

$$1. \sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{CONV if } p > 1 \\ \text{DIV if } p \leq 1 \end{cases} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^{5p}} \begin{cases} \text{CONV if } 5p > 1 \Leftrightarrow p > \frac{1}{5} \\ \text{DIV if } 5p \leq 1 \Leftrightarrow p \leq \frac{1}{5} \end{cases}$$

$$2. (i) \sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$$

$$(ii) \sum_{n=1}^{\infty} n^4 e^{n^5}$$

$$(iii) \sum_{n=2}^{\infty} \frac{\ln(n)}{n^3}$$

Use integral test. Recall, suppose

- $f(x)$ CONT, +ve, \downarrow on $[c, \infty)$

$$\text{• } f(n) = a_n$$

$$\text{Then, } ① \int_u^{\infty} f(x) dx \text{ CONV} \Rightarrow \sum_{n=u}^{\infty} a_n \text{ CONV}$$

$$② \int_u^{\infty} f(x) dx \text{ DIV} \Rightarrow \sum_{n=u}^{\infty} a_n \text{ DIV}$$

Observation(s)

- n^4 big
- e^{n^5} EVEN BIGGER

Is the sum of their product going to converge? NO!

$$\text{Consider } f(x) = \frac{\ln(x)}{x^3}$$

- CONT on $[2, \infty)$

- +ve on $[2, \infty)$

- $f'(x) < 0$

$$0 > \frac{\frac{1}{x} \cdot x^3 - 3x^2 \ln(x)}{(x^3)^2}$$

$$0 > \frac{x^2(1 - 3\ln(x))}{x^6}$$

$$x < 0 \text{ or}$$

$$1 - 3\ln(x) < 0$$

$$\frac{1}{3} < \ln(x)$$

$$1.3956 \approx e^{1/3} < x$$

$$\frac{e^{1/3}}{0} \quad \frac{1}{e^{1/3}}$$

Choose $[2, \infty)$ so $f(x)$ indeed decreasing on appropriate interval

Now,

$$\int_2^{\infty} \frac{\ln(x)}{x^3} dx$$

$$= \lim_{t \rightarrow \infty} \int_2^t \frac{\ln(x)}{x^3} dx$$

$$= \lim_{t \rightarrow \infty} \left[-\frac{\ln(x)}{(3-1)x^{3-1}} - \frac{1}{(3-1)x^{2-1}} \right]_2^t$$

$$= \lim_{t \rightarrow \infty} \left[\underbrace{-\frac{\ln(t)}{2t^2}}_0 - \underbrace{\frac{1}{4t^2}}_0 - \underbrace{\left[-\frac{\ln(2)}{2(2)^2} - \frac{1}{4(2)^2} \right]}_{\text{finite}} \right]$$

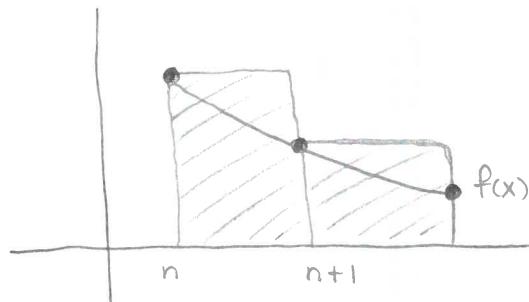
$\Rightarrow \text{CONV}$

Compute:

$$\begin{aligned} \int_2^{\infty} \frac{1}{x \ln(x)} dx &= \lim_{n \rightarrow \infty} \int_2^n \frac{1}{x \ln(x)} dx \\ &= \lim_{n \rightarrow \infty} \int_2^{\ln(n)} \frac{1}{u} du \quad \left\{ \begin{array}{l} u = \ln(x) \\ du = \frac{1}{x} dx \end{array} \right\} \\ &= \lim_{n \rightarrow \infty} \left[\ln(\ln(t)) - \ln(2) \right] = \infty \Rightarrow \text{DIV} \end{aligned}$$

3. According to integral estimate, what is smallest number of terms of series $\sum_{n=1}^{\infty} a_n$ would have to add so that the estimated sum has error < 0.0001 ?

Idea:



$$\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx$$

* Assume $\sum a_n$ conv by Integral Test

Certainly $\sum_{n=1}^{\infty} \frac{1}{n^6}$ is conv

- $\frac{1}{n^6} > 0$ on $[1, \infty)$ and cont

and DEC

- $\int_n^{\infty} \frac{1}{x^6} dx = -\frac{1}{5x^5} \Big|_n^{\infty} = \frac{1}{5n^5}$

By Remainder Test, want $R_n \leq \frac{1}{5n^5} \leq \underbrace{\frac{1}{10,000}}_{10,000 \leq 5n^5} = 0.0001$

$$10,000 \leq 5n^5$$

$$2000 \leq n^5$$

$$5 \approx 4.57 \leq n$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} \text{ conv as p-series as } p=4 > 1$$

$$\text{Compute } \int_n^{\infty} f(x) dx, \quad f(x) = \frac{1}{x^4}$$

$$\int_n^{\infty} \frac{1}{x^4} dx = \left. \frac{-1}{3x^3} \right|_n^{\infty} = \frac{1}{3n^3}$$

What value of n so that error < 0.00001 ?

$$\frac{1}{3n^3} < 0.00001 = \frac{1}{100,000}$$

$$\frac{100,000}{3} < n^3$$

$$\sqrt[3]{\frac{100,000}{3}} < n$$

$$\approx 32.18 < n$$

4. Comparison Test: $\sum a_n, \sum b_n$ series st $a_n > 0, b_n > 0$

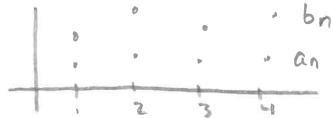
- (a) $\sum b_n$ CONV and $a_n \leq b_n \forall n$, then $\sum a_n$ CONV
 (b) $\sum b_n$ DIV and $a_n \geq b_n \forall n$, then $\sum a_n$ DIV

} "Discrete analogue" of comparison test for improper integrals

Intuition:

$\sum b_n$ CONV means $\{s_n = \sum_{i=1}^n a_i \text{ (partial sum)}\} \rightarrow s \in \mathbb{R}$ and so $\sum a_n = s$
 i.e. sequence of partial sums

$a_n \leq b_n$ means



$$(i) \sum_{n=1}^{\infty} \frac{4^{n+1}}{7^n + 2} = \sum_{n=1}^{\infty} \underbrace{\frac{4 \cdot 4^n}{7^n + 2}}_{a_n} \quad \text{Compare to } \frac{4 \cdot 4^n}{7^n} = b_n$$

$$a_n \leq b_n \text{ b/c } 7^n + 2 > 7^n \text{ so } 0 < \frac{4 \cdot 4^n}{7^n + 2} \leq 4 \cdot \left(\frac{4}{7}\right)^n$$

$$\text{and } \sum_{n=1}^{\infty} 4 \left(\frac{4}{7}\right)^n = \frac{4}{1 - (4/7)} \Rightarrow \text{CONV} \quad \left. \begin{array}{l} \text{Satisfies CT} \\ \text{so } \sum a_n \text{ CONV} \end{array} \right\}$$

$$4/7 \approx 0.5714$$

$$(ii) \sum_{n=1}^{\infty} \frac{\sqrt{n^5 - 2}}{n^4 + 3}$$

Suffices to look at "dominant terms"
 i.e. Look at how say $\sqrt{n^5}$ interacts
 with n^4

$$\frac{\sqrt{n^5}}{n^4} = \frac{n^{5/2}}{n^4} = n^{5/2 - 8/2} = n^{-3/2} = \frac{1}{n^{3/2}}$$

But also, $\frac{\sqrt{n^5 - 2}}{n^4 + 3} \leq \frac{\sqrt{n^5}}{n^4} = \left(\frac{1}{n^{3/2}}\right)$ CONV p-series
 $\left. \begin{array}{l} \text{p} > 1 \\ \text{so } \sum a_n \text{ CONV} \end{array} \right\}$

satisfies CT
 so $\sum a_n$ CONV

$$\textcircled{1} \quad \sqrt{n^5 - 2} < \sqrt{n^5} \text{ b/c } n^5 - 2 < n^5$$

$$\textcircled{2} \quad n^4 + 3 > n^4$$

$$(iii) \sum_{n=1}^{\infty} \frac{\arctan(n)}{n^4}$$

Well, $\arctan(n)$ bold above by $\pi/2$ as $n \rightarrow \infty$
 n^4 dominates $\arctan(n)$.

$$\text{Compare } \frac{\arctan(n)}{n^4} \leq \frac{\pi/2}{2 \underbrace{n^4}_{\text{CONV}}} \quad \left. \begin{array}{l} \text{Satisfies CT} \\ \text{so } \sum a_n \text{ CONV} \end{array} \right\}$$

5. (i) $\sum_{n=1}^{\infty} \frac{5^n}{7^n - 2}$

Tricky b/c of $7^n - 2$ in denominator so can't use $\frac{5^n}{7^n}$ in CT.
However, $\frac{5^n}{7^n - 2}$ and $\frac{5^n}{7^n}$ both have positive terms, let us try Limit Comparison Test (LCT).

LCT

$\sum a_n, \sum b_n$
series with positive terms

If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$

Then either both conv or DIV

(ii) $\sum_{n=1}^{\infty} \frac{n^5 + 2n}{n^8 + 7n + 5}$

$$\lim_{n \rightarrow \infty} \frac{\frac{5^n}{7^n - 2}}{\frac{5^n}{7^n}} = \lim_{n \rightarrow \infty} \frac{7^n}{7^n - 2} \stackrel{LH}{=} \lim_{n \rightarrow \infty} \frac{7^n \ln 7}{7^n \ln 7 - 2} = 1 > 0$$

$\sum a_n \text{ acts like } \sum b_n$

CONV as G.S.

* $\frac{d}{dx} a^x = a^x \ln(a)$

So $\sum_{n=1}^{\infty} \frac{5^n}{7^n - 2}$ CONV

Cases for 0 and ∞
in Ex 40, 41 of §11.4

Chris McLean, Lecture #9 has nice tabular summary

Suffices to look at dominant terms n^5 and n^8

Note that

$$\frac{n^5 + 2n}{n^8 + 7n + 5} \leq \frac{n^5}{n^8} \quad (\text{i.e. the linear terms cause this})$$

$$= \frac{1}{n^3} \quad (\text{conv})$$

so by CT, we have $\sum_{n=1}^{\infty} \frac{n^5 + 2n}{n^8 + 7n + 5}$ CONV

(iii) $\sum_{n=1}^{\infty} \frac{\sqrt{n^6 + 3n^4}}{n^5 + n}$

Hmm... $n^6 + 3n^4 > n^6$ for $n \geq 1$

$$\text{so } \frac{\sqrt{n^6 + 3n^4}}{n^5} > \frac{\sqrt{n^6}}{n^5} = \frac{n^{6/2}}{n^5} = \frac{n^3}{n^5} = \frac{1}{n^2}$$

conv p-series

$b_n = \frac{\text{highest power num}}{\text{highest power denom}}$

textbook

Try LCT: Take $a_n = \frac{\sqrt{n^6 + 3n^4}}{n^5 + n}$ and $b_n = \frac{\sqrt{n^6}}{n^5} = \frac{1}{n^2}$ CONV as p-series

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sqrt{n^6 + 3n^4}}{n^5 + n} \cdot n^2$$

$$= \lim_{n \rightarrow \infty} \frac{n\sqrt{n^6 + 3n^4}}{n^4 + 1}$$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{n^8 + 3n^8}/n^4}{(n^4 + 1)/n^4}$$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{\frac{n^8 + 3n^8}{n^8}}}{1 + \frac{1}{n^4}}$$

since $n^4 = \sqrt{n^8}$, $n > 0$

$$\Rightarrow = \lim_{n \rightarrow \infty} \frac{\sqrt{1 + \frac{3}{n^4}}}{1 + \frac{1}{n^4}} = 1$$

Note that
although

$$\frac{a_n}{b_n} \rightarrow 1 \text{ for ex}$$

$n \rightarrow \infty$, if b_n div then a_n div

6. (i) $\sum_{n=1}^{\infty} \arctan(n)$.

$\lim_{n \rightarrow \infty} \arctan(n) = \frac{\pi}{2} \neq 0 \Rightarrow \sum_{n=1}^{\infty} \arctan(n) \text{ DIV}$

(ii) $\sum_{n=1}^{\infty} \frac{5^{n+1}}{7^n} = \sum_{n=1}^{\infty} 5 \left(\frac{5}{7}\right)^n, \frac{5}{7} < 1 \Rightarrow \text{CONV as Geometric Series}$

(iii) $\sum_{n=1}^{\infty} \cos(1/n)$

Div Test

$\lim_{n \rightarrow \infty} \cos(1/n) = \cos(0) = 1 \neq 0 \Rightarrow \sum_{n=1}^{\infty} \cos(1/n) \text{ DIV}$

Given $\sum_{n=1}^{\infty} a_n$ and n -th partial sum $s_n = \frac{3n-4}{3n+4}$

(a) Find a_2

In general, note that $s_n = \sum_{i=1}^n a_i = a_1 + a_2 + \dots + a_n$ ①

So $s_{n-1} = \sum_{i=1}^{n-1} a_i = a_1 + a_2 + \dots + a_{n-1}$ ②

Now ① - ② $\Rightarrow s_n - s_{n-1} = a_n + a_{n+1} + \dots + a_{n-1} + a_n$ ③

$\underline{a_1 + a_2 + \dots + a_{n-1}}$ ④

$\overline{a_n}$

$$\boxed{a_n = s_n - s_{n-1}} \Rightarrow a_2 = s_2 - s_{2-1}$$

$= s_2 - s_1$

$= \frac{3(2)-4}{3(2)+4} - \frac{3(1)-4}{3(1)+4}$

(b) Find $\sum_{n=1}^{\infty} a_n = 1$

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{3n-4}{3n+4} = \frac{3}{3} = 1.$$

Facts $\sum a_n$ CONV $\Rightarrow \lim_{n \rightarrow \infty} a_n = 0$

$\lim_{n \rightarrow \infty} a_n \neq 0 \Rightarrow \sum a_n$ DIV (DIV TEST)
or DNE

$\sum a_n$ CONV by defn if seq of partial sums $\{s_n\}$

where $s_n := \sum_{i=1}^n a_i$ converges

$\{a_n\}$ infinite list

$\sum a_n$ infinite sum of list's terms!