Chapter 2 Problems

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1. **A tube of mass M and length l is free to swing around a pivot at one end. A mass m is positioned inside the (fricitonless) tube at this end. The tube is held horizontal and then released. Let θ be the angle of the tube with respect to the horizontal, and let x be the distance the mass has traveled along the tube. Find the Euler-Lagrange equations for θ and x, and then write them in terms of θ and $\nu = x/l$ (the fraction of the distance along the tube). These equations can only be solved numerically, and you must pick a numerical value for the ratio $r = m/M$ in order to do this. Write a prigram that produces the value of ν when the tube is vertical** $(\theta = \pi/2)$

0.1 Derivation of Equations of Motion

The total kinetic energy of the system will be the horizontal and vertical translational kinetic energies of the mass m, and the rotational kinetic energy of the tube.

Starting with the mass, consider its position x' and y':

$$\begin{aligned}
x'(t) &= x(t)\sin\theta(t) \\
y'(t) &= x(t)\cos\theta(t)
\end{aligned}$$

where x is the position the mass has fallen down the tube. Differentiating once and removing the explicit time dependence for brevity yields:

$$\begin{aligned}
\dot{x}' &= \dot{x}\sin\theta + x\cos\theta\dot{\theta} \\
\dot{y}' &= \dot{x}\cos\theta - x\sin\theta\dot{\theta}
\end{aligned}$$

Taking the squares of each:

$$\begin{aligned}
(\dot{x}')^2 &= \dot{x}^2\sin^2\theta + x^2\cos^2\theta\dot{\theta}^2 + 2\sin\theta\cos\theta\dot{x}\dot{\theta} \\
(\dot{y}')^2 &= \dot{x}^2\cos^2\theta + x^2\sin^2\theta\dot{\theta}^2 - 2\sin\theta\cos\theta\dot{x}\dot{\theta}
\end{aligned}$$

and adding:

$$\begin{aligned}
(\dot{x}')^2 + (\dot{y}')^2 &= \dot{x}^2\sin^2\theta + \dot{x}^2\cos^2\theta + x^2\cos^2\theta\dot{\theta}^2 + x^2\sin^2\theta\dot{\theta}^2 + 2\sin\theta\cos\theta\dot{x}\dot{\theta} - 2\sin\theta\cos\theta\dot{x}\dot{\theta} \\
&= \dot{x}^2 + x^2\dot{\theta}^2
\end{aligned}$$

corresponding to radial and tangential kinetic energies respectively. Therefore:

$$T_m = \frac{1}{2}m(\dot{x}^2 + x^2\dot{\theta}^2)$$

Now the moment of inertia for a tube rotating about its end is the same as a rod rotating about its end, and is given by:

$$I_M = \frac{1}{3}Ml^2$$

Therefore the total kinetic energies of the two masses are:

$$T = \frac{1}{2}m(\dot{x}^2 + x^2\dot{\theta}^2) + \frac{1}{6}Ml^2\dot{\theta}^2$$

The potential energy for m is simply:

$$V_m = -mgl\cos\theta$$

while the potnetial energy for m is the vertical component of the position of the center mass:

$$V_M = -\frac{Mgl}{2}\cos\theta$$

The Lagrangian $\mathcal{L}$ is:

$$\mathcal{L} = T - V = \frac{1}{2}m(\dot{x}^2 + x^2\dot{\theta}^2) + \frac{1}{6}Ml^2\dot{\theta}^2 + mgl\cos\theta + \frac{Mgl}{2}\cos\theta$$

Finding the Euler-Lagrange equations for x:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} = m\ddot{x}$$

$$\frac{\partial\mathcal{L}}{\partial x} = mx\dot{\theta}^2$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} = \frac{\partial\mathcal{L}}{\partial x} \Rightarrow \ddot{x} = x\dot{\theta}^2$$

and θ:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = m\frac{d}{dt}(x^2\dot{\theta}) + \frac{1}{3}Ml^2\ddot{\theta}$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = -gl(m + \frac{M}{2})\sin\theta$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{\partial\mathcal{L}}{\partial\theta} \Rightarrow m\frac{d}{dt}(x^2\dot{\theta}) + \frac{1}{3}Ml^2\ddot{\theta} = -gl(m + \frac{M}{2})\sin\theta$$

Therefore the Euler Lagrange equations are:

$$m\frac{d}{dt}(x^2\dot{\theta}) + \frac{1}{3}Ml^2\ddot{\theta} = -gl(m + \frac{M}{2})\sin\theta$$

$$\ddot{x} = x\dot{\theta}^2$$

Reparameterizing in terms of η and r:

$$r\frac{d}{dt}(\eta^2\dot{\theta}) + \frac{1}{3}\ddot{\theta} = -\frac{g}{l}(r + \frac{1}{2})\sin\theta \tag{1}$$

$$\ddot{\eta} = \eta\dot{\theta}^2 \tag{2}$$

are the Euler-Lagrange equations.

0.2 Approximations

0.2.1 Small mass approximation $m \ll M$

$m \ll M$ is equivalent to the condition $r \ll 1$. Let us assume that r is negligibly small. The equations of motion then read:

$$\ddot{\theta} = -\frac{3g}{2l}\sin\theta \tag{3}$$

$$\ddot{\eta} = \eta\dot{\theta}^2 \tag{4}$$

By our definition of θ the vertical position is at $\theta = 0$ so small oscillations about the vertical are given by:

$$\theta(t) = A\cos(\omega t + \phi)$$

where $\omega^2 = 3g/2l$.

Consider the initial conditions:

$$t_{\text{vertical}} = \frac{T}{2} = \frac{1}{2}\frac{2\pi}{\omega} = \frac{\pi}{\omega} = \frac{2\pi}{3}\sqrt{\frac{l}{g}}$$

$$\theta(t_{\text{vertical}}) = 0$$

Then:

$$\begin{aligned} 0 &= \cos(\omega(\frac{\pi}{\omega}) + \phi) \\ &= \cos(\pi + \phi) \\ &= \cos(\pi)\cos(\phi) - \sin(\pi)\sin(\phi) \\ &= \cos(\phi) \end{aligned}$$

therefore $\phi = \pi/2$ and the solution for θ can be rewritten without a phase shift as:

$$\theta(t) = A\sin(\omega t)$$

Taking a time derivative:

$$\dot{\theta}(t) = A\omega\cos(\omega t)$$

$$\dot{\theta}^2(t) = A^2\omega^2\cos^2(\omega t)$$

and substituting into 4:

$$\ddot{\eta} = A^2\omega^2\sin^2(\omega t)\eta$$

using the identity $\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos(2A)$, we can rewrite as:

$$\ddot{\eta} = A^2\omega^2(\frac{1}{2} - \frac{1}{2}\cos 2\omega t)\eta$$

This is a form of Mathieu's equation. Now consider $t = t_{\text{vertical}} + \delta$

$$\ddot{\eta}(\delta) = \; A^2\omega^2(\frac{1}{2} - \frac{1}{2}\cos 2\omega(t_{\text{vertical}} + \delta))\eta(\delta)$$

Consider the cos expression:

$$\begin{aligned} \cos 2\omega(t_{\text{vertical}} + \delta)) &= \cos(2\omega t_{\text{vertical}})\cos 2\omega\delta - \sin 2\omega t_{\text{vertical}} \sin 2\omega\delta \\ &= \cos 2\omega\delta \end{aligned}$$

Assume δ is small so that $\cos 2\omega\delta \approx 1 - 2\omega^2\delta^2$

$$\begin{aligned} \ddot{\eta} &= A^2\omega^2(\frac{1}{2} - \frac{1}{2}(1 - 2\omega^2\delta^2))\eta \\ &= A^2\omega^4\delta^2\eta \end{aligned}$$

0.2.2 Taylor Series Expansion

Taylor expand η and x as:

$$\eta(t) = \sum_{n=0}^{\infty} a_n(t - t_0)^n$$

and:

$$x(t) = \sum_{n=0}^{\infty} b_n(t - t_0)^n$$

Therefore: