Example on Multivariate Chain Rule (1AA3/12B, Winter 2020)

TA: A. Tan

$h(t) = f(x(t), y(t))$ where $x = 2e^t$ and $y = 3t$

$f_x(2,0) = 2$, $f_{xx}(2,0) = 3$, $f_{xy}(2,0) = 4$

$f_y(2,0) = 4$, $f_{yy}(2,0) = 4$

Find $\dfrac{d^2h}{dt^2}$ at $t=0$

Using $\boxed{2}$ Chain Rule case 1:

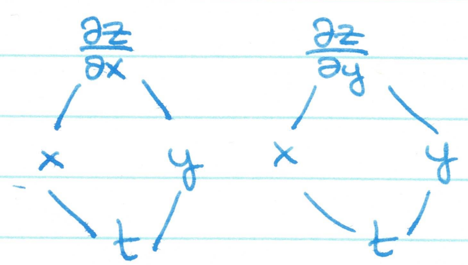
$$\frac{dh}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{d}{dt}\left(\frac{dh}{dt}\right) = \frac{d}{dt}\left[\frac{\partial z}{\partial x}\frac{dx}{dt}\right] + \frac{d}{dt}\left[\frac{\partial z}{\partial y}\frac{dy}{dt}\right]$$

$$= \underbrace{\frac{d}{dt}\left[\frac{\partial z}{\partial x}\right]}_{?}\cdot\underbrace{\frac{dx}{dt}}_{x'} + \underbrace{\frac{d}{dt}\left[\frac{dx}{dt}\right]}_{x''}\cdot\underbrace{\frac{\partial z}{\partial x}}_{f_x} + \underbrace{\frac{d}{dt}\left[\frac{\partial z}{\partial y}\right]}_{?}\cdot\underbrace{\frac{dy}{dt}}_{y'} + \underbrace{\frac{d}{dt}\left[\frac{dy}{dt}\right]}_{y''}\cdot\underbrace{\frac{\partial z}{\partial y}}_{f_y}$$

$$\frac{d}{dt}\left[\frac{\partial z}{\partial x}\right] = \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)}_{f_{xx}}\cdot\frac{dx}{dt} + \underbrace{\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)}_{f_{xy}}\cdot\frac{dy}{dt}$$

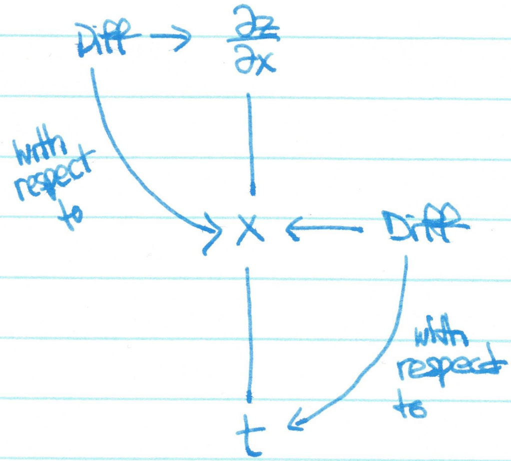
$$= f_{xx}\cdot x' + f_{xy}\cdot y'$$



$$\frac{d}{dt}\left[\frac{\partial z}{\partial y}\right] = \underbrace{\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right]}_{f_{xy}}\cdot\underbrace{\frac{dx}{dt}}_{x'} + \underbrace{\frac{\partial}{\partial y}\left[\frac{\partial z}{\partial y}\right]}_{f_{yy}}\cdot\underbrace{\frac{dy}{dt}}_{y'}$$

$$= f_{xy}\cdot x' + f_{yy}\cdot y'$$

A branch is read as



which gives

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial x}{\partial t}$$

Therefore,

$$\frac{d^2h}{dt^2} = \left[f_{xx}\cdot x' + f_{xy}\cdot y'\right]x' + x''\cdot f_x$$

$$+ \left[f_{xy}\cdot x' + f_{yy}\cdot y'\right]y' + y''\cdot f_y$$

At $(x_0,y_0) = (2,0)$ and $t=0$, we get

$$\left.\frac{d^2h}{dt^2}\right|_{(x_0,y_0)=(2,0),\ t=0} =$$

Example continued...

TA : A.Tan

$$\frac{d^2h}{dt^2} = [f_{xx} \cdot x' + f_{xy} \cdot y']\,x' + x'' \cdot f_x + [f_{xy} \cdot x' + f_{yy} \cdot y']\,y' + y'' \cdot f_y$$

$x = 2e^t, \quad x' = 2e^t, \quad x'' = 2e^t$

At $t = 0 \Rightarrow x = x' = x'' = 2e^0 = 2$

$y = 3t, \quad y' = 3, \quad y'' = 0$

At $t = 0 \Rightarrow y = 3(0) = 0$

$$\left.\frac{d^2h}{dt^2}\right|_{\substack{(2,0)\\ t=0}} = [3\times 2 + 4\times 3]\times 2 + 2\times 2 + [4\times 2 + 4\times 3]\times 3 + \cancel{0\times 4}$$

$$= [6+12]\times 2 + 4 + [8+12]\times 3$$

$$= 36 + 4 + 20\times 3$$

$$= 40 + 60$$

$$= 100.$$