ACM/IDS 104 APPLIED LINEAR ALGEBRA
PRACTICE PROBLEMS FOR LECTURE 2

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Each lecture is accompanied by two practice problems: a somewhat easier, more practical *Problem A*, and a more difficult, more conceptual *Problem B*. The main goal of the practice problems is threefold: to help you better understand the material covered in the corresponding lecture, to help you prepare to solve problems in problem sets and exams, and to accommodate the diversity of students' math backgrounds by providing both easier and more challenging problems. These problems are for self-practice: they will not be graded, and the solutions—posted on Piazza—also illustrate the expected level of rigor for problem sets and exams.

Problem 2A. Solving Linear Systems using the Permuted LU Decomposition
Consider the following system of linear equations $Ax = b$, where

$$A = \begin{bmatrix} 1 & 0 & 4 & 2025 \\ 1 & 2 & 5 & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 5 \\ 6 \\ 5 \end{bmatrix}. \tag{1}$$

(a) Find a permuted LU decomposition of matrix A, that is $PA = LU$, where P is a permutation matrix, L is special lower triangular, and U is in the row echelon form.
(b) What is the rank of matrix A?
(c) Does the system have no solution, unique solution, or infinitely many solutions?
(d) Find the general solution of $Ax = b$ using the permuted LU decomposition of A.

Problem 2B. Field of a Vector Space
In Lecture 2, we defined a vector space V over a field of real numbers. In other words, we assumed that the set of scalars $\mathbb{F}$ is the set of real numbers $\mathbb{R}$. However, more generally, $\mathbb{F}$ can be not only $\mathbb{R}$, but also the set of complex numbers $\mathbb{C}$ or any *field*, which is a set of objects in which we can add, subtract, multiply, and divide according to the usual laws of arithmetic. Interestingly, and somewhat counterintuitively, whether or not a set V is a vector space depends on the field $\mathbb{F}$ being considered. That is, V can be a vector space over one field but not over another. This problem illustrates this.

A complex square matrix $A \in \mathbb{M}_{n\times n}$ is called *Hermitian* if it equals its conjugate transpose, $A^H = A$, where $A^H := \bar{A}^T$. In other words, a square matrix $A = (a_{ij})$, where $a_{ij} \in \mathbb{C}$, is Hermitian if $a_{ij} = \bar{a}_{ji}$, where the overline denotes complex conjugation. Let $\mathcal{H}_n$ be the set of all $n \times n$ complex Hermitian matrices with the standard matrix addition and multiplication by scalars $\alpha \in \mathbb{F}$.

(a) Prove that if $\mathbb{F} = \mathbb{R}$, then $\mathcal{H}_n$ is a vector space.
(b) Prove that if $\mathbb{F} = \mathbb{C}$, then $\mathcal{H}_n$ is not a vector space.

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