

Calculus 4 Exercises!

1. Match each of the following functions with the corresponding contour plot and plot in \mathbb{R}^3 :

$$a(x, y) = \cos x \cos y e^{-\sqrt{x^2+y^2}/4}$$

$$b(x, y) = -\frac{xy^2}{x^2 + y^2}$$

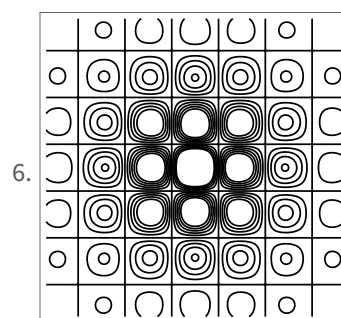
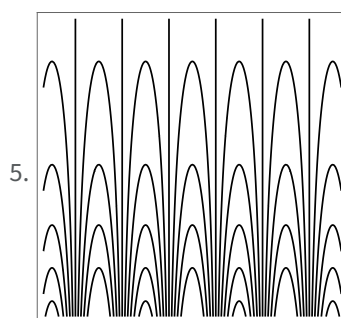
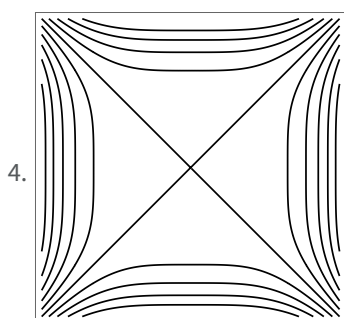
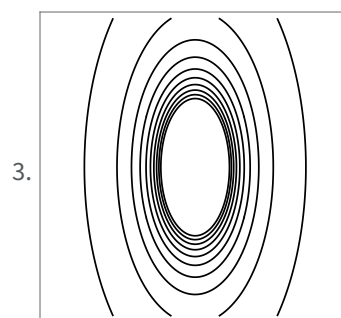
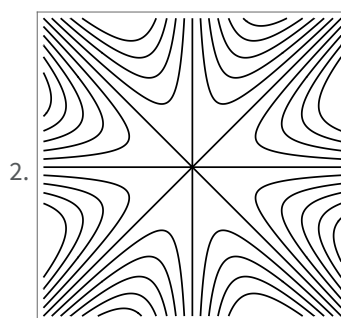
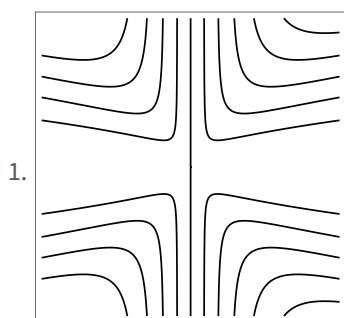
$$c(x, y) = \frac{1}{4x^2 + y^2 + 1}$$

$$d(x, y) = e^{-y/10} \cos x$$

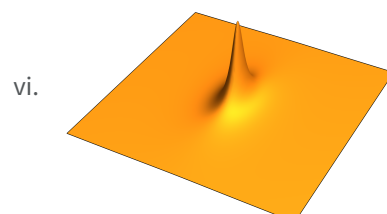
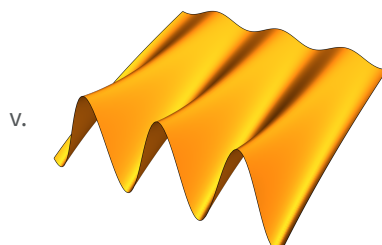
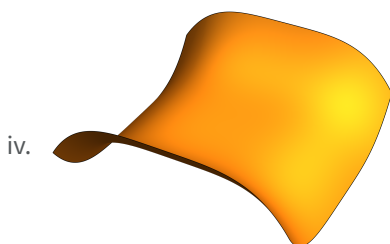
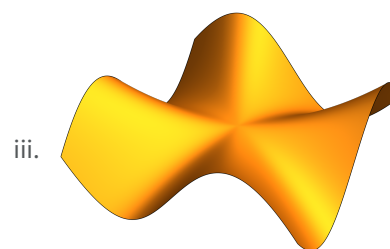
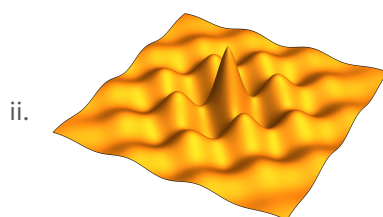
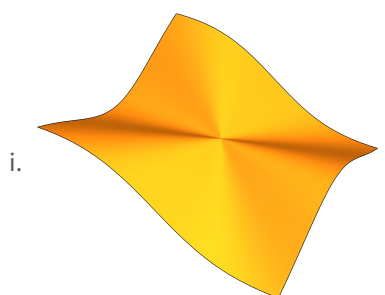
$$e(x, y) = xy \frac{x^2 - y^2}{x^2 + y^2}$$

$$f(x, y) = y^4 - x^4$$

Contour plots:



Plots in \mathbb{R}^3 :



2. Sketch the level curves $f(x, y) = c$ on the same set of axes for the given values of c :

- a. $f(x, y) = x + y - 1, c = -3, -2, -1, 0, 1, 2, 3$
- b. $f(x, y) = x^2 + y^2, c = 0, 1, 4, 9, 16, 25$
- c. $f(x, y) = xy, c = -9, -4, -1, 0, 1, 4, 9$
- d. $f(x, y) = \sqrt{25 - x^2 - y^2}, c = 0, 1, 2, 3, 4$

3. Sketch sample level curves and sketch the following functions in \mathbb{R}^3 :

- a. $f(x, y) = y^2$
- b. $f(x, y) = x^2 + y^2$
- c. $f(x, y) = 4 - x^2 - y^2$
- d. $f(x, y) = 4 - |x| - |y|$

4. Find these limits or explain why they do not exist:

- a. $\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$
- b. $\lim_{(x,y) \rightarrow (1, \pi/6)} \frac{x \sin y}{x^2 + 1}$
- c. $\lim_{(x,y) \rightarrow (1,1), x \neq y} \frac{x^2 - 2xy + y^2}{x - y}$
- d. $\lim_{\substack{(x,y) \rightarrow (0,0) \\ x \neq y, x \geq 0, y \geq 0}} \frac{x - y + 2\sqrt{x} - 2\sqrt{y}}{\sqrt{x} - \sqrt{y}}$
- e. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^4}{x^4 + y^2}$
- f. $\lim_{(x,y) \rightarrow (0,0)} \frac{x}{\sqrt{x^2 + y^2}}$
- g. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{\sqrt{x^4 + y^2}}$
- h. $\lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq y}} \frac{x^2 - y^2}{x - y}$
- i. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - xy^2}{x^2 + y^2}$
- j. $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2}$
- k. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$

5. At what points (x, y) in \mathbb{R}^2 are the following functions continuous?

- a. $\sin(x + y)$
- b. $\ln(x^2 + y^2)$
- c. $\frac{x^2 + y^2}{x^2 - 3x + 2}$
- d. $\frac{1}{x^2 - y}$

6. Find $\partial f / \partial x$, find $\partial f / \partial y$, and (in the cases where f is a function of z) find $\partial f / \partial z$:

- a. $f(x, y) = \ln(x + y)$
- b. $f(x, y) = e^{xy} \ln y$
- c. $f(x, y) = \int_x^y g(t) dt$ where g is continuous
- d. $f(x, y, z) = xy + xz + yz$
- e. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$
- f. $f(x, y, z) = \ln(x + 2y + 3z)$

7. Find all second order partial derivatives:

- a. $f(x, y) = x + y + xy$
- b. $f(x, y) = \sin(xy)$
- c. $f(x, y) = xy^2 + x^2y^3 + x^3y^4$

8. Find the value of $\partial z / \partial x$ at the point $(1, 1, 1)$ if the equation

$$xy + z^3x - 2yz = 0$$

defines z as a function of the independent variables x and y .

9. Show that each of the following functions satisfy the equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$:

- a. $f(x, y, z) = x^2 + y^2 - 2z^2$
- b. $f(x, y, z) = \ln \sqrt{x^2 + y^2}$
- c. $f(x, y, z) = \arctan(x/y)$

10. Show that each of the following functions satisfy the equation $\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$:

- a. $f(x, t) = \sin(x + ct)$

b. $f(x, t) = \tan(2x - 2ct)$

11. Assume that $w = f(s^3 + t^2)$ and $f'(x) = e^x$. Find $\partial w / \partial t$ and $\partial w / \partial s$.

12. Assume that $w = f(ts^2, s/t)$, $\partial f(x, y) / \partial x = xy$, and $\partial f(x, y) / \partial y = x^2/2$. Find $\partial w / \partial t$ and $\partial w / \partial s$.

13. Suppose that f is a function of u, v , and w where $u = x - y$, $v = y - z$, and $w = z - x$. Show that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0.$$

14. Suppose that we substitute the polar equations $x = r \cos \vartheta$, $y = r \sin \vartheta$ into $w = f(x, y)$.

a. Show that

$$\frac{\partial w}{\partial r} = f_x \cos \vartheta + f_y \sin \vartheta$$

and

$$\frac{1}{r} \frac{\partial w}{\partial \vartheta} = -f_x \sin \vartheta + f_y \cos \vartheta.$$

b. Solve the equations in part a. to express f_x and f_y in terms of $\partial w / \partial r$ and $\partial w / \partial \vartheta$.

c. Show that

$$(f_x)^2 + (f_y)^2 = \left(\frac{\partial w}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial w}{\partial \vartheta} \right)^2.$$

15. Show that if $w = f(u, v)$ satisfies

$$f_{uu} + f_{vv} = 0$$

and if $u = (x^2 - y^2)/2$ and $v = xy$, then w satisfies

$$w_{xx} + w_{yy} = 0.$$

16. Find the gradient of the function at the given point. Sketch the gradient and the level curve that passes through the point:

a. $f(x, y) = y - x$ at $(2, 1)$

b. $f(x, y) = \ln(x^2 + y^2)$ at $(1, 1)$

c. $f(x, y) = x^2/2 - y^2/2$ at $(\sqrt{2}, 1)$

17. Find the derivative of the function at P in the direction of \mathbf{u} :

a. $f(x, y) = 2x^2 + y^2$ with $P = (-1, 1)$ and $\mathbf{u} = \langle 3, -4 \rangle$

b. $f(x, y) = \frac{x-y}{xy+2}$ with $P = (1, -1)$ and $\mathbf{u} = \langle 12, 5 \rangle$

c. $f(x, y, z) = x^2 + 2y^2 - 3z^2$ with $P = (1, 1, 1)$ and $\mathbf{u} = \langle 1, 1, 1 \rangle$

18. Find the directions in which the functions increase and decrease most rapidly at P . Then find the derivatives of the functions in these directions.

a. $f(x, y) = x^2y + e^{xy} \sin y$ with $P = (1, 0)$

b. $f(x, y, z) = xe^y + z^2$ with $P = (1, \ln 2, 1/2)$

c. $f(x, y, z) = \ln(xy) + \ln(yz) + \ln(xz)$ with $P = (1, 1, 1)$

19. Is there a direction \mathbf{u} in which the rate of change of $f(x, y) = x^2 - 3xy + 4y^2$ at $(1, 2)$ equals 14? Why?

20. Find the tangent plane and the normal line for the surface at the point P :

a. $x^2 + y^2 + z^2 = 3$ at $P = (1, 1, 1)$

b. $x^2 + y^2 - z^2 = 18$ at $P = (3, 5, -4)$

c. $2z - x^2 = 0$ at $P = (2, 0, 2)$

d. $x^2 + y^2 - 2xy - x + 3y - z = -4$ at $P = (2, -3, 18)$

e. $z = \ln(x^2 + y^2)$ at $P = (1, 0, 0)$

f. $z = 4x^2 + y^2$ at $P = (1, 1, 5)$

21. Find all local maxima, local minima, and saddle points for these functions

a. $f(x, y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$

b. $f(x, y) = 5xy - 7x^2 + 3x - 6y + 2$

c. $f(x, y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$

d. $f(x, y) = x^3 + 3xy + y^3$

e. $f(x, y) = 4xy - x^4 - y^4$

f. $f(x, y) = \ln(x + y) + x^2 - y$

22. Find the absolute maxima and absolute minima of the function on the given domain:

a. $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ on the triangular region bounded by the lines $x = 0$, $y = 2$, $y = 2x$ in the first quadrant

b. $f(x, y) = x^2 + xy + y^2 - 6x$ on $[0, 5] \times [-3, 3]$

c. $f(x, y) = (4x - x^2) \cos y$ on $[1, 3] \times [-\frac{\pi}{4}, \frac{\pi}{4}]$

23. Find two numbers a and b with $a \leq b$ that maximizes

$$\int_a^b (6 - x - x^2) dx.$$

24. Find the maximum value of $xy + yz + xz$ where $x + y + z = 6$.

25. Find the minimum distance from $z = \sqrt{x^2 + y^2}$ to the point $(-6, 4, 0)$.

26. Find the points on the ellipse $x^2 + 2y^2 = 1$ where $f(x, y) = xy$ has its extreme values.

27. Find the extreme values of $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + y^2 - 10 = 0$.

28. Find the points on $x^2y = 2$ closest to the origin.

29. Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy -plane that are nearest and farthest from the origin.

30. Find the dimensions of the rectangle of largest perimeter that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$ with sides parallel to the coordinate axes. What is this perimeter?

31. Find the maximum and minimum values of $3x - y + 6$ subject to the constraint $x^2 + y^2 = 4$.

32. Maximize and minimize xyz^2 on $x^2 + y^2 + z^2 = 1$.

33. Evaluate the following integrals:

a. $\int_0^2 \int_{-1}^1 (x - y) dx dy$

b. $\int_0^1 \int_0^1 \left(1 - \frac{x^2 + y^2}{2}\right) dx dy$

c. $\int_0^1 \int_1^2 xye^x dx dy$

d. $\iint_R xy \cos y dA$ where $R = [-1, 1] \times [0, \pi]$

e. $\iint_R y \sin(x + y) dA$ where $R = [-\pi, 0] \times [0, \pi]$

f. $\iint_R \frac{y}{x^2y^2 + 1} dA$ where $R = [0, 1] \times [0, 1]$

g. $\iint_R \frac{1}{xy} dA$ where $R = [1, 2] \times [1, 2]$

34. Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below $[-1, 1] \times [-1, 1]$.

35. Sketch the described regions of integration in \mathbb{R}^2 :

a. $-2 \leq y \leq 2, y^2 \leq x \leq 4$

b. $0 \leq y \leq 1, y \leq x \leq 2y$

c. $1 \leq x \leq e^2, 0 \leq y \leq \ln x$.

36. Write an iterated integral for $\iint_R dA$ over the region R using both vertical cross sections and horizontal cross sections:

a. The triangle in the first quadrant of \mathbb{R}^2 bounded by the graphs of $x = 3$ and $y = 2x$.

b. The region in the first quadrant of \mathbb{R}^2 bounded by the lines $x = 2, y = 1$, and the graph of the function $y = e^x$.

c. The region bounded by $y = 3 - 2x, y = x$, and $x = 0$.

d. The region bounded by $y = x^2$ and $y = x + 2$.

37. Sketch the region and then evaluate the integral:

a. $\int_0^\pi \int_0^x x \sin y dy dx$

b. $\int_1^2 \int_y^{y^2} dx dy$

c. $\int_0^1 \int_0^{\sqrt{1-x^2}} 8y dy dx$

38. Reverse the order of integration and then evaluate the integral:

a. $\int_0^\pi \int_x^\pi \frac{\sin y}{y} dy dx$

b. $\int_0^1 \int_y^1 x^2 e^{xy} dx dy$

c. $\int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2} dx dy$

d. $\iint_R (y - 2x^2) dA$ where R is the region bounded by $|x| + |y| = 1$ in \mathbb{R}^2 .

39. Find the volume of the solid in the first octant bounded by the coordinate planes, the plane $x = 3$, and the parabolic cylinder $z = 4 - y^2$.

40. Find the area of the following regions using a double integral:

- The coordinate axes and the line $x + y = 2$.
- The parabola $x = -y^2$ and $y = 4$.
- The curve $y = e^x$ and the lines $y = 0$, $x = 0$, and $x = \ln 2$.
- The lines $y = 2x$, $y = x/2$, and $y = 3 - x$.

41. Change from Cartesian into polar and evaluate:

- $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} dy dx$
- $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2) dx dy$
- $\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy dx$
- $\int_{\sqrt{2}}^2 \int_0^x y dy dx$
- $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2} dy dx$
- $\int_0^1 \int_x^{\sqrt{2-x^2}} (x+2y) dy dx$
- $\int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2} dy dx$

42. The average value of a function $f(x, y)$ over the region R in \mathbb{R}^2 is given by

$$\frac{1}{\text{area}(R)} \iint_R f(x, y) dx dy.$$

Find the average values of $f(x, y) = \sqrt{a^2 - x^2 - y^2}$ over the region described by $x^2 + y^2 \leq a^2$.

43. Find the average distance from a point on the disk $x^2 + y^2 \leq a^2$ to the origin.

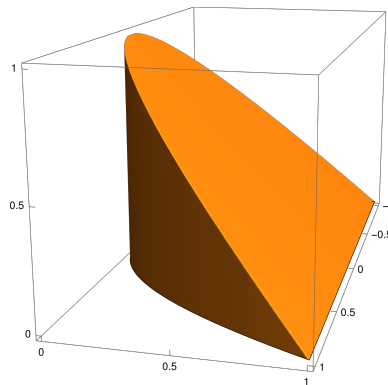
44. Let $I = \int_0^\infty e^{-x^2} dx$. Evaluate I^2 by noticing that

$$\begin{aligned} I^2 &= \left(\int_0^\infty e^{-x^2} dx \right) \left(\int_0^\infty e^{-y^2} dy \right) \\ &= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy \end{aligned}$$

and switching into polar.

45. Write six different iterated triple integrals for the volume of the following regions in \mathbb{R}^3 :

- The first octant enclosed by the cylinder $x^2 + z^2 = 4$ and the plane $y = 3$
- The region bounded by $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$.
- The region bounded in the first octant that satisfies $z + y \leq 1$ and $x^2 \leq y$:

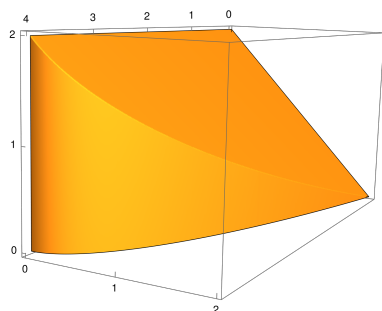


46. Evaluate the integrals:

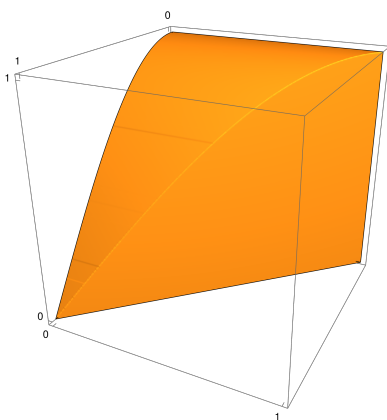
- $\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2) dz dy dx$
- $\int_0^e \int_0^{e^2} \int_0^{e^3} \frac{1}{xyz} dx dy dz$
- $\int_0^1 \int_0^{3-3x} \int_{x^2+3y^2}^{8-x^2-y^2} dz dy dx$
- $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2}} dz dy dx$
- $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} dz dy dx$
- $\int_0^\pi \int_0^\pi \int_0^\pi \cos(x+y+z) dz dy dx$

47. Find the volumes of the following regions in \mathbb{R}^3 :

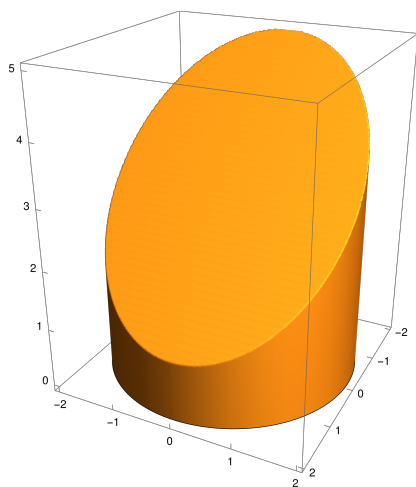
- The region in the first octant bounded by the coordinate planes, the plane $y + z = 2$, and the cylinder $x = 4 - y^2$:



- b. The tetrahedron in the first octant bounded by the coordinate planes and the plane passing through $(1, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 3)$.
- c. The region in the first octant bounded by the coordinate planes, the plane $y = 1 - x$ and the surface $z = \cos(\pi x/2)$ for $0 \leq x \leq 1$:



- d. The region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$:

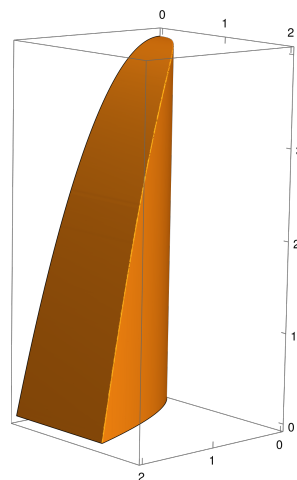


48. Find the center of mass of a thin plate of uniform density bounded by the lines $x = 0$, $y = x$, and the parabola $y = 2 - x^2$ in the first quadrant.

49. Find the center of mass of a thin plate of nonuniform density in the shape of a triangle bounded by the y -axis and the lines $y = x$ and $y = 2 - x$ if the density at the point (x, y) is $6x + 3y + 3$.

50. Find the mass of the solid and find the center of mass for the following regions in \mathbb{R}^3 :

- a. The solid region in the first octant bounded by the coordinate planes and the plane $x + y + z = 2$ where the solid has density given by $\delta(x, y, z) = 2x$.
- b. The solid region in the first octant bounded by the planes $y = 0$ and $z = 0$ and the surfaces $z = 4 - x^2$ and $x = y^2$ where the solid has density given by $\delta(x, y, z) = xy$:



- c. A solid cube in the first octant bounded by the planes $x = 1$, $y = 1$, and $z = 1$ where the density of the cube at (x, y, z) is given by $x + y + z + 1$.

51. Evaluate the integrals in cylindrical coordinates:

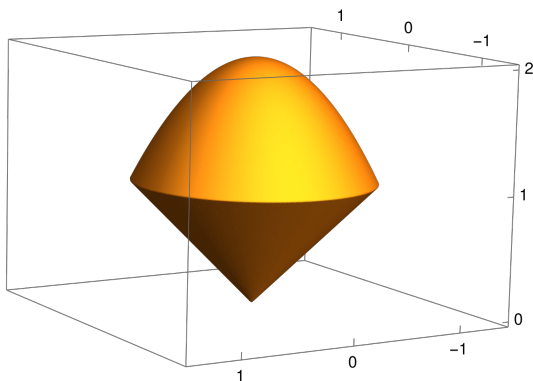
- a. $\int_0^{2\pi} \int_0^1 \int_r^{\sqrt{2-r^2}} dz r dr d\theta$
- b. $\int_0^{2\pi} \int_0^{\theta/(2\pi)} \int_0^{3+24r^2} dz r dr d\theta$
- c. $\int_0^{2\pi} \int_0^3 \int_0^{z/3} r^3 dr dz d\theta$

52. Convert

$$\int_{-1}^1 \int_0^{\sqrt{1-y^2}} \int_0^x (x^2 + y^2) dz dx dy$$

into cylindrical coordinates and evaluate.

53. Let R be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$:

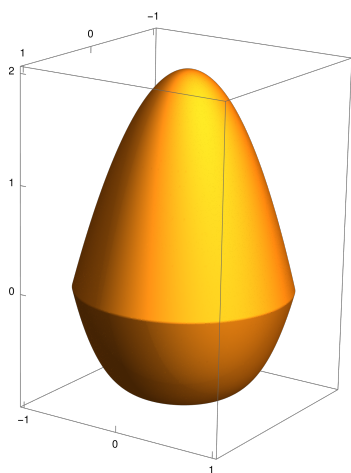


Set up the triple integrals that give the volume of R in these three orders: $dz dr d\theta$, $dr dz d\theta$ and $d\theta dz dr$.

54. Evaluate the integrals in spherical coordinates (some orders of integration may be easier than others):

- $\int_0^\pi \int_0^\pi \int_0^{2\sin\varphi} \rho^2 \sin\varphi d\rho d\varphi d\theta$
- $\int_0^{2\pi} \int_0^{\pi/4} \int_0^2 \rho^3 \cos\varphi \sin\varphi d\rho d\varphi d\theta$
- $\int_0^2 \int_{-\pi}^0 \int_{\pi/4}^{\pi/2} \rho^3 \sin(2\varphi) d\varphi d\theta d\rho$
- $\int_{\pi/6}^{\pi/2} \int_{-\pi/2}^{\pi/2} \int_{\csc\varphi}^2 5\rho^4 \sin^3\varphi d\rho d\theta d\varphi$

55. Find the volumes of the region above $z = (x^2 + y^2)^2 - 1$ and below $z = 4 - 4(x^2 + y^2)$:



56. Find the average value of the function $f(\rho, \varphi, \theta) = \rho$ over the solid ball described by $\rho \leq 1$ in spherical coordinates.

57. Let $u = x - y$ and $v = 2x + y$.

- a. Solve for x and y in terms of u and v .

b. Find the Jacobian $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$.

- c. Sketch the image of the triangle with corners $(0, 0)$, $(1, 1)$, and $(1, -2)$ in the x, y plane after the transformation into the u, v plane given by $u = x - y$ and $v = 2x + y$.

- d. Use this transformation to evaluate the integral

$$\iint_R (2x^2 - xy - y^2) dx dy$$

where R is the region in the first quadrant bounded by the lines $y = -2x + 4$, $y = -2x + 7$, $y = x - 2$, and $y = x + 1$.

58. Let $u = 3x + 2y$ and $v = x + 4y$.

- a. Solve for x and y in terms of u and v .

b. Find the Jacobian $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$.

- c. Sketch the image of the triangle bounded by the x -axis, the y -axis, and the line $x + y = 1$ in the x, y plane after the transformation into the u, v plane given by $u = 3x + 2y$ and $v = x + 4y$.

- d. Use this transformation to evaluate the integral

$$\iint_R (3x^2 + 14xy + 8y^2) dx dy$$

where R is the region in the first quadrant bounded by the lines $y = -(3/2)x + 1$, $y = -(3/2)x + 3$, $y = -(1/4)x$, and $y = -(1/4)x + 1$.

59. Find the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$ in the x, y -plane by using the transformation $x = au$, $y = bv$.

60. Use the transformation $x = u^2 - v^2$, $y = 2uv$ to evaluate

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2} dy dx.$$

61. Evaluate the following line integrals:

- a. $\int_C (x + y) ds$ where C is the straight line segment from $(0, 1, 0)$ to $(1, 0, 0)$.

b. $\int_C (x - y + z - 2) ds$ where C is the straight line segment from $(0, 1, 1)$ to $(1, 0, 1)$.

c. $\int_C (x + \sqrt{y} - z^2) ds$ where C is the path that starts at $(0, 0, 0)$ and then ends at $(1, 1, 1)$ by moving along $\langle t, t^2, 0 \rangle$ for $t \in [0, 1]$ and then moving along $\langle 1, 1, s \rangle$ for $s \in [0, 1]$.

d. $\int_C x ds$ where C is the straight line segment from $(0, 0)$ to $(4, 2)$ in \mathbb{R}^2 .

e. $\int_C x ds$ where C follows the parabolic curve $y = x^2$ from $(0, 0)$ to $(2, 4)$ in \mathbb{R}^2 .

f. $\int_C x^2 / (y^{4/3}) ds$ where C follows the path described by the parametric equations $x = t^2, y = t^3$ for $t \in [1, 2]$.

g. $\int_C \frac{1}{x^2 + y^2 + 1} ds$ where C travels counterclockwise along the perimeter of the square in \mathbb{R}^2 with corners $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$.

62. Find the gradient fields for the following functions:

a. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$

b. $f(x, y, z) = xy + yz + xz$

63. Find the line integrals of the potential function for the conservative vector field \mathbf{F} along both of the paths C_1 and C_2 where C_1 is the straight line segment from $(0, 0, 0)$ to $(1, 1, 1)$ and C_2 is the path described by $\langle t, t^2, t^3 \rangle$ for $t \in [0, 1]$:

a. $\mathbf{F} = \langle 3y, 3x, 4z \rangle$

b. $\mathbf{F} = \langle yz, xz, xy \rangle$

64. Find the work done by \mathbf{F} over the following curves:

a. $\mathbf{F} = \langle xy, y, -yz \rangle$ over $\mathbf{r}(t) = \langle t, t^2, t \rangle$ for $t \in [0, 1]$

b. $\mathbf{F} = \langle 2y, 3x, x + y \rangle$ over $\mathbf{r}(t) = \langle \cos t, \sin t, t/6 \rangle$ for $t \in [0, 2\pi]$

c. $\mathbf{F} = \langle xy, y - x \rangle$ over the straight line in \mathbb{R}^2 from $(1, 1)$ to $(2, 3)$

d. \mathbf{F} is the gradient of $f(x, y) = (x + y)^2$ over the path that travels counterclockwise once around the circle $x^2 + y^2 = 4$ in \mathbb{R}^2 that starts and ends at $(2, 0)$.

65. State whether or not the vector field is conservative:

a. $\mathbf{F} = \langle yz, xz, xy \rangle$

b. $\mathbf{F} = \langle y, x + z, -y \rangle$

c. $\mathbf{F} = \langle -y, x \rangle$

66. Find a potential function f for the field \mathbf{F} :

a. $\mathbf{F} = \langle 2x, 3y, 4z \rangle$

b. $\mathbf{F} = \langle y + z, x + z, x + y \rangle$

67. Show that the values of the follow integrals do not depend on the path from A to B :

a. $\int_A^B z^2 dx + 2y dy + 2xz dz$

b. $\int_A^B \frac{x dx + y dy + z dz}{\sqrt{x^2 + y^2 + z^2}}$

68. Find a potential function for the gravitational field

$$\mathbf{F} = -GmM \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}$$

where G , m , and M are constants. Then show that the work done by the gravitational field in moving a particle from a distance a to a distance b away from the origin is

$$GmM \left(\frac{1}{b} - \frac{1}{a} \right).$$

69. Find the work done by \mathbf{F} in moving a particle once counterclockwise around the given curve C :

a. $\mathbf{F} = \langle 2xy^3, 4x^2y^2 \rangle$ with C the boundary of the region in the first quadrant enclosed by the x -axis, the line $x = 1$, and the curve $y = x^3$

b. $\mathbf{F} = \langle 4x - 2y, 2x - 4y \rangle$ with C the boundary of the circle $(x - 2)^2 + (y - 2)^2 = 4$

70. Apply Green's theorem to evaluate the integrals:

a. $\oint_C (y^2 dx + x^2 dy)$ with C the triangle bounded by $x = 0$, $x + y = 1$, and $y = 0$

b. $\oint_C (3y dx + 2x dy)$ with C the boundary of $0 \leq x \leq \pi, 0 \leq y \leq \sin x$

71. Green's area formula says that the area of a region R enclosed by a simple closed curve C in \mathbb{R}^2 is given by

$$\frac{1}{2} \oint_C x dy - y dx.$$

Use this formula to find the areas of the regions enclosed by these curves:

- a. The circle $\mathbf{r}(t) = \langle a \cos t, a \sin t \rangle$ for $t \in [0, 2\pi]$
- b. The ellipse $\mathbf{r}(t) = \langle a \cos t, b \sin t \rangle$ for $t \in [0, 2\pi]$
- c. The asteroid $\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t \rangle$ for $t \in [0, 2\pi]$