Fun With Pipes

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While working on some plumping with my dad, questions related to the fluid dynamics of pipes arose. Having never studied fluids, I thought it would be interesting to see which equations I could derive from mechanical considerations of the fluids in the pipes.

1 Pipes of different radii

Consider two cylindrical pipes of radii r_1 and r_2 connected to one another. In them flows a fluid which is assumed to be incompressible and has density ρ. Furthermore, consider a slice of the fluid of extent x at two times, one before and one after the boundary as shown in Figure 1. Consider conservation of energy on this slice of the fluid:

$$\int F_1 dx + \frac{1}{2}m_1 v_1^2 + m_1 g h_1 = \int F_2 dx + \frac{1}{2}m_2 v_2^2 + m_2 g h_2$$

Rewriting the masses in terms of the densities and volumes using $m_i = \rho V_i = \pi\rho r_i^2 x$.

$$\int F_1 dx + \frac{1}{2}\rho\pi r_1^2 x v_1^2 + \pi\rho r_1^2 x g h_1 = \int F_2 dx + \frac{1}{2}\pi\rho r_2^2 x v_2^2 + \pi\rho r_2^2 x g h_2$$

Write the forces in terms of the pressures using $F_i = p_i A_i = \pi p_i r_i^2$. Assuming the pressures are constant:

$$\pi p_1 r_1^2 x + \frac{1}{2}\rho\pi r_1^2 x v_1^2 + \pi\rho r_1^2 x g h_1 = \pi p_2 r_2^2 x + \frac{1}{2}\pi\rho r_2^2 x v_2^2 + \pi\rho r_2^2 x g h_2$$

$$p_1 r_1^2 + \frac{1}{2}\rho r_1^2 v_1^2 + \rho r_1^2 g h_1 = p_2 r_2^2 + \frac{1}{2}\rho r_2^2 v_2^2 + \rho r_2^2 g h_2$$

$$r_1^2(p_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1) = r_2^2(p_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2)$$

Consider some special cases of the above equation.

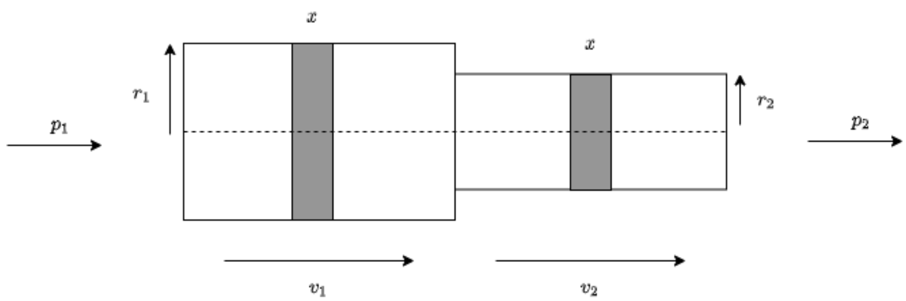

Figure 1:

1.1 $r_1 = r_2$

We regain Euler's equation:

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

1.2 Abscence of Gravity

$$r_1^2(p_1 + \frac{1}{2}\rho v_1^2) = r_2^2(p_2 + \frac{1}{2}\rho v_2^2)$$

This relationship is kinematical in nature. It would be nice to know how the fluid evolves through time.

2 $F = ma$ for the fluid

$$F = \frac{d}{dt}(mv)$$

Rewriting in terms of pressure and density:

$$-pA = \frac{d}{dt}(\rho V v)$$