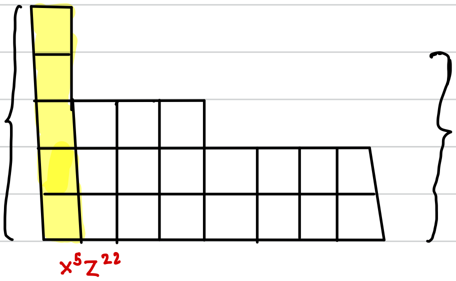


Recall: $\sum_{n=0}^{\infty} (\#\lambda\vdash n \text{ with } \max(\lambda) \leq k)\, z^n = \frac{1}{1-z}\cdot\frac{1}{1-z^2}\cdots\cdots\frac{1}{1-z^k}$

$= (1+z+z^2+\cdots\cdots)(1+z^2+(z^2)^2+\cdots\cdots)\cdots\cdots(1+z^k+(z^k)^2+\cdots\cdots)$

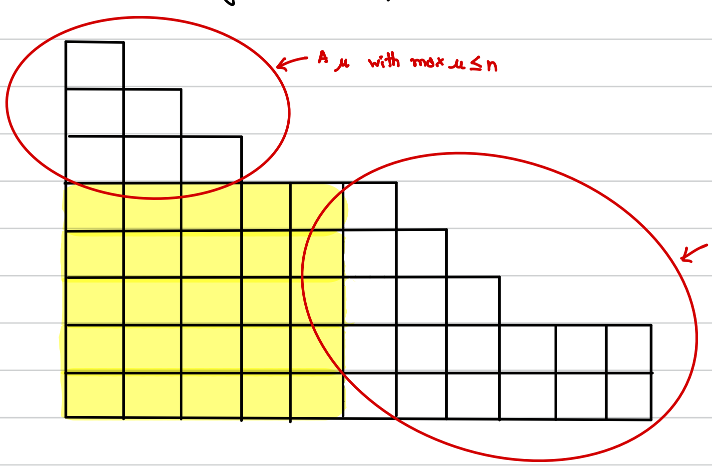
Theorem

$$\prod_{i=1}^{\infty}\frac{1}{1-z^i} = \sum_{n=0}^{\infty}\frac{z^{n^2}}{(1-z)^2(1-z^2)^2\cdots\cdots(1-z^n)^2}$$

$$= \sum_{n=0}^{\infty}(\#\lambda\vdash n)\, z^n$$

Proof

Let n be the maximum side length of a square that fits inside the diagram for λ.



Any λ can be created by

1. Pick box size n ← "$\sum_{n=0} z^{n^2}$"
2. Pick μ with max $\mu \leq n$. Put μ on top ← "$\frac{1}{(1-z)(1-z^2)\cdots\cdots(1-z^n)}$"
3. Pick α with $\ell(\alpha)\leq n$. Put α on side.

We have $\prod_{i=1}^{\infty}\frac{1}{1-z^i} = 1+z+2z^2+3z^3+5z^4+7z^5+\cdots\cdots$

and $\prod_{i=1}^{\infty}(1-z^i) = 1-z^1-z^2+z^5+z^7-z^{12}-z^{15}+z^{22}+z^{26}$

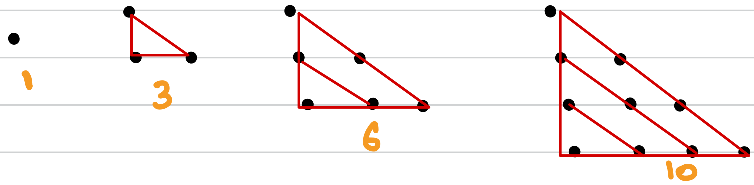
Observations

1. The signs are +, −, −, +, +, −, −, +, +, −, −, ...
2. The coefficient of z^n is +1, −1, 0.

What is the sequence 0, 1, 2, 5, 7, 12, 15, 22, 26, ... ?

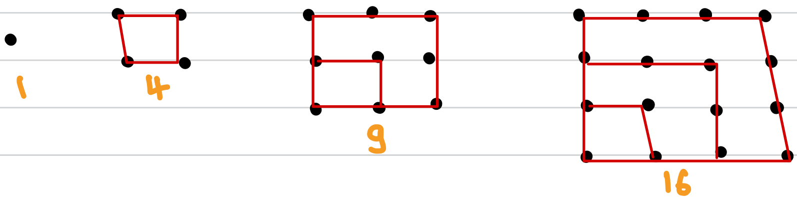
$k=0$, $k=1$, $k=-1$, $k=2$, $k=-2$, $k=3$, $k=-3$, $k=4$, $k=-4$

Triangle:



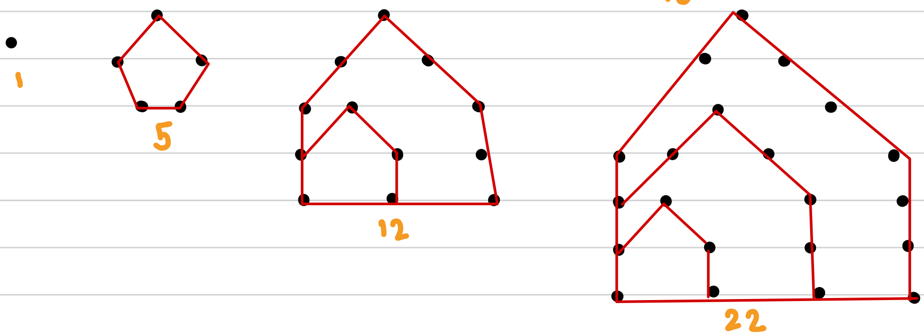
$\binom{k+1}{2}$

Square:



k^2

Pentagon



$\frac{k(3k-1)}{2}$

Theorem

Euler's pentagonal number theorem

$$\prod_{i=1}^{\infty}(1-z^i) = \sum_{k\in\mathbb{Z}}(-1)^k z^{k(3k-1)/2} = 1 - z^1 - z^2 + z^5 + z^7 - z^{12} - z^{15} + \ldots..$$

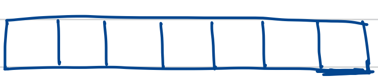
$k=0$, 1, -1, 2, -2, 3, -3

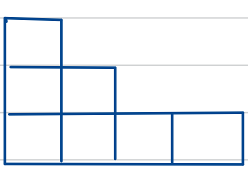
Proof

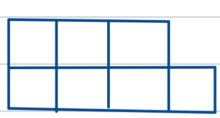
$$\prod_{i=1}^{\infty}(1-z^i) = \sum_{\lambda \text{ with distinct parts}} (-1)^{\ell(\lambda)} z^{|\lambda|}$$

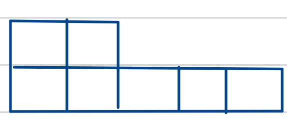
$(1-z^1)(1-z^2)(1-z^3)$

Ex: $n=7$

7  −

1, 2, 4  −

3, 4  +

2, 5  +

1, 6  +