Discrete Notes

Day 29: 11/13/23

~ Symmetric Functions ~

- Symmetric Functions in $x_1, \ldots, x_N$ are polynomials f such that $f(x_1, \ldots, x_N) = f(x_{\sigma(1)}, \ldots, x_{\sigma(N)})$ for $\sigma \in S_n$.

  ☆ Note: these are defined for finite variables, but N is large

Recall: 1) monomial $\rightarrow m_\lambda(x_1, \ldots, x_N) = x_1^{\lambda_1} x_2^{\lambda_2} \ldots + \ldots$

fewest terms possible to make it symmetric

example: $m_{(2,1,1)}(x_1, x_2, x_3, \ldots) = x_1^2 x_2^1 x_3^1 + x_1^1 x_2^2 x_3^1 + \ldots$

2) elementary $\rightarrow e_n(x_1, \ldots, x_N) =$ coefficient of z^n in $\prod(1+x_i z)$
$= m_{(1^n)}$

example: $e_3(x_1, x_2, x_3, \ldots) = x_1' x_2' x_3' + x_1 x_2 x_4 + x_1 x_2 x_5 + \ldots$

3) homogeneous $\rightarrow h_n(x_1, \ldots, x_N) =$ coefficient of z^n in $\prod \frac{1}{1-x_i z}$
$= \sum_{\lambda \vdash n} m_\lambda$

example: $h_3(x_1, x_2, x_3, \ldots) = x_1^3 + x_2^3 + x_3^3 + x_1^2 x_2 + \ldots + x_1 x_2 x_3 + \ldots$

New: 4) Power Symmetric $\rightarrow P_n(x_1, \ldots, x_N) = x_1^n + x_2^n + x_3^n + \ldots + x_N^n$
$= m_{(n)}$

example: $P_3(x_1, x_2, x_3, x_4) = x_1^3 + x_2^3 + x_3^3 + x_4^3$

Definition: A tableau (plural: tableaux) is a filling of the Young Diagram for $\lambda \vdash n$ w/ positive integers.

The **weight** of a tableau T is: $x_1^{(\# \text{ 1's in } T)} \, x_2^{(\# \text{ 2's in } T)} \cdots$

example: The weight of:
$$\begin{array}{r|cccccc}
3 & 1 & 1 & 2 & & & \\
5 & 3 & 3 & 3 & 3 & 3 & \\
6 & 1 & 1 & 1 & 2 & 3 & 1
\end{array}$$
is $x_1^6 x_2^2 x_3^6$

So, $e_n = \sum\limits_{\substack{\text{Tableau } T \text{ w/} \\ \text{shape } 1^n \text{ (vertical)} \\ \text{s.t. integers increase} \\ \text{from bottom to} \\ \text{top}}} \text{weight}(T)$

example: $e_3 = \text{weight}\left(\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}\right) + \text{weight}\left(\begin{array}{|c|}\hline 4 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}\right) + \text{weight}\left(\begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline 1 \\ \hline\end{array}\right) + \cdots$

$= x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + \cdots$

Note: numbers in boxes must be distinct for e

Then, $h_n = \sum\limits_{\substack{\text{Tableau } T \\ \text{w/ shape } (n) \\ \text{(horizontal) w/ non-decreasing} \\ \text{integers from } L \to R.}} \text{weight}(T)$

example: $h_3 = \text{weight}\left(\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline\end{array}\right) + \text{weight}\left(\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline\end{array}\right) + \text{weight}\left(\begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline\end{array}\right) + \cdots$

$= x_1^3 + x_1^2 x_2 + x_2^3 + \cdots$

Theorem: (Newton's Identity) $\sum_{i=0}^{n} (-1)^i e_i h_{n-i} = 0$ if $n \geq 1$.

Proof #1: $\sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} (-1)^i e_i h_{n-i} \right) z^n = \left(\sum_{n=0}^{\infty} (-1)^n e_n z^n \right) \left(\sum_{n=0}^{\infty} h_n z^n \right)$

$= \prod (1 + x_i(-z)) \cdot \prod \frac{1}{1 - x_i z}$ these cancel

$= 1$ ← the coefficient of $z^n = 0$,

so coeff. of z^n on LHS $= 0$ ☐

Proof #2: Consider pairs (T, S) where T is a tableau of shape 1^i and filling an increasing sequence (bottom to top) (from e_i part) and S is a tableau of shape $(n-i)$ and filling non-decreasing sequence (left to right) (from h_{n-i} part). The sign of a pair is $(-1)^i$ (i = height of T)

For example, $\left(\begin{array}{|c|} \hline 5 \\ \hline 4 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} , \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 4 & 4 & 4 \\ \hline \end{array} \right)$ is a + object bc height$(T) = 4$

Use sign-reversing involution! → move max. # of T to S or S to T

$\left(\begin{array}{|c|} \hline 4 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} , \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 4 & 4 & 4 & 5 \\ \hline \end{array} \right)$

If T and S have same max #...

$\left(\begin{array}{|c|} \hline 5 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} , \begin{array}{|c|c|} \hline 5 & 5 \\ \hline \end{array} \right) \to \left(\begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array} , \begin{array}{|c|c|c|} \hline 5 & 5 & 5 \\ \hline \end{array} \right)$ move max from T to S, since moving from S to T violates T being strictly increasing.

These pairs preserve weight, but + object and − object are always paired, so they cancel. So we get 0. ☐