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MATH 1AA3/1ZB3 Tutorial #4 - W2020

1. Conv or Div?

(i) $\sum_{n=1}^{\infty} (-1)^{n-1} e^{-n/3}$ (ii) $\sum_{n=1}^{\infty} (-1)^{n} \frac{n^2}{n^3+4}$ (iii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n+2)}$

<u>Soln</u> Notice that (i) - (iii) are Alternating Series.

(i) Let $b_n = e^{-n/3}$. Notice that $b_n = e^{-n/3} > 0$ $\boxed{b_n > 0}$

- $b_{n+1} = e^{-(n+1)/3} \leq b_n = e^{-n/3}$ for all n $\boxed{b_{n+1} \leq b_n \ \forall n}$

Why? Let $f(x) = e^{-x/3}$. Then $f'(x) = e^{-x/3} \cdot \left(-\frac{1}{3}\right) < 0 \ \forall x$, so we have that $f(x)$ is decreasing $\forall x$.

- $\lim_{n\to\infty} b_n = \lim_{n\to\infty} e^{-n/3} = 0.$ $\boxed{\lim_{n\to\infty} b_n = 0}$

Hence we may apply the Alternating Series Test (AST) to conclude that (i) is CONVERGENT.

(ii) Clearly, $b_n = \frac{n^2}{n^3+4}$ for $n \geq 1$.

- Let $f(x) = \frac{x^2}{x^3+4}$. Then $f'(x) < 0$

$$\Leftrightarrow \quad \frac{2x(x^3+4) - x^2(3x^2)}{(x^3+4)^2} \leq 0$$

$$\Leftrightarrow \quad \frac{x(2x^3+8-3x^3)}{(x^3+4)^2} \leq 0$$

$$\Leftrightarrow \quad x(8-x^3) \leq 0$$

$$\Leftrightarrow \quad x \leq 0 \ \text{ or } \ 8 - x^3 \leq 0$$

$$8 \leq x^3$$

$$2 \leq x$$

So, $b_{n+1} \leq b_n$ whenever $n \geq 2$.

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Soln Continued

- $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+4} = 0.$

  deg(num) < deg(denom)

Hence, we may conclude (ii) is CONVERGENT by AST since b_n satisfies
- $b_n > 0$
- $b_{n+1} \leq b_n \ \forall n \geq 2$
- $\lim_{n\to\infty} b_n = 0$

iii) Let $b_n = \frac{1}{\ln(n+2)}$. Again, clearly $b_n > 0$.

- Let $f(x) = \frac{1}{\ln(x+2)}$. Then

$$f'(x) < 0 \iff \frac{-1}{[\ln(x+2)]^2} \cdot \frac{1}{x+2} \cdot 1 = -\frac{1}{(x+2)[\ln(x+2)]^2} < 0$$

$$\iff -\frac{1}{(x+2)} < 0$$

$$\iff x + 2 > 0$$

$$\iff x > -2$$

Hence $f(x)$ decreasing when $x > -2$

i.e. $b_{n+1} \leq b_n$ when $n > -2$ so $n = 1$ starting is fine!

i.e. $b_{n+1} \leq b_n \ \forall n \geq 1$

- $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\ln(n+2)} = 0.$ B/c $\ln(n+2) \to \infty$ as $n \to \infty$

Hence we may conclude that (iii) is CONVERGENT by AST.

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2. What is the smallest number of terms of the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n3^n}$$

that would have to be added in order to estimate its sum with an absolute error that is <u>less</u> than .01?

<u>Soln</u> Note that

- Alternating series with $b_n > 0$
- Error <u>less</u> than $.01 = 10^{-2} = \frac{1}{100}$, i.e. Error $< \frac{1}{100}$. It is a strict inequality!
- We can show that $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n3^n}$ CONVERGES by AST.

$\rightarrow$ If $f(x) = \frac{1}{x3^x}$, then $f'(x) = \frac{-1}{(x3^x)^2} \cdot [1 \cdot 3^x + x \cdot 3^x \ln(3)]$.

So, $-[3^x + x \cdot 3^x \ln(3)] < 0$

$$3^x + x \cdot 3^x \ln(3) > 0$$

$$1 + x\ln(3) > 0$$

$$x > \frac{-1}{\ln(3)}$$ We're <u>OK</u> $\forall n \geq 1$.

$\rightarrow$ $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n3^n} = 0$ since $n3^n \to \infty$ as $n \to \infty$.

Since the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n3^n}$ is convergent by AST, we can use the Alternating Series Estimation Theorem (ASET).

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2. Soln Continued

Observe that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n3^n} = \underbrace{\underbrace{\frac{1}{1\cdot 3^1} - \frac{1}{2\cdot 3^2} + \underbrace{\frac{1}{3\cdot 3^3}}_{\approx 0.012}}_{s_3 = a_1 + a_2 + a_3} - \underbrace{\frac{1}{4\cdot 3^4}}_{\approx 0.003086}}_{s_4 = a_1 + a_2 + a_3 + a_4} + \cdots$$

By ASET, $|R_n| = |s - s_n| \le b_{n+1} < .01$ provided we choose $n = 3$ since

$b_{3+1} = b_4 = \frac{1}{4\cdot 3^4} \approx 0.003086 < .01.$

Other way, solve $b_{n+1} < .01$ for n!

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3. Which of the series are conditionally convergent?

(i) $\sum_{n=1}^{\infty} \frac{(-1)^n \arctan(n)}{n^2}$ (ii) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{\sqrt{n^4+3}}$ (iii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\cos(1/n)}$

Soln From text, page 738:

Defn We say $\sum a_n$ is conditionally convergent if it is convergent, but not absolutely convergent.

From text, page 737:

Defn We say $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

From text, page 738

Thm (3) If a series $\sum a_n$ is absolutely convergent, then it is convergent.

(i) $\sum_{n=1}^{\infty} \left| \frac{(-1)^n \arctan(n)}{n^2} \right| = \sum_{n=1}^{\infty} \frac{\arctan(n)}{n^2}$ b/c $\frac{\arctan(n)}{n^2} > 0$ for $n \geq 1$

But, $\frac{\arctan}{n^2} \leq \frac{\pi}{2} \cdot \frac{1}{n^2}$ and $\frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges as a p-series,

hence (i) is absolutely convergent by the Comparison Test.

(ii) $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1} n}{\sqrt{n^4+3}} \right| = \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4+3}}$, since $\frac{n}{\sqrt{n^4+3}} > 0$ for $n \geq 1$

Note that $\sqrt{n^4+3} > \sqrt{n^4}$, but $\frac{1}{\sqrt{n^4+3}} < \frac{1}{\sqrt{n^4}}$. So, multiplying by n yields

$$\frac{n}{\sqrt{n^4+3}} < \frac{n}{\sqrt{n^4}} = \frac{n}{n^2} = \frac{1}{n}$$

for which $\sum \frac{1}{n}$ is divergent. Therefore, it is <u>not</u> absolutely convergent.

This just means $\sum |a_n|$ is divergent. Absolute divergence is <u>not</u> a thing!

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3. Soln continued.

(ii) We saw that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{\sqrt{n^4+3}}$ is not absolutely convergent. Now, lets show decreasing... Let $f(x) = \frac{x}{\sqrt{x^4+3}}$. Then

$$f'(x) = \frac{1 \cdot (x^4+3)^{1/2} - x \cdot \frac{1}{2}(x^4+3)^{-1/2}(4x^3)}{x^4+3} < 0$$

$$\begin{aligned}
&\iff 2(x^4+3)^{1/2} - 4x^4(x^4+3)^{-1/2} < 0 \quad \text{(After multiplying by 2)} \\
&\iff (x^4+3)^{-1/2}\left[2(x^4+3) - 4x^4\right] < 0 \\
&\iff -2x^4 + 6 < 0 \\
&\iff \underbrace{1.31607}_{\text{Take } 2.} \approx \sqrt[4]{3} < x
\end{aligned}$$

So $b_{n+1} < b_n$ for $n \geq 2$

and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{\sqrt{n^4+2}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^4}} = \lim_{n\to\infty} \frac{1}{n} = 0$

Hence, by AST, we have convergence. So, conditionally convergent.

(iii) $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{\cos(1/n)} \right| = \sum_{n=1}^{\infty} \frac{1}{|\cos(1/n)|}$

But, $\lim_{n\to\infty} \frac{1}{|\cos(1/n)|} = \lim_{n\to\infty} \frac{1}{|\cos(0)|} = \lim_{n\to\infty} \frac{1}{1} = 1 \neq 0$

$\Rightarrow$ By Divergence Test that $\sum |a_n|$ is divergent.

Notice that $b_n = \frac{1}{\cos(1/n)} \to 1$ as $n \to \infty$. So, same argument as C. Mclean Lecture #6.

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4. Use the ratio test to determine convergence. (i) $\sum_{n=1}^{\infty} \frac{(n+3)!}{6^{n-1}}$ (ii) $\sum_{n=1}^{\infty} \frac{(-1)^n 6^{n+1}}{n^4}$ (iii) $\sum_{n=1}^{\infty} \frac{n^5 \ln n}{n!}$

Soln Ratio Test

$$\text{If } \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L = \begin{cases} <1 & \text{then } \sum a_n \text{ is absolutely convergent} \\ >1 \text{ or } =\infty & \text{then } \sum a_n \text{ is divergent} \\ =1 & \text{then Ratio Test is inconclusive} \end{cases}$$

(i) For simplicity, let $\mathcal{L} = \lim_{n\to\infty}$.

$$\mathcal{L} \left| \frac{a_{n+1}}{a_n} \right| = \mathcal{L} \left| \frac{((n+1)+3)!}{6^{(n+1)-1}} \cdot \frac{6^{n-1}}{(n+3)!} \right|$$

$$= \mathcal{L} \left| \frac{(n+4)\cancel{(n+3)!}}{\cancel{6^n}} \cdot \frac{\cancel{6^n} 6^{-1}}{\cancel{(n+3)!}} \right| \qquad \text{since } n! = n(n-1)!$$

$$= \mathcal{L} \left| \frac{1}{6}(n+4) \right| = \infty \implies \sum a_n \ \underline{\text{divergent}}$$

(ii) $$\mathcal{L} \left| \frac{a_{n+1}}{a_n} \right| = \mathcal{L} \left| \frac{(-1)^{n+1} 6^{(n+1)+1}}{(n+1)^4} \cdot \frac{n^4}{(-1)^n 6^{n+1}} \right|$$

$$= \mathcal{L} \left| \frac{\cancel{6^{n+1}} \cdot 6}{(n+1)^4} \cdot \frac{n^4}{\cancel{6^{n+1}}} \right| \qquad * \ |(-1)^{n+1}| = |(-1)^n| = 1$$

$$= 6 \cdot \underbrace{\mathcal{L} \left| \left(\frac{n}{n+1} \right)^4 \right|}_{\frac{n}{n+1} \to 1 \text{ as } n \to \infty} = 6 \cdot 1 > 1 \implies \sum a_n \ \underline{\text{divergent}}$$

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3. Soln Continued

(iii) $$\lim \left| \frac{(n+1)^5 \ln(n+1)}{(n+1)!} \cdot \frac{n!}{n^5 \ln(n)} \right|$$

$$= \lim \left| \frac{(n+1)^5}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{n^5} \cdot \frac{\ln(n+1)}{\ln(n)} \right|$$

$$= \underbrace{\lim \frac{(n+1)^4}{n^5}}_{\lim \frac{n^4}{n^5} = 0 \text{ as } n \to \infty} \cdot \underbrace{\lim \frac{\ln(n+1)}{\ln(n)}}_{\overset{LH}{=} \lim \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim \frac{n}{n+1} = 1} = 0 < 1 \Rightarrow \sum a_n \text{ is abs convergent}$$

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5. Consider the series $\sum_{n=1}^{\infty} \underbrace{\frac{4^n (3n)!}{2\cdot 6\cdot 10 \cdot \cdots \cdot (4n+2)}}_{a_n}$. Find a simplified expression for $\frac{a_{n+1}}{a_n}$.

Soln

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}(3(n+1))!}{\cancel{2}\prod_{i=1}^{n+1}(4i+2)} \cdot \frac{\cancel{2}\prod_{i=1}^{n}(4i+2)}{4^n (3n)!}$$

$$= \frac{4(3n+3)(3n+2)(3n+1)}{4(n+1)+2}$$

$$= \frac{4(3n+3)(3n+2)(3n+1)}{4n+6}$$

Key results:

- $(3(n+1))! = (3n+3)! = (3n+3)((3n+3)-1)!$
- $\underset{i=0}{2} \cdot \underset{i=1}{6} \cdot \underset{i=2}{10} \cdot \cdots \cdot \underset{i=n}{(4n+2)} = \underbrace{2\prod_{i=1}^{n}(4i+2)}_{\text{or use } \prod_{i=0}^{n}(4i+2)}$

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6. Find the interval of convergence of the following power series.

$$\sum_{n=1}^{\infty} \frac{(x-8)^n}{5^n n}$$

Soln

$$\lim_{n\to\infty}\left|\frac{(x-8)^{n+1}}{5^{n+1}(n+1)}\cdot\frac{5^n n}{(x-8)^n}\right| = \lim_{n\to\infty}\left|\frac{1}{5}\cdot\frac{(x-8)}{1}\cdot\frac{n}{n+1}\right|$$

$$= \frac{1}{5}|x-8|$$

By Ratio Test, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$ says $\sum a_n$ is absolutely convergent.

so,

$$\frac{1}{5}|x-8| < 1$$

$$\Leftrightarrow \quad |x-8| < 5 \qquad * \ |x|<r \Leftrightarrow -r<x<r$$

$$\Leftrightarrow \quad -5 < x-8 < 5$$

$$\Leftrightarrow \quad 3 < x < 13$$

should also check endpoints.

At $x = 3$: $\displaystyle\sum_{n=1}^{\infty}\frac{(3-8)^n}{5^n n} = \sum_{n=1}^{\infty}\frac{(-1)^n(5)^n}{5^n n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$

which converges by AST, i.e. $b_n = \frac{1}{n}$ has property that

- $b_{n+1} = \frac{1}{n+1} < b_n = \frac{1}{n}$,
- $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n} = 0.$

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6. Soln Continued

At $x = 13$: $\sum_{n=1}^{\infty} \frac{(13-8)^n}{5^n n} = \sum_{n=1}^{\infty} \frac{5^n}{5^n n} = \sum_{n=1}^{\infty} \frac{1}{n}$ which is the Harmonic series ($p=1$) which diverges!

Thus, the interval of convergence is $[3, 13)$.

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7. Find the radius of convergence of the following power series.

$$\sum_{n=1}^{\infty} \frac{(7x-8)^n}{n^7}$$

Soln $\lim_{n\to\infty} \left| \frac{(7x-8)^{n+1}}{(n+1)^7} \cdot \frac{n^7}{(7x-8)^n} \right| = \lim_{n\to\infty} \left| (7x-8) \cdot \underbrace{\frac{n^7}{(n+1)^7}}_{\to 1 \text{ as } n\to\infty} \right| = |7x-8|$

By the Ratio Test, for convergence (absolute), we require

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |7x-8| < 1$$

$\iff \left| 7\left(x - \frac{8}{7}\right) \right| < 1$ Has form $-R < x-a < R$

$\iff -\frac{1}{7} < x - \frac{8}{7} < \frac{1}{7}$

$\iff -\frac{1}{7} + \frac{8}{7} < x < \frac{1}{7} + \frac{8}{7}$

$\iff 1 < x < \frac{9}{7}$

Now, test endpoints as well.

$x = 1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^7}$ which converges by AST. $\to$ $b_n = 1/n^7 > 0$, $b_{n+1} < b_n$, $\lim_{n\to\infty} b_n = 0$

$x = \frac{9}{7}$: $\sum_{n=1}^{\infty} \frac{(1)^n}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^7}$ which is a convergent p-series.