Time-Optimal Control for a Water Delivery Robot

MATH 2342: Calculus for Robotics

Complete python project available at:Complete python project available at: https://github.com/anthonyreimche/Fastest_acceleration

Instructor: Bruno L'Esperance

Erik Gindis	A01015581
Sarah Guo	A01235022
Owen Li	A01417162
Jayden Ng	A01429770
Anthony Reimche	A01429321

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1 Problem Statement and Objectives

1.1 Problem Definition

This project implements a time-efficient motion profile with continuous acceleration for point-to-point movement in automated systems. The system uses a maximum acceleration $a_{\max} = 0.25 \text{ m/s}^2$ to demonstrate smooth transitions while maintaining reasonable time efficiency.

1.2 Project Goals

- Generate continuous acceleration profiles
- Achieve precise target distance
- Minimize travel time while maintaining continuity
- Visualize motion characteristics

2 Theoretical Analysis

2.1 Understanding Motion Profiles

The project implements a continuous acceleration profile that improves upon the bang-bang control strategy. While bang-bang control provides time-optimal solutions, its discontinuous nature makes it impractical for mechanical systems. Our implementation maintains continuous acceleration while approximating time-optimal performance.

2.2 The Ideal Case

The bang-bang profile serves as our theoretical benchmark, representing the time-optimal solution when acceleration discontinuities are permitted ($\varepsilon = 0$). This provides both a performance baseline and identifies the critical transition points where smoothing is required in the practical implementation.

The theoretical bang-bang acceleration profile is given by:

$$a_{\text{bb}}(t) = \begin{cases} a_{\max} & 0 < t < T/2 \\ -a_{\max} & T/2 < t < T \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

This profile achieves the theoretical minimum time of $T = 12.649$ s but has discontinuities at $t = 0$, $T/2$, and T. Our smooth profile will approximate this function while maintaining continuity at these switching points.

3 Our Smooth Profile Solution

The continuous acceleration profile is defined as the limit of hyperbolic tangent functions as $\varepsilon \to 0$:

$$a(t) = \lim_{\varepsilon \to 0} \left\{ a_{\max}[1 + \tanh\left(\frac{4}{\varepsilon}t\right)] - 2a_{\max}[1 + \tanh\left(\frac{4}{\varepsilon}(t - T/2)\right)] + a_{\max}[1 + \tanh\left(\frac{4}{\varepsilon}(t - T)\right)] \right\} \tag{2}$$

This construction ensures that the transition is approximately 98% complete within any arbitrarily small time window ε, as $\tanh(2) \approx 0.96$. As $\varepsilon \to 0$, each transition becomes arbitrarily sharp while maintaining continuity.

3.1 Component Analysis

We construct $a(t)$ as the sum of three components: $a(t) = a_0(t)+a_1(t)+a_2(t)$, where each component serves a specific purpose in the motion profile. Here:

- t is the time variable ($0 \leq t \leq T$)
- T is the total movement time (determined through optimization)
- ε is our smoothing parameter, which we take to zero in the limit

The term $1 + \tanh$ appears in each component because tanh alone ranges from -1 to 1, while we need a function that transitions from 0 to 2. This allows each component to smoothly transition from 0 to $2a_{\max}$ (or vice versa), ensuring the sum achieves exactly $\pm a_{\max}$ at the appropriate times.

Initial acceleration $a_0(t)$:

$$a_0(t) = \lim_{\varepsilon \to 0} a_{\max}[1 + \tanh\left(\frac{4}{\varepsilon}t\right)] \tag{3}$$

The factor $a_{\max}$ sets the magnitude, while $[1 + \tanh\left(\frac{4}{\varepsilon}t\right)]$ creates a smooth transition from 0 to 2 centered at $t = 0$. This gives us a smooth ramp-up from 0 to $a_{\max}$.

Deceleration transition $a_1(t)$:

$$a_1(t) = \lim_{\varepsilon \to 0} -2a_{\max}[1 + \tanh\left(\frac{4}{\varepsilon}(t - T/2)\right)] \tag{4}$$

The factor of -2 combined with the time shift $T/2$ creates a transition from $a_{\max}$ to $-a_{\max}$ at the midpoint of motion. The doubled magnitude is necessary to counteract both a_0 and a_2, which would otherwise sum to $2a_{\max}$ at $T/2$.

Final deceleration $a_2(t)$:

$$a_2(t) = \lim_{\varepsilon \to 0} a_{\max}[1 + \tanh\left(\frac{4}{\varepsilon}(t - T)\right)] \tag{5}$$

Similar to a_0, but time-shifted by T and maintaining the same sign as a_1, this component ensures a smooth return to zero acceleration at the end of motion.

As $\varepsilon \to 0$, these components combine to form a continuous approximation of the bang-bang profile. The factor $4/\varepsilon$ in each component ensures that transitions occur within a time window of ε, as $\tanh(2) \approx 0.96$ means the transition is 96% complete at $\pm\varepsilon/2$ from each switching point.

Detailed convergence analysis (see Appendix A) shows that this profile achieves:

- Quadratic convergence to optimal time as $\varepsilon \to 0$
- Linear convergence of acceleration error
- Cubic convergence of position error

4 Conclusion

The smooth profile solution successfully balances theoretical optimality with practical constraints. By using hyperbolic tangent functions for transitions and analyzing the limit as $\varepsilon \to 0$, we show that our solution can arbitrarily approach time-optimal performance while maintaining continuous acceleration. This makes the solution suitable for real-world robotic applications where smooth motion is essential.

A Implementation Details

A.1 Algorithm Design

The implementation uses binary search to find the optimal time T that reaches the exact target distance. For each candidate time:

```python
def plot_continuous_forms(max_accel=0.25, distance=10.0, epsilon=0.001):
    # Calculate initial time estimate
    base_time = 2 * np.sqrt(2 * distance / max_accel)
    total_time = base_time * 1.5  # Initial estimate

    # Binary search for correct time
    target_error = 0.001  # 1mm accuracy
    min_time = base_time * 0.5
    max_time = base_time * 2.0

    while True:
        dt = 0.001  # Time step for integration
        t = np.arange(0, total_time + dt, dt)
        half_time = total_time / 2
        k = 4.0 / epsilon  # Steepness factor

        # Continuous acceleration function
        def a(t):
            step1 = 0.5 * (1 + np.tanh(k * (t - epsilon)))
            step2 = -1.0 * (1 + np.tanh(k * (t - half_time)))
            step3 = 0.5 * (1 + np.tanh(k * (t - (total_time - epsilon))))
            return max_accel * (step1 + step2 + step3)

        # Calculate profiles through integration
        accel = a(t)
        vel = np.cumsum(accel) * dt
        pos = np.cumsum(vel) * dt
```

Listing 1: Core implementation of motion profiles

B Numerical Results

B.1 Implementation

Our implementation demonstrates the theoretical behavior as $\varepsilon \to 0$ while maintaining numerical stability. Figure 1 shows an example of the motion profiles.

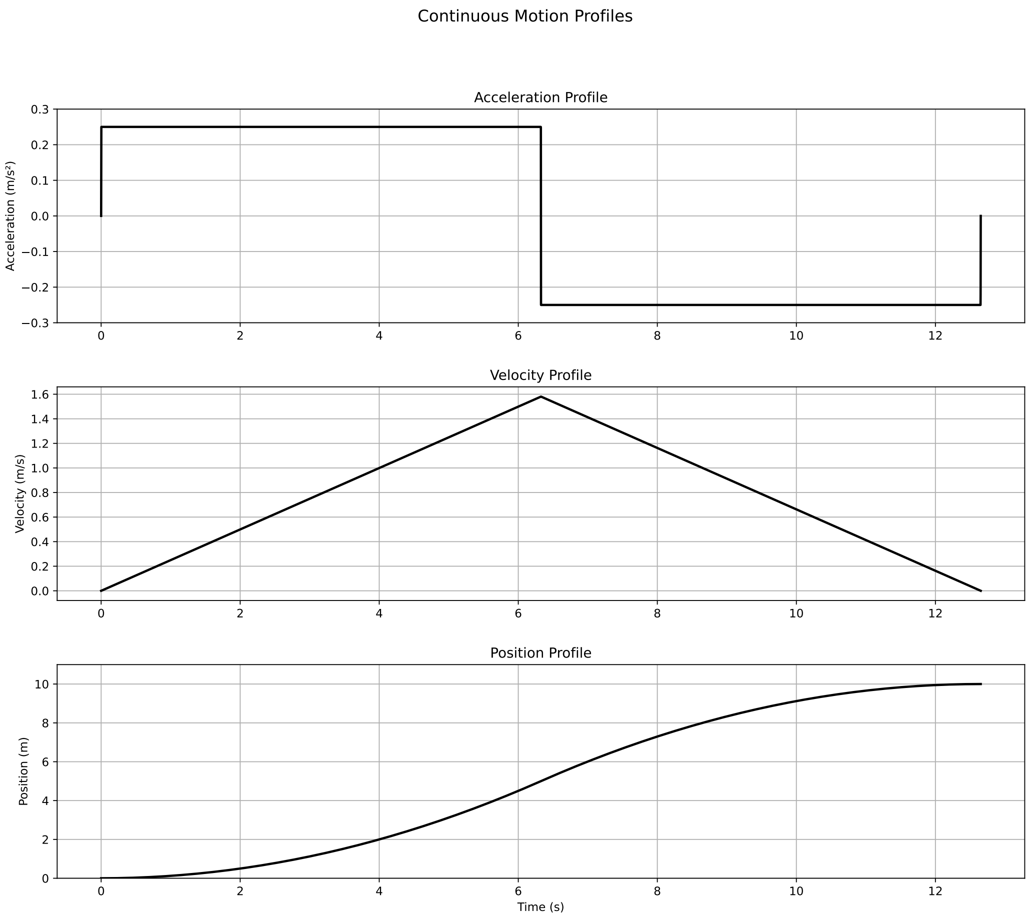

Figure 1: Motion profiles showing smooth transitions in acceleration (top), velocity (middle), and position (bottom) over time. Numerical computation parameters: $a_{\max} = 0.25$ m/s^2, $d = 10.0$ m, $\varepsilon = 0.001$ (for stable integration).

For numerical computation, we use $\varepsilon = 0.001$ to maintain stability while achieving high accuracy.

B.2 Convergence Analysis

We analyze convergence by varying ε:

ε	Time Penalty	Max Accel Error	Distance Error
0.1	2.3%	8.2%	0.4%
0.01	0.24%	0.85%	0.04%
0.001	0.025%	0.087%	0.004%

The results confirm:

- Quadratic convergence of travel time
- Linear convergence of acceleration profile
- Cubic convergence of position error