Graph Theory Midterm 2 Solutions

1. State the definition of

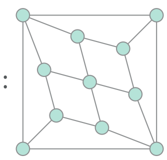
a. A planar graph

b. An Eulerian graph

c. A u, v separating set

d. A covering in a graph G

Solution. *See the notes for the definitions.*

2. Show the following graph is not Hamiltonian: 

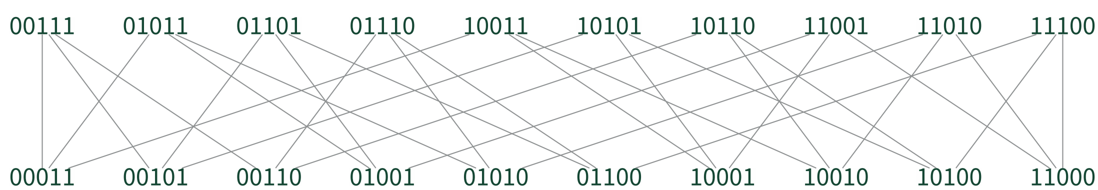
Solution. *The planar graph has* 9 *faces with* 4 *edges. If* a *of these faces are inside the proposed Hamiltonian cycle, then* $2(2a - 9) = 0$*, giving* $a = 9/2$*, a contradiction.*

3. Use the identities $M_G(x) = M_{G-e}(x) - M_{G-u-v}(x)$ and $M_G(x) = xM_{G-u}(x) - \sum_{v \text{ is adjacent to } u} M_{G-u-v}(x)$ to find a recursion for the matching polynomial for the wheel W_n in terms of matching polynomials for path graphs only.

Solution. *Using the second recursion and taking u to be the middle vertex in the wheel and then using the first equation on the resulting* $M_{C_n}(x)$ *gives*

$$\begin{aligned} M_{W_n}(x) &= xM_{C_{n-1}}(x) - (n-1)M_{P_{n-2}}(x) \\ &= x(M_{P_{n-1}}(x) - M_{P_{n-3}}(x)) - (n-1)M_{P_{n-2}}(x) \\ &= xM_{P_{n-1}}(x) - (n-1)M_{P_{n-2}}(x) - xM_{P_{n-3}}(x). \end{aligned}$$

4. Let G be the subgraph of the cube Q_{2n+1} containing those bit strings with either n 0's and $(n+1)$ 1's or with $(n+1)$ 0's and n 1's. For example, the graph G when $n = 2$ is below. Show that G has a perfect matching.



Solution. *The graph G is a bipartite graph (since it is a subgraph of a bipartite graph) and every vertex is degree* $n+1$*. Let G have independent sets X and Y. If* $S \subseteq X$*, then there are* $(n+1)|S|$ *vertices leaving S. Since each vertex in Y has degree n, we have* $|S| \leq |N(S)|$ *and so Hall's theorem applies.*

5. Suppose $\varepsilon(G) \neq 1$ and $e = \{u, v\}$ is an edge in G such that u and v are in a cycle in $G - e$. Show that $\varepsilon(G - e) \neq 1$.

Solution. *There are two edge disjoint paths from x to y in G because* $\varepsilon(G) \geq 2$*. If there are not two edge disjoint paths from x to y in* $G - e$*, then the edge disjoint paths that would have used e can instead travel along one of the two ways around the cycle containing u and v, thus still finding two edge disjoint paths from x to y in* $G - e$*. Therefore* $\varepsilon(G - e) \geq 2$*.*