Systems of equations

1. Use elementary row operations to put these matrices into row echelon form.

a. $\begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$

c. $\begin{bmatrix} 2-i & 2 \\ 1 & -i \end{bmatrix}$

d. $\begin{bmatrix} 0 & 1 & 5 \\ 0 & 2 & 4 \\ 0 & 5 & 1 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 1 & 5 & 1 & 2 \\ -1 & 2 & 4 & 2 & -1 \\ 3 & 5 & 1 & 3 & 0 \end{bmatrix}$

2. Find the general solutions to the systems with these augmented matrices by putting the matrix into Row Echelon Form:

a. $\begin{bmatrix} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

b. $\begin{bmatrix} 1 & -5 & 4 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -1 & 0 & 0 & 5 \\ 0 & 1 & -2 & 0 & 7 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix}$

e. $\begin{bmatrix} 1 & -3 & 0 & -5 \\ -3 & 7 & 0 & 9 \end{bmatrix}$

f. $\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

3. Solve the system of equations:

a.

$$\begin{aligned} x_2 + 5x_3 &= -4 \\ x_1 + 4x_2 + 3x_3 &= -2 \\ 2x_1 + 7x_2 + x_3 &= -2 \end{aligned}$$

b.

$$\begin{aligned} x_1 - 5x_2 + 4x_3 &= -3 \\ 2x_1 - 7x_2 + 3x_3 &= -2 \\ -2x_1 + x_2 + 7x_3 &= -1 \end{aligned}$$

c.

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ 2x_1 - 7x_2 &= -5 \end{aligned}$$

4. Determine if the following system is consistent. Do not completely solve the system.

$$\begin{aligned} x_1 - 6x_2 &= 5 \\ x_2 - 4x_3 + x_4 &= 0 \\ -x_1 + 6x_2 + x_3 + 5x_4 &= 3 \\ -x_2 + 5x_3 + 4x_4 &= 0 \end{aligned}$$

5. Do these three planes have at least one point in common? Why?

$$\begin{aligned} 2x_1 + 4x_2 + 4x_3 &= 4 \\ x_2 - 2x_3 &= -2 \\ 2x_1 + 3x_2 &= 0 \end{aligned}$$

6. True or false:

a. Every elementary row operation is reversible.

b. A 5×6 matrix has six rows.

c. Elementary row operations on an augmented matrix never change the solution set of the associated linear system.

7. Give an example of an inconsistent system (a system with no solution) of two equations in three unknowns.

Vectors and $A\mathbf{x} = \mathbf{b}$

8. Write the linear system

$$\begin{cases} 6x - 2y = 1, \\ -3x + y = 1. \end{cases}$$

as a multiplication of the form $A\mathbf{x} = \mathbf{b}$ and then verify that there are no solutions to this system.

9. Solve the following linear systems using elementary row operations:

a. $\begin{cases} 4x_1 - x_2 = 8, \\ 2x_1 + x_2 = 1. \end{cases}$

b. $\begin{bmatrix} 1 & 0 & 5 \\ 3 & -2 & 11 \\ 2 & -2 & 6 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$

c. $\begin{bmatrix} 1+i & 1-2i \\ -1+i & 2+i \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

10. Determine if $\mathbf{b}$ is a linear combination of the columns of A when:

a. $A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$.

b. $A = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$.

11. If A is an $m \times n$ matrix and $A\mathbf{x} = \mathbf{0}$ for vectors $\mathbf{x}$ and $\mathbf{0}$, then what dimensions must $\mathbf{x}$ and $\mathbf{0}$ be?

12. List 4 vectors in the span of $\mathbf{v}_1, \mathbf{v}_2$ in the cases below. For each example, show the weights on $\mathbf{v}_1$ and $\mathbf{v}_2$ used to generate the example vectors.

a. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$.

b. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -6 \\ -2 \\ 4 \end{bmatrix}$.

13. True or false:

a. Another notation for $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 2 \end{bmatrix}$.

b. An example of a linear combination of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is $\frac{1}{2}\mathbf{v}_1$.

c. Asking whether the linear system corresponding to the augmented matrix $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{bmatrix}$ has a solution is equivalent to asking if $\mathbf{b}$ is in the span of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

d. The coefficients $c_1, \ldots, c_n$ in a linear combination $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ cannot be all 0.

14. Write the system as a matrix equation $A\mathbf{x} = \mathbf{b}$:

a.
$$\begin{aligned} 5x_1 + x_2 - 3x_3 &= -2 \\ 7x_2 + x_3 &= 0 \end{aligned}$$

b.
$$\begin{aligned} 4x_1 - x_2 &= 9 \\ 7x_1 + x_2 &= 0 \\ 7x_1 + 3x_2 &= 1 \end{aligned}$$

15. Given the following examples of A and $\mathbf{b}$, solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$. Write the solutions as a vector.

a. $A = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 5 & 2 \\ -3 & -7 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -2 \\ 4 \\ 12 \end{bmatrix}$.

b. $A = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -4 & 2 \\ 5 & 2 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$.

16. Let $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 5 & -1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}$. Is $\mathbf{u}$ in the subset of $\mathbb{R}^3$ spanned by the columns of A? Why?

17. Can every vector in $\mathbb{R}^4$ be written as a linear combination of the columns in $\begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$? Do these columns span $\mathbb{R}^4$?

18. True or false:

a. A vector $\mathbf{b}$ is a linear combination of the columns of a matrix A if and only if the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution.

b. Any linear combination of vectors can always be written as $A\mathbf{x}$ for some matrix A and vector $\mathbf{x}$.

c. If $\mathbf{x}$ is a nontrivial solution to $A\mathbf{x} = \mathbf{0}$, then every entry in $\mathbf{x}$ is nonzero.

Linear Independence

19. Write the linear system

$$\begin{cases} x_1 + x_2 - x_3 + 5x_4 = 0, \\ 2x_2 - x_3 + 7x_4 = 0, \\ 4x_1 + 2x_2 - 3x_3 + 13x_4 = 0, \end{cases}$$

as a multiplication of the form $A\mathbf{x} = \mathbf{0}$ and then verify that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s \\ s-2t \\ 2s+3t \\ t \end{bmatrix}$$

is a solution to the system for any s and t.

20. Describe all solutions to $A\mathbf{x} = \mathbf{0}$ using parameters and vectors where A is each one of these matrices:

a. $\begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix}$

b. $\begin{bmatrix} 3 & -6 & 6 \\ -2 & 4 & -2 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -2 & 3 & -6 & 5 & 0 \\ 0 & 0 & 0 & 1 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

21. Let A be an $m \times n$ matrix and suppose $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$ such that $A\mathbf{v} = \mathbf{0}$ and $A\mathbf{w} = \mathbf{0}$; in other words, $\mathbf{v}$ and $\mathbf{w}$ are solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$. Show that $c\mathbf{v} + d\mathbf{w}$ is also a solution to $A\mathbf{x} = \mathbf{0}$.

22. Determine if the following vectors are linearly independent:

a. $\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$

b. $\begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -7 \\ 2 \\ 4 \end{bmatrix}.$

c. $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}.$

23. True or false:

a. The columns of A are linearly independent if the equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution.

b. If S is a linearly dependent set, then each vector in S is a linear combination of the other vectors in S.

c. The columns of any 4×5 matrix are linearly dependent.

d. If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent and if $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent, then $\mathbf{z}$ is in the span of $\mathbf{x}$ and $\mathbf{y}$.

e. If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent and if $\mathbf{z}$ is in the span of $\mathbf{x}$ and $\mathbf{y}$, then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent.

f. If a set in $\mathbb{R}^n$ is linearly dependent, then the set contains more than n vectors.

24. The following statements are either True (in all cases) or False. If the statement is False, give an example illustrating that it is false. If true, explain why.

a. If $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are vectors and if $\mathbf{x} = \mathbf{y} + 2\mathbf{z}$, then the set $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent.

b. If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{R}^5$ and $\mathbf{x}$ is not a scale multiple of $\mathbf{y}$, then $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent.

c. If $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are in $\mathbb{R}^3$ and $\mathbf{z}$ is not a linear combination of $\mathbf{x}$ and $\mathbf{y}$, then the set $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly independent.

d. If $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly independent, then so is $\{\mathbf{x}, \mathbf{y}\}$.

25. Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then so is $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}$.

Linear Maps

26. Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{e}_1$ to $\mathbf{y}_1$ and $\mathbf{e}_2$ to $\mathbf{y}_2$. Find the images of $\mathbf{e}_2 = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ under T.

27. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{x}$ to $x_1\mathbf{v}_1 + x_2\mathbf{v}_2$. Find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$.

28. True or false:

a. A linear transformation is a special type of function.

b. If A is a 3×5 matrix and T is a linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, then the domain of T is $\mathbb{R}^3$.

c. If A is a $m \times n$ matrix and T is a linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, then the range of T is $\mathbb{R}^m$.

d. Every linear transformation is a matrix transformation.

e. A linear transformation always sends the zero vector to the zero vector.

f. A linear transformation preserves the operations of vector addition and scalar multiplication.

29. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the function that sends $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to $\begin{bmatrix} x_1 \\ 0 \\ x_3 \end{bmatrix}$ for all real numbers x_1, x_2, x_3. Show that T is a linear transformation.

Matrix operations and Inverses

30. Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 0 & 3 \\ 5 & 1 & 1 \\ 4 & -4 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}.$$

Perform the following matrix operations if possible:

a. AB

b. BA

c. B^2

d. $B^\top B$

e. AC

f. DBC

g. CD

31. Let A be a $m \times n$ matrix and C an $r \times s$ matrix. What dimensions must B have so that ABC is defined?

32. Find A^2, A^3 and A^4 for

a. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ -1 & 0 & 2 \end{bmatrix}$

c. $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

33. Let A and B be $n \times n$ matrices. Show that

$$(A - B)^2 = A^2 - AB - BA + B^2.$$

34. Let $A = \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{bmatrix}$ show that that A satisfies

$$A^3 + A - 26I = 0$$

where I and 0 are the 3×3 identity and zero matrices.

35. Let $A = \begin{bmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Show that $A^4 = 0$.

36. A matrix A is symmetric if $A = A^\top$. Use properties of the transpose to show that

a. $AA^\top$ is symmetric for any matrix A

b. $A + A^\top$ is symmetric for any square matrix A

c. $(ABC)^\top = C^\top B^\top A^\top$.

37. Verify by matrix multiplication that these matrices are inverses, provided that $ad - bc \neq 0$:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

38. Find the inverse of the matrix if possible:

a. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 1+i \\ 1-i & 1 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & -2 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}$

39. Use the inverse matrix to solve the system:

a. $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 4 & 5 \\ 2 & 10 & 1 \\ 4 & 1 & 8 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

40. Let $A = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$. Show that $A^\top = A^{-1}$.

41. Suppose that A satisfies $A^n = 0$ for some positive integer n. Show that the inverse to $I - A$ is

$$I + A + A^2 + \cdots + A^{n-1}.$$

Characterizations of invertibility

42. Which of the following matrices are invertible? Why?

a. $\begin{bmatrix} 3 & 0 & 0 \\ -3 & -4 & 0 \\ 8 & 5 & -3 \end{bmatrix}$

b. $\begin{bmatrix} -5 & 1 & 4 \\ 0 & 0 & 0 \\ 1 & 4 & 9 \end{bmatrix}$

c. $\begin{bmatrix} 3 & 4 & 4 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 9 & 0 \end{bmatrix}$

43. For each of these statements, the matrix A is an $n \times n$ square matrix. True or false:

a. If the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then A can be row reduced to the identity matrix.

b. If the columns of A span $\mathbb{R}^n$, then the columns of A are linearly independent.

c. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}$.

d. If A is invertible, then so is $A^\top$.

e. If the columns of A are linearly independent, then the columns span $\mathbb{R}^n$.

44. Can a square matrix with two identical columns be invertible? Why or why not?

45. If a square matrix can be row reduced to find the identity matrix, what can be said about its columns?

46. The linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$ defined by $T(x_1, x_2) = (-5x_1 + 9x_2, 4x_1 - 7x_2)$ is invertible. Find a formula for the inverse T^{-1}.

Determinants

47. Calculate the determinant:

a. $\begin{vmatrix} 3 & 5 & 7 \\ -1 & 2 & 4 \\ 6 & 3 & -2 \end{vmatrix}$

b. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

c. $\begin{vmatrix} 3 & 5 & -1 & 2 \\ 2 & 1 & 5 & 2 \\ 3 & 2 & 5 & 7 \\ 1 & -1 & 2 & 1 \end{vmatrix}$

48. Let A be invertible. Show that $\det(A^{-1}) = \dfrac{1}{\det A}$.

49. Let A and B be $n \times n$ with $\det A = 5$ and $\det B = -4$. Evaluate the determinant:

a. $\det(AB)$

b. $\det(A^\top BA)$

c. $\det(A^{-1}BA)$

d. $\det(3A)$

e. $\det C$ where C is A with its first two columns interchanged

f. $\det C$ where C is A with its first row multiplied by 2

50. Let A satisfy $A^\top A = I$. Show that $\det A = \pm 1$.

Subspaces

51. Either show that S is a subspace of the vector space V or give an example showing why it is not:

a. $V = \mathbb{R}^3$, S is the set of vectors of the form $\begin{bmatrix} x \\ 2x \\ 3x \end{bmatrix}$.

b. $V = \mathbb{R}^4$, S is the set of vectors of the form $\begin{bmatrix} x \\ y \\ x \\ 0 \end{bmatrix}$.

c. $V = \mathbb{R}^4$, S is the set of vectors of the form $\begin{bmatrix} x \\ 1 \\ 2x \\ 0 \end{bmatrix}$.

d. $V = \mathbb{R}^n$, S is the set of solutions to $A\mathbf{x} = \mathbf{0}$ where A is a fixed $m \times n$ matrix.

e. V is the vector space of 2×2 matrices with entries in $\mathbb{R}$, S is the set of matrices A with $\det A = 1$.

f. V is the vector space of 3×3 matrices with entries in $\mathbb{R}$, S is the set of upper triangular matrices.

g. V is the vector space of $n \times n$ matrices with entries in $\mathbb{R}$, S is the set of invertible matrices.

h. V is the vector space of real valued functions with domain $\mathbb{R}$, S is the set of functions $f(x)$ that satisfy $f(3) = 0$.

i. V is the vector space of real valued functions with domain $\mathbb{R}$, S is the set of functions of the form $ax^2 + bx + c$ where a, b, c are real numbers.

j. V is the vector space of real valued functions with domain $\mathbb{R}$, S is the set of solutions to the differential equation $y''(x) + y(x) = 0$.

52. Find a set of vectors that span the subspace S of the vector space V:

a. V is the space of 2×3 matrices with entries in $\mathbb{R}$, S is the set of matrices with entries that sum to 0.

b. V is the space of $n \times n$ matrices with entries in $\mathbb{R}$, S is the set of upper triangular matrices.

c. V is $\mathbb{R}^3$, S is the set of solutions to $x - 2y - z = 0$.

d. V is the space of polynomials of degree 5 or less with coefficients in $\mathbb{R}$, S is the set of polynomials p that satisfy $p'(x) = 0$.

53. Find vectors that span the nullspace of the following matrices:

a. $\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & 2 \end{bmatrix}$

b. $\begin{bmatrix} 1 & -4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

54. Give an example of an 4×3 rank 1 matrix. Give an example of an 3×4 rank 2 matrix.

55. True or false:

a. The nullspace of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$.

b. If the columns of an $m \times n$ matrix A are linearly independent, the column space of A is R^m.

c. The set of all solutions to a homogeneous linear system is the nullspace of some matrix A.

Bases and dimension

56. Determine if the given set of vectors is a basis for the subspace S of the vector space V:

a. $V = \mathbb{R}^2, S = \mathbb{R}^2, \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$

b. $V = \mathbb{R}^3, S = \mathbb{R}^3, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$

c. V is space of 2×2 matrices with entries in $\mathbb{R}$, S is the subspace containing matrices with entries that sum to 0, $\begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}, \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}.$

57. Find a basis for the nullspace of the matrix (a basis for the subspace of $\mathbb{R}^n$ containing solutions to $A\mathbf{x} = \mathbf{0}$):

a. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & -1 & 4 \\ 2 & 3 & -2 \\ 1 & 2 & -2 \end{bmatrix}$

58. Find a basis and the dimension of the subspace S of the vector space V:

a. V is the set of real valued functions on $\mathbb{R}$, S is the set of solutions to $f''(x) = 0$.

b. V is the set of polynomials of degree 3 or less with coefficients in $\mathbb{R}$, S is the set of polynomials p that satisfy $p(-1) = 0$.

c. V is $\mathbb{R}^3$, S is the span of the vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} \right\}.$

d. V is $\mathbb{R}^3$, S is the span of $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} \right\}.$

e. V is the space of 2×2 matrices over $\mathbb{R}$, S is the span of $\left\{ \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 5 & -6 \\ -5 & 2 \end{bmatrix} \right\}.$

f. V is the space of 4×4 matrices over $\mathbb{R}$, S is the set of matrices A that satisfy $A^\top = -A$.

59. True or false:

a. If S is the span of some vectors, then those vectors are a basis for S.

b. The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

c. The rows of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

d. A basis is a linearly independent set that is as large as possible.

Eigenvalues and Eigenvectors

60. Find the eigenvalues and eigenvectors:

a. $\begin{bmatrix} 5 & -4 \\ 8 & -7 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 6 \\ 2 & -3 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

d. $\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$

e. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

61. Show that if λ is an eigenvalue for an invertible matrix A, then λ^{-1} is an eigenvalue for A^{-1}.

62. Show that if A is square, then A and $A^\top$ have the same eigenvalues.

63. True or false:

a. If $A\mathbf{x} = \lambda\mathbf{x}$ for some vector $\mathbf{x}$, then λ is an eigenvalue for A.

b. If $A\mathbf{x} = \lambda\mathbf{x}$ for some constant λ, then $\mathbf{x}$ is an eigenvector for A.

c. A number c is an eigenvalue for A if and only if the equation $(A - cI)\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

d. A matrix A is invertible if and only if 0 is not an eigenvalue of A.

64. Explain why a 2×2 matrix can have at most 2 distinct eigenvalues. Why can an $n \times n$ matrix have at most n distinct eigenvalues?

Diagonalization

65. Diagonalize the matrix A if possible: (provide a matrix S and D such that $A = S^{-1}DS$).

a. $\begin{bmatrix} -9 & 0 \\ 4 & -9 \end{bmatrix}$

b. $\begin{bmatrix} -7 & 4 \\ -4 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 7 \\ 1 & 1 & -3 \end{bmatrix}$

e. $\begin{bmatrix} -2 & 1 & 4 \\ -2 & 1 & 4 \\ -2 & 1 & 4 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

g. $\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

66. True or false:

a. A $n \times n$ matrix A is diagonalizable if and only if A has n distinct eigenvalues.

b. Let A be an $n \times n$ matrix A. If there are n eigenvalues of A that span $\mathbb{R}^n$, then A is diagonalizable.

c. If A is invertible, then A is diagonalizable.

67. Suppose A is invertible and diagonalizable. Explain why A^{-1} is also diagonalizable.

Inner products and projections

68. Find a unit vector in the direction of $\begin{bmatrix} 7/4 \\ 1/2 \\ 1 \end{bmatrix}$.

69. Find the distance between $\begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}$.

70. True or false:

a. $\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2$ for all vectors $\mathbf{x} \in \mathbb{R}^n$

b. If the distance from $\mathbf{u}$ to $\mathbf{v}$ equal the distance from $\mathbf{u}$ to $-\mathbf{v}$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

c. If A is an $n \times n$ symmetric square matrix (meaning $A = A^\top$), then the every vector in the column space of A is orthogonal to every vector in the nullspace of A. (Consider $\mathbf{y}^\top A\mathbf{x}$ where $\mathbf{y}$ is in the nullspace of A and $\mathbf{x}$ is any vector in $\mathbb{R}^n$).

d. $\mathbf{x} \cdot \mathbf{y} - \mathbf{y} \cdot \mathbf{x} = 0$.

e. If $c \in \mathbb{R}$, then $\|c\mathbf{x}\| = c\|\mathbf{x}\|$ for every $\mathbf{x} \in \mathbb{R}^n$.

f. If $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$, then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

g. $\mathbf{x} \cdot \mathbf{x} > 0$ for every $\mathbf{x} \in \mathbb{R}^n$

71. Verify that $\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 2\|\mathbf{y}\|^2$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

72. Suppose $\mathbf{x}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$. Show that $\mathbf{x}$ is orthogonal to every linear combination of $\mathbf{u}$ and $\mathbf{v}$.

73. Show that if $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are pairwise orthogonal vectors such that $\|\mathbf{u}_i\| = 1$ for all i, then

$$\|c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k\|^2 = c_1^2 + \cdots + c_k^2.$$

Gram-Schmidt and projections

74. Use the Gram-Schmidt procedure to orthogonalize the span of the vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}$.

75. Find the vector in the span of $\begin{bmatrix}4\\2\\2\\1\end{bmatrix}, \begin{bmatrix}0\\-2\\1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\0\\-2\\0\end{bmatrix}$ closest to $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$.

76. Find the projection matrix P for the span of the vectors $\begin{bmatrix}1\\-1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}$, and $\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$. Warning: to use

$$P = \frac{1}{\mathbf{u}_1^\top \mathbf{u}_1}\mathbf{u}_1\mathbf{u}_1^\top + \cdots + \frac{1}{\mathbf{u}_k^\top \mathbf{u}_k}\mathbf{u}_k\mathbf{u}_k^\top$$

the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ must be pairwise orthogonal.

77. Use the projection matrix P to find the vector $\mathbf{w}$ in the span of $\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}$ closest to $\begin{bmatrix}a\\b\\c\end{bmatrix}$.

78. Let P be the projection matrix onto the subspace S of $\mathbb{R}^n$ and let $\mathbf{x}, \mathbf{y}$ be any other vectors in $\mathbb{R}^n$. Explain why $(P\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (P\mathbf{y})$.

79. Let P be the projection matrix onto the span of $\mathbf{u}_1, \ldots, \mathbf{u}_k$. Let I be the identity matrix and define Q to be the matrix $I - P$. Show that these properties hold for Q:

a. $Q^\top = Q$

b. $Q^2 = Q$

c. $PQ = QP$