

Chapter 2 Problems

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1. A tube of mass M and length l is free to swing around a pivot at one end. A mass m is positioned inside the (frictionless) tube at this end. The tube is held horizontal and then released. Let θ be the angle of the tube with respect to the horizontal, and let x be the distance the mass has traveled along the tube. Find the Euler-Lagrange equations for θ and x , and then write them in terms of θ and $\nu = x/l$ (the fraction of the distance along the tube). These equations can only be solved numerically, and you must pick a numerical value for the ratio $r = m/M$ in order to do this. Write a program that produces the value of ν when the tube is vertical ($\theta = \pi/2$)

0.1 Derivation of Equations of Motion

The total kinetic energy of the system will be the horizontal and vertical translational kinetic energies of the mass m , and the rotational kinetic energy of the tube.

Starting with the mass, consider its position x' and y' :

$$\begin{aligned}x'(t) &= x(t) \sin \theta(t) \\ y'(t) &= x(t) \cos \theta(t)\end{aligned}$$

where x is the position the mass has fallen down the tube. Differentiating once and removing the explicit time dependence for brevity yields:

$$\begin{aligned}\dot{x}' &= \dot{x} \sin \theta + x \cos \theta \dot{\theta} \\ \dot{y}' &= \dot{x} \cos \theta - x \sin \theta \dot{\theta}\end{aligned}$$

Taking the squares of each:

$$\begin{aligned}(\dot{x}')^2 &= \dot{x}^2 \sin^2 \theta + x^2 \cos^2 \theta \dot{\theta}^2 + 2 \sin \theta \cos \theta \dot{x} \dot{\theta} \\ (\dot{y}')^2 &= \dot{x}^2 \cos^2 \theta + x^2 \sin^2 \theta \dot{\theta}^2 - 2 \sin \theta \cos \theta \dot{x} \dot{\theta}\end{aligned}$$

and adding:

$$\begin{aligned}(\dot{x}')^2 + (\dot{y}')^2 &= \dot{x}^2 \sin^2 \theta + \dot{x}^2 \cos^2 \theta + x^2 \cos^2 \theta \dot{\theta}^2 + x^2 \sin^2 \theta \dot{\theta}^2 + 2 \sin \theta \cos \theta \dot{x} \dot{\theta} - 2 \sin \theta \cos \theta \dot{x} \dot{\theta} \\ &= \dot{x}^2 + x^2 \dot{\theta}^2\end{aligned}$$

corresponding to radial and tangential kinetic energies respectively. Therefore:

$$T_m = \frac{1}{2} m (\dot{x}^2 + x^2 \dot{\theta}^2)$$

Now the moment of inertia for a tube rotating about its end is the same as a rod rotating about its end, and is given by:

$$I_M = \frac{1}{3} M l^2$$

Therefore the total kinetic energies of the two masses are:

$$T = \frac{1}{2}m(\dot{x}^2 + x^2\dot{\theta}^2) + \frac{1}{6}Ml^2\dot{\theta}^2$$

The potential energy for m is simply:

$$V_m = -mgx \cos \theta$$

while the potential energy for M is the vertical component of the position of the center mass:

$$V_M = -\frac{Mgl}{2} \cos \theta$$

The Lagrangian \mathcal{L} is:

$$\mathcal{L} = T - V = \frac{1}{2}m(\dot{x}^2 + x^2\dot{\theta}^2) + \frac{1}{6}Ml^2\dot{\theta}^2 + mgx \cos \theta + \frac{Mgl}{2} \cos \theta$$

Finding the Euler-Lagrange equations for x :

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\ddot{x}$$

$$\frac{\partial \mathcal{L}}{\partial x} = m\dot{x}\dot{\theta}^2 + mg \cos \theta$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x} \Rightarrow \ddot{x} = \dot{x}\dot{\theta}^2 + g \cos \theta$$

and θ :

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m \frac{d}{dt}(x^2\dot{\theta}) + \frac{1}{3}Ml^2\ddot{\theta}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -g(mx + \frac{Ml}{2}) \sin \theta$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta} \Rightarrow m \frac{d}{dt}(x^2\dot{\theta}) + \frac{1}{3}Ml^2\ddot{\theta} = -g(mx + \frac{Ml}{2}) \sin \theta$$

Therefore the Euler Lagrange equations are:

$$m \frac{d}{dt}(x^2\dot{\theta}) + \frac{1}{3}Ml^2\ddot{\theta} = -g(mx + \frac{Ml}{2}) \sin \theta$$

$$\ddot{x} = \dot{x}\dot{\theta}^2$$

Reparameterizing in terms of η and $r = \frac{m}{M}$:

$$r \frac{d}{dt}(\eta^2\dot{\theta}) + \frac{1}{3}\ddot{\theta} = -\frac{g}{l}(\frac{rx}{l} + \frac{1}{2}) \sin \theta \quad (1)$$

$$\ddot{\eta} = \eta\dot{\theta}^2 + g \cos \theta \quad (2)$$

are the Euler-Lagrange equations.

0.2 Approximations

0.2.1 Small mass approximation $m \ll M$

$m \ll M$ is equivalent to the condition $r \ll 1$. Let us assume that r is negligibly small. The equations of motion then read:

$$\ddot{\theta} = -\frac{3g}{2l} \sin \theta \quad (3)$$

$$\ddot{\eta} = \eta \dot{\theta}^2 \quad (4)$$

By our definition of θ the vertical position is at $\theta = 0$ so small oscillations about the vertical are given by:

$$\theta(t) = A \cos(\omega t + \phi)$$

where $\omega^2 = 3g/2l$.

Consider the initial conditions:

$$t_{\text{vertical}} = \frac{T}{2} = \frac{1}{2} \frac{2\pi}{\omega} = \frac{\pi}{\omega} = \frac{2\pi}{3} \sqrt{\frac{l}{g}}$$

$$\theta(t_{\text{vertical}}) = 0$$

Then:

$$\begin{aligned} 0 &= \cos\left(\omega\left(\frac{\pi}{\omega}\right) + \phi\right) \\ &= \cos(\pi + \phi) \\ &= \cos(\pi) \cos(\phi) - \sin(\pi) \sin(\phi) \\ &= \cos(\phi) \end{aligned}$$

therefore $\phi = \pi/2$ and the solution for θ can be rewritten without a phase shift as:

$$\theta(t) = A \sin(\omega t)$$

Taking a time derivative:

$$\dot{\theta}(t) = A\omega \cos(\omega t)$$

$$\dot{\theta}^2(t) = A^2\omega^2 \cos^2(\omega t)$$

and substituting into 4:

$$\ddot{\eta} = A^2\omega^2 \sin^2(\omega t)\eta$$

using the identity $\sin^2 A = \frac{1}{2} - \frac{1}{2} \cos(2A)$, we can rewrite as:

$$\ddot{\eta} = A^2\omega^2 \left(\frac{1}{2} - \frac{1}{2} \cos 2\omega t\right)\eta$$

This is a form of Mathieu's equation. Now consider $t = t_{\text{vertical}} + \delta$

$$\ddot{\eta}(\delta) = A^2\omega^2\left(\frac{1}{2} - \frac{1}{2}\cos 2\omega(t_{\text{vertical}} + \delta)\right)\eta(\delta)$$

Consider the cos expression:

$$\begin{aligned}\cos 2\omega(t_{\text{vertical}} + \delta) &= \cos(2\omega t_{\text{vertical}}) \cos 2\omega\delta - \sin 2\omega t_{\text{vertical}} \sin 2\omega\delta \\ &= \cos 2\omega\delta\end{aligned}$$

Assume δ is small so that $\cos 2\omega\delta \approx 1 - 2\omega^2\delta^2$

$$\begin{aligned}\ddot{\eta} &= A^2\omega^2\left(\frac{1}{2} - \frac{1}{2}(1 - 2\omega^2\delta^2)\right)\eta \\ &= A^2\omega^4\delta^2\eta\end{aligned}$$

0.2.2 Taylor Series Expansion

Taylor expand η and x as:

$$\eta(t) = \sum_{n=0}^{\infty} a_n(t - t_0)^n$$

and:

$$x(t) = \sum_{n=0}^{\infty} b_n(t - t_0)^n$$

Therefore: