Buffon's Needle Problem for a Rectangular Grid

Author(s): Robert Fakler

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Robert Fakler

Buffon's Needle Problem for a Rectangular Grid

If a needle of length l is dropped onto a floor constructed of evenly spaced wooden planks of width k, what is the probability that the needle crosses a crack between planks? This question is "Buffon's needle problem." Schroeder (1974), in an article in the *Mathematics Teacher,* considers this problem and shows that for the case $l \le k$, the solution is

$$p = \frac{2l}{k\pi}.$$

He observes that since classrooms no longer have wooden floors, the needle problem could be brought up to date by considering what happens when a needle of length l is dropped on a floor covered with square tiles of side k, where $l \le k$. He also asks what is the probability that the needle will cross a crack in the floor.

To solve this problem, it is necessary to represent the possible positions in which the needle could land, relative to the cracks in the floor, by a set of independent variables. How many such variables would be needed? Two variables—an x-coordinate and a y-coordinate—are required to specify the position of one point on the needle. But the needle could fall in infinitely many positions that have this point in the same position, relative to the cracks in the floor, but different "slopes." To determine a unique position of the needle, in addition to specifying the position of a particular point, we could specify the "slope" of the needle. This process would require a third variable. Thus we see that *three* independent variables are required to determine the result of a needle toss. In solving his variation of the needle problem, Schroeder mistakenly uses *two* variables to represent the set of all possible needle tosses and obtains an incorrect answer,

$$p = \frac{2l\sqrt{2}}{k\pi}.$$

Instead of the problem Schroeder proposed, we shall consider a more general problem, in which a needle of length l is tossed onto a rectangular grid composed of rectangles having dimensions $a \times b$. If a needle of length l is tossed onto a rectangular grid consisting of $a \times b$ rectangles, what is the probability that the needle will cross a grid line? We assume that $l \le a$ and $l \le b$. (If

$$a < l < \sqrt{a^2 + b^2}$$

or

$$b < l < \sqrt{a^2 + b^2},$$

the solution of this problem becomes more difficult, and we leave this case for the interested reader to pursue. If

$$l \ge \sqrt{a^2 + b^2},$$

then the needle will always cross a line, and so the probability that the needle will cross a line will be 1.)

Schroeder's problem is the special case where $a = b = k$. The outcome of each toss of the needle is determined by x, y, and θ, where x is the distance from the left end of the needle to the nearest vertical line to the right, y is the distance from the lower end of the needle to the nearest horizontal line above it, and θ is the angle whose initial side is the ray extending from the left end of the needle horizontally to the right and whose terminal side is the ray extending from the left end of the needle through the needle (see **fig. 1**). We notice that $0 \le x \le a$, $0 \le y \le b$, and $-\pi/2 < \theta \le \pi/2$. We can identify the set of all possible outcomes for our needle-tossing experiment with the points in the rectangular solid

$$S = \left\{(x, y, \theta) \,|\, 0 \le x \le a,\ 0 \le y \le b,\ \frac{-\pi}{2} < \theta \le \frac{\pi}{2}\right\}.$$

Let R be the subregion of S corresponding to the outcomes in which the needle crosses a line. Then the probability, p, of the needle's crossing a line will be

$$p = \frac{\text{volume of } R}{\text{volume of } S}.$$

If $0 \le \theta \le \pi/2$, the needle will cross a line if and only if $x \le l \cos \theta$ or $y \le l \sin \theta$. Let S^+ be the subregion of S consisting of those points in S for which

How do we solve the classic problem if we have tiles on the floor instead of floorboards?

Robert Fakler teaches at the University of Michigan—Dearborn, Dearborn, MI 48128. His interests include geometrical probability and its use to enhance mathematics teaching.

We can use the method of "volume by slicing"

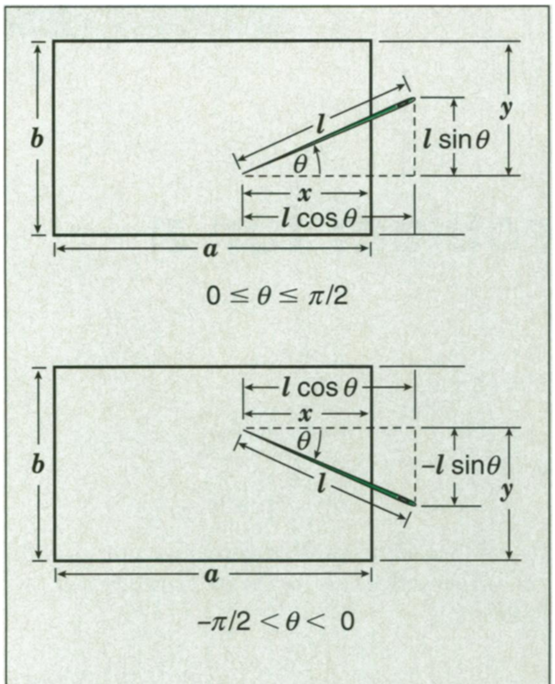

Fig. 1
Needle toss determined by x, y, and θ

$0 \le \theta \le \pi/2$. In this case, the needle tosses that result in the needle's crossing a line can be identified with the points in the region R^+ consisting of those points in S^+ that are inside the union of the cylinders $x = l\cos\theta$ and $y = l\sin\theta$. The regions R^+ and S^+ are shown in **figure 2.**

The region R^+ is the union of two right cylinders of heights a and b with bases B_1 and B_2, respectively. These cylinders will have volumes

$$(\text{area of } B_1)a$$

and

$$(\text{area of } B_2)b,$$

respectively. Let I be the region formed by the intersection of these cylinders. Then the volume of R^+ is

$$(\text{area of } B_1)a + (\text{area of } B_2)b - \text{volume of } I.$$

So

$$\begin{aligned}\text{area of } B_1 &= \int_0^{\pi/2} l\sin\theta\, d\theta \\ &= -l\cos\theta\Big|_0^{\pi/2} \\ &= l,\end{aligned}$$

and

$$\begin{aligned}\text{area of } B_2 &= \int_0^{\pi/2} l\cos\theta\, d\theta \\ &= l\sin\theta\Big|_0^{\pi/2} \\ &= l.\end{aligned}$$

To find the volume of I, we can use the method of "volumes by slicing." Notice that a typical cross section of I perpendicular to the θ-axis is a rectangle with dimensions $l\cos\theta$ and $l\sin\theta$. (See **fig. 3.**) Thus the area $A(\theta)$ of this cross section is given by

$$A(\theta) = (l\cos\theta)(l\sin\theta) = l^2\cos\theta\sin\theta,$$

and so

$$\begin{aligned}\text{volume of } I &= \int_0^{\pi/2} A(\theta)\, d\theta \\ &= \int_0^{\pi/2} l^2 \sin\theta\cos\theta\, d\theta \\ &= l^2\left(\frac{\sin^2\theta}{2}\right)\Big|_0^{\pi/2} \\ &= \frac{l^2}{2}.\end{aligned}$$

Therefore,

$$\text{volume of } R^+ = la + lb - \frac{l^2}{2}.$$

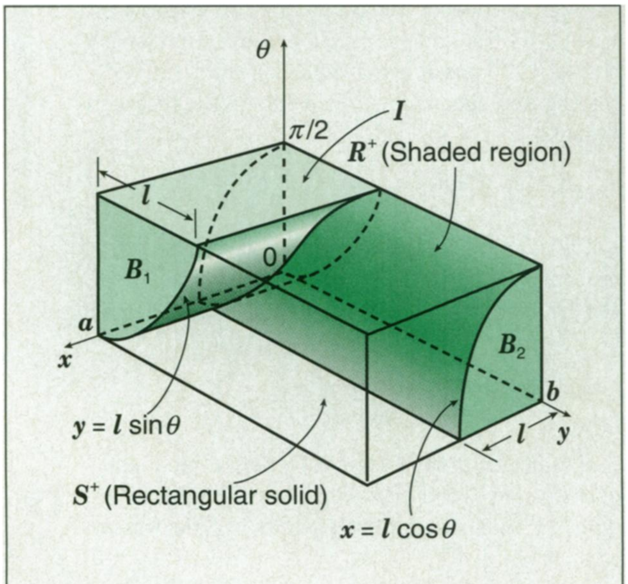

Fig. 2
The regions S^+ and R^+ representing all possible needle tosses and, respectively, those where the needle crosses a line, for $0 \le \theta \le \pi/2$

Next suppose that $-\pi/2 < \theta < 0$. Then the needle will cross a line, that is, (x, y, θ) will belong to R, if and only if $x < l\cos\theta$ or $y < -l\sin\theta$. Let R^- be the subregion of R consisting of those points where $-\pi/2 < \theta < 0$. Then R^- will consist of all points (x, y, θ)

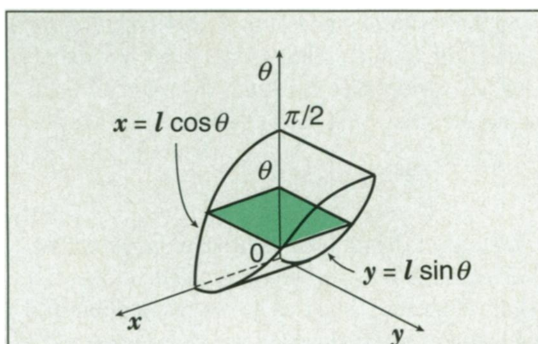

Fig. 3
Typical cross section of the region I

in R satisfying $-\pi/2 < \theta < 0$ and also $x < l\cos\theta$ or $y < -l\sin\theta$. We claim that the subregion of R^+ satisfying $0 < \theta < \pi/2$ is symmetric to R^- with respect to the xy-plane. To see this relationship, notice that $(x, y, -\theta)$ is the symmetric point to (x, y, θ) with respect to the xy-plane and that

$$\begin{aligned}
(x, y, -\theta) \in R^- & \\
&\Leftrightarrow (x < l\cos(-\theta) \text{ or } y < -l\sin(-\theta)) \\
&\qquad\qquad \text{and } -\frac{\pi}{2} < -\theta < 0 \\
&\Leftrightarrow (x < l\cos\theta \text{ or } y < l\sin\theta) \\
&\qquad\qquad \text{and } 0 < \theta < \frac{\pi}{2} \\
&\Leftrightarrow (x, y, \theta) \in R^+ \text{ and } 0 < \theta < \frac{\pi}{2}.
\end{aligned}$$

Therefore, the volume of the subregion of R^+ consisting of those points with $0 < \theta < \pi/2$ is the same as the volume of R^-. Since the volume of this subregion is also the same as the volume of R^+, we observe that

$$\text{volume of } R = 2(\text{volume of } R^+).$$

Also, we easily see that

$$\text{volume of } S^+ = 2(\text{volume of } S).$$

Since the volume of S^+ is $(\pi ab)/2$, we obtain

$$\begin{aligned}
p &= \frac{\text{volume of } R}{\text{volume of } S} \\
&= \frac{2(\text{volume of } R^+)}{2(\text{volume of } S^+)} \\
&= \frac{\text{volume of } R^+}{\text{volume of } S^+} \\
&= \frac{la + lb - \frac{l^2}{2}}{\frac{\pi ab}{2}} \\
&= \frac{2l(a+b) - l^2}{\pi ab}.
\end{aligned}$$

When the $a \times b$ rectangles are squares with side k, we have $a = b = k$ and so

$$p = \frac{4lk - l^2}{\pi k^2},$$

which is the correct solution for the needle problem with a square grid considered by Schroeder.

An interesting observation is that if we let one of the dimensions, say a, of the rectangles in our rectangular grid approach infinity, our solution should approach that for the needle problem with parallel lines, where b is the distance between adjacent lines. We see that this outcome indeed occurs, since

$$\begin{aligned}
\lim_{a\to\infty} \frac{2l(a+b) - l^2}{\pi ab} &= \lim_{a\to\infty} \frac{2l + \frac{2lb}{a} - \frac{l^2}{a}}{\pi b} \\
&= \frac{2l}{\pi b}.
\end{aligned}$$

PROGRAM 1

```
10   REM   VARIABLES
20   REM   PI. . . . THE CONSTANT PI
30   REM   N. . . . . NUMBER OF TOSSES
40   REM   S. . . . . NUMBER OF TIMES NEEDLE
                      CROSSES A LINE
50   REM   X. . . . . DISTANCE BETWEEN
                      LEFTMOST END OF
                      NEEDLE AND NEAREST
                      VERTICAL LINE TO THE
                      RIGHT
60   REM   Y. . . . . DISTANCE BETWEEN LOWER
                      END OF NEEDLE AND
                      NEAREST HORIZONTAL
                      LINE ABOVE IT
70   REM   THETA . ANGLE MADE BY NEEDLE
                      WITH A HORIZONTAL
                      LINE THROUGH ITS
                      LOWER END
80   REM   I. . . . . INDEX VARIABLE FOR
                      FOR-NEXT LOOP
90   LET   PI = 3.14159
100  PRINT "NO. OF TOSSES"; TAB (17);
           "PROBABILITY ESTIMATE"
110  FOR N = 1000 TO 10000 STEP 1000
           RANDOMIZE (N)
130        LET S = 0
140        FOR I = 1 TO N
150          LET X = 2 * RND
160          LET Y = 3 * RND
170          LET THETA = (PI/2) * RND
180          IF X <= COS (THETA) OR
               Y <= SIN (ABS(THETA))
               THEN S = S + 1
190        NEXT I
200        PRINT TAB (5) ; N; TAB (23) ; S/N
210  NEXT N
```

NO. OF TOSSES	PROBABILITY ESTIMATE
1000	.465
2000	.4805
3000	.4743334
4000	.4995
5000	.4552
6000	.4805
7000	.475
8000	.474375
9000	.471
10000	.4735

We can avoid calculus by using a computer simulation

Finally, we recall that our solution to the needle problem required the use of integration to obtain the volume of R. Another approach, which would avoid the use of integration, would be to write a computer program to simulate the needle-tossing experiment. We could have our program "toss" the needle a large number of times and obtain an empirical estimate of the probability of crossing a line by dividing the number of times the needle crosses a line by the number of times the needle is tossed. For example, suppose we take $l = 1$, $a = 2$, and $b = 3$. **Program 1** includes an IBM PC BASIC program that computes, in this case, empirical probability estimates starting with 1 000 tosses and continuing in steps of 1 000 up to 10 000 tosses, along with a sample run; the teacher should note that this program takes a long time to run. In this case, the mathematical probability of the needle's crossing a line is

$$p = \frac{2(1)(2+3) - 1^2}{\pi(2)(3)} = \frac{9}{6\pi} \approx 0.477.$$

REFERENCE

Schroeder, Lee L. "Buffon's Needle Problem: An Exciting Application of Many Mathematical Concepts." *Mathematics Teacher* 67 (February 1974):183–86.