

A Relativist's Toolkit Problems

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Chapter 1

Fundamentals

1. The surface of a two-dimensional cone is embedded in three-dimensional flat space. The cone has an opening angle of 2α . Points on the cone which all have the same distance r from the apex define a circle, and ϕ is the angle that runs along the circle.

- (a) Write down the metric of the cone, in terms of the coordinates r and ϕ .

The parameterization of the cone $X(r, \phi)$ is:

$$X(r, \phi) = \begin{bmatrix} r \sin \alpha \cos \phi \\ r \sin \alpha \sin \phi \\ r \cos \alpha \end{bmatrix}$$

Differentiating with respect to r and ϕ gives:

$$X_{,\phi} = \begin{bmatrix} -r \sin \alpha \sin \phi \\ r \sin \alpha \cos \phi \\ 0 \end{bmatrix}$$

and

$$X_{,r} = \begin{bmatrix} \sin \alpha \cos \phi \\ \sin \alpha \sin \phi \\ \cos \alpha \end{bmatrix}$$

The general form of the metric in two dimensions is:

$$ds^2 = E d\phi^2 + 2F d\phi dr + G dr^2$$

where:

$$E = X_{,\phi} \cdot X_{,\phi} = r^2 \sin^2 \alpha$$

$$F = X_{,\phi} \cdot X_{,r} = 0$$

$$G = X_{,r} \cdot X_{,r} = 1$$

therefore:

$$ds^2 = dr^2 + r^2 \sin^2 \alpha d\phi^2$$

- (b) **Find the coordinate transformation $x(r, \phi), y(r, \phi)$ that brings the metric into the form $ds^2 = dx^2 + dy^2$. Do these coordinates cover the entire two-dimensional plane?**

Consider:

$$x(r, \phi) = r \sin \alpha \cos \phi$$

$$y(r, \phi) = r \sin \alpha \sin \phi$$

Then:

$$\frac{y}{x} = \tan \phi \rightarrow \phi = \arctan \frac{y}{x}$$

and

$$r = \frac{x}{\sin \alpha \cos \phi} = \frac{x \sqrt{\left(\frac{y}{x}\right)^2 + 1}}{\sin \alpha} = \frac{\sqrt{x^2 + y^2}}{\sin \alpha}$$

likewise:

$$r = \frac{y}{\sin \alpha \sin \phi} = \frac{y \sqrt{\left(\frac{y}{x}\right)^2 + 1}}{\frac{y}{x} \sin \alpha} = \frac{\sqrt{x^2 + y^2}}{\sin \alpha}$$

Proving that this transformation does indeed result in a metric of the form $ds^2 = dx^2 + dy^2$:

$$\begin{aligned} d\phi &= \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \\ &= \frac{-1}{1 + \frac{x^2}{y^2}} dx + \frac{1}{1 + \frac{y^2}{x^2}} dy \\ &= \frac{-y^2}{x^2 + y^2} dx + \frac{x^2}{x^2 + y^2} dy \end{aligned}$$

$$d\phi^2 = \frac{y^4}{(x^2 + y^2)^2} dx^2 + \frac{x^4}{(x^2 + y^2)^2} dy^2 - 2 \frac{x^4 y^4}{(x^2 + y^2)^2} dx dy$$

$$\sin^2 \phi = \frac{y^2}{x^2 + y^2}$$

$$\begin{aligned} dr &= \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy \\ &= \frac{x}{\sin \alpha \sqrt{x^2 + y^2}} dx + \frac{y}{\sin \alpha \sqrt{x^2 + y^2}} dy \end{aligned}$$

- (c) **Prove that any vector parallel transported along a circle of constant r on the surface of the cone ends up rotated by an angle β after a complete trip. Express β in terms of α .**