MATH 1K03 Lecture 13 10/03/2018

Announcements

- Test 1 at 7:00pm – 8:00pm { ABB 102 if your TA is Anthony; ITB AB102 if your TA is Jordan
- Please do not forget your student ID and the correct calculator, i.e. Casio FX991MS or MS+

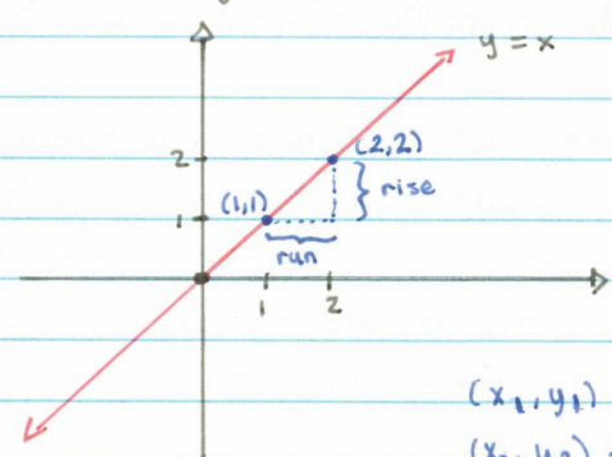
Introduction to Derivatives (+ more details!)

Done with limits, so now we're ready for derivatives, and calculus! So, what is calculus? Calculus is the study of change. In particular, rates of change. For simplicity, rate of change will be abbreviated as RC.

Ex
- Minimum wage $14 / hr ← constant RC
- Gas price $ amount / litre ← (temporarily) constant RC
- speed/velocity meters/second or m/s ← not necessarily constant RC

Recall, $\boxed{\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}}$ Δ means "delta" or "change in" here

Ex (Visualizing slope)



$(x_1, y_1) = (1,1)$

$(x_2, y_2) = (2,2)$

$y = 1 \cdot x + 0$ ← y-intercept

↑ slope $= \frac{\text{rise}}{\text{run}} = \frac{1}{1} = 1$

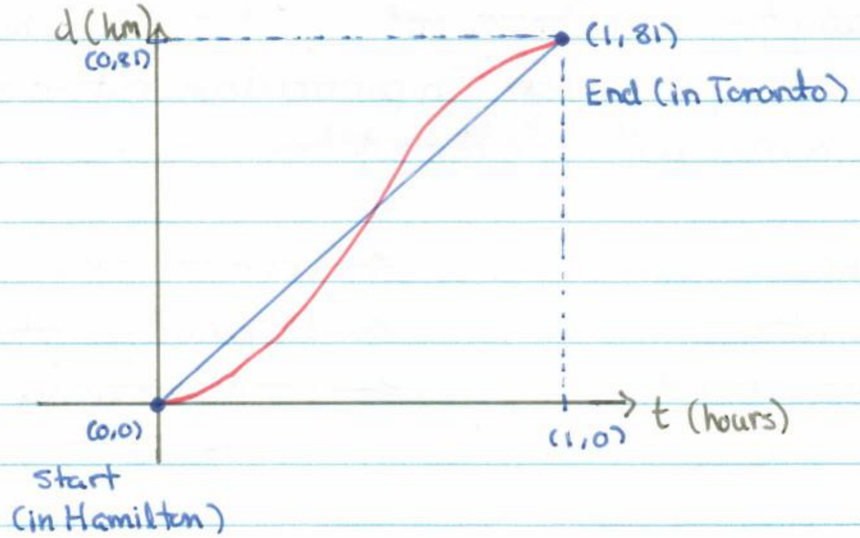
This is an example of constant RC. In general, linear functions always have constant RC.

However, this is NOT the case for non-linear functions.

Q How can we measure the RC of non-linear functions?

Sketch of the Answer

Ex Driving a car from Hamilton to Toronto (81 km) and the trip takes an hour to complete. What is the average speed during this time?



d - distance

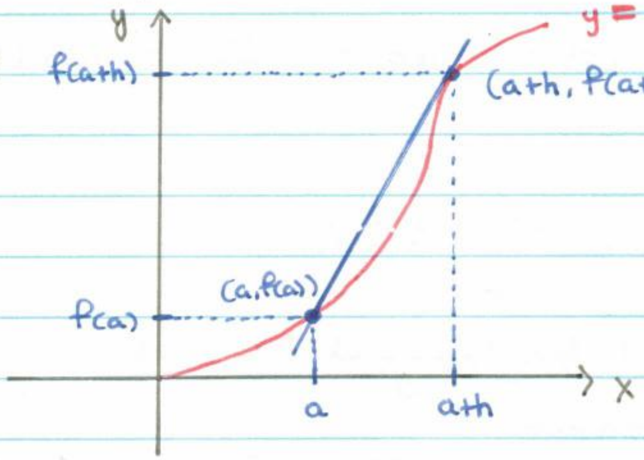
t - time

$$V_{avg} = \frac{\text{rise}}{\text{run}} = \frac{81\text{ km}}{1\text{ hr}}$$
$$\left(\frac{y_2 - y_1}{x_2 - x_1} \right)$$

We have illustrated an example of average RC (or ARC) here. The blue line connecting the two points (0,0) and (1,81) is known as a secant line. The slope of the secant line is equal to the ARC (or 81 km/hr in this case). Note, secant lines intersect at least two points.

Knowing ARC in a non-linear scenario is fine, but how about measuring RC at a particular "instant" in time? This would be a lot more useful in real-life applications! It is easy to see in the linear scenario, but we will require a bit more machinery for measuring RC at specific instances. Let's see this step-by-step, which I'll call S1, S2, S3, ...

(S1)

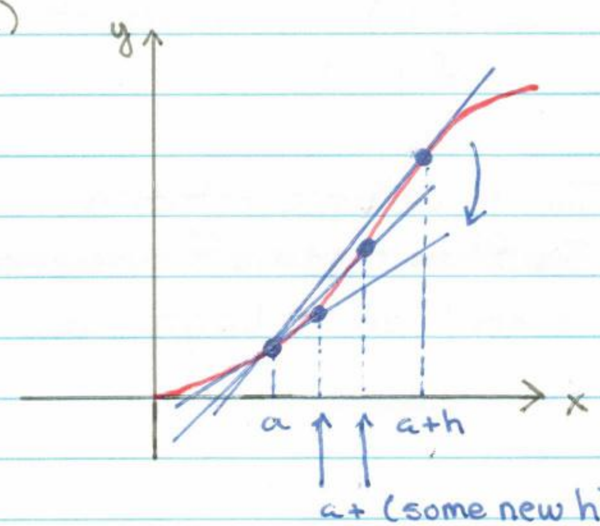


To approximate the RC at $x = a$, mark another point $a+h$ on the x-axis and connect the points $(a, f(a))$ and $(a+h, f(a+h))$. We are going to use secant lines for measuring the RC at $x = a$.

PAUSE! How did I get $f(a)$ and $f(a+h)$? These are just the y-coordinates associated to $x = a$ and $x = a+h$ and the pair $(a, f(a))$ and $(a+h, f(a+h))$, respectively.

What is h? You should think of h as some deviation or "step size" from a. In other words, if I am at $x = 2$ and you are at $x = 5$, this is like saying you are 3 steps away from me (i.e., $x = 2 + 3$, where $h = 3$).

(S2)



Decrease the deviation from a, i.e. decrease h. Each time this is done draw a new secant line. The slope of each secant line is given by

$$m_{secant} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(a+h) - f(a)}{h}$$

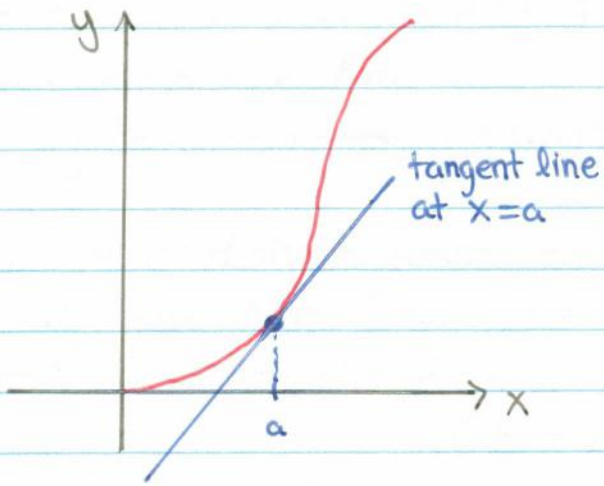
which is the difference quotient! Note that $x_2 - x_1 = (a+h) - a = h$.

(53) Remember limits? Let's use them. Let's make h VERY small (i.e. have $h \to 0$).

$$\lim_{h\to 0} \left(\text{slopes of secant lines}\right) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \text{slope of the tangent line at } x = a, \text{ or } m_{tangent}$$

So, by taking the limit of the slope of secant lines, we get the slope of the tangent line at x=a. (i.e. a line intersecting exactly one point). This slope of the tangent line gives us our instantaneous rate of change at $x = a$.



Slope of tangent line at $x = a$ has Synonyms:

- instantaneous rate of change at $x = a$
- derivative at $x = a$

Notation: $\boxed{f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}}$ ← $f'(a)$ or the derivative of f at a.

Can also use $\underbrace{\left.\frac{df}{dx}\right|_{x=a}}_{\text{derivative of } f \text{ evaluated at } x=a}$ ← This is Leibniz notation

The f' is read as "f prime" and is credited to Lagrange.

Ex Find the derivative of $f(x) = \sqrt{x}$ at $x = 1$.

Soln
$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{1+h} - \sqrt{1}}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{1+h} - 1}{h} \times \frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}$$
$$= \lim_{h\to 0} \frac{(1+h) + \sqrt{1+h} - \sqrt{1+h} - 1}{h(\sqrt{1+h}+1)}$$
$$= \lim_{h\to 0} \frac{\cancel{1} + h + \cancel{\sqrt{1+h}} - \cancel{\sqrt{1+h}} \cancel{-1}}{h(\sqrt{1+h}+1)}$$
$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{1+h}+1)}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{1+h}+1}$$
$$\overset{\text{DIRECT}}{\underset{\text{SUB}\,(h=0)}{=}} \frac{1}{\sqrt{1+(0)}+1}$$
$$= \boxed{\frac{1}{2}}$$

Extra Material (I rushed some of this)

Here is what I rushed through in class, but in more detail.

Defn The derivative of a function $f(x)$ with respect to x is the function

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}}$$

where $\frac{f(x+h) - f(x)}{h}$ is the difference quotient of $f(x)$.

This expression is the general version of what we saw previously. If $x = a$, then we get the $f'(a)$ expression.

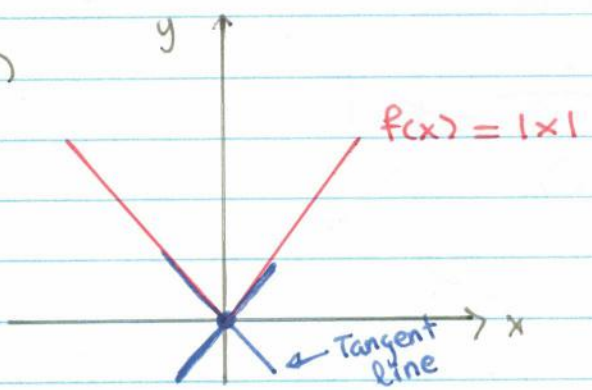
We have a catch... This calculation can fail!

The process of computing derivatives is called differentiation.

We say $f(x)$ is differentiable at $x=a$ provided $f'(a)$ exists, i.e. the limit exists!

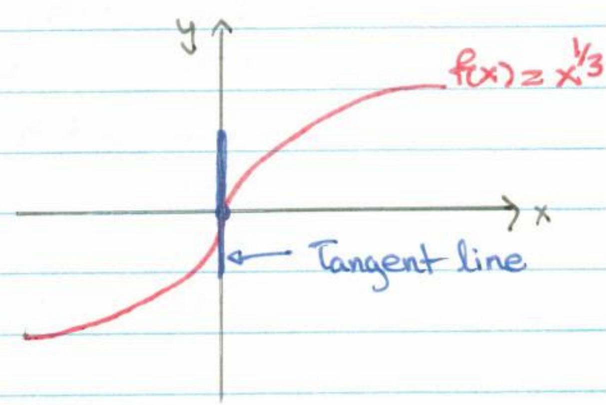
<u>Ex</u> (So when is a function NOT differentiable?)

(a)

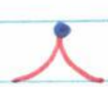


Not differentiable at $x=0$. In general, we can't differentiate at "corners" or "sharp points".

(b)



Not differentiable at $x=0$. In general, we cannot have vertical tangent lines.

There are more examples like "cusps"  or discontinuities } See textbook

Notice that the above functions are continuous. So the moral here is that although a function is continuous, that does NOT imply that it is differentiable everywhere.

A function is differentiable if it is differentiable everywhere on its domain.