Ex: Define $\varphi(e_n) = \begin{cases} (-1)^{n-1} & \text{if } n \le 2 \\ 0 & \text{if } n > 2 \end{cases}$ to be a homomorphism

Then $\varphi(h_n) = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,(n)}| \varphi(e_\lambda)$

$= \sum_{\lambda \vdash n} |B_{\lambda,(n)}|$

w/ parts of size 1 or 2 only

$\varphi(e_{\lambda_1})\varphi(e_{\lambda_2})\cdots$

$(-1)^{\lambda_1 - 1}(-1)^{\lambda_2 - 1}\cdots$

$(-1)^{n-\ell(\lambda)}$

so (-1)'s cancel

Object:

① Strip of length n

② Fill w/ bricks of length 1 or 2

$n=10$

Say the # of objects is f_n

Then $f_n = f_{n-2} + f_{n-1}$ and $f_0 = f_1 = 1$

start w/ brick length 2   start w/ brick length 1

Now apply φ to $\sum_{n=0}^{\infty} h_n z^n = \frac{1}{\sum_{n=0}^{\infty} (-1)^n e_n z^n}$

to get $\sum_{n=0}^{\infty} f_n z^n = \frac{1}{1-z-z^2}$

(Fibonacci Sequence)

Define: A word of length n w/ letters $\{1,\dots,k\}$ is a finite sequence $w_1 \cdots w_n$ w/ $w_i \in \{1,\dots,k\}$

Ex: A word of length 10 w/ letters $\{1,2,3\}$

3311231121

If w is a word, then $des(w) = (\#\, i$ w/ $w_i > w_{i+1})$

Ex: $w = 3311231121$

$des(w) = 3$

Note:
if $n > k$,
$\binom{k}{n} = 0$

Define $\varphi(e_n) = (-1)^{n-1}\binom{k}{n}(x-1)^{n-1}$ to be a homomorphism

Then $\varphi(h_n) = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,(n)}| \varphi(e_\lambda)$

$= \sum_{\lambda \vdash n} |B_{\lambda,(n)}| \binom{k}{\lambda_1}(x-1)^{\lambda_1 - 1}\binom{k}{\lambda_2}(x-1)^{\lambda_2 - 1}\cdots$

$= \sum_{w \text{ of length } n} x^{des(w)}$

Object:

① Strip of length n

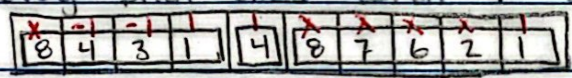
② Fill w/ bricks of λ length (arbitrary)

③ Write a decreasing sequence in each brick, with elements from $\{1, \dots, k\}$ [accounts for $\binom{k}{\lambda_i}$ for $i = 1, 2, \dots$]

④ At the top, write an x or -1

But, every brick ends with 1 [accounts for $(x-1)^{\lambda_i - 1}$ for $i = 1, 2, \dots$]

$n = 10,\ k = 8$



Involution

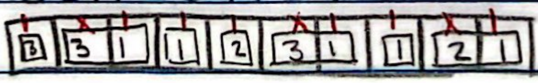
Scan L to R for
a -1 or bricks to combine
Break or combine

Fixed Point:

each x counts a descent

(cannot have a descent b/w bricks)

Ex: $w = 3311231121$ $\{1,2,3\}$, $n = 10, k = 3$



Scanned with CamScanner

Apply φ to $\sum_{n=0}^{\infty} h_n z^n = \frac{1}{\sum_{n=0}^{\infty}(-1)^n e_n z^n}$ to find

$$\sum_{n=0}^{\infty}\left(\sum_{w \text{ length } n} x^{\mathrm{des}(w)}\right) z^n = \frac{1}{1+\sum_{n=1}^{\infty}(-1)^n(-1)^{n-1}\binom{k}{n}(x-1)^{n-1} z^n} \cdot \frac{(x-1)}{(x-1)}$$

$$= \frac{x-1}{(x-1)-\sum_{n=1}^{\infty}\binom{k}{n}(z(x-1))^n}$$

$$= \frac{x-1}{x-((z(x-1))+1)^k}$$