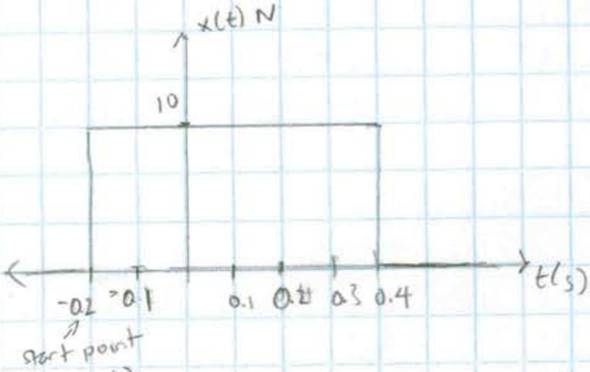
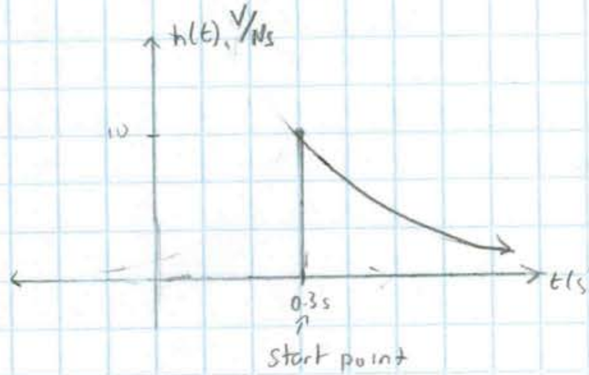
Homework Set 3

Anthony Pham
ME 579

3-1 a) $x(t) = 10\ N$ for $-0.2 \le t \le 0.4\ s$ $\qquad$ $h(t) = 0\ V/(Ns)$ for $t < 0.3\ s$

$x(t) = 0N$ for $t < -0.2$ and $t > 0.4$ $\qquad$ $h(t) = 10e^{-5(t-0.3)}\ V/(Ns)$ for $t \ge 0.3s$





b) Start point of $x(t)$: $t = -0.2s$

Start point of $h(t)$: $t = 0.3s$

$\Rightarrow y(t)$ starts at $x$start $+ h$ start $= -0.2s + 0.3s = \boxed{0.1s}$

c) 

① No overlap: $t < 0.1$

② Partial overlap: $0.1 < t < 0.7$

③ Full overlap over $x(t)$: $t > 0.7$

d) ① $y(t) = 0$

② $y(t) = \int_{-0.2}^{t-0.3} 10 \cdot 10e^{-5(t-\tau-0.3)} d\tau = 100e^{-5(t-0.3)} \int_{-0.2}^{t-0.3} e^{5\tau} d\tau$

$= 20e^{-5(t-0.3)} e^{5\tau}\Big|_{-0.2}^{t-0.3} = 20e^{-5(t-0.3)}\left[e^{5(t-0.3)} - e^{-1}\right]$

$\boxed{= 20 - 20e^{-5(t-0.1)} = y(t) \text{ for } 0.1 < t < 0.7}$

③ $y(t) = \int_{-0.2}^{0.4} 10 \cdot 10e^{-5(t-\tau-0.3)} d\tau = 100e^{-5(t-0.3)} \int_{-0.2}^{0.4} e^{5\tau} d\tau$

$= 20e^{-5(t-0.3)} e^{5\tau}\Big|_{0.2}^{0.4} = 20e^{-5(t-0.3)}\left[e^{-5(-0.4)} - e^{-5(0.2)}\right]$

$\boxed{= 20e^{-5(t-0.7)} - 20e^{-5(t-0.1)} \quad \text{for } t > 0.7}$

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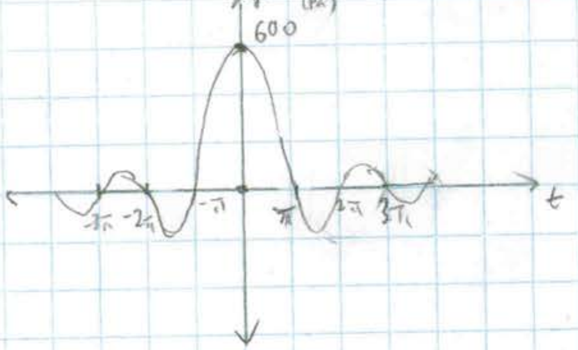
3-2 a) $X_s(f) = \frac{1}{\Delta} \sum_{k=-\infty}^{\infty} X(f - kf_s)$

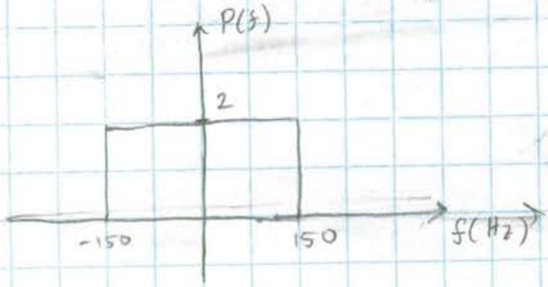
b) $p(t) = 600 \frac{\sin(\pi 300 t)}{\pi 300 t}$



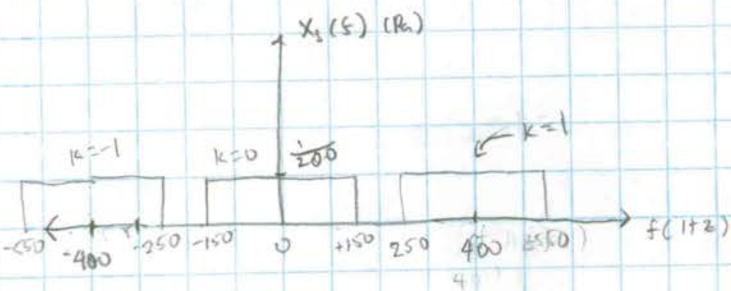
c) $P(f) = \int_{-\infty}^{\infty} p(t) e^{-j2\pi f t}\, dt = \int_{-\infty}^{\infty} 600 \cdot \frac{\sin(\pi 300 t)}{\pi 300 t} e^{-j2\pi f t}\, dt$

Let $T = 300$

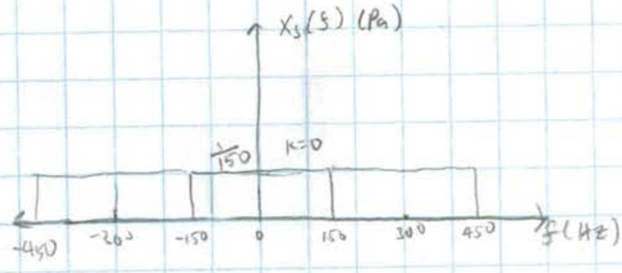
$\Rightarrow P(f) = 2\int_{-\infty}^{\infty} T \operatorname{sinc}(\pi T t) e^{-j2\pi f t} dt = \boxed{\begin{cases} 2 & \text{if } |f| \le 150 \\ 0 & \text{if } |f| \ge 150 \end{cases} = P(f)}$



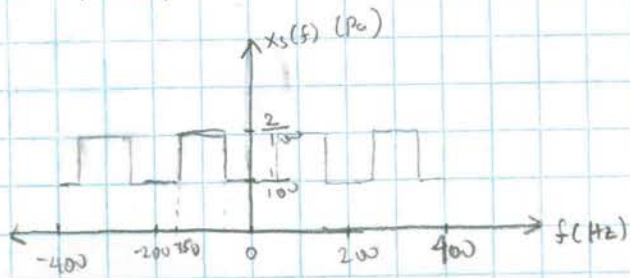
d) 400 samples per second $= f_s \Rightarrow \Delta = \frac{1}{400}$



300 samples per second $= f_s \Rightarrow \Delta = \frac{1}{300}$



200 samples per second $= f_s \Rightarrow \Delta = \frac{1}{100}$



3-2 e) $X_s(f) = \sum_{n=-\infty}^{\infty} x(n\Delta) e^{-j2\pi f n\Delta}$

Zero padding to 2N points:

$\Rightarrow \boxed{X_s(f) = \sum_{n=0}^{N-1} x(n\Delta) e^{-j2\pi f n\Delta}}$

$$X_s(f) = \sum_{n=0}^{2N-1} x(n\Delta) e^{-j2\pi f n\Delta}$$
$$= \sum_{n=0}^{N-1} x(n\Delta) e^{-j2\pi f n\Delta} + \cancel{\sum_{n=N}^{2N-1} x(n\Delta) e^{-j2\pi f n\Delta}}^{\,0}$$

$\Delta = \frac{1}{f_s}$

f) $X_s(k \cdot \frac{f_s}{N}) = \sum_{n=0}^{N-1} x(n\Delta) e^{-j2\pi (k\cdot \frac{f_s}{N}) n\Delta} = \boxed{\sum_{n=0}^{N-1} x(n\Delta) e^{-j2\pi \frac{nk}{N}}}$

g) A DFT ~~of~~ of the $X_s(f)$ in part (e) of an N-point sequence padded to 2N points will have a higher resolution than just the N-point sequence.

h) $x_n = Ae^{-\alpha\Delta n}$ Volts for $n \geq 0$; $= 0$ V for $n < 0$

$$X_s(f) = \sum_{n=0}^{\infty} Ae^{-\alpha\Delta n} e^{-j2\pi f\Delta n} = A\sum_{n=0}^{\infty} e^{-\Delta n(\alpha + j2\pi f)} = \frac{A}{1 - e^{-(\alpha + j2\pi f)\Delta}} \quad \text{if } N \to \infty$$

$1,\ N \to \infty$

$$X_s(f) = \sum_{n=0}^{N-1} Ae^{-\alpha\Delta n} e^{-j2\pi f\Delta n} = A\sum_{n=0}^{N-1} e^{-\Delta n(\alpha + j2\pi f)} = \frac{A(1 - e^{-(\alpha + j2\pi f)\Delta N})}{(1 - e^{-(\alpha + j2\pi f)\Delta})}$$

i) & j) on MATLAB attachment

3-3 a) Hann window: $w(t) = \frac{1}{2}\cos\left(\frac{2\pi}{T}t\right) + \frac{1}{2}$

$$W_{comp} = \frac{1}{T}\int_{-0.5T}^{0.5T} w^2(t)\, dt = \int_{-0.5T}^{0.5T} \frac{1}{4}\cos^2\left(\frac{2\pi}{T}t\right) + \frac{1}{2}\cos\left(\frac{2\pi}{T}t\right) + \frac{1}{4}\ dt$$

$$= \frac{1}{T}\int_{-0.5T}^{0.5T} \frac{1 + \cos\left(\frac{4\pi}{T}t\right)}{8}\, dt + \frac{T}{4\pi}\sin\left(\frac{2\pi}{T}t\right)\Big|_{-0.5T}^{0.5T} + \frac{1}{4}t\Big|_{-0.5T}^{0.5T}$$

$$= \frac{1}{T}\left[\frac{1}{8}t + \frac{T}{32\pi}\sin\left(\frac{4\pi}{T}t\right)\Big|_{-0.5T}^{0.5T} + \frac{T}{4\pi}\left(\cancel{\sin\pi}^{\,0} - \sin(-\pi)\right) + \frac{1}{8}T + \frac{1}{8}T\right]$$

$$= \frac{1}{T}\left[\frac{1}{8}T + \frac{T}{32\pi}\left(\sin 2\pi - \sin(-2\pi)\right) + \frac{1}{4}T\right] = \frac{1}{\cancel{T}}\left(\frac{1}{4} + \frac{1}{8}\right)\cancel{T} = \frac{1}{4} + \frac{1}{8} = \boxed{0.375}$$

$$\frac{1}{T}\int_{-0.5}^{0.5T} 1\, dt = \frac{1}{T}t\Big|_{-0.5T}^{0.5T} = 1$$

3-3 b) $\hat{B}_{xx}(f) = \frac{|X_T(f)|^2}{T \cdot w_{comp}} \approx \boxed{\frac{|\Delta X_k|^2}{N\Delta \cdot w_{comp}}}$

e) Hann window: $\frac{f_s}{N} \leq \frac{|f_a - f_b|}{M_{required}}$

Rectangular window: $\frac{f_s}{N} \leq \frac{|f_a - f_b|}{M_{required}}$

3-4 a) $x(t) = Ae^{-\alpha t}$ for $t \geq 0$; $x(t) = 0$ for $t < 0$

$h(t) = Be^{-\gamma t}$ for $t \geq 0$; $h(t) = 0$ for $t < 0$

$$X(f) = \int_{-\infty}^{\infty} Ae^{-\alpha t} e^{-j2\pi f t}\, dt = A\int_{0}^{\infty} e^{-(\alpha + j2\pi f)t}\, dt = A \cdot -\frac{e^{-(\alpha+j2\pi f)t}}{\alpha + j2\pi f}\Bigg|_0^{\infty} = \frac{A}{\alpha + j2\pi f}$$

$$H(f) = \frac{B}{\gamma + j2\pi f}$$

$\gamma^2 + (j2\pi f)^2 = \gamma^2 + 4\pi^2 f^2 = (\gamma + 4\pi f)(\gamma - 4\pi f)$

$$Y(f) = X(f) \cdot H(f) = \frac{AB}{(\gamma + j2\pi f)(\alpha + j2\pi f)} = \frac{C}{\gamma + j2\pi f} + \frac{D}{\alpha + j2\pi f} = \frac{-\frac{AB}{\gamma - \alpha}}{\gamma + j2\pi f} + \frac{\frac{AB}{\gamma - \alpha}}{\alpha + j2\pi f}$$

$C(\alpha + j2\pi f) + D(\gamma + j2\pi f) = AB$

$C\alpha + Cj2\pi f + D\gamma + Dj2\pi f = AB$

$(C+D)j2\pi f + C\alpha + D\gamma = AB$

$C + D = 0 \Rightarrow C = -D$

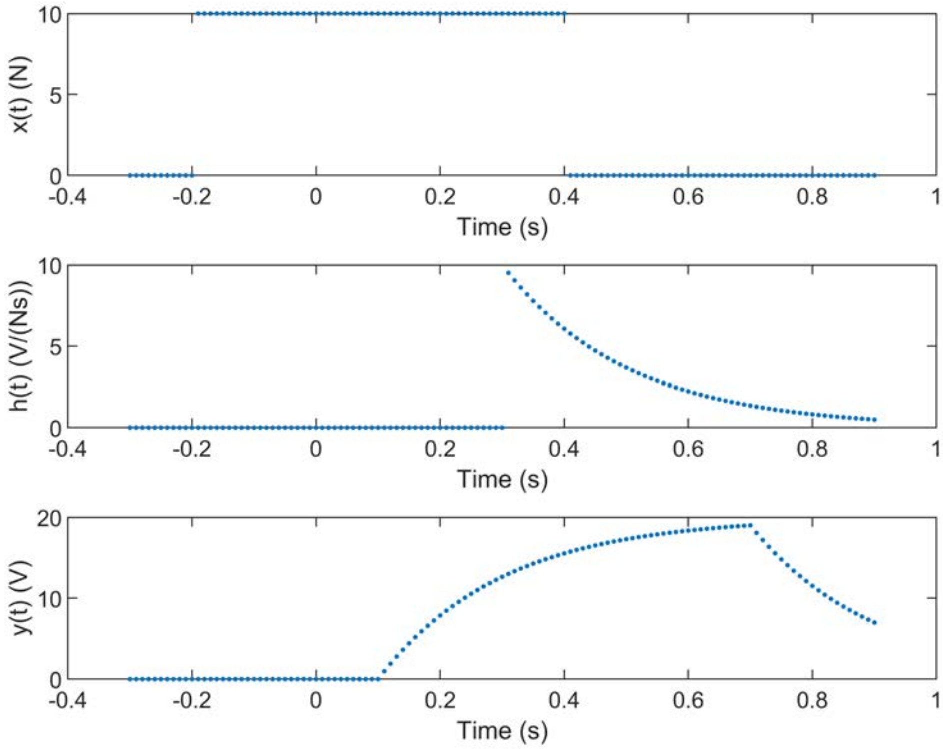
$C\alpha + D\gamma = AB \Rightarrow -D\alpha + D\gamma = AB \Rightarrow D = \frac{AB}{\gamma - \alpha} \Rightarrow C = -\frac{AB}{\gamma - \alpha}$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-\frac{AB}{\gamma-\alpha}}{\gamma + j2\pi f} + \frac{\frac{AB}{\gamma-\alpha}}{\alpha + j2\pi f}\, e^{-j2\pi f t}\, df = \frac{-AB}{2\pi(\gamma - \alpha)} e^{-\gamma t} + \frac{AB}{2\pi(\gamma - \alpha)} e^{-\alpha t}$$

$$= \frac{AB}{2\pi(\gamma - \alpha)}\left(e^{-\alpha t} - e^{-\gamma t}\right); \quad y(t) = 0 \text{ for } t < 0$$

Problem 3-1

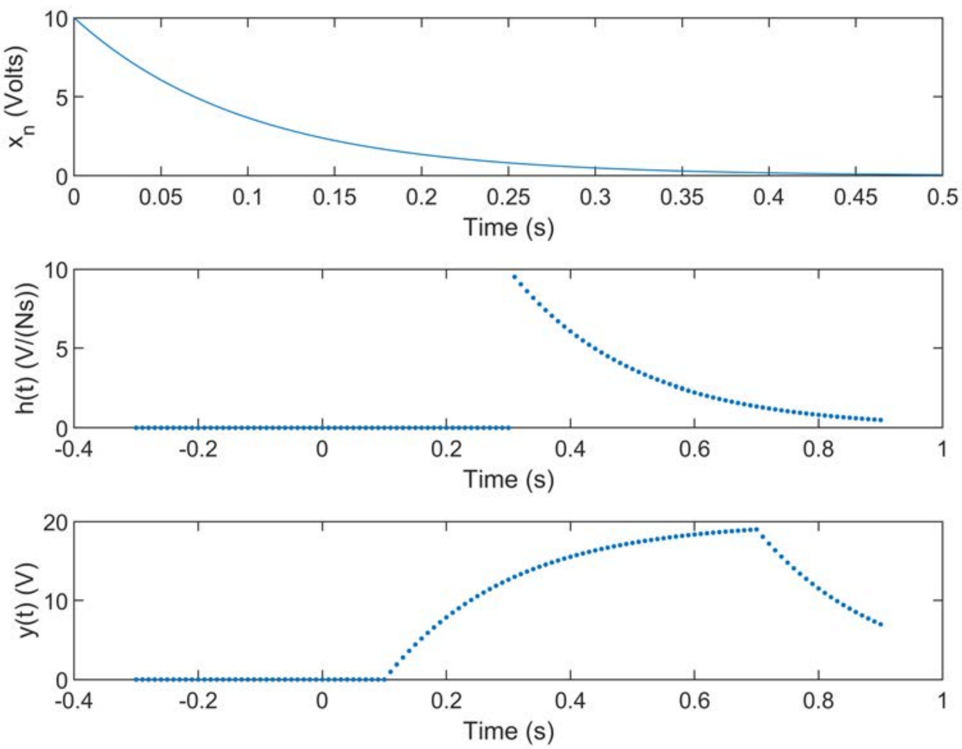
```
% 3-1e
t_start = -0.3;
t_end = 0.9;
delta = 0.01;
t = t_start:delta:t_end;

x_start = -0.2;
x_end = 0.4;

x = [zeros(1, length(t_start:delta:x_start)),...
    repmat(10, [1,length(x_start+delta:delta:x_end)]),...
    zeros(1, length(x_end+delta:delta:t_end))];

h_start = 0.3;
t_h = h_start+delta:delta:t_end;
h = [zeros(1, length(t_start:delta:h_start)),...
    10*exp(-5.*(t_h-0.3))];

y_t1 = 0.1; % First transition point of y
y_t2 = 0.7; % Second transition point of y
t_y1 = t_start:delta:y_t1;
t_y2 = y_t1+delta:delta:y_t2;
t_y3 = y_t2+delta:delta:t_end;

y_1 = zeros(1,length(t_y1));
y_2 = 20 - 20*exp(-5.*(t_y2-0.1));
y_3 = 20*exp(-5.*(t_y3-0.7)) - 20*exp(-5.*(t_y3-0.1));
y = [y_1 y_2 y_3];

figure;
subplot(3,1,1)
plot(t,x,'.');
xlabel('Time (s)')
ylabel('x(t) (N)')

subplot(3,1,2)
plot(t,h,'.');
xlabel('Time (s)')
ylabel('h(t) (V/(Ns))')

subplot(3,1,3)
plot(t,y,'.');
xlabel('Time (s)')
ylabel('y(t) (V)')
```



Problem 3-2

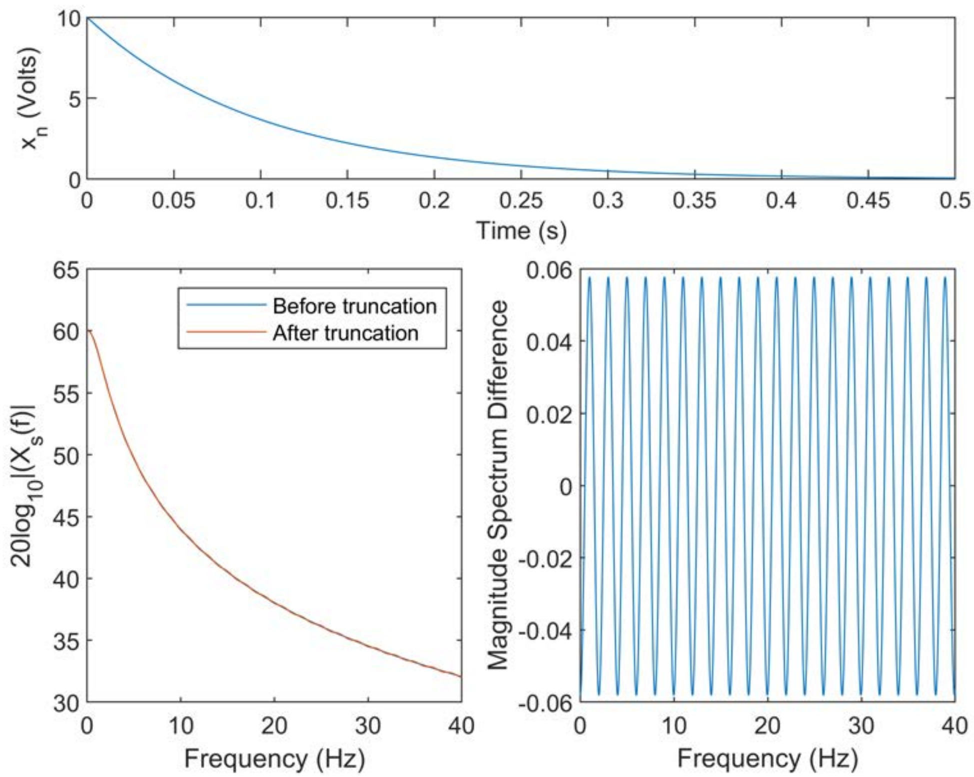
```
% 3-2i
clear

alpha = 10;
A = 10;
df = 0.01;
f_max = 40;
f_s = 1000;
delta = 1/f_s;
f = 0:df:f_max;
t_1 = 0:delta:0.5;
t_2 = 0:delta:0.2;

% Part (i)
n1 = 0:length(t_1)-1;
n2 = 0:length(t_2)-1;
x_n = @(n) A.*exp(-alpha*delta.*n);
x_n1 = x_n(n1);
x_n2 = x_n(n2);

figure(1);
subplot(3,2,[1 2])
plot(t_1, x_n1)
```

```
xlabel('Time (s)')
ylabel('x_n (Volts)')
ylim([0, A])
```

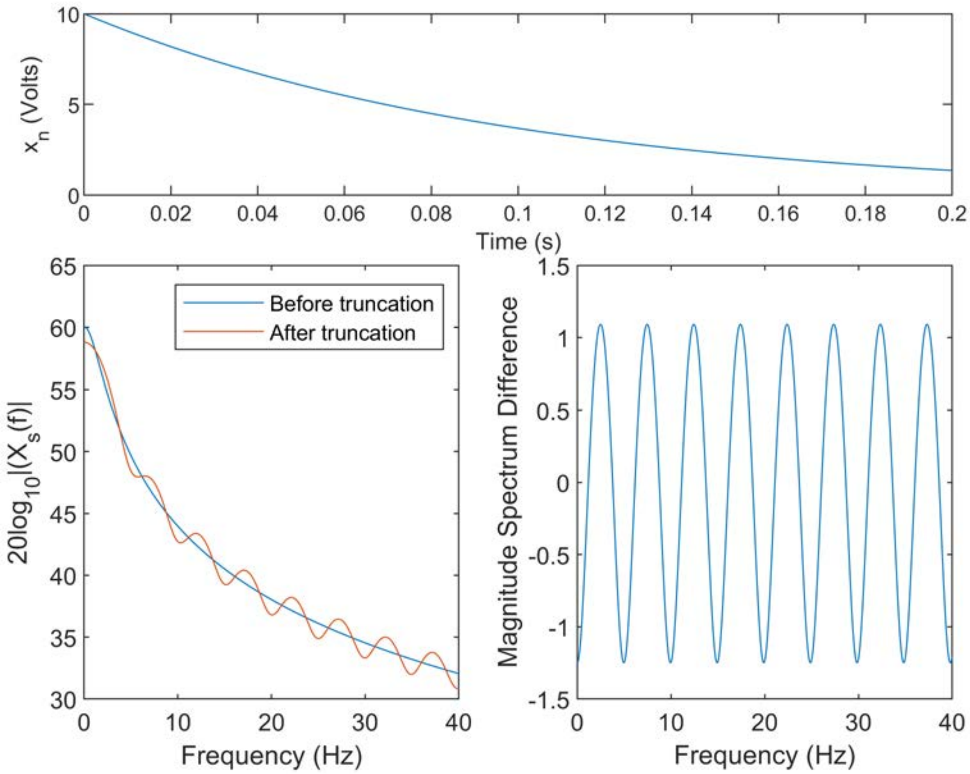



```
figure(2);
subplot(3,2,[1 2])
plot(t_2, x_n2)
xlabel('Time (s)')
ylabel('x_n (Volts)')
ylim([0, A])


% Part (ii)
X_inf = A./(1-exp(-(alpha+1j*2*pi.*f)*delta));
X_N = @(n) A.*(1-exp(-(alpha+1j*2*pi.*f)*delta*length(n)))./(1-exp(-(alpha+1j*2*pi.*f)*delta));
X_N1 = X_N(n1);
X_N2 = X_N(n2);

figure(1)
subplot(3,2,[3 5])
plot(f, 20*log10(abs(X_inf)), f,20*log10(abs(X_N1)));
legend('Before truncation', 'After truncation')
xlabel('Frequency (Hz)')
ylabel('20log_{10}|(X_s(f)|')

figure(2)
subplot(3,2,[3 5])
plot(f, 20*log10(abs(X_inf)),f,20*log10(abs(X_N2)))
```

```
legend('Before truncation', 'After truncation')
xlabel('Frequency (Hz)')
ylabel('20log_{10}|(X_s(f)|')

% Part (iii)

figure(1)
subplot(3,2,[4 6])
plot(f, 20*log10(abs(X_N1))-20*log10(abs(X_inf)));
xlabel('Frequency (Hz)')
ylabel('Magnitude Spectrum Difference')
```




```
figure(2)
subplot(3,2,[4 6])
plot(f, 20*log10(abs(X_N2))-20*log10(abs(X_inf)))
xlabel('Frequency (Hz)')
ylabel('Magnitude Spectrum Difference')
```

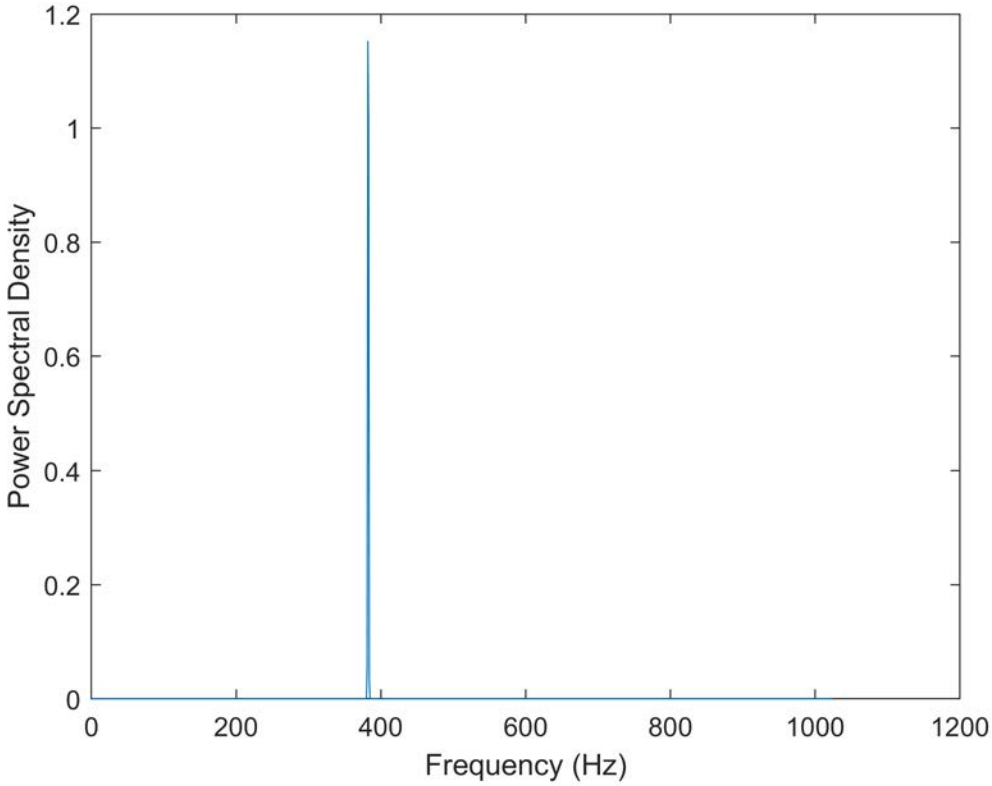


(3-2j) A larger t_0 reduces the difference between the spectrum before truncation and after truncation. Larger sampling rates increase the magnitude of X_s(f).

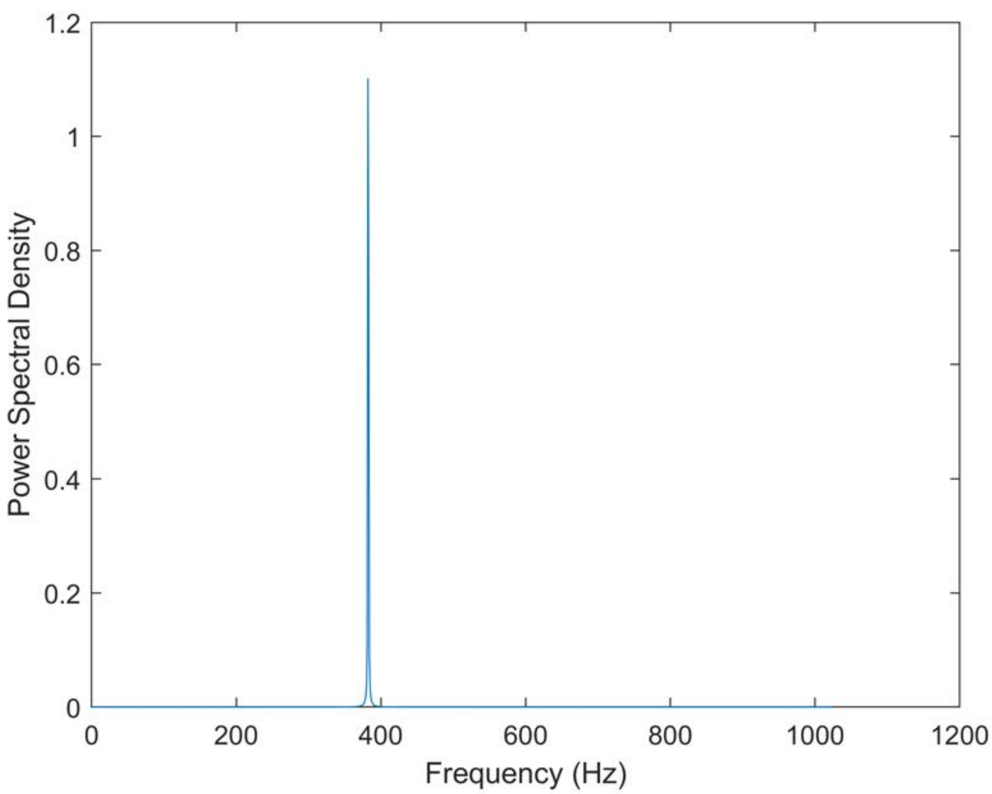
Problem 3-3

Problems 3-3c and 3-3d are in the function section.

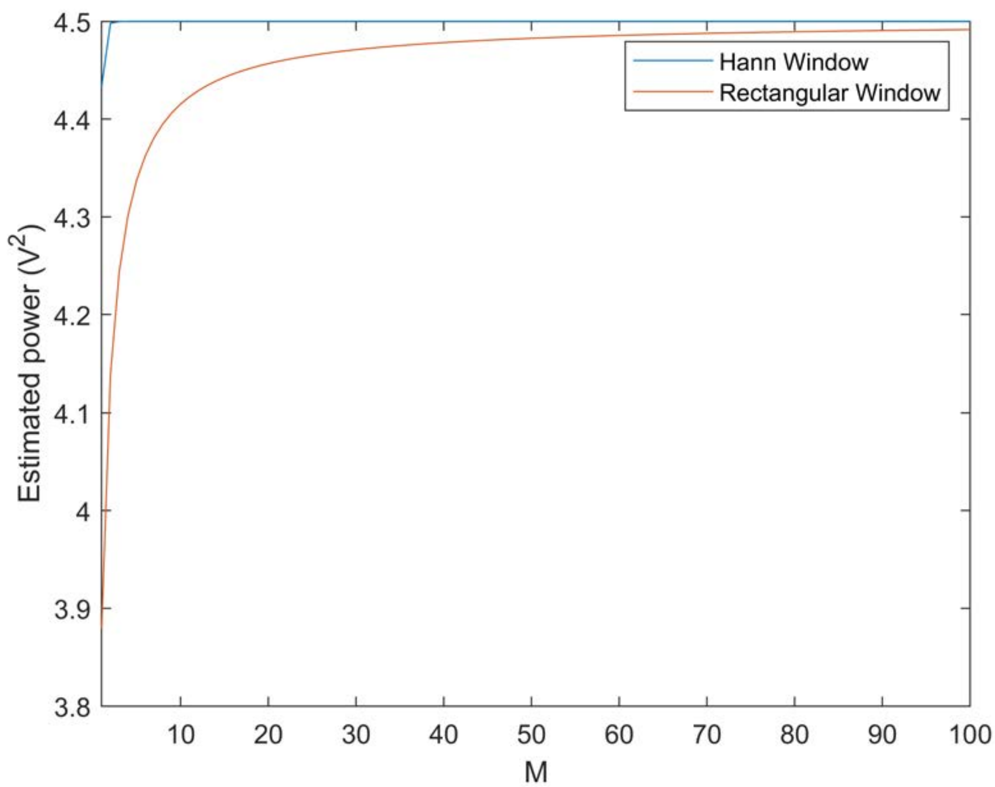
```
% Problem 3-3e
clear;
A = 3;
wave_freq = 2403; % rad/s
f_s = 2048; % samples/s
N = 2048; % sampling for 1s
dt = 1/f_s;

x = A*sin(wave_freq*dt*(0:N-1));

[f_hann, S_hat_hann, ~, ~] = psd_estimator(x, N, 'hann', f_s);
```




```
[f_rect, S_hat_rect, ~, ~] = psd_estimator(x, N, 'rect', f_s);
```



```
M = 1:100;
P_hann = zeros(max(M),1);
P_rect = zeros(max(M),1);

for i = M
    P_hann(i) = power_integrator(f_hann, S_hat_hann, 382, i, N, f_s);
    P_rect(i) = power_integrator(f_rect, S_hat_rect, 382, i, N, f_s);
end

figure;
plot(M,P_hann, M, P_rect)
xlim([1 max(M)])
xlabel('M')
ylabel('Estimated power (V^2)')
legend('Hann Window', 'Rectangular Window')
```

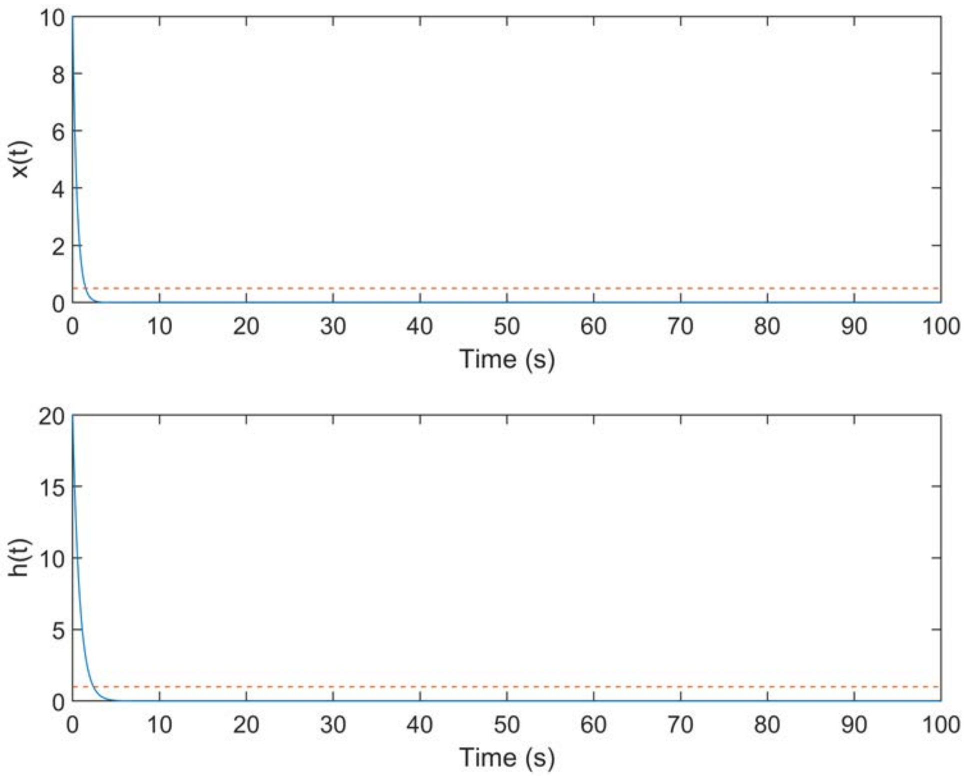


M_required for a Hann window is 2, while M_required for a rectangular window is around 20.

Problem 3-4

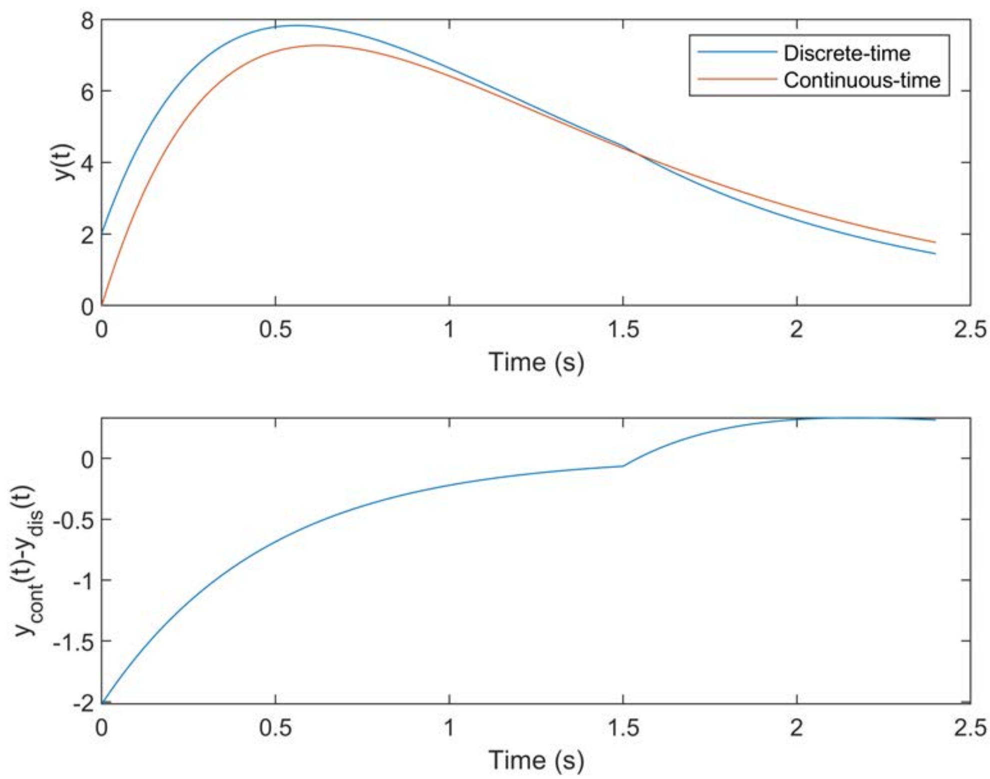
```
% 3-4c
clear;

A = 10;
alpha = 2;
f_s = 50;
B=20;
gamma = 1.25;
```

```
t = 0:1/f_s:100;

exponential = @(A,alpha,t) A*exp(-alpha.*t);

x_n = exponential(A,alpha,t); x_5pct = 0.05*max(x_n);
h_n = exponential(B,gamma,t); h_5pct = 0.05*max(h_n);

figure;
subplot(2,1,1)
plot(t, x_n, t, repmat(x_5pct, [1,length(t)]),'--');
xlabel('Time (s)')
ylabel('x(t)')


subplot(2,1,2)
plot(t, h_n, t, repmat(h_5pct, [1,length(t)]),'--')
xlabel('Time (s)')
ylabel('h(t)')
```



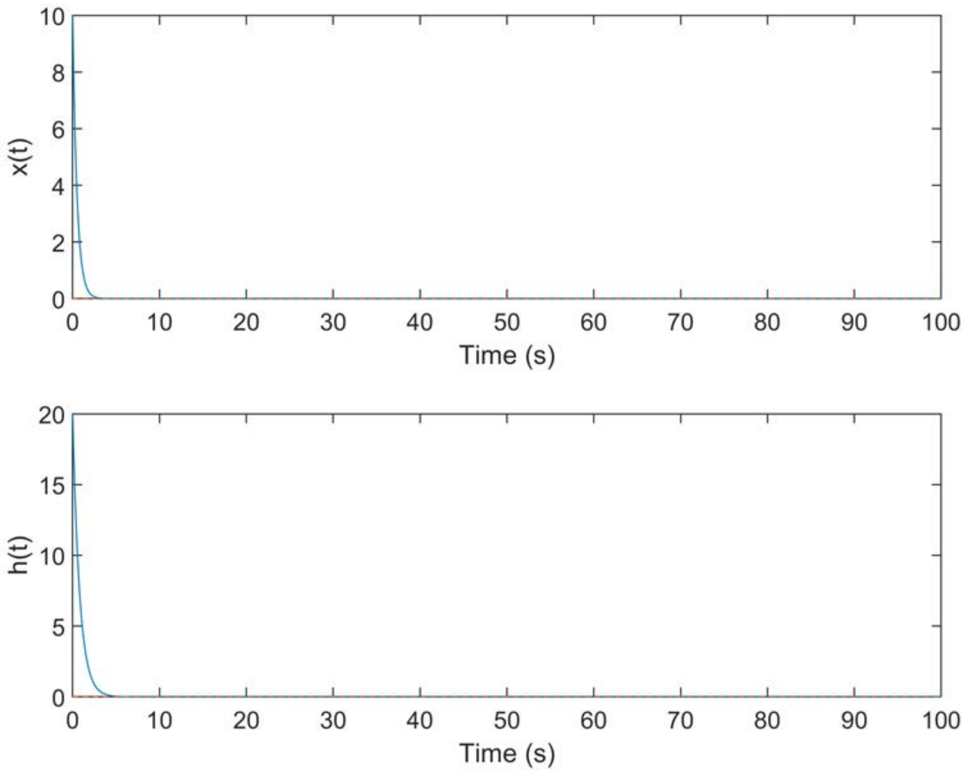

```
% Truncate at t = 1.5s for x, t = 2.4s for h
t_x = 0:1/f_s:1.5;
t_h = 0:1/f_s:2.4;
x_n = exponential(A,alpha,t_x);
h_n = exponential(B,gamma,t_h);

y_dis = convolution(f_s, x_n, h_n);
```

```
y_cont = A*B/(2*pi*(gamma-alpha))*(exp(-alpha.*t_h)-exp(-gamma.*t_h));

figure;
subplot(2,1,1)
plot(t_h, y_dis, t_h, y_cont);
xlabel('Time (s)')
ylabel('y(t)')
legend('Discrete-time', 'Continuous-time')

subplot(2,1,2)
plot(t_h, y_cont-y_dis)
xlabel('Time (s)')
ylabel('y_{cont}(t)-y_{dis}(t)')
```

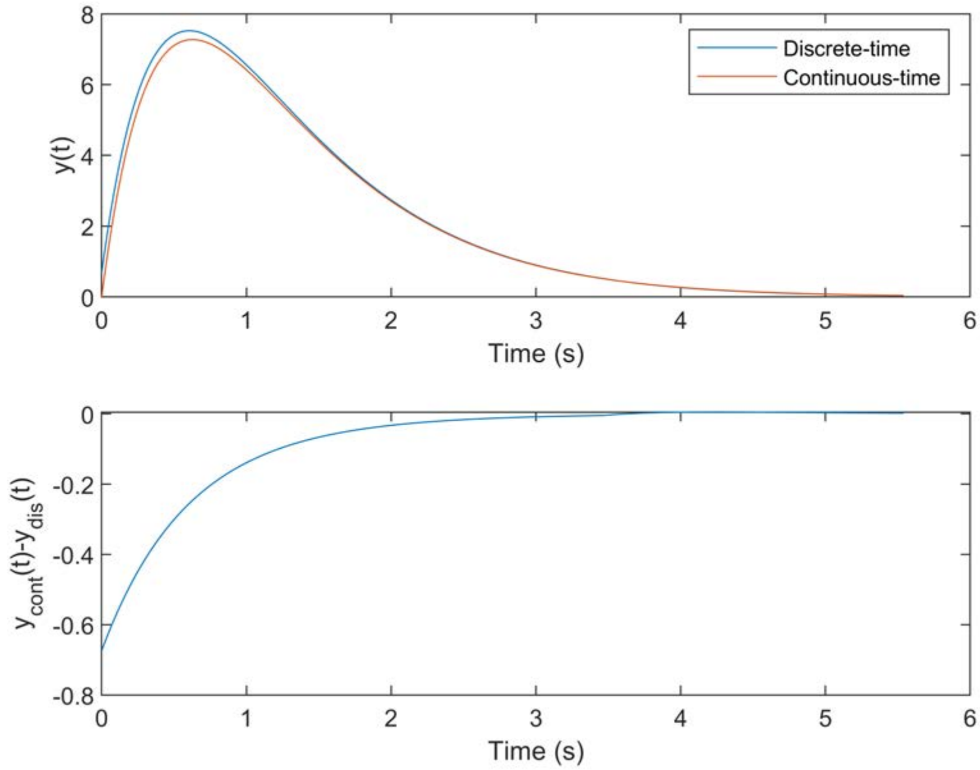



```
% Problem 3-4d
x_n = exponential(A,alpha,t); x_001pct = 0.001*max(x_n);
h_n = exponential(B,gamma,t); h_001pct = 0.001*max(h_n);

figure;
subplot(2,1,1)
plot(t, x_n, t, repmat(x_001pct, [1,length(t)]),'--');
xlabel('Time (s)')
ylabel('x(t)')


subplot(2,1,2)
plot(t, h_n, t, repmat(h_001pct, [1,length(t)]),'--')
xlabel('Time (s)')
```

```
ylabel('h(t)')
```




```
% Truncate at t = 3.46s for x, t = 5.54s for h
t_x = 0:1/f_s:3.46;
t_h = 0:1/f_s:5.54;
x_n = exponential(A,alpha,t_x);
h_n = exponential(B,gamma,t_h);

y_dis = convolution(f_s, x_n, h_n);

y_cont = A*B/(2*pi*(gamma-alpha))*(exp(-alpha.*t_h)-exp(-gamma.*t_h));

figure;
subplot(2,1,1)
plot(t_h, y_dis, t_h, y_cont);
xlabel('Time (s)')
ylabel('y(t)')
legend('Discrete-time', 'Continuous-time')

subplot(2,1,2)
plot(t_h, y_cont-y_dis)
xlabel('Time (s)')
ylabel('y_{cont}(t)-y_{dis}(t)')
```



(3-4e) The difference in the discrete-time and the continuous-time convolution results lies in the window used in sampling. The larger the window, the closer the discrete-time convolution result is to the continuous-time result as seen when comparing parts (c) and (d).

```
function [f, S_hat, N, f_s] = psd_estimator(x, N, window, f_s)

% Problem 3-3c

delta = 1/f_s;
T = N*delta;

% Using only N points in the signal
x = x(1:N);

if strcmpi(window, 'hann')
    w = 0.5*cos(2*pi.*(-N/2:N/2-1)*delta/T)+0.5;
    wcomp = 0.375;
elseif strcmpi(window, 'rect')
    w = ones(1,N);
    wcomp = 1;
end

% Apply window to vector
x_T = x.*w;

% Obtain Fourier transform of vector
```

```
X_k = fft(x_T, N);

f = 0:(f_s/N):0.5*f_s-1;
S_hat = (abs(delta.*X_k).^2./(N*delta*wcomp));
S_hat = S_hat(1:N/2);

figure;
plot(f, S_hat)
xlabel('Frequency (Hz)')
ylabel('Power Spectral Density')

end

function P = power_integrator(f, S_hat, peak_loc, M, N, f_s)
% Problem 3-3d

[~, peak_ind] = min(abs(f-peak_loc));
ind_vector = peak_ind-M:peak_ind+M;

P = 2*sum(S_hat(ind_vector))*f_s/N;

end

% Problem 3-4b
function y = convolution(f_s, x_n, h_n)
N = max(length(x_n), length(h_n));

x_k = fft(x_n,N);
h_k = fft(h_n,N);
y = ifft(x_k.*h_k)/2/pi/f_s;
end
```