Linear Analysis II Set 14

1. a. Show that the Fourier Transform of $e^{-|t|}$ is $\dfrac{1}{\pi(\omega^2+1)}$.

b. Using part a. and the Fourier relations, find the Fourier transform of $\dfrac{1}{t^2+1}$.

2. The one dimensional wave equation is the partial differential equation $u_{tt}(t,x) = k^2 u_{xx}(t,x)$ where k is a real number and $u(t,x)$ is a function of time t and one spacial dimension x. The wave equation models the displacement of a vibrating string at time t and location x. Consider the system

$$\begin{cases} u_{tt}(t,x) = k^2 u_{xx}(t,x), \\ u(0,x) = 1/(1+x^2), \\ u_t(0,x) = 0. \end{cases}$$

a. Take the Fourier transform of the system with respect to the variable x and then solve the resulting differential equation in the variable t to find $F[u(t,x)]$.

b. The Fourier relations give that $u(t,x) = \displaystyle\int_{-\infty}^{\infty} F[u(t,x)]e^{i\omega x}\, d\omega$. Evaluate this integral to find $u(t,x)$.

Hint: This can be done by writing the integral as $\int_{-\infty}^{\infty} = \int_{-\infty}^{0} + \int_{0}^{\infty}$ and then relating each of the two integrals to the Laplace transform with respect to the variable ω of the function $e^{-\omega}\cos(kt\omega)$. It is acceptable to leave the final answer involving the complex unit i, but the clever student may be able to find an answer that does not involve i.