~~PROBEM~~

PROBLEM SET 1

Problem 1

1)

Using marginalization,

$P(A) = \frac{1}{3}$, $P(B) = 1/4$, $P(C) = \frac{9}{16}$

$\left(= \sum_{B,C} P(A,B,C)\right)$ $= \left(\sum_{A,C} P(A,B,C)\right)$ $\left(= \sum_{A,B} P(A,B,C)\right)$

And,

$P(A,B) = \sum_{C} P(A,B,C) =$

A	B	P(A,B)
0	0	1/2
0	1	1/6
1	0	1/4
1	1	1/12

$P(A,C) = \sum_{B} P(A,B,C) =$

A	C	P(A,C)
0	0	~~8~~ 7/24
0	1	3/8
1	0	7/48
1	1	3/16

$P(B,C) = \sum_{A} P(A,B,C) =$

B	C	P(B,C)
0	0	3/8
0	1	3/8
1	0	1/16
1	1	3/16

$P(A,B) = \frac{1}{12} = P(A) \cdot P(B) = \frac{1}{3} \cdot \frac{1}{4}$ $\Rightarrow$ Independent!

$P(A,C) = \frac{\cancel{8}\,3}{16} = P(A) \cdot P(C) = \frac{1}{3} \cdot \frac{9}{16} = \frac{3}{16}$ $\Rightarrow$ Independent!

$P(B,C) = \frac{3}{16} \neq P(B) \cdot P(C) = \frac{1}{4} \cdot \frac{9}{16}$

$\Rightarrow$ Not Independent.

$\therefore$ G is:

$$\begin{array}{ccc} A & & A \\ B \rightarrow C & \text{OR} & B \leftarrow C \end{array}$$

2)

(a) All these conditional independence assumptions are using the non-descendants

$A \perp C, D, E, J, F, G, I \mid B, H.$

$B \perp H, I, C, J$

$C \perp A, B, E, J \mid H, I$

$D \perp A, E, J, I, H \mid B, C$

$E \perp A, H, C, I, G, F, D \mid B, J$

$F \perp B, E, A, I \mid D, C, H, J$

$G \perp A, B, C, D, E, J \mid F, H, I.$

$H \perp B, I, J, E$

$I \perp B, H, J, E$

(b)

$A \perp G \mid H, D, J, I$ Yes.

Paths,

$A \leftarrow H \rightarrow G$,
$A \leftarrow H \rightarrow C \rightarrow F \rightarrow G$
$A \leftarrow H \rightarrow F \rightarrow G$
$A \leftarrow H \rightarrow C \rightarrow D \rightarrow F \rightarrow G$
$A \leftarrow H \rightarrow C \leftarrow I \rightarrow G$

H given $\Rightarrow$ Blocked.
All ~~to~~ divergence conditions on H.

$A \leftarrow B \rightarrow D \rightarrow F \rightarrow G$ D given $\Rightarrow$ Blocked (Series)

$A \leftarrow B \rightarrow E \leftarrow J \rightarrow F \rightarrow G$ J given $\Rightarrow$ Blocked (Diverging)

(c) $E \perp D \mid B$ Yes

Paths,

$E \leftarrow B \rightarrow D$ — B given ⇒ Blocked (Diverging) and all other paths going through B are blocked because of the same reason.

$E \leftarrow J \rightarrow F \leftarrow D$ — F not given ⇒ Blocked (Converging) and all other paths going through F are blocked.

(d) $C \perp A \mid \{E, H, G\}$ No.

Paths,

$A \leftarrow H \rightarrow C$ — H given ⇒ Blocked (Diverging) and similarly all other paths through H.

$A \leftarrow B \rightarrow \underset{G}{\underset{\wr}{D}} \leftarrow C$ — G given ⇒ Not blocked. G is a descendent of D which acts as a converging node in the path.

(e) $J \perp A \mid E, F, D, H$ Yes.

Paths,

All paths through H would be blocked because of diverging condition.

$A \leftarrow B \rightarrow E \leftarrow J$ — E not given ⇒ Blocked (Converging)

$A \leftarrow B \rightarrow D \rightarrow F \leftarrow J$ — D is given ⇒ Blocked (Series)

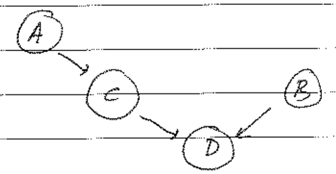
(f) $B \perp I \mid A, E$ No.

Paths,

$B \rightarrow E \leftarrow J \rightarrow F \rightarrow G \leftarrow I$ — E is given ⇒ Not blocked (Converging)

3) $\left.\begin{array}{l} A \text{ d-sep } B \\ \text{and } C \text{ d-sep } B \end{array}\right\} \Rightarrow A \text{ d-sep } C$? No.

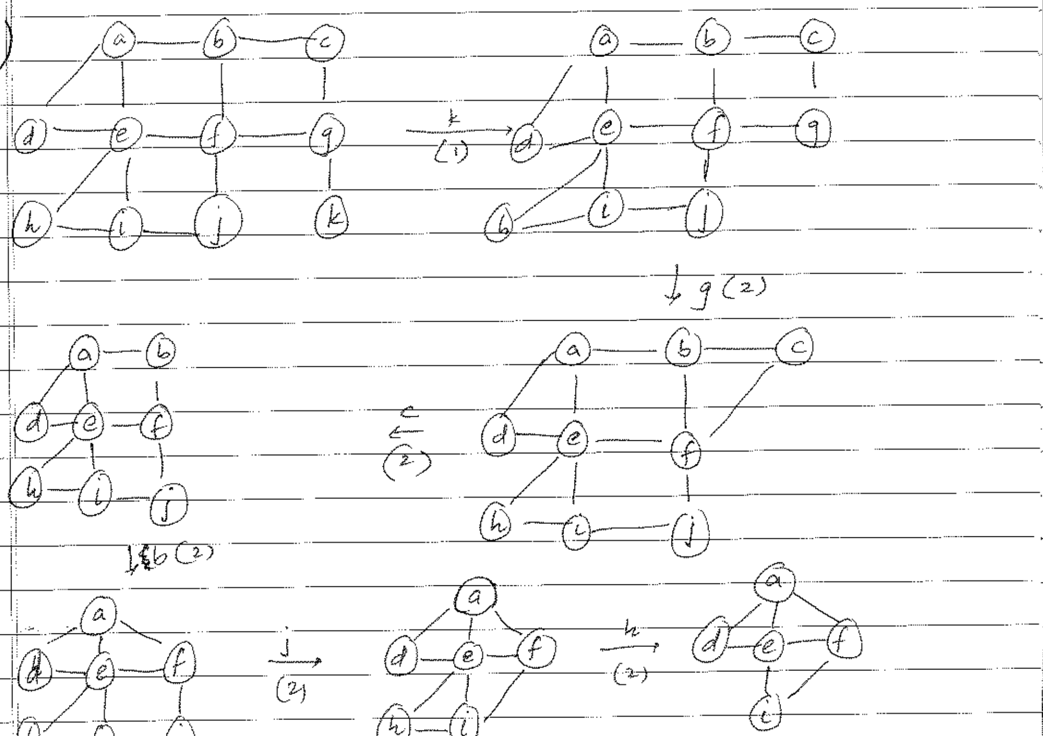
Example:

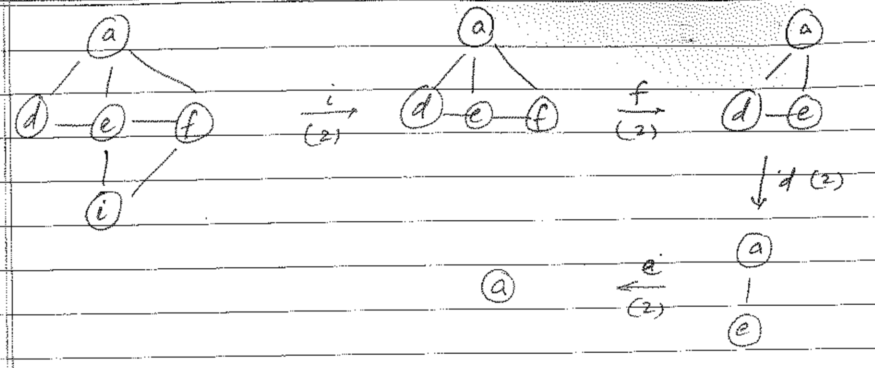


$A, C \perp B$ but $A \not\perp C$

Problem 2

1)





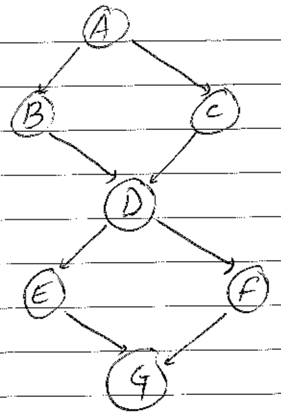
Tree width = largest clique created had size = 2.

2)* Let's say a graph has a maximal clique of size 3. which means that when we ~~decompose~~ eliminate / sum out any of the nodes of that ~~max~~ clique, a clique of size two will be made in even the most optimal order of elimination.

This can ~~be~~ be extended to a general case. For any ~~clique~~ maximal clique in a graph of size k, any node being eliminated from that clique will always generate a clique of size k-1, hence will always have a treewidth of k-1.

$\Rightarrow$ Treewidth = Size of maximal clique in graph (k) - 1.

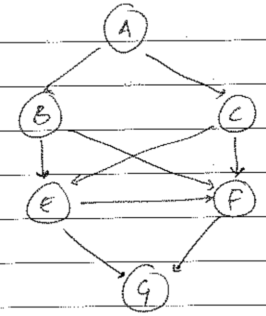
Problem 3



~~On removing~~ Before removing D,

B and F are dependent,
B and E are dependent.
C and F ~~and~~ are dependent
~~a~~ C and E are dependent
and E and F are dependent.

Removing D,



Problem 4

1. Represent the graph coloring problem as a Markov Random field on the undirected graph. Each edge has a constraint that the two nodes ~~is~~ along the edge do not have the same color.

or given $G = (V, E)$

2. $C \in V$ where for any $(i,j) \in E$ color of x_i is not the same as color of x_j.

$$P(x_V) = \frac{1}{Z} \prod_{(i,j)\in E} \mathbb{1}_{x_i \neq x_j}$$

3. For $k = 1$ $(k > 0)$

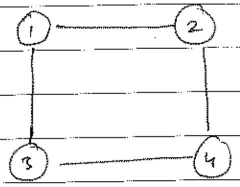
A clique of 2 nodes will not ~~be~~ have a valid probability distribution and no valid coloring.

For $k = 2$ $(k > 0)$

A clique of 3 nodes will not have a valid probability distribution and no valid ~~ordering~~ coloring.

In general for any $k > 0$
a clique of $k+1$ nodes will not have a valid probability distribution.

4. G :

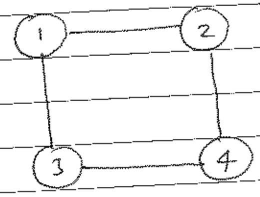


Z = # of all possible Colorings of the graph = 18

$$Z = \sum_{i \in V} \prod_{(i,j)\in E} \mathbb{1}_{x_i \neq x_j}$$

P.T.O →

Enumerate all possibilities.



1	2	3	4	Possible colorings
c_1	c_2	c_3	c_1	1
c_1	c_2	c_2	c_1/c_3	2
c_1	c_3	c_2	c_1	1
c_1	c_3	c_3	c_1/c_2	2
c_2	c_1	c_3	c_2	1
c_2	c_1	c_1	c_2/c_3	2
c_2	c_3	c_1	c_2	1
c_2	c_3	c_3	c_2/c_1	2
c_3	c_1	c_2	c_3	1
c_3	c_1	c_1	c_3/c_2	2
c_3	c_2	c_1	c_3	1
c_3	c_2	c_2	c_3/c_1	2

Total = 18