CSC236 Week 7

Larry Zhang

Test 1 result

- Class average: 20.7/30 (69%). One person got 30/30. Well done!
- Average for each question
  - Q1: 8.6/10
  - Q2: 6.8/10
  - Q3: 5.2/10
- Solutions are posted on the course web page.
- Remarking request:
  - Fill in the test remarking form: http://www.cs.toronto.edu/~ylzhang/csc236/files/csc236-test1-remarking.pdf
  - Attach it to your test and submit both to me by Nov 2nd, 2016.
- **Make sure your mark is correctly recorded on MarkUs**

If you didn’t do well...

- Reflections:
    - Did I spend enough time on this course?
    - Has my learning method been efficient?
- Suggestion: Arrange a meeting with Larry to discuss how to improve it for the rest of the course. This has been proven to be quite useful.
- It’s not too late if you start doing it right from now.
    - This test will only be 12%
- If you’re not sure whether you should drop the course, can also talk to Larry

learning continued...

Recap: last week

- We designed a couple algorithms using the divide-and-conquer technique
  - max_segment_sum()
  - MergeSort()
- Both of them had runtime like

$$T(n) = \begin{cases} c & n = 1 \\ 2T(n/2) + dn + e & n > 1 \end{cases}$$

- After applying the substitution method (5-6 steps), we figured out that T(n) is in O(nlogn)
- We said there is a quicker way to get O(nlogn) in just **1 step**.

Typical form of the runtime of divide-and-conquer algorithms

$$T(n) = aT(n/b) + \Theta(n^k)$$

- ***a*** is the number of recursive calls
- ***b*** is the rate at which subproblem size decreases
- ***k*** represents the runtime of the non-recursive part of the algorithm
  - like max_crossing in max_seg_sum, k=1
  - merge in MergeSort, k=1

There is a **theorem** that, just given the values of ***a***, ***b*** and ***k***, can **directly** give us the asymptotic bound on T(n).

- no need to do repeated substitution

Master Theorem

Master Theorem

Let $T(n)$ be defined by the recurrence $T(n) = aT(n/b) + \Theta(n^k)$, for some constants $a \geq 1$, $b > 1$, and $k \geq 0$. Then we can conclude the following about the asymptotic complexity of $T(n)$:

(1) If $k = \log_b a$, then $T(n) = O(n^k \log n)$.

(2) If $k < \log_b a$, then $T(n) = O(n^{\log_b a})$.

(3) If $k > \log_b a$, then $T(n) = O(n^k)$.

How to use master theorem

(1) If $k = \log_b a$, then $T(n) = O(n^k \log n)$.
(2) If $k < \log_b a$, then $T(n) = O(n^{\log_b a})$.
(3) If $k > \log_b a$, then $T(n) = O(n^k)$.

$$T(n) = aT(n/b) + n^k$$

- First make sure you can actually use the master theorem
  - for some recurrences you cannot use it, like T(n) = T(n-1) + 1
  - the recurrence must be of the above form
  - if cannot use master theorem
    - there are more powerful theorem's available, but they are not allowed in this course
    - use the good-old repeated substitution
- If master theorem apply, start by calculating **$\log_b a$**
- Then compare **$\log_b a$** to **k**, and decide which case it belongs to

Exercises

Consider the following functions. For each, decide which case of the master theorem applies (if any), and give the asymptotic worst-case runtime

EX 1

$$T(n) = aT(n/b) + \Theta(n^k)$$

(1) If $k = \log_b a$, then $T(n) = O(n^k \log n)$.

(2) If $k < \log_b a$, then $T(n) = O(n^{\log_b a})$.

(3) If $k > \log_b a$, then $T(n) = O(n^k)$.

$$T(n) = 2T(n/2) + dn + e$$

$$a = 2 \qquad b = 2 \qquad k = 1$$

$$\log_b a = 1 \quad = k$$

Case 1 $\qquad T(n) = O(n \log n)$

EX 2

$$T(n) = aT(n/b) + \Theta(n^k)$$

(1) If $k = \log_b a$, then $T(n) = O(n^k \log n)$.

(2) If $k < \log_b a$, then $T(n) = O(n^{\log_b a})$.

(3) If $k > \log_b a$, then $T(n) = O(n^k)$.

$$T(n) = 9T(n/3) + n$$

$a = 9 \qquad b = 3 \qquad k = 1$

$\log_b a = 2 > k$

Case 2 $\qquad T(n) = O(n^2)$

EX 3

$$T(n) = aT(n/b) + \Theta(n^k)$$

(1) If $k = \log_b a$, then $T(n) = O(n^k \log n)$.
(2) If $k < \log_b a$, then $T(n) = O(n^{\log_b a})$.
(3) If $k > \log_b a$, then $T(n) = O(n^k)$.

$$T(n) = 10T(n/3) + n$$

$$\log_b a = log_3 10 \approx 2.1 > k = 1$$

$$\text{Case 2}: T(n) = O(n^{log_3 10}) \approx O(n^{2.1})$$

EX 4

$$T(n) = aT(n/b) + \Theta(n^k)$$

(1) If $k = \log_b a$, then $T(n) = O(n^k \log n)$.

(2) If $k < \log_b a$, then $T(n) = O(n^{\log_b a})$.

(3) If $k > \log_b a$, then $T(n) = O(n^k)$.

$$T(n) = 10\,T(n/3) + n^4$$

$$\log_b a = \log_3 10 \approx 2.1 < k = 4$$

Case 3: $\quad T(n) = O(n^4)$

EX 5

$$T(n) = aT(n/b) + \Theta(n^k)$$

(1) If $k = \log_b a$, then $T(n) = O(n^k \log n)$.

(2) If $k < \log_b a$, then $T(n) = O(n^{\log_b a})$.

(3) If $k > \log_b a$, then $T(n) = O(n^k)$.

$$T(n) = T(2n/3) + 1$$

$$a = 1 \qquad b = 3/2 \qquad k = 0$$

$$\log_b a = \log_{3/2} 1 = 0 = k$$

Case 1: $T(n) = O(\log n)$

EX 6

$$T(n) = aT(n/b) + \Theta(n^k)$$

(1) If $k = \log_b a$, then $T(n) = O(n^k \log n)$.

(2) If $k < \log_b a$, then $T(n) = O(n^{\log_b a})$.

(3) If $k > \log_b a$, then $T(n) = O(n^k)$.

$$T(n) = T(n/2) + T(n/3) + n$$

Cannot use the master theorem directly, but can still do some bounding

$$T(n/3) \leq T(n/2)$$

$$T(n) = T(n/2) + T(n/3) + n \leq 2T(n/2) + n \quad = O(n \log n)$$

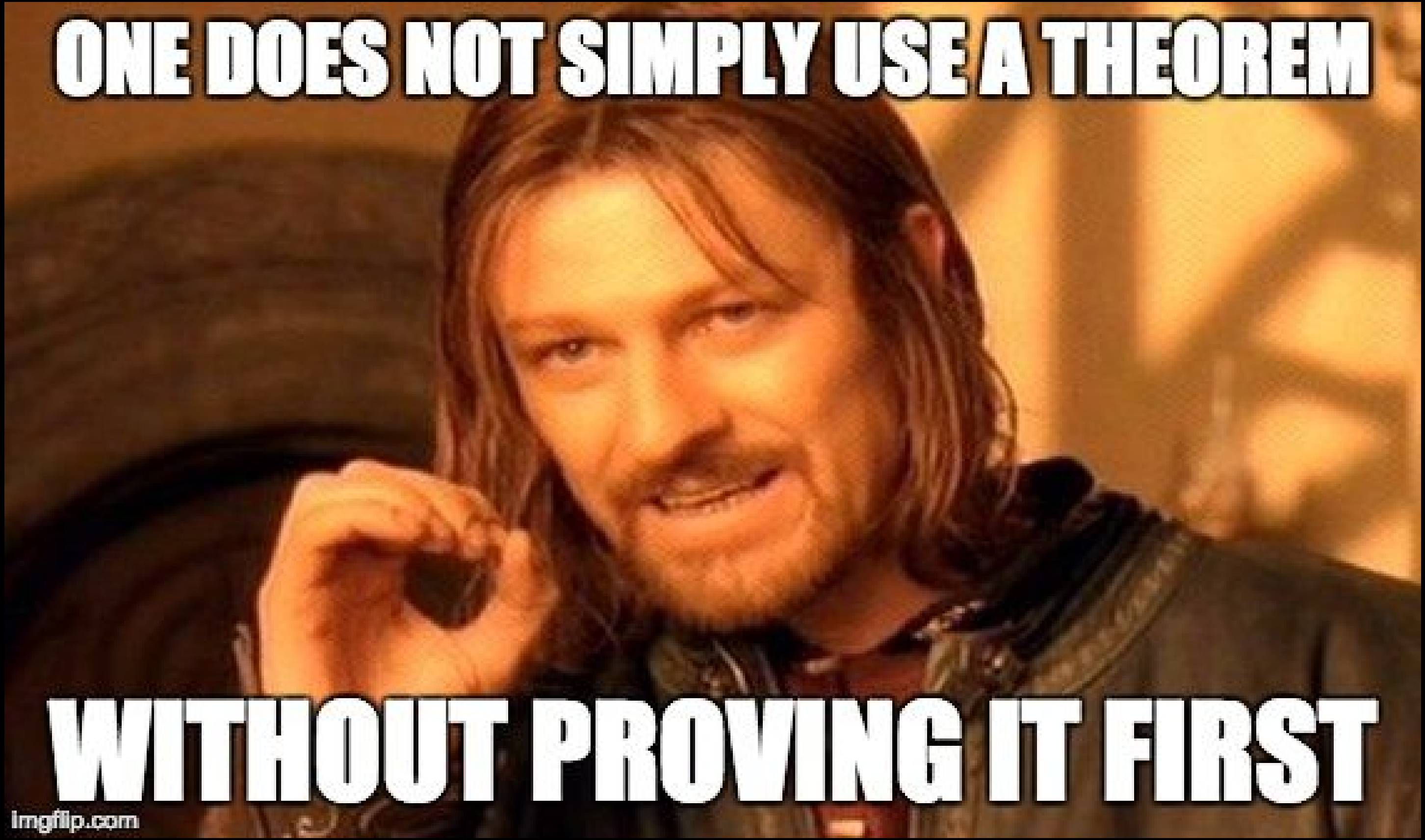
$$T(n) = O(n \log n)$$

Divide-and-Conquer + Master Theorem

The combination of the two gives you the ability to very quickly iterate between algorithm **design** and its runtime **analysis**.

Very **pro** way of algorithm development!

Now we know how to use the master theorem,
but ...


ONE DOES NOT SIMPLY USE A THEOREM
WITHOUT PROVING IT FIRST
imgflip.com

Prove the Master Theorem

Proof:

$$T(n) = \begin{cases} c, & \text{if } n = 1 \\ aT(n/b) + n^k, & \text{if } n > 1 \end{cases}$$

Basically, we want to find the closed form the above recurrence.

How?

Use repeated substitution!

$$T(n) = \begin{cases} c, & \text{if } n = 1 \\ aT(n/b) + n^k, & \text{if } n > 1 \end{cases}$$

Step 1: substitute a few times

$j = 1$

$$T(n) = aT(n/b) + n^k$$

$j = 2$

$$\begin{aligned} T(n) &= a(aT(n/b^2) + (n/b)^k) + n^k \\ &= a(aT(n/b^2) + (n^k/b^k)) + n^k \\ &= a^2T(n/b^2) + a(n^k/b^k) + n^k \\ &= a^2T(n/b^2) + n^k(1 + a/b^k) \end{aligned}$$

$$T(n) = \begin{cases} c, & \text{if } n = 1 \\ aT(n/b) + n^k, & \text{if } n > 1 \end{cases}$$

one more sub...

$$a^2 T(n/b^2) + n^k(1 + a/b^k)$$

$$\begin{aligned} j = 3 \qquad T(n) &= a^2(aT(n/b^3) + (n/b^2)^k) + n^k(1 + a/b^k) \\ &= a^2(aT(n/b^3) + (n^k/b^{2k})) + n^k(1 + a/b^k) \\ &= a^3 T(n/b^3) + a^2(n^k/b^{2k}) + n^k(1 + a/b^k) \\ &= a^3 T(n/b^3) + n^k(1 + a/b^k + (a/b^k)^2) \end{aligned}$$

Guess: $T(n) = a^j T(n/b^j) + n^k \sum_{i=0}^{j-1} (a/b^k)^i$

$$T(n) = a^j T(n/b^j) + n^k \sum_{i=0}^{j-1} (a/b^k)^i$$

$$T(n) = \begin{cases} c, & \text{if } n = 1 \\ aT(n/b) + n^k, & \text{if } n > 1 \end{cases}$$

solve j for base case

$$n/b^j = 1$$

$$n = b^j$$

$$j = \log_b n$$

plug in j

$$\begin{aligned} T(n) &= ca^j + n^k \sum_{i=0}^{j-1} (a/b^k)^i \\ &= ca^{\log_b n} + n^k \sum_{i=0}^{j-1} (a/b^k)^i \\ &= cn^{\log_b a} + n^k \sum_{i=0}^{j-1} (a/b^k)^i \end{aligned}$$

$$\boxed{a^{\log_b n} = a^{\frac{\log_a n}{\log_a b}} = a^{\log_a n \cdot \log_b a} = (a^{\log_a n})^{\log_b a} = n^{\log_b a}}$$

$$T(n) = cn^{\log_b a} + n^k \sum_{i=0}^{j-1} (a/b^k)^i$$

$j = \log_b n$

geometric series with common ratio **a/b^k**

If $a/b^k = 1$, then

- $a = b^k$, and $\log_b a = k$
- each term of the sum is equal to 1, so the entire sum is j

$$\begin{aligned} T(n) &= cn^{\log_b(b^k)} + n^k \sum_{i=0}^{j-1} 1^i \\ &= cn^k + n^k j \\ &= cn^k + n^k * \log_b(n) \\ &= O(n^k \log n) \end{aligned}$$

(1) If $k = \log_b a$, then $T(n) = O(n^k \log n)$.
(2) If $k < \log_b a$, then $T(n) = O(n^{\log_b a})$.
(3) If $k > \log_b a$, then $T(n) = O(n^k)$.

This is exactly **Case 1** of the master theorem!

$$T(n) = cn^{\log_b a} + n^k \sum_{i=0}^{j-1} (a/b^k)^i$$

$$\sum_{k=0}^{n-1} ar^k = a\,\frac{1-r^n}{1-r},$$

if $a/b^k \neq 1$ just do the sum of geometric series

$$\begin{aligned}
T(n) &= cn^{\log_b a} + n^k \sum_{i=0}^{j-1} (a/b^k)^i \\
&= cn^{\log_b a} + n^k (1 - (a/b^k)^j)/(1 - a/b^k) \\
&= cn^{\log_b a} + n^k (1 - (a^j/b^{jk}))/(1 - a/b^k) \\
&= cn^{\log_b a} + n^k (1 - (n^{\log_b a}/n^k))/(1 - a/b^k)
\end{aligned}$$

$$T(n) = cn^{\log_b a} + n^k(1 - (n^{\log_b a}/n^k))/(1 - a/b^k)$$

This is nasty. Let:

- $n_1 = n^{\log_b a}$
- $n_2 = n^k$
- $z = 1 - a/b^k$

We get:

$cn_1 + n_2(1 - (n_1/n_2))/z$

(1) If $k = \log_b a$, then $T(n) = O(n^k \log n)$.
(2) If $k < \log_b a$, then $T(n) = O(n^{\log_b a})$.
(3) If $k > \log_b a$, then $T(n) = O(n^k)$.

- $n_1 = n^{\log_b a}$
- $n_2 = n^k$
- $z = 1 - a/b^k$

$cn_1 + n_2(1 - (n_1/n_2))/z$

Write this so that it is of the form $n_1(\ldots) + n_2(\ldots)$

$$
\begin{aligned}
& cn_1 + n_2(1 - (n_1/n_2))/z \\
&= cn_1 + n_2((n_2 - n_1)/n_2)/z \\
&= cn_1 + n_2((n_2 - n_1)/n_2 z \\
&= cn_1 + (n_2 - n_1)/z \\
&= cn_1 + (n_2/z) - (n_1/z) \\
&= n_1(c - 1/z) + n_2(1/z)
\end{aligned}
$$

(1) If $k = \log_b a$, then $T(n) = O(n^k \log n)$.
(2) If $k < \log_b a$, then $T(n) = O(n^{\log_b a})$.
(3) If $k > \log_b a$, then $T(n) = O(n^k)$.

- $n_1 = n^{\log_b a}$
- $n_2 = n^k$
- $z = 1 - a/b^k$

$$T(n) = n_1(c - 1/z) + n_2(1/z)$$

$$= n^{\log_b a}(c - 1/z) + n^k(1/z)$$

if $k < \log_b a$, the first term is higher order $T(n) = O(n^{\log_b a})$

if $k > \log_b a$, the second term is higher order $T(n) = O(n^k)$

(1) If $k = \log_b a$, then $T(n) = O(n^k \log n)$.
(2) If $k < \log_b a$, then $T(n) = O(n^{\log_b a})$.
(3) If $k > \log_b a$, then $T(n) = O(n^k)$.

Summary

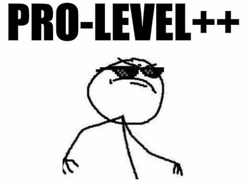
- Master theorem lets you go from the recurrence to the asymptotic bound very quickly, so you're more like a pro.
- It typically works well for divide-and-conquer algorithms
- But master theorem does not apply to all recurrences. When it does not apply, you can:
    - do some upper/lower bounding and get a potentially looser bound
    - use the substitution method
- The course notes have several interesting examples of using divide-and-conquer and the master theorem. Read them!

NEW TOPIC

Program Correctness

Program Correctness

- So far we have been studying the runtime of algorithms
- But what's more important than runtime is that the algorithms actually work **correctly**!
- How did you guarantee algorithm correctness in CSC148?
    - You used test cases. The more test cases you passed, the more confident you were about your code working correctly.
    - But you were never 100% sure that it is correct …
- In CSC236, we will learn to **formally prove** the correctness of your program, without using test cases.



Preconditions and Postconditions

- A **precondition** of a function is what the function **requires** of its parameters so that it can guarantee correct execution.
- A **postcondition** of a function is what the function **promises**, assuming that it was called in a way that satisfies the precondition.

```
def fact_rec(a):
    '''
    Pre: a is an integer >= 1
    Post: returns the factorial of a
    '''
    if a == 1:
        return a # path 1 ends
    else:
        return a * fact_rec(a - 1) # path 2 ends
```

Definition of program correctness

Let `f` be a function that has a given precondition and postcondition.

Then `f` is correct with respect to the precondition and postcondition if:

- for every call `f(I)` where `I` satisfies the precondition, `f(I)` **terminates** in a way that **satisfies** the postcondition

We will discuss later programs with loops.

Prove correctness for **recursive** programs

- For each **program path** from the first line to a **`return`** statement, we need to show that it terminates and satisfies the postcondition.
    - if there is no recursive call, analyze the code directly
    - if there are recursive calls
        - show that the precondition holds at the time of each recursive call
        - Show that the recursive call occurs on "smaller" values than the original call. (so it will terminate eventually)
        - You can then assume that the recursive call satisfies the postcondition
        - show that the postcondition is satisfied based on the assumption

E.g., assume f(n-1) is correct, then use it to show f(n) is correct. This is **induction**!

Examples

Prove the correctness of **fact_rec**

```
def fact_rec(a):
  '''
  Pre: a is an integer >= 1
  Post: returns the factorial of a
  '''
  if a == 1:
    return a # path 1 ends
  else:
    return a * fact_rec(a - 1) # path 2 ends
```

There are **two program paths** from the first line to a `return` statement.

Analyze them one by one.

We need to show:

For all integer **a >= 1**, `fact_rec(a)` terminates and returns the factorial of **a**.

Analyze Path 1

Path 1 has no recursive call.

- To get to this path, **a=1**
- Path 1 returns 1, which is exactly the factorial of **a** when **a=1**
- **Correct.**

```
def fact_rec(a):
  '''
  Pre: a is an integer >= 1
  Post: returns the factorial of a
  '''
  if a == 1:
    return a # path 1 ends
  else:
    return a * fact_rec(a - 1) # path 2 ends
```

Analyze Path 2

Path 2 has recursive calls

- Check that recursive calls have preconditions satisfied
  - to get to Path 2, **a >= 2**, so **a-1 >= 1**, precondition satisfied
- Check that the recursive calls are on “smaller” values.
  - **a-1** is smaller than **a**
  - check.

```
def fact_rec(a):
  '''
  Pre: a is an integer >= 1
  Post: returns the factorial of a
  '''
  if a == 1:
    return a # path 1 ends
  else:
    return a * fact_rec(a - 1) # path 2 ends
```

Analyze Path 2, continued

- Assume that the recursive call satisfies the postcondition
  - assume `fact_rec(a-1)` satisfies the postcondition
  - i.e., it returns **(a-1)!**, the factorial of **a-1**
- Show that postcondition is satisfied based on the assumption
  - show that `fact_rec(a)` satisfies the post condition.

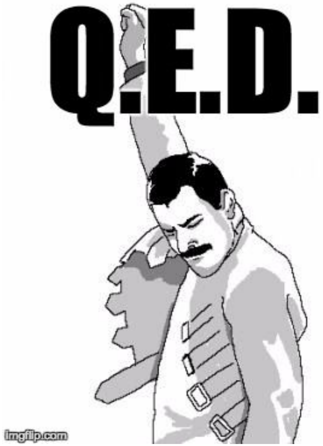
```
fact_rec(a) = a * (a-1)!   # by assumption
            = a!           # postcondition satisfied
```

```
def fact_rec(a):
  '''
  Pre: a is an integer >= 1
  Post: returns the factorial of a
  '''
  if a == 1:
    return a # path 1 ends
  else:
    return a * fact_rec(a - 1) # path 2 ends
```

```
def fact_rec(a):
  '''
  Pre: a is an integer >= 1
  Post: returns the factorial of a
  '''
  if a == 1:
    return a # path 1 ends
  else:
    return a * fact_rec(a - 1) # path 2 ends
```

All program paths have been shown to be terminating and satisfying the postcondition.

Therefore, the above `fact_rec` program is **correct**.



Example 2

Prove the correctness of **gcd_rec**

```
def gcd_rec(a, b):
  '''
  Pre: a and b are integers >= 1, and a >= b
  Post: returns the greatest common divisor of a and b
  '''
  if a == 1 or b == 1:
    return 1                 # path 1 ends
  elif a % b == 0:
    return b                 # path 2 ends
  else:
    return gcd_rec(b, a % b) # path 3 ends
```

Need to show

For all integers a, b ≥ 1 such that a ≥ b, `gcd_rec(a, b)` terminates and returns the greatest common divisor (GCD) of a and b.

- There are three program paths
- Analyze them separately.

```
def gcd_rec(a, b):
  '''
  Pre: a and b are integers >= 1, and a >= b
  Post: returns the greatest common divisor of a and b
  '''
  if a == 1 or b == 1:
    return 1                  # path 1 ends
  elif a % b == 0:
    return b                  # path 2 ends
  else:
    return gcd_rec(b, a % b) # path 3 ends
```

Analyze Path 1 & 2

Path 1:

- to get into this path, a == 1 or b == 1
- the GCD of 1 with other number must be 1, so “return 1” is correct.

Path 2:

- to get into this path, a % b = 0, i.e., b is a divisor of a.
- b is the largest divisor of itself, no larger divisor is possible.
- so “return b” is correct

```
def gcd_rec(a, b):
  '''
  Pre: a and b are integers >= 1, and a >= b
  Post: returns the greatest common divisor of a and b
  '''
  if a == 1 or b == 1:
    return 1                    # path 1 ends
  elif a % b == 0:
    return b                    # path 2 ends
  else:
    return gcd_rec(b, a % b) # path 3 ends
```

Analyze Path 3

```
def gcd_rec(a, b):
  '''
  Pre: a and b are integers >= 1, and a >= b
  Post: returns the greatest common divisor of a and b
  '''
  if a == 1 or b == 1:
    return 1                    # path 1 ends
  elif a % b == 0:
    return b                    # path 2 ends
  else:
    return gcd_rec(b, a % b) # path 3 ends
```

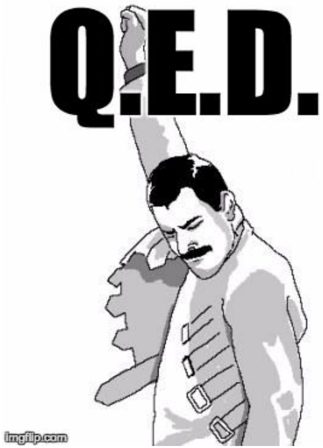
- Check that the recursive call satisfies the precondition
  - b >= 1, from the function precondition
  - a % b >= 1, because …
    - if a % b == 0, Path 2 would have applied
  - b >= a%b, because a%b <= b-1 by def
- Check that the recursive call is on smaller value
  - ARG1: b <= a, from the function precondition
  - ARG2: a%b < b, because a%b <= b-1 by def
  - So overall the input is smaller value

Analyze Path 3 continued

```
def gcd_rec(a, b):
  '''
  Pre: a and b are integers >= 1, and a >= b
  Post: returns the greatest common divisor of a and b
  '''
  if a == 1 or b == 1:
    return 1                   # path 1 ends
  elif a % b == 0:
    return b                   # path 2 ends
  else:
    return gcd_rec(b, a % b) # path 3 ends
```

- Assume the recursive call returns the correct value **GCD(b, a % b)**
- Use the math identity
  - GCD(a, b) = GCD(b, a % b)
  - “Euclidean method”
- GCD(a, b) is exactly the correct return value of the gcd_rec(a, b)
- So postcondition satisfied
- Path 3 proven correct.

gcd_rec is correct



Next week

- Correctness of programs with **loops**