ACM/IDS 104 APPLIED LINEAR ALGEBRA
PRACTICE PROBLEMS FOR LECTURE 3: SOLUTIONS

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Problem 3A. Linear Independence, Span, Basis
Consider the following four two-by-two matrices:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{1}$$

(a) Determine whether A_1, A_2, A_3, and A_4 are linearly dependent or not.
(b) Determine whether A_1, A_2, A_3, and A_4 span the vector space of all two-by-two matrices $\mathbb{M}_{2\times 2}$.
(c) Determine whether A_1, A_2, A_3, and A_4 form a basis of $\mathbb{M}_{2\times 2}$.

Solution:

(a) To determine whether the matrices are linearly dependent, we need to find scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that

$$\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4 = 0. \tag{2}$$

If this equation holds only if all $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, then they are linearly independent; otherwise, they are linearly dependent. Computing the linear combination, we have:

$$\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4 = \begin{bmatrix} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 & \alpha_2 + \alpha_4 \\ \alpha_3 + \alpha_4 & \alpha_1 + \alpha_2 + \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{3}$$

Looking at entries (1,1) and (2,2), we conclude that $\alpha_4 = 0$. This implies that $\alpha_2 = 0$ (entry (1,2)) and $\alpha_3 = 0$ (entry (2,1)). This means that $\alpha_1 = 0$ as well. Hence, A_1, A_2, A_3, and A_4 are linearly independent.

(b) To solve this problem, we can use the theorem proved in Lecture 3: if V is an n-dimensional vector space and vectors $v_1, \dots, v_n$ are linearly independent, then they span V and form a basis of V. In our case, $V = \mathbb{M}_{2\times 2}$ is 4-dimensional and A_1, A_2, A_3, A_4 are linearly independent. Therefore, they span $\mathbb{M}_{2\times 2}$ and form its basis.

Alternatively, we can solve this problem directly using the definition of span. To determine whether the matrices span $\mathbb{M}_{2\times 2}$, we need to check whether any two-by-two matrix can be represented as a linear combination of A_1, A_2, A_3, and A_4. In other words, if for any matrix $B \in \mathbb{M}_{2\times 2}$, we can find scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that

$$\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4 = B. \tag{4}$$

Let

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \tag{5}$$

be an arbitrary two-by-two matrix. Then (4) reduces to

$$\begin{bmatrix} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 & \alpha_2 + \alpha_4 \\ \alpha_3 + \alpha_4 & \alpha_1 + \alpha_2 + \alpha_3 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \tag{6}$$

This system of four equations is straightforward to solve:

$$\begin{aligned} \alpha_4 &= b_{11} - b_{22}, \\ \alpha_2 &= b_{12} - b_{11} + b_{22}, \\ \alpha_3 &= b_{21} - b_{11} + b_{22}, \\ \alpha_1 &= b_{11} - (b_{12} - b_{11} + b_{22}) - (b_{21} - b_{11} + b_{22}) - (b_{11} - b_{22}) = 2b_{11} - b_{12} - b_{21} - b_{22} \end{aligned} \tag{7}$$

A solution exists and, therefore, A_1, A_2, A_3, A_4 span $\mathbb{M}_{2\times 2}$.

(c) Since A_1, A_2, A_3, A_4 are linearly independent and span $\mathbb{M}_{2\times 2}$, they form a basis of $\mathbb{M}_{2\times 2}$.

Problem 3B. Linear Independence in Functional Spaces
Proving linear dependence or independence in functional vector spaces can be nontrivial. In this problem, we will derive one method that can help us to establish linear independence if the functions are sufficiently smooth.

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Let $f_1, \dots, f_n : I \to \mathbb{R}$ be a collection of n functions defined on an interval $I \subseteq \mathbb{R}$ that are differentiable at least $n-1$ times on I. Consider the following $n \times n$ matrix consisting of the derivatives of these functions:

$$W(x) = \begin{bmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ f_1''(x) & f_2''(x) & \dots & f_n''(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{bmatrix} \tag{8}$$

The determinant of this matrix, $\det W(x)$, is called the *Wronskian* of $f_1, \dots, f_n$.

(a) Prove that if $f_1, \dots, f_n$ are linearly dependent, then $\det W(x) = 0$ for all $x \in I$.
(b) Use the result of part (a) to show that functions $x, \log(x), \sin(x)$ are linearly independent.
(c) Can the result of part (a) be used to prove linear dependence of functions?

Solution:

(a) Since $f_1, \dots, f_n$ are linearly dependent, there exist constants $\alpha_1, \dots, \alpha_n$ (not all zero), such that

$$\alpha_1 f_1(x) + \ldots + \alpha_n f_n(x) = 0 \quad \text{for all } x \in I. \tag{9}$$

Differentiating this identity k times for $k = 1, \dots, n-1$, we have:

$$\alpha_1 f_1^{(k)}(x) + \ldots + \alpha_n f_n^{(k)}(x) = 0 \quad \text{for all } x \in I. \tag{10}$$

In matrix form,

$$\begin{bmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ f_1''(x) & f_2''(x) & \dots & f_n''(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad W(x)\alpha = 0. \tag{11}$$

Since this homogeneous linear system has a non-zero solution, the matrix $W(x)$ is singular for any $x \in I$. Therefore, $\det W(x) = 0$ for all $x \in I$.

(b) Here $I = (0, \infty)$. To prove linear independence of $x, \log(x), \sin(x)$, it is sufficient to show that $\det W(x_0) \neq 0$ for some $x_0 \in I$. For these three functions, the matrix $W(x)$ has the following form:

$$W(x) = \begin{bmatrix} x & \log(x) & \sin(x) \\ 1 & \frac{1}{x} & \cos(x) \\ 0 & -\frac{1}{x^2} & -\sin(x) \end{bmatrix} \tag{12}$$

For computational convenience, let's take $x_0 = \pi$. Then

$$W(\pi) = \begin{bmatrix} \pi & \log(\pi) & 0 \\ 1 & \frac{1}{\pi} & -1 \\ 0 & -\frac{1}{\pi^2} & 0 \end{bmatrix} \quad \Rightarrow \quad \det W(\pi) = -\frac{1}{\pi} \neq 0. \tag{13}$$

Since the Wronskian of $x, \log(x), \sin(x)$ is non-zero, the functions are linearly independent.

(c) No, we cannot use the property established in part (a) to prove linear dependence, we can use it only to prove linear independence. Namely, if $\det W(x_0) \neq 0$ at least for some $x_0 \in I$, then the functions must be linearly independent. But if $\det W(x) = 0$ for all $x \in I$, then the functions may still be linearly independent (i.e. they are not necessarily linearly dependent).

To see a concrete example, let $f_1(x) = x^2$ and $f_2(x) = x|x|$. Both functions are differentiable on $I = \mathbb{R}$.

$$\begin{aligned} &\text{If } x > 0 \quad \Rightarrow \quad W(x) = \begin{bmatrix} x^2 & x^2 \\ 2x & 2x \end{bmatrix} \quad \Rightarrow \quad \det W(x) = 0. \\ &\text{If } x < 0 \quad \Rightarrow \quad W(x) = \begin{bmatrix} x^2 & -x^2 \\ 2x & -2x \end{bmatrix} \quad \Rightarrow \quad \det W(x) = 0. \\ &\text{If } x = 0 \quad \Rightarrow \quad W(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \det W(x) = 0. \end{aligned} \tag{14}$$

So, for any $x \in I$ the Wronskian $\det W(x) = 0$.

And yet, the functions $f_1(x) = x^2$ and $f_2(x) = x|x|$ are linearly independent on $I = \mathbb{R}$ since for

$$\alpha_1 x^2 + \alpha_2 x|x| = 0 \tag{15}$$

to hold for all $x \in \mathbb{R}$, we must have $\alpha_1 = \alpha_2 = 0$. Indeed, for the identity to hold for positive x, we must have $\alpha_1 = -\alpha_2$. On the other hand, for the identity to hold for negative x, we must have $\alpha_1 = \alpha_2$. The only way to satisfy these requirements is to have $\alpha_1 = \alpha_2 = 0$.