ACM/IDS 104 APPLIED LINEAR ALGEBRA
PRACTICE PROBLEMS FOR LECTURE 2: SOLUTIONS

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Problem 2A. Solving Linear Systems using the Permuted LU Decomposition
Consider the following system of linear equations $Ax = b$, where

$$A = \begin{bmatrix} 1 & 0 & 4 & 2025 \\ 1 & 2 & 5 & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 5 \\ 6 \\ 5 \end{bmatrix}. \tag{1}$$

(a) Find a permuted LU decomposition of matrix A, that is $PA = LU$, where P is a permutation matrix, L is special lower triangular, and U is in the row echelon form.
(b) What is the rank of matrix A?
(c) Does the system have no solution, unique solution, or infinitely many solutions?
(d) Find the general solution of $Ax = b$ using the permuted LU decomposition of A.

Solution:
(a) Matrix A is the same as in Problem 1A with a new fourth column. As in solving part (b) of Problem 1A, let's first swap rows 2 and 3, and then perform the elementary row operations of type I to reduce A to its row echelon form U:

$$\begin{aligned} A = \begin{bmatrix} 1 & 0 & 4 & 2025 \\ 1 & 2 & 5 & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix} &\xrightarrow{a^2 \leftrightarrow a^3} \begin{bmatrix} 1 & 0 & 4 & 2025 \\ 1 & 1 & 4 & 0 \\ 1 & 2 & 5 & 0 \end{bmatrix} \xrightarrow{a^2 \to a^2 - a^1} \begin{bmatrix} 1 & 0 & 4 & 2025 \\ 0 & 1 & 0 & -2025 \\ 1 & 2 & 5 & 0 \end{bmatrix} \xrightarrow{a^3 \to a^3 - a^1} \\ \begin{bmatrix} 1 & 0 & 4 & 2025 \\ 0 & 1 & 0 & -2025 \\ 0 & 2 & 1 & -2025 \end{bmatrix} &\xrightarrow{a^3 \to a^3 - 2a^2} \begin{bmatrix} 1 & 0 & 4 & 2025 \\ 0 & 1 & 0 & -2025 \\ 0 & 0 & 1 & 2025 \end{bmatrix} = U \end{aligned} \tag{2}$$

Equivalently, in matrix form (using how row operations are realized by matrix multiplications):

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{E_{32}^{(-2)}, a^3 \to a^3 - 2a^2} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{E_{31}^{(-1)}, a^3 \to a^3 - a^1} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}^{(-1)}, a^2 \to a^2 - a^1} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P, a^2 \leftrightarrow a^3} \underbrace{\begin{bmatrix} 1 & 0 & 4 & 2025 \\ 1 & 2 & 5 & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 4 & 2025 \\ 0 & 1 & 0 & -2025 \\ 0 & 0 & 1 & 2025 \end{bmatrix}}_{U} \tag{3}$$

The lower triangular matrix L is

$$\begin{aligned} L = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1} &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}^{-1} \\ = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \end{aligned} \tag{4}$$

So, the permuted LU decomposition of A looks as follows:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & 0 & 4 & 2025 \\ 1 & 2 & 5 & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 & 4 & 2025 \\ 0 & 1 & 0 & -2025 \\ 0 & 0 & 1 & 2025 \end{bmatrix}}_{U} \tag{5}$$

(b) The number of pivots in the row echelon form U is 3. Therefore, the rank of A is 3.
(c) Since $r = \text{rank}(A) = 3$, the number of unknowns $n = 4$, and $r < n$, the system has infinitely many solutions (for any right-hand side vector b). The number of free variables is $n - r = 1$.

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(d) Following the algorithm described in Lecture 2, let's multiply both sides of $Ax = b$ by P:

$$PAx = Pb =: \tilde{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 6 \end{bmatrix} \tag{6}$$

Since $PA = LU$, we have

$$L \underbrace{Ux}_{y} = \tilde{b} \tag{7}$$

Since L is special lower triangular, we can solve $Ly = \tilde{b}$ for y using forward substitution:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 6 \end{bmatrix} \quad \Leftrightarrow \quad y = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} \tag{8}$$

The last step is to solve $Ux = y$ for x:

$$\begin{bmatrix} 1 & 0 & 4 & 2025 \\ 0 & 1 & 0 & -2025 \\ 0 & 0 & 1 & 2025 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} \tag{9}$$

Circled entries are pivots, and the corresponding variables x_1, x_2, and x_3 are *basic variables*. The last variable x_4 is *free*, meaning that it can take any value. We can express all basic variables in terms of the free variable using backward substitution:

$$\begin{aligned} x_3 &= 1 - 2025x_4, \\ x_2 &= 2025x_4, \\ x_1 &= 5 - 4x_3 - 2025x_4 = 5 - 4(1 - 2025x_4) - 2025x_4 = 1 + 6075x_4. \end{aligned} \tag{10}$$

So, the general solution of $Ax = b$ is given by (10), where x_4 is a free parameter.

Problem 2B. FIELD OF A VECTOR SPACE

In Lecture 2, we defined a vector space V over a field of real numbers. In other words, we assumed that the set of scalars $\mathbb{F}$ is the set of real numbers $\mathbb{R}$. However, more generally, $\mathbb{F}$ can be not only $\mathbb{R}$, but also the set of complex numbers $\mathbb{C}$ or any *field*, which is a set of objects in which we can add, subtract, multiply, and divide according to the usual laws of arithmetic. Interestingly, and somewhat counterintuitively, whether or not a set V is a vector space depends on the field $\mathbb{F}$ being considered. That is, V can be a vector space over one field but not over another. This problem illustrates this.

A complex square matrix $A \in \mathbb{M}_{n\times n}$ is called *Hermitian* if it equals its conjugate transpose, $A^H = A$, where $A^H := \bar{A}^T$. In other words, a square matrix $A = (a_{ij})$, where $a_{ij} \in \mathbb{C}$, is Hermitian if $a_{ij} = \bar{a}_{ji}$, where the overline denotes complex conjugation. Let $\mathcal{H}_n$ be the set of all $n \times n$ complex Hermitian matrices with the standard matrix addition and multiplication by scalars $\alpha \in \mathbb{F}$.

(a) Prove that if $\mathbb{F} = \mathbb{R}$, then $\mathcal{H}_n$ is a vector space.
(b) Prove that if $\mathbb{F} = \mathbb{C}$, then $\mathcal{H}_n$ is not a vector space.

Solution:

(a) First let's check that the addition and multiplication by scalars are well-defined in the sense that if $A, B \in \mathcal{H}_n$ then $(A + B) \in \mathcal{H}_n$ and if $A \in \mathcal{H}_n$ and $\alpha \in \mathbb{F} = \mathbb{R}$, then $\alpha A \in \mathcal{H}_n$. Indeed:

$$\begin{aligned} (A+B)^H &= \overline{A+B}^T = \bar{A}^T + \bar{B}^T = A + B \quad \Rightarrow \quad (A+B) \in \mathcal{H}_n \\ (\alpha A)^H &= \overline{\alpha A}^T = \alpha \bar{A}^T = \alpha A \quad \Rightarrow \quad \alpha A \in \mathcal{H}_n \end{aligned} \tag{11}$$

Now we need to check that the properties of the vector space operations do hold. Let $A, B, C \in \mathcal{H}_n$ and $\alpha, \beta \in \mathbb{F} = \mathbb{R}$. Then the following properties follow immediately from the definitions of addition and scalar multiplication:

$$\begin{aligned} &1.\ A+B=B+A \\ &2.\ (A+B)+C=A+(B+C) \\ &3.\ A+0=A \quad \text{(the zero vector is the zero matrix)} \\ &4.\ A+(-A)=0 \\ &5.\ (\alpha+\beta)A=\alpha A+\beta A \\ &6.\ \alpha(A+B)=\alpha A+\alpha B \\ &7.\ (\alpha\beta)A=\alpha(\beta A) \\ &8.\ 1A=A \end{aligned} \tag{12}$$

Hence, $\mathcal{H}_n$ is indeed a vector space over $\mathbb{R}$.

(b) Now, if the field $\mathbb{F}=\mathbb{C}$, then the second property in (11) does not hold any more. Indeed, if $A \in \mathcal{H}_n$ and $\alpha \in \mathbb{F}=\mathbb{C}$, then

$$(\alpha A)^H = \overline{\alpha A}^T = \bar{\alpha}\bar{A}^T = \bar{\alpha}A \neq \alpha A, \quad \text{for } \alpha \in \mathbb{C}\setminus\mathbb{R}. \tag{13}$$

For example,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathcal{H}_2, \quad \text{but} \quad iA = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \notin \mathcal{H}_2, \quad \text{since } (iA)^H = -iA \neq iA. \tag{14}$$

Hence, $\mathcal{H}_n$ is not a vector space over $\mathbb{C}$.