Category, space, type - Benjamin Antieau
08. Compactness

Besides the connectedness and T2, or Hausdorff, property, the third primary property commonly imposed on a topological space is compactness.

Definition 7.1 (Covers). Let X be a topological space and let $A \subseteq X$ be a subset. Let $\mathcal{C} = \{C_i\}_{i \in I} \subseteq \mathbf{P}(X)$ be a collection of subsets of X for some indexing set I. We say that $\mathcal{C}$ is a cover of A, or $\mathcal{C}$ covers A, if

$$A \subseteq \bigcup_{i \in I} C_i.$$

If the C_i are all open, then we say that $\mathcal{C}$ is an *open* cover of A. We will primarily be concerned with open covers.

Example 7.2. The set of objects $\{(-1,1),(0,2)\}$ is an open cover of $(0,1)$.

Example 7.3. If X is a topological space, an open cover of X is a collection of open subsets $\mathcal{C} = \{U_i\}_{i \in I}$ for some I such that every element of x appears in U_i for some $i \in I$.

Definition 7.4 (Subcovers). Let $\mathcal{C} = \{C_i\}_{i \in I}$ be a cover of a subset $A \subseteq X$. A subcover of $\mathcal{C}$ is a subcollection $\mathcal{D} \subseteq \mathcal{C}$ which is a cover of A.

Definition 7.5 (Finite covers). A cover $\mathcal{C}$ is finite if it contains finitely many elements.

Definition 7.6 (Compactness). Let X be a topological space. If every open cover $\mathcal{C}$ of X has a finite subcover $\mathcal{D} \subseteq \mathcal{C}$, then we say that X is compact. If $A \subseteq X$ is a subset, then A is compact if every open cover of A has a finite subcover. Equivalently, A with its subspace topology, is compact.

Remark 7.7. Sometimes in newer work this notion is called quasi-compactness.

Example 7.8. Finite topological spaces are compact.

Example 7.9. The discrete topology X^δ on a set X is compact if and only if X is finite.

Example 7.10. The trivial topology X^{triv} on a set X is compact.

Example 7.11. Metric spaces can be compact or not. For example, $\mathbf{R}$ is not compact, but any discrete finite space is compact, and metrizable.

Example 7.12. The cofinite topology on $\mathbf{N}$ is compact. Indeed, if $\mathcal{C}$ is an open cover of $\mathbf{N}$, then some open set $U_0 \in \mathcal{C}$ contains all but finitely many natural numbers, say $a_1, \dots, a_n$. Since $\mathcal{C}$ is a cover, there are $U_i \in \mathcal{C}$ for $i = 1, \dots, n$ such that $a_i \in U_i$. Then, $\{U_0, U_1, \dots, U_n\}$ is a finite subcover of $\mathcal{C}$ which covers $\mathbf{N}$.

Exercise 7.13. (a) Show that if $f\colon X \to Y$ is a continuous function is compact, then $f(X) \subseteq Y$ is compact (with either the quotient or subspace topology).

(b) Show that if $f\colon X \to Y$ is a continuous injection and Y is Hausdorff, then X is Hausdorff.

Proposition 7.14. *Let X be a Hausdorff space and let $A \subseteq X$ be compact. If $x \in X \setminus A$, then there are open subsets $U, V \subseteq X$ such that $x \in U$, $A \subseteq V$, and $U \cap V = \emptyset$.*

Proof. For each $a \in A$, since X is Hausdorff, we can choose U_a, V_a open subsets of X such that $x \in U$, $a \in V$, and $U \cap V = \emptyset$. The collection $\{V_a\}_{a \in A}$ is an open cover of A. Since A is compact, there is a finite subcover which covers A. Let $a_1, \dots, a_n$ be such that $V_{a_1}, \dots, V_{a_n}$ cover A. Then, $U = U_{a_1} \cap \cdots \cap U_{a_n}$ is open and contains a. Let $V = V_{a_1} \cup \cdots \cup V_{a_n}$. Then, $A \subseteq V$. Also, $U \cap V = \emptyset$, as desired. □

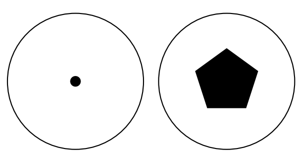

Figure 1: Diagram for Proposition 7.14.

Corollary 7.15. *Let X be Hausdorff and suppose that $A \subseteq X$ is a compact subset. Then, A is closed.*

Proof. Proposition 7.14 implies that if $x \in X \setminus A$, then there is $x \in U \subseteq X \setminus A$ where U is open. Thus, $X \setminus A$ is open, so A is closed. □

Lemma 7.16. *Suppose that X is a compact Hausdorff topological space. If $A \subseteq X$ is closed, then it is compact.*

Proof. Let $\mathcal{C}$ be an open cover of A. Then, $\{X \setminus A\} \cup \mathcal{C}$ is an open cover of X. It has a finite subcover and this finite subcover covers A. If it contains $X \setminus A$ we can throw it out and it will still cover A and it will be a finite subcover of $\mathcal{C}$. Thus, A is compact. $\square$

Theorem 7.17. *Let $f : X \to Y$ be a continuous bijection. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Proof. By Exercise 7.13, X is Hausdorff and Y is compact. Let $A \subseteq X$ be closed. Then, it is compact by Lemma 7.16. Thus, $f(A) \subseteq Y$ is compact by Exercise 7.13. Thus, $f(A)$ is closed by Corollary 7.15. It follows that f is closed. Hence, since f is a bijection, f is open. In otherwords, the inverse map is continuous, so that f is a homeomorphism. $\square$

Remark 7.18 (T_3 or regular Hausdorff). In a compact Hausdorff space, points and closed subsets (which are necessarily compact by Lemma 7.16) can be separated by disjoint open subsets using Proposition 7.14. This is another separation axiom which is called being *regular*. This does not by itself have a "T-number", but a regular Hausdorff space is called T_3.

Exercise 7.19. Let X be a Hausdorff space and let $A, B \subseteq X$ be disjoint compact subsets. Show there exist open sets U, V such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.

Remark 7.20 (Normal spaces). A topological space X is normal if for every pair $A, B \subseteq X$ of disjoint closed subsets, there exist open sets U, V such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$. Exercise 7.19 shows that a compact Hausdorff space is normal. Normal Hausdorff spaces are called T_4.

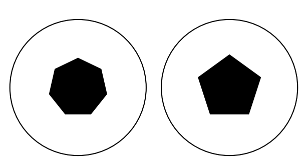

Figure 2: Picture of normality.

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