Chapter 2 Problems

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1. **A tube of mass M and length l is free to swing around a pivot at one end. A mass m is positioned inside the (frictionless) tube at this end. The tube is held horizontal and then released. Let θ be the angle of the tube with respect to the horizontal, and let x be the distance the mass has traveled along the tube. Find the Euler-Lagrange equations for θ and x, and then write them in terms of θ and $\nu = x/l$ (the fraction of the distance along the tube). These equations can only be solved numerically, and you must pick a numerical value for the ratio $r = m/M$ in order to do this. Write a prigram that produces the value of ν when the tube is vertical** $(\theta = \pi/2)$

0.1 Derivation of Equations of Motion

The total kinetic energy of the system will be the horizontal and vertical translational kinetic energies of the mass m, and the rotational kinetic energy of the tube.

Starting with the mass, consider its position x' and y':

$$x'(t) = x(t)\sin\theta(t)$$
$$y'(t) = x(t)\cos\theta(t)$$

where x is the position the mass has fallen down the tube. Differentiating once and removing the explicit time dependence for brevity yields:

$$\dot{x}' = \dot{x}\sin\theta + x\cos\theta\dot{\theta}$$
$$\dot{y}' = \dot{x}\cos\theta - x\sin\theta\dot{\theta}$$

Taking the squares of each:

$$(\dot{x}')^2 = \dot{x}^2\sin^2\theta + x^2\cos^2\theta\dot{\theta}^2 + 2\sin\theta\cos\theta\dot{x}\dot{\theta}$$
$$(\dot{y}')^2 = \dot{x}^2\cos^2\theta + x^2\sin^2\theta\dot{\theta}^2 - 2\sin\theta\cos\theta\dot{x}\dot{\theta}$$

and adding:

$$\begin{aligned}(\dot{x}')^2 + (\dot{y}')^2 &= \dot{x}^2\sin^2\theta + \dot{x}^2\cos^2\theta + x^2\cos^2\theta\dot{\theta}^2 + x^2\sin^2\theta\dot{\theta}^2 + 2\sin\theta\cos\theta\dot{x}\dot{\theta} - 2\sin\theta\cos\theta\dot{x}\dot{\theta} \\ &= \dot{x}^2 + x^2\dot{\theta}^2\end{aligned}$$

corresponding to radial and tangential kinetic energies respectively. Therefore:

$$T_m = \frac{1}{2}m(\dot{x}^2 + x^2\dot{\theta}^2)$$

Now the moment of inertia for a tube rotating about its end is the same as a rod rotating about its end, and is given by:

$$I_M = \frac{1}{3}Ml^2$$

Therefore the total kinetic energies of the two masses are:

$$T = \frac{1}{2}m(\dot{x}^2 + x^2\dot{\theta}^2) + \frac{1}{6}Ml^2\dot{\theta}^2$$

The potential energy for m is simply:

$$V_m = -mgx\cos\theta$$

while the potnetial energy for m is the vertical component of the position of the center mass:

$$V_M = -\frac{Mgl}{2}\cos\theta$$

The Lagrangian $\mathcal{L}$ is:

$$\mathcal{L} = T - V = \frac{1}{2}m(\dot{x}^2 + x^2\dot{\theta}^2) + \frac{1}{6}Ml^2\dot{\theta}^2 + mgx\cos\theta + \frac{Mgl}{2}\cos\theta$$

Finding the Euler-Lagrange equations for x:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} = m\ddot{x}$$

$$\frac{\partial\mathcal{L}}{\partial x} = mx\dot{\theta}^2 + mg\cos\theta$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} = \frac{\partial\mathcal{L}}{\partial x} \Rightarrow \ddot{x} = x\dot{\theta}^2 + g\cos\theta$$

and θ:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = m\frac{d}{dt}(x^2\dot{\theta}) + \frac{1}{3}Ml^2\ddot{\theta}$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = -g(mx + \frac{Ml}{2})\sin\theta$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{\partial\mathcal{L}}{\partial\theta} \Rightarrow m\frac{d}{dt}(x^2\dot{\theta}) + \frac{1}{3}Ml^2\ddot{\theta} = -g(mx + \frac{Ml}{2})\sin\theta$$

Therefore the Euler Lagrange equations are:

$$m\frac{d}{dt}(x^2\dot{\theta}) + \frac{1}{3}Ml^2\ddot{\theta} = -g(mx + \frac{Ml}{2})\sin\theta$$

$$\ddot{x} = x\dot{\theta}^2 + g\cos\theta$$

Reparameterizing in terms of η and $r = \frac{m}{M}$:

$$r\frac{d}{dt}(\eta^2\dot{\theta}) + \frac{1}{3}\ddot{\theta} = -\frac{g}{l}(\frac{rx}{l} + \frac{1}{2})\sin\theta \tag{1}$$

$$\ddot{\eta} = \eta\dot{\theta}^2 + \frac{g}{l}\cos\theta \tag{2}$$

are the Euler-Lagrange equations.

0.2 Dimensional analysis

η is dimensionless. The dimensional quantities in the problem are m, M, l, g. To get a dimensonless number, the mass terms would have to cancel in a ratio, but you could not cancel l with g. You need to introduce another quantity which has dimensions of time. Therefore we could also introduce the frequency ω and write:

$$\begin{aligned}[\eta] &\propto [m]^a[M]^b[l]^c[g]^d[\omega]^e \\ &\propto (\text{mass})^a(\text{mass})^b(\text{length})^c(\frac{\text{length}}{\text{time}^2})^d(\frac{1}{\text{time}})^e\end{aligned}$$

Giving the equations:

$$\begin{aligned}a + b = 0 &\Rightarrow a = -b \\ c + d = 0 &\Rightarrow c = -d \\ -2d - e = 0 &\Rightarrow e = -2d\end{aligned}$$

and implying:

$$\eta \propto \frac{m}{M}\frac{l}{g}\omega^2$$

However, the frequency of an oscillator is:

$$\omega^2 \propto \frac{g}{l}f^2(\theta)$$

So dimensional analysis would imply that:

$$\eta \propto \frac{m}{M}$$

meaning that it should not depend on the length of the tube.

0.3 Asymptotic Behaviour of Solutions

Does the mass exit the tube? If $\eta \propto m/M$ is correct, then the mass would exit the tube if $m > M$.

In this case, the mass exits the bottom of the tube and the system essentially decouples. After that, the mass is simply in free fall in a gravitational potential and the tube executes harmonic motion.

If the $\eta < 1$ when the tube is vertical, the mass never exits the tube.

0.4 Approximations

Though the equations of motion cannot be solved exactly, it would be nice to determine whether $\eta(t_{\text{vertical}})$ depends on l in some approximation. To determine this we need an equation for η by solving the η equation of motion.

The equations are coupled (i.e. θ appears in the equation of motion for η and η appears in the equation of motion for θ) so a useful approximation would be one that decouples at least one of them.

0.4.1 Small mass approximation $m \ll M$

$m \ll M$ is equivalent to the condition $r \ll 1$. Let us assume that r is negligibly small. The equations of motion then read:

$$\ddot{\theta} = -\frac{3g}{2l}\sin\theta \tag{3}$$

$$\ddot{\eta} = \eta\dot{\theta}^2 + \frac{g}{l}\cos\theta \tag{4}$$

By our definition of θ the vertical position is at $\theta = 0$ so small oscillations about the vertical are given by:

$$\theta(t) = A\cos(\omega t + \phi)$$

where $\omega^2 = 3g/2l$.

Consider the initial conditions:

$$t_{\text{vertical}} = \frac{T}{2} = \frac{1}{2}\frac{2\pi}{\omega} = \frac{\pi}{\omega} = \frac{2\pi}{3}\sqrt{\frac{l}{g}}$$

$$\theta(t_{\text{vertical}}) = 0$$

Then:

$$\begin{aligned}
0 &= \cos(\omega(\frac{\pi}{\omega}) + \phi) \\
&= \cos(\pi + \phi) \\
&= \cos(\pi)\cos(\phi) - \sin(\pi)\sin(\phi) \\
&= \cos(\phi)
\end{aligned}$$

therefore $\phi = \pi/2$ and the solution for θ can be rewritten without a phase shift as:

$$\theta(t) = A\sin(\omega t)$$

Taking a time derivative:

$$\dot{\theta}(t) = A\omega\cos(\omega t)$$

$$\dot{\theta}^2(t) = A^2\omega^2\cos^2(\omega t)$$

We are interested in the time $t = t_{\text{vertical}} + \delta$ where δ is small. In this small region of time:

$$\theta(\delta) = A\omega\delta$$

$$\dot{\theta}(\delta) = A\omega$$

and:

$$\ddot{\eta} = A^2\omega^2\eta + \frac{g}{l}$$

$a = -A^2\omega^2$, $C_0 = \frac{g}{l}$.

$$\eta(t) = (\frac{g}{-}$$

0.4.2 Taylor Series Expansion

Taylor expand η and x as:

$$\eta(t) = \sum_{n=0}^{\infty} a_n(t - t_0)^n$$

and:

$$x(t) = \sum_{n=0}^{\infty} b_n(t - t_0)^n$$

Therefore: