1. Evaluate the improper integral.

a. $$\underbrace{\int_e^\infty \frac{1}{x[\ln(x)]^2}\,dx}_{\text{Type I}} = \lim_{t\to\infty}\int_e^t \frac{1}{x[\ln(x)]^2}\,dx \qquad \left\{\begin{array}{l} u=\ln(x) \\ du=\frac{1}{x}\,dx \end{array}\right\}$$

$$= \lim_{t\to\infty}\int_1^{\ln(t)} \frac{1}{u^2}\,du$$

$$= \lim_{t\to\infty}\left[-\frac{1}{\ln(x)}\Big|_e^t\right]$$

$$= \lim_{t\to\infty}\left[\underbrace{-\frac{1}{\ln(t)}}_{\substack{\to 0 \\ \text{as } t\to\infty}} + \underbrace{\frac{1}{\ln(e)}}_{=1}\right]$$

$$= 0 + 1 = \boxed{1} \Rightarrow \boxed{\text{CONV}}$$

b. $$\int_{-1}^0 \frac{e^{1/x}}{x^3}\,dx = \lim_{t\to 0^-}\int_{-1}^t \frac{e^{1/x}}{x^3}\,dx$$

$$= \lim_{t\to 0^-}\int_{-1}^t e^{1/x}\cdot\frac{1}{x}\cdot\frac{1}{x^2}\,dx \qquad \left\{\begin{array}{l} y=1/x \\ dy=\frac{-1}{x^2}\,dx \\ -dy=\frac{1}{x^2}\,dx \end{array}\right\}$$

$$= \lim_{t\to 0^-}\int_{-1}^{1/t} e^y\cdot(-y\,dy)$$

$$= -\lim_{t\to 0^-}\int_{-1}^{1/t} ye^y\,dy \qquad \left\{\begin{array}{l} u=y,\ u'=1 \\ v'=e^y,\ v=e^y \end{array}\right\}$$

Aside: Recall Integration by Parts

$$\int uv' = uv - \int vu'$$

b. Cont...

$$= -\lim_{t\to 0^-} \int_{-1}^{1/t} y e^y \, dy \quad \left\{ \begin{matrix} u = y, & u' = 1 \\ v' = e^y, & v = e^y \end{matrix} \right\}$$

$$= -\lim_{t\to 0^-} \left[u e^u \Big|_{-1}^{1/t} - \int_{-1}^{1/t} e^y \, dy \right]$$

$$= -\lim_{t\to 0^-} \left[\frac{1}{t} e^{1/t} + e^{-1} - e^{1/t} + e^{-1} \right]$$

$$= \lim_{t\to 0^-} \left[\underbrace{e^{1/t}}_{\to 0 \text{ as } t \to 0^-} - \frac{1}{t} e^{1/t} - 2e^{-1} \right]$$

$$= \lim_{t\to 0^-} \left[-\frac{1}{t} e^{1/t} - 2e^{-1} \right]$$

Let $s = 1/t$

$$= \lim_{\substack{\cancel{s\to 0^-} \\ s \to -\infty}} \left[-s e^{s} - 2e^{-1} \right]$$

Rewrite $-se^{+s}$ as $\dfrac{-s}{e^{-s}}$

$$= \lim_{\substack{\cancel{s\to 0^-} \\ s \to -\infty}} \left[\frac{-s}{e^{-s}} - 2e^{-1} \right]$$

$$\overset{\text{L'Hôpital}}{=} \lim_{\cancel{s\to 0^-}} \left[\frac{-1}{\underbrace{-e^{-s}}_{\to 0 \text{ as } \cancel{s \to 0^-}\ s \to -\infty}} - 2e^{-1} \right] = \boxed{-2e^{-1}} \Longrightarrow \boxed{\text{CONV}}$$

2. "Inspection" Example: Does $\int^{\infty} \frac{\arctan(x)}{e^{-x}}$ conv or div?

i. Look at $\arctan(x)$ <u>first</u>

ii. It is bounded by $\pi/2$, i.e. $\arctan(x) \leq \pi/2$.

iii. It suffices to examine "e^{-x}". BUT, notice that

$$\int_1^{\infty} \frac{\arctan(x)}{e^{-x}}\,dx = \int_1^{\infty} \frac{e^{x}\arctan(x)}{1}\,dx$$

$$= \int_1^{\infty} e^{x}\arctan(x)\,dx$$

iv. By inspection on $e^{x}\arctan(x)$, as $x \to \infty$, we have that

$$e^{x}\arctan(x) \to \infty \cdot \pi/2$$

Thus, we must have divergence in

$$\int_1^{\infty} e^{x}\arctan(x)\,dx$$

However,

$$\int_1^{\infty} \frac{\arctan(x)}{e^{x}+2}\,dx$$ converges, but why?

3. (a) $\int_1^\infty \frac{1}{x^2+x}\,dx$ C or D?

Guess: We know that $\int_1^\infty \frac{1}{x^2}\,dx$ converges.

Maybe $\int_1^\infty \frac{1}{x^2+\text{"something"}}\,dx$ also converges?

$x \geq 1 \implies x^2 \geq x$

$\implies \frac{1}{x^2} \leq \frac{1}{x}$

Consider adding x to denom.

$$\frac{1}{x^2+x} \leq \frac{1}{x+x}$$

FACT

Although true, "$\frac{1}{x}$" NOT CONV $\longleftrightarrow$ $\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{CONV, \# if } p> \\ \text{DIV, } \pm\infty \text{ if } p\leq \end{cases}$

So, use $\frac{1}{x^2}$ instead, i.e.

$$\frac{1}{x^2+x} \leq \frac{1}{x^2}$$ which also follows that $$0 \leq \frac{1}{x^2+x} \leq \frac{1}{x^2}$$

In which we have that $\int_1^\infty \frac{1}{x^2}\,dx$ CONV by FACT

$$\implies \int_1^\infty \frac{1}{x^2+x}\,dx \text{ CONV by Comparison Theorem}$$

Recall Let f, g be cont. fct. for $x \geq a$. Then,

(1) If $\int_a^\infty f\,dx$ CONV, then $\int_a^\infty g\,dx$ CONV

(2) If $\int_a^\infty g\,dx$ DIV, then $\int_a^\infty f\,dx$ DIV

Cont... We had the comparison thm:

Thm (Comparison for Type I) Let $f(x)$, $g(x)$ be continuous functions on $[a, \infty)$ with $f(x) \geq g(x) \geq 0$. Then,

(a) If $\int_a^\infty f(x)\,dx$ CONV, then $\int_a^\infty g(x)$ CONV

(b) If $\int_a^\infty g(x)\,dx$ DIV, then $\int_a^\infty f(x)$ DIV

Idea:

(a) If you're smaller than something (bigger) that converges, then you also converge.

(b) If you're bigger than something (smaller) that diverges, then you also diverge.

Back to Ex 3. Let's do another problem:

(b) $$\int_2^\infty \frac{\sqrt{x^8-2}}{x^6}\,dx$$

Observe that $\sqrt{x^8} \geq \sqrt{x^8-2}$ for $x \geq 2$.

Since $x^6 > 0$ (again for $x \geq 2$), the inequality is preserved, i.e.

$$\underbrace{\frac{\sqrt{x^8}}{x^6}} \geq \frac{\sqrt{x^8-2}}{x^6} \quad (\geq 0)$$

$= \frac{x^4}{x^6} = \frac{1}{x^2}$ and $\int_2^\infty \frac{1}{x^2}$ CONVERGES! $\Rightarrow \int_2^\infty \frac{\sqrt{x^8-2}}{x^6}\,dx$

CONVERGES BY CT