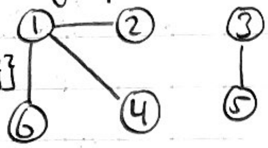
Recall: An exponential structure of size n:
Take a set partition of n and arrange elements in each set in some way
Let C_n = # ways to arrange a subset of size n. "Cards"
H_n = set of exponential structures of size n. "hands"

Example) $\underline{H_n}$
Labeled graphs

$\{\{1,2,4,6\}, \{3,5\}\}$



$\underline{C_n}$
# of connected graphs with n vertices

Permutations of n in cycle notation
$\{\{1,2,4,6\}, \{3,5\}\}$
$\rightarrow (1462)(35)$
$= (4621)(53)$

# of ways to create a single cycle with n elements.
$= (n-1)!$

Set partitions with no sets of size 1 allowed

$\begin{cases} 0 & \text{if } n=1 \\ 1 & \text{if } n\neq 1 \end{cases}$

$\{\{1\},\{2\},\{3\}\}$	0
$\{\{1,2\},\{3\}\}$	0
$\{\{1,3\},\{2\}\}$	0
$\{\{2,3\},\{1\}\}$	0
$\{\{1,2,3\}\}$	1

Example) Consider the set partitions of 3 and permutations.

$\{\{1\},\{2\},\{3\}\}$	$(1)(2)(3)$	$C_1 = 1$, $C_2 = 1$, $C_3 = 2$
$\{\{1,2\},\{3\}\}$	$(12)(3)$	
$\{\{1,3\},\{2\}\}$	$(13)(2)$	
$\{\{2,3\},\{1\}\}$	$(23)(1)$	
$\{\{1,2,3\}\}$	(123) and (132)	

Example)

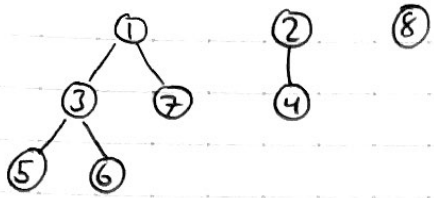
$\underline{H_n}$
Degree 2 graph with n vertices

$\{\{1,2,3,5\}, \{4,6,7\}\}$

$\underline{C_n}$
# of ways to take a set of size n and create a single cycle graph
"Circle graph"
$= \frac{(n-1)!}{2}$

H_n

Labeled forests on n vertices



C_n

trees on n vertices $= n^{n-2}$

Theorem) $\sum_{n=0}^{\infty}\left(\sum_{h\in H_n} y^{\#\text{cards in } h}\right)\frac{x^n}{n!} = e^{y\sum_{n=1}^{\infty}\frac{C_n}{n!}x^n}$

Example) Consider counting permutations of n in cycle notation.

$1 + y\frac{x^1}{1!} + (y+y^2)\frac{x^2}{2!} + (2y+3y^2+y^3)\frac{x^3}{3!} + \cdots$

$C_n = (n-1)!$

$= e^{y\sum_{n=1}^{\infty}\frac{(n-1)!}{n}x^n}$

$\sum_{n=1}^{\infty}\frac{x^n}{n} = \log\left(\frac{1}{1-x}\right)$

$= e^{y\log\left(\frac{1}{1-x}\right)}$

$= (1-x)^{-y}$

$(1)(2)(3)$ y^3

$(12)(3)$ y^2

$(13)(2)$ y^2

$(23)(1)$ y^2

$(123), (132)$ $y^1 + y^1$