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Last time: $b_n = \#$ of set partitions of n.

$$B(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n$$

$$= e^{-1} e^{e^x}$$

Let $b_{n,k} = \#$ of set partitions of n into exactly k sets

Ex: $b_{6,3}$

could be $\{\{1,2,3,4\},\{5\},\{6\}\}$

there are $\binom{6}{2}$ ways to have set sizes 4, 1, 1

there are $\binom{6}{3,2,1}$ ways to have set sizes 3, 2, 1

there are $\frac{\binom{6}{2,2,2}}{6}$ ways to have set sizes 2, 2, 2

$15 + 60 + 90 = 165$

Our recursion $b_{n+1,k} = b_{n,k-1} + k\, b_{n,k}$ "n+1" is all alone; n+1 is in a set with others

Let $B(x,y) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} b_{n,k}\, y^k\right) \frac{x^n}{n!}$ So $b_{n,k}$ is the coefficient of $y^k x^n \cdot n!$

$$= 1 + (1y^1)\frac{x^1}{1!} + (y^1 + y^2)\frac{x^2}{2!} + (y^1 + 3y^2 + y^3)\frac{x^3}{3!}$$

Note: $B(x,1) =$ the Bell numbers $= e^{-1} e^{e^x}$

$$B_x(x,y) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} b_{n,k}\, y^k\right) \frac{x^{n-1}}{(n-1)!} = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} b_{n,k}\, y^k\right) \frac{x^{n-1}}{(n-1)!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n+1} b_{n+1,k}\, y^k\right) \frac{x^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n+1} b_{n,k-1}\, y^k\right) \frac{x^n}{n!} + \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} k\, b_{n,k}\, y^k\right) \frac{x^n}{n!}$$

$$= y\, B(x,y) + y\, B_y(x,y)$$ b/c $k y^k = y \frac{\partial}{\partial y} y^k$

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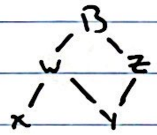
So we have a partial differential equation for $B(x,y)$

$B_x(x,y) = y\,B(x,y) + y\,B_y(x,y) \qquad B(x,1) = e^{-1}e^{e^x}$

Solving PDEs is hard and Prof. Mendes won't require it on the midterm.

Let $w(x,y) = ye^x$ and $z(x,y) = y$

So  $B_x = \frac{\partial B}{\partial w}\cdot\frac{\partial w}{\partial x} \qquad B_y = \frac{\partial B}{\partial w}\cdot\frac{\partial w}{\partial y} + \frac{\partial B}{\partial z}\cdot\frac{\partial z}{\partial y}$

Since $\frac{\partial w}{\partial x} = ye^x$, $\frac{\partial w}{\partial y} = e^x$, $\frac{\partial z}{\partial y} = 1$

We have $B_w\,y e^x = yB + y(B_w e^x + B_z)$

$\Rightarrow \cancel{B_w y e^x} = yB + \cancel{yB_w e^x} + yB_z$

$\Rightarrow 0 = yB + yB_z$

$\Rightarrow 0 = B + B_z$ since $y=0$ is uninteresting

$\Rightarrow B_z = -B$

Thus $B(w,z) = C(w)\,e^{-z}$

So $B(x,y) = C(ye^x)\,e^{-y}$

Using the initial ~~eq~~ condition plugging in $y=1$

So $B(x,1) = C(e^x)\,e^{-1}$ should equal $e^{e^x}e^{-1}$

Thus $C(x) = e^x$

Therefore $B(x,y) = e^{ye^x}e^{-y} = (e^y)^{e^x - 1}$