Misner, Thorne and Wheeler's Gravitation Problems

Anthony Steel

April 20, 2021

Chapter 1

Chapter 2

Chapter 3

The Electromagnetic Field

1. **Derive equations:**

$$\| F^\alpha_\beta \| = \begin{Vmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{Vmatrix} \tag{3.1}$$

and

$$\| F_{\alpha\beta} \| = \begin{Vmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{Vmatrix} \tag{3.2}$$

for the components of Faraday by comparing:

$$dp^\alpha / d\tau = eF^\alpha_\beta u^\beta \tag{3.3}$$

with

$$\frac{d\mathbf{p}}{d\tau} = \frac{1}{\sqrt{1-\mathbf{v}^2}} \frac{d\mathbf{p}}{dt} = \frac{e}{\sqrt{1-\mathbf{v}^2}}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = e(u^0\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{3.4}$$

$$\frac{dp^0}{d\tau} = \frac{1}{\sqrt{1-\mathbf{v}^2}} \frac{dE}{dt} = \frac{1}{\sqrt{1-\mathbf{v}^2}} e\mathbf{E} \cdot \mathbf{v} = e\mathbf{E} \cdot \mathbf{u} \tag{3.5}$$

and by using definition:

$$F_{\alpha\beta} = \eta_{\alpha\gamma} F^\gamma_\beta \tag{3.6}$$

Consider equation 3.3 for the index $\alpha = 0$:

$$\frac{dp^0}{d\tau} = e[F^0_0 u^0 + F^0_1 u^1 + F^0_2 u^2 + F^0_3 u^3]$$

Equate this with 3.5:

$$e[F^0_0 u^0 + F^0_1 u^1 + F^0_2 u^2 + F^0_3 u^3] = e\mathbf{E} \cdot \mathbf{u} = e[E_1 u^1 + E_2 u^2 + E_3 u^3]$$

It is clear that:

$$\begin{aligned}
F^0_0 &= 0 \\
F^0_1 u^1 &= E_1 u^1 \Rightarrow F^0_1 = E_1 \\
F^0_2 u^2 &= E_2 u^2 \Rightarrow F^0_2 = E_2 \\
F^0_3 u^3 &= E_3 u^3 \Rightarrow F^0_3 = E_3
\end{aligned}$$

Now equating equation 3.4 with the remaining components of equation 3.3:

$$\begin{aligned}
\frac{dp^1}{d\tau} &= e[F^1_0 u^0 + F^1_1 u^1 + F^1_2 u^2 + F^1_3 u^3] = e[E_1 u^0 + B_3 u^2 - B_2 u^3] \\
\frac{dp^2}{d\tau} &= e[F^2_0 u^0 + F^2_1 u^1 + F^2_2 u^2 + F^2_3 u^3] = e[E_2 u^0 + B_1 u^3 - B_3 u^2] \\
\frac{dp^3}{d\tau} &= e[F^3_0 u^0 + F^3_1 u^1 + F^3_2 u^2 + F^3_3 u^3] = e[E_3 u^0 + B_2 u^1 - B_1 u^2]
\end{aligned}$$

and equating components as before:

$$\begin{aligned}
F^1_0 u^0 &= E_1 u^0 \Rightarrow F^1_0 = E_1 \\
F^1_1 u^1 &= 0 \Rightarrow F^1_1 = 0 \\
F^1_2 u^2 &= B_3 u^2 \Rightarrow F^1_2 = B_3 \\
F^1_3 u^3 &= -B_2 u^3 \Rightarrow F^1_3 = -B_2 \\
F^2_0 u^0 &= E_2 u^0 \Rightarrow F^2_0 = E_2 \\
F^2_1 u^1 &= -B_3 u^1 \Rightarrow F^2_1 = -B_3 \\
F^2_2 u^2 &= 0 \Rightarrow F^2_2 = 0 \\
F^2_3 u^3 &= B_1 u^3 \Rightarrow F^2_3 = B_1 \\
F^3_0 u^0 &= E_3 u^0 \Rightarrow F^3_0 = E_3 \\
F^3_1 u^1 &= B_2 u^1 \Rightarrow F^3_1 = B_2 \\
F^3_2 u^2 &= -B_1 u^2 \Rightarrow F^3_2 = -B_1 \\
F^3_3 u^3 &= 0 \Rightarrow F^3_3 = 0
\end{aligned}$$

Collecting all of these components in matrix form and relabling indices with the following mapping:

$$\begin{aligned}
\alpha &= 1 \rightarrow x \\
\alpha &= 2 \rightarrow y \\
\alpha &= 3 \rightarrow z
\end{aligned}$$

gives equation 3.1:

$$|| \, F^\alpha_\beta \, || = \begin{Vmatrix} F^0_0 & F^0_1 & F^0_2 & F^0_3 \\ F^1_0 & F^1_1 & F^1_2 & F^1_3 \\ F^2_0 & F^2_1 & F^2_2 & F^2_3 \\ F^3_0 & F^3_1 & F^3_2 & F^3_3 \end{Vmatrix} = \begin{Vmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{Vmatrix}$$

Now equation 3.6 can be used to convert the mixed Faraday tensor to the fully covariant one. Remeber that for all components $\alpha \neq \beta$ the Minkowski metric is zero. Therefore the only non-zero components in the sums created by the summation convention in equation 3.6 are:

$$
\begin{aligned}
F_{00} = \eta_{00} F^0_0 &\Rightarrow F_{00} = -F^0_0 \\
F_{01} = \eta_{00} F^0_1 &\Rightarrow F_{01} = -F^0_1 \\
F_{02} = \eta_{00} F^0_2 &\Rightarrow F_{02} = -F^0_2 \\
F_{03} = \eta_{00} F^0_3 &\Rightarrow F_{03} = -F^0_3 \\
F_{10} = \eta_{11} F^1_0 &\Rightarrow F_{10} = F^1_0 \\
F_{11} = \eta_{11} F^1_1 &\Rightarrow F_{11} = F^1_1 \\
F_{12} = \eta_{11} F^1_2 &\Rightarrow F_{12} = F^1_2 \\
F_{13} = \eta_{11} F^1_3 &\Rightarrow F_{13} = F^1_3 \\
F_{20} = \eta_{22} F^2_0 &\Rightarrow F_{02} = F^2_0 \\
F_{21} = \eta_{22} F^2_1 &\Rightarrow F_{12} = F^2_1 \\
F_{22} = \eta_{22} F^2_2 &\Rightarrow F_{22} = F^2_2 \\
F_{23} = \eta_{22} F^2_3 &\Rightarrow F_{23} = F^3_2 \\
F_{30} = \eta_{33} F^3_0 &\Rightarrow F_{30} = F^3_0 \\
F_{31} = \eta_{33} F^3_1 &\Rightarrow F_{31} = F^3_1 \\
F_{32} = \eta_{33} F^3_2 &\Rightarrow F_{32} = F^3_2 \\
F_{33} = \eta_{33} F^3_3 &\Rightarrow F_{33} = F^3_3
\end{aligned}
$$

Collecting the components into matrix form recovers the fully covariant Faraday tensor:

$$
|| \; F_{\alpha\beta} \; ||= \begin{Vmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{Vmatrix} \tag{3.7}
$$

2. **From the transformation laws for components of vectors and 1-forms, derive the transformation law:**

$$
S^{\mu'\nu'}{}_{\lambda'} = S^{\alpha\beta}{}_{\gamma} \Lambda^{\mu'}_{\alpha} \Lambda^{\nu'}_{\beta} \Lambda^{\gamma}_{\lambda'}
$$

Consider the tensor $\mathbf{S}$ of rank $(2, 1)$, in geometric notation the transformation between two sets basis vectors and 1-forms reads:

$$
\mathbf{S}(\sigma, \rho, \nu) = \mathbf{S}(\sigma', \rho', \nu')
$$

In component form this reads:

$$
S^{\alpha\beta}{}_{\gamma} \sigma_\alpha \rho_\beta \nu^\gamma = S^{\mu'\nu'}{}_{\lambda'} \sigma_{\mu'} \rho_{\nu'} \nu^{\lambda'} \tag{3.8}
$$

Using the Lorentz transformation laws to transform one basis into the other for σ, ρ, ν gives:

$$\sigma_\alpha = \Lambda^{\mu'}_{\alpha} \sigma_{\mu'}$$
$$\rho_\beta = \Lambda^{\nu'}_{\beta} \rho_{\mu'}$$
$$\nu^\beta = \Lambda^{\gamma}_{\lambda'} \nu^{\lambda'}$$

and substituting these transformations into equation 3.8:

$$S^{\mu'\nu'}{}_{\lambda'} \sigma_{\mu'} \rho_{\nu'} \nu^{\lambda'} = S^{\alpha\beta}{}_{\gamma} (\Lambda^{\mu'}_{\alpha} \sigma_{\mu'})(\Lambda^{\nu'}_{\beta} \rho_{\mu'})(\Lambda^{\gamma}_{\lambda'} \nu^{\lambda'})$$

$$S^{\mu'\nu'}{}_{\lambda'} \sigma_{\mu'} \rho_{\nu'} \nu^{\lambda'} = S^{\alpha\beta}{}_{\gamma} \Lambda^{\mu'}_{\alpha} \Lambda^{\nu'}_{\beta} \Lambda^{\gamma}_{\lambda'} \sigma_{\mu'} \rho_{\mu'} \nu^{\lambda'}$$

Equating the components gives the desired transformation law:

$$S^{\mu'\nu'}{}_{\lambda'} = S^{\alpha\beta}{}_{\gamma} \Lambda^{\mu'}_{\alpha} \Lambda^{\nu'}_{\beta} \Lambda^{\gamma}_{\lambda'}$$

3. **Raising and lowering indices. Derive:**

$$S^{\alpha}{}_{\beta\gamma} = \eta_{\beta\mu} S^{\alpha\mu}{}_{\gamma} \tag{3.9}$$

and:

$$S^{\alpha\mu}{}_{\gamma} = \eta^{\mu\beta} S^{\alpha}{}_{\beta\gamma} \tag{3.10}$$

from:

4.

5.

6.

7. **Maxwell's Equations. Show, by explicit examination of components, that the geometric laws**

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0 \tag{3.11}$$

$$F^{\alpha\beta}{}_{,\beta} = 4\pi J^\alpha \tag{3.12}$$

do reduce to Maxwell's equations

$$\nabla \cdot \mathbf{B} = 0 \tag{3.13}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \tag{3.14}$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho \tag{3.15}$$

$$\frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -4\pi \mathbf{J} \tag{3.16}$$

8.

9.

10. **More differentiation. (a) Justify the formula,**

$$d(u^\mu u_\mu)/d\tau = 2u_\mu(du^\mu/d\tau),$$

by writing out the summation $u^\mu u_\mu = \eta_{\mu\nu}u^\mu u^\nu$ **explicitly** Writing out the components explicitly yields:

$$\begin{aligned} u^\mu u_\mu &= \eta_{00}u^0u^0 + \eta_{11}u^1u^1 + \eta_{22}u^2u^2 + \eta_{33}u^3u^3 \\ &= \eta_{00}(u^0)^2 + \eta_{11}(u^1)^2 + \eta_{22}(u^2)^2 + \eta_{33}(u^3)^2 \\ &= -(u^0)^2 + (u^1)^2 + (u^2)^2 + (u^3)^2 \end{aligned}$$

Taking a total derivative of the above with respect to τ:

$$\begin{aligned} \frac{d}{d\tau}(u^\mu u_\mu) &= \frac{d}{d\tau}(-(u^0)^2 + (u^1)^2 + (u^2)^2 + (u^3)^2) \\ &= -2u^0\frac{du^0}{d\tau} + 2u^1\frac{du^1}{d\tau} + 2u^2\frac{du^2}{d\tau} + 2u^3\frac{du^3}{d\tau} \\ &= 2[-u^0\frac{du^0}{d\tau} + u^1\frac{du^1}{d\tau} + u^2\frac{du^2}{d\tau} + u^3\frac{du^3}{d\tau}] \\ &= 2\eta_{\mu\nu}u^\mu\frac{du^\nu}{d\tau} \\ &= 2u_\mu\frac{du^\nu}{d\tau} \end{aligned}$$

Therefore the formula is justified.

b) Let δ indicate a variation or small change, and justify the formula:

$$\delta(F_{\alpha\beta}F^{\alpha\beta}) = 2F_{\alpha\beta}\delta F^{\alpha\beta}$$

The variation will obey the product rule as follows and rembering that the variation of the metric components would be zero:

$$\begin{aligned} \delta(F_{\alpha\beta}F^{\alpha\beta}) &= (\delta F_{\alpha\beta})F^{\alpha\beta} + F_{\alpha\beta}(\delta F^{\alpha\beta}) \\ &= (\delta(\eta_{\alpha\gamma}\eta_{\beta\nu}F^{\alpha\beta})F^{\gamma\nu}) + F_{\alpha\beta}(\delta F^{\alpha\beta}) \\ &= (\delta(\eta_{\alpha\gamma})\eta_{\beta\nu}F^{\alpha\beta} + \eta_{\alpha\gamma}\delta(\eta_{\beta\nu})F^{\alpha\beta} + \eta_{\alpha\gamma}\eta_{\beta\nu}\delta F^{\alpha\beta})F^{\gamma\nu}) + F_{\alpha\beta}(\delta F^{\alpha\beta}) \\ &= \eta_{\alpha\gamma}\eta_{\beta\nu}\delta F^{\alpha\beta}F^{\gamma\nu} + F_{\alpha\beta}(\delta F^{\alpha\beta}) \\ &= \delta F^{\alpha\beta}(\eta_{\alpha\gamma}\eta_{\beta\nu}F^{\gamma\nu}) + F_{\alpha\beta}(\delta F^{\alpha\beta}) \\ &= \delta F^{\alpha\beta}F_{\alpha\beta} + F_{\alpha\beta}\delta F^{\alpha\beta} \\ &= 2F_{\alpha\beta}\delta F^{\alpha\beta} \end{aligned}$$

Therefore the formula is justified.

c) Compute $(F_{\alpha\beta}F^{\alpha\beta})_{,\mu} =?$

$$\begin{aligned}
(F_{\alpha\beta}F^{\alpha\beta})_{,\mu} &= F_{\alpha\beta,\mu}F^{\alpha\beta} + F_{\alpha\beta}F^{\alpha\beta}{}_{,\mu} \\
&= (\eta_{\alpha\gamma}\eta_{\beta\nu}F^{\alpha\beta})_{,\mu}F^{\gamma\nu} + F_{\alpha\beta}F^{\alpha\beta}{}_{,\mu} \\
&= F^{\alpha\beta}{}_{,\mu}F_{\alpha\beta} + F_{\alpha\beta}F^{\alpha\beta}{}_{,\mu} \\
&= 2F_{\alpha\beta}F^{\alpha\beta}{}_{,\mu}
\end{aligned}$$

Chapter 4

Electromagnetism and Differential Forms

Chapter 5

Stress-Energy Tensor and Conservatoin Laws

Chapter 6

Accelerated Observers

1. **A TRIP TO THE GALACTIC NUCLEUS: Compute the proper time required for the occupants of a rocket schip to travel the** $\approx 30,000$ **light-years to get from the Earth to the center of the Galaxy. Asusme that they maintain an acceleration of one earth gravity** $(10^3$ **cm/sec**$^2)$ **for half the trip, and then decelerate at one earth gravity for the remaining of the half.**

   Qualitatively, the worldline of the traveller is pictured in the Figure 6.1. The travellers worldine is composed of two arcs of hyperbola AC and CB.

$$t = g^{-1} \sinh g\tau$$
$$x = g^{-1} \cosh g\tau$$
$$\tau = g^{-1} \sinh^{-1} gt$$
$$\begin{aligned} x &= g^{-1} \cosh(\sinh^{-1} gt) \\ &= g^{-1} \sqrt{1 + (gt)^2} \end{aligned}$$
$$dx = \frac{gt}{\sqrt{1 + (gt)^2}} dt$$

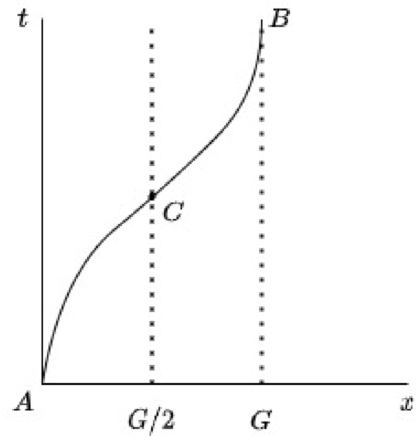

Figure 6.1: The worldline of the traveller is composed of two arcs of hyperbola AC and CB. G indicates the distance to the galactic center.

$$
\begin{aligned}
dt' &= \sqrt{dt^2 - dx^2} \\
&= \sqrt{dt^2 - \Big(\frac{gt}{\sqrt{1+(gt)^2}}\Big)^2 dt^2} \\
&= \sqrt{1 - \frac{(gt)^2}{1+(gt)^2}}dt \\
&= \sqrt{\frac{1+(gt)^2-(gt)^2}{1+(gt)^2}}dt \\
&= \frac{1}{\sqrt{1+(gt)^2}}dt
\end{aligned}
$$

Using an integral table:

$$t' = \int \frac{dt}{\sqrt{1+(gt)^2}} = \ln(gt + \sqrt{1+(gt)^2}) = \sinh^{-1}(gt)$$