

# Graph Theory Set 4

**16.** This exercise gives another proof of Cayley's theorem. Let  $t_n$  be the number of labeled trees on  $n$  vertices. Let  $A$  be the set of objects which can be created this way:

1. Select a labeled tree with  $n$  vertices.
2. Mark one vertex.
3. Label the  $n-1$  edges with  $1, \dots, n-1$  such that each edge has a different label.

For example, one possible element in  $A$  when  $n = 6$  is:



- a. By following these instructions, how many elements are in  $A$ ? (The answer should involve  $t_n$  since we are pretending that we do not know that  $t_n = n^{n-2}$ .)

There is another way to create elements in  $A$ :

1. Start with an empty graph with vertices  $1, \dots, n$ . Set  $i = 1$ . Mark every vertex.
2. Select any vertex, say  $v$ .
3. Select a marked vertex, say  $w$ .
4. Remove the mark on  $w$  and draw an edge with label  $i$  between  $v$  and  $w$ .
5. If there are at least two marked vertices, increment  $i$  by 1 and go back to step 2.. If not, stop.
- b. By following the 5 above steps, how many elements are in  $A$ ? Why does this prove that  $t_n = n^{n-2}$ ?

**17.** Suppose  $G$  has two spanning trees  $T_1$  and  $T_2$ . Let  $e$  be any edge in  $T_1$ . Show that there is an edge  $f$  in  $T_2$  such that the graph  $(T_1 - e) + f$  (remove  $e$  from  $T_1$  and include  $f$ ) is also a spanning tree.

**18.** Let  $\tau(G)$  be the number of spanning trees for  $G$  and let  $e$  be an edge in  $G$ . Show that  $\tau(G) = \tau(G - e) + \tau(G/e)$ .

**19.** It has been proved that if  $G$  is planar, then it can be drawn in the plane with straight line segments as edges. Exhibit such planar drawings for  $K_5 - e$  and  $K_{3,3} - e$ .

**20.** Remove and contract edges in the following graph to find  $K_{3,3}$ , showing that it is not planar.

