Homework Set 2

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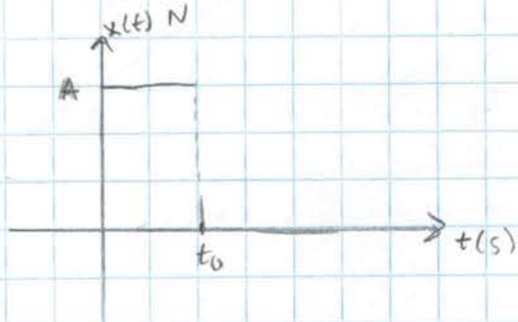
ME 579

2-1) a. $x(t) = \begin{cases} A \text{ Newtons for } 0 \le t \le t_0 \text{ seconds,} \\ 0 \text{ Newtons for } t > t_0 \text{ and } t < 0. \end{cases}$



b. $X(j2\pi f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt = \int_{0}^{t_0} A e^{-j2\pi f t}\, dt = \frac{-Ae^{-j2\pi f t}}{j2\pi f}\Big|_{0}^{t_0} =$

$$= \frac{-A}{j2\pi f} - \frac{Ae^{-j2\pi f t_0}}{j2\pi f} = \boxed{\frac{A(1-e^{-j2\pi f t_0})}{j2\pi f}} = \frac{A(1-\cos(2\pi f t_0) + j\sin(2\pi f t_0))}{j2\pi f}$$

At $f = 0$ Hz, $X(0) = 0$

$$|X(j2\pi f)| = \frac{A\sqrt{1-2\cos(2\pi f t_0)+\cos^2(2\pi f t_0)+\sin^2(2\pi f t_0)}}{2\pi f} = \boxed{\frac{A\sqrt{2-2\cos(2\pi f t_0)}}{2\pi f}}$$

$$\boxed{\angle X(j2\pi f) = \arctan\left(\frac{\sin(2\pi f t_0)}{-\cos(2\pi f t_0)+1}\right) - \frac{\pi}{2}}$$

c-e in MATLAB attachment

2-2) a. $y(t) = x(t) \cdot w(t)$

$$Y(f) = \int_{-\infty}^{\infty} x(t)\, w(t)\, e^{-ft}\, dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(g) e^{gt}\, dg \int_{-\infty}^{\infty} W(g_1) e^{g_1 t}\, dg_1\, e^{-ft}\, dt$$

$$= \int_{-\infty}^{\infty} X(g) \int_{-\infty}^{\infty} W(g_1) \int_{-\infty}^{\infty} e^{-(f-g-g_1)t}\, dt\, dg_1\, dg$$

$$= \int_{-\infty}^{\infty} X(g) \int_{-\infty}^{\infty} W(g_1)\, \underbrace{\delta(f-g-g_1)}_{= 0 \text{ for } g_1 = f-g}\, dg_1\, dg$$

$$= \int_{-\infty}^{\infty} X(g)\, W(f-g)\, dg \equiv X(f) * W(f)$$

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2-2) b. $x(t) \rightarrow X(f) = At_1 \frac{\sin(\pi f t_1)}{\pi f t_1} \Rightarrow x(t) = A \quad |t| < t_1/2$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 0 \quad |t| \ge t_1/2$

$p(t) \rightarrow P(f) = Bt_2 \frac{\sin(\pi f t_2)}{\pi f t_2} \Rightarrow p(t) = B \quad |t| < t_2/2$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad |t| \ge t_2/2$

Inverse Fourier transform of $AT \frac{\sin(\pi f T)}{\pi f T} = \begin{cases} A & |t| < T/2 \\ 0 & |t| \ge T/2 \end{cases}$

Convolution in frequency domain is multiplication in time:

$$X(f) * P(f) = \int_{-\infty}^{\infty} x(t) \cdot p(t) \cdot e^{-j2\pi f t}\, dt$$

$$t_1 > t_2 \rightarrow \; = \int_{-t_2/2}^{t_2/2} A \cdot B\, e^{-j2\pi f t}\, dt = AB \frac{e^{-j2\pi f t}}{-j2\pi f}\Big|_{-t_2/2}^{t_2/2} = \frac{AB(-e^{-j\pi f t_2} + e^{j\pi f t_2})}{j2\pi f}$$

$$\boxed{= AB\, t_2 \left(\frac{\sin(\pi f t_2)}{\pi f t_2} \right)}$$

2-3) a. $p(t) = \sum_{k=-\infty}^{\infty} C_k e^{+j2\pi f_k t}$, $\quad f_k = \frac{k}{T_p}$

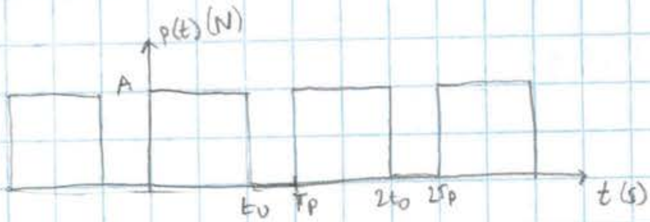
$$P(f) = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} C_k e^{+j2\pi f_k t} e^{-j2\pi f t}\, dt = \int_{\infty}^{\infty} \sum_{k=-\infty}^{\infty} C_k e^{j2\pi (f_k - f) t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} C_k \int_{-\infty}^{\infty} e^{j2\pi (f_k - f) t}\, dt = \boxed{\sum_{k=-\infty}^{\infty} C_k \delta(f_k - f) = P(f)}$$

b. $p(t) = \begin{cases} A \text{ Newtons for } 0 \le t \le t_0 \text{ seconds} \\ 0 \text{ Newtons for } t_0 < t < T_p \end{cases}$, $\quad p(t) = p(t + qT_p)$ where q is any integer

$$C_k = \frac{1}{T_p} \int_0^{T_p} p(t) e^{-j2\pi \frac{k}{T_p} t}\, dt = \frac{1}{T_p} \int_0^{t_0} A e^{-j2\pi \frac{k}{T_p} t}\, dt = \frac{A}{T_p} \frac{-e^{-j2\pi \frac{k}{T_p} t}}{j2\pi \frac{k}{T_p}} \Big|_0^{t_0}$$

$$\boxed{C_k = \frac{A}{T_p} \left(\frac{1 - e^{-j2\pi \frac{k}{T_p} t_0}}{j2\pi \frac{k}{T_p}} \right)}$$

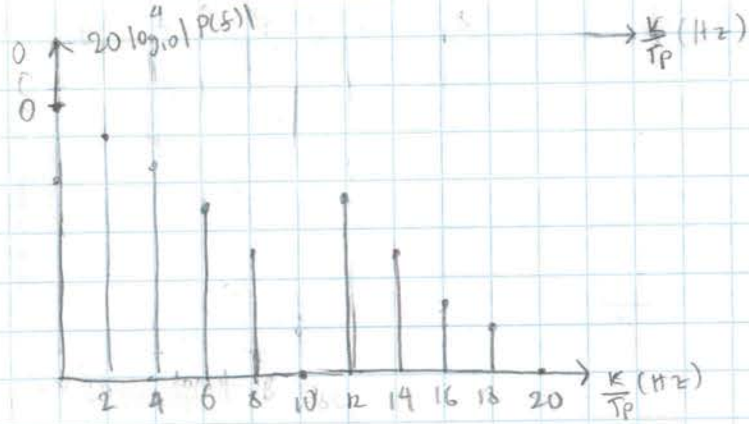


$$\boxed{P(f) = \sum_{k=-\infty}^{\infty} A \left(\frac{1 - e^{-j2\pi \frac{k}{T_p} t_0}}{j2\pi k} \right) \delta(f_k - f)}$$

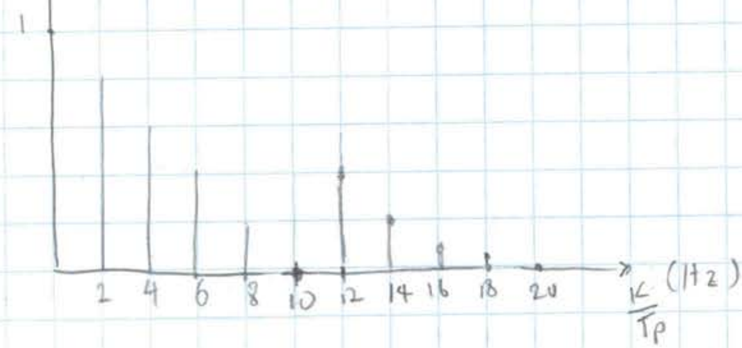
2-3) c. $A = 10\text{ V},\ t_0 = 0.1\text{ s},\ T_p = 0.5\text{ s}$

$$P(f) = \sum_{k=-\infty}^{\infty} 10\text{ V}\left(\frac{1-e^{-j2\pi\frac{k}{0.5}0.1}}{j2\pi k}\right)\delta\left(\frac{k}{0.5}-f\right) \qquad |P(f)| = \frac{10\sqrt{2-2\cos(0.4\pi k)}}{2\pi k}$$

Amplitude Density Spectrum



Amplitude Spectrum

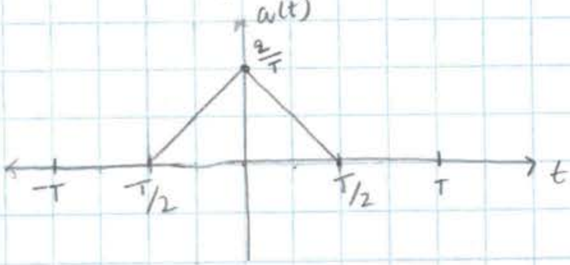


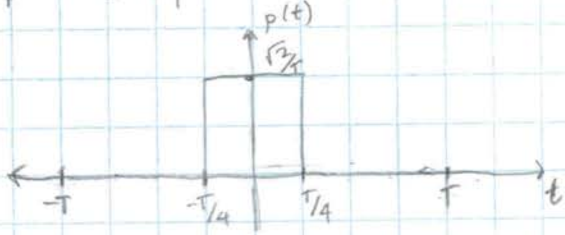
d. The spectrum of P(f) is almost identical to that of X(f), but P(f) is not continuous.

e. P(f) will be spaced out less (by half) in the x axis if T_p is doubled.

2-4) a. $w(t) = p(t) * p(t)$ and $p(t) = \sqrt{\frac{2}{T}}$ for $|t| < \frac{T}{4}$ and 0 elsewhere

$w(t) \triangleq \int_{-\infty}^{\infty} p(\tau)\, p(t-\tau)\, d\tau$





b. $W(f) = P(f) \cdot P(f)$ convolution in time domain is multiplication in frequency

$$P(f) = \int_{-\infty}^{\infty} p(t) e^{-j2\pi f t}\, dt = \int_{-T/4}^{T/4} \sqrt{\frac{2}{T}}\, e^{-j2\pi f t}\, dt = \left. \frac{\sqrt{\frac{2}{T}}\, e^{-j2\pi f t}}{-j2\pi f} \right|_{-T/4}^{T/4}$$

$$= \sqrt{\frac{2}{T}} \left(\frac{e^{j2\pi f T/4} - e^{-j2\pi f T/4}}{j2\pi f} \right) = \sqrt{\frac{2}{T}} \cdot \frac{T}{4}\, \frac{\sin(\pi f \frac{T}{4})}{\pi f \frac{T}{4}} = \sqrt{\frac{T}{8}}\, \frac{\sin(\pi f \frac{T}{4})}{\pi f \frac{T}{4}} = \sqrt{\frac{T}{8}}\, \mathrm{sinc}\left(\pi f \frac{T}{4}\right)$$

$$\boxed{W(f) = \frac{T}{8}\, \mathrm{sinc}^2\left(\pi f \frac{T}{4}\right)}$$

2-5) a. $y(t) = A \sin(2\pi f_c t + \phi(t))$ Volts $\qquad f(t) = f_c + \frac{1}{2\pi}\frac{d\phi}{dt}$ Hz

$\phi(t) = \gamma \frac{f_c}{f_m} \cos(2\pi f_m t)$, $A = 10V$, $f_c = 50Hz$

$\frac{d\phi}{dt} = -\gamma f_c 2\pi \sin(2\pi f_m t)$

For $f_m = 2Hz$ and $\gamma = 0.5$, $f(t) = 50 + \frac{1}{\cancel{2\pi}}(-0.5 \cdot 50 \cdot \cancel{2\pi} \cdot \sin(2\pi \cdot 2 \cdot t))$

$$\boxed{f(t) = 50 - 25 \sin(4\pi t) \text{ Hz}}$$

Highest value of instantaneous frequency is 75 Hz

→ Sampling at 750 Hz

Plots on MATLAB attachment

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2-5) c. $x(t) = 10\sin(at + bt^2)$

$\phi(t) = bt^2$ $\frac{d\phi}{dt} = 2bt$ $a = 2\pi f_c \Rightarrow f_c = \frac{a}{2\pi}$

$f(t) = f_c + \frac{1}{2\pi}2bt = \frac{a}{2\pi} + \frac{1}{2\pi}2bt = \boxed{\frac{a+2bt}{2\pi} = f(t)}$

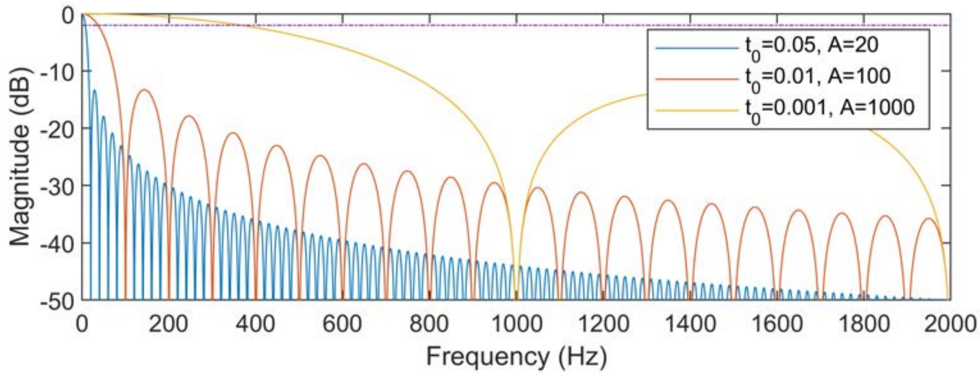
$f(0) = 10 = \frac{a}{2\pi} \Rightarrow a = 20\pi$

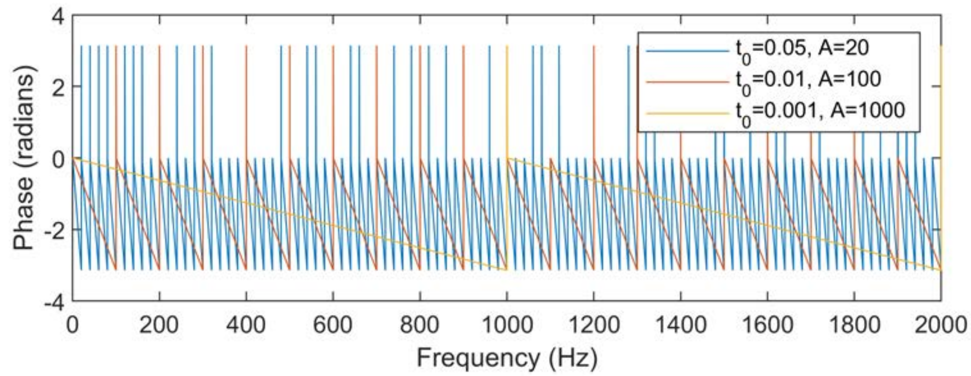
$f(10) = 100 = \frac{20\pi + 2b(10)}{2\pi} \Rightarrow 200\pi - 20\pi = 20b \Rightarrow \boxed{b = \frac{180\pi}{20} = 9\pi}$

Homework 2

Problem 2-1

```
% Setting up parameters
f = 0:0.01:2000; % Small frequency spacing for sufficient resolution on case 1
t0_1 = 0.05; t0_2 = 0.01; t0_3 = 0.001;
A_1 = 20; A_2 = 100; A_3 = 1000;

% Setting up function of X(j2pif):
X = @(t0,A) A*(1-cos(2*pi.*f*t0)+1j*sin(2*pi.*f*t0))./(1j*2*pi.*f);

X_1 = X(t0_1, A_1); mag_X_1 = 20*log10(abs(X_1)); ang_X_1 = angle(X_1);
X_2 = X(t0_2, A_2); mag_X_2 = 20*log10(abs(X_2)); ang_X_2 = angle(X_2);
X_3 = X(t0_3, A_3); mag_X_3 = 20*log10(abs(X_3)); ang_X_3 = angle(X_3);

bandwidth_estimate = abs(X_1(2))-3; % Units in dB
% Can use any equation since X_1(2)=X_2(2)=X_3(2), using index 2 term since f=0 is undefined
bandwidth_estimate = repmat(bandwidth_estimate, [1,length(f)]);

% Magnitude plots
subplot(2,1,1);
plot(f,mag_X_1,f,mag_X_2,f,mag_X_3, f, bandwidth_estimate, '-.');
xlabel('Frequency (Hz)');
ylabel('Magnitude (dB)');
ylim([-50 0]);
legend('t_0=0.05, A=20', 't_0=0.01, A=100', 't_0=0.001, A=1000');

% Phase plots (2-1e)
subplot(2,1,2);
plot(f,ang_X_1,f,ang_X_2,f,ang_X_3);
xlabel('Frequency (Hz)');
ylabel('Phase (radians)');
legend('t_0=0.05, A=20', 't_0=0.01, A=100', 't_0=0.001, A=1000');
```



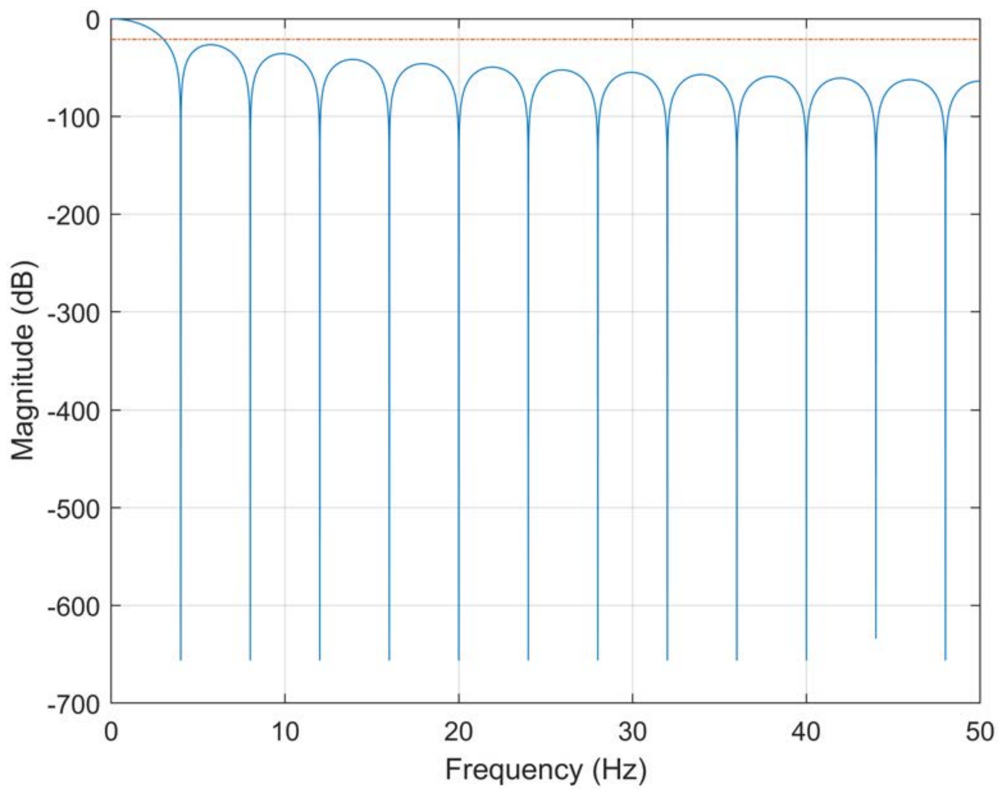


(2-1d) For (i), the frequency is 7.31 Hz. For (ii), the frequency is 36.56 Hz. For (iii), the frequency is 365.4 Hz.

(2-1e) A linear phase component is present due to the arctan(sin/(cos+1)) term. The gradient is different in each case due to the decreasing t_0 term, which decreases the rate at which the 2*pi*f*t_0 term changes. The jumps of pi radians in the phase spectra are caused by the limits of the limits of the arctan component of the phase equation (see 2-1b).

Problem 2-4

```
f = 0:0.01:50;

% Using T = 1
W = (1/8)*(sin(pi.*f*1/4)./(pi.*f*1/4)).^2; % Using sin(x)/x for sinc function
W_0 = 1/8; % sinc(0) = 1

bandwidth_estimate = 20*log10(abs(W_0))-3; % Units in dB
bandwidth_estimate = repmat(bandwidth_estimate, [1,length(f)]);

mag_W_ref = 20*log10(abs(W)/abs(W_0));


figure;
plot(f, mag_W_ref, f, bandwidth_estimate, '-.');
xlabel('Frequency (Hz)');
ylabel('Magnitude (dB)');
grid on;
```



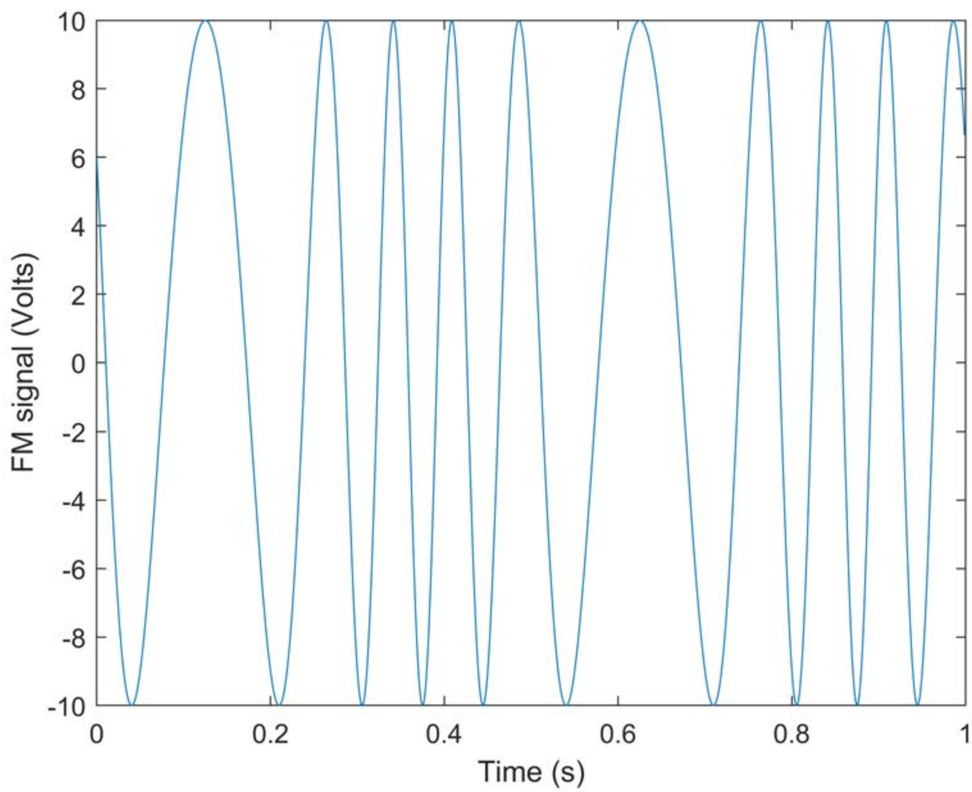
(2-4d) The frequency location of the 3 dB down point is 3 Hz. The amplitude of the highest sidelobe is -26.52 dB and its location is 5.69 Hz. The first zero crossing is located at 4 Hz. The dominant roll-off rate is 11 dB/octave at higher frequencies.

(2-4e) The high frequency roll-off of this window is higher than that of the rectangular window (11 dB for this window vs 6 dB for rectangular) but less than that of the Hann window (11 dB vs 18 dB for the Hann window).

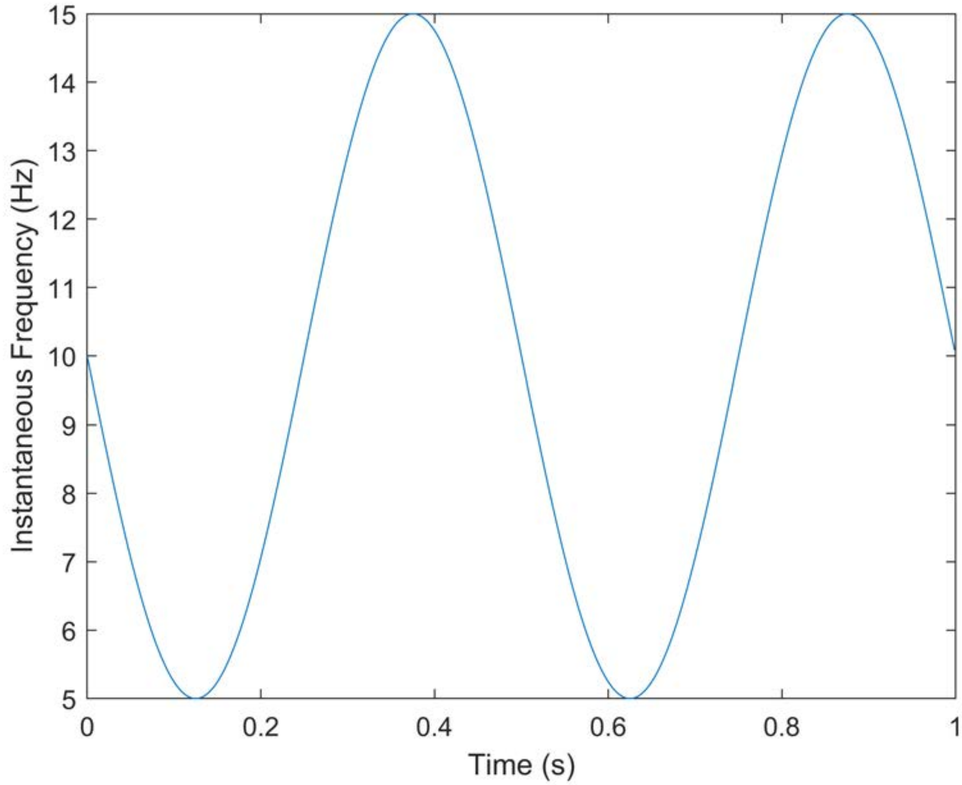
Problem 2-5

```
% Problem 2-5a
fs = 75 * 10; % Sampling frequency is 10 times highest value of instantaneous frequency, 75 Hz
dt = 1/fs;
t = 0:dt:1-dt;



phi = @(t, f_m, gam) (gam*f_c/f_m)*cos(2*pi*f_m.*t);

y = @(t, f_m, gam) A*sin(2*pi*f_c.*t + phi(t, f_m, gam));
f = @(t, f_m, gam) f_c + (1/(2*pi))*(-gam*f_c*2*pi*sin(2*pi*f_m.*t));

A = 10; f_c = 10; f_m = 2; gam = 0.5;

y_a = y(t, f_m, gam);
f_a = f(t, f_m, gam);
```

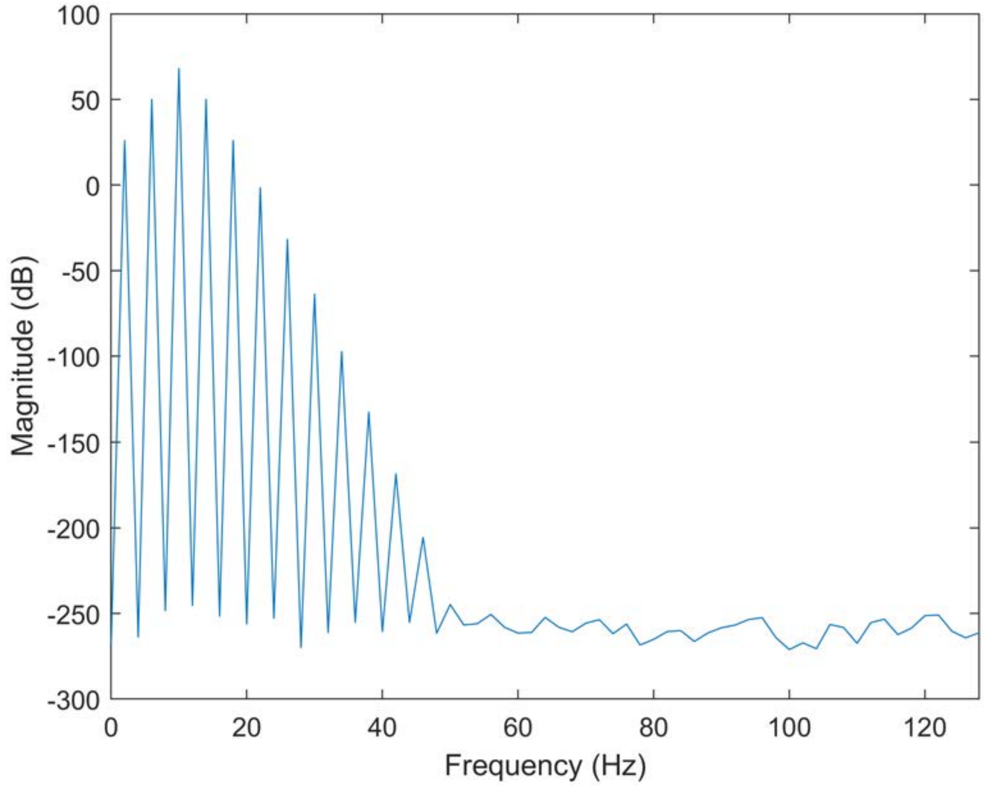
```
figure;
plot(t, y_a);
xlabel('Time (s)');
ylabel('FM signal (Volts)')
```



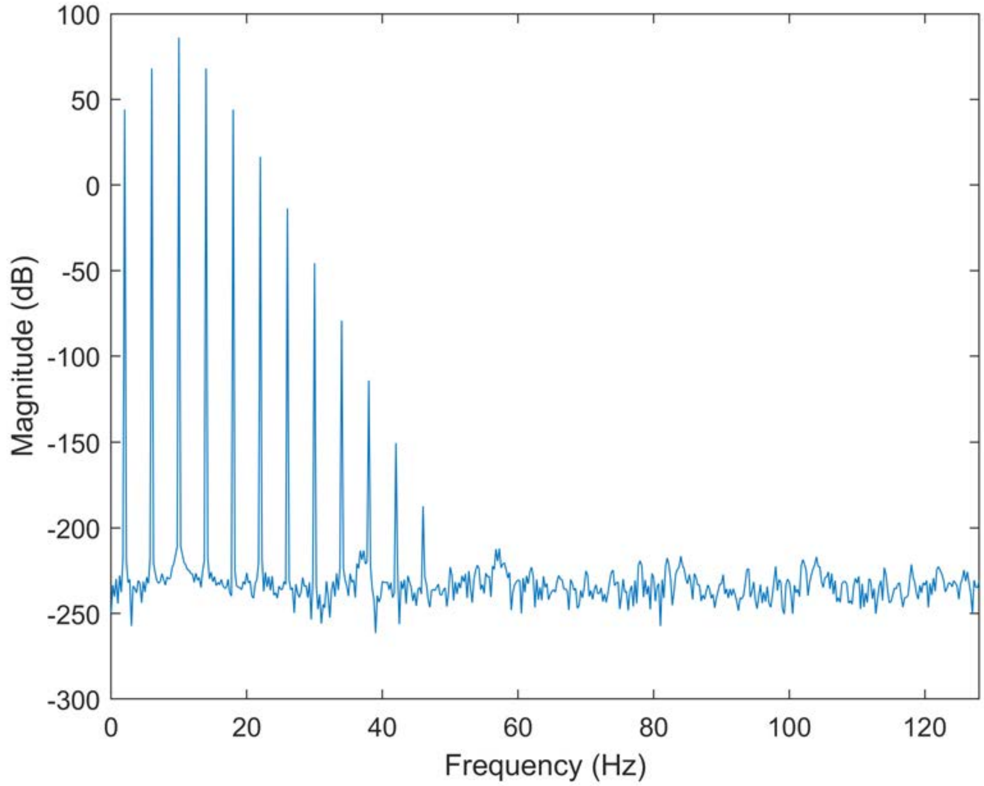

```
figure;
plot(t, f_a);
xlabel('Time (s)');
ylabel('Instantaneous Frequency (Hz)')
```



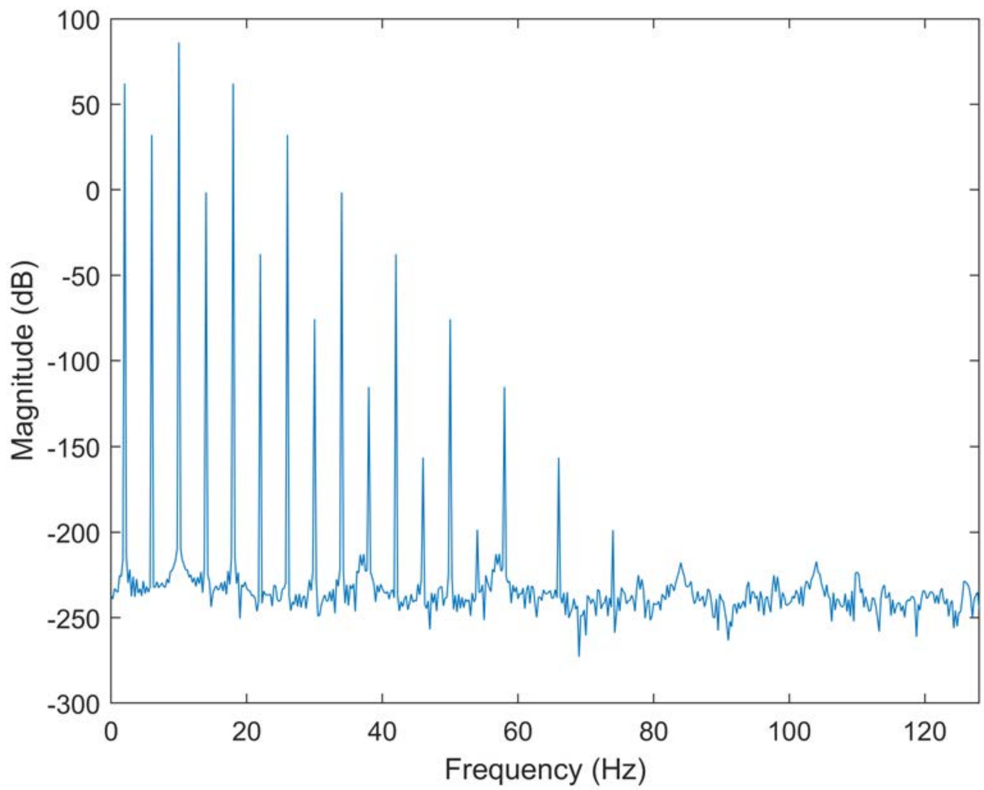

```
% Problem 2-5b
fs = 1024; % 1024 samples a second
dt = 1/fs;

% Part (i)
t_1 = 0:dt:0.5-dt;
f_m = 4; gam = 0.1;

fft_plotter(t_1, f_m, gam, fs);
```



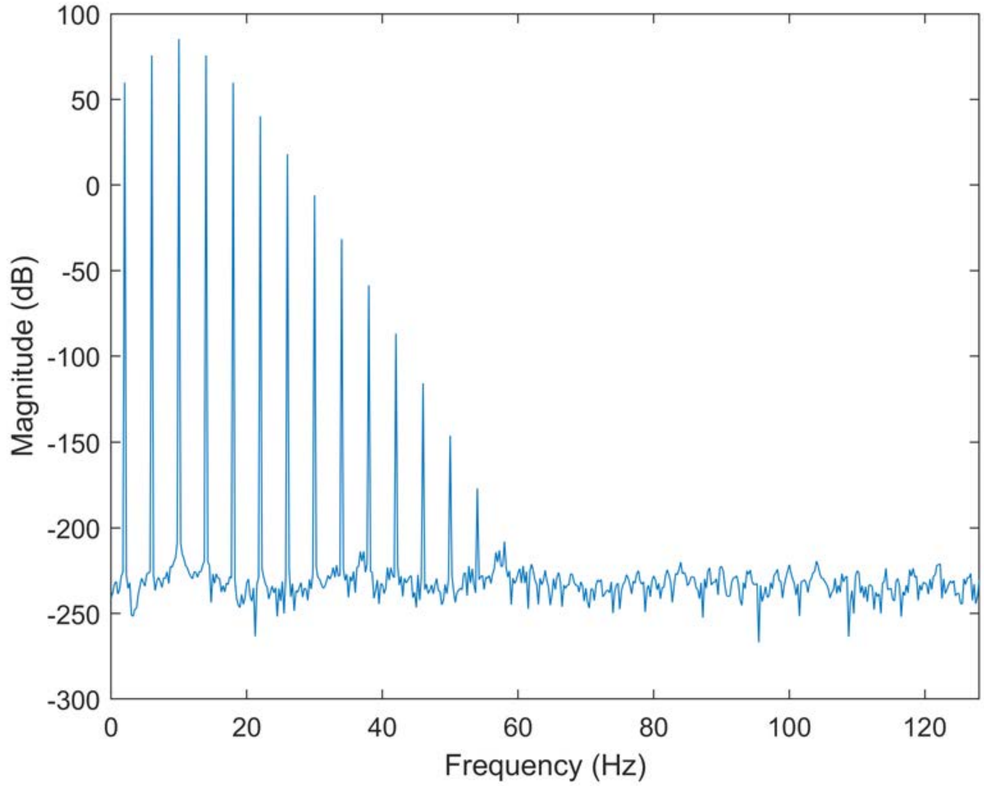

```
t_2 = 0:dt:4-dt; t_3 = t_2; t_4 = t_2;

% Part(ii)
f_m= 4; gam = 0.1;
fft_plotter(t_2, f_m, gam, fs);
```




```
% Part (iii)
f_m = 8; gam = 0.1;
fft_plotter(t_3, f_m, gam, fs);
```




```
% Part(iv)
f_m = 4; gam= 0.25;
fft_plotter(t_4, f_m, gam, fs);
```



A longer signal duration (comparing (i) and (ii)) leads to "sharper" peaks in the amplitude density spectrum. Increasing f_m (comparing (ii) and (iii)) decreases the amplitude density spectrum peak at every other peak. Increasing gamma (comparing (ii) and (iv)) increases the "width" of the spectrum.

```
function [f, P] = fft_plotter(t, f_m, gam, fs)
A = 10; f_c = 10;

phi = @(t, f_m, gam) (gam*f_c/f_m)*cos(2*pi*f_m.*t);
y_func = @(t, f_m, gam) A*sin(2*pi*f_c.*t + phi(t, f_m, gam));

y = y_func(t, f_m, gam);
Y = fft(y);
N = length(Y);
k = 0:(N-1)/2;
f = k.*fs/N;
P = Y(1:N/2);

figure;
plot(f, 20*log10(abs(P)));
xlim([0, 128]);
xlabel('Frequency (Hz)');
ylabel('Magnitude (dB)');
end
```