Discrete Mathematics Set 8

Math 435 and MATH 530: Complete 4 parts of the following exercises, with exercise 3 counting for 2 exercises.

1. Prove the following identities with either a generating function or with a bijection.

a. The number of integer partitions of n with no part divisible by d is equal to the number of integer partitions of n with no part repeated d or more times.

b. The number of integer partitions of n in which each part appears exactly $2, 3$, or 5 times equals the number integer partitions of n into parts which are congruent to $2, 3, 6, 9$, or 10 modulo 12.

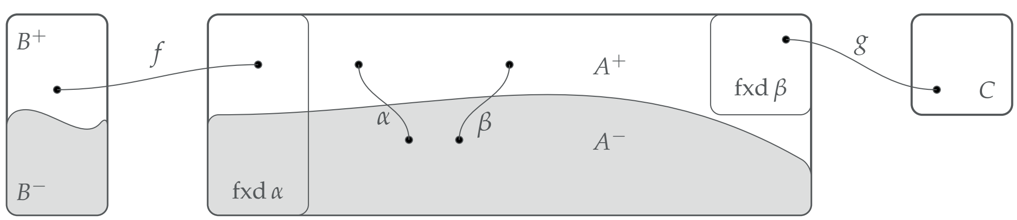
c. The number of integer partitions of n in which no part appears exactly once is equal to the number of integer partitions of n with no part equal to 1 and where consecutive integers do not both appear as parts.

d. The number of integer partitions of n in which no part appears exactly once is equal to the number of integer partitions of n where no part is congruent to 1 or 5 modulo 6.

e. The number of partitions of n in which only odd parts may be repeated is equal to the number of partitions of n in which no part appears more than 3 times.

2. Suppose A, B and C are finite sets such that

1. A is the disjoint union of two sets A^+ and A^-,
2. B is the disjoint union of two sets B^+ and B^-,
3. there is an involution $\alpha : A \to A$ such that $\alpha(A^+ \setminus \operatorname{fxd} \alpha) \subseteq A^-$,
4. there is a bijection $f : \operatorname{fxd} \alpha \to B$ such that $f(\operatorname{fxd} \alpha \cap A^+) = B^+$ and $f(\operatorname{fxd} \alpha \cap A^-) = B^-$,
5. there is an involution $\beta : A \to A$ such that $\operatorname{fxd} \beta \subseteq A^+$, and
6. there is a bijection $g : \operatorname{fxd} \beta \to C$.



Prove that there is an involution $\gamma : B \to B$ such that $\operatorname{fxd} \gamma \subseteq B^+$ and a bijection $h : \operatorname{fxd}(\gamma) \to C$.

3. Write Python or Mathematica code defining a function **`bijection_machine`**. The input is (λ, A, B) where

1. $A = (A_1, \ldots, A_k)$ and $B = (B_1, \ldots, B_k)$ are lists of pairwise disjoint lists such that the sum of the elements in A_i and B_i are the same for all i (these are "diseases"), and
2. λ is an integer partition without any diseases in A.

The output is the integer partition without any diseases in B as produced by Remmel's bijection machine.