Wald's General Relatvity Problems

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Chapter 1

Introduction

1. **Car and Garage Paradox. The lack of a notion of absolute simultaneity in special relativity leads to many supposed paradoxes. One of the most famous of these involves a car and a garage of equal proper length. The driver speeds toward the garage, and a doorman at the garage is instructed to slam the door shut as soon as the back end of the car enters the garage. According to the doorman, "the car Lorentz contracted and easily fitted into the garage when I slammed the door". According to the driver, "the garage Lorentz contracted and was too small for the car when I entered the garage." Draw a spacetime diagram showing the above events and explain what really happens. Is the doorman's statement correct? For definiteness, assume that the car crashes through the back wall of the garage without stopping or slowing down.**

   The doorman's statement is correct but the doorman and driver disagree on the order of events. In the doorman's frame, he sees the end of the car coincident with him and closes the door. In his frame, the car is Lorentz contracted so that it has not broken through the wall yet. To be a nice guy, he decides to send a message to the driver telling them to stop the car so that the car does not break through the wall.

   Let L be the proper length of the car and garage, v be the velocity of the car in the doorman's reference frame, and γ be the Lorentz factor.

   In the doorman's frame, he observes the car to be L/γ. When he closes the door, the car has not crashed through the end of the garage and the distance from the front of the car to the end of the garage is:

$$\Delta L = L(1 - \frac{1}{\gamma})$$

   as shown in 1.1. Assume for simplicity that there is a detector at the end of the garage that will absorb the signal from the doorman. If the

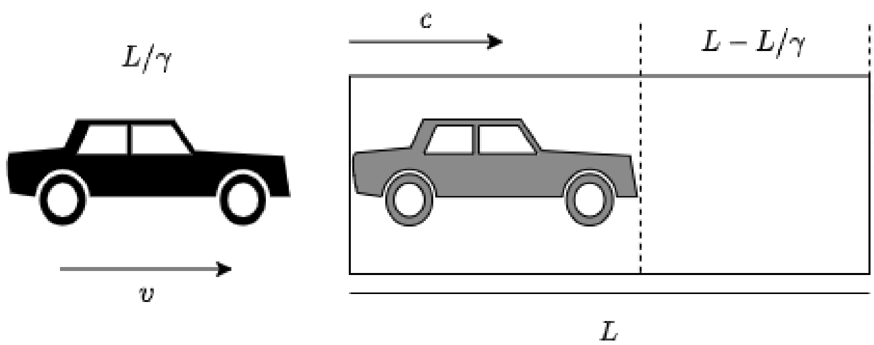

Figure 1.1: The car and garage paradox in the doorman's frame.

signal has been absorbed it is green and if not it is red. The driver simply observes the detector when he passes to see if he should stop or not.

When the doorman emits the light signal to tell the driver to stop the car, the light takes a time:

$$\Delta t_c = L/c$$

to reach the end of the garage. The car however takes:

$$\Delta t_v = \frac{\Delta L}{v} = \frac{L}{v}(1 - \frac{1}{\gamma})$$

If $\Delta t_c \leq \Delta t_v$ the car stops but if $\Delta t_c > \Delta t_v$ the car crashes through the end of the garage. Assume:

$$\begin{aligned}
\Delta t_c &> \Delta t_v \\
\frac{L}{c} &> \frac{L}{v}(1 - \frac{1}{\gamma}) \\
\frac{v}{c} &> (1 - \frac{1}{\gamma}) \\
\frac{v}{c} &> (1 - \sqrt{1 - \frac{v^2}{c^2}}) \\
\sqrt{1 - \frac{v^2}{c^2}} &> 1 - \frac{v}{c} \\
\sqrt{(1 - \frac{v}{c})(1 + \frac{v}{c})} &> 1 - \frac{v}{c} \\
\sqrt{1 + \frac{v}{c}} &> \sqrt{1 - \frac{v}{c}}
\end{aligned}$$

This is always true, so the car always crashes through the end of the garage before the doorman can tell the driver to stop. This resolves the apparent paradox.

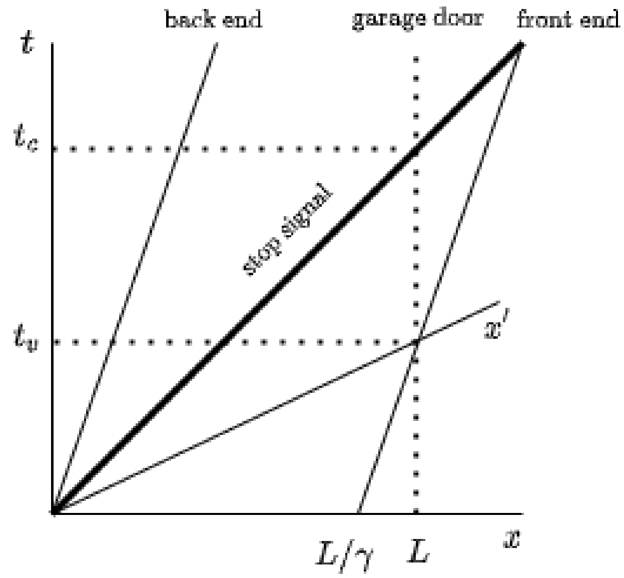

Figure 1.2: Minkowski diagram in the reference frame of the doorman

However, it is clear that the order of events is not the same. As mentioned in the problem definition, the driver sees the garage Lorentz contracted so he crashes through the garage and then observes the door close. In the doorman's frame the order of events is inverted.

In hindsight, this is obvious from constructing a correct Minkowski diagram as shown in 1.2.

Chapter 2

Manifolds and Tensors

1. (a)

   (b) **Show by explicit construction that two coordinate systems (as opposed to the six used in the text) suffice to cover S^2 (it is impossible to cover S^2 with a signle chart, as follows from the fact that S^2 is compact, but every open subset of $\mathbb{R}^2$ is noncompact see appendix A.)**

   Consider the collection of two subsets, O_N and O_S. O_N contains all the points in the set S^2 except the north pole $(0,0,1)$, and O_S contains all the points in S except the south pole $(0,0,-1)$. Together this collection contains every point in S^2 thereby satisfying property 1 required for a manifold.

   Next we must determine two maps, ψ_N and ψ_S for O_N and O_S respectively, and show that they are one-to-one and map S^2 to $\mathbb{R}^2$. To construct these maps consider Figure 2.1. Pictured are all the points in S^2 which lie in the X-Z plane. We will use this diagram to construct the map ψ.

   Draw a line from the north pole N that intersects the S^2 at a point P and the X axis at P'. The point P has coordinates $(x,0,z)$. The triangles NzP and NOP' can be related using similar triangles. This will give a relation between the point P and P'.

$$\frac{OP'}{NO} = \frac{zP}{Nz}$$
$$P' = \frac{x}{1-z}$$

   P' is equivalent to mapping the any point P into the $X-Y$ plane and we will denote this coordinate in the X direction X'. The same argument can be made to find Y' by replacing x in the above equation with y. Therefore:

$$O^N : S^2 \to \mathbb{R}^2 = (X', Y') = (\frac{x}{1-z}, \frac{y}{1-z})$$

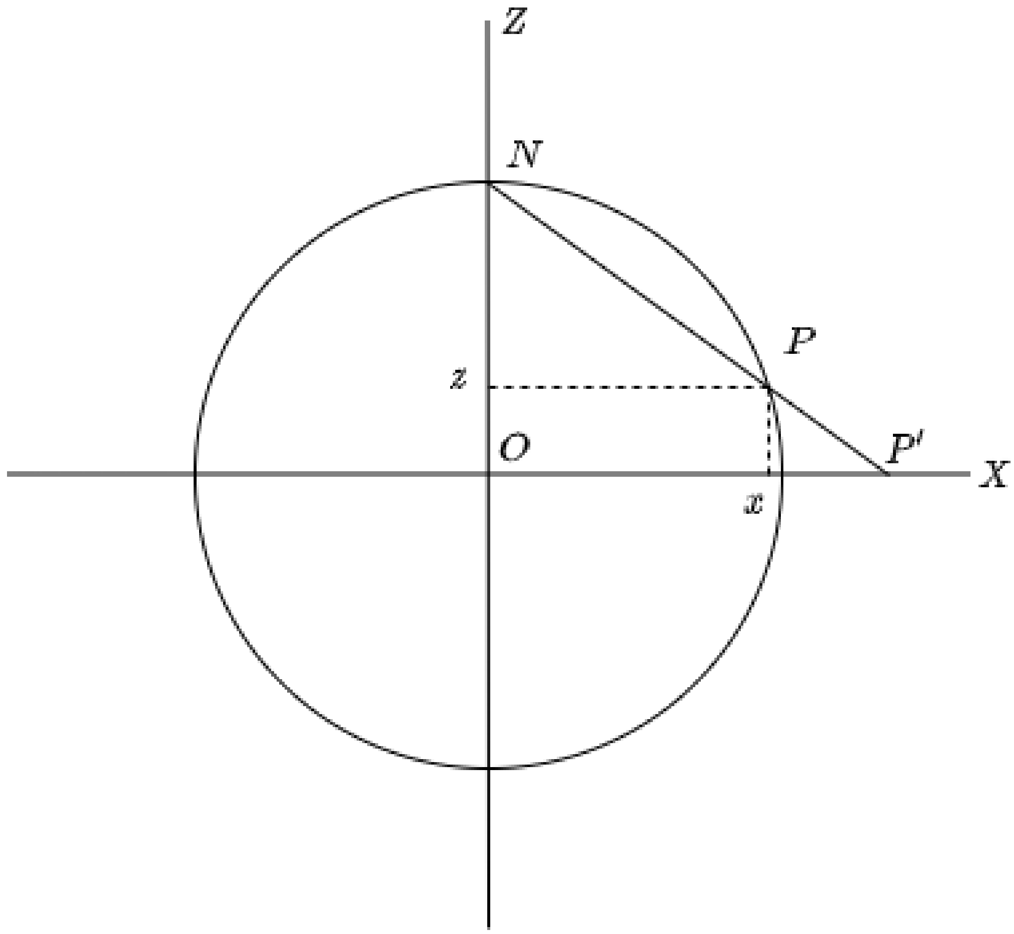

Figure 2.1: Projection of points on a circle into $\mathbb{R}^2$

ψ_N can be constructed in a similar manner but replacing z with $-z$:

$$(X', Y') = (\frac{x}{1+z}, \frac{y}{1+z})$$

Each of these maps are clearly one to one and onto.

$$(X, Y) = (\frac{x}{1-z}, \frac{y}{1-z})$$

$$(x, y, z) = (\frac{2X}{1+X^2+Y^2}. \frac{2Y}{1+X^2+Y^2}, \frac{-1+X^2+Y^2}{1+X^2+Y^2})$$

$\psi_N : O_N \to U_N \subset \mathbb{R}^2$ and $\psi_S : O_S \to U_S \subset \mathbb{R}^2$. In order to satisfy property 3 of manifolds we must show that $\psi_N \circ \psi_S^{-1}$ and $\psi_S \circ \psi_N^{-1}$ the subsets of the map are open and the maps are infinitely differentiable.

$$(X', Y') = (\frac{2X}{(1+X^2+Y^2)(1-\frac{-1+X^2+Y^2}{1+X^2+Y^2})}, \frac{2Y}{(1+X^2+Y^2)(1-\frac{-1+X^2+Y^2}{1+X^2+Y^2})})$$

$$(X', Y') = (\frac{2X}{1+X^2+Y^2+1-X^2-Y^2}, \frac{2Y}{1+X^2+Y^2+1-X^2-Y^2}) = (X, Y)$$

2. **Prove that any smooth function $F : \mathbb{R}^n \to \mathbb{R}$ can be written in the form equation (2.2.2)**

3. (a) **Verify that the commutator, defined by equation (2.2.14), satisfies the linearity and Leibnitz properties, and hence defines a vector field**

$$\begin{aligned}
[v, w](f+g) &= v(w(f+g)) - w(v(f+g)) \\
&= v(w(f) + w(g)) - w(v(f) + v(g)) \\
&= v(w(f)) + v(w(g)) - w(v(f)) - w(v(g)) \\
&= v(w(f)) - w(v(f)) + v(w(g)) - w(v(g)) \\
&= [v, w]f + [v, w]g
\end{aligned}$$

Therefore the commutator satisfies the linearity property in v. The same procedure can be applied symmetrically to w.

$$\begin{aligned}
[v, w](fg) &= v(w(fg)) - w(v(fg)) \\
&= v(w(f)g + fw(g)) - w(v(f)g + fv(g)) \\
&= v(w(f)g) + v(fw(g)) - w(v(f)g) - w(fv(g)) \\
&= v(w(f))g + w(f)v(g) + v(f)w(g) + fv(w(g)) \\
&- w(v(f))g - v(f)w(g) - w(f)v(g) - fw(v(g)) \\
&= v(w(f))g + f(v(w(g)) - w(v(f))g - fw(v(g)) \\
&= f(v(w(g)) - fw(v(g)) + v(w(f))g - w(v(f))g \\
&= f\Big((v(w(g)) - w(v(g))\Big) + g\Big(v(w(f)) - w(v(f))\Big) \\
&= f[v, w](g) + g[v, w](f)
\end{aligned}$$

(b) **Let X, Y, Z be smooth vector fields on a manifold M. Verify that their commutator satisfies the Jacobi identity:**

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$$

Expand $[[X,Y],Z]$:

$$\begin{aligned}
[[X,Y],Z](f) &= [X,Y]Z(f) - Z[X,Y](f) \\
&= X(Y(Z(f))) - Y(X(Z(f))) - Z(X(Y(f)) - Y(Z(f))) \\
&= X(Y(Z(f))) - Y(X(Z(f))) - Z(X(Y(f))) + Z(Y(Z(f))) \\
&= X(Y(Z(f))) - Y(X(Z(f))) - Z(X(Y(f))) + Z(Y(Z(f)))
\end{aligned}$$

Now cyclically permute X,Y,Z:

$$\begin{aligned}
&= \underbrace{X(Y(Z(f)))}_{a} - \underbrace{Y(X(Z(f)))}_{-g} - \underbrace{Z(X(Y(f)))}_{-e} + \underbrace{Z(Y(X(f)))}_{b} \\
&+ \underbrace{Y(Z(X(f)))}_{c} - \underbrace{Z(Y(X(f)))}_{-b} - \underbrace{X(Y(Z(f)))}_{-a} + \underbrace{X(Z(Y(f)))}_{d} \\
&+ \underbrace{Z(X(Y(f)))}_{e} - \underbrace{X(Z(Y(f)))}_{-d} - \underbrace{Y(Z(X(f)))}_{-c} + \underbrace{Y(X(Z(f)))}_{g} \\
&= 0
\end{aligned}$$

(c) **Let $Y_1, \ldots, Y_n$ be smooth vector fields on an n-dimensional M such that at each $p \in M$ they form a basis of the tangent space V_p. Then, at each point, we may expand each commutator $[Y_\alpha, Y_\beta]$ in this basis, thereby defining the functions $C^\gamma_{\alpha\beta} = -C^\gamma_{\beta\alpha}$ by**

$$[Y_\alpha, Y_\beta] = \sum_\gamma C^\gamma_{\alpha\beta} Y_\gamma$$

Use the Jacobi identity to derive an equation satisfied by $C^\gamma_{\alpha\beta}$.

Consider:

$$\begin{aligned}
[[Y_\alpha, Y_\beta], Y_\sigma] &= [C^\gamma_{\alpha\beta} Y_\gamma, Y_\sigma] \\
&= C^\gamma_{\alpha\beta} Y_\gamma Y_\sigma - Y_\sigma C^\gamma_{\alpha\beta} Y_\gamma \\
&= C^\gamma_{\alpha\beta} Y_\gamma Y_\sigma - C^\gamma_{\alpha\beta} Y_\sigma Y_\gamma \\
&= C^\gamma_{\alpha\beta} [Y_\gamma Y_\sigma - Y_\sigma Y_\gamma] \\
&= C^\gamma_{\alpha\beta} \Big(Y_\gamma Y_\sigma - Y_\sigma Y_\gamma \Big) \\
&= C^\gamma_{\alpha\beta} [Y_\gamma, Y_\sigma] \\
&= C^\gamma_{\alpha\beta} C^\epsilon_{\gamma\sigma} Y_\epsilon
\end{aligned}$$

Therefore the Jacobi identity gives:

$$\begin{aligned}
[[Y_\alpha, Y_\beta], Y_\sigma] + [[Y_\beta, Y_\sigma], Y_\alpha] + [[Y_\sigma, Y_\alpha], Y_\beta] = 0 \\
C^\gamma_{\alpha\beta} C^\epsilon_{\gamma\sigma} Y_\epsilon + C^\gamma_{\beta\sigma} C^\epsilon_{\gamma\alpha} Y_\epsilon + C^\gamma_{\sigma\alpha} C^\epsilon_{\gamma\beta} Y_\epsilon = 0 \\
(C^\gamma_{\alpha\beta} C^\epsilon_{\gamma\sigma} + C^\gamma_{\beta\sigma} C^\epsilon_{\gamma\alpha} + C^\gamma_{\sigma\alpha} C^\epsilon_{\gamma\beta}) Y_\epsilon = 0
\end{aligned}$$

Therefore the equations satisfied by the functions are:

$$(C^\gamma_{\alpha\beta} C^\epsilon_{\gamma\sigma} + C^\gamma_{\beta\sigma} C^\epsilon_{\gamma\alpha} + C^\gamma_{\sigma\alpha} C^\epsilon_{\gamma\beta}) = 0$$

4. (a) **Show that in any coordinate basis, the components of the commutator of two vector fields v and w are given by**

$$[v, w]^\mu = \sum_\nu \left(v^\nu \frac{\partial w^\mu}{\partial x^\nu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \right)$$

$$\begin{aligned}
[v, w](f) &= v(w(f)) - w(v(f)) \\
&= v(w^\mu \frac{\partial f}{\partial x^\mu}) - w(v^\mu \frac{\partial f}{\partial x^\mu}) \\
&= v^\nu \frac{\partial}{\partial x^\nu}(w^\mu \frac{\partial f}{\partial x^\mu}) - w^\nu \frac{\partial}{\partial x^\nu}(v^\mu \frac{\partial f}{\partial x^\mu}) \\
&= v^\nu \left(\frac{\partial w^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} + w^\mu \frac{\partial f}{\partial x^\nu x^\mu} \right) - w^\nu \left(\frac{\partial v^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} + v^\mu \frac{\partial f}{\partial x^\nu x^\mu} \right) \\
&= v^\nu \frac{\partial w^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} + v^\nu w^\mu \frac{\partial f}{\partial x^\nu x^\mu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} - w^\nu v^\mu \frac{\partial f}{\partial x^\nu x^\mu} \\
&= v^\nu \frac{\partial w^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} + \left(v^\nu w^\mu \frac{\partial f}{\partial x^\nu x^\mu} - w^\nu v^\mu \frac{\partial f}{\partial x^\nu x^\mu} \right)
\end{aligned}$$

Using the equality of mixed partial derivatives we can relabel the indices:

$$\begin{aligned}
&= v^\nu \frac{\partial w^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} + \left(v^\nu w^\mu \frac{\partial f}{\partial x^\nu x^\mu} - w^\mu v^\nu \frac{\partial f}{\partial x^\mu x^\nu} \right) \\
&= v^\nu \frac{\partial w^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \frac{\partial f}{\partial x^\mu} \\
&= \left(v^\nu \frac{\partial w^\mu}{\partial x^\nu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \right) \frac{\partial f}{\partial x^\mu} \\
&= [v, w]^\mu \frac{\partial f}{\partial x^\mu}
\end{aligned}$$

(b) **Let Y_1, …, Y_n be as in problem 3(c). Let Y^{1^*},…, Y^{n^*} be the dual basis. Show that the components $(Y^{\gamma^*})_\mu$ of Y^{γ^*} in any coordinate basis satisfy**

$$\frac{\partial (Y^{\gamma^*})_\mu}{\partial x^\nu} - \frac{\partial (Y^{\gamma^*})_\nu}{\partial x^\mu} = \sum_{\alpha,\beta} C^\gamma_{\alpha\beta} (Y^{\alpha^*})_\mu (Y^{\beta^*})_\nu$$

Considering the commutator used in problem 3(c):

$$[Y_\alpha, Y_\beta] = \sum_\gamma C^\gamma_{\alpha\beta} Y_\gamma$$

Act the commutator on a dual vector Y^{γ^*}:

$$[Y_\alpha, Y_\beta] Y^{\gamma^*} = \sum_\gamma C^\gamma_{\alpha\beta} Y_\gamma Y^{\gamma^*}$$

Start with the right hand side.

5.

6.

7.

8. (a) **The metric of flat, three-dimensional Euclidean space is:**

$$ds^2 = dx^2 + dy^2 + dz^2$$

Show that the metric components g_{uv} in spherical polar coordinates r, θ, ϕ defined by:

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ \cos\theta &= \frac{z}{r}, \\ \tan\phi &= \frac{y}{x} \end{aligned}$$

is given by:

$$s^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \phi^2$$

g_{uv} is a tensor of type $(0, 2)$ and therefore transforms as:

$$g_{\mu',\nu'} = g_{\mu,\nu} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}}$$

(see page 22 for the general *tensor transformation law*). The above equation uses Einstein index notation indicating that μ and ν are to to be summed from 1 to 3 and the free indices, μ' and ν', are enumerated through all possible combinations. Therefore the components that need to be calculated are:

$$\begin{array}{ccc} g_{r,r} & g_{r,\theta} & g_{r,\phi} \\ g_{\theta,r} & g_{\theta,\theta} & g_{\theta,\phi} \\ g_{\phi,r} & g_{\phi,\theta} & g_{\phi,\phi} \end{array}$$

Starting with:

$$g_{\mu',\nu'} = \sum_{\mu=1}^{3} \sum_{\nu=1}^{3} g_{\mu,\nu} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}}$$

$$= \sum_{\mu=1}^{3} g_{\mu,1} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{1}}{\partial x^{\nu'}} + g_{\mu,2} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{2}}{\partial x^{\nu'}} + g_{\mu,3} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{3}}{\partial x^{\nu'}}$$

$$\begin{aligned} &= g_{1,1} \frac{\partial x^{1}}{\partial x^{\mu'}} \frac{\partial x^{1}}{\partial x^{\nu'}} + g_{1,2} \frac{\partial x^{1}}{\partial x^{\mu'}} \frac{\partial x^{2}}{\partial x^{\nu'}} + g_{1,3} \frac{\partial x^{1}}{\partial x^{\mu'}} \frac{\partial x^{3}}{\partial x^{\nu'}} \\ &\quad g_{2,1} \frac{\partial x^{2}}{\partial x^{\mu'}} \frac{\partial x^{1}}{\partial x^{\nu'}} + g_{2,2} \frac{\partial x^{2}}{\partial x^{\mu'}} \frac{\partial x^{2}}{\partial x^{\nu'}} + g_{2,3} \frac{\partial x^{2}}{\partial x^{\mu'}} \frac{\partial x^{3}}{\partial x^{\nu'}} \\ &\quad g_{3,1} \frac{\partial x^{2}}{\partial x^{\mu'}} \frac{\partial x^{1}}{\partial x^{\nu'}} + g_{3,2} \frac{\partial x^{2}}{\partial x^{\mu'}} \frac{\partial x^{2}}{\partial x^{\nu'}} + g_{3,3} \frac{\partial x^{2}}{\partial x^{\mu'}} \frac{\partial x^{3}}{\partial x^{\nu'}} \end{aligned}$$

Substituting the notation for the indices in flat, orthonormal Euclidean space:

$$\begin{aligned} &= g_{x,x} \frac{\partial x}{\partial x^{\mu'}} \frac{\partial x}{\partial x^{\nu'}} + g_{x,y} \frac{\partial x}{\partial x^{\mu'}} \frac{\partial y}{\partial x^{\nu'}} + g_{x,z} \frac{\partial x}{\partial x^{\mu'}} \frac{\partial z}{\partial x^{\nu'}} \\ &\quad g_{y,x} \frac{\partial y}{\partial x^{\mu'}} \frac{\partial x}{\partial x^{\nu'}} + g_{y,y} \frac{\partial y}{\partial x^{\mu'}} \frac{\partial y}{\partial x^{\nu'}} + g_{y,z} \frac{\partial y}{\partial x^{\mu'}} \frac{\partial z}{\partial x^{\nu'}} \\ &\quad g_{z,x} \frac{\partial y}{\partial x^{\mu'}} \frac{\partial x}{\partial x^{\nu'}} + g_{z,y} \frac{\partial y}{\partial x^{\mu'}} \frac{\partial y}{\partial x^{\nu'}} + g_{z,z} \frac{\partial y}{\partial x^{\mu'}} \frac{\partial z}{\partial x^{\nu'}} \end{aligned}$$

The off diagonal elements of the Euclidean metric are zero:

$$g_{x,y} = g_{y,x} = g_{x,z} = g_{z,x} = g_{y,z} = g_{z,y} = 0$$

and the diagonal components are one:

$$g_{x,x} = g_{y,y} = g_{z,z} = 1$$

This reduces the above summation from nine expressions to the following three:

$$g_{\mu',\nu'} = \frac{\partial x}{\partial x^{\mu'}} \frac{\partial x}{\partial x^{\nu'}} + \frac{\partial y}{\partial x^{\mu'}} \frac{\partial y}{\partial x^{\nu'}} + \frac{\partial z}{\partial x^{\mu'}} \frac{\partial z}{\partial x^{\nu'}}$$

For indices where $\mu' = \nu'$

$$g_{\mu',\mu'} = \left(\frac{\partial x}{\partial x^{\mu'}} \right)^2 + \left(\frac{\partial y}{\partial x^{\mu'}} \right)^2 + \left(\frac{\partial z}{\partial x^{\mu'}} \right)^2$$

Therefore the six unique components that need to be calculated to find the components of the metric in spherical polar coorindates are:

$$\begin{aligned}
g_{r,r} &= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 \\
g_{r,\theta} = g_{\theta,r} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta} \\
g_{r,\phi} = g_{\phi,r} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \phi} \\
g_{\theta,\theta} &= \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 \\
g_{\theta,\phi} = g_{\phi,\theta} &= \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \phi} \\
g_{\phi,\phi} &= \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2
\end{aligned}$$

To take the above derivatives, find an equation for x, y, z in terms of r, θ, ϕ. Starting by finding x:

$$\begin{aligned}
r &= \sqrt{x^2+y^2+z^2} \rightarrow r^2 = x^2 + z^2 + y^2, \\
\cos\theta &= \frac{z}{r} \rightarrow z = r\cos\theta, \\
\tan\phi &= \frac{y}{x} \rightarrow y = x\tan\phi
\end{aligned}$$

Substituting the second and third equation into the first gives:

$$\begin{aligned}
r^2 &= x^2 + (r\cos\theta)^2 + (x\tan\phi)^2 \\
r^2 &= x^2 + r^2\cos^2\theta + x^2\tan^2\phi \\
r^2 - r^2\cos\theta &= x^2 + x^2\tan^2\phi \\
(1-\cos^2\theta)r^2 &= (1+\tan^2\phi)x^2 \\
r^2\sin^2\theta &= (1+\tan^2\phi)x^2 \\
x &= r\frac{\sin\theta}{\sqrt{1+\tan^2\phi}}
\end{aligned}$$

Therefore the equations for x, y, z in terms of r, θ, ϕ:

$$x = r\frac{\sin\theta}{\sqrt{1+\tan^2\phi}},\ \ y = r\tan\phi\frac{\sin\theta}{\sqrt{1+\tan^2\phi}},\ \ z = r\cos\theta$$

Find all the necessary derivatives:

$$\frac{\partial x}{\partial r} = \frac{\sin\theta}{\sqrt{1+\tan^2\phi}}$$
$$\frac{\partial x}{\partial \theta} = -r\frac{\cos\theta}{\sqrt{1+\tan^2\phi}}$$
$$\frac{\partial x}{\partial \phi} = -r\sin\theta\frac{\tan\phi\sec^2\phi}{(1+\tan^2\phi)^{\frac{3}{2}}}$$
$$\frac{\partial y}{\partial r} = \tan\phi\frac{\sin\theta}{\sqrt{1+\tan^2\phi}}$$
$$\frac{\partial y}{\partial \theta} = -r\tan\phi\frac{\cos\theta}{\sqrt{1+\tan^2\phi}}$$
$$\frac{\partial y}{\partial \phi} = -r\sin\theta\frac{\sec^2\phi}{(1+\tan^2\phi)^{\frac{3}{2}}}$$
$$\frac{\partial z}{\partial r} = \cos\theta$$
$$\frac{\partial z}{\partial \theta} = r\sin\theta$$
$$\frac{\partial z}{\partial \phi} = 0$$

Then compute the components of the metric in spherical polar coordinates:

$$\begin{aligned}
g_{r,r} &= \frac{\sin\theta^2}{1+\tan^2\phi} + \frac{\sin^2\theta}{1+\tan^2\phi}\tan^2\phi + \cos^2\theta \\
&= \frac{\sin\theta^2 + \sin^2\theta\tan^2\phi}{1+\tan^2\phi} + \frac{(1+\tan^2\phi)\cos^2\theta}{1+\tan^2\phi} \\
&= \frac{\sin\theta^2 + \sin^2\theta\tan^2\phi + (1+\tan^2\phi)\cos^2\theta}{1+\tan^2\phi} \\
&= \frac{\sin^2\theta + \cos^2\theta + \sin^2\theta\tan^2\phi + \tan^2\phi\cos^2\theta}{1+\tan^2\phi} \\
&= \frac{1+\tan^2\phi}{1+\tan^2\phi} \\
&= 1
\end{aligned}$$

$$
\begin{aligned}
g_{\theta,\theta} &= r^2 \frac{\cos^2\theta}{1+\tan^2\phi} + r^2 \frac{\cos^2\theta}{1+\tan^2\phi}\tan^2\phi + r^2\sin^2\theta \\
&= r^2 \frac{\cos^2\theta}{1+\tan^2\phi} + r^2 \frac{\cos^2\theta}{1+\tan^2\phi}\tan^2\phi + r^2\sin^2\theta \frac{1+\tan^2\phi}{1+\tan^2\phi} \\
&= r^2 \frac{\cos^2\theta + \cos^2\theta\tan^2\phi + \sin^2\theta + \tan^2\phi\sin^2\theta}{1+\tan^2\phi} \\
&= r^2 \frac{(\cos^2\theta+\sin^2\theta) + (\cos^2\theta+\sin^2\theta)\tan^2\phi}{1+\tan^2\phi} \\
&= r^2 \frac{1+\tan^2\phi}{1+\tan^2\phi} \\
&= r^2
\end{aligned}
$$

$$
\begin{aligned}
g_{\phi,\phi} &= r^2\sin^2\theta\left(\frac{1}{(1+\tan^2\phi)^3\cos^4\phi} + \frac{\sin^2\phi}{(1+\tan^2\phi)^3\cos^6 x}\right) \\
&= r^2\sin^2\theta\left(\frac{\cos^2\phi+\sin^2\phi}{(1+\tan^2\phi)^3\cos^6\phi}\right) \\
&= r^2\sin^2\theta\frac{1}{\left(\frac{\sin^2\phi+\cos^2\phi}{\cos^2\phi}\right)^3\cos^6\phi} \\
&= r^2\sin^2\theta
\end{aligned}
$$

$$
\begin{aligned}
g_{\theta,r} = g_{r,\theta} &= g_{x,x}\frac{\partial x}{\partial r}\frac{\partial x}{\partial\theta} + g_{y,y}\frac{\partial y}{\partial r}\frac{\partial y}{\partial\theta} + g_{z,z}\frac{\partial z}{\partial r}\frac{\partial z}{\partial\theta} \\
&= -r\frac{\sin\theta\cos\theta}{1+\tan^2\phi} - r\frac{\sin\theta\cos\theta}{1+\tan^2\phi}\tan^2\phi + r\sin\theta\cos\theta \\
&= -r\frac{\sin\theta\cos\theta}{1+\tan^2\phi}(1+\tan^2\phi) + r\sin\theta\cos\theta \\
&= -r\sin\theta\cos\theta + r\sin\theta\cos\theta \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
g_{r,\phi} = g_{\phi,r} &= g_{x,x}\frac{\partial x}{\partial r}\frac{\partial x}{\partial\phi} + g_{y,y}\frac{\partial y}{\partial r}\frac{\partial y}{\partial\phi} + g_{z,z}\frac{\partial z}{\partial r}\frac{\partial z}{\partial\phi} \\
&= -r\sin\theta\cos\theta\frac{\tan\phi\sec^2\phi}{(1+\tan^2\phi)^2} + r\cos\theta\sin\theta\frac{\tan\phi\sec^2\phi}{(1+\tan^2\phi)^2} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
g_{\theta,\phi} = g_{\phi,\theta} &= g_{x,x}\frac{\partial x}{\partial\theta}\frac{\partial x}{\partial\phi} + g_{y,y}\frac{\partial y}{\partial\theta}\frac{\partial y}{\partial\phi} + g_{z,z}\frac{\partial z}{\partial\theta}\frac{\partial z}{\partial\phi} \\
&= -r\frac{\cos\theta}{\sqrt{1+\tan^2\phi}}\left(-r\sin\theta\frac{\tan\phi\sec^2\phi}{(1+\tan^2\phi)^{\frac{3}{2}}}\right) - r\tan\phi\frac{\cos\theta}{\sqrt{1+\tan^2\phi}}\left(r\sin\theta\frac{\sec^2\phi}{(1+\tan^2\phi)^{\frac{3}{2}}}\right. \\
&= r^2\sin\theta\cos\theta\frac{\tan\phi\sec^2\phi}{(1+\tan^2\phi)^2} - r^2\sin\theta\cos\theta\frac{\tan\phi\sec^2\phi}{(1+\tan^2\phi)^2} \\
&= 0
\end{aligned}
$$

Therefore the metric components in spherical polar coordinates are:

$$g_{\mu,\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{bmatrix}$$

(b) **The spacetime metric of special relativity is**

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

Find the components, $g_{\mu\nu}$ and $g^{\mu\nu}$, of the metric and inverse metric in "rotating coordinates", defined by

$$\begin{aligned} t' &= t \\ x' &= (x^2 + y^2)^{\frac{1}{2}} \cos(\phi - wt) \\ y' &= (x^2 + y^2)^{\frac{1}{2}} \sin(\phi - wt) \\ z' &= z \end{aligned}$$

where $\tan \phi = \frac{y}{x}$

It is easier differentiate with respect to the primed coordinates so find $g^{\mu\nu}$ first. First writting the primed coordinates in terms of the unprimed:

$$\begin{aligned} t' &= t \\ x' &= (x^2 + y^2)^{\frac{1}{2}} \cos(\tan^{-1} \frac{y}{x} - wt) \\ y' &= (x^2 + y^2)^{\frac{1}{2}} \sin(\tan^{-1} \frac{y}{x} - wt) \\ z' &= z \end{aligned}$$

Find all the necessary derivatives:

$$\frac{\partial t'}{\partial t} = 1$$
$$\frac{\partial t'}{\partial x} = \frac{\partial t'}{\partial y} = \frac{\partial t}{\partial z} = 0$$

$$\frac{\partial x'}{\partial t} = -w\sqrt{x^2+y^2}\sin(\tan^{-1}\frac{y}{x} - wt)$$
$$\frac{\partial x'}{\partial x} = \frac{x\cos(\tan^{-1}\frac{y}{x} - \omega t) + y\sin(\tan^{-1}\frac{y}{x} - \omega t)}{\sqrt{x^2+y^2}}$$
$$\frac{\partial x'}{\partial y} = \frac{-x\sin(\tan^{-1}\frac{y}{x} - \omega t) + y\cos(\tan^{-1}\frac{y}{x} - \omega t)}{\sqrt{x^2+y^2}}$$
$$\frac{\partial x'}{\partial z} = 0$$

$$\frac{\partial y'}{\partial t} = -w\sqrt{x^2+y^2}\cos(\tan^{-1}\frac{y}{x} - wt)$$
$$\frac{\partial y'}{\partial x} = \frac{x\sin(\tan^{-1}\frac{y}{x} - \omega t) - y\cos(\tan^{-1}\frac{y}{x} - \omega t)}{\sqrt{x^2+y^2}}$$
$$\frac{\partial y'}{\partial y} = \frac{x\sin(\tan^{-1}\frac{y}{x} - \omega t) + y\cos(\tan^{-1}\frac{y}{x} - \omega t)}{\sqrt{x^2+y^2}}$$
$$\frac{\partial y'}{\partial z} = 0$$

$$\frac{\partial z'}{\partial t} = \frac{\partial z'}{\partial x} = \frac{\partial z'}{\partial y} = 0$$
$$\frac{\partial z'}{\partial z} = 1$$

$$\left(\frac{\partial x'}{\partial x}\right)^2 = \left(\frac{x\cos(\tan^{-1}\frac{y}{x} - \omega t) + y\sin(\tan^{-1}\frac{y}{x} - \omega t)}{\sqrt{x^2+y^2}}\right)^2 = \frac{(x^2+y^2)\sin^2(wt)}{(x^2+y^2)}$$
$$\left(\frac{\partial y'}{\partial y}\right)^2 = \left(\frac{x\cos(\tan^{-1}\frac{y}{x} - \omega t) + y\sin(\tan^{-1}\frac{y}{x} - \omega t)}{\sqrt{x^2+y^2}}\right)^2 = (x^2+y^2)\sin^2(wt)$$

Chapter 3

Curvature

1. **Let property (5) (the "torsion free" condition) be dropped from the definition of derivative operator ∇_α in section 3.1.**

   (a) **Show that there exists a tensor T^c_{ab} (called the *torsion tensor*) such that for all smooth functions, f, we have $\nabla_a\nabla_b f - \nabla_b\nabla_a f = -T^c_{ab}\nabla_c f$. (Hint: Repeat the derivation of eq [3.1.8], letting $\tilde{\nabla}$ be a torsion free derivative operator.)**

   Following the same line of reasoning as the derivation of eq [3.1.8] consider that given any two derivative operators $\tilde{\nabla}_a$ and ∇_a there exists a tensor field C^c_{ab} such that:

   $$\nabla_a\nabla_b f = \tilde{\nabla}_a\tilde{\nabla}_b f - C^c_{ab}\nabla_c f$$

   Let $\tilde{\nabla}_a$ be torsion-free but not ∇_a. Therefore by the definition of property (5):

   $$\begin{aligned}\nabla_a\nabla_b f - \nabla_b\nabla_a f &= \tilde{\nabla}_a\tilde{\nabla}_b f - C^c_{ab}\nabla_c f - \tilde{\nabla}_b\tilde{\nabla}_a f + C^c_{ba}\nabla_c f\\ &= (\tilde{\nabla}_a\tilde{\nabla}_b - \tilde{\nabla}_b\tilde{\nabla}_a)f - (c^c_{ab} - c^c_{ba})\nabla_c f\end{aligned}$$

   By the definition of torsion free $\tilde{\nabla}_a\tilde{\nabla}_b f = \tilde{\nabla}_b\tilde{\nabla}_a f$ so:

   $$\nabla_a\nabla_b f - \nabla_b\nabla_a f = -(C^c_{ab} - C^c_{ba})\nabla_c f$$

   Therefore there exists a tensor $T^c_{ab} = C^c_{ab} - C^c_{ba}$, called the torsion tensor, such that:

   $$\nabla_a\nabla_b f - \nabla_b\nabla_a f = -T^c_{ab}\nabla_c f$$

(b) **Show that for any smooth vector fields X^a, Y^a we have**

$$T^c_{ab} X^a Y^b = X^a \nabla_a Y^c - Y^a \nabla_a X^c - [X, Y]^c$$