

Linear Analysis II Set 7

1. a. Let k be a nonnegative integer. Show that the equation

$$\frac{d}{dx} \left(\sqrt{1-x^2} y' \right) = -k^2 \frac{y}{\sqrt{1-x^2}}.$$

is the same as the differential equation

$$(1-x^2)y'' - xy' + k^2y = 0.$$

- b. Define t_k to be the polynomial solution to the differential equation in a. that satisfies $t_k(1) = 1$. Find t_0, t_1, t_2 , and t_3 .
- c. Let $t_k(x)$ and $t_m(x)$ be two different polynomials as defined in part b.. This means that t_k and t_m satisfy the equations

$$\begin{aligned} \frac{d}{dx} \left(\sqrt{1-x^2} t_k' \right) &= -k^2 \frac{t_k}{\sqrt{1-x^2}}, \\ \frac{d}{dx} \left(\sqrt{1-x^2} t_m' \right) &= -m^2 \frac{t_m}{\sqrt{1-x^2}}. \end{aligned}$$

Multiply the first of these equations by t_m , multiply the second equation by t_k , and then subtract the two equations. After integrating both sides of the result from -1 to 1 , use integration by parts to show that

$$\int_{-1}^1 \frac{t_k(x)t_m(x)}{\sqrt{1-x^2}} dx = 0.$$

- d. Let $T_k(x) = \cos(k \arccos x)$. Show that $T_k(1) = 1$ and that

$$\frac{d}{dx} \left(\sqrt{1-x^2} T_k' \right) = -k^2 \frac{T_k}{\sqrt{1-x^2}}.$$

This means that $T_k(x) = t_k(x)$.

2. Let $p_k(x)$ be the k^{th} Legendre polynomial. Find the constants a_0, \dots, a_3 so that the approximation

$$u_0(x)x \approx a_0p_0(x) + a_1p_1(x) + a_2p_2(x) + a_3p_3(x)$$

on $[-1, 1]$ is as accurate as possible. (Here, $u_0(x)$ is the unit step function.) To do this, recall that

$$a_k = \frac{2k+1}{2} \int_{-1}^1 f(x)p_k(x) dx.$$