Discrete Mathematics Set 1

Math 435: Complete 6 parts of the following exercises. (For example, one option is {1, 2, 3a, 4, 6a, 6c}.)
Math 530: Complete exercises 1, 3, 5 and 6.

1. Using the Taylor series centered at $x = 0$, show that $(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n$ where $\binom{a}{n} = \frac{a(a-1)\cdots(a-n+1)}{n!}$.

2. Prove that $\frac{1}{(1-x)^a} = \sum_{n=0}^{\infty} \binom{a+n-1}{n} x^n$.

3. Verify the following identities involving the products of series:

a. $\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n$

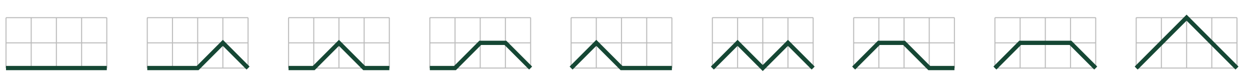
b. $\left(\sum_{n=0}^{\infty} a_n x^n\right)^k = \sum_{n=0}^{\infty} \left(\sum_{\substack{i_1,\dots,i_k \geq 0 \\ i_1+\cdots+i_k=n}} a_{i_1} \cdots a_{i_k}\right) x^n.$

4. By multiplying $(1+x)^a$ and $(1+x)^b$, prove that $\binom{a+b}{n} = \sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k}$ holds for all $a, b \in \mathbb{C}$.

5. Let a_n be the number of ways to tile a $2 \times n$ chessboard with dominoes of sizes 2×1 and 2×2. For example, there are 11 such tilings when $n = 4$:

Find a recurrence for a_n, the generating function for a_n, and a formula for a_n.

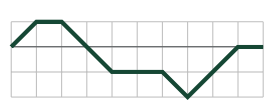
6. A Motzkin path of length n is a path in the plane which starts at $(0, 0)$, ends at $(n, 0)$, uses steps of the form $(1, 1)$, $(1, -1)$, and $(1, 0)$, and never travels below (but may touch) the x-axis. For example,



are the 9 Motzkin paths of length 4. Let m_n be the number of Motzkin paths of length n and let $M(x) = \sum_{n=0}^{\infty} m_n x^n$.

a. Show that $(M(x) - 1)/x = M(x) + xM(x)^2$ and then find an explicit formula for $M(x)$.

b. Let a_n be the number of paths in the plane which start at $(0, 0)$, end at $(0, n)$, and use steps of the form $(1, 1)$, $(1, -1)$, and $(1, 0)$. For example, one path when $n = 11$ is



By looking at the first time a path touches the x axis, show that $a_{n+2} = a_{n+1} + 2\sum_{k=0}^{n} m_k a_{n-k}$ for $n \geq 0$.

c. Show that $A(x) = 1/\sqrt{1 - 2x - 3x^2}$.

Discrete Mathematics Set 2

Math 435: Complete 8 parts of the following exercises.
Math 530: Complete exercises 8 and 9.

7. Let $b_{n,k}$ be the coefficient of $e^{e^x}e^{kx}$ in $\frac{d^n}{dx^n}\left(e^{e^x}\right)$. Prove that $b_{n,k}$ is the number of set partitions of n into k sets.

8. Let $b_{n,k}$ be the number of set partitions of n into k sets and $T_n(y) = \sum_{k=0}^{n} b_{n,k}y^k$. We have $\sum_{n=0}^{\infty} T_n(y)\frac{x^n}{n!} = e^{y(e^x-1)}$.

a. Show that $T_n(a+b) = \sum_{k=0}^{n} \binom{n}{k} T_k(a)T_{n-k}(b)$.

b. Show that $\sum_{n=0}^{\infty} T_{n+1}(y)\frac{x^n}{n!} = e^{y(e^x-1)}ye^x$ and use this to show $T_{n+1}(y) = y\sum_{k=0}^{n}\binom{n}{k}T_k(y)$.

9. An ordered set partition of n is an ordered list of disjoint nonempty sets with union $\{1,\ldots,n\}$. For example, there are 13 ordered set partitions of 3:

$$
\begin{array}{l}
(\{1,2,3\}),\\
(\{1\},\{2,3\}),\quad (\{2,3\},\{1\}),\quad (\{2\},\{1,3\}),\quad (\{1,3\},\{2\}),\quad (\{3\},\{1,2\}),\quad (\{1,2\},\{3\}),\\
(\{1\},\{2\},\{3\}),\quad (\{1\},\{3\},\{2\}),\quad (\{2\},\{1\},\{3\}),\quad (\{2\},\{3\},\{1\}),\quad (\{3\},\{1\},\{2\}),\quad (\{3\},\{2\},\{1\}).
\end{array}
$$

Let a_n be the number of ordered set partitions of n and let $A(x) = \sum_{n=0}^{\infty} a_n\frac{x^n}{n!}$.

a. Show that $a_0 = 1$ and $a_n = \sum_{k=1}^{n}\binom{n}{k}a_{n-k}$ for $n \geq 1$.

b. Show that $A(x) = 1/(2-e^x)$.

c. Expand $A(x)$ as a geometric series to show that $a_n = \frac{1}{2}\sum_{k=0}^{\infty}\frac{k^n}{2^k}$.

d. Let $a_{n,k}$ be the number of ordered set partitions of n into exactly k sets. Show that $a_{n+1} = ka_{n,k} + ka_{n,k-1}$.

e. Let $A(x,y) = \sum_{n=0}^{\infty}\left(\sum_{k=1}^{n} a_{n,k}y^k\right)\frac{x^n}{n!}$. Show that $A(x,y)$ satisfies $A_x = yA + (y+y^2)A_y$ and $A(x,1) = 1/(2-e^x)$.

f. Verify that $A(x,y) = 1/(1-y(e^x-1))$.

g. Let t_n be the total number of sets in all ordered set partitions of n. Find a generating function for $\sum_{n=0}^{\infty} t_n\frac{x^n}{n!}$.

h. (Optional!) Show the change of variables $w(x,y) = (1+1/y)e^{-x}$ and $z(x,y) = y$ can solve the PDE in part e.

Discrete Mathematics Set 3

Math 435: Complete 6 parts of the following exercises.
Math 530: Complete exercises 12 and 13.

10. Use the exponential formula to find a generating function for the number of permutations of n that do not have any cycles of size 1 (such a permutation is called a derangement). Use this generating function to find an explicit formula for the number of such permutations of n.

11. Find a generating function for the number of set partitions of n which have an even total number of sets, all of which are an even size. Write the answer in terms of the function $\cosh x = \sum_{n=0}^{\infty} x^{2n}/(2n)!$.

12. The generating function for the number of permutations of n with only even sized cycles is

$$\sqrt{\frac{1}{1-x^2}}. \tag{1}$$

(The number of such permutations is $1^2 \cdot 3^2 \cdot 5^2 \cdots (n-1)^2$ if n is even and 0 if n is odd.)

a. Use the exponential formula to prove that

$$\sum_{n=0}^{\infty} (\text{the number of permutations of } n \text{ with only odd sized cycles}) \frac{x^n}{n!} = (1+x)\sqrt{\frac{1}{1-x^2}}. \tag{2}$$

b. The coefficients of x^2 in (1) and (2) are the same. Therefore the number of permutations of $2n$ with only even sized cycles is equal to the number of permutations of $2n$ with only odd sized cycles. Find a bijection between these two sets of permutations.

13. Let L_n be the set of ordered lists of the form $(C_1, \ldots, C_m)$ where $C_1, \ldots, C_m$ are cards containing disjoint sets with union $\{1, \ldots, n\}$. This is similar to hands in the exponential formula with the difference being that hands are unordered and lists are ordered.

a. Let $C(x) = \sum_{n=1}^{\infty} |C_n| \frac{x^n}{n!}$. Show that $\sum_{n=0}^{\infty} \left(\sum_{\ell \in L_n} y^{(\text{number of cards in } \ell)} \right) \frac{x^n}{n!} = \frac{1}{1-yC(x)}$.

b. Use part a. of this exercise to find the result in part f. of Exercise 9 in Set 2.

c. A permutation of n with ordered cycles is a list $(\sigma_1, \ldots, \sigma_m)$ where $\sigma_1, \ldots, \sigma_m$ are the cycles in a permutation of n. Let $\mathcal{A}_n$ be the set of permutations of n with ordered cycles and find

$$\sum_{n=0}^{\infty} \left(\sum_{\ell \in \mathcal{A}_n} y^{(\text{number of cycles in } \ell)} \right) \frac{x^n}{n!}.$$

d. Let t_n be the total number of cards in all elements in L_n. Find a generating function involving $C(x)$ for $\sum_{n=0}^{\infty} t_n \frac{x^n}{n!}$.

Discrete Mathematics Set 4

Math 435: Complete 7 parts of the following exercises.
Math 530: Complete exercises 16, 17 and 18.

14. Show $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$. (This was used when introducing the Gamma function and Stirling's approximation.)

15. Let a_n, b_n, c_n and d_n be sequences of real numbers.

a. Show that $\lim_{n\to\infty} |a_n - b_n| = 0$ does not imply $a_n \sim b_n$.

b. Show that $a_n \sim b_n$ and $c_n \sim d_n$ does not imply $a_n + c_n \sim b_n + d_n$.

16. Let $\alpha > 0$ and n be a nonnegative integer.

a. Use induction to show that $\int_0^1 x^{\alpha-1}(1-x)^n\,dx = \frac{n!}{\alpha(\alpha+1)\cdots(\alpha+n)}$.

b. Assuming that the limit and integral can be interchanged, use $\lim\limits_{n\to\infty} \int_0^n x^{\alpha-1}\left(1-\frac{x}{n}\right)^n dx$ to show that

$$\Gamma(\alpha) = \lim_{n\to\infty} \frac{n!n^\alpha}{\alpha(\alpha+1)\cdots(\alpha+n)}.$$

c. (Optional!) Justify why the limit and integral can be interchanged in part b.

17. Let a_n be the number of ordered set partitions of n. Set 2 Exercise 9c gives $a_n = \frac{1}{2}\sum\limits_{k=0}^{\infty} k^n 2^{-k} \approx \frac{1}{2}\int_0^\infty x^n 2^{-x}\,dx$.
Use a substitution in the above integral and then use Stirling's approximation to find $a_n \approx \frac{\sqrt{2\pi n}}{\ln 4}\left(\frac{n}{e\ln 2}\right)^n$.

18. Let A_n be the set of paths in $\mathbb{R}^2$ which start at $(0,0)$, end at (n,n), and only use steps of the form $(1,0)$ or $(0,1)$. Denote the number of times the path $p \in A_n$ touches the line $y = x$ by $\operatorname{touch}(p)$. Let

$$A(x,t) = \sum_{n=0}^{\infty}\left(\sum_{p\in A_n} t^{\operatorname{touch}(p)}\right) x^n.$$

a. Let c_n be the n^{th} Catalan number. Show that

$$\sum_{p\in A_{n+1}} t^{\operatorname{touch}(p)} = 2t\sum_{k=0}^{n} c_k\left(\sum_{p\in A_{n-k}} t^{\operatorname{touch}(p)}\right).$$

b. Show that $A(x,t) = \frac{t}{1-t+2t\sqrt{\frac{1}{4}-x}}$.

c. With the help of the corollary to the first asymptotic result in video 17, find an asymptotic formula for the average number of times a path in A_n touches the line $y = x$.

Discrete Mathematics Set 5

Math 435: Complete 6 parts of the following exercises.
Math 530: Complete exercises 21—24.

19. Let a_n be as defined in Set 1 Exercise 6b. Find an asymptotic formula for a_n using the result in Set 1 Exercise 6c.

20. Let a_n be the number of ordered set partitions of n.

a. Use the generating function in Set 2 Exercise 9 to find an asymptotic formula for a_n given in Set 4 Exercise 17.

b. Using the ideas in Set 3 Exercise 13d, show that the average number of sets in an ordered set partition of n is approximately $n/\ln 4$.

21. Let a_n be the number of permutations of n with ordered cycles.

a. Use the generating function in Set 3 Exercise 13c to find an asymptotic formula for a_n.

b. Find an asymptotic formula for the average number of cycles in a permutation of n with ordered cycles.

22. Find an asymptotic formula for the probability that a permutation of n does not have a cycle of length 1, 2 or 3.

23. We define

1. the Chebyshev polynomial of the first kind $T_n(y)$ by $\sum_{n=0}^{\infty} T_n(y)x^n = \frac{1-yx}{1-2yx+x^2}$,
2. the Chebyshev polynomial of the second kind $U_n(y)$ by $\sum_{n=0}^{\infty} U_n(y)x^n = \frac{1}{1-2yx+x^2}$, and
3. the Legendre polynomial $P_n(y)$ by $\sum_{n=0}^{\infty} P_n(y)x^n = \frac{1}{\sqrt{1-2yx+x^2}}$.

Find asymptotic formulas for $T_n(5/4)$, $U_n(5/4)$, and $P_n(5/4)$.

24. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a complex valued function with nonnegative real coefficients $a_n \geq 0$. Suppose that a singularity of f with smallest complex magnitude has magnitude R (this means that R is the radius of convergence of $f(z)$ and that the series $f(z_0)$ diverges for all z_0 with $|z_0| > R$). This exercise will show that R is a singularity of f.

a. Show that $f(z) = \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} \binom{n}{k} a_n (R/2)^{n-k} \right) (z-R/2)^k$ in some neighborhood of $R/2$.

b. Looking for a contradiction, assume that R is not a singularity of f. This means that there is an $\varepsilon > 0$ such that the above expression is valid for $R+\varepsilon$. Take $R+\varepsilon$ in the above expression and prove that

$$f(R+\varepsilon) = \sum_{n=0}^{\infty} a_n (R+\varepsilon)^n.$$

Why is this a contradiction? Where was the hypothesis that $a_n \geq 0$ used?

Discrete Mathematics Set 6

Math 435: Complete 6 of the following exercises.
Math 530: Complete exercises 32 and 33 and 4 of the remaining exercises.

25. Find the average value of the descent statistic over all permutations of n.

26. Find the average value of the inversion statistic over all permutations of n.

27. Find $\lim_{q \to 1} \frac{d}{dq}[n]_q!$.

28. Find $\lim_{q \to -1} \frac{d}{dq}[n]_q!$.

29. Let $a_{n,k}$ be the number of permutations in S_n with exactly k descents. By inserting "$n+1$" into a permutation of n, show $a_{n+1,k} = (k+1)a_{n,k} + (n+1-k)a_{n,k-1}$.

30. Let $\varphi : S_n \to S_n$ be a bijection such that $\mathrm{des}(\sigma) = \mathrm{exc}(\varphi(\sigma))$ for all $\sigma \in S_n$. Write a computer program in Python or Mathematica (if you would like to use another programming language for any of the programming exercises, just ask me!) that inputs σ in one line notation and outputs $\varphi(\sigma)$ in one line notation.

31. Let $\varphi : S_n \to S_n$ be a bijection such that $\mathrm{inv}(\sigma) = \mathrm{maj}(\varphi(\sigma))$ for all $\sigma \in S_n$. Write a computer program in Python or Mathematica that inputs σ in one line notation and outputs $\varphi(\sigma)$ in one line notation.

32. Suppose that in one line notation, the permutation $\sigma \in S_n$ has σ_i in position i. Then the inverse permutation σ^{-1} written in one line notation has i in position σ_i. Show that $\mathrm{inv}(\sigma) = \mathrm{inv}(\sigma^{-1})$ for all $\sigma \in S_n$.

33. Using terminology from Abstract Algebra, prove that the sign of σ (as found by writing σ as a product of transpositions) is equal to $(-1)^{\mathrm{inv}(\sigma)}$.

Discrete Mathematics Set 7

Math 435: Complete 6 parts of the following exercises.
Math 530: Complete exercise 38 and 3 parts of the remaining exercises.

34. Find the average number of inversions in $R(0^k, 1^{n-k})$ and use the result to simplify $\lim_{q\to 1} \frac{d}{dq} \begin{bmatrix} n \\ k \end{bmatrix}_q$.

35. The q-multinomial coefficient $\begin{bmatrix} n \\ k_1, \ldots, k_\ell \end{bmatrix}_q$ is defined to be $\frac{[n]_q!}{[k_1]_q! \cdots [k_\ell]_q!}$ for $n = k_1 + \cdots + k_\ell$. Show that

$$\begin{bmatrix} n \\ k_1, \ldots, k_\ell \end{bmatrix}_q = \sum_{r \in R(1^{k_1}, \ldots, \ell^{k_\ell})} q^{\operatorname{inv}(r)}$$

where $R(1^{k_1}, \ldots, \ell^{k_\ell})$ denotes the set of rearrangements of k_1 1's, k_2 2's, etc.

36. Let φ be the bijection in the proof of $\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{r \in R(0^k, 1^{n-k})} q^{\mathrm{maj}(r)}$ found on the next page.

a. Find both $\varphi(110111011001)$ and $\varphi^{-1}(110111011001)$.

b. Write a computer program in Python or Mathematica that inputs r and outputs $\varphi(r)$.

37. Prove these identities without writing $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ as a fraction and manipulating powers of q. Instead, interpret both sides of the identity as rearrangements or integer partitions and show the result by double counting or a bijection.

a. (The q-Pascal identity) $\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$.

b. (The q-Vandermonde identity) $\begin{bmatrix} a+b \\ n \end{bmatrix}_q = \sum_{k=0}^{n} q^{(a-k)(n-k)} \begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} b \\ n-k \end{bmatrix}_q$.

c. (The q-binomial theorem) $(1 + xq^0) \cdots (1 + xq^{n-1}) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k$.

38. Let q a prime power, $\mathbb{F}_q$ be the finite field with q elements, and $\mathbb{F}_q^n$ be the n-dimensional vector space over $\mathbb{F}_q$.

a. Prove that the number of k dimensional subspaces in $\mathbb{F}_q^n$ is equal to $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

b. Let X be a vector space with a finite number of elements x. Show that there are

$$\begin{bmatrix} n \\ n-k \end{bmatrix}_q (x - q^0) \cdots (x - q^{k-1})$$

linear maps $L : \mathbb{F}_q^n \to X$ which have a null space of dimension $n-k$.

c. By counting linear maps $L : \mathbb{F}_q^n \to X$, prove the q-Cauchy identity:

$$x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (x - q^0) \cdots (x - q^{k-1}).$$

d. The identity in part c. has been shown true for prime powers q. How can we conclude that this identity is true for any complex number q?

Theorem 1. *If $0 \leq k \leq n$, then* $\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{r \in R(0^k, 1^{n-k})} q^{maj(r)}$.

Proof. We prove this result by defining a bijection $\varphi : R(0^k, 1^{n-k}) \to R(0^k, 1^{n-k})$ such that $\text{maj}(r) = \text{inv}(\varphi(r))$ for all $r \in R(0^k, 1^{n-k})$. This is enough because we have already shown that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{r \in R(0^k, 1^{n-k})} q^{\text{inv}(r)}.$$

We first define an auxiliary bijection $\Gamma : R(0^k, 1^{n-k}) \to R(0^k, 1^{n-k})$. If r ends with a 0, define $\Gamma(r)$ to be r with every consecutive substring of the form $11 \cdots 110$ changed to $011 \cdots 11$. If r ends with a 1, define $\Gamma(r)$ to be r with every consecutive substring of the form $00 \cdots 001$ changed to $100 \cdots 00$. For example,

$$\Gamma(1100011110100) = 0110001111010.$$

If r ends with a 0, then $\text{inv}(\Gamma(r)) = \text{inv}(r) - (n-k)$ because changing $11 \cdots 110$ into $011 \cdots 11$ for all 1's in r decreases the number of inversions in r by 1 for each of the $n-k$ 1's in r. Similarly, if r ends with a 1, then $\text{inv}(\Gamma(r)) = \text{inv}(r) + k$.

We can now define our main bijection φ. If r contains no 0's, then we define $\varphi(r) = r$. Otherwise, let w be r with the last 0 and all trailing 1's deleted so that r can be written as $w011 \cdots 11$. For any rearrangement $r \in R(0^k, 1^{n-k})$, define $\varphi(r)$ recursively by $\varphi(r) = \Gamma(\varphi(w))011 \cdots 11$. For example, it can be checked that

$$\varphi(10110100011) = 00111010011.$$

By definition, $\varphi(r)$ ends with a 0 if and only if r ends with a 0.

The fact that φ is a bijection follows from the fact that Γ is a bijection. To complete the proof, we will show that $\text{maj}(r) = \text{inv}(\varphi(r))$ by induction on the length of r. Suppose we add a 0 to the end of $r \in R(0^k, 1^{n-k})$. Then we have

$$\begin{aligned}
\text{inv}(\varphi(r0)) &= \text{inv}\left(\Gamma(\varphi(r))0\right) \\
&= \text{inv}\left(\Gamma(\varphi(r))\right) + (n-k) \\
&= \begin{cases} \text{inv}\left(\varphi(r)\right) - (n-k) + (n-k) & \text{if } \varphi(r) \text{ ends in } 0, \\ \text{inv}\left(\varphi(r)\right) + k + (n-k) & \text{if } \varphi(r) \text{ ends in } 1. \end{cases}
\end{aligned}$$

Using the induction hypothesis and the fact that $\varphi(r)$ ends in a 0 if and only if r does, this is equal to

$$\begin{cases} \text{maj}(r) & \text{if } r \text{ ends in } 0, \\ \text{maj}(r) + n & \text{if } r \text{ ends in } 1. \end{cases}$$

In both cases, this is equal to $\text{maj}(r0)$. We have shown that $\text{inv}(\varphi(r0)) = \text{maj}(r0)$.

Now suppose we add a 1 onto the end of r. Since $\varphi(r1) = \Gamma(\varphi(w))01 \cdots 11 = \varphi(r)1$, we have

$$\text{inv}(\varphi(r1)) = \text{inv}(\varphi(r)1) = \text{inv}(\varphi(r)) = \text{maj}(r) = \text{maj}(r1).$$

This completes the proof. $\square$

Discrete Mathematics Set 8

Math 435: Complete 6 parts of the following exercises.
Math 530: Complete exercise 41 and 3 parts of the remaining exercises.

39. Prove the following identities by either proving that they have the same generating functions or by proving them with a bijection.

a. Show that the number of integer partitions of n with no part divisible by d is equal to the number of integer partitions of n with no part repeated d or more times.

b. Prove that the number of integer integer partitions of n with both odd and distinct parts is equal to the number of integer integer partitions of n that are equal to their conjugate.

c. Prove that the number of integer partitions of n in which each part appears exactly $2, 3$, or 5 times equals the number integer partitions of n into parts which are congruent to $2, 3, 6, 9$, or 10 modulo 12.

d. Show that the number of integer partitions of n in which no part appears exactly once is equal to the number of integer partitions of n with no part equal to 1 and where consecutive integers do not both appear as parts.

e. Show that the number of integer partitions of n in which no part appears exactly once is equal to the number of integer partitions of n where no part is congruent to 1 or 5 modulo 6.

40. Prove that $\sum_{n=1}^{\infty} x^n y^n z(1+zx^1)\cdots(1+zx^{n-1}) = \sum_{n=1}^{\infty} \frac{x^{\binom{n+1}{2}} y^n z^n}{(1-yx^1)\cdots(1-yx^n)}$.

41. Let $p_k(n)$ be the number of integer partitions with $\ell(\lambda) = k$.

a. Show there are $\binom{n-1}{k-1}$ solutions to $x_1 + \cdots + x_k = n$ where $x_1, \ldots, x_k$ are positive integers. (One way is to use a “balls and bars” or “stars and bars” argument from an introductory combinatorics course.)

b. By considering rearrangements of the parts of partitions, show that $\binom{n-1}{k-1} \le k! p_k(n)$.

c. By making the parts of a partition distinct, show that $k! p_k(n) \le \binom{n+\binom{k}{2}-1}{k-1}$.

d. Show that $\binom{n+a-1}{k-1} \sim \frac{n^{k-1}}{(k-1)!}$ for any nonnegative integer a and then show that $p_k(n) \sim \frac{n^{k-1}}{k!(k-1)!}$.

Discrete Mathematics Set 9

Math 435: Complete 6 parts of the following exercises.
Math 530: Complete exercises 43, 44, 46a and ((46b, 46c, and 46d) or (46e, 46f, and 46g)).

42. Recover Euler's pentagonal number theorem by taking $y = -q$ and $x = q^3$ in Jacobi's triple product

$$(1+y)\prod_{n=1}^{\infty}(1-x^n)(1+yx^n)(1+y^{-1}x^n) = \sum_{k\in\mathbb{Z}} y^k x^{k(k-1)/2}.$$

43. Use the function $x^2 \prod_{n=1}^{\infty}(1-x^n)^6$ to show that the number of integer partitions of $7n+5$ is divisible by 7.

44. This exercise proves a finite version of Jacobi's triple product identity.

a. Prove $\prod_{i=1}^{n}(1+yq^{i-1})(1+y^{-1}q^i) = \sum_{k=-n}^{n} y^k q^{k(k-1)/2} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q.$

b. Take $\lim_{n\to\infty}$ of the above expression to find the full Jacobi triple product.

45. Suppose $F(x,y)$ is an infinite product in two indeterminates y and x. Euler's device refers to this process of turning $F(x,y)$ into a sum:

1. Find an equation providing a relationship between $F(x,xy)$ and $F(x,y)$.
2. Assume that $F(x,y) = \sum_{n=0}^{\infty} a_n(x)y^n$ and plug this into the equation found in step 1.
3. Compare coefficients of y^n to find a recursion for $a_n(x)$.
4. Iterate this recursion to find a formula for $a_n(x)$.

Show $F(x,y) = \prod_{i=0}^{\infty}\frac{1}{1-yx^i}$ satisfies $F(x,xy) = (1-y)F(x,y)$ and then express $F(x,y)$ as a sum using Euler's device.

46. Let $F(x,y)$ be the function that satisfies the recursion $F(x,y) = F(x,xy) + xyF(x,x^2y)$ and $F(x,0) = 1$.

a. Use Euler's device to show $F(x,y) = \sum_{n=0}^{\infty}\frac{y^n x^{n^2}}{(1-x)\cdots(1-x^n)}$.

b. Use the result in the theorem on the next page to show $F(q,1) = \left(\sum_{k\in\mathbb{Z}}(-1)^n q^{n(5n-1)/2}\right)\prod_{i=1}^{\infty}\frac{1}{1-q^i}$.

c. Take $y = -q^2$ and $x = q^5$ in Jacobi's triple product to show $F(q,1) = \prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$.

d. Show that (the number of integer partitions of n with parts differing by at least 2) is equal to (the number of integer partitions of n with parts congruent to $\pm 1 \bmod 5$). (Hint: $1+3+5+\cdots+(2n-1) = n^2$.)

e. Use the result in the theorem on the next page to show $F(q,q) = \left(\sum_{k\in\mathbb{Z}}(-1)^n q^{n(5n+3)/2}\right)\prod_{i=1}^{\infty}\frac{1}{1-q^i}$.

f. Take $y = -q^4$ and $x = q^5$ in Jacobi's triple product to show $F(q,q) = \prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}$.

g. Show that (the number of integer partitions of n with parts differing by at least 2 and no part of size 1) is equal to (the number of integer partitions of n with parts congruent to $\pm 2 \bmod 5$). (Hint: $2+\cdots+2n = n^2+n$.)

Theorem. *The function*

$$F(x,y)=\left(1+\sum_{n=1}^{\infty}(-1)^n y^{2n}x^{n(5n-1)/2}\left(1-yx^{2n}\right)\frac{(1-yx)\cdots\left(1-yx^{n-1}\right)}{(1-x)\cdots(1-x^n)}\right)\prod_{i=1}^{\infty}\frac{1}{1-yx^i} \tag{3}$$

satisfies $F(x,y)=F(x,xy)+xyF(x,x^2y)$ *and* $F(x,0)=1$.

Proof. Using $\left(1-yx^{2n}\right)=(1-x^n)+x^n(1-yx^n)$, we have that

$$\begin{aligned}F(x,y)\prod_{i=1}^{\infty}\left(1-yx^i\right)&=1+\sum_{n=1}^{\infty}(-1)^n y^{2n}x^{n(5n-1)/2}\frac{(1-yx)\cdots\left(1-yx^{n-1}\right)}{(1-x)\cdots\left(1-x^{n-1}\right)}\\&\quad+\sum_{n=1}^{\infty}(-1)^n y^{2n}x^{n(5n+1)/2+n}\frac{(1-yx)\cdots(1-yx^n)}{(1-x)\cdots(1-x^n)}\\&=\sum_{n=0}^{\infty}(-1)^n y^{2n}x^{n(5n+1)/2}\left(1-y^2x^{4n+2}\right)\frac{(1-yx)\cdots(1-yx^n)}{(1-x)\cdots(1-x^n)}.\end{aligned} \tag{4}$$

where we changed the first line into the second by reindexing the first infinite sum. Therefore, using (4) to simplify the first product and (3) to simplify the second product,

$$(F(x,y)-F(x,xy))\prod_{i=1}^{\infty}\left(1-yx^i\right)=\sum_{n=0}^{\infty}(-1)^n y^{2n}x^{n(5n+1)/2}\frac{(1-yx)\cdots(1-yx^n)}{(1-x)\cdots(1-x^n)}\left(1-y^2x^{4n+2}-x^n\left(1-yx^{2n+1}\right)\right).$$

Using $\left(1-y^2x^{4n+2}-x^n\left(1-yx^{2n+1}\right)\right)=(1-x^n)+yx^{3n+1}\left(1-yx^{n+1}\right)$, the above expression is equal to

$$\sum_{n=1}^{\infty}(-1)^n y^{2n}x^{(5n+1)/2}\frac{(1-yx)\cdots(1-yx^n)}{(1-x)\cdots\left(1-x^{n-1}\right)}+yx\sum_{n=0}^{\infty}(-1)^n y^{2n}x^{n(5n+7)/2}\frac{(1-yx)\cdots\left(1-yx^{n+1}\right)}{(1-x)\cdots(1-x^n)},$$

which in turn, by reindeying the first sum, is equal to

$$yx\sum_{n=0}(-1)^n y^{2n}x^{n(5n+7)/2}\left(1-yx^{2n+2}\right)\frac{(1-yx)\cdots\left(1-yx^{n+1}\right)}{(1-x)\cdots(1-x^n)}=yx\left(\prod_{i=1}^{\infty}\left(1-yx^i\right)\right)F(x,x^2y).$$

The last step used (3) again. Thus we have proved

$$(F(x,y)-F(x,xy))\prod_{i=1}^{\infty}\left(1-yx^i\right)=xyF(x,x^2y)\prod_{i=1}^{\infty}\left(1-yx^i\right),$$

which implies the desired result. $\square$

Discrete Mathematics Set 10

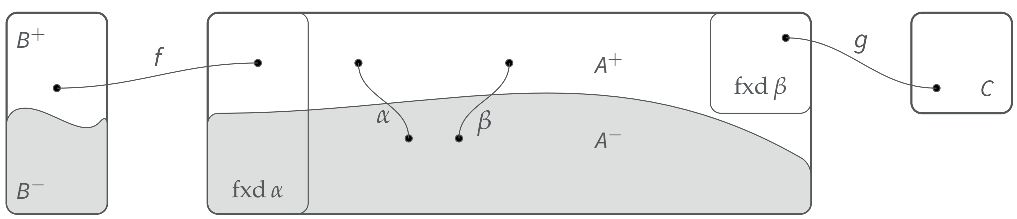
Math 435: Complete either exercises 47–50 or exercise 51.
Math 530: Complete either exercises 47–50 or exercise 51.

47. Provide a bijection to prove that (the number of partitions of n in which only odd parts may be repeated) is equal to (the number of partitions of n in which no part appears more than 3 times). Give an explicit, nontrivial example of your bijection.

48. Show that the number of partitions of n in which 3 consecutive parts may not repeated equals the number of partitions of n in which 3 consecutive even parts do not appear.

49. Suppose A, B and C are finite sets such that

1. A is the disjoint union of two sets A^+ and A^-,
2. B is the disjoint union of two sets B^+ and B^-,
3. there is an involution $\alpha : A \to A$ such that $\alpha(A^+ \setminus \operatorname{fxd} \alpha) \subseteq A^-$,
4. there is a bijection $f : \operatorname{fxd} \alpha \to B$ such that $f(\operatorname{fxd} \alpha \cap A^+) = B^+$ and $f(\operatorname{fxd} \alpha \cap A^-) = B^-$,
5. there is an involution $\beta : A \to A$ such that $\operatorname{fxd} \beta \subseteq A^+$, and
6. there is a bijection $g : \operatorname{fxd} \beta \to C$.



Prove that there is an involution $\gamma : B \to B$ such that $\operatorname{fxd} \gamma \subseteq B^+$ and a bijection $h : \operatorname{fxd}(\gamma) \to C$.

50. Let $A_1, \ldots, A_n$ be finite sets. Prove the principle of inclusion/exclusion:

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{S \subseteq \{1,\ldots,n\}} (-1)^{|S|-1} \left| \bigcap_{i \in S} A_i \right|$$

by way of a sign reversing involution.

51. Write Python or Mathematica code defining a function `bijection_machine`. The input is (λ, A, B) where

1. $A = (A_1, \ldots, A_k)$ and $B = (B_1, \ldots, B_k)$ are lists of pairwise disjoint lists such that the sum of the elements in A_i and B_i are the same for all i (these are “diseases”), and
2. λ is an integer partition without any diseases in A.

The output is the integer partition without any diseases in B as produced by Remmel’s bijection machine.

Discrete Mathematics Set 11

Math 435: Complete 4 parts of the following exercises.
Math 530: Complete exercise 53, 54, and 55.

52. Let $K_{\lambda,\mu}$ be the number of column strict tableau T of shape $\lambda \vdash n$ and content $\mu = (\mu_1, \mu_2, \ldots) \vdash n$ (the content means that there are μ_1 copies of 1 in T, μ_2 copies of 2 in T, and so on). The Kostka matrix K_n is matrix with rows and columns indexed by integer partitions of n with the row λ and column μ entry equal to $K_{\mu,\lambda}$. For example, when $n = 4$, this matrix is

$$\begin{array}{cc} & \begin{array}{ccccc} (4) & (3,1) & (2^2) & (2,1^2) & (1^4) \end{array} \\ \begin{array}{c} (4) \\ (3,1) \\ (2^2) \\ (2,1^2) \\ (1^4) \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & 3 & 2 & 3 & 1 \end{bmatrix} \end{array}.$$

a. Find K_5 and (using a machine) find K_5^{-1}.

b. Let a_λ be the vector in $\mathbb{R}^{p(n)}$ with a 1 in the λ entry and 0 elsewhere. What do the matrix multiplications $K_n a_\lambda$ and $K_n^{-1} a_\lambda$ mean in terms of the monomial and Schur symmetric functions?

53. Let RCS_λ denote the set of reverse column strict tableaux; that is, all tableaux where the integer labeling weakly decreases in rows and strictly decreases up columns. Show that $s_\lambda = \sum_{RCS_\lambda} w(T)$ for any $\lambda \vdash n$. For example, here are all elements in $RCS_{(2,1)}$ that are filled with integers ≤ 3:

$$\begin{array}{|c|c} \hline 2 \\ \hline 3 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 1 \\ \hline 3 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 1 \\ \hline 2 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 1 \\ \hline 3 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 1 \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 1 \\ \hline 3 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 2 \\ \hline 3 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 2 \\ \hline 3 & \multicolumn{1}{c|}{3} \\ \hline \end{array}.$$

54. Prove that the power symmetric polynomial $p_\lambda(x_1, \ldots, x_N)$, the homogeneous symmetric polynomial $h_\lambda(x_1, \ldots, x_N)$, and the elementary symmetric polynomial $e_\lambda(x_1, \ldots, x_N)$ are indeed symmetric polynomials.

55. An alternating polynomial f in $x_1, \ldots, x_n$ is a polynomial such that for all $\sigma = \sigma_1 \cdots \sigma_n \in S_n$,

$$f(x_1, \ldots, x_n) = \text{sign}(\sigma) f(x_{\sigma_1}, \ldots, x_{\sigma_n}).$$

a. Show that an alternating polynomial is divisible by $\Delta = \prod_{i<j}(x_i - x_j)$.

b. Let $\mathcal{A}_k$ be the vector space of alternating polynomials with every term degree k. Show that division by Δ is a vector space isomorphism between $\mathcal{A}_{n+\binom{n}{2}}$ and Λ_n. (Therefore understanding Λ_n is the same as understanding $\mathcal{A}_{n+\binom{n}{2}}$.)

Discrete Mathematics Set 12

Math 435: Complete 7 parts of the following exercises.
Math 530: Complete 7 parts of the following exercises.

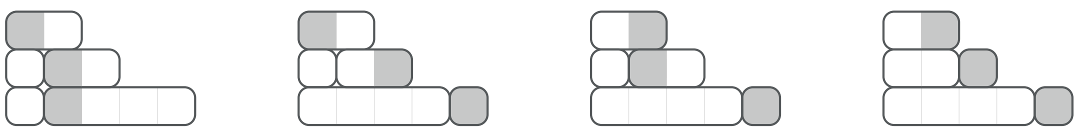
56. Prove that the coefficient of m_λ in h_μ is the number of matrices with nonnegative integer entries with row sum λ and column sum μ.

57. Let $\mu \vdash n$.

a. Use a similar proof as used to prove $h_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,\mu}| e_\lambda$ to prove $e_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,\mu}| h_\lambda$.

b. Let $p(n)$ be the number of integer partitions of n and let B_n be the $p(n) \times p(n)$ matrix with row μ, column λ entry equal to $(-1)^{n-\ell(\lambda)} |B_{\lambda,\mu}|$. Why does part a. imply $B_n^{-1} = B_n$?

58. A weighted brick tabloid of content λ and shape μ is the usual brick tabloid of content λ and shape μ but with one cell in the final brick in each row shaded. Let $WB_{\lambda,\mu}$ be the set of all weighted brick tabloids of content λ and shape μ. Here are 4 of the 30 examples of weighted brick tabloids found on the next page:



a. Use a similar proof as used to prove $h_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,\mu}| e_\lambda$ to prove $p_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |WB_{\lambda,\mu}| e_\lambda$.

b. Prove $p_\mu = \sum_{\lambda \vdash n} (-1)^{\ell(\mu)-\ell(\lambda)} |WB_{\lambda,\mu}| h_\lambda$.

c. By counting weighted brick tabloids, find the 5×5 matrix with row and columns indexed by integer partitions of 4 and with row μ and column λ entry equal to $(-1)^{n-\ell(\lambda)} |WB_{\lambda,\mu}|$. Why does this matrix verify that $\{p_\lambda : \lambda \vdash 4\}$ is a basis for Λ_4? More generally, why is $\{p_\lambda : \lambda \vdash n\}$ a basis for Λ_n?

59. Prove these identities are true for $n \geq 1$ using bijections or sign reversing involutions:

a. $p_n = \sum_{i=0}^{n-1} (-1)^i s_{(n-i,1^i)}$.

b. $\sum_{i=0}^{n-1} h_i p_{n-i} = n h_n$.

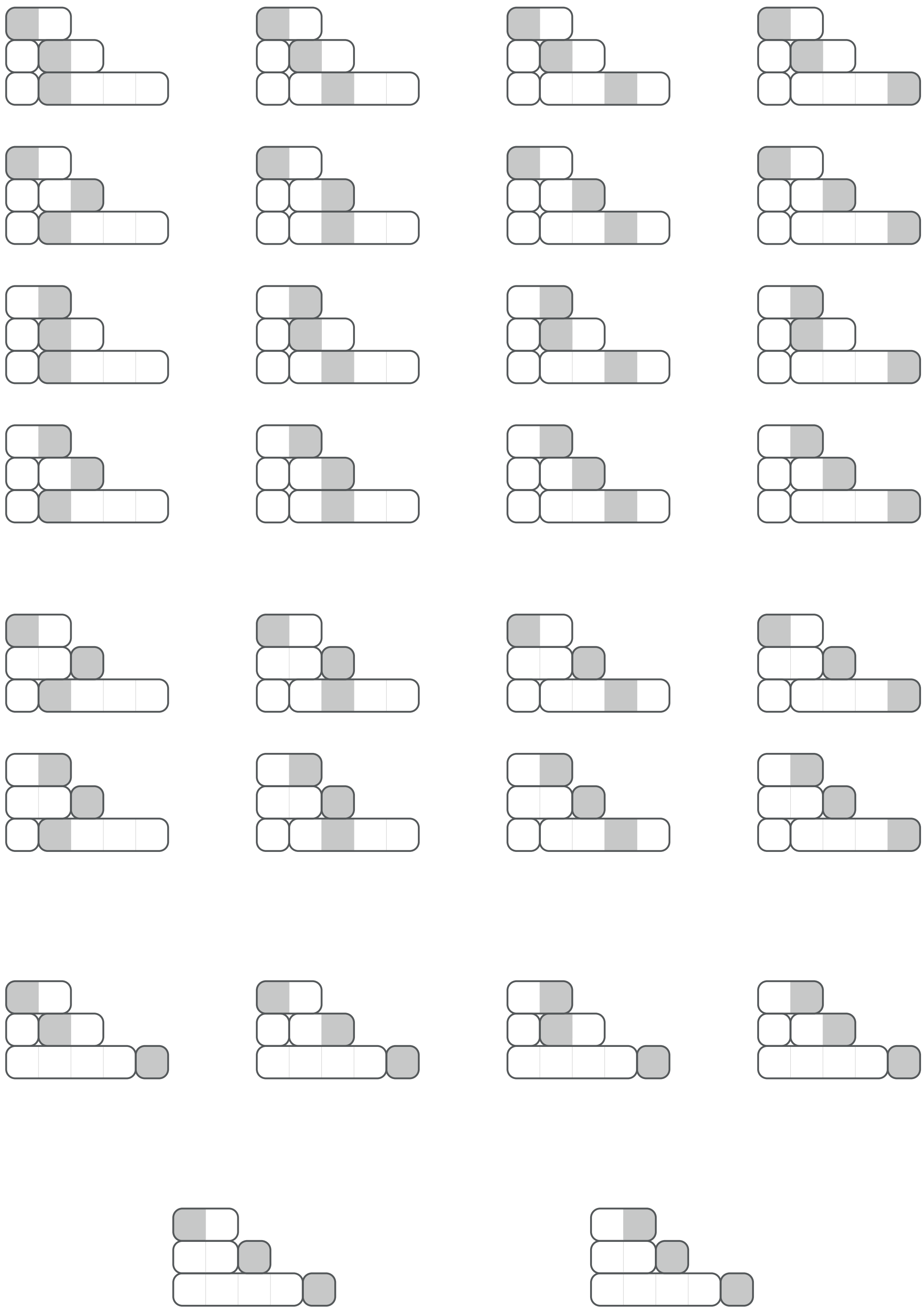
c. $\sum_{i=0}^{n-1} (-1)^i e_i p_{n-i} = (-1)^{n-1} n e_n$.

60. Let $p_n = p_{(n)}(x_1, \ldots, x_N)$ be the power symmetric polynomial in $x_1, \ldots, x_N$, let $h_n = h_{(n)}(x_1, \ldots, x_N)$ be the homogeneous symmetric polynomial, and let $H(t) = \sum_{n=0}^{\infty} h_n t^n$. Show $\sum_{n=1}^{\infty} \frac{p_n}{n} t^n = \ln H(t)$ and $\sum_{n=1}^{\infty} p_n t^n = \frac{tH'(t)}{H(t)}$.

61. Show that $\sum_{n=0}^{\infty} \sum_{\lambda \vdash n} h_\lambda(x_1, \ldots, x_N) m_\lambda(y_1, \ldots, y_M) = \prod_{j=1}^{M} \prod_{i=1}^{N} \frac{1}{1 - x_i y_j}$.

Hint: Multiply sums of the form $\prod_{i=1}^{N} \frac{1}{1 - x_i y_j} = \sum_{n=0}^{\infty} h_n(x_1, \ldots, x_N) y_j^n$.

All 30 weighted brick tabloids of shape $(5,3,2)$ and content $(4,2,2,1,1)$.



Discrete Mathematics Set 13

Math 435: Complete 4 parts of the following exercises.
Math 530: Complete exercise 65 and two parts of the remaining exercises.

62. Define a ring homomorphism φ on Λ by $\varphi(e_n) = (-1)^{n-1}/n!$ for $n \geq 1$. Use $\varphi(h_n)$ to find the generating function for the number of ordered set partitions of n first found in Exercise 9 in Set 2.

63. Define a ring homomorphism φ on Λ by $\varphi(e_n) = (-1)^{n-1}k(x-1)^{n-1}$ for $n \geq 1$. Use $\varphi(h_n)$ to find the generating function for

$$\sum_{w \in \{1,\ldots,k\}^n} x^{\mathrm{equals}(w)}$$

where $\mathrm{equals}(w)$ denotes the number of times there are consecutive equal integers in a word $w \in \{1, \ldots, k\}^n$.

64. Define a ring homomorphism φ on Λ by $\varphi(e_n) = \begin{cases} 1 & \text{if } n = 0 \text{ or } n = 2, \\ 2x & \text{if } n = 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$

a. Recall from exercise 23 the definitions of the Chebyshev polynomial of the first kind $T_n(x)$ and the Chebyshev polynomial of the second kind $U_n(x)$. Show that $\varphi(p_n) = 2T_n(x)$ for $n \geq 1$ and $\varphi(h_n) = U_n(x)$ for $n \geq 0$. It may help to use an identity found in Exercise 60.

b. Use previously established relationships between e_n, h_n, and p_n (such as those in Exercise 59) to show these identities hold for $n \geq 3$:

   i. $U_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} (-1)^i (2x)^{n-2i}$

   ii. $U_n(x) = \frac{2}{n} \sum_{i=0}^{n-1} U_i(x) T_{n-i}(x)$

   iii. $U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0$

   iv. $T_n(x) - 2xT_{n-1}(x) + T_{n-2}(x) = 0$

65. Define a ring homomorphism φ on Λ by $\varphi(e_n) = \begin{cases} (-1)^{k+k(3k-1)/2} & \text{if } n = k(3k-1)/2 \text{ for some } k \in \mathbb{Z}, \\ 0 & \text{if not.} \end{cases}$

a. Show that $\varphi(h_n) = p(n)$ where $p(n)$ is the number of integer partitions of n.

b. Apply φ to the generating function for p_n/n in Exercise 60 to show that $\varphi(p_n) = \sigma(n)$ where $\sigma(n)$ is the sum of the positive integer divisors of n.

c. Use an identity found in Exercise 59 to show that $p(n) = \frac{1}{n} \sum_{i=1}^{n} \sigma(i) p(n-i)$, thereby giving a recursion for the number of integer partitions of n. Calculate $p(7)$ using this recursion.

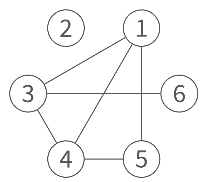
Discrete Mathematics Set 14

Math 435: Complete 3 of the following exercises.
Math 530: Complete exercise 70 and 2 exercises in {67, 68, 71, 72}.

66. How many ways are there to color the vertices of a cube using N colors (two colorings are the same if the cube can be rotated to turn one coloring into another)? How many ways are there to color the vertices if 3 vertices must be red and 5 must be black?

67. How many ways are there to color the edges of a cube using N colors (two colorings are the same if the cube can be rotated to turn one coloring into another)? How many ways are there to color the vertices if 6 edges must be red and 6 must be black?

68. Let E be the set of two element subsets of $\{1, \ldots, n\}$. A simple graph on n vertices corresponds to a coloring of E which uses two different colors: a set $\{i, j\}$ is colored q if the edge between i and j appears in a simple graph and 1 if not. For example, the graph



corresponds to coloring each of $\{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$, $\{3, 4\}$, $\{3, 6\}$, and $\{4, 5\}$ with q and all other elements of E with 1. In this way, the number of edges in the graph is the number of times q is used in the coloring.

By defining $\sigma\{i, j\} = \{\sigma(i), \sigma(j)\}$ for all $\sigma \in S_n$, the symmetric group S_n acts on elements of E. Find

$$\sum_{\text{inequivalent 2 colorings } f \text{ of } E} q^{\text{the number of times color } q \text{ is used in } f}$$

when $n = 4$. Using the language of graph theory, we are finding

$$\sum_{\text{nonisomorphic simple graphs } g \text{ on 4 vertices}} q^{\text{the number edges in } g}.$$

69. Show that the cycle index polynomial satisfies $Z_{G \times H} = Z_G Z_H$.

70. Let C_n be the cyclic group of order n (the group generated by rotations of an n–sided regular polygon) and let D_n be the dihedral group of order $2n$ (the group generated by rotations and reflections of an n–sided regular polygon). Show that the cycle index polynomials for these groups are

$$Z_{C_n} = \frac{1}{n} \sum_{i=1}^{n} (p_{n/\gcd(i,n)})^{\gcd(i,n)} \qquad \text{and} \qquad Z_{D_n} = \frac{1}{2} Z_{C_n} + \begin{cases} p_1 p_2^{(n-1)/2}/2 & \text{if } n \text{ is odd,} \\ (p_2^{n/2} + p_1^2 p_2^{(n-2)/2})/4 & \text{if } n \text{ is even.} \end{cases}$$

where $\gcd(i, n)$ is the greatest common divisor of i and n.

71. Show the cycle index polynomial for the symmetric group S_n is the homogeneous symmetric function h_n.

72. Let A_n be the alternating group, the subgroup of S_n containing all permutations σ with an even number of even sized cycles. Show that the cycle index polynomial for the alternating group A_n is $h_n + e_n$.