Calculus 3 Exercises!

Polynomial Approximations

1. Find the degree 5 Taylor polynomial at $x = 0$ for each of these functions:

a. $\cos x$

b. $1 - 3x^2 + 2x^3 + x^7 + 4x^{10}$

c. $(1 - 2x)^2 + (1 - 3x)^3 + x^{1000}$

d. $\dfrac{1}{\sqrt{1-x}}$

e. $(1 + x)^\pi$

f. $\sin(4x)$

g. e^x

h. $e^{\pi x}$

i. $(1 + x)^\pi + \sin(4x)$

j. $(1 - x)^{-3}$

k. The function $f(x)$ which satisfies $f(0) = 1$ and

$$f'(x) = f(x/2).$$

(Don't be intimidated by the abstraction here, just the usual thing: take derivatives using the chain rule and then plug in 0.)

2. Which degree 5 polynomial best approximates $\sqrt{1+x}$ at $x = 0$? Use this polynomial evaluated at $x = 1$ to find an approximation of the value of $\sqrt{2}$. Use a calculator to determine the (absolute) error in using this approximation.

3. Throughout this exercise, let $f(x) = \dfrac{1}{3-x}$.

a. Find an M such that $|f^{(n+1)}(x)| < M$ for all x in $[-1, 1]$. (The value of M involves n.)

b. Use the formula

$$\text{Error} \le \frac{M}{n!}|a|^{n+1}$$

where M is the value found in part a. to find a bound on the error when using the degree n Taylor polynomial to approximate the value of $f(1)$.

4. Let $f(x) = \cos 2x$.

a. Find an M such that $|f^{(n+1)}(x)| < M$ for all x in $[-\pi, \pi]$. (The value of M involves n.)

b. Use the formula

$$\text{Error} \le \frac{M}{n!}|a|^{n+1}$$

where M is the value found in part a. to find a bound on the error when using the degree n Taylor polynomial to approximate the value of $f(1)$.

5. The approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

is the best degree 8 polynomial approximation for $\sin x$ at $x = 0$. Show that the error in using this approximation is less than 0.1 when $-\pi \le x \le \pi$.

6. Let $f(x) = \ln\left(\dfrac{1}{1-x}\right)$.

a. Find the degree n Taylor polynomial at $x = 0$ for $f(x)$.

b. Show that if x is in $\left[-\frac{1}{2}, \frac{1}{2}\right]$, then

$$\left|f^{(n+1)}(x)\right| \le 2^{n+1}n!.$$

c. Show that the error when approximating $\ln 2$ by taking $x = 1/2$ in the polynomial in part a. is at most $1/(n+1)$. How large should n be in order to make $1/(n+1) < 0.05$?

d. Using part a., approximate the value of $\ln 2$ so that the error is smaller than 0.05. (Leave your answer as a sum of fractions.)

7. Find the degree 5 Taylor polynomial for

a. $\cos x$ at $x = \pi/2$.

b. $1 - 3x^2 + 2x^3 + x^7 + 4x^{10}$ at $x = 1$.

c. $\dfrac{1}{\sqrt{1-x}}$ at $x = -2$.

d. $(1 - x)^{-3}$ at $x = 2$.

e. $\sqrt{3+x}$ at $x = 1$.

f. The function $f(x)$ which satisfies $f'(x) = 2f(x)$ and $f(1) = -1$ at $x = 1$.

Infinite Series

8. Simplify these sums (or write "divergent!" if the sum does not exist):

a. $\sum_{n=0}^{\infty}(0.7)^n$.

b. $\sum_{n=2}^{\infty}\frac{3^n}{5^{n-1}}$.

c. $\sum_{n=1}^{\infty}\frac{5^n+6^n}{7^n}$.

d. $\sum_{n=0}^{\infty}\frac{5^n 6^n}{7^n}$.

e. $9.99999\cdots = 9+0.9+0.09+0.009+\cdots$.

9. For which values of x do these sums converge? What functions are they equal to when they do converge?

a. $\sum_{n=0}^{\infty}(x+1)^n$.

b. $\sum_{n=0}^{\infty}(2x)^n$.

c. $\sum_{n=1}^{\infty}2x^n$.

d. $\sum_{n=2}^{\infty}(3x-2)^n$.

10. What percentage of the area in the following square is green?

11. Do the following series converge or diverge? Give a reason why your answer is correct.

a. $\sum_{n=1}^{\infty}\frac{1+n^2}{1+n^4}$

b. $\sum_{n=1}^{\infty}\frac{1}{n^{3+\sin n}}$

c. $\sum_{n=1}^{\infty}\frac{2+\sin n}{2^n}$

d. $\sum_{n=1}^{\infty}\frac{n+2}{(n+1)^3}$

e. $\sum_{n=1}^{\infty}\frac{1+3^n}{1+2^n}$

f. $\sum_{n=1}^{\infty}\frac{1}{2n+5}$

g. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3+1}}$

h. $\sum_{n=1}^{\infty}\frac{1}{n^{1.000000000000000000001}}$

i. $\sum_{n=1}^{\infty}\frac{n^2-1}{3n^4+1}$

j. $\sum_{n=1}^{\infty}\frac{n}{\sqrt{n^2+2}}$

k. $\sum_{n=1}^{\infty}\frac{100^n}{n!}$

l. $\sum_{n=0}^{\infty}\frac{2n}{\sqrt{n}+1}$

m. $\sum_{n=0}^{\infty}\frac{(n!)^n}{n^{4n}}$

n. $\sum_{n=2}^{\infty}\frac{1}{(\ln n)^n}$

o. $\sum_{n=0}^{\infty}n^2e^{-n}$

p. $\sum_{n=0}^{\infty}\frac{(n!)^2}{2^{n^2}}$

12. For which values of x do the following series converge?

a. $\sum_{n=0}^{\infty}(-1)^n(\ln n)x^n$

b. $\sum_{n=0}^{\infty}\frac{x^{2n}}{n^4}$

c. $1+\frac{1\cdot 4}{1\cdot 3}x+\frac{1\cdot 4\cdot 7}{1\cdot 3\cdot 5}x^2+\frac{1\cdot 4\cdot 7\cdot 10}{1\cdot 3\cdot 5\cdot 7}x^3+\cdots$

d. $\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{1\cdot 3\cdot 5\cdots(2n+1)}$

e. $\sum_{n=0}^{\infty}3^{\sqrt{n}}x^n$

f. $1+\frac{1}{1\cdot 5}x+\frac{1}{1\cdot 5\cdot 9}x^2+\cdots$

13. Do the following alternating series converge or diverge? Please provide a reason why your answer is correct.

a. $\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}$

b. $\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\sqrt{n}}$

c. $\sum_{n=1}^{\infty}\frac{(-1)^n 5^n}{4^n}$

d. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{0.0000000000000000001}}$

e. $\sum_{n=1}^{\infty}(-1)^n n^n$

f. $\sum_{n=1}^{\infty}(-1)^n$

14. Approximate the sum of each of the following series to within $1/100$ of the true value. You may leave your answer as a sum of fractions.

a. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n!}}.$

b. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}.$

c. $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{n^2}{2^n}.$

Power series

15. Every human is born knowing these series:

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!} \qquad \text{(true for all } x)$$

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n \qquad \text{(true for all } -1<x<1)$$

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} \qquad \text{(true for all } x)$$

By differentiating, integrating, or otherwise manipulating one of the above series, find the series representations for each of the functions below. Include the values of x for which equality holds.

a. $\frac{\sin(x^2)}{x}$

b. $\frac{1}{1+x}$

c. $\frac{1}{1-x^2}$

d. $\frac{e^{-x^2}-1}{x^2}$

e. $x^3\cos(x^2)$

f. $\int\frac{\sin x}{x}\,dx$

g. $\int e^{-x^3}\,dx$

h. $\frac{\arctan x-x}{x^2}$

i. $\frac{d}{dx}\left(\frac{1-\cos\left(\sqrt{x}\right)}{x}\right)$

16. By multiplication or division of known series, find the first 4 terms in the Taylor series for:

a. $e^{2x}\sin(x/2)$

b. $e^{-x^2}/(1-x)$

c. $(\arctan x)^2$

d. $1/\cos x$

17. a. Find the interval and radius of convergence for

$$y=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n}.$$

b. Show that y satisfies the differential equation

$$x^2y'' + xy' + x^2y = 0.$$

(Take two derivatives of y, plug it into the differential equation, and show that everything simplifies to 0.)

Parametric Equations

18. Plot these parametric curves (with starting and ending points and an arrow indicating direction):

a. $\begin{cases} x = 3t-5, \\ y = 2t+1 \end{cases}$ for $t \in (-\infty, \infty)$

b. $\begin{cases} x = t^2-2, \\ y = 5-2t \end{cases}$ for $t \in [-3, 4]$

c. $\begin{cases} x = t^2, \\ y = t^3 \end{cases}$ for $t \in [-1, 1]$

d. $\begin{cases} x = 2\cos(3t), \\ y = 3\sin(3t) \end{cases}$ for $t \in [-\pi/2, 3\pi/2]$

e. $\begin{cases} x = \ln t, \\ y = \sqrt{t} \end{cases}$ for $t \in [1, \infty)$

19. Find the line tangent to the curves at the indicated point:

a. $\begin{cases} x = 6\sin t \\ y = t^2+t \end{cases}$ at the point found when $t = 1$.

b. $\begin{cases} x = \cos t + \cos(2t) \\ y = \sin t + \sin(2t) \end{cases}$ at the point $(-1, 1)$.

20. Find the first and second derivatives of these parametric curves. For which values of t is the parametric equation concave up?

a. $\begin{cases} x = 2\sin t \\ y = 3\cos t \end{cases}$

b. $\begin{cases} x = t^3-12t \\ y = t^2-1 \end{cases}$

21. Find the exact length of the curve:

a. $\begin{cases} x = 1+3t^2, \\ y = 4+2t^3 \end{cases}$ for $t \in [0, 1]$

b. $\begin{cases} x = e^t+e^{-t}, \\ y = 5-2t \end{cases}$ for $t \in [0, 3]$

c. $\begin{cases} x = e^t\cos t, \\ y = e^t\sin t \end{cases}$ for $t \in [0, \pi]$

22. Consider the parametric equations

$$\begin{cases} x = \displaystyle\int_0^t \frac{\cos u}{1+u^2}\,du, \\ y = \displaystyle\int_0^t \frac{\sin u}{1+u^2}\,du \end{cases}$$

for $t \in [0, \infty)$. What is the first positive value of t for which this curve has a vertical tangent line? What is the length of the curve from $(0, 0)$ to this value?

Polar Equations

23. Plot these polar functions:

a. $r = \theta$ for $\theta \in [-\pi, \pi]$,

b. $r = \sin\theta$ for $\theta \in [0, \pi]$.

c. $r = 1-2\cos\theta$ for $\theta \in [0, 2\pi]$.

24. Find the equation of the line tangent to the polar curve at the given point:

a. $r = 2\sin 2\theta$ at $\theta = 3\pi/4$.

b. $r = 1/\theta$ at the x, y coordinate $(0, 2/\pi)$.

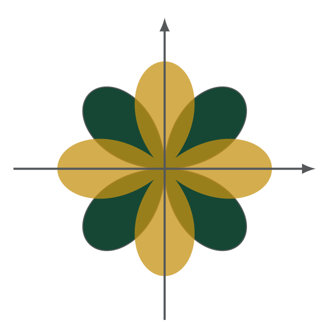
25. Find the points on the polar curve where the tangent line has a horizontal or a vertical tangent:

a. $r = 1+\cos\theta$.

b. $r = 4$

26. Find the area swept out by the polar equation $r = \sqrt{\theta}$ for $\theta \in [0, 2\pi]$.

27. Find the area enclosed by the graph of $r = \sin(2\theta)$ but outside the graph of $r = \cos(2\theta)$:



28. Find the exact length of the polar curve

a. $r = 3\sin\theta$ for $\theta \in [0, \pi/3]$.

b. $r = e^{2\theta}$ for $\theta \in [0, 2\pi]$.

Vectors in $\mathbb{R}^3$

29. Draw the points in $\mathbb{R}^3$ represented by these relations:

a. $x^2 + z^2 \leq 3$

b. $(x-1)^2 + y^2 + (z+1)^2 = 1$

c. $x^2 + y^2 + z^2 - 6x + 4y - 2z = 11$.

30. The vector $\mathbf{v}$ lies in the first quadrant of $\mathbb{R}^2$, has $|\mathbf{v}| = 4$, and makes an angle of $\pi/3$ with the x-axis. Write $\mathbf{v}$ as $\langle a, b\rangle$ for some real numbers a and b.

31. Do the following operations on the vectors $\mathbf{u} = \langle 3, 1, 2\rangle$, $\mathbf{v} = \langle 2, 0, -1\rangle$, and $\mathbf{w} = \langle 1, 1, 1\rangle$:

a. Find a vector in the same direction as $\mathbf{u} + \mathbf{v}$ but has length 2.

b. Find the angle between $\mathbf{u}$ and $\mathbf{v}$ and the angle between $\mathbf{u}$ and $\mathbf{w}$.

c. Find the cross products $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$.

d. $|\mathbf{u} \times (2\mathbf{v} - \mathbf{w})|$.

e. Find two unit vectors in a direction orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

32. Find the cross product of $\langle t, t^2, t^3\rangle$ and $\langle 1, 2t, 3t^2\rangle$ and show that it is orthogonal to both vectors.

33. Find all vectors $\mathbf{u}$ and $\mathbf{v}$ such that

$$|\mathbf{u} \times \mathbf{v}| = \mathbf{u} \cdot \mathbf{v}.$$

34. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors. Which of these operations make sense?

a. $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$

b. $(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

c. $(\mathbf{u} \cdot \mathbf{v})|\mathbf{w}|$

d. $(\mathbf{u} \cdot \mathbf{v}) + \mathbf{w}$

e. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w}$

f. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

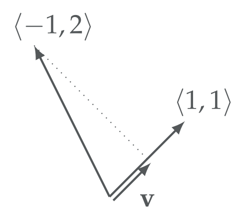
g. $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$

h. $(\mathbf{u} \cdot \mathbf{v}) \times \mathbf{w}$

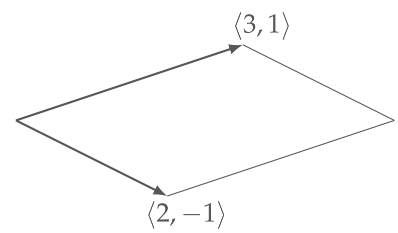
35. Show, for any general vectors in $\mathbb{R}^3$, that

$$(\mathbf{v} \times \mathbf{u}) \cdot \mathbf{u} = 0.$$

36. Find the vector $\mathbf{v}$ depicted here:



37. Find the area of this parallelogram:



38. Generalizing question 37, find a formula for the area of the parallelogram defined by the two vectors $\mathbf{v}$ and $\mathbf{u}$ in $\mathbb{R}^3$.

Lines and Planes

39. Find the parametric equations for the lines described below:

a. The line passing through the point $(2, 3, -1)$ and parallel to $\langle 1, 0, 1\rangle$.

b. The line passing through the point $(0, 3, -1)$ and perpendicular to both $\langle 2, 2, 1\rangle$ and $\langle 1, -2, 1\rangle$.

c. The line passing through the points $(0, 1, -1)$ and $(2, 2, 2)$.

d. The line of intersection between the planes $x + y + z = 1$ and $x + z = 0$.

40. Find the equation for the planes described below:

a. The plane passing through $(1, -1, 1)$ and perpendicular to the vector $\langle 1, 2, 3\rangle$.

b. The plane passing through the origin in $\mathbb{R}^3$ and parallel to the plane $2x - y + z = 3$.

c. The plane that contains the line

$$\begin{cases} x = 3 + 2t, \\ y = t, \\ z = 8 - t, \end{cases}$$

for $t \in \mathbb{R}$ and is parallel to $2x + 4y + 8z = 17$.

d. The plane which passes through the points $(1, 2, 3)$, $(4, 5, 6)$, and $(7, 8, 10)$.

e. The plane which passes through the point $(1, 2, 3)$ and contains the line

$$\begin{cases} x = 3t, \\ y = 1 + t, \\ z = 2 - t, \end{cases}$$

for $t \in \mathbb{R}$.

f. The plane containing all points equidistant from the points $(1, 0, -2)$ and $(3, 4, 0)$.

Vector Valued Functions

41. Sketch the curve described by the vector valued function:

a. $\mathbf{r}(t) = \langle \sin t, t \rangle$

b. $\mathbf{r}(t) = \langle 1, \cos t, 2 \sin t \rangle$

42. Show that the curve described by $\mathbf{r}(t) = \langle t \cos t, t \sin t, t \rangle$ lies on the cone $z^2 = x^2 + y^2$ and use this fact to sketch the curve.

43. At which points do $\mathbf{r}(t) = \langle \sin t, \cos t, t \rangle$ and $x^2 + y^2 + z^2 = 5$ intersect?

44. Find the unit tangent vector $\mathbf{T}(t)$ at the indicated point

a. $\mathbf{r}(t) = \left\langle te^{-t}, 2 \arctan t, 2e^t \right\rangle$ at $t = 0$.

b. $\mathbf{r}(t) = \langle \cos t, 3t, 2 \sin 2t \rangle$ at $t = 0$.

c. $\mathbf{r}(t) = \langle 2 \sin t, \tan t, 2 \cos t \rangle$ at $t = \pi/4$.

45. If $\mathbf{r}(t) = \left\langle t, t^2, t^3 \right\rangle$, find $\mathbf{r}'(t)$, $\mathbf{T}(1)$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$.

46. Find the parametric equations for the line tangent to the curve at the given point:

a. $\mathbf{r}(t) = \left\langle e^{-t} \cos t, e^{-t} \sin t, e^{-t} \right\rangle$ at $(1, 0, 1)$

b. $\mathbf{r}(t) = \left\langle \ln t, 2\sqrt{t}, t^2 \right\rangle$ at $(0, 2, 1)$.

47. Find the length of the curve described by $\mathbf{r}(t) = \langle a \sin t, bt, a \cos t \rangle$ for $t \in [-10, 10]$.

48. Find the unit tangent vector $\mathbf{T}$, the unit normal vector $\mathbf{N}$, and the curvature κ for $\mathbf{r}(t) = \langle 2 \sin t, 5t, 2 \cos t \rangle$

49. Find the curvature of the curve defined by the function $y = \cos x$.

50. Find the unit tangent vector $\mathbf{T}$, the unit normal vector $\mathbf{N}$, and the binomial vector $\mathbf{B}$ at the point $(1, 2/3, 1)$ for $\mathbf{r}(t) = \langle t^2, 2t^3/3, t \rangle$.

51. The DNA molecule has the shape of a double helix. The radius of each helix is nearly 10 angstroms (1 angstrom is 10^{-8}cm). Each helix rises about 34 angstroms during a complete turn, and there are 2.9×10^8 complete turns. Estimate the length of each helix.

52. Let k be any number. At what point does the graph of e^{kx} have maximum curvature?