Theorem 1. *We have*

$$\prod_{n=1}^{\infty}(1-z^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)z^{k(k+1)/2}.$$

Proof. Using Jacobi's triple product identity, we have

$$\begin{aligned}
\prod_{n=1}^{\infty}(1-z^n)^3 &= \lim_{x\to-1}\prod_{n=1}^{\infty}(1-z^n)(1+xz^n)(1+x^{-1}z^n) \\
&= \lim_{x\to-1}\frac{\sum_{k\in\mathbb{Z}} x^k z^{k(k-1)/2}}{1+x} \\
&= \lim_{x\to-1}\frac{\sum_{k=1}^{\infty} x^k z^{k(k-1)/2} + \sum_{n=0}^{\infty} x^{-n} z^{-n(-n-1)/2}}{1+x} \\
&= \lim_{x\to-1}\frac{\sum_{n=0}^{\infty}(x^{n+1}+x^{-n})z^{-n(-n-1)/2}}{1+x}.
\end{aligned}$$

Now, since $((-1)^{n+1}+(-1)^{-n})$ is zero for all integers n, we have a limit that is an indeterminate form, ripe for the application of L'Hôpital's rule. Doing this gives

$$\lim_{x\to-1}\frac{\sum_{n=0}^{\infty}((n+1)(x)^{n+1}-n(x)^{-n})z^{-n(-n-1)/2}}{1} = \sum_{n=0}^{\infty}(-1)^n(2n+1)z^{n(n+1)/2},$$

as needed. ∎