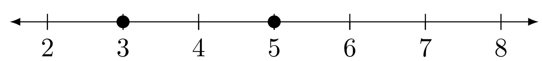
1. **("2nd Sample Test # 1 - Winter Session, 2017)**

19. If $\sum_{n=0}^{\infty} c_n(x-5)^n$ converges when $x = 3$, what can you say about the convergence of the following series?

(i) $\sum_{n=0}^{\infty} c_n(-1)^n$ (ii) $\sum_{n=0}^{\infty} c_n 2^n$

(a) convergent, nothing **(b)** divergent, nothing **(c)** convergent, convergent
(d) nothing, nothing **(e)** convergent, divergent

Solution. The power series is centred at $a = 5$ and has convergence point $x = 3$. Since 3 is about 2 units from 5, we can consider the case where we have radius of convergence $r = 2$. Visually, we have the following diagram of the interval of convergence.



We consider this case as it has the least degree of freedom. In other words, the worst case scenario because more degrees of freedom means a wider interval and more possibilities of convergence!

It is clear that if $x = 3$ is an endpoint, then anywhere between 3 and 5, inclusive, will also converge. However, if we look toward the right of 5 and move 2 units, up to before $x = 7$, we run into trouble. Since our radius is $r = 2$, that makes 7 an endpoint, which is not necessarily convergent, remember? We definitely need more information and we do not have the luxury of that since c_n is unknown. Please, please, please, do not make any assumptions about c_n! Treat it is an arbitrary power series coefficient, and leave it that way.

Now, let's go through the cases.

Case (i) Notice that "$x - 5$" is replaced by "-1". This means we are testing convergence when $x - 5 = -1 \iff x = 4$. Since 4 sits nicely within our interval of convergence, this series will be convergent.

Case (ii) Notice that "$x - 5$" is replaced by "2". This means we are testing convergence when $x - 5 = 2 \iff x = 7$. As mentioned before, $x = 7$ is an endpoint, which is not necessarily convergent. We need more information. So, we can safely say we cannot infer anything simply because we lack information.

$\boxed{\textbf{Answer to 19: (a)}}$

2. **("Post Test # 2 Problem Sampler" - April 2, 2020)**

12. Let

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+2y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases},$$

and let

$$g(x,y) = \begin{cases} \frac{3xy}{\sqrt{2x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

Which of the above functions are continuous at $(0,0)$?
(a) f only **(b)** g only **(c)** f and g **(d)** neither

Solution to (b). Remember that a function $z = f(x,y)$ is continuous if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

This statement has three parts to it: $f(x_0,y_0)$ is defined, the limit exists, and the limit is equal to $f(x_0,y_0)$. Applying this to (b), we have that $g(0,0)$ is defined and is equal to 0. This leaves for us to determine if the limit exists, and is equal to $g(0,0)$.

Let us try a couple of paths to see if there is any hope of discontinuity or to see if there is a pattern that may suggest possible continuity.

Path 1: Let $y = mx$, $m \in \mathbb{R}$. Then $g(x)$ reduces to

$$g(x,y) = g(x,mx) = g(x) = \frac{3x^2}{\sqrt{(2+m^2)x^2}} = \frac{3x}{\sqrt{2+m^2}}$$

Taking the limit as $x \to 0$, we see that

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{x\to 0} g(x) = 0$$

Path 2: Let $y = x^2$. Then $g(x)$ reduces to

$$g(x,y) = g(x,x^2) = g(x) = \frac{3x^3}{\sqrt{2x^2+x^4}} = \frac{3x^2}{\sqrt{2+x^2}}$$

Taking the limit as $x \to 0$, we see that

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{x\to 0} g(x) = 0$$

I meant to do a couple, but path 1 actually took care of a bunch (infinitely many), and path 2 all agree the limit exists and is equal to 0. So, we may have reasonable grounds to believe it really is continuous at $(0, 0)$. Remember once again everyone, paths **CANNOT** prove continuity. They are used to disprove it.

I will now invoke the Squeeze Theorem to solve this problem. Recall, that we require two candidate functions $f(x, y)$ and $h(x, y)$ with the same limit L as $(x, y) \to (x_0, y_0)$ such that $f(x, y) \leq g(x, y) \leq h(x, y)$. Then, we conclude that the limit of $g(x, y)$ is also L as $(x, y) \to (x_0, y_0)$. Building inequalities can be difficult, especially when we have to worry about signs. Yes, I am talking about the x and y in the numerator. Depending on where x and y are approaching from, this can affect our limit. In order to account for all cases, it is convenient to use absolute values. In case you have forgotten, the absolute value is given by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Let $f(x, y) = 0$ and $h(x, y) = 3|xy|$. Observe that

$$0 \leq \frac{3|xy|}{\sqrt{2x^2 + y^2}} \leq 3|xy|$$

The reader should verify this inequality as an exercise and that it is also equal to $g(x, y)$. After the reader has done so, we may easily see that upon taking the limit of $f(x, y)$ and $h(x, y)$ as $(x, y) \to (0, 0)$, we get $L = 0$. This forces the middle function, or $g(x, y)$, to also have limit $L = 0$.

Answer to 12: (b)

3. **("Post Test # 2 Problem Sampler" - April 2, 2020)**

10. Draw a contour plot of $f(x, y) = \ln(y - 1 - 2\sin x)$.

Solution. Set $f(x, y) = c$, a constant. Then,

$$c = \ln(y - 1 - 2\sin x) \iff e^c = y - 1 - 2\sin x \iff y = 2\sin x + 1 + K, \text{ where } K = e^c$$

These level curves are based off "$2\sin x + 1$" which translates (up or down) by a factor of K. It is clear that $K > 0$ since $e^c > 0$ for all $c \in \mathbb{R}$.

Remark. If you want to visualize this in $\mathbb{R}^3$, then for each K considered, you are tracking the elevation along that level curve. So, if you get increasing elevations, pull it up from the ground. Conversely, you could put it into the ground. What shall it be?

$\boxed{\textbf{Answer to 10: (e)}}$

4. **("Post Test # 2 Problem Sampler" - April 2, 2020)**

2. Let $w(s,t) = F(u(s,t), v(s,t))$, where F, u, and v are differentiable, $u(1,0) = 2$, $u_s(1,0) = -2$, $u_t(1,0) = 6$, $v(1,0) = 3$, $v_s(1,0) = 5$, $v_t(1,0) = 4$, $F_u(2,3) = -1$, and $F_v(2,3) = 10$. Find $w_t(1,0)$.
(a) 41 **(b)** 5 **(c)** 16 **(d)** 28 **(e)** 34

Solution. By Case 2 of the Multivariate Chain Rule, we have

$$\begin{aligned} w_t &= \frac{\partial F}{\partial t} \text{ (or } \frac{\partial w}{\partial t} \text{ if you prefer)} \\ &= \frac{\partial F}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial t} \end{aligned}$$

where

$$\begin{aligned} u_t(1,0) &= 6; v_t(1,0) = 4 \\ \frac{\partial F}{\partial u}(2,3) &= F_u(2,3) = -1; \frac{\partial F}{\partial v}(2,3) = F_v(2,3) = 10; \\ u(1,0) &= 2, v(1,0) = 3 \end{aligned}$$

Hence

$$w_t(1,0) = F_u(2,3) \cdot u_t(1,0) + F_v(2,3) \cdot v_t(1,0) = (-1)(6) + (10)(4) = 34$$

$$\boxed{\textbf{Answer to 2: (e)}}$$

5. **("Post Test # 2 Problem Sampler" - April 2, 2020)**

8. If $z = x^2 - xy + 3y^2$ and (x, y) changes from $(3, -1)$ to $(2.96, -0.95)$, find the value of the differential dz.
(a) $-\frac{73}{100}$ **(b)** $\frac{17}{100}$ **(c)** $\frac{73}{100}$ **(d)** $-\frac{17}{100}$ **(e)** $\frac{17}{10}$

Solution. Recall, the differential of z, dz, is given by

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

where $dx = \Delta x = x - x_0$ and $dy = \Delta y = y - y_0$.

From above, we have

$$\frac{\partial f}{\partial x} = 2x - y; \frac{\partial f}{\partial y} = 6y - x$$

with

$$\frac{\partial f}{\partial x}(3, -1) = 7; \frac{\partial f}{\partial y} = -9; dx = 2.96 - 3 = -0.04; dy = -0.95 - (-1) = 0.05$$

Hence

$$dz = (7)(-0.04) + (-9)(-0.05) = \frac{7}{1} \cdot \frac{(-4)}{100} + \frac{(-9)}{1} \cdot \frac{5}{100} = -\frac{73}{100}$$

$\boxed{\textbf{Answer to 8: (a)}}$

6. **("Post Test # 2 Problem Sampler" - April 2, 2020)**

20. If

$$e^{xyz} = x + y + z$$

then $\frac{\partial z}{\partial x}$ is equal to

(a) $\dfrac{-1}{1-e^{-xyz}}$ **(b)** $\dfrac{yze^{xyz}-1}{1-xye^{xyz}}$ **(c)** $\dfrac{yze^{xyz}-1-y}{1-xye^{xyz}}$

(d) $yze^{xyz}-1-y$ **(e)** $\dfrac{-xye^{xyz}/z^2-1-y}{1-e^{-xyz}}$

Solution. Assume $z = f(x,y)$. That is, z is an implicit function of x and y, the independent variables. Applying the derivative with respect to x to both sides and re-arranging we obtain

$$\begin{aligned}
\frac{\partial}{\partial x}e^{xyz} &= \frac{\partial}{\partial x}(x+y+z) \\
e^{xyz}\cdot\left(\frac{\partial}{\partial x}(xyz)\right) &= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x} + \frac{\partial z}{\partial x} \\
e^{xyz}\cdot\left(\frac{\partial x}{\partial x}\cdot yz + \frac{\partial y}{\partial x}\cdot xz + \frac{\partial z}{\partial x}\cdot xy\right) &= 1 + 0 + z_x \\
(yz + xy\cdot z_x)e^{xyz} &= 1 + z_x \\
yze^{xyz} + xy\cdot z_x\cdot e^{xyz} &= 1 + z_x \\
xy\cdot z_x\cdot e^{xyz} - z_x &= 1 - yze^{xyz} \\
z_x &= \frac{1-yze^{xyz}}{xye^{xyz}-1} \\
z_x &= \frac{(-1)(yze^{xyz}-1)}{(-1)(1-xye^{xyz})} \\
z_x &= \frac{yze^{xyz}-1}{1-xye^{xyz}}
\end{aligned}$$

$\boxed{\textbf{Answer to 20: (b)}}$

7. **("For the April 2020 Exam Practice")**

30. The parametric curve $x = \cos t$, $y = \sin t \cos t$ has two tangent lines at $(0,0)$. Find their slopes.
(a) $\frac{1}{2}, -1$ **(b)** $-\frac{1}{2}, 1$ **(c)** $-1, 1$ **(d)** $0, 1$ **(e)** $0, -1$

Solution. Let us begin by computing the respective derivatives.

$$\frac{dx}{dt} = -\sin t; \frac{dy}{dt} = \cos^2 t - \sin^2 t; \frac{dy}{dx} = \frac{\cos^2 t - \sin^2 t}{-\sin t} = \frac{\sin^2 t - \cos^2 t}{\sin t}$$

We want to determine the value of t such that $x = 0 = \cos t$. We know that this occurs whenever $t =$ "odd" $\cdot \frac{\pi}{2} = (2n+1) \cdot \frac{\pi}{2}$ for $k \in \mathbb{Z}$ (symbol for the set of integers). For $t \in [0, 2\pi]$, we have $t = \pi/2$ and $t = 3\pi/2$ and the two will yield $(x, y) = (0, 0)$. In trignometric parametrizations, the period (or interval in this case) tells you about repetitive properties produced by a particle (or point) on the curve. Since the usual period is unaffected via transformations, then we are clear to use it.

$$\text{At } t = \pi/2 : \frac{dy}{dx} = \frac{1-0}{1} = 1$$
$$\text{At } t = 3\pi/2 : \frac{dy}{dx} = \frac{1-0}{-1} = -1$$

$$\boxed{\textbf{Answer to 30: (c)}}$$

8. **("For the April 2020 Exam Practice")**

36. Find the length of the following parametric curve

$$x = e^t + e^{-t}, \quad y = 5 - 2t \qquad 0 \le t \le 1$$

(a) $\frac{1}{2}e(e-1)$ **(b)** $e(e-1)$ **(c)** $e^2 - 1$ **(d)** $e - \frac{1}{e}$ **(e)** $\frac{1}{2}(e^2 - e^{-2} + 4)$

Solution. By definition, we have

$$
\begin{aligned}
L &= \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_0^1 \sqrt{(e^t - e^{-t})^2 + 4}\, dt \\
&= \int_0^1 \sqrt{e^{2t} - 2e^t e^{-t} + e^{-2t} + 4}\, dt \\
&= \int_0^1 \sqrt{e^{2t} + e^{-2t} + 2}\, dt \\
&= \int_0^1 \sqrt{\frac{e^{4t} + 2e^{2t} + 1}{e^{2t}}}\, dt \\
&= \int_0^1 \sqrt{\frac{(e^{2t}+1)^2}{(e^t)^2}}\, dt \\
&= \int_0^1 e^{-t}(e^{2t} + 1)\, dt \\
&= \int_0^1 e^t + e^{-t}\, dt \\
&= e^t - e^{-t} \Big|_0^1 \\
&= e^1 - e^{-1} - (e^0 - e^0) \\
&= e - \frac{1}{e}
\end{aligned}
$$

$\boxed{\textbf{Answer to 36: (d)}}$