Ma 6a PS1

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§1 Find all $s, t \in \mathbb{Z}$ such that $\gcd(240, 84) = 240s + 84t$.

Apply the Euclidean algorithm:

$$\begin{aligned} 240 &= 84 \cdot 2 + 72, \\ 84 &= 72 \cdot 1 + 12, \\ 72 &= 12 \cdot 6 + 0, \end{aligned}$$

hence $\gcd(240, 84) = 12$.

Express 12 as a linear combination of 240 and 84:

$$12 = 84 - 72 = 84 - (240 - 84 \cdot 2) = 84 \cdot 3 - 240.$$

Thus one solution is $(s_0, t_0) = (-1, 3)$, i.e.

$$12 = 240(-1) + 84(3).$$

All integer solutions are obtained by adding multiples of $\frac{84}{12} = 7$ to s and subtracting the corresponding multiples of $\frac{240}{12} = 20$ from t:

$$\boxed{s = -1 + 7k, \qquad t = 3 - 20k, \qquad k \in \mathbb{Z}}.$$

§2 Show that for all $a, b, n \in \mathbb{N}$, we have $\gcd(an, bn) = n \gcd(a, b)$.

Let $d = \gcd(a, b)$. By Bezouts identity, there exist $x, y \in \mathbb{Z}$ such that

$$ax + by = d.$$

Multiplying by n gives

$$(an)x + (bn)y = nd,$$

so nd is a common divisor of an and bn, hence $nd \leq \gcd(an, bn)$.

Conversely, let $g = \gcd(an, bn)$. Again by Bezout, there exist $u, v \in \mathbb{Z}$ with

$$(an)u + (bn)v = g.$$

Factor n:

$$n(au + bv) = g \quad \Rightarrow \quad n \mid g.$$

Set $t = g/n \in \mathbb{N}$. Since $t = \frac{g}{n}$ divides both a and b, $t \mid d$. On the other hand, from $ax + by = d$ it follows that every common divisor of a and b divides d, so d is the greatest such that $d \geq t$. Combining $t \mid d$ with $d \geq t$ and $nd \leq g = nt$ forces $d = t$ and hence $g = nd$.

$$\boxed{\text{Therefore, } \gcd(an,\, bn) = n \gcd(a,\, b) \text{ for all } a, b, n \in \mathbb{N}.}$$

§3 Show that for all $a, b, n \in \mathbb{N}^*$, if $n \mid ab$ and $\gcd(a, n) = 1$, then $n \mid b$.

By Bezout, there exist $x, y \in \mathbb{Z}$ w/

$$ax + ny = 1.$$

Multiplying by b gives

$$abx + n(by) = b.$$

Both terms on the left are divisible by n (since $n \mid ab$ and trivially $n \mid n(by)$), hence $\boxed{n \mid b.}$

§4 Prove that a natural number written in base 10 is divisible by 9 if and only if the sum of its digits is divisible by 9.

Let N have decimal digits $d_0, d_1, \dots, d_k$ such that

$$N = \sum_{i=0}^{k} d_i \, 10^i.$$

Since $10 \equiv 1 \pmod 9$, it follows that $10^i \equiv 1 \pmod 9$ for every i, hence

$$N \equiv \sum_{i=0}^{k} d_i \pmod 9.$$

Therefore, $9 \mid N \iff 9 \mid \sum_{i=0}^{k} d_i$,

i.e., a base-10 number is divisible by 9 exactly when the sum of its digits is divisible by 9.

§5 Let F_n be the Fibonacci sequence, with $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Fix $m \in \mathbb{N}^*$. Show that the sequence $F_n \bmod m$ is periodic.

Let $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$. Let's consider the pair sequence

$$P_n = \big(F_n \bmod m,\ F_{n+1} \bmod m\big) \in (\mathbb{Z}/m\mathbb{Z})^2.$$

The Fibonacci sequence is an updating linear transformations follows:

$$P_{n+1} = T(P_n), \qquad T(x,y) = (y,\ x+y) \pmod{m}.$$

Pigeonhole. The set $(\mathbb{Z}/m\mathbb{Z})^2$ has m^2 elements; hence among $P_0, P_1, \ldots, P_{m^2}$ two must be equal. Therefore the sequence of pairs eventually repeats.

No pre-period (therefore invertible). The map T is invertible under mod m, with inverse

$$T^{-1}(x,y) = (y-x,\ x) \pmod{m},$$

since

$$T\big(y-x, x\big) = (x,\ (y-x)+x) = (x,y).$$

Therefore we can say if $P_i = P_j$ for some $i < j$, applying T^{-1} repeatedly yields $P_{i-1} = P_{j-1}$, $P_{i-2} = P_{j-2}$, $\ldots$, and $P_0 = P_{j-i}$ by repated induction. Hence the first repeated pair is $P_0 = (0,1)$, and the sequence is purely periodic.

Therefore there exists a least $T \geq 1$ such that

$$(F_{n+T}, F_{n+1+T}) \equiv (F_n, F_{n+1}) \pmod{m} \quad \text{for all } n,$$

or,

$$F_{n+T} \equiv F_n \pmod{m} \quad \text{for all } n.$$

$$\boxed{\text{Finally, we conclude that a period exists with the bound } T \leq m^2, \text{ so } \{F_n \bmod m\} \text{ is periodic.}}$$