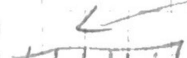
Thm. $\sum_{i=0}^{n-1} h_i p_{n-i} = n h_n$ for $n \geq 1$

Σ weight(T) with sum over constant fillings of

Σ weight(T) with sum over nondecreasing filling of



Pf

Consider pairs (S, T) where S has shape i and nondecreasing filling and T has shape $n-i$ with constant filling

Ex: ($\boxed{1\,|\,1\,|\,1\,|\,2\,|\,2\,|\,4\,|\,5}$, $\boxed{2\,|\,2\,|\,2}$) $\longmapsto$ $\boxed{1\,|\,1\,|\,2\,|\,2\,|\,2^{*}\,|\,2\,|\,2\,|\,4\,|\,5}$

We uncover a bijective way to get every weight and there are n ways (star) to get the same weight

There are n ways to place this star, each corresponding to a different pair.

Thm. $\sum_{i=0}^{n-1} (-1)^i e_i p_{n-i} = (-1)^{n-1} n e_n$

$\sum$ weight(T) w/ T shape (1^i) and increasing

Pf

Ex: $\left(\begin{array}{|c|} 11 \\ 9 \\ 5 \\ 3 \\ 1 \end{array}^{-} , \boxed{3\,|\,3\,|\,3\,|\,3\,|\,3} \right) \longleftrightarrow \left(\begin{array}{|c|} 11 \\ 9 \\ 5 \\ 1 \end{array}^{+} , \boxed{3\,|\,3\,|\,3\,|\,3\,|\,3\,|\,3} \right)$

Fixed $\left(\begin{array}{|c|} 7 \\ 5 \\ 4 \\ 3 \\ 1 \end{array} , \boxed{2}^{*} \right) \longleftrightarrow \begin{array}{|c|} 7 \\ 5 \\ 4 \\ 3 \\ 2^{*} \\ 1 \end{array}$

sign $= (-1)^{n-1}$

n-ways to choose a single cell

Let $C_\lambda = \{ \sigma \in S_n \mid \text{cycle type}(\sigma) = \lambda \}$

Then $n!\, h_n = \sum_{\lambda \vdash n} |C_\lambda|\, p_{\lambda_1} p_{\lambda_2} \cdots$

Ex: $C_{(2,1,1)} = \{ (12)(3)(4), (13)(2)(4), (14)(2)(3), (23)(1)(4), (24)(1)(3), (34)(1)(2) \}$

$p_{\lambda_1} p_{\lambda_2} p_{\lambda_3} = p_2 p_1 p_1$

max number above 1 and everything to the right

Pf. $(3)(4)(10\ 1\ 9)(7\ 6)(8)(2)(5) \leftarrow \sigma \in S_n$ $n!$

$\boxed{1\,|\,1\,|\,1\,|\,1\,|\,1\,|\,1\,|\,3\,|\,3\,|\,3\,|\,4\,|\,4} \leftarrow h_n$

T of shape n weakly increasing

$\lambda = (3, 2, 1, 1, 1, 1, 1)$

$\updownarrow$

$\left(\boxed{1}\ \boxed{1}\ \boxed{1\,|\,1\,|\,1}\ \boxed{3\,|\,3}\ \boxed{3}\ \boxed{4}\ \boxed{4}, \ (3)(4)(10\ 1\ 9)(7\ 6)(8)(2)(5) \right)$