# Discrete Mathematics Set 4

**Math 435**: Complete 7 parts of the following exercises.
**Math 530**: Exercises 1, 2, and any two of the remaining exercises.

**1.** Let $A_n$ be the set of paths in $\mathbb{R}^2$ which start at $(0,0)$, end at $(n,n)$, and only use steps of the form $(1,0)$ or $(0,1)$.

  a. Show that there are $\binom{2n}{n}$ elements in $A_n$.

  b. Denote the number of times the path $p \in A_n$ touches the line $y = x$ by touch$(p)$. Let

$$A(x,t) = \sum_{n=0}^{\infty} \left( \sum_{p \in A_n} t^{\text{touch}(p)} \right) x^n$$

  and let $c_n$ be the $n^{th}$ Catalan number. Show that

$$\sum_{p \in A_{n+1}} t^{\text{touch}(p)} = 2t \sum_{k=0}^{n} c_k \left( \sum_{p \in A_{n-k}} t^{\text{touch}(p)} \right).$$

  c. Show that $A(x,t) = \dfrac{t}{1 - t + 2t\sqrt{\frac{1}{4} - x}}$.

  d. Find an asymptotic formula for the average number of times a path in $A_n$ touches the line $y = x$.

**2.** Let $M(x) = \sum_{n=0}^{\infty} m_n x^n$ be the generating function for $m_n$, the number of Motzkin paths of length $n$, as defined in Set 1 Exercise 6.

  a. Find $\lim_{x \to 1/3} (1/3 - x)^{1/2} x M'(x)$ and use this result to find an asymptotic formula for $m_n$. (As always, calculations can be done using a computer algebra system if desired.)

  b. Let $a_n$ be as defined in Set 1 Exercise 6b. Find an asymptotic formula for $a_n$ and use it to find an asymptotic formula for the probability that a random path in the plane that from $(0,0)$ to $(n,n)$ using steps of the form $(1,1), (1,-1)$ and $(1,0)$ is a Motzkin path.

**3.** Find an asymptotic formula for the average number of sets in an ordered set partition of $n$ (see Set 2 Exercise 3, Set 3 Exercise 1b, and Set 3 Exercise 1d).

**4.** Find an asymptotic formula for the average number of cycles in a permutation of $n$ with ordered cycles (see Set 3 Exercise 1c and Set 3 Exercise 1d.)

**5.** Find an asymptotic formula for the number of permutations of $n$ that do not have any cycles of length $1, 2$ or $3$.

**6.** Define

  1. the Chebyshev polynomial of the first kind $T_n(y)$ by $\sum_{n=0}^{\infty} T_n(y) x^n = \dfrac{1 - yx}{1 - 2yx + x^2}$,

  2. the Chebyshev polynomial of the second kind $U_n(y)$ by $\sum_{n=0}^{\infty} U_n(y) x^n = \dfrac{1}{1 - 2yx + x^2}$, and

3. the Legendre polynomial $P_n(y)$ by $\sum_{n=0}^{\infty} P_n(y)x^n = \dfrac{1}{\sqrt{1 - 2yx + x^2}}$.

Find asymptotic formulas for $T_n(5/4)$, $U_n(5/4)$, and $P_n(5/4)$.