Prof. Mendes’ Top 10 Graph Theory Things

Theorems

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8. **Hoffman and Wilf's bounds**

The chromatic number satisfies $1 - \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} \leq \chi(G) \leq \lambda_{\text{max}} + 1$.

7. **Tutte's layout**

If $v_1, \ldots, v_k$ form a face in a planar G, then Tutte layout with $v_1, \ldots, v_k$ as vertices of a polygon gives a planar embedding.

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5. **The four color theorem**

If G is planar, then $\chi(G) \leq 4$.

4. **Menger's theorems and Max Flow/Min Cut**

The maximum value for a flow from u to v in a network is equal to the minimum weight u, v disconnecting set.

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If $0, \mu_2, \ldots, \mu_n$ are the eigenvalues for $L(G)$, then the number of spanning trees for G is $\mu_2 \cdots \mu_n / n$.

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2. **Cayley's formula**

There are n^{n-2} labeled trees on n vertices.

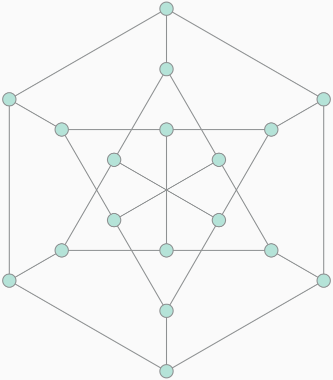
Prof. Mendes' Top 10 Graph Theorems

1. Euler's equation

If G is planar, then $V - E + F = 2$.

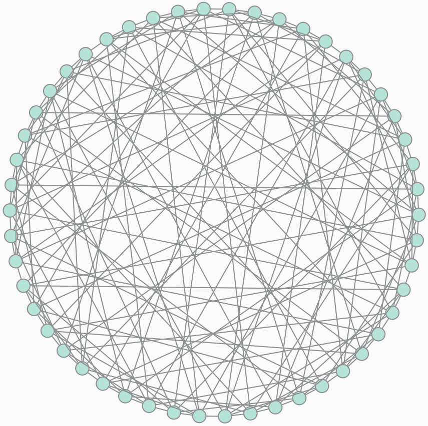
Graphs

10. The Pappus graph



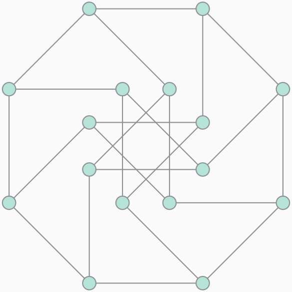
A smallest 3-regular graph with crossing number 5.

9. The Hoffman-Singleton graph



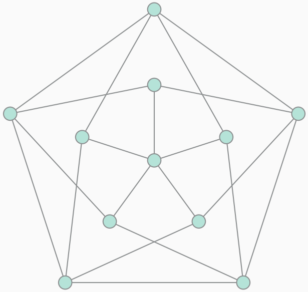
The largest graph with diameter 2 and smallest cycle size 5.

8. The Möbius–Kantor graph



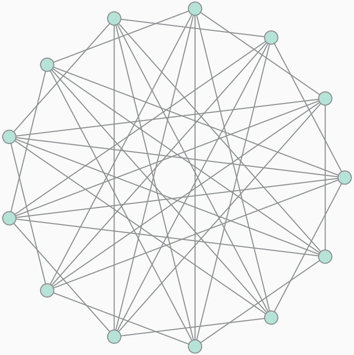
The smallest 3-regular graph with crossing number 4.

7. The Grötzsch graph



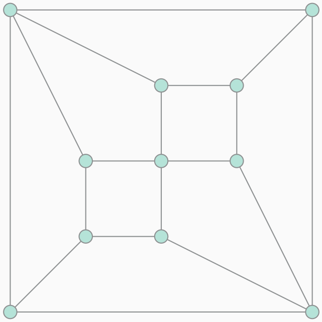
The smallest triangle free graph with chromatic number 4.

6. The Paley 13 graph



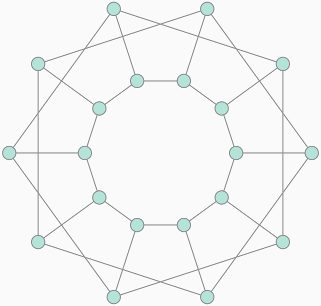
Vertices in $\mathbb{F}_{13}$ and edges $\{a, b\}$ when $a - b$ is a square in $\mathbb{F}_{13}$.

5. **The Herschel graph**



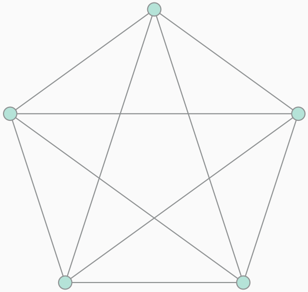
The smallest non-Hamiltonian graph of a convex polyhedron.

4. The Dodecahedron graph



The platonic solid graph with faces of maximum size.

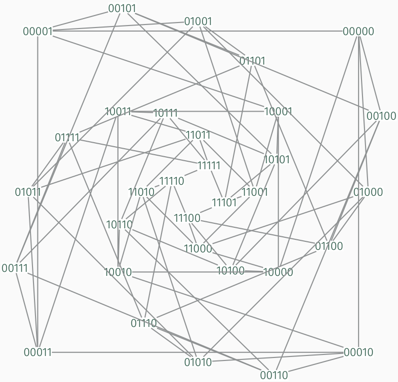
3. The complete graph K_5



The smallest non-planar graph.

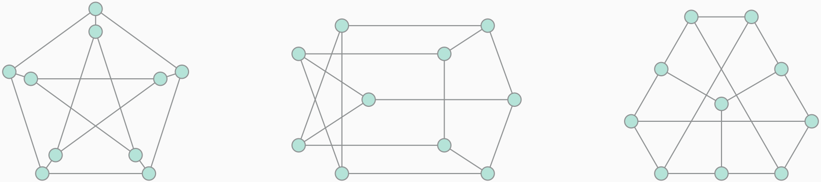
Prof. Mendes' Top 10 Graphs

2. The cube Q_5



Hamiltonian cycles are Gray codes.

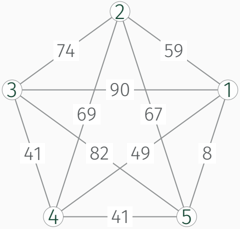
1. The Petersen graph



The smallest non-Hamiltonian graph such that $G - v$ is Hamiltonian for any vertex v.

Open Problems

10. What are optimal strategies for the hider in the "hide and seek" game on a weighted K_n?



An example hider strategy: Hide at vertex $2, 3, 4, 5$ with probabilities $0.484, 0.227, 0.226, 0.061$.

Prof. Mendes' Top 10 Open Problems

9. Does a 57-regular, diameter 2, smallest cycle 5 graph exist?

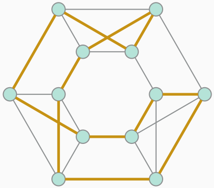
An example would have 3250 vertices.

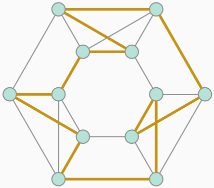
8. What is the crossing number for K_n?

The conjectured crossing number is $\frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$.

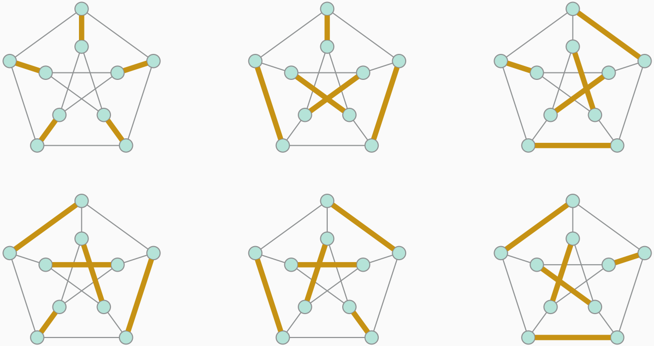
The largest known crossing number for K_n is K_{10} with 60.

7. No 4-regular graph has exactly one Hamiltonian cycle.

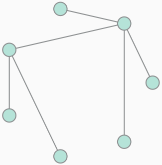




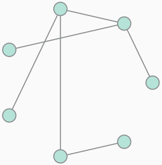
6. A 3-regular and bridgeless graph has 6 perfect matchings such that each edge is contained in exactly two matchings.



5. If T is a tree and $L(T)$ is the line graph of T, can we identify T from the sequence $|T|, |L(T)|, |L(L(T))|, \ldots$?

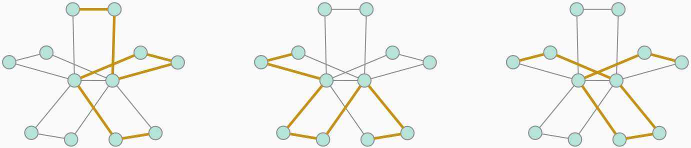


gives $7, 6, 9, 21, 84, \ldots$



gives $7, 6, 7, 12, 33, \ldots$

4. Do any three longest paths in a connected graph share a common vertex?

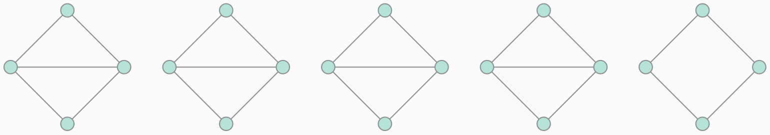


3. What is the smallest R needed so that every graph with R vertices either has a K_5 subgraph or a K_5^c subgraph?

Current bounds give $43 \leq R \leq 48$.

Prof. Mendes' Top 10 Open Problems

2. Can G be reconstructed from the multiset $\{G - v : \text{vertex } v\}$?



The above multiset of graphs is $\{W_5 - v\}$.

Prof. Mendes' Top 10 Open Problems

1. **The Traveling salesman problem**

Can a polynomial time algorithm produce a Hamiltonian cycle of minimum weight in a weighted K_n? Is $P = NP$?

An optimal path for TSP for US cities with pop. ≥ 500 in 1998: