Discrete Mathematics Set 2

Math 435: Complete 10 parts of the following exercises
Math 530: Exercises 2 and 3

1. Let $b_{n,k}$ be the number of set partitions of n into k sets and $T_n(y) = \sum_{k=0}^{n} b_{n,k} y^k$. We have $\sum_{n=0}^{\infty} T_n(y) \frac{x^n}{n!} = e^{y(e^x-1)}$.

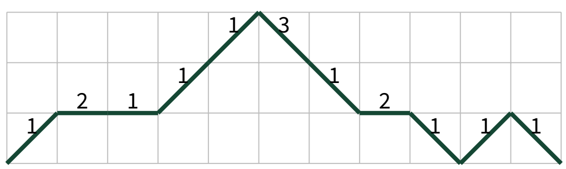
a. Show that $T_n(a+b) = \sum_{k=0}^{n} \binom{n}{k} T_k(a) T_{n-k}(b)$.

b. Show that $\sum_{n=0}^{\infty} T_{n+1}(y) \frac{x^n}{n!} = e^{y(e^x-1)} y e^x$ and use this to show $T_{n+1}(y) = y \sum_{k=0}^{n} \binom{n}{k} T_k(y)$.

2. Let a_n be the number of Motzkin paths with edges labeled according to the following rules:

1. A step of the form $(1,1)$ is always labeled 1.
2. A step of the form $(1,0)$ at height $y=k$ is labeled with an integer in $\{1, \ldots, k+1\}$.
3. A step of the form $(1,-1)$ going from height $y=k$ to height $y=k-1$ is labeled with an integer in $\{1, \ldots, k\}$.

One example of an labeled Motzkin path of length 11 is here:



a. Show that a_n is equal to the number of set partitions of n by defining a bijection that pairs each labeled Motzkin paths with a set partition.

b. Let $a_{n,k}$ be the number of labeled Motzkin paths that satisfy the rules above but start at $(0,k)$, end at (n,k), and never travel below the line $y=k$. For example, $a_{n,0} = a_n$. Show that $a_{0,k} = 1$, $a_{1,k} = (k+1)$, and

$$a_{n,k} = (k+1) a_{n-1,k} + (k+1) \sum_{i=2}^{n} a_{i-2,k+1} a_{n-i,k}$$

for $n \geq 2$.

c. Let $A_k(x) = \sum_{n=0}^{\infty} a_{n,k} x^n$ and show that

$$A_k(x) = \frac{1}{1-(k+1)x-(k+1)x^2 A_{k+1}}.$$

Starting with $A_0(x)$ and iterating this expression, we find the amazing fact that the generating function for the number of set partitions of n can be expressed as this infinite continued fraction:

$$A_0(x) = \cfrac{1}{1-x-\cfrac{x^2}{1-2x-\cfrac{2x^2}{1-3x-\cfrac{3x^2}{1-4x-\cfrac{4x^2}{\ddots}}}}}$$

d. The Fibonacci sequence satisfies $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Find the generating function $F(x) = \sum_{n=0}^{\infty} f_n x^n$ for the Fibonacci sequence.

e. Use the continued fraction expression to show that a_n and f_n have the same parity (the same modulo 2).

3. An ordered set partition of n is an ordered list of disjoint nonempty sets with union $\{1, \ldots, n\}$. For example, there are 13 ordered set partitions of 3:

$$(\{1,2,3\}),$$
$$(\{1\},\{2,3\}),\quad (\{2,3\},\{1\}),\quad (\{2\},\{1,3\}),\quad (\{1,3\},\{2\}),\quad (\{3\},\{1,2\}),\quad (\{1,2\},\{3\}),$$
$$(\{1\},\{2\},\{3\}),\quad (\{1\},\{3\},\{2\}),\quad (\{2\},\{1\},\{3\}),\quad (\{2\},\{3\},\{1\}),\quad (\{3\},\{1\},\{2\}),\quad (\{3\},\{2\},\{1\}).$$

Let a_n be the number of ordered set partitions of n and let $A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$.

a. Show that $a_0 = 1$ and $a_n = \sum_{k=1}^{n} \binom{n}{k} a_{n-k}$ for $n \geq 1$.

b. Show that $A(x) = 1/(2 - e^x)$.

c. Expand $A(x)$ as a geometric series to show that $a_n = \frac{1}{2} \sum_{k=0}^{\infty} \frac{k^n}{2^k}$.

d. Let $a_{n,k}$ be the number of ordered set partitions of n into exactly k sets. Show that $a_{n+1,k} = k a_{n,k} + k a_{n,k-1}$.

e. Let $A(x,y) = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n} a_{n,k} y^k \right) \frac{x^n}{n!}$. Show that $A(x,y)$ satisfies $A_x = yA + (y + y^2) A_y$ and $A(x,1) = 1/(2 - e^x)$.

f. Use the change of variables $w(x,y) = (1 + 1/y)e^{-x}$ and $z(x,y) = y$ to solve the PDE in part e.

g. Let t_n be the total number of sets in all ordered set partitions of n. Find a generating function for $\sum_{n=0}^{\infty} t_n \frac{x^n}{n!}$.

4. Use the exponential formula to find a generating function for the number of permutations of n that do not have any cycles of size 1 (such a permutation is called a derangement). Use this generating function to find an explicit formula for the number of such permutations of n.

5. Find a generating function for the number of set partitions of n which have an even total number of sets, all of which are an even size. Write the answer in terms of $\cosh x = \sum_{n=0}^{\infty} x^{2n}/(2n)!$.