# Discrete Mathematics Set 1

**Math 435**: Complete 6 parts of the following exercises. (For example, one option is {1, 2, 3a, 4, 6a, 6c}.)
**Math 530**: Complete exercises 1, 3, 5 and 6.

**1.** Using the Taylor series centered at $x = 0$, show that $(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n$ where $\binom{a}{n} = \dfrac{a(a-1)\cdots(a-n+1)}{n!}$.

**2.** Prove that $\dfrac{1}{(1-x)^a} = \sum_{n=0}^{\infty} \binom{a+n-1}{n} x^n$.

**3.** Verify the following identities involving the products of series:

a. $\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)x^n$

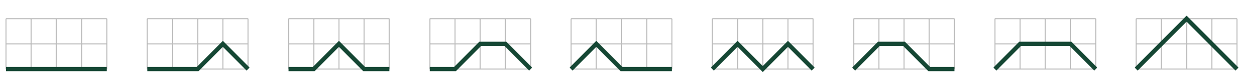
b. $\left(\sum_{n=0}^{\infty} a_n x^n\right)^k = \sum_{n=0}^{\infty}\left(\sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k=n}} a_{i_1}\cdots a_{i_k}\right)x^n.$

**4.** By multiplying $(1+x)^a$ and $(1+x)^b$, prove that $\binom{a+b}{n} = \sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k}$ holds for all $a, b \in \mathbb{C}$.

**5.** Let $a_n$ be the number of ways to tile a $2 \times n$ chessboard with dominoes of sizes $2 \times 1$ and $2 \times 2$. For example, there are 11 such tilings when $n = 4$:

Find a recurrence for $a_n$, the generating function for $a_n$, and a formula for $a_n$.

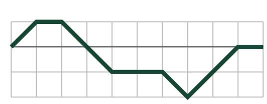
**6.** A Motzkin path of length $n$ is a path in the plane which starts at $(0,0)$, ends at $(n,0)$, uses steps of the form $(1,1)$, $(1,-1)$, and $(1,0)$, and never travels below (but may touch) the $x$-axis. For example,



are the 9 Motzkin paths of length 4. Let $m_n$ be the number of Motzkin paths of length $n$ and let $M(x) = \sum_{n=0}^{\infty} m_n x^n$.

a. Show that $(M(x)-1)/x = M(x) + xM(x)^2$ and then find an explicit formula for $M(x)$.

b. Let $a_n$ be the number of paths in the plane which start at $(0,0)$, end at $(0,n)$, and use steps of the form $(1,1)$, $(1,-1)$, and $(1,0)$. For example, one path when $n = 11$ is



By looking at the first time a path touches the $x$ axis, show that $a_{n+2} = a_{n+1} + 2\sum_{k=0}^{n} m_k a_{n-k}$ for $n \geq 0$.

c. Show that $A(x) = 1/\sqrt{1-2x-3x^2}$.

# Discrete Mathematics Set 2

**Math 435**: Complete 8 parts of the following exercises.
**Math 530**: Complete exercises 8 and 9.

**7.** Let $b_{n,k}$ be the coefficient of $e^{e^x}e^{kx}$ in $\dfrac{d^n}{dx^n}\left(e^{e^x}\right)$. Prove that $b_{n,k}$ is the number of set partitions of $n$ into $k$ sets.

**8.** Let $b_{n,k}$ be the number of set partitions of $n$ into $k$ sets and $T_n(y) = \sum_{k=0}^{n} b_{n,k}y^k$. We have $\sum_{n=0}^{\infty} T_n(y)\dfrac{x^n}{n!} = e^{y(e^x-1)}$.

a. Show that $T_n(a+b) = \sum_{k=0}^{n} \binom{n}{k} T_k(a)T_{n-k}(b)$.

b. Show that $\sum_{n=0}^{\infty} T_{n+1}(y)\dfrac{x^n}{n!} = e^{y(e^x-1)}ye^x$ and use this to show $T_{n+1}(y) = y\sum_{k=0}^{n}\binom{n}{k}T_k(y)$.

**9.** An ordered set partition of $n$ is an ordered list of disjoint nonempty sets with union $\{1,\ldots,n\}$. For example, there are 13 ordered set partitions of 3:

$$
\begin{gathered}
(\{1,2,3\}), \\
(\{1\},\{2,3\}), \quad (\{2,3\},\{1\}), \quad (\{2\},\{1,3\}), \quad (\{1,3\},\{2\}), \quad (\{3\},\{1,2\}), \quad (\{1,2\},\{3\}), \\
(\{1\},\{2\},\{3\}), \quad (\{1\},\{3\},\{2\}), \quad (\{2\},\{1\},\{3\}), \quad (\{2\},\{3\},\{1\}), \quad (\{3\},\{1\},\{2\}), \quad (\{3\},\{2\},\{1\}).
\end{gathered}
$$

Let $a_n$ be the number of ordered set partitions of $n$ and let $A(x) = \sum_{n=0}^{\infty} a_n\dfrac{x^n}{n!}$.

a. Show that $a_0 = 1$ and $a_n = \sum_{k=1}^{n}\binom{n}{k}a_{n-k}$ for $n \geq 1$.

b. Show that $A(x) = 1/(2-e^x)$.

c. Expand $A(x)$ as a geometric series to show that $a_n = \dfrac{1}{2}\sum_{k=0}^{\infty}\dfrac{k^n}{2^k}$.

d. Let $a_{n,k}$ be the number of ordered set partitions of $n$ into exactly $k$ sets. Show that $a_{n+1,k} = ka_{n,k} + ka_{n,k-1}$.

e. Let $A(x,y) = \sum_{n=0}^{\infty}\left(\sum_{k=1}^{n} a_{n,k}y^k\right)\dfrac{x^n}{n!}$. Show that $A(x,y)$ satisfies $A_x = yA + (y+y^2)A_y$ and $A(x,1) = 1/(2-e^x)$.

f. Verify that $A(x,y) = 1/(1-y(e^x-1))$.

g. Let $t_n$ be the total number of sets in all ordered set partitions of $n$. Find a generating function for $\sum_{n=0}^{\infty} t_n\dfrac{x^n}{n!}$.

h. (Optional!) Show the change of variables $w(x,y) = (1+1/y)e^{-x}$ and $z(x,y) = y$ can solve the PDE in part e.

# Discrete Mathematics Set 3

**Math 435**: Complete 6 parts of the following exercises.
**Math 530**: Complete exercises 12 and 13.

**10.** Use the exponential formula to find a generating function for the number of permutations of $n$ that do not have any cycles of size 1 (such a permutation is called a derangement). Use this generating function to find an explicit formula for the number of such permutations of $n$.

**11.** Find a generating function for the number of set partitions of $n$ which have an even total number of sets, all of which are an even size. Write the answer in terms of the function $\cosh x = \sum_{n=0}^{\infty} x^{2n}/(2n)!$.

**12.** The generating function for the number of permutations of $n$ with only even sized cycles is

$$\sqrt{\frac{1}{1-x^2}}. \tag{1}$$

(The number of such permutations is $1^2 \cdot 3^2 \cdot 5^2 \cdots (n-1)^2$ if $n$ is even and 0 if $n$ is odd.)

a. Use the exponential formula to prove that

$$\sum_{n=0}^{\infty} (\text{the number of permutations of } n \text{ with only odd sized cycles}) \frac{x^n}{n!} = (1+x)\sqrt{\frac{1}{1-x^2}}. \tag{2}$$

b. The coefficients of $x^2$ in (1) and (2) are the same. Therefore the number of permutations of $2n$ with only even sized cycles is equal to the number of permutations of $2n$ with only odd sized cycles. Find a bijection between these two sets of permutations.

**13.** Let $L_n$ be the set of ordered lists of the form $(C_1, \ldots, C_m)$ where $C_1, \ldots, C_m$ are cards containing disjoint sets with union $\{1, \ldots, n\}$. This is similar to hands in the exponential formula with the difference being that hands are unordered and lists are ordered.

a. Let $C(x) = \sum_{n=1}^{\infty} |C_n| \frac{x^n}{n!}$. Show that $\sum_{n=0}^{\infty} \left( \sum_{\ell \in L_n} y^{(\text{number of cards in } \ell)} \right) \frac{x^n}{n!} = \frac{1}{1-yC(x)}$.

b. Use part a. of this exercise to find the result in part f. of Exercise 9 in Set 2.

c. A permutation of $n$ with ordered cycles is a list $(\sigma_1, \ldots, \sigma_m)$ where $\sigma_1, \ldots, \sigma_m$ are the cycles in a permutation of $n$. Let $\mathcal{A}_n$ be the set of permutations of $n$ with ordered cycles and find

$$\sum_{n=0}^{\infty} \left( \sum_{\ell \in \mathcal{A}_n} y^{(\text{number of cycles in } \ell)} \right) \frac{x^n}{n!}.$$

d. Let $t_n$ be the total number of cards in all elements in $L_n$. Find a generating function involving $C(x)$ for $\sum_{n=0}^{\infty} t_n \frac{x^n}{n!}$.

# Discrete Mathematics Set 4

**Math 435**: Complete 7 parts of the following exercises.
**Math 530**: Complete exercises 16, 17 and 18.

**14.** Show $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$. (This was used when introducing the Gamma function and Stirling's approximation.)

**15.** Let $a_n, b_n, c_n$ and $d_n$ be sequences of real numbers.

a. Show that $\lim_{n\to\infty} |a_n - b_n| = 0$ does not imply $a_n \sim b_n$.

b. Show that $a_n \sim b_n$ and $c_n \sim d_n$ does not imply $a_n + c_n \sim b_n + d_n$.

**16.** Let $\alpha > 0$ and $n$ be a nonnegative integer.

a. Use induction to show that $\int_0^1 x^{\alpha-1}(1-x)^n\,dx = \frac{n!}{\alpha(\alpha+1)\cdots(\alpha+n)}$.

b. Assuming that the limit and integral can be interchanged, use $\lim\limits_{n\to\infty} \int_0^n x^{\alpha-1}\left(1-\frac{x}{n}\right)^n dx$ to show that

$$\Gamma(\alpha) = \lim_{n\to\infty} \frac{n!n^\alpha}{\alpha(\alpha+1)\cdots(\alpha+n)}.$$

c. (Optional!) Justify why the limit and integral can be interchanged in part b.

**17.** Let $a_n$ be the number of ordered set partitions of $n$. Set 2 Exercise 9c gives $a_n = \frac{1}{2}\sum\limits_{k=0}^{\infty} k^n 2^{-k} \approx \frac{1}{2}\int_0^\infty x^n 2^{-x}\,dx$.

Use a substitution in the above integral and then use Stirling's approximation to find $a_n \approx \frac{\sqrt{2\pi n}}{\ln 4}\left(\frac{n}{e\ln 2}\right)^n$.

**18.** Let $A_n$ be the set of paths in $\mathbb{R}^2$ which start at $(0,0)$, end at $(n,n)$, and only use steps of the form $(1,0)$ or $(0,1)$. Denote the number of times the path $p \in A_n$ touches the line $y = x$ by $\mathrm{touch}(p)$. Let

$$A(x,t) = \sum_{n=0}^{\infty}\left(\sum_{p\in A_n} t^{\mathrm{touch}(p)}\right)x^n.$$

a. Let $c_n$ be the $n^{th}$ Catalan number. Show that

$$\sum_{p\in A_{n+1}} t^{\mathrm{touch}(p)} = 2t\sum_{k=0}^{n} c_k\left(\sum_{p\in A_{n-k}} t^{\mathrm{touch}(p)}\right).$$

b. Show that $A(x,t) = \frac{t}{1-t+2t\sqrt{\frac{1}{4}-x}}$.

c. With the help of the corollary to the first asymptotic result in video 17, find an asymptotic formula for the average number of times a path in $A_n$ touches the line $y = x$.

# Discrete Mathematics Set 5

**Math 435**: Complete 6 parts of the following exercises.
**Math 530**: Complete exercises 21—24.

**19.** Let $a_n$ be as defined in Set 1 Exercise 6b. Find an asymptotic formula for $a_n$ using the result in Set 1 Exercise 6c.

**20.** Let $a_n$ be the number of ordered set partitions of $n$.

a. Use the generating function in Set 2 Exercise 9 to find an asymptotic formula for $a_n$ given in Set 4 Exercise 17.

b. Using the ideas in Set 3 Exercise 13d, show that the average number of sets in an ordered set partition of $n$ is approximately $n/\ln 4$.

**21.** Let $a_n$ be the number of permutations of $n$ with ordered cycles.

a. Use the generating function in Set 3 Exercise 13c to find an asymptotic formula for $a_n$.

b. Find an asymptotic formula for the average number of cycles in a permutation of $n$ with ordered cycles.

**22.** Find an asymptotic formula for the probability that a permutation of $n$ does not have a cycle of length $1, 2$ or $3$.

**23.** We define

1. the Chebyshev polynomial of the first kind $T_n(y)$ by $\sum_{n=0}^{\infty} T_n(y)x^n = \dfrac{1-yx}{1-2yx+x^2}$,
2. the Chebyshev polynomial of the second kind $U_n(y)$ by $\sum_{n=0}^{\infty} U_n(y)x^n = \dfrac{1}{1-2yx+x^2}$, and
3. the Legendre polynomial $P_n(y)$ by $\sum_{n=0}^{\infty} P_n(y)x^n = \dfrac{1}{\sqrt{1-2yx+x^2}}$.

Find asymptotic formulas for $T_n(5/4)$, $U_n(5/4)$, and $P_n(5/4)$.

**24.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a complex valued function with nonnegative real coefficients $a_n \geq 0$. Suppose that a singularity of $f$ with smallest complex magnitude has magnitude $R$ (this means that $R$ is the radius of convergence of $f(z)$ and that the series $f(z_0)$ diverges for all $z_0$ with $|z_0| > R$). This exercise will show that $R$ is a singularity of $f$.

a. Show that $f(z) = \sum_{k=0}^{\infty} \left( \sum_{n=k}^{\infty} \binom{n}{k} a_n (R/2)^{n-k} \right) (z - R/2)^k$ in some neighborhood of $R/2$.

b. Looking for a contradiction, assume that $R$ is not a singularity of $f$. This means that there is an $\varepsilon > 0$ such that the above expression is valid for $R+\varepsilon$. Take $R+\varepsilon$ in the above expression and prove that

$$f(R+\varepsilon) = \sum_{n=0}^{\infty} a_n (R+\varepsilon)^n.$$

Why is this a contradiction? Where was the hypothesis that $a_n \geq 0$ used?

# Discrete Mathematics Set 6

**Math 435**: Complete 6 of the following exercises.
**Math 530**: Complete exercises 32 and 33 and 4 of the remaining exercises.

**25.** Find the average value of the descent statistic over all permutations of $n$.

**26.** Find the average value of the inversion statistic over all permutations of $n$.

**27.** Find $\lim_{q\to 1} \frac{d}{dq}[n]_q!$.

**28.** Find $\lim_{q\to -1} \frac{d}{dq}[n]_q!$.

**29.** Let $a_{n,k}$ be the number of permutations in $S_n$ with exactly $k$ descents. By inserting "$n+1$" into a permutation of $n$, show $a_{n+1,k} = (k+1)a_{n,k} + (n+1-k)a_{n,k-1}$.

**30.** Let $\varphi : S_n \to S_n$ be a bijection such that $\mathrm{des}(\sigma) = \mathrm{exc}(\varphi(\sigma))$ for all $\sigma \in S_n$. Write a computer program in Python or Mathematica (if you would like to use another programming language for any of the programming exercises, just ask me!) that inputs $\sigma$ in one line notation and outputs $\varphi(\sigma)$ in one line notation.

**31.** Let $\varphi : S_n \to S_n$ be a bijection such that $\mathrm{inv}(\sigma) = \mathrm{maj}(\varphi(\sigma))$ for all $\sigma \in S_n$. Write a computer program in Python or Mathematica that inputs $\sigma$ in one line notation and outputs $\varphi(\sigma)$ in one line notation.

**32.** Suppose that in one line notation, the permutation $\sigma \in S_n$ has $\sigma_i$ in position $i$. Then the inverse permutation $\sigma^{-1}$ written in one line notation has $i$ in position $\sigma_i$. Show that $\mathrm{inv}(\sigma) = \mathrm{inv}(\sigma^{-1})$ for all $\sigma \in S_n$.

**33.** Using terminology from Abstract Algebra, prove that the sign of $\sigma$ (as found by writing $\sigma$ as a product of transpositions) is equal to $(-1)^{\mathrm{inv}(\sigma)}$.

# Discrete Mathematics Set 7

**Math 435**: Complete 6 parts of the following exercises.
**Math 530**: Complete exercise 38 and 3 parts of the remaining exercises.

**34.** Find the average number of inversions in $R(0^k, 1^{n-k})$ and use the result to simplify $\lim_{q\to 1} \frac{d}{dq} \begin{bmatrix} n \\ k \end{bmatrix}_q$.

**35.** The $q$-multinomial coefficient $\begin{bmatrix} n \\ k_1, \dots, k_\ell \end{bmatrix}_q$ is defined to be $\frac{[n]_q!}{[k_1]_q! \cdots [k_\ell]_q!}$ for $n = k_1 + \cdots + k_\ell$. Show that

$$\begin{bmatrix} n \\ k_1, \dots, k_\ell \end{bmatrix}_q = \sum_{r \in R(1^{k_1}, \dots, \ell^{k_\ell})} q^{\operatorname{inv}(r)}$$

where $R(1^{k_1}, \dots, \ell^{k_\ell})$ denotes the set of rearrangements of $k_1$ 1's, $k_2$ 2's, etc.

**36.** Let $\varphi$ be the bijection in the proof of $\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{r \in R(0^k, 1^{n-k})} q^{\mathsf{maj}(r)}$ found on the next page.

a. Find both $\varphi(110111011001)$ and $\varphi^{-1}(110111011001)$.

b. Write a computer program in Python or Mathematica that inputs $r$ and outputs $\varphi(r)$.

**37.** Prove these identities without writing $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ as a fraction and manipulating powers of $q$. Instead, interpret both sides of the identity as rearrangements or integer partitions and show the result by double counting or a bijection.

a. (The $q$-Pascal identity) $\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$.

b. (The $q$-Vandermonde identity) $\begin{bmatrix} a+b \\ n \end{bmatrix}_q = \sum_{k=0}^{n} q^{(a-k)(n-k)} \begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} b \\ n-k \end{bmatrix}_q$.

c. (The $q$-binomial theorem) $(1 + xq^0) \cdots (1 + xq^{n-1}) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k$.

**38.** Let $q$ a prime power, $\mathbb{F}_q$ be the finite field with $q$ elements, and $\mathbb{F}_q^n$ be the $n$-dimensional vector space over $\mathbb{F}_q$.

a. Prove that the number of $k$ dimensional subspaces in $\mathbb{F}_q^n$ is equal to $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

b. Let $X$ be a vector space with a finite number of elements $x$. Show that there are

$$\begin{bmatrix} n \\ n-k \end{bmatrix}_q (x - q^0) \cdots (x - q^{k-1})$$

linear maps $L : \mathbb{F}_q^n \to X$ which have a null space of dimension $n - k$.

c. By counting linear maps $L : \mathbb{F}_q^n \to X$, prove the $q$-Cauchy identity:

$$x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (x - q^0) \cdots (x - q^{k-1}).$$

d. The identity in part c. has been shown true for prime powers $q$. How can we conclude that this identity is true for any complex number $q$?

**Theorem 1.** *If $0 \le k \le n$, then* $\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{r \in R(0^k, 1^{n-k})} q^{maj(r)}$.

*Proof.* We prove this result by defining a bijection $\varphi : R(0^k, 1^{n-k}) \to R(0^k, 1^{n-k})$ such that $\mathrm{maj}(r) = \mathrm{inv}(\varphi(r))$ for all $r \in R(0^k, 1^{n-k})$. This is enough because we have already shown that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{r \in R(0^k, 1^{n-k})} q^{\mathrm{inv}(r)}.$$

We first define an auxiliary bijection $\Gamma : R(0^k, 1^{n-k}) \to R(0^k, 1^{n-k})$. If $r$ ends with a 0, define $\Gamma(r)$ to be $r$ with every consecutive substring of the form $11 \cdots 110$ changed to $011 \cdots 11$. If $r$ ends with a 1, define $\Gamma(r)$ to be $r$ with every consecutive substring of the form $00 \cdots 001$ changed to $100 \cdots 00$. For example,

$$\Gamma(1100011110100) = 0110001111010.$$

If $r$ ends with a 0, then $\mathrm{inv}(\Gamma(r)) = \mathrm{inv}(r) - (n-k)$ because changing $11 \cdots 110$ into $011 \cdots 11$ for all 1's in $r$ decreases the number of inversions in $r$ by 1 for each of the $n-k$ 1's in $r$. Similarly, if $r$ ends with a 1, then $\mathrm{inv}(\Gamma(r)) = \mathrm{inv}(r) + k$.

We can now define our main bijection $\varphi$. If $r$ contains no 0's, then we define $\varphi(r) = r$. Otherwise, let $w$ be $r$ with the last 0 and all trailing 1's deleted so that $r$ can be written as $w011 \cdots 11$. For any rearrangement $r \in R(0^k, 1^{n-k})$, define $\varphi(r)$ recursively by $\varphi(r) = \Gamma(\varphi(w))011 \cdots 11$. For example, it can be checked that

$$\varphi(10110100011) = 00111010011.$$

By definition, $\varphi(r)$ ends with a 0 if and only if $r$ ends with a 0.

The fact that $\varphi$ is a bijection follows from the fact that $\Gamma$ is a bijection. To complete the proof, we will show that $\mathrm{maj}(r) = \mathrm{inv}(\varphi(r))$ by induction on the length of $r$. Suppose we add a 0 to the end of $r \in R(0^k, 1^{n-k})$. Then we have

$$\begin{aligned} \mathrm{inv}(\varphi(r0)) &= \mathrm{inv}\left(\Gamma(\varphi(r))0\right) \\ &= \mathrm{inv}\left(\Gamma(\varphi(r))\right) + (n-k) \\ &= \begin{cases} \mathrm{inv}\left(\varphi(r)\right) - (n-k) + (n-k) & \text{if } \varphi(r) \text{ ends in 0,} \\ \mathrm{inv}\left(\varphi(r)\right) + k + (n-k) & \text{if } \varphi(r) \text{ ends in 1.} \end{cases} \end{aligned}$$

Using the induction hypothesis and the fact that $\varphi(r)$ ends in a 0 if and only if $r$ does, this is equal to

$$\begin{cases} \mathrm{maj}(r) & \text{if } r \text{ ends in 0,} \\ \mathrm{maj}(r) + n & \text{if } r \text{ ends in 1.} \end{cases}$$

In both cases, this is equal to $\mathrm{maj}(r0)$. We have shown that $\mathrm{inv}(\varphi(r0)) = \mathrm{maj}(r0)$.

Now suppose we add a 1 onto the end of $r$. Since $\varphi(r1) = \Gamma(\varphi(w))01 \cdots 11 = \varphi(r)1$, we have

$$\mathrm{inv}(\varphi(r1)) = \mathrm{inv}(\varphi(r)1) = \mathrm{inv}(\varphi(r)) = \mathrm{maj}(r) = \mathrm{maj}(r1).$$

This completes the proof. $\square$

# Discrete Mathematics Set 8

**Math 435**: Complete 6 parts of the following exercises.
**Math 530**: Complete exercise 41 and 3 parts of the remaining exercises.

**39.** Prove the following identities by either proving that they have the same generating functions or by proving them with a bijection.

a. Show that the number of integer partitions of $n$ with no part divisible by $d$ is equal to the number of integer partitions of $n$ with no part repeated $d$ or more times.

b. Prove that the number of integer integer partitions of $n$ with both odd and distinct parts is equal to the number of integer integer partitions of $n$ that are equal to their conjugate.

c. Prove that the number of integer partitions of $n$ in which each part appears exactly 2, 3, or 5 times equals the number integer partitions of $n$ into parts which are congruent to 2, 3, 6, 9, or 10 modulo 12.

d. Show that the number of integer partitions of $n$ in which no part appears exactly once is equal to the number of integer partitions of $n$ with no part equal to 1 and where consecutive integers do not both appear as parts.

e. Show that the number of integer partitions of $n$ in which no part appears exactly once is equal to the number of integer partitions of $n$ where no part is congruent to 1 or 5 modulo 6.

**40.** Prove that $\sum_{n=1}^{\infty} x^n y^n z(1+zx^1)\cdots(1+zx^{n-1}) = \sum_{n=1}^{\infty} \dfrac{x^{\binom{n+1}{2}} y^n z^n}{(1-yx^1)\cdots(1-yx^n)}$.

**41.** Let $p_k(n)$ be the number of integer partitions with $\ell(\lambda)=k$.

a. Show there are $\binom{n-1}{k-1}$ solutions to $x_1+\cdots+x_k=n$ where $x_1,\ldots,x_k$ are positive integers. (One way is to use a “balls and bars” or “stars and bars” argument from an introductory combinatorics course.)

b. By considering rearrangements of the parts of partitions, show that $\binom{n-1}{k-1} \le k!p_k(n)$.

c. By making the parts of a partition distinct, show that $k!p_k(n) \le \binom{n+\binom{k}{2}-1}{k-1}$.

d. Show that $\binom{n+a-1}{k-1} \sim \dfrac{n^{k-1}}{(k-1)!}$ for any nonnegative integer $a$ and then show that $p_k(n) \sim \dfrac{n^{k-1}}{k!(k-1)!}$.

# Discrete Mathematics Set 9

**Math 435**: Complete 6 parts of the following exercises.
**Math 530**: Complete exercises 43, 44, 46a and ((46b, 46c, and 46d) or (46e, 46f, and 46g)).

**42.** Recover Euler's pentagonal number theorem by taking $y = -q$ and $x = q^3$ in Jacobi's triple product

$$(1+y)\prod_{n=1}^{\infty}(1-x^n)(1+yx^n)(1+y^{-1}x^n) = \sum_{k\in\mathbb{Z}} y^k x^{k(k-1)/2}.$$

**43.** Use the function $x^2 \prod_{n=1}^{\infty}(1-x^n)^6$ to show that the number of integer partitions of $7n+5$ is divisible by 7.

**44.** This exercise proves a finite version of Jacobi's triple product identity.

a. Prove $\prod_{i=1}^{n}(1+yq^{i-1})(1+y^{-1}q^i) = \sum_{k=-n}^{n} y^k q^{k(k-1)/2}\begin{bmatrix}2n\\n+k\end{bmatrix}_q$.

b. Take $\lim_{n\to\infty}$ of the above expression to find the full Jacobi triple product.

**45.** Suppose $F(x,y)$ is an infinite product in two indeterminates $y$ and $x$. Euler's device refers to this process of turning $F(x,y)$ into a sum:

1. Find an equation providing a relationship between $F(x,xy)$ and $F(x,y)$.
2. Assume that $F(x,y) = \sum_{n=0}^{\infty} a_n(x)y^n$ and plug this into the equation found in step 1.
3. Compare coefficients of $y^n$ to find a recursion for $a_n(x)$.
4. Iterate this recursion to find a formula for $a_n(x)$.

Show $F(x,y) = \prod_{i=0}^{\infty}\frac{1}{1-yx^i}$ satisfies $F(x,xy) = (1-y)F(x,y)$ and then express $F(x,y)$ as a sum using Euler's device.

**46.** Let $F(x,y)$ be the function that satisfies the recursion $F(x,y) = F(x,xy) + xyF(x,x^2y)$ and $F(x,0) = 1$.

a. Use Euler's device to show $F(x,y) = \sum_{n=0}^{\infty}\frac{y^n x^{n^2}}{(1-x)\cdots(1-x^n)}$.

b. Use the result in the theorem on the next page to show $F(q,1) = \left(\sum_{k\in\mathbb{Z}}(-1)^n q^{n(5n-1)/2}\right)\prod_{i=1}^{\infty}\frac{1}{1-q^i}$.

c. Take $y = -q^2$ and $x = q^5$ in Jacobi's triple product to show $F(q,1) = \prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$.

d. Show that (the number of integer partitions of $n$ with parts differing by at least 2) is equal to (the number of integer partitions of $n$ with parts congruent to $\pm 1 \bmod 5$). (Hint: $1+3+5+\cdots+(2n-1) = n^2$.)

e. Use the result in the theorem on the next page to show $F(q,q) = \left(\sum_{k\in\mathbb{Z}}(-1)^n q^{n(5n+3)/2}\right)\prod_{i=1}^{\infty}\frac{1}{1-q^i}$.

f. Take $y = -q^4$ and $x = q^5$ in Jacobi's triple product to show $F(q,q) = \prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}$.

g. Show that (the number of integer partitions of $n$ with parts differing by at least 2 and no part of size 1) is equal to (the number of integer partitions of $n$ with parts congruent to $\pm 2 \bmod 5$). (Hint: $2+\cdots+2n = n^2+n$.)

**Theorem.** *The function*

$$F(x,y)=\left(1+\sum_{n=1}^{\infty}(-1)^n y^{2n}x^{n(5n-1)/2}\left(1-yx^{2n}\right)\frac{(1-yx)\cdots(1-yx^{n-1})}{(1-x)\cdots(1-x^n)}\right)\prod_{i=1}^{\infty}\frac{1}{1-yx^i} \tag{3}$$

*satisfies $F(x,y)=F(x,xy)+xyF(x,x^2y)$ and $F(x,0)=1$.*

*Proof.* Using $\left(1-yx^{2n}\right)=(1-x^n)+x^n\left(1-yx^n\right)$, we have that

$$\begin{aligned}F(x,y)\prod_{i=1}^{\infty}\left(1-yx^i\right)&=1+\sum_{n=1}^{\infty}(-1)^n y^{2n}x^{n(5n-1)/2}\frac{(1-yx)\cdots(1-yx^{n-1})}{(1-x)\cdots(1-x^{n-1})}\\&\quad+\sum_{n=1}^{\infty}(-1)^n y^{2n}x^{n(5n+1)/2+n}\frac{(1-yx)\cdots(1-yx^{n})}{(1-x)\cdots(1-x^{n})}\\&=\sum_{n=0}^{\infty}(-1)^n y^{2n}x^{n(5n+1)/2}\left(1-y^2x^{4n+2}\right)\frac{(1-yx)\cdots(1-yx^{n})}{(1-x)\cdots(1-x^{n})}.\end{aligned} \tag{4}$$

where we changed the first line into the second by reindexing the first infinite sum. Therefore, using (4) to simplify the first product and (3) to simplify the second product,

$$\left(F(x,y)-F(x,xy)\right)\prod_{i=1}^{\infty}\left(1-yx^i\right)=\sum_{n=0}^{\infty}(-1)^n y^{2n}x^{n(5n+1)/2}\frac{(1-yx)\cdots(1-yx^{n})}{(1-x)\cdots(1-x^{n})}\left(1-y^2x^{4n+2}-x^n\left(1-yx^{2n+1}\right)\right).$$

Using $\left(1-y^2x^{4n+2}-x^n\left(1-yx^{2n+1}\right)\right)=(1-x^n)+yx^{3n+1}\left(1-yx^{n+1}\right)$, the above expression is equal to

$$\sum_{n=1}^{\infty}(-1)^n y^{2n}x^{(5n+1)/2}\frac{(1-yx)\cdots(1-yx^{n})}{(1-x)\cdots(1-x^{n-1})}+yx\sum_{n=0}^{\infty}(-1)^n y^{2n}x^{n(5n+7)/2}\frac{(1-yx)\cdots(1-yx^{n+1})}{(1-x)\cdots(1-x^{n})},$$

which in turn, by reindeying the first sum, is equal to

$$yx\sum_{n=0}(-1)^n y^{2n}x^{n(5n+7)/2}\left(1-yx^{2n+2}\right)\frac{(1-yx)\cdots(1-yx^{n+1})}{(1-x)\cdots(1-x^{n})}=yx\left(\prod_{i=1}^{\infty}\left(1-yx^i\right)\right)F(x,x^2y).$$

The last step used (3) again. Thus we have proved

$$\left(F(x,y)-F(x,xy)\right)\prod_{i=1}^{\infty}\left(1-yx^i\right)=xyF(x,x^2y)\prod_{i=1}^{\infty}\left(1-yx^i\right),$$

which implies the desired result. $\square$

# Discrete Mathematics Set 10

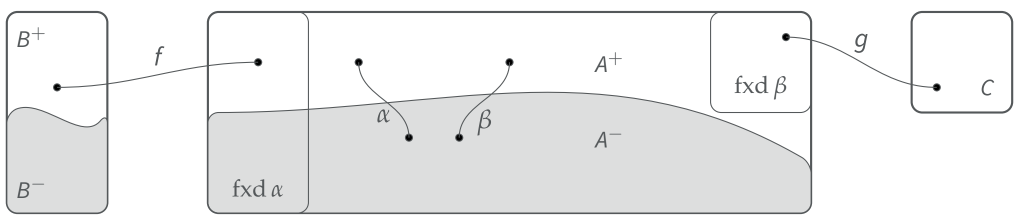
**Math 435**: Complete either exercises 47–50 or exercise 51.
**Math 530**: Complete either exercises 47–50 or exercise 51.

**47.** Provide a bijection to prove that (the number of partitions of $n$ in which only odd parts may be repeated) is equal to (the number of partitions of $n$ in which no part appears more than 3 times). Give an explicit, nontrivial example of your bijection.

**48.** Show that the number of partitions of $n$ in which 3 consecutive parts may not repeated equals the number of partitions of $n$ in which 3 consecutive even parts do not appear.

**49.** Suppose $A$, $B$ and $C$ are finite sets such that

1. $A$ is the disjoint union of two sets $A^+$ and $A^-$,
2. $B$ is the disjoint union of two sets $B^+$ and $B^-$,
3. there is an involution $\alpha : A \to A$ such that $\alpha(A^+ \setminus \operatorname{fxd} \alpha) \subseteq A^-$,
4. there is a bijection $f : \operatorname{fxd} \alpha \to B$ such that $f(\operatorname{fxd} \alpha \cap A^+) = B^+$ and $f(\operatorname{fxd} \alpha \cap A^-) = B^-$,
5. there is an involution $\beta : A \to A$ such that $\operatorname{fxd} \beta \subseteq A^+$, and
6. there is a bijection $g : \operatorname{fxd} \beta \to C$.




Prove that there is an involution $\gamma : B \to B$ such that $\operatorname{fxd} \gamma \subseteq B^+$ and a bijection $h : \operatorname{fxd}(\gamma) \to C$.

**50.** Let $A_1, \ldots, A_n$ be finite sets. Prove the principle of inclusion/exclusion:

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{S \subseteq \{1,\ldots,n\}} (-1)^{|S|-1} \left| \bigcap_{i \in S} A_i \right|$$

by way of a sign reversing involution.

**51.** Write Python or Mathematica code defining a function `bijection_machine`. The input is $(\lambda, A, B)$ where

1. $A = (A_1, \ldots, A_k)$ and $B = (B_1, \ldots, B_k)$ are lists of pairwise disjoint lists such that the sum of the elements in $A_i$ and $B_i$ are the same for all $i$ (these are "diseases"), and
2. $\lambda$ is an integer partition without any diseases in $A$.

The output is the integer partition without any diseases in $B$ as produced by Remmel's bijection machine.

# Discrete Mathematics Set 11

**Math 435**: Complete 4 parts of the following exercises.
**Math 530**: Complete exercise 53, 54, and 55.

**52.** Let $K_{\lambda,\mu}$ be the number of column strict tableau $T$ of shape $\lambda \vdash n$ and content $\mu = (\mu_1, \mu_2, \dots) \vdash n$ (the content means that there are $\mu_1$ copies of 1 in $T$, $\mu_2$ copies of 2 in $T$, and so on). The Kostka matrix $K_n$ is matrix with rows and columns indexed by integer partitions of $n$ with the row $\lambda$ and column $\mu$ entry equal to $K_{\mu,\lambda}$. For example, when $n = 4$, this matrix is

$$\begin{array}{c c} & \begin{array}{ccccc} (4) & (3,1) & (2^2) & (2,1^2) & (1^4) \end{array} \\ \begin{array}{c} (4) \\ (3,1) \\ (2^2) \\ (2,1^2) \\ (1^4) \end{array} & \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & 3 & 2 & 3 & 1 \end{array}\right] \end{array}.$$

a. Find $K_5$ and (using a machine) find $K_5^{-1}$.

b. Let $a_\lambda$ be the vector in $\mathbb{R}^{p(n)}$ with a 1 in the $\lambda$ entry and 0 elsewhere. What do the matrix multiplications $K_n a_\lambda$ and $K_n^{-1} a_\lambda$ mean in terms of the monomial and Schur symmetric functions?

**53.** Let $RCS_\lambda$ denote the set of reverse column strict tableaux; that is, all tableaux where the integer labeling weakly decreases in rows and strictly decreases up columns. Show that $s_\lambda = \sum_{RCS_\lambda} w(T)$ for any $\lambda \vdash n$. For example, here are all elements in $RCS_{(2,1)}$ that are filled with integers $\leq 3$:

$$\begin{array}{|c|c} \hline 2 & \\ \hline 3 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 1 & \\ \hline 3 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 1 & \\ \hline 2 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 1 & \\ \hline 3 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 1 & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 1 & \\ \hline 3 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 2 & \\ \hline 3 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 2 & \\ \hline 3 & \multicolumn{1}{c|}{3} \\ \hline \end{array} .$$

**54.** Prove that the power symmetric polynomial $p_\lambda(x_1, \dots, x_N)$, the homogeneous symmetric polynomial $h_\lambda(x_1, \dots, x_N)$, and the elementary symmetric polynomial $e_\lambda(x_1, \dots, x_N)$ are indeed symmetric polynomials.

**55.** An alternating polynomial $f$ in $x_1, \dots, x_n$ is a polynomial such that for all $\sigma = \sigma_1 \cdots \sigma_n \in S_n$,

$$f(x_1, \dots, x_n) = \operatorname{sign}(\sigma) f(x_{\sigma_1}, \dots, x_{\sigma_n}).$$

a. Show that an alternating polynomial is divisible by $\Delta = \prod_{i<j}(x_i - x_j)$.

b. Let $\mathcal{A}_k$ be the vector space of alternating polynomials with every term degree $k$. Show that division by $\Delta$ is a vector space isomorphism between $\mathcal{A}_{n+\binom{n}{2}}$ and $\Lambda_n$. (Therefore understanding $\Lambda_n$ is the same as understanding $\mathcal{A}_{n+\binom{n}{2}}$.)

# Discrete Mathematics Set 12

**Math 435**: Complete 7 parts of the following exercises.
**Math 530**: Complete 7 parts of the following exercises.

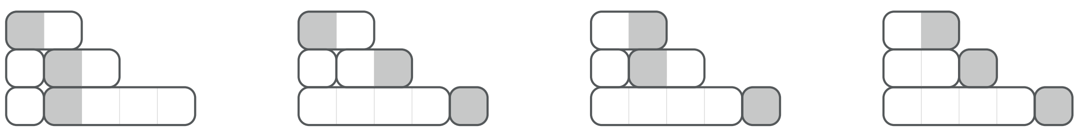
**56.** Prove that the coefficient of $m_\lambda$ in $h_\mu$ is the number of matrices with nonnegative integer entries with row sum $\lambda$ and column sum $\mu$.

**57.** Let $\mu \vdash n$.

a. Use a similar proof as used to prove $h_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,\mu}| e_\lambda$ to prove $e_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,\mu}| h_\lambda$.

b. Let $p(n)$ be the number of integer partitions of $n$ and let $B_n$ be the $p(n) \times p(n)$ matrix with row $\mu$, column $\lambda$ entry equal to $(-1)^{n-\ell(\lambda)}|B_{\lambda,\mu}|$. Why does part a. imply $B_n^{-1} = B_n$?

**58.** A weighted brick tabloid of content $\lambda$ and shape $\mu$ is the usual brick tabloid of content $\lambda$ and shape $\mu$ but with one cell in the final brick in each row shaded. Let $WB_{\lambda,\mu}$ be the set of all weighted brick tabloids of content $\lambda$ and shape $\mu$. Here are 4 of the 30 examples of weighted brick tabloids found on the next page:



a. Use a similar proof as used to prove $h_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,\mu}| e_\lambda$ to prove $p_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |WB_{\lambda,\mu}| e_\lambda$.

b. Prove $p_\mu = \sum_{\lambda \vdash n} (-1)^{\ell(\mu)-\ell(\lambda)} |WB_{\lambda,\mu}| h_\lambda$.

c. By counting weighted brick tabloids, find the $5 \times 5$ matrix with row and columns indexed by integer partitions of 4 and with row $\mu$ and column $\lambda$ entry equal to $(-1)^{n-\ell(\lambda)}|WB_{\lambda,\mu}|$. Why does this matrix verify that $\{p_\lambda : \lambda \vdash 4\}$ is a basis for $\Lambda_4$? More generally, why is $\{p_\lambda : \lambda \vdash n\}$ a basis for $\Lambda_n$?

**59.** Prove these identities are true for $n \geq 1$ using bijections or sign reversing involutions:

a. $p_n = \sum_{i=0}^{n-1} (-1)^i s_{(n-i,1^i)}$.

b. $\sum_{i=0}^{n-1} h_i p_{n-i} = n h_n$.

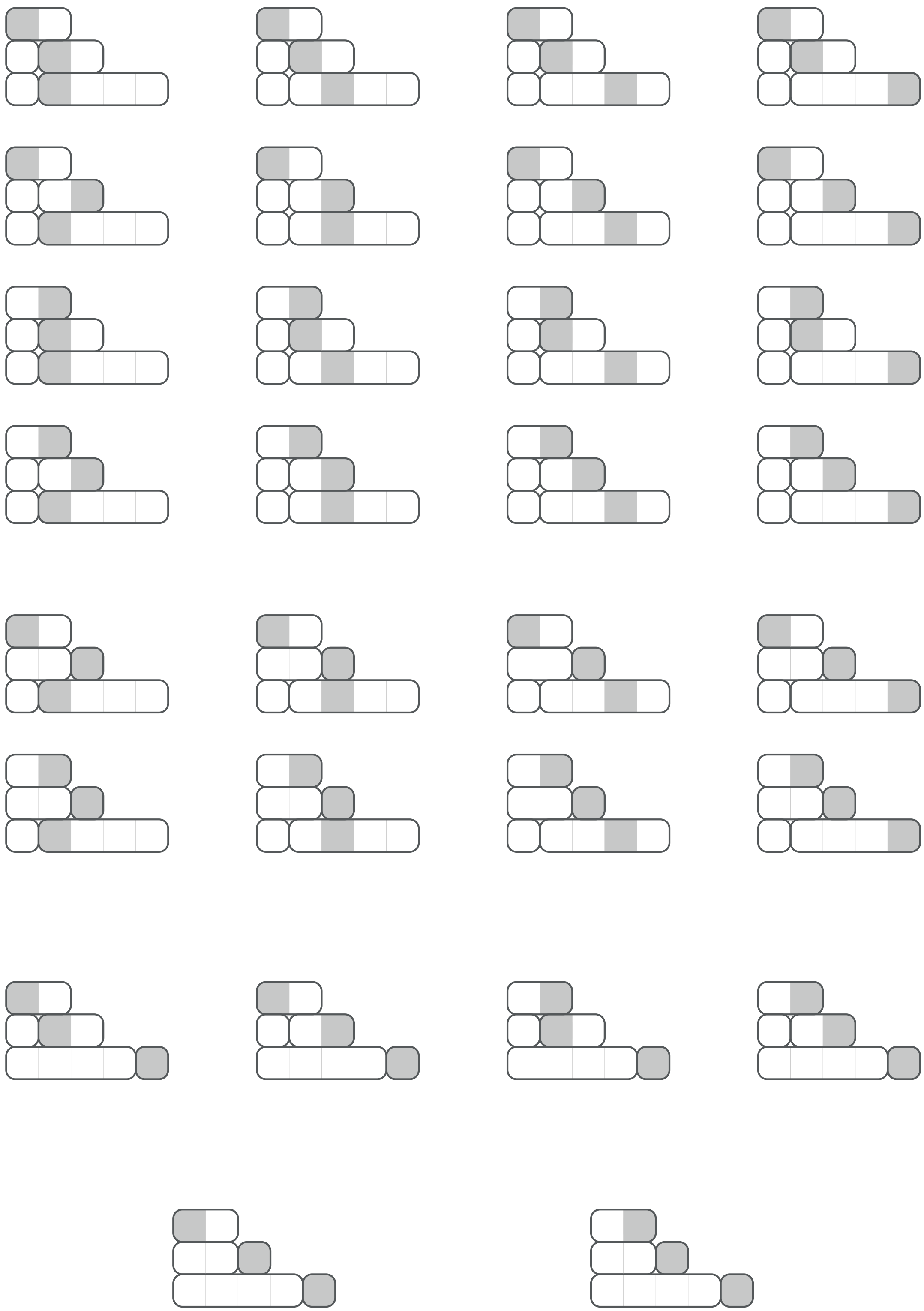
c. $\sum_{i=0}^{n-1} (-1)^i e_i p_{n-i} = (-1)^{n-1} n e_n$.

**60.** Let $p_n = p_{(n)}(x_1, \ldots, x_N)$ be the power symmetric polynomial in $x_1, \ldots, x_N$, let $h_n = h_{(n)}(x_1, \ldots, x_N)$ be the homogeneous symmetric polynomial, and let $H(t) = \sum_{n=0}^{\infty} h_n t^n$. Show $\sum_{n=1}^{\infty} \frac{p_n}{n} t^n = \ln H(t)$ and $\sum_{n=1}^{\infty} p_n t^n = \frac{tH'(t)}{H(t)}$.

**61.** Show that $\sum_{n=0}^{\infty} \sum_{\lambda \vdash n} h_\lambda(x_1, \ldots, x_N) m_\lambda(y_1, \ldots, y_M) = \prod_{j=1}^{M} \prod_{i=1}^{N} \frac{1}{1 - x_i y_j}$.

Hint: Multiply sums of the form $\prod_{i=1}^{N} \frac{1}{1 - x_i y_j} = \sum_{n=0}^{\infty} h_n(x_1, \ldots, x_N) y_j^n$.

All 30 weighted brick tabloids of shape $(5,3,2)$ and content $(4,2,2,1,1)$.



# Discrete Mathematics Set 13

**Math 435**: Complete 4 parts of the following exercises.
**Math 530**: Complete exercise 65 and two parts of the remaining exercises.

**62.** Define a ring homomorphism $\varphi$ on $\Lambda$ by $\varphi(e_n) = (-1)^{n-1}/n!$ for $n \geq 1$. Use $\varphi(h_n)$ to find the generating function for the number of ordered set partitions of $n$ first found in Exercise 9 in Set 2.

**63.** Define a ring homomorphism $\varphi$ on $\Lambda$ by $\varphi(e_n) = (-1)^{n-1}k(x-1)^{n-1}$ for $n \geq 1$. Use $\varphi(h_n)$ to find the generating function for

$$\sum_{w \in \{1,\ldots,k\}^n} x^{\text{equals}(w)}$$

where equals$(w)$ denotes the number of times there are consecutive equal integers in a word $w \in \{1,\ldots,k\}^n$.

**64.** Define a ring homomorphism $\varphi$ on $\Lambda$ by $\varphi(e_n) = \begin{cases} 1 & \text{if } n = 0 \text{ or } n = 2, \\ 2x & \text{if } n = 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$

a. Recall from exercise 23 the definitions of the Chebyshev polynomial of the first kind $T_n(x)$ and the Chebyshev polynomial of the second kind $U_n(x)$. Show that $\varphi(p_n) = 2T_n(x)$ for $n \geq 1$ and $\varphi(h_n) = U_n(x)$ for $n \geq 0$. It may help to use an identity found in Exercise 60.

b. Use previously established relationships between $e_n, h_n$, and $p_n$ (such as those in Exercise 59) to show these identities hold for $n \geq 3$:

   i. $U_n(x) = \displaystyle\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} (-1)^i (2x)^{n-2i}$

   ii. $U_n(x) = \displaystyle\frac{2}{n} \sum_{i=0}^{n-1} U_i(x) T_{n-i}(x)$

   iii. $U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0$

   iv. $T_n(x) - 2xT_{n-1}(x) + T_{n-2}(x) = 0$

**65.** Define a ring homomorphism $\varphi$ on $\Lambda$ by $\varphi(e_n) = \begin{cases} (-1)^{k+k(3k-1)/2} & \text{if } n = k(3k-1)/2 \text{ for some } k \in \mathbb{Z}, \\ 0 & \text{if not.} \end{cases}$

a. Show that $\varphi(h_n) = p(n)$ where $p(n)$ is the number of integer partitions of $n$.

b. Apply $\varphi$ to the generating function for $p_n/n$ in Exercise 60 to show that $\varphi(p_n) = \sigma(n)$ where $\sigma(n)$ is the sum of the positive integer divisors of $n$.

c. Use an identity found in Exercise 59 to show that $p(n) = \displaystyle\frac{1}{n} \sum_{i=1}^{n} \sigma(i) p(n-i)$, thereby giving a recursion for the number of integer partitions of $n$. Calculate $p(7)$ using this recursion.

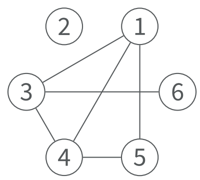
# Discrete Mathematics Set 14

**Math 435**: Complete 3 of the following exercises.
**Math 530**: Complete exercise 70 and 2 exercises in {67, 68, 71, 72}.

**66.** How many ways are there to color the vertices of a cube using $N$ colors (two colorings are the same if the cube can be rotated to turn one coloring into another)? How many ways are there to color the vertices if 3 vertices must be red and 5 must be black?

**67.** How many ways are there to color the edges of a cube using $N$ colors (two colorings are the same if the cube can be rotated to turn one coloring into another)? How many ways are there to color the vertices if 6 edges must be red and 6 must be black?

**68.** Let $E$ be the set of two element subsets of $\{1, \ldots, n\}$. A simple graph on $n$ vertices corresponds to a coloring of $E$ which uses two different colors: a set $\{i, j\}$ is colored $q$ if the edge between $i$ and $j$ appears in a simple graph and 1 if not. For example, the graph




corresponds to coloring each of $\{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$, $\{3, 4\}$, $\{3, 6\}$, and $\{4, 5\}$ with $q$ and all other elements of $E$ with 1. In this way, the number of edges in the graph is the number of times $q$ is used in the coloring.

By defining $\sigma\{i, j\} = \{\sigma(i), \sigma(j)\}$ for all $\sigma \in S_n$, the symmetric group $S_n$ acts on elements of $E$. Find

$$\sum_{\text{inequivalent 2 colorings } f \text{ of } E} q^{\text{the number of times color } q \text{ is used in } f}$$

when $n = 4$. Using the language of graph theory, we are finding

$$\sum_{\text{nonisomorphic simple graphs } g \text{ on 4 vertices}} q^{\text{the number edges in } g}.$$

**69.** Show that the cycle index polynomial satisfies $Z_{G \times H} = Z_G Z_H$.

**70.** Let $C_n$ be the cyclic group of order $n$ (the group generated by rotations of an $n$–sided regular polygon) and let $D_n$ be the dihedral group of order $2n$ (the group generated by rotations and reflections of an $n$–sided regular polygon). Show that the cycle index polynomials for these groups are

$$Z_{C_n} = \frac{1}{n} \sum_{i=1}^{n} (p_{n/\gcd(i,n)})^{\gcd(i,n)} \qquad \text{and} \qquad Z_{D_n} = \frac{1}{2} Z_{C_n} + \begin{cases} p_1 p_2^{(n-1)/2}/2 & \text{if } n \text{ is odd,} \\ (p_2^{n/2} + p_1^2 p_2^{(n-2)/2})/4 & \text{if } n \text{ is even.} \end{cases}$$

where $\gcd(i, n)$ is the greatest common divisor of $i$ and $n$.

**71.** Show the cycle index polynomial for the symmetric group $S_n$ is the homogeneous symmetric function $h_n$.

**72.** Let $A_n$ be the alternating group, the subgroup of $S_n$ containing all permutations $\sigma$ with an even number of even sized cycles. Show that the cycle index polynomial for the alternating group $A_n$ is $h_n + e_n$.