# Chapter 2 Problems

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August 17, 2020

1. (a)

   (b) **Show by explicit construction that two coordinate systems (as opposed to the six used in the text) suffice to cover $S^2$ (it is impossible to cover $S^2$ with a signle chart, as follows from the fact that $S^2$ is compact, but every open subset of $\mathbb{R}^2$ is noncompact see appendix A.)**

   Consider the collection of two subsets, $O_N$ and $O_S$. $O_N$ contains all the points in the set $S^2$ except the north pole $(0,0,1)$, and $O_S$ contains all the points in $S$ except the south pole $(0,0,-1)$. Together this collection contains every point in $S^2$ thereby satisfying property 1 required for a manifold.

   Next we must determine two maps, $\psi_N$ and $\psi_S$ for $O_N$ and $O_S$ respectively, and show that they are one-to-one and map $S^2$ to $\mathbb{R}^2$. To construct these maps consider Figure 1. Pictured are all the points in $S^2$ which lie in the $X$-$Z$ plane. We will use this diagram to construct the map $\psi$.

   Draw a line from the north pole $N$ that intersects the $S^2$ at a point $P$ and the $X$ axis at $P'$. The point $P$ has coordinates $(x,0,z)$. The triangles $NzP$ and $NOP'$ can be related using similar triangles. This will give a relation between the point $P$ and $P'$.

$$\frac{OP'}{NO} = \frac{zP}{Nz}$$
$$P' = \frac{x}{1-z}$$

   $P'$ is equivalent to mapping the any point $P$ into the $X-Y$ plane and we will denote this coordinate in the $X$ direction $X'$. The same argument can be made to find $Y'$ by replacing $x$ in the above equation with $y$. Therefore:

$$O^N : S^2 \to \mathbb{R}^2 = (X', Y') = (\frac{x}{1-z}, \frac{y}{1-z})$$

   $\psi_N$ can be constructed in a similar manner but replacing $z$ with $-z$:

$$(X', Y') = (\frac{x}{1+z}, \frac{y}{1+z})$$

   Each of these maps are clearly one to one and onto.

$$(X, Y) = (\frac{x}{1-z}, \frac{y}{1-z})$$

$$(x,y,z) = (\frac{2X}{1+X^2+Y^2}. \frac{2Y}{1+X^2+Y^2}, \frac{-1+X^2+Y^2}{1+X^2+Y^2})$$

   $\psi_N : O_N \to U_N \subset \mathbb{R}^2$ and $\psi_S : O_S \to U_S \subset \mathbb{R}^2$. In order to satisfy property 3 of manifolds we must show that $\psi_N \circ \psi_S^{-1}$ and $\psi_S \circ \psi_N^{-1}$ the

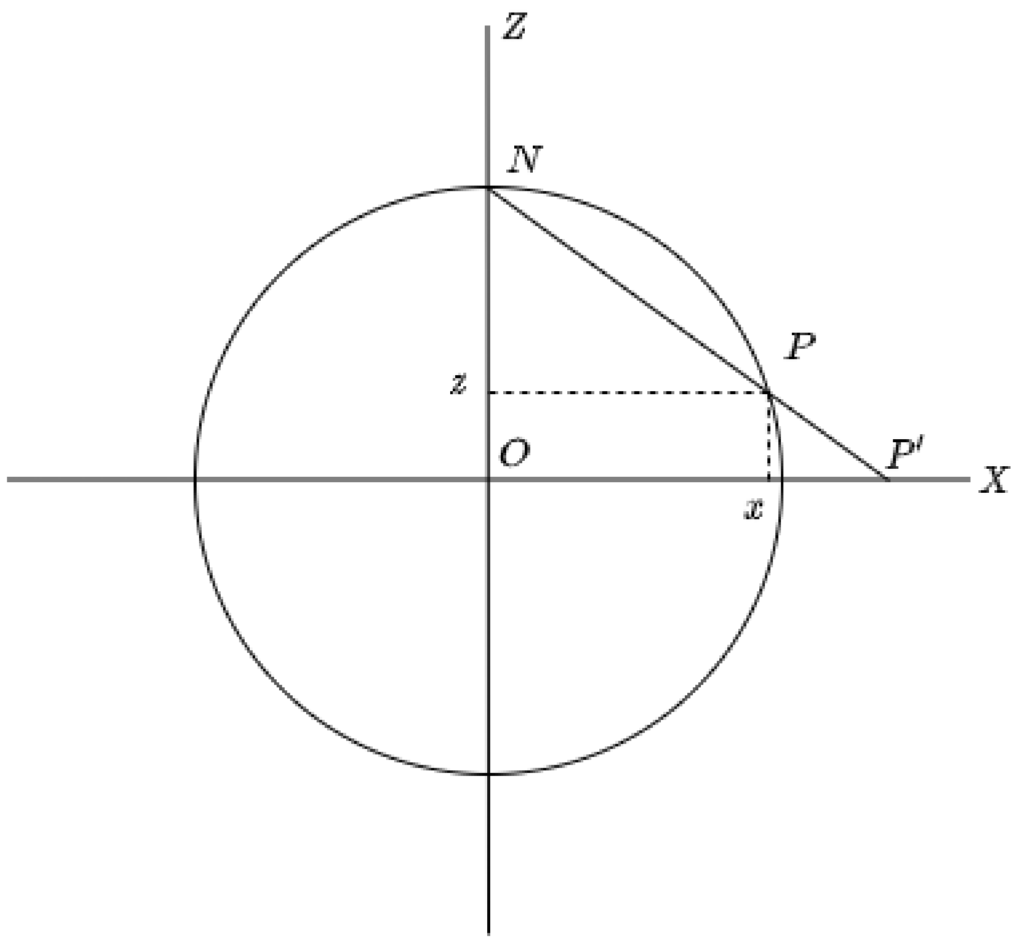


Figure 1: Projection of points on a circle into $\mathbb{R}^2$

subsets of the map are open and the maps are infinitely differentiable.

$$(X', Y') = (\frac{2X}{(1+X^2+Y^2)(1-\frac{-1+X^2+Y^2}{1+X^2+Y^2})}, \frac{2Y}{(1+X^2+Y^2)(1-\frac{-1+X^2+Y^2}{1+X^2+Y^2})})$$

$$(X', Y') = (\frac{2X}{1+X^2+Y^2+1-X^2-Y^2}, \frac{2Y}{1+X^2+Y^2+1-X^2-Y^2}) = (X, Y)$$

2. **Prove that any smooth function $F : \mathbb{R}^n \to \mathbb{R}$ can be written in the form equation (2.2.2)**

3. (a) **Verify that the commutator, defined by equation (2.2.14), satisfies the linearity and Leibnitz properties, and hence defines a vector field**

$$\begin{aligned}
[v,w](f+g) &= v(w(f+g)) - w(v(f+g)) \\
&= v(w(f)+w(g)) - w(v(f)+v(g)) \\
&= v(w(f)) + v(w(g)) - w(v(f)) - w(v(g)) \\
&= v(w(f)) - w(v(f)) + v(w(g)) - w(v(g)) \\
&= [v,w]f + [v,w]g
\end{aligned}$$

Therefore the commutator satisfies the linearity property in $v$. The same procedure can be applied symmetrically to $w$.

$$\begin{aligned}
[v,w](fg) &= v(w(fg)) - w(v(fg)) \\
&= v(w(f)g + fw(g)) - w(v(f)g + fv(g)) \\
&= v(w(f)g) + v(fw(g)) - w(v(f)g) - w(fv(g)) \\
&= v(w(f))g + w(f)v(g) + v(f)w(g) + fv(w(g)) \\
&- w(v(f))g - v(f)w(g) - w(f)v(g) - fw(v(g)) \\
&= v(w(f))g + f(v(w(g)) - w(v(f))g - fw(v(g)) \\
&= f(v(w(g)) - fw(v(g)) + v(w(f))g - w(v(f))g \\
&= f\Big((v(w(g)) - w(v(g))\Big) + g\Big(v(w(f)) - w(v(f))\Big) \\
&= f[v,w](g) + g[v,w](f)
\end{aligned}$$

(b) **Let $X$, $Y$, $Z$ be smooth vector fields on a manifold $M$. Verify that their commutator satisfies the Jacobi identity:**

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$$

Expand $[[X,Y],Z]$:

$$\begin{aligned}
[[X,Y],Z](f) &= [X,Y]Z(f) - Z[X,Y](f) \\
&= X(Y(Z(f))) - Y(X(Z(f))) - Z(X(Y(f)) - Y(Z(f))) \\
&= X(Y(Z(f))) - Y(X(Z(f))) - Z(X(Y(f))) + Z(Y(Z(f))) \\
&= X(Y(Z(f))) - Y(X(Z(f))) - Z(X(Y(f))) + Z(Y(Z(f)))
\end{aligned}$$

Now cyclically permute $X$,$Y$,$Z$:

$$
\begin{aligned}
&= \underbrace{X(Y(Z(f)))}_{a} - \underbrace{Y(X(Z(f)))}_{-g} - \underbrace{Z(X(Y(f)))}_{-e} + \underbrace{Z(Y(X(f)))}_{b} \\
&+ \underbrace{Y(Z(X(f)))}_{c} - \underbrace{Z(Y(X(f)))}_{-b} - \underbrace{X(Y(Z(f)))}_{-a} + \underbrace{X(Z(Y(f)))}_{d} \\
&+ \underbrace{Z(X(Y(f)))}_{e} - \underbrace{X(Z(Y(f)))}_{-d} - \underbrace{Y(Z(X(f)))}_{-c} + \underbrace{Y(X(Z(f)))}_{g} \\
&= 0
\end{aligned}
$$

(c) **Let $Y_1, \ldots, Y_n$ be smooth vector fields on an n-dimensional $M$ such that at each $p \in M$ they form a basis of the tangent space $V_p$. Then, at each point, we may expand each commutator $[Y_\alpha, Y_\beta]$ in this basis, thereby defining the functions $C^\gamma_{\alpha\beta} = -C^\gamma_{\beta\alpha}$ by**

$$[Y_\alpha, Y_\beta] = \sum_\gamma C^\gamma_{\alpha\beta} Y_\gamma$$

**Use the Jacobi identity to derive an equation satisfied by $C^\gamma_{\alpha\beta}$.**

Consider:

$$
\begin{aligned}
[[Y_\alpha, Y_\beta], Y_\sigma] &= [C^\gamma_{\alpha\beta} Y_\gamma, Y_\sigma] \\
&= C^\gamma_{\alpha\beta} Y_\gamma Y_\sigma - Y_\sigma C^\gamma_{\alpha\beta} Y_\gamma \\
&= C^\gamma_{\alpha\beta} Y_\gamma Y_\sigma - C^\gamma_{\alpha\beta} Y_\sigma Y_\gamma \\
&= C^\gamma_{\alpha\beta} [Y_\gamma Y_\sigma - Y_\sigma Y_\gamma] \\
&= C^\gamma_{\alpha\beta} \Big( Y_\gamma Y_\sigma - Y_\sigma Y_\gamma \Big) \\
&= C^\gamma_{\alpha\beta} [Y_\gamma, Y_\sigma] \\
&= C^\gamma_{\alpha\beta} C^\epsilon_{\gamma\sigma} Y_\epsilon
\end{aligned}
$$

Therefore the Jacobi identity gives:

$$
\begin{aligned}
[[Y_\alpha, Y_\beta], Y_\sigma] + [[Y_\beta, Y_\sigma], Y_\alpha] + [[Y_\sigma, Y_\alpha], Y_\beta] &= 0 \\
C^\gamma_{\alpha\beta} C^\epsilon_{\gamma\sigma} Y_\epsilon + C^\gamma_{\beta\sigma} C^\epsilon_{\gamma\alpha} Y_\epsilon + C^\gamma_{\sigma\alpha} C^\epsilon_{\gamma\beta} Y_\epsilon &= 0 \\
(C^\gamma_{\alpha\beta} C^\epsilon_{\gamma\sigma} + C^\gamma_{\beta\sigma} C^\epsilon_{\gamma\alpha} + C^\gamma_{\sigma\alpha} C^\epsilon_{\gamma\beta}) Y_\epsilon &= 0
\end{aligned}
$$

Therefore the equations satisfied by the functions are:

$$(C^\gamma_{\alpha\beta} C^\epsilon_{\gamma\sigma} + C^\gamma_{\beta\sigma} C^\epsilon_{\gamma\alpha} + C^\gamma_{\sigma\alpha} C^\epsilon_{\gamma\beta}) = 0$$

4. (a) **Show that in any coordinate basis, the components of the commutator of two vector fields $v$ and $w$ are given by**

$$[v, w]^\mu = \sum_\nu \left( v^\nu \frac{\partial w^\mu}{\partial x^\nu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \right)$$

$$
\begin{aligned}
[v,w](f) &= v(w(f)) - w(v(f)) \\
&= v(w^\mu \frac{\partial f}{\partial x^\mu}) - w(v^\mu \frac{\partial f}{\partial x^\mu}) \\
&= v^\nu \frac{\partial}{\partial x^\nu}(w^\mu \frac{\partial f}{\partial x^\mu}) - w^\nu \frac{\partial}{\partial x^\nu}(v^\mu \frac{\partial f}{\partial x^\mu}) \\
&= v^\nu \Big(\frac{\partial w^\mu}{\partial x^\nu}\frac{\partial f}{\partial x^\mu} + w^\mu \frac{\partial f}{\partial x^\nu x^\mu}\Big) - w^\nu \Big(\frac{\partial v^\mu}{\partial x^\nu}\frac{\partial f}{\partial x^\mu} + v^\mu \frac{\partial f}{\partial x^\nu x^\mu}\Big) \\
&= v^\nu \frac{\partial w^\mu}{\partial x^\nu}\frac{\partial f}{\partial x^\mu} + v^\nu w^\mu \frac{\partial f}{\partial x^\nu x^\mu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu}\frac{\partial f}{\partial x^\mu} - w^\nu v^\mu \frac{\partial f}{\partial x^\nu x^\mu} \\
&= v^\nu \frac{\partial w^\mu}{\partial x^\nu}\frac{\partial f}{\partial x^\mu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu}\frac{\partial f}{\partial x^\mu} + \Big(v^\nu w^\mu \frac{\partial f}{\partial x^\nu x^\mu} - w^\nu v^\mu \frac{\partial f}{\partial x^\nu x^\mu}\Big)
\end{aligned}
$$

Using the equality of mixed partial derivatives we can relabel the indices:

$$
\begin{aligned}
&= v^\nu \frac{\partial w^\mu}{\partial x^\nu}\frac{\partial f}{\partial x^\mu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu}\frac{\partial f}{\partial x^\mu} + \Big(v^\nu w^\mu \frac{\partial f}{\partial x^\nu x^\mu} - w^\mu v^\nu \frac{\partial f}{\partial x^\mu x^\nu}\Big) \\
&= v^\nu \frac{\partial w^\mu}{\partial x^\nu}\frac{\partial f}{\partial x^\mu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu}\frac{\partial f}{\partial x^\mu} \\
&= \Big(v^\nu \frac{\partial w^\mu}{\partial x^\nu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu}\Big)\frac{\partial f}{\partial x^\mu} \\
&= [v,w]^\mu \frac{\partial f}{\partial x^\mu}
\end{aligned}
$$

(b) **Let $Y_1$, …, $Y_n$ be as in problem 3(c). Let $Y^{1^*}$,…, $Y^{n^*}$ be the dual basis. Show that the components $(Y^{\gamma^*})_\mu$ of $Y^{\gamma^*}$ in any coordinate basis satisfy**

$$\frac{\partial (Y^{\gamma^*})_\mu}{\partial x^\nu} - \frac{\partial (Y^{\gamma^*})_\nu}{\partial x^\mu} = \sum_{\alpha,\beta} C^\gamma_{\alpha\beta}(Y^{\alpha^*})_\mu (Y^{\beta^*})_\nu$$

Considering the commutator used in problem 3(c):

$$[Y_\alpha, Y_\beta] = \sum_\gamma C^\gamma_{\alpha\beta} Y_\gamma$$

Act the commutator on a dual vector $Y^{\gamma^*}$:

$$[Y_\alpha, Y_\beta]Y^{\gamma^*} = \sum_\gamma C^\gamma_{\alpha\beta} Y_\gamma Y^{\gamma^*}$$

Start with the right hand side.

5.

6.

7.

8. (a) **The metric of flat, three-dimensional Euclidean space is:**

$$ds^2 = dx^2 + dy^2 + dz^2$$

**Show that the metric components $g_{uv}$ in spherical polar coordinates $r, \theta, \phi$ defined by:**

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ \cos\theta &= \frac{z}{r}, \\ \tan\phi &= \frac{y}{x} \end{aligned}$$

**is given by:**

$$s^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \phi^2$$

$g_{uv}$ is a tensor of type $(0, 2)$ and therefore transforms as:

$$g_{\mu',\nu'} = g_{\mu,\nu} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}}$$

(see page 22 for the general *tensor transformation law*). The above equation uses Einstein index notation indicating that $\mu$ and $\nu$ are to to be summed from 1 to 3 and the free indices, $\mu'$ and $\nu'$, are enumerated through all possible combinations. Therefore the components that need to be calculated are:

$$\begin{matrix} g_{r,r} & g_{r,\theta} & g_{r,\phi} \\ g_{\theta,r} & g_{\theta,\theta} & g_{\theta,\phi} \\ g_{\phi,r} & g_{\phi,\theta} & g_{\phi,\phi} \end{matrix}$$

Starting with:

$$g_{\mu',\nu'} = \sum_{\mu=1}^{3} \sum_{\nu=1}^{3} g_{\mu,\nu} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}}$$

$$= \sum_{\mu=1}^{3} g_{\mu,1} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{\mu,2} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{\mu,3} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}}$$

$$\begin{aligned} = g_{1,1} \frac{\partial x^1}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{1,2} \frac{\partial x^1}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{1,3} \frac{\partial x^1}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}} \\ g_{2,1} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{2,2} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{2,3} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}} \\ g_{3,1} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{3,2} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{3,3} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}} \end{aligned}$$

Substituting the notation for the indices in flat, orthonormal Euclidean space:

$$= g_{x,x}\frac{\partial x}{\partial x^{\mu'}}\frac{\partial x}{\partial x^{\nu'}} + g_{x,y}\frac{\partial x}{\partial x^{\mu'}}\frac{\partial y}{\partial x^{\nu'}} + g_{x,z}\frac{\partial x}{\partial x^{\mu'}}\frac{\partial z}{\partial x^{\nu'}}$$
$$g_{y,x}\frac{\partial y}{\partial x^{\mu'}}\frac{\partial x}{\partial x^{\nu'}} + g_{y,y}\frac{\partial y}{\partial x^{\mu'}}\frac{\partial y}{\partial x^{\nu'}} + g_{y,z}\frac{\partial y}{\partial x^{\mu'}}\frac{\partial z}{\partial x^{\nu'}}$$
$$g_{z,x}\frac{\partial y}{\partial x^{\mu'}}\frac{\partial x}{\partial x^{\nu'}} + g_{z,y}\frac{\partial y}{\partial x^{\mu'}}\frac{\partial y}{\partial x^{\nu'}} + g_{z,z}\frac{\partial y}{\partial x^{\mu'}}\frac{\partial z}{\partial x^{\nu'}}$$

The off diagonal elements of the Euclidean metric are zero:

$$g_{x,y} = g_{y,x} = g_{x,z} = g_{z,x} = g_{y,z} = g_{z,y} = 0$$

and the diagonal components are one:

$$g_{x,x} = g_{y,y} = g_{z,z} = 1$$

This reduces the above summation from nine expressions to the following three:

$$g_{\mu',\nu'} = \frac{\partial x}{\partial x^{\mu'}}\frac{\partial x}{\partial x^{\nu'}} + \frac{\partial y}{\partial x^{\mu'}}\frac{\partial y}{\partial x^{\nu'}} + \frac{\partial z}{\partial x^{\mu'}}\frac{\partial z}{\partial x^{\nu'}}$$

For indices where $\mu' = \nu'$

$$g_{\mu',\mu'} = \left(\frac{\partial x}{\partial x^{\mu'}}\right)^2 + \left(\frac{\partial y}{\partial x^{\mu'}}\right)^2 + \left(\frac{\partial z}{\partial x^{\mu'}}\right)^2$$

Therefore the six unique components that need to be calculated to find the components of the metric in spherical polar coorindates are:

$$g_{r,r} = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2$$
$$g_{r,\theta} = g_{\theta,r} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta}$$
$$g_{r,\phi} = g_{\phi,r} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \phi}$$
$$g_{\theta,\theta} = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2$$
$$g_{\theta,\phi} = g_{\phi,\theta} = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \phi}$$
$$g_{\phi,\phi} = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2$$

To take the above derivatives, find an equation for $x$, $y$, $z$ in terms of $r$, $\theta$, $\phi$. Starting by finding $x$:

$$
\begin{aligned}
&r = \sqrt{x^2 + y^2 + z^2} \to r^2 = x^2 + z^2 + y^2, \\
&\cos\theta = \frac{z}{r} \to z = r\cos\theta, \\
&\tan\phi = \frac{y}{x} \to y = x\tan\phi
\end{aligned}
$$

Substituting the second and third equation into the first gives:

$$
\begin{aligned}
r^2 &= x^2 + (r\cos\theta)^2 + (x\tan\phi)^2 \\
r^2 &= x^2 + r^2\cos^2\theta + x^2\tan^2\phi \\
r^2 - r^2\cos\theta &= x^2 + x^2\tan^2\phi \\
(1 - \cos^2\theta)r^2 &= (1 + \tan^2\phi)x^2 \\
r^2\sin^2\theta &= (1 + \tan^2\phi)x^2 \\
x &= r\frac{\sin\theta}{\sqrt{1 + \tan^2\phi}}
\end{aligned}
$$

Therefore the equations for $x$, $y$, $z$ in terms of $r$, $\theta$, $\phi$:

$$
x = r\frac{\sin\theta}{\sqrt{1 + \tan^2\phi}}, \quad y = r\tan\phi\frac{\sin\theta}{\sqrt{1 + \tan^2\phi}}, \quad z = r\cos\theta
$$

Find all the necessary derivatives:

$$
\begin{aligned}
\frac{\partial x}{\partial r} &= \frac{\sin\theta}{\sqrt{1 + \tan^2\phi}} \\
\frac{\partial x}{\partial \theta} &= -r\frac{\cos\theta}{\sqrt{1 + \tan^2\phi}} \\
\frac{\partial x}{\partial \phi} &= -r\sin\theta\frac{\tan\phi\sec^2\phi}{(1 + \tan^2\phi)^{\frac{3}{2}}} \\
\frac{\partial y}{\partial r} &= \tan\phi\frac{\sin\theta}{\sqrt{1 + \tan^2\phi}} \\
\frac{\partial y}{\partial \theta} &= -r\tan\phi\frac{\cos\theta}{\sqrt{1 + \tan^2\phi}} \\
\frac{\partial y}{\partial \phi} &= -r\sin\theta\frac{\sec^2\phi}{(1 + \tan^2\phi)^{\frac{3}{2}}} \\
\frac{\partial z}{\partial r} &= \cos\theta \\
\frac{\partial z}{\partial \theta} &= r\sin\theta \\
\frac{\partial z}{\partial \phi} &= 0
\end{aligned}
$$

Then compute the components of the metric in spherical polar coordinates:

$$
\begin{aligned}
g_{r,r} &= \frac{\sin\theta^2}{1+\tan^2\phi} + \frac{\sin^2\theta}{1+\tan^2\phi}\tan^2\phi + \cos^2\theta \\
&= \frac{\sin\theta^2 + \sin^2\theta\tan^2\phi}{1+\tan^2\phi} + \frac{(1+\tan^2\phi)\cos^2\theta}{1+\tan^2\phi} \\
&= \frac{\sin\theta^2 + \sin^2\theta\tan^2\phi + (1+\tan^2\phi)\cos^2\theta}{1+\tan^2\phi} \\
&= \frac{\sin^2\theta + \cos^2\theta + \sin^2\theta\tan^2\phi + \tan^2\phi\cos^2\theta}{1+\tan^2\phi} \\
&= \frac{1+\tan^2\phi}{1+\tan^2\phi} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
g_{\theta,\theta} &= r^2\frac{\cos^2\theta}{1+\tan^2\phi} + r^2\frac{\cos^2\theta}{1+\tan^2\phi}\tan^2\phi + r^2\sin^2\theta \\
&= r^2\frac{\cos^2\theta}{1+\tan^2\phi} + r^2\frac{\cos^2\theta}{1+\tan^2\phi}\tan^2\phi + r^2\sin^2\theta\frac{1+\tan^2\phi}{1+\tan^2\phi} \\
&= r^2\frac{\cos^2\theta + \cos^2\theta\tan^2\phi + \sin^2\theta + \tan^2\phi\sin^2\theta}{1+\tan^2\phi} \\
&= r^2\frac{(\cos^2\theta + \sin^2\theta) + (\cos^2\theta + \sin^2\theta)\tan^2\phi}{1+\tan^2\phi} \\
&= r^2\frac{1+\tan^2\phi}{1+\tan^2\phi} \\
&= r^2
\end{aligned}
$$

$$
\begin{aligned}
g_{\phi,\phi} &= r^2\sin^2\theta\left(\frac{1}{(1+\tan^2\phi)^3\cos^4\phi} + \frac{\sin^2\phi}{(1+\tan^2\phi)^3\cos^6 x}\right) \\
&= r^2\sin^2\theta\left(\frac{\cos^2\phi + \sin^2\phi}{(1+\tan^2\phi)^3\cos^6\phi}\right) \\
&= r^2\sin^2\theta\frac{1}{\left(\frac{\sin^2\phi+\cos^2\phi}{\cos^2\phi}\right)^3\cos^6\phi} \\
&= r^2\sin^2\theta
\end{aligned}
$$

$$
\begin{aligned}
g_{\theta,r} = g_{r,\theta} &= g_{x,x}\frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + g_{y,y}\frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} + g_{z,z}\frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta} \\
&= -r\frac{\sin\theta\cos\theta}{1+\tan^2\phi} - r\frac{\sin\theta\cos\theta}{1+\tan^2\phi}\tan^2\phi + r\sin\theta\cos\theta \\
&= -r\frac{\sin\theta\cos\theta}{1+\tan^2\phi}(1+\tan^2\phi) + r\sin\theta\cos\theta \\
&= -r\sin\theta\cos\theta + r\sin\theta\cos\theta \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
g_{r,\phi} = g_{\phi,r} &= g_{x,x}\frac{\partial x}{\partial r}\frac{\partial x}{\partial \phi} + g_{y,y}\frac{\partial y}{\partial r}\frac{\partial y}{\partial \phi} + g_{z,z}\frac{\partial z}{\partial r}\frac{\partial z}{\partial \phi} \\
&= -r\sin\theta\cos\theta\frac{\tan\phi\sec^2\phi}{(1+\tan^2\phi)^2} + r\cos\theta\sin\theta\frac{\tan\phi\sec^2\phi}{(1+\tan^2\phi)^2} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
g_{\theta,\phi} = g_{\phi,\theta} &= g_{x,x}\frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \phi} + g_{y,y}\frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \phi} + g_{z,z}\frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \phi} \\
&= -r\frac{\cos\theta}{\sqrt{1+\tan^2\phi}}\left(-r\sin\theta\frac{\tan\phi\sec^2\phi}{(1+\tan^2\phi)^{\frac{3}{2}}}\right) - r\tan\phi\frac{\cos\theta}{\sqrt{1+\tan^2\phi}}\left(r\sin\theta\frac{\sec^2\phi}{(1+\tan^2\phi)^{\frac{3}{2}}}\right. \\
&= r^2\sin\theta\cos\theta\frac{\tan\phi\sec^2\phi}{(1+\tan^2\phi)^2} - r^2\sin\theta\cos\theta\frac{\tan\phi\sec^2\phi}{(1+\tan^2\phi)^2} \\
&= 0
\end{aligned}
$$

Therefore the metric components in spherical polar coordinates are:

$$
g_{\mu,\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}
$$

(b) **The spacetime metric of special relativity is**

$$
ds^2 = -dt^2 + dx^2 + dy^2 + dz^2
$$

**Find the components, $g_{\mu\nu}$ and $g^{\mu\nu}$, of the metric and inverse metric in "rotating coordinates", defined by**

$$
\begin{aligned}
t' &= t \\
x' &= (x^2+y^2)^{\frac{1}{2}}\cos(\phi - wt) \\
y' &= (x^2+y^2)^{\frac{1}{2}}\sin(\phi - wt) \\
z' &= z
\end{aligned}
$$

**where** $\tan\phi = \frac{y}{x}$
It is easier differentiate with respect to the primed coordinates so

find $g^{\mu\nu}$ first. First writting the primed coordinates in terms of the unprimed:

$$
\begin{aligned}
t' &= t \\
x' &= (x^2+y^2)^{\frac{1}{2}} \cos(\tan^{-1}\frac{y}{x} - wt) \\
y' &= (x^2+y^2)^{\frac{1}{2}} \sin(\tan^{-1}\frac{y}{x} - wt) \\
z' &= z
\end{aligned}
$$

Find all the necessary derivatives:

$$
\begin{aligned}
\frac{\partial t'}{\partial t} &= 1 \\
\frac{\partial t'}{\partial x} &= \frac{\partial t'}{\partial y} = \frac{\partial t}{\partial z} = 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial x'}{\partial t} &= -w\sqrt{x^2+y^2}\sin(\tan^{-1}\frac{y}{x} - wt) \\
\frac{\partial x'}{\partial x} &= \frac{x\cos(\tan^{-1}\frac{y}{x} - \omega t) + y\sin(\tan^{-1}\frac{y}{x} - \omega t)}{\sqrt{x^2+y^2}} \\
\frac{\partial x'}{\partial y} &= \frac{-x\sin(\tan^{-1}\frac{y}{x} - \omega t) + y\cos(\tan^{-1}\frac{y}{x} - \omega t)}{\sqrt{x^2+y^2}} \\
\frac{\partial x'}{\partial z} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial y'}{\partial t} &= -w\sqrt{x^2+y^2}\cos(\tan^{-1}\frac{y}{x} - wt) \\
\frac{\partial y'}{\partial x} &= \frac{x\sin(\tan^{-1}\frac{y}{x} - \omega t) - y\cos(\tan^{-1}\frac{y}{x} - \omega t)}{\sqrt{x^2+y^2}} \\
\frac{\partial y'}{\partial y} &= \frac{x\sin(\tan^{-1}\frac{y}{x} - \omega t) + y\cos(\tan^{-1}\frac{y}{x} - \omega t)}{\sqrt{x^2+y^2}} \\
\frac{\partial y'}{\partial z} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial z'}{\partial t} &= \frac{\partial z'}{\partial x} = \frac{\partial z'}{\partial y} = 0 \\
\frac{\partial z'}{\partial z} &= 1
\end{aligned}
$$

$$
\left(\frac{\partial x'}{\partial x}\right)^2 = \left(\frac{x\cos(\tan^{-1}\frac{y}{x} - \omega t) + y\sin(\tan^{-1}\frac{y}{x} - \omega t)}{\sqrt{x^2+y^2}}\right)^2 = \frac{(x^2+y^2)\sin^2(wt)}{(x^2+y^2)}
$$

$$\left(\frac{\partial y'}{\partial y}\right)^2 = \left(\frac{x\cos(\tan^{-1}\frac{y}{x} - \omega t) + y\sin(\tan^{-1}\frac{y}{x} - \omega t)}{\sqrt{x^2+y^2}}\right)^2 = (x^2+y^2)\sin^2(wt)$$

## 0.1 Appendix A - Tensor Expansion

The metric is a rank (0,2) tensor so the transformation of the components between basis is given by:

$$g_{\mu',\nu'} = g_{\mu,\nu}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}$$

This equation is a shorthand for the following:

$$\begin{aligned}
g_{\mu',\nu'} &= \sum_{\mu=1}^{3}\sum_{\nu=1}^{3} g_{\mu,\nu}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}} \\
g_{\mu',\nu'} &= \sum_{\mu=1}^{3} g_{\mu,1}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^1}{\partial x^{\nu'}} + g_{\mu,2}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^2}{\partial x^{\nu'}} + g_{\mu,3}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^3}{\partial x^{\nu'}} \\
g_{\mu',\nu'} &= g_{1,1}\frac{\partial x^1}{\partial x^{\mu'}}\frac{\partial x^1}{\partial x^{\nu'}} + g_{1,2}\frac{\partial x^1}{\partial x^{\mu'}}\frac{\partial x^2}{\partial x^{\nu'}} + g_{1,3}\frac{\partial x^1}{\partial x^{\mu'}}\frac{\partial x^3}{\partial x^{\nu'}} \\
& g_{2,1}\frac{\partial x^2}{\partial x^{\mu'}}\frac{\partial x^1}{\partial x^{\nu'}} + g_{2,2}\frac{\partial x^2}{\partial x^{\mu'}}\frac{\partial x^2}{\partial x^{\nu'}} + g_{2,3}\frac{\partial x^2}{\partial x^{\mu'}}\frac{\partial x^3}{\partial x^{\nu'}} \\
& g_{3,1}\frac{\partial x^2}{\partial x^{\mu'}}\frac{\partial x^1}{\partial x^{\nu'}} + g_{3,2}\frac{\partial x^2}{\partial x^{\mu'}}\frac{\partial x^2}{\partial x^{\nu'}} + g_{3,3}\frac{\partial x^2}{\partial x^{\mu'}}\frac{\partial x^3}{\partial x^{\nu'}}
\end{aligned}$$

The full expansion for $\mu', \nu'$ is:

$$\begin{aligned}
g_{1',1'} = & g_{1,1}\frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^1}{\partial x^{1'}} + g_{1,2}\frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{1'}} + g_{1,3}\frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^3}{\partial x^{1'}} \\
& g_{2,1}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^1}{\partial x^{1'}} + g_{2,2}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{1'}} + g_{2,3}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^3}{\partial x^{1'}} \\
& g_{3,1}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^1}{\partial x^{1'}} + g_{3,2}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{1'}} + g_{3,3}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^3}{\partial x^{1'}}
\end{aligned}$$

$$\begin{aligned}
g_{1',2'} = & g_{1,1}\frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^1}{\partial x^{2'}} + g_{1,2}\frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{2'}} + g_{1,3}\frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^3}{\partial x^{2'}} \\
& g_{2,1}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^1}{\partial x^{2'}} + g_{2,2}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{2'}} + g_{2,3}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^3}{\partial x^{2'}} \\
& g_{3,1}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^1}{\partial x^{2'}} + g_{3,2}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{2'}} + g_{3,3}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^3}{\partial x^{2'}} \\
g_{1',3'} = & g_{1,1}\frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^1}{\partial x^{3'}} + g_{1,2}\frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{3'}} + g_{1,3}\frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^3}{\partial x^{3'}} \\
& g_{2,1}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^1}{\partial x^{3'}} + g_{2,2}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{3'}} + g_{2,3}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^3}{\partial x^{3'}} \\
& g_{3,1}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^1}{\partial x^{3'}} + g_{3,2}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{3'}} + g_{3,3}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^3}{\partial x^{3'}}
\end{aligned}$$

$$
\begin{aligned}
g_{2',1'} = {} & g_{1,1}\frac{\partial x^1}{\partial x^{2'}}\frac{\partial x^1}{\partial x^{1'}} + g_{1,2}\frac{\partial x^1}{\partial x^{2'}}\frac{\partial x^2}{\partial x^{1'}} + g_{1,3}\frac{\partial x^1}{\partial x^{2'}}\frac{\partial x^3}{\partial x^{1'}} \\
& g_{2,1}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^1}{\partial x^{1'}} + g_{2,2}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^2}{\partial x^{1'}} + g_{2,3}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^3}{\partial x^{1'}} \\
& g_{3,1}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^1}{\partial x^{1'}} + g_{3,2}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^2}{\partial x^{1'}} + g_{3,3}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^3}{\partial x^{1'}}
\end{aligned}
$$

$$
\begin{aligned}
g_{2',2'} = {} & g_{1,1}\frac{\partial x^1}{\partial x^{2'}}\frac{\partial x^1}{\partial x^{2'}} + g_{1,2}\frac{\partial x^1}{\partial x^{2'}}\frac{\partial x^2}{\partial x^{2'}} + g_{1,3}\frac{\partial x^1}{\partial x^{2'}}\frac{\partial x^3}{\partial x^{2'}} \\
& g_{2,1}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^1}{\partial x^{2'}} + g_{2,2}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^2}{\partial x^{2'}} + g_{2,3}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^3}{\partial x^{2'}} \\
& g_{3,1}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^1}{\partial x^{2'}} + g_{3,2}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^2}{\partial x^{2'}} + g_{3,3}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^3}{\partial x^{2'}} \\
g_{2',3'} = {} & g_{1,1}\frac{\partial x^1}{\partial x^{2'}}\frac{\partial x^1}{\partial x^{3'}} + g_{1,2}\frac{\partial x^1}{\partial x^{2'}}\frac{\partial x^2}{\partial x^{3'}} + g_{1,3}\frac{\partial x^1}{\partial x^{2'}}\frac{\partial x^3}{\partial x^{3'}} \\
& g_{2,1}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^1}{\partial x^{3'}} + g_{2,2}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^2}{\partial x^{3'}} + g_{2,3}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^3}{\partial x^{3'}} \\
& g_{3,1}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^1}{\partial x^{3'}} + g_{3,2}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^2}{\partial x^{3'}} + g_{3,3}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^3}{\partial x^{3'}} \\
g_{3',1'} = {} & g_{1,1}\frac{\partial x^1}{\partial x^{3'}}\frac{\partial x^1}{\partial x^{1'}} + g_{1,2}\frac{\partial x^1}{\partial x^{3'}}\frac{\partial x^2}{\partial x^{1'}} + g_{1,3}\frac{\partial x^1}{\partial x^{3'}}\frac{\partial x^3}{\partial x^{1'}} \\
& g_{2,1}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^1}{\partial x^{1'}} + g_{2,2}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^2}{\partial x^{1'}} + g_{2,3}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^3}{\partial x^{1'}} \\
& g_{3,1}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^1}{\partial x^{1'}} + g_{3,2}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^2}{\partial x^{1'}} + g_{3,3}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^3}{\partial x^{1'}} \\
g_{3',2'} = {} & g_{1,1}\frac{\partial x^1}{\partial x^{3'}}\frac{\partial x^1}{\partial x^{2'}} + g_{1,2}\frac{\partial x^1}{\partial x^{3'}}\frac{\partial x^2}{\partial x^{2'}} + g_{1,3}\frac{\partial x^1}{\partial x^{3'}}\frac{\partial x^3}{\partial x^{2'}} \\
& g_{2,1}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^1}{\partial x^{2'}} + g_{2,2}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^2}{\partial x^{2'}} + g_{2,3}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^3}{\partial x^{2'}} \\
& g_{3,1}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^1}{\partial x^{2'}} + g_{3,2}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^2}{\partial x^{2'}} + g_{3,3}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^3}{\partial x^{2'}} \\
g_{3',3'} = {} & g_{1,1}\frac{\partial x^1}{\partial x^{3'}}\frac{\partial x^1}{\partial x^{3'}} + g_{1,2}\frac{\partial x^1}{\partial x^{3'}}\frac{\partial x^2}{\partial x^{3'}} + g_{1,3}\frac{\partial x^1}{\partial x^{3'}}\frac{\partial x^3}{\partial x^{3'}} \\
& g_{2,1}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^1}{\partial x^{3'}} + g_{2,2}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^2}{\partial x^{3'}} + g_{2,3}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^3}{\partial x^{3'}} \\
& g_{3,1}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^1}{\partial x^{3'}} + g_{3,2}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^2}{\partial x^{3'}} + g_{3,3}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^3}{\partial x^{3'}}
\end{aligned}
$$

The tensor transformation law for a tensor of rank (0, 2) in 3 space represents 9 components. Each components contains a summation of 9 expressions. Each expression has 3 terms. So in total the equation is representative of 243 terms.