Discrete Mathematics Set 4

Math 435: Complete 7 parts of the following exercises.
Math 530: Exercises 1, 2, and any two of the remaining exercises.

1. Let A_n be the set of paths in $\mathbb{R}^2$ which start at $(0,0)$, end at (n,n), and only use steps of the form $(1,0)$ or $(0,1)$.

a. Show that there are $\binom{2n}{n}$ elements in A_n.

b. Denote the number of times the path $p \in A_n$ touches the line $y = x$ by touch(p). Let

$$A(x,t) = \sum_{n=0}^{\infty} \left(\sum_{p \in A_n} t^{\text{touch}(p)} \right) x^n$$

and let c_n be the n^{th} Catalan number. Show that

$$\sum_{p \in A_{n+1}} t^{\text{touch}(p)} = 2t \sum_{k=0}^{n} c_k \left(\sum_{p \in A_{n-k}} t^{\text{touch}(p)} \right).$$

c. Show that $A(x,t) = \dfrac{t}{1 - t + 2t\sqrt{\frac{1}{4} - x}}$.

d. Find an asymptotic formula for the average number of times a path in C_n touches the line $y = x$.

2. Let $M(x) = \sum_{n=0}^{\infty} m_n x^n$ be the generating function for m_n, the number of Motzkin paths of length n, as defined in Set 1 Exercise 6.

a. Show that $\lim_{x \to 1/3} (1/3 - x)^{1/2} x M'(x)$ and use this result to find an asymptotic formula for m_n. (As always, calculations can be done using a computer algebra system if desired.)

b. Let a_n be as defined in Set 1 Exercise 6b. Find an asymptotic formula for a_n and use it to find an asymptotic formula for the probability that a random path in the plane that from $(0,0)$ to (n,n) using steps of the form $(1,1)$, $(1,-1)$ and $(1,0)$ is a Motzkin path.

3. Find an asymptotic formula for the average number of sets in an ordered set partition of n (see Set 2 Exercise 3, Set 3 Exercise 1b, and Set 3 Exercise 1d).

4. Find an asymptotic formula for the average number of cycles in a permutation of n with ordered cycles (see Set 3 Exercise 1c and Set 3 Exercise 1d.)

5. Find an asymptotic formula for the number of permutations of n that do not have any cycles of length $1, 2$ or 3.

6. Define

1. the Chebyshev polynomial of the first kind $T_n(y)$ by $\sum_{n=0}^{\infty} T_n(y) x^n = \dfrac{1 - yx}{1 - 2yx + x^2}$,

2. the Chebyshev polynomial of the second kind $U_n(y)$ by $\sum_{n=0}^{\infty} U_n(y) x^n = \dfrac{1}{1 - 2yx + x^2}$, and

3. the Legendre polynomial $P_n(y)$ by $\sum_{n=0}^{\infty} P_n(y)x^n = \dfrac{1}{\sqrt{1-2yx+x^2}}$.

Find asymptotic formulas for $T_n(5/4)$, $U_n(5/4)$, and $P_n(5/4)$.