

Discrete Mathematics Set 4

Math 435: Complete 7 parts of the following exercises.

Math 530: Exercises 1, 2, and any two of the remaining exercises.

1. Let A_n be the set of paths in \mathbb{R}^2 which start at $(0, 0)$, end at (n, n) , and only use steps of the form $(1, 0)$ or $(0, 1)$.

a. Show that there are $\binom{2n}{n}$ elements in A_n .

b. Denote the number of times the path $p \in A_n$ touches the line $y = x$ by $\text{touch}(p)$. Let

$$A(x, t) = \sum_{n=0}^{\infty} \left(\sum_{p \in A_n} t^{\text{touch}(p)} \right) x^n$$

and let c_n be the n^{th} Catalan number. Show that

$$\sum_{p \in A_{n+1}} t^{\text{touch}(p)} = 2t \sum_{k=0}^n c_k \left(\sum_{p \in A_{n-k}} t^{\text{touch}(p)} \right).$$

c. Show that $A(x, t) = \frac{t}{1 - t + 2t\sqrt{\frac{1}{4} - x}}$.

d. Find an asymptotic formula for the average number of times a path in A_n touches the line $y = x$.

2. Let $M(x) = \sum_{n=0}^{\infty} m_n x^n$ be the generating function for m_n , the number of Motzkin paths of length n , as defined in Set 1 Exercise 6.

a. Find $\lim_{x \rightarrow 1/3} (1/3 - x)^{1/2} x M'(x)$ and use this result to find an asymptotic formula for m_n . (As always, calculations can be done using a computer algebra system if desired.)

b. Let a_n be as defined in Set 1 Exercise 6b. Find an asymptotic formula for a_n and use it to find an asymptotic formula for the probability that a random path in the plane that from $(0, 0)$ to $(n, 0)$ using steps of the form $(1, 1)$, $(1, -1)$ and $(1, 0)$ is a Motzkin path.

3. Find an asymptotic formula for the average number of sets in an ordered set partition of n (see Set 2 Exercise 3, Set 3 Exercise 1b, and Set 3 Exercise 1d).

4. Find an asymptotic formula for the average number of cycles in a permutation of n with ordered cycles (see Set 3 Exercise 1c and Set 3 Exercise 1d.)

5. Find an asymptotic formula for the number of permutations of n that do not have any cycles of length 1, 2 or 3.

6. Define

1. the Chebyshev polynomial of the first kind $T_n(y)$ by $\sum_{n=0}^{\infty} T_n(y) x^n = \frac{1 - yx}{1 - 2yx + x^2}$,

2. the Chebyshev polynomial of the second kind $U_n(y)$ by $\sum_{n=0}^{\infty} U_n(y) x^n = \frac{1}{1 - 2yx + x^2}$, and

3. the Legendre polynomial $P_n(y)$ by $\sum_{n=0}^{\infty} P_n(y)x^n = \frac{1}{\sqrt{1-2yx+x^2}}$.

Find asymptotic formulas for $T_n(5/4)$, $U_n(5/4)$, and $P_n(5/4)$.