Homework Set 1

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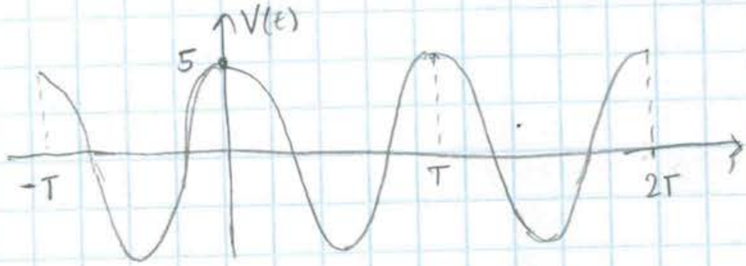
ME579

1-1 a) $x(t) = 5\cos(2\pi \frac{1}{2T} t)$ volts for $-0.025 \le t \le 0.025$ seconds
&
$x(t) = x(t+qT)$ for any integer q, and T is period



$T_p = 0.025s - (-0.025)s = 0.05s$

$f_1 = \frac{1}{T} = 20\ Hz$ $\quad f_k = k \cdot \frac{1}{T} = 20k\ Hz$

b) $C_k = \frac{1}{T_p}\int_{period} x(t) e^{-j\frac{2\pi k}{T_p}t}\, dt = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} 5\cos(\frac{\pi}{T_p}t) e^{-j\frac{2\pi k}{T_p}t}\, dt$

Integration by parts

$$= \frac{5 e^{-j\frac{2\pi k}{T_p}t}}{T_p\left(\frac{4\pi^2 k^2}{T_p^2} + \frac{\pi^2}{T_p^2}\right)}\left(\frac{\pi}{T_p}\sin(\frac{\pi}{T_p}t) - j\frac{2\pi k}{T_p}\cos(\frac{\pi}{T_p}t)\right)\Bigg|_{-T_p/2}^{T_p/2}$$

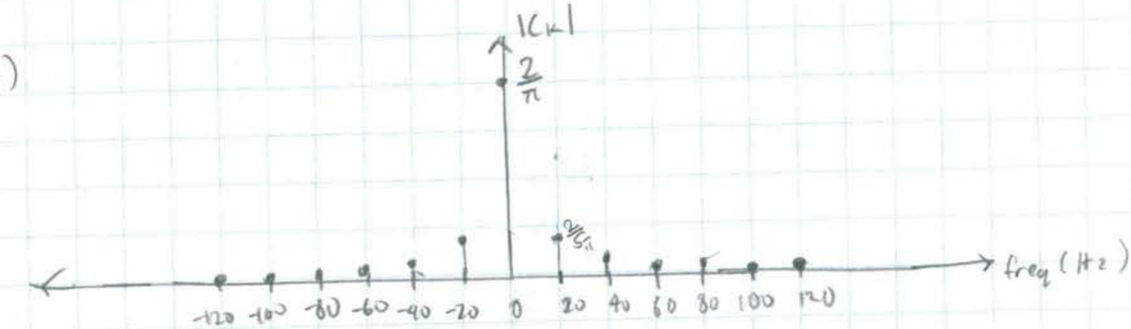
$$= \frac{5 e^{-j\frac{2\pi k}{T_p}\frac{T_p}{2}}}{\pi(1+4k^2)}\left(\frac{\pi}{T_p}\sin(\frac{\pi}{T_p}\cdot\frac{T_p}{2}) - j\frac{2\pi k}{T_p}\cos(\frac{\pi}{T_p}\frac{T_p}{2})\right) - \frac{e^{j\pi k}}{\pi(1+4k^2)}\left(\sin(\frac{-\pi}{2}) - j\,2k\cos(\frac{\pi}{2})\right)$$

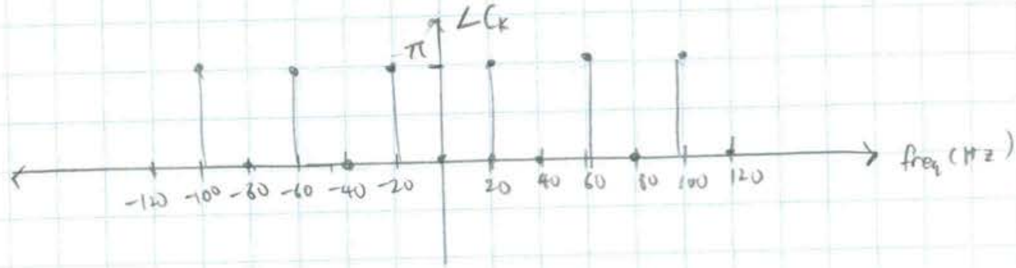
$$= \frac{e^{-j\pi k}(-1-0) - e^{j\pi k}(-1-0)}{\pi(1+4k^2)} = \boxed{\frac{e^{-j\pi k} + e^{j\pi k}}{\pi(1+4k^2)} = C_k,\ C_0 = \frac{1+1}{2\pi} = \frac{2}{\pi}}$$

b)

k	f_k	C_k	$\lvert C_k \rvert$	$\angle C_k$
0	0	$\frac{2}{\pi}$	$\frac{2}{\pi}$	0
1	20 Hz	$\frac{-2}{5\pi}$	$\frac{2}{5\pi}$	π
2	40 Hz	$2/17\pi$	$2/17\pi$	0
3	60 Hz	$-2/37\pi$	$2/37\pi$	π
4	80 Hz	$2/65\pi$	$2/65\pi$	0
5	100 Hz	$-2/101\pi$	$2/101\pi$	π
6	120 Hz	$2/145\pi$	$2/145\pi$	0

1-1 b)





c) On MATLAB attachment

1-2 a) $C_k e^{j2\pi\frac{k}{T}t} + C_{-k} e^{-j2\pi\frac{k}{T}t} = A_k \cos(2\pi\frac{k}{T}t) + B_k \sin(2\pi\frac{k}{T}t)$

Let $2\pi\frac{k}{T} = \omega_k$ & assume $C_{-k} = C_k^*$

Euler's Formula: $e^{i\phi} = \cos\phi + i\sin\phi$

$\Rightarrow C_k e^{j\omega_k t} + C_k^* e^{-j\omega_k t} = A_k \cos(\omega_k t) + B_k \sin(\omega_k t)$

$C_k[\cos(\omega_k t) + j\sin(\omega_k t)] + C_k^*[\overset{\text{even function}}{\cos(\omega_k t)} - j\overset{\text{odd function}}{\sin(\omega_k t)}] = A_k \cos(\omega_k t) + B_k \sin(\omega_k t)$

$(C_k + C_k^*)\cos(\omega_k t) + (C_k - C_k^*) j\sin(\omega_k t) = A_k \cos(\omega_k t) + B_k \sin(\omega_k t)$

$2Re(C_k)\cos(\omega_k t) + 2\,Im(C_k) j\sin(\omega_k t) = A_k \cos(\omega_k t) + B_k \sin(\omega_k t)$

$A_k = 2 \cdot Re(C_k)$, $B_k = 2j\,Im(C_k)$

$\Rightarrow Re(C_k) = \frac{A_k}{2}$, $Im(C_k) = \frac{B_k}{2j} = -j\frac{B_k}{2}$

$\Rightarrow \boxed{C_k = \frac{A_k}{2} - j\frac{B_k}{2}}$

1-2 a) $C_k = \frac{5-3jk}{1-2k^2}$, $k \neq 0$, and $C_0 = 1$

$$C_k = \frac{A_k}{2} - j\frac{B_k}{2} = \frac{5-3jk}{1-2k^2} \Rightarrow \boxed{\begin{array}{l} \frac{A_k}{2} = \frac{5}{1-2k^2} \Rightarrow A_k = \frac{10}{1-2k^2} \Rightarrow \frac{A_0}{2} = \frac{10}{2} = 5 \\ j\frac{B_k}{2} = \frac{-3jk}{1-2k^2} \Rightarrow B_k = \frac{-6k}{1-2k^2} \end{array}}$$

b) $gc_k = \cos\left(2\pi \frac{k}{T_p} t\right)$, $k = 0, 1, 2, \dots$ and $gs_k = \sin\left(2\pi \frac{k}{T_p} t\right)$, $k = 1, 2, \dots$

$$\frac{2}{T_p}\int_0^{T_p} gc_k(t)\cdot gc_{k_1}(t)\,dt = \frac{2}{T_p}\int_0^{T_p} \cos\left(2\pi\frac{k}{T_p}t\right)\cos\left(2\pi\frac{k_1}{T_p}t\right)dt$$

Let $\omega_k = \frac{2\pi k}{T_p}$, $\omega_{k_1} = \frac{2\pi k_1}{T_p}$

$$= \frac{2}{T_p}\int_0^{T_p}\cos(\omega_k t)\cos(\omega_{k_1} t)\,dt = \frac{2}{T_p}\int_0^{T_p}\frac{\cos(\omega_k t + \omega_{k_1} t) + \cos(\omega_k t - \omega_{k_1} t)}{2}\,dt$$

$$= \frac{1}{T_p}\left[\int_0^{T_p}\cos([\omega_k+\omega_{k_1}]t)\,dt + \int_0^{T_p}\cos([\omega_k-\omega_{k_1}]t)\,dt\right] \boxed{\xRightarrow{\text{if } k=k_1} \frac{1}{T_p}\frac{T_p}{4\pi k_1}\cdot\cancelto{0}{\sin(4\pi k_1)} + 1 = 1}$$

if $k \neq k_1$

$$\boxed{\Rightarrow \frac{1}{T_p}\left(\frac{1}{\omega_k+\omega_{k_1}}\sin\left(\frac{2\pi(k+k_1)}{\cancel{T_p}}\cancel{T_p}\right) + \frac{1}{\omega_k-\omega_{k_1}}\sin\left(\frac{2\pi(k-k_1)\cancel{T_p}}{\cancel{T_p}}\right)\right) = 0 \text{ since } k \text{ is integer}}$$

$$\frac{2}{T_p}\int_0^{T_p} gs_k(t)\cdot gs_{k_1}(t) = \frac{2}{T_p}\int_0^{T_p}\sin(\omega_k t)\sin(\omega_{k_1} t)\,dt$$

$$= \frac{1}{T_p}\int_0^{T_p}\cos((\omega_k-\omega_{k_1})t) - \cos((\omega_k+\omega_{k_1})t)\,dt \xRightarrow{\text{If } k=k_1} 1 - \frac{1}{T_p}\left(\frac{1}{\omega_k+\omega_{k_1}}\underbrace{\cancel{\sin((\omega_k+\omega_{k_1})T_p)}}_{0 \text{ since } k \text{ is integer}}\right) = 1$$

If $k \neq k_1$

$$\Rightarrow \frac{1}{T_p}\left[\frac{1}{\omega_k-\omega_{k_1}}\cancelto{0}{\sin((\omega_k-\omega_{k_1})T_p)} - \frac{1}{\omega_k+\omega_{k_1}}\cancelto{0}{\sin((\omega_k+\omega_{k_1})T_p)}\right] = 0$$

$$\frac{2}{T_p}\int_0^{T_p} gc_k(t)\cdot gs_{k_1}(t)\,dt = \frac{2}{T_p}\int_0^{T_p}\cos(\omega_k t)\sin(\omega_{k_1} t)\,dt$$

$$= \frac{1}{T_p}\int_0^{T_p}\left[\sin((\omega_k+\omega_{k_1})t) - \sin((\omega_k-\omega_{k_1})t)\right]dt \boxed{\xRightarrow{\text{If } k=k_1} \frac{1}{T_p}\cdot\left(\frac{-1}{2\omega_{k_1}}\cos(2\omega_{k_1}T_p) + \frac{1}{2\omega_{k_1}}\right) = 0}$$

If $k \neq k_1$

$$\boxed{\Rightarrow \frac{1}{T_p}\left(\frac{-1}{\omega_k+\omega_{k_1}}\cancelto{1}{\cos((\omega_k+\omega_{k_1})T_p)} + \frac{1}{\omega_k+\omega_{k_1}} + \frac{1}{\omega_k-\omega_{k_1}}\cancelto{1}{\cos((\omega_k-\omega_{k_1})T_p)} - \frac{1}{\omega_k-\omega_{k_1}}\right) = 0}$$

1-2 c) $C_k = \frac{1}{T_p}\int_0^{T_p} x(t) e^{-j\frac{2\pi k}{T_p}t}\,dt$ Let $\omega_k = \frac{2\pi k}{T_p}$

$= \frac{1}{T_p}\int_0^{T_p} x(t)\left(\cos(\omega_k t) - j\sin(\omega_k t)\right)dt$

$C_{-k} = \frac{1}{T_p}\int_0^{T_p} x(t) e^{j\frac{2\pi k}{T_p}t}\,dt =$

$= \frac{1}{T_p}\int_0^{T_p} x(t)\left(\cos(\omega_k t) + j\sin(\omega_k t)\right)dt$

$\Rightarrow C_k = C_{-k}^*$ if $x(t)$ is a real periodic function of time

$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{+j\frac{2\pi k}{T_p}t} = \quad x(-t) = \sum_{k=-\infty}^{\infty} C_k e^{-j\frac{2\pi k}{T_p}t}$

$\sum_{k=-\infty}^{\infty} C_k e^{+j\frac{2\pi k}{T_p}t} = \sum_{k=-\infty}^{\infty} C_k e^{-j\frac{2\pi k}{T_p}t} = \sum_{k=-\infty}^{\infty} C_k^* e^{-j\frac{2\pi k}{T_p}t}$

$\Rightarrow C_k\left(\cos(\omega_k t) + j\cancel{\sin(\omega_k t)}\right) = C_k^*\left(\cos(\omega_k t) - j\cancel{\sin(\omega_k t)}\right)$ 0 $x(t)$ is real

$C_k\cos(\omega_k t) = C_k^*(\omega_k t) \Rightarrow$ only true if C_k has only real parts

1-3 c) $C_k = \frac{A_k}{2} - j\frac{B_k}{2}$ $\quad x(t) = 10\sin(2\pi 14 t) + 6\cos(2\pi 21 t)$

$= 10\sin(2\pi(2\cdot 7)t) + 6\cos(2\pi(3\cdot 7)t)$

$k=3 \Rightarrow A_k = 6, B_k = 0$

$k=2 \Rightarrow B_k = 10, A_k = 0$

$x(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} A_k\cos\left(2\pi\frac{k}{T_p}t\right) + B_k\sin\left(2\pi\frac{k}{T_p}t\right) = 6\cos\left(2\pi\cdot\frac{3}{\frac{1}{7}}t\right) + 10\sin\left(2\pi\cdot\frac{2}{\frac{1}{7}}t\right)$

$A_k = 0$ everywhere except $k=3$

$B_k = 0$ everywhere except $k=2$

$$\Rightarrow \boxed{C_k = \begin{cases} \pm 5j & \text{if } k = \pm 2 \\ 3 & \text{if } k = \pm 3 \\ 0 & \text{elsewhere} \end{cases}}$$

d) The results shown in C_k obtained from my dft model are very different from the C_k obtained from the continuous signal - this is because the sampling frequencies used in part (b) were insufficiently high to avoid aliasing the signal.

1-4) a) The fundamental frequency f_1 of the signal is 100 Hz and the period T_p of the signal is 1/100 s.

b)

k	f_k (Hz)	C_k (volts)
0	0	$-2j$
2	200	$\sqrt{2} - j\sqrt{2}$
3	300	-5
5	500	$7j$

1-4 (c)-(e) on ~~an~~ m file attachment

1-5 a) (i) $x_{am}(t) = [M_0 + m(t)] \cdot x(t)$

$x(t) = A_1 \cos(2\pi f_{x_1} t) + A_3 \cos(2\pi 3 f_{x_1} t)$

$M_0 \neq 0,\ m(t) = M_1 \sin(2\pi f_{m_1} t) + M_2 \sin(2\pi 2 f_{m_1} t), \quad f_{m_1} \ll f_{x_1}$

$$m(t)x(t) = (A_1 \cos(2\pi f_{x_1} t) + A_3 \cos(2\pi 3 f_{x_1} t))(M_1 \sin(2\pi f_{m_1} t) + M_2 \sin(2\pi 2 f_{m_1} t))$$

$$= A_1M_1 \cos(2\pi f_{x_1} t)\sin(2\pi f_{m_1} t) + A_1M_2 \cos(2\pi f_{x_1} t)\sin(2\pi 2 f_{m_1} t) + A_3M_1 \cos(2\pi 3 f_{x_1} t)\sin(2\pi f_{m_1} t)$$
$$+ A_3M_2 \cos(2\pi 3 f_{x_1} t)\sin(2\pi 2 f_{m_1} t)$$

$$m(t)x(t) = \frac{A_1M_1}{2}[\sin(2\pi f_{x_1} t + 2\pi f_{m_1} t) - \sin(2\pi f_{x_1} t - 2\pi f_{m_1} t)]$$

$$+ \frac{A_1M_2}{2}[\sin(2\pi f_{x_1} t + 2\pi 2 f_{m_1} t) - \sin(2\pi f_{x_1} t - 2\pi 2 f_{m_1} t)]$$

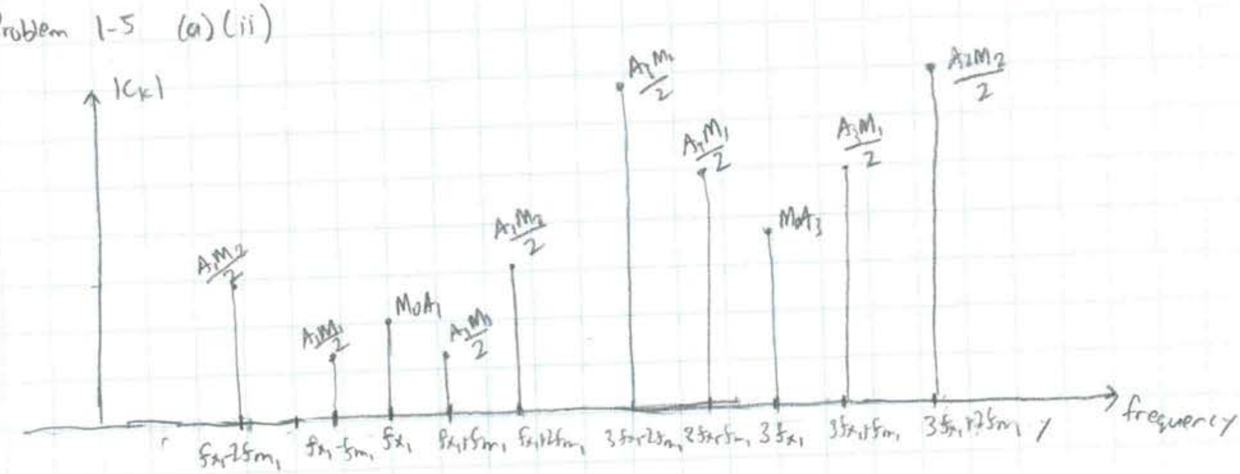
$$+ \frac{A_3M_1}{2}[\sin(2\pi 3 f_{x_1} t + 2\pi f_{m_1} t) - \sin(2\pi 3 f_{x_1} t - 2\pi f_{m_1} t)]$$

$$+ \frac{A_3M_2}{2}[\sin(2\pi 3 f_{x_1} t + 2\pi 2 f_{m_1} t) - \sin(2\pi 3 f_{x_1} t - 2\pi 2 f_{m_1} t)]$$

$$\boxed{x_{am}(t) = M_0(A_1 \cos(2\pi f_{x_1} t) + A_3 \cos(2\pi 3 f_{x_1} t)) + m(t)x(t)}$$

$\boxed{\text{Frequencies present: } f_{x_1},\ 3f_{x_1},\ f_{x_1}+f_{m_1},\ f_{x_1}-f_{m_1},\ f_{x_1} \pm 2f_{m_1},\ 3f_{x_1} \pm f_{m_1},\ 3f_{x_1} \pm 2f_{m_1}}$

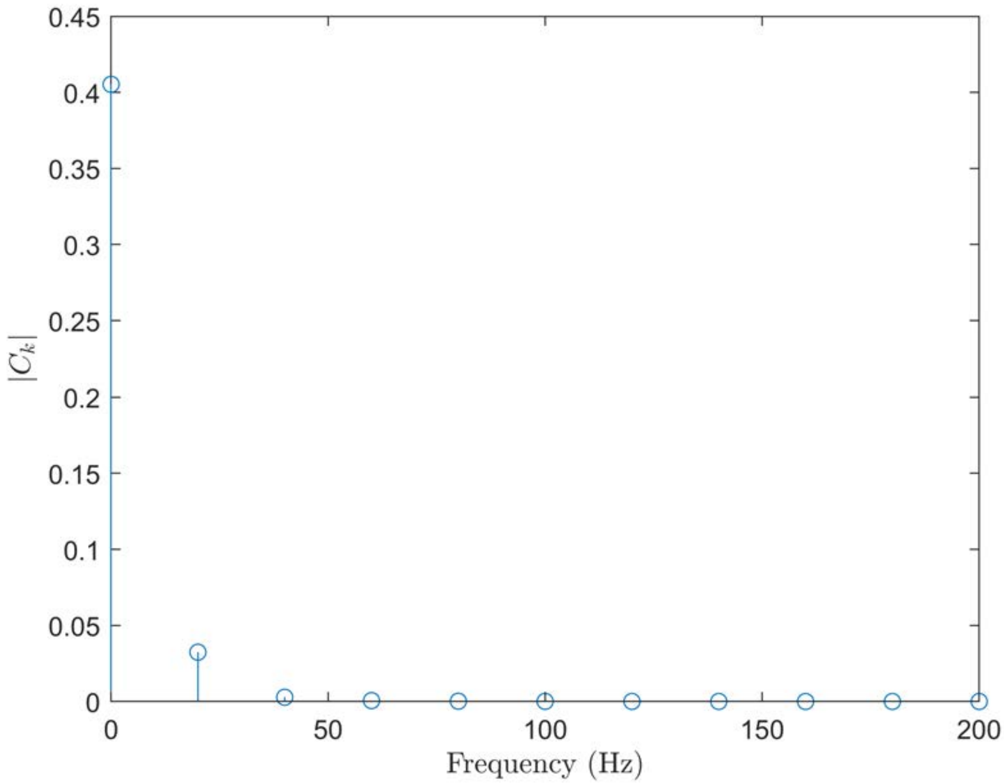
Problem 1-5 (a) (ii)



Plotting: f_{x_1}, $3f_{x_1}$, $f_{x_1} \pm f_{m_1}$, $f_{x_1} \pm 2f_{m_1}$, $3f_{x_1} \pm f_{m_1}$, $3f_{x_1} \pm 2f_{m_1}$

b) x_{cm} will always be periodic if the frequency components in the signal are integer multiples of the fundamental frequency since the terms in x_{cm} will share a greatest common denominator.

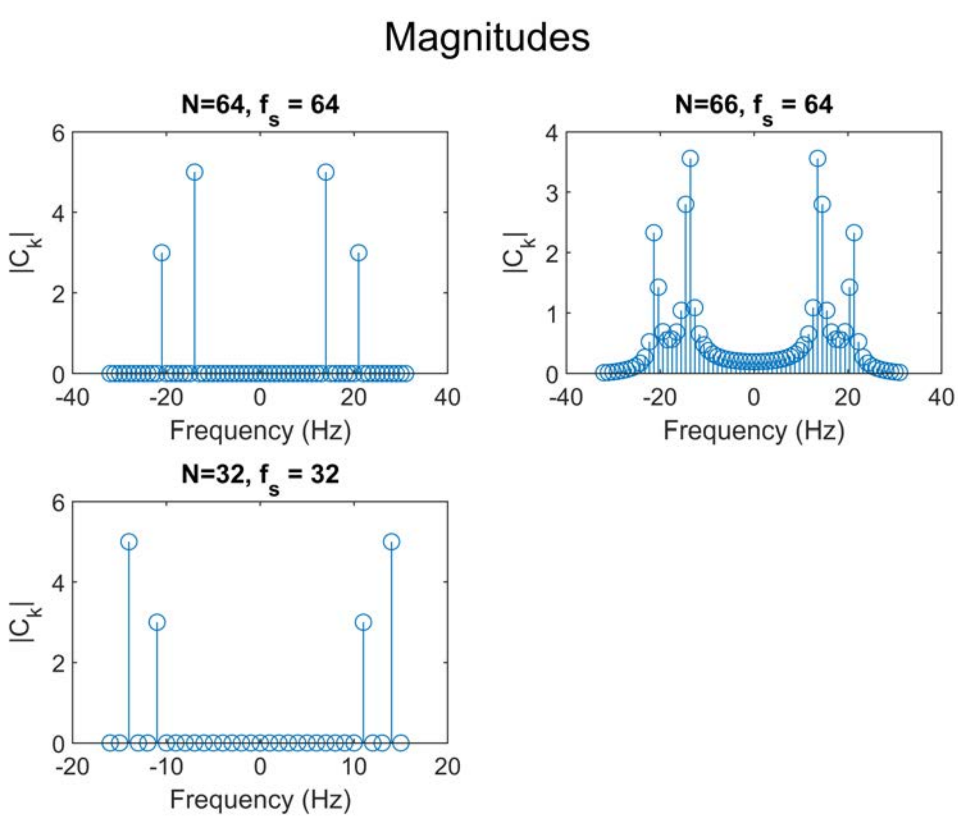
```
clf;
clear;
%% Problem 1-1c
f = 0:20:200; % Natural frequency is 20, so frequency spectrum should be divided into intervals
k = 0:1:(200/20); % Want 11 points from 0 to 200
C_k = ((-1).^(-k) + (-1).^k)./(pi*(1+4*k.^2)); % e^-j*pi*k replaced with (-1)^(-k), e^j*pi*k re
P = [C_k(1)^2 2*C_k(2:end).^2]; % One-sided power spectrum, C_0 not included in the *2 operatio

Figure1 = figure(1);
set(Figure1, 'defaulttextinterpreter', 'latex')
stem(f,P) % Plotting the one-sided power spectrum using formula 2*C_k^2
xlim([f(1), f(end)])
xlabel('Frequency (Hz)')
ylabel('$|C_k|$')
```



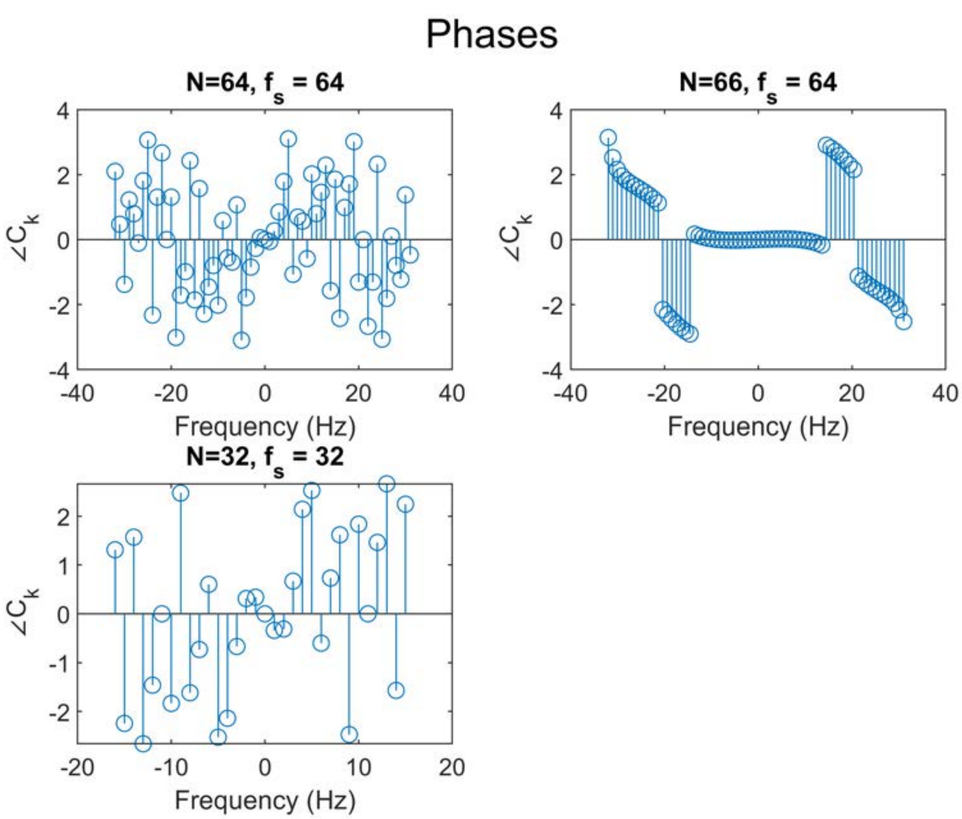
Problem 1-3b

```
clear
t_start = 0;

% Part (i)
N_1 = 64;
f_s1 = 64;
[C_k1, f_k1] = problem_3_program(t_start,f_s1,N_1);

% Part (ii)
```

```
N_2 = 66;
f_s2 = 64;
[C_k2, f_k2] = problem_3_program(t_start,f_s2,N_2);

% Part (iii)
N_3 = 32;
f_s3 = 32;
[C_k3, f_k3] = problem_3_program(t_start, f_s3, N_3);


% Plotting magnitudes
figure(2);
sgtitle('Magnitudes')
subplot(2,2,1)
stem(f_k1, abs(C_k1))
title('N=64, f_s = 64')
xlabel('Frequency (Hz)')
ylabel('|C_k|')
subplot(2,2,2)
stem(f_k2, abs(C_k2))
title('N=66, f_s = 64')
xlabel('Frequency (Hz)')
ylabel('|C_k|')
subplot(2,2,3)
stem(f_k3,abs(C_k3))
title('N=32, f_s = 32')
xlabel('Frequency (Hz)')
ylabel('|C_k|')
```



```
orient TALL

% Plotting Phases
figure(3);
sgtitle('Phases')
subplot(2,2,1)
stem(f_k1, angle(C_k1))
title('N=64, f_s = 64')
xlabel('Frequency (Hz)')
ylabel('\angle{C_k}')
subplot(2,2,2)
stem(f_k2, angle(C_k2))
title('N=66, f_s = 64')
xlabel('Frequency (Hz)')
ylabel('\angle{C_k}')
subplot(2,2,3)
stem(f_k3,angle(C_k3))
title('N=32, f_s = 32')
xlabel('Frequency (Hz)')
ylabel('\angle{C_k}')
```

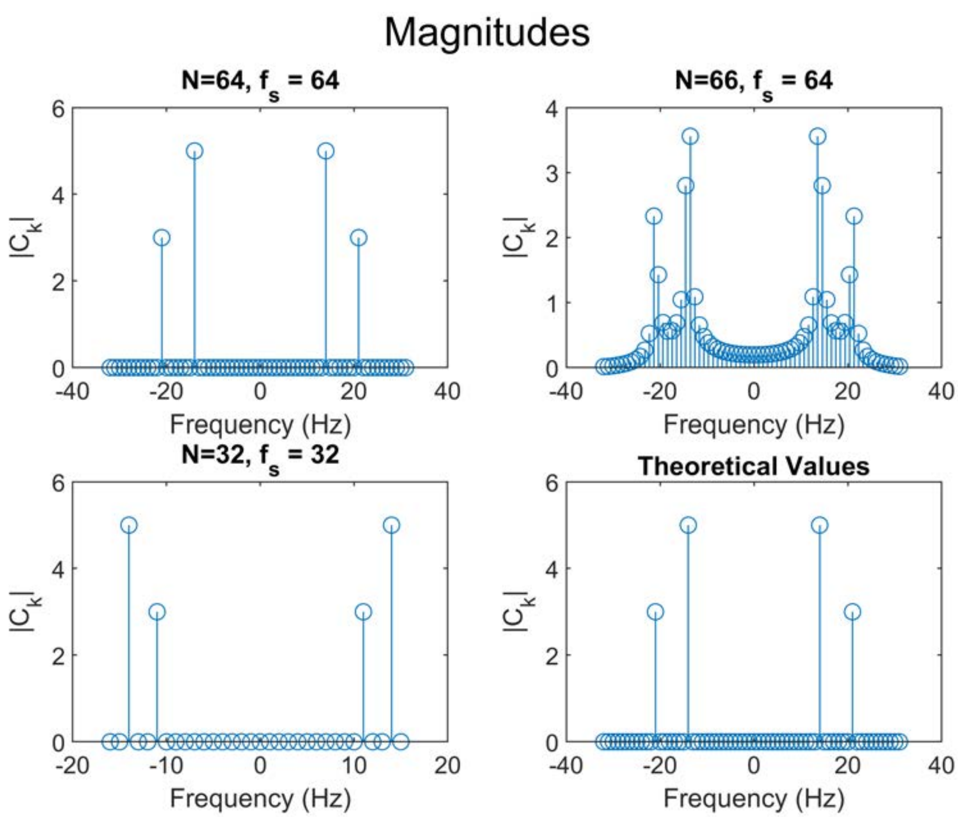



```
orient TALL
```

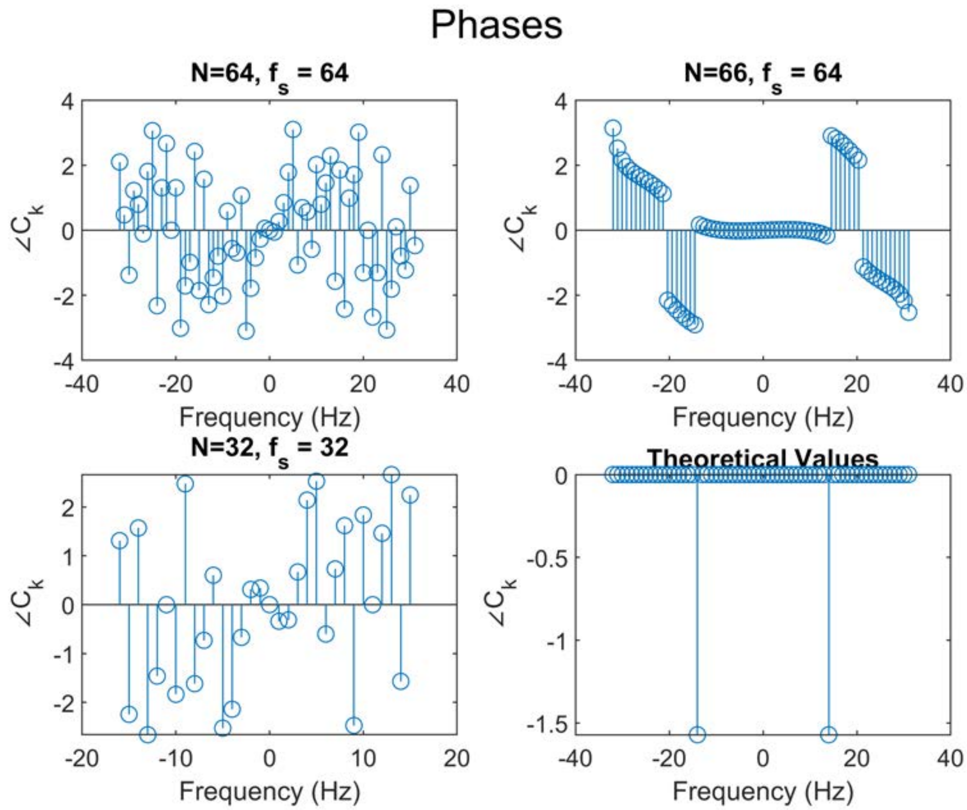
Problem 1-3c

The fundamental frequency of the continuous signal is 7 Hz and the period is 0.1429 second.

```
C_k = zeros(1,length(f_k1));
C_k(f_k1==-21) = 3; C_k(f_k1==21) = 3;
C_k(f_k1==-14) = -5j; C_k(f_k1==14) = -5j;
figure(2);
subplot(2,2,4)
stem(f_k1, abs(C_k))
title('Theoretical Values')
xlabel('Frequency (Hz)')
ylabel('|C_k|')
sgtitle('Magnitudes')
```




```
figure(3);
subplot(2,2,4)
stem(f_k1, angle(C_k))
title('Theoretical Values')
xlabel('Frequency (Hz)')
ylabel('\angle{C_k}')
sgtitle('Phases')
```

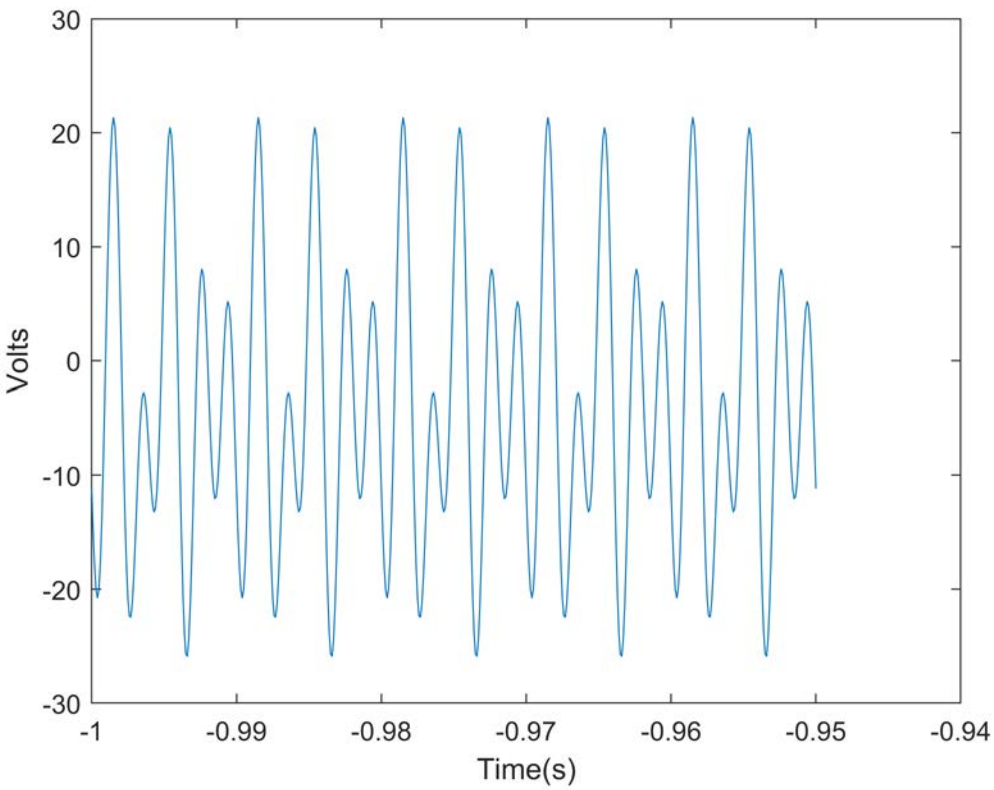


Problem 1-4

```
% Part d
clear;
f_k = [0 100 200 300 400 500];
C_k = [-2 0 sqrt(2)-1j*sqrt(2) -5 0 7j];
f_1 = 100;
num_periods = 5;
t0 = -1;
[t,x] = problem_4c_program(f_k, C_k, t0, f_1, num_periods);
figure;
plot(t,x)
```

```
Warning: Imaginary parts of complex X and/or Y arguments ignored.
```

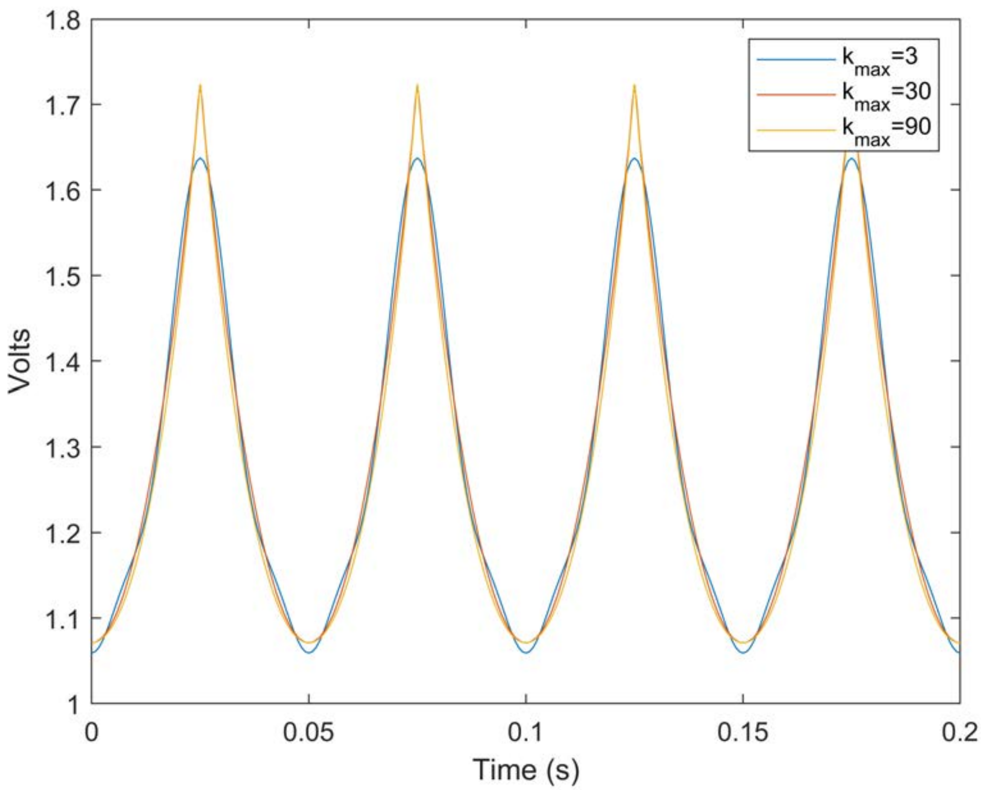
```
xlabel('Time(s)')
ylabel('Volts')
```




```
% Part e
figure;
k1 = 0:3;
k2 = 0:30;
k3 = 0:90;

f_k1 = 20*k1;
f_k2 = 20*k2;
f_k3 = 20*k3;

C_k1 = [exp(-1j*pi.*k1) + exp(1j*pi.*k1)]./(pi*(1+4.*k1.^2));
C_k2 = [exp(-1j*pi.*k2) + exp(1j*pi.*k2)]./(pi*(1+4.*k2.^2));
C_k3 = [exp(-1j*pi.*k3) + exp(1j*pi.*k3)]./(pi*(1+4.*k3.^2));

[t1,x1] = problem_4c_program(f_k1, C_k1, 0, 20, 4);
[t2,x2] = problem_4c_program(f_k2, C_k2, 0, 20, 4);
[t3,x3] = problem_4c_program(f_k3, C_k3, 0, 20, 4);

figure;
plot(t1,abs(x1), t2,abs(x2), t3,x3)
```

```
Warning: Imaginary parts of complex X and/or Y arguments ignored.
```

```
legend('k_{max}=3', 'k_{max}=30', 'k_{max}=90')
xlabel('Time (s)')
ylabel('Volts')
```



The 3 results are similar to each other. However, the results with a higher maximum value of k is less smooth and has greater resolution. These are not close to the true signal at all, likely due incorrect Fourier Series coefficients produced in Problem 1.

```
function [t,x] = problem_4c_program(f_k, C_k, t0, f_1, num_periods)
T_p = 1/f_1;
k = 0:length(f_k)-1;
f_s = 20*max(f_k);
t = t0:1/f_s:t0+num_periods*T_p;
for i = 1:length(t)
    x(i) = sum(2*C_k.*exp(1j*2*pi*k.*t(i)/T_p)); % k indexing starts from 0 in my equation, so
end
end

function [c_k, f_k] = problem_3_program(t_start, f_s, N)
t_end = t_start+N*1/f_s;
t = t_start:1/f_s:t_end-1/f_s;
n = 0:N-1;
x_n = 10*sin(2*pi*14*t)+ 6*cos(2*pi*21*t);
[x_k, f_k] = mydft(x_n, n, N, f_s);
c_k = x_k/N;
end
```