

Chapter 2 Problems

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1. (a)

- (b) **Show by explicit construction that two coordinate systems (as opposed to the six used in the text) suffice to cover S^2 (it is impossible to cover S^2 with a single chart, as follows from the fact that S^2 is compact, but every open subset of \mathbb{R}^2 is noncompact see appendix A.)**

Consider the collection of two subsets, O_N and O_S . O_N contains all the points in the set S^2 except the north pole $(0, 0, 1)$, and O_S contains all the points in S except the south pole $(0, 0, -1)$. Together this collection contains every point in S^2 thereby satisfying property 1 required for a manifold.

Next we must determine two maps, ψ_N and ψ_S for O_N and O_S respectively, and show that they are one-to-one and map S^2 to \mathbb{R}^2 . To construct these maps consider Figure ?? . Pictured are all the points in S^2 which lie in the X - Z plane. We will use this diagram to construct the map ψ .

Draw a line from the north pole N that intersects the S^2 at a point P and the X axis at P' . The point P has coordinates $(x, 0, z)$. The triangles NzP and NOP' can be related using similar triangles. This will give a relation between the point P and P' .

$$\frac{OP'}{NO} = \frac{zP}{Nz}$$

$$P' = \frac{x}{1-z}$$

P' is equivalent to mapping the any point P into the $X - Y$ plane and we will denote this coordinate in the X direction X' . The same argument can be made to find Y' by replacing x in the above equation with y . Therefore:

$$O^N : S^2 \rightarrow \mathbb{R}^2 = (X', Y') = \left(\frac{x}{1-z}, \frac{y}{1-z} \right)$$

ψ_N can be constructed in a similar manner but replacing z with $-z$:

$$(X', Y') = \left(\frac{x}{1+z}, \frac{y}{1+z} \right)$$

Each of these maps are clearly one to one and onto.

$$(X, Y) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right)$$

$$(x, y, z) = \left(\frac{2X}{1+X^2+Y^2}, \frac{2Y}{1+X^2+Y^2}, \frac{-1+X^2+Y^2}{1+X^2+Y^2} \right)$$

$\psi_N : O_N \rightarrow U_N \subset \mathbb{R}^2$ and $\psi_S : O_S \rightarrow U_S \subset \mathbb{R}^2$. In order to satisfy property 3 of manifolds we must show that $\psi_N \circ \psi_S^{-1}$ and $\psi_S \circ \psi_N^{-1}$ the

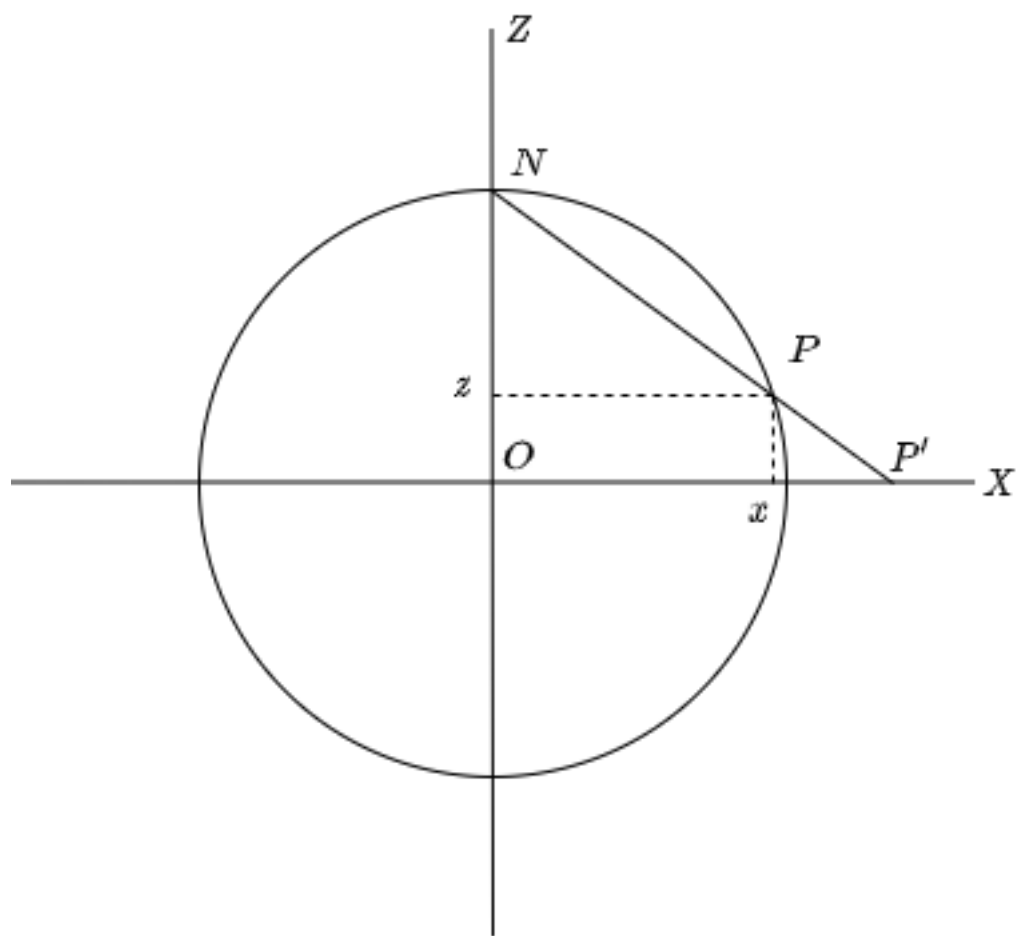


Figure 1: Projection of points on a circle into \mathbb{R}^2

subsets of the map are open and the maps are infinitely differentiable.

$$(X', Y') = \left(\frac{2X}{(1 + X^2 + Y^2)(1 - \frac{-1+X^2+Y^2}{1+X^2+Y^2})}, \frac{2Y}{(1 + X^2 + Y^2)(1 - \frac{-1+X^2+Y^2}{1+X^2+Y^2})} \right)$$

$$(X', Y') = \left(\frac{2X}{1 + X^2 + Y^2 + 1 - X^2 - Y^2}, \frac{2Y}{1 + X^2 + Y^2 + 1 - X^2 - Y^2} \right) = (X, Y)$$

2. **Prove that any smooth function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ can be written in the form equation (2.2.2)**
3. (a) **Verify that the commutator, defined by equation (2.2.14), satisfies the linearity and Leibnitz properties, and hence defines a vector field**

$$\begin{aligned} [v, w](f + g) &= v(w(f + g)) - w(v(f + g)) \\ &= v(w(f) + w(g)) - w(v(f) + v(g)) \\ &= v(w(f)) + v(w(g)) - w(v(f)) - w(v(g)) \\ &= v(w(f)) - w(v(f)) + v(w(g)) - w(v(g)) \\ &= [v, w]f + [v, w]g \end{aligned}$$

Therefore the commutator satisfies the linearity property in v . The same procedure can be applied symmetrically to w .

$$\begin{aligned} [v, w](fg) &= v(w(fg)) - w(v(fg)) \\ &= v(w(f)g + fw(g)) - w(v(f)g + fv(g)) \\ &= v(w(f)g) + v(fw(g)) - w(v(f)g) - w(fv(g)) \\ &= v(w(f))g + w(f)v(g) + v(f)w(g) + fw(v(g)) \\ &\quad - w(v(f))g - v(f)w(g) - w(f)v(g) - fw(v(g)) \\ &= v(w(f))g + f(v(w(g)) - w(v(f))g - fw(v(g)) \\ &= f(v(w(g)) - fw(v(g)) + v(w(f))g - w(v(f))g \\ &= f\left((v(w(g)) - w(v(g)))\right) + g\left(v(w(f)) - w(v(f))\right) \\ &= f[v, w](g) + g[v, w](f) \end{aligned}$$

- (b) **Let X, Y, Z be smooth vector fields on a manifold M . Verify that their commutator satisfies the Jacobi identity:**

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

Expand $[[X, Y], Z]$:

$$\begin{aligned} [[X, Y], Z](f) &= [X, Y]Z(f) - Z[X, Y](f) \\ &= X(Y(Z(f))) - Y(X(Z(f))) - Z(X(Y(f)) - Y(Z(f))) \\ &= X(Y(Z(f))) - Y(X(Z(f))) - Z(X(Y(f))) + Z(Y(Z(f))) \\ &= X(Y(Z(f))) - Y(X(Z(f))) - Z(X(Y(f))) + Z(Y(Z(f))) \end{aligned}$$

Now cyclically permute X, Y, Z :

$$\begin{aligned}
&= \underbrace{X(Y(Z(f)))}_a - \underbrace{Y(X(Z(f)))}_{-g} - \underbrace{Z(X(Y(f)))}_{-e} + \underbrace{Z(Y(X(f)))}_b \\
&+ \underbrace{Y(Z(X(f)))}_c - \underbrace{Z(Y(X(f)))}_{-b} - \underbrace{X(Y(Z(f)))}_{-a} + \underbrace{X(Z(Y(f)))}_d \\
&+ \underbrace{Z(X(Y(f)))}_e - \underbrace{X(Z(Y(f)))}_{-d} - \underbrace{Y(Z(X(f)))}_{-c} + \underbrace{Y(X(Z(f)))}_g \\
&= 0
\end{aligned}$$

- (c) Let Y_1, \dots, Y_n be smooth vector fields on an n -dimensional M such that at each $p \in M$ they form a basis of the tangent space V_p . Then, at each point, we may expand each commutator $[Y_\alpha, Y_\beta]$ in this basis, thereby defining the functions $C_{\alpha\beta}^\gamma = -C_{\beta\alpha}^\gamma$ by

$$[Y_\alpha, Y_\beta] = \sum_{\gamma} C_{\alpha\beta}^\gamma Y_\gamma$$

Use the Jacobi identity to derive an equation satisfied by $C_{\alpha\beta}^\gamma$.

Consider:

$$\begin{aligned}
[[Y_\alpha, Y_\beta], Y_\sigma] &= [C_{\alpha\beta}^\gamma Y_\gamma, Y_\sigma] \\
&= C_{\alpha\beta}^\gamma Y_\gamma Y_\sigma - Y_\sigma C_{\alpha\beta}^\gamma Y_\gamma \\
&= C_{\alpha\beta}^\gamma Y_\gamma Y_\sigma - C_{\alpha\beta}^\gamma Y_\sigma Y_\gamma \\
&= C_{\alpha\beta}^\gamma [Y_\gamma Y_\sigma - Y_\sigma Y_\gamma] \\
&= C_{\alpha\beta}^\gamma (Y_\gamma Y_\sigma - Y_\sigma Y_\gamma) \\
&= C_{\alpha\beta}^\gamma [Y_\gamma, Y_\sigma] \\
&= C_{\alpha\beta}^\gamma C_{\gamma\sigma}^\epsilon Y_\epsilon
\end{aligned}$$

Therefore the Jacobi identity gives:

$$\begin{aligned}
[[Y_\alpha, Y_\beta], Y_\sigma] + [[Y_\beta, Y_\sigma], Y_\alpha] + [[Y_\sigma, Y_\alpha], Y_\beta] &= 0 \\
C_{\alpha\beta}^\gamma C_{\gamma\sigma}^\epsilon Y_\epsilon + C_{\beta\sigma}^\gamma C_{\gamma\alpha}^\epsilon Y_\epsilon + C_{\sigma\alpha}^\gamma C_{\gamma\beta}^\epsilon Y_\epsilon &= 0 \\
(C_{\alpha\beta}^\gamma C_{\gamma\sigma}^\epsilon + C_{\beta\sigma}^\gamma C_{\gamma\alpha}^\epsilon + C_{\sigma\alpha}^\gamma C_{\gamma\beta}^\epsilon) Y_\epsilon &= 0
\end{aligned}$$

Therefore the equations satisfied by the functions are:

$$(C_{\alpha\beta}^\gamma C_{\gamma\sigma}^\epsilon + C_{\beta\sigma}^\gamma C_{\gamma\alpha}^\epsilon + C_{\sigma\alpha}^\gamma C_{\gamma\beta}^\epsilon) = 0$$

4. (a) Show that in any coordinate basis, the components of the commutator of two vector fields v and w are given by

$$[v, w]^\mu = \sum_{\nu} \left(v^\nu \frac{\partial w^\mu}{\partial x^\nu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \right)$$

$$\begin{aligned}
[v, w](f) &= v(w(f)) - w(v(f)) \\
&= v\left(w^\mu \frac{\partial f}{\partial x^\mu}\right) - w\left(v^\mu \frac{\partial f}{\partial x^\mu}\right) \\
&= v^\nu \frac{\partial}{\partial x^\nu} \left(w^\mu \frac{\partial f}{\partial x^\mu}\right) - w^\nu \frac{\partial}{\partial x^\nu} \left(v^\mu \frac{\partial f}{\partial x^\mu}\right) \\
&= v^\nu \left(\frac{\partial w^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} + w^\mu \frac{\partial^2 f}{\partial x^\nu \partial x^\mu}\right) - w^\nu \left(\frac{\partial v^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} + v^\mu \frac{\partial^2 f}{\partial x^\nu \partial x^\mu}\right) \\
&= v^\nu \frac{\partial w^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} + v^\nu w^\mu \frac{\partial^2 f}{\partial x^\nu \partial x^\mu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} - w^\nu v^\mu \frac{\partial^2 f}{\partial x^\nu \partial x^\mu} \\
&= v^\nu \frac{\partial w^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} + \left(v^\nu w^\mu \frac{\partial^2 f}{\partial x^\nu \partial x^\mu} - w^\nu v^\mu \frac{\partial^2 f}{\partial x^\nu \partial x^\mu}\right)
\end{aligned}$$

Using the equality of mixed partial derivatives we can relabel the indices:

$$\begin{aligned}
&= v^\nu \frac{\partial w^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} + \left(v^\nu w^\mu \frac{\partial^2 f}{\partial x^\nu \partial x^\mu} - w^\nu v^\mu \frac{\partial^2 f}{\partial x^\mu \partial x^\nu}\right) \\
&= v^\nu \frac{\partial w^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} \\
&= \left(v^\nu \frac{\partial w^\mu}{\partial x^\nu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu}\right) \frac{\partial f}{\partial x^\mu} \\
&= [v, w]^\mu \frac{\partial f}{\partial x^\mu}
\end{aligned}$$

- (b) Let Y_1, \dots, Y_n be as in problem 3(c). Let Y^{1*}, \dots, Y^{n*} be the dual basis. Show that the components $(Y^{\gamma*})_\mu$ of $Y^{\gamma*}$ in any coordinate basis satisfy

$$\frac{\partial (Y^{\gamma*})_\mu}{\partial x^\nu} - \frac{\partial (Y^{\gamma*})_\nu}{\partial x^\mu} = \sum_{\alpha, \beta} C_{\alpha\beta}^\gamma (Y^{\alpha*})_\mu (Y^{\beta*})_\nu$$

Considering the commutator used in problem 3(c):

$$[Y_\alpha, Y_\beta] = \sum_\gamma C_{\alpha\beta}^\gamma Y_\gamma$$

Act the commutator on a dual vector $Y^{\gamma*}$:

$$[Y_\alpha, Y_\beta] Y^{\gamma*} = \sum_\gamma C_{\alpha\beta}^\gamma Y_\gamma Y^{\gamma*}$$

Start with the right hand side.

5.

6.

7.

8. (a) The metric of flat, three-dimensional Euclidean space is:

$$ds^2 = dx^2 + dy^2 + dz^2$$

Show that the metric components g_{uv} in spherical polar coordinates r, θ, ϕ defined by:

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ \cos \theta &= \frac{z}{r}, \\ \tan \phi &= \frac{y}{x} \end{aligned}$$

is given by:

$$s^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

g_{uv} is a tensor of type $(0, 2)$ and therefore transforms as:

$$g_{\mu', \nu'} = g_{\mu, \nu} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}}$$

(see page 22 for the general *tensor transformation law*). The above equation uses Einstein index notation indicating that μ and ν are to be summed from 1 to 3 and the free indices, μ' and ν' , are enumerated through all possible combinations. Therefore the components that need to be calculated are:

$$\begin{array}{ccc} g_{r,r} & g_{r,\theta} & g_{r,\phi} \\ g_{\theta,r} & g_{\theta,\theta} & g_{\theta,\phi} \\ g_{\phi,r} & g_{\phi,\theta} & g_{\phi,\phi} \end{array}$$

Starting with:

$$\begin{aligned} g_{\mu', \nu'} &= \sum_{\mu=1}^3 \sum_{\nu=1}^3 g_{\mu, \nu} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \\ &= \sum_{\mu=1}^3 g_{\mu,1} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{\mu,2} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{\mu,3} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}} \\ &= g_{1,1} \frac{\partial x^1}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{1,2} \frac{\partial x^1}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{1,3} \frac{\partial x^1}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}} \\ &\quad g_{2,1} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{2,2} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{2,3} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}} \\ &\quad g_{3,1} \frac{\partial x^3}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{3,2} \frac{\partial x^3}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{3,3} \frac{\partial x^3}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}} \end{aligned}$$

Substituting the notation for the indices in flat, orthonormal Euclidean space:

$$\begin{aligned}
&= g_{x,x} \frac{\partial x}{\partial x^{\mu'}} \frac{\partial x}{\partial x^{\nu'}} + g_{x,y} \frac{\partial x}{\partial x^{\mu'}} \frac{\partial y}{\partial x^{\nu'}} + g_{x,z} \frac{\partial x}{\partial x^{\mu'}} \frac{\partial z}{\partial x^{\nu'}} \\
&\quad g_{y,x} \frac{\partial y}{\partial x^{\mu'}} \frac{\partial x}{\partial x^{\nu'}} + g_{y,y} \frac{\partial y}{\partial x^{\mu'}} \frac{\partial y}{\partial x^{\nu'}} + g_{y,z} \frac{\partial y}{\partial x^{\mu'}} \frac{\partial z}{\partial x^{\nu'}} \\
&\quad g_{z,x} \frac{\partial z}{\partial x^{\mu'}} \frac{\partial x}{\partial x^{\nu'}} + g_{z,y} \frac{\partial z}{\partial x^{\mu'}} \frac{\partial y}{\partial x^{\nu'}} + g_{z,z} \frac{\partial z}{\partial x^{\mu'}} \frac{\partial z}{\partial x^{\nu'}}
\end{aligned}$$

The off diagonal elements of the Euclidean metric are zero:

$$g_{x,y} = g_{y,x} = g_{x,z} = g_{z,x} = g_{y,z} = g_{z,y} = 0$$

and the diagonal components are one:

$$g_{x,x} = g_{y,y} = g_{z,z} = 1$$

This reduces the above summation from nine expressions to the following three:

$$g_{\mu',\nu'} = \frac{\partial x}{\partial x^{\mu'}} \frac{\partial x}{\partial x^{\nu'}} + \frac{\partial y}{\partial x^{\mu'}} \frac{\partial y}{\partial x^{\nu'}} + \frac{\partial z}{\partial x^{\mu'}} \frac{\partial z}{\partial x^{\nu'}}$$

For indices where $\mu' = \nu'$

$$g_{\mu',\mu'} = \left(\frac{\partial x}{\partial x^{\mu'}} \right)^2 + \left(\frac{\partial y}{\partial x^{\mu'}} \right)^2 + \left(\frac{\partial z}{\partial x^{\mu'}} \right)^2$$

Therefore the six unique components that need to be calculated to find the components of the metric in spherical polar coordinates are:

$$\begin{aligned}
g_{r,r} &= \left(\frac{\partial x}{\partial r} \right)^2 + \left(\frac{\partial y}{\partial r} \right)^2 + \left(\frac{\partial z}{\partial r} \right)^2 \\
g_{r,\theta} &= g_{\theta,r} = \frac{\partial x}{\partial r} \frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r} \frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r} \frac{\partial z}{\partial \theta} \\
g_{r,\phi} &= g_{\phi,r} = \frac{\partial x}{\partial r} \frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial r} \frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial r} \frac{\partial z}{\partial \phi} \\
g_{\theta,\theta} &= \left(\frac{\partial x}{\partial \theta} \right)^2 + \left(\frac{\partial y}{\partial \theta} \right)^2 + \left(\frac{\partial z}{\partial \theta} \right)^2 \\
g_{\theta,\phi} &= g_{\phi,\theta} = \frac{\partial x}{\partial \theta} \frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \theta} \frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial \theta} \frac{\partial z}{\partial \phi} \\
g_{\phi,\phi} &= \left(\frac{\partial x}{\partial \phi} \right)^2 + \left(\frac{\partial y}{\partial \phi} \right)^2 + \left(\frac{\partial z}{\partial \phi} \right)^2
\end{aligned}$$

To take the above derivatives, find an equation for x, y, z in terms of r, θ, ϕ . Starting by finding x :

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \rightarrow r^2 = x^2 + z^2 + y^2, \\ \cos \theta &= \frac{z}{r} \rightarrow z = r \cos \theta, \\ \tan \phi &= \frac{y}{x} \rightarrow y = x \tan \phi \end{aligned}$$

Substituting the second and third equation into the first gives:

$$\begin{aligned} r^2 &= x^2 + (r \cos \theta)^2 + (x \tan \phi)^2 \\ r^2 &= x^2 + r^2 \cos^2 \theta + x^2 \tan^2 \phi \\ r^2 - r^2 \cos^2 \theta &= x^2 + x^2 \tan^2 \phi \\ (1 - \cos^2 \theta) r^2 &= (1 + \tan^2 \phi) x^2 \\ r^2 \sin^2 \theta &= (1 + \tan^2 \phi) x^2 \\ x &= r \frac{\sin \theta}{\sqrt{1 + \tan^2 \phi}} \end{aligned}$$

Therefore the equations for x, y, z in terms of r, θ, ϕ :

$$x = r \frac{\sin \theta}{\sqrt{1 + \tan^2 \phi}}, \quad y = r \tan \phi \frac{\sin \theta}{\sqrt{1 + \tan^2 \phi}}, \quad z = r \cos \theta$$

Find all the necessary derivatives:

$$\begin{aligned} \frac{\partial x}{\partial r} &= \frac{\sin \theta}{\sqrt{1 + \tan^2 \phi}} \\ \frac{\partial x}{\partial \theta} &= -r \frac{\cos \theta}{\sqrt{1 + \tan^2 \phi}} \\ \frac{\partial x}{\partial \phi} &= -r \sin \theta \frac{\tan \phi \sec^2 \phi}{(1 + \tan^2 \phi)^{\frac{3}{2}}} \\ \frac{\partial y}{\partial r} &= \tan \phi \frac{\sin \theta}{\sqrt{1 + \tan^2 \phi}} \\ \frac{\partial y}{\partial \theta} &= -r \tan \phi \frac{\cos \theta}{\sqrt{1 + \tan^2 \phi}} \\ \frac{\partial y}{\partial \phi} &= -r \sin \theta \frac{\sec^2 \phi}{(1 + \tan^2 \phi)^{\frac{3}{2}}} \\ \frac{\partial z}{\partial r} &= \cos \theta \\ \frac{\partial z}{\partial \theta} &= -r \sin \theta \\ \frac{\partial z}{\partial \phi} &= 0 \end{aligned}$$

Then compute the components of the metric in spherical polar coordinates:

$$\begin{aligned}
g_{r,r} &= \frac{\sin^2 \theta}{1 + \tan^2 \phi} + \frac{\sin^2 \theta}{1 + \tan^2 \phi} \tan^2 \phi + \cos^2 \theta \\
&= \frac{\sin^2 \theta + \sin^2 \theta \tan^2 \phi}{1 + \tan^2 \phi} + \frac{(1 + \tan^2 \phi) \cos^2 \theta}{1 + \tan^2 \phi} \\
&= \frac{\sin^2 \theta + \sin^2 \theta \tan^2 \phi + (1 + \tan^2 \phi) \cos^2 \theta}{1 + \tan^2 \phi} \\
&= \frac{\sin^2 \theta + \cos^2 \theta + \sin^2 \theta \tan^2 \phi + \tan^2 \phi \cos^2 \theta}{1 + \tan^2 \phi} \\
&= \frac{1 + \tan^2 \phi}{1 + \tan^2 \phi} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
g_{\theta,\theta} &= r^2 \frac{\cos^2 \theta}{1 + \tan^2 \phi} + r^2 \frac{\cos^2 \theta}{1 + \tan^2 \phi} \tan^2 \phi + r^2 \sin^2 \theta \\
&= r^2 \frac{\cos^2 \theta}{1 + \tan^2 \phi} + r^2 \frac{\cos^2 \theta}{1 + \tan^2 \phi} \tan^2 \phi + r^2 \sin^2 \theta \frac{1 + \tan^2 \phi}{1 + \tan^2 \phi} \\
&= r^2 \frac{\cos^2 \theta + \cos^2 \theta \tan^2 \phi + \sin^2 \theta + \tan^2 \phi \sin^2 \theta}{1 + \tan^2 \phi} \\
&= r^2 \frac{(\cos^2 \theta + \sin^2 \theta) + (\cos^2 \theta + \sin^2 \theta) \tan^2 \phi}{1 + \tan^2 \phi} \\
&= r^2 \frac{1 + \tan^2 \phi}{1 + \tan^2 \phi} \\
&= r^2
\end{aligned}$$

$$\begin{aligned}
g_{\phi,\phi} &= r^2 \sin^2 \theta \left(\frac{1}{(1 + \tan^2 \phi)^3 \cos^4 \phi} + \frac{\sin^2 \phi}{(1 + \tan^2 \phi)^3 \cos^6 \phi} \right) \\
&= r^2 \sin^2 \theta \left(\frac{\cos^2 \phi + \sin^2 \phi}{(1 + \tan^2 \phi)^3 \cos^6 \phi} \right) \\
&= r^2 \sin^2 \theta \frac{1}{\left(\frac{\sin^2 \phi + \cos^2 \phi}{\cos^2 \phi} \right)^3 \cos^6 \phi} \\
&= r^2 \sin^2 \theta
\end{aligned}$$

$$\begin{aligned}
g_{\theta,r} = g_{r,\theta} &= g_{x,x} \frac{\partial x}{\partial r} \frac{\partial x}{\partial \theta} + g_{y,y} \frac{\partial y}{\partial r} \frac{\partial y}{\partial \theta} + g_{z,z} \frac{\partial z}{\partial r} \frac{\partial z}{\partial \theta} \\
&= -r \frac{\sin \theta \cos \theta}{1 + \tan^2 \phi} - r \frac{\sin \theta \cos \theta}{1 + \tan^2 \phi} \tan^2 \phi + r \sin \theta \cos \theta \\
&= -r \frac{\sin \theta \cos \theta}{1 + \tan^2 \phi} (1 + \tan^2 \phi) + r \sin \theta \cos \theta \\
&= -r \sin \theta \cos \theta + r \sin \theta \cos \theta \\
&= 0
\end{aligned}$$

$$\begin{aligned}
g_{r,\phi} = g_{\phi,r} &= g_{x,x} \frac{\partial x}{\partial r} \frac{\partial x}{\partial \phi} + g_{y,y} \frac{\partial y}{\partial r} \frac{\partial y}{\partial \phi} + g_{z,z} \frac{\partial z}{\partial r} \frac{\partial z}{\partial \phi} \\
&= -r \sin \theta \cos \theta \frac{\tan \phi \sec^2 \phi}{(1 + \tan^2 \phi)^2} + r \cos \theta \sin \theta \frac{\tan \phi \sec^2 \phi}{(1 + \tan^2 \phi)^2} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
g_{\theta,\phi} = g_{\phi,\theta} &= g_{x,x} \frac{\partial x}{\partial \theta} \frac{\partial x}{\partial \phi} + g_{y,y} \frac{\partial y}{\partial \theta} \frac{\partial y}{\partial \phi} + g_{z,z} \frac{\partial z}{\partial \theta} \frac{\partial z}{\partial \phi} \\
&= -r \frac{\cos \theta}{\sqrt{1 + \tan^2 \phi}} \left(-r \sin \theta \frac{\tan \phi \sec^2 \phi}{(1 + \tan^2 \phi)^{\frac{3}{2}}} \right) - r \tan \phi \frac{\cos \theta}{\sqrt{1 + \tan^2 \phi}} \left(r \sin \theta \frac{\sec^2 \phi}{(1 + \tan^2 \phi)^{\frac{3}{2}}} \right) \\
&= r^2 \sin \theta \cos \theta \frac{\tan \phi \sec^2 \phi}{(1 + \tan^2 \phi)^2} - r^2 \sin \theta \cos \theta \frac{\tan \phi \sec^2 \phi}{(1 + \tan^2 \phi)^2} \\
&= 0
\end{aligned}$$

Therefore the metric components in spherical polar coordinates are:

$$g_{\mu,\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{bmatrix}$$

(b) **The spacetime metric of special relativity is**

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

Find the components, $g_{\mu\nu}$ and $g^{\mu\nu}$, of the metric and inverse metric in "rotating coordinates", defined by

$$\begin{aligned}
t' &= t \\
x' &= (x^2 + y^2)^{\frac{1}{2}} \cos(\phi - wt) \\
y' &= (x^2 + y^2)^{\frac{1}{2}} \sin(\phi - wt) \\
z' &= z
\end{aligned}$$

where $\tan \phi = \frac{y}{x}$

It is easier differentiate with respect to the primed coordinates so

find $g^{\mu\nu}$ first. First writing the primed coordinates in terms of the unprimed:

$$\begin{aligned}t' &= t \\x' &= (x^2 + y^2)^{\frac{1}{2}} \cos(\tan^{-1} \frac{y}{x} - \omega t) \\y' &= (x^2 + y^2)^{\frac{1}{2}} \sin(\tan^{-1} \frac{y}{x} - \omega t) \\z' &= z\end{aligned}$$

Find all the necessary derivatives:

$$\begin{aligned}\frac{\partial t'}{\partial t} &= 1 \\ \frac{\partial t'}{\partial x} &= \frac{\partial t'}{\partial y} = \frac{\partial t}{\partial z} = 0\end{aligned}$$

$$\begin{aligned}\frac{\partial x'}{\partial t} &= -w\sqrt{x^2 + y^2} \sin(\tan^{-1} \frac{y}{x} - \omega t) \\ \frac{\partial x'}{\partial x} &= \frac{x \cos(\tan^{-1} \frac{y}{x} - \omega t) + y \sin(\tan^{-1} \frac{y}{x} - \omega t)}{\sqrt{x^2 + y^2}} \\ \frac{\partial x'}{\partial y} &= \frac{-x \sin(\tan^{-1} \frac{y}{x} - \omega t) + y \cos(\tan^{-1} \frac{y}{x} - \omega t)}{\sqrt{x^2 + y^2}} \\ \frac{\partial x'}{\partial z} &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial y'}{\partial t} &= -w\sqrt{x^2 + y^2} \cos(\tan^{-1} \frac{y}{x} - \omega t) \\ \frac{\partial y'}{\partial x} &= \frac{x \sin(\tan^{-1} \frac{y}{x} - \omega t) - y \cos(\tan^{-1} \frac{y}{x} - \omega t)}{\sqrt{x^2 + y^2}} \\ \frac{\partial y'}{\partial y} &= \frac{x \sin(\tan^{-1} \frac{y}{x} - \omega t) + y \cos(\tan^{-1} \frac{y}{x} - \omega t)}{\sqrt{x^2 + y^2}} \\ \frac{\partial y'}{\partial z} &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial z'}{\partial t} &= \frac{\partial z'}{\partial x} = \frac{\partial z'}{\partial y} = 0 \\ \frac{\partial z'}{\partial z} &= 1\end{aligned}$$

$$\left(\frac{\partial x'}{\partial x}\right)^2 = \left(\frac{x \cos(\tan^{-1} \frac{y}{x} - \omega t) + y \sin(\tan^{-1} \frac{y}{x} - \omega t)}{\sqrt{x^2 + y^2}}\right)^2 = \frac{(x^2 + y^2) \sin^2(\omega t)}{(x^2 + y^2)}$$

$$\left(\frac{\partial y'}{\partial y}\right)^2 = \left(\frac{x \cos(\tan^{-1} \frac{y}{x} - \omega t) + y \sin(\tan^{-1} \frac{y}{x} - \omega t)}{\sqrt{x^2 + y^2}}\right)^2 = (x^2 + y^2) \sin^2(\omega t)$$

0.1 Appendix A - Tensor Expansion

The metric is a rank (0,2) tensor so the transformation of the components between basis is given by:

$$g_{\mu', \nu'} = g_{\mu, \nu} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}}$$

This equation is a shorthand for the following:

$$\begin{aligned} g_{\mu', \nu'} &= \sum_{\mu=1}^3 \sum_{\nu=1}^3 g_{\mu, \nu} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \\ g_{\mu', \nu'} &= \sum_{\mu=1}^3 g_{\mu, 1} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{\mu, 2} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{\mu, 3} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}} \\ g_{\mu', \nu'} &= g_{1,1} \frac{\partial x^1}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{1,2} \frac{\partial x^1}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{1,3} \frac{\partial x^1}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}} \\ &\quad g_{2,1} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{2,2} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{2,3} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}} \\ &\quad g_{3,1} \frac{\partial x^3}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{3,2} \frac{\partial x^3}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{3,3} \frac{\partial x^3}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}} \end{aligned}$$

The full expansion for μ', ν' is:

$$\begin{aligned} g_{1', 1'} &= g_{1,1} \frac{\partial x^1}{\partial x^{1'}} \frac{\partial x^1}{\partial x^{1'}} + g_{1,2} \frac{\partial x^1}{\partial x^{1'}} \frac{\partial x^2}{\partial x^{1'}} + g_{1,3} \frac{\partial x^1}{\partial x^{1'}} \frac{\partial x^3}{\partial x^{1'}} \\ &\quad g_{2,1} \frac{\partial x^2}{\partial x^{1'}} \frac{\partial x^1}{\partial x^{1'}} + g_{2,2} \frac{\partial x^2}{\partial x^{1'}} \frac{\partial x^2}{\partial x^{1'}} + g_{2,3} \frac{\partial x^2}{\partial x^{1'}} \frac{\partial x^3}{\partial x^{1'}} \\ &\quad g_{3,1} \frac{\partial x^3}{\partial x^{1'}} \frac{\partial x^1}{\partial x^{1'}} + g_{3,2} \frac{\partial x^3}{\partial x^{1'}} \frac{\partial x^2}{\partial x^{1'}} + g_{3,3} \frac{\partial x^3}{\partial x^{1'}} \frac{\partial x^3}{\partial x^{1'}} \\ g_{1', 2'} &= g_{1,1} \frac{\partial x^1}{\partial x^{1'}} \frac{\partial x^1}{\partial x^{2'}} + g_{1,2} \frac{\partial x^1}{\partial x^{1'}} \frac{\partial x^2}{\partial x^{2'}} + g_{1,3} \frac{\partial x^1}{\partial x^{1'}} \frac{\partial x^3}{\partial x^{2'}} \\ &\quad g_{2,1} \frac{\partial x^2}{\partial x^{1'}} \frac{\partial x^1}{\partial x^{2'}} + g_{2,2} \frac{\partial x^2}{\partial x^{1'}} \frac{\partial x^2}{\partial x^{2'}} + g_{2,3} \frac{\partial x^2}{\partial x^{1'}} \frac{\partial x^3}{\partial x^{2'}} \\ &\quad g_{3,1} \frac{\partial x^3}{\partial x^{1'}} \frac{\partial x^1}{\partial x^{2'}} + g_{3,2} \frac{\partial x^3}{\partial x^{1'}} \frac{\partial x^2}{\partial x^{2'}} + g_{3,3} \frac{\partial x^3}{\partial x^{1'}} \frac{\partial x^3}{\partial x^{2'}} \\ g_{1', 3'} &= g_{1,1} \frac{\partial x^1}{\partial x^{1'}} \frac{\partial x^1}{\partial x^{3'}} + g_{1,2} \frac{\partial x^1}{\partial x^{1'}} \frac{\partial x^2}{\partial x^{3'}} + g_{1,3} \frac{\partial x^1}{\partial x^{1'}} \frac{\partial x^3}{\partial x^{3'}} \\ &\quad g_{2,1} \frac{\partial x^2}{\partial x^{1'}} \frac{\partial x^1}{\partial x^{3'}} + g_{2,2} \frac{\partial x^2}{\partial x^{1'}} \frac{\partial x^2}{\partial x^{3'}} + g_{2,3} \frac{\partial x^2}{\partial x^{1'}} \frac{\partial x^3}{\partial x^{3'}} \\ &\quad g_{3,1} \frac{\partial x^3}{\partial x^{1'}} \frac{\partial x^1}{\partial x^{3'}} + g_{3,2} \frac{\partial x^3}{\partial x^{1'}} \frac{\partial x^2}{\partial x^{3'}} + g_{3,3} \frac{\partial x^3}{\partial x^{1'}} \frac{\partial x^3}{\partial x^{3'}} \end{aligned}$$

$$\begin{aligned} g_{3',3'} &= g_{1,1} \frac{\partial x^1}{\partial x^{3'}} \frac{\partial x^1}{\partial x^{3'}} + g_{1,2} \frac{\partial x^1}{\partial x^{3'}} \frac{\partial x^2}{\partial x^{3'}} + g_{1,3} \frac{\partial x^1}{\partial x^{3'}} \frac{\partial x^3}{\partial x^{3'}} \\ &\quad g_{2,1} \frac{\partial x^2}{\partial x^{3'}} \frac{\partial x^1}{\partial x^{3'}} + g_{2,2} \frac{\partial x^2}{\partial x^{3'}} \frac{\partial x^2}{\partial x^{3'}} + g_{2,3} \frac{\partial x^2}{\partial x^{3'}} \frac{\partial x^3}{\partial x^{3'}} \\ &\quad g_{3,1} \frac{\partial x^2}{\partial x^{3'}} \frac{\partial x^1}{\partial x^{3'}} + g_{3,2} \frac{\partial x^2}{\partial x^{3'}} \frac{\partial x^2}{\partial x^{3'}} + g_{3,3} \frac{\partial x^2}{\partial x^{3'}} \frac{\partial x^3}{\partial x^{3'}} \end{aligned}$$

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