line notation.

Let a_{2n} = # of alternating permutations in S_{2n}

Theorem: Let φ be a homomorphism defined by

$$\varphi(e_n) = \frac{(-1)^{n-1}}{n!} f(n) \text{ where } f(n) = \begin{cases} (-1)^{\frac{n}{2}-1} & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$$

then $(2n)!\,\varphi(h_{2n}) = a_{2n}$

$\boxed{\text{mini Ex:}}$ when $n=2$

$$(2\cdot 2)!\,\varphi(h_4) = 4!\,\varphi(h_4)$$

$$= 4!\Big[(-1)^{4-1}\varphi(e_{(4)}) + (-1)^{4-2}\cdot 2\cdot \varphi(e_{(3,1)}) + (-1)^{4-2}\cdot \varphi(e_{(2,2)}) + (-1)^{4-3}\cdot 3\,\varphi(e_{(2,1,1)}) + (-1)^{4-4}\cdot 1\cdot \varphi(e_{(1^4)})\Big]$$

$$= 4!\left[(-1)^3 \frac{(-1)^{4-1}}{4!}\cdot(-1)^{\frac{4}{2}-1} + 0 + (-1)^2 (e_2)^2 + 0 + 0\right]$$

$$= 4!\left((-1)^3 \frac{(-1)^3}{4!}(-1)^1 + \left(\left(\frac{(-1)^{2-1}}{2!}\right)\cdot(-1)^{\frac{2}{2}-1}\right)^2\right)$$

$$= 4!\left(-\frac{1}{4!} + \left(\frac{-1}{2!}\right)^2\right)$$

$$= 4!\left(-\frac{1}{4!} + \left(\frac{-1}{2!}\right)\left(\frac{-1}{2!}\right)\right)$$

$$= -1 + 6 = 5$$

Proof: $(2n)!\,\varphi(h_{2n}) = (2n)! \sum_{\lambda\vdash 2n} (-1)^{2n-\ell(\lambda)} |B_{\lambda,(2n)}|\, \varphi(e_\lambda)$

$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots$

$$= (2n)! \sum_{\lambda\vdash 2n} (-1)^{2n-\ell(\lambda)} |B_{\lambda_1,(2n)}| \frac{(-1)^{\lambda_1-1}}{\lambda_1!} f(\lambda_1) \frac{(-1)^{\lambda_2-1}}{\lambda_2!} f(\lambda_2) \frac{(-1)^{\lambda_3-1}}{\lambda_3!} f(\lambda_3)$$

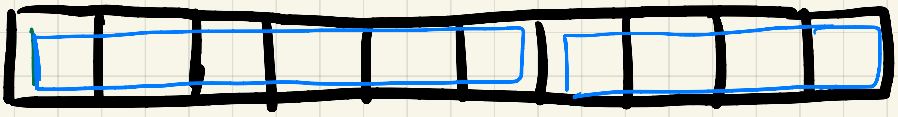
$$= \sum_{\lambda\vdash 2n} \binom{2n}{\lambda_1,\lambda_2,\dots} |B_{\lambda,(2n)}|\, f(\lambda_1) f(\lambda_2) \cdots$$

Create an object:

(1) Draw a strip of 2n cells split into bricks of even length.
since $f(n)=0$ if n odd

* uses $\sum_{\lambda\vdash 2n}$ & $|B_{\lambda,(2n)}|$.

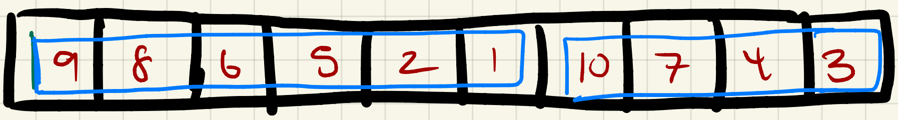
(1) $n=5$, 10 cells



(2) Fill bricks w/ decreasing sequence w/ union $1,2,\dots,2n$

* uses $\binom{2n}{\lambda_1,\lambda_2,\dots}$

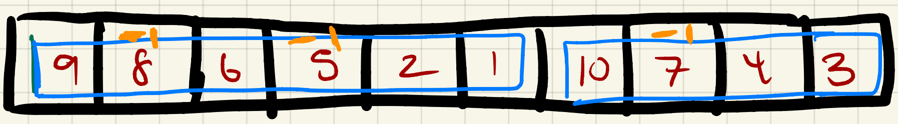
(2)



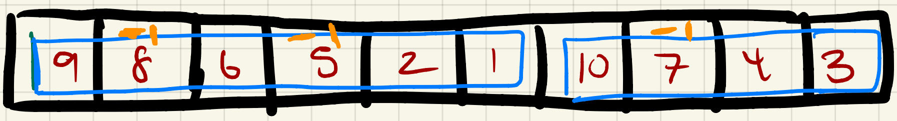
(3) Place -1 in every other cell except final cell in each brick.

* uses $f(\lambda_1) f(\lambda_2) \cdots$

(3)



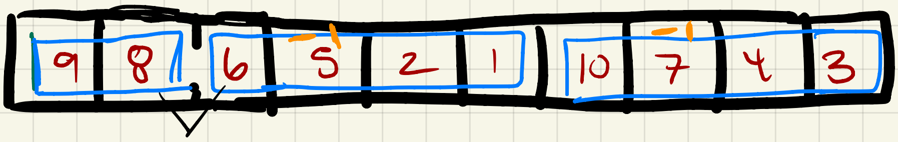
Using the object, use involution defined on Friday.

③ 

use $f(x_1) f(x_2) \ldots$

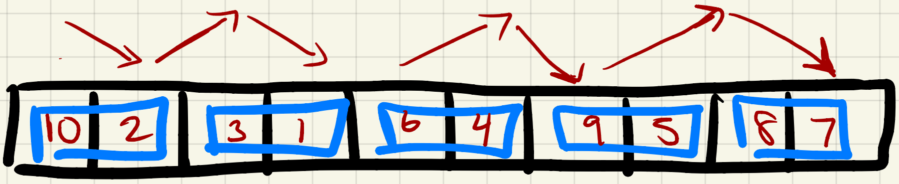
Using involution

Involution: Scan left to right for a -1 OR two bricks that can be glued together. Cut / glue accordingly.



can combine them since this is a desc.

Fixed Point:



These fixed points are counted by a_{2n}. □

Recall...

$$\sum_{n=0}^{\infty} h_n z^n = \frac{1}{1+\sum_{n=1}^{\infty} (-1)^n e_n z^n}$$

Now apply φ to this to get...

$$\sum_{n=0}^{\infty} \frac{1}{(2n)!} a_{2n} z^{2n} = \frac{1}{1+\sum_{n=1}^{\infty} (-1)^n \frac{(-1)^{n-1}}{n!} f(n)}$$

If n is odd term=0, nice!

$$= \frac{1}{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}}$$

$$= \frac{1}{\cos(z)}$$

* can modify proof to prove odd alternating permutations.