CSC236 Week 2

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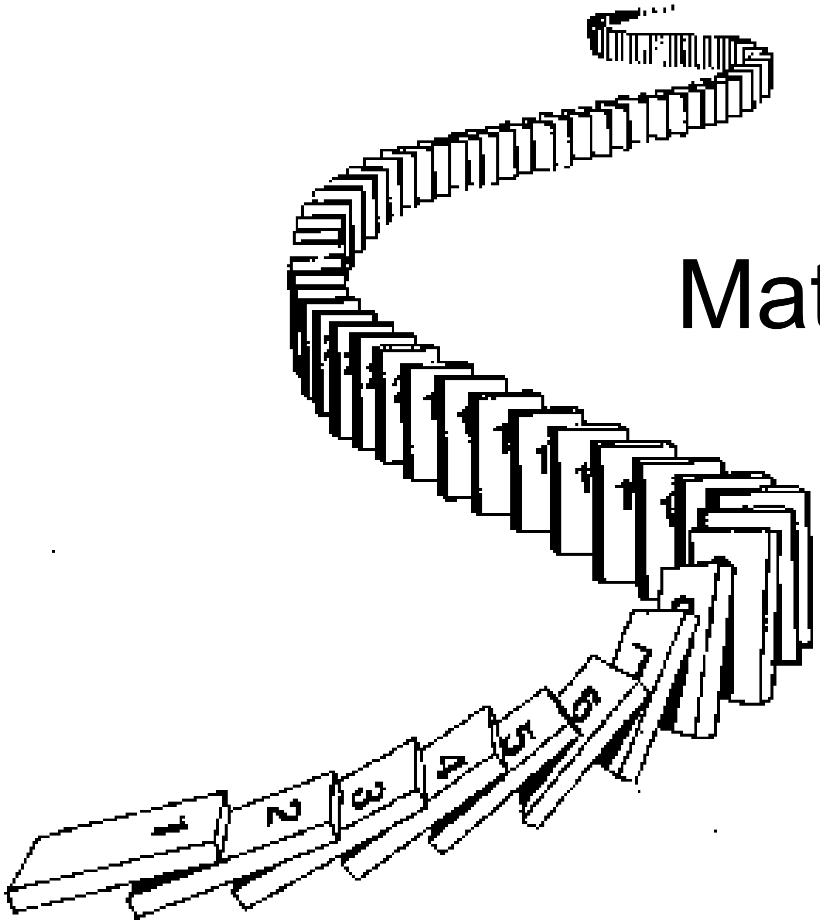
Announcements

- Tutorials start this week.
- Problem Set 1 will be out by Friday



Mathematical Induction

Mathematical Induction

- It’s a proof technique
- It’s an old proof technique
  - People (like Plato) started using it around 300BC
- It’s a powerful, sometimes magical proof technique
  - Can prove very complicated statements using very simple arguments.
- CS people like it a lot
  - Many computer problems have natural structures for inductions to work

Mathematical induction typically establishes a statement for **natural numbers**, e.g.,

$$\forall n \in \mathbb{N}, \boxed{P(n)}$$

This is a **predicate**

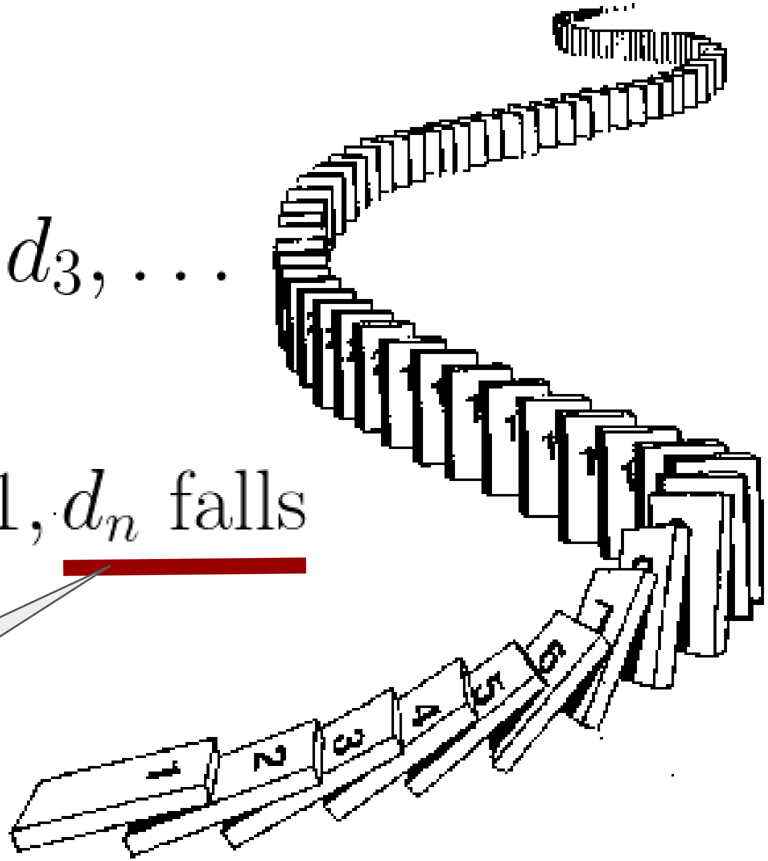
Predicate *P(n)*

- A predicate is a parameterized logical statement
- It takes ***n*** has input, and outputs either **True** or **False**
- Can be seen as a Python function with boolean return values
- Examples:
    - ***P(n)***: n is an even number
    - ***P(n)***: the sum 1+2+...+n is equal to n(n+1)/2
    - ***P(n)***: the n-th domino will fall
    - ...

Let’s try to prove the following

Name the dominoes $d_1, d_2, d_3, \ldots$

Prove: if d_1 falls, then $\forall n \geq 1, \underbrace{d_n \text{ falls}}_{P(n)}$



Prove: if d_1 falls, then $\forall n \geq 1, d_n$ falls

An naive approach

Assume d1 falls,

then d2 falls, because d1 hits d2

then d3 falls, because d2 hits d3

then d4 falls, because d3 hits d4

....

keep doing this until all n >= 1 been enumerated

which means never

There must be a better way!

The induction idea

In order to be convinced that all dominoes will fall.

We just need to know:

(1) The first domino falls, i.e., **d[1]** falls

(2) Every domino will kick down the next one, i.e., **d[k]** falls implies **d[k+1]** must ball, **for all k >= 1**.

To prove the statement, just show (1) and (2) and you’re done! No need to check them one by one.

Principle of **Simple Induction**

- Simple induction is also called **weak induction**
- Besides **simple induction**, there are other types of mathematical inductions such as **complete induction** and **structural induction**, which we will learn later.

Principle of Simple Induction

If

(i) If $P(b)$ is True,

Base Case

(ii) And $P(n) \Rightarrow P(n+1)$ is True for all $n \geq b$,

Induction Step

then

$P(n)$ is True for **all** integers $n \geq b$.

Recipe for writing a proof using simple induction

Step 1: **Define the predicate** P(n)

Don't forget this step!

Step 2: **Base Case**: show that P(b) is True, e.g., b = 0

Step 3: **Induction Step**: show P(k) => P(k+1), for k >= b, i.e.,

a. Assume P(k) is True (**Inductive Hypothesis**)

b. Show that P(k+1) is True

Done.

Examples

Example 1

Prove that for every natural number $n \geq 0$, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

Step 1:

Define the predicate

$$P(n): \quad \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Prove that for every natural number $n \geq 0$, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

Example 1

Step 2:

Base case:

$$P(n) : \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

When n = 0, need to show...

P(0) is True

$$P(0) : \sum_{i=0}^{0} i = \frac{0(0+1)}{2} = 0$$

This is True, so based case done.

Prove that for every natural number $n \geq 0$, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

Example 1

Step 3: Induction Step

$$P(n) : \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Need to show:

P(k) => P(k+1)

***I.H.**: Assume P(k):* $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Want to show *P(k+1):* $\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$.

Prove that for every natural number $n \geq 0$, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

Example 1

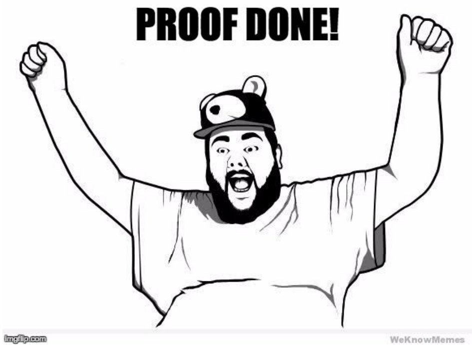
Calculations

$$\sum_{i=0}^{k+1} i = \left(\sum_{i=0}^{k} i\right) + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1) \qquad \text{\# By I.H.}$$

$$= (k+1)\left(\frac{k}{2} + 1\right)$$

$$= \frac{(k+1)(k+2)}{2}$$



$$P(n) : \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

I.H.: $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}.$

$$\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}.$$

Example 1: Write up the proof

Proof:

1. Define the predicate as $P(n) : \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$
2. Base case: for n = 0, P(0) is true because of the following equality

$$\sum_{i=0}^{0} i = \frac{0(0+1)}{2} = 0$$

3. Induction step: Assume P(k), i.e., $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

   then P(k+1) is true because of the following calculation:

   (calculations omitted, refer to the last page)

Complete the proof. / Q.E.D. / $\blacksquare$ ...

Note

- Make sure your induction proof always has the **three** steps explicitly written down
  - Define predicate
  - Base case
  - Induction step
    - Assume …, then ...
- Elegant proofs always have **structures**, like a poem, a painting, a building, a symphony.

Example 2

A real-life problem

The kingdom of Inductionland has only two types of coins circulating: **6 cents** and **11 cents**. The King claims that, using these two types of coins, their citizens can make any amount greater than or equal to **60 cents**.

Prove it for the King.

Step 1: Define the Predicate

P(n):

n can be made using **6** and **11**

We want to prove: $\forall n \geq 60, P(n)$

Step 2: Base Case

n = 60

P(60) is true because…

60 can be made by 6x10

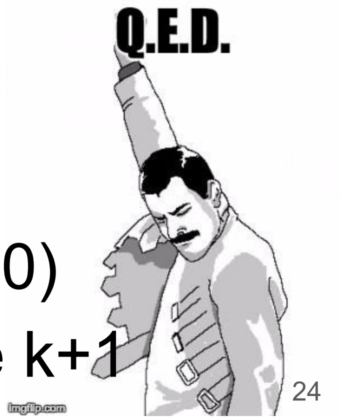
Step 3: Induction Step

Assume **P(k)**, i.e., **k** can be made using 6 and 11

then need to prove **P(k+1)**, i.e., need to make **k+1**.

In other words, given the combination that makes **k**, how to modify it so that it makes **k+1**?

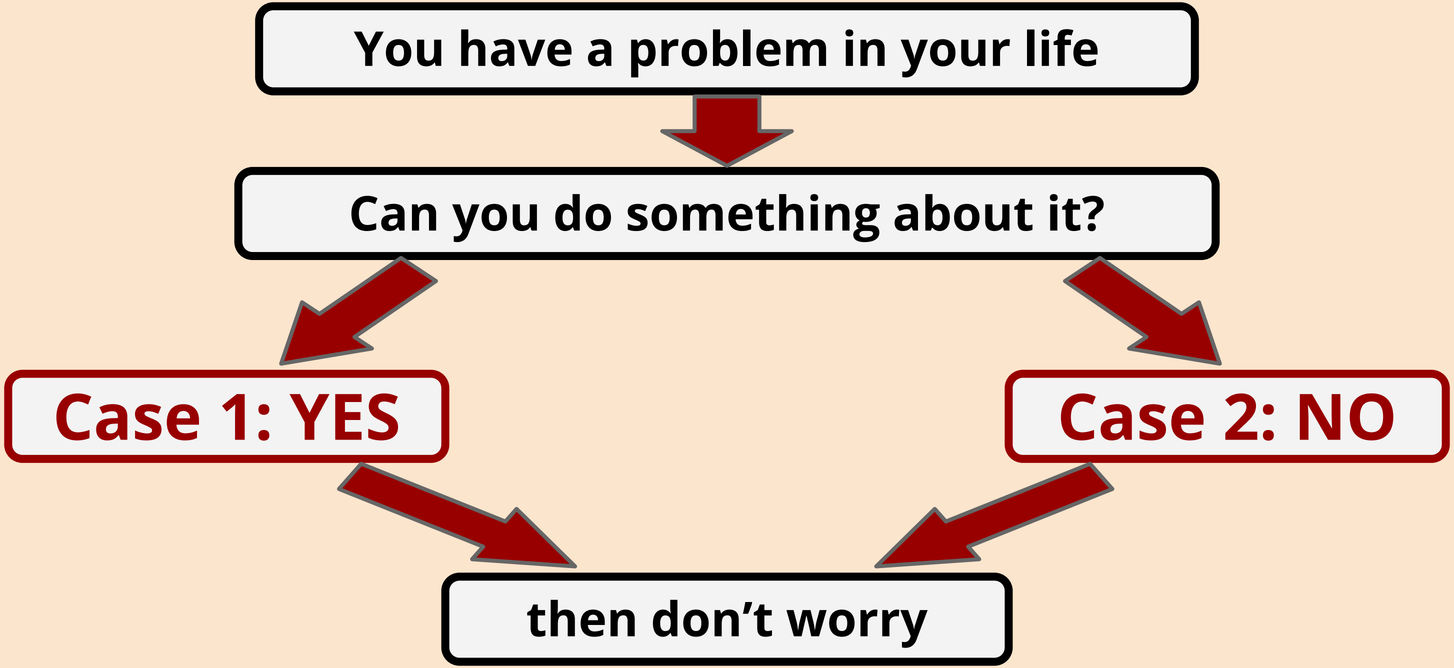
- Case 1: **at least one** **11** is used when making **k**
  - Replace **11** with **6x2=12**, then we have **k+1**
- Case 2: **no** **11** is used when making **k**
  - There must be least nine **6**'s (because k >= 60)
  - Replace **6x9=54** with **11x5=55**, then we have k+1



Interlude: proof by cases

- ➔ **split** your argument into differences cases
- ➔ prove the conclusion **for each** case

Prove: If you have a problem in your life, then don't worry.



What makes it a valid proof?

The union of the cases are covering ALL possibilities.

Example 3

What happens if we skip the base case?

Prove by induction that
$\forall n \geq 0, \sum_{t=0}^{n} 2^t = 2^{n+1}$

Step 1: Define the predicate $P(n) : \sum_{t=0}^{n} 2^t = 2^{n+1}$

Step 2: Base Case (skipped on purpose...)

Prove by induction that

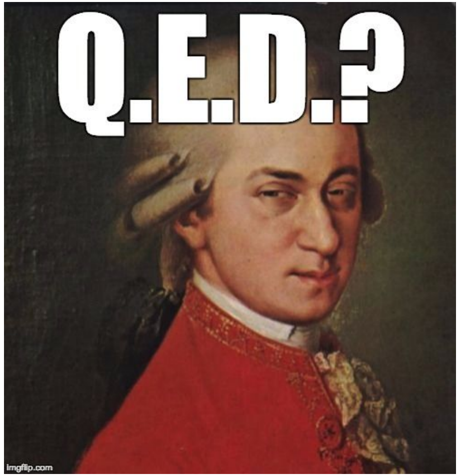
$$\forall n \geq 0, \sum_{t=0}^{n} 2^t = 2^{n+1}$$

Using k and k+1, or n and n+1, both are fine.

Step 3: Induction Step

Suppose that $n \geq 0$ and that $\sum_{t=0}^{n} 2^t = 2^{n+1}$

$$\begin{aligned}\sum_{t=0}^{n+1} 2^t &= \sum_{t=0}^{n} 2^t + 2^{n+1} \\ &= 2^{n+1} + 2^{n+1} \quad \text{\# By I.H.} \\ &= 2 * 2^{n+1} \\ &= 2^{n+2}\end{aligned}$$



Prove by induction that
$\forall n \geq 0, \sum_{t=0}^{n} 2^t = 2^{n+1}$

Verification

- $n = 2$, $1 + 2 + 4 = 7$, not $2^3 = 8$
- $n = 1$, $1 + 2 = 3$, not $2^2 = 4$
- $n = 0$, left side is 1, not 2 (**base case not satisfied!**)

Skipping base case caused **proving a FALSE statement**

Never skip the base case!

Summary: How to do simple induction right

- Always follow the three steps
- Don’t miss any step
- In all steps, be mathematically precise

A proof that is NOT mathematically precise

Prove the all numbers >= 0 are a whole lot less than a million

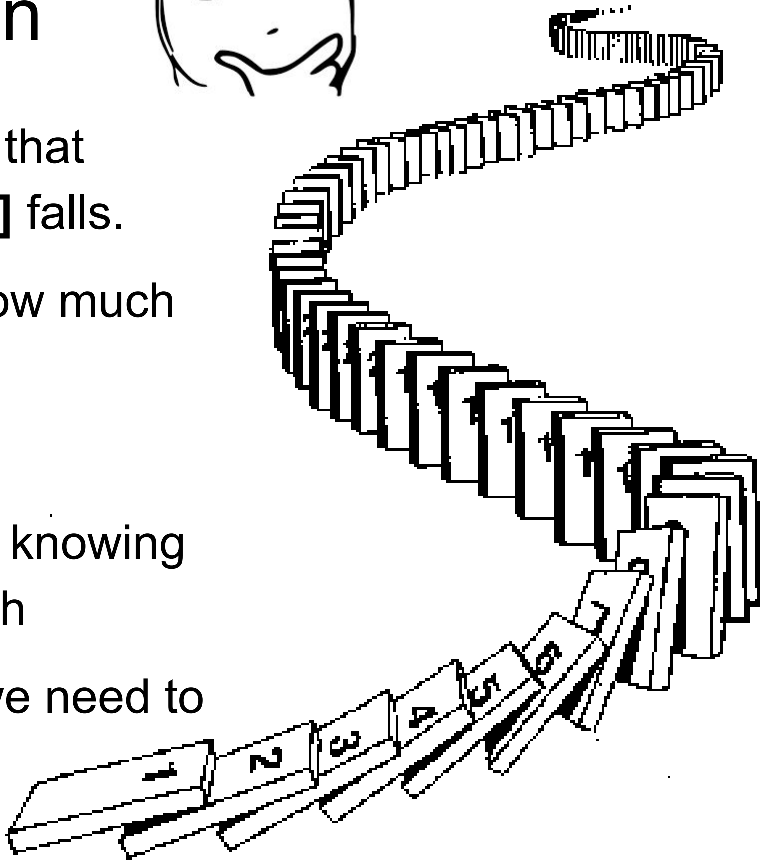
Proof:

1. Define predicate P(n): n is a whole lot less than a million
2. Base Case: n = 0
   - P(0) is true because 0 is whole lot less than a million
3. Induction Step: Assume P(k) is true
   - P(k+1) is true, because if k is a whole lot less than a million then k+1 is just slightly larger than k, so it must also be a whole lot less than a million.

Simple induction is great,
but sometimes it is not enough

Think about the dominoes again

- What simple induction says is that, to show that **d[236]** falls, all I need to know is that **d[235]** falls.
- But by knowing **d[235]** falls, we actually know much more…
- We also know **d[1]** to **d[234]** all fall
  - We didn’t use this information because knowing that **d[235]** falls happened to be enough
  - But sometimes it is NOT enough and we need to use **all the information** we know.



In other words

What we did in **simple induction**

- Suppose **P(0)** is True
- Then we use **P(0)** to prove **P(1)** is True
- Then we use **P(1)** to prove **P(2)** is true.
- Then we use **P(2)** to prove **P(3)** is true
- ……

- Suppose **P(0)** is True
- Then we can use **P(0)** to prove **P(1)** is True
- Then we can use **both P(0) and P(1)** to prove **P(2)** is true.
- Then we can use **P(0), P(1) and P(2)** to prove **P(3)** is true
- ……
- This is called **complete (strong) induction**.

Complete (Strong) Induction

$\wedge$ means "and"

Principle of Complete Induction

(i) If $P(b)$ is True,

(ii) And $\underbrace{P(b) \wedge P(b+1) \wedge \ldots \wedge P(n-1)}_{\text{Induction Hypothesis}} \Rightarrow P(n)$ is True for all $n > b$,

Then $P(n)$ is True for **all** integers $n \geq b$.

Notice the detail with n-1 and n, n > b and n >= b.
Exercise: rewrite it into an equivalent form using P(n+1)

Example 1

Prime or Product of Primes

Prove that every natural number greater than 1 can be written as a product of primes.

For example:

2 = 2
3 = 3
4 = 2 x 2
5 = 5
6 = 2 x 3
28 = 2 x 2 x 7
236 = 2 x 2 x 59

Let's try simple induction ...

Define predicate P(n): n can be decomposed into a product of primes

Base case: n=2

2 is already a product of primes (2 is prime), so we're done.

Induction Step:

Assume **n >= 2** and that **n** can be written as a product of primes.
Need to prove that **n+1** can be written as a product of primes...

Imagine that we know that 8 can be written as a product of primes. (2x2x2)
How does this help us decompose 9 into a product of primes? (3x3)
Not obvious!

Problem: There is no obvious relation between the decomposition of k and the decomposition of k+1. Simple induction not working!

Use Complete Induction

Define predicate P(n): n can be decomposed into a product of primes. (same as before)

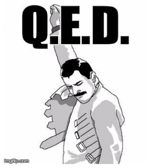
Base case: n=2, 2 is already a product of primes (2 is prime), so we're done. (same as before)

Induction Step:

Assume ***P(2) ∧ P(3) ∧ P(4) ∧ … ∧ P(n-1)***, i..e, all numbers from 2 to n-1 can be written as a product of primes. (**Induction Hypothesis of Complete Induction**)

Now need to show ***P(n)***, i.e., ***n*** can be written as a product of primes

- Case 1: n is prime …
  - then n is already a product of primes, done
- Case 2: n is composite (not prime) …
  - then n can be written as **n = a x b**, where a & b satisfies **2 <= a,b <= n-1**
  - According to **I.H.**, each of ***a*** and ***b*** can be written as a product of primes.
  - So n = **a x b** can be written as a product of primes.



Takeaways

- If jumping "**one number back**" is sufficient to prove the claim for the next number, then use **simple** induction
- If jumping **further back** is necessary, then use **complete** induction
- The structure/steps of complete induction is **very similar** to that of simple induction; the only difference is how the **induction hypothesis** is made.

Example 2

To be continued...