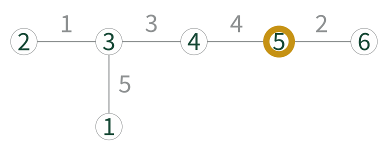
Graph Theory Set 4

16. This exercise gives another proof of Cayley's theorem. Let t_n be the number of labeled trees on n vertices. Let A be the set of objects which can be created this way:

1. Select a labeled tree with n vertices.
2. Mark one vertex.
3. Label the $n-1$ edges with $1, \ldots, n-1$ such that each edge has a different label.

For example, one possible element in A when $n = 6$ is:



a. By following these instructions, how many elements are in A? (The answer should involve t_n since we are pretending that we do not know that $t_n = n^{n-2}$.)

There is another way to create elements in A:

1. Start with an empty graph with vertices $1, \ldots, n$. Set $i = 1$. Mark every vertex.
2. Select any vertex, say v.
3. Select a marked vertex in different component than v, say w.
4. Remove the mark on w and draw an edge with label i between v and w.
5. If there is more than one component, increment i by 1 and go back to step 2.. If not, stop.

b. By following the 5 above steps, how many elements are in A? Why does this prove that $t_n = n^{n-2}$?

17. Suppose G has two spanning trees T_1 and T_2. Let e be any edge in T_1. Show that there is an edge f in T_2 such that the graph $(T - e) + f$ (remove e from T_1 and include f) is also a spanning tree.

18. Let $\tau(G)$ be the number of spanning trees for G and let e be an edge in G not on a triangle. Show that $\tau(G) = \tau(G - e) + \tau(G/e)$.

19. It has been proved that if G is planar, then it can be drawn in the plane with straight line segments as edges. Exhibit such planar drawings for $K_5 - e$ and $K_{3,3} - e$.

20. Remove and contract edges in the following graph to find $K_{3,3}$, showing that it is not planar.