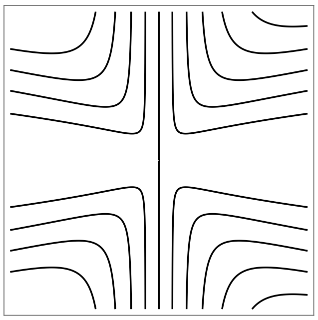
Calculus 4 Exercises!

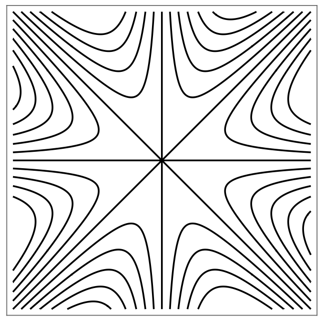
1. Match each of the following functions with the corresponding contour plot and plot in $\mathbb{R}^3$:

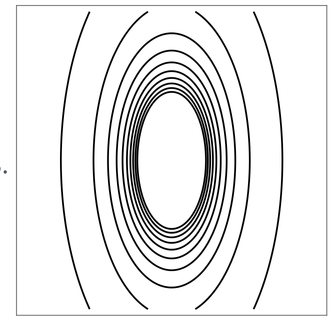
$$a(x,y) = \cos x \cos y \, e^{-\sqrt{x^2+y^2}/4} \qquad b(x,y) = -\frac{xy^2}{x^2+y^2} \qquad c(x,y) = \frac{1}{4x^2+y^2+1}$$

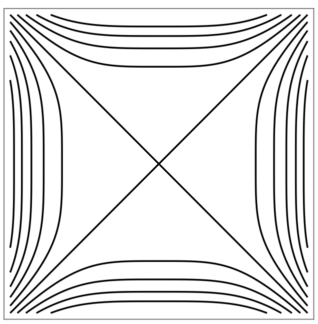
$$d(x,y) = e^{-y/10}\cos x \qquad e(x,y) = xy\frac{x^2-y^2}{x^2+y^2} \qquad f(x,y) = y^4 - x^4$$

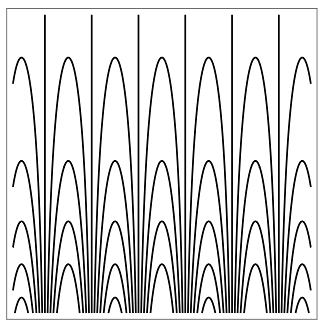
Contour plots:

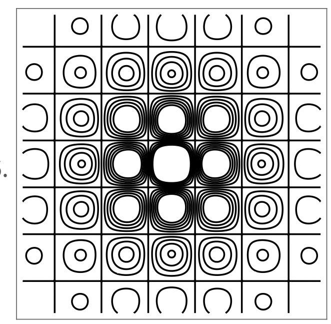
1. 

2. 

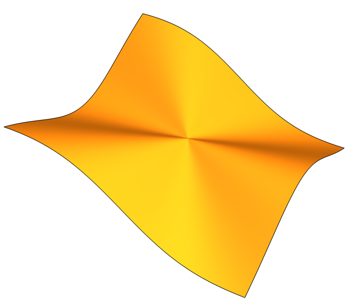
3. 

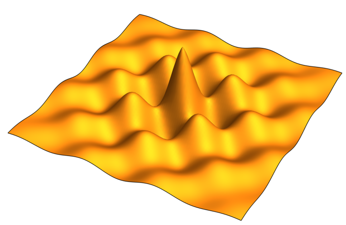
4. 

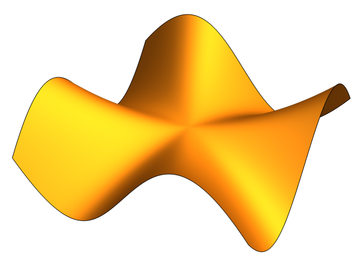
5. 

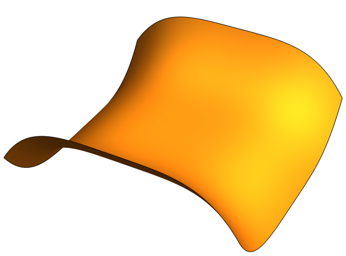
6. 

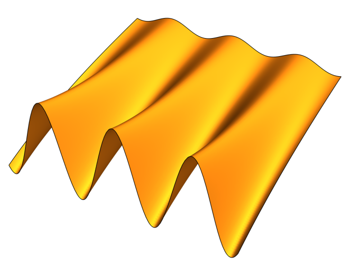
Plots in $\mathbb{R}^3$:

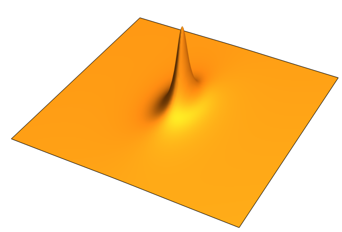
i. 

ii. 

iii. 

iv. 

v. 

vi. 

2. Sketch the level curves $f(x,y)=c$ on the same set of axes for the given values of c:

a. $f(x,y)=x+y-1, c=-3,-2,-1,0,1,2,3$

b. $f(x,y)=x^2+y^2, c=0,1,4,9,16,25$

c. $f(x,y)=xy, c=-9,-4,-1,0,1,4,9$

d. $f(x,y)=\sqrt{25-x^2-y^2}, c=0,1,2,3,4$

3. Sketch sample level curves and sketch the following functions in $\mathbb{R}^3$:

a. $f(x,y)=y^2$

b. $f(x,y)=x^2+y^2$

c. $f(x,y)=4-x^2-y^2$

d. $f(x,y)=4-|x|-|y|$

4. Find these limits or explain why they do not exist:

a. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2-y^2+5}{x^2+y^2+2}$

b. $\displaystyle\lim_{(x,y)\to(1,\pi/6)} \frac{x\sin y}{x^2+1}$

c. $\displaystyle\lim_{(x,y)\to(1,1),x\neq y} \frac{x^2-2xy+y^2}{x-y}$

d. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq y,x\geq 0,y\geq 0}} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

e. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}$

f. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$

g. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{\sqrt{x^4+y^2}}$

h. $\displaystyle\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-y^2}{x-y}$

i. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3-xy^2}{x^2+y^2}$

j. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

k. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

5. At what points (x,y) in $\mathbb{R}^2$ are the following functions continuous?

a. $\sin(x+y)$

b. $\ln(x^2+y^2)$

c. $\dfrac{x^2+y^2}{x^2-3x+2}$

d. $\dfrac{1}{x^2-y}$

6. Find $\partial f/\partial x$, find $\partial f/\partial y$, and (in the cases where f is a function of z) find $\partial f/\partial z$:

a. $f(x,y)=\ln(x+y)$

b. $f(x,y)=e^{xy}\ln y$

c. $f(x,y)=\int_x^y g(t)\,dt$ where g is continuous

d. $f(x,y,z)=xy+xz+yz$

e. $f(x,y,z)=(x^2+y^2+z^2)^{-1/2}$

f. $f(x,y,z)=\ln(x+2y+3z)$

7. Find all second order partial derivatives:

a. $f(x,y)=x+y+xy$

b. $f(x,y)=\sin(xy)$

c. $f(x,y)=xy^2+x^2y^3+x^3y^4$

8. Find the value of $\partial z/\partial x$ at the point $(1,1,1)$ if the equation

$$xy+z^3x-2yz=0$$

defines z as a function of the independent variables x and y.

9. Show that each of the following functions satisfy the equation $\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}+\dfrac{\partial^2 f}{\partial z^2}=0$:

a. $f(x,y,z)=x^2+y^2-2z^2$

b. $f(x,y,z)=\ln\sqrt{x^2+y^2}$

c. $f(x,y,z)=\arctan(x/y)$

10. Show that each of the following functions satisfy the equation $\dfrac{\partial^2 f}{\partial t^2}=c^2\dfrac{\partial^2 f}{\partial x^2}$:

a. $f(x,t)=\sin(x+ct)$

b. $f(x,t) = \tan(2x - 2ct)$

11. Assume that $w = f(s^3 + t^2)$ and $f'(x) = e^x$. Find $\partial w/\partial t$ and $\partial w/\partial s$.

12. Assume that $w = f(ts^2, s/t)$, $\partial f(x,y)/\partial x = xy$, and $\partial f(x,y)/\partial y = x^2/2$. Find $\partial w/\partial t$ and $\partial w/\partial s$.

13. Suppose that f is a function of u, v, and w where $u = x - y$, $v = y - z$, and $w = z - x$. Show that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0.$$

14. Suppose that we substitute the polar equations $x = r\cos\vartheta$ $y = r\sin\vartheta$ into $w = f(x,y)$.

a. Show that

$$\frac{\partial w}{\partial r} = f_x \cos\vartheta + f_y \sin\vartheta$$

and

$$\frac{1}{r}\frac{\partial w}{\partial \vartheta} = -f_x \sin\vartheta + f_y \cos\vartheta.$$

b. Solve the equations in part a. to express f_x and f_y in terms of $\partial w/\partial r$ and $\partial w/\partial \vartheta$.

c. Show that

$$(f_x)^2 + (f_y)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \vartheta}\right)^2.$$

15. Show that if $w = f(u,v)$ satisfies

$$f_{uu} + f_{vv} = 0$$

and if $u = (x^2 - y^2)/2$ and $v = xy$, then w satisfies

$$w_{xx} + w_{yy} = 0.$$

16. Find the gradient of the function at the given point. Sketch the gradient and the level curve that passes through the point:

a. $f(x,y) = y - x$ at $(2,1)$

b. $f(x,y) = \ln(x^2 + y^2)$ at $(1,1)$

c. $f(x,y) = x^2/2 - y^2/2$ at $(\sqrt{2}, 1)$

17. Find the derivative of the function at P in the direction of $\mathbf{u}$:

a. $f(x,y) = 2x^2 + y^2$ with $P = (-1,1)$ and $\mathbf{u} = \langle 3, -4\rangle$

b. $f(x,y) = \dfrac{x-y}{xy+2}$ with $P = (1,-1)$ and $\mathbf{u} = \langle 12, 5\rangle$

c. $f(x,y,z) = x^2 + 2y^2 - 3z^2$ with $P = (1,1,1)$ and $\mathbf{u} = \langle 1,1,1\rangle$

18. Find the directions in which the functions increase and decrease most rapidly at P Then find the derivatives of the functions in these directions.

a. $f(x,y) = x^2y + e^{xy}\sin y$ with $P = (1,0)$

b. $f(x,y,z) = xe^y + z^2$ with $P = (1, \ln 2, 1/2)$

c. $f(x,y,z) = \ln(xy) + \ln(yz) + \ln(xy)$ with $P = (1,1,1)$

19. Is there a direction $\mathbf{u}$ in which the rate of change of $f(x,y) = x^2 - 3xy + 4y^2$ at $(1,2)$ equals 14? Why?

20. Find the tangent plane and the normal line for the surface at the point P:

a. $x^2 + y^2 + z^2 = 3$ at $P = (1,1,1)$

b. $x^2 + y^2 - z^2 = 18$ at $P = (3,5,-4)$

c. $2z - x^2 = 0$ at $P = (2,0,2)$

d. $x^2 + y^2 - 2xy - x + 3y - z = -4$ at $P = (2,-3,18)$

e. $z = \ln(x^2 + y^2)$ at $P = (1,0,0)$

f. $z = 4x^2 + y^2$ at $P = (1,1,5)$

21. Find all local maxima, local minima, and saddle points for these functions

a. $f(x,y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$

b. $f(x,y) = 5xy - 7x^2 + 3x - 6y + 2$

c. $f(x,y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$

d. $f(x,y) = x^3 + 3xy + y^3$

e. $f(x,y) = 4xy - x^4 - y^4$

f. $f(x,y) = \ln(x+y) + x^2 - y$

22. Find the absolute maxima and absolute minima of the function on the given domain:

a. $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on the triangular region bounded by the lines $x = 0$, $y = 2$, $y = 2x$ in the first quadrant

b. $f(x,y) = x^2 + xy + y^2 - 6x$ on $[0,5] \times [-3,3]$

c. $f(x,y) = (4x - x^2)\cos y$ on $[1,3] \times [-\frac{\pi}{4}, \frac{\pi}{4}]$

23. Find two numbers a and b with $a \leq b$ that maximizes

$$\int_a^b (6 - x - x^2)\,dx.$$

24. Find the maximum value of $xy + yz + xz$ where $x + y + z = 6$.

25. Find the minimum distance from $z = \sqrt{x^2 + y^2}$ to the point $(-6, 4, 0)$.

26. Find the points on the ellipse $x^2 + 2y^2 = 1$ where $f(x,y) = xy$ has its extreme values.

27. Find the extreme values of $f(x,y) = xy$ subject to the constraint $g(x,y) = x^2 + y^2 - 10 = 0$.

28. Find the points on $x^2 y = 2$ closest to the origin.

29. Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy-plane that are nearest and farthest from the origin.

30. Find the dimensions of the rectangle of largest perimeter that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$ with sides parallel to the coordinate axes. What is this perimeter?

31. Find the maximum and minimum values of $3x - y + 6$ subject to the constraint $x^2 + y^2 = 4$.

32. Maximize and minimize xyz^2 on $x^2 + y^2 + z^2 = 1$.

33. Evaluate the following integrals:

a. $\displaystyle\int_0^2 \int_{-1}^1 (x - y)\,dx\,dy$

b. $\displaystyle\int_0^1 \int_0^1 \left(1 - \frac{x^2 + y^2}{2}\right) dx\,dy$

c. $\displaystyle\int_0^1 \int_1^2 xye^x\,dx\,dy$

d. $\displaystyle\iint_R xy\cos y\,dA$ where $R = [-1,1] \times [0,\pi]$

e. $\displaystyle\iint_R y\sin(x+y)\,dA$ where $R = [-\pi, 0] \times [0,\pi]$

f. $\displaystyle\iint_R \frac{y}{x^2y^2 + 1}\,dA$ where $R = [0,1] \times [0,1]$

g. $\displaystyle\iint_R \frac{1}{xy}\,dA$ where $R = [1,2] \times [1,2]$

34. Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below $[-1,1] \times [-1,1]$.

35. Sketch the described regions of integration in $\mathbb{R}^2$:

a. $-2 \leq y \leq 2, y^2 \leq x \leq 4$

b. $0 \leq y \leq 1, y \leq x \leq 2y$

c. $1 \leq x \leq e^2, 0 \leq y \leq \ln x.$

36. Write an iterated integral for $\iint_R dA$ over the region R using both vertical cross sections and horizontal cross sections:

a. The triangle in the first quadrant of $\mathbb{R}^2$ bounded by the graphs of $x = 3$ and $y = 2x$.

b. The region in the first quadrant of $\mathbb{R}^2$ bounded by the lines $x = 2, y = 1$, and the graph of the function $y = e^x$.

c. The region bounded by $y = 3 - 2x$, $y = x$, and $x = 0$.

d. The region bounded by $y = x^2$ and $y = x + 2$.

37. Sketch the region and then evaluate the integral:

a. $\displaystyle\int_0^\pi \int_0^x x\sin y\,dy\,dx$

b. $\displaystyle\int_1^2 \int_y^{y^2} dx\,dy$

c. $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} 8y\,dy\,dx$

38. Reverse the order of integration and then evaluate the integral:

a. $\displaystyle\int_0^\pi \int_x^\pi \frac{\sin y}{y}\,dy\,dx$

b. $\displaystyle\int_0^1 \int_y^1 x^2 e^{xy}\,dx\,dy$

c. $\displaystyle\int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2}\,dx\,dy$

d. $\displaystyle\iint_R (y - 2x^2)\,dA$ where R is the region bounded by $|x| + |y| = 1$ in $\mathbb{R}^2$.

39. Find the volume of the solid in the first octant bounded by the coordinate planes, the plane $x = 3$, and the parabolic cylinder $z = 4 - y^2$.

40. Find the area of the following regions using a double integral:

a. The coordinate axes and the line $x + y = 2$.

b. The parabola $x = -y^2$ and $y = 4$.

c. The curve $y = e^x$ and the lines $y = 0$, $x = 0$, and $x = \ln 2$.

d. The lines $y = 2x$, $y = x/2$, and $y = 3 - x$.

41. Change from Cartesian into polar and evaluate:

a. $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dy\,dx$

b. $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} (x^2+y^2)\,dx\,dy$

c. $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$

d. $\displaystyle\int_{\sqrt{2}}^{2}\int_{0}^{x} y\,dy\,dx$

e. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}\,dy\,dx$

f. $\displaystyle\int_{0}^{1}\int_{x}^{\sqrt{2-x^2}} (x+2y)\,dy\,dx$

g. $\displaystyle\int_{0}^{\infty}\int_{0}^{\infty} \frac{1}{(1+x^2+y^2)^2}\,dy\,dx$

42. The average value of a function $f(x,y)$ over the region R in $\mathbb{R}^2$ is given by

$$\frac{1}{\text{area}(R)}\iint_R f(x,y)\,dx\,dy.$$

Find the average values of $f(x,y) = \sqrt{a^2-x^2-y^2}$ over the region described by $x^2+y^2 \le a^2$.

43. Find the average distance from a point on the disk $x^2+y^2 \le a^2$ to the origin.

44. Let $I = \int_0^\infty e^{-x^2}\,dx$. Evaluate I^2 by noticing that

$$\begin{aligned} I^2 &= \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right) \\ &= \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy \end{aligned}$$

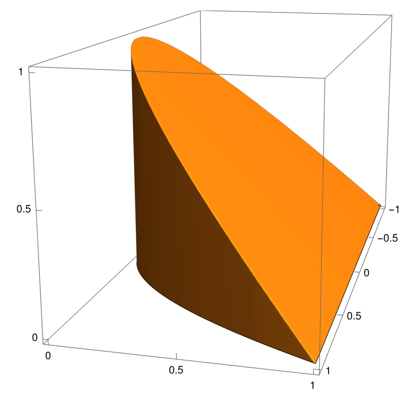
and switching into polar.

45. Write six different iterated triple integrals for the volume of the following regions in $\mathbb{R}^3$:

a. The first octant enclosed by the cylinder $x^2+z^2 = 4$ and the plane $y = 3$

b. The region bounded by $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$.

c. The region bounded in the first octant that satisfies $z + y \le 1$ and $x^2 \le y$:



46. Evaluate the integrals:

a. $\displaystyle\int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\,dz\,dy\,dx$

b. $\displaystyle\int_0^e\int_0^{e^2}\int_0^{e^3} \frac{1}{xyz}\,dx\,dy\,dz$

c. $\displaystyle\int_0^1\int_0^{3-3x}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$

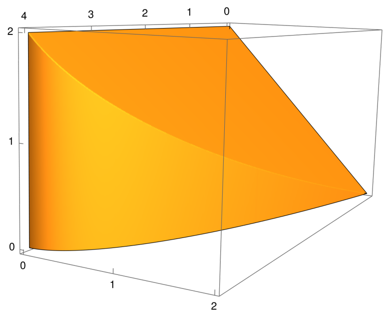
d. $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2}} dz\,dy\,dx$

e. $\displaystyle\int_0^1\int_0^{2-x}\int_0^{2-x-y} dz\,dy\,dx$

f. $\displaystyle\int_0^\pi\int_0^\pi\int_0^\pi \cos(x+y+z)\,dz\,dy\,dx$

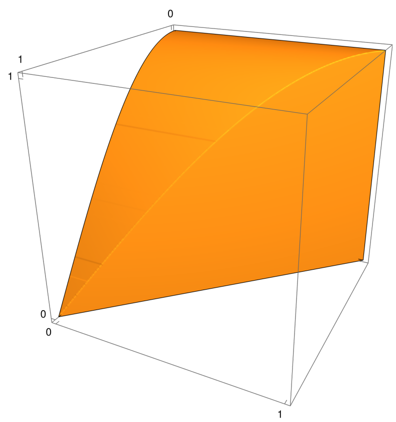
47. Find the volumes of the following regions in $\mathbb{R}^3$:

a. The region in the first octant bounded by the coordinate planes, the plane $y + z = 2$, and the cylinder $x = 4 - y^2$:

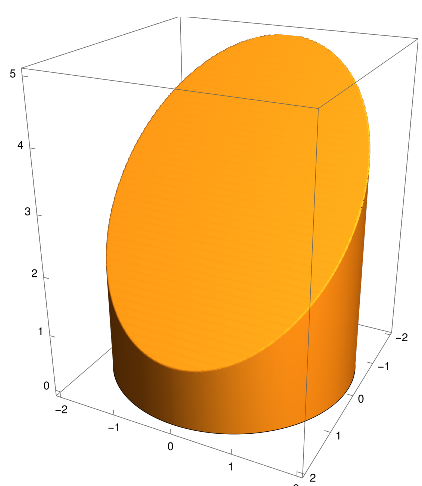


b. The tetrahedron in the first octant bounded by the coordinate planes and the plane passing through $(1,0,0)$, $(0,2,0)$ and $(0,0,3)$.

c. The region in the first octant bounded the by the coordinate planes, the plane $y = 1 - x$ and the surface $z = \cos(\pi x/2)$ for $0 \leq x \leq 1$:



d. The region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$:



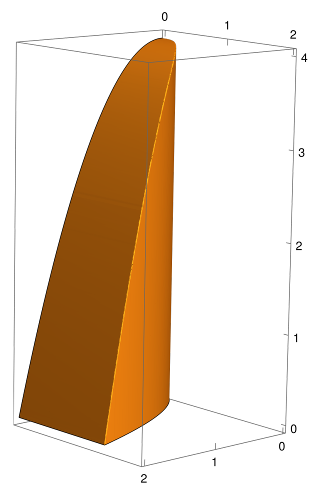
48. Find the center of mass of a thin plate of uniform density bounded by the lines $x = 0$, $y = x$, and the parabola $y = 2 - x^2$ in the first quadrant.

49. Find the center of mass of a thin plate of nonuniform density in the shape of a triangle bounded by the y-axis and the lines $y = x$ and $y = 2 - x$ if the density at the point (x, y) is $6x + 3y + 3$.

50. Find the mass of the solid and find the center of mass for the following regions in $\mathbb{R}^3$:

a. The solid region in the first octant bounded by the coordinate planes and the plane $x + y + z = 2$ where the solid has density given by $\delta(x, y, z) = 2x$.

b. The solid region in the first octant bounded by the planes $y = 0$ and $z = 0$ and the surfaces $z = 4 - x^2 4$ and $x = y^2$ where the solid has density given by $\delta(x, y, z) = xy$:



c. A solid cube in the first octant bounded by the planes $x = 1$, $y = 1$, and $z = 1$ where the density of the cube at (x, y, z) is given by $x + y + z + 1$.

51. Evaluate the integrals in cylindrical coordinates:

a. $\displaystyle\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz r\, dr\, d\vartheta$

b. $\displaystyle\int_0^{2\pi}\int_0^{\vartheta/(2\pi)}\int_0^{3+24r^2} dz r\, dr\, d\vartheta$

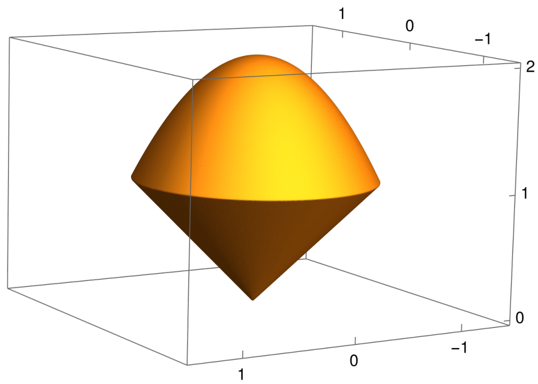
c. $\displaystyle\int_0^{2\pi}\int_0^3\int_0^{z/3} r^3\, dr\, dz\, d\vartheta$

52. Convert

$$\int_{-1}^{1}\int_0^{\sqrt{1-y^2}}\int_0^x (x^2 + y^2)\, dz\, dx\, dy$$

into cylindrical coordinates and evaluate.

53. Let R be the region bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$:



Set up the triple integrals that give the volume of R in these three orders: $dz\,dr\,d\vartheta$, $dr\,dz\,d\vartheta$ and $d\vartheta\,dz\,dr$.

54. Evaluate the integrals in spherical coordinates (some orders of integration may be easier than others):

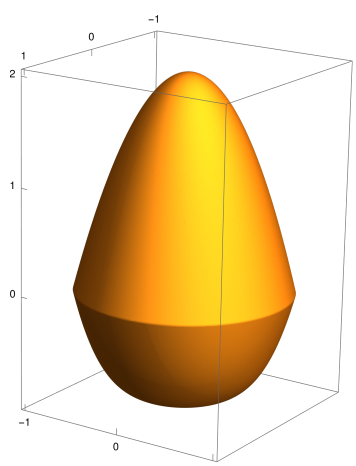
a. $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\vartheta$

b. $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} \rho^3 \cos\varphi \sin\varphi\, d\rho\, d\varphi\, d\vartheta$

c. $\displaystyle\int_0^{2}\int_{-\pi}^{0}\int_{\pi/4}^{\pi/2} \rho^3 \sin(2\varphi)\, d\varphi\, d\vartheta\, d\rho$

d. $\displaystyle\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\csc\varphi}^{2} 5\rho^4 \sin^3\varphi\, d\rho\, d\vartheta\, d\varphi$

55. Find the volumes of the region above $z = (x^2+y^2)^2 - 1$ and below $z = 4 - 4(x^2+y^2)$:



56. Find the average value of the function $f(\rho,\varphi,\vartheta) = \rho$ over the solid ball described by $\rho \le 1$ in spherical coordinates.

57. Let $u = x - y$ and $v = 2x + y$.

a. Solve for x and y in terms of u and v.

b. Find the Jacobian $\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$.

c. Sketch the image of the triangle with corners $(0,0)$, $(1,1)$, and $(1,-2)$ in the x,y plane after the transformation into the u,v plane given by $u = x - y$ and $v = 2x + y$.

d. Use this transformation to evaluate the integral

$$\iint_R (2x^2 - xy - y^2)\, dx\, dy$$

where R is the region in the first quadrant bounded by the lines $y = -2x + 4$, $y = -2x + 7$, $y = x - 2$, and $y = x + 1$.

58. Let $u = 3x + 2y$ and $v = x + 4y$.

a. Solve for x and y in terms of u and v.

b. Find the Jacobian $\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$.

c. Sketch the image of the triangle bounded by the x-axis, the y-axis, and the line $x + y = 1$ in the x,y plane after the transformation into the u,v plane given by $u = 3x + 2y$ and $v = x + 4y$.

d. Use this transformation to evaluate the integral

$$\iint_R (3x^2 + 14xy + 8y^2)\, dx\, dy$$

where R is the region in the first quadrant bounded by the lines $y = -(3/2)x + 1$, $y = -(3/2)x + 3$, $y = -(1/4)x$, and $y = -(1/4)x + 1$.

59. Find the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$ in the x,y-plane by using the transformation $x = au$, $y = bv$.

60. Use the transformation $x = u^2 - v^2$, $y = 2uv$ to evaluate

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2+y^2}\, dy\, dx.$$

61. Evaluate the following line integrals:

a. $\displaystyle\int_C (x+y)\, ds$ where C is the straight line segment from $(0,1,0)$ to $(1,0,0)$.

b. $\int_C (x - y + z - 2)\, ds$ where C is the straight line segment from $(0,1,1)$ to $(1,0,1)$.

c. $\int_C (x + \sqrt{y} - z^2)\, ds$ where C is the path that starts at $(0,0,0)$ and then ends at $(1,1,1)$ by moving along $\langle t, t^2, 0 \rangle$ for $t \in [0,1]$ and then moving along $\langle 1, 1, s \rangle$ for $s \in [0,1]$.

d. $\int_C x\, ds$ where C is the straight line segment from $(0,0)$ to $(4,2)$ in $\mathbb{R}^2$.

e. $\int_C x\, ds$ where C follows the parabolic curve $y = x^2$ from $(0,0)$ to $(2,4)$ in $\mathbb{R}^2$.

f. $\int_C x^2/(y^{4/3})\, ds$ where C follows the path described by the parametric equations $x = t^2, y = t^3$ for $t \in [1,2]$.

g. $\int_C \frac{1}{x^2 + y^2 + 1}\, ds$ where C travels counterclockwise along the perimeter of the square in $\mathbb{R}^2$ with corners $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$.

62. Find the gradient fields for the following functions:

a. $f(x,y,z) = (x^2 + y^2 + z^2)^{-1/2}$

b. $f(x,y,z) = xy + yz + xz$

63. Find the line integrals of $\mathbf{F}$ along both of the paths C_1 and C_2 where C_1 is the straight line segment from $(0,0,0)$ to $(1,1,1)$ and C_2 is the path described by $\langle t, t^2, t^3 \rangle$ for $t \in [0,1]$:

a. $\mathbf{F} = \langle 3y, 2x, 4z \rangle$

b. $\mathbf{F} = \langle xy, yz, xz \rangle$

64. Find the work done by $\mathbf{F}$ over the following curves:

a. $\mathbf{F} = \langle xy, y, -yz \rangle$ over $\mathbf{r}(t) = \langle t, t^2, t \rangle$ for $t \in [0,1]$

b. $\mathbf{F} = \langle 2y, 3x, x + y \rangle$ over $\mathbf{r}(t) = \langle \cos t, \sin t, t/6 \rangle$ for $t \in [0, 2\pi]$

c. $\mathbf{F} = \langle xy, y - x \rangle$ over the straight line in $\mathbb{R}^2$ from $(1,1)$ to $(2,3)$

d. $\mathbf{F}$ is the gradient of $f(x,y) = (x+y)^2$ over the path that travels counterclockwise once around the circle $x^2 + y^2 = 4$ in $\mathbb{R}^2$ that starts and ends at $(2,0)$.

65. State whether or not the vector field is conservative:

a. $\mathbf{F} = \langle yz, xz, xy \rangle$

b. $\mathbf{F} = \langle y, x + z, -y \rangle$

c. $\mathbf{F} = \langle -y, x \rangle$

66. Find a potential function f for the field $\mathbf{F}$:

a. $\mathbf{F} = \langle 2x, 3y, 4z \rangle$

b. $\mathbf{F} = \langle y + z, x + z, x + y \rangle$

67. Show that the values of the follow integrals do not depend on the path from A to B:

a. $\int_A^B z^2\, dx + 2y\, dy + 2xz\, dz$

b. $\int_A^B \frac{x\, dx + y\, dy + z\, dz}{\sqrt{x^2 + y^2 + z^2}}$

68. Find a potential function for the gravitational field

$$\mathbf{F} = -GmM \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}$$

where G, m, and M are constants. Then show that the word done by the gravitational field in moving a particle from a distance a to a distance b away from the origin is

$$GmM \left(\frac{1}{b} - \frac{1}{a} \right).$$

69. Find the work done by $\mathbf{F}$ in moving a particle once counterclockwise around the given curve C:

a. $\mathbf{F} = \langle 2xy^3, 4x^2y^2 \rangle$ with C the boundary of the region in the first quadrant enclosed by the x-axis, the line $x = 1$, and the curve $y = x^3$

b. $\mathbf{F} = \langle 4x - 2y, 2x - 4y \rangle$ with C the boundary of the circle $(x-2)^2 + (y-2)^2 = 4$

70. Apply Green's theorem to evaluate the integrals:

a. $\oint_C (y^2\, dx + x^2\, dy)$ with C the triangle bounded by $x = 0$, $x + y = 1$, and $y = 0$

b. $\oint_C (3y\, dx + 2x\, dy)$ with C the boundary of $0 \leq x \leq \pi, 0 \leq y \leq \sin x$

71. Green's area formula says that the area of a region R enclosed by a simple closed curve C in $\mathbb{R}^2$ is given by

$$\frac{1}{2} \oint_C x\, dy - y\, dx.$$

Use this formula to find the areas of the regions enclosed by these curves:

a. The circle $\mathbf{r}(t) = \langle a\cos t, a\sin t\rangle$ for $t \in [0, 2\pi]$

b. The ellipse $\mathbf{r}(t) = \langle a\cos t, b\sin t\rangle$ for $t \in [0, 2\pi]$

c. The asteroid $\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t\rangle$ for $t \in [0, 2\pi]$