`Category, space, type - Benjamin Antieau`
`10. Alexandrov spaces`

Recall that an Alexandrov space is one where arbitrary intersections of open sets are open or, equivalently, if arbitrary unions of closed sets are closed.

The purpose of this section is to prove the following theorem which relates the specialization partial order $\leqslant$ on a T_0 topological space to the space itself.

`Theorem 10.1.` (a) *For any T_0 topological space X, the identity on X defines a continuous map $X^{\leqslant} \to X$.*

(b) *If X is a T_0 topological space, the identity map $X^{\leqslant} \to X$ is a homeomorphism if and only if X is Alexandrov.*

(c) *If $(P, \leqslant)$ is a poset, then the specialization partial order, say $\leqslant'$, on $P^{\leqslant}$ agrees with $\leqslant$.*

(d) *If $X \to Y$ is a continuous map of T_0 topological spaces, then $(X, \leqslant) \to (Y, \leqslant)$ is a map of posets.*

(e) *If $(P, \leqslant) \to (Q, \leqslant)$ is a map of posets, then $P^{\leqslant} \to Q^{\leqslant}$ is continuous.*

Thus, in some sense, we see an equivalence between the notions of a T_0 Alexandroff space and a poset. We will formalize this in the next lecture.

Proof of Theorem 10.1(a). Let X be a T_0 topological space. If $U \subseteq X$ is open, then we must see that it is open in the topology $X^{\leqslant}$. The opens in $X^{\leqslant}$ are precisely the downsets, the subsets V of X such that if $x \leqslant y$ and $y \in V$, then $x \in V$. Thus, suppose that $x \leqslant y$ in X and that $y \in U$. By definition of the specialization partial order, it follows that x is in every open set containing y. In particular, since $y \in U$, we have $x \in U$. Thus, U is a downset and the identity $X^{\leqslant} \to X$ is continuous. □

Proof of Theorem 10.1(b). Suppose that X is a T_0 topological space. If $X^{\leqslant} \to X$ is a homeomorphism, then X is indeed Alexandrov because we observed that $X^{\leqslant}$ is Alexandrov in Definition 8.16. Conversely, suppose that X is Alexandrov. The proof of part (a) shows that if $U \subseteq X$ is open, then U is a downset. We must show conversely that if $U \subseteq X$ is a downset with respect to the specialization partial order $\leqslant$, then U is open.

Let $u \in X$. The *principal downset* associated to u is $X_u = \{x \in X | x \leqslant u\}$. Note that since U is a downset we must have an equality

$$U = \bigcup_{u \in U} X_u.$$

Thus, it suffices to see that each X_u is open in X. By definition, X_u consists of the points x such that x is in every open set containing u. Suppose that y is not in X_u. Then, since X is T_0, there is an open set V_y such that $u \in V_y$ and $y \notin V_y$. Since V_y is a downset, it follows that $X_u \subseteq V_y$. Since X is Alexandrof,

$$V = \bigcap_{y \notin X_u} U_y$$

is open. We have $X_u \subseteq V$ and no element not in X_u is in V. Thus, $X_u = V$ and X_u is open, as desired. □

Proof of Theorem 10.1(c). Let $(P, \leqslant)$ be a poset and consider the specialization order $\leqslant'$ on the topological space $P^{\leqslant}$. The open subsets of $P^{\leqslant}$ are the downsets. Thus, $x \leqslant' y$ if and only if every downset of y contains x. In particular, $x \leqslant y$. Conversely, if $x \leqslant y$, then every downset containing y contains x, so $x \leqslant' y$. □

Proof of Theorem 10.1(d). Fix a continuous map $f \colon X \to Y$ of T_0 topological spaces. We want to show that if $x \leqslant y$, then $f(x) \leqslant f(y)$. But, if U is an open neighborhood of $f(y)$, then $f^{-1}(U)$ is an open neighborhood of y and so contains x. Thus, $f(x) \in U$. □

Proof of Theorem 10.1(e). Let $f \colon P \to Q$ be a map of posets. If $U \subseteq Q$ is a downset, we must show that $f^{-1}(U)$ is a downset. If $y \in f^{-1}(U)$ and $x \leqslant y$, then $f(x) \leqslant f(y)$. But, $f(y) \in U$ and U is a downset, so $f(x) \in U$. Thus ,$x \in f^{-1}(U)$. □

`Exercise 10.2.` Formulate and prove the analogue of Theorem 10.1 for all Alexandrov spaces by dropping the T_0 hypotheses and using the specialization preorder $\lesssim$ and your solution to Exercise 8.17.

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