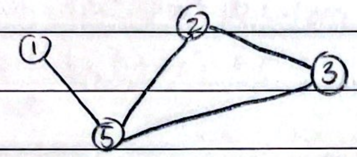
Day 10: 10/6/23

example: How many labeled graphs have n vertices?

$\rightarrow 2^{\binom{n}{2}}$ → There are $\binom{n}{2}$ pairs for edges and there are 2 options for each pair: ① to be connected ② not connected

How many connected labeled graphs?



exponential formula gives: $\displaystyle\sum_{n=0}^{\infty} \underbrace{2^{\binom{n}{2}}}_{\#\text{ of hands } H_n} \frac{x^n}{n!} = e^{\sum_{n=1}^{\infty} \frac{C_n}{n!} x^n}$

C_n = # of connected labeled graphs

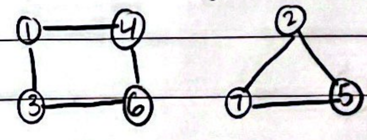
$y=1$ counts # of total hands

ln both sides

So, $\displaystyle\sum_{n=1}^{\infty} \frac{C_n}{n!} x^n = \ln\left(\sum_{n=0}^{\infty} \frac{2^{\binom{n}{2}}}{n!} x^n\right)$

example: to find $n=100$, use a computer to solve right side from $n=0$ to $n=100$, then the x^{100} term coeff. is $\frac{C_n}{n!} = \frac{C_{100}}{100!}$

Example: How many 2-regular graphs on n vertices?



exp. structure: $\displaystyle\sum_{n=0}^{\infty} \left(\begin{array}{c}\#\text{ 2 reg. graphs}\\ \text{on } n \text{ vertices}\end{array}\right) \frac{x^n}{n!} = e^{\sum_{n=1}^{\infty} \frac{C_n}{n!} x^n}$

C_n = # ways to build one piece of overall structure

$y=1$ means total count

$$= e^{\sum_{n=3}^{\infty} \frac{\frac{(n-1)!}{2}}{n!} x^n}$$

starts @ n=3 because n=1 vertex has none
n=2 vertex has none

$$= e^{\sum_{n=1}^{\infty} \frac{1}{2}\frac{x^n}{n} \underbrace{- \frac{x}{2} - \frac{x^2}{4}}}$$

subtract n=1 and n=2 terms to start sum @ n=1

$$= e^{\frac{1}{2}\ln\left(\frac{1}{1-x}\right) - \frac{x}{2} - \frac{x^2}{4}} = \boxed{\frac{e^{-\frac{x}{2} - \frac{x^2}{4}}}{\sqrt{1-x}}}$$

<u>Definition:</u> Let $\alpha > 0$. The <u>gamma function</u> is:

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x}\, dx$$

<u>example:</u> $\Gamma(1) = \int_0^{\infty} x^{1-1} e^{-x}\, dx = 1$

<u>Theorem:</u> $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ only when $\alpha > 0$

<u>Proof:</u> $\Gamma(\alpha+1) = \int_0^{\infty} x^{\alpha} e^{-x}\, dx$

$$\left\{ \begin{array}{ll} \text{IBP} & \\ u = x^{\alpha} & v = -e^{-x} \\ du = \alpha x^{\alpha-1} & dv = e^{-x}\,dx \end{array} \right.$$

$$= \underbrace{-x^{\alpha} e^{-x}\Big|_0^{\infty}}_{\text{goes to } 0} + \alpha \int_0^{\infty} x^{\alpha-1} e^{-x}\, dx$$

$$= \alpha \int_0^{\infty} x^{\alpha-1} e^{-x}\, dx$$

$$= \alpha\Gamma(\alpha) \checkmark$$

<u>Corollary:</u> $\Gamma(n) = (n-1)!$ if $n \in \mathbb{Z}$

Proof: $\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = \dots = (n-1)(n-2)\dots\Gamma(1)$

$= (n-1)!$ ✓

Theorem: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Proof: $\Gamma(\frac{1}{2}) = \int_0^\infty \frac{1}{\sqrt{x}} e^{-x} dx \quad = 2\int_0^\infty \cancel{\sqrt{x}} \frac{1}{\cancel{\sqrt{x}}} e^{-u^2} du$

$u = \sqrt{x} \rightarrow x = u^2$

$du = \frac{1}{2}x^{-1/2} dx$

$dx = 2\sqrt{x}\, du$

$= 2\int_0^\infty e^{-u^2} du = \sqrt{\pi}$

Theorem: (Stirling's Approximation) $\alpha > 0$, $\Gamma(\alpha+1) \sim \sqrt{2\pi\alpha}\left(\frac{\alpha}{e}\right)^\alpha$

(Proof on Monday)

Example for Notation: we let $f(n) \sim g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$

* the $\sim$ means "asymptotic too"
  - equivalence relations

examples: $n^3 + 2n + 1 \sim n^3$ since $\lim_{n\to\infty} \frac{n^3+2n+1}{n^3} = \lim_{n\to\infty} \frac{1 + \overset{0}{\cancel{\frac{2}{n^2}}} + \overset{0}{\cancel{\frac{1}{n^3}}}}{1} = 1$

$n^2 2^n + 4n^3 \sim n^2 2^n$

$1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} \sim e$

Wallace Product: $\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot \dots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot \dots}$

- If $\alpha = n \in \mathbb{Z}$, then $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

example: $\frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\frac{2n!}{(n!)^2} \sim \frac{1}{n+1}\frac{\sqrt{2\pi\cdot 2n}\left(\frac{2\cancel{n}}{\cancel{e}}\right)^{2n}}{\left(\sqrt{2\pi n}\left(\frac{\cancel{n}}{\cancel{e}}\right)^n\right)^2} \sim \frac{2^{2n}}{(n+1)\sqrt{\pi n}} \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$