Representation theory example

There are 10 subsets of $\{1, \ldots, 5\}$ of size 2:

$$\begin{aligned} p_1 &= \{1, 2\} \\ p_2 &= \{1, 3\} \\ p_3 &= \{1, 4\} \\ p_4 &= \{1, 5\} \\ p_5 &= \{2, 3\} \\ p_6 &= \{2, 4\} \\ p_7 &= \{2, 5\} \\ p_8 &= \{3, 4\} \\ p_9 &= \{3, 5\} \\ p_{10} &= \{4, 5\} \end{aligned}$$

The symmetric group acts on this set: $(1\ 2\ 4)\ p_1 = p_6$.

Representation theory example

This gives a matrix representation:

$$X((1\ 2\ 4)) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Representation theory example

How does this representation break into irreducibles?

We first calculate the character values of χ^X counting fixed subsets:

	$C_{(5)}$	$C_{(4,1)}$	$C_{(3,2)}$	$C_{(3,1^2)}$	$C_{(2^2,1)}$	$C_{(2,1^3)}$	$C_{(1^5)}$
χ^X	0	0	1	1	2	4	10

Representation theory example

Then we take inner products using the character table for S_n, found using rim hook tableaux:

$$\begin{aligned}
\langle \chi^X, \chi^{(5)} \rangle &= \frac{1}{5!}\left(|C_{(5)}| \cdot \chi^X_{(5)} \cdot \chi^{(5)}_{(5)} + \cdots + |C_{(1^5)}| \cdot \chi^X_{(1^5)} \cdot \chi^{(5)}_{(1^5)} \right) = 1 \\
\langle \chi^X, \chi^{(4,1)} \rangle &= 1 \\
\langle \chi^X, \chi^{(3,2)} \rangle &= 1 \\
\langle \chi^X, \chi^{(3,1^2)} \rangle &= 0 \\
\langle \chi^X, \chi^{(2^2,1)} \rangle &= 0 \\
\langle \chi^X, \chi^{(2,1^3)} \rangle &= 0 \\
\langle \chi^X, \chi^{(1^5)} \rangle &= 0
\end{aligned}$$

This shows $\chi^X = \chi^{(5)} + \chi^{(4,1)} + \chi^{(3,2)}$.

Representation theory example

We can check that the degrees are correct in

$$\chi^X = \chi^{(5)} + \chi^{(4,1)} + \chi^{(3,2)}$$

because the hook length formula gives the degrees are 1, 4, and 5, which sum to 10.