Graph Theory Notes

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Abstract

These notes introduce graph theory at a level appropriate for an undergraduate mathematics course.

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— 1 —

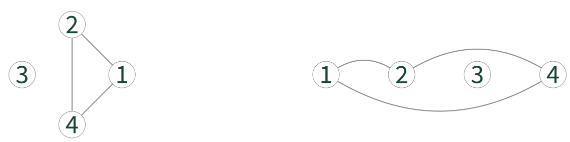
First definitions

This section contains basic graph theory definitions.

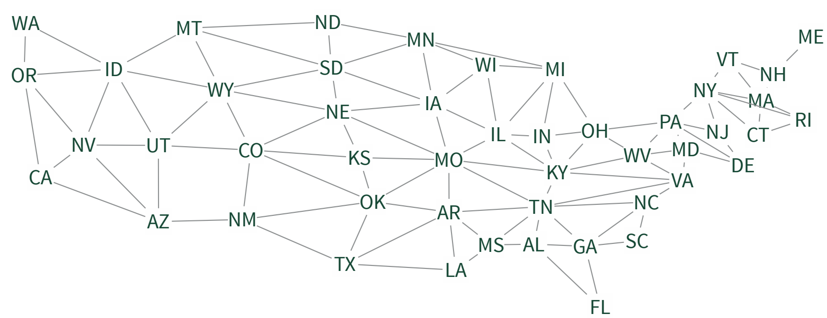
Definition. *A **graph** $G = (V, E)$ is an ordered pair where V is a finite set of elements called **vertices** and E is a subset of $\{\{u, v\} : u, v \in V\}$. The singular form of the plural “vertices” is **vertex** and elements in E are **edges**.*

Graphs be drawn by writing down the vertices and then using lines that connect vertices to indicate edges. There are many ways to draw the same graph. Artistic license can be taken with how to arrange the vertices and edges as to best illustrate properties of the graph.

Example 1. If $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2\}, \{1, 4\}, \{2, 4\}\}$, then $G = (V, E)$ is a graph. This graph is shown twice below, drawn in two different ways:



Example 2. A graph G can be defined by letting V be the set of the 48 contiguous US states and by letting $E = \{\{A, B\} :$ states A and B share a border$\}$.

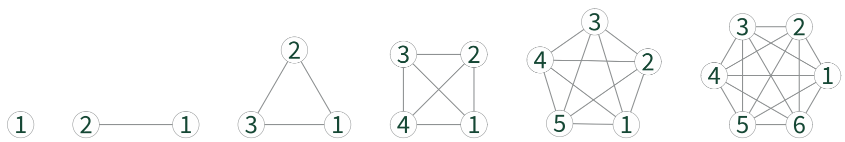


Graphs are an abstraction of the a relationship between elements in a set. Other “real world” examples of graphs are:

vertex set	edge set relation
social media accounts	friendship
atoms in a molecule	chemical bonds
computer folders in a file system	containment

The creative reader can think of an endless number of examples of graphs!

Definition. *The **complete graph** K_n is the graph that has vertices $\{1, \dots, n\}$ and that has all possible edges. Below are $K_1, \dots, K_6$:*



Theorem 3. *The complete graph K_n has $\binom{n}{2} = n(n-1)/2$ edges.*

Proof. The edges in K_n are sets of two elements of the form $\{i, j\}$ with $1 \le i < j \le n$. There are n ways to select i from $1, \dots, n$ and, for each choice of i, there are $n - 1$ ways to select j from the remaining numbers. Therefore there are $n(n-1)$ ways to select ordered pairs (i, j). Since this process counts (i, j) and (j, i) as different, we divide by 2 to count each set of the form $\{i, j\}$ with $1 \le i < j \le n$ once. This shows the number of edges is indeed $n(n-1)/2$. □

Theorem 4. *There are $2^{\binom{n}{2}}$ possible graphs with n vertices.*

Proof. For each one of the $\binom{n}{2}$ possible edges in a graph on n vertices, make one of two choices: either include the edge in the graph or do not include the edge. □

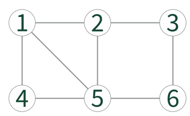
Definition. *The **complement** of the graph G, denoted G^c, is the graph with the same vertex set as G but with edge set $\{\{u, v\} : \{u, v\}$ is not an edge in $G\}$.*

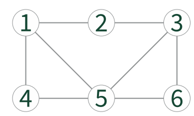
Example 5. These two graphs are complements of one another:

It follows that if G has n vertices, then G and G^c have a combined $\binom{n}{2}$ edges since together they have every possible edge in a complete graph. It also can be seen that $(G^c)^c = G$.

Definition. *Edges are **incident** if they share a vertex. Vertices u, v are **adjacent** in a graph G if $\{u, v\}$ is an edge. The **degree** of a vertex v is the number of vertices adjacent to v. The **degree sequence** of a graph G is a list of the degrees of the vertices in G in weakly decreasing order.*

Example 6. The degree sequence for the two different graphs shown below are both equal to $(4, 3, 3, 2, 2, 2)$.





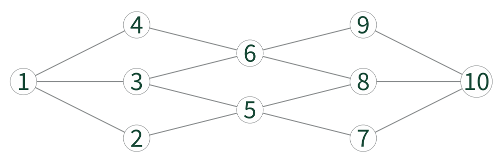
Can you find a graph with degree sequence $(3, 3, 3, 3, 1, 1)$?

Theorem 7 (Euler's handshaking lemma)**.** *If G has E edges and degree sequence $(d_1, \ldots, d_n)$, then $d_1 + \cdots + d_n = 2E$.*

Proof. Let $\{u, v\}$ be an edge in G and let d_u and d_v be the degrees of vertices u and v. The edge $\{u, v\}$ is counted twice in $d_1 + \cdots + d_n$; once with the d_u term and once with the d_v term. Thus the sum $d_1 + \cdots + d_n$ counts each edge exactly twice. $\square$

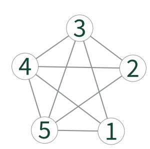
Definition. *A **path** from vertex u to vertex v in G is a sequence of distinct vertices $u = x_1, x_2, \ldots, x_n = v$ in G such that x_i and x_{i+1} are adjacent for $i = 1, \ldots, n-1$.*

Example 8. The sequence $1, 2, 5, 8, 10$ is a path from 1 to 10 in this graph



Definition. *If e is an edge in G, we let $G - e$ be the graph with e removed. If v is a vertex in G, we let $G - v$ be the graph with v and any edges containing v removed. A **subgraph** of G is any graph formed by removing a set of vertices from G.*

Example 9. We have $K_5 - \{1, 2\} =$ is equal to

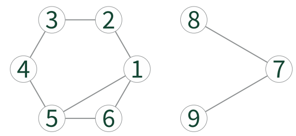


and $K_5 - 5 = K_4$. More generally, K_m is a subgraph of K_n whenever $m \leq n$.

Definition. *A graph G is **connected** if there is a path from u to v for all vertices u and v in G. A **component** of G is a maximal connected subgraph of G.*

This graph in Example 8 is connected because there is a path between every possible pair of vertices. Any connected graph has only one component.

Example 10. The following graph not connected because there is no path from 1 to 7. It has two components.



Theorem 11. *If G has n vertices and more than $\binom{n-1}{2}$ edges, then G is connected.*

Proof. Suppose G is not connected and has component G_1 with k vertices where $1 \leq k \leq n/2$. The graph G_1 has at most $\binom{k}{2}$ edges, the part of G that does not contain G_1 has at most $\binom{n-k}{2}$ edges, so the number of edges in G is at most

$$\binom{k}{2} + \binom{n-k}{2} = \frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} = k^2 - kn + \frac{n^2-n}{2}.$$

The first derivative of this expression with respect to k is $2k - n$ and the second derivative is 2, meaning that this expression is minimized when $k = n/2$ and maximized at the endpoint $k = 1$. Taking $k = 1$, we see that the number of edges in G is at most $\binom{n-1}{2}$, as needed. $\square$

Definition. *Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are* ***isomorphic*** *if there is a bijection $f : V_1 \to V_2$ such that u, v are adjacent in G_1 if and only if $f(u), f(v)$ are adjacent in G_2. This bijection f is an* ***isomorphism****.*

Example 12. These two graphs are isomorphic:

An isomorphism f could be $f(1) = a, f(2) = b, \ldots, f(8) = h$.

Example 13. These two graphs are isomorphic:

An isomorphism f could be $f(1) = 1, f(2) = 3, f(3) = 5, f(4) = 2, f(5) = 4, f(6) = 6$.

Isomorphic graphs are the same with the exception that the labels on the graphs are different. If G_1 and G_2 are isomorphic, then the graphs G_1 and G_2 have all of the same properties. In particular, isomorphic graphs have the same degree sequence. However, two graphs having the same degree sequence does not mean that they are isomorphic. For example, the two graphs in Example 6 have the same degree sequence but are not isomorphic because the first graph has adjacent degree 2 vertices but the second graph does not. There are no known efficient algorithms that can quickly determine if two graphs are isomorphic.

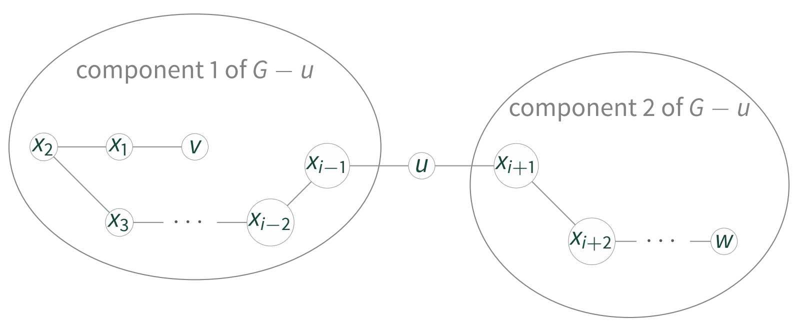
Definition. *The **Path graph** P_n is the graph that has vertices $\{1, \dots, n\}$ and edges $\{\{1,2\}, \{2,3\}, \dots, \{n-1, n\}\}$. Below is P_{10}:*

Theorem 14. *Let G be a graph with at least 2 vertices. If both G and G^c are connected, then G has a subgraph isomorphic to P_4.*

Proof. Suppose the theorem is not true and let G be a graph with the minimum number of vertices such that G is connected, G^c is connected, and G does not have a subgraph isomorphic to P_4.

Let u be a vertex in G. Then $G - u$ does not have a subgraph isomorphic to P_4 and, since G was the least counterexample to the theorem, either $G - u$ or $(G - u)^c$ is not connected. Without loss of generality suppose $G - u$ is not connected.

Since G^c is connected, u cannot be adjacent to every other vertex in G. Thus there is a v such that u and v are not adjacent in G. Take w such that v and w are not adjacent in $G - u$, possible since $G - u$ is not connected. Let $v, x_1, \dots, x_k, w$ be a path of minimum length from v to w in G. It follows that u must be along this path, say that $u = x_i$. A depiction of this path is here:



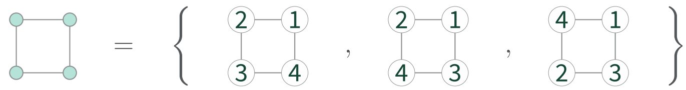
We claim that the subgraph of G containing $x_{i-2}, x_{i-1}, u, x_{i+1}$ is isomorphic to P_4. Indeed, since v and u are not adjacent it cannot be the case that $v = x_{i-1}$ and so this path is actually length 4. Furthermore, neither x_{i-2} nor x_{i-1} can be adjacent to x_{i+1} because otherwise $G - u$ would be connected. Lastly, x_{i-2} cannot be adjacent to u because we chose our path to have minimum possible length.

We have shown that G has a subgraph isomorphic to P_4, our contradiction. $\square$

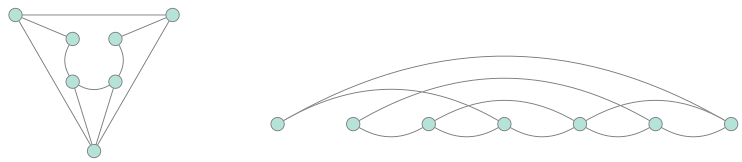
Definition. *An **unlabeled graph** with n vertices is the set of all graphs which are isomorphic to some graph with vertex set $\{1, \dots, n\}$.*

Using fancy language, an unlabeled graph is an equivalence class under the equivalence relation given by graph isomorphism. We will use the term graph to mean both labeled and unlabeled graphs. The type of graph should be clear from context. We can draw an unlabeled graph in the same manner as a labeled graph but without the labels.

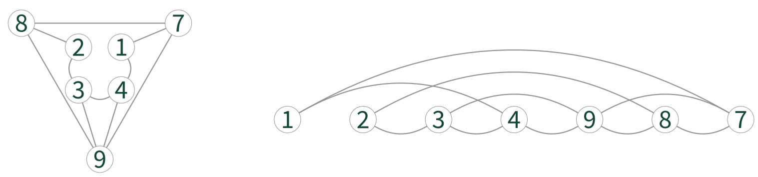
Example 15. An example of an unlabeled graph is



Example 16. These two labeled graphs are the same



because the unlabeled vertices in both graphs can be labeled to create the same labeled graph:



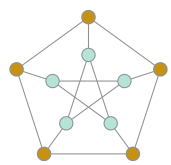
Example 17. There are 11 possible unlabeled graphs with 4 vertices:

Unlabeled graphs are drawn when we are studying a property of graphs for which the labeling does not matter. For example, the length of the longest path in a graph does not depend on the labels, so when studying properties of longest paths we may draw unlabeled versions of graphs.

— 2 —
Vertex colorings

Definition. *An r-**coloring** of $G = (V, E)$ is a function $f : V \to \{1, \dots, r\}$.*

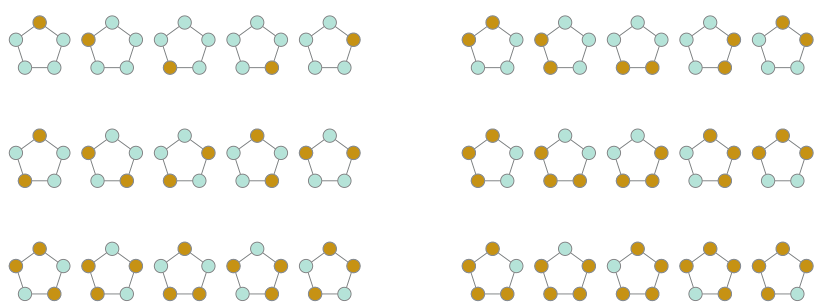
Example 18. A 2-coloring of the Petersen graph is



where we are representing 1 with the color ● and 2 with ●.

Theorem 19 (Fermat's little theorem)**.** *If p is a prime number and r a positive integer, then $r^p - r$ is divisible by p.*

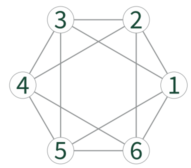
Proof. There are $r^P - r$ possible r-colorings of the cycle graph C_p that use at least two colors because there are r^p total colorings (making one choice of r colors for each of the p vertices) and r of these use a single color. Sort this collection of colored graphs into groups by rotational symmetry. For example, when $p = 5$ and $r = 2$, we have

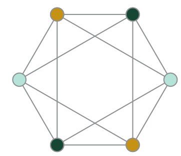


Since p is prime, each collection of graphs grouped by rotational symmetry will have exactly p elements. Therefore $r^p - r$ is divisible by p. □

Definition. *A coloring f is **proper** if $f(u) \neq f(v)$ for all adjacent vertices u and v.*

Example 20. A proper 3-coloring of the graph on the left is shown on the right





where we are representing 1 with the color , 2 with , and 3 with . This is a proper coloring because adjacent vertices never have the same color.

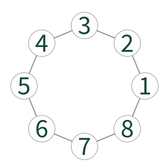
Theorem 21. *There are $r(r-1)\cdots(r-n+1)$ proper r-colorings of K_n.*

Proof. There are r choices on the color of vertex 1. No matter the choice of color for vertex 1 there are $r-1$ ways to color vertex 2. No matter the previous choices of colors, there are $r-2$ ways to color vertex 3, and so on, until we find there are $r-(n-1)$ ways to color vertex n. Thus there are $r(r-1)\cdots(r-n+1)$ proper r-colorings of K_n. $\square$

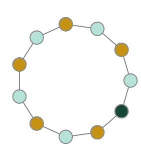
Definition. *The **chromatic number** of a graph G, denoted $\chi(G)$, is the least r for which there exists a proper r-coloring of G.*

First observations about the chromatic number are that $\chi(K_n) = n$, $\chi(G) \geq \chi(G-v)$ for any vertex v, $\chi(G) \geq \chi(G-e)$ for any edge e, and if G is disconnected with components $G_1, \dots, G_k$, then $\chi(G) = \max\{\chi(G_1), \dots, \chi(G_k)\}$.

Definition. *The **cycle graph** C_n is the path graph P_n with the additional edge between vertices 1 and n. Below is C_8:*



Example 22. We have $\chi(C_{2n}) = 2$ and $\chi(C_{2n+1}) = 3$ for all $n \geq 1$. In the case of an even cycle we can color vertices in an alternating pattern, only using 2 colors. In the case of an odd cycle we can again alternate colors but will need to use a third color, see the case of C_{11} below as an example:



Theorem 23. *If Δ is the maximum degree of a vertex in G, then $\chi(G) \leq \Delta + 1$.*

Proof. We proceed by induction on the number of vertices in G with the assertion clearly true if G has only one vertex.

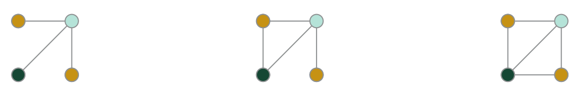
Let v be a vertex of degree Δ. By the induction hypothesis there is a proper coloring of $G-v$ that uses at most $\Delta+1$ colors. Use this coloring of to create a proper coloring of G that uses at most $\Delta + 1$ colors by coloring vertex v a different color than the vertices adjacent to v. □

It was relatively easy to find the upper bound for $\chi(G)$ in Theorem 23. We improve this bound for graphs other than K_n and C_{2n+1} in our next theorem. The proof is significantly more involved than those we have encountered so far.

Theorem 24 (Brooks)**.** *Let Δ be the maximum degree of a vertex in G. If G is a connected graph that is not a complete graph nor an odd cycle, then $\chi(G) \leq \Delta$.*

Proof. The only connected graph with $\Delta = 0$ is K_1, the only connected graph with $\Delta = 1$ is K_2, and the only connected graphs with $\Delta = 2$ are of the form C_n or $C_n - e$ for an edge e. In all of these cases the theorem is quickly seen to be true, so we assume from here on that $\Delta \geq 3$.

We proceed by induction on the number of vertices in G. In the base step when G has 4 vertices, is not equal to K_4, and has $\Delta \geq 3$, it follows that G is one these graphs, shown colored with Δ colors:

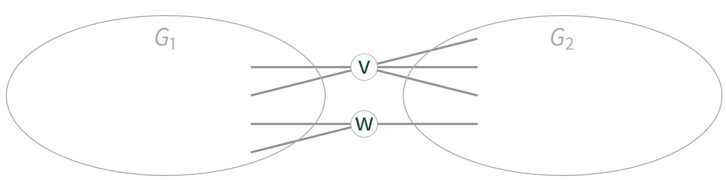


Case 1: There is a vertex v such that $G - v$ is not connected.

Let $G_1, \ldots, G_k$ be the components of $G - v$ and let $G_i + v$ be the subgraph of G containing v and G_i. By induction, the graphs $G_1 + v, \ldots, G_k + v$ can all be properly colored using at most Δ colors. Without loss of generality, select proper colorings of these graphs such that v is always the same color. This gives us a proper coloring of G that uses at most Δ colors, proving that $\chi(G) \leq \Delta$ in this case.

Case 2: There are nonadjacent vertices v and w such that $G - v - w$ is not connected.

Let G_1 be a component of $G - v - w$ and let G_2 be graph created by deleting the vertices in G_1 from $G - v - w$.



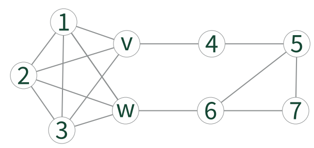
We can assume there is at least one edge from v to G_1 and at least one edge from v to G_2 because otherwise we would be back in Case 1. Similarly there is at least one edge from w to G_1 and from w to G_2.

The plan is to properly color the vertices in the subgraphs $G_1 + v + w$ and $G_2 + v + w$ with at most Δ colors by induction and then to combine these colorings, producing a proper coloring of G. This plan will succeed when both subgraph colorings color v and w the same color or when both subgraph colorings color

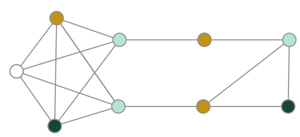
v and w different colors, in which case we can re-color if necessary to ensure that v and w are colors 1 and 2 respectively in both colorings.

Let H_1 be the graph $G_1 + v + w$ with the added edge $\{v, w\}$ and let H_2 be the graph $G_2 + v + w$ with the added edge $\{v, w\}$. The maximum degree in H_1 or H_2 is still less than or equal to Δ because of our assumption that there are edges from v to G_1 and G_2 and edges from w to G_1 and G_2.

If one of these graphs, say H_1, is a complete graph, then both v and w are degree 1 vertices in H_2. One example of a graph G in this situation when $\Delta = 4$ is shown here:



In this situation merge v and w in H_2 into a single vertex and then properly color the resulting graph by induction using at most Δ colors. This coloring can now be combined with the coloring of $G_1 + v + w$ that colors vertices v and w the same color and the remaining $\Delta - 1$ vertices different colors. This coloring applied to the above example gives



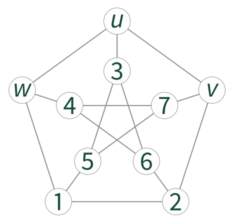
If neither H_1 nor H_2 are complete graphs, then by induction we can properly color both H_1 and H_2 (if either is an odd cycle we can still color by our assumption that $\Delta \geq 3$). These colorings both have v and w different colors and so these colorings can be combined to create a proper coloring of G with at most Δ colors.

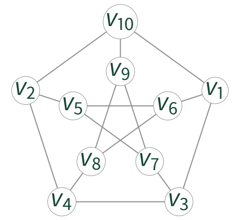
Case 3: Every pair of nonadjacent vertices v and w leaves $G - v - w$ connected.

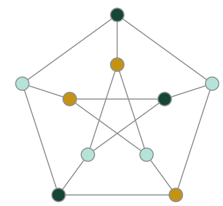
Let u be a vertex of degree Δ in G. Since G is not a complete graph there are nonadjacent vertices v and w that are both adjacent to u. Let $v_1 = v$ and $v_2 = w$. Since $G - v - w$ is connected, the remaining $n - 2$ vertices can be listed in some order $v_3, \ldots, v_n$ such that $v_n = u$ and such that v_i is adjacent to at least one of the vertices $v_{i+1}, \ldots, v_n$.

Properly color the vertices $v_1, \ldots, v_n$ greedily, meaning to color the vertices in sequence, using the least possible available color at each step.

As an example when $\Delta = 3$, consider the Petersen graph with vertices u, v, and w chosen as shown on the left. The vertices $v_1, \ldots, v_{10}$ can be labeled as shown in the middle, and the greedy coloring is shown on the right where we are representing 1 with the color ●, 2 with ●, and 3 with ● :





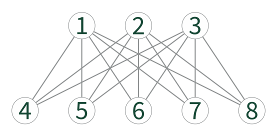


This labeling scheme colors v and w the same color. Furthermore, since every vertex except u has at most $\Delta - 1$ adjacent vertices preceding it in the greedy coloring scheme, we will never need to use more than Δ colors for the vertices $v_3, \ldots, v_{n-1}$. Lastly, since v and w are adjacent to u, the vertex $v_n = u$ can be properly colored so that the entire graph uses at most Δ colors. This completes the proof. □

Definition. *A graph G is **bipartite** if $\chi(G) \leq 2$. If G is a bipartite graph that is properly colored with a set X of vertices colored red and a set Y colored blue, then X and Y are **independent** sets of vertices.*

Example 25. Even cycle graphs are bipartite and odd cycle graphs are not.

Definition. *The **complete bipartite graph** $K_{m,n}$ is the graph with vertices $1, \ldots, n$ and with edge set $\{\{a, b\} : 1 \leq a \leq m$ and $m + 1 \leq b \leq n\}$. Below is $K_{3,5}$.*



Definition. *A **cycle** of length n in a graph G is a path $v_1, \ldots, v_n$ in G such that v_1 and v_n are adjacent.*

A cycle of length n in a graph G does not mean that G has a C_n subgraph.

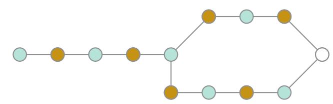
Example 26. The complete bipartite graph $K_{3,5}$ has a cycle of length 6, namely the path $1, 4, 2, 5, 3, 6, 1$, but it does not have a cycle of length 5.

Theorem 27. *A graph G is bipartite if and only if G does not have a cycle of an odd length.*

Proof. If G is bipartite, then it cannot contain an odd cycle because then $\chi(G) \geq 3$.

Assume G is connected and has no cycles of an odd length. Pick a vertex u and color it color 1. Take v to be another vertex in G. If the shortest path from u to v has an odd number of vertices, color it color 1, otherwise color v color 2.

This coloring scheme will fail only if adjacent verities are assigned the same color, as depicted here where we are representing 1 with the color and 2 with the color :



This can only happen when G has an odd cycle. □

Definition. *The chromatic polynomial for a graph G, denoted $P_G(x)$, is the polynomial such that $P_G(r)$ is equal to the number of proper r-colorings for $r = 0, 1, 2, \ldots$.*

Example 28. Theorem 21 gives that the chromatic polynomial for the complete graph is $P_{K_n}(x) = x(x-1)\cdots(x-n+1)$.

Example 29. The chromatic polynomial for the path graph is $P_{P_n}(x) = x(x-1)^{n-1}$ because there are x choices for the color of vertex 1 and then $x-1$ choices for the colors of the remaining vertices.

Theorem 30. *If e is an edge in G, then $P_G(x) = P_{G-e}(x) - P_{G/e}(x)$ where G/e is the graph G with the edge e contracted, meaning that if $e = \{v, w\}$, then the vertices v and w are merged into a single vertex:*

Proof. In the graph $G-e$, the vertices in the contracted edge can be the same color in $P_{G/e}(x)$ ways and different colors in $P_G(x)$ ways. Thus $P_{G-e}(x) = P_{G/e}(x) + P_G(x)$. □

Example 31. The chromatic polynomial for the cycle graph for $n \geq 2$ is equal to

$$P_{C_n}(x) = (x-1)^n + (-1)^n(x-1).$$

This can be proved using Theorem 30 and induction: $P_{C_2}(x) = (x-1)^2 + (x-1) = x(x-1)$ is correct and

$$\begin{aligned} P_{C_n}(x) &= P_{C_n - e}(x) - P_{C_n/e}(x) \\ &= P_{P_n}(x) - P_{C_{n-1}}(x) \\ &= x(x-1)^n - ((x-1)^{n-1} + (-1)^{n-1}(x-1)) \\ &= (x-1)^n + (-1)^n(x-1), \end{aligned}$$

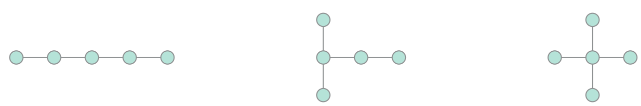
as needed.

— 3 —

Trees

Definition. *A **tree** is a connected graph without a cycle. A **leaf** is a degree* 1 *vertex in a tree.*

Example 32. The three unlabeled trees with 5 vertices are:

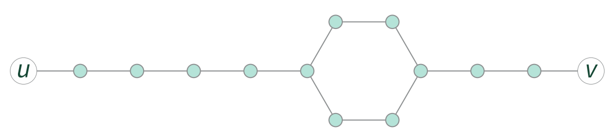


These trees have 2, 3, and 4 leaves, respectively.

As a corollary of Theorem 27, trees are bipartite.

Theorem 33. *A graph is a tree if and only if there is a unique path between any pair of vertices.*

Proof. If two paths from vertices u and v existed, the graph would contain a cycle, as shown below. □



Theorem 34. *Every tree with at least two vertices has at least two leaves.*

Proof. Let P be a longest path in a tree. Suppose that P is a path from vertex u to vertex v. If either u or v had degree 2, then the path P could be extended by at least one vertex, contradicting the fact that P is the longest path. □

Theorem 35. *A graph T is a tree with n vertices if and only if the chromatic polynomial $P_T(x) = x(x-1)^{n-1}$.*

Proof. Assume that T is a tree. We use induction on the number of edges in T with the assertion true if T has no edges (and thus T is a single vertex). Let e be an edge incident to a leaf. Theorem 30 gives

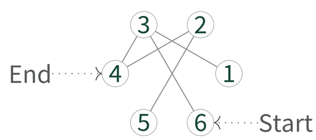
$$P_T(x) = P_{T-e}(x) + P_{T/e}(x) = x(x(x-1)^{n-2}) - x(x-1)^{n-2} = x(x-1)^{n-1}$$

where we use induction twice: once when noticing that $T - e$ consists of a lone vertex that can be colored one of x colors and a tree with $n - 1$ vertices and once on the tree T/e with $n - 1$ vertices.

Now assume that $P_T(x) = x(x-1)^{n-1}$. If T were not connected, then x^2 would divide $P_T(x)$. If T contained a degree 2 vertex, then $P_T(x)$ would be divisible by $(x-2)$. Thus T is a tree. $\square$

Theorem 36 (Cayley)**.** *There are n^{n-2} labeled trees with n vertices.*

Proof. Let t_n be the number of labeled trees with n vertices. We will show that $n^2 t_n = n^n$. Start by selecting a tree in one of t_n ways. Select one of the n vertices in the tree as a "Start" vertex and select of the n vertices as an "End" vertex. The start and end vertices can be the same. For example,



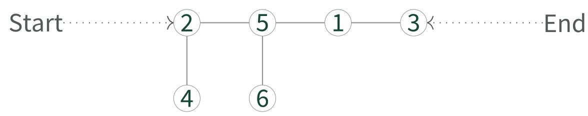
We will turn this tree with Start and End vertices into a list of n integers with each integer a member of $\{1, \ldots, n\}$. Since there are n^n such lists, this would prove the theorem. To change the tree into the list,

1. Let P be the unique path from Start to End in the tree.
2. In the list positions found in P, write the path P from left to right.
3. In the remaining positions i, place the second vertex on the path from i to P.

For example, using the tree displayed above, the path P from Start to End contains the vertices $6, 3, 4$. So, after step 2, our list is $(\ ,\ ,6,3,\ ,4)$. Then, position 1 on the list is filled with 3 because the second vertex on the path from 1 to P is 3. Applying this logic to the other two empty positions arrives at the list $(3, 4, 6, 3, 2, 4)$.

This process is bijective (meaning that each list corresponds to one and only one tree with Start and End labels). Suppose f is the function from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$ such that $f(i)$ is the integer in position i on the list. The path P can be reconstructed from f since an integer i is in the path P if iterating f eventually gives i. This is because once we are on the path P, iterating f will keep us on the path and if we are not on P, then applying f will move us one step closer to P at each iteration. Once the path is identified, the remaining part of the tree can be easily reconstructed.

For example, if our list is $(2, 5, 1, 2, 3, 5)$, then 2 is on the path P because $f(2) = 5$, $f(5) = 3$, $f(3) = 1$, $f(1) = 2$. Similarly, $5, 1$ and 3 are also on the path P. Reading positions $1, 2, 3$ and 5 on the list gives that P is $2, 5, 1, 3$. The other positions on the list give how to connect the other vertices to the tree; 4 is adjacent to 2 and 6 is adjacent to 5. The reconstructed tree is:



We now have shown that $n^2 t_n = n^2$, as needed. □

Theorem 37. *The expected number of leaves on a labeled tree with n vertices is approximately n/e.*

Proof. There are $(n-1)(n-1)^{n-3} = (n-1)^{n-2}$ labeled trees with the vertex labeled 1 as a leaf because we may take any one of the $(n-1)^{n-3}$ labeled trees on the vertices $2, \ldots, n$ and attach 1 to any one of the $n-1$ vertices. Since there are a total of n^{n-2} trees, the probability that vertex 1 (or any other vertex) is a leaf is

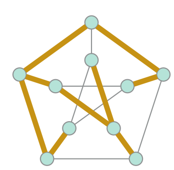
$$\frac{(n-1)^{n-2}}{n^{n-2}} = \left(1 - \frac{1}{n}\right)^n \left(1 - \frac{1}{n}\right)^{-2}$$

We recall from Calculus that $(1 - 1/n)^n$ has limit $1/e$ and that $(1 - 1/n)^{-2}$ has limit 1. Therefore the probability that any one of the n vertices is a leaf is approximately $1/e$, showing that there are approximately n/e leaves on a labeled tree. □

Definition. *A* ***spanning tree*** *for a connected graph G is a tree found from removing edges from G.*

Since a tree must be connected, a spanning tree is the graph with the minimum number of edges that connects all vertices in G.

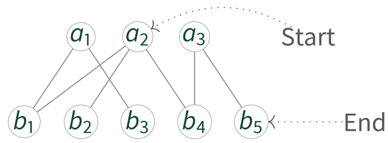
Example 38. The edges in a spanning tree for the Petersen graph are colored gold:



Cayley's theorem can be restated to say that there are n^{n-2} spanning trees for the complete graph K_n. Our proof of Cayley's theorem can be modified to give a similar result for spanning trees in the complete bipartite graph $K_{m,n}$.

Theorem 39. *There are $n^{m-1}m^{n-1}$ spanning trees for $K_{m,n}$.*

Proof. Begin by labeling the m vertices in $K_{m,n}$ with $a_1, \ldots, a_m$ and labeling the n vertices in $K_{m,n}$ with $b_1, \ldots, b_n$. Select a spanning tree for $K_{m,n}$. Select a "Start" vertex among the "a" vertices and select an "End" vertex among the "b" vertices. If $t_{m,n}$ denotes the number of spanning trees for $K_{m,n}$, then There are $mnt_{m,n}$ ways to make these choices. For example, one possible choice when $m = 3$ and $n = 5$ is:



Create two lists, one with positions $a_1, \ldots, a_m$ and the other with positions $b_1, \ldots, b_n$. Let P be the unique path from Start to End.

1. Write the "b" elements from P from left to right in the positions $a_1, \ldots, a_m$ that are also in P. Similarly, write the "a" elements from P from left to right in the positions $b_1, \ldots, b_n$ that are also in P.

2. In the remaining positions a_i, write the second vertex on the path from a_i to P. Similarly, in the remaining positions b_i, write the second vertex on the path from b_i to P.

For example, using the tree shown above, the two lists have these positions that need to be filled in:

$$\Big(\underset{a_1}{\underline{\qquad}}, \underset{a_2}{\underline{\qquad}}, \underset{a_3}{\underline{\qquad}}\Big) \qquad \Big(\underset{b_1}{\underline{\qquad}}, \underset{b_2}{\underline{\qquad}}, \underset{b_3}{\underline{\qquad}}, \underset{b_4}{\underline{\qquad}}, \underset{b_5}{\underline{\qquad}}\Big).$$

The path P is a_2, b_4, a_3, b_5, so after completing the first step in the above instructions, we find

$$\Big(\underset{a_1}{\underline{\qquad}}, \underset{a_2}{\underline{\;b_4\;}}, \underset{a_3}{\underline{\;b_5\;}}\Big) \qquad \Big(\underset{b_1}{\underline{\qquad}}, \underset{b_2}{\underline{\qquad}}, \underset{b_3}{\underline{\qquad}}, \underset{b_4}{\underline{\;a_2\;}}, \underset{b_5}{\underline{\;a_3\;}}\Big).$$

Following the second set of instructions gives

$$\Big(\underset{a_1}{\underline{\;b_1\;}}, \underset{a_2}{\underline{\;b_4\;}}, \underset{a_3}{\underline{\;b_5\;}}\Big) \qquad \Big(\underset{b_1}{\underline{\;a_2\;}}, \underset{b_2}{\underline{\;a_2\;}}, \underset{b_3}{\underline{\;a_1\;}}, \underset{b_4}{\underline{\;a_2\;}}, \underset{b_5}{\underline{\;a_3\;}}\Big).$$

The first list of length m can contain elements in $\{b_1, \ldots, b_n\}$ and so there are m^n such lists. Similarly, there are n^m possible choices for the second list. Therefore, there are $m^n n^m$ possible pairs of lists.

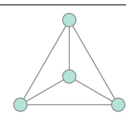
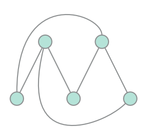
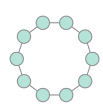
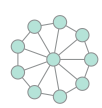
Showing that this process is bijective is so similar to the ideas in the proof of Theorem 36 that it is left to the reader. This shows that $nmt_{m,n} = n^m m^n$. $\square$

— 4 —

Planarity

Definition. *A graph is **planar** if it can be drawn on a plane without any edges crossing. If a graph G is drawn in this way, then the regions in the plane (including the "outside" region) bounded by edges are **faces**.*

Example 40. The following table gives examples of various planar graphs:

Graph	Planar drawing	# vertices	# edges	# faces
K_4		4	6	4
$K_{2,3}$		5	6	3
C_n		n	n	2
W_n		n	$2n-2$	n

Theorem 41 (Euler)**.** *If G is planar, connected, and has V vertices, E edges and F faces, then $V - E + F = 2$.*

Proof. We prove this by induction on the number of edges e with the assertion true when G has only one vertex and no edges.

If G is a tree, then $E = V - 1$, and $F = 1$, so $V - E + F = 2$. If G is not a tree, then an edge on a cycle separates two faces, so removing this edge reduces the number of faces by 1, leaving $V - E + F$ unchanged. We are now done by induction. $\square$

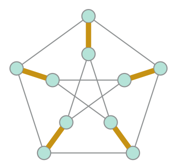
Theorem 42. *If G is planar with $V \geq 3$ vertices and E edges, then $E \leq 3V - 6$. If G happens to also be bipartite, then $E \leq 2V - 4$.*

Proof. Each face is surrounded by at least 3 edges, so $3F \leq 2E$ where F is the number of faces. Using this in $V - E + F = 2$, we have $E = V + F - 2 \leq V + 2E/3 - 2$, which gives $E \leq 3V - 6$. If G is bipartite, then each face is surrounded by at least 4 edges, giving $4F \leq 2E$, which gives the result $E \leq 2V - 4$ when used in Euler's formula. □

As a corollary of Theorem 42, the graphs K_5 and $K_{3,3}$ are not planar. Indeed, K_5 has $E = 10$ but $3V - 6 = 9$ and $K_{3,3}$ has $E = 9$ but $2V - 4 = 8$. Theorem 43 below shows that these two non-planar graphs must appear inside every non-planar graph.

Theorem 43. *The graph G is planar if and only if either K_5 or $K_{3,3}$ can be found from contracting edges, removing edges, and/or removing vertices in G.*

For example, the Petersen graph is not planar because the gold edges shown below can be contracted to find K_5:



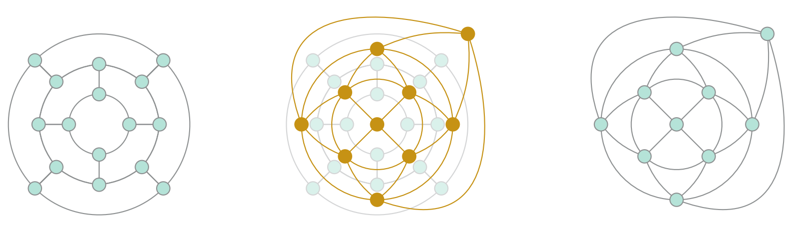
The proof of Theorem 43 is relatively long and technical, and, although it is within the capabilities of the reader, we choose to omit the proof.

Theorem 44. *If G is planar, then there is a vertex with degree* 5 *or less.*

Proof. If every vertex has degree 6 or more, then $2E = 6V$ means that $E \geq 3V > 3v - 6$, contradicting Theorem 42. □

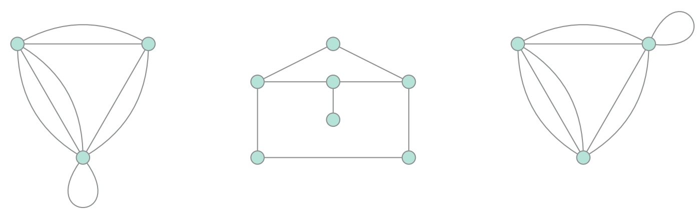
Definition. *The **dual** of a planar graph G, denoted G^*, is a graph found by drawing G in the plane without edge crossings and then taking the vertices of G^* as the faces of G with $\{f_1, f_2\}$ equal to an edge in G^* whenever faces f_1 and f_2 share an edge in G.*

Example 45. Below we display a graph G on the left, a dual G^* on the right, and a depiction of how to find a dual in the middle.



There may be many different dual graphs for a single graph G. Furthermore, a dual graph can have **loops** (an edge from a vertex to itself) or **multiple edges** between vertices. Theorems 41, 42, 43 and 44 all still hold for graphs with loops or multiple edges.

Example 46. The graphs on the right and left are duals of the graph in the center:



The graph on the right is the dual of a different planar embedding of the graph in the center, the embedding found by redrawing the degree 1 vertex inside of the triangle.

Theorem 47 (The five color theorem)**.** *If G is planar and without loops and G^* is a dual of G, then $\chi(G^*) \le 5$.*

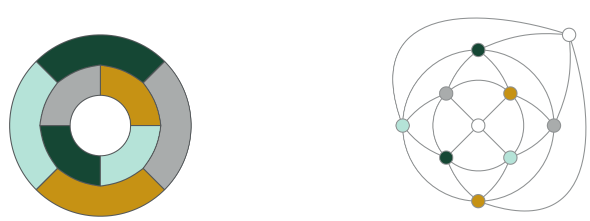
Proof. We proceed by induction on the number of vertices in G with the assertion true if G has a single vertex.

Since G is planar, so is G^*. By Theorem 44, G^* has a vertex v of degree 5 or less. By induction, $\chi(G^* - v) \le 5$, so there is a proper coloring of $G^* - v$ that uses at most 5 colors.

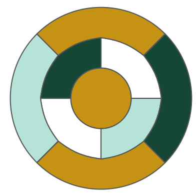
If the degree of v is less than 5, then it can be colored a color different than its neighbors. If the degree of v is 5 and all five vertices adjacent to v are different colors, there must be two such adjacent vertices u and w such that $\{u, w\}$ is not an edge because otherwise K_5 would be a subgraph of G^*.

Consider the graph G^* with the edges $\{v, u\}$ and $\{v, w\}$ contracted. By induction, the resulting graph can be properly colored using at most 5 colors. Reinstate the contracted edges and assign u and w the color of v. Now v is adjacent to vertices of only 4 different colors, meaning that G^* can be properly colored using at most 5 colors. □

Theorem 47 can be interpreted to say that the faces in a planar graph without loops can be colored with at most five different colors such that faces that share an edge are different colors. For example, below we show a coloring of the faces of the planar graph in Example 45 next to the corresponding proper coloring of the vertices in the dual graph. One of the colors we use is white, the color of the outside face.



A closer inspection of this graph reveals that it can be properly colored using four colors instead of five:



Indeed, Theorem 47 can be strengthened to the Four Color Theorem, stated here.

Theorem 48 (The four color theorem)**.** *If G is planar and without loops and G^* is a dual of G, then $\chi(G^*) \leq 4$.*

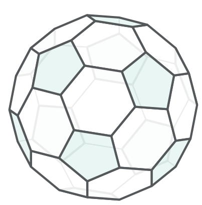
The proof of the four color theorem is famously difficult and no simple proof is known. All current versions of the proof reduce the situation to a careful analysis of a large but finite number of specific graphs which are then checked brute force by a computer. The proof of the four color theorem in 1979 was the first example of a proof that required a computer to complete and sparked a debate on the future role humans in mathematical theorem proving.

Definition. *A subset $C \subseteq \mathbb{R}^3$ is* ***convex*** *if the line segment from $\mathbf{x}$ to $\mathbf{y}$ is completely contained in C for every $\mathbf{x}, \mathbf{y} \in C$. The* ***convex hull*** *of a subset $S \subseteq \mathbb{R}^3$ is the intersection of all convex sets C which contain S.*

Example 49. A sphere is convex but like a banana, you are not convex.

Definition. *A convex polyhedron is the convex hull of a finite number of points in $\mathbb{R}^3$, called vertices, such that not all vertices are coplanar and no one vertex lies in the convex hull of the other vertices.*

Example 50. A soccer ball is a convex polyhedron, with the vertices each point of intersection of a hexagon and a pentagon.



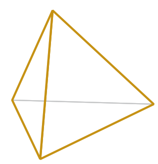
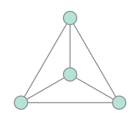
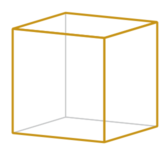
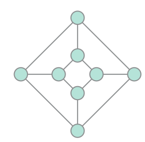
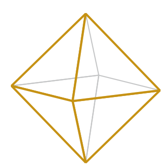
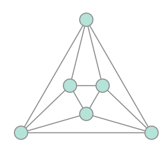
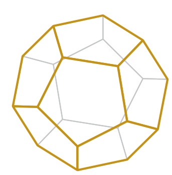
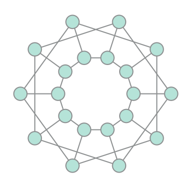
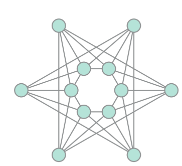
Each convex polyhedron with V vertices has an associated graph with V vertices, found by connecting two vertices in the graph if vertices in the convex polyhedron are connected by an edge. This way our definitions of vertex, edge, and face for convex polyhedra are the natural ones.

Theorem 51. *The graph of a convex polyhedron is planar.*

Proof. Enclose the convex polyhedron in large sphere. Consider the shadows cast on the sphere by the vertices and edges of P by a light source in the center of P. This is the graph of P drawn without edge crossings on a sphere, which can be used to draw the graph in the plane by using a map projection. □

Definition. *A **Platonic solid** is a convex polyhedron such that the same number of edges meet at each vertex and faces are congruent regular polygons.*

This table records five examples of Platonic solids. The graphs are planar even though we choose not to always exhibit a planar embedding.

Solid	V	E	F	Drawing	Graph
Tetrahedron	4	6	4		
Cube	8	12	6		
Octahedron	6	12	8		
Dodecahedron	20	30	12		
Icosahedron	12	30	20		

Theorem 52. *There are exactly five Platonic solids.*

Proof. Suppose a Platonic solid has V vertices, E edges, and F faces such that each face is a regular p-gon and each vertex joins q edges. Then we have $pF = 2E$ by counting the edges bordering each face and we have $qV = 2E$ by counting degrees. Since $V - E + F = 2$, we have $2E/q - E + 2E/p = 2$. Dividing by $2E$ gives the identity $1/E = 1/p + 1/q - 1/2$, which must be a positive number.

The values of p and q must be at least 3. If either p or q is greater than 5, the quantity $1/p+1/q-1/2$ is less than or equal to 0. The only values of p and q between 3 and 5 that make $1/p + 1/q - 1/2$ positive are recorded in this table:

p	q	$\frac{1}{p} + \frac{1}{q} - \frac{1}{2}$	Platonic Solid
3	3	$1/6$	tetrahedron
4	3	$1/12$	cube
3	4	$1/12$	octahedron
3	5	$1/30$	icosahedron
5	3	$1/30$	dodecahedron

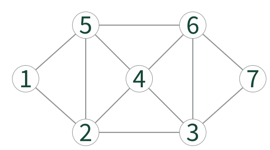
These are the five platonic solids. $\square$

— 5 —

Eulerian and Hamiltonian paths

Definition. *A **walk** is a sequence of vertices $v_1, \ldots, v_n$ such that v_i and v_{i+1} are adjacent for $i = 1, \ldots, n-1$. The difference between a walk and a path is that vertices in a path must be distinct. A **trail** is a walk where every edge is distinct. A graph is **Eulerian** if there is a trail that starts and ends at the same vertex and that uses every edge in the graph.*

Example 53. The following graph is Eulerian



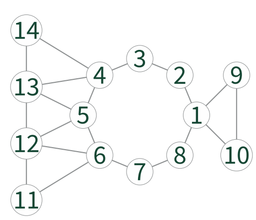
because of the trail 1, 5, 6, 7, 3, 6, 4, 3, 2, 5, 4, 2, 1.

Theorem 54. *A connected graph is Eulerian if and only if every vertex degree is even.*

Proof. If the graph is Eulerian, then the degree of a vertex v increases by 2 each time the Eulerian trail passes through v.

Now suppose every vertex in a connected graph G has an even degree. We show that G is Eulerian by induction on the number of edges in G.

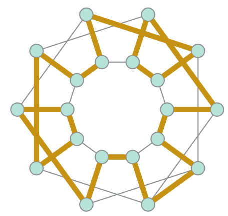
A cycle C must exist in G because otherwise G would be a tree and then have a degree 1 vertex, which is not even. Remove the edges in C from G. By induction, each connected component is Eulerian. Create an Eulerian trail for G by traveling around C, taking detours using the Eulerian trails along each component along the way. For example, if C is the cycle 1, 2, 3, 4, 5, 6, 7, 8, 1 in the graph below,



then an Eulerian trail is 1, 9, 10, 1, 2, 3, 4, 14, 13, 12, 11, 6, 12, 5, 13, 4, 5, 6, 7, 8, 1. □

Definition. *A graph is **Hamiltonian** if there is a cycle that contains every vertex.*

Example 55. The graph of the dodecahedron is Hamiltonian:



Theorem 56 (Bondy-Chvátal)**.** *Let G be a graph with $n \geq 3$ vertices and let u and v be non-adjacent vertices such that the sum of the degrees of u and v is at least n. Then G is Hamiltonian if and only if $G + \{u, v\}$ is Hamiltonian.*

Proof. If G is Hamiltonian, then $G + \{u, v\}$ is also clearly Hamiltonian.

The reverse implication is proved by contradiction. Assume $C = u, x_2, \ldots, x_{n-1}, v, u$ is a Hamiltonian cycle in $G + \{u, v\}$ and assume that G is not Hamiltonian. If some vertex x_i in this cycle C is adjacent to u, then x_{i-1} cannot be adjacent to v because otherwise $u, x_i, \ldots, x_{n-1}, v, x_{i-1}, \ldots, x_2, u$ would be a Hamiltonian cycle in G. Therefore the sum of the degrees of u and v is at most $n - 1$, a contradiction. □

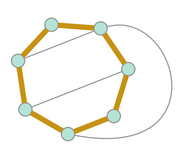
As a corollary of Theorem 56, if G has n vertices and all degrees are at least $n/2$, then G is Hamiltonian. This follows because we can keep adding edges between non-adjacent vertices until we find a complete graph. In general, deciding whether or not a given graph is Hamiltonian is a difficult problem (it is in a class of problems known as NP-complete), so theorems where certain conditions imply that graphs are Hamiltonian are probably the best we can hope for. Another example of such a theorem is given next.

Theorem 57. *Let C be a Hamiltonian cycle in a planar graph, let inside(i) be the number of i-edged faces inside C and outside(i) be the number of i-edged faces outside C. Then*

$$\sum_i (i-2)\left(\mathit{inside}(i) - \mathit{outside}(i)\right) = 0.$$

Proof. If C contains x inside chords, then there are $x + 1 = \sum_i \text{inside}(i)$ inside faces. Counting the edges around interior faces gives $\sum_i i\text{inside}(i) = 2x + n$ where the graph has n vertices. Combining these last two equations gives $\sum_i (i-2)\text{inside}(i) = n - 2$. Applying the same logic to outside edges gives $\sum_i (i-2)\text{outside}(i) = n - 2$. The statement in the theorem follows. □

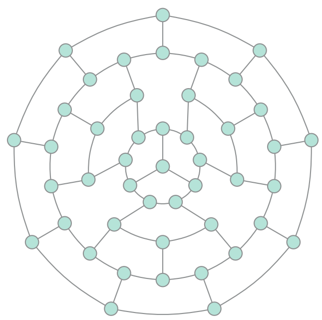
Example 58. As an example of Theorem 57, consider



This graph is Hamiltonian with the cycle highlighted in gold. As Theorem 57 says, the last column in the table below sums to 0.

i	inside(i)	outside(i)	$(i-2)\,(\text{inside}(i)-\text{outside}(i))$
3	1	0	1
4	2	1	2
5	0	1	-3

Example 59. Theorem 57 can be used to show that certain planar graphs are not Hamiltonian. Consider the following graph with 21 faces with 5 edges, 3 faces with 8 edges, and 1 face with 9 edges:

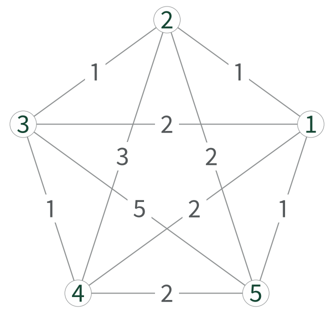


If the graph had a Hamiltonian cycle, then the face with 9 egdes would be an outside face and the last column in the below table would sum to 0 for some integers a, b:

i	inside(i)	outside(i)	$(i-2)\,(\text{inside}(i)-\text{outside}(i))$
5	a	$21-a$	$3(2a-21)$
8	b	$3-b$	$6(2b-3)$
9	0	1	-7

Setting the sum of the last column equal to 0 and simplifying gives $6a+12b=88$. There are no integer solutions to this equation because the left side is divisible by 3 but 88 is not. Thus the graph is not Hamiltonian.

The famous Traveling Salesman Problem is closely related to the problem of deciding whether or not a graph has a Hamiltonian cycle. Assign positive values to the edges in a complete graph K_n. These weights represent the cost to use the edge. The Traveling Salesman Problem asks to find a minimum weight Hamiltonian cycle. For example, the weighted complete graph K_5 shown here



has minimum weight Hamiltonian cycle given by 1, 2, 3, 4, 5, 1. It is notoriously difficult to find an exact solution to a large Traveling Salesman Problem, but there are approximate heuristic solutions that can be found quickly which are within small percentages of the exact solution.

— 6 —

Connectivity

Definition. *A **disconnecting set** D of edges in a graph G is a set of edges such that $G - D$ is disconnected. The **edge connectivity** $\varepsilon(G)$ is the smallest size of a disconnecting set.*

Example 60. We see that $\varepsilon(K_n) = n - 1$, $\varepsilon(C_n) = 2$, and $\varepsilon(T) = 1$ for trees T.

Definition. *A **bridge** is an edge such that $\{e\}$ is a disconnecting set.*

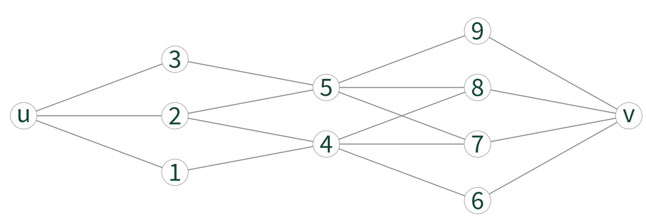
Theorem 61. *Let G be connected with edge $e = \{u, v\}$. Then e is the only path from u to v if and only if e is a bridge.*

Proof. If e is the only path from u to v, then $G - e$ has no path from u to v and is therefore disconnected.

On the other hand, suppose G_1, G_2 are distinct components of $G - e$ such that u is in G_1. Suppose w is in G_2 and let P be a path from u to w in G. The path P must use the edge e, meaning that v must be the second vertex on the path P, showing that v is in C_2. Thus there are no other paths from u to v other than the path u, v that uses the edge e. □

Definition. *A u, v-**disconnecting set** is a set of E of edges in G such that u and v are in different components of $G - E$.*

Example 62. If G is the graph shown below,

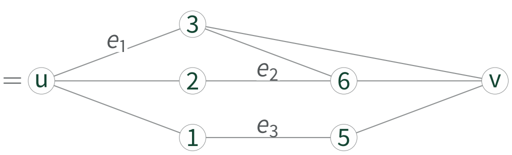


then the minimum size of a u, v disconnecting set is 3 because the $\{u, 1\}$, $\{u, 2\}$, $\{u, 3\}$ edges can be removed to disconnect u and v. In this example there also happen to be 3 paths from u to v that are edge disjoint (meaning that each edge is used at most once in any of the paths): $u, 1, 4, 6, v$ and $u, 2, 4, 7, v$ and $u, 3, 5, 8, v$.

Theorem 63 (Menger, edge version)**.** *The maximum number of edge disjoint paths from u to v is equal to the minimum number of edges in a u, v disconnecting set.*

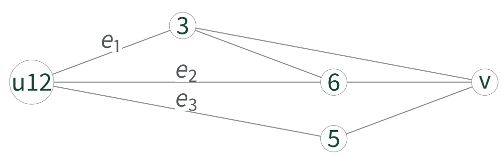
Proof. We prove this by induction on the number of edges in G. The theorem is true if G has no edges.

Case 1: There is a minimal u, v disconnecting set E that has an edge not incident to u and an edge not incident to v.

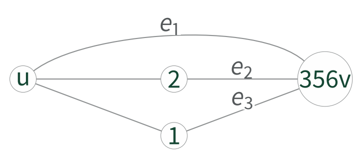
$G =$ 

The above graph G depicts one such situation, with $E = \{e_1, e_2, e_3\}$.

Let G_u be the graph G with all vertices in the component of $G - E$ containing u merged. For example, using the above G, we have

$G_u =$ 

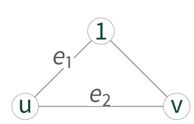
and

$G_v =$ 

The set E is still a disconnecting set for both G_u and G_v of minimum size, so by induction both G_u and G_v have the correct number of edge disjoint paths from u to v. Combine these paths in the natural way to find the correct number of edge disjoint paths from u to v in G.

In the above examples, the edge disjoint paths from $u12$ to v in G_u are $u12, 3, v$ and $u12, 6, v$ and $u12, 5, v$. The edge disjoint paths from u to $356v$ in G_v are $u, 356v$ and $u, 2, 356v$ and $u, 1, 356v$. Combining these paths gives the edge disjoint paths $u, 3, v$ and $u, 2, 6, v$ and $u, 1, 5, v$.

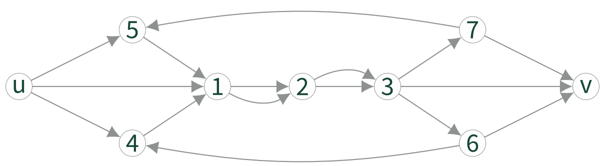
Case 2: Suppose we are not in Case 1.



There cannot be 2 edges on a path from u to v since otherwise not every edge in a u, v disconnecting set would need an edge incident to u or v. Remove a path P from u to v. There is now one less edge needed in a u, v disconnecting set and one less path from u to v. We are now done by induction. □

Definition. *A **directed graph**, or **digraph**, is a graph where each edge is given a direction. A **simple** graph is a graph that is not a directed graph, has no loops or multiple edges, and does not have weighted edges. Usually the unqualified term "graph" refers to a simple graph unless otherwise stated.*

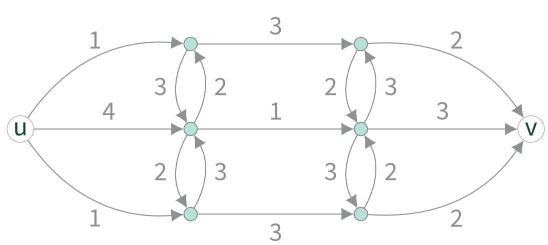
Example 64. A directed graph with multiple edges is shown below:



The proof of Theorem 63 still holds for directed graphs with multiple edges. So, for instance, the graph in Example 64 has a minimum of two edges in a u, v disconnecting set because if the two edges between vertices 1 and 2 are removed then there is not a path from 1 to 2 (and therefore the resulting graph is disconnected). Two edge disjoint paths from u to v are $u, 5, 1, 2, 3, 7, v$ and $u, 1, 2, 3, 6, v$.

Definition. *A **network** is a directed graph where nonnegative rational weights are assigned to each edge. The **in-degree** of a vertex v in a network is the sum of the weights of the edges that point to v and the **out-degree** is the sum of the weights of the edges that leave v.*

Example 65. A network is shown below:

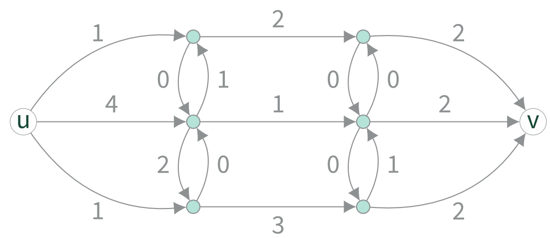


Such a graph can be used to model the number of cars that can drive from one city to another in an hour, travel times, the capacity of a water filled pipes, or the weights on the edges can represent multiple edges between nodes.

Definition. *A **flow** from u to v in a network N is a network N′ such that the weight on an edge in N′ is less than or equal to the weight on the corresponding edge in N and such that the in-degree and out-degree are the same for all vertices in except for u and v. The **flow value** is the out-degree of u in N′.*

Using the analogy of the edge weights in a network representing the amount of water that can move through a pipe, the flow of the network models water flowing from u to v.

Example 66. A flow from u to v for the graph in Example 65 is shown below:



The flow value is 6 because the out-degree of u (and the in-degree of v) is 6. We also see that there is a u, v disconnecting set with edge weights that sum to 6 because the edges surrounding u can be deleted to disconnect the graph.

Theorem 67 (Max-Flow Min-Cut)**.** *Let N be a network with vertices u, v. The maximum value for a flow from u to v is equal to the minimum weight u, v disconnecting set.*

Proof. If not all edge weights are integers, then we can multiply all rational edge weights by a large constant to find an equivalent problem involving integer edge weights.

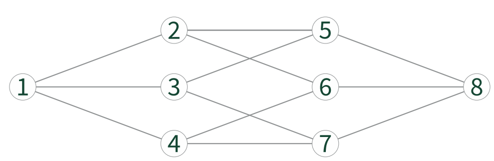
When all edge weights are integers, the network can be considered a directed graph with multiple edges where the edge weight in the network represents the number of edges between vertices. The value of a maximum flow from u to v is then the number of edge disjoint paths from u to v. By Theorem 63, this is also the size of a u, v disconnecting set, which is equal to the minimum weight u, v disconnecting set in the network. □

Definition. *A **separating set** for graph G that is not a complete graph is a set of vertices V such that $G - V$ is not connected. The **vertex connectivity** $\kappa(G)$ is the minimum size of a separating set and we set $\kappa(K_{n-1}) = n - 1$. A u, v-**separating set** is a separating set V such that u and v are in different components of $G - V$.*

Example 68. We have $\kappa(C_n) = 2$ and $\kappa(T) = 1$ for any tree T.

It can be seen that $\kappa(G) \leq \varepsilon(G)$ for any graph G. Indeed, if E is a set of edges such that $G - E$ is disconnected, then taking V to be a set of vertices such that $\{u, v\}$ is an edge in E causes $G - V$ to be disconnected.

Example 69. A minimal size 1, 8-separating set for the graph shown below is $\{2, 3, 4\}$.



This is also the maximum number of paths from 1 to 8 that are vertex disjoint: $1, 2, 5, 8$ and $1, 3, 7, 8$ and $1, 4, 6, 8$. With the exception of the start and end vertices, these three paths do not share a vertex.

Theorem 70 (Menger, vertex version)**.** *The maximum number of vertex disjoint paths from u to v is equal to the minimum number of vertices in a u, v-separating set.*

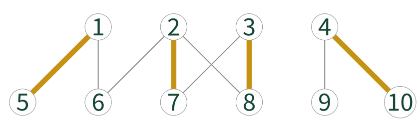
The proof of the vertex version of Menger's theorem is very similar to the proof of the edge version of Menger's theorem given in Theorem 63 with every instance if "edge" replaced with "vertex". We leave the details to the reader as an exercise.

— 7 —

Matchings

Definition. *A **matching** in a graph G is a subset M of edges such that no two edges in M are incident. The matching **saturates** a subset X of vertices in G if every vertex in X is incident to an edge in M.*

Example 71. The edges in an example of a matching in a bipartite graph is highlighted below:



This matching saturates the set $\{1, 2, 3, 4\}$.

Theorem 72 (Hall)**.** *Let G be a bipartite graph with independent sets X and Y. For a subset S of vertices, let $N(S)$ be the set of vertices in G that are adjacent to a vertex in S. Then there is a matching for G that saturates X if and only if $|S| \leq |N(S)|$ for all $S \subseteq X$.*

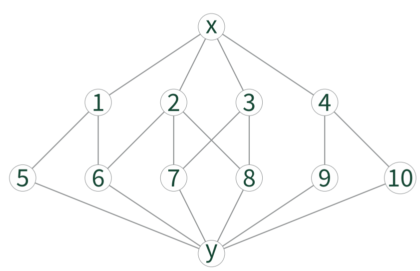
Example 73. Before its proof we illustrate the statement of Hall's matching theorem by continuing Example 71. The set $X = \{1, 2, 3, 4\}$ and so there are 15 nontrivial subsets $S \subseteq X$ to check:

S	$N(S)$	$\lvert S\rvert \leq \lvert N(S)\rvert$?
$\{1\}$	$\{5, 6\}$	Yes
$\{2\}$	$\{6, 7, 8\}$	Yes
$\{3\}$	$\{7, 8\}$	Yes
$\{4\}$	$\{9, 10\}$	Yes
$\{1, 2\}$	$\{5, 6, 7, 8\}$	Yes
$\{1, 3\}$	$\{5, 6, 7, 8\}$	Yes
$\{1, 4\}$	$\{5, 6, 9, 10\}$	Yes
$\{2, 3\}$	$\{6, 7, 8\}$	Yes
$\{2, 4\}$	$\{6, 7, 8, 9, 10\}$	Yes
$\{3, 4\}$	$\{7, 8, 9, 10\}$	Yes
$\{1, 2, 3\}$	$\{5, 6, 7, 8\}$	Yes
$\{1, 2, 4\}$	$\{5, 6, 7, 8, 9, 10\}$	Yes
$\{1, 3, 4\}$	$\{5, 6, 7, 8, 9, 10\}$	Yes
$\{2, 3, 4\}$	$\{6, 7, 8, 9, 10\}$	Yes
$\{1, 2, 3, 4\}$	$\{5, 6, 7, 8, 9, 10\}$	Yes

In all cases we have $|S| \leq |N(S)|$, so Hall's matching condition says that there is a matching that saturates $\{1, 2, 3, 4\}$. On the other hand, there is not a matching that saturates $\{5, 6, 7, 8, 9, 10\}$ because if $S = \{9, 10\}$, then $N(S) = \{4\}$ and so the inequality $|S| \leq |N(S)|$ does not hold.

Proof. Suppose that a matching that saturates X exists and let $S \subseteq X$. Each vertex in S is matched with a unique vertex in $N(S)$, and so $|S| \leq |N(S)|$.

Now suppose that $|S| \leq |N(S)|$ for all subsets S of vertices in X. Let G' be the graph G with two extra vertices: a vertex x that is adjacent to every vertex in X and a vertex y that is adjacent to every vertex in Y. For instance, the graph G' for the graph shown in Example 71 is



Let A be a subset of X and B be a subset of Y such that the union $A \cup B$ is an x, y separating set. This means that there is not an edge that connects a vertex in $X - A$ to a vertex in $Y - B$ and so $N(X - A)$ must be a subset of B. Therefore, using the hypothesis that $|S| \leq |N(S)|$ in the case where $S = X - A$, we have

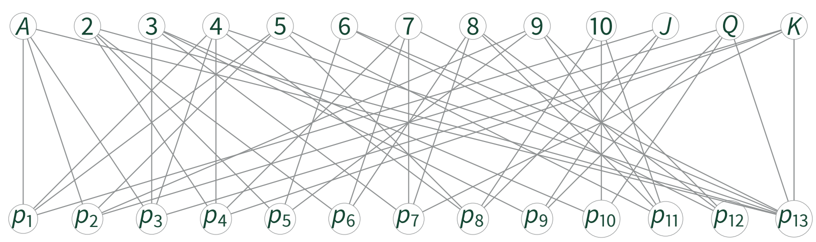
$$|X| - |A| = |X - A| \leq |N(X - A)| \leq |B|.$$

This implies $|X| \leq |A| + |B| = |A \cup B|$, meaning that the size of any x, y separating set must at least as large as the number of vertices in X. By the vertex version of

Menger's theorem (Theorem 70), there are at least $|X|$ vertex disjoint paths from x to y. These vertex disjoint paths correspond to a matching that saturates X. $\square$

Example 74. A standard deck of playing cards is shuffled and sorted into 13 piles of 4 cards. Why is it possible to take one card from each pile to form the set $\{A, \ldots, K\}$?

Create a bipartite graph with one set of vertices given by the cards $A, \ldots, K$, the second set of vertices given by the piles $p_1, \ldots, p_{13}$, and with an edge from a card to a pile if that card appears in the pile. For instance, one such graph is shown here

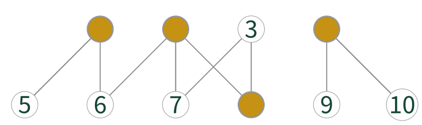


In this example, an A appears in piles p_1, p_2, p_3 and p_{13}, a 2 appears in piles p_4, p_5, p_6 and p_{13}, and so on.

If S is a subset of $\{A, \ldots, K\}$, then there are $4|S|$ cards that must appear in at least $|S|$ different piles since each pile contains 4 cards. Thus $|S| \leq |N(S)|$ and we have verified Hall's matching condition. The matching that saturates $\{A, \ldots, K\}$ corresponds to the desired ability to take one card from each pile to form the set $\{A, \ldots, K\}$.

Definition. *A* ***covering*** *for a graph G is a set of vertices X such that every edge in G is incident to a vertex in X.*

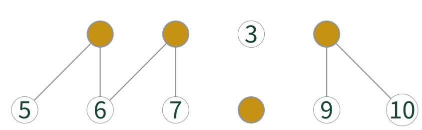
Example 75. Below we indicate a covering for the bipartite graph in Example 71:



Theorem 76 (Kőnig)**.** *The maximum number of edges in a matching in a bipartite graph is equal to the minimum number of vertices in a covering.*

Proof. Let Q be a vertex cover that uses the minimum number of vertices. If M is any matching, then $|M| \leq |Q|$ because each edge in a matching is incident to at least one vertex in Q. To complete the proof we will show that there is a matching M with $|M| = |Q|$, showing that equality can be achieved.

Let G have independent sets X and Y. Let G_x be the graph G but with the edges in G that connect vertices in $Q \cap X$ with vertices in $Y - Q$. For example, the graph G_x coming from Example 75 is shown below

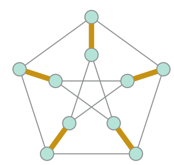


Let S be a subset of vertices in $Q \cap X$ in G_x. It follows that $N(S)$ is a subset of $Y - Q$ in G_x that satisfies $|S| \leq |N(S)|$ because otherwise we can replace S with $N(S)$ in Q to find a covering for G that uses fewer vertices than Q. Thus, by Hall's matching condition (our Theorem 72), there is a matching M_x that saturates $Q \cap X$.

Using similar logic on the graph G_y formed using in G that connect vertices in $Q \cap Y$ with vertices in $X - Q$, we find a matching M_y that saturates $Q \cap Y$. Our desired matching is $M_x \cup M_y$. □

Definition. *A matching M is **perfect** if every vertex is incident to an edge in M.*

Example 77. A perfect matching for the Petersen graph is indicated below:



Theorem 78 (Tutte)**.** *For any subset S of vertices in a graph G, let $\mathrm{odd}_G(S)$ denote the number of components of $G - S$ that have an odd number of vertices. Then G has a perfect matching if and only if $\mathrm{odd}_G(S) \leq |S|$ for all subsets S of vertices.*

Proof. Assume that G has a perfect matching. Each of the odd components in $G - S$ must have vertices matched to distinct vertices in S, and so $\mathrm{odd}_G(S) \leq |S|$ for all subsets S of vertices.

Now assume that $\mathrm{odd}_G(S) \leq |S|$ for all subsets S of vertices. We will prove that G has a perfect matching using induction on the number of vertices in G.

By taking S as the empty set, we see that $\mathrm{odd}_G(S) \leq 0$, meaning that G must have an even number of vertices. By counting vertices it follows that the parity of $|S|$ and $\mathrm{odd}_G(S)$ are the same for any subset S and thus $|S|$ and $\mathrm{odd}_G(S)$ cannot differ by 1. This permits us to can break the problem into the following two cases.

Case 1: Every subset S of vertices satisfies $\mathrm{odd}_G(S) \leq |S| - 2$.

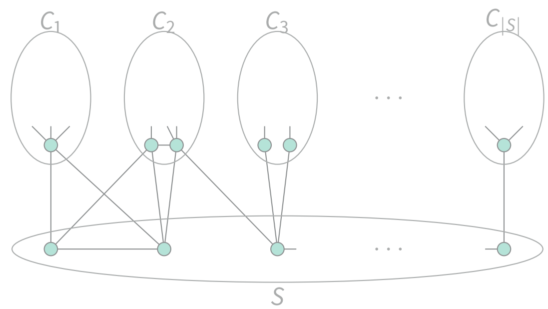
Let u and v be adjacent vertices in G, let $G' = G - u - v$, and let T be any subset of vertices in G'.

$$\mathrm{odd}_{G'}(T) = \mathrm{odd}_G(T \cup \{u, v\}) \leq |T \cup \{u, v\}| - 2 = |T|$$

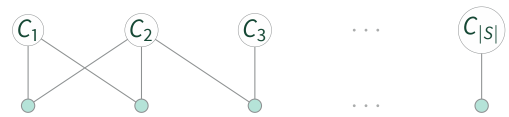
and so we can form a matching by matching u and v and then matching vertices in G' by induction.

Case 2: There is a subset S of vertices such that $\mathrm{odd}_G(S) = |S|$.

Take such an S with the maximum possible number of vertices. Each component $C_1, \ldots, C_{|S|}$ of $G - S$ must have an odd number of vertices because otherwise we can remove one vertex from an even sized component and add it to S, thereby increasing the size of S.



Let G' be the bipartite graph with independent sets $\{C_1, \ldots, C_{|S|}\}$ and S and with an edge between C_i and v_j in G' if there is an edge from component C_i to v_j in G.



Let T be a subset of $\{C_1, \ldots, C_{|S|}\}$ and let $N(T)$ be the vertices in S that are adjacent to a vertex in T in G'. We have $|T| \leq \text{odd}_G(N(T))$ because each component in T is an odd sized component counted by $\text{odd}_G(N(T))$. Using the set $N(T)$ in the hypothesis of the theorem, we have

$$|T| \leq \text{odd}_G(N(T)) \leq |N(T)|$$

for all subsets T of $\{C_1, \ldots, C_{|S|}\}$. By Hall's matching condition there is a matching M' for G' that saturates $\{C_1, \ldots, C_{|S|}\}$ and, since $\text{odd}_G(S) = |S|$, this matching also saturates S.

Let C be a graph found by taking one of the components C_i in $G - S$ and removing the vertex v found in the matching M'. To extend the matching M' to a matching for G we need to find a matching for C. This can be done by induction provided $\text{odd}_C(U) \leq |U|$ for all subsets U of vertices in C.

Suppose to the contrary that $\text{odd}_C(U) > |U|$. Since these quantities have the same parity they cannot differ by 1 and so $\text{odd}_C(U) \geq |U| + 2$. Thus we have

$$\begin{aligned}\text{odd}_G(S \cup U \cup \{v\}) &= \text{odd}_G(S) + \text{odd}_C(U) - 1\\ &\geq |S| + |U| + 1\\ &= |S \cup U \cup \{v\}|,\end{aligned}$$

meaning that S is not the set with the maximum number of vertices that satisfies $\text{odd}_G(S) = S$ as it could be replaced by $S \cup U \cup \{v\}$. This completes the proof. $\square$

Example 79. Let G be a graph such that every vertex has degree 3 and such that $\varepsilon(G) \geq 2$. We can use Theorem 78 to show that G has a perfect matching.

Let S be any set of vertices and let H be an odd component of $G - S$. We have

$$(\text{the sum of degrees in } H) = 3|H| - (\text{the number of edges from } H \text{ to } S)$$

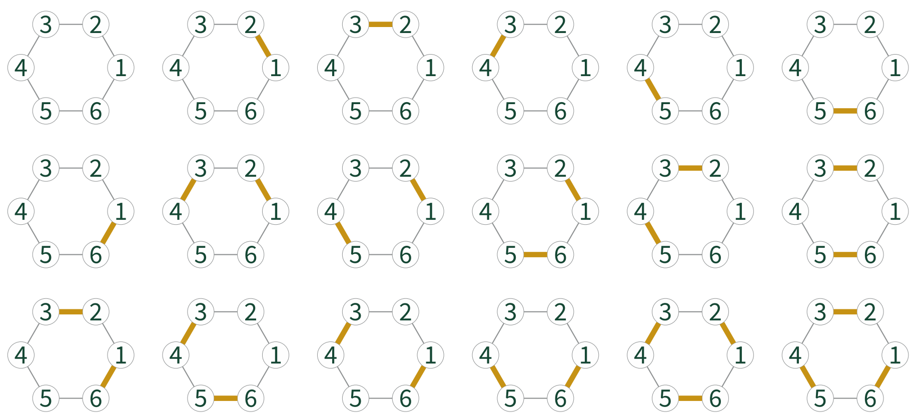
Since (the sum of degrees in H) is even and $3|H|$ is odd, there are an odd number of edges from H to S. Using the fact that $\varepsilon(G) \geq 2$, there must be at least 3 edges between H and S.

Since each odd component connects at least 3 times with S and since every vertex in S has degree 3, we have $\text{odd}_G(S) \leq |S|$, as needed.

Definition. *Let $m_G(k)$ denote the number of matchings for G that have exactly k edges. The* ***matching polynomial*** *for the graph G with n vertices is*

$$M_G(x) = \sum_k (-1)^k m_G(k) x^{n-2k}.$$

Example 80. All possible matchings of C_6 are shown below:



One of these matchings has 0 edges, 6 matchings have 1 edge, 9 matchings have 2 edges, and 2 matchings have 3 edges. Therefore the matching polynomial for C_6 is

$$M_{C_6}(x) = x^6 - 6x^4 + 9x^2 + 2.$$

Theorem 81. *Suppose G has $n \geq 3$ vertices and $e = \{u, v\}$ is an edge in G. Then*

$$M_G(x) = M_{G-e}(x) - M_{G-u-v}(x).$$

Proof. By counting whether or not e is used in a matching, we have

$$m_G(k) = m_{G-e}(k) + m_{G-u-v}(k-1)$$

and therefore

$$\begin{aligned}
M_G(x) &= \sum_k (-1)^k m_G(k) x^{n-2k} \\
&= \sum_k (-1)^k m_{G-e}(k) x^{n-2k} + \sum_k (-1)^k m_{G-u--v}(k-1) x^{n-2k} \\
&= M_{G-e}(x) - \sum_k (-1)^{k-1} m_{G-u-v}(k-1) x^{(n-2)-2(k-1)} \\
&= M_{G-e}(x) - M_{G-u-v}(x).
\end{aligned}$$

□

Example 82. Using Theorem 81 on the first edge in a path graph, we have that the matching polynomial for P_{n+1} satisfies the recursion

$$M_{P_{n+1}}(x) = xM_{P_n}(x) - M_{P_{n-1}}(x)$$

for $n \geq 2$ with the initial conditions that $M_{P_0}(x) = 1$ and $M_{P_1}(x) = x$. These polynomials are related to the Chebyshev polynomials of the second kind, defined by the recursion

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$$

with $U_0(x) = 1$ and $U_1(x) = 2x$. Comparing these recursions shows $M_{P_n}(x) = U_n(x/2)$.

Example 83. Using Theorem 81 on cycle graphs, we have $M_{C_n}(x) = M_{P_n}(x) - M_{P_{n-2}}(x)$ for $n \geq 3$. These polynomials are related to the Chebyshev polynomials of the first kind, defined by

$$T_n(x) = \frac{1}{2}\left(U_n(x) - U_{n-2}(x)\right).$$

Comparing these expressions shows that $M_{C_n}(x) = 2T_n(x/2)$.

Theorem 84. *Let G have $n \geq 3$ vertices. Then for any vertex u we have*

$$M_G(x) = xM_{G-u}(x) - \sum_{v \text{ is adjacent to } u} M_{G-u-v}(x).$$

Proof. If u has degree 0, then $M_G(x) = xM_{G-u}(x)$. We continue by induction on the degree of u.

If $e = \{u, w\}$ is an edge in G, then by Theorem 81 we have

$$\begin{aligned}
M_G(x) &= M_{G-e}(x) - M_{G-u-w}(x) \\
&= xM_{G-u}(x) - \sum_{v \neq w \text{ is adjacent to } u} M_{G-u-v}(x) - M_{G-u-w}(x) \\
&= xM_{G-u}(x) - \sum_{v \text{ is adjacent to } u} M_{G-u-v}(x).
\end{aligned}$$

□

Example 85. Using Theorem 84 on complete graphs, we have $M_{K_0}(x) = 1$, $M_{K_1}(x) = x$, and $M_{K_n}(x) = xM_{K_{n-1}}(x) - nM_{K_{n-2}}(x)$ for $n \geq 2$. These polynomials are related to the probabilist's Hermite polynomials, defined by

$$H_n(x) = xH_{n-1}(x) - nH_{n-2}(x)$$

with the same initial conditions as $M_{K_n}(x)$. Comparing these expressions shows that $M_{K_n}(x) = H_n(x)$.

— 8 —

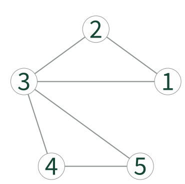
The adjacency matrix

In this chapter we assume knowledge of basic operations in matrix algebra that are usually found in a first course on the topic: Matrix multiplication, properties of transposes, linear independence, finding eigenvalues and eigenvectors, the characteristic polynomial, and diagonalization.

The most interesting connections between graph theory and matrix algebra use theorems that students who have only taken a single matrix algebra course may not have seen yet. When such a theorem is needed, we will simply state the theorem without proof. These proofs of such theorems can be found in most matrix algebra texts.

Definition. *Let G have vertices $v_1, \ldots, v_n$. The* ***adjacency matrix*** *$A(G)$ is the $n \times n$ matrix with i, j entry equal to 1 if there is an edge from v_j to v_i and 0 otherwise. If we have a graph with directed or weighted edges, then this i, j entry is the weight of the edge from v_j to v_i.*

Example 86. A graph G and its adjacency matrix $A(G)$ are shown below:



$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Theorem 87. *The i, j entry of $A(G)^k$ is the number of walks of length k that start at v_i and end at v_j.*

Proof. We show this by induction on k with the assertion true when $k = 1$. If we let $A_{i,j}$ denote the i, j entry of the matrix A, then the definition of matrix multiplication gives that the i, j entry of $A(G)^{k+1} = A(G)A(G)^k$ is

$$\begin{aligned} &\sum_{\ell=1}^{n} A(G)_{i,\ell} (A(G))^k_{\ell,j} \\ &\quad = \sum_{\ell=1}^{n} \left(\begin{cases} 1 & \text{if } v_i, v_\ell \text{ are adjacent} \\ 0 & \text{if not} \end{cases} \right) (\text{\# walks of length } k \text{ from } v_\ell \text{ to } v_j) \\ &\quad = (\text{\# walks of length } k+1 \text{ from } v_i \text{ to } v_j) \end{aligned}$$

□

Example 88. Continuing the example in Example 86, we have

$$A(G)^2 = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 4 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix} \qquad \text{and} \qquad A(G)^3 = \begin{bmatrix} 2 & 3 & 5 & 2 & 2 \\ 3 & 2 & 5 & 2 & 2 \\ 5 & 5 & 4 & 5 & 5 \\ 2 & 2 & 5 & 2 & 3 \\ 2 & 2 & 5 & 3 & 2 \end{bmatrix}$$

This says, for example, there are 4 walks of length 2 from vertex 3 back to vertex 3 in G and there are 2 walks of length 3 from vertex 1 vertex 5.

Theorem 89. *Let $\mathrm{tr}(A)$ denote the matrix trace (the sum of the diagonal entries in A). Then*

a. $\mathrm{tr}(A(G)^2)/2$ *is equal to the number of edges in G, and*

b. $\mathrm{tr}(A(G)^3)/6$ *is equal to the number of triangles (cycles of length 3) in G.*

Proof. The i, i diagonal entry in $A(G)^2$ gives the number of paths from v_i back to itself, which counts each edge incident to v_i. Summing the diagonal elements in $A(G)^2$ therefore counts every edge twice.

Similarly, every walk from v_i to itself of length 3 counts a triangle. Each triangle is counted six times in the trace of $A(G)^3$, twice for each of the three vertices in the triangle. □

If G is a simple graph, then the adjacency matrix $A(G)$ is a real symmetric matrix (meaning $A(G)^\top = A(G)$). Real symmetric matrices are the easiest class of matrices to understand. There are a number of theorems that give great information about real symmetric matrices. One such theorem is the spectral theorem, stated below without proof.

Theorem 90 (Spectral theorem)**.** *If A is a real valued symmetric matrix, then all eigenvalues of A are real and there is an orthonormal basis of eigenvectors. This implies A can be diagonalized using an orthogonal matrix P, which says that*

$$A(G) = P \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} P^{-1}$$

where $\lambda_1, \ldots, \lambda_n$ are the real eigenvalues of A and where $P^{-1} = P^\top$.

Definition. *A graph G has **eigenvalue** λ and **eigenvector** $\mathbf{v}$ if λ is an eigenvalue and $\mathbf{v}$ is an eigenvector for the adjacency matrix $A(G)$. This means that $A(G)\mathbf{v} = \lambda\mathbf{v}$.*

Example 91. Continuing the example in Example 86 and doing the calculations on a computer algebra system, we find the eigenvalues for G are

$$\frac{1+\sqrt{17}}{2}, 1, -1, -1, \frac{1-\sqrt{17}}{2}.$$

If we take the matrices P and D to equal

$$P = \begin{bmatrix} 1 & -1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & 1 \\ -\frac{1-\sqrt{17}}{2} & 0 & 0 & 0 & -\frac{1+\sqrt{17}}{2} \\ 1 & 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} \frac{1+\sqrt{17}}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-\sqrt{17}}{2} \end{bmatrix},$$

Then we have $A(G) = PDP^{-1}$. Now that we have diagonalized the adjacency matrix it is relatively easy to find powers of the matrix A. Indeed,

$$A(G)^k = PD^kP^{-1} = P \begin{bmatrix} \left(\frac{1+\sqrt{17}}{2}\right)^k & 0 & 0 & 0 & 0 \\ 0 & 1^k & 0 & 0 & 0 \\ 0 & 0 & (-1)^k & 0 & 0 \\ 0 & 0 & 0 & (-1)^k & 0 \\ 0 & 0 & 0 & 0 & \left(\frac{1-\sqrt{17}}{2}\right)^k \end{bmatrix} P^{-1}$$

By explicitly doing the above matrix multiplication, we can find formulas for the number of walks from one vertex to another. For example, using a computer algebra system, we find that the row 1 column 3 entry of $A(G)^k$ is

$$\frac{1}{\sqrt{17}}\left(\left(\frac{1+\sqrt{17}}{2}\right)^k - \left(\frac{1-\sqrt{17}}{2}\right)^k\right),$$

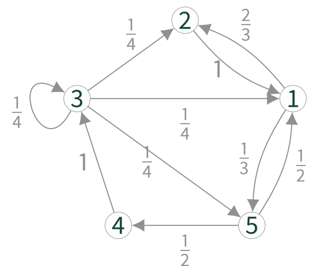
so this gives the number of walks of length k From vertex 1 to vertex 3 in G. As another example, the trace of $A(G)^k$ gives the number of walks of length k that start and end at the same vertex. Since the trace of a matrix satisfies $\operatorname{tr}\left(PD^kP^{-1}\right) = \operatorname{tr} D^k$, the number of walks of length k that start and end at the same vertex is

$$\left(\frac{1+\sqrt{17}}{2}\right)^k + 1^k + (-1)^k + (-1)^k + \left(\frac{1-\sqrt{17}}{2}\right)^k.$$

Theorem 92. *If G eigenvalues $\lambda_1, \ldots, \lambda_n$, then the number of walks of length k that start and end at the same vertex is $\lambda_1^k + \cdots + \lambda_n^k$.*

Proof. Since $A(G)$ is symmetric, it can be diagonalized. Let P be a matrix such that $A(G) = PDP^{-1}$ where D is a diagonal matrix with the eigenvalues $\lambda_1, \ldots, \lambda_n$ along the diagonal. Then $A(G)^k = (PDP^{-1})^k = PD^kP^{-1}$ has trace equal to $\operatorname{tr} D^k$. The matrix D^k has trace $\lambda_1^k + \cdots + \lambda_n^k$, so we are done by Theorem 87. □

Example 93. Theorem 92 still holds for directed or graphs with weighted edges, provided the adjacency matrix is still diagonalizable, a fairly common situation. For example, the network N shown below



has adjacency matrix

$$A(N) = \begin{bmatrix} 0 & 1 & 1/4 & 0 & 1/2 \\ 2/3 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1/2 \\ 1/3 & 0 & 1/4 & 0 & 0 \end{bmatrix}$$

that is diagonalizable with eigenvalues that are approximately

$$1, 0.361613, -0.899806, -0.105904 + 0.341818i, -0.105904 - 0.341818i$$

and so the result in Theorem 92 holds for this graph. This means that $\lambda_1^k + \cdots + \lambda_n^k$ is an integer for all nonnegative integers k, even though there are complex valued eigenvalues.

Definition. *A network is **strongly connected** if for every pair of vertices u, v there exists a walk from u to v in which no edge has weight 0. A **probability vector** is a vector that has nonnegative components that sum to 1.*

The Perron-Frobenius theorem gives interesting information about the eigenvalues and eigenvectors of square matrices with nonnegative entries, which is relevant since the adjacency matrix for any network is such a matrix. We state the next theorem without proof.

Theorem 94 (Perron-Frobenius)**.** *If A is the adjacency matrix for a strongly connected network, then the following statements are true.*

a. *There is a positive real number $\lambda_{\max}$ called the **Perron value** such that $\lambda_{\max}$ is an eigenvalue for A and such that every eigenvalue λ of A satisfies $|\lambda| \leq \lambda_{\max}$.*

b. *There is a vector $\mathbf{v}$ with strictly positive entries called the **Perron vector** such that $\mathbf{v}$ is both a probability vector and an eigenvector with eigenvalue $\lambda_{\max}$.*

c. *Any other eigenvector of A with eigenvalue $\lambda_{\max}$ is a scalar multiple of the Perron vector.*

d. *No other eigenvector of A besides scalar multiples of the Perron vector can have all positive components.*

Example 95. The graph in Example 86 has largest eigenvalue $(1+\sqrt{17})/2$ and so this is the Perron value. The Perron vector is

$$\frac{1}{16}\begin{bmatrix} 7-\sqrt{17} \\ 7-\sqrt{17} \\ 4\sqrt{17}-12 \\ 7-\sqrt{17} \\ 7-\sqrt{17} \end{bmatrix} \approx \begin{bmatrix} 0.18 \\ 0.18 \\ 0.28 \\ 0.18 \\ 0.18 \end{bmatrix}.$$

Example 96. The graph in Example 93 has largest eigenvalue 1 and so this is the Perron value. All other eigenvalues have complex magnitude less than 1. The Perron vector is

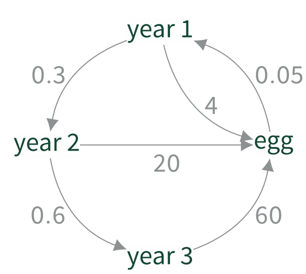
$$\frac{1}{39}\begin{bmatrix} 15 \\ 11 \\ 4 \\ 3 \\ 6 \end{bmatrix} \approx \begin{bmatrix} 0.385 \\ 0.282 \\ 0.103 \\ 0.077 \\ 0.153 \end{bmatrix}.$$

If A is the adjacency matrix for a strongly connected network with Perron value $\lambda_{\max}$ and Perron vector $\mathbf{v}$, then $\mathbf{v}$ satisfies $A\mathbf{v}/\lambda_{\max} = \mathbf{v}$, meaning that $\mathbf{v}$ is a fixed point under multiplication by $A/\lambda_{\max}$. This is part of the reason why the Perron vector arises in applications as it can be found by repeatedly multiplying by $A/\lambda_{\max}$. Indeed, if $\mathbf{y} = \lim_{k\to\infty} A^k\mathbf{x}/\lambda_{\max}^k$ exists for some vector $\mathbf{x}$, then this limit is a multiple of the Perron vector. This is because

$$\frac{A\mathbf{y}}{\lambda_{\max}} = \lim_{k\to\infty}\frac{A^{k+1}\mathbf{x}}{\lambda_{\max}^{k+1}} = \mathbf{y},$$

meaning that $\mathbf{y}$ is an eigenvector for A with eigenvalue $\lambda_{\max}$, and so $\mathbf{y}$ is a scalar multiple of the Perron vector.

Example 97. Columbia river salmon can live for three years. A one year old salmon produces 4 eggs on average, two year old salmon produce 20, and three year old salmon produce 60. The weights on the remaining edges in the network below give the percentage of salmon that make it to the next age:

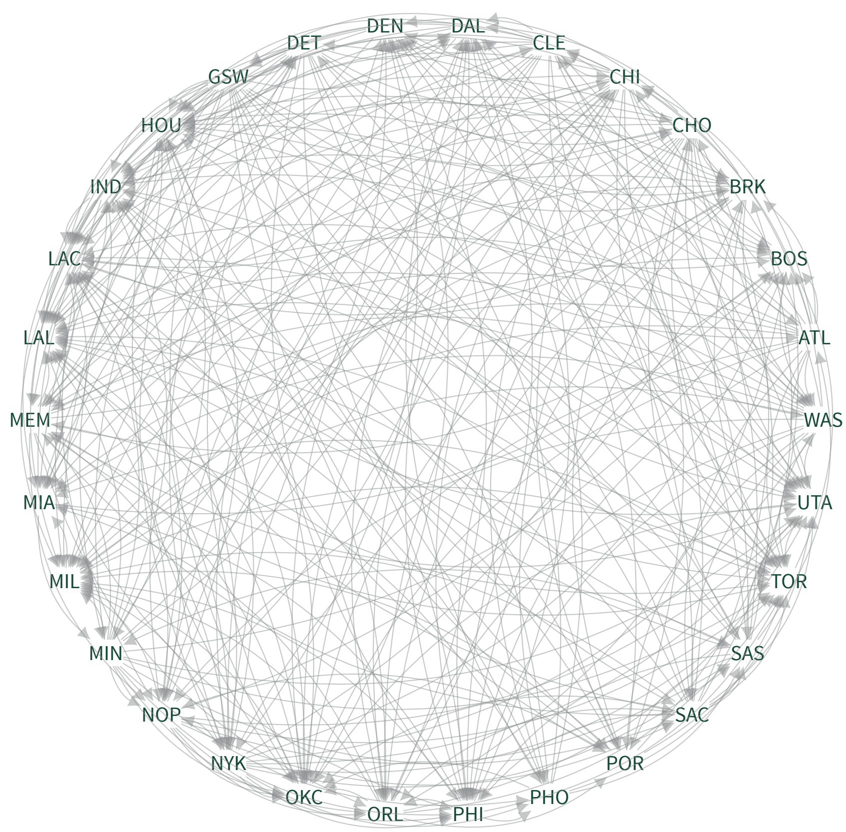


The adjacency matrix $\begin{bmatrix} 0 & 4 & 20 & 60 \\ 0.05 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \end{bmatrix}$ has Perron vector $\approx \begin{bmatrix} 0.932 \\ 0.046 \\ 0.014 \\ 0.008 \end{bmatrix}$ with Perron value ≈ 1.01187.

The Perron vector gives the long term age distribution of salmon. A salmon picked at random with will be an egg with probability 0.932, a year 1 salmon with probability 0.046, a year 2 salmon with probability 0.014, and a year 3 salmon with probability 0.008. If we do not wish to count an egg as a salmon, then rescaling the probabilities gives 0.68 for year 1, 0.2 for year 2, and 0.12 for year 3.

The Perron value tells us the rate at which the salmon population is growing. Each year there are approximately 1.187% more salmon than the previous year.

Example 98. The Perron vector can be used to rank sports teams. Let G be the graph with nodes the 30 NBA basketball teams. Draw an edge from team i to team j if team j has a better record in the games that the two teams have played. Do not draw an edge if they have not played or if they have split the games they have played. For example, this graph for the 2019–2020 Covid-19 shortened season is shown below:



Create a network from this graph by such that if there are n out edges leaving team i in this graph, then each of these edges are weighted $1/n$. Consider a **random walk** in this network, meaning that we start at a single team in the graph and then repeatedly follow edges with the probabilities given by the edge weights.

Since each step in this walk moves from a losing team to a winning team, we can expect to land on better teams more often in this random walk. The Perron vector gives us the limiting probabilities that we would land on each team in the random walk, so the Perron vector gives us our ranking of teams. The approximate ranking for the 2019–2020 regular season is shown below:

0.075	LAC	0.040	UTA	0.021	SAC
0.071	MIL	0.039	OKC	0.017	CLE
0.069	HOU	0.030	SAS	0.016	WAS
0.065	LAL	0.028	PHI	0.016	ORL
0.062	MIA	0.028	NOP	0.015	MEM
0.058	TOR	0.027	DET	0.014	PHO
0.053	DAL	0.026	POR	0.011	NYK
0.051	DEN	0.023	CHI	0.005	CHO
0.047	IND	0.023	BRK	0.003	ATL
0.044	BOS	0.022	MIN	0.002	GSW

If the nodes in a graph are web pages and the edges between pages indicate links, then a slightly modified version of the ranking method using the Perron vector in the NBA basketball example was famously used by Google to rank web pages by importance when sorting search results.

Theorem 99. *A graph G with n vertices is bipartite if and only if there is a relabeling of the vertices such that the adjacency matrix has the form*

$$\begin{bmatrix} 0 & C \\ C^\top & 0 \end{bmatrix}$$

for some $k \times (n-k)$ matrix C where 0 is the matrix of 0's.

Proof. Assume that G is bipartite. By possibly relabeling the vertices we can assume that the independent sets are vertices labeled $1, \dots, k$ and $k+1, \dots, n$ for some k. Then, since there are no edges that connect vertices within the independent sets, the adjacency matrix for G has the desired form.

Now assume that the the adjacency matrix for G has the desired form. This means that there are no edges within the independent sets of vertices labeled $1, \dots, k$ and $k+1, \dots, n$ for some k, as needed. $\square$

Theorem 100. *Let G be a connected graph with Perron value $\lambda_{\max}$ and Perron vector $\mathbf{v}$. Then G is bipartite if and only if $-\lambda_{\max}$ is an eigenvalue of G.*

Proof. Suppose G is bipartite. Possibly relabel the vertices of G so that the adjacency matrix A for G has the form in the statement of Theorem 99 for some $k \times (n-k)$ matrix C. Suppose λ is an eigenvalue for A with eigenvector $\mathbf{x} = \begin{bmatrix} \mathbf{y} & \mathbf{z} \end{bmatrix}^\top$ where $\mathbf{y}$ is a $k \times 1$ vector and $\mathbf{z}$ is an $(n-k) \times 1$ vector.

Then the equation $A\mathbf{x} = \lambda\mathbf{x}$ implies $C\mathbf{z} = \lambda\mathbf{y}$ and $C^\top\mathbf{y} = \lambda\mathbf{z}$. Then we have

$$A\begin{bmatrix} -\mathbf{y} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} 0 & C \\ C^\top & 0 \end{bmatrix}\begin{bmatrix} -\mathbf{y} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} C\mathbf{z} \\ -C^\top\mathbf{y} \end{bmatrix} = \begin{bmatrix} \lambda\mathbf{y} \\ -\lambda\mathbf{z} \end{bmatrix} = -\lambda\begin{bmatrix} -\mathbf{y} \\ \mathbf{z} \end{bmatrix},$$

showing that $-\lambda$ is an eigenvalue of G. In particular, $-\lambda_{\max}$ is an eigenvalue of G.

Now suppose $-\lambda_{\max}$ is an eigenvalue of G with eigenvector $\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top$ and let A be the adjacency matrix for G. Then if $|\mathbf{x}|$ denotes the vector $\begin{bmatrix} |x_1| & \cdots & |x_n| \end{bmatrix}^\top$, we have

$$\lambda_{\max}|\mathbf{x}| = |-\lambda_{\max}\mathbf{x}| = |A\mathbf{x}| \leq A|\mathbf{x}|.$$

The inequality in the above equation means that each component of $\lambda_{\max}|\mathbf{x}|$ is less than or equal to the corresponding component in $A\mathbf{x}$. However, this inequality is actually an equality because if there is a component for which the strict inequality holds then we would have

$$\lambda_{\max}\mathbf{v}^\top|\mathbf{x}| < \mathbf{v}^\top A|\mathbf{x}| = \left(|\mathbf{x}|^\top A^\top \mathbf{v}\right)^\top = \left(|\mathbf{x}|^\top \lambda_{\max}\mathbf{v}\right)^\top = \lambda_{\max}\mathbf{v}^\top|\mathbf{x}|$$

because the vector $\mathbf{v}$ has strictly positive components. This cannot happen and so we have $A|\mathbf{x}| = \lambda_{\max}|\mathbf{x}|$.

Therefore $|\mathbf{x}|$ is a scalar multiple of the Perron vector and cannot have a component equal to 0. By possibly relabeling the vertices of G we can assume without loss of generality that $x_1, \ldots, x_k$ are all positive and $x_{k+1}, \ldots, x_n$ are all negative. Let

$$\mathbf{x}^+ = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}, \qquad \mathbf{x}^- = \begin{bmatrix} x_{k+1} \\ \vdots \\ x_n \end{bmatrix}, \qquad \text{and} \qquad A = \begin{bmatrix} B & C \\ C^\top & D \end{bmatrix}$$

where B, C, and D are block matrices such that B is a $k \times k$ matrix, C is a $k \times (n-k)$ matrix, and D is $(n-k) \times (n-k)$.

Using block matrix multiplication, the equation $A\mathbf{x} = -\lambda_{\max}\mathbf{x}$ tells us that

$$B\mathbf{x}^+ + C\mathbf{x}^- = -\lambda_{\max}\mathbf{x}^+ \qquad \text{and} \qquad C^\top\mathbf{x}^+ + D\mathbf{x}^- = -\lambda_{\max}\mathbf{x}^-.$$

The equation $A|\mathbf{x}| = \lambda_{\max}|\mathbf{x}|$ with the observation that $|\mathbf{x}| = \begin{bmatrix} \mathbf{x}^+ & -\mathbf{x}^- \end{bmatrix}^\top$ gives

$$B\mathbf{x}^+ - C\mathbf{x}^- = \lambda_{\max}\mathbf{x}^+ \qquad \text{and} \qquad C^\top\mathbf{x}^+ - D\mathbf{x}^- = \lambda_{\max}\mathbf{x}^-.$$

Combining these two expressions shows $B\mathbf{x}^+ = 0$ and $-D\mathbf{x}^- = 0$. Since B and D are matrices with nonnegative entries and since $\mathbf{x}^+$ and $-\mathbf{x}^-$ have strictly positive entries, the matrices B and D must be zero matrices. By Theorem 99, G is bipartite. $\square$

The proof of Theorem 100 says that if G is bipartite, then the positive and negative entries of the eigenvector that corresponds to eigenvalue $-\lambda_{\max}$ partition the graph into the independent sets.

The next well-known linear algebra theorem that we state without proof is used with some frequency when finding bounds on the eigenvalues for real symmetric matrices.

Theorem 101 (Courant-Fischer)**.** *If A is a real symmetric matrix with eigenvalues $\lambda_{\max} \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_{\min}$, then*

$$\lambda_{\max} = \max \mathbf{x}^\top A\mathbf{x} \qquad \textit{and} \qquad \lambda_{\min} = \min \mathbf{x}^\top A\mathbf{x}$$

where the maximum and minimum is taken over unit vectors $\mathbf{x}$ *(that is,* $\mathbf{x}^\top\mathbf{x} = 1$*). Furthermore, if* $\lambda_{\max}$ *has eigenvector* $\mathbf{v}_1$ *and* $\lambda_{\min}$ *has eigenvector* $\mathbf{v}_n$*, then the second largest and second smallest eigenvalues satisfy*

$$\lambda_2 = \max \mathbf{x}^\top A\mathbf{x} \qquad \textit{and} \qquad \lambda_{n-1} = \min \mathbf{x}^\top A\mathbf{x}$$

where the maximum and minimum is taken over all unit vectors $\mathbf{x}$ *that are also orthogonal to* $\mathbf{v}_1$ *and* $\mathbf{v}_n$*, respectively (that is,* $\mathbf{x}^\top\mathbf{v}_1 = 0$ *for the maximum and* $\mathbf{x}^\top\mathbf{v}_n = 0$ *for the minimum).*

One of our first applications of Theorem 101 is a bound on the maximum eigenvalue of G, the content of our next theorem.

Theorem 102. *Let* $\lambda_{\max}$ *be the maximum eigenvalue for a graph* G *with* n *vertices and let* d *be the maximum degree in* G*. Then*

$$(\textit{the average vertex degree in } G) \le \lambda_{\max} \le d.$$

Furthermore, under the added hypothesis that G *is connected, the equality* $\lambda_{\max} = d$ *holds if and only if every vertex in* G *has degree* d*.*

Proof. Let 1 be the vector of all 1's and A be the adjacency matrix for G. Using the unit vector $(1/\sqrt{n})1$ in Theorem 101, we have

$$\lambda_{\max} \ge \left(\frac{1}{\sqrt{n}}1\right)^\top A\left(\frac{1}{\sqrt{n}}1\right) = \frac{\text{the sum of the entries in } A}{n},$$

which is equal to the average vertex degree, showing the lower bound on $\lambda_{\max}$.

Let $\mathbf{v} = \begin{bmatrix} v_1 & \cdots & v_n\end{bmatrix}^\top$ be the Perron vector for A. Suppose that i is the index such that v_i is a maximum component of $\mathbf{v}$. Then we have

$$\lambda_{\max} v_i = (\text{component } i \text{ in } A\mathbf{v}) = \sum_{\text{vertex } j \text{ is adjacent to vertex } i} v_j \le dv_i,$$

showing that $\lambda_{\max} \le d$ as needed.

If $\lambda_{\max} = d$, the then above inequality is an equality, implying $v_i = v_j$ for all vertices j that are adjacent to i. Repeating the above argument with i replaced with a vertex j adjacent to i shows that all vertices k adjacent to j also have $v_i = v_k$. Assuming G is connected, continuing in this manner shows that every coordinate in the Perron vector is the same and therefore $\mathbf{v} = 1/n$. The equation $A1/n = (d/n)1$ now implies that each row of A sums to d, meaning that every vertex in G has degree d.

On the other hand, if every vertex in a connected graph G has degree d, then $A1/n = (d/n)1$, showing that the Perron value $\lambda_{\max} = d$. □

The intuition behind Theorem 102 is that a random walk of length k grows at a rate asymptotic to $c\lambda_{\max}^k$ for some positive constant c. Such a random walk has m choices to leave a vertex of degree m, so if every vertex has degree d, then the random walk grows at a rate asymptotic to d^k, giving evidence that $\lambda_{\max} = d$. If not every vertex has degree d, then at least the random walk grows at a rate asymptotic to a^k where a is the average degree, giving evidence that $a \le \lambda_{\max}$.

Theorem 103 (Wilf)**.** *If $\lambda_{\max}$ is the largest eigenvalue for G, then the chromatic number satisfies $\chi(G) \leq \lambda_{\max} + 1$.*

Proof. Let H be a $\chi(G)$-critical subgraph of G (see our exercise on critical subgraphs) and let $\lambda_{\max}(H)$ be the largest eigenvalue for H. By the exercises, the minimum degree in H is at least $\chi(G) - 1$. Theorem 102 now gives

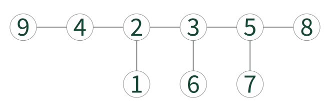
$$\chi(G) - 1 \leq (\text{the average degree in } H) \leq \lambda_{\max}(H) \leq \lambda_{\max}(G)$$

where the last inequality the content of our exercise on bounding the eigenvalues of subgraphs. □

Theorem 103 provides an upper bound on the chromatic number. Since upper bounds can be found by simply providing some random proper coloring of the graph, lower bounds are generally more interesting. Lower bounds on the chromatic number that involve the eigenvalues of the adjacency matrix exist, as we state in the next Theorem. The proof relies on more specialized techniques in linear algebra and so it is omitted.

Theorem 104 (Hoffman)**.** *If $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum eigenvalues of G, then $1 - \lambda_{\max}/\lambda_{\min} \leq \chi(G)$.*

Example 105. The matching polynomial for the tree T shown below



is $M_T(x) = x^9 - 8x^7 + 18x^5 - 12x^3 + 2x$. The characteristic polynomial for the adjacency matrix A for T is equal to $\det(A(T) - xI)$ where I is the identity matrix. When this calculation is carried out, we find the characteristic polynomial is $-x^9 + 8x^7 - 18x^5 + 12x^3 - 2x$. The characteristic polynomial is to equal $M_T(-x)$

Theorem 106 shows that the relationship between the matching polynomial for the tree and the characteristic polynomial for the adjacency matrix in Example 105 was not an accident. The proof is not difficult for those who have seen the determinant written as a sum over permutations in the symmetric group, but we choose to omit the proof because introducing the background material needed for the proof is beyond the scope of this course.

Theorem 106. *Let $M_G(x)$ be the matching polynomial for G. Then G has no cycles if and only if $M_G(-x)$ is the characteristic polynomial for the adjacency matrix for G.*

This chapter has shown how results from matrix algebra can be applied to the adjacency matrix to learn about the graph. The eigenvalues and eigenvectors for the graph play an interesting role in the subject. We end this chapter by describing one more result, relating the number of distinct eigenvectors to the diameter of the graph.

Definition. *The **distance** between vertices u and v is the length of the shortest path from u to v. The **diameter** of G is the largest distance between two vertices in G.*

Example 107. The diameter of C_{12} is 6. The eigenvalues of C_{12} are

$$2, \sqrt{3}, \sqrt{3}, 1, 1, 0, 0, -1, -1, -\sqrt{3}, -\sqrt{3}, -2.$$

There are 12 eigenvalues but only 7 distinct eigenvalues.

More generally, the diameter of C_n is $n/2$ if n is even and $(n-1)/2$ if n is odd and it can be shown that the number of distinct eigenvalues of C_n is $n/2+1$ if n is even and $(n-1)/2+1$ if n is odd.

The proof of Theorem 109 relies on yet another result from matrix algebra that gives evidence that real symmetric matrices are the best possible matrices to understand.

Theorem 108. *If A is a real symmetric matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, then*

$$(A-\lambda_1 I)\cdots(A-\lambda_k I) = 0$$

and no polynomial $p(x)$ with a degree smaller than k has $p(A) = 0$. In other words, the minimal polynomial for A is $(x-\lambda_1)\cdots(x-\lambda_k)$.

Theorem 109. *The diameter of a connected graph G is less than the number of distinct eigenvalues of G.*

Proof. Suppose the adjacency matrix A for G has k distinct eigenvalues. Theorem 108 implies that A^k is a linear combination of $I, A^1, \ldots, A^{k-1}$, meaning that there are constants $c_0, \ldots, c_{k-1}$ such that

$$A^k = c_0 I + c_1 A^1 + \cdots + c_{k-1}A^{k-1}.$$

It follows that A^m is also a linear combination of $I, A^1, \ldots, A^{m-1}$ for all $m \geq k$ because

$$A^{k+(m-k)} = c_0 A^{m-k} + c_1 A^{m-k+1} + \cdots + c_{k-1}A^{m-1}.$$

Thinking about walks in G, if the diameter of G is d, then there are vertices u and v such that the u, v entry of A^d is nonzero but that the u, v entry in each of $I, A^1, \ldots, A^{d-1}$ is 0. This means that A^d cannot be a linear combination of $I, A^1, \ldots, A^{d-1}$ and therefore $d < k$. □

— 9 —

The Laplacian

The adjacency matrix is good matrix to use when understanding walks in a graph, but for many other purposes the Laplacian matrix is a better tool.

Definition. *Let G be a simple graph and let D be a directed graph created by arbitrarily assigning a direction to each edge in G. An **incidence matrix** Q for G is the matrix with rows indexed by edges, columns indexed by vertices, and with v, e entry equal to*

$$\begin{cases} 1 & \textit{if } e \textit{ points to } v \textit{ in } D \\ -1 & \textit{if } e \textit{ leaves } v \textit{ in } D \\ 0 & \textit{otherwise}. \end{cases}$$

Example 110. If the graph G has its edges directed as shown below,

then listing the edges in the order $\{1,2\}, \{2,3\}, \{3,1\}, \{3,4\}, \{4,5\}, \{5,3\}$ gives the incidence matrix

$$Q = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

.

Definition. *The **Laplacian** for the graph G is the matrix $L = Q^\top Q$ for some incidence matrix Q.*

Example 111. Calculating $Q^\top Q$ using the matrix in Example 110, the Laplacian is

$$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

The eigenvalues for this Laplacian are $5, 3, 3, 1, 0$.

Theorem 112. *First observations about the Laplacian L for a graph on n vertices are:*

a. *If D is the $n \times n$ diagonal matrix with the vertex degrees along the diagonal and $A(G)$ is the adjacency matrix for G, then $L = D - A(G)$. This implies that the Laplacian does not depend on how the edges were directed when finding Q.*

b. *The Laplacian L is a real valued symmetric matrix.*

c. *If $\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top$, then $\mathbf{x}^\top L\mathbf{x} = \displaystyle\sum_{\{i,j\} \text{ is an edge}} (x_i - x_j)^2$.*

d. *The smallest eigenvalue $\mu_{\min}$ of L is equal to 0 with eigenvector $(1/\sqrt{n})\mathbb{1}$ where $\mathbb{1}$ is the vector of all 1's.*

Proof. Statement **a.** comes from writing $L = Q^\top Q$ for some incidence matrix Q and then using the definition of matrix multiplication. Statement **b.** is true because $L^\top = \left(Q^\top Q\right)^\top = Q^\top Q = L$.

As for statement **c.**, we have

$$\mathbf{x}^\top L\mathbf{x} = \mathbf{x}^\top Q^\top Q\mathbf{x} = (Q\mathbf{x})^\top (Q\mathbf{x}) \, .$$

The vector $Q\mathbf{x}$ is indexed by edges and has edge $e = \{i, j\}$ entry equal to $\pm(x_i - x_j)$. Thus $(Q\mathbf{x})^\top (Q\mathbf{x})$ gives the squared length of this vector, which is the desired expression.

Finally, for statement **d.**, the rows of $L = D - A(G)$ sum to 0 and so $(1/\sqrt{n})L\mathbb{1} = 0$, showing that $(1/\sqrt{n})\mathbb{1}$ is an eigenvector with eigenvalue 0. Statement **c.** combined with Theorem 101 gives that the minimum eigenvalue is at least

$$\mathbf{x}^\top L\mathbf{x} = \sum_{\text{edges } \{i,j\}} (x_i - x_j)^2$$

for all unit vectors $\mathbf{x}$, which must be nonnegative, and so it is 0. $\square$

Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ be the eigenvalues of the Laplacian L. The second smallest eigenvalue μ_2 provides enough interesting information about the connectivity of the graph that it warrants its own definition.

Definition. *The **algebraic connectivity** of a graph G, denoted $\mu_2(G)$, is the second smallest eigenvalue of the Laplacian matrix for G.*

Since the vector $\mathbb{1}$ is an eigenvector with eigenvalue 0 for the Laplacian of a graph with n vertices, Theorem 101 tells us that the algebraic connectivity μ_2 satisfies

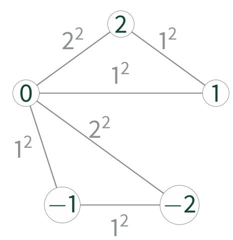
$$\mu_2 = \min \mathbf{x}^\top L\mathbf{x} = \min \sum_{\{i,j\} \text{ is an edge}} (x_i - x_j)^2$$

where the minimization is over unit vectors $\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top$ such that $\mathbf{x}^\top \mathbb{1} = 0$ (or, equivalently, the components of the unit vector sum to 0). Therefore a common approach to finding an upper bound on μ_2 is to assign the real number x_i to vertex i in the graph such that $x_1 + \cdots + x_n = 0$. Then

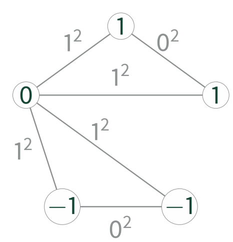
$$\mu_2 \leq \frac{1}{\mathbf{x}^\top \mathbf{x}} \sum_{\{i,j\} \text{ is an edge}} (x_i - x_j)^2$$

where the division by $\mathbf{x}^\top \mathbf{x}$ is present for the situation where $\mathbf{x}$ is not a unit vector.

Example 113. We show how to find an upper bound on the algebraic multiplicity in the graph in Example 110 by placing real numbers $x_1, \ldots, x_5$ that sum to 0 in for the vertices of the graph and label the edges with $(x_i - x_j)^2$. One arbitrary choice is



and so $\mu_2 \leq (1^2 + 1^2 + 1^2 + 1^2 + 2^2 + 2^2)/(0^2 + 1^2 + 1^2 + 2^2 + 2^2) = 6/5$. Another choice is



and so $\mu_2 \leq 4/4 = 1$. This is an optimal labeling because $\begin{bmatrix} 1 & 1 & 0 & -1 & -1 \end{bmatrix}^\top$ is an eigenvector corresponding to the second smallest eigenvalue for the Laplacian matrix for the graph in this example.

Theorem 114. *If G is a graph with n vertices and S is a subgraph of G that has k vertices, then*

$$\mu_2 \leq \frac{n}{k(n-k)} E(S, G - S).$$

where $E(S, G - S)$ denotes the number of edges between vertices in S and $G - S$.

Proof. Define $\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top$ such that $x_i = \begin{cases} n - k & \text{if } i \text{ is a vertex in } S, \\ -k & \text{if } i \text{ is a vertex in } G - S. \end{cases}$
Then we have $x_1 + \cdots + x_n = k(n-k) + (-k)(n-k) = 0$ and

$$(x_i - x_j)^2 = \begin{cases} 0 & \text{if both } i \text{ and } j \text{ are in } S, \\ 0 & \text{if both } i \text{ and } j \text{ are in } G - S, \\ n^2 & \text{otherwise.} \end{cases}$$

Therefore

$$
\begin{aligned}
\mu_2 &\le \frac{1}{\mathbf{x}^\top \mathbf{x}} \sum_{\{i,j\} \text{ is an edge}} (x_i - x_j)^2 \\
&= \frac{1}{k(n-k)^2 + k^2(n-k)} n^2 E(S, G-S) \\
&= \frac{n}{k(n-k)} E(S, G-S).
\end{aligned}
$$

□

Theorem 114 says that a high algebraic connectivity means that there are many edges between and set of vertices and the complement set of vertices. Conversely, a low algebraic connectivity means that it is relatively easy to disconnect the graph. Indeed, as a corollary of Theorem 114, the algebraic connectivity of G is 0 if G is not connected. Indeed, it can be shown the number of components of G is the multiplicity of 0 as an eigenvalue of the Laplacian. The next two theorems reinforce this intuition.

Theorem 115. *If v is a vertex in a graph G with n vertices, then $\mu_2(G) \le \mu_2(G-v)+1$.*

Proof. Let G' be the graph created by possibly adding edges to G such that v connected to all other vertices. The Laplacian matrix satisfies

$$
L(G') = \begin{bmatrix} L(G-v)+I & -\mathbf{1} \\ -\mathbf{1}^\top & n-1 \end{bmatrix}
$$

where this is a block matrix, I is the identity matrix, $\mathbf{1}$ is the vector of all 1's, and where we are assuming without loss of generality that vertex v is written last.

Let $\mathbf{v}$ be an eigenvector for $L(G-v)$ with eigenvalue $\mu_2(G-v)$. Then

$$
L(G') \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix} = (\mu_2(G-v)+1) \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix},
$$

showing that $\mu_2(G-v)+1$ is an eigenvalue for $L(G')$. This eigenvalue cannot be the smallest eigenvalue for $L(G')$ because it is positive (since $\mu_2(G-v) \ge 0$), and so it is at least the second smallest eigenvalue. Using our exercise on how removing an edge can change the algebraic connectivity, we now have

$$
\mu_2(G) \le \mu_2(G') \le \mu_2(G-v)+1.
$$

□

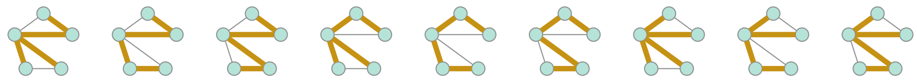
Theorem 116. *If $\kappa(G)$ is the vertex connectivity of G, then $\mu_2(G) \le \kappa(G)$.*

Proof. Suppose $V = \{v_1, \ldots, v_{\kappa(G)}\}$ is a minimum set of vertices such that $G - V$ is not connected. Repeatedly using Theorem 115 gives

$$
\mu_2(G) \le \mu_2(G-v_1)+1 \le \cdots \le \mu_G(G - v_1 - \cdots - v_{\kappa(G)}) + \kappa(G) = \kappa(G).
$$

□

Example 117. There are 9 spanning trees for the graph in Example 110:



In this example we see that when we multiply the nonzero eigenvalues for the Laplacian and divide by the number of vertices, we also find $5 \cdot 3 \cdot 3 \cdot 1/5 = 9$.

Theorem 118 (Kirchhoff's matrix tree theorem)**.** *For any matrix A, let $A^{(i,j)}$ denote the matrix found by deleting the row i and column j in A. If $\tau(G)$ is the number of spanning trees for G, then $\tau(G) = \det(L(G)^{(1,1)})$.*

Proof. We proceed by induction on the number of edges and vertices in G. If G has no edges, then $L(G)$ is the zero matrix and so both $\tau(G)$ and $\det(L(G)^{(1,1)})$ are equal to 0. If G has no vertices, then the theorem is vacuously true.

By possibly reordering vertices we can assume without loss of generality that e is an edge that connects vertex 1 and vertex 2 in G. The Laplacian is a block matrix of the form

$$L(G) = \begin{bmatrix} d_1 & -1 & \mathbf{u}^\top \\ -1 & d_2 & \mathbf{v}^\top \\ \mathbf{u} & \mathbf{v} & L_1 \end{bmatrix}$$

where d_1 is the degree of vertex 1, d_2 is the degree of vertex 2, $\mathbf{v} = \begin{bmatrix} v_1 & \cdots & v_{n-2} \end{bmatrix}^\top$ and $\mathbf{u}$ are $n-2$ dimensional vectors containing 0's or -1's, and L_1 is the $(n-2) \times (n-2)$ submatrix found in the bottom right corner of $L(G)$. This matrix L_1 is the Laplacian for the graph without the first two vertices. Then we have

$$L(G-e) = \begin{bmatrix} d_1 - 1 & 0 & \mathbf{u}^\top \\ 0 & d_2 - 1 & \mathbf{v}^\top \\ \mathbf{u} & \mathbf{v} & L_1 \end{bmatrix} \quad \text{and} \quad L(G/e) = \begin{bmatrix} d_1 + d_2 - 2 & \mathbf{w}^\top \\ \mathbf{w} & L_1 \end{bmatrix}$$

for some vector $\mathbf{w}$. From this we have

$$L(G)^{(1,1)} = \begin{bmatrix} d_2 & \mathbf{v}^\top \\ \mathbf{v} & L_1 \end{bmatrix}, \quad L(G-e)^{(1,1)} = \begin{bmatrix} d_2 - 1 & \mathbf{v}^\top \\ \mathbf{v} & L_1 \end{bmatrix}, \quad \text{and} \quad L(G/e)^{(1,1)} = L_1.$$

Taking the determinant using the cofactor expansion along the first row of the matrix, we see by induction that

$$\begin{aligned} \det(L(G)^{(1,1)}) &= d_2 \det L_1 - \sum_{i=1}^{n-2} (-1)^i v_i \det(L_1^{(1,i)}) \\ &= (d_2 - 1) \det L_1 - \sum_{i=1}^{n-2} (-1)^i v_i \det(L_1^{(1,i)}) + \det L_1 \\ &= \det(L(G-e)^{(1,1)}) + \det(L(G/e)^{(1,1)}) \\ &= \tau(G-e) + \tau(G/e) \end{aligned}$$

where the last line follows by induction. By our exercise on spanning trees, we have $\det(L(G)^{(1,1)}) = \tau(G)$, as needed. □

It is not difficult to adjust the above proof to show that the result in Theorem 118 still holds if the graph G is allowed to have multiple edges between vertices. Using determinants can be awkward, so the result in Theorem 118 can be rephrased in terms of the eigenvalues of the Laplacian matrix, as shown in Theorem 119.

Theorem 119. *If $0, \mu_2, \ldots, \mu_n$ are the eigenvalues for the Laplacian matrix of a graph with n vertices, then $\tau(G) = \mu_2 \cdots \mu_n / n$.*

Proof. The characteristic polynomial for the Laplacian is equal to

$$\det(L - xI) = (-x)(\mu_2 - x) \cdots (\mu_n - x),$$

and so the coefficient of x in this polynomial is $-\mu_2 \cdots \mu_n$.

Adding a multiple of a row (or column) to another row (or column) does not change the determinant of a matrix. Since the columns of L sum to 0, change $L - xI$ by adding rows $2, \ldots, n$ to the first row, to find

$$\det(L - xI) = \begin{bmatrix} -x & (-x)1^\top \\ \mathbf{v} & L^{(1,1)} - xI \end{bmatrix} = (-x)\det \begin{bmatrix} 1 & 1^\top \\ \mathbf{v} & L^{(1,1)} - xI \end{bmatrix}$$

where $\mathbf{v}$ is a vector of 0's and (-1)'s. Thus the coefficient of x in this polynomial is

$$-\det \begin{bmatrix} 1 & 1^\top \\ \mathbf{v} & L^{(1,1)} \end{bmatrix}$$

Since the rows of L sum to 0, adding columns $2, \ldots, n$ in the above matrix to the first column gives that the above determinant is equal to

$$-\det \begin{bmatrix} n & 1^\top \\ 0 & L^{(1,1)} \end{bmatrix} = -n \det(L^{(1,1)})$$

where the determinant was calculated using the cofactor expansion along the first column. Theorem 118 gives $-\mu_2 \cdots \mu_n = -n \det(L^{(1,1)}) = -n\tau(G)$, as needed. $\square$

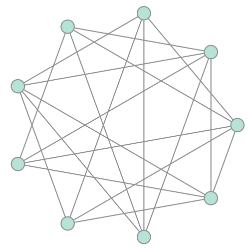
The Laplacian matrix can be used to help with graph visualization. Since the Laplacian is a real valued symmetric matrix, it has an orthogonal basis of eigenvectors. Two or three of these eigenvectors can be used as an axis system for representing the graph in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Suppose that $\mu_2, \ldots, \mu_n$ are the eigenvalues for the Laplacian matrix for a graph. If $\mathbf{x} = \begin{bmatrix} x_1 \cdots x_n \end{bmatrix}^\top$ is an eigenvector of length 1 with eigenvalue μ_2, then

$$\mu_2 = \mathbf{x}^\top L \mathbf{x} = \sum_{\{i,j\} \text{ is an edge}} (x_i - x_j)^2$$

is minimum over unit vectors that are orthogonal to 1. Therefore adjacent vertices will have relatively close values of x_i and x_j as to make $(x_i - x_j)^2$ small. Similarly, if $\mathbf{y} = \begin{bmatrix} y_1 \cdots y_n \end{bmatrix}^\top$ is an eigenvector of length 1 with eigenvalue μ_3, then $\mathbf{y}$ minimizes $\sum_{\{i,j\} \text{ is an edge}} (y_i - y_j)^2$ over all possible vectors that are orthogonal to both 1 and $\mathbf{x}$, meaning that adjacent vertices will have relatively close values of y_i and y_j. Continuing in this manner suggests that a third vector to use in an axis system is an eigenvector $\mathbf{z}$ corresponding to μ_4.

Example 120. A graph and its Laplacian are shown below:

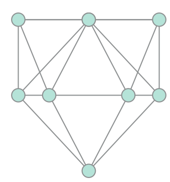


$$\begin{bmatrix} 5 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 \\ 0 & 5 & 0 & -1 & -1 & -1 & 0 & -1 & -1 \\ -1 & 0 & 4 & 0 & -1 & 0 & -1 & -1 & 0 \\ -1 & -1 & 0 & 4 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 & 5 & 0 & -1 & -1 & -1 \\ -1 & -1 & 0 & -1 & 0 & 4 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 & 0 & 4 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & 0 & 0 & 4 & 0 \\ -1 & -1 & 0 & 0 & -1 & -1 & -1 & 0 & 5 \end{bmatrix}$$

To get a better visualization of the graph we find that the eigenvectors $\mathbf{x}$ and $\mathbf{y}$ corresponding to the eigenvalues $\mu_2 \approx 2.44$ and $\mu_3 = 3$ are

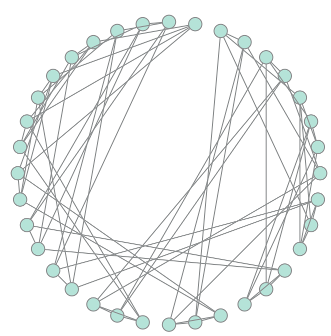
$$\begin{aligned} \mathbf{x} &\approx \begin{bmatrix} 0.00 & -0.26 & 0.47 & -0.47 & 0.26 & -0.47 & 0.47 & 0.00 & 0.00 \end{bmatrix}^\top, \\ \mathbf{y} &\approx \begin{bmatrix} 0.33 & -0.17 & -0.17 & -0.17 & -0.17 & 0.33 & 0.33 & -0.67 & 0.33 \end{bmatrix}^\top. \end{aligned}$$

Taking the vertex coordinates as the ordered pairs of the form (x_i, y_i) gives the visualization shown below:

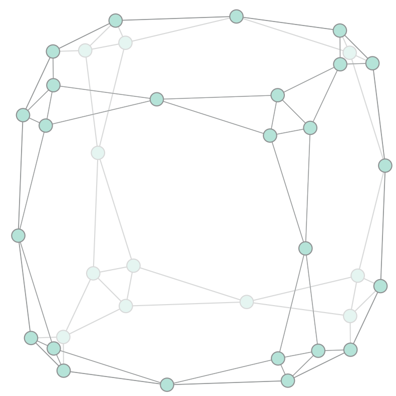


This representation more clearly shows the overall graph structure.

Example 121. The graph shown below



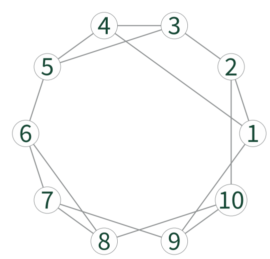
has eigenvalues for the Laplacian matrix that are approximately equal to 0, 0.44, 0.44, 0.44, 1, 1, ... The eigenvalues μ_2, μ_3 and μ_4 are all the same, indicating that there are is some symmetry among the eigenvectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ corresponding to these three smallest nonzero eigenvalues. To get a better visualization of the graph we place vertices in $\mathbb{R}^3$ at the coordinates of the form (x_i, y_i, z_i) to find



which reveals a cube-like structure to the graph. To emphasize, this representation of graph was created only from the coordinates of three of the eigenvectors of the Laplacian matrix and without any knowledge of the geometry of the graph.

Definition. *A **Tutte layout** of a graph G is found by fixing the position of some vertices and then placing the remaining vertices at the average coordinate of adjacent vertices.*

Example 122. Consider the graph shown below:



Place vertices $1, 2, 3, 4$ at the corners of a square, say at $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$. We can create a linear systems of equations to determine the placement of the remaining vertices in a Tutte layout. If (x_i, y_i) is the coordinate of vertex i for $i = 5, \ldots, 10$, then we have

$$\begin{aligned}
(x_5, y_5) &= \left((1,1) + (0,1) + (x_6, y_6)\right)/3 \\
(x_6, y_6) &= \left((x_5, y_5) + (x_7, y_7) + (x_8, y_8)\right)/3 \\
(x_7, y_7) &= \left((x_6, y_6) + (x_8, y_8) + (x_9, y_9)\right)/3 \\
(x_8, y_8) &= \left((x_6, y_6) + (x_7, y_7) + (x_{10}, y_{10})\right)/3 \\
(x_9, y_9) &= \left((0,0) + (x_7, y_7) + (x_{10}, y_{10})\right)/3 \\
(x_{10}, y_{10}) &= \left((1,0) + (x_8, y_8) + (x_9, y_9)\right)/3
\end{aligned}$$

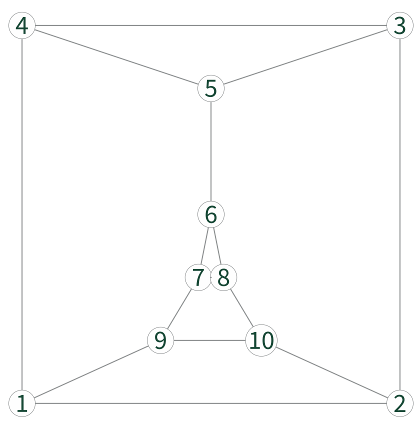
Rearranging terms and writing as a matrix multiplication gives

$$\begin{bmatrix} 3 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & -1 & 3 & 0 & -1 \\ 0 & 0 & -1 & 0 & 3 & -1 \\ 0 & 0 & 0 & -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_5 & y_5 \\ x_6 & y_6 \\ x_7 & y_7 \\ x_8 & y_8 \\ x_9 & y_9 \\ x_{10} & y_{10} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

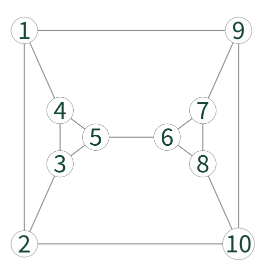
The matrix on the left is the lower right 6×6 block of the Laplacian matrix. The matrix on the right is the lower left 6×4 block of the adjacency matrix for G. The matrix on the left happens to be invertible and so multiplying by the inverse gives the unique solution

$$\begin{bmatrix} x_5 & y_5 \\ x_6 & y_6 \\ x_7 & y_7 \\ x_8 & y_8 \\ x_9 & y_9 \\ x_{10} & y_{10} \end{bmatrix} = \begin{bmatrix} 1/2 & 5/6 \\ 1/2 & 1/2 \\ 7/15 & 1/3 \\ 8/15 & 1/3 \\ 11/30 & 1/6 \\ 19/30 & 1/6 \end{bmatrix}.$$

Using these coordinates to plot the remaining vertices, we find



Fixing the positions of other choices of vertices gives an alternative embedding of the graph. For example, fixing the positions of vertices 1, 2, 10 and 9 at the corners of a square gives the Tutte layout shown below:



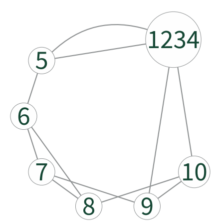
Theorem 123. *If G is a connected graph with n vertices and the position of k vertices are fixed with $2 \leq k \leq n-1$, then the positions of the remaining vertices are uniquely determined in a Tutte layout.*

Proof. Without loss of generality, assume that vertices $1, \ldots, k$ are fixed such that vertex i is placed at coordinate (x_i, y_i). The Laplacian matrix for G has the form

$$L(G) = \begin{bmatrix} L_1 & -A^\top \\ -A & L_2 \end{bmatrix}$$

where L_1 is a $k \times k$ block matrix, L_2 is a $(n-k) \times (n-k)$ block matrix, and A is the $(n-k) \times k$ submatrix of the adjacency matrix found in the bottom right corner.

Consider the graph G' created by contracting all edges in $1, \ldots, k$ while maintaining edges between vertices in $1, \ldots, k$ and $k+1, \ldots, n$, possibly resulting in a graph with multiple edges. For instance, if $k = 4$ in Example 122, then G' is shown below.



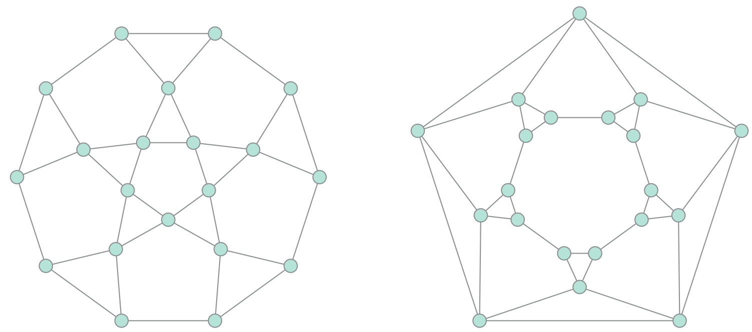
The Laplacian for G' has the form $L(G') = \begin{bmatrix} d & \mathbf{v}^\top \\ \mathbf{v} & L_2 \end{bmatrix}$ where d is the degree of the vertex created from contracting the vertices $1, \ldots, k$ and $\mathbf{v}$ is the sum of the columns in $-A$. The matrix-tree theorem gives that $\det L_2$ is the number of spanning trees for G', which is nonzero since G' is connected. This implies that L_2 is invertible.

Since the coordinates of vertices $k+1, \ldots, n$ are found at the average coordinate of adjacent vertices, the system of equations that determine the coordinates of vertices $k+1, \ldots, n$ is

$$\begin{bmatrix} -A & L_2 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} = 0, \qquad \text{or equivalently} \qquad L_2 \begin{bmatrix} x_{k+1} & y_{k+1} \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} = A \begin{bmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_k & y_k \end{bmatrix}.$$

Multiplying through by L_2^{-1} gives the unique solution. $\square$

Example 124. The following are two embeddings of the same planar graph created by evenly spacing vertices in a face around a circle and then positioning the remaining vertices using the Tutte layout.

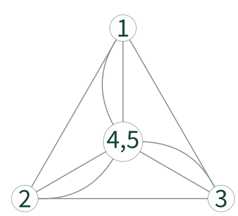


In both depictions we find a straight line embedding of G that does not contain any edge crossings.

We end this introduction into graph layouts by stating without proof a well-known theorem that tells us that the Tutte layout has nice properties for a well connected planar graph.

Theorem 125 (Tutte)**.** *If G is a planar graph with vertex connectivity $\kappa(G) \geq 3$ and $v_1, \ldots, v_k$ are the vertices surrounding a face in G, then the Tutte layout found by placing $v_1, \ldots, v_k$ at the vertices of a regular polygon provides an straight line embedding of G that does not contain any edge crossings.*

Unfortunately the Tutte layout can assign multiple vertices to the same coordinates. For example, one Tutte layout for $K_5 - \{4, 5\}$ is



where the vertices 4 and 5 have the same position. Using a different face other than the face containing $1, 2$ and 3 does, however, provide a Tutte layout that does not have two vertices with the same position: