Linear Analysis II Set 1

1. Use the known Laplace transforms of t^n, e^{at}, $\cos(at)$, and $\sin(at)$ that are given in the table of Laplace transforms on our web site to find

$$\mathcal{L}\left[\sin(\sqrt{3}t) - 7\sin^2(3t) - e^{4t} + 2 + t^{101}\right].$$

Hint: $\sin^2 x = (1 - \cos(2x))/2$.

2. Use integration by parts to show that $\mathcal{L}[t^r] = \frac{r}{s}\mathcal{L}[t^{r-1}]$ holds for all $r > 0$.

3. Here are some well known series:

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \qquad \sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}, \qquad \ln\left(\frac{1}{1-t}\right) = \sum_{n=1}^{\infty} \frac{t^n}{n}, \qquad \arctan t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)}.$$

Find the following Laplace transforms by writing the function as a series, using $\mathcal{L}[t^n] = n!/s^{n+1}$ on each term, and then identifying the result as a variation on one of the above series:

a. $\mathcal{L}\left[\frac{\sin t}{t}\right]$

b. $\mathcal{L}\left[\frac{e^t - 1}{t}\right]$

c. $\mathcal{L}[J(\sqrt{t})]$ where $J(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{t}{2}\right)^{2n} = 1 - \frac{1}{(1!)^2 2^2}t^2 + \frac{1}{(2!)^2 2^4}t^4 - \frac{1}{(3!)^2 2^6}t^6 + \cdots$.

4. a. Use the substitution $t = \frac{x^2}{s}$ to show that $\mathcal{L}\left[\frac{1}{\sqrt{t}}\right] = \frac{2}{\sqrt{s}}\int_0^{\infty} e^{-x^2}\,dx$.

b. Show that $\mathcal{L}\left[\frac{1}{\sqrt{t}}\right]^2 = \frac{4}{s}\int_0^{\infty}\int_0^{\infty} e^{-x^2-y^2}\,dx\,dy$.

c. Using polar coordinates to integrate the above result, find $\mathcal{L}\left[\frac{1}{\sqrt{t}}\right]$.

d. Find $\mathcal{L}[\sqrt{t}]$ and $\mathcal{L}[t^{3/2}]$ using this exercise and Exercise 2.