Graph Theory Set 7

32. A basketball tournament has $2n$ teams. Each team plays one game a day over the span of $2n-1$ days, playing every other team exactly once. Each game ends with a winner and a loser (there are no ties). Show that a winning team can be chosen on each day without selecting the same winning team twice.

Hint: Create a graph with vertices the $(2n-1)$ game days and the $2n$ teams.

33. An $m \times n$ latin rectangle is an $m \times n$ matrix with $m \leq n$ is a matrix with entries in $\{1, \ldots, n\}$ such that no two entries in any row or any column are equal. For example,

$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 3 & 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{bmatrix}.$$

Show that a row can be added to an $(m-1) \times n$ latin rectangle to create an $m \times n$ latin rectangle.

34. An $n \times n$ doubly stochastic matrix is a matrix of nonnegative real numbers such that each row and each column sums to 1. For example,

$$\begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 3/4 & 0 \end{bmatrix}.$$

From an $n \times n$ doubly stochastic matrix M, create a bipartite graph G with independent sets the rows and columns of M and edges between row i and column j if the i, j entry of M is nonzero. Show that G has a perfect matching.

35. Find the minimum number of vertices in a covering for the cube graph Q_n.

36. Let G be a graph with n vertices.

a. Show that both sides of the equation

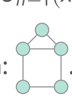
$$(n-2k)m_G(k) = \sum_{\text{vertices } v} m_{G-v}(k)$$

are equal to the number of ways to select a vertex v and a matching M such that v is not incident to an edge in M.

b. Show that $\dfrac{d}{dx} M_G(x) = \displaystyle\sum_{\text{vertices } v} M_{G-v}(x)$.

c. Recall $M_{K_n}(x) = H_n(x)$ where $H_n(x)$ is the probabilist's Hermite polynomial. Show that $H_n'(x) = nH_{n-1}(x)$ holds for $n \geq 1$.

d. Recall $M_{P_n}(x) = U_n(x/2)$ and $M_{C_n}(x) = 2T_n(x/2)$ where $T_n(x)$ and $U_n(x)$ are the Chebyshev polynomials of the first and second kind. Show $T_n'(x) = nU_{n-1}(x)$.

37. Show that $K_{2,3}$ has the same matching polynomial as the **house graph**: .