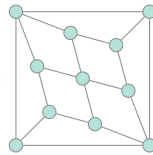


# Graph Theory Midterm 2 Solutions

1. State the definition of

- a. A planar graph
- b. An Eulerian graph
- c. A  $u, v$  separating set
- d. A covering in a graph  $G$

**Solution.** See the notes for the definitions.



2. Show the following graph is not Hamiltonian:

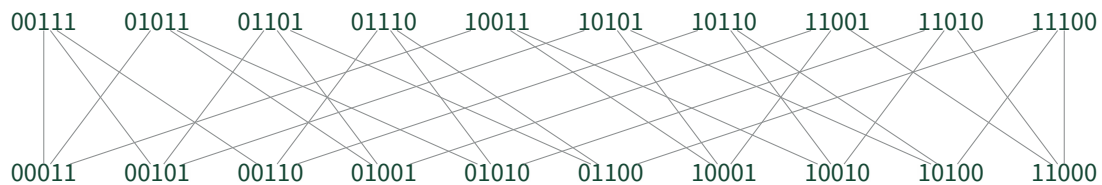
**Solution.** The planar graph has 9 faces with 4 edges. If  $a$  of these faces are inside the proposed Hamiltonian cycle, then  $2(2a - 9) = 0$ , giving  $a = 9/2$ , a contradiction.

3. Use the identities  $M_G(x) = M_{G-e}(x) - M_{G-u-v}(x)$  and  $M_G(x) = xM_{G-u}(x) - \sum_{v \text{ is adjacent to } u} M_{G-u-v}(x)$  to find a recursion for the matching polynomial for the wheel  $W_n$  in terms of matching polynomials for path graphs only.

**Solution.** Using the second recursion and taking  $u$  to be the middle vertex in the wheel and then using the first equation on the resulting  $M_{C_n}(x)$  gives

$$\begin{aligned} M_{W_n}(x) &= xM_{C_{n-1}}(x) - (n-1)M_{P_{n-2}}(x) \\ &= x(M_{P_{n-1}}(x) - M_{P_{n-3}}(x)) - (n-1)M_{P_{n-2}}(x) \\ &= xM_{P_{n-1}}(x) - (n-1)M_{P_{n-2}}(x) - xM_{P_{n-3}}(x). \end{aligned}$$

4. Let  $G$  be the subgraph of the cube  $Q_{2n+1}$  containing those bit strings with either  $n$  0's and  $(n+1)$  1's or with  $(n+1)$  1's and  $n$  0's. For example, the graph  $G$  when  $n = 2$  is below. Show that  $G$  has a perfect matching.



**Solution.** The graph  $G$  is a bipartite graph (since it is a subgraph of a bipartite graph) and every vertex is degree  $n$ . Let  $G$  have independent sets  $X$  and  $Y$ . If  $S \subseteq X$ , then there are  $n|S|$  vertices leaving  $S$ . Since each vertex in  $Y$  has degree  $n$ , we have  $|S| \leq |N(S)|$  and so Hall's theorem applies.

5. Suppose  $\varepsilon(G) \neq 1$  and  $e = \{u, v\}$  is an edge in  $G$  such that  $u$  and  $v$  are in a cycle in  $G - e$ . Show that  $\varepsilon(G - e) \neq 1$ .

**Solution.** There are two edge disjoint paths from  $x$  to  $y$  in  $G$  because  $\varepsilon(G) \geq 2$ . If there are not two edge disjoint paths from  $x$  to  $y$  in  $G - e$ , then the edge disjoint paths that would have used  $e$  can instead travel along one of the two ways around the cycle containing  $u$  and  $v$ , thus still finding two edge disjoint paths from  $x$  to  $y$  in  $G - e$ . Therefore  $\varepsilon(G - e) \geq 2$ .