Q. How to turn a generating function of $\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$ into a g.f of $\sum_{n=0}^{\infty} a_n x^n$?

A. Do the Laplace Transform

For $f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$, $\int_0^{\infty} f(x) e^{-sx}\, dx = \mathcal{L}\{f(x)\} = \sum_{n=0}^{\infty} a_n x^n$

Given $f(x) = x^n$, (1) $\mathcal{L}\{f(x)\} = \frac{n!}{s^{n+1}}$ by integration by parts

So $f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!}(x^n) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{n!}{s^{n+1}}\right) = \frac{1}{s}\sum_{n=0}^{\infty} a_n \left(\frac{1}{s}\right)^n$

Then let $x = \frac{1}{s}$ (2) and divide by x (3).

Ex. $\sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$. $\mathcal{L}\{e^x\} = \frac{1}{s-1}$. Then $\frac{1}{\frac{1}{x} - 1} = \frac{x}{1-x} \Rightarrow \frac{1}{x}\left(\frac{x}{1-x}\right) = \sum_{n=0}^{\infty} (1) x^n$

(1) (1) (2) (2) (3)

$= \frac{1}{1-x}$ ✓

Def The q-analogues of n, $n!$, $\binom{n}{k}$ for $n=0,1,2,\dots$ are

$$[n]_q = q^0 + q^1 + \dots + q^{n-1} = \frac{1-q^n}{1-q}$$

$$[n]_q! = [n]_q [n-1]_q (\dots) [1]_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q ! = \frac{[n]_q!}{[k]_q! \, [n-k]_q!}$$

Ex. $[5] = 1+q+q^2+q^3+q^4$

$[3]! = [3][2][1] = (1+q+q^2)(1+q)(1) = 1+2q+2q^2+q^3$

$[4]! = [4]\,[3]! = (1+q+q^2+q^3)(1+2q+2q^2+q^3) = 1+3q+5q^2+6q^3+5q^4+3q^5+q^6$

$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{[4]!}{[2]!\,[2]!} = \frac{1+3q+\dots+q^3}{(1+q)(1+q)} = 1+q+2q^2+q^3+q^4$

Thm. $[n]! = \sum_{\sigma \in S_n} q^{inv(\sigma)}$

Ex: S_3 =

(123)	0	$1q^0$
(132)	1	$\}2q$
(213)	1	
(231)	2	$\}2q^2$
(312)	2	
(321)	3	$1q^3$

Pf. By induction, $[n]! = [n][n-1]! = (1+q+\dots+q^{n-1}) \sum_{\sigma \in S_n} q^{inv(\sigma)}$
multiplying requires a choice of a power and a $\sigma \in S_n$.
Let's just pick one q^j and $\sigma \in S_{n-1}$. Place n in position j, reading backwards

$$\underset{7}{\_}\ 1\ \underset{6}{\_}\ 3\ \underset{5}{\_}\ 2\ \underset{4}{\_}\ 4\ \underset{3}{\_}\ 7\ \underset{2}{\_}\ 5\ \underset{1}{\_}\ 6\ \underset{0}{\_} \qquad j=3 \Rightarrow$$

So $q^j \cdot q^n = q^{n+j}$, so increments the inversions by j, which is correct

Thm. $[n]! = \sum_{\sigma \in S_n} q^{maj(\sigma)}$