

# A Relativist's Toolkit Problems

Anthony Steel

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# Chapter 1

## Fundamentals

1. The surface of a two-dimensional cone is embedded in three-dimensional flat space. The cone has an opening angle of  $2\alpha$ . Points on the cone which all have the same distance  $r$  from the apex define a circle, and  $\phi$  is the angle that runs along the circle.

- (a) Write down the metric of the cone, in terms of the coordinates  $r$  and  $\phi$ . Consider the metric for 3-space in spherical polar coordinates:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

On a cone, the coordinate  $\theta$  is half of the opening angle and is a constant implying:  $\theta = \alpha$  and  $d\theta = d\alpha = 0$ . Therefore the metric of the cone is:

$$ds^2 = dr^2 + r^2 \sin^2 \alpha d\phi^2$$

Shown another way, consider the parameterization of the cone  $X(r, \phi)$

$$X(r, \phi) = \begin{bmatrix} r \sin \alpha \cos \phi \\ r \sin \alpha \sin \phi \\ r \cos \alpha \end{bmatrix}$$

Differentiating with respect to  $r$  and  $\phi$  gives:

$$X_{,\phi} = \begin{bmatrix} -r \sin \alpha \sin \phi \\ r \sin \alpha \cos \phi \\ 0 \end{bmatrix}$$

and

$$X_{,r} = \begin{bmatrix} \sin \alpha \cos \phi \\ \sin \alpha \sin \phi \\ \cos \alpha \end{bmatrix}$$

The general form of the metric in two dimensions is:

$$ds^2 = E d\phi^2 + 2F d\phi dr + G dr^2$$

where:

$$E = X_{,\phi} \cdot X_{,\phi} = r \sin^2 \alpha$$

$$F = X_{,\phi} \cdot X_{,r} = 0$$

$$G = X_{,r} \cdot X_{,r} = 1$$

therefore:

$$ds^2 = dr^2 + r^2 \sin^2 \alpha d\phi^2$$

- (b) **Find the coordinate transformation  $x(r, \phi)$ ,  $y(r, \phi)$  that brings the metric into the form  $ds^2 = dx^2 + dy^2$ . Do these coordinates cover the entire two-dimensional plane?**

Consider the differentials of the transformation  $x(r, \phi)$  and  $y(r, \phi)$ :

$$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \phi} d\phi$$

$$dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \phi} d\phi$$

Therefore:

$$\begin{aligned} dx^2 + dy^2 &= \left[ \left( \frac{\partial x}{\partial r} \right)^2 + \left( \frac{\partial y}{\partial r} \right)^2 \right] dr^2 + \left[ \left( \frac{\partial x}{\partial \phi} \right)^2 + \left( \frac{\partial y}{\partial \phi} \right)^2 \right] d\phi^2 + \left( \frac{\partial x}{\partial r} \frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial r} \frac{\partial y}{\partial \phi} \right) dr d\phi \\ &= dr^2 + r^2 \sin^2 \alpha d\phi^2 \end{aligned}$$

Which gives three equations:

$$\left( \frac{\partial x}{\partial r} \right)^2 + \left( \frac{\partial y}{\partial r} \right)^2 = 1 \quad (1.1)$$

$$\left( \frac{\partial x}{\partial \phi} \right)^2 + \left( \frac{\partial y}{\partial \phi} \right)^2 = r^2 \sin^2 \alpha \quad (1.2)$$

$$\left( \frac{\partial x}{\partial r} \frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial r} \frac{\partial y}{\partial \phi} \right) = 0 \rightarrow \frac{\partial x}{\partial r} \frac{\partial x}{\partial \phi} = - \frac{\partial y}{\partial r} \frac{\partial y}{\partial \phi} \quad (1.3)$$

Make the following ansatz:

$$x(r, \phi) = r \sin \phi$$

$$y(r, \phi) = r \cos \phi$$

$$\frac{\partial x}{\partial r} = \sin \phi$$

$$\frac{\partial y}{\partial r} = \cos \phi$$

$$\left( \frac{\partial x}{\partial r} \right)^2 + \left( \frac{\partial y}{\partial r} \right)^2 = \sin^2 \phi + \cos^2 \phi = 1$$

Therefore equation 1.1 is satisfied. To satisfy equation 1.2 modify the equations to:

$$x(r, \phi) = r \sin(\phi \sin \alpha)$$

$$y(r, \phi) = r \cos(\phi \sin \alpha)$$

leaving the results of equation 1.1 unchanged.

$$\begin{aligned}\frac{\partial x}{\partial \phi} &= r \sin \alpha \cos(\phi \sin \alpha) \\ \frac{\partial y}{\partial \phi} &= -r \sin \alpha \sin(\phi \sin \alpha) \\ \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 &= r^2 \sin^2 \alpha \sin^2(\phi \sin \alpha) + r^2 \sin^2 \alpha \cos^2(\phi \sin \alpha) \\ &= r^2 \sin^2 \alpha\end{aligned}$$

Therefore 1.2 is satisfied. Checking the last equation:

$$\begin{aligned}\frac{\partial x}{\partial r} \frac{\partial x}{\partial \phi} &= -\frac{\partial y}{\partial r} \frac{\partial y}{\partial \phi} \\ r \sin \alpha \sin(\phi \sin \alpha) \cos(\phi \sin \alpha) &= -(-r \sin \alpha \cos(\phi \sin \alpha) \sin(\phi \sin \alpha))\end{aligned}$$

Therefore the transformations that bring the metric into the form  $ds^2 = dx^2 + dy^2$  are:

$$\begin{aligned}x(r, \phi) &= r \sin(\phi \sin \alpha) \\ y(r, \phi) &= r \cos(\phi \sin \alpha)\end{aligned}$$

Proof:

$$\begin{aligned}dx &= \sin(\phi \sin \alpha)dr + r \sin \alpha \cos(\phi \sin \alpha)d\phi \\ dy &= \cos(\phi \sin \alpha)dr - r \sin \alpha \sin(\phi \sin \alpha)d\phi\end{aligned}$$

and:

$$\begin{aligned}ds^2 &= dx^2 + dy^2 \\ &= \sin^2(\phi \sin \alpha)dr^2 + r^2 \sin^2 \alpha \cos^2(\phi \sin \alpha)d\phi^2 + 2r \sin \alpha \sin(\phi \sin \alpha) \cos(\phi \sin \alpha)drd\phi \\ &\quad + \cos^2(\phi \sin \alpha)dr^2 + r^2 \sin^2 \alpha \sin^2(\phi \sin \alpha)d\phi^2 - 2r \sin \alpha \sin(\phi \sin \alpha) \cos(\phi \sin \alpha)drd\phi \\ &= dr^2 + r^2 \sin^2 \alpha d\phi^2\end{aligned}$$

Consider the inverted transformation:

$$\begin{aligned}\phi &= \frac{1}{\sin \alpha} \arctan\left(\frac{y}{x}\right) \\ r &= \sqrt{x^2 + y^2}\end{aligned}$$

In order for these coordinates to cover the entire plane, the range of  $\phi(x, y)$  and  $r(x, y)$  must be  $0 \leq \phi < 2\pi$  and  $0 \leq r < \infty$ . The range of  $\arctan$  is  $-\pi/2 < \phi < \pi/2$ .

- (c) **Prove that any vector parallel transported along a circle of constant  $r$  on the surface of the cone ends up rotated by an angle  $\beta$  after a complete trip. Express  $\beta$  in terms of  $\alpha$ .**