`Category, space, type - Benjamin Antieau`
`03. Continuous functions`

`Suggested reading.` Read Section 12 from Munkres; review exercises 13.1 and 13.4.

Topological spaces are related to each other by certain functions, namely continuous functions. The definition below is motivated by considerations from calculus and analysis. However, the reader does not need experience with these to understand it.

`Definition 3.1 (Continuous function).` Let X and Y be topological spaces. A continuous function $X \to Y$ is a function from the underlying set of X to the underlying set of Y such that $f^{-1}(U) \subseteq X$ is open whenever $U \subseteq Y$ is open. Sometimes, we call these continuous functions simply "maps".

`Notation 3.2.` If X and Y are topological spaces, let $C(X,Y)$ denote the set of continuous functions from X to Y.

Here are some extremal cases of continuous functions.

`Example 3.3.` Any function $X \to Y^{\mathrm{triv}}$ is continuous.

`Example 3.4.` Any function $X^{\delta} \to Y$ is continuous.

`Example 3.5.` Consider the identity function $\mathrm{id}\colon X \to X$. If we view it as a function $X^{\delta} \to X^{\mathrm{triv}}$, it is continuous. If it is a function $X^{\mathrm{triv}} \to X^{\delta}$ and if X has at least 2 elements, then it is not continuous.

`Remark 3.6.` For any topological space X there are continuous functions

$$X^{\delta} \to X \to X^{\mathrm{triv}}.$$

On the left we have the finest topology on X, the one with the most open subsets. On the right we have the coarsest topology on X, with the fewest. Any other topology sits in between these two. These continuous functions are both bijections on the underlying sets of points.

`Example 3.7.` Let (X,d) and (Y,d') be metric spaces. Say that a function $f\colon X \to Y$ is continuous (with respect to d and d') if for every $x \in X$ and every $\epsilon > 0$ there exists $\delta > 0$ such that if $d(x,y) < \delta$, then $d'(f(x),f(y)) < \epsilon$.

The following exercise translates between continuity as learned in calculus and the definition of continuity given above.

`Exercise 3.8.` If (X, d) is a metric space, let $\mathcal{U}_d \subseteq \mathbf{P}(X)$ be the set of d-open subsets. Now, let (X, d) and (Y, d') be metric spaces. Prove that a function $X \to Y$ is continuous with respect to the metrics d and d' if and only if it is continuous with respect to the associated topologies $\mathcal{U}_d$ and $\mathcal{U}_{d'}$ in the sense of Definition 1.5.

Recall that given a set X, the set of functions $X \to \{0, 1\}$ is in bijection with $\mathbf{P}(X)$, the power set of X.

`Exercise 3.9.` Let X be a topological space.

(a) Show that the set of continuous functions $X \to \{0, 1\}^\delta$ is in bijection to the set of subsets of X which are both open and closed.

(b) Let T be the Sierpiński space of Example 1.11. Show that the set of continuous functions $X \to T$ is in bijection to the set of open subsets of X.

(c) Show that the set of continuous functions $X \to \{0, 1\}^{\mathrm{triv}}$ is in bijection to the set of subsets of X.

`Exercise 3.10.` Let X be a finite topological space and let Y be a metric space. Let $i\colon Y^\delta \to Y$ be the canonical continuous bijection. Show that any continuous function $f\colon X \to Y$ factors as $f = i \circ f'$ where $f'\colon X \to Y^\delta$.

`Lemma 3.11.` *Let X, Y, Z be topological spaces. If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous functions, then so is $g \circ f\colon X \to Z$.*

Proof. Let $U \subseteq Z$ be open. We have to show that $(g \circ f)^{-1}(U) \subseteq X$ is open. But, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. Since g is continuous, $g^{-1}(U) \subseteq Y$ is open. Since f is continuous, $f^{-1}(g^{-1}(U))$ is open, as desired. □

`Example 3.12.` For any topological space X, the identity $\mathrm{id}_X\colon X \to X$ is continuous.

`Construction 3.13.` Let **Top** be the category of topological spaces. The objects are topological spaces. Given two objects (topological spaces) X, Y, we set $\mathrm{Hom}_{\mathbf{Top}}(X, Y) = C(X, Y)$, the set of continuous functions. By Lemma 3.11, set-theoretic composition gives a well-defined function $C(Y,Z) \times C(X,Y) \to C(X,Z)$, or $\mathrm{Hom}_{\mathbf{Top}}(Y,Z) \times \mathrm{Hom}_{\mathbf{Top}}(X,Y) \to \mathrm{Hom}_{\mathbf{Top}}(X,Z)$. It is associative because composition of functions between sets is associative. It is unital because identity functions are continuous and a continuous function is determined by what it does on points, so that $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$ for any continuous function $f\colon X \to Y$.

The category **Top** looks in some ways like **Set**. The objects X are, after all, sets with certain extra structure, the specification of a topology. The morphisms $f\colon X \to Y$ are functions on the sets underlying X and Y which satisfy a special property related to the specified topologies. One starts to get a feeling for the difference by the following phenomenon.

`Example 3.14 (Non-isomorphic bijections).` Let X and Y be topological spaces and let $f\colon X \to Y$ be a continuous function. If f is a bijection at the level of underlying sets, is it an isomorphism in the category **Top**? Recall that this means that it has an inverse $g\colon Y \to X$, a continuous function such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. Let X be a set. Write $f\colon X^{\delta} \to X^{\mathrm{triv}}$ for the identity function, but viewed as a continuous function between these two, usually different, topological spaces. Evidently, f is a bijection at the level of the underlying sets. Its inverse, g, if it exists would be a continuous function $g\colon X^{\mathrm{triv}} \to X^{\delta}$. By looking at underlying sets, we would have $g(x) = x$ for all x. Thus, we ask if this is continuous. If X has at least two elements, it is not continuous. Thus, there are lots of non-isomorphic continuous bijections.

`Definition 3.15.` Isomorphisms in the category **Top** are called homeomorphisms.

`Remark 3.16.` A topological property of a topological space is one which is invariant under isomorphism. That is to say that a topological space X satisfies property $\mathbf{T}$ if and only if any topological space Y homoeomorphic to X also satisfies property $\mathbf{T}$. We have not many of these properties yet. However, the properties of having the discrete topology or the trivial topology are both topological properties. The property of having 1 as an element of X is not a topological property because we can simply relable the elements of X to obtain a homeomorphic, but not equal, topological space Y where $1 \notin Y$.

`Remark 3.17.` A general goal of this course is to classify topological spaces, at least those satisfying certain topological properties, up to homeomorphism.

`Example 3.18.` There are three topological spaces with 2 elements up to homemorphism: $\{0,1\}^{\delta}$, the Sierpinski space

$$T = (\{0,1\}, \{\emptyset, \{0\}, \{0,1\}\}),$$

and $\{0,1\}^{\mathrm{triv}}$. The identity functions are continuous maps $\{0,1\}^{\delta} \to T \to \{0,1\}^{\mathrm{triv}}$. Any topological space with 2 elements is homeomorphic to one of these and no two of these are homeomorphic.

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