

Linear Analysis II Exercise Set 13

1. Show that the Fourier transform of the second derivative $F[f''(t)]$ is equal to $-\omega^2 F[f(t)]$.

2. a. Show that the Fourier Transform of $e^{-|t|}$ is $\frac{1}{\pi(\omega^2 + 1)}$.

b. Using the inverse Fourier Transform and part a., find the Fourier transform of $\frac{1}{t^2 + 1}$.

3. The one dimensional wave equation is the partial differential equation $u_{tt}(t, x) = k^2 u_{xx}(t, x)$ where k is a real number and $u(t, x)$ is a function of time t and one spacial dimension x . The wave equation models the displacement of a vibrating string at time t and location x .

By taking partial derivatives and plugging in the function

$$u(t, x) = \frac{f(x + kt) + f(x - kt)}{2} + \frac{1}{2k} \int_{x-kt}^{x+kt} g(s) ds$$

into the partial differential equation, show that the above function is a solution to the wave equation

$$\begin{cases} u_{tt}(t, x) = k^2 u_{xx}(t, x) \\ u(0, x) = f(x), u_t(0, x) = g(x). \end{cases}$$

4. Consider the wave equation

$$\begin{cases} u_{tt}(t, x) = k^2 u_{xx}(t, x), \\ u(0, x) = 1/(1 + x^2), \\ u_t(0, x) = 0. \end{cases}$$

a. Take the Fourier transform of the system with respect to the variable x and then solve the resulting differential equation in the variable t to find $F[u(t, x)]$.

b. The inverse Fourier transform gives $u(t, x) = \int_{-\infty}^{\infty} F[u(t, x)] e^{i\omega x} d\omega$. Evaluate this to find $u(t, x)$.

Hint: This can be done by writing the integral as $\int_{-\infty}^{\infty} = \int_{-\infty}^0 + \int_0^{\infty}$ and then relating each of the two integrals to the Laplace transform with respect to the variable ω of the function $e^{-\omega} \cos(kt\omega)$. It is acceptable to leave the final answer involving the complex unit i , but the clever student may be able to find an answer that does not involve i .