# Misner, Thorne and Wheeler's Gravitation Problems

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# Chapter 1

# Chapter 2

# Chapter 3

# The Electromagnetic Field

1. **Derive equations:**

$$\| F^{\alpha}_{\beta} \| = \begin{Vmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{Vmatrix} \tag{3.1}$$

**and**

$$\| F_{\alpha\beta} \| = \begin{Vmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{Vmatrix} \tag{3.2}$$

**for the components of Faraday by comparing:**

$$dp^{\alpha}/d\tau = eF^{\alpha}_{\beta}u^{\beta} \tag{3.3}$$

**with**

$$\frac{d\mathbf{p}}{d\tau} = \frac{1}{\sqrt{1-\mathbf{v}^2}}\frac{d\mathbf{p}}{dt} = \frac{e}{\sqrt{1-\mathbf{v}^2}}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = e(u^0\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{3.4}$$

$$\frac{dp^0}{d\tau} = \frac{1}{\sqrt{1-\mathbf{v}^2}}\frac{dE}{dt} = \frac{1}{\sqrt{1-\mathbf{v}^2}}e\mathbf{E}\cdot\mathbf{v} = e\mathbf{E}\cdot\mathbf{u} \tag{3.5}$$

**and by using definition:**

$$F_{\alpha\beta} = \eta_{\alpha\gamma}F^{\gamma}_{\beta} \tag{3.6}$$

Consider equation 3.3 for the index $\alpha = 0$:

$$\frac{dp^0}{d\tau} = e[F^0_0u^0 + F^0_1u^1 + F^0_2u^2 + F^0_3u^3]$$

Equate this with 3.5:

$$e[F^0_0u^0 + F^0_1u^1 + F^0_2u^2 + F^0_3u^3] = e\mathbf{E}\cdot\mathbf{u} = e[E_1u^1 + E_2u^2 + E_3u^3]$$

It is clear that:

$$\begin{aligned}
F^0_0 &= 0 \\
F^0_1 u^1 &= E_1 u^1 \Rightarrow F^0_1 = E_1 \\
F^0_2 u^2 &= E_2 u^2 \Rightarrow F^0_2 = E_2 \\
F^0_3 u^3 &= E_3 u^3 \Rightarrow F^0_3 = E_3
\end{aligned}$$

Now equating equation 3.4 with the remaining components of equation 3.3:

$$\frac{dp^1}{d\tau} = e[F^1_0 u^0 + F^1_1 u^1 + F^1_2 u^2 + F^1_3 u^3] = e[E_1 u^0 + B_3 u^2 - B_2 u^3]$$

$$\frac{dp^2}{d\tau} = e[F^2_0 u^0 + F^2_1 u^1 + F^2_2 u^2 + F^2_3 u^3] = e[E_2 u^0 + B_1 u^3 - B_3 u^2]$$

$$\frac{dp^3}{d\tau} = e[F^3_0 u^0 + F^3_1 u^1 + F^3_2 u^2 + F^3_3 u^3] = e[E_3 u^0 + B_2 u^1 - B_1 u^2]$$

and equating components as before:

$$\begin{aligned}
F^1_0 u^0 &= E_1 u^0 \Rightarrow F^1_0 = E_1 \\
F^1_1 u^1 &= 0 \Rightarrow F^1_1 = 0 \\
F^1_2 u^2 &= B_3 u^2 \Rightarrow F^1_2 = B_3 \\
F^1_3 u^3 &= -B_2 u^3 \Rightarrow F^1_3 = -B_2 \\
F^2_0 u^0 &= E_2 u^0 \Rightarrow F^2_0 = E_2 \\
F^2_1 u^1 &= -B_3 u^1 \Rightarrow F^2_1 = -B_3 \\
F^2_2 u^2 &= 0 \Rightarrow F^2_2 = 0 \\
F^2_3 u^3 &= B_1 u^3 \Rightarrow F^2_3 = B_1 \\
F^3_0 u^0 &= E_3 u^0 \Rightarrow F^3_0 = E_3 \\
F^3_1 u^1 &= B_2 u^1 \Rightarrow F^3_1 = B_2 \\
F^3_2 u^2 &= -B_1 u^2 \Rightarrow F^3_2 = -B_1 \\
F^3_3 u^3 &= 0 \Rightarrow F^3_3 = 0
\end{aligned}$$

Collecting all of these components in matrix form and relabling indices with the following mapping:

$$\begin{aligned}
\alpha &= 1 \to x \\
\alpha &= 2 \to y \\
\alpha &= 3 \to z
\end{aligned}$$

gives equation 3.1:

$$|| \; F^\alpha_\beta \; || = \begin{Vmatrix} F^0_0 & F^0_1 & F^0_2 & F^0_3 \\ F^1_0 & F^1_1 & F^1_2 & F^1_3 \\ F^2_0 & F^2_1 & F^2_2 & F^2_3 \\ F^3_0 & F^3_1 & F^3_2 & F^3_3 \end{Vmatrix} = \begin{Vmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{Vmatrix}$$

Now equation 3.6 can be used to convert the mixed Faraday tensor to the fully covariant one. Remeber that for all components $\alpha \neq \beta$ the Minkowski metric is zero. Therefore the only non-zero components in the sums created by the summation convention in equation 3.6 are:

$$
\begin{aligned}
F_{00} = \eta_{00}F^0_0 &\Rightarrow F_{00} = -F^0_0 \\
F_{01} = \eta_{00}F^0_1 &\Rightarrow F_{01} = -F^0_1 \\
F_{02} = \eta_{00}F^0_2 &\Rightarrow F_{02} = -F^0_2 \\
F_{03} = \eta_{00}F^0_3 &\Rightarrow F_{03} = -F^0_3 \\
F_{10} = \eta_{11}F^1_0 &\Rightarrow F_{10} = F^1_0 \\
F_{11} = \eta_{11}F^1_1 &\Rightarrow F_{11} = F^1_1 \\
F_{12} = \eta_{11}F^1_2 &\Rightarrow F_{12} = F^1_2 \\
F_{13} = \eta_{11}F^1_3 &\Rightarrow F_{13} = F^1_3 \\
F_{20} = \eta_{22}F^2_0 &\Rightarrow F_{02} = F^2_0 \\
F_{21} = \eta_{22}F^2_1 &\Rightarrow F_{12} = F^2_1 \\
F_{22} = \eta_{22}F^2_2 &\Rightarrow F_{22} = F^2_2 \\
F_{23} = \eta_{22}F^2_3 &\Rightarrow F_{23} = F^3_2 \\
F_{30} = \eta_{33}F^3_0 &\Rightarrow F_{30} = F^3_0 \\
F_{31} = \eta_{33}F^3_1 &\Rightarrow F_{31} = F^3_1 \\
F_{32} = \eta_{33}F^3_2 &\Rightarrow F_{32} = F^3_2 \\
F_{33} = \eta_{33}F^3_3 &\Rightarrow F_{33} = F^3_3
\end{aligned}
$$

Collecting the components into matrix form recovers the fully covariant Faraday tensor:

$$
|| \ F_{\alpha\beta} \ ||= \begin{Vmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{Vmatrix} \tag{3.7}
$$

2. **From the transformation laws for components of vectors and 1-forms, derive the transformation law:**

$$S^{\mu'\nu'}{}_{\lambda'} = S^{\alpha\beta}{}_{\gamma}\Lambda^{\mu'}_{\alpha}\Lambda^{\nu'}_{\beta}\Lambda^{\gamma}_{\lambda'}$$

Consider the tensor $\mathbf{S}$ of rank $(2,1)$, in geometric notation the transformation between two sets basis vectors and 1-forms reads:

$$\mathbf{S}(\sigma, \rho, \nu) = \mathbf{S}(\sigma', \rho', \nu')$$

In component form this reads:

$$S^{\alpha\beta}{}_{\gamma}\sigma_\alpha\rho_\beta\nu^\gamma = S^{\mu'\nu'}{}_{\lambda'}\sigma_{\mu'}\rho_{\nu'}\nu^{\lambda'} \tag{3.8}$$

Using the Lorentz transformation laws to transform one basis into the other for $\sigma$, $\rho$, $\nu$ gives:

$$\sigma_\alpha = \Lambda^{\mu'}_{\alpha}\sigma_{\mu'}$$
$$\rho_\beta = \Lambda^{\nu'}_{\beta}\rho_{\mu'}$$
$$\nu^\beta = \Lambda^{\gamma}_{\lambda'}\nu^{\lambda'}$$

and substituting these transformations into equation 3.8:

$$S^{\mu'\nu'}{}_{\lambda'}\sigma_{\mu'}\rho_{\nu'}\nu^{\lambda'} = S^{\alpha\beta}{}_{\gamma}(\Lambda^{\mu'}_{\alpha}\sigma_{\mu'})(\Lambda^{\nu'}_{\beta}\rho_{\mu'})(\Lambda^{\gamma}_{\lambda'}\nu^{\lambda'})$$

$$S^{\mu'\nu'}{}_{\lambda'}\sigma_{\mu'}\rho_{\nu'}\nu^{\lambda'} = S^{\alpha\beta}{}_{\gamma}\Lambda^{\mu'}_{\alpha}\Lambda^{\nu'}_{\beta}\Lambda^{\gamma}_{\lambda'}\sigma_{\mu'}\rho_{\mu'}\nu^{\lambda'}$$

Equating the components gives the desired transformation law:

$$S^{\mu'\nu'}{}_{\lambda'} = S^{\alpha\beta}{}_{\gamma}\Lambda^{\mu'}_{\alpha}\Lambda^{\nu'}_{\beta}\Lambda^{\gamma}_{\lambda'}$$

3. **Raising and lowering indices. Derive:**

$$S^{\alpha}_{\beta\,\gamma} = \eta_{\beta\mu}S^{\alpha\mu}{}_{\gamma} \tag{3.9}$$

**and:**

$$S^{\alpha\mu}{}_{\gamma} = \eta^{\mu\beta}S^{\alpha}_{\beta\,\gamma} \tag{3.10}$$

**from:**

4.

5.

6.

7.

8.

9.

10. **More differentiation. (a) Justify the formula,**

$$d(u^\mu u_\mu)/d\tau = 2u_\mu(du^\mu/d\tau),$$

**by writing out the summation $u^\mu u_\mu = \eta_{\mu\nu}u^\mu u^\nu$ explicitly** Writing out the components explicitly yields:

$$\begin{aligned} u^\mu u_\mu &= \eta_{00}u^0u^0 + \eta_{11}u^1u^1 + \eta_{22}u^2u^2 + \eta_{33}u^3u^3 \\ &= \eta_{00}(u^0)^2 + \eta_{11}(u^1)^2 + \eta_{22}(u^2)^2 + \eta_{33}(u^3)^2 \\ &= -(u^0)^2 + (u^1)^2 + (u^2)^2 + (u^3)^2 \end{aligned}$$

Taking a total derivative of the above with respect to $\tau$:

$$\begin{aligned}
\frac{d}{d\tau}(u^\mu u_\mu) &= \frac{d}{d\tau}(-(u^0)^2 + (u^1)^2 + (u^2)^2 + (u^3)^2) \\
&= -2u^0\frac{du^0}{d\tau} + 2u^1\frac{du^1}{d\tau} + 2u^2\frac{du^2}{d\tau} + 2u^3\frac{du^3}{d\tau} \\
&= 2[-u^0\frac{du^0}{d\tau} + u^1\frac{du^1}{d\tau} + u^2\frac{du^2}{d\tau} + u^3\frac{du^3}{d\tau}] \\
&= 2\eta_{\mu\nu}u^\mu\frac{du^\nu}{d\tau} \\
&= 2u_\mu\frac{du^\nu}{d\tau}
\end{aligned}$$

Therefore the formula is justified.