Problem Set 2 Solutions

Problem 1.

See the attached sample solution by Jinghui.
Problem 1.2 may also be solved directly by a linear programming solver (e.g., linprog in matlab); see *prob1.m* by Changbin Li.

Problem 2.

Grading criteria:
75% (60 out of 80 pts) based on whether the algorithms appear to have been implemented correctly (message passing, Bethe free energy calculation, etc.);
25% (20 out of 80) based on test cases; sumprod and maxprod are run on 5 MRFs structures for 10 tests, each worth 2 pts. The code *ps2_tests.m* is used for automatically grading the tests.
It's worth noting that many people got the correct Bethe free energy with w=[0 0 0] (used in sumprod tests), but not with non-uniform weights; this likely means a bug in the calculation of the inner-product term (sum sum tau(xc) log phi(xc) …) in the Bethe free energy (no points are deducted for this bug).

The 5 MRFs for the edge-coloring problem have the following graphs:

```
% trees
%1
% A--B
%2
%     D
%   / | \
% A   B   C
%3
%     D
%   / | \
% A   B   C—E
% loopys
%4
%     A
%    /  \
%    B--C
%5
%     A
%   / | \
% B   |   C
%   \ | /
%     D
```

Homework 2
Jinghui Guo

Problem 1: MAP IP vs. LP

1. The weighted edge coloring problem

$$\mathrm{P(x)} = \frac{1}{Z}\prod_{(i,j)\in E}\psi_{ij}(x_i,x_j)\prod_{i\in V}1_{\forall j,k\in N(i)x_{ij}\neq x_{ik}}$$

$$= \frac{1}{Z}\prod_{(i,j)\in E} exp(w_{x_{ij}})\prod_{i\in V}1_{\forall j,k\in N(i)x_{ij}\neq x_{ik}}$$

Where Z is the normalizing constant

$$\mathrm{Z} = \sum_{x_{ij}\in\{1,\ldots,k\}}\prod_{(i,j)\in E}\exp\left(w_{x_{ij}}\right)\prod_{i\in V}1_{\forall j,k\in N(i)x_{ij}\neq x_{ik}}$$

$$\log \mathrm{P(x)} = -\log Z + \sum_{(i,j)\in E} w_{x_{ij}}\log e + \sum_{i\in V}\log 1_{\forall j,k\in N(i)x_{ij}\neq x_{ik}}$$

We want to maximize the value of log P(x). The first part log Z is a constant; the last part always stays 0 because all the valid assignments will evaluate to log 1 = 0. So maximizing log P(x) becomes the problem

$$\max\sum_{(i,j)\in E} w_{x_{ij}}\log e = \sum_{(i,j)\in E} x_{ij}\log e$$

Since we have 2n nodes in a circle, the number of edges is 2n. The MAP assignment in this case should be coloring all the 2t – 1 edges with k^{th} color while coloring all the 2t edges with $(k - 1)^{th}$ color, where $t \in \{1,\ldots,n\}$.

2. MAP IP & LP

Suppose $x_i \in E$ is the i^{th} edge in the graph G. $x_i \cap x_j \in \mathrm{V}$ is the vertex that link the edge x_i and x_j. The potential function will be for each edge $x_i \in \mathrm{E}, \phi_i(x_i) = \exp\left(w_{x_i}\right)$. We still have $w_{x_i} = x_i$.

$$\max\sum_{x_i\in E}\sum_{x_i}\tau_i(x_i)w_{x_i} + \sum_{x_i\cap x_j\in \mathrm{V}}\sum_{x_i,x_j}\tau_{ij}(x_i,x_j)\log 1_{x_i\neq x_j}$$

For the MAP IP problem, the constraints are

$$\sum_{x_i}\tau_i(x_i) = 1 \qquad for\ all\ x_i \in E$$

$$\sum_{x_j}\tau_{ij}(x_i,x_j) = \tau_i(x_i) \qquad for\ all\ x_i\cap x_j \in \mathrm{V}$$

$$\tau_i(x_i) \in \{0,1\} \qquad for\ all\ x_i \in E$$

$$\tau_{ij}\left(x_i,x_j\right) \in \{0,1\} \qquad for\ all\ x_i\cap x_j \in \mathrm{V}$$

The first part $\log Z$ is a constant; the last part always stays 0 since all the valid assignments will evaluate to log1 = 0. So, MAP IP problem becomes the problem

$$max\sum_{x_i\in E}\sum_{x_i}\tau_i(x_i)w_{x_i} = max\sum_{x_i\in E}\sum_{x_i}\tau_i(x_i)x_i$$

Plug in all the valid assignments: (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,2,1), (3,1,2) and we got the same results 3 * 1 + 2 * 1 + 1 * 1 = 6.

For the MAP LP problem, the constraints are

$$\sum_{x_i} \tau_i(x_i) = 1 \qquad\qquad for\ all\ x_i \in E$$
$$\sum_{x_j} \tau_{ij}(x_i, x_j) = \tau_i(x_i) \qquad\qquad for\ all\ x_i \cap x_j \in \mathrm{V}$$
$$\tau_i(x_i) \in [0,1] \qquad\qquad for\ all\ x_i \in E$$
$$\tau_{ij}\left(x_i, x_j\right) \in [0,1] \qquad\qquad for\ all\ x_i \cap x_j \in \mathrm{V}$$

Solve this linear programming, we get

$$\tau_i(x_i = 1) = 0$$
$$\tau_i(x_i = 2) = 1/2$$
$$\tau_i(x_i = 3) = 1/2$$

The max LP solution is (1 * 0 + 2 * ½ + 3 * ½) * 3 = 7.5.