UNIVERSITY OF TORONTO MISSISSAUGA
DECEMBER 2015 FINAL EXAMINATION

MAT102H5F - Introduction to Mathematical Proofs

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Duration: 2 hours

Aids: None

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Please note, you ***CANNOT*** *petition to re-write an examination once the exam has begun.*

ISSUED TO STUDENT

INSTRUCTIONS

1. There are two parts to this examination:
   PART I (40 marks): Ten short questions. Each question is worth 4 marks.
   PART II (60 marks): Six written questions. Each question is worth 10 marks.
2. This examination has 9 different pages including this page. Make sure your copy of the examination has 9 different pages and **sign** at the top of this page. You can use the back of page 9 for rough work.

Good Luck!

Question	Part I	Part II Q#1	Part II Q#2	Part II Q#3	Part II Q#4	Part II Q#5	Part II Q#6	TOTAL
Marks	20 /40	0 /10	4 /10	5 /10	6 /10	5 /10	10 /10	50 /100

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PART I (40 marks)

Answer the following short questions. Each question is worth **4 marks.**

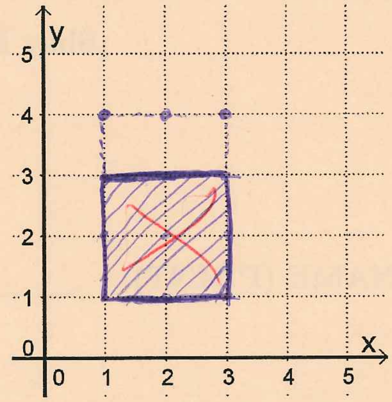
No explanation is needed.

1. Let D be the following subset of $\mathbb{R}$:

$$D=[1,3]\cup\{4\}$$

Draw, on the provided grid, the set $D\times D$.

Use **solid** and **dotted lines** to indicate whether the boundary is or is not part of the set.



2. **Complete the following sentence:**

The **image** of the function $f:\mathbb{N}\times\mathbb{N}\to\mathbb{R}$, $f(n,m)=(-1)^n+(-1)^m$ is $(0,\infty)$.

3. **Complete the addition table** on the right for a field with **four elements** $F=\{0,1,a,b\}$.

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

4. **Consider the following statement:**

P = "There exists an integer M , such that $x^2\leq M$ for all real numbers x ."

(a) Write the statement P using the **logic symbols**.

$(\exists M\in\mathbb{Z})(\forall x\in\mathbb{R})(x^2\leq M)$

(b) Write **the negation** of P using the **logic symbols** (but without using the symbol '$\neg$').

$(\forall M\in\mathbb{Z})(\exists x\in\mathbb{R})(x^2>M)$

(c) Which statement is true: P or $\neg P$? **Circle the correct answer.**

P is TRUE $\neg P$ is TRUE

Continued on page 3 . . .

5. **Complete with a single number**: gcd(24 , 54+24^7) = 3 ______ .

6. **Describe** (in words) the set of all **integers** m , for which the equation $3\,x+m{\cdot}y=1$ has **integer solutions**.

m must be an integer such that gcd(3, m) = 1. This means 3 and m must be relitively prime.

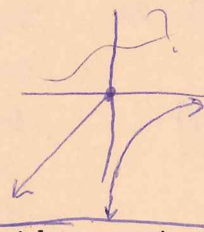
In Questions 7-10, there is only ONE correct answer. Circle it!

7. Let P and Q be two statements.

If P is **true** and Q is **false**, then which of the following statements is TRUE ?

- $P \wedge Q$
- $P \Rightarrow Q$
- $Q \Rightarrow P$
- $(\neg P) \vee Q$

8. The function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} x & \text{if } x \le 0 \\ \dfrac{1}{x} & \text{if } x > 0 \end{cases}$ is...



- ~~a surjection but not an injection.~~
- an injection but not a surjection.
- a bijection.
- neither a surjection nor an injection.

This is neither surjective or injective.

9. Which of the following sets is **countable**?

- $\mathbb{R} - \mathbb{Q}$
- $(0\,,\infty)$
- $P(\mathbb{Q})$
- $\mathbb{Z} - \mathbb{N}$

10. Consider the following **equivalence relation** on $\mathbb{Z}$:

$$a \sim b \qquad \text{if and only if} \qquad a-b \text{ is divisible by } 5 .$$

Then the set $\{\ldots, -7\,, -2\,, 3\,, 8\,, 13\,, 18\,,\ldots\}$ is the **equivalence class** of ...

- 22
- 33
- −11
- −15

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PART II (60 marks)

Answer the following questions. Each question is worth **10 marks**.

Provide **complete solutions and justify your arguments**.

1. Is the following statement **TRUE** or **FALSE**?

" For any two sets A and B : if $A-B=B-A$, then $A=B$. "

If the statement is **true**, **prove it!** If the statement is **false**, **provide a counter-example**.

a) $A-B \subseteq B-A$

If $A=B$,

$x \in A$

$x \in B$

$A-B=\emptyset$

$A-B \subseteq B-A$

If $A=B$, x would be both an element of A and B. This means if A and B are subtracted, an empty set is formed. If the sets are equal, the order of subtraction can be changed to achive the same answer.

b) $A-B \supseteq B-A$

If $A=B$

$y \in B$

$y \in A$

$B-A=\emptyset$

$B-A \subseteq A-B$

Following the same idea, $B-A$ can be written as $A-B$, making $A=B$

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$\boxed{\text{So The statement holds true, } A-B=\cancel{B-A} \text{ if } A=B}$

not what the question is asking?

Continued on page 5 ...

2. Prove, **by induction**, the following identity for all $n \in \mathbb{N}$:

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{4n^3 - n}{3} \quad .$$

Base case:

$n=1$

$(2(1)-1)^2 = \dfrac{4(1)^3 - 1}{3}$

$1 = \dfrac{3}{3}$

$1 = 1$ ∴ True

WHY?

Induction.

Assume $\dfrac{4k^3-k}{3}$ holds true for $k \in \mathbb{N}$. Prove this holds true for $k+1$

$1^2 + 3^2 + 5^2 + (2k-1)^2 + (2(k+1)-1)^2 = \dfrac{4(k+1)^3 - (k+1)}{3}$

I.H. $\rightarrow$ $\dfrac{4k^3-k}{3} + (2(k+1)-1)^2 = \dfrac{4(k+1)^3-(k+1)}{3}$

$4k^3 - k + 3(2(k+1)-1)^2 = 4(k+1)^3 - (k+1)$

$4k^3 - k + 3(2k+1)^2 = 4(k+1)^3 - (k+1)$

$4k^3 - k + 3(4k^2+4k+1) = 4(4k^2+2k+1)(k+1) - (k+1)$

~~$4k^3 + 12k^2 + 23k + 12 = 4k^3 + 4k^2 + \ldots - (k+1)$~~

~~$4k^3 + 12k^2 + 23k + 12$~~

$4k^3 - k + 12k^2 + 12k + 3 = (4k^2 + 8k + 4)(k+1) - k - 1$

$4k^3 + 12k^2 + 11k + 3 = 4k^3 + 4k^2 + 8k^2 + 8k + 4k + 4 - k - 1$

$4k^3 + 12k^2 + 11k + 3 = 4k^3 + 12k^2 + 11k + 3$

∴ LS = RS

$\boxed{\text{∴ By PMI, the statement holds true for } n \in \mathbb{N}}$

Continued on page 6

3. Prove, that for any **odd integer** k , the number $\sqrt{2k}$ is **irrational**.

(Hint: Use contradiction.)

~~Proof by contradiction. lets show~~

lets assume $\sqrt{2k}$ is rational for odd values of k. let $m \in \mathbb{Z}$

$\frac{p}{q} = \sqrt{2(2m-1)}$ $p, q \in \mathbb{Z}$

$\frac{p^2}{q^2} = 2(2m-1)$

WHY?

$p^2 = q^2(4m-2)$ ∴ p is a multiple of $(4m-2)$

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$k(4m-2)^2 = q^2(4m-2)$, $k \in \mathbb{Z}$

?

∴ q is also a multiple of $(4m-2)$

There is a contradiction since the fraction $\frac{p}{q}$ is not in lowest terms

∴ $\sqrt{2k}$ is irrational when k is an odd number.

Continued on page 7 . . .

4. Define a relation R on the set of all real number as follows:

$$(x, y) \in R \quad \text{if and only if} \quad |x+y| = |x| + |y| .$$

Is this relation **reflexive**? Is it **symmetric**? Is it **transitive**?

Is this an **equivalence relation**? **Explain.**

a)

Reflexive: let $x=y$.

$|x+x| = |x| + |x|$

$|2x| \neq 2|x|$

$\therefore$ The relation is not reflexive.

Symmetric: $|x+y| = |x|+|y|$

$|y+x| = |x|+|y|$

$|x+y| = |y+x|$

addition laws allow us to switch the position of 2 numbers and have the same answere

$\therefore$ Symetric.

3

3

Transitive:

$|x+y| = |x| + |y| \Rightarrow |x+y| - |y| = |x|$

$|y+z| = |y| + |z| \Rightarrow |y+z| - |y| = |z|$

$|x+z| \overset{?}{=} |x| + |z|$?

$|x+z| = |x+y| - |y| + |y+z| - |y|$

$|x+z| \neq |x+y| + |y+z| - 2|y|$

$\therefore$ The relation is not transitive

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$\boxed{\therefore \text{ The relation is only symetric}}$

b) This is not an equivalence relation since it does not satisfy refliviity and transitivity. An equivalene relation requires all 3.

Continued on page 8 ...

5. Consider the function $f\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$, $f(a,b) = 12^a \cdot 18^b$.

(a) Show that f is an **injection** (one-to-one).

let $a=1$ let $b=1$

$f(1,b_1) = f(1,b_2)$ $f(a_1,1) = f(a_2,1)$

$12 \cdot 18^{b_1} = 12 \cdot 18^{b_2}$ $12^{a_1} \cdot 18 = 12^{a_2} \cdot 18$

$18^{b_1} = 18^{b_2}$ $12^{a_1} = 12^{a_2}$

So $b_1 = b_2$ So $a_1 = a_2$

So The function f is an injection.

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(b) Is f a **surjection** (onto)? Explain.

No. by substituting $a,b=1$ (the smallest case) we ge $f(1,1)=12\cdot18$

This means that $f(a,b) \geq 12\cdot18$. All natural numbers less than $12\cdot18$ will not be mapped. So the function is not a surjection.

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6. Consider the function $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = 2x + 1$.

Prove, by induction, that for any $n \in \mathbb{N}$, $g^n(x) = 2^n \cdot x + 2^n - 1$.

Note: g^n is the function obtained by composing n copies of g: $g^n = \underbrace{g \circ g \circ \cdots \circ g}_{n \text{ times}}$.

base case:

$n=1$

$g^1(x) = 2^1 \cdot x + 2^1 - 1$

$= 2x + 1$

∴ True.

Induction:

assume $g^k(x) = 2^k \cdot x + 2^k - 1$, for $k \in \mathbb{N}$. Prove this for $k+1$

$g^k(2x+1) = g^{k+1}(x)$

I.H $\to$ $2^k \cdot (2x+1) + 2^k - 1 = 2^{k+1} \cdot x + 2^{k+1} - 1$

$2^k \cdot (2x+1) + 2^k = 2^{k+1} \cdot x + 2^{k+1}$

$2^k 2x + 2^k + 2^k = 2^{k+1} x + 2^{k+1}$

$2^k(2x+1+1) = 2^{k+1}(x+1)$

$2^k(2x+2) = 2^{k+1}(x+1)$

$2^k 2(x+1) = 2^{k+1}(x+1)$

$2^{k+1}(x+1) = 2^{k+1}(x+1)$

$\boxed{\text{∴ by PMI, the statement holds true for } n \in \mathbb{N}}$