# ACM/IDS 104 APPLIED LINEAR ALGEBRA
# PRACTICE PROBLEMS FOR LECTURE 4: SOLUTIONS

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**Problem 4A.** Fundamental Subspaces
Consider the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 \end{bmatrix} \tag{1}$$

(a) Find the four fundamental subspaces, $\ker A$, $\operatorname{im} A$, $\operatorname{coker} A$, and $\operatorname{coim} A$, using their definitions. Here, to "find" means to describe the subspaces explicitly by constructing their bases.
(b) Check that the dimensions of the four fundamental subspaces satisfy the relationships predicted by the Fundamental Theorem of Linear Algebra.

**Solution:**
(a) By definition, the kernel of $A$ is a subspace of $\mathbb{R}^4$ consisting of all vectors $x \in \mathbb{R}^4$ such that $Ax = 0$. The most systematic way to find $\ker A$ is first to reduce $A$ to its row echelon form $U$, and then find $\ker U$, which coincides with $\ker A$. Indeed, since $PA = LU$, we have that $A = NU$, where $N = P^{-1}L$ is a nonsingular matrix, and, therefore,

$$x \in \ker A \;\Leftrightarrow\; Ax = 0 \;\Leftrightarrow\; NUx = 0 \;\Leftrightarrow\; Ux = 0 \;\Leftrightarrow\; x \in \ker U. \tag{2}$$

Finding $U$ is straightforward:

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{a^3 \to a^3 - a^1} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{a^3 \to a^3 - a^2} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U \tag{3}$$

The circled entries are pivots. Therefore, $\operatorname{rank} A = 2$ (we will need this number for part (b) anyway), $x_1$ and $x_2$ are basic variables, and $x_3$ and $x_4$ are free variables. The general solution of $Ux = 0$ looks, therefore, as follows:

$$\begin{aligned} x_2 &= -x_3 - x_4, \\ x_1 &= -x_3 - 2x_4, \end{aligned} \tag{4}$$

where $x_3$ and $x_4$ are arbitrary. As we discussed in Lecture 4, the basis of $\ker U (= \ker A)$ can be constructed by setting the free variables to $x_3 = 1, x_4 = 0$ and then to $x_3 = 0, x_4 = 1$, and computing the corresponding values of the basic variables $x_1$ and $x_2$ from the above relationships. Therefore,

$$\ker A = \operatorname{span}\{v_1, v_2\}, \quad \text{where } v_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \tag{5}$$

By definition, the image of $A$ is the column space of $A$, that is, the span of all columns of $A$. The four columns $a_1, a_2, a_3, a_4$ of $A$ are linearly dependent, since any four vectors in $\mathbb{R}^3$ are linearly dependent. In fact, $a_3 = a_1 + a_2$ and $a_4 = 2a_1 + a_2$. But $a_1$ and $a_2$ are linearly independent. Therefore,

$$\operatorname{im} A = \operatorname{span}\{a_1, a_2\}, \quad \text{where } a_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } a_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \tag{6}$$

We can find $\operatorname{coker} A = \ker A^T$ by analogy with how we found $\ker A$. The row echelon form of $A^T$ is

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \xrightarrow{a^3 \to a^3 - a^1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \xrightarrow{a^4 \to a^4 - 2a^1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{a^3 \to a^3 - a^2,\ a^4 \to a^4 - a^2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = W \tag{7}$$

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The general solution of $Wx = 0$, where $x \in \mathbb{R}^3$, looks as follows:

$$\begin{aligned} x_2 &= -x_3, \\ x_1 &= -x_3, \end{aligned} \tag{8}$$

where $x_3$ is arbitrary. Therefore,

$$\operatorname{coker} A = \operatorname{span}\{w_1\}, \quad \text{where } w_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \tag{9}$$

By definition, the coimage of $A$ is the row space of $A$, that is, the span of all rows of $A$. Note that $a^1, a^2, a^3$ are linearly dependent, since $a^3 = a^1 + a^2$. But $a^1$ and $a^2$ are linearly independent. Therefore,

$$\operatorname{coim} A = \operatorname{span}\{a^1, a^2\}, \quad \text{where } a^1 = \begin{bmatrix} 1 & 0 & 1 & 2 \end{bmatrix} \text{ and } a^2 = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix} \tag{10}$$

(b) In this problem, $m = 3$, $n = 4$, $r = \operatorname{rank} A = 2$, $\dim\ker A = 2$, $\dim\operatorname{im} A = 2$, $\dim\operatorname{coker} A = 1$, and $\dim\operatorname{coim} A = 2$. The relationships between these numbers predicted by the Fundamental Theorem of Linear Algebra hold:

$$\begin{aligned} \dim\ker A &= n - r = 4 - 2 = 2, \\ \dim\operatorname{im} A &= r = 2, \\ \dim\operatorname{coim} A &= r = 2, \\ \dim\operatorname{coker} A &= m - r = 3 - 2 = 1. \end{aligned} \tag{11}$$

**Problem 4B.** Frobenius Inner Product and Frobenius Norm

The vector space $\mathbb{M}_{m\times n}$ of all $m \times n$ matrices admits many inner products. One of the most useful ones, which appears in theorems and applications, is the Frobenius inner product[1].

(a) Prove that the function $\langle\cdot,\cdot\rangle_F : \mathbb{M}_{m\times n} \times \mathbb{M}_{m\times n} \to \mathbb{R}$ defined by

$$\langle A, B\rangle_F = \operatorname{tr}(A^T B), \quad \text{for all } A, B \in \mathbb{M}_{m\times n} \tag{12}$$

is an inner product (called the Frobenius inner product) on the vector space $\mathbb{M}_{m\times n}$.

(b) Let $\|\cdot\|_F$ be the Frobenius norm induced by the Frobenius inner product. Prove that this norm is invariant under transposition, that is $\|A^T\|_F = \|A\|_F$.

**Solution:**

(a) Let's check that $\langle\cdot,\cdot\rangle_F$ satisfies all three properties of inner products. First, it is bilinear:

$$\begin{aligned} \langle \alpha_1 A_1 + \alpha_2 A_2, B\rangle_F &= \operatorname{tr}((\alpha_1 A_1 + \alpha_2 A_2)^T B) = \operatorname{tr}(\alpha_1 A_1^T B + \alpha_2 A_2^T B) = \alpha_1 \operatorname{tr}(A_1^T B) + \alpha_2 \operatorname{tr}(A_2^T B) \\ &= \alpha_1 \langle A_1, B\rangle_F + \alpha_2 \langle A_2, B\rangle_F, \\ \langle A, \beta_1 B_1 + \beta_2 B_2\rangle_F &= \operatorname{tr}(A^T(\beta_1 B_1 + \beta_2 B_2)) = \operatorname{tr}(\beta_1 A^T B_1 + \beta_2 A^T B_2) = \beta_1 \operatorname{tr}(A^T B_1) + \beta_2 \operatorname{tr}(A^T B_2) \\ &= \beta_1 \langle A, B_1\rangle_F + \beta_2 \langle A, B_2\rangle_F. \end{aligned} \tag{13}$$

To prove that $\langle\cdot,\cdot\rangle_F$ is symmetric, we first establish the invariance of the trace under transposition: if $C$ is an $n \times n$ square matrix, then $\operatorname{tr}(C^T) = \operatorname{tr}(C)$. This is obvious, but here is a formal justification:

$$\operatorname{tr}(C^T) = \sum_{i=1}^n (C^T)_{ii} = \sum_{i=1}^n (C)_{ii} = \operatorname{tr}(C). \tag{14}$$

Now, the symmetry of $\langle\cdot,\cdot\rangle_F$ follows directly:

$$\langle B, A\rangle_F = \operatorname{tr}(B^T A) = \operatorname{tr}\left((B^T A)^T\right) = \operatorname{tr}\left(A^T (B^T)^T\right) = \operatorname{tr}(A^T B) = \langle A, B\rangle_F. \tag{15}$$

Finally, positive-definiteness:

$$\langle A, A\rangle_F = \operatorname{tr}(A^T A) = \sum_{i=1}^n \left(A^T A\right)_{ii} = \sum_{i=1}^n \sum_{j=1}^m (A^T)_{ij} A_{ji} = \sum_{i=1}^n \sum_{j=1}^m A_{ji}^2 \geq 0 \tag{16}$$

Moreover, $\langle A, A\rangle_F = 0$ if and only if $A_{ji} = 0$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$, that is, if and only if $A = 0$.

It is illuminating to write the expression for the Frobenius inner product in terms of the entries of matrices:

$$\langle A, B\rangle_F = \operatorname{tr}(A^T B) = \sum_{i=1}^n \left(A^T B\right)_{ii} = \sum_{i=1}^n \sum_{j=1}^m (A^T)_{ij} B_{ji} = \sum_{i=1}^n \sum_{j=1}^m A_{ji} B_{ji}. \tag{17}$$

[1] Also known as the Hilbert–Schmidt inner product.

In other words, this inner product multiplies all of the entries of $A$ by the corresponding entries of $B$ and adds them up, just like the dot product on $\mathbb{R}^{m\times n}$. The Frobenius inner product is simply what we get if we forget about the shape of $A$ and $B$ and just take their dot product as if they were vectors in $\mathbb{R}^{m\times n}$.

(b) By definition, the Frobenius norm is the norm on $\mathbb{M}_{m\times n}$ induced by the Frobenius inner product:

$$\|A\|_F = \sqrt{\langle A, A\rangle_F} = \sqrt{\mathrm{tr}(A^T A)} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} A_{ji}^2}. \tag{18}$$

It is straightforward to see that it is invariant under transposition:

$$\|A^T\|_F = \sqrt{\sum_{s=1}^{m}\sum_{k=1}^{n} \left((A^T)_{ks}\right)^2} = \sqrt{\sum_{s=1}^{m}\sum_{k=1}^{n} A_{sk}^2} = \sqrt{\sum_{k=1}^{n}\sum_{s=1}^{m} A_{sk}^2} = \|A\|_F. \tag{19}$$