1 Section 11.9

1. **(Representation of Functions as Power Series)** Find a power series representation for the function $f(x) = x^2/(x^2+6)$.

   Solution. Recall, the function $g(x) = 1/(1-x)$ may be expressed as an infinite geometric series of the form

$$g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \ |x| < 1$$

   To express $f(x)$ as a power series, we want to re-write it in such a way that allows us to incorporate $g(x)$. The easiest way to do so is by the following factorization.

$$f(x) = x^2 \cdot \frac{1}{x^2+6} = \frac{x^2}{6} \cdot \frac{1}{1-(-x^2/6)}$$

   Notice that the right-most expression is of the form $g(-x^2/6)$, which converges for $|-x^2/6| < 1$, or $|x| < \sqrt{6}$. Applying $g(-x^2/6)$ and simplifying, we have that

$$\begin{aligned} f(x) &= \frac{x^2}{6} \cdot \frac{1}{1-(-x^2/6)} \\ &= \frac{x^2}{6} \cdot \sum_{n=0}^{\infty} \left(-\frac{x^2}{6}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{x^2}{6} \cdot (-1)^n \cdot \frac{x^{2n}}{6^n} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{6^{n+1}}, \ |x| < \sqrt{6} \end{aligned}$$

2. **(11.9, Exercise 10.)** For $a > 0$, determine a power series representation for the function

$$f(x) = \frac{x+a}{x^2+a^2}$$

   Solution.

   Notice that we can rewrite $f(x)$ as

$$\begin{aligned} f(x) &= \frac{x+a}{x^2+a^2} \\ &= \frac{x}{x^2+a^2} + \frac{a}{x^2+a^2} \\ &= x \cdot \frac{1}{a^2-(-x^2)} + a \cdot \frac{1}{a^2-(-x^2)} \\ &= \frac{x}{a^2} \cdot \frac{1}{1-(-x^2/a^2)} + \frac{a}{a^2} \cdot \frac{1}{1-(-x^2/a^2)} \end{aligned}$$

The right-fractional part can be written in power series form provided

$$\left|\frac{-x^2}{a^2}\right| < 1 \iff \left|\frac{x^2}{a^2}\right| < 1 \iff |x^2| < |a^2| \iff |x| < a.$$

Putting it all together, we therefore have that

$$\begin{aligned} f(x) &= \frac{x+a}{x^2+a^2} \\ &= \frac{x}{x^2+a^2} + \frac{a}{x^2+a^2} \\ &= x \cdot \frac{1}{a^2-(-x^2)} + a \cdot \frac{1}{a^2-(-x^2)} \\ &= \frac{x}{a^2} \cdot \frac{1}{1-(-x^2/a^2)} + \frac{a}{a^2} \cdot \frac{1}{1-(-x^2/a^2)} \\ &= \frac{x}{a^2} \cdot \sum_{n=0}^{\infty} \left(\frac{-x^2}{a^2}\right)^n + \frac{a}{a^2} \cdot \sum_{n=0}^{\infty} \left(\frac{-x^2}{a^2}\right)^n \\ &= \sum_{n=0}^{\infty} (-1)^n \left(\frac{x^{2n+1}}{a^{2n+2}}\right) + \sum_{n=0}^{\infty} (-1)^n \left(\frac{x^{2n}}{a^{2n+1}}\right)^n, \; |x| < a \end{aligned}$$

From the fourth to fifth step, we were able to go from an algebraic expression to a sum provided $|x| < a$. Thus, there is no need to continue to simplify algebraically. That is, we can leave it as a sum of two sums where both converge for $|x| < a$.

3. **(Differentiation and Integration of Power Series)** Let us get a glimpse of how power series representations could be of use to us.

   (a) Find a power series representation for $\ln(1-x^8)$.

   (b) Use (a) to evaluate $\int \ln(1-x^8)\, dx$.

   Solution.

   (a) Observe that

$$\begin{aligned} \frac{d}{dx} \ln(1-x^8) &= \frac{1}{1-x^8} \cdot (-8x^7) \\ \frac{d}{dx} \ln(1-x^8) &= \frac{-8x^7}{1-x^8} \end{aligned}$$

   Next, we integrate both sides. Notice that by integrating, we recover $\ln(1-x^8)$. What we want to do now is express the right-hand-side by a power series representation.

$$\begin{aligned}
\int \frac{d}{dx}\ln(1-x^8)\,dx &= \int \frac{-8x^7}{1-x^8}\,dx \\
\ln(1-x^8) &= \int (-8x^7)\cdot\frac{1}{1-x^8}\,dx,\ |x^8|<1 \\
&= \int (-8x^7)\cdot\sum_{n=0}^{\infty}(x^8)^n\,dx \\
&= (-8)\int\left[\sum_{n=0}^{\infty}x^{8n+7}\,dx\right] \\
&= (-8)\sum_{n=0}^{\infty}\left[\int x^{8n+7}\,dx\right] \\
&= \sum_{n=0}^{\infty}\frac{-8}{8n+8}x^{8n+8} \\
&= -\sum_{n=0}^{\infty}\frac{x^{8n+8}}{n+1}
\end{aligned}$$

Thus, we have that

$$\ln(1-x^8) = -\sum_{n=0}^{\infty}\frac{x^{8n+8}}{n+1},\ |x|<1$$

You may be wondering where the constants of integration are. For simplicity, we can set $x=0$ to see that $C=0$. As an exercise, verify that this is true!

(b) From (a), we replace $\ln(1-x^8)$ with its power series representation.

$$\begin{aligned}
\int \ln(1-x^8)\,dx &= \int\left[-\sum_{n=0}^{\infty}\frac{x^{8n+8}}{n+1}\right]dx \\
&= (-1)\sum_{n=0}^{\infty}\left[\int\frac{x^{8n+8}}{n+1}\,dx\right] \\
&= (-1)\sum_{n=0}^{\infty}\frac{x^{8n+9}}{(n+1)(8n+9)}
\end{aligned}$$

Thus, we have that

$$\int \ln(1-x^8)\,dx = (-1)\sum_{n=0}^{\infty}\frac{x^{8n+9}}{(n+1)(8n+9)},\ |x|<1$$

Upon differentiation or integration, the new expression will have the same radius of convergence as the original expression.

2 Section 11.10

1. Given $f(x) = xe^x$ and $a = 0$, determine $T_4(x)$.

   Solution. Recall, we can write $T_n(x)$ as

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

   For $n = 4$, $T_4(x)$ is a degree 4 polynomial. The derivatives are given by

$$\begin{aligned} f^{(0)}(x) &= xe^x, \\ f^{(1)}(x) &= e^x + xe^x = (1+x)e^x, \\ f^{(2)}(x) &= e^x + (1+x)e^x = (2+x)e^x, \\ &\vdots \\ f^{(n)}(x) &= (n+x)e^x \end{aligned}$$

   For $a = 0$, the coefficients are given by

$$\begin{aligned} c_0 &= \frac{f^{(0)}(0)}{0!} = \frac{0 \cdot e^0}{0!} = 0, \\ c_1 &= \frac{f^{(1)}(0)}{1!} = \frac{(1+0)e^0}{1!} = 1, \\ c_2 &= \frac{f^{(2)}(0)}{2!} = \frac{(2+0) \cdot e^0}{2!} = 1, \\ c_3 &= \frac{f^{(3)}(0)}{3!} = \frac{(3+0) \cdot e^0}{3!} = \frac{1}{2}, \\ c_4 &= \frac{f^{(4)}(0)}{4!} = \frac{(4+0) \cdot e^0}{4!} = \frac{1}{6} \end{aligned}$$

   Hence, $T_4(x)$ is given by

$$T_4(x) = \sum_{i=0}^{4} \frac{f^{(i)}(a)}{i!}(x-a)^i = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$$

2. Derive the Maclaurin Series for $f(x) = \cos x$ using the definition. Do not show that $R_n(x) \to 0$.

   Solution.

   Observe the behavior of the derivatives of $\cos x$.

$$\begin{aligned}
f^{(0)}(x) &= \cos x, \\
f^{(1)}(x) &= -\sin x, \\
f^{(2)}(x) &= -\cos x, \\
f^{(3)}(x) &= \sin x, \\
f^{(4)}(x) &= \cos x, \\
f^{(5)}(x) &= -\sin x, \\
f^{(6)}(x) &= -\cos x, \\
f^{(7)}(x) &= \sin x, \\
f^{(8)}(x) &= \cos x \\
&\vdots
\end{aligned}$$

We see a periodic pattern occurring after every three derivatives. Now, apply the definition of the Maclaurin Series and simplify accordingly.

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \\
&= \frac{\cos 0}{0!} - \frac{\sin 0}{1!}x - \frac{\cos 0}{2!}x^2 + \frac{\sin 0}{3!}x^3 + \frac{\cos 0}{4!}x^4 - \frac{\sin 0}{5!}x^5 - \frac{\cos 0}{6!}x^6 + \frac{\sin 0}{7!}x^7 + \frac{\cos 0}{8!}x^8 + \cdots \\
&= \frac{1}{0!} - \frac{0}{1!}x - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 - \frac{0}{5!}x^5 - \frac{1}{6!}x^6 + \frac{0}{7!}x^7 + \frac{1}{8!}x^8 + \cdots \\
&= \frac{1}{0!} - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots \\
&= \sum_{n=0}^{\infty} (-1)^n \, \frac{x^{2n}}{(2n)!}, \; R = \infty
\end{aligned}$$

Therefore, the Maclaurin Series for $f(x) = \cos x$ is given by

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \, \frac{x^{2n}}{(2n)!}, \; R = \infty$$

3. Suppose we approximate $f(x) = 1/x^2$ on the interval $I = [0.6, 1.4]$ using the fourth degree Taylor polynomial $T_4(x)$ centered at $a = 1$. Determine the smallest value of M that satisfies Taylor's Inequality for the estimate of the error of this approximation.

   Solution. I am going to take an informal approach to this solution. So, let us converse as if we were in front of one another. At first glance, this problem seems to be quite technical. As a study tip, it is good practice to write the definitions of the words you do not know. I will leave it to the reader to look up the exact statement for Taylor's Inequality. So, pause here. After you have done that, I want to direct your attention to the assumption.

"If $|f^{(n+1)}(x)| \leq M$, for some $M > 0$, on $|x-a| \leq d \ldots$"

Now, pause and try to translate this word-by-word in plain English. Okay, now let us compare your translation to mine.

Translation. If the absolute value of the $(n+1)$th derivative is less than or equal to some positive value M for every x belonging the interval $I = [a-d, a+d]$. Here, you can take $y = |f^{(n+1)}|$. Then, we are simply comparing all y-values up to some M.

Q. What must M be?

A. The value of M, informally, must be the maximum value of the absolute value of the $(n+1)$th derivative on the interval I. More formally, we define M here as

$$M := \sup_{x \,\in\, I} |f^{(n+1)}(x)|$$

Since we are approximating by $T_4(x)$, this means $n = 4$. Thus, we are looking for the maximum value of $|f^{(5)}(x)|$ on $[0.6, 1.4]$.

$$\begin{aligned}
f^{(0)}(x) &= 1/x^2 \\
f^{(1)}(x) &= -2/x^3 \\
f^{(2)}(x) &= 6/x^4 \\
f^{(3)}(x) &= -24/x^5 \\
f^{(4)}(x) &= 120/x^6 \\
f^{(5)}(x) &= -720/x^7 \\
f^{(6)}(x) &= -720(-7)/x^8
\end{aligned}$$

You might be wondering why we are calculating a 6th derivative. This is because I want to know a bit about the behavior of $f^{(5)}(x)$. Since $f^{(6)}(x)$ is positive for all x, we have that $f^{(5)}(x)$ is a strictly increasing function and concave down. Since I is a closed interval, the maximum and minimum occur at the end points. In this case, $x = 0.6$ would yield the maximum value. I will leave it to the reader to verify this argument. Therefore, we choose

$$M = \left| -\frac{720}{(6/10)^7} \right| = \frac{720 \cdot 10^7}{6^7}$$

Why is it the smallest value? We want to think of M as a least upper bound. So, we could have a collection of things that could serve as upper bounds or "ceilings", but we choose the smallest out of them. In this case, we obtain a local solution in the interval I, which effectively serves as our least upper bound.