

$$1. \sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \rightarrow \text{CONV} & \text{if } p > 1 \\ \rightarrow \text{DIV} & \text{if } p \leq 1 \end{cases} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^{5p}} \begin{cases} \rightarrow \text{CONV} & \text{if } 5p > 1 \Leftrightarrow p > 1/5 \\ \rightarrow \text{DIV} & \text{if } 5p \leq 1 \Leftrightarrow p \leq 1/5 \end{cases}$$

$$2. (i) \sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$$

Use integral test. Recall, suppose

• $f(x)$ CONT, +ve, \downarrow on $[c, \infty)$

• $f(n) = a_n$

Then, ① $\int_c^{\infty} f(x) dx$ CONV $\Rightarrow \sum_{n=c}^{\infty} a_n$ CONV

② $\int_c^{\infty} f(x) dx$ DIV $\Rightarrow \sum_{n=c}^{\infty} a_n$ DIV

Consider $f(x) = \frac{1}{x \ln(x)}$, $2 \leq x < \infty$.

• (clearly $\frac{1}{x \ln(x)}$ CONT on $[2, \infty)$)

and $\frac{1}{x \ln(x)} > 0$ ✓

• Now, to show $f(x) \downarrow$ on $[2, \infty)$

just look at when $f'(x) < 0$

$$f'(x) < 0 \Leftrightarrow -(\ln(x))^{-2} \cdot [1 \cdot \ln(x) + x \cdot \frac{1}{x}] < 0$$

$$= \frac{-1}{[\ln(x)]^2} \cdot [\ln(x) + 1] < 0$$

$$\Leftrightarrow -[\ln(x) + 1] < 0$$

$$\Leftrightarrow \ln(x) > -1$$

$$\Leftrightarrow x > 1/e \quad \checkmark$$

Compute:

$$\int_2^{\infty} \frac{1}{x \ln(x)} dx = \lim_{n \rightarrow \infty} \int_2^n \frac{1}{x \ln(x)} dx$$

$$= \lim_{n \rightarrow \infty} \int_2^{\ln(t)} \frac{1}{u} du \quad \left\{ \begin{array}{l} u = \ln(x) \\ du = \frac{1}{x} dx \end{array} \right.$$

$$= \lim_{n \rightarrow \infty} [\ln(\ln(t)) - \ln(2)] = \infty \Rightarrow \text{DIV}$$

$$(ii) \sum_{n=1}^{\infty} n^4 e^{n^5}$$

Observations

• n^4 big

• e^{n^5} EVEN BIGGER

Is the sum of their product going to converge? NO!

$$(iii) \sum_{n=2}^{\infty} \frac{\ln(n)}{n^3}$$

Consider $f(x) = \frac{\ln(x)}{x^3}$

• CONT on $[2, \infty)$

• +ve on $[2, \infty)$

• $f'(x) < 0$

$$0 > \frac{\frac{1}{x} \cdot x^3 - 3x^2 \ln(x)}{(x^3)^2}$$

$$0 > \frac{x^2(1 - 3\ln(x))}{x^6}$$

$$x < 0 \quad \text{or}$$

$$1 - 3\ln(x) < 0$$

$$\frac{1}{3} < \ln(x)$$

$$1.3956 \approx e^{1/3} < x \quad \checkmark$$

$$\begin{array}{c} \odot \odot \odot \\ | \\ 0 \end{array} \quad \begin{array}{c} \odot \odot \odot \\ | \\ e^{1/3} \end{array}$$

Choose $[2, \infty)$ so $f(x)$ indeed decreasing on appropriate interval

Now,

$$\int_2^{\infty} \frac{\ln(x)}{x^3} dx$$

$$= \lim_{t \rightarrow \infty} \int_2^t \frac{\ln(x)}{x^3} dx$$

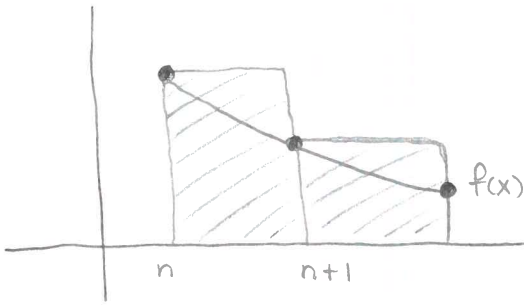
$$= \lim_{t \rightarrow \infty} \left[\frac{-\ln(x)}{(3-1)x^{3-1}} - \frac{1}{(3-1)^2 x^{3-1}} \right]_2^t$$

$$= \lim_{t \rightarrow \infty} \left[\underbrace{\frac{-\ln(t)}{2t^2}}_{\rightarrow 0 \text{ as } t \rightarrow \infty} - \underbrace{\frac{1}{4t^2}}_{\rightarrow 0 \text{ as } t \rightarrow \infty} - \left[\frac{-\ln(2)}{2(2)^2} - \frac{1}{4(2)^2} \right] \right]$$

\Rightarrow CONV

3. According to integral estimate, what is smallest number of terms of series would have to add so that the estimated sum has error < 0.0001 ?

Idea:



$$\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx$$

* Assume $\sum a_n$ conv by Integral Test

Certainly $\sum_{n=1}^{\infty} \frac{1}{n^6}$ is CONV

• $\frac{1}{n^6} > 0$ on $[1, \infty)$ and cont
and DEC

• $\int_n^{\infty} \frac{1}{x^6} dx = -\frac{1}{5x^5} \Big|_n^{\infty} = \frac{1}{5n^5}$

By Remainder Test, want $R_n \leq \frac{1}{5n^5} \leq \frac{1}{10,000} = 0.0001$

$$10,000 \leq 5n^5$$

$$2000 \leq n^5$$

$$5 \approx 4.57 \leq n$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} \text{ conv as p-series as } p=4 > 1$$

Compute $\int_n^{\infty} f(x) dx$, $f(x) = \frac{1}{x^4}$

$$\int_n^{\infty} \frac{1}{x^4} dx = \left. \frac{-1}{3x^3} \right|_n^{\infty} = \frac{1}{3n^3}$$

What value of n so that error < 0.00001 ?

$$\frac{1}{3n^3} < 0.00001 = \frac{1}{100,000}$$

$$\frac{100,000}{3} < n^3$$

$$\sqrt[3]{\frac{100,000}{3}} < n$$

$$\approx 32.18 < n$$

4. Comparison Test: $\sum a_n, \sum b_n$ series st $a_n > 0, b_n > 0$

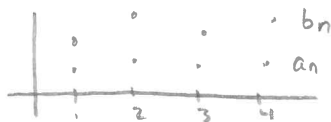
(a) $\sum b_n$ CONV and $a_n \leq b_n \forall n$, then $\sum a_n$ CONV

(b) $\sum b_n$ DIV and $a_n \geq b_n \forall n$, then $\sum a_n$ DIV

"Discrete analogue" of comparison test for improper integrals

Intuition: $\sum b_n$ CONV means $\{s_n = \sum_{i=1}^n a_i \text{ (partial sum)}\} \rightarrow s \in \mathbb{R}$ and so $\sum a_n = s$
 \uparrow sequence of partial sums

$a_n \leq b_n$ means



(i) $\sum_{n=1}^{\infty} \frac{4^{n+1}}{7^{n+2}} = \sum_{n=1}^{\infty} \underbrace{\frac{4 \cdot 4^n}{7^{n+2}}}_{a_n}$ Compare to $\frac{4 \cdot 4^n}{7^n} = b_n$

$a_n \leq b_n$ b/c $7^{n+2} > 7^n$ so $0 < \frac{4 \cdot 4^n}{7^{n+2}} \leq 4 \cdot \left(\frac{4}{7}\right)^n$

and $\sum_{n=1}^{\infty} 4 \left(\frac{4}{7}\right)^n = \frac{4}{1 - (4/7)} \Rightarrow \text{CONV}$
 $\frac{4}{7} \approx 0.5714$

Satisfies CT
so $\sum a_n$ CONV

(ii) $\sum_{n=1}^{\infty} \frac{\sqrt{n^5 - 2}}{n^4 + 3}$

Suffices to look at "dominant terms"
i.e. Look at how say $\sqrt{n^5}$ interacts with n^4

$$\frac{\sqrt{n^5}}{n^4} = \frac{n^{5/2}}{n^4} = n^{5/2 - 8/2} = n^{-3/2} = \frac{1}{n^{3/2}}$$

But also, $\frac{\sqrt{n^5 - 2}}{n^4 + 3} \leq \frac{\sqrt{n^5}}{n^4} = \left(\frac{1}{n^{3/2}}\right)$ CONV p-series $p > 1$

Satisfies CT
so $\sum a_n$ CONV

① $\sqrt{n^5 - 2} < \sqrt{n^5}$ b/c $n^5 - 2 < n^5$

② $n^4 + 3 > n^4$

(iii) $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^4}$

Well, $\arctan(n)$ bdd above by $\pi/2$ as $n \rightarrow \infty$
 n^4 dominates $\arctan(n)$.

Compare $\frac{\arctan(n)}{n^4} \leq \frac{\pi/2}{n^4}$
 $\underbrace{\frac{1}{n^4}}_{\text{CONV}}$

Satisfies CT
so $\sum a_n$ CONV

5. (i) $\sum_{n=1}^{\infty} \frac{5^n}{7^n - 2}$

Tricky b/c of $7^n - 2$ in denominator so can't use $5^n/7^n$ in CT.
However, $5^n/7^{n-2}$ and $5^n/7^n$ both have positive terms, let us try Limit Comparison Test (LCT).

LCT

$\sum a_n, \sum b_n$
series with positive terms

If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$

Then either both conv or DIV

$$\lim_{n \rightarrow \infty} \frac{5^n/7^{n-2}}{5^n/7^n} = \lim_{n \rightarrow \infty} \frac{7^n}{7^n - 2} \stackrel{LH}{=} \lim_{n \rightarrow \infty} \frac{7^n \ln 7}{7^n \ln 7} = 1 > 0$$

$\sum a_n$ acts like $\sum b_n$
 $a_n \approx b_n$

* $\frac{d}{dx} a^x = a^x \ln(a)$

So $\sum_{n=1}^{\infty} \frac{5^n}{7^n - 2}$ CONV

Cases for 0 and ∞
in Ex 40, 41 of §11.4

(Chris McLean, Lecture #9 has nice tabular summary)

Suffices to look at dominant terms n^5 and n^8
Note that

$$\frac{n^5 + 2n}{n^8 + 7n + 5} \leq \frac{n^5}{n^8} \quad (\text{i.e. the linear terms cause this})$$

$$= \frac{1}{n^3} \quad (\text{conv})$$

so by CT, we have $\sum_{n=1}^{\infty} \frac{n^5 + 2n}{n^8 + 7n + 5}$ CONV

(iii) $\sum_{n=1}^{\infty} \frac{\sqrt{n^6 + 3n}}{n^5 + n}$

Hmm... $n^6 + 3n > n^6$ for $n \geq 1$

so $\frac{\sqrt{n^6 + 3n}}{n^5} > \frac{\sqrt{n^6}}{n^5} = \frac{n^{6/2}}{n^5} = \frac{n^3}{n^5} = \frac{1}{n^2}$

conv p-series

Further, $\frac{\sqrt{n^6 + 3n}}{n^5 + n} < \frac{\sqrt{n^6 + 3n}}{n^5}$

Not going to get much farther here

$b_n = \frac{\text{highest pwr num}}{\text{highest pwr denom}}$
textbook

Try LCT: Take $a_n = \frac{\sqrt{n^6 + 3n}}{n^5 + n}$ and $b_n = \frac{\sqrt{n^6}}{n^5} = \frac{1}{n^2}$ CONV as p-series
DIV o/w example is troublesome

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sqrt{n^6 + 3n}}{n^5 + n} \cdot n^2$$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{n^6 + 3n}}{n^4 + 1}$$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{n^8 + 3n^3}/n^4}{(n^4 + 1)/n^4}$$

since $n^4 = \sqrt{n^8}, n > 0$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{\frac{n^8 + 3n^3}{n^8}}}{1 + \frac{1}{n^4}}$$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{1 + \frac{3}{n^5}}}{1 + \frac{1}{n^4}} = 1$$

Note that
although

$\frac{a_n}{b_n} \rightarrow 1$ for ex
 $n \rightarrow \infty$, if b_n div then

Div Test (also for lim DNE)

A.Tan

6. (i) $\sum_{n=1}^{\infty} \arctan(n)$

$$\lim_{n \rightarrow \infty} \arctan(n) = \pi/2 \neq 0 \Rightarrow \sum_{n=1}^{\infty} \arctan(n) \text{ DIV}$$

(ii) $\sum_{n=1}^{\infty} \frac{5^{n+1}}{7^n} = \sum_{n=1}^{\infty} 5 \left(\frac{5}{7}\right)^n, \frac{5}{7} < 1 \Rightarrow \text{CONV as Geometric Series}$

(iii) $\sum_{n=1}^{\infty} \cos(1/n)$

Div Test

$$\lim_{n \rightarrow \infty} \cos(1/n) = \cos(0) = 1 \neq 0 \Rightarrow \sum_{n=1}^{\infty} \cos(1/n) \text{ DIV}$$

Given $\sum_{n=1}^{\infty} a_n$ and n -th partial sum $s_n = \frac{3n-4}{3n+4}$

(a) Find a_2

In general, note that $s_n = \sum_{i=1}^n a_i = a_1 + a_2 + \dots + a_n$ ①

So $s_{n-1} = \sum_{i=1}^{n-1} a_i = a_1 + a_2 + \dots + a_{n-1}$ ②

Now ① - ② $\Rightarrow s_n - s_{n-1} = \cancel{a_1} + \cancel{a_2} + \dots + \cancel{a_{n-1}} + a_n$ ③

④ $\cancel{a_1} + \cancel{a_2} + \dots + \cancel{a_{n-1}}$ ④

a_n

$\boxed{a_n = s_n - s_{n-1}} \Rightarrow a_2 = s_2 - s_{2-1}$

$$= s_2 - s_1$$

$$= \frac{3(2)-4}{3(2)+4} - \frac{3(1)-4}{3(1)+4}$$

(b) Find $\sum_{n=1}^{\infty} a_n = 1$

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{3n-4}{3n+4} = \frac{3}{3} = 1.$$

Facts $\sum a_n \text{ CONV} \Rightarrow \lim_{n \rightarrow \infty} a_n = 0$

$\lim_{n \rightarrow \infty} a_n \neq 0 \Rightarrow \sum a_n \text{ DIV (DIV TEST)}$
or DNE

$\sum a_n \text{ CONV}$ by defn if seq of partial sums $\{s_n\}$
where $s_n := \sum_{i=1}^n a_i$ converges

$\{a_n\}$ infinite list

$\sum a_n$ infinite sum of list's terms!