ACM/IDS 104 APPLIED LINEAR ALGEBRA
PRACTICE PROBLEMS FOR LECTURE 1: SOLUTIONS

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Problem 1A. Gaussian Elimination
Let A be the following nonsingular matrix:

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix} \tag{1}$$

(a) Reduce A to an upper triangular matrix U with non-zero diagonal elements using elementary row operations.
(b) Find a permutation matrix P and a special lower triangular matrix L such that $PA = LU$.

Solution:
(a) Here we go:

$$\begin{aligned} A = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix} &\xrightarrow{a^2 \to a^2 - a^1} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 1 \\ 1 & 1 & 4 \end{bmatrix} \xrightarrow{a^3 \to a^3 - a^1} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{a^2 \leftrightarrow a^3} \\ &\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{a^3 \to a^3 - 2a^2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U \end{aligned} \tag{2}$$

(b) We know that it is possible to reduce A to U by first performing type II operations (swapping of rows) and then type I operation ($a^i \to a^i + \alpha a^j$). In part (a) we saw that we needed to swap rows 2 and 3 in the middle of the process. So, now let's start with this swap:

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix} \xrightarrow{a^2 \leftrightarrow a^3} \underbrace{\begin{bmatrix} 1 & 0 & 4 \\ 1 & 1 & 4 \\ 1 & 2 & 5 \end{bmatrix}}_{A_1} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P_{23}} A \tag{3}$$

Here, P_{23} is the permutation matrix that realizes the permutation of rows 2 and 3. Now we can apply to matrix A_1 the elementary row operations of type I:

$$\begin{aligned} A_1 = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 1 & 4 \\ 1 & 2 & 5 \end{bmatrix} &\xrightarrow{a^2 \to a^2 - a^1} \underbrace{\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 1 & 2 & 5 \end{bmatrix}}_{A_2} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}^{(-1)}} A_1 \\ A_2 = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 1 & 2 & 5 \end{bmatrix} &\xrightarrow{a^3 \to a^3 - a^1} \underbrace{\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}}_{A_3} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{E_{31}^{(-1)}} A_2 \\ A_3 = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} &\xrightarrow{a^3 \to a^3 - 2a^2} \underbrace{\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{E_{32}^{(-2)}} A_3 \end{aligned} \tag{4}$$

So, we have:

$$U = E_{32}^{(-2)} A_3 = E_{32}^{(-2)} E_{31}^{(-1)} A_2 = E_{32}^{(-2)} E_{31}^{(-1)} E_{21}^{(-1)} A_1 = \underbrace{E_{32}^{(-2)} E_{31}^{(-1)} E_{21}^{(-1)}}_{E} \underbrace{P_{23}}_{P} A. \tag{5}$$

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To find the permuted LU decomposition, $PA = LU$, we need to invert the special lower triangular matrix E:

$$\begin{aligned} L = E^{-1} &= (E_{32}^{(-2)}E_{31}^{(-1)}E_{21}^{(-1)})^{-1} = (E_{21}^{(-1)})^{-1}(E_{31}^{(-1)})^{-1}(E_{32}^{(-2)})^{-1} = E_{21}^{(1)}E_{31}^{(1)}E_{32}^{(2)} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \end{aligned} \tag{6}$$

So, the permuted LU decomposition of A looks as follows:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \tag{7}$$

Problem 1B. Uniqueness of the Permuted LU Decomposition for Fixed P
Let A be a nonsingular matrix. In general, the permuted LU decomposition $PA = LU$, where P is a permutation matrix, L is special lower triangular, and U is upper triangular with non-zero diagonal elements, is not unique[1]. However, if P is fixed, then the decomposition is unique. Namely, prove that if

$$PA = LU \quad \text{and} \quad PA = \tilde{L}\tilde{U}, \tag{8}$$

where $L, \tilde{L}$ are special lower triangular and $U, \tilde{U}$ are upper triangular with non-zero diagonal elements, then $L = \tilde{L}$ and $U = \tilde{U}$.

Solution: Since $PA = LU$ and $PA = \tilde{L}\tilde{U}$, we have that

$$LU = \tilde{L}\tilde{U}. \tag{9}$$

All four matrices $L, U, \tilde{L}, \tilde{U}$ are nonsingular. Therefore,

$$\tilde{L}^{-1}L = \tilde{U}U^{-1} =: D \tag{10}$$

Let's prove that if $\tilde{L}$ is special lower triangular, then so is $\tilde{L}^{-1}$. Indeed, since $\tilde{L}\tilde{L}^{-1} = I$, to find the j^{th} column of $\tilde{L}^{-1}$, denoted $(\tilde{L}^{-1})_j$, we need to solve $\tilde{L}(\tilde{L}^{-1})_j = e_j$, where e_j is the j^{th} column of the identity matrix. We can easily find $(\tilde{L}^{-1})_j$ using forward substitution. Because $\tilde{L}$ is special lower triangular and the first $(j-1)$ components of e_j are zeros, the first $(j-1)$ components of $(\tilde{L}^{-1})_j$ are also zeros. Because $\tilde{L}$ is special lower triangular and the j^{th} component of e_j is one, the j^{th} component of $(\tilde{L}^{-1})_j$ is also one. The remaining components of $(\tilde{L}^{-1})_j$ are uniquely determined from $\tilde{L}(\tilde{L}^{-1})_j = e_j$. Thus, each column of $\tilde{L}^{-1}$ has zeros above the diagonal and a 1 on the diagonal. Hence, $\tilde{L}^{-1}$ is special lower triangular.

Now let's prove that the product $M = AB$ of two special lower triangular matrices A and B is also special lower triangular. The (i, j) entry of M is

$$M_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}. \tag{11}$$

Suppose $i < j$. Since A and B are lower triangular, $A_{ik} = 0$ for $k > i$ and $B_{kj} = 0$ for $k < j$. In this case, $A_{ik}B_{kj} = 0$ for all $k = 1, \ldots, n$, and, therefore, $M_{ij} = 0$ and M is lower triangular.

Suppose $i = j$. Then

$$M_{ii} = \sum_{k=1}^{n} A_{ik}B_{ki} = \underbrace{A_{ii}B_{ii}}_{1} + \sum_{k \neq i} \underbrace{A_{ik}B_{ki}}_{0} = 1. \tag{12}$$

So, all diagonal elements of M are ones, and, therefore, M is special lower triangular.

This means that the matrix $D = \tilde{L}^{-1}L$ is special lower triangular.

Similarly, we can show that if U is upper triangular with non-zero diagonal elements, then so is U^{-1}. And the product of two upper triangular matrices with non-zero diagonal elements is also upper triangular with non-zero diagonal elements. Therefore, $D = \tilde{U}U^{-1}$ is upper triangular with non-zero diagonal elements.

This leaves only one option for D: if it is both special lower triangular and upper triangular, then it must be the identity matrix. Hence, $L = \tilde{L}$ and $U = \tilde{U}$.

[1]It can be shown that there exist other $\tilde{P}, \tilde{L}, \tilde{U}$ such that $\tilde{P}A = \tilde{L}\tilde{U}$ precisely when $\tilde{P}A$ has all leading principal minors (determinants of the top-left $k \times k$ submatrices, $k = 1, \ldots, n$) non-zero.