Misner, Thorne and Wheeler's Gravitation Problems

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Chapter 1

Chapter 2

Chapter 3

The Electromagnetic Field

1. **Derive equations:**

$$\| F^{\alpha}_{\beta} \| = \begin{Vmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{Vmatrix} \tag{3.1}$$

and

$$\| F_{\alpha\beta} \| = \begin{Vmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{Vmatrix} \tag{3.2}$$

for the components of Faraday by comparing:

$$dp^{\alpha}/d\tau = eF^{\alpha}_{\beta}u^{\beta} \tag{3.3}$$

with

$$\frac{d\mathbf{p}}{d\tau} = \frac{1}{\sqrt{1-\mathbf{v}^2}}\frac{d\mathbf{p}}{dt} = \frac{e}{\sqrt{1-\mathbf{v}^2}}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = e(u^0\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{3.4}$$

$$\frac{dp^0}{d\tau} = \frac{1}{\sqrt{1-\mathbf{v}^2}}\frac{dE}{dt} = \frac{1}{\sqrt{1-\mathbf{v}^2}}e\mathbf{E} \cdot \mathbf{v} = e\mathbf{E} \cdot \mathbf{u} \tag{3.5}$$

and by using definition:

$$F_{\alpha\beta} = \eta_{\alpha\gamma}F^{\gamma}_{\beta} \tag{3.6}$$

Consider equation 3.3 for the index $\alpha = 0$:

$$\frac{dp^0}{d\tau} = e[F^0_0 u^0 + F^0_1 u^1 + F^0_2 u^2 + F^0_3 u^3]$$

Equate this with 3.5:

$$e[F^0_0 u^0 + F^0_1 u^1 + F^0_2 u^2 + F^0_3 u^3] = e\mathbf{E} \cdot \mathbf{u} = e[E_1 u^1 + E_2 u^2 + E_3 u^3]$$

It is clear that:

$$\begin{aligned}
F^0_0 &= 0 \\
F^0_1 u^1 &= E_1 u^1 \Rightarrow F^0_1 = E_1 \\
F^0_2 u^2 &= E_2 u^2 \Rightarrow F^0_2 = E_2 \\
F^0_3 u^3 &= E_3 u^3 \Rightarrow F^0_3 = E_3
\end{aligned}$$

Now equating equation 3.4 with the remaining components of equation 3.3:

$$\frac{dp^1}{d\tau} = e[F^1_0 u^0 + F^1_1 u^1 + F^1_2 u^2 + F^1_3 u^3] = e[E_1 u^0 + B_3 u^2 - B_2 u^3]$$
$$\frac{dp^2}{d\tau} = e[F^2_0 u^0 + F^2_1 u^1 + F^2_2 u^2 + F^2_3 u^3] = e[E_2 u^0 + B_1 u^3 - B_3 u^2]$$
$$\frac{dp^3}{d\tau} = e[F^3_0 u^0 + F^3_1 u^1 + F^3_2 u^2 + F^3_3 u^3] = e[E_3 u^0 + B_2 u^1 - B_1 u^2]$$

and equating components as before:

$$\begin{aligned}
F^1_0 u^0 &= E_1 u^0 \Rightarrow F^1_0 = E_1 \\
F^1_1 u^1 &= 0 \Rightarrow F^1_1 = 0 \\
F^1_2 u^2 &= B_3 u^2 \Rightarrow F^1_2 = B_3 \\
F^1_3 u^3 &= -B_2 u^3 \Rightarrow F^1_3 = -B_2 \\
F^2_0 u^0 &= E_2 u^0 \Rightarrow F^2_0 = E_2 \\
F^2_1 u^1 &= -B_3 u^1 \Rightarrow F^2_1 = -B_3 \\
F^2_2 u^2 &= 0 \Rightarrow F^2_2 = 0 \\
F^2_3 u^3 &= B_1 u^3 \Rightarrow F^2_3 = B_1 \\
F^3_0 u^0 &= E_3 u^0 \Rightarrow F^3_0 = E_3 \\
F^3_1 u^1 &= B_2 u^1 \Rightarrow F^3_1 = B_2 \\
F^3_2 u^2 &= -B_1 u^2 \Rightarrow F^3_2 = -B_1 \\
F^3_3 u^3 &= 0 \Rightarrow F^3_3 = 0
\end{aligned}$$

Collecting all of these components in matrix form and relabling indices with the following mapping:

$$\begin{aligned}
\alpha &= 1 \rightarrow x \\
\alpha &= 2 \rightarrow y \\
\alpha &= 3 \rightarrow z
\end{aligned}$$

gives equation 3.1:

$$\| F^\alpha_\beta \| = \begin{Vmatrix} F^0_0 & F^0_1 & F^0_2 & F^0_3 \\ F^1_0 & F^1_1 & F^1_2 & F^1_3 \\ F^2_0 & F^2_1 & F^2_2 & F^2_3 \\ F^3_0 & F^3_1 & F^3_2 & F^3_3 \end{Vmatrix} = \begin{Vmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{Vmatrix}$$

Now equation 3.6 can be used to convert the mixed Faraday tensor to the fully covariant one. Remeber that for all components $\alpha \neq \beta$ the Minkowski metric is zero. Therefore the only non-zero components in the sums created by the summation convention in equation 3.6 are:

$$
\begin{aligned}
F_{00} &= \eta_{00} F^0_0 \Rightarrow F_{00} = -F^0_0 \\
F_{01} &= \eta_{00} F^0_1 \Rightarrow F_{01} = -F^0_1 \\
F_{02} &= \eta_{00} F^0_2 \Rightarrow F_{02} = -F^0_2 \\
F_{03} &= \eta_{00} F^0_3 \Rightarrow F_{03} = -F^0_3 \\
F_{10} &= \eta_{11} F^1_0 \Rightarrow F_{10} = F^1_0 \\
F_{11} &= \eta_{11} F^1_1 \Rightarrow F_{11} = F^1_1 \\
F_{12} &= \eta_{11} F^1_2 \Rightarrow F_{12} = F^1_2 \\
F_{13} &= \eta_{11} F^1_3 \Rightarrow F_{13} = F^1_3 \\
F_{20} &= \eta_{22} F^2_0 \Rightarrow F_{02} = F^2_0 \\
F_{21} &= \eta_{22} F^2_1 \Rightarrow F_{12} = F^2_1 \\
F_{22} &= \eta_{22} F^2_2 \Rightarrow F_{22} = F^2_2 \\
F_{23} &= \eta_{22} F^2_3 \Rightarrow F_{23} = F^3_2 \\
F_{30} &= \eta_{33} F^3_0 \Rightarrow F_{30} = F^3_0 \\
F_{31} &= \eta_{33} F^3_1 \Rightarrow F_{31} = F^3_1 \\
F_{32} &= \eta_{33} F^3_2 \Rightarrow F_{32} = F^3_2 \\
F_{33} &= \eta_{33} F^3_3 \Rightarrow F_{33} = F^3_3
\end{aligned}
$$

Collecting the components into matrix form recovers the fully covariant Faraday tensor:

$$
|| \; F_{\alpha\beta} \; || = \begin{Vmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{Vmatrix} \tag{3.7}
$$

2. **From the transformation laws for components of vectors and 1-forms, derive the transformation law:**

$$
S^{\mu'\nu'}{}_{\lambda'} = S^{\alpha\beta}{}_{\gamma} \Lambda^{\mu'}_{\alpha} \Lambda^{\nu'}_{\beta} \Lambda^{\gamma}_{\lambda'}
$$

Consider the tensor $\mathbf{S}$ of rank $(2,1)$, in geometric notation the transformation between two sets basis vectors and 1-forms reads:

$$
\mathbf{S}(\sigma, \rho, \nu) = \mathbf{S}(\sigma', \rho', \nu')
$$

In component form this reads:

$$
S^{\alpha\beta}{}_{\gamma} \sigma_\alpha \rho_\beta \nu^\gamma = S^{\mu'\nu'}{}_{\lambda'} \sigma_{\mu'} \rho_{\nu'} \nu^{\lambda'} \tag{3.8}
$$

Using the Lorentz transformation laws to transform one basis into the other for σ, ρ, ν gives:

$$\sigma_\alpha = \Lambda^{\mu'}_{\alpha}\sigma_{\mu'}$$
$$\rho_\beta = \Lambda^{\nu'}_{\beta}\rho_{\mu'}$$
$$\nu^\beta = \Lambda^{\gamma}_{\lambda'}\nu^{\lambda'}$$

and substituting these transformations into equation 3.8:

$$S^{\mu'\nu'}{}_{\lambda'}\sigma_{\mu'}\rho_{\nu'}\nu^{\lambda'} = S^{\alpha\beta}{}_{\gamma}(\Lambda^{\mu'}_{\alpha}\sigma_{\mu'})(\Lambda^{\nu'}_{\beta}\rho_{\mu'})(\Lambda^{\gamma}_{\lambda'}\nu^{\lambda'})$$

$$S^{\mu'\nu'}{}_{\lambda'}\sigma_{\mu'}\rho_{\nu'}\nu^{\lambda'} = S^{\alpha\beta}{}_{\gamma}\Lambda^{\mu'}_{\alpha}\Lambda^{\nu'}_{\beta}\Lambda^{\gamma}_{\lambda'}\sigma_{\mu'}\rho_{\mu'}\nu^{\lambda'}$$

Equating the components gives the desired transformation law:

$$S^{\mu'\nu'}{}_{\lambda'} = S^{\alpha\beta}{}_{\gamma}\Lambda^{\mu'}_{\alpha}\Lambda^{\nu'}_{\beta}\Lambda^{\gamma}_{\lambda'}$$

3. **Raising and lowering indices. Derive:**

$$S^{\alpha}_{\beta\,\gamma} = \eta_{\beta\mu}S^{\alpha\mu}{}_{\gamma} \tag{3.9}$$

and:

$$S^{\alpha\mu}{}_{\gamma} = \eta^{\mu\beta}S^{\alpha}_{\beta\,\gamma} \tag{3.10}$$

from:

4.

5.

6.

7. **Maxwell's Equations. Show, by explicit examination of components, that the geometric laws**

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0 \tag{3.11}$$

$$F^{\alpha\beta}_{,\beta} = 4\pi J^\alpha \tag{3.12}$$

do reduce to Maxwell's equations

$$\nabla\cdot\mathbf{B} = 0 \tag{3.13}$$

$$\frac{\partial\mathbf{B}}{\partial t} + \nabla\times\mathbf{E} = 0 \tag{3.14}$$

$$\nabla\cdot\mathbf{E} = 4\pi\rho \tag{3.15}$$

$$\frac{\partial\mathbf{E}}{\partial t} - \nabla\times\mathbf{B} = -4\pi\mathbf{J} \tag{3.16}$$

8.

9.

10. **More differentiation. (a) Justify the formula,**

$$d(u^\mu u_\mu)/d\tau = 2u_\mu(du^\mu/d\tau),$$

by writing out the summation $u^\mu u_\mu = \eta_{\mu\nu}u^\mu u^\nu$ **explicitly** Writing out the components explicitly yields:

$$\begin{aligned} u^\mu u_\mu &= \eta_{00}u^0u^0 + \eta_{11}u^1u^1 + \eta_{22}u^2u^2 + \eta_{33}u^3u^3 \\ &= \eta_{00}(u^0)^2 + \eta_{11}(u^1)^2 + \eta_{22}(u^2)^2 + \eta_{33}(u^3)^2 \\ &= -(u^0)^2 + (u^1)^2 + (u^2)^2 + (u^3)^2 \end{aligned}$$

Taking a total derivative of the above with respect to τ:

$$\begin{aligned} \frac{d}{d\tau}(u^\mu u_\mu) &= \frac{d}{d\tau}(-(u^0)^2 + (u^1)^2 + (u^2)^2 + (u^3)^2) \\ &= -2u^0\frac{du^0}{d\tau} + 2u^1\frac{du^1}{d\tau} + 2u^2\frac{du^2}{d\tau} + 2u^3\frac{du^3}{d\tau} \\ &= 2[-u^0\frac{du^0}{d\tau} + u^1\frac{du^1}{d\tau} + u^2\frac{du^2}{d\tau} + u^3\frac{du^3}{d\tau}] \\ &= 2\eta_{\mu\nu}u^\mu\frac{du^\nu}{d\tau} \\ &= 2u_\mu\frac{du^\nu}{d\tau} \end{aligned}$$

Therefore the formula is justified.

b) Let δ indicate a variation or small change, and justify the formula:

$$\delta(F_{\alpha\beta}F^{\alpha\beta}) = 2F_{\alpha\beta}\delta F^{\alpha\beta}$$

The variation will obey the product rule as follows and rembering that the variation of the metric components would be zero:

$$\begin{aligned} \delta(F_{\alpha\beta}F^{\alpha\beta}) &= (\delta F_{\alpha\beta})F^{\alpha\beta} + F_{\alpha\beta}(\delta F^{\alpha\beta}) \\ &= (\delta(\eta_{\alpha\gamma}\eta_{\beta\nu}F^{\alpha\beta})F^{\gamma\nu}) + F_{\alpha\beta}(\delta F^{\alpha\beta}) \\ &= (\delta(\eta_{\alpha\gamma})\eta_{\beta\nu}F^{\alpha\beta} + \eta_{\alpha\gamma}\delta(\eta_{\beta\nu})F^{\alpha\beta} + \eta_{\alpha\gamma}\eta_{\beta\nu}\delta F^{\alpha\beta})F^{\gamma\nu}) + F_{\alpha\beta}(\delta F^{\alpha\beta}) \\ &= \eta_{\alpha\gamma}\eta_{\beta\nu}\delta F^{\alpha\beta}F^{\gamma\nu} + F_{\alpha\beta}(\delta F^{\alpha\beta}) \\ &= \delta F^{\alpha\beta}(\eta_{\alpha\gamma}\eta_{\beta\nu}F^{\gamma\nu}) + F_{\alpha\beta}(\delta F^{\alpha\beta}) \\ &= \delta F^{\alpha\beta}F_{\alpha\beta} + F_{\alpha\beta}\delta F^{\alpha\beta} \\ &= 2F_{\alpha\beta}\delta F^{\alpha\beta} \end{aligned}$$

Therefore the formula is justified.

c) Compute $(F_{\alpha\beta}F^{\alpha\beta})_{,\mu} = ?$

$$\begin{aligned}(F_{\alpha\beta}F^{\alpha\beta})_{,\mu} &= F_{\alpha\beta,\mu}F^{\alpha\beta} + F_{\alpha\beta}F^{\alpha\beta}{}_{,\mu} \\ &= (\eta_{\alpha\gamma}\eta_{\beta\nu}F^{\alpha\beta})_{,\mu}F^{\gamma\nu} + F_{\alpha\beta}F^{\alpha\beta}{}_{,\mu} \\ &= F^{\alpha\beta}{}_{,\mu}F_{\alpha\beta} + F_{\alpha\beta}F^{\alpha\beta}{}_{,\mu} \\ &= 2F_{\alpha\beta}F^{\alpha\beta}{}_{,\mu}\end{aligned}$$

Chapter 4

Electromagnetism and Differential Forms

Chapter 5

Stress-Energy Tensor and Conservatoin Laws

Chapter 6

Accelerated Observers

1. **A TRIP TO THE GALACTIC NUCLEUS**

   Compute the proper time required for the occupants of a rocket schip to travel the $\approx 30,000$ light-years to get from the Earth to the center of the Galaxy. Asusme that they maintain an acceleration of one earth gravity (10^3 cm/sec^2) for half the trip, and then decelerate at one earth gravity for the remaining of the half.

   Qualitatively, the worldline of the traveller is pictured in the Figure 6.1. The travellers worldine is composed of two arcs of hyperbola AC and CB. It is clear geometrically that the total time required for the trip to the center will be double the time to the middle of galaxy because of the symmetry of accelerations/decelerations. The benefit of handling the situation this way is we can use the equations derived for a positive acceleration.

   As stated:

   $$t = g^{-1} \sinh g\tau \tag{6.1}$$

   $$x = g^{-1} \cosh g\tau \tag{6.2}$$

   describe the worldline of an accelerated observer in the reference frame of an inertial observer with respect to x^1, the direction of acceleration. Invert Equation 6.2:

   $$\tau = g^{-1} \cosh^{-1} gx$$

   Reinserting dimensional units gives:

   $$\tau = cg^{-1} \cosh^{-1} gxc^{-2}$$

   Substituting $x = 15000$ light-years gives $\tau \approx 12.25$ years for half the trip, and $\tau \approx 24.5$ years for the full trip.

2. **ROCKET PAYLOAD**

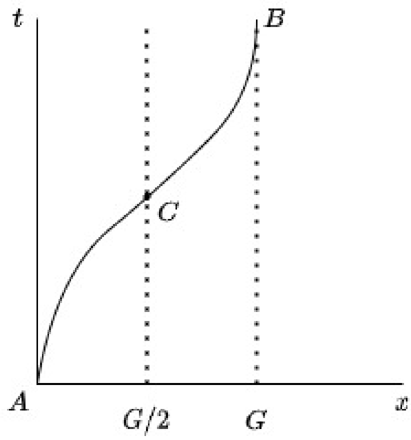

Figure 6.1: The worldline of the traveller is composed of two arcs of hyperbola AC and CB. G indicates the distance to the galactic center.

What fraction of the initial mass of the rocket can be payload for the journey considered in exercise 1? Assume an ideal rocket that converts rest mass into radiation and ejects the radiation out the back of the rocket with 100 per cent efficiency and perfect collimation.

Consider the 4-momentum of the rocket:

$$mu^{\alpha} = m(u^t, u^x)$$

u^x is the momentum in the direction of travel and u^t is mass-energy of the rocket. The change in the mass-energy is the radiation ejected from the rocket, and it is equal to the negative of the change in momentum of the rocket. A decrease in mass-energy results in an increase of momentum in the x direction and vice-versa. Therefore:

$$\begin{aligned}
\frac{d}{d\tau}(mu^t) &= -\frac{d}{d\tau}(mu^x) \\
\frac{dm}{d\tau}u^t + m\frac{du^t}{d\tau} &= -(\frac{dm}{d\tau}u^x + m\frac{du^x}{d\tau}) \\
(u^t + u^x)\frac{dm}{d\tau} &= -m(\frac{du^t}{d\tau} + \frac{du^x}{d\tau}) \\
(u^t + u^x)\frac{dm}{d\tau} &= -m\frac{d}{d\tau}(u^t + u^x) \\
\frac{1}{m}\frac{dm}{d\tau} &= \frac{1}{(u^t + u^x)}\frac{d}{d\tau}(u^t + u^x)
\end{aligned}$$

This equation can be integrated as:

$$\ln \frac{m}{m_0} = -\ln(u^t + u^x)$$
$$\frac{m}{m_0} = \frac{1}{u^t + u^x}$$
$$m = \frac{1}{u^t + u^x} m_0$$

Remembering that:

$$u^t = \frac{dt}{d\tau} = \frac{d}{d\tau}(g^{-1}\sinh g\tau) = \cosh g\tau$$

$$u^x = \frac{dx}{d\tau} = \frac{d}{d\tau}(g^{-1}\cosh g\tau) = \sinh g\tau$$

Substituting and symplifying:

$$\begin{aligned} m &= \frac{1}{\cosh g\tau + \sinh g\tau} m_0 \\ &= \frac{2}{e^{g\tau} + e^{-g\tau} + e^{g\tau} - e^{-g\tau}} m_0 \\ &= \frac{2}{2e^{g\tau}} m_0 \\ &= e^{-g\tau} m_0 \end{aligned}$$

which for $\tau \approx 12.25$ years times 2 gives:

$$m = 6.5 * 10^{-6} m_0$$

3. **TWIN PARADOX**

(a) Show that, of all timelike word lines connecting two events A and B, the one with the longest lapse of proper time is the unacccelerated one. Hint: perform the calculation in the inertial frame of the unaccelerated world line.

Consider the proper time defined by:

$$\tau = \int_A^B \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

The problem amounts to varying the proper time and showing that :

$$\delta\tau = 0$$

implies no 4-acceleration (i.e. $d^2x^\nu/d\tau^2 = 0$). This problem should be familiar from the calculus of variations. The condition for $\delta\tau = 0$ is that L must satisfy Lagrange's equations:

$$\frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}^\gamma} = \frac{\partial L}{\partial x^\gamma}$$

where $\dot{x}^\gamma = dx^\gamma/d\lambda$ and :

$$L = \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$$

Therefore:

$$\begin{aligned}
\frac{\partial L}{\partial x^\gamma} &= -\frac{1}{2L}\frac{\partial}{\partial x^\gamma}\Big(\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\Big) \\
&= -\frac{1}{2L}\Big(\frac{\partial \eta_{\mu\nu}}{\partial x^\gamma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + 2\eta_{\mu\nu}\frac{\partial}{\partial x^\gamma}\Big(\frac{dx^\mu}{d\lambda}\Big)\frac{dx^\nu}{d\lambda}\Big) \\
&= -\frac{1}{2L}\Big(\frac{\partial \eta_{\mu\nu}}{\partial x^\gamma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + 2\eta_{\mu\nu}\frac{d}{d\lambda}\Big(\frac{\partial x^\mu}{\partial x^\gamma}\Big)\frac{dx^\nu}{d\lambda}\Big) \\
&= -\frac{1}{2L}\Big(\frac{\partial \eta_{\mu\nu}}{\partial x^\gamma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + 2\eta_{\mu\nu}\frac{d}{d\lambda}\delta^\nu_\gamma\frac{dx^\nu}{d\lambda}\Big) \\
&= -\frac{1}{2L}\frac{\partial \eta_{\mu\nu}}{\partial x^\gamma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\gamma} &= \frac{d}{d\lambda}\Big(-\frac{1}{2L}\frac{\partial}{\partial \dot{x}^\gamma}\Big(\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\Big)\Big) \\
&= \frac{d}{d\lambda}\Big(-\frac{1}{2L}\Big(\frac{\partial \eta_{\mu\nu}}{\partial \dot{x}^\gamma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + \eta_{\mu\nu}\frac{\partial \dot{x}^\mu}{\partial \dot{x}^\gamma}\dot{x}^\nu + \eta_{\mu\nu}\dot{x}^\mu\frac{\partial \dot{x}^\nu}{\partial \dot{x}^\gamma}\Big)\Big) \\
&= \frac{d}{d\lambda}\Big(-\frac{1}{2L}\Big(\frac{\partial \eta_{\mu\nu}}{\partial \dot{x}^\gamma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + \eta_{\mu\nu}\delta^\mu_\gamma\dot{x}^\nu + \eta_{\mu\nu}\dot{x}^\mu\delta^\nu_\gamma\Big)\Big) \\
&= \frac{d}{d\lambda}\Big(-\frac{1}{2L}\Big(\frac{\partial \eta_{\mu\nu}}{\partial \dot{x}^\gamma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + 2\eta_{\gamma\nu}\dot{x}^\nu\Big)\Big) \\
&= -\frac{d}{d\lambda}\Big(\frac{1}{L}\eta_{\gamma\nu}\dot{x}^\nu\Big)
\end{aligned}$$

Now from the equation for τ it is clear that:

$$\frac{d\tau}{d\lambda} = L$$

and therefore for a function $f(\tau(\lambda))$:

$$\frac{df}{d\lambda} = \frac{df}{d\tau}\frac{d\tau}{d\lambda} = L\frac{df}{d\tau}$$

We can use this relationship to exchange differentiation with respect to λ for differentiation with respect to τ, such that:

$$\begin{aligned}
\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\gamma} &= -\frac{d}{d\lambda}\Big(\frac{1}{L}\eta_{\gamma\nu}\frac{dx^\nu}{d\lambda}\Big) \\
&= -L\frac{d}{d\tau}\Big(\frac{1}{L}\eta_{\gamma\nu}\Big(L\frac{dx^\nu}{d\tau}\Big)\Big) \\
&= -L\eta_{\gamma\nu}\frac{d^2x^\nu}{d\tau^2}
\end{aligned}$$

Combining the two results gives:

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\gamma} = \frac{\partial L}{\partial x^\gamma} \Rightarrow L\eta_{\gamma\nu}\frac{d^2x^\nu}{d\tau^2} = 0$$

Because $\eta_{\gamma\nu}$ is non-zero and L is non-zero for timelike curves, the condition for $\delta\tau$ to be zero is:

$$\frac{d^2x^\nu}{d\tau^2} = 0$$

We know that this is a maximum not a minimum because the worldline is timelike, meaning:

$$d\tau^2 < 0$$

Chapter 7

Incompatibility of Gravity and Special Relativity

Chapter 8

Differential Geometry: An Overview

Chapter 9

Differential Topology

1. **COMPONENT MANIPULATIONS**

   Derive equations:

$$u^{\alpha} = \langle \boldsymbol{\omega}^{\alpha} \boldsymbol{u} \rangle \tag{9.1}$$

$$\sigma_{\beta} = \langle \boldsymbol{\sigma}, \boldsymbol{e}_{\beta} \rangle \tag{9.2}$$

$$\langle \boldsymbol{\sigma}, \boldsymbol{u} \rangle = \sigma_{\alpha} u^{\alpha} \tag{9.3}$$

$$\boldsymbol{\omega}^{\alpha'} = L^{\alpha'}_{\beta} \boldsymbol{\omega}^{\beta} \tag{9.4}$$

$$\sigma_{\alpha'} = \sigma_{\beta} L^{\beta}_{\alpha'} \tag{9.5}$$

Consider:

$$\begin{aligned} \langle \boldsymbol{\omega}^{\alpha}, \boldsymbol{u} \rangle &= u^{\beta} \langle \boldsymbol{\omega}^{\alpha}, \boldsymbol{e}_{\beta} \rangle \\ &= u^{\beta} \delta^{\alpha}_{\beta} \\ &= u^{\alpha} \end{aligned}$$

$$\begin{aligned} \langle \boldsymbol{\sigma}, \boldsymbol{e}_{\beta} \rangle &= \sigma_{\alpha} \langle \boldsymbol{\omega}^{\alpha}, \boldsymbol{e}_{\boldsymbol{\beta}} \rangle \\ &= \sigma_{\alpha} \delta^{\alpha}_{\beta} \\ &= \sigma_{\beta} \end{aligned}$$

$$\begin{aligned} \langle \boldsymbol{\sigma}, \boldsymbol{u} \rangle &= \sigma_{\alpha} u^{\beta} \langle \boldsymbol{\omega}^{\alpha}, \boldsymbol{e}_{\alpha} \rangle \\ &= \sigma_{\alpha} u^{\beta} \delta^{\alpha}_{\beta} \\ &= \sigma_{\beta} u^{\beta} \end{aligned}$$

$$\begin{aligned} \boldsymbol{\sigma} &= \sigma_{\alpha'} \boldsymbol{\omega}^{\alpha'} \\ &= \sigma_{\beta} \boldsymbol{\omega}^{\beta} \\ &= \sigma_{\gamma} \delta^{\gamma}_{\beta} \boldsymbol{\omega}^{\beta} \\ &= \sigma_{\gamma} L^{\gamma}_{\alpha'} L^{\alpha'}_{\beta} \boldsymbol{\omega}^{\beta} \end{aligned}$$

From inspection of components, it is clear that:

$$\sigma_{\alpha'} = \sigma_\gamma L^\gamma_{\alpha'}$$

$$\boldsymbol{\omega}^{\alpha'} = L^{\alpha'}_\beta \boldsymbol{\omega}^\beta$$

2. **COMPONENTS OF GRADIENT, AND DUALITY OF CORDINATE BASES**

   In an arbitrary basis, define $f_{,\alpha}$ by the expansion:

$$\boldsymbol{d}f = f_{,\alpha}\boldsymbol{\omega}^\alpha$$

   Then combine equations:

$$\sigma_\beta = \langle \boldsymbol{\sigma}, \boldsymbol{e}_\beta \rangle$$

   and:

$$\langle \boldsymbol{d}f, \boldsymbol{u} \rangle = \partial_{\boldsymbol{u}} f = \boldsymbol{u}[f]$$

   to obtain the method:

$$f_{,\alpha} = \partial_\alpha f = \boldsymbol{e}_\alpha[f]$$

   Finally, combine equations:

$$\langle \boldsymbol{d}f, \boldsymbol{u} \rangle = \partial_{\boldsymbol{u}} f = \boldsymbol{u}[f]$$

   and:

$$f_{,\alpha} = \partial_\alpha f = \boldsymbol{e}_\alpha[f]$$

   to show that the bases $\boldsymbol{d}x^\alpha$ and $\partial/\partial x^\beta$ are the duals of each other.
   Consider:

$$\begin{aligned}\langle \boldsymbol{d}f, \boldsymbol{u} \rangle &= f_{,\alpha} u^\beta \langle \boldsymbol{\omega}^\alpha, \boldsymbol{e}_\beta \rangle \\ &= f_{,\alpha} u^\beta \delta^\alpha_\beta \\ &= f_{,\alpha} u^\alpha\end{aligned}$$