Chapter 5 - Risk Budgeting Approach

Or called the '**risk parity**' approach. In contrast to the weight budgeting or the performance budgeting, the risk budgeting apporach requires less discretionary inputs, mainly the risk profiles of the assets and the planned risk budgets allocated to those assets.

Given $\mathbb{R}(w)$ as a risk measure for a portfolio with the weight vector $w = [w_1, ..., w_n]'$, let's define the marginal contribution to risk from asset i ($\mathbb{MCR}_i$) and the contribution to risk from asset i ($\mathbb{CR}_i$) as follows:

$$\mathbb{MCR}_i = \frac{\partial \mathbb{R}(w)}{\partial w_i},$$

$$\mathbb{CR}_i = w_i \cdot \mathbb{MCR}_i = w_i \frac{\partial \mathbb{R}(w)}{\partial w_i}.$$

The risk measure $\mathbb{R}(w)$ satisfies the Euler decomposition if: $\mathbb{R}(w) = \sum_{i=1}^{n} \mathbb{CR}_i$.

Note: the Euler decomposition holds for any risk measure that is continuous, differentiable and homogeneous of degree 1, i.e., $\mathbb{R}(cw) = c\mathbb{R}(w)$ for all positive scalar constant c.

Traditional risk measures such as volatility, VaR, the Expected Shortfall all satisfy the Euler decomposition (but the variance doesn't). <u>In this chapter, we assume volatility as our default risk measure.</u>

Section A: Risk Contribution of Portfolio Assets

Assuming portfolio returns are Gaussian distributed, let's consider the case of two assets:

$$\sigma(w) = \sqrt{w_1^2\sigma_{11} + 2w_1w_2\sigma_{12} + w_2^2\sigma_{22}},$$

where σ_{ij} is the covariance between asset i and asset j.

We can compute the marginal risk of the two assets as follows:

$$\mathbb{MCR}_1 = \frac{\partial \sigma(w)}{\partial w_1} = \frac{w_1\sigma_{11} + w_2\sigma_{12}}{\sigma(w)}, \text{ and } \mathbb{MCR}_2 = \frac{\partial \sigma(w)}{\partial w_2} = \frac{w_2\sigma_{22} + w_1\sigma_{12}}{\sigma(w)}.$$

And thus we can calculate their risk contributions:

$$\mathbb{CR}_1 = w_1 \cdot \mathbb{MCR}_1 = w_1 \cdot \frac{\partial \sigma(w)}{\partial w_1} = \frac{w_1^2\sigma_{11} + w_1w_2\sigma_{12}}{\sigma(w)}, \text{ and } \mathbb{CR}_2 = w_2 \cdot \mathbb{MCR}_2 = w_2 \cdot \frac{\partial \sigma(w)}{\partial w_2} = \frac{w_2^2\sigma_{22} + w_1w_2\sigma_{12}}{\sigma(w)}.$$

Therefore, we can prove the risk measure of volatility satisfies the Euler decomposition under the condition of two assets:

$$\mathbb{CR}_1 + \mathbb{CR}_2 = \frac{w_1^2\sigma_{11} + 2w_1w_2\sigma_{12} + w_2^2\sigma_{22}}{\sigma(w)} = \frac{\sigma(w)^2}{\sigma(w)} = \sigma(w).$$

We can easily expand to the general case of n assets, and have the following representations:

$$\mathbb{MCR}_i = \frac{\sum_{j=1}^{n} w_j\sigma_{ij}}{\sigma(w)}, \text{ and } \mathbb{CR}_i = w_i \cdot \mathbb{MCR}_i.$$

And the Euler decomposition holds for the volatility measure under the condition of n assets:

$$\sum_{i=1}^{n} \mathbb{CR}_i = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} w_iw_j\sigma_{ij}}{\sigma(w)} = \frac{\sigma(w)^2}{\sigma(w)} = \sigma(w).$$

Let's denote the covariance matrix among the n assets as Σ, the marginal risk vector as $\mathbb{MCR} = [\mathbb{MCR}_1, ..., \mathbb{MCR}_n]'$, and the risk contribution vector as $\mathbb{CR} = [\mathbb{CR}_1, ..., \mathbb{CR}_n]'$, then *under the risk measure of volatility* we have:

$$\sigma(w) = \sqrt{w'\Sigma w},$$

$$\mathbb{MCR} = \frac{\Sigma w}{\sigma(w)},$$

$$\mathbb{CR} = \frac{\mathrm{diag}(w)\Sigma w}{\sigma(w)},$$

where $\mathrm{diag}(w)$ is a $n \times n$ diagnoal matrix whose i-th element is w_i. (Show the summation of all elements on the vector $\mathbb{CR}$ equals $\sigma(w)$).

```
%%%%%%%%% An Example to Calculate Asset Class Risk Contributions %%%%%%%%%%

%   load monthly return information for asset classes in the CAPM model
[blah,blah,rawraw]=xlsread('CAPM_AssetClasses.xls','MonthlyReturns');
ac_name=rawraw(1,2:end);
ac_mret=cell2mat(rawraw(3:242,2:end));

%   the number of months
num_month=size(ac_mret,1);

%   the asset classes loaded
ac_name=ac_name';

%   number of asset classes
num_ac=length(ac_name);

%   load the market portfolio
%   note: in the CAPM model, the market portfolio is the equilibrium
%   portfolio
w_mkt=xlsread('CAPM_AssetClasses.xls','MarketPortfolio','B3:M3');
w_mkt=w_mkt';

%   calculate the historical covariance matrix
Sigma=nancov(ac_mret);

%   the volatility of market portfolio
sigma_mkt=sqrt(w_mkt'*Sigma*w_mkt);

%   calculate marginal risk and risk contribution vectors
mcr_vec=Sigma*w_mkt/sigma_mkt;
cr_vec=diag(w_mkt)*mcr_vec;

%   risk contribution in percentage
cr_prc=cr_vec/sigma_mkt;

sum(cr_prc)
```

```
ans = 1.0000
```

```
array2table([w_mkt,mcr_vec,cr_vec,cr_vec/sigma_mkt],'RowNames',ac_name,'VariableNames',...
    {'Weight','MCR','CR','CRInPerc'})
```

ans = 12×4 table

	Weight	MCR	CR	CRInPerc
1 Fixed-Income: Treasury Inflation Protected Securitie...	0.0400	0.3669	0.0147	0.0055
2 Fixed-Income: U.S. Treasury Bonds and Investment G...	0.2920	0.1033	0.0302	0.0114
3 Fixed-Income: U.S. High Yield Corp Bonds	0.0300	1.9319	0.0580	0.0219
4 Fixed-Income: Non U.S. Sovereign Bonds	0.0380	0.6954	0.0264	0.0100
5 Equities: U.S. Large Cap Growth	0.1660	4.3089	0.7153	0.2701
6 Equities: U.S. Large Cap Value	0.1660	3.7575	0.6238	0.2356
7 Equities: U.S. Small Cap Growth	0.0240	5.2728	0.1265	0.0478

8 Equities: U.S. Small Cap Value	0.0240	4.2148	0.1012	0.0382
9 Equities: Developed Countries Non-US	0.1200	4.4044	0.5285	0.1996
10 Equities: Emerging Markets	0.0450	5.4202	0.2439	0.0921

```
%   let's calculate the total risk contribtuion (in %) from fixed-income
%   and equities
fprintf('Fixed-Income: Total Weight=%.2f%%, Total Risk Contribution=%.2f%%.\n',...
    100*sum(w_mkt(1:4)),100*sum(cr_prc(1:4)));
```

```
Fixed-Income: Total Weight=40.00%, Total Risk Contribution=4.88%.
```

```
fprintf('Equities: Total Weight=%.2f%%, Total Risk Contribution=%.2f%%.\n',...
    100*sum(w_mkt(5:10)),100*sum(cr_prc(5:10)));
```

```
Equities: Total Weight=54.50%, Total Risk Contribution=88.34%.
```

Remarks

We've already had the following representations from previous discussions:

- $\sigma(w) = \sqrt{w'\Sigma w}$
- $\mathbb{MCR} = \dfrac{\Sigma w}{\sigma(w)}$
- $\mathbb{CR} = \dfrac{\text{diag}(w)\Sigma w}{\sigma(w)}$

Recall the calculation of a security's beta to a portfolio: $\beta_i = \dfrac{Cov(r_i, r_{port})}{Var(r_{port})}$.

For the i-th security in the portfolio $w = (w_1, ..., w_n)'$, let's calculate the security's beta w.r.t. the portfolio w:

$$\beta_i = \frac{Cov(r_i, \sum_{j=1}^{n} w_j r_j)}{\sigma(w)^2} = \frac{\sum_{j=1}^{n} w_j \sigma_{ij}}{\sigma(w)^2} = \frac{(\Sigma w)_i}{\sigma(w)^2},$$

which implies:

1. $\mathbb{MCR}_i = \beta_i \cdot \sigma(w)$
2. $\mathbb{CR}_i = w_i \cdot \beta_i \cdot \sigma(w)$
3. $\mathbb{PCR}_i = w_i \cdot \beta_i$, where $\mathbb{PCR}$ is the percentage contribution to risk.

<u>Corollary A</u>: in a long-only portfolio, i.e., all weights are non-negative, a security's percentage risk contribution is higher (lower) than its weight in the portfolio if and only if its beta to the portfolio is greater (smaller) than 1.

<u>Corollary B</u>: a security has zero risk contribution to the portfolio if and only if at least one of the two conditions exist: 1) its weight in the portfolio is zero; 2) its beta to the portfolio is zero.

```
%%%%%%%%%%%%%%%%%%%%%%%
%   let's calculate securities' betas to the market portfolio and use the
%   new formulas to calculate risk contribution measures

%   we'll caculate each security's beta to the market portfolio in the most
%   traditional way, i.e.,

%   step 1: calculate the portfolio's return time series

%   step 2: calculate the covariance matrix between the security's returns
%   and the portfolio's reutrns

%   step 3: beta is then calculated as the covariance between the two over
%   the variance of the portfolio's returns.
%%%%%%%%%%%%%%%%%%%%%%%
```

```
%   portfolio return time series
port_ts=ac_mret*w_mkt;

%   calculate beta for each security
beta_vec=nan(num_ac,1);

for i=1:num_ac

    %   covariance matrix
    tmp_cov=nancov(ac_mret(:,i),port_ts);

    %   beta
    beta_vec(i)=tmp_cov(1,2)/tmp_cov(2,2);
end

%   let's re-calculate the marginal contribution to risk, and compare with
%   the previously-calculated values
mcr_vec_new=sigma_mkt*beta_vec;
[mcr_vec_new,mcr_vec]
```

```
ans = 12×2
    0.3669    0.3669
    0.1033    0.1033
    1.9319    1.9319
    0.6954    0.6954
    4.3089    4.3089
    3.7575    3.7575
    5.2728    5.2728
    4.2148    4.2148
    4.4044    4.4044
    5.4202    5.4202
      ⋮
```

```
%   let's re-calculate the contribution to risk, and compare with
%   the previously-calculated values
cr_vec_new=w_mkt.*(sigma_mkt*beta_vec);
[cr_vec_new,cr_vec]
```

```
ans = 12×2
    0.0147    0.0147
    0.0302    0.0302
    0.0580    0.0580
    0.0264    0.0264
    0.7153    0.7153
    0.6238    0.6238
    0.1265    0.1265
    0.1012    0.1012
    0.5285    0.5285
    0.2439    0.2439
      ⋮
```

```
%   show the weight, beta, and percentage contribution to risk
%   show the beta vector
array2table([w_mkt,beta_vec,w_mkt.*beta_vec, cr_vec/sigma_mkt],'RowNames',ac_name,'VariableNames',...
    {'Weight','Beta','WtXBeta','CRInPerc'})
```

ans = 12×4 table

	Weight	Beta	WtXBeta	CRInPerc
1 Fixed-Income: Treasury Inflation Protected Securitie...	0.0400	0.1386	0.0055	0.0055
2 Fixed-Income: U.S. Treasury Bonds and Investment G...	0.2920	0.0390	0.0114	0.0114

3 Fixed-Income: U.S. High Yield Corp Bonds	0.0300	0.7296	0.0219	0.0219
4 Fixed-Income: Non U.S. Sovereign Bonds	0.0380	0.2626	0.0100	0.0100
5 Equities: U.S. Large Cap Growth	0.1660	1.6273	0.2701	0.2701
6 Equities: U.S. Large Cap Value	0.1660	1.4191	0.2356	0.2356
7 Equities: U.S. Small Cap Growth	0.0240	1.9913	0.0478	0.0478
8 Equities: U.S. Small Cap Value	0.0240	1.5918	0.0382	0.0382
9 Equities: Developed Countries Non-US	0.1200	1.6634	0.1996	0.1996
10 Equities: Emerging Markets	0.0450	2.0470	0.0921	0.0921

We can also write out the security i's beta to the portfolio w as follows:

$$\beta_i = \frac{\rho_{i,w} \cdot \sigma_i \cdot \sigma(w)}{\sigma(w)^2} = \frac{\rho_{i,w} \cdot \sigma_i}{\sigma(w)},$$

where $\rho_{i,w}$ is the security's correlation with the portfolio. Therefore, we can re-write risk contribution metrics as follows:

1. $\mathbb{MCR}_i = \rho_{i,w} \cdot \sigma_i$
2. $\mathbb{CR}_i = w_i \cdot \rho_{i,w} \cdot \sigma_i$
3. $\mathbb{PCR}_i = \frac{w_i \cdot \rho_{i,w} \cdot \sigma_i}{\sigma(w)}$, where $\mathbb{PCR}$ is the percentage contribution to risk.

Because $\rho_{i,w}$ is a correlation coefficient, it must be within $[-1, 1]$. Therefore,

<u>Corollary C</u>: *in a long-only portfolio, i.e., all weights are non-negative, a security's marginal contribution to risk is always capped by its standalone risk; its contribution to risk is always capped by its standalone contribution to risk (defined as its weight times its standalone risk).*

```
%   let's calculate each security's standalone risk and its coefficient with the portfolio
risk_vec=nan(num_ac,1);
corr_vec=nan(num_ac,1);

for i=1:num_ac

    %   covariance matrix
    tmp_cov=nancov(ac_mret(:,i),port_ts);

    %   standalone risk
    risk_vec(i)=sqrt(tmp_cov(1,1));

    %   coefficient
    corr_vec(i)=tmp_cov(1,2)/sqrt(tmp_cov(1,1)*tmp_cov(2,2));
end

%   let's only show the marginal contribution to the risk
[corr_vec.*risk_vec,mcr_vec]
```

```
ans = 12×2
    0.3669    0.3669
    0.1033    0.1033
    1.9319    1.9319
    0.6954    0.6954
    4.3089    4.3089
    3.7575    3.7575
    5.2728    5.2728
    4.2148    4.2148
    4.4044    4.4044
    5.4202    5.4202
      ⋮
```

```
array2table([corr_vec,risk_vec, mcr_vec],'RowNames',ac_name,'VariableNames',...
    {'Correlation','StandaloneRisk','MCR'})
```

ans = 12×3 table

	Correlation	StandaloneRisk	MCR
1 Fixed-Income: Treasury Inflation Protected Securitie...	0.2243	1.6355	0.3669
2 Fixed-Income: U.S. Treasury Bonds and Investment G...	0.1059	0.9760	0.1033
3 Fixed-Income: U.S. High Yield Corp Bonds	0.7490	2.5792	1.9319
4 Fixed-Income: Non U.S. Sovereign Bonds	0.2976	2.3366	0.6954
5 Equities: U.S. Large Cap Growth	0.8964	4.8069	4.3089
6 Equities: U.S. Large Cap Value	0.8950	4.1982	3.7575
7 Equities: U.S. Small Cap Growth	0.8197	6.4326	5.2728
8 Equities: U.S. Small Cap Value	0.8160	5.1652	4.2148
9 Equities: Developed Countries Non-US	0.9349	4.7111	4.4044
10 Equities: Emerging Markets	0.8608	6.2965	5.4202

Section B: The Risk Budgeting Portfolio

A risk budgeting problem considers a vector of risk budgets $\mathbf{b} = [b_1, ..., b_n]$, and aims to find a portfolio with weight vector w, so that $\mathbb{CR}_i = b_i \cdot \sigma(w)$, $\forall i = 1, ..., n$.

Note: in this chapter, we always assume $b_i > 0$ for any i, otherwise, we simply take the asset with 0 risk budgeting out of the investable universe.

With the exception of small trivial cases (e.g., $n = 2$, or the strong assumptions on the correlation matrix), it is impossible to find an analytical solution to a risk budgeting problem. However, we can always use optimization to find the solution:

$$\textbf{minimize} \quad \sum_{i=1}^{n} (\mathbb{CR}_i - b_i \cdot \sigma(w))^2$$

$$\textbf{subject to} \quad w^T \mathbf{1} = 1, \mathbf{0} \leq w \leq \mathbf{1}$$

The question is whether the solution from the above optimization could **make the objective function equal 0**, in which case, we can easily see $\mathbb{CR}_i = b_i \cdot \sigma(w)$ for any i, and the risk budgeting portfolio given $\mathbf{b}$ has been located.

In the example below, for the sake of reducing the optimization complexity, we change the objective function to: $\sum_{i=1}^{n} (\mathbb{CR}_i \cdot \sigma(w) - b_i \cdot \sigma(w)^2)^2$.

```
%%%%%%%%%%%%%%%%    Example to run the optimization given any risk budget

%   get a random risk budgeting vector b
b_vec=rand(num_ac,1);
b_vec=b_vec/sum(b_vec);

%   define the objective function
%   let's multiply the portfolio's variance
obj_func=@(w) (diag(w)*Sigma*w-b_vec.*(w'*Sigma*w))'*(diag(w)*Sigma*w-b_vec.*(w'*Sigma*w));

%   the constraints
%   A*w<=b,Aeq*w=beq
A=[eye(num_ac);-eye(num_ac)];
b=[ones(num_ac,1);zeros(num_ac,1)];
Aeq=ones(1,num_ac);
beq=1;

%   run the optimization
w0=rand(num_ac,1);
w0=w0/sum(w0);
```

```
[w_opt,fval]=fmincon(obj_func,w0,A,b,Aeq,beq)
```

```
Local minimum found that satisfies the constraints.

Optimization completed because the objective function is non-decreasing in
feasible directions, to within the default value of the optimality tolerance,
and constraints are satisfied to within the default value of the constraint tolerance.

<stopping criteria details>
w_opt = 12×1
    0.2824
    0.1485
    0.0610
    0.0615
    0.0261
    0.0383
    0.0457
    0.0067
    0.1109
    0.0886
      ⋮

fval = 5.8829e-12
```

```
%   let's check if the solution satisfies our conditions
%   in case the optimizer stopped too early
sum(w_opt)
```

```
ans = 1
```

```
%   calculate the risk contributions
sigma=sqrt(w_opt'*Sigma*w_opt);
cr_opt=diag(w_opt)*Sigma*w_opt/sigma;

%   the realized risk contribution in percentage, and compare it with the
%   budgeted risk
b_realized=cr_opt/sigma;
[b_realized,b_vec]
```

```
ans = 12×2
    0.0881    0.0881
    0.0167    0.0167
    0.0526    0.0526
    0.0270    0.0270
    0.0403    0.0403
    0.0526    0.0526
    0.0915    0.0915
    0.0109    0.0109
    0.1980    0.1980
    0.2072    0.2072
      ⋮
```

Existence and Uniqueness of the Risk Budgeting Portfolio

From the above example, it seems like given any (**positive-value-only**) risk budgeting vector $\mathbf{b}$, we can always find a solution portfolio with exactly the desired risk budgeting distribution across assets.

Can we prove existence and uniqueness of the risk budgeting portfolio given strictly positive budgeting vector $\mathbf{b}$*?*

Answer: Yes.

To prove existence and uniqueness, we need to consider a new optimization problem (Bruder and Roncalli, 2012):

$$\textbf{minimize} \quad \sigma(w)$$

subject to $\sum_{i=1}^{n} b_i \cdot \ln w_i \geq c,\ w \geq \mathbf{0}$

where c is an arbitrary constant.

The above is a traditional convex optimization problem with convex domain, and **the existence and uniqueness of its solution is guaranteed**. In the following, we show that the risk budgeting portfolio can be located based on this solution.

The associated Lagrange function of the above optimization problem is:

$$\mathscr{L}(w; \lambda, \lambda_c) = \sigma(w) - \lambda_c(\sum_{i=1}^{n} b_i \cdot \ln w_i - c) - \lambda' w,$$

and the solution w^* must satisfy the first-order condition:

$$\frac{\partial \mathscr{L}(w; \lambda, \lambda_c)}{\partial w_i} = \frac{\partial \sigma(w)}{\partial w_i} - \frac{\lambda_c b_i}{w_i} - \lambda_i = 0, \forall i.$$

The KKT conditions for this optimization problem is:

1. $\lambda_c(\sum_{i=1}^{n} b_i \cdot \ln w_i - c) = 0$ and $\lambda_c \geq 0$
2. $\lambda_i w_i = 0$ and $\lambda_i \geq 0, \forall i.$

Because $\ln w_i$ is not defined for $w_i = 0$, therefore we have $\lambda = \mathbf{0}$ from condition 2. And if $\lambda_c = 0$, then from the first-order condition (combined with the knowledge that $\lambda = \mathbf{0}$) would lead to $\frac{\partial \sigma(w)}{\partial w_i} = 0, \forall i$, which further leads to $\sigma(w) = \sum_{i=1}^{n} w_i \frac{\partial \sigma(w)}{\partial w_i} = 0$, i.e., it's equivalent to say $w^* = \mathbf{0}$, which could not be the solution of the optimization problem.

In summary, from KKT conditions, we have observed that $\lambda = \mathbf{0}$, and λ_c is a positive constant. Therefore, from the first-order condition:

$$\frac{\partial \sigma(w)}{\partial w_i} = \frac{\lambda_c b_i}{w_i} \Rightarrow w_i \frac{\partial \sigma(w)}{\partial w_i} = b_i \lambda_c, \forall i.$$

Given $\sum_{i=1}^{n} b_i = 1$ and $\sum_{i=1}^{n} w_i \frac{\partial \sigma(w)}{\partial w_i} = \sigma(w)$, we can see $\lambda_c = \sigma(w)$, and we have completed our proof that the solution of the new optimization problem results in the risk budgeting portfolio?

Wait !!!!!

In the optimization problem, we didn't add the constraint of net leverage of 1, i.e., $w'\mathbf{1} = 1$. Indeed, we only need to normalize w^*, so that the weigth vector $\frac{w^*}{\sum_{i=1}^{n} w_i^*}$ is the weight vector for the risk budgeting portfolio with net leverage of 1. (Easy to prove)

Note: the above analysis is only valid when the risk budget vector b is strictly positive. Although we've described in previous section that an asset with non-zero allocation may also have zero risk contribution (when its beta to the portfolio is zero), <u>from the investor's point of view, if an asset is allocated with zero risk budget, the asset should be removed from the investable universe of interest</u>. Therefore, we always assume strictly positive risk budget b.

Let's run the new optimization to locate the risk budgeting portfolio:

minimize $\sigma(w)$

subject to $\sum_{i=1}^{n} b_i \cdot \ln w_i \geq c,\ w \geq \mathbf{0}$

```
%   define the new objective function
obj_func2=@(w) sqrt(w'*Sigma*w);

%   an arbitrary constant c
const=1;
```

```
%   the constraints
%   A*w<=b,Aeq*w=beq
A=[-eye(num_ac)];
b=[zeros(num_ac,1)];
Aeq=[];
beq=[];


%   run the optimization
w0=rand(num_ac,1);
w0=w0/sum(w0);

[w_opt2,fval]=fmincon(obj_func2,w0,A,b,Aeq,beq,[],[],@(w) nonlcon(w,b_vec,const));
```

```
Local minimum found that satisfies the constraints.

Optimization completed because the objective function is non-decreasing in
feasible directions, to within the default value of the optimality tolerance,
and constraints are satisfied to within the default value of the constraint tolerance.

<stopping criteria details>
```

```
%   normalize to 1 so that net leverage equals 1
w_opt2=w_opt2/sum(w_opt2);

%   compare the solution with the solution from the previous optimization
%   problem
[w_opt,w_opt2]
```

```
ans = 12×2

    0.2824    0.2824
    0.1485    0.1485
    0.0610    0.0610
    0.0615    0.0615
    0.0261    0.0261
    0.0383    0.0383
    0.0457    0.0457
    0.0067    0.0067
    0.1109    0.1109
    0.0886    0.0886
      ⋮
```

Risk Parity Portfolio

A portfolio is a risk parity portfolio, or an *equal risk contribution portfolio* (ERC), if the risk budgets are the same across different assets:

$$b_i = \frac{1}{n}, \text{ for } i = 1, ..., n$$

if there are n assets in the universe of interest. It can be viewed as a neutral portfolio when the portfolio manager has no views on the risk budgets. Specifically, Maillard et al. (2010) show that the risk parity portfolio corresponds to the tangency portfolio in the case where the Sharpe ratio is the same for all assets and the correlation matrix is a uniform matrix (i.e., except for values of 1 on the diagonal of the correlation matrix, all other elements are the same).

We need to run optimization to find the risk parity portfolio in general case, however, here are two special cases for which we can easily locate the risk parity weight vector .

(Assuming the weight vector is non-negative and sums up to 1)

Special Case A - *when there are only two assets, i.e.,* $n = 2$

From Section A in the beginning of the chapter, we have found when there are only two assets:

$\mathbb{CR}_1 = w_1 \cdot \mathbb{MCR}_1 = w_1 \cdot \frac{\partial \sigma(w)}{\partial w_1} = \frac{w_1^2\sigma_{11} + w_1 w_2 \sigma_{12}}{\sigma(w)}$, and $\mathbb{CR}_2 = w_2 \cdot \mathbb{MCR}_2 = w_2 \cdot \frac{\partial \sigma(w)}{\partial w_2} = \frac{w_2^2\sigma_{22} + w_1 w_2 \sigma_{12}}{\sigma(w)}$,

where $\sigma_{11} = \sigma_1^2$, $\sigma_{22} = \sigma_2^2$.

To locate the risk parity portfolio, we need to find (w_1, w_2) so that $\mathbb{CR}_1 = \mathbb{CR}_2$, and,

$$\Rightarrow \frac{w_1^2\sigma_{11} + w_1 w_2 \sigma_{12}}{\sigma(w)} = \frac{w_2^2\sigma_{22} + w_1 w_2 \sigma_{12}}{\sigma(w)}$$

$$\Rightarrow w_1^2\sigma_{11} = w_2^2\sigma_{22}$$

$$\Rightarrow w_1\sigma_1 = w_2\sigma_2$$

$$\Rightarrow w_1 = \frac{1/\sigma_1}{1/\sigma_1 + 1/\sigma_2}, \text{ and } w_2 = \frac{1/\sigma_2}{1/\sigma_1 + 1/\sigma_2}.$$

(Try to prove the last step by yourself using the known condition that $w_1 + w_2 = 1$)

<u>*Special Case B*</u> - *when the correlation matrix is a uniform matrix (i.e., $\rho_{ij} = \rho$, for $i \neq j$)*

The contribution to risk vector is $\mathbb{CR} = \frac{\text{diag}(w)\Sigma w}{\sigma(w)}$, and given the assumption that the correlation matrix is a uniform matrix (i.e., $\sigma_{ij} = \Sigma_{i,j} = \rho\sigma_i\sigma_j$, when $i \neq j$), we have for any i: $\mathbb{CR}_i \cdot \sigma(w) = w_i \sum_{j=1}^{n} \sigma_{ij} w_j = w_i^2\sigma_i^2 + \rho \cdot \sum_{j \neq i} \sigma_i \sigma_j w_i w_j$.

We have one candidate for the above risk parity problem: $w_i\sigma_i = c$, $\forall i$, where c is a constant so that the weight vector $w = (w_1, ..., w_n)'$ sums up to 1. It's easy to prove that this solution leads to $\mathbb{CR}_i \cdot \sigma(w) = c^2(1 + (n-1) \cdot \rho)$, $\forall i$, which is an equal risk contribution outcome. Therefore, combined with the uniquenss of solution for any risk budgeting problem, the candidate is exactly our solution, and we have the risk parity portfolio with the vector w where:

$$w_i = \frac{1/\sigma_i}{\sum_{j=1}^{n} 1/\sigma_j}, \text{ for } i = 1, ..., n. \qquad (*)$$

For the above two special cases, their solutions have the same structure. We also call the portfolio whose weight vector satisfies (*) as **the volatility parity portfolio**.

```
%   let's solve for the risk parity portfolio and the volatility parity portfolio
%   and compare them

%%%%%%%%%
%   to solve for the Risk Parity portfolio, we need to run optimization

%   run the optimization but need to change the risk budget vector
b_erc=ones(num_ac,1);
b_erc=b_erc/sum(b_erc);

w0=ones(num_ac,1)/num_ac;
[w_rp,fval]=fmincon(obj_func2,w0,A,b,Aeq,beq,[],[],@(w) nonlcon(w,b_erc,const));
```

```
Local minimum found that satisfies the constraints.

Optimization completed because the objective function is non-decreasing in
feasible directions, to within the default value of the optimality tolerance,
and constraints are satisfied to within the default value of the constraint tolerance.

<stopping criteria details>
```

```
%   remember to normalize the weight vector
w_rp=w_rp/sum(w_rp);
```

```
%   check the risk contribution vector
sigma_rp=sqrt(w_rp'*Sigma*w_rp);
cr_for_rp=diag(w_rp)*Sigma*w_rp/sigma_rp;
pcr_for_rp=cr_for_rp/sigma_rp;

%%%%%%%%%%
%   now let's get the volatility parity portfolio

%   security volatility vetor
risk_vec=sqrt(diag(Sigma));

%   the weight is inversely proportional to its volatility level
w_vp=1./risk_vec;
w_vp=w_vp/sum(w_vp);

%   check the risk contribution vector
sigma_vp=sqrt(w_vp'*Sigma*w_vp);
cr_for_vp=diag(w_vp)*Sigma*w_vp/sigma_vp;
pcr_for_vp=cr_for_vp/sigma_vp;

array2table([w_rp,pcr_for_rp,w_vp,pcr_for_vp],'RowNames',ac_name,'VariableNames',...
    {'WtRiskParity','PCR1','WtVolParity','PCR2'})
```

ans = 12×4 table

	WtRiskParity	PCR1	WtVolParity	PCR2
1 Fixed-Income: Treasury Inflation Protected Securitie...	0.1630	0.0833	0.1553	0.0531
2 Fixed-Income: U.S. Treasury Bonds and Investment G...	0.3531	0.0833	0.2602	0.0366
3 Fixed-Income: U.S. High Yield Corp Bonds	0.0709	0.0833	0.0984	0.0999
4 Fixed-Income: Non U.S. Sovereign Bonds	0.1143	0.0833	0.1087	0.0559
5 Equities: U.S. Large Cap Growth	0.0416	0.0833	0.0528	0.0967
6 Equities: U.S. Large Cap Value	0.0455	0.0833	0.0605	0.1006
7 Equities: U.S. Small Cap Growth	0.0319	0.0833	0.0395	0.0947
8 Equities: U.S. Small Cap Value	0.0378	0.0833	0.0492	0.0987
9 Equities: Developed Countries Non-US	0.0361	0.0833	0.0539	0.1097
10 Equities: Emerging Markets	0.0284	0.0833	0.0403	0.1041

When n is large, searching for the risk parity solution could be computational intensive. The volatility parity weighting serves as a good proxy for risk parity weighting in most situations.

Appendix: Diversification Measures

Diversification Ratio measures a portfolio's risk reduction from diversification among constituent securities:

$$DR(w) = \frac{\sum_{i=1}^{n} w_i \sigma_i}{\sqrt{w' \Sigma w}}.$$

The higher the DR value, the more diversified the portfolio is.

Other Measures: Herfindahl Index, Shannon's Entropy, etc. They can measure any distribution's dispersion.