 A. Tan

1. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ → CONV if $p>1$, → DIV if $p \le 1$ $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{1}{n^{5p}}$ → CONV if $5p > 1 \iff p > 1/5$, → DIV if $5p \le 1 \iff p \le 1/5$

2. (i) $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

Use integral test. Recall, suppose
- $f(x)$ CONT, +ve, ↘ on $[c,\infty)$
- $f(n) = a_n$

Then, ① $\int_k^{\infty} f(x)\,dx$ CONV $\Rightarrow \sum_{n=k}^{\infty} a_n$ CONV

② $\int_k^{\infty} f(x)\,dx$ DIV $\Rightarrow \sum_{n=k}^{\infty} a_n$ DIV

Consider $f(x) = \frac{1}{x\ln(x)}$, $2 \le x < \infty$.

- Clearly $\frac{1}{x\ln(x)}$ CONT on $[2,\infty)$ and $\frac{1}{x\ln(x)} > 0$ ✓
- Now, to show $f(x)$ ↘ on $[2,\infty)$ just look at when $f'(x) < 0$

$$f'(x) < 0 \iff -(x\ln(x))^{-2}\cdot\left[1\cdot\ln(x) + x\cdot\frac{1}{x}\right] < 0$$
$$= \frac{-1}{[x\ln(x)]^2}\cdot[\ln(x)+1] < 0$$
$$\iff -[\ln(x)+1] < 0$$
$$\iff \ln(x) > -1$$
$$\iff x > 1/e \quad ✓$$

Compute:
$$\int_2^{\infty} \frac{1}{x\ln(x)}dx = \lim_{n\to\infty}\int_2^t \frac{1}{x\ln(x)}dx$$
$$= \lim_{n\to\infty}\int_{\ln(2)}^{\ln(t)} \frac{1}{u}du \quad \left\{\begin{array}{l} u = \ln(x) \\ du = \frac{1}{x}dx\end{array}\right\}$$
$$= \lim_{n\to\infty}\left[\ln(\ln(t)) - \ln(2)\right] = \infty \Rightarrow \text{DIV}$$

(ii) $\sum_{n=1}^{\infty} n^4 e^{n^5}$

Observation(s)
- n^4 big
- e^{n^5} EVEN BIGGER

Is the sum of their product going to converge? NO!

(iii) $\sum_{n=2}^{\infty} \frac{\ln(n)}{n^3}$

Consider $f(x) = \frac{\ln(x)}{x^3}$

- CONT on $[2,\infty)$
- +ve on $[2,\infty)$
- $f'(x) < 0$

$$0 > \frac{\frac{1}{x}\cdot x^3 - 3x^2\ln(x)}{(x^3)^2}$$
$$0 > \frac{x^2(1-3\ln(x))}{x^6}$$

$x<0$ or

$$1 - 3\ln(x) < 0$$
$$\frac{1}{3} < \ln(x)$$
$$1.3956 \approx e^{1/3} < x \quad ✓$$

⊖⊖⊖ | 0 | ⊖⊖⊖ | $e^{1/3}$

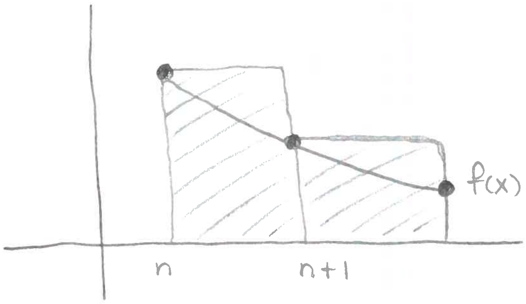
Choose $[2,\infty)$ so $f(x)$ indeed decreasing on appropriate interval

Now,
$$\int_2^{\infty} \frac{\ln(x)}{x^3}dx$$
$$= \lim_{t\to\infty}\int_2^t \frac{\ln(x)}{x^3}dx$$
$$= \lim_{t\to\infty} \frac{-\ln(x)}{(3-1)x^{3-1}} - \frac{1}{(3-1)^2x^{3-1}}\Bigg|_2^t$$
$$= \lim_{t\to\infty} \underbrace{-\frac{\ln(t)}{2t^2}}_{\to 0} - \underbrace{\frac{1}{4t^2}}_{\to 0 \text{ as } t\to\infty} - \underbrace{\left[-\frac{\ln(2)}{2(2)^2} - \frac{1}{4(2)^2}\right]}_{\text{finite}}$$
$\Rightarrow$ CONV

 A. Tan

3. According to integral estimate, what is smallest number of terms of series $\sum_{n=1}^{\infty} \frac{1}{\,}$ would have to add so that the estimated sum has error < 0.0001?

Idea:



$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

* Assume $\sum a_n$ conv by Integral Test

Certainly $\sum_{n=1}^{\infty} \frac{1}{n^6}$ is CONV

- $\frac{1}{n^6} > 0$ on $[1, \infty)$ and cont and DEC
- $\int_n^{\infty} \frac{1}{x^6}\,dx = -\frac{1}{5x^5}\Big|_n^{\infty} = \frac{1}{5n^5}$

By Remainder Test, want $R_n \leq \underbrace{\frac{1}{5n^5} \leq \frac{1}{10{,}000}}$ $= 0.0001$

$$10{,}000 \leq 5n^5$$

$$2000 \leq n^5$$

$$5 \approx 4.57 \leq n$$

 A. Tan

$\sum_{n=1}^{\infty} \frac{1}{n^4}$ CONV as p-series as $p = 4 > 1$

Compute $\int_n^{\infty} f(x)\,dx$, $f(x) = \frac{1}{x^4}$

$$\int_n^{\infty} \frac{1}{x^4}\,dx = \left.\frac{-1}{3x^3}\right|_n^{\infty} = \frac{1}{3n^3}$$

What value of n so that error < 0.00001?

$$\frac{1}{3n^3} < 0.00001 = \frac{1}{100,000}$$

$$\frac{100,000}{3} < n^3$$

$$\sqrt[3]{\frac{100,000}{3}} < n$$

$$\approx 32.18 < n$$

4. Comparison Test: $\sum a_n, \sum b_n$ series st $a_n > 0,\ b_n > 0$

(a) $\sum b_n$ CONV and $a_n \le b_n\ \forall n$, then $\sum a_n$ CONV

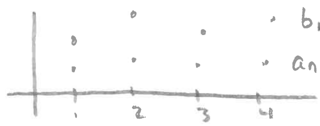
(b) $\sum b_n$ DIV and $a_n \ge b_n\ \forall n$, then $\sum a_n$ DIV

"Discrete analogue" of comparison test for improper integrals

Intuition: $\sum b_n$ CONV means $\{\underbrace{S_n = \sum_{i=1}^{n} a_i \text{ (partial sum)}}\} \longrightarrow s \in \mathbb{R}$ and so $\sum a_n = s$

sequence of partial sums

$a_n \le b_n$ means



(i) $$\sum_{n=1}^{\infty} \frac{4^{n+1}}{7^n+2} = \sum_{n=1}^{\infty} \underbrace{\frac{4 \cdot 4^n}{7^n+2}}_{a_n}$$ Compare to $\dfrac{4\cdot 4^n}{7^n} = b_n$

$a_n \le b_n$ b/c $7^n + 2 > 7^n$ so $0 < \dfrac{4\cdot 4^n}{7^n+2} \le 4\cdot\left(\dfrac{4}{7}\right)^n$

and $\displaystyle\sum_{n=1}^{\infty} 4\underbrace{\left(\frac{4}{7}\right)}_{4/7 \approx 0.5714}^{n} = \frac{4}{1-(4/7)} \Rightarrow$ CONV

Satisfies CT so $\sum a_n$ CONV

(ii) $$\sum_{n=1}^{\infty} \frac{\sqrt{n^5-2}}{n^4+3}$$

Suffices to look at "dominant terms"
i.e. Look at how say $\sqrt{n^5}$ interacts with n^4

$$\frac{\sqrt{n^5}}{n^4} = \frac{n^{5/2}}{n^4} = n^{5/2 - 8/2} = n^{-3/2} = \frac{1}{n^{3/2}}$$

But also, $\dfrac{\sqrt{n^5-2}}{n^4+3} \le \dfrac{\sqrt{n^5}}{n^4} = \dfrac{1}{n^{3/2}}$ CONV p-series $p>1$

Satisfies CT so $\sum a_n$ CONV

① $\sqrt{n^5-2} < \sqrt{n^5}$ b/c $n^5 - 2 < n^5$

② $n^4 + 3 > n^4$

(iii) $$\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^4}$$

Well, $\arctan(n)$ bdd above by $\pi/2$ as $n \to \infty$
n^4 dominates $\arctan(n)$.

Compare $\dfrac{\arctan(n)}{n^4} \le \underbrace{\dfrac{\pi}{2}\dfrac{1}{n^4}}_{\text{CONV}}$

Satisfies CT so $\sum a_n$ CONV

 A.Tan

5. (i) $\sum_{n=1}^{\infty} \frac{5^n}{7^n-2}$

Tricky b/c of 7^n-2 in denominator so can't use $5^n/7^n$ in CT. However, $5^n/7^n-2$ and $5^n/7^n$ both have positive terms, let us try Limit Comparison Test (LCT).

$$\lim_{n\to\infty} \frac{5^n/7^n-2}{5^n/7^n} = \lim_{n\to\infty} \frac{7^n}{7^n-2} \overset{LH}{=} \lim_{n\to\infty} \frac{\cancel{7^n}\ln 7}{\cancel{7^n}\ln 7} = 1 > 0$$

$5^n/7^n$ CONV as G.S,

Σa_n acts like Σb_n

$a_n \approx k b_n$

Cases for 0 and ∞ in Ex 40, 41 of §11.4

$*$ $\frac{d}{dx} a^x = a^x \ln(a)$

So $\sum_{n=1}^{\infty} \frac{5^n}{7^n-2}$ CONV

LCT

$\Sigma a_n, \Sigma b_n$ series with positive terms

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$

Then either both CONV or DIV

Chris McLean, Lecture #9 has nice tabular summary

(ii) $\sum_{n=1}^{\infty} \frac{n^5+2n}{n^8+7n+5}$

Suffices to look at dominant terms n^5 and n^8

Note that

$$\frac{n^5+2n}{n^8+7n+5} \leq \frac{n^5}{n^8} \quad \text{(i.e. the linear terms cause this)}$$

$$= \underline{\frac{1}{n^3}} \quad \text{(CONV)}$$

so by CT, we have $\sum_{n=1}^{\infty} \frac{n^5+2n}{n^8+7n+5}$ CONV

(iii) $\sum_{n=1}^{\infty} \frac{\sqrt{n^6+3n}}{n^5+n}$

Hmm... $n^6+3n > n^6$ for $n \geq 1$

$$\text{so } \frac{\sqrt{n^6+3n}}{n^5} > \frac{\sqrt{n^6}}{n^5} = \frac{n^{6/2}}{n^5} = \frac{n^3}{n^5} = \underline{\frac{1}{n^2}}$$

CONV p-series

Further, $\frac{\sqrt{n^6+3n}}{n^5+n} < \frac{\sqrt{n^6+3n}}{n^5}$ Not going to get much farther here

Try LCT: Take $a_n = \frac{\sqrt{n^6+3n}}{n^5+n}$ and $b_n = \frac{\sqrt{n^6}}{n^5} = \frac{1}{n^2}$ CONV as p-series

DIV o/w example is troublesome

$b_n = \frac{\text{highest pwr num}}{\text{highest pwr denom}}$

textbook

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n^6+3n}}{n^5+n} \cdot n^2$$

$$= \lim_{n\to\infty} \frac{n\sqrt{n^6+3n}}{n^4+1}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^8+3n^3}/n^4}{(n^4+1)/n^4} \quad \text{since } n^4 = \sqrt{n^8},\ n>0$$

$$= \lim_{n\to\infty} \frac{\sqrt{\frac{n^8+3n^3}{n^8}}}{1+\frac{1}{n^4}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{1+\frac{3}{n^5}}}{1+\frac{1}{n^4}} = 1$$

Note that although $\frac{a_n}{b_n} \to 1$ for ex $n\to\infty$, if b_n div then a_n div

Div Test (also for lim DNE)

A. Tan

6. (i) $\sum_{n=1}^{\infty} \arctan(n)$. $\lim_{n\to\infty} \arctan(n) = \pi/2 \neq 0 \Rightarrow \sum_{n=1}^{\infty} \arctan(n)$ DIV

(ii) $\sum_{n=1}^{\infty} \frac{5^{n+1}}{7^n} = \sum_{n=1}^{\infty} 5\left(\frac{5}{7}\right)^n, \quad \frac{5}{7} < 1 \Rightarrow$ CONV as Geometric Series

(iii) $\sum_{n=1}^{\infty} \cos(1/n)$ Div Test

$\lim_{n\to\infty} \cos(1/n) = \cos(0) = 1 \neq 0 \Rightarrow \sum_{n=1}^{\infty} \cos(1/n)$ DIV

7. Given $\sum_{n=1}^{\infty} a_n$ and n-th partial sum $s_n = \frac{3n-4}{3n+4}$

(a) Find a_2

In general, note that $s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \dots + a_n$ ①

So $s_{n-1} = \sum_{i=1}^{n-1} a_i = a_1 + a_2 + \dots + a_{n-1}$ ②

Now ① − ② $\Rightarrow s_n - s_{n-1} = \cancel{a_1} + \cancel{a_2} + \dots + \cancel{a_{n-1}} + a_n$ ①

$\ominus\ \cancel{a_1} + \cancel{a_2} + \dots + \cancel{a_{n-1}}$ ②

a_n

$\boxed{a_n = s_n - s_{n-1}} \Rightarrow a_2 = s_2 - s_{2-1}$

$= s_2 - s_1$

$= \frac{3(2)-4}{3(2)+4} - \frac{3(1)-4}{3(1)+4}$

(b) Find $\sum_{n=1}^{\infty} a_n = 1$

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{3n-4}{3n+4} = \frac{3}{3} = 1.$

 A. Tan

Facts $\sum a_n$ CONV $\Rightarrow \lim_{n\to\infty} a_n = 0$

$\lim_{n\to\infty} a_n \neq 0$ or DNE $\Rightarrow \sum a_n$ DIV (DIV TEST)

$\sum a_n$ CONV by defn if seq of partial sums $\{s_n\}$ where $s_n := \sum_{i=1}^{n} a_i$ converges

$\{a_n\}$ infinite list

$\sum a_n$ infinite sum of list's terms!