

Misner, Thorne and Wheeler's Gravitation Problems

Anthony Steel

April 22, 2021

Chapter 1

Chapter 2

Chapter 3

The Electromagnetic Field

1. Derive equations:

$$|| F_{\beta}^{\alpha} || = \begin{vmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{vmatrix} \quad (3.1)$$

and

$$|| F_{\alpha\beta} || = \begin{vmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{vmatrix} \quad (3.2)$$

for the components of Faraday by comparing:

$$dp^{\alpha}/d\tau = eF_{\beta}^{\alpha}u^{\beta} \quad (3.3)$$

with

$$\frac{d\mathbf{p}}{d\tau} = \frac{1}{\sqrt{1-\mathbf{v}^2}} \frac{d\mathbf{p}}{dt} = \frac{e}{\sqrt{1-\mathbf{v}^2}} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) = e(u^0 \mathbf{E} + \mathbf{u} \times \mathbf{B}) \quad (3.4)$$

$$\frac{dp^0}{d\tau} = \frac{1}{\sqrt{1-\mathbf{v}^2}} \frac{dE}{dt} = \frac{1}{\sqrt{1-\mathbf{v}^2}} e \mathbf{E} \cdot \mathbf{v} = e \mathbf{E} \cdot \mathbf{u} \quad (3.5)$$

and by using definition:

$$F_{\alpha\beta} = \eta_{\alpha\gamma} F_{\beta}^{\gamma} \quad (3.6)$$

Consider equation 3.3 for the index $\alpha = 0$:

$$\frac{dp^0}{d\tau} = e[F_0^0 u^0 + F_1^0 u^1 + F_2^0 u^2 + F_3^0 u^3]$$

Equate this with 3.5:

$$e[F_0^0 u^0 + F_1^0 u^1 + F_2^0 u^2 + F_3^0 u^3] = e \mathbf{E} \cdot \mathbf{u} = e[E_1 u^1 + E_2 u^2 + E_3 u^3]$$

It is clear that:

$$\begin{aligned}
F_0^0 &= 0 \\
F_1^0 u^1 &= E_1 u^1 \Rightarrow F_1^0 = E_1 \\
F_2^0 u^2 &= E_2 u^2 \Rightarrow F_2^0 = E_2 \\
F_3^0 u^3 &= E_3 u^3 \Rightarrow F_3^0 = E_3
\end{aligned}$$

Now equating equation 3.4 with the remaining components of equation 3.3:

$$\begin{aligned}
\frac{dp^1}{d\tau} &= e[F_0^1 u^0 + F_1^1 u^1 + F_2^1 u^2 + F_3^1 u^3] = e[E_1 u^0 + B_3 u^2 - B_2 u^3] \\
\frac{dp^2}{d\tau} &= e[F_0^2 u^0 + F_1^2 u^1 + F_2^2 u^2 + F_3^2 u^3] = e[E_2 u^0 + B_1 u^3 - B_3 u^2] \\
\frac{dp^3}{d\tau} &= e[F_0^3 u^0 + F_1^3 u^1 + F_2^3 u^2 + F_3^3 u^3] = e[E_3 u^0 + B_2 u^1 - B_1 u^2]
\end{aligned}$$

and equating components as before:

$$\begin{aligned}
F_0^1 u^0 &= E_1 u^0 \Rightarrow F_0^1 = E_1 \\
F_1^1 u^1 &= 0 \Rightarrow F_1^1 = 0 \\
F_2^1 u^2 &= B_3 u^2 \Rightarrow F_2^1 = B_3 \\
F_3^1 u^3 &= -B_2 u^3 \Rightarrow F_3^1 = -B_2 \\
F_0^2 u^0 &= E_2 u^0 \Rightarrow F_0^2 = E_2 \\
F_1^2 u^1 &= -B_3 u^1 \Rightarrow F_1^2 = -B_3 \\
F_2^2 u^2 &= 0 \Rightarrow F_2^2 = 0 \\
F_3^2 u^3 &= B_1 u^3 \Rightarrow F_3^2 = B_1 \\
F_0^3 u^0 &= E_3 u^0 \Rightarrow F_0^3 = E_3 \\
F_1^3 u^1 &= B_2 u^1 \Rightarrow F_1^3 = B_2 \\
F_2^3 u^2 &= -B_1 u^2 \Rightarrow F_2^3 = -B_1 \\
F_3^3 u^3 &= 0 \Rightarrow F_3^3 = 0
\end{aligned}$$

Collecting all of these components in matrix form and relabelling indices with the following mapping:

$$\begin{aligned}
\alpha = 1 &\rightarrow x \\
\alpha = 2 &\rightarrow y \\
\alpha = 3 &\rightarrow z
\end{aligned}$$

gives equation 3.1:

$$\| F_\beta^\alpha \| = \begin{vmatrix} F_0^0 & F_1^0 & F_2^0 & F_3^0 \\ F_0^1 & F_1^1 & F_2^1 & F_3^1 \\ F_0^2 & F_1^2 & F_2^2 & F_3^2 \\ F_0^3 & F_1^3 & F_2^3 & F_3^3 \end{vmatrix} = \begin{vmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{vmatrix}$$

Now equation 3.6 can be used to convert the mixed Faraday tensor to the fully covariant one. Remember that for all components $\alpha \neq \beta$ the Minkowski metric is zero. Therefore the only non-zero components in the sums created by the summation convention in equation 3.6 are:

$$\begin{aligned}
F_{00} &= \eta_{00}F_0^0 \Rightarrow F_{00} = -F_0^0 \\
F_{01} &= \eta_{00}F_1^0 \Rightarrow F_{01} = -F_1^0 \\
F_{02} &= \eta_{00}F_2^0 \Rightarrow F_{02} = -F_2^0 \\
F_{03} &= \eta_{00}F_3^0 \Rightarrow F_{03} = -F_3^0 \\
F_{10} &= \eta_{11}F_0^1 \Rightarrow F_{10} = F_0^1 \\
F_{11} &= \eta_{11}F_1^1 \Rightarrow F_{11} = F_1^1 \\
F_{12} &= \eta_{11}F_2^1 \Rightarrow F_{12} = F_2^1 \\
F_{13} &= \eta_{11}F_3^1 \Rightarrow F_{13} = F_3^1 \\
F_{20} &= \eta_{22}F_0^2 \Rightarrow F_{02} = F_0^2 \\
F_{21} &= \eta_{22}F_1^2 \Rightarrow F_{12} = F_1^2 \\
F_{22} &= \eta_{22}F_2^2 \Rightarrow F_{22} = F_2^2 \\
F_{23} &= \eta_{22}F_3^2 \Rightarrow F_{23} = F_2^3 \\
F_{30} &= \eta_{33}F_0^3 \Rightarrow F_{30} = F_0^3 \\
F_{31} &= \eta_{33}F_1^3 \Rightarrow F_{31} = F_1^3 \\
F_{32} &= \eta_{33}F_2^3 \Rightarrow F_{32} = F_2^3 \\
F_{33} &= \eta_{33}F_3^3 \Rightarrow F_{33} = F_3^3
\end{aligned}$$

Collecting the components into matrix form recovers the fully covariant Faraday tensor:

$$|| F_{\alpha\beta} || = \begin{vmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{vmatrix} \quad (3.7)$$

2. From the transformation laws for components of vectors and 1-forms, derive the transformation law:

$$S^{\mu'\nu'}_{\lambda'} = S^{\alpha\beta}_{\gamma} \Lambda^{\mu'}_{\alpha} \Lambda^{\nu'}_{\beta} \Lambda^{\gamma}_{\lambda'}$$

Consider the tensor \mathbf{S} of rank $(2, 1)$, in geometric notation the transformation between two sets basis vectors and 1-forms reads:

$$\mathbf{S}(\sigma, \rho, \nu) = \mathbf{S}(\sigma', \rho', \nu')$$

In component form this reads:

$$S^{\alpha\beta}_{\gamma} \sigma_{\alpha} \rho_{\beta} \nu^{\gamma} = S^{\mu'\nu'}_{\lambda'} \sigma_{\mu'} \rho_{\nu'} \nu^{\lambda'} \quad (3.8)$$

Using the Lorentz transformation laws to transform one basis into the other for σ , ρ , ν gives:

$$\begin{aligned}\sigma_\alpha &= \Lambda_\alpha^{\mu'} \sigma_{\mu'} \\ \rho_\beta &= \Lambda_\beta^{\nu'} \rho_{\nu'} \\ \nu^\beta &= \Lambda_{\lambda'}^\gamma \nu^{\lambda'}\end{aligned}$$

and substituting these transformations into equation 3.8:

$$\begin{aligned}S^{\mu'\nu'}_{\lambda'} \sigma_{\mu'} \rho_{\nu'} \nu^{\lambda'} &= S^{\alpha\beta}_\gamma (\Lambda_\alpha^{\mu'} \sigma_{\mu'}) (\Lambda_\beta^{\nu'} \rho_{\nu'}) (\Lambda_{\lambda'}^\gamma \nu^{\lambda'}) \\ S^{\mu'\nu'}_{\lambda'} \sigma_{\mu'} \rho_{\nu'} \nu^{\lambda'} &= S^{\alpha\beta}_\gamma \Lambda_\alpha^{\mu'} \Lambda_\beta^{\nu'} \Lambda_{\lambda'}^\gamma \sigma_{\mu'} \rho_{\nu'} \nu^{\lambda'}\end{aligned}$$

Equating the components gives the desired transformation law:

$$S^{\mu'\nu'}_{\lambda'} = S^{\alpha\beta}_\gamma \Lambda_\alpha^{\mu'} \Lambda_\beta^{\nu'} \Lambda_{\lambda'}^\gamma$$

3. Raising and lowering indices. Derive:

$$S^\alpha_{\beta\gamma} = \eta_{\beta\mu} S^{\alpha\mu}_\gamma \quad (3.9)$$

and:

$$S^{\alpha\mu}_\gamma = \eta^{\mu\beta} S^\alpha_{\beta\gamma} \quad (3.10)$$

from:

4.

5.

6.

7. Maxwell's Equations. Show, by explicit examination of components, that the geometric laws

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0 \quad (3.11)$$

$$F^{\alpha\beta}_{,\beta} = 4\pi J^\alpha \quad (3.12)$$

do reduce to Maxwell's equations

$$\nabla \cdot \mathbf{B} = 0 \quad (3.13)$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \quad (3.14)$$

$$\nabla \cdot \mathbf{E} = 4\pi \rho \quad (3.15)$$

$$\frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -4\pi \mathbf{J} \quad (3.16)$$

8.

9.

10. **More differentiation. (a) Justify the formula,**

$$d(u^\mu u_\mu)/d\tau = 2u_\mu(du^\mu/d\tau),$$

by writing out the summation $u^\mu u_\mu = \eta_{\mu\nu} u^\mu u^\nu$ **explicitly** Writing out the components explicitly yields:

$$\begin{aligned} u^\mu u_\mu &= \eta_{00} u^0 u^0 + \eta_{11} u^1 u^1 + \eta_{22} u^2 u^2 + \eta_{33} u^3 u^3 \\ &= \eta_{00} (u^0)^2 + \eta_{11} (u^1)^2 + \eta_{22} (u^2)^2 + \eta_{33} (u^3)^2 \\ &= -(u^0)^2 + (u^1)^2 + (u^2)^2 + (u^3)^2 \end{aligned}$$

Taking a total derivative of the above with respect to τ :

$$\begin{aligned} \frac{d}{d\tau}(u^\mu u_\mu) &= \frac{d}{d\tau}(-(u^0)^2 + (u^1)^2 + (u^2)^2 + (u^3)^2) \\ &= -2u^0 \frac{du^0}{d\tau} + 2u^1 \frac{du^1}{d\tau} + 2u^2 \frac{du^2}{d\tau} + 2u^3 \frac{du^3}{d\tau} \\ &= 2[-u^0 \frac{du^0}{d\tau} + u^1 \frac{du^1}{d\tau} + u^2 \frac{du^2}{d\tau} + u^3 \frac{du^3}{d\tau}] \\ &= 2\eta_{\mu\nu} u^\mu \frac{du^\nu}{d\tau} \\ &= 2u_\mu \frac{du^\mu}{d\tau} \end{aligned}$$

Therefore the formula is justified.

b) Let δ indicate a variation or small change, and justify the formula:

$$\delta(F_{\alpha\beta} F^{\alpha\beta}) = 2F_{\alpha\beta} \delta F^{\alpha\beta}$$

The variation will obey the product rule as follows and remembering that the variation of the metric components would be zero:

$$\begin{aligned} \delta(F_{\alpha\beta} F^{\alpha\beta}) &= (\delta F_{\alpha\beta}) F^{\alpha\beta} + F_{\alpha\beta} (\delta F^{\alpha\beta}) \\ &= (\delta(\eta_{\alpha\gamma} \eta_{\beta\nu} F^{\alpha\beta}) F^{\gamma\nu}) + F_{\alpha\beta} (\delta F^{\alpha\beta}) \\ &= (\delta(\eta_{\alpha\gamma}) \eta_{\beta\nu} F^{\alpha\beta} + \eta_{\alpha\gamma} \delta(\eta_{\beta\nu}) F^{\alpha\beta} + \eta_{\alpha\gamma} \eta_{\beta\nu} \delta F^{\alpha\beta}) F^{\gamma\nu} + F_{\alpha\beta} (\delta F^{\alpha\beta}) \\ &= \eta_{\alpha\gamma} \eta_{\beta\nu} \delta F^{\alpha\beta} F^{\gamma\nu} + F_{\alpha\beta} (\delta F^{\alpha\beta}) \\ &= \delta F^{\alpha\beta} (\eta_{\alpha\gamma} \eta_{\beta\nu} F^{\gamma\nu}) + F_{\alpha\beta} (\delta F^{\alpha\beta}) \\ &= \delta F^{\alpha\beta} F_{\alpha\beta} + F_{\alpha\beta} \delta F^{\alpha\beta} \\ &= 2F_{\alpha\beta} \delta F^{\alpha\beta} \end{aligned}$$

Therefore the formula is justified.

c) Compute $(F_{\alpha\beta}F^{\alpha\beta})_{,\mu}=?$

$$\begin{aligned}
 (F_{\alpha\beta}F^{\alpha\beta})_{,\mu} &= F_{\alpha\beta,\mu}F^{\alpha\beta} + F_{\alpha\beta}F^{\alpha\beta}_{,\mu} \\
 &= (\eta_{\alpha\gamma}\eta_{\beta\nu}F^{\alpha\beta})_{,\mu}F^{\gamma\nu} + F_{\alpha\beta}F^{\alpha\beta}_{,\mu} \\
 &= F^{\alpha\beta}_{,\mu}F_{\alpha\beta} + F_{\alpha\beta}F^{\alpha\beta}_{,\mu} \\
 &= 2F_{\alpha\beta}F^{\alpha\beta}_{,\mu}
 \end{aligned}$$

Chapter 4

Electromagnetism and Differential Forms

Chapter 5

Stress-Energy Tensor and Conservation Laws

Chapter 6

Accelerated Observers

1. **A TRIP TO THE GALACTIC NUCLEUS:** Compute the proper time required for the occupants of a rocket ship to travel the $\approx 30,000$ light-years to get from the Earth to the center of the Galaxy. Assume that they maintain an acceleration of one earth gravity (10^3 cm/sec^2) for half the trip, and then decelerate at one earth gravity for the remaining of the half.

Qualitatively, the worldline of the traveller is pictured in the Figure 6.1. The travellers worldline is composed of two arcs of hyperbola AC and CB .

$$\begin{aligned}t &= g^{-1} \sinh g\tau \\x &= g^{-1} \cosh g\tau \\ \tau &= g^{-1} \sinh^{-1} gt \\x &= g^{-1} \cosh(\sinh^{-1} gt) \\&= g^{-1} \sqrt{1 + (gt)^2} \\dx &= \frac{gt}{\sqrt{1 + (gt)^2}} dt\end{aligned}$$

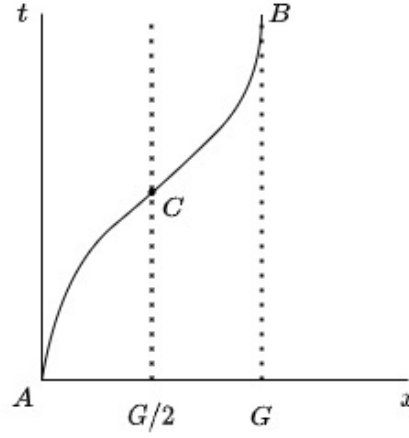


Figure 6.1: The worldline of the traveller is composed of two arcs of hyperbola AC and CB . G indicates the distance to the galactic center.

$$\begin{aligned}
 dt' &= \sqrt{dt^2 - dx^2} \\
 &= \sqrt{dt^2 - \left(\frac{gt}{\sqrt{1+(gt)^2}}\right)^2 dt^2} \\
 &= \sqrt{1 - \frac{(gt)^2}{1+(gt)^2}} dt \\
 &= \sqrt{\frac{1+(gt)^2 - (gt)^2}{1+(gt)^2}} dt \\
 &= \frac{1}{\sqrt{1+(gt)^2}} dt
 \end{aligned}$$

Using an integral table:

$$t' = \int \frac{dt}{\sqrt{1+(gt)^2}} = \ln(gt + \sqrt{1+(gt)^2}) = \sinh^{-1}(gt)$$

Chapter 7

Incompatibility of Gravity and Special Relativity

Chapter 8

Differential Geometry: An Overview

Chapter 9

Differential Topology

1. COMPONENT MANIPULATIONS

Derive equations:

$$u^\alpha = \langle \omega^\alpha, \mathbf{u} \rangle \quad (9.1)$$

$$\sigma_\beta = \langle \sigma, \mathbf{e}_\beta \rangle \quad (9.2)$$

$$\langle \sigma, \mathbf{u} \rangle = \sigma_\alpha u^\alpha \quad (9.3)$$

$$\omega^{\alpha'} = L_{\beta}^{\alpha'} \omega^\beta \quad (9.4)$$

$$\sigma_{\alpha'} = \sigma_\beta L_{\alpha'}^\beta \quad (9.5)$$

Consider:

$$\begin{aligned} \langle \omega^\alpha, \mathbf{u} \rangle &= u^\beta \langle \omega^\alpha, \mathbf{e}_\beta \rangle \\ &= u^\beta \delta_\beta^\alpha \\ &= u^\alpha \end{aligned}$$

$$\begin{aligned} \langle \sigma, \mathbf{e}_\beta \rangle &= \sigma_\alpha \langle \omega^\alpha, \mathbf{e}_\beta \rangle \\ &= \sigma_\alpha \delta_\beta^\alpha \\ &= \sigma_\beta \end{aligned}$$

$$\begin{aligned} \langle \sigma, \mathbf{u} \rangle &= \sigma_\alpha u^\beta \langle \omega^\alpha, \mathbf{e}_\beta \rangle \\ &= \sigma_\alpha u^\beta \delta_\beta^\alpha \\ &= \sigma_\beta u^\beta \end{aligned}$$

$$\begin{aligned} \sigma &= \sigma_{\alpha'} \omega^{\alpha'} \\ &= \sigma_\beta \omega^\beta \\ &= \sigma_\gamma \delta_\beta^\gamma \omega^\beta \\ &= \sigma_\gamma L_{\alpha'}^\gamma L_{\beta}^{\alpha'} \omega^\beta \end{aligned}$$

From inspection of components, it is clear that:

$$\sigma_{\alpha'} = \sigma_{\gamma} L_{\alpha'}^{\gamma}$$

$$\omega^{\alpha'} = L_{\beta}^{\alpha'} \omega^{\beta}$$