F6 · <u>Theorems</u>: Let $p(n) = \#\lambda \vdash n$.

★ $p(5n-1) \equiv 0 \mod 5$.

$p(7n+5) \equiv 0 \mod 7$.

$p(11n+6) \equiv 0 \mod 11$.

$p(11^3 \cdot 13n + 237) \equiv 0 \mod 13$.

$p(999959^4 \cdot 29n + 28995221336976431135321647) \equiv 0 \mod 29$.

Theorem: $(a_1 + \dots + a_n)^p \equiv (a_1^p + \dots + a_n^p) \bmod p$ if p is prime.

Proof: $(a_1 + \dots + a_n)^p = \sum_{i_1 + \dots + i_n = p} \underbrace{\binom{p}{i_1, i_2, \dots, i_n}}_{\frac{p!}{i_1! \cdots i_n!}} a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n}$ ← all of these terms have factor of p except when one $i_j = p$

$\equiv (a_1^p + \dots + a_n^p) \bmod p$ ☒

Corollary: $n^p \equiv n \bmod p$ if p is prime.

prove by setting $a_1, \dots, a_n = 1$

Corollary: Let $f(z) = \sum a_n z^n$ w/ $a_n \in \mathbb{Z}$.

Then
$$
\begin{aligned}
f(z)^p &\equiv \sum (a_n z^n)^p \\
&= \sum a_n^p z^{np} \\
&\equiv \sum a_n (z^p)^n \\
&= f(z^p)
\end{aligned}
$$

Recall ① $\prod_{i=1}^{\infty} (1 - z^i) = \sum_{k \in \mathbb{Z}} (-1)^k z^{\frac{k(3k-1)}{2}}$

Jacobi's triple product $(1+x) \prod_{n=1}^{\infty} (1-z)^n (1+xz)^n (1+x^{-1}z)^n = \sum_{k \in \mathbb{Z}} x^k z^{\frac{k(k-1)}{2}}$

② $\prod_{n=1}^{\infty} (1 - z^n)^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) z^{\frac{n(n+1)}{2}}$ ← triangular #

[Proven in online notes]

Theorem: $p(5n-1) \equiv 0 \bmod 5$.

Proof:
$$
\begin{aligned}
\sum_{n=1}^{\infty} p(n-1) z^n &= z \sum_{n=1}^{\infty} p(n-1) z^{n-1} \\
&= z \prod_{i=1}^{\infty} \frac{1}{1-z^i} \\
&= z \underbrace{\prod_{i=1}^{\infty} (1-z^i)^4}_{F(z)} \prod_{i=1}^{\infty} \left(\frac{1}{1-z^i}\right)^5
\end{aligned}
$$

Therefore, $p(5n-1) = F(z) \prod_{i=1}^{\infty} \left(\frac{1}{1-z^i}\right)^5 \Big|_{z^{5n}}$ ← "extract the coefficient of z^{5n}"

$$\equiv \sum_{j=0}^{5n} F(z)\Big|_{z^j} \cdot \prod_{i=1}^{\infty}\left(\frac{1}{1-z^{5i}}\right)\Big|_{z^{5n-j}}$$

If $j \not\equiv 0 \mod 5$, then this expression is $\equiv 0 \mod 5$ (by the second component)

If $j \equiv 0 \mod 5$, consider

$$F(z)\Big|_{z^j} = z\prod_{i=1}^{\infty}(1-z^i)\prod_{i=1}^{\infty}(1-z^i)^3\Big|_{z^j}$$

$$= z\left(\sum_{k\in\mathbb{Z}}(-1)^k z^{\frac{k(3k-1)}{2}}\right)\left(\sum_{n=0}^{\infty}(-1)^n(2n+1)z^{\frac{n(n+1)}{2}}\right)\Big|_{z^j}$$

$n=2$

$k \backslash n$	0	1	2	3	4
0	1	2	4	2	1
$k=1 \rightarrow$ 1	2	3	0	3	2
2	1	2	4	2	1
3	3	4	1	4	3
4	3	4	1	4	3

$z^{1+\frac{k(3k-1)}{2}+\frac{n(n+1)}{2}}$

power of z (mod 5)

$\equiv 0 \mod 5$

(by above table, since $j \equiv 0 \mod 5$, only possible case is $k=1$ & $n=2$, BUT when $n=2$, $(2n+1)=5$ so still $\equiv 0 \mod 5$)

$\boxtimes$

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