Linear Analysis II Set 13

1. Show that the Fourier transform of the second derivative $F[f''(t)]$ is equal to $-\omega^2 F[f(t)]$.

2. a. Show that the Fourier Transform of $e^{-|t|}$ is $\dfrac{1}{\pi(\omega^2+1)}$.

b. Using the inverse Fourier Transform and part a., find the Fourier transform of $\dfrac{1}{t^2+1}$.

3. The one dimensional wave equation is the partial differential equation $u_{tt}(t,x) = k^2 u_{xx}(t,x)$ where k is a real number and $u(t,x)$ is a function of time t and one spacial dimension x. The wave equation models the displacement of a vibrating string at time t and location x.

By taking partial derivatives and plugging in the function

$$u(t,x) = \frac{f(x+kt)+f(x-kt)}{2} + \frac{1}{2k}\int_{x-kt}^{x+kt} g(s)\,ds$$

into the partial differential equation, show that the above function is a solution to the wave equation

$$\begin{cases} u_{tt}(t,x) = k^2 u_{xx}(t,x) \\ u(0,x) = f(x), u_t(0,x) = g(x). \end{cases}$$

4. Consider the wave equation

$$\begin{cases} u_{tt}(t,x) = k^2 u_{xx}(t,x), \\ u(0,x) = 1/(1+x^2), \\ u_t(0,x) = 0. \end{cases}$$

a. Take the Fourier transform of the system with respect to the variable x and then solve the resulting differential equation in the variable t to find $F[u(t,x)]$.

b. The inverse Fourier transform gives $u(t,x) = \displaystyle\int_{-\infty}^{\infty} F[u(t,x)]e^{i\omega x}\,d\omega$. Evaluate this to find $u(t,x)$.

Hint: This can be done by writing the integral as $\int_{-\infty}^{\infty} = \int_{-\infty}^{0} + \int_{0}^{\infty}$ and then relating each of the two integrals to the Laplace transform with respect to the variable ω of the function $e^{-\omega}\cos(kt\omega)$. It is acceptable to leave the final answer involving the complex unit i, but the clever student may be able to find an answer that does not involve i.