THE BÉNABOU-ROUBAUD THEOREM

Abstract. This is a writeup of a proof of the Bénabou-Roubaud theorem using string diagrams which are generated by the Haskell program `tikzsd`.

1. Introduction

This is a writeup of a proof of the Bénabou-Roubaud theorem [1]. It was written as an example of using the `tikzsd` Haskell program to generate complicated string diagrams. No originality is in the argument claimed: the general outline of the proof comes from a draft report of Jovana Obradović [2], though some of the details are different.

2. An example of using `tikzsd`

We encourage anyone looking at this document as an example of using `tikzsd` to look at the source files

- `benabou-roubaud.tex` is the main LaTeX file compiled using `pdflatex`.
- `preamble.tex` is the preamble, which is included into `benabou-roubaud.tex` via `\input{preamble}`.
- In the `source-files` folder, the two files
  1. `br-diagrams-1.tzsd`
  2. `br-diagrams-2.tzsd`

  are the files which are used to generate the other files in this folder. The other files in this folder are automatically generated by running the commands

  ```
  $ tikzsd br-diagrams-1.tzsd
  $ tikzsd br-diagrams-2.tzsd
  ```

In `preamble.tex` the relevant lines are

```
\usepackage{tikz}
\usetikzlibrary{shapes}
\tikzset{utriangle/.style = ...}
\tikzset{btriange/.style= ...}
```

The first line imports the `tikz`, which is needed to process the `tikzpicture` environments created by `tikzsd`. The second line says to use the tikzlibrary `shapes` which allows us to define of non-default shapes. The other two lines define styles called `utriangle` and `btriangle`, which two shapes for a TikZ node: a triangle facing up and a triangle facing down.

In `benabou-roubaud.tex`, we import the preamble with `\input{preamble}`. We then have the lines

```
\makeatletter
\def\input@path{{./source-files/}}
\makeatother
```

which tells `pdflatex` to search the folder `source-files` for the file `file.tex` when processing `\input{file}`.

For the rest of the workflow, we used `tikzsd` on the file `br-diagrams-1.tzsd` or `br-diagrams-2.tzsd` te generate `.tex` files in the folder `source-files`. Then in `benabou-roubaud.tex`, we `\include` the `.tex` file where we want the corresponding string diagram.

3. Adjoints, Monads and Algebras for Monads

We quickly review the definition of these objects and draw the associated string diagrams.

3.1. **Definition.** An *adjoint pair* is a tuple $(\mathcal{C}, \mathcal{D}, F, G, \eta, \varepsilon)$ where

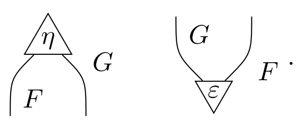
- $\mathcal{C}, \mathcal{D}$ are categories,
- $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ are functors,
- $\eta : 1_{\mathcal{C}} \to G \circ F$, $\varepsilon : F \circ G \to 1_{\mathcal{D}}$ are natural transformations

such that

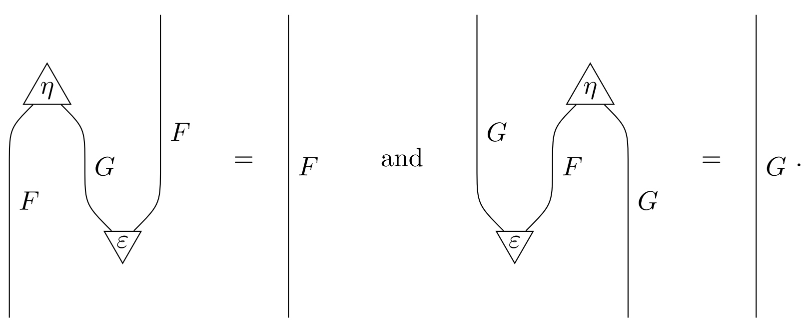
- $\varepsilon F \circ F\eta = 1_F$,
- $G\varepsilon \circ \eta G = 1_G$.

The functors F and G are called the *left adjoint* and *right adjoint*, respectively. The natural transformations η and ε are called the *unit* and *counit*, respectively. Adjoints sometimes denoted by $F \dashv G$, with the unit and counit left implicit.

The natural transformations η and ε are represented by the string diagrams



The two equalities can be written as



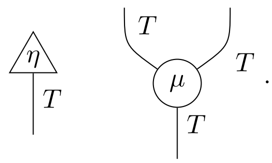
3.2. **Definition.** A *monad* is a tuple $(\mathcal{C}, T, \eta, \mu)$ where

- $\mathcal{C}$ is a category,
- $T : \mathcal{C} \to \mathcal{C}$ is an endofunctor,
- $\eta : 1_{\mathcal{C}} \to T$ and $\mu : T^2 \to T$ are natural transformations

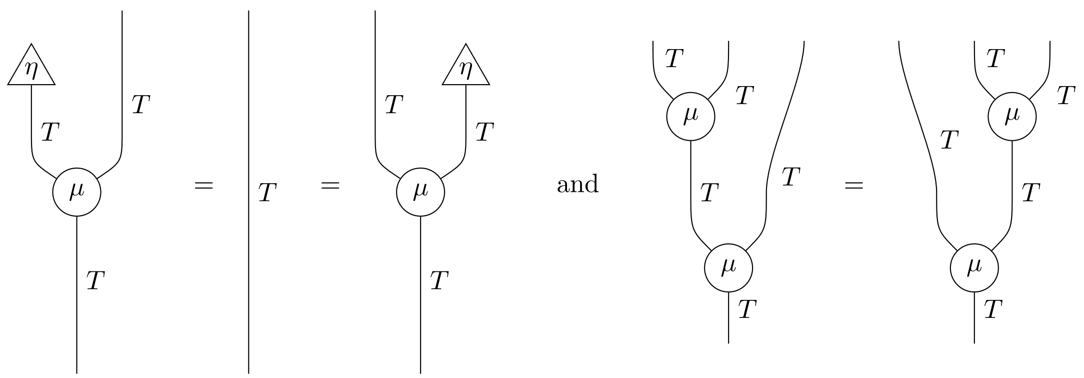
such that

- $\mu \circ T\eta = 1_T = \mu \circ \eta T$,
- $\mu \circ T\mu = \mu \circ \mu T$.

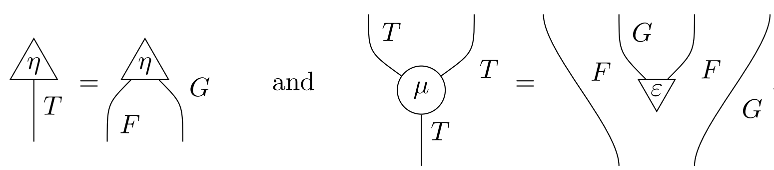
The natural transformations η and μ are represented by string diagrams



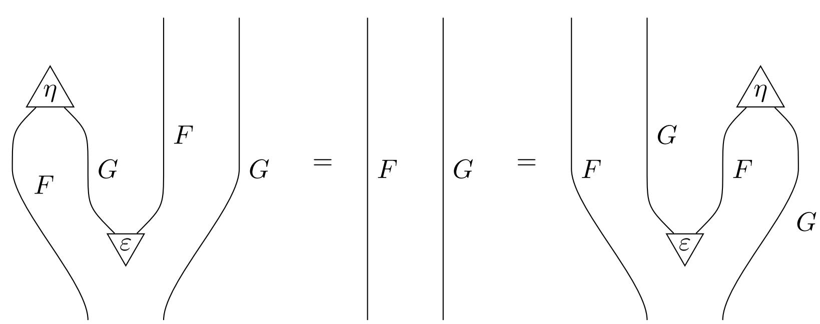
The equalities can then be written as



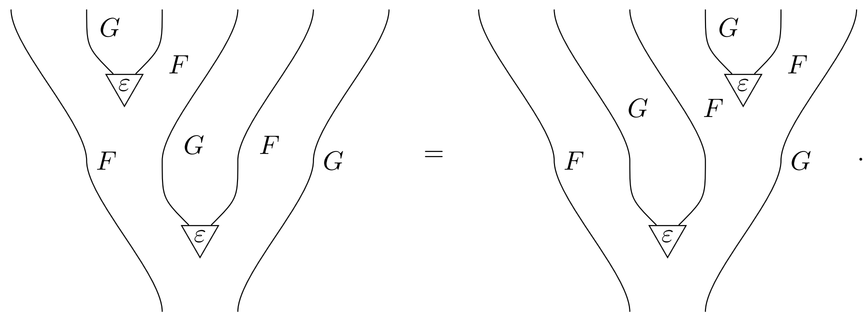
3.3. **Example.** If $(\mathcal{C}, \mathcal{D}, F, G, \eta, \varepsilon)$ is an adjoint pair, then $T = G \circ F$ is an endofunctor of $\mathcal{C}$ which can be made into a monad via the following diagrams:



Note here that there is no ambiguity between the two meanings of η. One can check the axioms:



and



3.4. **Definition.** Let $(\mathcal{C}, T, \eta, \mu)$ be a monad. An algebra over this monad is a tuple (c, α) where

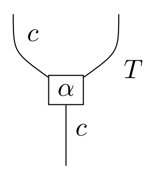
- c is an object of $\mathcal{C}$,
- $\alpha : T(c) \to c$ is a morphism

such that

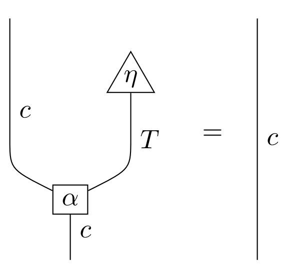
- $\alpha \circ \eta_c = \mathrm{id}_c$,
- $\alpha \circ T\alpha = \alpha \circ \mu_c$.

If (c, α) and (c', α') are algebras over the monad, then a T-algebra morphism $(c, \alpha) \to (c', \alpha')$ is a morphism $f : c \to c'$ such that $\alpha' \circ T(f) = f \circ \alpha$. We denote the category of algebras over the monad T by $\mathcal{C}^T$.

We can identify objects $c \in \mathrm{Obj}(\mathcal{C})$ with functors from the singleton category to $\mathcal{C}$ and morphisms between objects as natural transformations between the corresponding functors from the singleton category. Using this identification, the morphism h can be represented by the string diagram



and the two equalities can be written as



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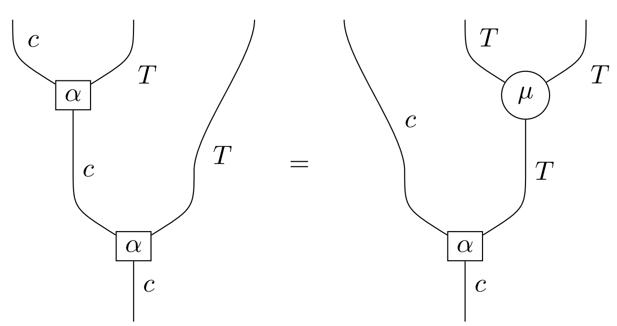
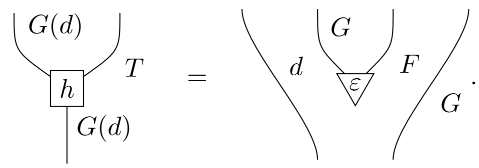
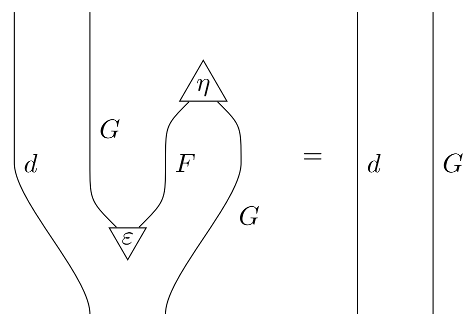


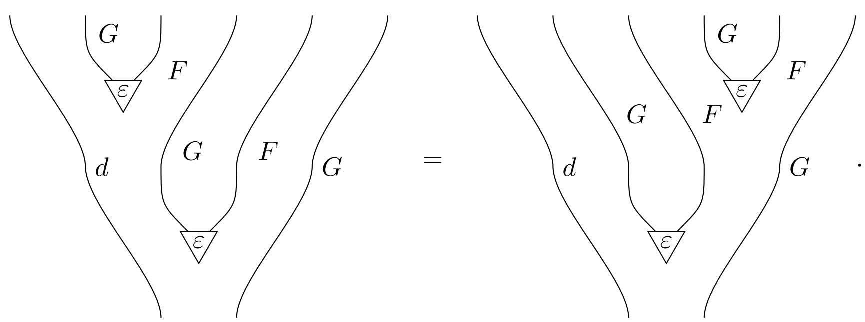
3.5. **Definition.** Let $(\mathcal{C}, \mathcal{D}, F, G, \eta, \varepsilon)$ be an adjoint pair, and let $(\mathcal{C}, T, \eta, \mu)$ be the corresponding algebra on $\mathcal{C}$ as in Example 3.3. Then for every object $d \in \mathrm{Obj}(\mathcal{D})$, the object $G(d) \in \mathrm{Obj}(\mathcal{C})$ where we define



We check the equalities



and



This map $d \mapsto G(d)$ with its algebra structure defines a functor from the category $\mathcal{D}$ to the category $\mathcal{C}^T$ of T-algebras.

We say that the adjunction $(\mathcal{C}, \mathcal{D}, F, G, \eta, \varepsilon)$ is *monadic* if the morphism $\mathcal{D} \to \mathcal{C}^T$ is an equivalence of categories.

4. Statement of the Bénabou-Roubaud Theorem

In this section, we review the notions of fibrations, bifibrations and descent data. We then use these concepts to state the B'enabou-Roubaud theorem.

4.1. **Definition.** Let $\phi : \mathcal{F} \to \mathcal{C}$ be a functor. We say that a morphism $m : a \to b$ in $\mathcal{F}$ *lies over* the morphism $f : c \to d$ in $\mathcal{C}$ if $\phi(m) = f$.

Given $c \in \mathrm{Obj}(\mathcal{C})$, the *fiber category* over c, denoted $\mathcal{F}(c)$, is the category

- whose objects are the objects a of $\mathcal{F}$ with $\phi(a) = c$,
- whose morphisms $f : a \to a'$ between two objects a, a' are the morphisms between these two objects in $\mathcal{F}$ which lie over id_c.

4.2. **Definition.** Let $\phi : \mathcal{F} \to \mathcal{C}$ be a functor. Let $m : a \to b$ in $\mathcal{F}$ be a morphism lying over $f : c \to d$.

- We say that m is a *cartesian morphism* if for every morphism $n : a' \to b$ in $\mathcal{F}$ with the same target as m and also lying over f, there exists a unique morphism $\gamma : a' \to a$ which lies over id_c and satisfies $m \circ \gamma = n$.
- Dually, we say that m is a *co-cartesian morphism* if for every morphism $n : a \to b'$ in $\mathcal{F}$ with the same source as m and also lying over f, there exists a unique morphism $\gamma : b \to b'$ which lies over id_d and satisfies $\gamma \circ m = n$.

4.3. **Definition.** A *fibration over* $\mathcal{C}$ is a functor $\phi : \mathcal{F} \to \mathcal{C}$ such that

- any composition of cartesian morphisms is cartesian.
- for every pair $(b, f : c \to d)$ where b is an object of $\mathcal{F}$ and $f : c \to d$ is a morphism in $\mathcal{C}$ with $d = \phi(b)$, there exists a cartesian morphism $m : a \to b$ lying over f.

If $m : f^*b \to b$ is such a cartesian morphism, we shall call (f^*b, m) an *inverse image* of b under f. By the properties of cartesian morphisms, if (f^*b, m) and $((f^*b)', m')$ are two inverse images of b under f, then there exists a unique isomorphism $\iota : (f^*b)' \to f^*b$ lying over id_c satisfying $m' = m \circ \iota$.

If $\phi : \mathcal{F} \to \mathcal{C}$ is a fibration, then a *choice of inverse images* is a choice of inverse image $(f^*b, m_{f^*b} : f^*b \to b)$ for every pair (b, f) as above.

4.4. *Remark.* Fibrations over $\mathcal{C}$ are models of 2-presheaves on $\mathcal{C}$, i.e. contravariant 2-functors from $\mathcal{C}$ to the 2-category of small categories. Given a fibration $\phi : \mathcal{F} \to \mathcal{C}$ such that the fiber categories $\mathcal{F}(c)$ are small, along with a choice of inverse images $(f, b) \mapsto (f^*b, m_{f^*b})$, we can define a 2-functor as follows:

- At the level of objects, we send $c \in \mathrm{Obj}(\mathcal{C})$ to the category $\mathcal{F}(c)$.
- At the level of morphisms, if $f : c \to d$ is a morphism in $\mathcal{C}$ and b is an object in $\mathcal{F}(d)$, then f^*b is an object of $\mathcal{F}(c)$. Moreover, if $h : b \to b'$ is a morphism in $\mathcal{F}(d)$, then there exists a unique morphism $f^*(h) : f^*b \to f^*b'$ such that $m_{f^*b'} \circ f^*(h) = h \circ m_{f^*b}$. This defines a functor $\mathcal{F}(d) \to \mathcal{F}(c)$.
- At the level of 2-morphisms, if $f = h \circ g$ in $\mathcal{C}$ and d is the target of this morphism, then for any object b of $\mathcal{F}(d)$, both (f^*b, m_{f^*b}) and $(h^*g^*b, m_{g^*b} \circ m_{h^*g^*b})$ are inverse images of b under $f = h \circ g$. There is then a unique isomorphism $\iota_b : f^*b \to h^*g^*b$ with $m_{f^*b} = m_{g^*b} \circ m_{h^*g^*b} \circ \iota_b$. We have thus described a natural isomorphism between f^* and h^*g^*.

These definitions will give us a contravariant 2-functor from $\mathcal{C}$ to the 2-category of small categories. Note the fact that in this last point, the functors f^* and h^*g^* are not the same but rather are naturally isomorphic. This is what makes this into a 2-functor and not a functor.

On the other hand, one can go backwards: given a 2-functor from $\mathcal{C}$ to the 2-category of small categories, it is possible construct an associated fiber category over $\mathcal{C}$. We skip this construction. Going from a fibration to a 2-functor back to a fibration yields a fibration over $\mathcal{C}$ which is equivalent (in a suitable sense) to the original fibration. Similarly, going from a 2-functor to a fibration back to a 2-functor yields a 2-functor which is equivalent to the orginial one.

4.5. **Definition.** Dually, a *op-fibration* is a functor $\phi : \mathcal{F} \to \mathcal{C}$ such that

- the composition of co-cartesian morphisms is co-cartesian.
- for every pair $(a, f : c \to d)$ where a is an object of $\mathcal{F}$ and $f : c \to d$ is a morphism in $\mathcal{C}$ with $d = \phi(a)$, there exists a co-cartesian morphism $m : a \to b$ lying over f.

Note that $\phi : \mathcal{F} \to \mathcal{C}$ being an op-fibration is equivalent to $\phi^{\mathrm{op}} : \mathcal{F}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ being a fibration.

If $m : a \to f_*a$ is a co-cartesian morphism lying over f, we say that (f_*a, m) is a direct image of a under f. Similar to before, two direct images are canonically isomorphic, and we can similarly give the notion of a *choice of direct images.* Moreover, given a choice of direct images, we can define direct image functors $f_* : \mathcal{F}(c) \to \mathcal{F}(d)$ for every morphism $f : c \to d$ in $\mathcal{C}$.

4.6. **Definition.** A *bifibration* $\phi : \mathcal{F} \to \mathcal{C}$ is a morphism of categories such that both $\phi : \mathcal{F} \to \mathcal{C}$ and $\phi^{\mathrm{op}} : \mathcal{F}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ are fibrations, or equivalently, if ϕ is both a fibration and an op-fibration.

4.7. **Proposition.** *Suppose $\phi : \mathcal{F} \to \mathcal{C}$ is a bifibration. Pick any choice of inverse images and direct images. For morphisms $f : c \to d$ in $\mathcal{C}$, let $f^* : \mathcal{F}(d) \to \mathcal{F}(c)$ and $f_* : \mathcal{F}(c) \to \mathcal{F}(d)$ be the corresponding inverse and direct image functors. Then for any $a \in \mathrm{Obj}(\mathcal{F}(c))$ and $b \in \mathrm{Obj}(\mathcal{F}(d))$,*

$$\mathrm{Hom}_{\mathcal{F}(d)}(f_*a, b) \cong \mathrm{Hom}_f(a, b) \cong \mathrm{Hom}_{\mathcal{F}(c)}(a, f^*b).$$

Here $\mathrm{Hom}_f(a, b)$ denotes the set of morphisms $a \to b$ in $\mathcal{F}$ which lie over f. In particular, we have an adjoint pair $f_* \dashv f^*$.

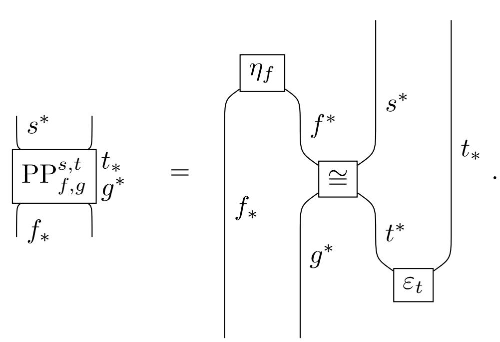
The given isomorphisms are easy to check from the definitions of pullback and pushforward morphisms. If $\phi : \mathcal{F} \to \mathcal{C}$ is a bifibration and $f : a \to b$ is a functor in $\mathcal{C}$, we shall denote the unit and counit of the adjunction $f_* \dashv f^*$ by η_f and ε_f, respectively.

4.8. *Remark.* One sees that bifibrations $\phi : \mathcal{F} \to \mathcal{C}$ give a contravariant 2-functors from $\mathcal{C}$ to the 2-category of small categories with right adjoints as the morphisms. On the other hand, the fibration formed from a contravariant 2-functor from $\mathcal{C}$ to the 2-category of small categories with right adjoints as the morphisms is actually a bifibration. So bifibrations are models of such contravariant functors.

4.9. **Definition.** Suppose $\phi : \mathcal{F} \to \mathcal{C}$ is a bifibration, and pick a choice of pullbacks and pushforwards. For any commuting square

$$\begin{tikzcd} a \arrow[r, "s"] \arrow[d, "t"'] & b \arrow[d, "f"] \\ c \arrow[r, "g"] & d \end{tikzcd}$$

in $\mathcal{C}$, we define the *push-pull natural transformation* $\mathrm{PP}^{s,t}_{f,g} : t_*s^* \to g^*f_*$ by the string diagram



The center of the second diagram is the isomorphism $s^*f^* \cong t^*g^*$ coming from the fact that $f \circ s = g \circ t$.

4.10. **Definition** (Beck-Chevalley Condition)**.** Let $\phi : \mathcal{F} \to \mathcal{C}$ be a bifibration. Pick any choice of inverse and direct images. We say that the bifibration satisfies the *Beck-Chevalley condition* if for every pullback square

$$\begin{tikzcd} a \arrow[r, "s"] \arrow[d, "t"] & b \arrow[d, "f"] \\ c \arrow[r, "g"] & d \end{tikzcd}$$

in $\mathcal{C}$, the push-pull natural transformation $\mathrm{PP}^{s,t}_{f,g}$ is a natural isomorphism.

Finally, the last element we need to state the Bénabou-Roubaud theorem is the notion of descent data.

4.11. **Definition** (Descent Data)**.** Suppose $\phi : \mathcal{F} \to \mathcal{C}$ is a fibration and suppose $\mathcal{C}$ has all pullbacks. Let $f : b \to a$ be a morphism in $\mathcal{C}$. Let $p_1, p_2 : a \times_b a \to a$ denote the two projections, and $\pi_1, \pi_2, \pi_3 : a \times_b a \times_b a \to a$ denote the three projections. Write $\pi_{ij} = (\pi_i, \pi_j) : a \times_b a \times_b a \to a \times_b a$ for $i, j \in \{1, 2, 3\}$.

A *descent datum* relative to f is a pair (x, β) where

- x is an object of $\mathcal{F}(a)$,
- $\beta : p_1^*x \to p_2^*x$ is an isomorphism,

such that the following holds:

- if $\tau_{ij} : \pi_i^*x \to \pi_j^*x$ is the composite isomorphism
$$\pi_i^*x \cong \pi_{ij}^*p_0^*x \xrightarrow{\beta} \pi_{ij}^*p_1^*x \cong \pi_j^*x$$
then $\tau_{13} = \tau_{23} \circ \tau_{12}$.

The collection of descent data relative to f are the objects of a category $\mathrm{Desc}(f)$. The collection of morphisms from (x, β_x) to (y, β_y) is the collection of morphisms $h : x \to y$ in $\mathcal{F}(a)$ such that $p_1^*(h) \circ \beta_x = \beta_y \circ p_0^*(h)$.

4.12. *Remark.* The category of descent data relative to f is a model for the 2-limit of the diagram

$$\mathcal{F}(b) \rightrightarrows \mathcal{F}(b \times_a b) \overset{\longrightarrow}{\rightrightarrows} \mathcal{F}(b \times_a b \times_a b).$$

Here, the two morphisms $\mathcal{F}(b) \to \mathcal{F}(b \times_a b)$ are p_2^* and p_1^* and the three morphisms $\mathcal{F}(b \times_a b) \to \mathcal{F}(b \times_a b \times_a b)$ are π_{23}^*, π_{13}^* and π_{12}^*.

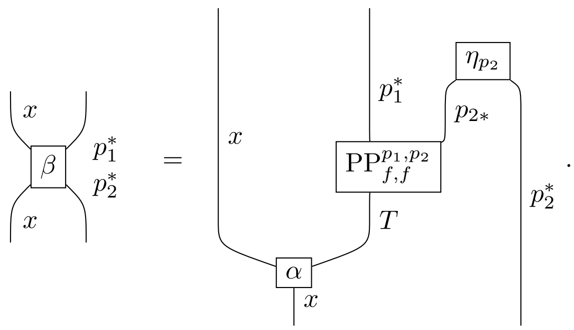
4.13. **Example.** There is a natural functor $\mathcal{F}(b) \to \mathrm{Desc}(f)$. If y is an object of $\mathcal{F}(b)$, then f^*y is an object of $\mathcal{F}(a)$. Moreover, since $p_1 \circ f = p_2 \circ f$ as morphisms $a \times_b a \to b$, there is a natural isomorphism $\iota : p_1^*f^* \cong p_2^*f^*$. Then (f^*y, ι_y) will be a descent datum along f. This will define our functor $\mathcal{F}(b) \to \mathrm{Desc}(f)$.

4.14. **Definition.** If $\phi : \mathcal{F} \to \mathcal{C}$ is a fibration. Suppose $\mathcal{C}$ has all pullbacks, and let $f : a \to b$ be a morphism in $\mathcal{C}$. We say that *descent along f is effective* if the natural functor $\mathcal{F}(b) \to \mathrm{Desc}(f)$ is an equivalence of categories.

We can now state the Bénabou-Roubaud result.

4.15. **Theorem** (Bénabou-Roubaud)**.** *Let $\phi : \mathcal{F} \to \mathcal{C}$ be a bifibration, and suppose $\mathcal{C}$ has all pullbacks. For any morphism $f : b \to a$ in $\mathcal{C}$, let $(\mathcal{F}(b), T, \eta_f, \mu)$ be the monad defined by the adjoint pair $f_* \dashv f^*$. Then*

(1) *for any T-algebra $(x, \alpha : Tx \to x)$, the pair $(x, \beta : p_1^* x \to p_2^* x)$ is a descent datum along f, where β is defined by the string diagram*



This defines a functor

$$\mathcal{F}(b)^T \to \mathrm{Desc}(f).$$

(2) *There are natural morphisms $\mathcal{F}(a) \to \mathcal{F}(b)^T$ and $\mathcal{F}(a) \to \mathrm{Desc}(f)$ coming from the properties of adjunctions and the properties of descent data. The diagram*

$$\begin{tikzcd}
\mathcal{F}(a) \arrow[d] \arrow[dr] & \\
\mathcal{F}(b)^T \arrow[r] & \mathrm{Desc}(f)
\end{tikzcd}$$

commutes up to a canonical natural isomorphism.

(3) *If $\phi : \mathcal{F} \to \mathcal{C}$ satisfies the Beck-Chevalley condition, then the functor $\mathcal{F}(a)^T \to \mathrm{Desc}(f)$ is an equivalence of categories.*

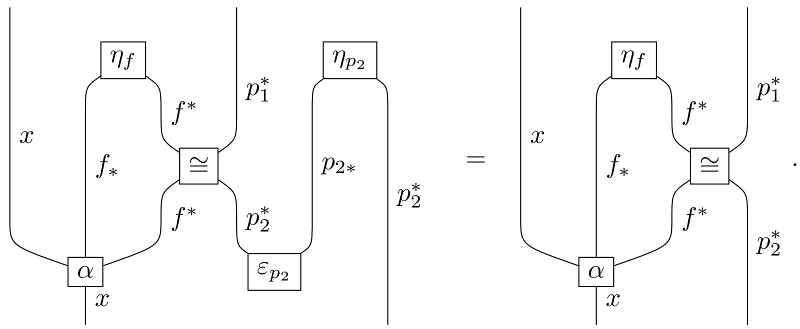
4.16. *Remark.* Actually, under our particular definition of $\mathrm{Desc}(f)$ in given in 4.11, along with our definitions of the functors into $\mathrm{Desc}(f)$, the diagram in (2) commutes, and when the Beck-Chevalley condition holds, the functor $\mathcal{F}(a)^T \to \mathrm{Desc}(f)$ is an isomorphism. However, if we think of $\mathrm{Desc}(f)$ as the 2-limit of a certain diagram, then $\mathrm{Desc}(f)$ is only defined up to equivalence of categories, and it is only possible to specify functors into $\mathrm{Desc}(f)$ up to a canonical natural isomorphism. This is why we have stated the theorem in the way that we did.

4.17. **Corollary.** *Suppose we are in the situation of the Bénabou-Roubaud theorem, and our bifibration satisfies the Beck-Chevalley condition. If $f_* \dashv f^*$ is monadic, then descent along f is effective.*

5. The functor $\mathcal{F}(b)^T \to \mathrm{Desc}(f)$

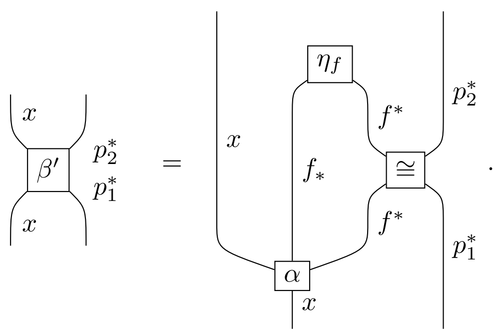
In this section we show parts (1) and (2) of Theorem 4.15.

For this section, let $(x, \alpha : Tx \to x)$ be a T-algebra and let $(x, \beta : p^* x \to p_2^* x)$ be defined as in Theorem 4.15(1) Expanding out $\mathrm{PP}^{p_1,p_2}_{f,f}$ in the definition of β, we see that it is equal to

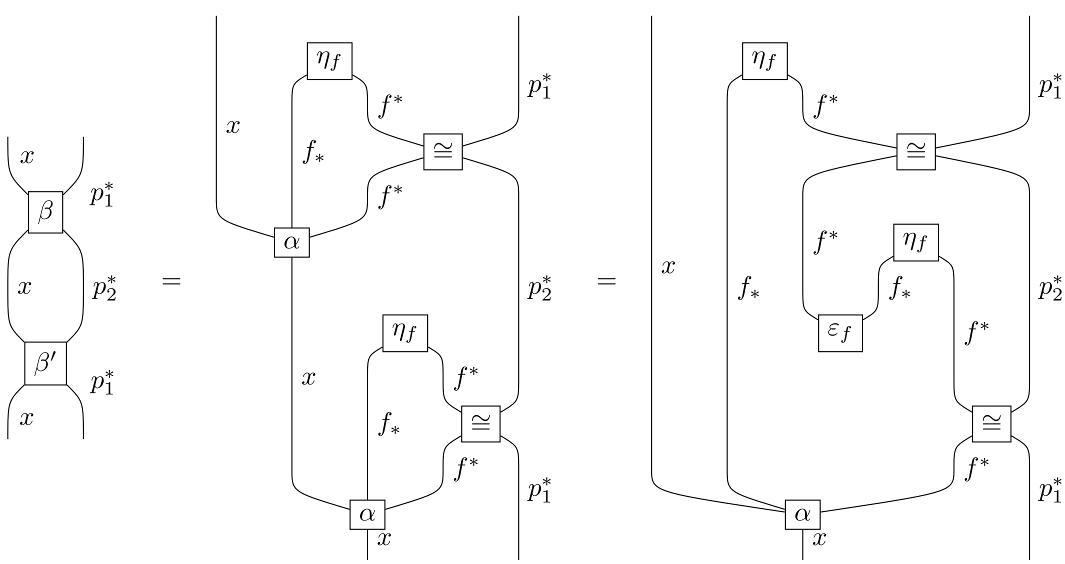


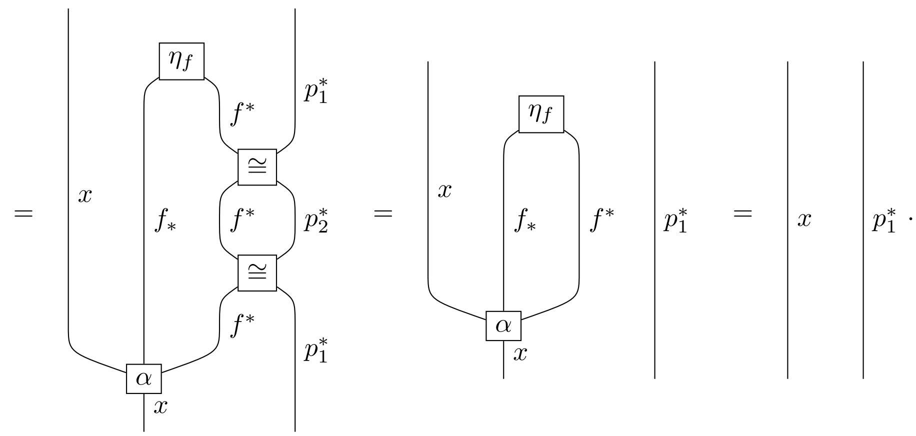
5.1. **Proposition.** *β is an isomorphism.*

Let β' be the morphism $p_2^*x \to p_1^*x$ gotten by reversing the roles of p_1 and p_2 in the definition of β, i.e.



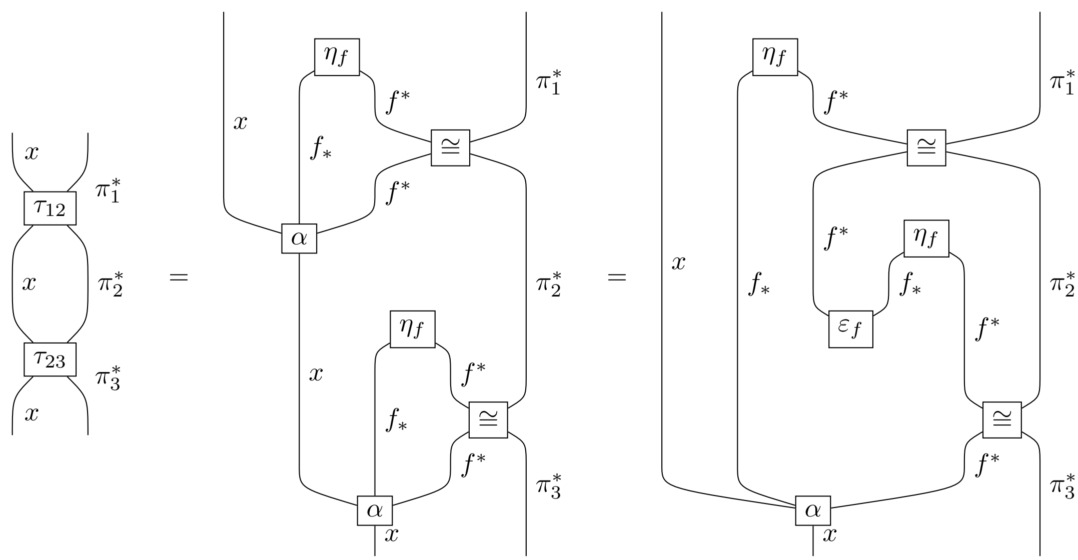
We show that β' is an inverse to β. The composition $\beta' \circ \beta$ is given by

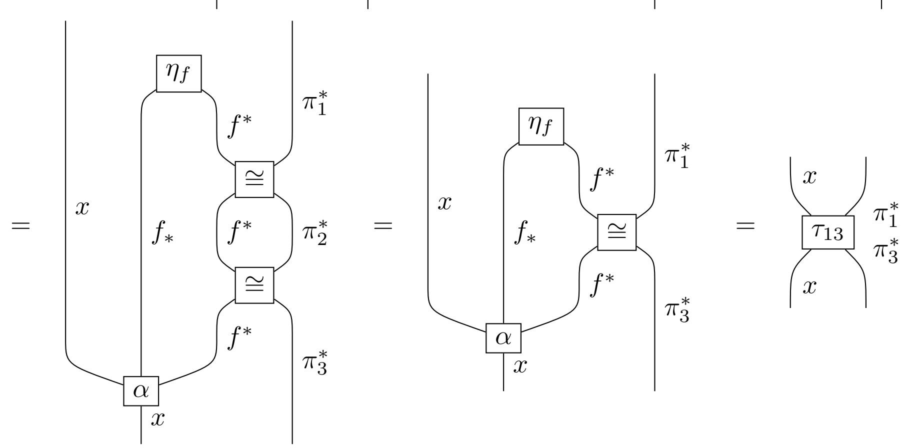




We thus see that $\beta' \circ \beta = \mathrm{id}_{p_1^* x}$. Reversing the roles of p_1 and p_2 shows that $\beta \circ \beta' = \mathrm{id}_{p_2^* x}$, so we conclude that β' is an inverse to β.

5.2. **Proposition.** *In the notation of Definition 4.11, $\tau_{13} = \tau_{23} \circ \tau_{12}$.*





The above string diagrams shows the proposition. Here, we've used the fact that the composition $\pi_1^* f^* \xrightarrow{\cong} \pi_2^* f^* \xrightarrow{\cong} \pi_3^* f^*$ is equal to $\pi_1^* f^* \xrightarrow{\cong} \pi_3^* f^*$ using the properties of fibrations: $\pi_1^* f^* z$ and $\pi_3^* f^* z$ both come from cartesian arrows, and there is exactly one morphism between them making the diagram with these cartesian arrows commute.

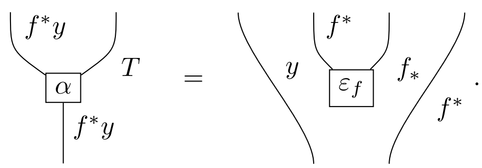
We have thus shown that our definition of $(x, \alpha) \mapsto (x, \beta)$ defines a functor $\mathcal{F}(b)^T \to \mathrm{Desc}(f)$. It just remains to show:

5.3. **Proposition.** *The diagram*

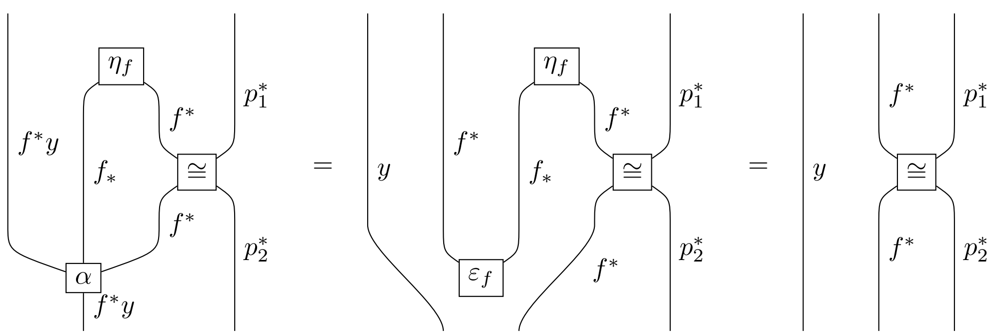
$$\begin{tikzcd} \mathcal{F}(a) \arrow[d] \arrow[dr] & \\ \mathcal{F}(b)^T \arrow[r] & \mathrm{Desc}(f) \end{tikzcd}$$

commutes.

Let y be an object of $\mathcal{F}(a)$. Then y maps to (f^*y, α) in $\mathcal{F}(b)^T$, where



We thus see that the associated β is



which is exactly the isomorphism $p_1^* f^* y \to p_2^* f^* y$ in the map $\mathcal{F}(a) \to \mathrm{Desc}(f)$. This shows the commutativity of the diagram.

6. A functor from $\mathrm{Desc}(f)$ to an auxiliary category $\mathcal{F}(b)^{p_{2*}p_1^*}$

In this section, we define a category $\mathcal{F}(b)^{p_{2*}p_1^*}$, along with a functor $\mathrm{Desc}(f) \to \mathcal{F}(b)^{p_{2*}p_1^*}$.

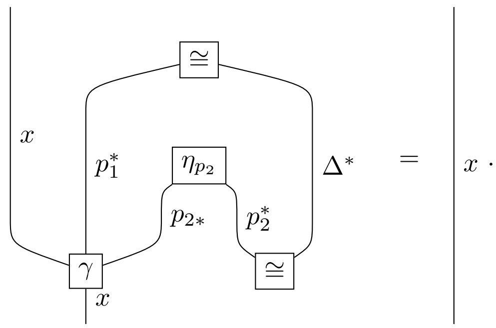
In the following, let $\Delta : b \to b \times_a b$ be the diagonal morphism.

6.1. **Definition.** Let $\mathcal{F}(b)^{p_{2*}p_1^*}$ benote the category of pairs (x, γ) where

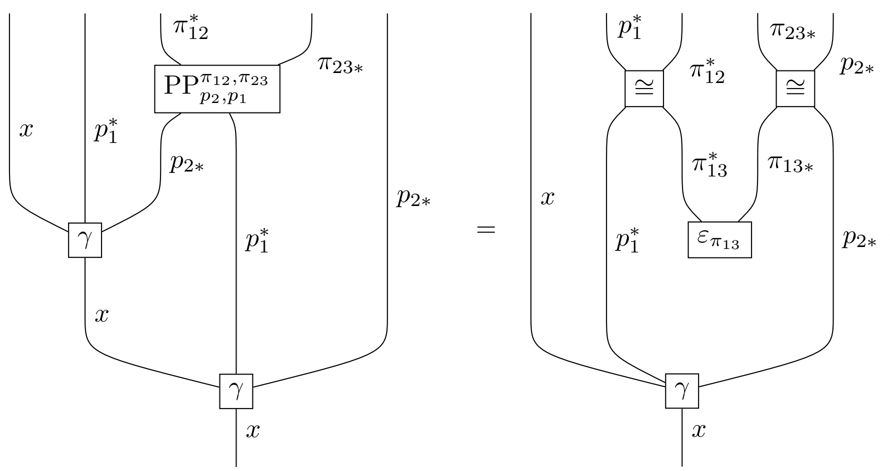
- x is an object of $\mathcal{F}(b)$
- $\gamma : p_{2*}p_1^* x \to x$ is a morphism in $\mathcal{F}(b)$

subject to the following two axioms:

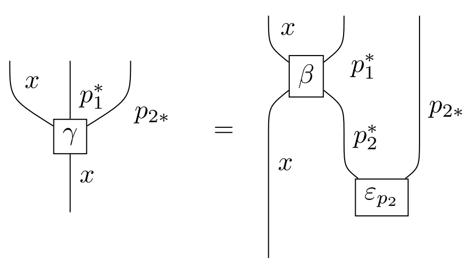
(1)



(2)

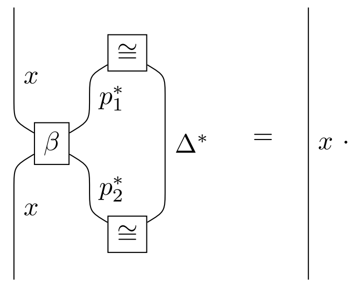


6.2. **Definition.** We define a functor $\mathrm{Desc}(f) \to \mathcal{F}(b)^{p_{2*}p_1^*}$ by sending $(x, \beta : p_1^*x \to p_2^*x)$ to $(x, \gamma : p_{2*}p_1^*x \to x)$ where γ is defined by



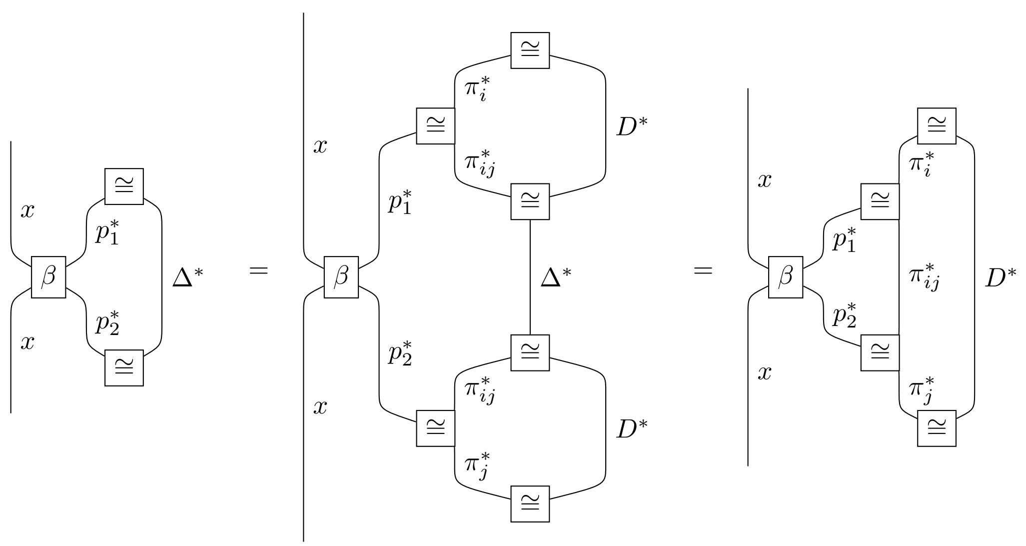
We need to check that this γ satisfies the two axioms.
To check the first, we need the following lemma:

6.3. **Lemma.** *Suppose* $(x, \beta) \in \mathrm{Desc}(f)$*. Then*

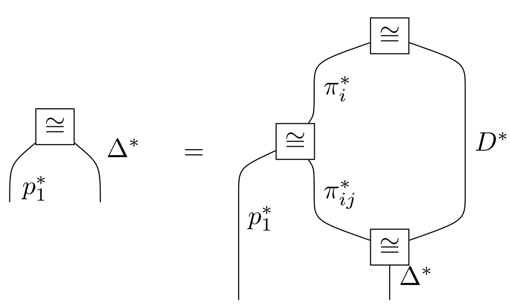


Proof. Let $\iota : x \to x$ be the morphism defined by the left diagram above. Since β is an isomorphism, we see immediately that ι is also an isomorphism. To show that $\iota = \mathrm{id}_x$, it thus suffices to show that $\iota \circ \iota = \iota$.

Let $D : b \to b \times_a b \times_a b$ be the diagonal morphism. Note that for any $1 \leq i < j \leq 3$, we have

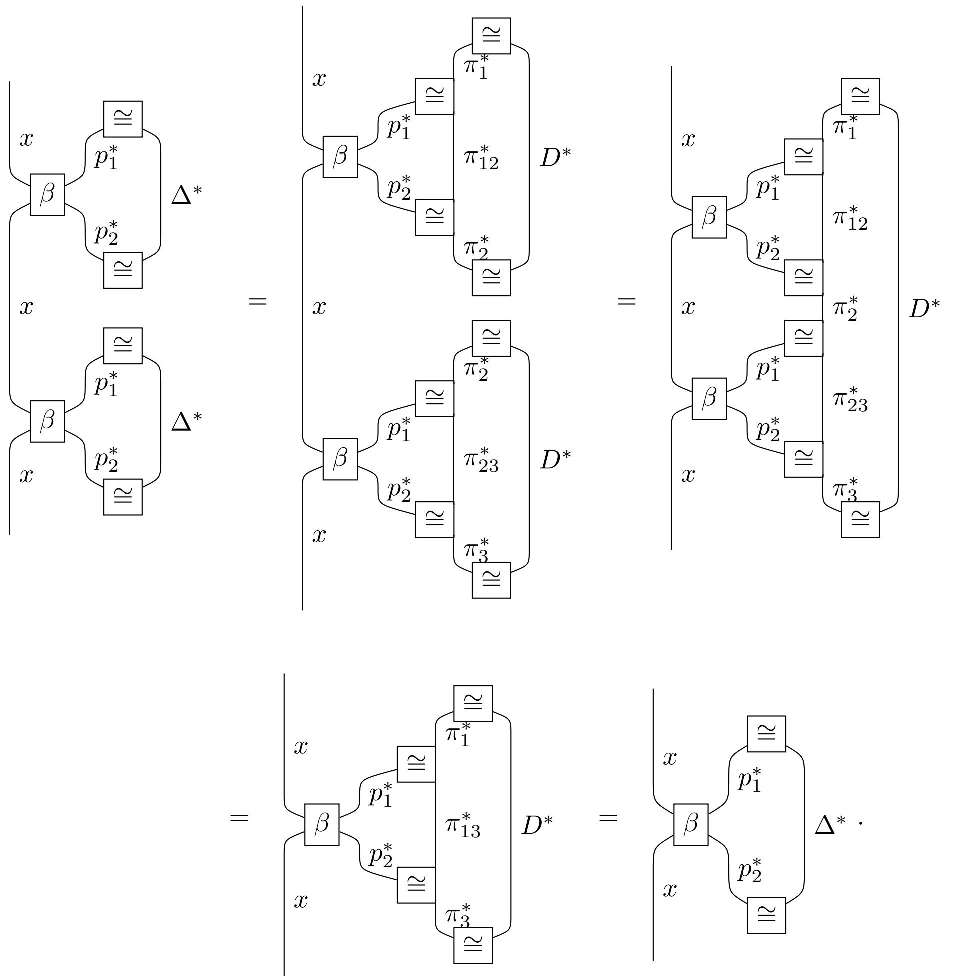


Here we note that



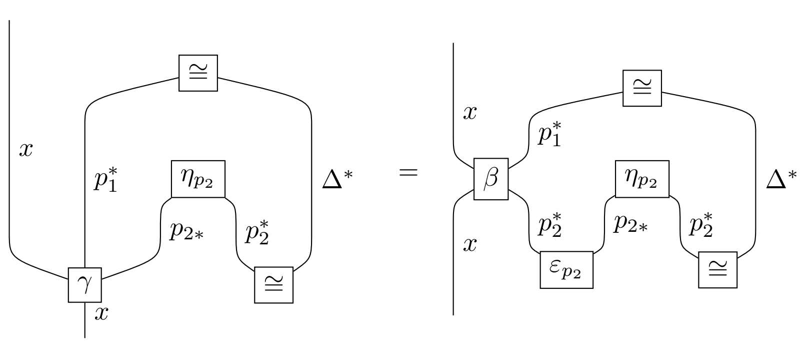
because both are equal to the unique natural transformation $\mathrm{id}_{\mathcal{F}(b)} \to \Delta^* p_1^*$ coming from the properties of cartesian arrows. A similar identity going the other way was used to deduce the first equality above.

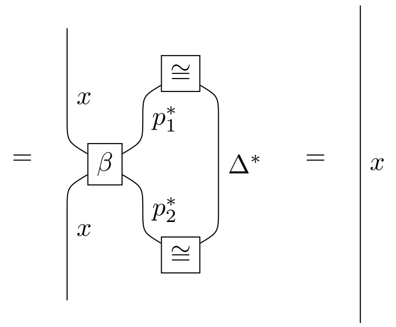
We then see that



Where the third equality comes from the axiom $\tau_{13} = \tau_{23} \circ \tau_{12}$ of descent data. This shows that $\iota \circ \iota = \iota$, thus proving the lemma. □

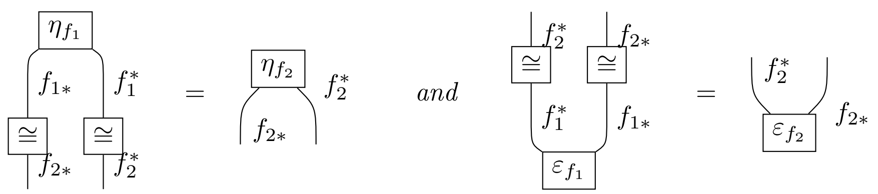
That our definition of γ satisfies the first axiom is now easy to show since





To show the second axiom, we use the following lemma:

6.4. **Lemma.** *Suppose $\phi : \mathcal{F} \to \mathcal{C}$ is a bifibration and let $f : b \to a$ be a morphism in $\mathcal{C}$. Let $f_1^*, f_2^* : \mathcal{F}(a) \to \mathcal{F}(b)$ and $f_{1*}, f_{2*} : \mathcal{F}(b) \to \mathcal{F}(a)$ be two inverse image and direct image functors resulting from two different choices of inverse images and direct images. Then*

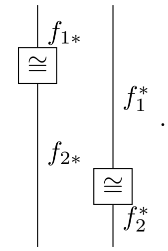


Proof. We show the first equality, as the second equality is dual. Let x be an object of $\mathcal{F}(b)$, and let $\alpha_1 : x \to f_{1*}x$, $\alpha_2 : x \to f_{2*}x$ be the cocartesian morphisms used when defining the functors f_{1*} and f_{2*}, respectively. Let $\beta_{11} : f_1^* f_{1*}x \to f_{1*}x$, $\beta_{12} : f_1^* f_{2*}x \to f_{2*}x$, and $\beta_{22} : f_2^* f_{2*}x \to f_{2*}x$ be the cartesian morphisms used when defining the functors f_1^*, f_1^* and f_2^*, respectively.

Then $\eta_1 := \eta_{f_1,x} : x \to f_1^* f_{1*}x$ and $\eta_2 := \eta_{f_2,x} : x \to f_2^* f_{2*}x$ are the unique morphisms such that $\alpha_1 = \beta_1 \circ \eta_1$ and $\alpha_2 = \beta_2 \circ \eta_2$. Let $i : f_{1*}x \to f_{2*}x$ be the morphism such that $\alpha_2 = i \circ \alpha_1$, and $j : f_1^* f_{2*}x \to f_2^* f_{2*}x$ such that $\beta_{12} = \beta_{22} \circ j$. Then the diagram

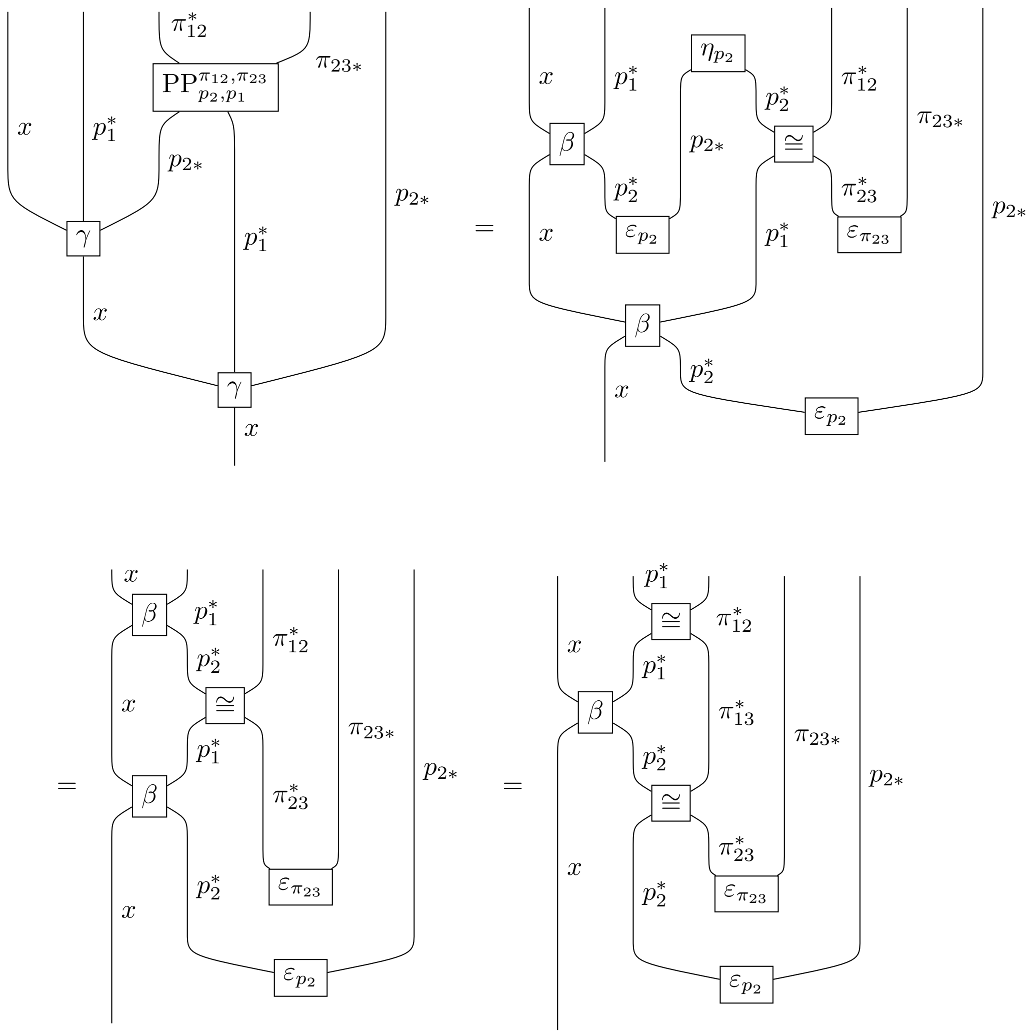
$$\begin{tikzcd}
 & x \arrow[dl, "\eta_1"'] \arrow[dr, "\alpha_1"] & \\
f_1^* f_{1*}x \arrow[rr, "\beta_{11}"] \arrow[d, "f_1^* i"] & & f_{1*}x \arrow[dd, "i"] \\
f_1^* f_{2*}x \arrow[d, "j"] \arrow[drr, "\beta_{12}"] & & \\
f_2^* f_{2*x} \arrow[rr, "\beta_{22}"] & & f_{2*}x
\end{tikzcd}$$

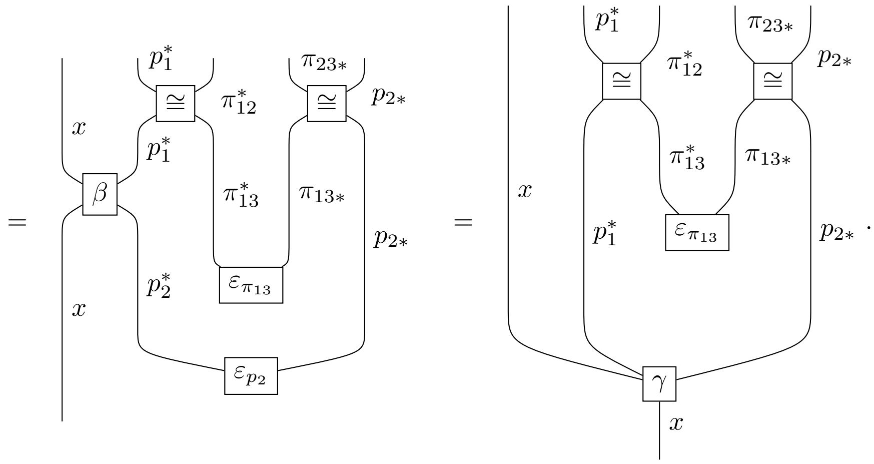
commutes since each of the three subdiagrams commute. As $\alpha_2 = i \circ \alpha_1$ and η_2 is the unique morphism such that $\alpha_2 = \beta_2 \circ \eta_2$, we see that $\eta_2 = j \circ (f_1^* i) \circ \eta_1$. But the morphism $j \circ (f_1^* i) : f_1^* f_{1*}x \to f_2^* f_{2*}x$ is precisely the morphism represented by the string diagram



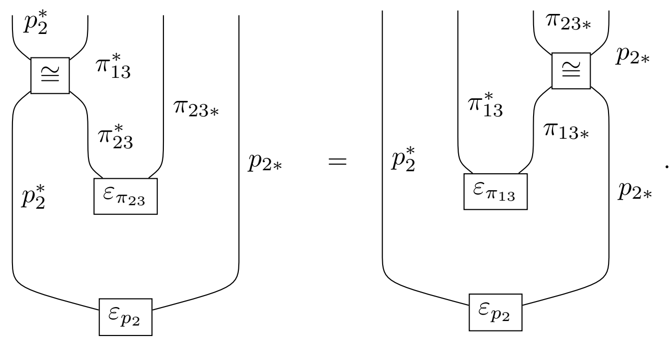
This shows the desired equality. $\square$

We can now check that our definition of γ satisfies the second axiom.





Here, we've used the axiom $\tau_{13} = \tau_{23} \circ \tau_{12}$ on the second line. To get from the second to the third line, we've also applied Lemma 6.4 to the adjoints $\pi_{2*}\pi_{13*} \dashv \pi_{13}^* p_2^*$ and $\pi_{2*}\pi_{23*} \dashv \pi_{23}^* p_2^*$ to show that

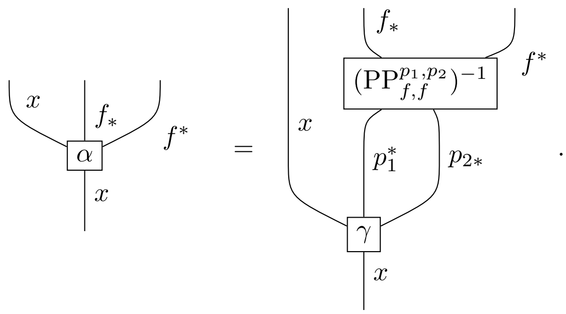


Since both axioms hold for (x, γ), we thus see that the map $(x, \beta) \mapsto (x, \gamma)$ is a functor $\mathrm{Desc}(f) \to \mathcal{F}(b)^{p_{2*}p_1^*}$.

7. A functor $\mathcal{F}(b)^{p_{2*}p_1^*} \to \mathcal{F}(b)^T$ under Beck-Chevalley

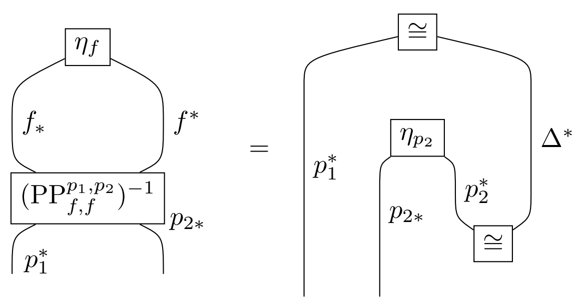
In this section, we assume that our bifibration $\phi : \mathcal{F} \to \mathcal{C}$ satisfies the Beck-Chevalley condition. Under this assumption, we can then define a functor $\mathcal{F}(b)^{p_{2*}p_1^*} \to \mathcal{F}(b)^T$. The three functors $\mathcal{F}(b)^T \to \mathrm{Desc}(f)$, $\mathrm{Desc}(f) \to \mathcal{F}(b)^{p_{2*}p_1^*}$ and $\mathcal{F}(b)^{p_{2*}p_1^*} \to \mathcal{F}(b)^T$ will then form the sides of a triangle such that starting at any corner and going around the triangle results in the identity functor. This will show that the three categories are isomorphic, and implies the last part of the Bénabou-Roubaud theorem.

7.1. **Definition.** Assuming the Beck-Chevalley condition holds, we define a functor $\mathcal{F}(b)^{p_{2*}p_1^*} \to \mathcal{F}(b)^T$ by sending the pair (x, γ) to (x, α) where $\alpha : f^* f_* x \to x$ is defined by

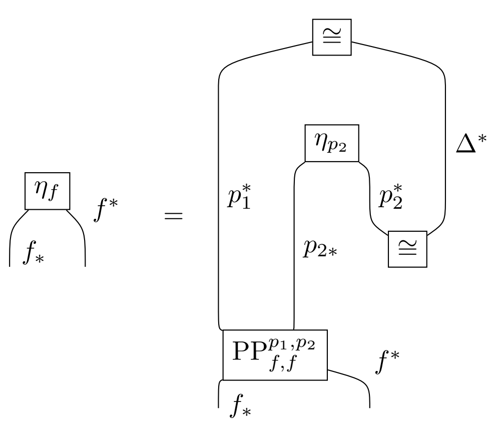


We need to check that (x, α) is a T-algebra.

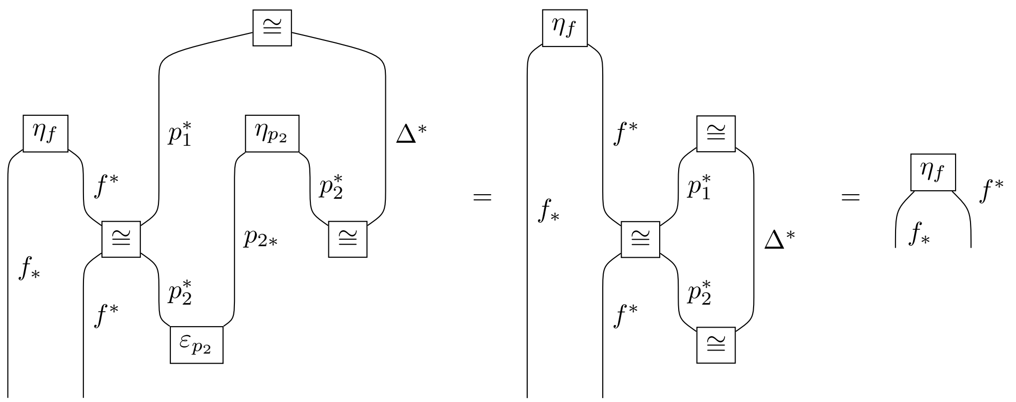
To check the first axiom of T-algebras, it suffices to show that



But this is equivalent to

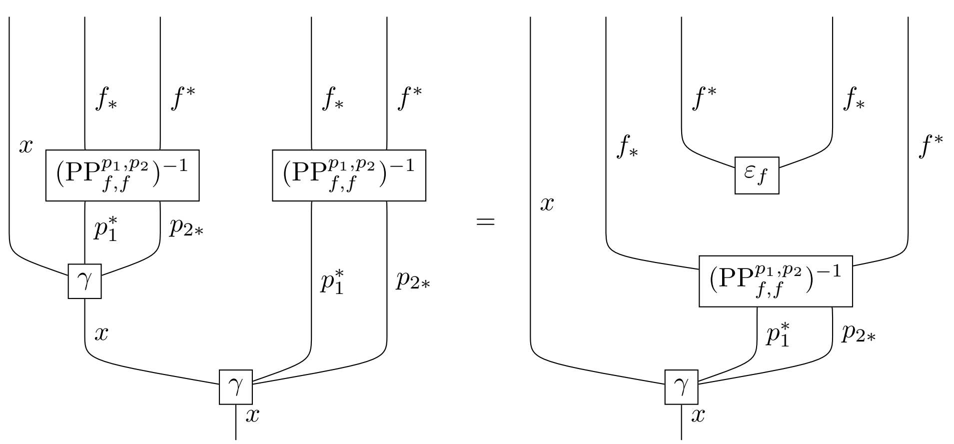


Expanding out the push-pull morphism on the right, we see that the right side is equal to

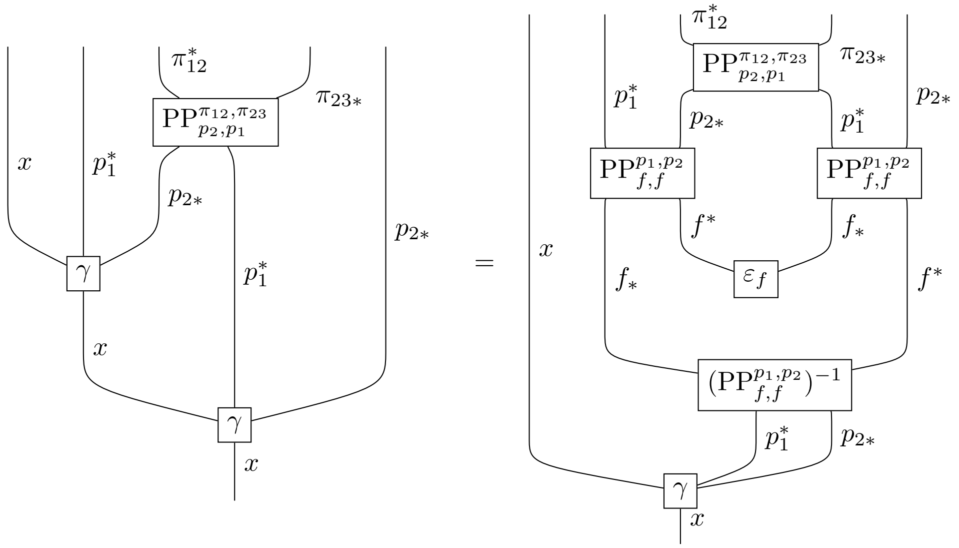


Here, we've used the fact that the compositon $f^*x \xrightarrow{\cong} \Delta^* p_1^* f^* x \xrightarrow{\cong} \Delta^* p_2^* f^* x \xrightarrow{\cong} f^*x$ is the unique morphism $f^*x \to f^*x$ coming from the fact that f^*x is an inverse image of x under f, i.e. it is the identity. We see that the desired equality holds, so (x, α) satisfies the first axiom of T-algebras.

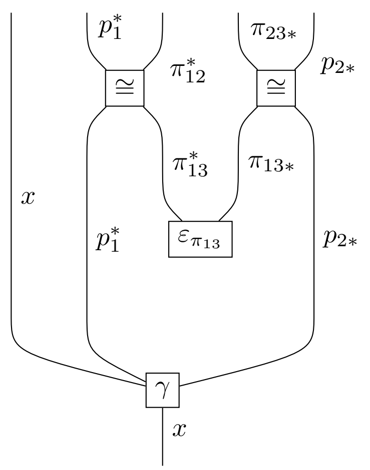
To see that (x, α) satisfies the second axiom for T-algebras, we need to show that



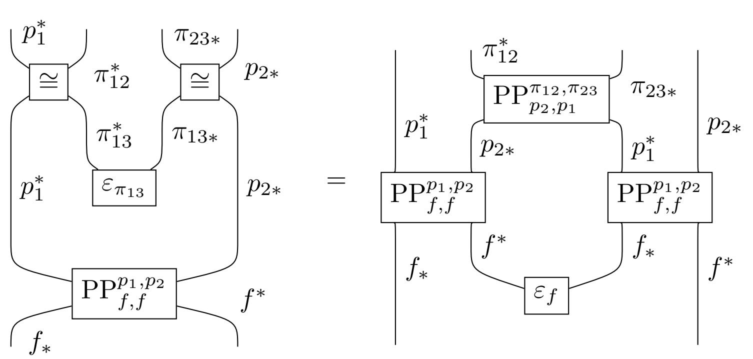
Since the Beck-Chevalley condition holds, we know that $\mathrm{PP}_{f,f}^{p_1,p_2}$ and $\mathrm{PP}_{p_2,p_1}^{\pi_{12},\pi_{23}}$ are isomorphisms, so this is equivalent to



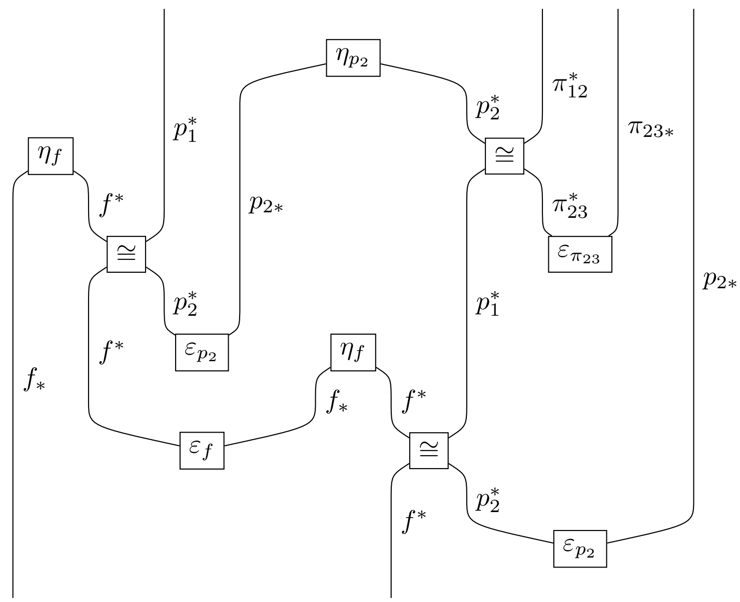
But by the second axiom for objects in $\mathcal{F}(b)^{p_{2*}p_1^*}$, the left side is equal to



so we see that it suffices to show that



Expanding out the push-pull morphisms in the second diagram, we get



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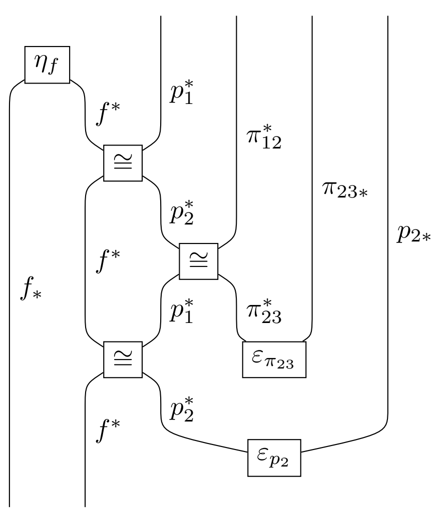

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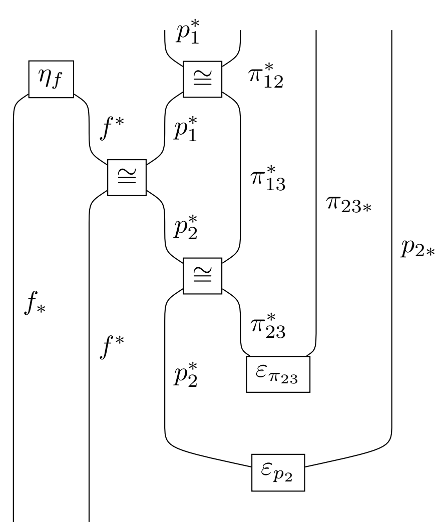

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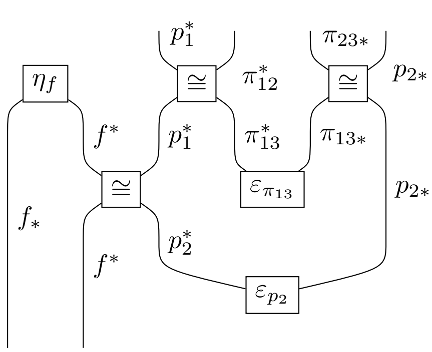

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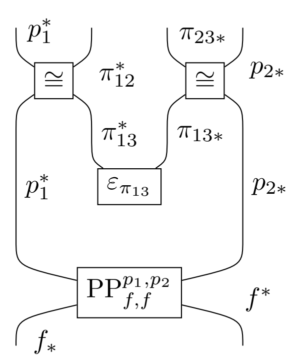

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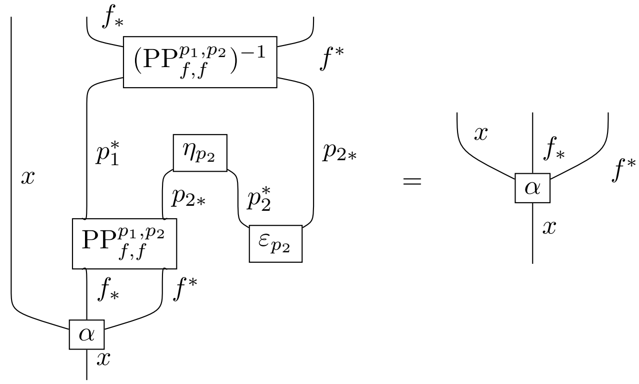
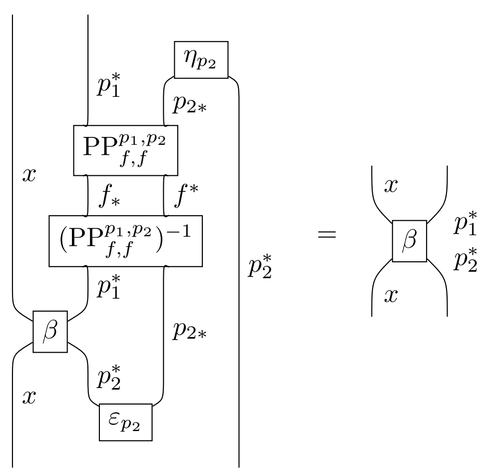
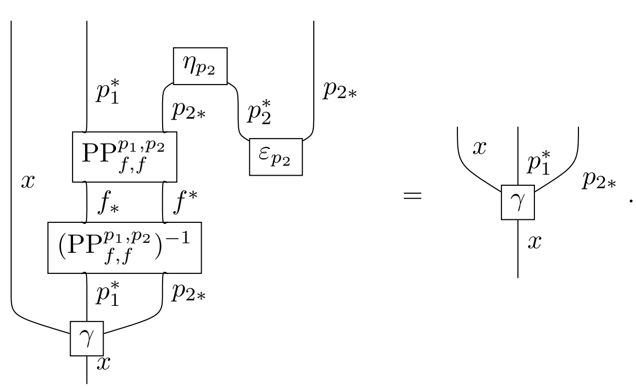
On the second line above, we've used the fact that there is a unique morphism $\pi_{12}^* p_1^* f^* \to \pi_{23}^* p_2^* f^*$ making the diagram with the associated cartesian arrows commute. To get from the second to the third line, we've used Lemma 6.4 applied to the adjoints $\pi_{2*}\pi_{13*} \dashv \pi_{13}^* p_2^*$ and $\pi_{2*}\pi_{23*} \dashv \pi_{23}^* p_2^*$ as in the proof of the second axiom when defining the functor $\mathrm{Desc}(f) \to \mathcal{F}(b)^{p_{2*}p_1^*}$. We thus conclude that (x, α) satisfies the second axiom of a T-algebra, and our map $(x, \gamma) \mapsto (x, \alpha)$ is a functor $\mathcal{F}(b)^{p_{2*}p_1^*} \to \mathcal{F}(b)^T$.

Finally, we check that the three functors define isomorphisms:

- 

- 

- 

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[1] Jean Bénabou, Jacques Roubaud, *Monades et descente*, C.R. Acad. Sc. Paris, t. 270 (12 Janvier 1970), Serie A, 96-98.

[2] Jovana Obradović, *The Bénabou-Roubaud result via string diagrams*. Draft report. `https://www.dropbox.com/s/236deicmr3636d5/draft.pdf?dl=0` Accessed April 2021.