Chapter 2 Problems

Anthony Steel

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1. **The metric of flat, three-dimensional Euclidean space is:**

$$ds^2 = dx^2 + dy^2 + dz^2$$

Show that the metric components g_{uv} in spherical polar coordinates r, θ, ϕ defined by:

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\cos\theta = \frac{z}{r},$$
$$\tan\phi = \frac{y}{x}$$

is given by:

$$s^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\phi^2$$

g_{uv} is a tensor of type $(0,2)$ and therefore transforms as:

$$g_{\mu',\nu'} = g_{\mu,\nu} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}}$$

(see page 22 for the general *tensor transformation law*). The above equation uses Einstein index notation indicating that μ and ν are to to be summed from 1 to 3 and the free indices, μ' and ν', are enumerated through all possible combinations:

$$
\begin{array}{ccc}
g_{1'1'} & g_{1'2'} & g_{1'3'} \\
g_{2'1'} & g_{2'2'} & g_{2'3'} \\
g_{3'1'} & g_{3'2'} & g_{3'3'}
\end{array}
$$

Starting with:

$$g_{1',1'} = \sum_{\mu=1}^{3}\sum_{\nu=1}^{3} g_{\mu,\nu} \frac{\partial x^\mu}{\partial x^{1'}} \frac{\partial x^\nu}{\partial x^{1'}}$$

$$= \sum_{\mu=1}^{3} g_{\mu,1} \frac{\partial x^\mu}{\partial x^{1'}} \frac{\partial x^1}{\partial x^{1'}} + g_{\mu,2} \frac{\partial x^\mu}{\partial x^{1'}} \frac{\partial x^2}{\partial x^{1'}} + g_{\mu,3} \frac{\partial x^\mu}{\partial x^{1'}} \frac{\partial x^3}{\partial x^{1'}}$$

The off diagonal elements of the Euclidean metric written in matrix for is an identity matrix. This reduces the above summation from a possible 9 components to the following three:

$$g_{1',1'} = \frac{\partial x^1}{\partial x^{1'}} \frac{\partial x^1}{\partial x^{1'}} + \frac{\partial x^2}{\partial x^{1'}} \frac{\partial x^2}{\partial x^{1'}} + \frac{\partial x^3}{\partial x^{1'}} \frac{\partial x^3}{\partial x^{1'}} = \left(\frac{\partial x^1}{\partial x^{1'}}\right)^2 + \left(\frac{\partial x^2}{\partial x^{1'}}\right)^2 + \left(\frac{\partial x^3}{\partial x^{1'}}\right)^2$$

Converting the notation to reflect the choices of basis:

$$g_{r,r} = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2$$

$$g_{\theta,\theta} = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2$$

$$g_{\phi,\phi} = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2$$

First, find an equation for x in terms of r. Rearranging:

$$r = \sqrt{x^2 + y^2 + z^2} \to r^2 = x^2 + z^2 + y^2,$$

$$\cos\theta = \frac{z}{r} \to z = r\cos\theta,$$

$$\tan\phi = \frac{y}{x} \to y = x\tan\phi$$

Substituting the second and third equation into the first gives:

$$r^2 = x^2 + (r\cos\theta)^2 + (x\tan\phi)^2$$
$$r^2 = x^2 + r^2\cos^2\theta + x^2\tan^2\phi$$
$$r^2 - r^2\cos\theta = x^2 + x^2\tan^2\phi$$
$$(1 - \cos^2\theta)r^2 = (1 + \tan^2\phi)x^2$$
$$r^2\sin^2\theta = (1 + \tan^2\phi)x^2$$
$$x = r\frac{\sin\theta}{\sqrt{1 + \tan^2\phi}}$$

Differentiating:

$$\frac{\partial x}{\partial r} = \frac{\sin\theta}{\sqrt{1 + \tan^2\phi}},$$

$$\frac{\partial x}{\partial \theta} = -r\frac{\cos\theta}{\sqrt{1 + \tan^2\phi}}$$

$$\frac{\partial x}{\partial \phi} = -r\frac{\sin\theta}{(1 + \tan^2\phi)^{\frac{3}{2}}}\tan\phi\frac{1}{\cos^2\phi}$$

$$= -r\frac{\sin\theta}{(1 + \tan^2\phi)^{\frac{3}{2}}}\frac{\sin\phi}{\cos^3\phi}$$

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Second, find an equation for y in terms of r, and then differentiate:

$$y = x \tan \phi$$

$$= r \frac{\sin \theta}{\sqrt{1 + \tan^2 \phi}} \tan \phi$$

$$\frac{\partial y}{\partial r} = \frac{\sin \theta}{\sqrt{1 + \tan^2 \phi}} \tan \phi$$

$$\frac{\partial y}{\partial \theta} = -r \frac{\cos \theta}{\sqrt{1 + \tan^2 \phi}} \tan \phi$$

$$\frac{\partial y}{\partial \phi} = \frac{-r \sin \theta}{(1 + \tan^2 \phi)^{\frac{3}{2}}} \frac{1}{\cos^2 \phi}$$

Lastly, z in terms of r simply follows from the definition of them transformation:

$$z = r \cos \theta \rightarrow \frac{\partial z}{\partial r} = \cos \theta, \frac{\partial z}{\partial \theta} = r \sin \theta, \frac{\partial z}{\partial \phi} = 0$$

Therefore:

$$g_{r,r} = \frac{\sin \theta^2}{1 + \tan^2 \phi} + \frac{\sin^2 \theta}{1 + \tan^2 \phi} \tan^2 \phi + \cos^2 \theta$$

$$g_{r,r} = \frac{\sin \theta^2 + \sin^2 \theta \tan^2 \phi}{1 + \tan^2 \phi} + \frac{(1 + \tan^2 \phi) \cos^2 \theta}{1 + \tan^2 \phi}$$

$$g_{r,r} = \frac{\sin \theta^2 + \sin^2 \theta \tan^2 \phi + (1 + \tan^2 \phi) \cos^2 \theta}{1 + \tan^2 \phi}$$

$$g_{r,r} = \frac{\sin^2 \theta + \cos^2 \theta + \sin^2 \theta \tan^2 \phi + \tan^2 \phi \cos^2 \theta}{1 + \tan^2 \phi}$$

$$g_{r,r} = \frac{1 + \tan^2 \phi}{1 + \tan^2 \phi}$$

$$g_{r,r} = 1$$

$$g_{\theta,\theta} = r^2 \frac{\cos^2 \theta}{1 + \tan^2 \phi} + r^2 \frac{\cos^2 \theta}{1 + \tan^2 \phi} \tan^2 \phi + r^2 \sin^2 \theta$$

$$g_{\theta,\theta} = r^2 \frac{\cos^2 \theta}{1 + \tan^2 \phi} + r^2 \frac{\cos^2 \theta}{1 + \tan^2 \phi} \tan^2 \phi + r^2 \sin^2 \theta \frac{1 + \tan^2 \phi}{1 + \tan^2 \phi}$$

$$g_{\theta,\theta} = r^2 \frac{\cos^2 \theta + \cos^2 \theta \tan^2 \phi + \sin^2 \theta + \tan^2 \phi \sin^2 \theta}{1 + \tan^2 \phi}$$

$$g_{\theta,\theta} = r^2 \frac{(\cos^2 \theta + \sin^2 \theta) + (\cos^2 \theta + \sin^2 \theta) \tan^2 \phi}{1 + \tan^2 \phi}$$

$$g_{\theta,\theta} = r^2 \frac{1 + \tan^2 \phi}{1 + \tan^2 \phi}$$

$$g_{\theta,\theta} = r^2$$

$$g_{\phi,\phi} = r^2 \sin^2 \theta \left(\frac{1}{(1 + \tan^2 \phi)^3 \cos^4 \phi} + \frac{\sin^2 \phi}{(1 + \tan^2 \phi)^3 \cos^6 x} \right)$$

$$= r^2 \sin^2 \theta \left(\frac{\cos^2 \phi + \sin^2 \phi}{(1 + \tan^2 \phi)^3 \cos^6 \phi} \right)$$

$$= r^2 \sin^2 \theta \frac{1}{\left(\frac{\sin^2 \phi + \cos^2 \phi}{\cos^2 \phi} \right)^3 \cos^6 \phi}$$

$$= r^2 \sin^2 \theta$$

Therefore the metric components are:

$$g_{\mu,\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{bmatrix}$$