Recall. ① **Monomial Symmetric Function**

$m_\lambda(x_1, x_2, x_3, \dots) = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_\ell^{\lambda_\ell} + \cdots$ (All monomial combos)

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$.

② $\{m_\lambda : \lambda \vdash n\}$ is a **basis** for vector space of degree-n symmetric functions.

③ **Elementary Symmetric Function**

$$e_n(x_1, \dots) = \sum_{\substack{\text{Tableaux } T \\ \text{shape } 1^n \\ \text{inc labels } (\uparrow)}} \text{weight}(T)$$

Ex. $e_3(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4$

$\begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array} \quad \begin{array}{|c|}\hline 4\\\hline 2\\\hline 1\\\hline\end{array} \quad \begin{array}{|c|}\hline 4\\\hline 3\\\hline 1\\\hline\end{array} \quad \begin{array}{|c|}\hline 4\\\hline 3\\\hline 2\\\hline\end{array}$

$= m_{(1^3)}$

④ h_n: $\begin{array}{|c|c|c|}\hline 1 & 1 & 2\\\hline\end{array}$, ...

⑤ $p_n = x_1^n + x_2^n + \cdots$

Define for any $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$:

① $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots$

② $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$

③ $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$

Ex. $e_{(3,3,2)}(x_1, x_2, \dots) = e_3 \cdot e_3 \cdot e_2$

$= (x_1 x_2 x_3 + x_1 x_2 x_4 + \cdots)(x_1 x_2 x_3 + x_1 x_2 x_4 + \cdots)(x_1 x_2 + x_1 x_3 + x_1 x_4 + \cdots)$

$= 1 \cdot x_1^3 x_2^3 x_3^2 + \cdots 2 \cdot x_1^3 x_2^3 x_3 x_4 + \cdots + \underline{x_1^3 x_2^2 x_3^2 x_4}$

$= 1 \cdot m_{(3,3,2)} + 2 \cdot m_{(3,3,1,1)}$

Counting # ways to construct $x_1^3 x_2^2 x_3^2 x_4$ from 3 products.

Organizing into matrices, with rows (vars) & cols (e_n's).

$$\begin{array}{cccccc} & & e_3 & e_3 & e_2 & & \\ x_1 & \Bigg[& 1 & 1 & 1 & \Bigg] & \rightarrow 3 \\ x_2 & & 1 & 1 & 0 & & \rightarrow 2 \\ x_3 & & 1 & 1 & 0 & & \rightarrow 2 \\ x_4 & & 0 & 0 & 1 & & \rightarrow 1 \\ & & \downarrow & \downarrow & \downarrow & & \\ & & 3 & 3 & 2 & & \end{array}$$

Since row/col sums constant, organize count w/ these matrices:

$$\begin{array}{cccc} & 3 & 3 & 2 \\ 3 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array} \qquad \begin{array}{cccc} & 3 & 3 & 2 \\ 3 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}$$

$$\begin{array}{cccc} & 3 & 3 & 2 \\ 3 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{array} \qquad \begin{array}{cccc} & 3 & 3 & 2 \\ 3 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}$$

5 ways: $\therefore$ $5 \cdot m_{(3,2,2,1)}$

THM. Let $\lambda, \mu \vdash n$. Then the coefficient of m_λ in e_μ is the number of 0-1 (binary) matrices with row sum $= \lambda$ & col sum $= \mu$.

Ex. $(n=4)$

μ

		(4)	$(3,1)$	$(2,2)$	$(2,1^2)$	(1^4)
	(4)	0	0	0	0	1
	$(3,1)$	0	0	0	1	4
λ	$(2,2)$	0	0	1	2	6
	$(2,1^2)$	0	1	2	5	12
	(1^4)	1	4	6	12	24

$\lambda = (3,1)$, $\mu = (2,1^2)$

$$\begin{array}{cc} & \begin{array}{ccc} 2 & 1 & 1 \end{array} \\ \begin{array}{c} 3 \\ 1 \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

$\lambda = (2,1^2)$, $\mu = (2,1^2)$

$$\begin{array}{cc} & \begin{array}{ccc} 2 & 1 & 1 \end{array} \\ \begin{array}{c} 2 \\ 1 \\ 1 \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccc} 2 & 1 & 1 \end{array} \\ \begin{array}{c} 2 \\ 1 \\ 1 \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccc} 2 & 1 & 1 \end{array} \\ \begin{array}{c} 2 \\ 1 \\ 1 \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccc} 2 & 1 & 1 \end{array} \\ \begin{array}{c} 2 \\ 1 \\ 1 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccc} 2 & 1 & 1 \end{array} \\ \begin{array}{c} 2 \\ 1 \\ 1 \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

$\lambda = (1^4)$, $\mu = (3,1)$

$$\begin{array}{c|cc} & 3 & 1 \\ \hline 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{array} \qquad \begin{array}{c|cc} & 3 & 1 \\ \hline 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \begin{array}{c|cc} & 3 & 1 \\ \hline 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}$$

$$\begin{array}{c|cc} & 3 & 1 \\ \hline 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{array} \qquad \therefore 4$$

μ

λ	(4)	$(3,1)$	$(2,2)$	$(2,1^2)$	(1^4)
(4)	0	0	0	0	1
$(3,1)$	0	0	0	1	4
$(2,2)$	0	0	1	2	6
$(2,1^2)$	0	1	2	5	12
(1^4)	1	4	6	12	24

Observations.

- ↳ Symmetric.
- ↳ 0's up until approach conjugate of λ/μ
- ↳ Matrix is invertible: $\det = \pm 1$.
- ↳ Change of Basis from e_λ to m_λ