Linear Independence

1. Describe all solutions to $A\mathbf{x} = \mathbf{0}$ using parameters and vectors where A is each one of these matrices:

a. $\begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix}$

b. $\begin{bmatrix} 3 & -6 & 6 \\ -2 & 4 & -2 \end{bmatrix}$nn

c. $\begin{bmatrix} 1 & -2 & 3 & -6 & 5 & 0 \\ 0 & 0 & 0 & 1 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

2. Let A be an $m \times n$ matrix and suppose $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$ such that $A\mathbf{v} = \mathbf{0}$ and $A\mathbf{w} = \mathbf{0}$; in other words, $\mathbf{v}$ and $\mathbf{w}$ are solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$. Show that $c\mathbf{v} + d\mathbf{w}$ is also a solution to $A\mathbf{x} = \mathbf{0}$.

3. Determine if the following vectors are linearly independent:

a. $\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$

b. $\begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -7 \\ 2 \\ 4 \end{bmatrix}.$

c. $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}.$

4. True or false:

a. The columns of A are linearly independent if the equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution.

b. If S is a linearly dependent set, then each vector in S is a linear combination of the other vectors in S.

c. The columns of any 4×5 matrix are linearly dependent.

d. If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent and if $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent, then $\mathbf{z}$ is in the span of $\mathbf{x}$ and $\mathbf{y}$.

e. If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent and if $\mathbf{z}$ is in the span of $\mathbf{x}$ and $\mathbf{y}$, then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent.

f. If a set in $\mathbb{R}^n$ is linearly dependent, then the set contains more than n vectors.

5. The following statements are either True (in all cases) or False. If the statement is False, give an example illustrating that it is false. If true, explain why.

a. If $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are linearly independent and if $\mathbf{x} = \mathbf{y} + 2\mathbf{z}$, then the set $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent.

b. If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{R}^5$ and $\mathbf{x}$ is not a scale multiple of $\mathbf{y}$, then $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent.

c. If $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are in $\mathbb{R}^3$ and $\mathbf{z}$ is not a linear combination of $\mathbf{x}$ and $\mathbf{y}$, then the set $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly independent.

d. If $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly independent, then so is $\{\mathbf{x}, \mathbf{y}\}$.