A Relativist's Toolkit
Problems

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Chapter 1

Fundamentals

1. **The surface of a two-dimensional cone is embedded in three-dimensional flat space. The cone has an opening angle of 2α. Points on the cone which all have the same distance r from the apex define a circle, and ϕ is the angle that runs along the circle.**

   (a) **Write down the metric of the cone, in terms of the coordinates r and ϕ.** Consider the metric for 3-space in spherical polar coordinates:
   $$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$
   On a cone, the cooridnate θ is half of the opening angle and is a constant implying: $\theta = \alpha$ and $d\theta = d\alpha = 0$. Therefore the metric of the cone is:
   $$ds^2 = dr^2 + r^2 \sin^2\alpha d\phi^2$$
   Shown another way, consider the paramaterization of the cone $X(r, \phi)$
   $$X(r,\phi) = \begin{bmatrix} r\sin\alpha\cos\phi \\ r\sin\alpha\sin\phi \\ r\cos\alpha \end{bmatrix}$$
   Differentiating with respect to r and ϕ gives:
   $$X_{,\phi} = \begin{bmatrix} -r\sin\alpha\sin\phi \\ r\sin\alpha\cos\phi \\ 0 \end{bmatrix}$$
   and
   $$X_{,r} = \begin{bmatrix} \sin\alpha\cos\phi \\ \sin\alpha\sin\phi \\ \cos\alpha \end{bmatrix}$$
   The general form of the metric in two dimensions is:
   $$ds^2 = Ed\phi^2 + 2Fd\phi dr + Gdr^2$$

where:

$$E = X_{,\phi} \cdot X_{,\phi} = r \sin^2 \phi$$
$$F = X_{,\phi} \cdot X_{,r} = 0$$
$$G = X_{,r} \cdot X_{,r} = 1$$

therefore:

$$ds^2 = dr^2 + r \sin^2 \phi d\phi^2$$

(b) **Find the coordinate transformation $x(r, \phi)$, $y(r, \phi)$ that brings the metric into the form $ds^2 = dx^2 + dy^2$. Do these coordinates cover the entire two-dimensional plane?**

Consider:

$$x(r, \phi) = r \sin \alpha \cos \phi$$
$$y(r, \phi) = r \sin \alpha \sin \phi$$

Then:

$$\frac{y}{x} = \tan \phi \to \phi = \arctan \frac{y}{x}$$

and

$$r = \frac{x}{\sin \alpha \cos \phi} = \frac{x\sqrt{(\frac{y}{x})^2 + 1}}{\sin \alpha} = \frac{\sqrt{x^2 + y^2}}{\sin \alpha}$$

likewise:

$$r = \frac{y}{\sin \alpha \sin \phi} = \frac{y\sqrt{(\frac{y}{x})^2 + 1}}{\frac{y}{x} \sin \alpha} = \frac{\sqrt{x^2 + y^2}}{\sin \alpha}$$

Proving that this transformation does indeed result in a metric of the form $ds^2 = dx^2 + dy^2$:

$$\begin{aligned} d\phi &= \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \\ &= \frac{-1}{1 + \frac{x^2}{y^2}} dx + \frac{1}{1 + \frac{y^2}{x^2}} dy \\ &= \frac{-y^2}{x^2 + y^2} dx + \frac{x^2}{x^2 + y^2} dy \end{aligned}$$

$$d\phi^2 = \frac{y^4}{(x^2 + y^2)^2} dx^2 + \frac{x^4}{(x^2 + y^2)^2} dy^2 - 2\frac{x^4 y^4}{(x^2 + y^2)^2} dx dy$$

$$\sin^2 \phi = \frac{y^2}{x^2 + y^2}$$

$$\begin{aligned} dr &= \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy \\ &= \frac{x}{\sin \alpha \sqrt{x^2 + y^2}} dx + \frac{y}{\sin \alpha \sqrt{x^2 + y^2}} dy \end{aligned}$$

(c) **Prove that any vector parallel transported along a circle of constant r on the surface of the cone ends up rotated by an angle β after a complete trip. Express β in terms of α.**