# Misner, Thorne and Wheeler's Gravitation Problems

Anthony Steel

April 19, 2021

# Chapter 1

# Chapter 2

# Chapter 3

# The Electromagnetic Field

1. **Derive equations:**

$$\| F^{\alpha}_{\beta} \| = \begin{Vmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{Vmatrix} \tag{3.1}$$

**and**

$$\| F_{\alpha\beta} \| = \begin{Vmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{Vmatrix} \tag{3.2}$$

**for the components of Faraday by comparing:**

$$dp^{\alpha}/d\tau = eF^{\alpha}_{\beta}u^{\beta} \tag{3.3}$$

**with**

$$\frac{d\mathbf{p}}{d\tau} = \frac{1}{\sqrt{1-\mathbf{v}^2}}\frac{d\mathbf{p}}{dt} = \frac{e}{\sqrt{1-\mathbf{v}^2}}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = e(u^0\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{3.4}$$

$$\frac{dp^0}{d\tau} = \frac{1}{\sqrt{1-\mathbf{v}^2}}\frac{dE}{dt} = \frac{1}{\sqrt{1-\mathbf{v}^2}}e\mathbf{E} \cdot \mathbf{v} = e\mathbf{E} \cdot \mathbf{u} \tag{3.5}$$

**and by using definition:**

$$F_{\alpha\beta} = \eta_{\alpha\gamma}F^{\gamma}_{\beta} \tag{3.6}$$

Consider equation 3.3 for the index $\alpha = 0$:

$$\frac{dp^0}{d\tau} = e[F^0_0 u^0 + F^0_1 u^1 + F^0_2 u^2 + F^0_3 u^3]$$

Equate this with 3.5:

$$e[F^0_0 u^0 + F^0_1 u^1 + F^0_2 u^2 + F^0_3 u^3] = e\mathbf{E} \cdot \mathbf{u} = e[E_1 u^1 + E_2 u^2 + E_3 u^3]$$

It is clear that:

$$F_0^0 = 0$$
$$F_1^0 u^1 = E_1 u^1 \Rightarrow F_1^0 = E_1$$
$$F_2^0 u^2 = E_2 u^2 \Rightarrow F_2^0 = E_2$$
$$F_3^0 u^3 = E_3 u^3 \Rightarrow F_3^0 = E_3$$

Now equating equation 3.4 with the remaining components of equation 3.3:

$$\frac{dp^1}{d\tau} = e[F_0^1 u^0 + F_1^1 u^1 + F_2^1 u^2 + F_3^1 u^3] = e[E_1 u^0 + B_3 u^2 - B_2 u^3]$$

$$\frac{dp^2}{d\tau} = e[F_0^2 u^0 + F_1^2 u^1 + F_2^2 u^2 + F_3^2 u^3] = e[E_2 u^0 + B_1 u^3 - B_3 u^2]$$

$$\frac{dp^3}{d\tau} = e[F_0^3 u^0 + F_1^3 u^1 + F_2^3 u^2 + F_3^3 u^3] = e[E_3 u^0 + B_2 u^1 - B_1 u^2]$$

and equating components as before:

$$F_0^1 u^0 = E_1 u^0 \Rightarrow F_0^1 = E_1$$
$$F_1^1 u^1 = 0 \Rightarrow F_1^1 = 0$$
$$F_2^1 u^2 = B_3 u^2 \Rightarrow F_2^1 = B_3$$
$$F_3^1 u^3 = -B_2 u^3 \Rightarrow F_3^1 = -B_2$$
$$F_0^2 u^0 = E_2 u^0 \Rightarrow F_0^2 = E_2$$
$$F_1^2 u^1 = -B_3 u^1 \Rightarrow F_1^2 = -B_3$$
$$F_2^2 u^2 = 0 \Rightarrow F_2^2 = 0$$
$$F_3^2 u^3 = B_1 u^3 \Rightarrow F_3^2 = B_1$$
$$F_0^3 u^0 = E_3 u^0 \Rightarrow F_0^3 = E_3$$
$$F_1^3 u^1 = B_2 u^1 \Rightarrow F_1^3 = B_2$$
$$F_2^3 u^2 = -B_1 u^2 \Rightarrow F_2^3 = -B_1$$
$$F_3^3 u^3 = 0 \Rightarrow F_3^3 = 0$$

Collecting all of these components in matrix form and relabling indices with the following mapping:

$$\alpha = 1 \to x$$
$$\alpha = 2 \to y$$
$$\alpha = 3 \to z$$

gives equation 3.1:

$$\| F_\beta^\alpha \| = \begin{Vmatrix} F_0^0 & F_1^0 & F_2^0 & F_3^0 \\ F_0^1 & F_1^1 & F_2^1 & F_3^1 \\ F_0^2 & F_1^2 & F_2^2 & F_3^2 \\ F_0^3 & F_1^3 & F_2^3 & F_3^3 \end{Vmatrix} = \begin{Vmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{Vmatrix}$$

Now equation 3.6 can be used to convert the mixed Faraday tensor to the fully covariant one. Remeber that for all components $\alpha \neq \beta$ the Minkowski metric is zero. Therefore the only non-zero components in the sums created by the summation convention in equation 3.6 are:

$$F_{00} = \eta_{00} F^0_0 \Rightarrow F_{00} = -F^0_0$$
$$F_{01} = \eta_{00} F^0_1 \Rightarrow F_{01} = -F^0_1$$
$$F_{02} = \eta_{00} F^0_2 \Rightarrow F_{02} = -F^0_2$$
$$F_{03} = \eta_{00} F^0_3 \Rightarrow F_{03} = -F^0_3$$
$$F_{10} = \eta_{11} F^1_0 \Rightarrow F_{10} = F^1_0$$
$$F_{11} = \eta_{11} F^1_1 \Rightarrow F_{11} = F^1_1$$
$$F_{12} = \eta_{11} F^1_2 \Rightarrow F_{12} = F^1_2$$
$$F_{13} = \eta_{11} F^1_3 \Rightarrow F_{13} = F^1_3$$
$$F_{20} = \eta_{22} F^2_0 \Rightarrow F_{02} = F^2_0$$
$$F_{21} = \eta_{22} F^2_1 \Rightarrow F_{12} = F^2_1$$
$$F_{22} = \eta_{22} F^2_2 \Rightarrow F_{22} = F^2_2$$
$$F_{23} = \eta_{22} F^2_3 \Rightarrow F_{23} = F^3_2$$
$$F_{30} = \eta_{33} F^3_0 \Rightarrow F_{30} = F^3_0$$
$$F_{31} = \eta_{33} F^3_1 \Rightarrow F_{31} = F^3_1$$
$$F_{32} = \eta_{33} F^3_2 \Rightarrow F_{32} = F^3_2$$
$$F_{33} = \eta_{33} F^3_3 \Rightarrow F_{33} = F^3_3$$

Collecting the components into matrix form recovers the fully covariant Faraday tensor:

$$\| F_{\alpha\beta} \| = \begin{Vmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{Vmatrix} \tag{3.7}$$

2. **From the transformation laws for components of vectors and 1-forms, derive the transformation law:**

$$S^{\mu'\nu'}{}_{\lambda'} = S^{\alpha\beta}{}_{\gamma} \Lambda^{\mu'}_{\alpha} \Lambda^{\nu'}_{\beta} \Lambda^{\gamma}_{\lambda'}$$

Consider the tensor **S** of rank $(2,1)$, in geometric notation the transformation between two sets basis vectors and 1-forms reads:

$$\mathbf{S}(\sigma, \rho, \nu) = \mathbf{S}(\sigma', \rho', \nu')$$

In component form this reads:

$$S^{\alpha\beta}{}_{\gamma} \sigma_{\alpha} \rho_{\beta} \nu^{\gamma} = S^{\mu'\nu'}{}_{\lambda'} \sigma_{\mu'} \rho_{\nu'} \nu^{\lambda'} \tag{3.8}$$

Using the Lorentz transformation laws to transform one basis into the other for $\sigma$, $\rho$, $\nu$ gives:

$$\sigma_\alpha = \Lambda_\alpha^{\mu'} \sigma_{\mu'}$$

$$\rho_\beta = \Lambda_\beta^{\nu'} \rho_{\mu'}$$

$$\nu^\beta = \Lambda_{\lambda'}^\gamma \nu^{\lambda'}$$

and substituting these transformations into equation 3.8:

$$S^{\mu'\nu'}{}_{\lambda'}\sigma_{\mu'}\rho_{\nu'}\nu^{\lambda'} = S^{\alpha\beta}{}_\gamma(\Lambda_\alpha^{\mu'}\sigma_{\mu'})(\Lambda_\beta^{\nu'}\rho_{\mu'})(\Lambda_{\lambda'}^\gamma\nu^{\lambda'})$$

$$S^{\mu'\nu'}{}_{\lambda'}\sigma_{\mu'}\rho_{\nu'}\nu^{\lambda'} = S^{\alpha\beta}{}_\gamma\Lambda_\alpha^{\mu'}\Lambda_\beta^{\nu'}\Lambda_{\lambda'}^\gamma\sigma_{\mu'}\rho_{\mu'}\nu^{\lambda'}$$

Equating the components gives the desired transformation law:

$$S^{\mu'\nu'}{}_{\lambda'} = S^{\alpha\beta}{}_\gamma\Lambda_\alpha^{\mu'}\Lambda_\beta^{\nu'}\Lambda_{\lambda'}^\gamma$$

3. **Raising and lowering indices. Derive:**

$$S^\alpha_{\beta\gamma} = \eta_{\beta\mu}S^{\alpha\mu}{}_\gamma \tag{3.9}$$

   **and:**

$$S^{\alpha\mu}{}_\gamma = \eta^{\mu\beta}S^\alpha_{\beta\gamma} \tag{3.10}$$

   **from:**

4.

5.

6.

7. **Maxwell's Equations. Show, by explicit examination of components, that the geometric laws**

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0 \tag{3.11}$$

$$F^{\alpha\beta}{}_{,\beta} = 4\pi J^\alpha \tag{3.12}$$

   **do reduce to Maxwell's equations**

$$\nabla \cdot \mathbf{B} = 0 \tag{3.13}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \tag{3.14}$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho \tag{3.15}$$

$$\frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -4\pi\mathbf{J} \tag{3.16}$$

8.

9.

10. **More differentiation. (a) Justify the formula,**

$$d(u^\mu u_\mu)/d\tau = 2u_\mu(du^\mu/d\tau),$$

**by writing out the summation** $u^\mu u_\mu = \eta_{\mu\nu} u^\mu u^\nu$ **explicitly** Writing out the components explicitly yields:

$$
\begin{aligned}
u^\mu u_\mu &= \eta_{00} u^0 u^0 + \eta_{11} u^1 u^1 + \eta_{22} u^2 u^2 + \eta_{33} u^3 u^3 \\
&= \eta_{00}(u^0)^2 + \eta_{11}(u^1)^2 + \eta_{22}(u^2)^2 + \eta_{33}(u^3)^2 \\
&= -(u^0)^2 + (u^1)^2 + (u^2)^2 + (u^3)^2
\end{aligned}
$$

Taking a total derivative of the above with respect to $\tau$:

$$
\begin{aligned}
\frac{d}{d\tau}(u^\mu u_\mu) &= \frac{d}{d\tau}\left(-(u^0)^2 + (u^1)^2 + (u^2)^2 + (u^3)^2\right) \\
&= -2u^0 \frac{du^0}{d\tau} + 2u^1 \frac{du^1}{d\tau} + 2u^2 \frac{du^2}{d\tau} + 2u^3 \frac{du^3}{d\tau} \\
&= 2\left[-u^0 \frac{du^0}{d\tau} + u^1 \frac{du^1}{d\tau} + u^2 \frac{du^2}{d\tau} + u^3 \frac{du^3}{d\tau}\right] \\
&= 2\eta_{\mu\nu} u^\mu \frac{du^\nu}{d\tau} \\
&= 2u_\mu \frac{du^\nu}{d\tau}
\end{aligned}
$$

Therefore the formula is justified.

**b) Let $\delta$ indicate a variation or small change, and justify the formula:**

$$\delta(F_{\alpha\beta} F^{\alpha\beta}) = 2F_{\alpha\beta} \delta F^{\alpha\beta}$$

The variation will obey the product rule as follows and remembering that the variation of the metric components would be zero:

$$
\begin{aligned}
\delta(F_{\alpha\beta} F^{\alpha\beta}) &= (\delta F_{\alpha\beta}) F^{\alpha\beta} + F_{\alpha\beta}(\delta F^{\alpha\beta}) \\
&= (\delta(\eta_{\alpha\gamma} \eta_{\beta\nu} F^{\alpha\beta}) F^{\gamma\nu}) + F_{\alpha\beta}(\delta F^{\alpha\beta}) \\
&= (\delta(\eta_{\alpha\gamma}) \eta_{\beta\nu} F^{\alpha\beta} + \eta_{\alpha\gamma} \delta(\eta_{\beta\nu}) F^{\alpha\beta} + \eta_{\alpha\gamma} \eta_{\beta\nu} \delta F^{\alpha\beta}) F^{\gamma\nu}) + F_{\alpha\beta}(\delta F^{\alpha\beta}) \\
&= \eta_{\alpha\gamma} \eta_{\beta\nu} \delta F^{\alpha\beta} F^{\gamma\nu} + F_{\alpha\beta}(\delta F^{\alpha\beta}) \\
&= \delta F^{\alpha\beta}(\eta_{\alpha\gamma} \eta_{\beta\nu} F^{\gamma\nu}) + F_{\alpha\beta}(\delta F^{\alpha\beta}) \\
&= \delta F^{\alpha\beta} F_{\alpha\beta} + F_{\alpha\beta} \delta F^{\alpha\beta} \\
&= 2F_{\alpha\beta} \delta F^{\alpha\beta}
\end{aligned}
$$

Therefore the formula is justified.

**c) Compute** $(F_{\alpha\beta}F^{\alpha\beta})_{,\mu} =?$

$$
\begin{aligned}
(F_{\alpha\beta}F^{\alpha\beta})_{,\mu} &= F_{\alpha\beta,\mu}F^{\alpha\beta} + F_{\alpha\beta}F^{\alpha\beta}{}_{,\mu} \\
&= (\eta_{\alpha\gamma}\eta_{\beta\nu}F^{\alpha\beta})_{,\mu}F^{\gamma\nu} + F_{\alpha\beta}F^{\alpha\beta}{}_{,\mu} \\
&= F^{\alpha\beta}{}_{,\mu}F_{\alpha\beta} + F_{\alpha\beta}F^{\alpha\beta}{}_{,\mu} \\
&= 2F_{\alpha\beta}F^{\alpha\beta}{}_{,\mu}
\end{aligned}
$$