Discrete Mathematics Set 1

Math 435: Complete 6 parts of the following exercises.
Math 530: Complete exercises 4, 5 and 6.

1. Using the Taylor series centered at $x = 0$, show that $(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n$ where $\binom{a}{n} = \frac{a(a-1)\cdots(a-n+1)}{n!}$.

2. Verify the following identities involving the products of series:

a. $\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n$

b. $\left(\sum_{n=0}^{\infty} a_n x^n\right)^k = \sum_{n=0}^{\infty}\left(\sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k=n}} a_{i_1}\cdots a_{i_k}\right) x^n.$

3. By multiplying $(1+x)^a$ and $(1+x)^b$, prove that $\binom{a+b}{n} = \sum_{k=0}^{n}\binom{a}{k}\binom{b}{n-k}$ holds for all $a, b \in \mathbb{C}$.

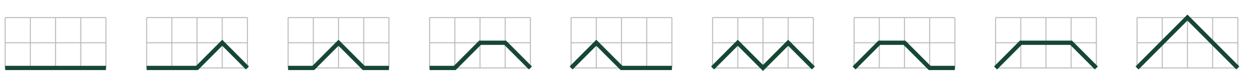
4. Prove that $\frac{1}{(1-x)^a} = \sum_{n=0}^{\infty}\binom{a+n-1}{n} x^n$.

5. Let a_n be the number of words $w = w_1 w_2 \cdots w_n$ of length n with letters in $\{1,2,3\}$ such that $w_1 + \cdots + w_n$ is even. For example, $a_3 = 13$:

112, 121, 123, 132, 211, 213, 222, 231, 233, 312, 321, 323, 332

Find a recurrence for a_n, the generating function for a_n, and a formula for a_n.

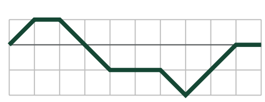
6. A Motzkin path of length n is a path in the plane which starts at $(0,0)$, ends at $(n,0)$, uses steps of the form $(1,1)$, $(1,-1)$, and $(1,0)$, and never travels below (but may touch) the x-axis. For example,



are the 9 Motzkin paths of length 4. Let m_n be the number of Motzkin paths of length n and let $M(x) = \sum_{n=0}^{\infty} m_n x^n$.

a. Show that $(M(x)-1)/x = M(x) + xM(x)^2$ and then find an explicit formula for $M(x)$.

b. Let a_n be the number of paths in the plane which start at $(0,0)$, end at $(0,n)$, and use steps of the form $(1,1)$, $(1,-1)$, and $(1,0)$. For example, one path when $n = 11$ is



By looking at the first time a path touches the x axis, show that $a_{n+2} = a_{n+1} + 2\sum_{k=0}^{n} m_k a_{n-k}$ for $n \geq 0$.

c. Show that $A(x) = \sum_{n=0}^{\infty} a_n x^n = 1/\sqrt{1-2x-3x^2}$.