

# Linear Analysis II Exercise Set 1

1. Use the known Laplace transforms of  $t^n$ ,  $e^{at}$ ,  $\cos(at)$ , and  $\sin(at)$  that are given in the table of Laplace transforms on our web site to find

$$\mathcal{L} \left[ \sin(\sqrt{3}t) - 7 \sin^2(3t) - e^{4t} + 2 + t^{101} \right].$$

Hint:  $\sin^2 x = (1 - \cos(2x))/2$ .

2. Use integration by parts to show that  $\mathcal{L}[t^r] = \frac{r}{s} \mathcal{L}[t^{r-1}]$  holds for all  $r > 0$ .

3. Here are some well known series:

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \quad \sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}, \quad \ln \left( \frac{1}{1-t} \right) = \sum_{n=1}^{\infty} \frac{t^n}{n}, \quad \arctan t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)}.$$

Find the following Laplace transforms by writing the function as a series, using  $\mathcal{L}[t^n] = n!/s^{n+1}$  on each term, and then identifying the result as a variation on one of the above series:

a.  $\mathcal{L} \left[ \frac{\sin t}{t} \right]$

b.  $\mathcal{L} \left[ \frac{e^t - 1}{t} \right]$

c.  $\mathcal{L}[J(\sqrt{t})]$  where  $J(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left( \frac{t}{2} \right)^{2n} = 1 - \frac{1}{(1!)^2 2^2} t^2 + \frac{1}{(2!)^2 2^4} t^4 - \frac{1}{(3!)^2 2^6} t^6 + \dots$

4. a. Use the substitution  $t = \frac{x^2}{s}$  to show that  $\mathcal{L} \left[ \frac{1}{\sqrt{t}} \right] = \frac{2}{\sqrt{s}} \int_0^{\infty} e^{-x^2} dx$ .

b. Show that  $\mathcal{L} \left[ \frac{1}{\sqrt{t}} \right]^2 = \frac{4}{s} \int_0^{\infty} \int_0^{\infty} e^{-x^2 - y^2} dx dy$ .

c. Using polar coordinates to integrate the above result, find  $\mathcal{L} \left[ \frac{1}{\sqrt{t}} \right]$ .

d. Find  $\mathcal{L}[\sqrt{t}]$  and  $\mathcal{L}[t^{3/2}]$  using this exercise and Exercise 2.