Minimal Surfaces in $\mathbb{R}^3$

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Surfaces and graphs on a domain

- Suppose we have a closed, bounded domain $D \subset \mathbb{R}^2$
- Let $x = x(u, v), y = y(u, v), z = z(u, v)$ taking points in the $u - v$ plane into the $x - y - z$ space
- A surface in $\mathbb{R}^3$ on that domain can be represented by $S = \{(x, y, z) : (x, y) \in D\}$
- Now, suppose we can write $z = f(x, y)$, then the graph of f is $G_f = \{(x, y, z) : (x, y) \in D, z = f(x, y)\}$

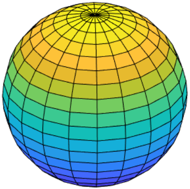

Figure: $x^2 + y^2 + z^2 = 1$

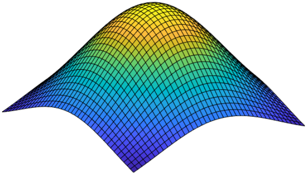

Figure: $z = e^{-x^2-y^2}$

Minimal surfaces and application

- Minimal surfaces are surfaces in space which locally minimize the area, in the sense that any small enough piece of the surface has the smallest area among all surfaces with the same boundary.
- Application: soap films
- Take a given boundary and dip it into a bubble solution, what is the bubble that's going to be created given that boundary?

Minimal graph equation

- Area functional:

$$Area(f) = \int_D \|f_x \times f_y\| = \int_D \sqrt{(1 + f_x^2 + f_y^2)}\, dx\, dy$$

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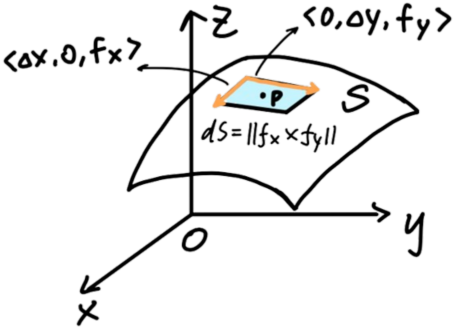

Figure: Obtain total surface area of S by integrating area of tangent plane at each point p.

Minimal graph equation cont.

- Pick a $\mathscr{C}^1$ function $h : \bar{D} \to \mathbb{R}$ vanishing on the boundary. Consider deformation: $Area(f + sh)$ for small $s \in \mathbb{R}$.
- Then $Area(f) \leq Area(f + sh)$ if $Area$ attains a minimum at f
- Want: $\frac{d}{ds}\Big|_{s=0} Area(f + sh) = 0$.
- Minimal graph equation:

$$(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy} = 0$$

Measuring a curve's curvature using a circle of best fit

- Pick a desired curve on the surface at a point, say $(0, 0)$
- Locally represent curve on its tangent plane
- Find circle of best fit with those points
- Curvature $\kappa = 1/r$

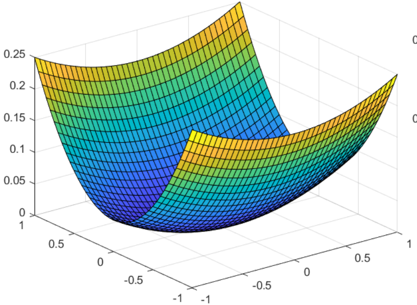

Figure: Graph of $\frac{x^2}{20} + \frac{y^2}{5} = z$

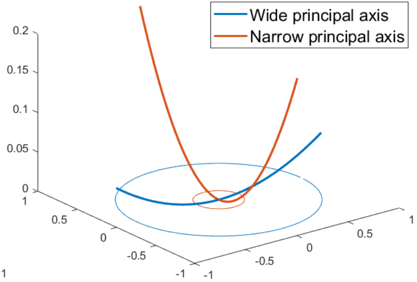

Figure: Principal axes and circles of best fit

Vanishing mean curvature

- Mean curvature H is defined as $H = \kappa_1 + \kappa_2$ at a point p , where κ_1 and κ_2 are the principal curvatures at p
- A minimal surface S has vanishing mean curvature at every point $p \in S$, or $H = 0$ at every point
- Curve lies below tangent plane $\implies$ negative curvature, curve lies above the tangent plane $\implies$ positive curvature

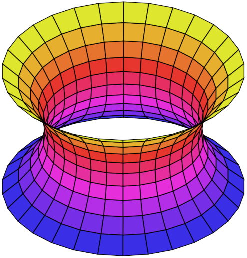

Figure: Catenoid, a minimal surface

A closer look at the catenoid

- Principal curves at a point are the curves with maximum and minimum curvatures
- The catenoid's principal curves at any point are the catenary and the circle
- These curves lie on opposite sides of the tangent plane, and the max and min curvatures cancel out, leaving $H = 0$

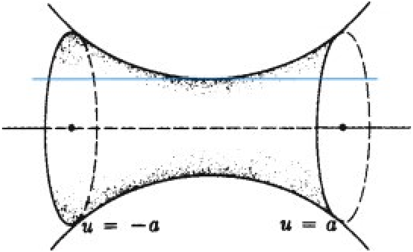

Figure: Catenoid, the surface of $x = u$, $y = a \cosh \frac{u}{a} sin v$, $z = a \cosh \frac{u}{a} \cos u$

Minimal surface and conformal parameterization

- An **immersion** $F : D \to \mathbb{R}^3$ from a domain $D \in \mathbb{R}^2$ is a smooth map such that at every point $p \in D$, dF_p is injective.
- Such an immersion is said to be **conformal** if it preserves angles between curves at every point.
- A conformal immersion $F : D \to \mathbb{R}^3$ gives a minimal surface if and only if the Laplacian

$$\Delta F = (\Delta x, \Delta y, \Delta z) = 0$$

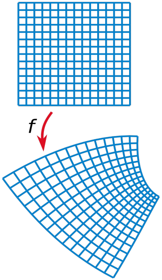

Figure: A conformal map preserves angle

References

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Frank Morgan (2000)
Geometric Measure Theory

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Illustration of a conformal map.
https://commons.wikimedia.org/wiki/File:Conformal_map.svg