Linear Analysis II Exercise Set 11

1. Find the function $f(x) = mx + b$ that best fits the data $(0,0), (-1,1), (1,2)$. Solve the problem two ways; both with and without using the normal equation (the equation $(V^\top V)^{-1}V^\top \mathbf{a}$).

2. Find the g in the span of $\{1, x\}$ closest to x^2 on $PS[-1,1]$.

3.
   a. Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be pairwise orthogonal vectors in $\mathbb{R}^n$ that all have length 1. Let U to be the square matrix with columns $\mathbf{u}_1, \ldots, \mathbf{u}_n$. Show that $U^\top U = I_n$, the $n \times n$ identity matrix.

   b. Let P be the projection matrix onto the span of $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$. Explain why PU is the matrix with columns $\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{0}, \ldots, \mathbf{0}$.

   c. Explain why $P = U \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} U^\top$ where the middle matrix is an $n \times n$ block matrix with top left $k \times k$ block the identity matrix.

4. Find the function g in the span of $1, \cos x$ and $\sin x$ in $PS[-\pi, \pi]$ that is closest to x.

5. For any two functions f and g in $PS[a,b]$, verify that

$$0 \leq \frac{1}{2} \int_a^b \int_a^b [f(x)g(y) - g(x)f(y)]^2 \, dx \, dy = \|f\|^2 \|g\|^2 - \langle f, g \rangle^2.$$

Use this fact to establish the inequality $|\langle f, g \rangle| \leq \|f\| \|g\|$.

6. For any two functions f and g in $PS[a,b]$, show that $\|f + g\| \leq \|f\| + \|g\|$. (Hint: Verify that $\|f + g\|^2 = \|f\|^2 + 2\langle f, g \rangle + \|g\|^2$ and then use the above exercise)

7. Verify that $2\|f\|^2 + 2\|g\|^2 = \|f + g\|^2 + \|f - g\|^2$ for any functions f, g in $PS[a,b]$.