Thm The coefficient of m_λ in e_μ is the # of 0-1 matrices with row sum λ and column sum μ.

EX $(n=4)$

$$
\begin{array}{cc}
 & \mu \\
\lambda &
\begin{array}{c|ccccc|}
 & (4) & (3,1) & (2,2) & (2,1^2) & (1^4) \\
\hline
(4) & 0 & 0 & 0 & 0 & 1 \\
(3,1) & 0 & 0 & 0 & 1 & 4 \\
(2,2) & 0 & 0 & 1 & 2 & 6 \\
(2,1^2) & 0 & 1 & 2 & 5 & 12 \\
(1^4) & 1 & 4 & 6 & 12 & 24
\end{array} = M
\end{array}
$$

This matrix is a change of basis from the e's to m's

$(\det M = \pm 1)$

EX Express $e_{(2,1,1)}$ in the monomial basis

The answer is $M \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 5 \\ 12 \end{bmatrix}$

$\to$ meaning $e_2 \cdot e_1 \cdot e_1 = m_{(3,1)} + 2m_{(2,2)} + 5\, m_{(2,1,1)} + 12\, m_{(1,1,1,1)}$

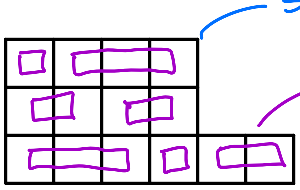
Fundamental Theorem of Symmetric Functions

$\{e_\lambda \mid \lambda \vdash n\}$ is a basis for the vector space of symmetric functions of degree n.

Def Let $B_{\lambda,\mu}$ be the set of "brick tabloids" of content λ and shape μ.

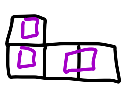
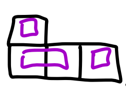
EX

shape $\mu = (6,4,4)$



content $(3,3,2,2,2,1,1) = \lambda$

(lengths of bricks)

EX $B_{(2,1,1),(3,1)} = \{$  ,  $\}$

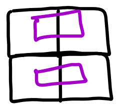
Thm $h_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \underbrace{|B_{\lambda,\mu}|}_{\text{\# of brick tabloids}} e_\lambda$

<u>Ex</u> $h_{(2,2)}(x_1, x_2, x_3, x_4) = h_2 h_2 = (x_1^2 + x_2^2 + \cdots + x_1 x_2 + \cdots)^2$

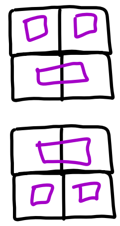
$$= x_1^4 + \cdots + 2x_1 x_2^3 + \cdots + 3x_1^2 x_2^2 + \cdots + 6x_1^2 x_2 x_3 + \cdots$$
$$+ 6x_1 x_2 x_3 x_4 + \cdots$$

?

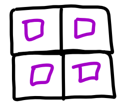
$$h_{(2,2)} = (-1)^{4-2} \cdot 1 \cdot e_{(2,2)} + (-1)^{4-3} \cdot 2 \cdot e_{(2,1,1)} + (-1)^{4-4} \cdot 1 \cdot e_{(1^4)}$$



$\lambda = (2,2)$



$\lambda = (2,1,1)$

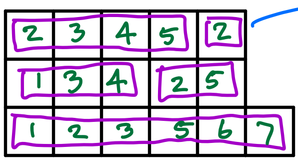


$\lambda = (1^4)$

$$= e_2 \cdot e_2 - 2e_2 e_1 e_1 + e_1 e_1 e_1 e_1$$

$$= (x_1 x_2 + x_1 x_3 + x_2 x_3 + \cdots)^2 - 2(x_1 x_2 + x_1 x_3 + x_2 x_3 + \cdots)(x_1 + x_2 + \cdots)^2 + (x_1 + x_2 + \cdots)^4$$

<u>Pf</u> Interpret the R.H.S. with objects that look like



shape μ

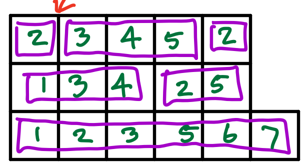
$\lambda = (6,4,3,2,1)$

<u>Sign:</u> $(-1)^{16-5} = -1$

scan L → R
and top → bottom

looking for a brick
of length > 1
or do reverse

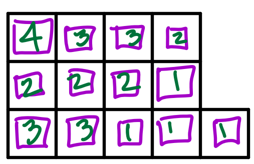
cut brick



<u>Sign</u>: +1

e_λ's are the sequences in bricks

<u>Fixed point</u>



weakly decreasing sequence! → h_μ

<u>Sign</u>: +1