9-21 Notes

Generating functions

Def The generating function for $a_0, a_1, a_2 \dots$ is

$$a_0 x^0 + a_1 x^1 + a_2 x^2 \dots = \sum_{n=0}^{\infty} a_n x^n$$

Ex The generating fn. for $1, 1, \frac{1}{2!}, \frac{1}{3!}, \dots$ is $\sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$

The generating fn. for $1, 1, 1, \dots$ is $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ (if $|x|<1$, but this is not important for us, since we do not care about the value of x)

The g.f. for $\binom{r}{0}, \binom{r}{1}, \binom{r}{2} \dots$ is $\sum_{n=0}^{\infty} \binom{r}{n} x^n = (1+x)^r$ Binomial Thm.

where $\binom{r}{n} = \frac{r(r-1)(r-2)\dots(r-(n+1))}{n!}$, $n \in \{0,1,2\dots\}$, r can be <u>any number</u>

For example, $\binom{\pi}{3} = \frac{\pi(\pi-1)(\pi-2)}{3!}$, $\binom{5}{4} = \frac{5 \cdot 4}{2!} = \frac{5!}{2!\,3!}$

Ex Define a sequence by $a_0 = 1$, and $a_n = 3^n - 2a_{n-1}$ for $n \geq 1$

Then, we have $1, 1, 7, 13, 55, \dots$

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + \sum_{n=1}^{\infty} a_n x^n = 1 + \sum_{n=1}^{\infty} (3^n - 2a_{n-1}) x^n = 1 + \sum_{n=1}^{\infty} 3^n x^n - 2\sum_{n=1}^{\infty} a_{n-1} x^n$

$$A(x) = 1 + \left(\frac{1}{1-3x} - 1\right) - 2x A(x)$$

Recall $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

So, $A(x) = \frac{1}{(1-3x)(1+2x)} = \frac{3/5}{1-3x} + \frac{2/5}{1+2x}$

$$= \frac{3}{5} \sum_{n=0}^{\infty} 3^n x^n + \frac{2}{5} \sum_{n=0}^{\infty} (-2)^n x^n$$

So, $\boxed{a_n = \frac{3}{5} \cdot 3^n + \frac{2}{5} \cdot (-2)^n}$