Chapter 2 Problems

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April 11, 2020

1. **The metric of flat, three-dimensional Euclidean space is:**

$$ds^2 = dx^2 + dy^2 + dz^2$$

Show that the metric components g_{uv} in spherical polar coordinates r, θ, ϕ defined by:

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ \cos\theta &= \frac{z}{r}, \\ \tan\phi &= \frac{y}{x} \end{aligned}$$

is given by:

$$s^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \phi^2$$

g_{uv} is a tensor of type $(0, 2)$ and therefore transforms as:

$$g_{\mu',\nu'} = g_{\mu,\nu} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}}$$

(see page 22 for the general *tensor transformation law*). The above equation uses Einstein index notation indicating that μ and ν are to to be summed from 1 to 3 and the free indices, μ' and ν', are enumerated through all possible combinations. Therefore the components that need to be calculated are:

$$\begin{matrix} g_{r,r} & g_{r,\theta} & g_{r,\phi} \\ g_{\theta,r} & g_{\theta,\theta} & g_{\theta,\phi} \\ g_{\phi,r} & g_{\phi,\theta} & g_{\phi,\phi} \end{matrix}$$

Starting with:

$$g_{\mu',\nu'} = \sum_{\mu=1}^{3} \sum_{\nu=1}^{3} g_{\mu,\nu} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}}$$

$$= \sum_{\mu=1}^{3} g_{\mu,1} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{\mu,2} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{\mu,3} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}}$$

$$\begin{aligned} = g_{1,1} \frac{\partial x^1}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{1,2} \frac{\partial x^1}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{1,3} \frac{\partial x^1}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}} \\ g_{2,1} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{2,2} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{2,3} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}} \\ g_{3,1} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{3,2} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{3,3} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}} \end{aligned}$$

Substituting the notation for the indices in flat, orthonormal Euclidean space:

$$
\begin{aligned}
&= g_{x,x}\frac{\partial x}{\partial x^{\mu'}}\frac{\partial x}{\partial x^{\nu'}} + g_{x,y}\frac{\partial x}{\partial x^{\mu'}}\frac{\partial y}{\partial x^{\nu'}} + g_{x,z}\frac{\partial x}{\partial x^{\mu'}}\frac{\partial z}{\partial x^{\nu'}} \\
&\quad g_{y,x}\frac{\partial y}{\partial x^{\mu'}}\frac{\partial x}{\partial x^{\nu'}} + g_{y,y}\frac{\partial y}{\partial x^{\mu'}}\frac{\partial y}{\partial x^{\nu'}} + g_{y,z}\frac{\partial y}{\partial x^{\mu'}}\frac{\partial z}{\partial x^{\nu'}} \\
&\quad g_{z,x}\frac{\partial y}{\partial x^{\mu'}}\frac{\partial x}{\partial x^{\nu'}} + g_{z,y}\frac{\partial y}{\partial x^{\mu'}}\frac{\partial y}{\partial x^{\nu'}} + g_{z,z}\frac{\partial y}{\partial x^{\mu'}}\frac{\partial z}{\partial x^{\nu'}}
\end{aligned}
$$

The off diagonal elements of the Euclidean metric are zero:

$$g_{x,y} = g_{y,x} = g_{x,z} = g_{z,x} = g_{y,z} = g_{z,y} = 0$$

and the diagonal components are one:

$$g_{x,x} = g_{y,y} = g_{z,z} = 1$$

This reduces the above summation from nine expressions to the following three:

$$g_{\mu',\nu'} = \frac{\partial x}{\partial x^{\mu'}}\frac{\partial x}{\partial x^{\nu'}} + \frac{\partial y}{\partial x^{\mu'}}\frac{\partial y}{\partial x^{\nu'}} + \frac{\partial z}{\partial x^{\mu'}}\frac{\partial z}{\partial x^{\nu'}}$$

For indices where $\mu' = \nu'$

$$g_{\mu',\mu'} = \left(\frac{\partial x}{\partial x^{\mu'}}\right)^2 + \left(\frac{\partial y}{\partial x^{\mu'}}\right)^2 + \left(\frac{\partial z}{\partial x^{\mu'}}\right)^2$$

Therefore the six unique components that need to be calculated to find the components of the metric in spherical polar coorindates are:

$$
\begin{aligned}
g_{r,r} &= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 \\
g_{r,\theta} = g_{\theta,r} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta} \\
g_{r,\phi} = g_{\phi,r} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \phi} \\
g_{\theta,\theta} &= \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 \\
g_{\theta,\phi} = g_{\phi,\theta} &= \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \phi} \\
g_{\phi,\phi} &= \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2
\end{aligned}
$$

To take the above derivatives, find an equation for x, y, z in terms of r, θ, ϕ. Starting by finding x:

$$
\begin{aligned}
&r = \sqrt{x^2 + y^2 + z^2} \rightarrow r^2 = x^2 + z^2 + y^2, \\
&\cos\theta = \frac{z}{r} \rightarrow z = r\cos\theta, \\
&\tan\phi = \frac{y}{x} \rightarrow y = x\tan\phi
\end{aligned}
$$

Substituting the second and third equation into the first gives:

$$
\begin{aligned}
r^2 &= x^2 + (r\cos\theta)^2 + (x\tan\phi)^2 \\
r^2 &= x^2 + r^2\cos^2\theta + x^2\tan^2\phi \\
r^2 - r^2\cos\theta &= x^2 + x^2\tan^2\phi \\
(1-\cos^2\theta)r^2 &= (1+\tan^2\phi)x^2 \\
r^2\sin^2\theta &= (1+\tan^2\phi)x^2 \\
x &= r\frac{\sin\theta}{\sqrt{1+\tan^2\phi}}
\end{aligned}
$$

Therefore the equations for x, y, z in terms of r, θ, ϕ:

$$
x = r\frac{\sin\theta}{\sqrt{1+\tan^2\phi}},\quad y = r\tan\phi\frac{\sin\theta}{\sqrt{1+\tan^2\phi}},\quad z = r\cos\theta
$$

Find all the necessary derivatives:

$$
\begin{aligned}
\frac{\partial x}{\partial r} &= \frac{\sin\theta}{\sqrt{1+\tan^2\phi}} \\
\frac{\partial x}{\partial \theta} &= -r\frac{\cos\theta}{\sqrt{1+\tan^2\phi}} \\
\frac{\partial x}{\partial \phi} &= -r\sin\theta\frac{\tan\phi\sec^2\phi}{(1+\tan^2\phi)^{\frac{3}{2}}} \\
\frac{\partial y}{\partial r} &= \tan\phi\frac{\sin\theta}{\sqrt{1+\tan^2\phi}} \\
\frac{\partial y}{\partial \theta} &= -r\tan\phi\frac{\cos\theta}{\sqrt{1+\tan^2\phi}} \\
\frac{\partial y}{\partial \phi} &= -r\sin\theta\frac{\sec^2\phi}{(1+\tan^2\phi)^{\frac{3}{2}}} \\
\frac{\partial z}{\partial r} &= \cos\theta \\
\frac{\partial z}{\partial \theta} &= r\sin\theta \\
\frac{\partial z}{\partial \phi} &= 0
\end{aligned}
$$

Then compute the components of the metric in spherical polar coordinates:

$$
\begin{aligned}
g_{r,r} &= \frac{\sin\theta^2}{1+\tan^2\phi} + \frac{\sin^2\theta}{1+\tan^2\phi}\tan^2\phi + \cos^2\theta \\
&= \frac{\sin\theta^2 + \sin^2\theta\tan^2\phi}{1+\tan^2\phi} + \frac{(1+\tan^2\phi)\cos^2\theta}{1+\tan^2\phi} \\
&= \frac{\sin\theta^2 + \sin^2\theta\tan^2\phi + (1+\tan^2\phi)\cos^2\theta}{1+\tan^2\phi} \\
&= \frac{\sin^2\theta + \cos^2\theta + \sin^2\theta\tan^2\phi + \tan^2\phi\cos^2\theta}{1+\tan^2\phi} \\
&= \frac{1+\tan^2\phi}{1+\tan^2\phi} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
g_{\theta,\theta} &= r^2\frac{\cos^2\theta}{1+\tan^2\phi} + r^2\frac{\cos^2\theta}{1+\tan^2\phi}\tan^2\phi + r^2\sin^2\theta \\
&= r^2\frac{\cos^2\theta}{1+\tan^2\phi} + r^2\frac{\cos^2\theta}{1+\tan^2\phi}\tan^2\phi + r^2\sin^2\theta\frac{1+\tan^2\phi}{1+\tan^2\phi} \\
&= r^2\frac{\cos^2\theta + \cos^2\theta\tan^2\phi + \sin^2\theta + \tan^2\phi\sin^2\theta}{1+\tan^2\phi} \\
&= r^2\frac{(\cos^2\theta + \sin^2\theta) + (\cos^2\theta+\sin^2\theta)\tan^2\phi}{1+\tan^2\phi} \\
&= r^2\frac{1+\tan^2\phi}{1+\tan^2\phi} \\
&= r^2
\end{aligned}
$$

$$
\begin{aligned}
g_{\phi,\phi} &= r^2\sin^2\theta\left(\frac{1}{(1+\tan^2\phi)^3\cos^4\phi} + \frac{\sin^2\phi}{(1+\tan^2\phi)^3\cos^6 x}\right) \\
&= r^2\sin^2\theta\left(\frac{\cos^2\phi+\sin^2\phi}{(1+\tan^2\phi)^3\cos^6\phi}\right) \\
&= r^2\sin^2\theta\frac{1}{\left(\frac{\sin^2\phi+\cos^2\phi}{\cos^2\phi}\right)^3\cos^6\phi} \\
&= r^2\sin^2\theta
\end{aligned}
$$

$$
\begin{aligned}
g_{\theta,r} = g_{r,\theta} &= g_{x,x}\frac{\partial x}{\partial r}\frac{\partial x}{\partial\theta} + g_{y,y}\frac{\partial y}{\partial r}\frac{\partial y}{\partial\theta} + g_{z,z}\frac{\partial z}{\partial r}\frac{\partial z}{\partial\theta} \\
&= -r\frac{\sin\theta\cos\theta}{1+\tan^2\phi} - r\frac{\sin\theta\cos\theta}{1+\tan^2\phi}\tan^2\phi + r\sin\theta\cos\theta \\
&= -r\frac{\sin\theta\cos\theta}{1+\tan^2\phi}(1+\tan^2\phi) + r\sin\theta\cos\theta \\
&= -r\sin\theta\cos\theta + r\sin\theta\cos\theta \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
g_{r,\phi} = g_{\phi,r} &= g_{x,x}\frac{\partial x}{\partial r}\frac{\partial x}{\partial \phi} + g_{y,y}\frac{\partial y}{\partial r}\frac{\partial y}{\partial \phi} + g_{z,z}\frac{\partial z}{\partial r}\frac{\partial z}{\partial \phi} \\
&= -r\sin\theta\cos\theta\frac{\tan\phi\sec^2\phi}{(1+\tan^2\phi)^2} + r\cos\theta\sin\theta\frac{\tan\phi\sec^2\phi}{(1+\tan^2\phi)^2} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
g_{\theta,\phi} = g_{\phi,\theta} &= g_{x,x}\frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \phi} + g_{y,y}\frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \phi} + g_{z,z}\frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \phi} \\
&= -r\frac{\cos\theta}{\sqrt{1+\tan^2\phi}}\left(-r\sin\theta\frac{\tan\phi\sec^2\phi}{(1+\tan^2\phi)^{\frac{3}{2}}}\right) - r\tan\phi\frac{\cos\theta}{\sqrt{1+\tan^2\phi}}\left(r\sin\theta\frac{\sec^2\phi}{(1+\tan^2\phi)^{\frac{3}{2}}}\right) \\
&= r^2\sin\theta\cos\theta\frac{\tan\phi\sec^2\phi}{(1+\tan^2\phi)^2} - r^2\sin\theta\cos\theta\frac{\tan\phi\sec^2\phi}{(1+\tan^2\phi)^2} \\
&= 0
\end{aligned}
$$

Therefore the metric components in spherical polar coordinates are:

$$
g_{\mu,\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}
$$

1 Appendix A - Tensor Expansion

The metric is a rank (0,2) tensor so the transformation of the components between basis is given by:

$$
g_{\mu',\nu'} = g_{\mu,\nu}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}
$$

This equation is a shorthand for the following:

$$
\begin{aligned}
g_{\mu',\nu'} &= \sum_{\mu=1}^{3}\sum_{\nu=1}^{3} g_{\mu,\nu}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}} \\
g_{\mu',\nu'} &= \sum_{\mu=1}^{3} g_{\mu,1}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^1}{\partial x^{\nu'}} + g_{\mu,2}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^2}{\partial x^{\nu'}} + g_{\mu,3}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^3}{\partial x^{\nu'}} \\
g_{\mu',\nu'} &= g_{1,1}\frac{\partial x^1}{\partial x^{\mu'}}\frac{\partial x^1}{\partial x^{\nu'}} + g_{1,2}\frac{\partial x^1}{\partial x^{\mu'}}\frac{\partial x^2}{\partial x^{\nu'}} + g_{1,3}\frac{\partial x^1}{\partial x^{\mu'}}\frac{\partial x^3}{\partial x^{\nu'}} \\
&\quad g_{2,1}\frac{\partial x^2}{\partial x^{\mu'}}\frac{\partial x^1}{\partial x^{\nu'}} + g_{2,2}\frac{\partial x^2}{\partial x^{\mu'}}\frac{\partial x^2}{\partial x^{\nu'}} + g_{2,3}\frac{\partial x^2}{\partial x^{\mu'}}\frac{\partial x^3}{\partial x^{\nu'}} \\
&\quad g_{3,1}\frac{\partial x^2}{\partial x^{\mu'}}\frac{\partial x^1}{\partial x^{\nu'}} + g_{3,2}\frac{\partial x^2}{\partial x^{\mu'}}\frac{\partial x^2}{\partial x^{\nu'}} + g_{3,3}\frac{\partial x^2}{\partial x^{\mu'}}\frac{\partial x^3}{\partial x^{\nu'}}
\end{aligned}
$$

The full expansion for μ', ν' is:

$$
\begin{aligned}
g_{1',1'} &= g_{1,1}\frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^1}{\partial x^{1'}} + g_{1,2}\frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{1'}} + g_{1,3}\frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^3}{\partial x^{1'}} \\
&\quad g_{2,1}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^1}{\partial x^{1'}} + g_{2,2}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{1'}} + g_{2,3}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^3}{\partial x^{1'}} \\
&\quad g_{3,1}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^1}{\partial x^{1'}} + g_{3,2}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{1'}} + g_{3,3}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^3}{\partial x^{1'}}
\end{aligned}
$$

$$
\begin{aligned}
g_{1',2'} = {} & g_{1,1}\frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^1}{\partial x^{2'}} + g_{1,2}\frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{2'}} + g_{1,3}\frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^3}{\partial x^{2'}} \\
& g_{2,1}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^1}{\partial x^{2'}} + g_{2,2}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{2'}} + g_{2,3}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^3}{\partial x^{2'}} \\
& g_{3,1}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^1}{\partial x^{2'}} + g_{3,2}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{2'}} + g_{3,3}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^3}{\partial x^{2'}}
\end{aligned}
$$

$$
\begin{aligned}
g_{1',3'} = {} & g_{1,1}\frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^1}{\partial x^{3'}} + g_{1,2}\frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{3'}} + g_{1,3}\frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^3}{\partial x^{3'}} \\
& g_{2,1}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^1}{\partial x^{3'}} + g_{2,2}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{3'}} + g_{2,3}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^3}{\partial x^{3'}} \\
& g_{3,1}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^1}{\partial x^{3'}} + g_{3,2}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{3'}} + g_{3,3}\frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^3}{\partial x^{3'}}
\end{aligned}
$$

$$
\begin{aligned}
g_{2',1'} = {} & g_{1,1}\frac{\partial x^1}{\partial x^{2'}}\frac{\partial x^1}{\partial x^{1'}} + g_{1,2}\frac{\partial x^1}{\partial x^{2'}}\frac{\partial x^2}{\partial x^{1'}} + g_{1,3}\frac{\partial x^1}{\partial x^{2'}}\frac{\partial x^3}{\partial x^{1'}} \\
& g_{2,1}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^1}{\partial x^{1'}} + g_{2,2}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^2}{\partial x^{1'}} + g_{2,3}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^3}{\partial x^{1'}} \\
& g_{3,1}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^1}{\partial x^{1'}} + g_{3,2}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^2}{\partial x^{1'}} + g_{3,3}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^3}{\partial x^{1'}}
\end{aligned}
$$

$$
\begin{aligned}
g_{2',2'} = {} & g_{1,1}\frac{\partial x^1}{\partial x^{2'}}\frac{\partial x^1}{\partial x^{2'}} + g_{1,2}\frac{\partial x^1}{\partial x^{2'}}\frac{\partial x^2}{\partial x^{2'}} + g_{1,3}\frac{\partial x^1}{\partial x^{2'}}\frac{\partial x^3}{\partial x^{2'}} \\
& g_{2,1}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^1}{\partial x^{2'}} + g_{2,2}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^2}{\partial x^{2'}} + g_{2,3}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^3}{\partial x^{2'}} \\
& g_{3,1}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^1}{\partial x^{2'}} + g_{3,2}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^2}{\partial x^{2'}} + g_{3,3}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^3}{\partial x^{2'}}
\end{aligned}
$$

$$
\begin{aligned}
g_{2',3'} = {} & g_{1,1}\frac{\partial x^1}{\partial x^{2'}}\frac{\partial x^1}{\partial x^{3'}} + g_{1,2}\frac{\partial x^1}{\partial x^{2'}}\frac{\partial x^2}{\partial x^{3'}} + g_{1,3}\frac{\partial x^1}{\partial x^{2'}}\frac{\partial x^3}{\partial x^{3'}} \\
& g_{2,1}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^1}{\partial x^{3'}} + g_{2,2}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^2}{\partial x^{3'}} + g_{2,3}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^3}{\partial x^{3'}} \\
& g_{3,1}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^1}{\partial x^{3'}} + g_{3,2}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^2}{\partial x^{3'}} + g_{3,3}\frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^3}{\partial x^{3'}}
\end{aligned}
$$

$$
\begin{aligned}
g_{3',1'} = {} & g_{1,1}\frac{\partial x^1}{\partial x^{3'}}\frac{\partial x^1}{\partial x^{1'}} + g_{1,2}\frac{\partial x^1}{\partial x^{3'}}\frac{\partial x^2}{\partial x^{1'}} + g_{1,3}\frac{\partial x^1}{\partial x^{3'}}\frac{\partial x^3}{\partial x^{1'}} \\
& g_{2,1}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^1}{\partial x^{1'}} + g_{2,2}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^2}{\partial x^{1'}} + g_{2,3}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^3}{\partial x^{1'}} \\
& g_{3,1}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^1}{\partial x^{1'}} + g_{3,2}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^2}{\partial x^{1'}} + g_{3,3}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^3}{\partial x^{1'}}
\end{aligned}
$$

$$
\begin{aligned}
g_{3',2'} = {} & g_{1,1}\frac{\partial x^1}{\partial x^{3'}}\frac{\partial x^1}{\partial x^{2'}} + g_{1,2}\frac{\partial x^1}{\partial x^{3'}}\frac{\partial x^2}{\partial x^{2'}} + g_{1,3}\frac{\partial x^1}{\partial x^{3'}}\frac{\partial x^3}{\partial x^{2'}} \\
& g_{2,1}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^1}{\partial x^{2'}} + g_{2,2}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^2}{\partial x^{2'}} + g_{2,3}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^3}{\partial x^{2'}} \\
& g_{3,1}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^1}{\partial x^{2'}} + g_{3,2}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^2}{\partial x^{2'}} + g_{3,3}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^3}{\partial x^{2'}}
\end{aligned}
$$

$$
\begin{aligned}
g_{3',3'} = g_{1,1}\frac{\partial x^1}{\partial x^{3'}}\frac{\partial x^1}{\partial x^{3'}} + g_{1,2}\frac{\partial x^1}{\partial x^{3'}}\frac{\partial x^2}{\partial x^{3'}} + g_{1,3}\frac{\partial x^1}{\partial x^{3'}}\frac{\partial x^3}{\partial x^{3'}} \\
g_{2,1}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^1}{\partial x^{3'}} + g_{2,2}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^2}{\partial x^{3'}} + g_{2,3}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^3}{\partial x^{3'}} \\
g_{3,1}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^1}{\partial x^{3'}} + g_{3,2}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^2}{\partial x^{3'}} + g_{3,3}\frac{\partial x^2}{\partial x^{3'}}\frac{\partial x^3}{\partial x^{3'}}
\end{aligned}
$$

The tensor transformation law for a tensor of rank (0, 2) in 3 space represents 9 components. Each components contains a summation of 9 expressions. Each expression has 3 terms. So in total the equation is representative of 243 terms.