Discrete Mathematics Set 6

Math 435: Complete 5 parts of the following exercises.
Math 530: Exercise 2 and 4 parts of the remaining exercises.

1. Prove these identities without writing $\left[{n \atop k}\right]_q$ as a fraction and manipulating powers of q. Instead, interpret both sides of the identity as rearrangements or integer partitions and show the result by double counting or a bijection.

a. (The q-Pascal identity) $\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$

b. (The q-Vandermonde identity) $\begin{bmatrix} a+b \\ n \end{bmatrix}_q = \sum_{k=0}^{n} q^{(a-k)(n-k)} \begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} b \\ n-k \end{bmatrix}_q.$

c. (The q-binomial theorem) $(1+xq^0)\cdots(1+xq^{n-1}) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k.$

2. Let q be a prime power, $\mathbb{F}_q$ be the finite field with q elements, and $\mathbb{F}_q^n$ be the n-dimensional vector space over $\mathbb{F}_q$.

a. Prove that the number of k dimensional subspaces in $\mathbb{F}_q^n$ is equal to $\begin{bmatrix} n \\ k \end{bmatrix}_q.$

(For this you may wish to use the definition of the q-analogues.)

b. Let X be a vector space with a finite number of elements x. Show that there are

$$\begin{bmatrix} n \\ n-k \end{bmatrix}_q (x-q^0)\cdots(x-q^{k-1}) = \begin{bmatrix} n \\ k \end{bmatrix}_q (x-q^0)\cdots(x-q^{k-1})$$

linear maps $L : \mathbb{F}_q^n \to X$ which have a nullspace of dimension $n-k$.

c. By counting linear maps $L : \mathbb{F}_q^n \to X$, prove the q-Cauchy identity:

$$x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (x-q^0)\cdots(x-q^{k-1}).$$

d. The identity in part c. has been shown true for prime powers q. How can we conclude that this identity is true for any complex number q?

3. Let $p_k(n)$ be the number of integer partitions of n with $\ell(\lambda) = k$.

a. Show there are $\binom{n-1}{k-1}$ solutions to $x_1 + \cdots + x_k = n$ where $x_1, \ldots, x_k$ are positive integers. (One way is to use a "balls and bars" or "stars and bars" argument from an introductory combinatorics course.)

b. By considering rearrangements of the parts of partitions, show that $\binom{n-1}{k-1} \le k!p_k(n)$.

c. By making the parts of a partition distinct, show that $k!p_k(n) \le \binom{n+\binom{k}{2}-1}{k-1}$.

d. Show that $\binom{n+a-1}{k-1} \sim \frac{n^{k-1}}{(k-1)!}$ for any nonnegative integer a and then show $p_k(n) \sim \frac{n^{k-1}}{k!(k-1)!}$.