ACM/IDS 104 APPLIED LINEAR ALGEBRA
PROBLEM SET 2

Please submit your solution as a single PDF file, that contains both the written-up and published code parts, via Gradescope by **9pm Tuesday, October 21**. An example of the submission process is shown here: `https://www.gradescope.com/get_started#student-submission`

- For theoretical problems, please use a pen, not a pencil: it is hard to read scanned submission written by a pencil.
- For coding problems, please convert your MATLAB livescripts (`.mlx`) to PDF by selecting `Live Editor` $\to$ `Save` $\to$ `Export to PDF` and merge them with the rest of your solution.
- After uploading your submission to Gradescope, please label all pages.

Problem 1. (10 POINTS) SUBSPACES
Which of the following subsets $W \subset V$ are *subspaces* of the vector spaces V? Explain why.

(a) (1 point) $V = \mathbb{M}_{n\times n}$ and W is a subset of all singular matrices, $W = \{A \mid \det A = 0\}$.

(b) (1 point) The *trace* of a square matrix $A \in V = \mathbb{M}_{n\times n}$ is the sum of its diagonal entries:

$$\mathrm{tr}A = \sum_{i=1}^{n} a_{ii}. \tag{1}$$

Let W be a subset of all trace zero matrices, $W = \{A \mid \mathrm{tr}A = 0\}$.

(c) (1 point) Let $V = C([0,1])$ be the vector space of all continuous functions on the interval $[0,1]$, and $W = \{f(x) \in V \mid f(0)f(1) = 1\}$.

(d) (1 points) The vector space V is again $C([0,1])$. The subset $W \subset V$ consists of all functions satisfying the following condition:

$$f\left(\frac{1}{2}\right) = \int_0^1 f(t)dt. \tag{2}$$

(e) (1 point) A *planar vector field* is a function which assigns a vector

$$v(x,y) = \begin{bmatrix} v_1(x,y) \\ v_2(x,y) \end{bmatrix} \tag{3}$$

to each point $(x,y) \in \mathbb{R}^2$. Let V be the vector space of all planar vector fields. A planar vector field is called *incompressible* if it has zero divergence:

$$\nabla \cdot v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0. \tag{4}$$

Let $W \subset V$ be the subset of all incompressible vector fields.

(f) (2 points) Let V_1 and V_2 be subspaces of a vector space V. Is the union $V_1 \cup V_2$ necessarily a subspace of V?

(g) (3 points) Let V_1 and V_2 be subspaces of a vector space V. Is the intersection $V_1 \cap V_2$ necessarily a subspace of V?

Problem 2. (10 POINTS) POLYNOMIALS AND EXPONENTIALS
Let $\mathcal{P}^{(2)}$ be the vector space of quadratic polynomials. Let

$$p_1(x) = x^2 - 3, \quad p_2(x) = 2 - x, \quad p_3(x) = (x+1)^2. \tag{5}$$

(a) (2 points) Determine whether p_1, p_2 and p_3 are linearly independent.

(b) (2 points) Do they span $\mathcal{P}^{(2)}$?

(c) (3 points) Do they form a basis of $\mathcal{P}^{(2)}$? If yes, find the coordinates of $q(x) = 1$ in that basis.

Now consider the following functions:

$$f_1(x) = e^x, \quad f_2(x) = xe^x, \quad f_3(x) = x^2e^x. \tag{6}$$

(d) (3 points) Determine whether f_1, f_2 and f_3 are linearly independent.

Problem 3. (10 POINTS) FIBONACCI SEQUENCES
When learning abstract concepts, such as vector spaces, it is handy to have some weird examples at hand that illustrate the concept. These examples will a) develop you abstract thinking and b) motivate you to look for vector spaces in other examples that you will encounter in your study or research[1]. In lectures, we discussed one such example: $(Q, \oplus, *)$. Here is another one.

The Fibonacci sequence is a famous[2] sequence in mathematics and engineering:

$$\mathfrak{f}^* = (1, 1, 2, 3, 5, 8, 13, \dots). \tag{7}$$

The first two numbers of the sequence are $x_1 = x_2 = 1$ and each subsequent number is the sum of the previous two:

$$x_n = x_{n-1} + x_{n-2}, \quad n = 3, \dots \tag{8}$$

The sequence $\mathfrak{f}^*$ in (7) is the original Fibonacci sequence. Let us define a generalized Fibanaci sequence $\mathfrak{f}$ as follows:

$$\begin{aligned} &\mathfrak{f} = (x_1, x_2, x_3, \dots) \\ &x_1 \text{ and } x_2 \text{ are arbitrary real numbers (not necessarily 1)} \\ &x_n = x_{n-1} + x_{n-2}, \quad n = 3, \dots \end{aligned} \tag{9}$$

That is, in a generalized Fibonacci sequence we can start from any first two numbers. Let $\mathcal{F}$ denote the set of all generalized Fibonacci sequences. Let's define two natural operations on $\mathcal{F}$:

$$\begin{aligned} &\text{Addition: } \mathfrak{f}_1 = (x_1, x_2 \dots), \;\; \mathfrak{f}_2 = (y_1, y_2, \dots), \;\; \mathfrak{f}_1 + \mathfrak{f}_2 = (x_1 + y_1, x_2 + y_2 \dots) \\ &\text{Scalar multiplication: } \mathfrak{f} = (x_1, x_2, \dots), \;\; \alpha \in \mathbb{R}, \;\; \alpha\mathfrak{f} = (\alpha x_1, \alpha x_2, \dots). \end{aligned} \tag{10}$$

(a) (4 points) Show that the set of generalized Fibonacci sequences $\mathcal{F}$ is a vector space.
(b) (4 points) Find its dimension and a basis.
(c) (2 points) Find the coordinates of the original sequence $\mathfrak{f}^*$ in that basis.

Problem 4. (10 POINTS) FUNDAMENTAL MATRIX SUBSPACES
Let's start with a low-dimensional example. Consider the following matrix:

$$A = \begin{bmatrix} 2 & 0 \\ 2 & 2 \\ 20 & 25 \end{bmatrix} \tag{11}$$

(a) (2 points) Find the dimensions of the kernel, cokernel, image, and coimage of A.
(b) (3 points) Find bases (if they exist) for the four fundamental subspaces of A.

Now let's move on to a higher-dimensional problem.

(c) (5 points) Find bases for the kernel, image, cokernel, and coimage of the following matrix:

$$B = \begin{bmatrix} 1 & 2 & \dots & n \\ n+1 & n+2 & \dots & 2n \\ \vdots & \vdots & & \vdots \\ n^2-n+1 & n^2-n+2 & \dots & n^2 \end{bmatrix} \tag{12}$$

Hint: The proof of the Fundamental Theorem of Linear Algebra (see pages 20–21 of the Lecture Notes) may be helpful for part (c), but it is not necessary.

Problem 5. (10 POINTS) FUNDAMENTAL MATRIX SUBSPACES: IMPLEMENTATION
Complete Problem 5 in `PS2.mlx`.

[1] And as soon as you identify that a set is actually a vector space, you can employ the powerful machinery of linear algebra to attack the underlying problem.

[2] Read `https://en.wikipedia.org/wiki/Fibonacci_number` for information and watch `https://youtu.be/O2wU-HT7FiM` for inspiration.