Chapter 2 Problems

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1. **The metric of flat, three-dimensional Euclidean space is:**

$$ds^2 = dx^2 + dy^2 + dz^2$$

Show that the metric components g_{uv} in spherical polar coordinates r, θ, ϕ defined by:

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\cos\theta = \frac{z}{r},$$
$$\tan\phi = \frac{y}{x}$$

is given by:

$$s^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \phi^2$$

g_{uv} is a tensor of type $(0, 2)$ and therefore transforms as:

$$g_{\mu',\nu'} = g_{\mu,\nu} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}}$$

(see page 22 for the general *tensor transformation law*). The above equation uses Einstein index notation indicating that μ and ν are to to be summed from 1 to 3 and the free indices, μ' and ν', are enumerated through all possible combinations:

$$\begin{matrix} g_{1'1'} & g_{1'2'} & g_{1'3'} \\ g_{2'1'} & g_{2'2'} & g_{2'3'} \\ g_{3'1'} & g_{3'2'} & g_{3'3'} \end{matrix}$$

Starting with:

$$g_{1',1'} = \sum_{\mu=1}^{3} \sum_{\nu=1}^{3} g_{\mu,\nu} \frac{\partial x^{\mu}}{\partial x^{1'}} \frac{\partial x^{\nu}}{\partial x^{1'}}$$

$$= \sum_{\mu=1}^{3} g_{\mu,1} \frac{\partial x^{\mu}}{\partial x^{1'}} \frac{\partial x^{1}}{\partial x^{1'}} + g_{\mu,2} \frac{\partial x^{\mu}}{\partial x^{1'}} \frac{\partial x^{2}}{\partial x^{1'}} + g_{\mu,3} \frac{\partial x^{\mu}}{\partial x^{1'}} \frac{\partial x^{3}}{\partial x^{1'}}$$

The off diagonal elements of the Euclidean metric written in matrix for is an identity matrix. This reduces the above summation from a possible 9 components to the following three:

$$g_{1',1'} = \frac{\partial x^1}{\partial x^{1'}} \frac{\partial x^1}{\partial x^{1'}} + \frac{\partial x^2}{\partial x^{1'}} \frac{\partial x^2}{\partial x^{1'}} + \frac{\partial x^3}{\partial x^{1'}} \frac{\partial x^3}{\partial x^{1'}} = \left(\frac{\partial x^1}{\partial x^{1'}}\right)^2 + \left(\frac{\partial x^2}{\partial x^{1'}}\right)^2 + \left(\frac{\partial x^3}{\partial x^{1'}}\right)^2$$

Converting the notation to reflect the choices of basis:

$$g_{r,r} = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2$$
$$g_{\theta,\theta} = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2$$
$$g_{\phi,\phi} = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2$$

First, find an equation for x in terms of r. Rearranging:

$$\begin{aligned}
r &= \sqrt{x^2+y^2+z^2} \to r^2 = x^2+z^2+y^2,\\
\cos\theta &= \frac{z}{r} \to z = r\cos\theta,\\
\tan\phi &= \frac{y}{x} \to y = x\tan\phi
\end{aligned}$$

Substituting the second and third equation into the first gives:

$$\begin{aligned}
r^2 &= x^2 + (r\cos\theta)^2 + (x\tan\phi)^2\\
r^2 &= x^2 + r^2\cos^2\theta + x^2\tan^2\phi\\
r^2 - r^2\cos\theta &= x^2 + x^2\tan^2\phi\\
(1-\cos^2\theta)r^2 &= (1+\tan^2\phi)x^2\\
r^2\sin^2\theta &= (1+\tan^2\phi)x^2\\
x &= r\sqrt{\frac{\sin\theta^2}{1+\tan^2\phi}}
\end{aligned}$$

Differentiating:

$$\begin{aligned}
\frac{\partial x}{\partial r} &= \sqrt{\frac{\sin^2\theta}{1+\tan^2\phi}},\\
\frac{\partial x}{\partial \theta} &= -r\frac{\sqrt{1+\tan^2\phi}}{\sin}\frac{\sin\theta\cos\theta}{1+\tan^2\phi}\\
&= -r\frac{\cos\theta}{\sqrt{1+\tan^2\phi}}.
\end{aligned}$$

Second, find an equation for y in terms of r, and then differentiate:

$$\begin{aligned}
y &= x\tan\phi\\
&= r\sqrt{\frac{\sin\theta^2}{1+\tan^2\phi}}\tan\phi\\
\frac{\partial y}{\partial r} &= \sqrt{\frac{\sin^2\theta}{1+\tan^2\phi}}\tan\phi
\end{aligned}$$

Lastly, z in terms of r simply follows from the definitiono of them transformation:

$$z = r\cos\theta \to \frac{\partial z}{\partial r} = \cos\theta$$

Therefore:

$$
\begin{aligned}
g_{rr} &= \frac{\sin\theta^2}{1+\tan^2\phi} + \frac{\sin^2\theta}{1+\tan^2\phi}\tan^2\phi + \cos^2\theta \\
g_{rr} &= \frac{\sin\theta^2 + \sin^2\theta\tan^2\phi}{1+\tan^2\phi} + \frac{(1+\tan^2\phi)\cos^2\theta}{1+\tan^2\phi} \\
g_{rr} &= \frac{\sin\theta^2 + \sin^2\theta\tan^2\phi + (1+\tan^2\phi)\cos^2\theta}{1+\tan^2\phi} \\
g_{rr} &= \frac{\sin^2\theta + \cos^2\theta + \sin^2\theta\tan^2\phi + \tan^2\phi\cos^2\theta}{1+\tan^2\phi} \\
g_{rr} &= \frac{1+\tan^2\phi}{1+\tan^2\phi} \\
g_{rr} &= 1 \\
&.
\end{aligned}
$$