

# Euler's pentagonal number thm.

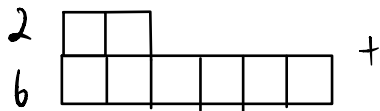
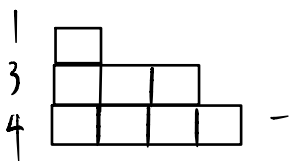
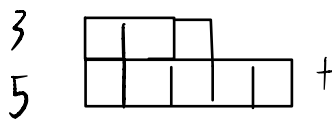
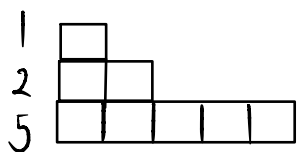
$$\prod_{i=1}^{\infty} (1 - z^i) = \sum_{k \in \mathbb{Z}} (-1)^k z^{k(3k-1)/2}$$

Pf.  $\prod_{i=1}^{\infty} (1 - z^i) = \sum_{\substack{\lambda \text{ w/} \\ \text{distinct} \\ \text{parts}}} (-1)^{\ell(\lambda)} z^{|\lambda|}$

Ex.  $n=7$

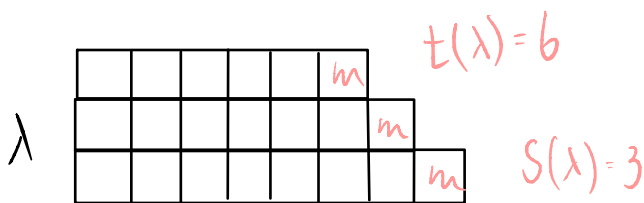
$$\begin{array}{lcl}
 7 & \begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array} & - & \begin{array}{l} 3 \\ 4 \end{array} \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \end{array} + \\
 \begin{array}{l} 1 \\ 2 \\ 4 \end{array} \begin{array}{|c|c|c|c|c|} \hline \square \\ \hline \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \end{array} & - & \begin{array}{l} 2 \\ 5 \end{array} \begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square \\ \hline \end{array} + \\
 & & & & & \begin{array}{l} 1 \\ 6 \end{array} \begin{array}{|c|c|c|c|c|c|c|} \hline \square \\ \hline \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array} +
 \end{array}$$

$$\begin{array}{lcl}
 n=8 & 8 & \begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array} - & \begin{array}{l} 1 \\ 7 \end{array} \begin{array}{|c|c|c|c|c|c|c|c|} \hline \square \\ \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array} +
 \end{array}$$

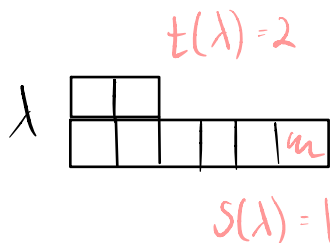
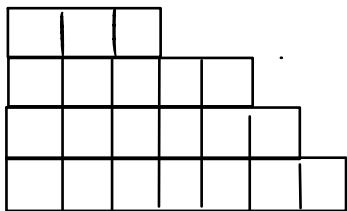


Let  $t(\lambda)$  = length of top row of  $\lambda$ .

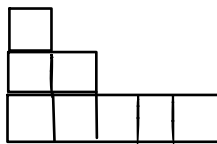
$s(\lambda)$  = length of "staircase" on right of diagram of  $\lambda$ .



$\updownarrow \psi$



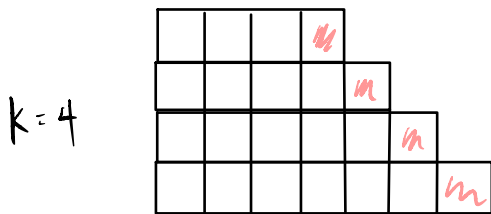
$\updownarrow \psi$



Define  $\varphi$  s.t.  $t(\lambda) \leq s(\lambda)$ . Let  $\varphi(\lambda)$  be  $\lambda$  w/ top row removed & place beside staircase. If  $s(\lambda) < t(\lambda)$ , remove staircase & put on top of  $\lambda$ .

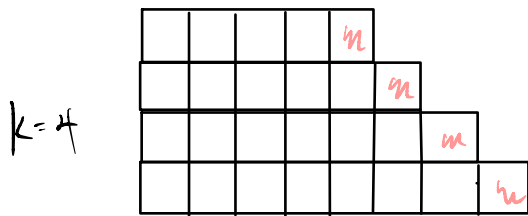
This will almost always pair + object w/- object.

Case 1:  $\ell(\lambda) = k$ ,  $t(\lambda) = k$ ,  $s(\lambda) = k$



$$k^2 + \binom{k}{2} = \frac{k(3k-1)}{2}$$

Case 2:  $\ell(\lambda) = k$ ,  $t(\lambda) = k+1$ ,  $s(\lambda) = k$



$$k(k+1) + \binom{k}{2} = \frac{(-k)(3(-k)-1)}{2}$$

