ACM/IDS 104 APPLIED LINEAR ALGEBRA
PRACTICE PROBLEMS FOR LECTURE 3

KONSTANTIN M. ZUEV

Each lecture is accompanied by two practice problems: a somewhat easier, more practical *Problem A*, and a more difficult, more conceptual *Problem B*. The main goal of the practice problems is threefold: to help you better understand the material covered in the corresponding lecture, to help you prepare to solve problems in problem sets and exams, and to accommodate the diversity of students' math backgrounds by providing both easier and more challenging problems. These problems are for self-practice: they will not be graded, and the solutions—posted on Piazza—also illustrate the expected level of rigor for problem sets and exams.

Problem 3A. Linear Independence, Span, Basis
Consider the following four two-by-two matrices:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{1}$$

(a) Determine whether A_1, A_2, A_3, and A_4 are linearly dependent or not.
(b) Determine whether A_1, A_2, A_3, and A_4 span the vector space of all two-by-two matrices $\mathbb{M}_{2\times 2}$.
(c) Determine whether A_1, A_2, A_3, and A_4 form a basis of $\mathbb{M}_{2\times 2}$.

Problem 3B. Linear Independence in Functional Spaces
Proving linear dependence or independence in functional vector spaces can be nontrivial. In this problem, we will derive one method that can help us to establish linear independence if the functions are sufficiently smooth.

Let $f_1, \dots, f_n : I \to \mathbb{R}$ be a collection of n functions defined on an interval $I \subseteq \mathbb{R}$ that are differentiable at least $n-1$ times on I. Consider the following $n \times n$ matrix consisting of the derivatives of these functions:

$$W(x) = \begin{bmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ f_1''(x) & f_2''(x) & \dots & f_n''(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{bmatrix} \tag{2}$$

The determinant of this matrix, $\det W(x)$, is called the *Wronskian* of $f_1, \dots, f_n$.

(a) Prove that if $f_1, \dots, f_n$ are linearly dependent, then $\det W(x) = 0$ for all $x \in I$.
(b) Use the result of part (a) to show that functions $x, \log(x), \sin(x)$ are linearly independent.
(c) Can the result of part (a) be used to prove linear dependence of functions?

E-mail address: kostia@caltech.edu.