ACM/IDS 104 APPLIED LINEAR ALGEBRA
PRACTICE PROBLEMS FOR LECTURE 4

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Each lecture is accompanied by two practice problems: a somewhat easier, more practical *Problem A*, and a more difficult, more conceptual *Problem B*. The main goal of the practice problems is threefold: to help you better understand the material covered in the corresponding lecture, to help you prepare to solve problems in problem sets and exams, and to accommodate the diversity of students' math backgrounds by providing both easier and more challenging problems. These problems are for self-practice: they will not be graded, and the solutions—posted on Piazza—also illustrate the expected level of rigor for problem sets and exams.

Problem 4A. Fundamental Subspaces
Consider the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 \end{bmatrix} \tag{1}$$

(a) Find the four fundamental subspaces, $\ker A$, $\operatorname{im} A$, $\operatorname{coker} A$, and $\operatorname{coim} A$, using their definitions. Here, to "find" means to describe the subspaces explicitly by constructing their bases.
(b) Check that the dimensions of the four fundamental subspaces satisfy the relationships predicted by the Fundamental Theorem of Linear Algebra.

Problem 4B. Frobenius Inner Product and Frobenius Norm
The vector space $\mathbb{M}_{m\times n}$ of all $m \times n$ matrices admits many inner products. One of the most useful ones, which appears in theorems and applications, is the Frobenius inner product[1].

(a) Prove that the function $\langle\cdot,\cdot\rangle_F : \mathbb{M}_{m\times n} \times \mathbb{M}_{m\times n} \to \mathbb{R}$ defined by

$$\langle A, B\rangle_F = \operatorname{tr}(A^T B), \quad \text{for all } A, B \in \mathbb{M}_{m\times n} \tag{2}$$

is an inner product (called the Frobenius inner product) on the vector space $\mathbb{M}_{m\times n}$.
(b) Let $\|\cdot\|_F$ be the Frobenius norm induced by the Frobenius inner product. Prove that this norm is invariant under transposition, that is $\|A^T\|_F = \|A\|_F$.

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[1]Also known as the Hilbert–Schmidt inner product.