ACM/IDS 104 APPLIED LINEAR ALGEBRA
PROBLEM SET 1

Please submit your solution as a single PDF file, that contains both the written-up and published code parts, via Gradescope by **9pm Tuesday, October 14**. An example of the submission process is shown here: `https://www.gradescope.com/get_started#student-submission`

- For theoretical problems, please use a pen, not a pencil: it is hard to read scanned submission written by a pencil.
- For coding problems, please convert your MATLAB livescripts (`.mlx`) to PDF by selecting `Live Editor` → `Save` → `Export to PDF` and merge them with the rest of your solution.
- After uploading your submission to Gradescope, please label all pages.

Problem 1. (10 POINTS) MATRIX MULTIPLICATION
Let $A \in \mathbb{M}_{m\times p}$ and $B \in \mathbb{M}_{p\times n}$ be $m \times p$ and $p \times n$ matrices respectively. Then, by definition, the (i, j) entry of $C = AB$ is the *dot product* of the i^{th} row of A and the j^{th} column of B:

$$c_{ij} = a^i \cdot b_j = \sum_{k=1}^{p} a_{ik} b_{kj}. \tag{1}$$

Interestingly, we can also compute AB by multiplying columns of A by rows of B. Show that

$$AB = \sum_{k=1}^{p} a_k b^k, \tag{2}$$

where each term $a_k b^k$ is a matrix of size $m \times n$. In particular, if B is a column vector $B = x \in \mathbb{R}^p$, then

$$Ax = \sum_{k=1}^{p} x_k a_k. \tag{3}$$

is a linear combination of columns $a_1, \dots, a_p$ of A with coefficients $x_1, \dots, x_p$ (components of x). We will use this important interpretation of matrix multiplication in lectures.

Problem 2. (10 POINTS) MATRIX IDENTITIES
It this problem, we will explore how familiar properties of numbers and functions look for matrices.

(a) (3 points) If a and b are non-zero numbers, then

$$\frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab}. \tag{4}$$

Let A and B be nonsingular matrices of the same size. Find an analog of (4) for $A^{-1} - B^{-1}$. That is, formulate an intelligent guess for

$$A^{-1} - B^{-1} = \dots \tag{5}$$

and prove it.

(b) (3 points) Let $A(t)$ be a family of nonsingular matrices depending smoothly on the real parameter $t \in \mathbb{R}$. Find and prove a formula for the derivative of the inverse $(A^{-1}(t))'$ in terms of the derivative $A'(t)$ and the inverse $A^{-1}(t)$. Check that the derived formula works in a special case, where $A(t)$ is a family of 1-by-1 matrices, that is, a family of functions.

(c) (4 points) Let $A(t)$ be a family of square matrices depending smoothly on the real parameter $t \in \mathbb{R}$. Find and prove a formula for the derivative of $A^2(t)$.

Problem 3. (10 POINTS) PERMUTED LU DECOMPOSITION
Let A_n be the $n \times n$ coefficient matrix of a linear system which we obtained in Lecture 1 by discretizing the Poisson equation. That is, let A_n be the $n \times n$ tridiagonal matrix with all 2's along the main diagonal, and all -1's along the sub- and super-diagonals. Find a permuted LU decomposition of matrix A_n, that is $P_n A_n = L_n U_n$, where P_n is a permutation matrix, L_n is special lower triangular, and U_n is upper triangular with non-zero diagonal elements.

Hint: It may be helpful to solve Practice Problems 1A and 2A before solving this problem.

Problem 4. (10 POINTS) TYPES OF MATRICES
In this course, we will discuss many different types of matrices which occur in various applications. In this problem, we will briefly explore the properties of orthogonal and skew-symmetric matrices.

The Gaussian elimination method results in the permuted LU decomposition, $PA = LU$, which is fundamental for solving linear systems. Another important factorization of a nonsingular matrix is the so-called QR factorization (which we will discuss in lectures), which can also be used for solving square linear systems (this approach is more computationally expensive, less prone to inaccuracies, and more numerically stable). The QR factorization is based on the notion of an *orthogonal* matrix. A square matrix A is called orthogonal if its inverse coincides with its transpose: $A^T = A^{-1}$.

(a) (3 points) Show that any permutation matrix P is orthogonal.
(b) (3 points) Is an orthogonal matrix necessarily a permutation matrix?

Other important classes of matrices are *symmetric* and *skew-symmetric* matrices. A square matrix A is symmetric if $A^T = A$, and is skew-symmetric if $A^T = -A$.

(c) (4 points) Show that every square matrix A can be written as a sum of a symmetric matrix S and a skew-symmetric matrix J:
$$A = S + J, \quad S^T = S, \quad J^T = -J. \tag{6}$$
Prove that this decomposition is unique and find the expressions for S and J in terms of A.

Problem 5. (10 POINTS) SOLVING LINEAR SYSTEMS
Let B be the following $n \times n$ matrix:
$$B = \begin{bmatrix} 1 & 2 & \dots & n \\ n+1 & n+2 & \dots & 2n \\ \vdots & \vdots & & \vdots \\ n^2-n+1 & n^2-n+2 & \dots & n^2 \end{bmatrix} \tag{7}$$

(a) (5 points) Find the rank of B and complete Problem 5a in `PS1.mlx`.
Remark: You don't need MATLAB to find the answer, but you can use it for making the right guess. We have provided a way of checking your answer in Problem 5a of `PS1.mlx`. In your solutions, please explain why your answer is correct for any n, and not only the values explored in MATLAB.
(b) (5 points) Complete Problem 5b in `PS1.mlx`. Remember to report the non-zero components in your solutions.