# 1 TA Information

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# 2 Study Tips

1. Practice, practice, practice.
2. Time yourself and be strict!
3. Comment line-by-line to understand more difficult calculations.
4. Talk to your professors and TA's.

# 3 Some Key Ideas

In this section, I addressed concepts which students found confusing. Here are some summaries to clarify.

## 3.1 First and Second Derivative Tests

- First Derivative Test
  - Generally, the "go-to" option.
  - You're solving $f'(x) = 0$ for critical points. In other words, the values of $x$ which allow $f'(x) = 0$.
  - Can determine intervals of increasing/decreasing which then also tells you about local maxima and minima.
  - If the sign of $f'(x)$ changes over consecutive intervals, this is your indication. Let $A$ and $B$ be two consecutive intervals. If $f'(x) > 0$ in interval $A$ and $f'(x) < 0$ in interval $B$, we have that $f(x)$ changes from increasing to decreasing. Hence we have a local maximum at the critical point. To find local minimums, you reverse the wording. (See the textbook/class notes)
- Second Derivative Test
  - You're solving $f'(x) = 0$ for critical points. In other words, the values of $x$ which allow $f'(x) = 0$.
  - You would then plug each of these $x$ into $f''(x)$ and see what sign you get. If $f''(x) > 0$, then you have a local minimum. If $f''(x) < 0$, then you have a local maximum.
  - However, if $f''(x) = 0$, then the test is inconclusive and you have to resort back to the first derivative test.

### 3.2 Concavity and Inflection Points

I will attempt to describe the procedure here. Now, be careful because now, we are looking to solve the equation

$$f''(x) = 0.$$

Then, like the first derivative test, we can determine intervals of concavity (concave up/down) which then also tells you about inflection points. Let $A$ and $B$ be two consecutive intervals. If the sign of $f''(x)$ changes over consecutive intervals, this is your indication. That is, If $f''(x) > 0$ in interval $A$ and $f''(x) < 0$ in interval $B$, we have that $f(x)$ changes from concave up to concave down. Hence we have an inflection point at the critical point. The reverse wording also applies. (See the textbook/class notes)

### 3.3 Anti-Derivatives

### 3.4 What is an Anti-Derivative?

In 5.1, you begin to study the other half of Calculus. We call this half the integral calculus portion. Now, the main idea here is the concept of an anti-derivative. I mentioned this quite a bit, but it is absolutely crucial to understand this definition.

An anti-derivative of $f(x)$ is a function $F(x)$ such that $f(x) = F'(x)$.

Ok so, this may be a bit technical, but I will attempt to break it down. Essentially, all you're trying to do is, given a function $f$, we can find another function $F$ such that if I take the derivative of $F$, I get $f$.

### 3.5 Guess-and-Check

Since substitution and onwards is not covered on the first test, then that leaves the most basic technique of anti-differentiation which is guess-and-check. In order to even do this, we require an understanding of the definition above.

**Example 1.** *Say we want to find the anti-derivative of $f(x) = e^x$. Is there a function such that if I differentiate it, I get back $e^x$? You may guess $e^x$ and in fact, you're actually right. To check, we know the derivative of $e^x$ is just itself. However, we can do a little better. If we add any constant, these are also valid solutions. So an anti-derivative of $f(x) = e^x$ is of the form $F(x) = e^x + C$, where $C$ is any constant.*

**Example 2.** *What is the antiderivative of $x^2$? Take $F(x) = (1/3)x^3$. Indeed, this solution works since $F'(x) = (1/3)(3)x^{3-1} = x^2$. More generally, the anti-derivative has the form $F(X) = (1/3)x^3 + C$, where $C$ is any constant.*

**Example 3.** *What is the antiderivative of $1/x$? The only function we know that has derivative $1/x$ is $\ln(x)$. However, note that the domain of $1/x$ considers all non-zero real values of $x$. To* *ensure that $f(x) = F'(x)$, we need to add an absolute value to $\ln(x)$. That is, we want $\ln|x|$. So, the anti-derivative of $1/x$ is of the form $F(x) = \ln|x| + C$, where $C$ is any constant.*

## 4 Problems

**Problem 1.** *Find the value of $a$ if*

$$\left(\frac{x^2}{x^{-5/2}}\right)^{-1/4}\sqrt{x^3} = x^a.$$

**Solution 1.** *For simplicity, abbreviate the left-hand side as LHS. Then applying exponent rules, the LHS simplifies to*

$$\begin{aligned}
\left(\frac{x^2}{x^{-5/2}}\right)^{-1/4}\sqrt{x^3} &= \left(\frac{x^2}{x^{-5/2}}\right)^{-1/4}x^{3/2} \\
&= (x^{9/2})^{-1/4}x^{3/2} \\
&= x^{-9/8}x^{3/2} \\
&= x^{3/8}.
\end{aligned}$$

*For the LHS to be equal to the right-hand side, or RHS, we must have $a = 3/8$.*

**Problem 2.** *Find all real values of $x$ that satisfy the following equation.*

$$\left(\frac{1}{25}\right)^{30-x^2} = 5^{14x}.$$

**Solution 2.** *Again, we simplify the LHS.*

$$\left(\frac{1}{25}\right)^{30-x^2} = (5^{-2})^{30-x^2} = 5^{-2(30-x^2)} = 5^{-60+x^2}.$$

*For LHS to equal RHS, we require that*

$$\begin{aligned}
-60 + 2x^2 = 14x &\iff 2x^2 - 14x - 60 = 0 \\
&\iff x^2 - 7x - 30 = 0 \\
&\iff (x+3)(x-10) = 0.
\end{aligned}$$

*In other words, $x = -3$ or $x = 10$.*

**Problem 3.** *Solve the following equation for $x$.*

$$\frac{17e^{9x}}{e^{9x}+9} = 8.$$

**Solution 3.**

$$\begin{aligned}
\frac{17e^{9x}}{e^{9x}+9} &= 8 \\
17e^{9x} &= 8(e^{9x}+9) \\
17e^{9x} &= 8e^{9x}+72 \\
17e^{9x}-8e^{9x} &= 72 \\
9e^{9x} &= 72 \\
e^{9x} &= 8 \\
\ln e^{9x} &= \ln 8 \\
9x &= \ln 8 \\
x &= \frac{\ln 8}{9}.
\end{aligned}$$

**Problem 4.** *Let $f(x) = xe^{1-2x^2}$. Find the largest interval(s) on which $f$ is increasing.*

**Solution 4.** *Let's compute the first derivative.*

$$\begin{aligned}
f'(x) &= 1 \cdot e^{1-2x^2} + xe^{1-2x^2}(0-4x) \\
&= e^{1-2x^2} - 4x^2e^{1-2x^2} \\
&= e^{1-2x^2}(1-4x^2).
\end{aligned}$$

*Upon solving $f'(x) = 0$, we have that*

$$f'(x) = 0 = e^{1-2x^2}(1-4x^2) \iff 1-4x^2 = 0 \iff x = \pm 1/2$$

*since $e^{1-2x^2} > 0$ for all values of $x$. Looking at these critical points, we obtain 3 intervals: $(-\infty, -1/2), (-1/2, 1/2)$, and $(1/2, \infty)$. To test for increasing and decreasing, we look at the sign of $f'(x)$ in each interval at some chosen point (this can be totally random so don't worry). Since $f'(x) < 0$ on $(-\infty, -1/2)$ and $(1/2, \infty)$ and $f'(x) > 0$ on $(-1/2, 1/2)$, we have that $f(x)$ is increasing on $(-1/2, 1/2)$.*

**Problem 5.** *Find the equation of the line that is tangent to the graph of the function*

$$f(x) = ln(5x+3)$$

*at the point $(0, \ln 3)$.*

**Solution 5.** *Since we are given a point $x$ with its associated $y$, we don't have to worry abotu finding $y$. All that is left is for us to compute the slope at $x = 0$, or find the value of the derivative at*

$x = 0$.

$$\begin{aligned} f'(x) &= \frac{1}{5x+3} \cdot \frac{d}{dx}(5x+3) \\ &= \frac{1}{5x+3} \cdot (5) \\ &= \frac{5}{5x+3}. \end{aligned}$$

*At $x = 0$, we have $f'(0) = 5/(5(0)+3) = 5/3$. Putting everything together, we get*

$$\begin{aligned} y &= mx + b \\ \ln 3 &= (5/3)(0) + b \\ b &= \ln 3. \end{aligned}$$

*Hence, the equation of the tangent line to $(0, \ln 3)$ is given by $y = (5/3)x + \ln 3$.*

**Problem 6.** *Find the smallest value (i.e., the absolute minimum) of the function*

$$f(x) = x \ln x$$

*on the interval $1 \leq x \leq 2$.*

**Solution 6.** *Start by taking the derivative of $f(x)$.*

$$\begin{aligned} f'(x) &= 1 \cdot \ln x + x \cdot \frac{1}{x} \\ &= \ln x + 1. \end{aligned}$$

*Set $f'(x) = 0$ to solve for the critical points. That is,*

$$f'(x) = 0 = \ln x + 1 \iff \ln x = -1 \iff x = e^{-1}.$$

*However, notice that $e^{-1} \approx 0.3679$. So this critical point does not lie in the interval $1 \leq x \leq 2$, and thus we cannot consider it. It suffices from here to test the end points of the interval. So, $f(1) = (1)\ln(1) = 0$ and $f(2) = (2)\ln(2)$. Since $0 < 2\ln(2)$, we have that the absolute minimum is $f(1) = 0$.*

**Problem 7.** *Compute the anti-derivative of $f(x) = x^5 + 3x^2$.*

**Solution 7.**

$$\begin{aligned}\int f(x)dx &= \int x^5 + 3x^2 dx \\ &= \int x^5 dx + \int 3x^2 dx \\ &= \int x^5 dx + 3\int x^2 dx \\ &= \frac{1}{6}x^6 + 3 \cdot \frac{1}{3}x^3 + C \\ &= \frac{1}{6}x^6 + x^3 + C.\end{aligned}$$

*Check:*

$$\frac{d}{dx}\left(\frac{1}{6}x^6 + x^3 + C\right) = \frac{1}{6} \cdot 6x^{6-1} + 3x^{3-1} + 0 = x^5 + 3x^2.$$

**Problem 8.** *Compute the anti-derivative of $x^n$, where $n \neq -1$.*

**Solution 8.**

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C.$$

*Check:*

$$\frac{d}{dx}\left(\frac{1}{n+1}x^{n+1} + C\right) = \frac{1}{n+1}(n+1)x^{(n+1)-1} + 0.$$

**Problem 9.** *Compute*

$$\int \frac{2\sqrt{x} + \sqrt[3]{x}}{\sqrt{x}} dx.$$

**Solution 9.**

$$\begin{aligned}\int \frac{2\sqrt{x} + \sqrt[3]{x}}{\sqrt{x}} dx &= \int \frac{2x^{1/2} + x^{1/3}}{x^{1/2}} dx \\ &= \int \frac{2x^{1/2}}{x^{1/2}} dx + \int \frac{x^{1/3}}{x^{1/2}} dx \\ &= \int 2dx + \int x^{-1/6} dx \\ &= 2x + \frac{1}{-1/6+1}x^{-1/6+1} + C \\ &= 2x + \frac{1}{5/6}x^{5/6} + C \\ &= 2x + \frac{6}{5}x^{5/6} + C.\end{aligned}$$