Scherrer's Quantum Mechanics Problems

Anthony Steel

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Chapter 1

The Origin Of Quantum Mechanics

1. **Assume that a human body emits blackbody radiation at the standard body temperature.**

   (a) **Estimate how much energy is radiated by the body in one hour.**
   The power emmitted by a blackbody is given by:

   $$P = \sigma A T^4$$

   where A is the surface area of the body, T is the temperature and σ is the Stefan-Boltzmann constant. Therefore the energy radiated by a body in a given time interval Δt:

   $$E = \sigma A T^4 \Delta t$$

   The surface area of the human body is approximately 2m^2, the average body temperature is $36.1°\text{C} = 309.25\text{K}$, and there are 3600s in an hour. Therefore:

   $$\begin{aligned} E_{\text{hour}} &= \left(5.67 * 10^{-8} \frac{\text{J}}{\text{s m}^2\text{K}^4}\right)(2\text{m}^2)(309.25\text{K})^4(3600\text{s}) \\ &= 3733\text{kJ} \end{aligned}$$

   (b) **At what wavelength does this radiation reach a maximum**
   The formula for the maximum wavelength is:

   $$\lambda_{\text{peak}} = \frac{w}{T}$$

   where $w = 2.90 * 10^{-3}\text{m K}$ and $T = 309.25\text{K}$ as before. Therefore the maximum wavelength is:

   $$\lambda_{\text{peak}} = \frac{2.9 * 10^{-3}\text{m K}}{309.25\text{K}} = 9.37 * 10^{-6}\text{m}$$

2. **A distant red star is observed to have a blackbody spectrum with a maximum at a wavelength of 3500Å[1Å$= 10^{-10}$ m]. What is the temperature of the star?**

   Inverting the formula from the pervious question:

$$T = \frac{w}{\lambda_{\text{peak}}}$$

   giving:

$$T = \frac{2.9 * 10^{-3}\text{m K}}{3500 * 10^{-10}\text{m}} = 51428\text{K}$$

3. **The universe is filled with blackbody radiation at a temperature of 2.7K left over from the Big Bang. [This radiation was disvoered in 1965 by Bell Laboratory scientists who thought at one point that they were seeing interference from pigeon droppings on their microwave reciever.**

   (a) **What is the total energy density of this radiation?**

   The energy density of the radiation is given by:

$$\rho = aT^4$$

   where $a = 7.56 * 10^{-16} \frac{\text{J}}{\text{m}^3\text{K}^4}$. Therefore:

$$\rho = 7.56 * 10^{-16} \frac{\text{J}}{\text{m}^3\text{K}^4} * (2.7\text{K})^4 = 4.01 * 10^{-14} \frac{\text{J}}{\text{m}^3}$$

   (b) **What is the total energy density with wavelengths between 1mm and 1.01mm? Is the Rayleigh-Jeans formula a good approximation at these wavelengths?**

Chapter 2

Math Interlude A: Complex Numbers and Linear Operators

1. **Eavluate all of th efollowing and express all of your final answers in the form $a + bi$:**

   (a) $i(2-3i)(3+5i)$

   $$\begin{aligned} i(2-3i)(3+5i) &= i(6+10i-9i+15) \\ &= i(21+i) \\ &= -1+21i \end{aligned}$$

   (b) $i/i-1$

   $$\begin{aligned} i/i-1 &= e^{i\pi/2}/(\sqrt{2}e^{3i\pi/4}) \\ &= \frac{1}{\sqrt{2}}e^{-i\pi/4} \end{aligned}$$

   (c) $(1+i)^{30}$

   $$\begin{aligned} (1+i)^{30} &= (\sqrt{2}e^{i\pi/4})^{30} \\ &= 2^{15}e^{15i\pi/2} \\ &= 2^{15}e^{3i\pi/2} \\ &= -2^{15}i \end{aligned}$$

2.

3.

4. **Suppose that a complex number z has the property that $z^* = z$. What does this indicate about z?**

   This indicates that z is a real number.

5. **Reduce i^i to a real number**

$$i^i = (e^{i\pi/2})^i = e^{-\pi/2}$$

6. **What is wrong with the following argument?**

$$\begin{aligned} \sqrt{\frac{1}{-1}} &= \frac{\sqrt{1}}{\sqrt{-1}} \\ \sqrt{-1} &= \frac{1}{i} \\ i &= \frac{1}{i} \\ (i)(i) &= 1 \\ -1 &= 1 \end{aligned}$$

   The first line is a false equivalence.

$$\sqrt{\frac{1}{-1}} = \sqrt{-1} = i = e^{i\pi/2}$$

   and

$$\frac{\sqrt{1}}{\sqrt{-1}} = \frac{1}{\sqrt{-1}} = e^{-i\pi/2}$$

7. **Determine which of the following are linear operators, and which are not.**

   (a) **The parity operator $\Pi[f(x)] = f(-x)$.**

$$\begin{aligned} \Pi[f(x) + g(x)] &= f(-x) + g(-x) \\ &= \Pi[f(x)] + \Pi[g(x)] \end{aligned}$$

$$\begin{aligned} \Pi[cf(x)] &= cf(-x) \\ &= c\Pi[f(x)] \end{aligned}$$

   Therefore the parity operator is linear.

   (b) **The transformation operator $T[f(x)] = f(x+1)$.**

$$\begin{aligned} T[f(x) + g(x)] &= f(x+1) + g(x+1) \\ &= T[f(x)] + T[g(x)] \end{aligned}$$

$$\begin{aligned} T[cf(x)] &= cf(x+1) \\ &= cT[f(x)] \end{aligned}$$

Therefore the transformation operator is linear.

(c) **The operator $L[f(x)] = f(x) + 1$**

$$L[f(x) + g(x)] = f(x) + g(x) + 1 \neq L[f(x)] + L[g(x)]$$

$$L[cf(x)] = cf(x) + 1 \neq cL[f(x)]$$

Therefore the operator is not linear.

8. **Consider the identity operator I, defined by $I[f(x)] = f(x)$.**

(a) **Show that I is a linear operator.**

$$I[f(x) + g(x)] = f(x) + g(x) = I[f(x)] + I[g(x)]$$

$$I[cf(x)] = cf(x) = cI[f(x)]$$

Therefore the identity operator is linear.

(b) **Find the eigenfunctions and corresponding eigenvalues of I**

The eigenfunctions are given by:

$$I[f(x)] = f(x) = cf(x)$$

Therefore every function is an eigen function of the identity operator with eigenvalue $c = 1$.

9. **Suppose that the function $f(x)$ is an eigenfunction of the linear operator P with eigenvalue p, and $f(x)$ is also an eigenfunction of the linear operator Q with eigenvalue q. Show that $PQ[f(x)] = QP[f(x)]$, where $PQ[f(x)]$ means to first apply the operator Q to $f(x)$, and then apply P to the result.**

$$\begin{aligned} PQ[f(x)] &= P[Q[f(x)] \\ &= P[qf(x)] \\ &= pqf(x) \end{aligned}$$

$$\begin{aligned} QP[f(x)] &= Q[P[f(x)] \\ &= Q[pf(x)] \\ &= qpf(x) \end{aligned}$$

Because the eigenvalues are real numbers they comute. Thefore:

$$PQ[f(x)] = QP[f(x)]$$

10. **Consider the square of the derivative operator**

(a) **Show that D^2 is a linear operator**

$$\begin{aligned} D^2[f(x)+g(x)] &= \frac{d^2}{dx^2}\Big(f(x)+g(x)\Big) \\ &= \frac{d^2 f}{dx^2} + \frac{d^2 g}{dx^2} \\ &= D^2[f(x)] + D^2[g(x)] \end{aligned}$$

$$\begin{aligned} D^2[cf(x)] &= \frac{d^2}{dx^2}\Big(cf(x)\Big) \\ &= c\frac{d^2 f}{dx^2} \\ &= cD^2[f(x)] \end{aligned}$$

Therfore D^2 is a linear operator.

(b) The eigenfunctions of D^2 are given by:

$$D^2[f(x)] = \frac{d^2 f}{dx^2} = cf(x)$$

which is equivalent to solving the homogenous linear second-order differential equation:

$$\frac{d^2 f}{dx^2} - cf(x) = 0$$

This equation has the solution:

$$f(x) = Ae^{i\sqrt{c}x} + Be^{-i\sqrt{c}x}$$

(c) **Give an example of an eigenfunction of D^2 which is not an eigenfunction of D**

$$f(x) = A\cos(\sqrt{c}x)$$

11. **Let $f(x)$ be an eigenfunction of a linear operator L with eigenvalue a. Show that $cf(x)$ (where c is a constant) is an eigenfunction of L with eigenvalue a.**