Math 344 Midterm 1 Solutions

1. Find $\mathcal{L}^{-1}\left[\dfrac{2s}{s^2+2s+4}+\dfrac{e^{-2s}}{(s+1)}+\dfrac{1}{s^{3/2}}+\dfrac{\mathcal{L}[f(t)]}{s^2}\right]$. (Leave the last term as a convolution.)

Solution. $2e^{-t}\cos\sqrt{3}t-\dfrac{2}{\sqrt{3}}e^{-t}\sin\sqrt{3}t+u_2(t)e^{-(t-2)}+\dfrac{2}{\sqrt{\pi}}\sqrt{t}+f(t)*t$.

2. Solve $y'+y=f(t)$ where $f(t)=\begin{cases}e^t & \text{if } 0\le t<1,\\ 0 & \text{if } 1\le t\end{cases}$ with $y(0)=1$.

Solution. $\dfrac{1}{2}e^{-t}+\dfrac{1}{2}e^{t}+\dfrac{1}{2}u_1(t)e^{2-t}-\dfrac{1}{2}u_1(t)e^{t}$.

3. Solve the system $\begin{cases}x'=-y+\delta(t-1),\\ y'=x\end{cases}$ where $x(0)=0$ and $y(0)=0$.

Solution. $x(t)=u_1(t)\sin(t-1), y(t)=u_1(t)\cos(t-1)$.

4. Solve $x^2y''-2xy'+3y=0$.

Solution. $C_1x^{3/2}\cos\left(\dfrac{\sqrt{3}}{2}\ln x\right)+C_2x^{3/2}\sin\left(\dfrac{\sqrt{3}}{2}\ln x\right)$.

5. Find the two series solutions to $y''+xy'+y=0$ up to the x^4 term.

Solution. $a_0\left(1-\dfrac{x^2}{2}+\dfrac{x^4}{8}+\cdots\right)+a_1\left(x-\dfrac{x^3}{3}+\dfrac{x^5}{15}+\cdots\right)$

Table of Laplace Transforms

$f(t)$	$\mathcal{L}[f(t)]$	
$f(t)$	$\int_0^\infty f(t)e^{-st}\,dt$	Definition of Laplace transform
t^n	$\frac{n!}{s^{n+1}}$	Valid for $n = 0, 1, 2, \ldots$
t^r	$\frac{r}{s}\mathcal{L}[t^{r-1}]$	Valid for $r > 0$
$t^{-1/2}$	$\sqrt{\frac{\pi}{s}}$	
e^{at}	$\frac{1}{s-a}$	
$\cos at$	$\frac{s}{s^2+a^2}$	
$\sin at$	$\frac{a}{s^2+a^2}$	
$\frac{\sin at}{t}$	$\arctan\left(\frac{a}{s}\right)$	
$\frac{e^{at}-1}{t}$	$\ln\left(\frac{s}{s-a}\right)$	
$f'(t)$	$s\mathcal{L}[f(t)] - f(0)$	First derivative in t
$f''(t)$	$s^2\mathcal{L}[f(t)] - sf(0) - f'(0)$	Second derivative in t
$e^{at}f(t)$	$F(s-a)$ where $F(s) = \mathcal{L}[f(t)]$	Shifting Theorem 1
$u_a(t)f(t-a)$	$e^{-as}\mathcal{L}[f(t)]$	Shifting Theorem 2
$\delta(t-a)$	e^{-as}	Dirac delta function
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}\mathcal{L}[f(t)]$	Derivatives in s
$f(t) * g(t)$	$\mathcal{L}[f(t)]\mathcal{L}[g(t)]$	The Convolution Theorem