General Notes

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Analysis Notes

Mostly just review, a couple of "new" things on uniform convergence, integration, and basic point-set topology. I use a lot of UCSD Math 142B materials, and Stanford Math 61CM and 171 materials. Rudin and Abbott are the primary texts used here for reference.

Resources used (these are hyperlinks):

- Elementary Analysis - The Theory of Calculus [Ross]
- Understanding Analysis [Abbott]
- Principles of Mathematical Analysis [Rudin]
- Foundations of Mathematical Analysis [JP]

Chapter 1. MATH 115 Review
Various definitions and theorems from analysis. Some from Ross, some from external notes.

Recall the following (not stated explicitly): Field properties and properties of the reals, maxima and minima, suprema and infima, limit theorems, lim sup and lim inf, infinite series and Cauchy criterion, convergence tests (comparison, ratio, root, integral), continuity and continuity theorems, uniform continuity and functional limits.

Triangle Inequality - For all $a, b \in \mathbb{R}$: $|a + b| \leq |a| + |b|$.

Completeness Axiom - Every nonempty subset $S \subset \mathbb{R}$ that is bounded above has a least upper bound. In other words, sup S exists and is a real number.

Def. (Convergence of a Sequence) - A sequence (s_n) converges to a real number s if, for every positive number ϵ, there exists an $N \in \mathbb{N}$ such that whenever $n > N$, it follows that $|s_n - s| < \epsilon$.

Def. (Divergent Sequences) - For a sequence (s_n), we write $\lim s_n = +\infty$ provided for each $M > 0$ there is a number N such that $n > N$ implies $s_n > M$.

Def. (Cauchy Sequences) - A sequence (s_n) is a Cauchy sequence if for every $\epsilon > 0$ there exists a number N such that $m, n > N$ implies $|s_n - s_m| < \epsilon$.

Thm. (A Couple Statements on Convergent Sequences) -

- All bounded monotone sequences converge.
- Convergent sequences are bounded.
- A sequence is convergent iff it is a Cauchy sequence.

Thm. (Bolzano-Weierstrass Theorem) - Every bounded sequence has a convergent subsequence.

Def. (Geometric Series) - A series $\sum_{n=0}^{\infty} ar^n$ is a geometric series which converges for $|r| < 1$, and is equal to $\frac{a}{1-r}$.

Def. (p-Series) - The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$.

Thm. - If a series $\sum a_n$ converges, then $\lim a_n = 0$.

Thm. (Alternating Series Thm) - If $a_1 \geq a_2 \geq \ldots \geq a_n \geq \ldots \geq 0$ and $\lim_{n\to\infty} a_n = 0$, then the alternating series $\sum (-1)^{n+1} a_n$ converges. Morever, the partial sums $s_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$ satisfying $|s - s_n| \leq a_n$ for all n.

Thm. (Continuity) - Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. Then f is continuous at x_0 in f's domain iff for each $\epsilon > 0$, there exists $\delta > 0$ s.t. $x \in$ domain of f and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \epsilon$.

Thm. (Extreme Value Theorem) - Let f be a continuous real-valued function on a closed interval $[a, b]$. f is a bounded function. Moreover, f assumes its max and min values on $[a, b]$ i.e. there exist $x_0, y_0 \in [a, b]$ s.t. $f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in [a, b]$.

Thm. (Intermediate Value Theorem) - Let f be a continuous real-valued function on an interval I, then f has the intermediate value property on I: Whenever $a, b \in I$, $a < b$ and y lies between $f(a)$ and $f(b)$, then there exists at least one $x \in (a, b)$ s.t. $f(x) = y$.

Def. (Uniform Continuity) - Let f be a real-valued function defined on a set $S \subseteq \mathbb{R}$. Then f is uniformly continuous on S if for each $\epsilon > 0$, there exists $\delta > 0$ s.t. $x, y \in S$ and $|x - y| < \delta$ imply $|f(x) - f(y)| < \epsilon$. We say f is uniformly continuous if f is uniformly continuous on the domain of f.

Thm. - If f is uniformly continuous on a set S and (s_n) is a Cauchy sequence in S, then $(f(s_n))$ is a Cauchy sequence.

Thm. - A real-valued function f on (a, b) is uniformly continuous on (a, b) if and only if it can be extended to a continuous function $\tilde{f}$ on $[a, b]$.

Def. (Functional Limits) -

- For $a \in \mathbb{R}$ and a function f, write $\lim_{x \to a} f(x) = L$ if $\lim_{x \to a^S} f(x) = L$ for some set $S = J \backslash \{a\}$, where J is an open interval containing a. This is the *two-sided limit* (or just limit) of f at a.

- For $a \in \mathbb{R}$ and a function f, write $\lim_{x \to a^+} f(x) = L$ provided $\lim_{x \to a^S} f(x) = L$ for some open interval $S = (a, b)$. This is the *right-hand limit* of f at a.

- For $a \in \mathbb{R}$ and a function f, write $\lim_{x \to a^-} f(x) = L$ provided $\lim_{x \to a^S} f(x) = L$ for some open interval $S = (c, a)$. This is the *left-hand limit* of f at a.

- For a function f, write $\lim_{x \to \infty} f(x) = L$ provided $\lim_{x \to \infty^S} f(x) = L$ for some interval $S = (c, \infty)$. Likewise, write $\lim_{x \to -\infty} f(x) = L$ provided $\lim_{x \to -\infty^S} f(x) = L$ for some interval $S = (-\infty, b)$.

Thm. - Let f be a function defined on $S \subseteq \mathbb{R}$, let $a \in \mathbb{R}$ be a limit of some sequence in S, and L be a real number. Then $\lim_{x \to a^S} f(x) = L$ iff for each $\epsilon > 0$, there exists $\delta > 0$ such that $x \in S$ and $|x - a| < \delta$ imply $|f(x) - L| < \epsilon$.

Thm. - Let f be a function defined on $J \backslash \{a\}$ for some open interval J containing a. Then $\lim_{x \to a} f(x)$ exists iff the limits $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ both exist and are equal.

Chapter 2. Basic Topology and Metric Spaces

Metric Spaces

Def. (Metric Space) - Let S be a set, and suppose d is a function defined for all pairs (x, y) of elements from S satisfying:

1. $d(x, x) = 0$ for all $x \in S$ and $d(x, y) > 0$ for distinct x, y in S.
2. $d(x, y) = d(y, x)$ for all $x, y \in S$.
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in S$. (Triangle Inequality)

The function d is called a *metric* on S. A *metric space* S is a set S included with a metric on it. The metric space is the pair (S, d).

Ex. 8.2.1 [Abbott] - Decide which of the following are metrics on $X = \mathbb{R}^2$:

(a) $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

(b)

(c)

Def. - A sequence (s_n) in a metric space (S, d) *converges to* s in S if $\lim_{n\to\infty} d(s_n, s) = 0$. A sequence (s_n) in S is a *Cauchy sequence* if for each $\epsilon > 0$, there exists an N s.t.

$$m, n > N \text{ implies } d(s_m, s_n) < \epsilon.$$

The metric space (S, d) is *complete* if every Cauchy sequence in S converges to some element in S.

Chapter 3. Differentiation

TBD

Chapter 4. Integration

TBD

Chapter 5. Sequences and Series of Functions

Here I use a mix of Abbott and Rudin. Some of this is "newer material."

Power Series

Thm. - For the power series $\sum a_n x^n$, let

$$\beta = \limsup |a_n|^{1/n} \text{ and } R = \frac{1}{\beta}.$$

Then

1. The power series converges for $|x| < R$;
2. The power series diverges for $|x| > R$.

R is known as the *radius of convergence* for the power series.