A Relativist's Toolkit
Problems

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Chapter 1

Fundamentals

1. **The surface of a two-dimensional cone is embedded in three-dimensional flat space. The cone has an opening angle of 2α. Points on the cone which all have the same distance r from the apex define a circle, and ϕ is the angle that runs along the circle.**

   (a) **Write down the metric of the cone, in terms of the coordinates r and ϕ.** Consider the metric for 3-space in spherical polar coordinates:

   $$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

   On a cone, the cooridnate θ is half of the opening angle and is a constant implying: $\theta = \alpha$ and $d\theta = d\alpha = 0$. Therefore the metric of the cone is:

   $$ds^2 = dr^2 + r^2 \sin^2 \alpha d\phi^2$$

   Shown another way, consider the paramaterization of the cone $X(r, \phi)$

   $$X(r, \phi) = \begin{bmatrix} r \sin \alpha \cos \phi \\ r \sin \alpha \sin \phi \\ r \cos \alpha \end{bmatrix}$$

   Differentiating with respect to r and ϕ gives:

   $$X_{,\phi} = \begin{bmatrix} -r \sin \alpha \sin \phi \\ r \sin \alpha \cos \phi \\ 0 \end{bmatrix}$$

   and

   $$X_{,r} = \begin{bmatrix} \sin \alpha \cos \phi \\ \sin \alpha \sin \phi \\ \cos \alpha \end{bmatrix}$$

   The general form of the metric in two dimensions is:

   $$ds^2 = E d\phi^2 + 2F d\phi dr + G dr^2$$

where:

$$
\begin{aligned}
E &= X_{,\phi} \cdot X_{,\phi} = r \sin^2 \alpha \\
F &= X_{,\phi} \cdot X_{,r} = 0 \\
G &= X_{,r} \cdot X_{,r} = 1
\end{aligned}
$$

therefore:

$$ds^2 = dr^2 + r^2 \sin^2 \alpha^2 d\phi^2$$

(b) **Find the coordinate transformation $x(r, \phi)$, $y(r, \phi)$ that brings the metric into the form $ds^2 = dx^2+dy^2$. Do these coordinates cover the entire two-dimensional plane?**

Consider the differentials of the transformation $x(r, \phi)$ and $y(r, \phi)$:

$$
\begin{aligned}
dx &= \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \phi} d\phi \\
dy &= \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \phi} d\phi
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
dx^2 + dy^2 &= \left[\left(\frac{\partial x}{\partial r} \right)^2 + \left(\frac{\partial y}{\partial r} \right)^2 \right] dr^2 + \left[\left(\frac{\partial x}{\partial \phi} \right)^2 + \left(\frac{\partial y}{\partial \phi} \right)^2 \right] d\phi^2 + \left(\frac{\partial x}{\partial r} \frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial r} \frac{\partial y}{\partial \phi} \right) dr d\phi \\
&= dr^2 + r^2 \sin^2 \alpha d\phi^2
\end{aligned}
$$

Which gives three equations:

$$\left(\frac{\partial x}{\partial r} \right)^2 + \left(\frac{\partial y}{\partial r} \right)^2 = 1 \tag{1.1}$$

$$\left(\frac{\partial x}{\partial \phi} \right)^2 + \left(\frac{\partial y}{\partial \phi} \right)^2 = r^2 \sin^2 \alpha \tag{1.2}$$

$$\left(\frac{\partial x}{\partial r} \frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial r} \frac{\partial y}{\partial \phi} \right) = 0 \rightarrow \frac{\partial x}{\partial r} \frac{\partial x}{\partial \phi} = - \frac{\partial y}{\partial r} \frac{\partial y}{\partial \phi} \tag{1.3}$$

Make the following ansatz:

$$
\begin{aligned}
x(r, \phi) &= r \sin \phi \\
y(r, \phi) &= r \cos \phi
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial x}{\partial r} &= \sin \phi \\
\frac{\partial y}{\partial r} &= \cos \phi \\
\left(\frac{\partial x}{\partial r} \right)^2 + \left(\frac{\partial y}{\partial r} \right)^2 &= \sin^2 \phi + \cos^2 \phi = 1
\end{aligned}
$$

Therefore equation 1.1 is satisfied. To satisfy equation 1.2 modify the equations to:

$$
\begin{aligned}
x(r, \phi) &= r \sin(\phi \sin \alpha) \\
y(r, \phi) &= r \cos(\phi \sin \alpha)
\end{aligned}
$$

leaving the results of equation 1.1 unchanged.

$$\frac{\partial x}{\partial \phi} = r\sin\alpha\cos(\phi\sin\alpha)$$
$$\frac{\partial y}{\partial \phi} = -r\sin\alpha\sin(\phi\sin\alpha)$$
$$\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 = r^2\sin^2\alpha\sin^2(\phi\sin\alpha) + r^2\sin^2\alpha\cos^2(\phi\cos\alpha)$$
$$= r^2\sin^2\alpha$$

Therefore 1.2 is satisfied. Checking the last equation:

$$\frac{\partial x}{\partial r}\frac{\partial x}{\partial \phi} = -\frac{\partial y}{\partial r}\frac{\partial y}{\partial \phi}$$
$$r\sin\alpha\sin(\phi\sin\alpha)\cos(\phi\sin\alpha) = -(-r\sin\alpha\cos(\phi\sin\alpha)\sin(\phi\sin\alpha))$$

Therefore the transformations that bring the metric into the form $ds^2 = dx^2 + dy^2$ are:

$$x(r,\phi) = r\sin(\phi\sin\alpha)$$
$$y(r,\phi) = r\cos(\phi\sin\alpha)$$

Proof:

$$dx = \sin(\phi\sin\alpha)dr + r\sin\alpha\cos(\phi\sin\alpha)d\phi$$
$$dy = \cos(\phi\sin\alpha)dr - r\sin\alpha\sin(\phi\sin\alpha)d\phi$$

and:

$$\begin{aligned} ds^2 &= dx^2 + dy^2 \\ &= \sin^2(\phi\sin\alpha)dr^2 + r^2\sin^2\alpha\cos^2(\phi\sin\alpha)d\phi^2 + 2r\sin\alpha\sin(\phi\sin\alpha)\cos(\phi\sin\alpha)drd\phi \\ &+ \cos^2(\phi\sin\alpha)dr^2 + r^2\sin^2\alpha\sin^2(\phi\sin\alpha)d\phi^2 - 2rs\sin\alpha\sin(\phi\sin\alpha)\cos(\phi\sin\alpha)drd\phi \\ &= dr^2 + r^2\sin^2\alpha d\phi^2 \end{aligned}$$

Consider the inverted transfromation:

$$\phi = \frac{1}{\sin\alpha}\arctan\left(\frac{y}{x}\right)$$
$$r = \sqrt{x^2+y^2}$$

In order for these coordinates to cover the entire plane, the range of $\phi(x,y)$ and $r(x,y)$ must be $0 \leq \phi < 2\pi$ and $0 \leq r < \infty$. The range of arctan is $-\pi/2 < \phi < \pi/2$. Check this later.

(c) **Prove that any vector parallel transported along a circle of constant r on the surface of the cone ends up rotated by an angle β after a complete trip. Express β in terms of α.** The equation for parallel transport is:

$$t^a\nabla_a v^b = 0$$

where t^a is the tangent along the curve and v^b is the vector being parallelly transported.

$$t^a \partial_a v^b + t^a \Gamma^b_{ac} v^c = 0$$

$$t^r \partial_r v^b + t^\phi \partial_\phi v^b + t^r \Gamma^b_{rr} v^r + t^r \Gamma^b_{r\phi} v^\phi + t^\phi \Gamma^b_{\phi r} v^r + t^\phi \Gamma^b_{\phi\phi} v^\phi = 0$$

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd} (\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab})$$

$$\Gamma^c_{r\phi} = \frac{1}{2} g^{cd} (\partial_r g_{\phi d} + \partial_b g_{ad} - \partial_d g_{ab})$$