

## Chapter 2 Problems

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- 2.
3. (a) **Verify that the commutator, defined by equation (2.2.14), satisfies the linearity and Leibnitz properties, and hence defines a vector field**

$$\begin{aligned}
[v, w](f + g) &= v(w(f + g)) - w(v(f + g)) \\
&= v(w(f) + w(g)) - w(v(f) + v(g)) \\
&= v(w(f)) + v(w(g)) - w(v(f)) - w(v(g)) \\
&= v(w(f)) - w(v(f)) + v(w(g)) - w(v(g)) \\
&= [v, w]f + [v, w]g
\end{aligned}$$

Therefore the commutator satisfies the linearity property in  $v$ . The same procedure can be applied symmetrically to  $w$ .

$$\begin{aligned}
[v, w](fg) &= v(w(fg)) - w(v(fg)) \\
&= v(w(f)g + fw(g)) - w(v(f)g + fv(g)) \\
&= v(w(f)g) + v(fw(g)) - w(v(f)g) - w(fv(g)) \\
&= v(w(f))g + w(f)v(g) + v(f)w(g) + fw(v(g)) \\
&\quad - w(v(f))g - v(f)w(g) - w(f)v(g) - fw(v(g)) \\
&= v(w(f))g + f(v(w(g)) - w(v(f))g - fw(v(g)) \\
&= f(v(w(g)) - fw(v(g)) + v(w(f))g - w(v(f))g \\
&= f\left((v(w(g)) - w(v(g)))\right) + g\left(v(w(f)) - w(v(f))\right) \\
&= f[v, w](g) + g[v, w](f)
\end{aligned}$$

- (b) **Let  $X, Y, Z$  be smooth vector fields on a manifold  $M$ . Verify that their commutator satisfies the Jacobi identity:**

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

Expand  $[[X, Y], Z]$ :

$$\begin{aligned}
[[X, Y], Z](f) &= [X, Y]Z(f) - Z[X, Y](f) \\
&= X(Y(Z(f))) - Y(X(Z(f))) - Z(X(Y(f)) - Y(Z(f))) \\
&= X(Y(Z(f))) - Y(X(Z(f))) - Z(X(Y(f))) + Z(Y(Z(f))) \\
&= X(Y(Z(f))) - Y(X(Z(f))) - Z(X(Y(f))) + Z(Y(Z(f)))
\end{aligned}$$

Now cyclically permute  $X, Y, Z$ :

$$\begin{aligned}
&= \underbrace{X(Y(Z(f)))}_a - \underbrace{Y(X(Z(f)))}_{-g} - \underbrace{Z(X(Y(f)))}_{-e} + \underbrace{Z(Y(X(f)))}_b \\
&+ \underbrace{Y(Z(X(f)))}_c - \underbrace{Z(Y(X(f)))}_{-b} - \underbrace{X(Y(Z(f)))}_{-a} + \underbrace{X(Z(Y(f)))}_d \\
&+ \underbrace{Z(X(Y(f)))}_e - \underbrace{X(Z(Y(f)))}_{-d} - \underbrace{Y(Z(X(f)))}_{-c} + \underbrace{Y(X(Z(f)))}_g \\
&= 0
\end{aligned}$$

- (c) Let  $Y_1, \dots, Y_n$  be smooth vector fields on an  $n$ -dimensional  $M$  such that at each  $p \in M$  they form a basis of the tangent space  $V_p$ . Then, at each point, we may expand each commutator  $[Y_\alpha, Y_\beta]$  in this basis, thereby defining the functions  $C_{\alpha\beta}^\gamma = -C_{\beta\alpha}^\gamma$  by

$$[Y_\alpha, Y_\beta] = \sum_{\gamma} C_{\alpha\beta}^\gamma Y_\gamma$$

Use the Jacobi identity to derive an equation satisfied by  $C_{\alpha\beta}^\gamma$ .

Consider:

$$\begin{aligned}
[[Y_\alpha, Y_\beta], Y_\sigma] &= [C_{\alpha\beta}^\gamma Y_\gamma, Y_\sigma] \\
&= C_{\alpha\beta}^\gamma Y_\gamma Y_\sigma - Y_\sigma C_{\alpha\beta}^\gamma Y_\gamma \\
&= C_{\alpha\beta}^\gamma Y_\gamma Y_\sigma - C_{\alpha\beta}^\gamma Y_\sigma Y_\gamma \\
&= C_{\alpha\beta}^\gamma [Y_\gamma Y_\sigma - Y_\sigma Y_\gamma] \\
&= C_{\alpha\beta}^\gamma (Y_\gamma Y_\sigma - Y_\sigma Y_\gamma) \\
&= C_{\alpha\beta}^\gamma [Y_\gamma, Y_\sigma] \\
&= C_{\alpha\beta}^\gamma C_{\gamma\sigma}^\epsilon Y_\epsilon
\end{aligned}$$

Therefore the Jacobi identity gives:

$$\begin{aligned}
[[Y_\alpha, Y_\beta], Y_\sigma] + [[Y_\beta, Y_\sigma], Y_\alpha] + [[Y_\sigma, Y_\alpha], Y_\beta] &= 0 \\
C_{\alpha\beta}^\gamma C_{\gamma\sigma}^\epsilon Y_\epsilon + C_{\beta\sigma}^\gamma C_{\gamma\alpha}^\epsilon Y_\epsilon + C_{\sigma\alpha}^\gamma C_{\gamma\beta}^\epsilon Y_\epsilon &= 0 \\
(C_{\alpha\beta}^\gamma C_{\gamma\sigma}^\epsilon + C_{\beta\sigma}^\gamma C_{\gamma\alpha}^\epsilon + C_{\sigma\alpha}^\gamma C_{\gamma\beta}^\epsilon) Y_\epsilon &= 0
\end{aligned}$$

Therefore the equations satisfied by the functions are:

$$(C_{\alpha\beta}^\gamma C_{\gamma\sigma}^\epsilon + C_{\beta\sigma}^\gamma C_{\gamma\alpha}^\epsilon + C_{\sigma\alpha}^\gamma C_{\gamma\beta}^\epsilon) = 0$$

4. (a) Show that in any coordinate basis, the components of the commutator of two vector fields  $v$  and  $w$  are given by

$$[v, w]^\mu = \sum_{\nu} \left( v^\nu \frac{\partial w^\mu}{\partial x^\nu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \right)$$

$$\begin{aligned}
[v, w](f) &= v(w(f)) - w(v(f)) \\
&= v(w^\mu \frac{\partial f}{\partial x^\mu}) - w(v^\mu \frac{\partial f}{\partial x^\mu}) \\
&= v^\nu \frac{\partial}{\partial x^\nu} (w^\mu \frac{\partial f}{\partial x^\mu}) - w^\nu \frac{\partial}{\partial x^\nu} (v^\mu \frac{\partial f}{\partial x^\mu}) \\
&= v^\nu \left( \frac{\partial w^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} + w^\mu \frac{\partial^2 f}{\partial x^\nu \partial x^\mu} \right) - w^\nu \left( \frac{\partial v^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} + v^\mu \frac{\partial^2 f}{\partial x^\nu \partial x^\mu} \right) \\
&= v^\nu \frac{\partial w^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} + v^\nu w^\mu \frac{\partial^2 f}{\partial x^\nu \partial x^\mu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} - w^\nu v^\mu \frac{\partial^2 f}{\partial x^\nu \partial x^\mu} \\
&= v^\nu \frac{\partial w^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} + \left( v^\nu w^\mu \frac{\partial^2 f}{\partial x^\nu \partial x^\mu} - w^\nu v^\mu \frac{\partial^2 f}{\partial x^\nu \partial x^\mu} \right)
\end{aligned}$$

Using the equality of mixed partial derivatives we can relabel the indices:

$$\begin{aligned}
&= v^\nu \frac{\partial w^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} + \left( v^\nu w^\mu \frac{\partial^2 f}{\partial x^\nu \partial x^\mu} - w^\mu v^\nu \frac{\partial^2 f}{\partial x^\mu \partial x^\nu} \right) \\
&= v^\nu \frac{\partial w^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} \\
&= \left( v^\nu \frac{\partial w^\mu}{\partial x^\nu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \right) \frac{\partial f}{\partial x^\mu} \\
&= [v, w]^\mu \frac{\partial f}{\partial x^\mu}
\end{aligned}$$

- (b) Let  $Y_1, \dots, Y_n$  be as in problem 3(c). Let  $Y^{1*}, \dots, Y^{n*}$  be the dual basis. Show that the components  $(Y^{\gamma*})_\mu$  of  $Y^{\gamma*}$  in any coordinate basis satisfy

$$\frac{\partial (Y^{\gamma*})_\mu}{\partial x^\nu} - \frac{\partial (Y^{\gamma*})_\nu}{\partial x^\mu} = \sum_{\alpha, \beta} C_{\alpha\beta}^\gamma (Y^{\alpha*})_\mu (Y^{\beta*})_\nu$$

$$[Y_\alpha, Y_\beta] = \sum_\gamma C_{\alpha\beta}^\gamma Y_\gamma$$

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8. (a) The metric of flat, three-dimensional Euclidean space is:

$$ds^2 = dx^2 + dy^2 + dz^2$$

Show that the metric components  $g_{uv}$  in spherical polar coordinates  $r, \theta, \phi$  defined by:

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ \cos \theta &= \frac{z}{r}, \\ \tan \phi &= \frac{y}{x} \end{aligned}$$

is given by:

$$s^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \phi^2$$

$g_{uv}$  is a tensor of type  $(0, 2)$  and therefore transforms as:

$$g_{\mu', \nu'} = g_{\mu, \nu} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}}$$

(see page 22 for the general *tensor transformation law*). The above equation uses Einstein index notation indicating that  $\mu$  and  $\nu$  are to be summed from 1 to 3 and the free indices,  $\mu'$  and  $\nu'$ , are enumerated through all possible combinations. Therefore the components that need to be calculated are:

$$\begin{array}{ccc} g_{r,r} & g_{r,\theta} & g_{r,\phi} \\ g_{\theta,r} & g_{\theta,\theta} & g_{\theta,\phi} \\ g_{\phi,r} & g_{\phi,\theta} & g_{\phi,\phi} \end{array}$$

Starting with:

$$\begin{aligned} g_{\mu', \nu'} &= \sum_{\mu=1}^3 \sum_{\nu=1}^3 g_{\mu, \nu} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \\ &= \sum_{\mu=1}^3 g_{\mu, 1} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{\mu, 2} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{\mu, 3} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}} \\ &= g_{1,1} \frac{\partial x^1}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{1,2} \frac{\partial x^1}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{1,3} \frac{\partial x^1}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}} \\ &\quad g_{2,1} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{2,2} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{2,3} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}} \\ &\quad g_{3,1} \frac{\partial x^3}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{3,2} \frac{\partial x^3}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{3,3} \frac{\partial x^3}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}} \end{aligned}$$

Substituting the notation for the indices in flat, orthonormal Euclidean space:

$$\begin{aligned} &= g_{x,x} \frac{\partial x}{\partial x^{\mu'}} \frac{\partial x}{\partial x^{\nu'}} + g_{x,y} \frac{\partial x}{\partial x^{\mu'}} \frac{\partial y}{\partial x^{\nu'}} + g_{x,z} \frac{\partial x}{\partial x^{\mu'}} \frac{\partial z}{\partial x^{\nu'}} \\ &\quad g_{y,x} \frac{\partial y}{\partial x^{\mu'}} \frac{\partial x}{\partial x^{\nu'}} + g_{y,y} \frac{\partial y}{\partial x^{\mu'}} \frac{\partial y}{\partial x^{\nu'}} + g_{y,z} \frac{\partial y}{\partial x^{\mu'}} \frac{\partial z}{\partial x^{\nu'}} \\ &\quad g_{z,x} \frac{\partial z}{\partial x^{\mu'}} \frac{\partial x}{\partial x^{\nu'}} + g_{z,y} \frac{\partial z}{\partial x^{\mu'}} \frac{\partial y}{\partial x^{\nu'}} + g_{z,z} \frac{\partial z}{\partial x^{\mu'}} \frac{\partial z}{\partial x^{\nu'}} \end{aligned}$$

The off diagonal elements of the Euclidean metric are zero:

$$g_{x,y} = g_{y,x} = g_{x,z} = g_{z,x} = g_{y,z} = g_{z,y} = 0$$

and the diagonal components are one:

$$g_{x,x} = g_{y,y} = g_{z,z} = 1$$

This reduces the above summation from nine expressions to the following three:

$$g_{\mu',\nu'} = \frac{\partial x}{\partial x^{\mu'}} \frac{\partial x}{\partial x^{\nu'}} + \frac{\partial y}{\partial x^{\mu'}} \frac{\partial y}{\partial x^{\nu'}} + \frac{\partial z}{\partial x^{\mu'}} \frac{\partial z}{\partial x^{\nu'}}$$

For indices where  $\mu' = \nu'$

$$g_{\mu',\mu'} = \left( \frac{\partial x}{\partial x^{\mu'}} \right)^2 + \left( \frac{\partial y}{\partial x^{\mu'}} \right)^2 + \left( \frac{\partial z}{\partial x^{\mu'}} \right)^2$$

Therefore the six unique components that need to be calculated to find the components of the metric in spherical polar coordinates are:

$$\begin{aligned} g_{r,r} &= \left( \frac{\partial x}{\partial r} \right)^2 + \left( \frac{\partial y}{\partial r} \right)^2 + \left( \frac{\partial z}{\partial r} \right)^2 \\ g_{r,\theta} = g_{\theta,r} &= \frac{\partial x}{\partial r} \frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r} \frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r} \frac{\partial z}{\partial \theta} \\ g_{r,\phi} = g_{\phi,r} &= \frac{\partial x}{\partial r} \frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial r} \frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial r} \frac{\partial z}{\partial \phi} \\ g_{\theta,\theta} &= \left( \frac{\partial x}{\partial \theta} \right)^2 + \left( \frac{\partial y}{\partial \theta} \right)^2 + \left( \frac{\partial z}{\partial \theta} \right)^2 \\ g_{\theta,\phi} = g_{\phi,\theta} &= \frac{\partial x}{\partial \theta} \frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \theta} \frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial \theta} \frac{\partial z}{\partial \phi} \\ g_{\phi,\phi} &= \left( \frac{\partial x}{\partial \phi} \right)^2 + \left( \frac{\partial y}{\partial \phi} \right)^2 + \left( \frac{\partial z}{\partial \phi} \right)^2 \end{aligned}$$

To take the above derivatives, find an equation for  $x$ ,  $y$ ,  $z$  in terms of  $r$ ,  $\theta$ ,  $\phi$ . Starting by finding  $x$ :

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \rightarrow r^2 = x^2 + y^2 + z^2, \\ \cos \theta &= \frac{z}{r} \rightarrow z = r \cos \theta, \\ \tan \phi &= \frac{y}{x} \rightarrow y = x \tan \phi \end{aligned}$$

Substituting the second and third equation into the first gives:

$$\begin{aligned}
r^2 &= x^2 + (r \cos \theta)^2 + (x \tan \phi)^2 \\
r^2 &= x^2 + r^2 \cos^2 \theta + x^2 \tan^2 \phi \\
r^2 - r^2 \cos^2 \theta &= x^2 + x^2 \tan^2 \phi \\
(1 - \cos^2 \theta) r^2 &= (1 + \tan^2 \phi) x^2 \\
r^2 \sin^2 \theta &= (1 + \tan^2 \phi) x^2 \\
x &= r \frac{\sin \theta}{\sqrt{1 + \tan^2 \phi}}
\end{aligned}$$

Therefore the equations for  $x, y, z$  in terms of  $r, \theta, \phi$ :

$$x = r \frac{\sin \theta}{\sqrt{1 + \tan^2 \phi}}, \quad y = r \tan \phi \frac{\sin \theta}{\sqrt{1 + \tan^2 \phi}}, \quad z = r \cos \theta$$

Find all the necessary derivatives:

$$\begin{aligned}
\frac{\partial x}{\partial r} &= \frac{\sin \theta}{\sqrt{1 + \tan^2 \phi}} \\
\frac{\partial x}{\partial \theta} &= -r \frac{\cos \theta}{\sqrt{1 + \tan^2 \phi}} \\
\frac{\partial x}{\partial \phi} &= -r \sin \theta \frac{\tan \phi \sec^2 \phi}{(1 + \tan^2 \phi)^{\frac{3}{2}}} \\
\frac{\partial y}{\partial r} &= \tan \phi \frac{\sin \theta}{\sqrt{1 + \tan^2 \phi}} \\
\frac{\partial y}{\partial \theta} &= -r \tan \phi \frac{\cos \theta}{\sqrt{1 + \tan^2 \phi}} \\
\frac{\partial y}{\partial \phi} &= -r \sin \theta \frac{\sec^2 \phi}{(1 + \tan^2 \phi)^{\frac{3}{2}}} \\
\frac{\partial z}{\partial r} &= \cos \theta \\
\frac{\partial z}{\partial \theta} &= r \sin \theta \\
\frac{\partial z}{\partial \phi} &= 0
\end{aligned}$$

Then compute the components of the metric in spherical polar coor-

dinates:

$$\begin{aligned}
g_{r,r} &= \frac{\sin^2 \theta}{1 + \tan^2 \phi} + \frac{\sin^2 \theta}{1 + \tan^2 \phi} \tan^2 \phi + \cos^2 \theta \\
&= \frac{\sin^2 \theta + \sin^2 \theta \tan^2 \phi}{1 + \tan^2 \phi} + \frac{(1 + \tan^2 \phi) \cos^2 \theta}{1 + \tan^2 \phi} \\
&= \frac{\sin^2 \theta + \sin^2 \theta \tan^2 \phi + (1 + \tan^2 \phi) \cos^2 \theta}{1 + \tan^2 \phi} \\
&= \frac{\sin^2 \theta + \cos^2 \theta + \sin^2 \theta \tan^2 \phi + \tan^2 \phi \cos^2 \theta}{1 + \tan^2 \phi} \\
&= \frac{1 + \tan^2 \phi}{1 + \tan^2 \phi} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
g_{\theta,\theta} &= r^2 \frac{\cos^2 \theta}{1 + \tan^2 \phi} + r^2 \frac{\cos^2 \theta}{1 + \tan^2 \phi} \tan^2 \phi + r^2 \sin^2 \theta \\
&= r^2 \frac{\cos^2 \theta}{1 + \tan^2 \phi} + r^2 \frac{\cos^2 \theta}{1 + \tan^2 \phi} \tan^2 \phi + r^2 \sin^2 \theta \frac{1 + \tan^2 \phi}{1 + \tan^2 \phi} \\
&= r^2 \frac{\cos^2 \theta + \cos^2 \theta \tan^2 \phi + \sin^2 \theta + \tan^2 \phi \sin^2 \theta}{1 + \tan^2 \phi} \\
&= r^2 \frac{(\cos^2 \theta + \sin^2 \theta) + (\cos^2 \theta + \sin^2 \theta) \tan^2 \phi}{1 + \tan^2 \phi} \\
&= r^2 \frac{1 + \tan^2 \phi}{1 + \tan^2 \phi} \\
&= r^2
\end{aligned}$$

$$\begin{aligned}
g_{\phi,\phi} &= r^2 \sin^2 \theta \left( \frac{1}{(1 + \tan^2 \phi)^3 \cos^4 \phi} + \frac{\sin^2 \phi}{(1 + \tan^2 \phi)^3 \cos^6 \phi} \right) \\
&= r^2 \sin^2 \theta \left( \frac{\cos^2 \phi + \sin^2 \phi}{(1 + \tan^2 \phi)^3 \cos^6 \phi} \right) \\
&= r^2 \sin^2 \theta \frac{1}{\left( \frac{\sin^2 \phi + \cos^2 \phi}{\cos^2 \phi} \right)^3 \cos^6 \phi} \\
&= r^2 \sin^2 \theta
\end{aligned}$$

$$\begin{aligned}
g_{\theta,r} = g_{r,\theta} &= g_{x,x} \frac{\partial x}{\partial r} \frac{\partial x}{\partial \theta} + g_{y,y} \frac{\partial y}{\partial r} \frac{\partial y}{\partial \theta} + g_{z,z} \frac{\partial z}{\partial r} \frac{\partial z}{\partial \theta} \\
&= -r \frac{\sin \theta \cos \theta}{1 + \tan^2 \phi} - r \frac{\sin \theta \cos \theta}{1 + \tan^2 \phi} \tan^2 \phi + r \sin \theta \cos \theta \\
&= -r \frac{\sin \theta \cos \theta}{1 + \tan^2 \phi} (1 + \tan^2 \phi) + r \sin \theta \cos \theta \\
&= -r \sin \theta \cos \theta + r \sin \theta \cos \theta \\
&= 0
\end{aligned}$$



$$\begin{aligned}
g_{r,\phi} = g_{\phi,r} &= g_{x,x} \frac{\partial x}{\partial r} \frac{\partial x}{\partial \phi} + g_{y,y} \frac{\partial y}{\partial r} \frac{\partial y}{\partial \phi} + g_{z,z} \frac{\partial z}{\partial r} \frac{\partial z}{\partial \phi} \\
&= -r \sin \theta \cos \theta \frac{\tan \phi \sec^2 \phi}{(1 + \tan^2 \phi)^2} + r \cos \theta \sin \theta \frac{\tan \phi \sec^2 \phi}{(1 + \tan^2 \phi)^2} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
g_{\theta,\phi} = g_{\phi,\theta} &= g_{x,x} \frac{\partial x}{\partial \theta} \frac{\partial x}{\partial \phi} + g_{y,y} \frac{\partial y}{\partial \theta} \frac{\partial y}{\partial \phi} + g_{z,z} \frac{\partial z}{\partial \theta} \frac{\partial z}{\partial \phi} \\
&= -r \frac{\cos \theta}{\sqrt{1 + \tan^2 \phi}} \left( -r \sin \theta \frac{\tan \phi \sec^2 \phi}{(1 + \tan^2 \phi)^{\frac{3}{2}}} \right) - r \tan \phi \frac{\cos \theta}{\sqrt{1 + \tan^2 \phi}} \left( r \sin \theta \frac{\sec^2 \phi}{(1 + \tan^2 \phi)^{\frac{3}{2}}} \right) \\
&= r^2 \sin \theta \cos \theta \frac{\tan \phi \sec^2 \phi}{(1 + \tan^2 \phi)^2} - r^2 \sin \theta \cos \theta \frac{\tan \phi \sec^2 \phi}{(1 + \tan^2 \phi)^2} \\
&= 0
\end{aligned}$$

Therefore the metric components in spherical polar coordinates are:

$$g_{\mu,\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{bmatrix}$$

(b) **The spacetime metric of special relativity is**

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

**Find the components,  $g_{\mu\nu}$  and  $g^{\mu\nu}$ , of the metric and inverse metric in "rotating coordinates", defined by**

$$\begin{aligned}
t' &= t \\
x' &= (x^2 + y^2)^{\frac{1}{2}} \cos(\phi - wt) \\
y' &= (x^2 + y^2)^{\frac{1}{2}} \sin(\phi - wt) \\
z' &= z
\end{aligned}$$

**where  $\tan \phi = \frac{y}{x}$**

It is easier differentiate with respect to the primed coordinates so find  $g^{\mu\nu}$  first. First writting the primed coordinates in terms of the unprimed:

$$\begin{aligned}
t' &= t \\
x' &= (x^2 + y^2)^{\frac{1}{2}} \cos(\tan^{-1} \frac{y}{x} - wt) \\
y' &= (x^2 + y^2)^{\frac{1}{2}} \sin(\tan^{-1} \frac{y}{x} - wt) \\
z' &= z
\end{aligned}$$

Find all the necessary derivatives:

$$\begin{aligned}\frac{\partial t'}{\partial t} &= 1 \\ \frac{\partial t'}{\partial x} &= \frac{\partial t'}{\partial y} = \frac{\partial t}{\partial z} = 0\end{aligned}$$

$$\begin{aligned}\frac{\partial x'}{\partial t} &= -w\sqrt{x^2 + y^2} \sin(\tan^{-1} \frac{y}{x} - wt) \\ \frac{\partial x'}{\partial x} &= \frac{x \cos(\tan^{-1} \frac{y}{x} - wt) + y \sin(\tan^{-1} \frac{y}{x} - wt)}{\sqrt{x^2 + y^2}} \\ \frac{\partial x'}{\partial y} &= \frac{-x \sin(\tan^{-1} \frac{y}{x} - wt) + y \cos(\tan^{-1} \frac{y}{x} - wt)}{\sqrt{x^2 + y^2}} \\ \frac{\partial x'}{\partial z} &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial y'}{\partial t} &= -w\sqrt{x^2 + y^2} \cos(\tan^{-1} \frac{y}{x} - wt) \\ \frac{\partial y'}{\partial x} &= \frac{x \sin(\tan^{-1} \frac{y}{x} - wt) - y \cos(\tan^{-1} \frac{y}{x} - wt)}{\sqrt{x^2 + y^2}} \\ \frac{\partial y'}{\partial y} &= \frac{x \sin(\tan^{-1} \frac{y}{x} - wt) + y \cos(\tan^{-1} \frac{y}{x} - wt)}{\sqrt{x^2 + y^2}} \\ \frac{\partial y'}{\partial z} &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial z'}{\partial t} &= \frac{\partial z'}{\partial x} = \frac{\partial z'}{\partial y} = 0 \\ \frac{\partial z'}{\partial z} &= 1\end{aligned}$$

$$\begin{aligned}\left(\frac{\partial x'}{\partial x}\right)^2 &= \left(\frac{x \cos(\tan^{-1} \frac{y}{x} - wt) + y \sin(\tan^{-1} \frac{y}{x} - wt)}{\sqrt{x^2 + y^2}}\right)^2 = \frac{(x^2 + y^2) \sin^2(wt)}{(x^2 + y^2)} \\ \left(\frac{\partial y'}{\partial y}\right)^2 &= \left(\frac{x \cos(\tan^{-1} \frac{y}{x} - wt) + y \sin(\tan^{-1} \frac{y}{x} - wt)}{\sqrt{x^2 + y^2}}\right)^2 = (x^2 + y^2) \sin^2(wt)\end{aligned}$$

## 0.1 Appendix A - Tensor Expansion

The metric is a rank (0,2) tensor so the transformation of the components between basis is given by:

$$g_{\mu', \nu'} = g_{\mu, \nu} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}}$$

This equation is a shorthand for the following:

$$\begin{aligned} g_{\mu', \nu'} &= \sum_{\mu=1}^3 \sum_{\nu=1}^3 g_{\mu, \nu} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \\ g_{\mu', \nu'} &= \sum_{\mu=1}^3 g_{\mu, 1} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{\mu, 2} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{\mu, 3} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}} \\ g_{\mu', \nu'} &= g_{1,1} \frac{\partial x^1}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{1,2} \frac{\partial x^1}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{1,3} \frac{\partial x^1}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}} \\ &\quad g_{2,1} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{2,2} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{2,3} \frac{\partial x^2}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}} \\ &\quad g_{3,1} \frac{\partial x^3}{\partial x^{\mu'}} \frac{\partial x^1}{\partial x^{\nu'}} + g_{3,2} \frac{\partial x^3}{\partial x^{\mu'}} \frac{\partial x^2}{\partial x^{\nu'}} + g_{3,3} \frac{\partial x^3}{\partial x^{\mu'}} \frac{\partial x^3}{\partial x^{\nu'}} \end{aligned}$$

The full expansion for  $\mu', \nu'$  is:

[illegible]

The tensor transformation law for a tensor of rank  $(0, 2)$  in 3 space represents 9 components. Each components contains a summation of 9 expressions. Each expression has 3 terms. So in total the equation is representative of 243 terms.