MATH 1M03 Test #2, Review Session

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Office hours: 2:30p - 6:30p Fri in Math Help Centre

① Substitution (or some articles/books say "u-substitution")

$$\int f(x)\,dx \longrightarrow \int g(u)\,du$$

> we are replacing x's with u's and (hopefully) things cancel nicely

> There is something in my expression that looks like a derivative of u

② Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

> We declare <u>one function to be u</u> and <u>another to be dv</u>

Eventually you'll a set-up like

$$\left\{ \begin{matrix} u = ? \,, & du = ? \\ dv = ? \,, & v = ? \end{matrix} \right\}$$

Alternatively, if you prefer

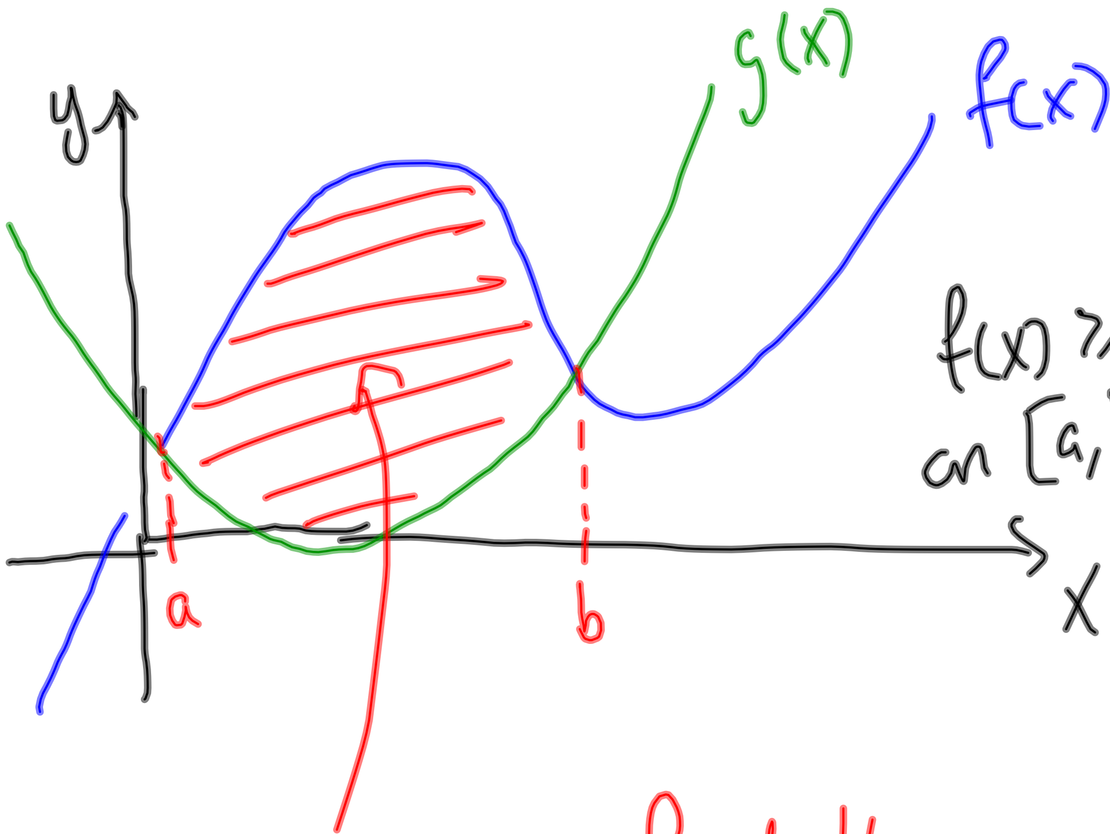
$$\left\{ \begin{matrix} u = ? \,, & u' = ? \\ v' = ? \,, & v = ? \end{matrix} \right\}$$

3) For a closed interval $[a,b]$ (or alternatively $a \le x \le b$)

the average value is given by

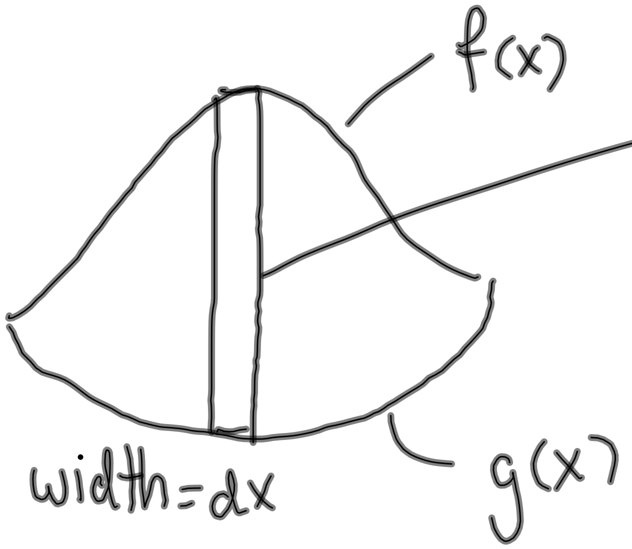
$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

4 Area between curves



$f(x) \geq g(x)$
on $[a,b]$

How do I find the area of this region?



$= \underline{f(x) - g(x)}$

≥ 0

b/c $f(x) \geq g(x)$

After cutting "a bunch of times"

Area between $f(x)$ and $g(x)$ on $[a, b]$ = "The continuous sum of rectangles between a and b"

$$= \int_a^b [f(x) - g(x)]\,dx$$

2nd Sample Test #2

2. The slope of a graph at any x is given by $f'(x) = e^2 - e^x$ and the function passes $(0,1)$. Find $f(x)$.

Thoughts:

- Want $f(x)$ given $f'(x)$...

$$f'(x) \xrightarrow{\int dx} f(x)$$

- $f(0) = 1$ (same as $(0,1)$)

Soln

$$\int e^2 - e^x \, dx = \underbrace{e^2 x - e^x}_{f(x)} + C$$

$$f(0) = 1$$

$$1 = \cancel{e^2(0)} - e^0 + C$$

$$C = 2$$

$$\therefore f(x) = e^2 x - e^x + 2 \quad \boxed{E}$$

7. $\displaystyle\int \frac{e^{2x}}{\sqrt{3+e^{2x}}}\,dx = \int \frac{\cancel{e^{2x}}}{\sqrt{u}} \cdot \frac{du}{2\cancel{e^{2x}}}$

$$\left\{\begin{aligned} u &= 3+e^{2x} \\ \frac{du}{dx} &= \frac{2e^{2x}}{1} \\ dx &= \frac{du}{2e^{2x}} \end{aligned}\right\}$$

$= \displaystyle\int \frac{1}{2} \cdot u^{-1/2}\,du$

$= \dfrac{1}{2} \cdot \dfrac{1}{-1/2+1} u^{-1/2+1} + C$

$= \dfrac{1}{\cancel{2}} \cdot \cancel{2} u^{1/2} + C$

$= (3+e^{2x})^{1/2} + C$

$\boxed{C}$

Ex (Not in sample test, but it's a tricky problem)

$$\int_1^3 t^5 \sqrt{t^3+4}\, dt$$

Pause and compute $\int t^5 \sqrt{t^3+4}\, dt$

Set-up: $u = t^3 + 4$

$$\frac{du}{dt} = 3t^2$$

Modify set-up:

$$t^3 = u - 4$$

$$\frac{du}{3} = t^2\, dt$$

$$\int t^5 \sqrt{t^3+4}\, dt$$

$$= \int t^2 t^3 \sqrt{t^3+4}\, dt$$

$$= \int \underbrace{t^3}_{u-4} \underbrace{\sqrt{t^3+4}}_{u} \underbrace{t^2\, dt}_{du/3}$$

$$= \int (u-4)\, u^{1/2}\, \frac{du}{3}$$

$$= \frac{1}{3} \int \left(u^{3/2} - 4u^{1/2}\right) du$$

$$= \frac{1}{3}\left[\frac{1}{1+3/2}u^{3/2+1} - \frac{4}{1+1/2}u^{1/2+1}\right] + C$$

$$= \frac{1}{3}\left[\frac{2}{5}u^{5/2} - 4\cdot\frac{2}{3}u^{3/2}\right] + C$$

$$\underbrace{= \frac{1}{3}\left[\frac{2}{5}(t^3+4)^{5/2} - \frac{8}{3}(t^3+4)^{3/2}\right] + C}_{F(t)}$$

$$\int_1^3 t^5\sqrt{t^3+4}\,dt = F(3) - F(1)$$

Ans: $\frac{2}{15}(31)^{5/2} - \frac{2}{15}(5)^{5/2} - \frac{8}{9}(31)^{3/2} + \frac{8}{9}(5)^{3/2}$

18. Evaluate $\int_1^2 x^2 e^x \, dx$

For now, forget about $\int_1^2$ and focus on $\int f \, dx$ (indefinite int)

OLD

$$\int x^2 e^x \, dx \quad \left\{ \begin{array}{l} u = x^2, \; du = 2x \\ dv = e^x, \; v = e^x \end{array} \right\}$$

$$= uv - \int v \, du$$

OLD

NEW

$$= x^2 e^x - \int 2x e^x \, dx \quad \left\{ \begin{array}{l} u = x, \; du = 1 \\ dv = e^x, \; v = e^x \end{array} \right\}$$

$$= x^2 e^x - 2\left(\int x e^x \, dx \right)$$

$$= x^2e^x - 2\left(uv - \int v\,du\right)$$

NEW

$$= x^2e^x - 2\left(xe^x - \int e^x\,dx\right)$$

$$= \underbrace{x^2e^x - 2\left(xe^x - e^x\right) + C}_{F(x)}$$

Then $\displaystyle\int_1^2 x^2e^x\,dx = F(2) - F(1)$

$$= 2e^2 - e \quad \boxed{C}$$

19. A manufacturer determines that when x hundred units of a partialr commodity are produced, the profit generated is P(x) thousand dollars, where

$$P(x) = \frac{500 \ln(x+1)}{(x+1)^2}$$

What is the average profit over the production range $0 \le x \le 10$?

a

b

$$\text{☆} = \frac{1}{b-a}\int_a^b P(x)\,dx$$

$$= \frac{1}{10}\int_0^{10} \frac{500\ln(x+1)}{(x+1)^2}\,dx$$

$$= 50\int_0^{10} \frac{\ln(x+1)}{(x+1)^2}\,dx$$

PAUSE

$$50\int \frac{\ln(x+1)}{(x+1)^2}\,dx$$

$$= 50\int \frac{\ln u}{u^2}\,du \quad \left\{ \begin{array}{l} u = x+1 \\ \frac{du}{dx} = 1 \\ \text{or } du = dx \end{array} \right\}$$

$$\underbrace{\ln u}_{v} \cdot \underbrace{\frac{1}{u^2}}_{dw}$$

$$\left\{ \begin{array}{l} v = \ln u, \quad dv = 1/u \\ dw = 1/u^2, \quad w = -1/u \end{array} \right\}$$

$$= 50\left(vw - \int w \, dv \right)$$

$$= 50\left(-\frac{\ln u}{u} - \int -\frac{1}{u} \cdot \frac{1}{u} \, du \right)$$

$$= 50\left(-\frac{\ln u}{u} + \int \frac{1}{u^2}\,du\right)$$

$$= 50\left(-\frac{\ln u}{u} - \frac{1}{u}\right) + C$$

$$= 50\left(-\frac{\ln(x+1)}{x+1} - \frac{1}{x+1}\right) + C$$

$$= \underbrace{-\frac{50}{x+1}\left(\ln(x+1)+1\right)}_{\hat{P}(x)} + \boxed{C}$$

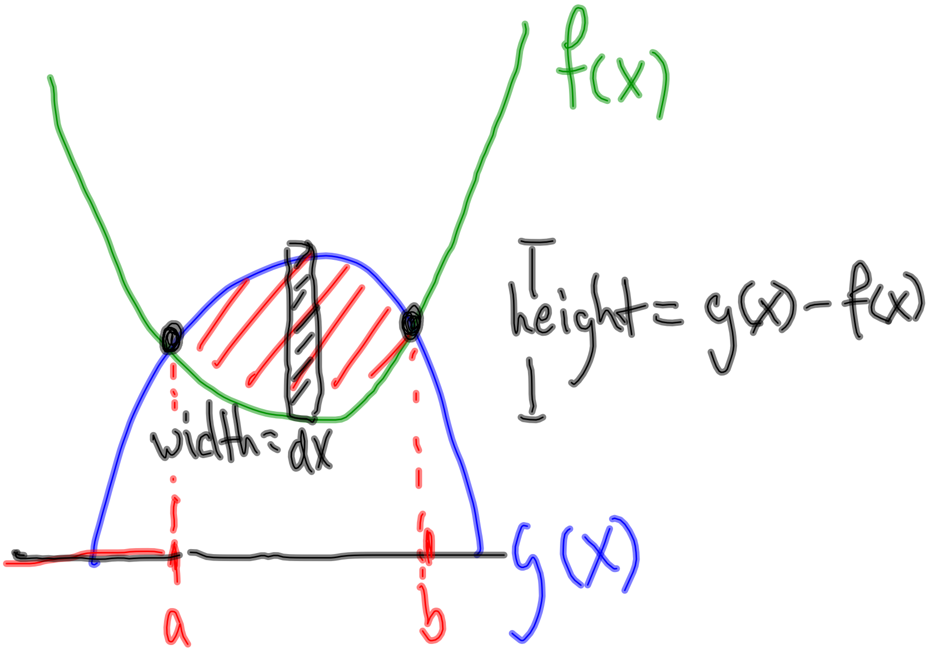
Ans: $\hat{P}(10) - \hat{P}(0) \approx 34\,555.02$

(after mult by 1000)

15. Find the area of the region bounded by $f(x) = x^2 + 9x - 28$

$g(x) = -x^2 + 5x + 2$

Soln Expect a picture like



- $g(x) \geq f(x)$
- Points of Intersection

$$f(x) = g(x)$$

$$\Leftrightarrow x^2 + 9x - 28 = -x^2 + 5x + 2$$

$$\Leftrightarrow 0 = x^2 + x^2 + 9x - 5x - 28 - 2$$

$$\Leftrightarrow \frac{0}{2} = \frac{2x^2}{2} + \frac{4x}{2} - \frac{30}{2}$$

$$\Leftrightarrow 0 = x^2 + 2x - 15$$

$$\Leftrightarrow 0 = (x-3)(x+5)$$

$$\Leftrightarrow x = \underbrace{-5}_{a} \text{ or } \underbrace{3}_{b}$$

Evaluate:

Area between $f(x)$ and $g(x)$

$$= \int_{-5}^{3} \text{height} \cdot \text{width}$$

$$= \int_{-5}^{3} [g(x) - f(x)]\, dx$$

$$= \int_{-5}^{3} [-2x^2 - 4x + 30]\, dx$$

$$= -\frac{2x^3}{3} - 2x^2 + 30x \Big|_{-5}^{3}$$

$$= \left[\frac{-2(3)^3}{3} - 2(3)^2 + 30(3)\right] - \left[\frac{-2(-5)^3}{3} - 2(-5)^2 + 30(-5)\right]$$

$$\approx 170.67 \quad \boxed{E}$$

Ex (Not from sample) Determine if the following integrals converge or diverge.

(a) $$\int_4^\infty \frac{2}{x}\,dx = \lim_{t\to\infty}\int_4^t \frac{2}{x}\,dx$$

$$= 2\lim_{t\to\infty}\int_4^t \frac{1}{x}\,dx$$

$$= 2\lim_{t\to\infty}\left[\ln|x|\,\Big|_4^t\right]$$

$$= 2 \lim_{t \to \infty} \left[\overset{\infty \text{ (slowly)}}{\cancel{\ln|t|}} - \ln|4| \right]$$

$$= \infty \implies \text{DIV}$$

(b) $$\int_{9}^{\infty} \frac{5}{x^{3/2}} \, dx$$

$$= 5 \lim_{t \to \infty} \int_{9}^{t} x^{-3/2} \, dx$$

$$= 5 \lim_{t \to \infty} \left[\frac{1}{1+(-3/2)} x^{-3/2+1} \Big|_{9}^{t} \right]$$

$$= 5 \lim_{t\to\infty} \left[-2x^{-1/2} \Big|_9^t \right]$$

$$= -10 \lim_{t\to\infty} \left[\frac{1}{x^{1/2}} \Big|_9^t \right]$$

$$= -10 \lim_{t\to\infty} \left[\cancel{\frac{1}{t^{1/2}}}^{\,0} - \frac{1}{3} \right]$$

$$= -10\left[0 - \frac{1}{3}\right]$$

$$= \frac{10}{3}$$

$\Rightarrow \lim = a$ #

$\Rightarrow$ CONV

Further,

$$\int_{-\infty}^{a} f(x)\,dx = \lim_{t \to -\infty} \int_{t}^{a} f(x)\,dx$$

CONV if limit exists

DIV if limit DNE

or

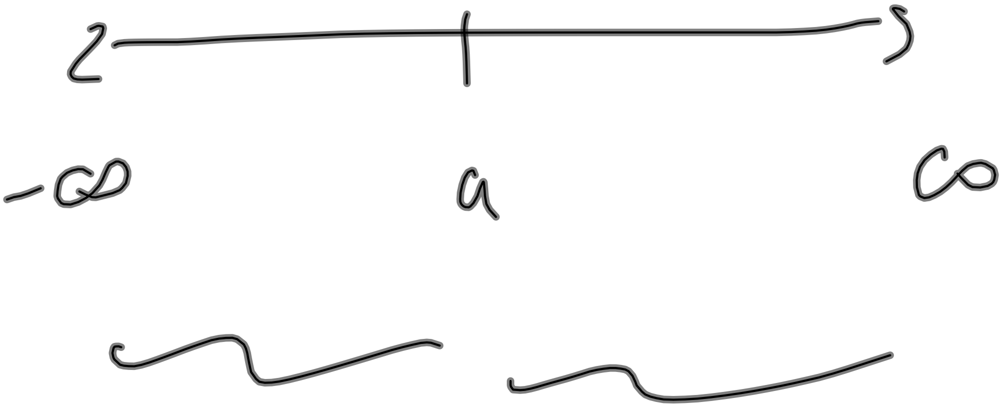
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$$

$$\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$$

$$= \underbrace{\lim_{t\to-\infty}\int_{t}^{a} f(x)\,dx}_{A} + \underbrace{\lim_{s\to\infty}\int_{a}^{s} f(x)\,dx}_{B}$$

If $\int_{-\infty}^{\infty} f(x)\,dx$ CONV,

then A and B CONV


$-\infty$
a
∞