Graph Theory Set 8

38. Cal Poly has six colleges: Agriculture (cagr), Architecture (caed), Engineering (ceng), Liberal Arts (cla), Business (ocob), and Science & Math (csm). The probability that a freshman will change majors from college j to a major in college i is:

	cagr	caed	ceng	cla	ocob	csm
cagr	0	0	1/913	2/593	1/516	3/511
caed	2/753	0	0	0	0	0
ceng	2/753	0	0	0	0	4/511
cla	6/753	0	0	0	1/516	5/511
ocob	3/753	0	0	1/593	0	1/511
csm	11/753	0	5/913	1/593	2/516	0

Consider this matrix as the adjacency matrix of a network. Find (using a machine) and interpret the Perron value and the Perron vector in this setting.

39. A d-**regular** graph is a graph with every vertex degree d. Let G be a connected d-regular graph.

- **a.** Let 1 be the vector of all 1's. Show that d is the Perron value and $(1/n)1$ is the Perron vector for G.
- **b.** The Courant-Fischer theorem implies that if $\mathbf{v}$ is an eigenvector that is not a multiple of the Perron vector, then $\mathbf{v}$ is orthogonal to 1, meaning that $1^\top \mathbf{v} = 0$. Show that if $\lambda \neq d$ is an eigenvalue for G, then $-1-\lambda$ is an eigenvalue for G^c.

40. Find the eigenvalues for the complete graph K_n.

41.

- **a.** Find the eigenvalues and eigenvectors for the cube Q_1.
- **b.** Show that the vertices of Q_{n+1} can ordered so that $A(Q_{n+1}) = \begin{bmatrix} A(Q_n) & I \\ I & A(Q_n) \end{bmatrix}$.
- **c.** Show that if λ is an eigenvalue for Q_n with eigenvector $\mathbf{x}$, then $\lambda \pm 1$ are eigenvalues for Q_{n+1} with eigenvectors $\begin{bmatrix} \mathbf{x} \\ \pm\mathbf{x}. \end{bmatrix}$.
- **d.** Show that $n-2i$ appears as an eigenvalue for Q_n with multiplicity $\binom{n}{i}$.

  Hint: It may be helpful to use the identity $\binom{n-1}{i} + \binom{n-1}{i-1} = \binom{n}{i}$.
- **e.** Prove the identities $\displaystyle\sum_{i=0}^{n} \binom{n}{i} (n-2i)^2 = n2^n$ and $\displaystyle\sum_{i=0}^{n} \binom{n}{i} (n-2i)^3 = 0$.

42. Let $\lambda_{\max}(G)$ denote the largest eigenvalue of a graph G. Use the Courant-Fischer theorem to show that if H is subgraph of G, then $\lambda_{\max}(H) \leq \lambda_{\max}(G)$.