Graph Theory Midterm 1 Review

Definitions: complete graph K_n, r-coloring, cycle graph C_n, path graph P_n, adjacent, bipartite, chromatic number $\chi(G)$, chromatic polynomial $P_G(x)$, complement, complete bipartite graph $K_{m,n}$, component, connected, cycle, degree, degree sequence, edge, graph, greedy coloring, incident, independent sets, isomorphic graphs, line graph, path, proper r-coloring, self-complementary, spanning tree, subgraph, unlabeled graph, vertex.

Theorems:

- K_n has $\binom{n}{2}$ edges.
- There are $2^{\binom{n}{2}}$ graphs on n vertices.
- The sum of the degree sequence is twice the number of edges.
- If G has n vertices and more than $\binom{n-1}{2}$ edges, then G is connected.
- If G and G^c are connected, then G has a P_4 subgraph.
- If p is prime, then $r^p - r$ is divisible by p for integer r.
- If G is not K_n or C_{2n+1}, then $\chi(G)$ is not more than the largest degree in G.
- G is bipartite if and only if G does not have an odd cycle.
- $P_G(x) = P_{G-e}(x) - P_{G/e}(x)$.
- $P_G(C_n) = (x-1)^{n-1} + (-1)^n(x-1)$.
- $P_G(K_n) = x(x-1)\cdots(x-n+1)$.
- $P_G(x) = x^{(\text{number of vertices})} - (\text{number of edges})x^{(\text{number of vertices})-1} + \cdots \pm ax^{(\text{number of components})}$.
- Let T be a graph with n vertices. The following statements are equivalent:
  - **a.** T is a tree.
  - **b.** T is connected and has no cycles (of length ≥ 3).
  - **c.** T is connected and has $n-1$ edges.
  - **d.** there is a unique path of distinct vertices between every pair of vertices u and v in T.
  - **e.** $P_T(x) = x(x-1)^{n-1}$.
- There are n^{n-2} trees.
- There are $n^{m-1}m^{n-1}$ spanning trees for $K_{m,n}$.

Extra exercises:

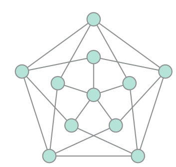
1. If G has n vertices with degree sequence $(d_1, \ldots, d_n)$, then what is the degree sequence of the complement graph G^c?

2. Let G be a graph with m vertices of degree 1 and let H be the graph found after removing all degree 1 vertices from G. Explain why the chromatic polynomial $P_G(x) = (x-1)^m P_H(x)$.

3. Show that a graph cannot have an odd number of vertices with an odd degree.

4. Find all connected unlabeled graphs with degree sequence $(3, 3, 2, 2, 1, 1)$.

5. Find the chromatic number for the **Grötzsch graph**:



6. Suppose T is a tree such that every vertex adjacent to a leaf has degree at least 3. Show that two leaves have a common adjacent vertex.

7. Find the number of spanning trees for: