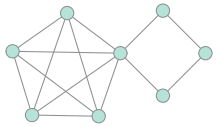
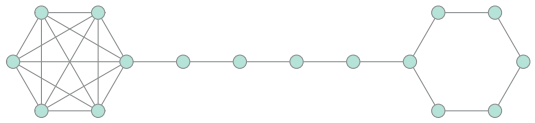
Graph Theory Set 3

11. A **coalescence** of the graphs G_1 and G_2 is a graph created by merging a vertex in G_1 with a vertex in G_2. For example, a coalescence of K_5 and C_4 is



a. Let G be a coalescence of G_1 and G_2. Explain why $P_G(x) = P_{G_1}(x)P_{G_2}(x)/x$.

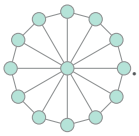
b. Find the chromatic polynomial for the following graph.



12. The **join** of the graphs G and H, denoted $G \vee H$, is the graph created by connecting every vertex in G to every vertex of H.

a. Show that $\chi(G \vee H) = \chi(G) + \chi(H)$.

b. Show that $P_{K_n \vee G}(x) = x(x-1)\cdots(x-n+1)P_G(x-n)$.

c. The **wheel graph** W_n is $K_1 \vee C_{n-1}$. Here is W_{13}:  . Find $P_{W_n}(x)$.

13. A graph G is **k-critical** if the chromatic number $\chi(G) = k$ and $\chi(G - v) = k - 1$ for any vertex v in G. The graph G is **critical** if it is k-critical for some k.

a. Why must every graph with $\chi(G) = k$ have a k-critical subgraph?

b. Show that the join $G \vee H$ is critical if and only if both G and H are critical.

c. Show that the odd cycles are the only 3-critical graphs.

d. Show that the even wheel graphs W_{2n} are 4-critical. (Open problem: Find a good characterization of all 4-critical graphs.)

e. Suppose G is k-critical. Why does every vertex have degree at least $k - 1$?

f. Suppose $\chi(G) = k$. Why must G have at least k vertices with degree $\geq k - 1$?