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Unobserved heterogeneity and the statistical analysis of highway accident data

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ABSTRACT

Highway accidents are complex events that involve a variety of human responses to external stimuli, as well as complex interactions between the vehicle, roadway features/condition, traffic-related factors, and environmental conditions. In addition, there are complexities involved in energy dissipation (once an accident has occurred) that relate to vehicle design, impact angles, the physiological characteristics of involved humans, and other factors. With such a complex process, it is impossible to have access to all of the data that could potentially determine the likelihood of a highway accident or its resulting injury severity. The absence of such important data can potentially present serious specification problems for traditional statistical analyses that can lead to biased and inconsistent parameter estimates, erroneous inferences and erroneous accident predictions. This paper presents a detailed discussion of this problem (typically referred to as unobserved heterogeneity) in the context of accident data and analysis. Various statistical approaches available to address this unobserved heterogeneity are presented along with their strengths and weaknesses. The paper concludes with a summary of the fundamental issues and directions for future methodological work that addresses unobserved heterogeneity.

1. Introduction

Accidents, and specifically highway-vehicle accidents, cost the lives of roughly one and a quarter million people worldwide every year. In addition, highway-traffic injuries are globally the leading cause of death among people 15 to 29 years old with over 300,000 deaths (World Health Organization, 2015). From a policy and engineering perspective, perhaps the most challenging element of these numbers is their persistence and the inability of advanced vehicle safety features, advances in highway design, and various safety-countermeasure policies to drastically lower these numbers.

Without doubt, efforts to improve highway safety are complicated by the behavior of individual vehicle operators which can vary widely across the population and can be inherently difficult to predict and/or modify. This is in contrast to other transportation modes (such as air and water transport) where fewer operators mean the human element can be more

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tightly controlled through licensing standards and other safety protocols. On highways, individual vehicle operators have a wide range of physical and mental abilities, different perceptions of risk, different reactions to external stimuli, and their operating abilities may be further complicated by varying degrees of self-inflicted impaired driving (alcohol and drug consumption). Engineering a safe transportation system with this level of behavioral variance is virtually impossible. This safety problem is one of the leading factors in the current move toward autonomous (connected and automated) vehicles that can remove the human element, potentially leading to huge advances in safety by making safety largely a function of engineered systems (hardware and software) where variance in performance, and ultimately safety, can be more tightly controlled and predicted.

Even after the introduction of autonomous vehicles in mainstream traffic, which unquestionably has the potential to substantially reduce variation in human elements, there will still remain variations in the effects of many other factors that influence the likelihood and resulting injury severity of highway accidents. For example, on any highway in the world, one will find considerable variation in vehicle attributes including mass, occupant protection, safety features, vehicle accident-energy dissipation features, and so on. In addition, there are variations in roadway characteristics such as pavement friction, proximity and types of objects just off the roadway, median design, guardrail design, and other infrastructure-related elements. Finally, there are variations in environmental conditions such as lighting, temperature, and precipitation, all of which will affect both the likelihood and resulting injury-severity of accidents. The entire process is further complicated by the variance in individual vehicle operators' physiologies and responses to vehicle characteristics, roadway characteristics and environmental conditions.

Existing data bases, which typically extract data from police accident reports, local weather stations, and state highway-asset-management databases, contain a wealth of information, especially after an accident has occurred, when injury-severity levels, safety-feature deployment, and many other factors are reported. However, these conventional databases only cover a small fraction of the large number of elements that define human behavior, vehicle and roadway characteristics, traffic characteristics, and environmental conditions that determine the likelihood of an accident and its resulting injury severity. Many other elements remain unobserved to the analyst. For example, weather and lighting conditions change continually over time as do the driver reactions to these conditions. In conventional databases, analysts will not have access to these data. Once an accident has occurred, the characteristics of energy dissipation through the vehicle structure and the resulting effect on individuals, which may vary widely based on which of the vehicle safety features deployed as well as bone mass, overall health, physical dimensions, and so on, will be largely unknown to the analyst.[1]

In light of the inherent deficiencies of current data sources (and likely deficiencies in future data sources), statistical and econometric methods have been developed to address this issue as unobserved heterogeneity (variations in the effect of variables across the sample population that are unknown to the analyst). The intent of these "heterogeneity" models is to allow analysts to make more accurate inferences by explicitly accounting for observation-specific variations in the effects of influential factors (which we will refer to in this paper as unobserved heterogeneity).

Our paper begins with a quick review of the statistical consequences of ignoring unobserved heterogeneity in highway accident data (Section 2). The paper then moves on to a presentation and discussion of various statistical/econometric methods (heterogeneity models) that have been applied in the accident analysis literature to date, including random parameter models (Section 3), latent class models (Section 4), joint latent-class/random-parameters models (Section 5), Markov-switching models (Section 6), unobserved heterogeneity in multivariate models (Section 7), and omitted variable and transferability issues relating to unobserved heterogeneity (Section 8). The paper concludes with a summary and insights for future work (Section 9).

2. The need to account for unobserved heterogeneity

The statistical analysis of accident data typically addresses the likelihood of an accident and its resulting injury severity (see Lord and Mannering, 2010, Savolainen et al., 2011, Mannering and Bhat, 2014 for reviews of studies that have addressed the likelihood and severity of an accident). The likelihood of an accident is often analyzed by considering the number of observed accidents occurring on a defined spatial entity over a specified time period; for example, the number of accidents per month occurring over a specified highway segment (of known distance) or at a highway intersection. Once an accident is observed, the injury severities of involved individuals are often modeled as discrete outcomes (for example, no injury, possible injury, evident injury, disabling injury, fatality).

With commonly collected data, some of the many factors affecting the likelihood of an accident and the resulting injury severity are not likely to be available to the analyst. These factors (which constitute unobserved heterogeneity) can introduce variation in the impact of the effect of observed variables on accident likelihood and injury severity. For example, consider gender as an observed human element that affects injury-severity outcomes. While there are clearly physiological differences between men and women (justifying the use of an indicator variable such as 1 for male and 0 otherwise), there is also great variation across people of the same gender, including differences in height, weight, bone density and other factors

[1] New data sources, such as those from naturalistic driving where many vehicle and human functions are monitored continuously, will help provide additional influential data but will still not approach the detail of data needed to fully model the likelihood and severity of accidents.

Table 1
Examples of explanatory variables with possible heterogeneous effects.

Possible explanatory variable	Why unobserved heterogeneity may make the influence of the variable different from one observation to the next
Human elements (influencing injury severity)[a]	
Gender	While there are clearly physiological differences between men and women (justifying the use of an indicator variable such as 1 for male and 0 otherwise), there is also great variation across people in the same gender including differences in height, weight, bone density and other factors that are generally unavailable to the analyst.
Age	Age is correlated with an individuals' physical characteristics and also with their reaction times, risk-taking behavior, and so on, all of which may influence injury severity. However, age is just a proxy for these factors (which analysts do not observe and often cannot measure) so the effect of age on injury severities may vary among individuals of the same age. Because age is sometimes included as a grouped indicator variable (for example, 1 if age is between 16 and 24 years old and 0 otherwise), the heterogeneous effects may be even more pronounced.
Driver-passenger-related behavior variables	The contributing cause of an accident can be a source of significant heterogeneity. For example, failure to yield can be a cited cause, but the kinematics behind this type of cause can vary greatly from one accident to the next. A behavioral variable that can cause substantial unobserved heterogeneity is the number of occupants in the vehicle. Two accidents with identical numbers of occupants can be influenced by heterogeneity due to the relationship of occupants, their collective perceptions of risk, and how that influences the operation of the vehicle (Xiong and Mannering, 2013, estimate a model that addresses this type of unobserved heterogeneity).
Vehicle characteristics (influencing injury severity)[a]	
Vehicle-type and model year	Vehicle-type indicator variables (such as whether the accident-involved vehicle was a passenger car, truck, sport-utility vehicle, etc.) can be used as a proxy for vehicle mass and other design elements. However, there is great variability within the vehicle-type designations. Model year information can serve as a proxy for the safety features and design standards that may be incorporated in the vehicle, but once again there is great unobserved variability across vehicles in the same model year. In addition, human elements enter the mix since different types of people may choose certain vehicle types and model years. For example, the safest drivers may choose to own newer cars with the latest safety features. So the model-year variable may be capturing both vehicle characteristics and human characteristics, and again there would be unobserved heterogeneity in both of these across observations.[b]
Safety-feature indicators	Variables indicating whether air-bags deployed, safety belts were used by occupants, and so on may also be used to explain injury-severity levels. However, these features may have different levels of effectiveness based on the physical characteristics of the occupants (height, weight, health conditions) which may not be known to the analyst (these are not available in typical databases). This would introduce considerable variability in the effectiveness of these features across observations. Human elements may also enter the influence of these variables since the most cautious drivers are more likely to own cars with airbags and use safety restraints. There is also the fact that airbags deploy when high decelerations are detected. Thus airbag deployment is only going to occur in accidents requiring significant energy dissipation (which is unobserved in typical databases) further adding to the heterogeneous effects of airbag deployment across observations.[b]
Roadway characteristics (influencing the likelihood of accident and injury severity)	
Median barrier presence and type indicators and median width	From an accident injury-severity perspective, the presence of a median barrier and/or the type of median will have considerably different effects depending on angle of impact, type of vehicle, speed at impact, and other factors that are not likely to be known to the analyst. In addition, the effects of median width may also vary depending on the soil types and conditions at the time of the accident which may affect vehicle speeds and vehicle recovery probabilities.[c] The roadside variables in particular are usually included in statistical models in the form of indicator variables, while in reality, they are usually active over only some portion of the total highway-segment length. This can result in a continuous measure that can have different slopes for variables that are active over different portions of the highway-segment length. However, accounting for this in a continuous manner alone may not quantify the unobserved heterogeneity completely. For example, the location by length of median barrier will also matter. This type of information is not easy to assemble from conventional databases. Roadside measurements in particular are viewed with high importance in modern safety databases, but their potential for capturing unobserved heterogeneity has not been fully explored.[d]
Shoulder and lane widths	The impact that varying lane and shoulder widths may have on the likelihood of an accident and its resulting injury severity could vary widely from one roadway segment to the next and be a function of time-varying traffic and weather conditions, as well as driver reactions to variations in these widths, which may not be known to the analyst. This in turn may cause the influence of these widths to vary across observations.

Table 1 (*continued*)

Possible explanatory variable	Why unobserved heterogeneity may make the influence of the variable different from one observation to the next
Horizontal and vertical curves and their characteristics	The presence of horizontal and vertical curves and their characteristics (such as radius, length and so on) may have heterogeneous effects across observations due to unobserved time-varying traffic and weather conditions as well as heterogeneous reactions of drivers to these curves. Such highway alignment variables have been found to be random parameters in recent studies.[e]
Number of ramps in a roadway segment	The number of ramps can be used as a proxy for potential disruptions in the traffic stream caused by merging and diverging vehicles. The effect of an individual ramp is likely to be a function of a variety of traffic and environmental conditions that are unknown to the analyst, thus causing heterogeneous effects across observations. Some studies have examined the effect of traffic disruptions by evaluating the heterogeneity of interchange and overpass parameters.[e]
Pavement measurements (friction measurements, pavement conditions)	A wide variety of physical pavement measurements may be available. However, there is a possibility of a variation in the effects of these variables across the population for a variety of reasons. For example, a poor pavement surface may cause a heterogeneous response in driver behavior (some drivers may slow down and other drivers may not) and this response (unobserved by the analyst) may vary from one roadway segment to the next.
Traffic characteristics (influencing the likelihood of accident)	
Traffic volume	Traffic volume is a variable common to many models, particularly those attempting to model the likelihood of an accident. However, the effect of volume on accident likelihood can be influenced be a variety of time-varying environmental characteristics as well variations in driver behavior in response to traffic. If the unobserved heterogeneity associated with these variations is ignored it is possible the analyst could conclude that the relationship between traffic volume and accident likelihood is non-linear, but this may not be the case if unobserved heterogeneity were properly taken into account.
Traffic vehicle mix	The mix of trucks and passenger cars in the vehicle stream has been found in many studies to influence the likelihood of an accident. Still, there is a possibility of heterogeneous effects across observations in time-varying environmental characteristics and variations in human responses (in response to environmental variations or variations in vehicle mixes), which will not be observed by the analyst.
Speed-related measurements	The speed limit is often used as an explanatory variable in accident-likelihood analyses. This measure is the posted speed limit on a particular highway segment, or the speed limit on adjacent segments if not immediately available for the highway segment in question. However, use of adjacent-segment information creates potential for unobserved heterogeneity, especially in speed-limit transition zones. To avoid this, some studies have used the highway's design speed, but this too can introduce unobserved heterogeneity since actual vehicle operating speeds may differ substantially on highways with the same design speed.
Naturalistic driving data	Naturalistic driving data use in-vehicle instrumentation to more precisely capture information relating to the likelihood of an accident. Instrumented vehicles collect vehicle kinematics data such as lane offsetting, pitch, yaw, and roll measurements at certain frequencies. Matched with roadway-geometric data, one would expect that with these measurements, the unobserved heterogeneity effects would be minimized. However, these vehicle-specific measurements can vary considerably from one vehicle to the next and thus the effect of vehicle-specific information can be heterogeneous across the vehicle population.[f]
Environmental characteristics (influencing the likelihood of accident and injury severity)	
Time of day, raining or snowing, lighting conditions	A wide variety of environmental have been found to affect both the likelihood of an accident and the resulting injury severity once an accident has occurred. However, there is great potential for heterogeneous effects of such environmental conditions. For example, snow accumulations of the same amount may have much different impacts in different geographic areas due to different driver responses to adverse weather (the degree to which they adjust their speeds in response to inclement conditions) and this will affect individual injury severities as well. All of these are likely to be different in individual vehicle accidents too, resulting in unobserved factors influencing the resulting injury severity in accidents.

[a] Typically observable only after an accident has occurred (these data are usually gathered from police accident reports).

[b] As discussion indicates, there are also potential issue of endogeneity here. Please see Mannering and Bhat (2014) for a discussion of this point.

[c] In addition, from an accident frequency perspective, some caution should be exercised since median barriers, for example, may be more likely to be placed at high-accident locations.

[d] Some work on using the actual length of features in roadway segments (instead of indicator variables) has been published in Venkataraman et al. (2011, 2013). The studies show that the use of these types of variables improves the statistical fit compared to the use of indicator variables, but that parameter heterogeneity does not necessarily disappear.

[e] Venkataraman et al. (2011, 2013, 2014) address alignment heterogeneity in a multitude of contexts including non-interchange and interchange segments in urban and rural contexts.

[f] Studies by Shankar et al. (2008) and Jovanis et al. (2011) indicate the limited utility of the vehicle kinematics measurements in the development of rich accident prediction models. The conclusion of these studies is that vehicle kinematics measurements contribute to a high degree of heterogeneity and maybe not be reliable in their raw form for the estimation of accident likelihood models.

that are generally unavailable to the analyst (and are not controlled for, even if other observed variables are included).

As another example of unobserved variation, consider the effect of the nature of an accident on injury severity. Assume for now that all accidents are either angle accidents or head-on accidents (the same discussion extends in a straightforward way to the more realistic case that considers additional types of accidents). As suggested by Castro et al. (2013), some angle accidents may lead to injury severities of those involved that may be far more severe than head-on accidents, even if the majority of angle accidents lead to a lesser degree of injury severity. More generally, the vehicle-to-vehicle kinematic interactions relating to vehicle speed differences, differences in vehicle size, variations in vehicle impact locations, variations in structural integrity of the vehicles, and variations in angle of impact all comprise a significant portion of heterogeneity in collision-type effects. Such interactions are impossible to measure in a comprehensive manner.

As a third example, consider the effect of an observed binary roadway lighting indicator variable (one if roadway lighting is present and zero otherwise). Unobserved factors are likely to influence the impact of this indicator due to variations across roadway segments in lighting type, the ambient lighting from land uses nearby, as well as the light-output and types of lighting used. Recent studies have demonstrated such heterogeneous effects (Venkataraman et al., 2011, 2013, 2014).

Table 1 provides a description of the potential heterogeneous effects of some other commonly available explanatory variables for modeling the likelihood and injury severity of highway accidents.

If unobserved heterogeneity is ignored, and the effects of observable variables is restricted to be the same across all observations, the model will be misspecified and the estimated parameters will, in general, be biased and inefficient, which could in turn lead to erroneous inferences and predictions. As an example, consider traffic volume and its effect on the likelihood of an accident. As discussed in Table 1, there are compelling reasons to believe that the effect of traffic volume on accident occurrences would vary from one roadway entity (highway segment or intersection) to the next as a result of unobserved time-varying environmental characteristics and unobserved variations in driver responses to traffic and these conditions. However, if the analyst were to ignore the possibility of a heterogeneous effect of this variable across roadway entities, multiple incorrect conclusions could be drawn from the resulting bias in parameter estimate such as believing that the effect of traffic volume on accident likelihood is non-linear (that is, increases in traffic volumes at higher levels of congestion do not increase accident likelihoods at the same rate as traffic-volume increases at lower levels of traffic congestion). However, without explicitly accounting for unobserved heterogeneity, it is impossible to discern whether the effect of traffic volume on accident likelihood is truly non-linear or if it just appears to be non-linear due to ignoring unobserved heterogeneity (that is, the apparent non-linearity is actually tracking unobserved heterogeneity in the data and not true non-linearities).[2]

Of the various approaches to account for unobserved heterogeneity, perhaps the so-called random parameters approach has been the most widely adopted. The idea with a traditional random parameters approach is that the heterogeneity from one data observation to the next is accounted for by allowing potentially every estimated parameter in the model to vary across observations according to an analyst-specified continuous distribution (such as the normal distribution). The estimation of a traditional random parameters model thus requires a parametric assumption (assumed distribution for the variation in parameters across observations). While the individual parameters estimated in the model can have different distributions, and a variety of distributions can be tested to determine which provides the best overall statistical fit, there are still potential problems with adopting parametric assumptions. For example, it may be difficult for conventional distributions to track heterogeneity in the population if there are groups of observations with similar parameters, which may result in a complex multimodal distribution with varying skewness and kurtosis.

Another popular approach for addressing heterogeneity is to assume finite mixtures (latent classes) where instead of having the heterogeneity vary across individual observations, the estimation approach seeks to identify groups of observations with homogeneous variable effects within each group. This approach is semi-parametric because it does not impose a parametric assumption on the distribution of parameter heterogeneity (the approach still requires a parametric model structure such as a negative binomial, logit, and so on). The disadvantage of this approach is that identifying the many groups that may exist in the data can be computationally cumbersome[3] and the procedure makes the assumption of parameter homogeneity within the identified groups.

A combination of the two above approaches has also been considered in the literature where the number of latent classes (mass points) are specified and then the parameters are allowed to vary across observations within each identified latent class. This combined approach allows a more sophisticated representation of unobserved heterogeneity because it can track variations across groups of data and individual observations.

There also exist temporal and spatial elements in accident data that are often overlooked in accident studies. That is, accidents are rare events and, to get a sufficient number of observations, they are often aggregated over time (for example, accidents per month) and/or space (accidents over a length of roadway segment). This creates additional unobserved

[2] It should be mentioned here that models that can account for unobserved heterogeneity can usually be compared statistically with those that do not (for example by using a likelihood ratio test). It is also true that the use of an inappropriate functional form for the effect of a variable can be picked up, incorrectly, as unobserved heterogeneity. So, if traffic volume actually were to have a non-linear effect on accident occurrence, and the analyst failed to capture this non-linearity, it can show up incorrectly as unobserved heterogeneity. In many safety applications, even after specifying the appropriate functional form for the effects of exogenous variables, there will very likely remain unobserved heterogeneity effects of the variable. Proper specification for the effects of observed explanatory variables and accounting for potential unobserved heterogeneity are both needed for a correct model specification.

[3] In most applications, after specifying more than 3 or 4 mass points (latent classes), model convergence may become very difficult.

Table 2
Summary of research accounting for unobserved heterogeneity in the analysis of accident-likelihood data.[a]

Methodological approach	Previous research
Random parameters count models[b]	Anastasopoulos and Mannering (2009); El-Basyouny and Sayed (2009); Granowski and Manner (2011);Venkataraman et al. (2011); Ukkusuri et al. (2011); Mitra and Washington (2012); Wu et al. (2013); Bullough et al. (2013); Castro et al., 2012, Narayanamoorthy et al. (2013); Bhat et al. (2014a, 2014b); Venkataraman et al. (2013); Chen and Tarko (2014); Venkataraman et al. (2014); Barua et al., (2015); Coruh et al. (2015); Barua et al. (2016); Buddhavarapu et al. (2016)
Random parameters tobit model	Anastasopoulos et al. (2012)
Random parameters generalized count models	Castro et al. (2012); Narayanamoorthy et al. (2013); Bhat et al. (2014a)
Latent-class (finite mixture) models	Park and Lord (2009); Park et al. (2010); Peng and Lord (2011); Zou et al. (2013); Zou et al. (2014); Yasmin et al. (2014); Buddhavarapu et al. (2016)
Markov switching count models	Malyshkina et al. (2009); Malyshkina and Mannering (2010a)
Bivariate/multivariate models with random parameters	Dong et al. (2014); Barua et al. (2015, 2016)

[a] It is important to mention here that the various models listed in this table (to address unobserved heterogeneity) often do not lend themselves to direct conventional statistical comparison (for example, random parameters and latent class approaches cannot be directly compared with a conventional method such as a likelihood ratio test).

[b] There has been an abundance of work that has addressed spatial and temporal unobserved heterogeneity and random effects (as a special restrictive case of the random parameters formulation where only the constant term varies across alternatives). These include the studies of Shankar et al. (1998); Miaou and Lord (2003); Flahaut et al. (2003); MacNab (2004); Miaou and Lord (2003); Miaou et al. (2005); Wang and Abdel-Aty (2006); Aguero-Valverde and Jovanis (2006); Kim et al. (2007); Aguero-Valverde and Jovanis (2008); Li et al. (2008); Sittikariya and Shankar (2009); Guo et al. (2010); Aguero-Valverde and Jovanis (2010); Ahmed et al. (2011); Usman et al. (2012); Mitra and Washington (2012); Yu et al. (2013); Deublein et al. (2013); Yu and Abdel-Aty (2013); Aguero-Valverde (2013); Mohammadi et al. (2014); and Xie et al. (2014).

heterogeneity issues that may be time or space dependent. Methods such as Markov-switching models have been used to address the unobserved heterogeneity issue over time and more advanced correlation structures have been used to link accident observations spatially.

Table 2 presents categorized common methodological approaches for addressing unobserved heterogeneity with regard to the likelihood of an accident, along with a list of research studies that have applied these approaches. Table 3 presents categorized common methodological approaches for addressing unobserved heterogeneity with regard to an accident's resulting injury severity, along with a list of research studies that have applied these approaches. A brief presentation of the more common methodological approaches presented in Tables 2 and 3 is provided in the following sections.

3. Random parameters formulations

3.1. Random effects versus random parameters

Before proceeding to random parameters model formulations, we first clarify terminology issues related to "random effects" and "random parameters". In many econometric treatments of the subject, the entry way to random parameters models is to first bring up panel data, discuss the so-called fixed-effects and random-effects estimators, and then proceed to introduce random parameters models. However, while the fixed-effects and random-effects models typically necessitate panel data, this is not the case with random parameters models. In particular, the fixed-effects and random effects approaches are two different ways to introduce unobserved individual-specific heterogeneity in the constant terms with panel data. In a random-effects model, the unobserved individual-specific heterogeneity is assumed to be completely unrelated to the explanatory-variable vector, which is a rather strong assumption. In a fixed-effects model, this assumption is relaxed, but the fixed effects model poses the incidental parameters problem that renders the usual maximum likelihood estimator inconsistent because the number of observations generated by the same entity (for example, accidents per some time period for the same roadway entity) is fixed and very few in number (see Greene and Hensher, 2010, page 60 for a discussion of this issue). In contrast to the fixed and random effects models, random parameters models can be estimated even with cross-sectional data as well as panel data. With panel data, one can allow random parameters not only in the response to explanatory variables (as in cross-sectional data), but also incorporate the typical panel random effect. Other possibilities exist as well, such as the flexibility to estimate an individual-specific as well as an observation-specific random parameter vector on the explanatory variables (see, for example, Bhat and Sidharthan, 2011). In the rest of this paper, we will motivate much of the discussion on random parameters from the standpoint of a cross-sectional notation set-up, though the concepts are readily extendible to panel data.

Table 3
Summary of research accounting for unobserved heterogeneity in the analysis of accident injury-severity data.[a].

Methodological approach	Previous research
Random parameters multinomial logit (mixed logit model)	Milton et al. (2008); Kim et al. (2008); Kim et al. (2010); Malyshkina and Mannering (2010b); Kim et al. (2010); Anastasopoulos and Mannering (2011); Moore et al. (2011); Ye and Lord (2011); Morgan and Mannering (2011); Chiou et al. (2013); Kim et al. (2013); Aziz et al. (2013); Manner and Wunsch-Ziegler (2013); Yasmin and Eluru (2013); Ye and Lord (2014); Cerwick et al. (2014); Behnood and Mannering (2015, 2016)
Random parameters ordered probability models	Eluru and Bhat (2007); Eluru et al. (2008); Zoi et al. (2010); Paleti et al. (2010); Xiong and Mannering (2013); Xiong et al. (2014); Yasmin et al. (2015); Eluru and Yasmin (2015)
Latent-class (finite mixture) models	Xie et al. (2012); Eluru et al. (2012); Xiong and Mannering (2013); Xiong et al. (2014); Yasmin. et al. (2013); Yasmin et al. (2014); Cerwick et al. (2014); Shaheed and Gritza (2014); Behnood et al. (2014); Behnood and Mannering (2016)
Latent-class models with random parameters within class	Xiong and Mannering (2013)
Markov switching models	Malyshkina and Mannering (2009); Xiong et al. (2014)
Markov switching with random parameters	Xiong et al. (2014)
Bivariate/multivariate models with random parameters	Abay et al., 2013, Russo et al. (2014)

[a] It is important to mention here that the various models listed in this table (to address unobserved heterogeneity) often do not lend themselves to direct conventional statistical comparison (for example, random parameters and latent class approaches cannot be directly compared with a conventional method such as a likelihood ratio test).

3.2. Random parameters accident likelihood models

The likelihood of an accident can be studied using a number of statistical techniques including traditional count-data models, zero-inflated count data models (which consider the possibility of a two-state process, one a near safe zero-accident state and the other a normal count state with non-negative integers), duration models (reframing observed accident counts into the time between accidents occurring on a specified roadway segment), generalized count models (through reframing count data as originating from a generalized ordered model set-up), or tobit regression models (arriving at a censored continuous variable by converting accident counts into accident rates by dividing observed accident counts over some time period by the traffic over that time period time the length of roadway being considered).

The application of traditional count-data involves determining the number of accidents that occur over some pre-determined space (a roadway entity such as an intersection or a segment of specified length) and time (such as a month or a year).[4] This results in a non-negative integer that is well suited to traditional count-data models. The most popular count-data approach is based on Poisson regression or its derivatives which include the negative binomial and zero-inflated models (see Washington et al., 2011). For the basic Poisson model, the probability $P(n_i)$ of road entity $\boldsymbol{i}$ (for example, and intersection or highway segment) having n_i accidents is,

$$\text{Prob}\,(n_i) = \frac{e^{-\lambda_i}(\lambda_i)^{n_i}}{n_i!}, \tag{1}$$

where λ_i is the Poisson parameter for highway entity $\boldsymbol{i}$. The Poisson regression specifies the Poisson parameter λ_i (which is also the expected number of accidents for entity i) as a function of explanatory variables by typically using a log-linear function,

$$\lambda_i = \exp(\boldsymbol{b}'\boldsymbol{x_i}), \tag{2}$$

where $\boldsymbol{x_i}$ is a vector of explanatory variables (now including a constant) and $\boldsymbol{b}$ is a vector of estimable parameters (Washington et al., 2011).

Depending on the data, a Poisson model may not always be appropriate because the Poisson distribution restricts the mean and variance to be equal ($E[n_i]=VAR[n_i]$). If this equality does not hold, the data are said to be underdispersed ($E[n_i]>VAR[n_i]$) or overdispersed ($E[n_i]<VAR[n_i]$), and the standard errors of the estimated parameter vector will be incorrect and incorrect inferences could be drawn. To account for the possibility of overdispersion (which is more commonly encountered in accident count data), the negative binomial model is derived by rewriting,

$$\lambda_i = \exp(\boldsymbol{b}'\boldsymbol{x_i} + \varepsilon_i), \tag{3}$$

[4] Other methods consider the time between accidents instead of counts over some pre-specified time period (Mannering, 1993).

where $\exp(\varepsilon_i)$ is a Gamma-distributed error term with mean 1 and variance α.[5,6] The addition of the $\exp(\varepsilon_i)$ term allows the variance to differ from the mean as $VAR[n_i]=E[n_i][1+\alpha E[n_i]]=E[n_i]+\alpha E[n_i]^2$. The negative binomial probability density is,

$$P(n_i) = \left(\frac{1/\alpha}{(1/\alpha)+\lambda_i}\right)^{1/\alpha} \frac{\Gamma\left[(1/\alpha)+n_i\right]}{\Gamma(1/\alpha)n_i!}\left(\frac{\lambda_i}{(1/\alpha)+\lambda_i}\right)^{n_i}, \tag{4}$$

where $\Gamma(.)$ is a gamma function. The Poisson regression is a limiting model of the negative binomial regression as α approaches zero. Thus, if α is significantly different from zero, the negative binomial is appropriate and if it is not, the Poisson model is appropriate (Washington et al., 2011).

To account for unobserved heterogeneity in response to the non-constant explanatory variables in count models, random parameters approaches have been developed and are available in standard software packages (see, for example, Greene, 2012).[7] To allow for such random parameters in count-data models, each estimable parameter on explanatory variable l in the vector $\boldsymbol{x_i}$ can be written as,

$$\beta_{il} = b_l + \varphi_{il}, \tag{5}$$

where β_{il} is the parameter on the lth explanatory variable for observation i, b_l is the mean parameter estimate across all observations for the lth explanatory variable, and φ_{il} is a randomly distributed scalar term that captures unobserved heterogeneity across observations, and the term can assume an analyst-specified distribution (such as the normal distribution or others).

With Eq. (5), the analyst can test for random parameters, using a specified distribution, across all observations i for each included explanatory variable (various distributions can be specified to determine the best statistical fit such as normal, lognormal, triangular, uniform and Weibull distributions). If the variance of the chosen distribution is not significantly different from zero, it suggests that a conventional fixed parameter (one parameter estimate for all observations) is statistically appropriate. Thus the model is likely to consist of a combination of fixed and random parameters across the included explanatory variables.

It is also important to note that random parameters models can be readily structured to account for heterogeneity among analyst-specified groups of observations instead of individual observations.[8] For example, instead of estimating separate parameter vectors for accidents on the individual approaches to an intersection, a single parameter vector may be estimated for all approaches to a specific intersection (see Wu et al., 2013). This is done simply by re-writing Eq. (5) as $\beta_{il} = b_l + \varphi_{gl} \ \ \forall \ i \in \text{group } g$, where φ_{gl} is now the group-specific random term that generates unobserved heterogeneity across groups in response to the lth explanatory variable. These analyst-specified groups can account for forms of both spatial and temporal effects.

The estimation of random parameters models is typically achieved with maximum simulated likelihood (for more on this technique, see Bhat, 2001, 2003; Train, 2009). However, Bhat (2011) has more recently proposed a maximum approximate composite marginal likelihood approach that he shows to be much more computationally efficient and even more accurate than traditional maximum simulated likelihood approaches for most random parameters models. Again, Table 2 lists random parameters count models that have been successfully applied in accident studies. This basic random parameters formulation can be readily extended to other accident-likelihood modeling methods such as zero-inflated count models (Shankar et al., 1997), duration models (Mannering, 1993), and tobit regressions (Anastasopoulos et al., 2012). A relatively recent development in count models that facilitates the introduction of unobserved heterogeneity and many other generalizations is the insight that any count data model structure can be recast as a restricted version of a generalized ordered-response model (see Castro et al. 2012, Bhat et al., 2014a, 2014b).

3.3. Random parameters injury severity models

Along similar lines to those above, injury severity models (which seek to study the probability of discrete injury outcomes such as no injury, possible injury, evident injury, disabling injury and fatality) can address unobserved heterogeneity with parameters that vary across observations. A common example of such a model is the random parameters multinomial logit model (also referred to as the mixed logit model). To see this model, define a function S_{ik} that determines the probability that accident i will result in injury-severity level k as,

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[5] Although uncommon, it is possible for the data to be underdispersed in which case the negative binomial is not appropriate and other models must be used (see Lord and Mannering, 2010, for a full discussion of this point as well as methodological alternatives).

[6] Note that we are able to accommodate a random-effects type specification in a cross-sectional count data model because of the functional form adopted for count models. This is easiest seen in the reframing of a count model as a generalized ordered-response model, where the λ_i term (which includes the error term ε_i in the negative binomial model) appears in the threshold part, while the original error term leading to the probability expression in any count model originates in the typical latent regression part (see Bhat, 2015).

[7] An alternative to a random parameters approach in the negative-binomial case would be to allow the dispersion parameter α to vary as a function of the mean, λ (see Cameron and Trivedi, 1986). However, this would be more restrictive in terms of its ability to account for heterogeneity across observations.

[8] Such grouping of observations often forms the basis of what are commonly called multilevel models in the literature. Multilevel model terminology simply refers to a modeling approach that partitions the data and potentially accounts for heterogeneity within these partitions.

$$S_{ik} = \alpha_k + \boldsymbol{\beta}_{\boldsymbol{i}}'\boldsymbol{x}_{\boldsymbol{ik}} + \varepsilon_{ik};\ \ \boldsymbol{\beta}_{\boldsymbol{i}} = \boldsymbol{b} + \tilde{\boldsymbol{\beta}}_{\boldsymbol{i}}, \tag{6}$$

where α_k is now a constant specific to injury-severity level k (with one of them set to zero for identification), $\boldsymbol{x}_{\boldsymbol{ik}}$ is an $(L \times 1)$-column vector of exogenous attributes specific to accident i and injury-severity level k, $\boldsymbol{\beta}_{\boldsymbol{i}}$ is an accident-specific $(L \times 1)$-column vector of corresponding parameters that varies across accidents based on unobserved accident-specific attributes, and ε_{ik} is assumed to be an independently and identically distributed (across injury severity levels and accidents) standard extreme-value error term. If $\boldsymbol{\beta}_{\boldsymbol{i}} = \boldsymbol{b}\ \forall\ i$, this implies no accident-specific unobserved heterogeneity, and the resulting model form is the standard multinomial logit (McFadden, 1981). However, if accident-specific unobserved heterogeneity is allowed, and the $\boldsymbol{\beta}_{\boldsymbol{i}}$ vector has a continuous density function $\text{Prob}(\boldsymbol{\beta}_{\boldsymbol{i}} = \boldsymbol{\beta}) = f(\boldsymbol{\beta}|\boldsymbol{\varphi})$, where $\boldsymbol{\varphi}$ is a vector of parameters characterizing the chosen density function (such as the location and scale). The resulting random parameters multinomial logit injury-severity probabilities are (see Bhat, 1998, McFadden and Train, 2000; Train, 2009),

$$P_i(k) = \int \frac{e^{\alpha_k + \boldsymbol{\beta}'\boldsymbol{x}_{\boldsymbol{ik}}}}{\sum_m e^{\alpha_m + \boldsymbol{\beta}'\boldsymbol{x}_{\boldsymbol{im}}}} f(\boldsymbol{\beta}|\boldsymbol{\varphi})\, d\boldsymbol{\beta}, \tag{7}$$

where $P_i(k)$ is the probability of injury severity k. As noted above with count-data models, if the elements related to scale in the vector $\boldsymbol{\varphi}$ are determined to be significantly different from zero, there will be accident-specific variations of the effect of one or more elements of the explanatory variable vector $\boldsymbol{x}_{\boldsymbol{ik}}$ on injury severity. As with other random parameters models, maximum simulated likelihood (MSL) is typically used to estimate mixed logit models.[9] Bhat (2011) and Bhat and Sidharthan (2011) have shown how the maximum approximate composite marginal likelihood (MACML) estimation of a normally mixed multinomial probit model offers substantially more computational efficiency as well as superior accuracy in recovering parameters relative to the maximum simulated likelihood (MSL). They demonstrate this through the estimation of a normally mixed multinomial logit model, and this opens up a new direction for estimating random parameters multinomial models in the safety area.

In addition to the random parameters multinomial model discussed above, random parameters can be readily introduced in other models that have been historically used to analyze accident-injury severities, including ordered probability models (models that account for the ordering of severity levels from lower to higher injury levels). Further, in these ordered probability models, unobserved heterogeneity can be introduced in both the latent regression as well as in the thresholds, as in Eluru et al. (2008). Savolainen et al. (2011), Castro et al. (2013) and Mannering and Bhat (2014) are good sources of review of this literature.

3.4. Random parameters models with correlated parameters

Almost all research in the accident field to date has assumed that the unobserved heterogeneity captured by random parameters is independent. That is, there is no allowance for correlation among the distribution of two or more random parameters in the model. In reality, there may be correlation among random parameters. As an example, consider unobserved heterogeneity caused by weather events and drivers' heterogeneous responses to these events. In this case, one might expect the effect of precipitation to influence the likelihood and severity of accidents differently across observations as drivers respond differently, and one might also expect the effect of pavement condition (coefficient of friction or roughness) to do the same. However, there is likely a correlation between these two sources of heterogeneity due to the interactive effects of precipitation and pavement condition. Accounting for correlation among random parameters can be achieved, for example, by assuming a multivariate normal distribution for $\boldsymbol{\beta}_{\boldsymbol{i}}$ and writing,

$$\boldsymbol{\beta}_{\boldsymbol{i}} = \boldsymbol{b} + \mathbf{C}\boldsymbol{\varphi}_i, \tag{8}$$

where $\boldsymbol{\beta}_{\boldsymbol{i}}$ is a vector of random parameters corresponding to explanatory variables for observation i, $\boldsymbol{b}$ is the mean parameter estimate across all observations, $\mathbf{C}$ is a lower triangular matrix that engenders correlation among the elements of the parameter vector $\boldsymbol{\beta}_{\boldsymbol{i}}$, and φ_i is a randomly and independently distributed uncorrelated vector term. Allowing for correlation among random parameters can complicate the interpretation of results, but explicitly considering correlation among random parameters can provide additional insights.[10]

3.5. Random parameters models with means (and variances) as functions of explanatory variables

As shown in Eqs. (5) and (8), the most common application of random parameters models is to assume that there is a

[9] In this case, logit probabilities shown in Eq. (11) are approximated by drawing values of $\boldsymbol{\beta}$ from $f(\boldsymbol{\beta}|\boldsymbol{\varphi})$ for given values of $\boldsymbol{\varphi}$. Research by Bhat (2000) and Bhat (2001) has shown that an efficient way of drawing to compute logit probabilities is to use a Halton sequence approach (for more on the Halton sequence, see Halton, 1960). As with count-data models, a variety of functional forms can be considered including normal, lognormal, triangular, uniform and Weibull distributions.

[10] This issue is important in the case of multiple random parameters where the parameters are not all necessarily normally distributed. It must be noted that empirically, it is rare to see a non-correlated model perform as well as a correlated-parameters random parameters model in safety applications. The correlated-parameters approach also has a high degree of sensitivity to the sparse indicator-variable problem (where indicator variables with low densities are used in the model). However, this issue can be mitigated by omitting sparse indicators in order to make estimation and convergence feasible.

single mean (b_l in Eq. (5) and $\boldsymbol{b}$ in Eq. (8)) across the population. Eq. (8) can be generalized to allow for the possibility that the mean may vary from one observation to the next as a function of observed explanatory variables (we use Eq. (8) instead of Eq. (5) to continue to allow for the possibility of correlated random parameters). To allow the means of random parameters to vary as a function of explanatory variables, Eq. (8) can be re-written as,

$$\boldsymbol{\beta_i} = \boldsymbol{b} + \boldsymbol{\Theta}\,\boldsymbol{z_i} + \mathbf{C}\boldsymbol{\varphi}_i, \tag{9}$$

where $\mathbf{z_i}$ is a $(\tilde{L} \times 1)$-vector of explanatory variables from observation i that influence the mean of the random parameter vector, $\boldsymbol{\Theta}$ is an $(L \times \tilde{L})$ matrix of estimable parameters (each row of $\boldsymbol{\Theta}$ corresponds to the loadings of a specific element of the $\boldsymbol{\beta_i}$ vector on the $\boldsymbol{z_i}$ vector; if a specific column entry in a row of $\boldsymbol{\Theta}$ is zero, it implies that there is no shift in the mean of the corresponding row element of the $\boldsymbol{\beta_i}$ vector due to the row element of the $\boldsymbol{z_i}$ vector corresponding to the column under consideration). Note that such a specification is equivalent to simply including an appropriate interaction term within the systematic specification of the model. For example, in the injury severity model of Eq. (6), substituting Eq. (9) for $\boldsymbol{\beta_i}$ is equivalent to having a random parameter vector with a fixed mean on the variable vector $\boldsymbol{x_{ik}}$ along with appropriate interactions of exogenous variables. In general, the analyst should always consider the variations in the effect of a variable due to observed factors before considering unobserved heterogeneity.

There have been several empirical studies that have addressed this issue in the accident literature. For example, Kim et al. (2013) found that, while newer vehicles reduced injury severity probabilities in single-vehicle crashes, this reduction was less for men than for women (they explain that this could be because men drive more aggressively). This is an example of the "newer vehicle" variable being interacted with the gender of the driver to shift the mean of the effect of a "newer vehicle" (relative to an "older vehicle") on injury severity. However, doing so does not influence the level of variation in the amount of unobserved heterogeneity itself. This can be noted from the fact that the unobserved heterogeneity portion $\mathbf{C}\boldsymbol{\varphi}_i$ in Eq. (9) remains unaffected when the mean is being shifted. But, in the example above, it is possible that when women drive newer vehicles, there is less variation (due to unobserved heterogeneity) in the injury severity sustained. In contrast, among men, this variation may be much higher because of a larger range of variance in aggressiveness. An approach to accommodate a shift in the variance (of unobserved heterogeneity) in responsiveness to "newer vehicles" across men and women is to write the standard deviation of the error terms in $\boldsymbol{\varphi}_i$ corresponding to the "newer vehicle" variable as a function of gender.[11]

In addition to injury-severity analysis, heterogeneity in the mean of a random parameter has also been explored in accident-likelihood contexts. For example, Venkataraman et al. (2014) explore a multitude of heterogeneous mean effects on the likelihood of accident occurrence.

4. Latent class (finite mixture) models

As discussed in Section 2 of this paper, there are potential drawbacks of random parameters models in capturing unobserved heterogeneity in that the analyst must assume a distribution for the parameters across the population and the possibility that parameters may vary across unobserved groups of observations instead of across individual observations. The approach to latent class models is the same for models addressing the likelihood of an accident as well as its resulting severity. As an example, consider a model where the probability of belonging to a latent class is specified by a multinomial logit model with Greene and Hensher (2003),

$$P_i(c) = \frac{e^{\gamma' \boldsymbol{z_{ic}}}}{\sum_g e^{\gamma' \boldsymbol{z_{ig}}}}, \tag{10}$$

where $P_i(c)$ is the probability of observation i belonging to latent class c, $\boldsymbol{z_{ic}}$ is a vector of explanatory variables specific to observation i and latent class c (including a constant for all latent classes except one) and $\boldsymbol{\gamma}$ is a vector of estimable parameters. With Eq. (10), models of both the likelihood of the accident and its resulting severity can readily be written and estimated. For example, if an injury severity model is estimated as a multinomial logit model the conditional severity model would be,

$$P_i(k)|c = \frac{e^{\alpha_{kc} + \boldsymbol{b'_c}\boldsymbol{x_{ik}}}}{\sum_m e^{\alpha_{mc} + \boldsymbol{b'_c}\boldsymbol{x_{im}}}}, \tag{11}$$

where $P_i(k)|c$ is the probability of an accident injury-severity level k for accident i if accident i were a member of unobserved class c, α_{kc} is a constant specific to injury severity level k for latent class c (with α_{kc} set to zero for one of the alternatives in each class c for identification), $\boldsymbol{x_{ik}}$ is as defined in the context of Eq. (6), and $\boldsymbol{b_c}$ is a class-specific set of fixed parameters. The unconditional probability for a specific accident i resulting in injury severity k would then be,

[11] Such a varaince shift has seldom been pursued in the accident literature, though the concept has been applied in non-accident contexts (see, for example, Bhat, 1997 and Bhat and Zhao, 2002).

$$P_i(k) = \sum_c P_i(c) \times \left[P_i(k)|c\right]. \tag{12}$$

Estimation of latent class models generally require the analyst to specify the number of classes (mass points) so, much like explanatory variable selection, the appropriate number of classes needs to be determined as part of the model-estimation process. As shown in Tables 2 and 3, latent class models have become an increasingly popular method of accounting for unobserved heterogeneity the study of the likelihood and severity of an accident.

5. Latent class models with random parameters within classes

Both latent class and random parameters models have their drawbacks. For example, random parameters models require distributional assumptions and may have difficulty tracking groups of observations with shared unobserved heterogeneity. Latent class models may have difficulty in accounting for unobserved heterogeneity within the identified latent classes. An approach that generalizes the latent class models to allow random parameters within each class can easily be envisioned. For example, in the case of injury severity, the multinomial logit model could readily be replaced with the random parameters logit model so Eq. (11) becomes (with Eqs. (10) and (12) still applying as before),

$$P_i(k)|c = \int \frac{e^{\alpha_{kc}+\boldsymbol{\beta'_c x_{ik}}}}{\sum_m e^{\alpha_{mc}+\boldsymbol{\beta'_c x_{im}}}} f\left(\boldsymbol{\beta_c}|\boldsymbol{\varphi}_c\right) d\boldsymbol{\beta_c}, \tag{13}$$

where $\boldsymbol{\varphi}_c$ is a class-specific vector of moment parameters characterizing the chosen density function. From an estimation perspective, allowing for the possibility of random parameters within each latent classes can seriously complicate the estimation process. In fact, due to the complexity of the estimation process Bayesian methods are typically used requiring a Markov Chain Monte Carlo (MCMC) algorithm with sampling provisions for model identification (see Xiong and Mannering, 2013, for an application of this joint latent class/random parameters approach to accident injury severity). Buddhavarapu et al. (2016) demonstrate a similar application to the crash likelihood context accounting for spatial dependencies of crash counts.

6. Temporal heterogeneity and Markov switching models

Highway accidents are relatively rare events and thus an accumulation over time is often used in analysis. For example, accident likelihoods on a specified segment of highway may be modeled as count data in the form of observed accidents per week or month. This introduces the potential for temporal heterogeneity where unobserved factors may vary from one time period to the next. This unobserved temporal heterogeneity could include factors such as weather-related factors that may not be observable to the analyst. Statistically, the presence of time-varying unobserved heterogeneity could lead to biased parameter estimates and erroneous inferences when variability over time is present (Xiong et al., 2014).

One way of addressing this temporal heterogeneity is using hidden-state Markov switching models which can account for unobserved heterogeneity across time periods by assuming that the likelihood of accident occurrence and the injury severities of observed accidents transition between two or more states over time. Theoretically, there are a number of reasons why multiple hidden states could exist and manifest itself as temporal unobserved heterogeneity, including variations in drivers' responses to weather conditions (not necessarily observed by the analyst) that change over time. The transition from one state to the next is often assumed not to depend on explanatory variables, although the transition probabilities could theoretically be made to be some function of observable variables.

Recently applied Markov-switching models in accident research (Malyshkina et al., 2009; Malyshkina and Mannering, 2009; Malyshkina and Mannering, 2010a, 2010b; Xiong et al., 2014) assume that temporal heterogeneity follows a stationary multiple-state Markov chain process. For example, if two hidden states are assumed to exist ($s_t = 0$ and $s_t = 1$) the time-dependent transition probabilities can be written as,

$$P\left(s_{t+1} = 1|s_t = 0\right) = p_{0\to 1}, \text{ and } P\left(s_{t+1} = 0|s_t = 1\right) = p_{1\to 0}, \tag{14}$$

where $P(s_{t+1} = 1|s_t = 0)$ is the conditional probability of $s_{t+1} = 1$ at time $t+1$ given that the observation is in state $s_t = 0$ at time t, $P(s_{t+1} = 0|s_t = 1)$ is the conditional probability of $s_{t+1} = 0$ at time $t+1$ given that the observation is in state $s_t = 1$ at time t, and the transition probabilities $p_{0\to 1}$ and $p_{1\to 0}$ can be estimated from the accident data.[12] Estimation of Markov-switching models can be complex, and typically requires Bayesian Markov Chain Monte Carlo (MCMC) methods to sample the hidden states. However, the potential to track temporal unobserved heterogeneity in data that are typically viewed as cross-sectional, makes Markov-switching models a very powerful tool that can yield important new insights into the

[12] As mentioned in the text and emphasized again here, existing applications of Markov switching models in accident analysis have not considered the transition probabilities as a function of explanatory variables. While the modeling approach can be readily extended to allow transition probabilities to be a function of explanatory variables, additional complexities in model estimation and identification can be problematic.

likelihood of accidents and their resulting injury severities. Markov switching models can also be combined with other methods of heterogeneity modeling to arrive at a more complete characterization of unobserved heterogeneity. For example, Xiong et al. (2014) estimate a Markov switching ordered probability model for accident injury severity with random parameters across observations.

7. Unobserved heterogeneity and multivariate models

Multivariate models can be encountered when studying the likelihood of an accident and/or its resulting severity. Multivariate models can result from correlations that emerge from a variety of sources. For example, in considering the likelihood of accidents resulting in different injury-severity levels, one may speculate that the factors that affect the likelihood of accidents resulting in severe injuries are fundamentally different than those that generate the likelihood of accidents resulting in no injuries. This may be due to how specific roadway-design characteristics interact with the likelihoods of specific injury-severity levels. If this is the case, one may consider estimating separate accident likelihood models (such as separate count-data models) for each discrete severity outcome (such as no injury, possible injury, evident injury, disabling injury, fatality). However, estimation of separate models in this case can be problematic because unobserved factors are likely to impact multiple accident counts, of different severity levels, simultaneously for each roadway entity being considered (for example, counts by roadway segment or intersection). In addition, if accident count data are collected on specific roadway entities over multiple time periods (for example months or years), unobserved factors will result in a temporal correlation in the number of accidents at the roadway entity over time. This temporal dependency can be combined with spatial dependencies (correlation in observed factors among spatially adjacent roadway entities) to produce multivariate models of very large dimension (see, for example, Narayanamoorthy et al., 2013 and Bhat et al., 2014b).

With regard to injury-severity data, multivariate issues can also arise with vehicle accidents that involve multiple occupant injuries from the same accident. In such cases, the different occupants may experience different levels of injury severity, but the unobserved factors influencing these injury levels (such as energy dissipated during the accident, structural features of the vehicle(s) involved, and so on) would be correlated (see, for example, Abay et al., 2013, Eluru et al., 2010, Yasmin et al., 2014 and Russo et al., 2014).

Accounting for unobserved heterogeneity (such as using random parameters and potentially latent-class approaches) in a multivariate framework can complicate the error-term structure and estimation process. Still, as shown in Tables 2 and 3, a few studies have considered random parameters in multivariate models. A particularly appealing way to combine unobserved heterogeneity effects with a multivariate outcome context (with the outcomes being of different types, including continuous, count, nominal, ordered, and grouped outcomes) is based on identifying stochastic latent constructs (for example, unobserved driver-specific psychological factors). These factors can be viewed as having an influence on multiple safety-related variables. Bhat proposes such a formulation and refers to this as a generalized heterogeneous data model (GHDM). Recent applications of this approach to the accident literature include Bhat and Dubey (2014) and Lavieri et al. (2016). The approach also provides a convenient way to incorporate variable endogeneity in multivariate models with unobserved heterogeneity, offering the opportunity to extend earlier work in the field such as Abay et al. (2013) and Paleti et al. (2010).

8. Unobserved heterogeneity, omitted variables bias and transferability

A major concern in safety analysis (and other fields as well) is that detailed data relating to the many factors that are likely to affect the likelihood and severity of an accident are often not available to the analyst. In the absence of complete data the analyst may estimate models that obviously exclude important explanatory variables which will produce an omitted variables bias which is likely to result in biased and inconsistent parameter estimates. With statistical approaches that account for unobserved heterogeneity, these omitted explanatory variables become part of the unobserved heterogeneity. While random parameters, latent class, and other unobserved heterogeneity approaches will mitigate the adverse impacts of omitting significant explanatory variables, the resulting model estimates will not be able to track the unobserved heterogeneity as well as when having the significant omitted variables included in the specification. Thus leaving out important explanatory variables still remains a problem even with advanced approaches to account for unobserved heterogeneity.

A criticism often leveled against the estimation of models that account for unobserved heterogeneity, such as random parameters models, is that the results will not be transferable to different locations since the individual parameter vector associated with each observation is unique to that observation. This is true, but finding significant random parameters (parameters that produce statistically significant standard deviations for the analyst specified distributions) means that unobserved heterogeneity is present on individual observations. If a fixed-parameters model is used for such data, the unobserved heterogeneity does not simply disappear. In fact, the fixed-parameters model will be estimated with a persistent bias and transferability will be problematic since this bias will be a function of unobserved heterogeneity. Finding significant random parameters suggests spatial (and potentially temporal) transferability problems regardless of the estimation method used.

Table 4
Summary of unobserved heterogeneity modeling approaches.

Methodological Approach	Advantages	Disadvantages
Random parameters	• Accounts for heterogeneity across individual observations or analyst specified groups of observations	• Analyst must make a parametric assumption relating to the distribution of heterogeneity across observations • Does not address possible temporal heterogeneity • Can pose convergence problems in the presence of some indicator (zero or one) variables
Latent-class (finite mixture) models	• Accounts for the possibility of common parameters among unobserved groups (classes) of observations • Does not require a parametric assumption relating to the distribution of unobserved heterogeneity in the data	• Can be difficult to extend beyond a few latent classes • Class membership specifications can be simplistic with few explanatory variables, providing little insight into class distinctions • Does not address possible heterogeneity within identified data classes
Latent-class models with random parameters within class	• Gives the advantage of both the semi-parametric latent classes and fully parametric random parameters	• Estimation can be cumbersome • Does not address possible temporal heterogeneity
Markov switching models	• Multi-state approach can account for time-varying heterogeneity	• Limited in accounting for heterogeneity across observations • Difficult to extend beyond two states • Restrictions may apply on state transition probabilities
Markov switching models with random parameters/latent classes	• Extremely flexible approach for accounting for unobserved heterogeneity	• Very complex estimation process • Difficult to extend beyond two states • Restrictions may need to be placed on state transition probabilities

9. Summary and conclusions

Due to the complexity of highway accidents (which involve complex interactions among human, vehicle, roadway, traffic and environmental elements), it is impossible to have access to all of the data that could potentially determine the likelihood of a highway accident or its resulting injury severity. This presents a problem with the conventional statistical analyses of accident data that can result in bias and inefficient model estimation, and erroneous inferences and predictions. This in turn can lead to the implementation of ineffective and potentially counter-productive safety policies and countermeasures.

As discussed at length in the current paper, relatively recent advances in statistical and econometric methods have allowed analysts to study conventional and emerging accident-data sources in new ways by addressing issues relating to unobserved heterogeneity. Table 4 summarizes the unobserved heterogeneity methods discussed in the current paper, along with a brief description of their strengths and weaknesses. As shown earlier in Tables 2 and 3, a number of recent accident-analysis research efforts have applied these methods to address unobserved heterogeneity, thus allowing important new insights to be extracted from existing accident data.

However, statistical approaches that address unobserved heterogeneity tend to be more complex from a model-estimation perspective. Also, the various models that address unobserved heterogeneity are often not nested, so direct conventional statistical comparison between models is often not possible (for example, random parameters and latent class approaches cannot be directly compared with a conventional method such as a likelihood ratio test). This often presents the analyst with difficult decisions that weigh model complexity and associated computational issues against the potential improvements in statistical fit.

As can be seen in Table 4, and from the discussions in this paper, no one approach to addressing unobserved heterogeneity is necessarily superior. In addition, any rigorous comparison between two or more approaches is likely to be data-specific because different patterns of heterogeneity are captured better by different heterogeneity modeling approaches, and these heterogeneity patterns are likely to vary from one data set to the next. There are substantial opportunities for applying existing methods that address unobserved heterogeneity as well as developing new methods that may be combinations of random parameters, latent class, Markov-switching. Because complex approaches are needed to account for complex unobserved heterogeneity, which are often present in accident data bases, continuing advances in estimation techniques and computational power will be needed to continue empirical advances in addressing unobserved heterogeneity in accident data. Since accident data are composed of both time varying and time invariant heterogeneity components, estimation techniques providing insights into the distinctive effects of these components will be required.

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