

Calculus 3 Exercises!

Polynomial Approximations

1. Find the degree 5 Taylor polynomial at $x = 0$ for each of these functions:

- $\cos x$
- $1 - 3x^2 + 2x^3 + x^7 + 4x^{10}$
- $(1 - 2x)^2 + (1 - 3x)^3 + x^{1000}$
- $\frac{1}{\sqrt{1-x}}$
- $(1+x)^\pi$
- $\sin(4x)$
- e^x
- $e^{\pi x}$
- $(1+x)^\pi + \sin(4x)$
- $(1-x)^{-3}$
- The function $f(x)$ which satisfies $f(0) = 1$ and

$$f'(x) = f(x/2).$$

(Don't be intimidated by the abstraction here, just the usual thing: take derivatives using the chain rule and then plug in 0.)

2. Which degree 5 polynomial best approximates $\sqrt{1+x}$ at $x = 0$? Use this polynomial evaluated at $x = 1$ to find an approximation of the value of $\sqrt{2}$. Use a calculator to determine the (absolute) error in using this approximation.

3. Throughout this exercise, let $f(x) = \frac{1}{3-x}$.

- Find an M such that $|f^{(n+1)}(x)| < M$ for all x in $[-1, 1]$. (The value of M involves n .)
- Use the formula

$$\text{Error} \leq \frac{M}{n!} |a|^{n+1}$$

where M is the value found in part a. to find a bound on the error when using the degree n Taylor polynomial to approximate the value of $f(1)$.

4. Let $f(x) = \cos 2x$.

- Find an M such that $|f^{(n+1)}(x)| < M$ for all x in $[-\pi, \pi]$. (The value of M involves n .)
- Use the formula

$$\text{Error} \leq \frac{M}{n!} |a|^{n+1}$$

where M is the value found in part a. to find a bound on the error when using the degree n Taylor polynomial to approximate the value of $f(1)$.

5. The approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

is the best degree 8 polynomial approximation for $\sin x$ at $x = 0$. Show that the error in using this approximation is less than 0.1 when $-\pi \leq x \leq \pi$.

6. Let $f(x) = \ln\left(\frac{1}{1-x}\right)$.

- Find the degree n Taylor polynomial at $x = 0$ for $f(x)$.
- Show that if x is in $\left[-\frac{1}{2}, \frac{1}{2}\right]$, then

$$|f^{(n+1)}(x)| \leq 2^{n+1} n!.$$

- Show that the error when approximating $\ln 2$ by taking $x = 1/2$ in the polynomial in part a. is at most $1/(n+1)$. How large should n be in order to make $1/(n+1) < 0.05$?
- Using part a., approximate the value of $\ln 2$ so that the error is smaller than 0.05. (Leave your answer as a sum of fractions.)

7. Find the degree 5 Taylor polynomial for

- $\cos x$ at $x = \pi/2$.
- $1 - 3x^2 + 2x^3 + x^7 + 4x^{10}$ at $x = 1$.
- $\frac{1}{\sqrt{1-x}}$ at $x = -2$.
- $(1-x)^{-3}$ at $x = 2$.
- $\sqrt{3+x}$ at $x = 1$.
- The function $f(x)$ which satisfies $f'(x) = 2f(x)$ and $f(1) = -1$ at $x = 1$.

Infinite Series

8. Simplify these sums (or write “divergent!” if the sum does not exist):

a. $\sum_{n=0}^{\infty} (0.7)^n$.

b. $\sum_{n=2}^{\infty} \frac{3^n}{5^{n-1}}.$

c. $\sum_{n=1}^{\infty} \frac{5^n + 6^n}{7^n}$.

d. $\sum_{n=0}^{\infty} \frac{5^n 6^n}{7^n}$.

e. $9.99999 \dots = 9 + 0.9 + 0.09 + 0.009 + \dots$

9. For which values of x do these sums converge? What functions are they equal to when they do converge?

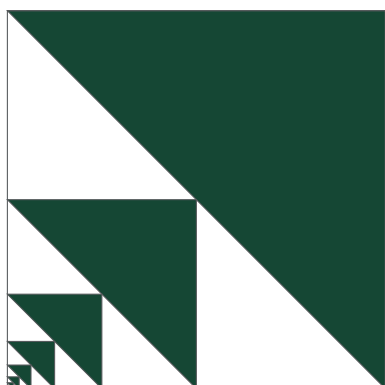
a. $\sum_{n=0}^{\infty} (x+1)^n.$

b. $\sum_{n=0}^{\infty} (2x)^n.$

c. $\sum_{n=1}^{\infty} 2x^n.$

d. $\sum_{n=2}^{\infty} (3x - 2)^n.$

10. What percentage of the area in the following square is green?



11. Do the following series converge or diverge? Give a reason why your answer is correct.

a. $\sum_{n=1}^{\infty} \frac{1+n^2}{1+n^4}$

b. $\sum_{n=1}^{\infty} \frac{1}{n^{3+\sin n}}$

c. $\sum_{n=1}^{\infty} \frac{2 + \sin n}{2^n}$

d. $\sum_{n=1}^{\infty} \frac{n+2}{(n+1)^3}$

e. $\sum_{n=1}^{\infty} \frac{1+3^n}{1+2^n}$

f. $\sum_{n=1}^{\infty} \frac{1}{2n+5}$

$$\text{eg. } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$$

[illegible]

i. $\sum_{n=1}^{\infty} \frac{n^2 - 1}{3n^4 + 1}$

j. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2 + 2}}$

k. $\sum_{n=1}^{\infty} \frac{100^n}{n!}$

l. $\sum_{n=0}^{\infty} \frac{2n}{\sqrt{n}+1}$

m. $\sum_{n=0}^{\infty} n^2 e^{-n}$

n. $\sum_{n=0}^{\infty} \frac{(n!)^2}{2^{n^2}}$

12. For which values of x do the following series converge?

a. $\sum_{n=0}^{\infty} (-1)^n (\ln n) x^n$

b. $\sum_{n=0}^{\infty} \frac{x^{2n}}{n^4}$

c. $1 + \frac{1 \cdot 4}{1 \cdot 3}x + \frac{1 \cdot 4 \cdot 7}{1 \cdot 3 \cdot 5}x^2 + \frac{1 \cdot 4 \cdot 7 \cdot 10}{1 \cdot 3 \cdot 5 \cdot 7}x^3 + \dots$

d. $\sum_{n=0}^{\infty} 3^{\sqrt{n}} x^n$

e. $1 + \frac{1}{1.5}x + \frac{1}{1.5 \cdot 9}x^2 + \dots$

13. Do the following alternating series converge or diverge? Please provide a reason why your answer is correct.

a. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$

b. $\sum_{n=1}^{\infty} \frac{(-1)^n}{1 + \sqrt{n}}$

c. $\sum_{n=1}^{\infty} \frac{(-1)^n 5^n}{4^n}$

[illegible]

e. $\sum_{n=1}^{\infty} (-1)^n n^n$

f. $\sum_{n=1}^{\infty} (-1)^n$

14. Approximate the sum of each of the following series to within $1/100$ of the true value. You may leave your answer as a sum of fractions.

a. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n!}}$.

b. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

c. $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{n^2}{2^n}.$

Power series

15. Every human is born knowing these series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (\text{true for all } x)$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (\text{true for all } -1 < x < 1)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad (\text{true for all } x)$$

By differentiating, integrating, or otherwise manipulating one of the above series, find the series representations for each of the functions below. Include the values of x for which equality holds.

a. $\frac{\sin(x^2)}{x}$

b. $\frac{1}{1+x}$

c. $\frac{1}{1-x^2}$

d. $\frac{e^{-x^2} - 1}{x^2}$

e. $x^3 \cos(x^2)$

f. $\int \frac{\sin x}{x} dx$

g. $\int e^{-x^3} dx$

h. $\frac{\arctan x - x}{x^2}$

i. $\frac{d}{dx} \left(\frac{1 - \cos(\sqrt{x})}{x} \right)$

16. By multiplication or division of known series, find the first 4 terms in the Taylor series for:

a. $e^{2x} \sin(x/2)$

b. $e^{-x^2}/(1-x)$

c. $(\arctan x)^2$

d. $1/\cos x$

17. a. Find the interval and radius of convergence for

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}.$$

b. Show that y satisfies the differential equation

$$x^2 y'' + xy' + x^2 y = 0.$$

(Take two derivatives of y , plug it into the differential equation, and show that everything simplifies to 0.)

Parametric Equations

18. Plot these parametric curves (with starting and ending points and an arrow indicating direction):

a. $\begin{cases} x = 3t - 5, \\ y = 2t + 1 \end{cases} \text{ for } t \in (-\infty, \infty)$

- b. $\begin{cases} x = t^2 - 2, \\ y = 5 - 2t \end{cases} \text{ for } t \in [-3, 4]$
- c. $\begin{cases} x = t^2, \\ y = t^3 \end{cases} \text{ for } t \in [-1, 1]$
- d. $\begin{cases} x = 2 \cos(3t), \\ y = 3 \sin(3t) \end{cases} \text{ for } t \in [-\pi/2, 3\pi/2]$
- e. $\begin{cases} x = \ln t, \\ y = \sqrt{t} \end{cases} \text{ for } t \in [1, \infty)$

19. Find the line tangent to the curves at the indicated point:

- a. $\begin{cases} x = 6 \sin t \\ y = t^2 + t \end{cases} \text{ at the point found when } t = 1.$
- b. $\begin{cases} x = \cos t + \cos(2t) \\ y = \sin t + \sin(2t) \end{cases} \text{ at the point } (-1, 1).$

20. Find the first and second derivatives of these parametric curves. For which values of t is the parametric equation concave up?

- a. $\begin{cases} x = 2 \sin t \\ y = 3 \cos t \end{cases}$
- b. $\begin{cases} x = t^3 - 12t \\ y = t^2 - 1 \end{cases}$

21. Find the exact length of the curve:

- a. $\begin{cases} x = 1 + 3t^2, \\ y = 4 + 2t^3 \end{cases} \text{ for } t \in [0, 1]$
- b. $\begin{cases} x = e^t + e^{-t}, \\ y = 5 - 2t \end{cases} \text{ for } t \in [0, 3]$
- c. $\begin{cases} x = e^t \cos t, \\ y = e^t \sin t \end{cases} \text{ for } t \in [0, \pi]$

22. Consider the parametric equations

$$\begin{cases} x = \int_0^t \frac{\cos u}{1+u^2} du, \\ y = \int_0^t \frac{\sin u}{1+u^2} du \end{cases}$$

for $t \in [0, \infty)$. What is the first positive value of t for which this curve has a vertical tangent line? What is the length of the curve from $(0, 0)$ to this value?

Polar Equations

23. Plot these polar functions:

- a. $r = \theta$ for $\theta \in [-\pi, \pi]$,
- b. $r = \sin \theta$ for $\theta \in [0, \pi]$.
- c. $r = 1 - 2 \cos \theta$ for $\theta \in [0, 2\pi]$.

24. Find the equation of the line tangent to the polar curve at the given point:

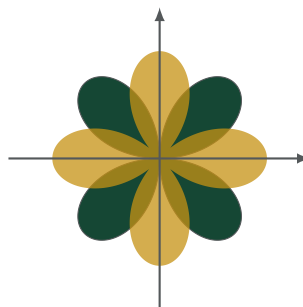
- a. $r = 2 \sin 2\theta$ at $\theta = 3\pi/4$.
- b. $r = 1/\theta$ at the x, y coordinate $(0, 2/\pi)$.

25. Find the points on the polar curve where the tangent line has a horizontal or a vertical tangent:

- a. $r = 1 + \cos \theta$.
- b. $r = 4$

26. Find the area swept out by the polar equation $r = \sqrt{\theta}$ for $\theta \in [0, 2\pi]$.

27. Find the area enclosed by the graph of $r = \sin(2\theta)$ but outside the graph of $r = \cos(2\theta)$:



28. Find the exact length of the polar curve

- a. $r = 3 \sin \theta$ for $\theta \in [0, \pi/3]$.
- b. $r = e^{2\theta}$ for $\theta \in [0, 2\pi]$.

Vectors in \mathbb{R}^3

29. Draw the points in \mathbb{R}^3 represented by these relations:

- a. $x^2 + z^2 \leq 3$
- b. $(x - 1)^2 + y^2 + (z + 1)^2 = 1$
- c. $x^2 + y^2 + z^2 - 6x + 4y - 2z = 11$.

30. The vector \mathbf{v} lies in the first quadrant of \mathbb{R}^2 , has $|\mathbf{v}| = 4$, and makes an angle of $\pi/3$ with the x -axis. Write \mathbf{v} as $\langle a, b \rangle$ for some real numbers a and b .

31. Do the following operations on the vectors $\mathbf{u} = \langle 3, 1, 2 \rangle$, $\mathbf{v} = \langle 2, 0, -1 \rangle$, and $\mathbf{w} = \langle 1, 1, 1 \rangle$:

- Find a vector in the same direction as $\mathbf{u} + \mathbf{v}$ but has length 2.
- Find the angle between \mathbf{u} and \mathbf{v} and the angle between \mathbf{u} and \mathbf{w} .
- Find the cross products $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$.
- $|\mathbf{u} \times (2\mathbf{v} - \mathbf{w})|$.
- Find two unit vectors in a direction orthogonal to both \mathbf{u} and \mathbf{v} .

32. Find the cross product of $\langle t, t^2, t^3 \rangle$ and $\langle 1, 2t, 3t^2 \rangle$ and show that it is orthogonal to both vectors.

33. Find all vectors \mathbf{u} and \mathbf{v} such that

$$|\mathbf{u} \times \mathbf{v}| = \mathbf{u} \cdot \mathbf{v}.$$

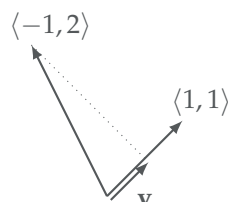
34. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors. Which of these operations make sense?

- $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$
- $(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$
- $(\mathbf{u} \cdot \mathbf{v})|\mathbf{w}|$
- $(\mathbf{u} \cdot \mathbf{v}) + \mathbf{w}$
- $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w}$
- $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$
- $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$
- $(\mathbf{u} \cdot \mathbf{v}) \times \mathbf{w}$

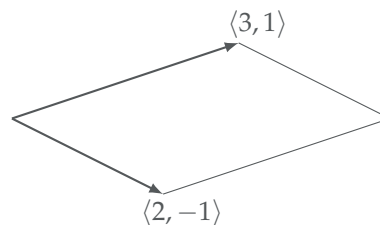
35. Show, for any general vectors in \mathbb{R}^3 , that

$$(\mathbf{v} \times \mathbf{u}) \cdot \mathbf{u} = 0.$$

36. Find the vector \mathbf{v} depicted here:



37. Find the area of this parallelogram:



38. Generalizing question 37, find a formula for the area of the parallelogram defined by the two vectors \mathbf{v} and \mathbf{u} in \mathbb{R}^3 .

Lines and Planes

39. Find the parametric equations for the lines described below:

- The line passing through the point $(2, 3, -1)$ and parallel to $\langle 1, 0, 1 \rangle$.
- The line passing through the point $(0, 3, -1)$ and perpendicular to both $\langle 2, 2, 1 \rangle$ and $\langle 1, -2, 1 \rangle$.
- The line passing through the points $(0, 1, -1)$ and $(2, 2, 2)$.
- The line of intersection between the planes $x + y + z = 1$ and $x + z = 0$.

40. Find the equation for the planes described below:

- The plane passing through $(1, -1, 1)$ and perpendicular to the vector $\langle 1, 2, 3 \rangle$.
- The plane passing through the origin in \mathbb{R}^3 and parallel to the plane $2x - y + z = 3$.
- The plane that contains the line

$$\begin{cases} x = 3 + 2t, \\ y = t, \\ z = 8 - t, \end{cases}$$

for $t \in \mathbb{R}$ and is parallel to $2x + 4y + 8z = 17$.

- The plane which passes through the points $(1, 2, 3)$, $(4, 5, 6)$, and $(7, 8, 10)$.
- The plane which passes through the point $(1, 2, 3)$ and contains the line

$$\begin{cases} x = 3t, \\ y = 1 + t, \\ z = 2 - t, \end{cases}$$

for $t \in \mathbb{R}$.

- f. The plane containing all points equidistant from the points $(1, 0, -2)$ and $(3, 4, 0)$.

angstroms during a complete turn, and there are 2.9×10^8 complete turns. Estimate the length of each helix.

Vector Valued Functions

41. Sketch the curve described by the vector valued function:

- a. $\mathbf{r}(t) = \langle \sin t, t \rangle$
 b. $\mathbf{r}(t) = \langle 1, \cos t, 2 \sin t \rangle$

42. Show that the curve described by $\mathbf{r}(t) = \langle t \cos t, t \sin t, t \rangle$ lies on the cone $z^2 = x^2 + y^2$ and use this fact to sketch the curve.

43. At which points do $\mathbf{r}(t) = \langle \sin t, \cos t, t \rangle$ and $x^2 + y^2 + z^2 = 5$ intersect?

44. Find the unit tangent vector $\mathbf{T}(t)$ at the indicated point

- a. $\mathbf{r}(t) = \langle te^{-t}, 2 \arctan t, 2e^t \rangle$ at $t = 0$.
 b. $\mathbf{r}(t) = \langle \cos t, 3t, 2 \sin 2t \rangle$ at $t = 0$.
 c. $\mathbf{r}(t) = \langle 2 \sin t, \tan t, 2 \cos t \rangle$ at $t = \pi/4$.

45. If $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, find $\mathbf{r}'(t)$, $\mathbf{T}(1)$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$.

46. Find the parametric equations for the line tangent to the curve at the given point:

- a. $\mathbf{r}(t) = \langle e^{-t} \cos t, e^{-t} \sin t, e^{-t} \rangle$ at $(1, 0, 1)$
 b. $\mathbf{r}(t) = \langle \ln t, 2\sqrt{t}, t^2 \rangle$ at $(0, 2, 1)$.

47. Find the length of the curve described by $\mathbf{r}(t) = \langle a \sin t, bt, a \cos t \rangle$ for $t \in [-10, 10]$.

48. Find the unit tangent vector \mathbf{T} , the unit normal vector \mathbf{N} , and the curvature κ for $\mathbf{r}(t) = \langle 2 \sin t, 5t, 2 \cos t \rangle$

49. Find the curvature of the curve defined by the function $y = \cos x$.

50. Find the unit tangent vector \mathbf{T} , the unit normal vector \mathbf{N} , and the binomial vector \mathbf{B} at the point $(1, 2/3, 1)$ for $\mathbf{r}(t) = \langle t^2, 2t^3/3, t \rangle$.

51. The DNA molecule has the shape of a double helix. The radius of each helix is nearly 10 angstroms (1 angstrom is 10^{-8} cm). Each helix rises about 34

52. Let k be any number. At what point does the graph of e^{kx} have maximum curvature?