Asyptotic Expansion & Complex Analyticity

Theorem: Let $R>0$, $\alpha>0$, $c>0$. Then if $\frac{c}{(R-x)^\alpha}=\sum_{n=0}^{\infty}a_nx^n$, then $a_n\sim\frac{cn^{\alpha-1}}{R^{n+\alpha}\Gamma(\alpha)}$

↳ Proof: We have $\frac{c}{(R-x)^\alpha}=\sum_{n=0}^{\infty}a_nx^n$

$$\Rightarrow \frac{c}{R^\alpha}\frac{1}{(1-x/R)^\alpha}=\frac{c}{R^\alpha}\sum_{n=0}^{\infty}\binom{-\alpha}{n}(-1)^n\frac{1}{R^n}x^n \quad \leftarrow \text{from binomial theorem } (1+x)^\beta=\sum_{n=0}^{\infty}\binom{\beta}{n}x^n$$

So the coefficient of x^n is:

$$\frac{c}{R^{\alpha+n}}\cancel{(-1)^n}\frac{(\overset{+}{\cancel{-}}\alpha)(\overset{+}{\cancel{-}}\alpha\overset{+}{\cancel{-}}1)\cdots(\overset{+}{\cancel{-}}\alpha\overset{+}{\cancel{-}}n\overset{-}{\cancel{+}}1)}{n!}$$

$$=\frac{c}{R^{\alpha+n}}\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)} \leftarrow \Gamma(\alpha+1)=\alpha\Gamma(\alpha)$$

$$\sim\frac{c}{R^{\alpha+n}}\frac{\sqrt{2\pi(\alpha+n-1)}\left(\frac{\alpha+n-1}{e}\right)^{\alpha+n-1}}{\Gamma(\alpha)\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} \leftarrow \text{using Stirling's Formula}$$

$$=\frac{c}{R^{\alpha+n}\Gamma(\alpha)}\underbrace{\sqrt{\frac{\alpha+n-1}{n}}}_{\to 1}\cancel{e^{1-\alpha}}\underbrace{\cancel{\left(\frac{\alpha+n-1}{n}\right)^n}}_{=\left(1+\frac{\alpha-1}{n}\right)^n\to e^{\alpha-1}}\underbrace{(\alpha+n-1)^{\alpha-1}}_{\to n^{\alpha-1}}$$

$$\sim\frac{c}{R^{\alpha+n}\Gamma(\alpha)}n^{\alpha-1}$$

$\boxtimes$

Corollary: Let $R>$, $\alpha>0$, $c>0$. Then if a_n satisfies $\frac{c}{(R-x)^\alpha}+\frac{c_1}{(R-x)^{\alpha-1}}+\dots+\frac{c_k}{(R-x)^{\alpha-k}}=\sum_{n=0}^{\infty}a_nx^n$, we have $a_n\sim\frac{cn^{\alpha-1}}{R^{n+\alpha}\Gamma(\alpha)}$.

↳ Proof idea: Use the previous theorem on each term and then simplify.

Def: A complex-valued function $f(z)$ is **analytic at z_0** if $\exists\ \varepsilon>0$ and $a_0,a_1,\ldots\in\mathbb{C}$ such that $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$ for $|z-z_0|<\varepsilon$.

Def: A complex-valued function $f(z)$ has a **singularity at z_0** if f is not analytic at z_0.

↳ In this course, singularities will come from dividing by zero.

Def: A singularity of $f(z)$ at $z_0=R$ is removable if $\lim_{z\to R}(R-z)^\alpha f(z)=c$ with $c\neq 0,\infty$.

Ex: $\dfrac{3}{(1-x)^{1/2}}$ Here $c=3$, $R=1$, $\alpha=1/2$

$\Rightarrow \sim \dfrac{3}{\sqrt{n}\sqrt{\pi}} = \dfrac{3}{\sqrt{n\pi}}$

Ex: e^z, $\sin(z)$, $3z^2+1$ are all analytic on all of $\mathbb{C}$

Ex: $f(z) = \dfrac{z}{(1-z)(2-z)}$

has singularities at 1 and 2

Ex: $f(z) = \sqrt{4-z}$ has a singularity at $z_0 = 4$.

Observe $\dfrac{df}{dz}(z) = -\dfrac{1}{2\sqrt{4-z}}$

Ex: $\dfrac{1}{1-\sin(z)}$ has singularities at $\left\{\frac{\pi}{2} + 2k\pi : k \in \mathbb{Z}\right\}$

Ex: removable singularity

$f(x) = \dfrac{1}{(1-x)^2(2-x)}$ has singularities at 1, 2

$\lim_{x\to 1} (1-x)^2 f(x) = 1 \Rightarrow 1$ is a removable singularity