Def A **set partition** of n is a set of nonempty disjoint sets w/ union $\{1, \cdots, n\}$

Ex. $n=5$: $\{\{1\}, \{2,3,4\}, \{5\}\}$

$\{\{1,2,5\}, \{3,4\}\} = \{\{4,3\}, \{1,5,2\}\}$

$\{\{1,2,3,4,5\}\}$ $\cdots$

Let b_n = # of set points of n (Bell #s)

our sequence: $1, 1, 2, 5, \cdots$

Thm $b_{n+1} = \sum_{k=0}^{n} \binom{n}{k} b_{n-k}$ for $n \geq 0$

Pf. Select the subset of $\{1, \cdots, n\}$ in which to place "$n+1$"

Select a set partition w/ leftover numbers.

Ex. $n=5$: $\{1,3,6\}, \{2,4\}, \{5\}$

Let $B(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n$

$$= 1 + x + \frac{2}{2!}x^2 + \frac{5}{3!}x^3 + \cdots$$

$$B'(x) = \sum_{n=0}^{\infty} n \frac{b_n}{n!} x^{n-1} = \sum_{n=1}^{\infty} \frac{b_n}{(n-1)!} x^{n-1} = \sum_{n=0}^{\infty} \frac{b_{n+1}}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\cancel{n!}}{k!(n-k)!} \frac{b_{n-k}}{\cancel{n!}} x^n$$

$$= \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{b_n}{n!} x^n \right) = e^x B(x)$$

our d.e. is $\begin{cases} B'(x) = e^x B(x) \\ B(x) = 0 \end{cases}$

$$\frac{1}{B} B' = e^x$$

$$\Rightarrow \int \frac{1}{B} \, dB = \int e^x \, dx$$

$$\ln B = e^x + c$$

$$B = e^{e^x + c}$$

The initial condition gives $B(x) = e^{-1} e^{e^x}$

$$B(x) = e^{-1} \sum_{n=0}^{\infty} \frac{(e^x)^n}{n!} = \frac{1}{e} \sum_{n=0}^{\infty} \frac{e^{nx}}{n!}$$

$$= \frac{1}{e} \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!}$$

$$= \frac{1}{e} \sum_{n=0}^{\infty} \left(\sum_{n=0}^{\infty} \frac{n^k}{n!} \right) \frac{x^k}{k!}$$

Conclusion: $D_k = \frac{1}{e} \sum_{n=0}^{\infty} \frac{n^k}{n!}$