Discrete Mathematics Set 5

Math 435: Complete 6 parts of the following exercises.
Math 530: Exercises 4, 5, 6, 9a, and any 2 of the remaining exercises.

1. Find $\lim_{q\to 1} \frac{d}{dq}[n]_q!$.

2. Find $\lim_{q\to 1} \frac{d}{dq}\begin{bmatrix} n \\ k \end{bmatrix}_q$.

3. Let $a_{n,k}$ be the number of permutations in S_n with exactly k descents. Show that

$$a_{n+1,k} = (k+1)a_{n,k} + (n+1-k)a_{n,k-1}.$$

4. Suppose that in one line notation, the permutation $\sigma \in S_n$ has σ_i in position i. Then the inverse permutation σ^{-1} written in one line notation has i in position σ_i. Show that $\text{inv}(\sigma) = \text{inv}(\sigma^{-1})$ for all $\sigma \in S_n$.

5. Using terminology from Abstract Algebra, prove that the sign of σ (as found by writing σ as a product of transpositions) is equal to $(-1)^{\text{inv}(\sigma)}$.

6. The q-multinomial coefficient $\begin{bmatrix} n \\ k_1,\ldots,k_\ell \end{bmatrix}_q$ is defined to be $\frac{[n]_q!}{[k_1]_q!\cdots[k_\ell]_q!}$ for $n = k_1 + \cdots + k_\ell$. Show that

$$\begin{bmatrix} n \\ k_1,\ldots,k_\ell \end{bmatrix}_q = \sum_{r\in R(1^{k_1},\ldots,\ell^{k_\ell})} q^{\text{inv}(r)}$$

where $R(1^{k_1},\ldots,\ell^{k_\ell})$ denotes the set of rearrangements of k_1 1's, k_2 2's, etc.

7. Let $\varphi : S_n \to S_n$ be a bijection such that $\text{des}(\sigma) = \text{exc}(\varphi(\sigma))$ for all $\sigma \in S_n$. Write a computer program in Python or Mathematica that inputs σ in one line notation and outputs $\varphi(\sigma)$ in one line notation.

8. Let $\varphi : S_n \to S_n$ be a bijection such that $\text{inv}(\sigma) = \text{maj}(\varphi(\sigma))$ for all $\sigma \in S_n$. Write a computer program in Python or Mathematica that inputs σ in one line notation and outputs $\varphi(\sigma)$ in one line notation.

9. Let φ be the bijection in the proof of $\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{r\in R(0^k,1^{n-k})} q^{\text{maj}(r)}$ found on the next page.

a. Find both $\varphi(110111011001)$ and $\varphi^{-1}(110111011001)$.

b. Write a computer program in Python or Mathematica that inputs r and outputs $\varphi(r)$.

Theorem. *If $0 \leq k \leq n$, then* $\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{r \in R(0^k,1^{n-k})} q^{maj(r)}$.

Proof. We prove this result by defining a bijection $\varphi : R(0^k, 1^{n-k}) \to R(0^k, 1^{n-k})$ such that $\text{maj}(r) = \text{inv}(\varphi(r))$ for all $r \in R(0^k, 1^{n-k})$. This is enough because we have already shown that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{r \in R(0^k,1^{n-k})} q^{\text{inv}(r)}.$$

We first define an auxiliary bijection $\Gamma : R(0^k, 1^{n-k}) \to R(0^k, 1^{n-k})$. If r ends with a 0, define $\Gamma(r)$ to be r with every consecutive substring of the form $11 \cdots 110$ changed to $011 \cdots 11$. If r ends with a 1, define $\Gamma(r)$ to be r with every consecutive substring of the form $00 \cdots 001$ changed to $100 \cdots 00$. For example,

$$\Gamma(1100011110100) = 0110001111010.$$

If r ends with a 0, then $\text{inv}(\Gamma(r)) = \text{inv}(r) - (n-k)$ because changing $11 \cdots 110$ into $011 \cdots 11$ for all 1's in r decreases the number of inversions in r by 1 for each of the $n-k$ 1's in r. Similarly, if r ends with a 1, then $\text{inv}(\Gamma(r)) = \text{inv}(r) + k$.

We can now define our main bijection φ. If r contains no 0's, then we define $\varphi(r) = r$. Otherwise, let w be r with the last 0 and all trailing 1's deleted so that r can be written as $w011 \cdots 11$. For any rearrangement $r \in R(0^k, 1^{n-k})$, define $\varphi(r)$ recursively by $\varphi(r) = \Gamma(\varphi(w))011 \cdots 11$. For example, it can be checked that

$$\varphi(10110100011) = 00111010011.$$

By definition, $\varphi(r)$ ends with a 0 if and only if r ends with a 0.

The fact that φ is a bijection follows from the fact that Γ is a bijection. To complete the proof, we will show that $\text{maj}(r) = \text{inv}(\varphi(r))$ by induction on the length of r. Suppose we add a 0 to the end of $r \in R(0^k, 1^{n-k})$. Then we have

$$\begin{aligned} \text{inv}(\varphi(r0)) &= \text{inv}\left(\Gamma(\varphi(r))0\right) \\ &= \text{inv}\left(\Gamma(\varphi(r))\right) + (n-k) \\ &= \begin{cases} \text{inv}\left(\varphi(r)\right) - (n-k) + (n-k) & \text{if } \varphi(r) \text{ ends in } 0, \\ \text{inv}\left(\varphi(r)\right) + k + (n-k) & \text{if } \varphi(r) \text{ ends in } 1. \end{cases} \end{aligned}$$

Using the induction hypothesis and the fact that $\varphi(r)$ ends in a 0 if and only if r does, this is equal to

$$\begin{cases} \text{maj}(r) & \text{if } r \text{ ends in } 0, \\ \text{maj}(r) + n & \text{if } r \text{ ends in } 1. \end{cases}$$

In both cases, this is equal to $\text{maj}(r0)$. We have shown that $\text{inv}(\varphi(r0)) = \text{maj}(r0)$.

Now suppose we add a 1 onto the end of r. Since $\varphi(r1) = \Gamma(\varphi(w))01 \cdots 11 = \varphi(r)1$, we have

$$\text{inv}(\varphi(r1)) = \text{inv}(\varphi(r)1) = \text{inv}(\varphi(r)) = \text{maj}(r) = \text{maj}(r1).$$

This completes the proof. $\square$