Math 521 - Derived algebraic geometry

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Fall 2020

Contents

1 Week 1: Simplicial commutative rings (16–22 Sept.) 2
1.1 Simplicial objects . . . 3
1.2 Simplicial commutative rings . . . 7

2 Week 2: Animated commutative rings (23–29 Sept.) 12
2.1 Cocompletion of 1-categories . . . 12
2.2 Cocompletion of ∞-categories . . . 16
2.3 Animation . . . 18
2.4 A tiny model categorical digression . . . 21
2.5 Animated commutative rings . . . 21

3 Week 3: The cotangent complex (30 Sept.–6 Oct.) 23
3.1 Kähler differentials . . . 23
3.2 The cotangent complex . . . 26
3.3 The universal property . . . 28
3.4 A little deformation theory . . . 29
3.5 Properties of the cotangent complex . . . 30
3.6 Finiteness . . . 31

4 Week 4: Derived de Rham cohomology (10 Oct.– 16 Oct.) 32
4.1 The discrete de Rham complex . . . 32
4.2 Derived commutative versus $\mathbb{E}_\infty$-k-algebras . . . 34
4.3 Derived de Rham cohomology . . . 34
4.4 The conjugate filtration . . . 35
4.5 The Hodge filtration . . . 37
4.6 Hartshorne's algebraic de Rham cohomology . . . 38
4.7 Décalage . . . 38
4.8 Hacks . . . 41

5 Week 5: circle actions (17–23 Oct.) 42
5.1 Stable ∞-categories and t-structures . . . 42
5.2 Parametrized spectra . . . 46
5.3 Group actions . . . 48
5.4 Circle actions . . . 49

6 Week 6: The HKR theorem (24–30 Oct.) 50
6.1 Hochschild homology . . . 50
6.2 The HKR theorem . . . 54
6.3 Descent . . . 58
6.4 Appendix: signs . . . 58
6.4.1 The symmetric monoidal category of chain complexes . . . 58
6.4.2 Differential graded derivations . . . 60
6.4.3 The cyclic bar complex . . . 62
6.4.4 From Hochschild homology to differential forms . . . 63

7 Week 7: The de Rham filtration on periodic cyclic homology (31 Oct.–6 Nov.) 65
7.1 Cyclic homology and negative cyclic homology . . . 66
7.2 Periodic cyclic homology . . . 67
7.3 The idea of the motivic filtration on $\mathrm{HC}^-(R/k)$ and $\mathrm{HP}(R/k)$ . . . 70
7.4 Spectral sequences . . . 72
7.5 The motivic filtration . . . 74
7.6 Degeneration in characteristic 0 . . . 75

8 Week 8: Derived stacks (7–13 Nov.) 75
8.1 The classical functor of points picture . . . 76
8.2 n-geometric derived stacks . . . 80
8.3 Relative n-stacks . . . 82
8.4 The classical locus . . . 83
8.5 Quasicoherent sheaves on derived stacks . . . 84
8.6 The cotangent complex of a derived stack . . . 85
8.7 Properties of geometric derived stacks . . . 86

9 Week 9: Artin–Lurie representability (14–24 Nov.) 86
9.1 Artin–Lurie representability: statement of result . . . 86
9.2 Toën-Vaquié representability . . . 88
9.3 Local moduli . . . 91
9.4 Generalized Toën–Vaquié representability . . . 92
9.5 Example: $\mathbf{Bun}_G$ . . . 93
9.6 Example: the Picard stack . . . 93

1 Week 1: Simplicial commutative rings (16–22 Sept.)

Algebraic geometry is built up out of commutative rings. Specifically, any scheme X is covered by affine open subschemes $\operatorname{Spec} R \subseteq X$. Moreover, the functor $\mathrm{Sch} \to \mathrm{Fun}(\mathrm{Aff}^{\mathrm{op}}, \mathrm{Set})$ which assigns to a scheme X its functor of points

$$X(\operatorname{Spec} R) = \mathrm{Hom}_{\mathrm{Sch}}(\operatorname{Spec} R, X)$$

is fully faithful and it is not difficult to identify the image. Namely, a presheaf is in in the image if and only if it is a sheaf for the big Zariski topology on all affine schemes and moreover it is covered by affine schemes via sufficiently geometric morphisms.

The passage to derived algebraic geometry we will discuss in this class is conceptually quite simple. We replace the category of affine schemes $\mathrm{Aff} = \mathrm{CAlg}^{\mathrm{op}}$ with a suitable category of derived affine schemes, defined as the opposite of an ∞-category of derived commutative rings. There are three primary choices one can make as the category of derived commutative rings:

(i) connective $\mathbb{E}_\infty$-ring spectra,

(ii) animated commutative rings (the rings formerly known as simplicial commutative), or

(iii) connective $\mathbb{Q}$-cdgas.

Here, **connective** means that $\pi_i X = 0$ or $\mathrm{H}_i X = 0$ for all $i < 0$, depending on the context. The list ranges from most general to least, but also from most technically demanding to least. We make the *Goldilocks choice* and cover animated commutative rings in this course. This is sufficient for arithmetic applications of DAG, but not enough for homotopical applications like topological modular forms.

We will get to the process of animation in Week 2. In Week 1, we cover the notion of simplicial sets and simplicial commutative rings. The latter provide a 1-categorical model for working with animated commutative rings which is sometimes helpful.

Notation 1.1.
- The symbol $=$ will mean a statement about elements in a set.
- The symbol $\cong$ will mean an isomorphism in a 1-category.
- The symbol $\simeq$ will mean an equivalence in an ∞-category, a model category, or between two categories.

1.1 Simplicial objects

References 1.2. Goerss–Jardine [21], May [43], and Weibel [64, Chap. 8].

Definition 1.3. We let Δ be the 1-category of finite nonempty totally ordered sets. This is called the **simplex category**.

Notation 1.4. Every object of Δ is isomorphic to $[n] = \{0 < 1 < \cdots < n\}$ for some n. Thus, we will often work just with the subcategory of Δ given by the objects $[n]$. Each morphism $[m] \to [n]$ in Δ has a unique epi-mono factorization

$$[m] \to [p] \to [n]$$

into a surjection followed by an injection. Moreover, each surjection factors into a sequence of surjections

$$[m] \to [m-1] \to \cdots \to [p]$$

of **degeneracy maps** and each injection factors into a sequence of injections

$$[p] \to [p+1] \to \cdots \to [n]$$

of **face maps**. Each degeneracy collapses two adjacent points to a single point and each face map skips some element of the target. More specifically, for $0 \leqslant i \leqslant n$, we let

$$\partial^i \colon [n-1] \to [n]$$

be the unique map that misses i and

$$\sigma^i \colon [n+1] \to [n]$$

be the unique map where $i \in [n]$ is the image of i and $i+1$. The category Δ is generated as a category by the objects $[n]$ for $n \geqslant 0$, the morphisms ∂^i and σ^j, and the following relations:

$$\begin{aligned}
\partial^j \partial^i &= \partial^i \partial^{j-1} && \text{if } i < j, \\
\sigma^j \sigma^i &= \sigma^i \sigma^{j+1} && \text{if } i \leqslant j, \\
\sigma^j \partial^i &= \begin{cases} \partial^i \sigma^{j-1} & \text{if } i < j, \\ \text{identity} & \text{if } i = j \text{ or } i = j+1, \\ \partial^{i-1} \sigma^j & \text{if } i > j+1. \end{cases}
\end{aligned}$$

Definition 1.5 ((Co)simplicial objects)**.** Let $\mathcal{C}$ be a category (or ∞-category). A **simplicial object** in $\mathcal{C}$ is a functor $X\colon \Delta^{\mathrm{op}} \to \mathcal{C}$. We will typically denote such functors as $X_\bullet$ with the convention that $X_n = X([n])$. The functor category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ is called the (∞-)category of simplicial objects in $\mathcal{C}$ and is denoted $\mathrm{s}\mathcal{C}$. Similarly, a **cosimplicial object** in $\mathcal{C}$ is a functor $\Delta \to \mathcal{C}$, which we will write as $X^\bullet$ with $X^n = X([n])$. The category of cosimplicial objects in $\mathcal{C}$ is denoted by $\mathrm{c}\mathcal{C}$. Note that $c\mathcal{C} \simeq (\mathrm{s}\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$.

Remark 1.6. To specify a cosimplicial object in a category $\mathcal{C}$, it is enough to specify objects X^n together with face and degeneracy maps $\partial^i\colon X^{n-1} \to X^n$ and $\sigma^i\colon X^{n+1} \to X^n$ for $0 \leqslant i \leqslant n$ satisfying the relations of Notation 1.4. Similarly, to specify a simplicial object, one specifies objects X_n and face and degeneracy maps $\partial_i\colon X_n \to X_{n-1}$ and $\sigma_i\colon X_n \to X_{n+1}$ satisfying the opposite relations.

Example 1.7 (The categorical simplex)**.** Each totally ordered set S can be viewed naturally as a category and each order-preserving map of totally ordered sets can be viewed as a functor. This allows us to view Δ as a full subcategory of the 1-category of small categories Cat_1. We let $\Delta^\bullet_{\mathrm{cat}}$ be the resulting cosimplicial category.

Example 1.8 (The topological simplex)**.** Let Top denote the 1-category of topological spaces and continuous functions. For each integer $n \geqslant 0$ there is a topological n-simplex

$$\Delta^n_{\mathrm{top}} = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : x_i \geqslant 0 \text{ for } 0 \leqslant i \leqslant n \text{ and } \sum_{i=0}^n x_i = 1\},$$

where Δ^n is given the subspace topology. Let $\alpha\colon [m] \to [n]$ be a map in Δ. We can define a map $\alpha_{\mathrm{top}}\colon \Delta^m_{\mathrm{top}} \to \Delta^n_{\mathrm{top}}$ as

$$\alpha_{\mathrm{top}}(x_0, \dots, x_m)_i = \sum_{j \in \alpha^{-1}(i)} x_j,$$

where we adopt the convention that the sum is 0 if the indexing set is empty (so i is not in the image of α). We leave it to the reader to check that $\alpha \mapsto \alpha_{\mathrm{top}}$ makes $[n] \mapsto \Delta^n_{\mathrm{top}}$ into a cosimplicial topological space $\Delta^\bullet_{\mathrm{top}}$, i.e., an object of cTop.

Exercise 1.9 (The algebraic simplex)**.** Construct a cosimplicial scheme $\mathbb{A}^\bullet$ with $\mathbb{A}^n$ affine n-space (say over $\mathbb{Z}$) and with the 'same' face and degeneracy maps as in the topological simplex.

Example 1.10 (The simplicial simplex)**.** Given $[n] \in \Delta$, there is a simplicial set Δ^n defined by $\Delta^n = \mathrm{Hom}_\Delta(-, [n])$. In fact, this assignment gives a functor $\Delta \to \mathrm{sSet}$, or in other words a cosimplicial simplicial set.

Simplicial abelian groups are especially important; from them one can cook up explicit chain complexes.

Example 1.11 (The (un)normalized chain complex)**.** Let $\mathcal{A}$ be an abelian category and let $X_\bullet \in \mathrm{s}\mathcal{A}$ be a simplicial object in $\mathcal{A}$. Attached to $X_\bullet$ are two chain complexes in $\mathcal{A}$, the **unnormalized chain complex** $C(X_\bullet)$ and the **normalized chain complex** $N(X_\bullet)$. The unnormalized chain complex

$$0 \leftarrow X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$$

has $C(X_\bullet)_n = X_n$ for $n \geqslant 0$ and $C(X_\bullet)_n = 0$ for $n < 0$ with differentials $d = \sum_{i=0}^n (-1)^i \partial_i\colon X_n \to X_{n-1}$. The normalized chain complex $N(X_\bullet)$ is the subcomplex of $C(X_\bullet)$ where $N(X_\bullet)_n \subseteq C(X_\bullet)_n$ is

$$\bigcap_{i=0}^{n-1} \ker(\partial_i\colon X_n \to X_{n-1}).$$

The differential $N(X_\bullet)_n \to N(X_\bullet)_{n-1}$ is given by $(-1)^n \partial_n$.

Exercise 1.12. Check that $d \circ d = 0$ in the unnormalized chain complex $C(X_\bullet)$ of a simplicial abelian group.

Exercise 1.13. Let M be an abelian group and let $M_\bullet$ be the corresponding constant object of s$\mathcal{A}$b. Prove that there is a natural quasi-isomorphism $C(M_\bullet) \to M$, where we view M as a chain complex concentrated in degree 0.

Lemma 1.14. *The inclusion map $N(X_\bullet) \to C(X_\bullet)$ is a quasi-isomorphism.*

Proof. See [21, Thm. III.2.1] or [64, Thm. 8.3.8]. □

The Dold–Kan correspondence says that the process of taking the normalized chain complex induces an equivalence of categories.

Theorem 1.15 (The Dold–Kan correspondence)**.** *The normalized chain complex functor $N\colon \mathrm{s}\mathcal{A} \to \mathrm{Ch}_{\geqslant 0}(\mathcal{A})$ is an equivalence of categories. The inverse functor is given by $K\colon \mathrm{Ch}_{\geqslant 0}(\mathcal{A}) \to \mathrm{s}\mathcal{A}$ with*

$$K(Y_\bullet)_n \cong \bigoplus_{\eta\colon [n] \twoheadrightarrow [p]} Y_p[\eta],$$

where the direct sum ranges over all surjections from $[n]$ to $[p]$, and where for a map $\alpha\colon [m] \to [n]$ the induced map $K(Y_\bullet)_n \to K(Y_\bullet)_m$ is determined on the summand $Y_p[\eta]$ of $K(Y_\bullet)_n$ corresponding to $\eta\colon [n] \twoheadrightarrow [p]$ by taking the epi-mono factorization $[m] \twoheadrightarrow [q] \hookrightarrow [p]$ and letting ϵ denote the surjection $[m] \twoheadrightarrow [q]$, then one fixes the map $Y_p[\eta] \to Y_q[\epsilon]$ to be zero, except in the following two cases:

1. *if $p = q$, one takes the map to be the identity and*
2. *if $p = q + 1$ and $\epsilon = \partial^p$ one takes the map to be the differential $Y_p[\eta] \cong Y_p \to Y_{p-1} \cong Y_q[\epsilon]$ of the chain complex $Y_\bullet$.*

Proof. See [21, Cor. III.2.3] or [64, Cor. 8.4.3]. □

Remark 1.16. Let $\mathcal{A}$ be an abelian category with enough projectives.[1] Say that a map $X_\bullet \to Y_\bullet$ in s$\mathcal{A}$ is a weak equivalence if $C(X_\bullet) \to C(Y_\bullet)$ is a quasi-isomorphism (or equivalently by Lemma 1.14 that $N(X_\bullet) \to N(Y_\bullet)$ is a quasi-isomorphism). The Dold–Kan correspondence provides an equivalence of ∞-categorical localizations

$$\mathrm{s}\mathcal{A}[W^{-1}] \simeq \mathrm{Ch}_{\geqslant 0}[\mathrm{qiso}^{-1}].$$

Thus, these localization both give explicit presentations of the ∞-category $\mathcal{D}(\mathcal{A})_{\geqslant 0} \subseteq \mathcal{D}(\mathcal{A})$ of objects of $\mathcal{D}(\mathcal{A})$ with homology concentrated in non-negative degrees.

There are three general constructions of simplicial objects we will use. They come from cosimplicial objects, bar constructions, or comonads.

Example 1.17 (Simplicial sets from cosimplicial objects)**.** Let $X^\bullet$ be a cosimplicial object in $\mathcal{C}$ and let $Y \in \mathcal{C}$. We obtain a simplicial set by

$$[n] \mapsto \mathrm{Hom}_{\mathcal{C}}(X^n, Y).$$

Example 1.18 (The singular simplicial set)**.** Let Y be a topological space. Examples 1.8 and 1.17 give a simplicial set

$$\mathrm{Sing}_\bullet(Y)\colon [n] \mapsto \mathrm{Hom}_{\mathrm{Top}}(\Delta^n_{\mathrm{top}}, Y).$$

[1] Having enough projectives is required to ensure that the ∞-categorical localizations below live in the same set-theoretic universe as $\mathcal{A}$ and are not too big.

The **set of n-simplices** is the set of continuous maps $\Delta^n_{\text{top}} \to Y$, which should be familiar from the definition of singular homology. Indeed, we can use the functor $\mathrm{Set} \to \mathcal{A}\mathrm{b}$ sending a set S to the free abelian group $\mathbb{Z}S = \oplus_{s\in S}\mathbb{Z}$ on S to get $\mathbb{Z}\mathrm{Sing}_\bullet(Y)$, a simplicial abelian group. Taking unnormalized chains, we obtain a chain complex $C(\mathbb{Z}\mathrm{Sing}_\bullet(Y))$ whose homology groups are the **singular homology groups** $\mathrm{H}_*(Y, \mathbb{Z})$ of the space Y by definition.

Example 1.19 (The singular simplicial category)**.** We can view the totally ordered set $[n] = \{0 < 1 < \cdots < n\}$ as a category Δ^n_{cat}, and the maps in Δ are naturally functors. This defines a cosimplicial category $\Delta^\bullet_{\text{cat}}$. Given any category $\mathcal{C}$, we thus obtain a simplicial category by taking $\mathrm{Fun}(\Delta^\bullet_{\text{cat}}, \mathcal{C})$. For example, $\mathrm{Fun}(\Delta^0, \mathcal{C}) \simeq \mathcal{C}$ and $\mathrm{Fun}(\Delta^1, \mathcal{C})$ is the category of arrows in $\mathcal{C}$. The 1-skeleton of this simplicial object looks like

$$\mathrm{Fun}(\Delta^1, \mathcal{C}) \overset{\rightrightarrows}{\leftarrow} \mathcal{C}.$$

The face maps ∂_0 and ∂_1 are given by taking target and source of a morphism, while the degeneracy σ_0 is given by taking an object to the identity arrow. We leave it to the reader to describe what the functors do on morphisms. This construction is important in the complete Segal space approach to ∞-categories.

Definition 1.20 ((Co)monads)**.** Let $\mathcal{C}$ be a category. A **monad** on $\mathcal{C}$ is an algebra object T of the monoidal category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ of functors on $\mathcal{C}$. In particular, there is a unit natural transformation $\eta\colon \mathrm{id}_{\mathcal{C}} \to T$ and a multiplication $\mu\colon T \otimes T = T^{\circ 2} \to T$ satisfying the usual associativity and unit conditions. A **comonad** is a coalgebra object of $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$.

Exercise 1.21 ((Co)monads from adjoint functors)**.** If $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ are adjoint functors, then $G \circ F$ is a monad on $\mathcal{C}$ and $F \circ G$ gives a comonad on $\mathcal{D}$. Spell out the (co)multiplication and (co)unit maps.

Construction 1.22 (Simplicial objects from comonads)**.** Given a comonad T on $\mathcal{C}$ with comultiplication $\Delta\colon T \to T^{\circ 2}$ and counit map $\epsilon\colon T \to \mathrm{id}_{\mathcal{C}}$ we can construct canonical simplicial objects of $\mathcal{C}$ associated to any object $X \in \mathcal{C}$. Specifically, we let

$$T_\bullet X$$

be the simplicial object with

$$T_n X = T^{\circ(n+1)} X$$

where the face maps are given by the natural transformations

$$\partial_i = T^{\circ i} \circ \epsilon \circ T^{\circ(n-i)}\colon T^{\circ(n+1)} \to T^{\circ n}$$

and the degeneracy maps are given by the natural transformations

$$\sigma_i = T^{\circ i} \circ \Delta \circ T^{\circ(n-i)}\colon T^{\circ(n+1)} \to T^{\circ(n+2)}.$$

Example 1.23 (Čech complexes)**.** Let $f\colon Y \to X$ be a map of sets. Consider the adjunction

$$f_!\colon \mathrm{Set}_{/Y} \rightleftarrows \mathrm{Set}_{/X} \colon f^*$$

where the left adjoint $f_!$ sends a set $Z \to Y$ to the composition $Z \to Y \to X$ and the right adjoint f^* sends $W \to X$ to $W \times_X Y \to Y$. The resulting comonad T on $\mathrm{Set}_{/X}$ is given by $T(W \to X) \cong (W \times_X Y \to X)$. In particular, applying the construction above in the special case of the comonad T and the final object $X \to X$, we obtain the usual **Čech complex** of the morphism f.

Remark 1.24. Similarly, given a monad, one obtains cosimplicial objects.

Example 1.25. Let G be a group. We construct a simplicial set $\mathrm{B}_\bullet G$ as follows. Set $\mathrm{B}_n G = G^{\times n}$. The face maps

$$G^{\times n} \to G^{\times(n-1)}$$

are given by

$$\partial_i(g_1, \dots g_n) = \begin{cases} (g_2, \dots, g_n) & \text{if } i = 0, \\ (g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_n) & \text{if } 1 \leqslant i \leqslant n-1, \\ (g_1, \dots, g_{n-1}) & \text{if } i = n \end{cases}$$

and the degeneracy maps $G^{\times n} \to G^{\times(n+1)}$ are given by

$$\sigma_i(g_1, \dots, g_n) = (g_1, \dots, g_i, 1, g_{i+1}, \dots, g_n).$$

Exercise 1.26. Check that $\mathrm{B}_\bullet G$ is indeed a simplicial object.

1.2 Simplicial commutative rings

References 1.27. Quillen [52, 50, 51], Illusie [29], and Lurie [37]. For some reason, Quillen never published [50], but it is summarized in [51].

Definition 1.28. Let k be a commutative ring. We let sCAlg_k denote the 1-category of simplicial commutative k-algebras, i.e., $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{CAlg}_k)$.

There are adjunctions

$$\mathrm{sSet} \rightleftarrows \mathrm{sMod}_k \rightleftarrows \mathrm{sCAlg}_k.$$

We let $kX_\bullet$ denote the free simplicial k-module on a simplicial set $X_\bullet$. Similarly, let $k[X_\bullet]$ be the free simplicial commutative k-algebra on a simplicial set $X_\bullet$. If $M_\bullet$ is a simplicial k-module, we let $\mathrm{Sym}(M_\bullet)$ denote the free simplicial k-algebra on $M_\bullet$. These are the left adjoints in the adjunctions above.

We will use these adjunctions as well as the Quillen model category structure on sSet to build a homotopy theory for sCAlg_k.

Definition 1.29 (Weak homotopy equivalences)**.** A continuous map $f\colon X \to Y$ of topological spaces is a **weak homotopy equivalence** if it induces a bijection $\pi_0(f)\colon \pi_0(X) \cong \pi_0(Y)$ on path-connected components and if for each $x \in X$ the induced maps $\pi_i(X, x) \to \pi_i(Y, f(x))$ are isomorphisms for all integers $i \geqslant 1$.

Definition 1.30 (Geometric realizations)**.** Consider the cosimplicial topological space $\Delta^\bullet_{\mathrm{top}}\colon \Delta \to \mathrm{Top}$. Since Top has all colimits, we can construct a left Kan extension

$$\begin{tikzcd} \Delta \arrow[r, "\Delta^\bullet_{\mathrm{top}}"] \arrow[d, "j"'] & \mathrm{Top} \\ \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set}). \arrow[ur, "|-|"'] & \end{tikzcd}$$

Here, $j\colon \Delta \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$ is the (fully faithful) Yoneda embedding with $j([n]) = \mathrm{Hom}_\Delta(-, [n])$. We let $\Delta^n = j([n])$. Note that $[n] \mapsto \Delta^n$ defines a cosimplicial simplicial set. The functor $|-|\colon \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Top}$ is called the **geometric realization** of simplicial sets. Given a simplicial set $X_\bullet$, the geometric realization may be computed as

$$|X_\bullet| \cong \operatorname*{colim}_{\Delta^n \to X_\bullet} \Delta^n_{\mathrm{top}},$$

where the colimit is over the simplex category $\Delta_{/X_\bullet}$ of $X_\bullet$.

Remark 1.31. Note that by the Yoneda lemma we have $\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, X_\bullet) \cong X_n$.

Exercise 1.32. Show that $|\Delta^n| \cong \Delta^n_{\mathrm{top}}$.

Remark 1.33. Let G be a group. The geometric realization $|\mathrm{B}_\bullet G|$ of the simplicial set $\mathrm{B}_\bullet G$ is a $K(G,1)$-space; that is, it is a path connected space (so $\pi_0|\mathrm{B}_\bullet G| \simeq *$) with $\pi_1|\mathrm{B}_\bullet G| \cong G$ and $\pi_i|\mathrm{B}_\bullet G| \simeq 0$ for $i > 1$. This space is often denoted by $\mathrm{B}G$. It is the **classifying space** of G and classifies locally trivial principal G-bundles.

Definition 1.34 (Weak equivalences)**.** A map $f\colon X_\bullet \to Y_\bullet$ of simplicial sets is a **weak equivalence** if $|f|\colon |X_\bullet| \to |Y_\bullet|$ is a weak homotopy equivalence of topological spaces.

Construction 1.35 (Model category structure on sSet)**.** There is a model category structure on sSet with the following classes of weak equivalences, cofibrations, and fibrations:

- *weak equivalences:* the weak equivalences as defined above;
- *cofibrations:* pointwise monomorphisms, i.e., maps $f\colon X_\bullet \to Y_\bullet$ such that $X_n \to Y_n$ is an injection for each n;
- *fibrations:* the **Kan fibrations**; i.e., the maps $f\colon X_\bullet \to Y_\bullet$ with the right lifting property with respect to $\Lambda^n_i \subseteq \Delta^n$ for all n, i, where Λ^n_i is the union of the $\partial_j(\Delta^{n-1}) \subseteq \Delta^n$ for $j \neq i$.

This is the Quillen model category structure on sSet

Remark 1.36. The 1-categorical localization of sSet at the weak equivalences agrees with the ordinary homotopy category of CW complexes and weak homotopy equivalences. The ∞-categorical localization $\mathrm{sSet}[W^{-1}]$ of sSet at the weak equivalences is called the **∞-category of spaces** or the **∞-category of animae**. The homotopy category of $\mathrm{sSet}[W^{-1}]$ is the 1-categorical localization.

Exercise 1.37. Show that if X is a topological space, then $\mathrm{Sing}_\bullet(X)$ is a **Kan complex**: the map $\mathrm{Sing}_\bullet(X) \to *$ is a Kan fibration.

Construction 1.38 (Model category structure on sMod_k)**.** There is a model category structure on sMod_k where the weak equivalences and fibrations are the maps where the underlying map of simplicial sets is a weak equivalence or fibration, respectively. The cofibrations are then determined. Note however that if $X_\bullet \to Y_\bullet$ is a cofibration in sSet, then $kX_\bullet \to kY_\bullet$ is a cofibration in sMod_k.

Exercise 1.39. Show that if $X_\bullet \to Y_\bullet$ is a pointwise surjective map of simplicial k-modules, then it is a fibration. Show that if this is the case, then the induced map $N(X_\bullet) \to N(Y_\bullet)$ of normalized chain complexes is surjective in each degree. Moreover, show that $X_\bullet \to Y_\bullet$ is a fibration if and only if $N(X_\bullet)_n \to N(Y_\bullet)_n$ is a surjection for each integer $n \geqslant 1$. See [21, Lem. 2.11].

Construction 1.40 (Model category structure on sCAlg_k)**.** As for the case of sMod_k, there is a model category structure on sCAlg_k where the weak equivalences and fibrations are the maps where the underlying map of simplicial sets (or simplicial k-modules) is a weak equivalence or fibration. The cofibrations are determined. An important case is if $X_\bullet \to Y_\bullet$ is a cofibration in sSet, then $k[X_\bullet] \to k[Y_\bullet]$ is a cofibration in sCAlg_k.

Remark 1.41. It is possible, using the Dold–Kan correspondence, to prove that if $X_\bullet$ is a simplicial k-module, then

$$\pi_i(|X_\bullet|) \cong \mathrm{H}_i(N(X_\bullet)) \cong \mathrm{H}_i(C(X_\bullet)),$$

where $N(-)$ and $C(-)$ denote the normalized and unnormalized chain complexes (see for example [64, Sec. 8.3-4]). It follows that the notion of weak equivalence on sMod_k agrees via the Dold–Kan correspondence to quasi-isomorphism on $\mathrm{Ch}(\mathrm{Mod}_k)_{\geqslant 0}$. It follows that if $f\colon R_\bullet \to S_\bullet$ is a map of simplicial commutative k-algebras, then f is a quasi-isomorphism if and only if the map of chain complexes $C(f)\colon C(R_\bullet) \to C(S_\bullet)$ is a quasi-isomorphism.

Remark 1.42. Monoidality of the Dold–Kan correspondence is always a subtle question, especially for commutative algebra objects, which requires the use of the $\mathbb{E}_\infty$-operad to make precise. However, the normalized chain complex functor $N\colon \mathrm{sMod}_k \to \mathrm{Ch}(\mathrm{Mod}_k)_{\geqslant 0}$ is lax symmetric monoidal by the Eilenberg–Zilber map [64, Sec. 8.5] (also called the Eilenberg–Mac Lane map or shuffle map), which implies it induces functor $N\colon \mathrm{Alg}(\mathrm{sMod}_k) \to \mathrm{Alg}(\mathrm{Ch}(\mathrm{Mod}_k)_{\geqslant 0})$. But, it is not hard to see that $\mathrm{Alg}(\mathrm{sMod}_k) \simeq \mathrm{sAlg}_k$ while $\mathrm{Alg}(\mathrm{Ch}(\mathrm{Mod}_k)_{\geqslant 0})$ is the category of differential graded k-algebras $A_\bullet$ with $A_i = 0$ for $i < 0$. Thus, we obtain a functor

$$N\colon \mathrm{sAlg}_k \to \mathrm{dga}_{k,\geqslant 0}.$$

Definition 1.43. Let $R_\bullet$ be a simplicial commutative k-algebra. The homotopy groups $\pi_i R_\bullet$ of $R_\bullet$ are defined to be the homotopy groups $\pi_i|R_\bullet|$ of the geometric realization of the underlying simplicial set or equivalently (by Remark 1.41) the homology groups $\mathrm{H}_i(N(R_\bullet))$ of the underlying simplicial k-module. In particular, they are k-modules.

Exercise 1.44. Let $R_\bullet$ be a simplicial commutative ring. Show that $\mathrm{im}(\partial_0 - \partial_1\colon R_1 \to R_0)$ is an ideal I of R_0 and that $\pi_0 R_\bullet \cong R_0/I$.

The shuffle product of Remark 1.42 implies that $N(R_\bullet)$ is naturally a differential graded k-algebra. It follows that $\pi_* R_\bullet$ forms a graded associative k-algebra, meaning that there are k-linear products $\pi_i R_\bullet \times \pi_j R_\bullet \to \pi_{i+j} R_\bullet$ and a unit in $\pi_0 R$ satisfying all the usual unit and associativity axioms. However, something better is true, as a result of the commutative part of simplicial commutative. So far, the graded associativity would hold equally well for a simplicial k-algebra.

Lemma 1.45. *If $R_\bullet$ is a simplicial commutative ring, then the graded ring $\pi_* R_\bullet$ is graded commutative, meaning that $xy = (-1)^{ij} yx$ for $x \in \pi_i R_\bullet$ and $y \in \pi_j R_\bullet$.*

Proof. The reader will have to believe me that we can define the product on $\pi_* R_\bullet$ in another way. Specifically, the multiplication map $R_\bullet \times R_\bullet \to R_\bullet$ (which is a map of simplicial sets) gives a map

$$|R_\bullet \times R_\bullet| \to |R_\bullet|.$$

Now, $|R_\bullet \times R_\bullet| \simeq |R_\bullet| \times |R_\bullet|$.[2] Moreover, $|R_\bullet|$ is pointed by $0 \in |R_\bullet|$ via the 0-simplex $\Delta^0 \to R_\bullet$ corresponding to $0 \in R_0$. The multiplication map factors through $|R_\bullet| \wedge |R_\bullet| \to |R_\bullet|$ since multiplying by 0 is 0. Now, given $S^i \xrightarrow{x} |R_\bullet|$ and $S^j \xrightarrow{y} |R_\bullet|$, the product xy is classified by

$$S^i \wedge S^j \xrightarrow{x \wedge y} |R_\bullet| \wedge |R_\bullet| \to |R_\bullet|.$$

The opposite multiplication is given by the composition

$$S^j \wedge S^i \xrightarrow{\text{switch}} S^i \wedge S^j \xrightarrow{x \wedge y} |R_\bullet| \wedge |R_\bullet| \to |R_\bullet|.$$

Now, it is a standard check that the degree of the switch map $S^j \wedge S^i \to S^i \wedge S^j$ is $(-1)^{ij}$ (see for example [27, Sec. 2.2]). □

[2] In fact, if Top is replaced by a *convenient category of topological spaces*, then geometric realization preserves finite products. In general, one has to worry about the topology on the product space. See `https://ncatlab.org/nlab/show/geometric+realization#GeometricRealizationIsLeftExact`. In any case, geometric realization does preserve finite products up to weak homotopy equivalence.

Remark 1.46. In fact, $\pi_* R_\bullet$ has even more structure when $R_\bullet$ is simplicial commutative. The even homotopy ring $\pi_{2*} R_\bullet$, which is commutative, admits the structure of a graded divided power algebra, meaning that there are operations $\lambda_n \colon \pi_{2i} R_\bullet \to \pi_{2in} R_\bullet$ generalizing the operations $x \mapsto \frac{x^n}{n!}$ and satisfying a variety of related axioms. The usual proof of this uses explicit meddling with the shuffle map.

Challenge 1.47. Find a purely monadic construction of the divided power structure on $\pi_{2*} R_\bullet$. This is clearest when $R_\bullet$ is the free simplicial commutative ring on $M[2]$ where M is a flat $\mathbb{Z}$-module and $M[2]$ denotes M as a chain complex sitting in degree 2. We implicitly apply the Dold–Kan correspondence to view $M[2]$ as a simplicial abelian group and then apply the free commutative ring functor degree-wise to obtain $\mathrm{LSym}_{\mathbb{Z}}(M[2])$. A classical décalage argument, due to Illusie perhaps, which we will cover in Week 3, implies that $\pi_{2*} \mathrm{LSym}_{\mathbb{Z}}(M[2])$ has a natural divided power structure. But, I am not sure how this works in general.

Remark 1.48. The adjunctions

$$\mathrm{sSet} \rightleftarrows \mathrm{sMod}_k \rightleftarrows \mathrm{sCAlg}_k$$

form Quillen adjunctions. Indeed, the right adjoints preserve weak equivalences and fibrations and hence in particular acyclic fibrations. This implies that the left adjoints preserve cofibrations and acyclic cofibrations and also weak equivalences between cofibration objects. In particular, one obtains adjunctions on the ∞-categorical localizations.

Exercise 1.49. Let $R, S \in \mathrm{CAlg}_k$ be two commutative k-algebras. Prove that the pushout $R \coprod S$ in CAlg_k is computed by the tensor product $R \otimes_k S$.

Exercise 1.50. Use Exercise 1.49 to prove that if $R_\bullet$ and $S_\bullet$ are simplicial commutative k-algebras, then the pushout $R_\bullet \coprod S_\bullet$ in sCAlg_k is $R_\bullet \otimes_k S_\bullet$, the pointwise tensor product.

Remark 1.51 (Variant over a simplicial commutative ring)**.** Let $R_\bullet \in \mathrm{sCAlg}_k$ be a simplicial commutative k-algebra. Define $\mathrm{sCAlg}_{R_\bullet}$ to be the comma category $(\mathrm{sCAlg}_k)_{R_\bullet/}$ of simplicial commutative k-algebras under $R_\bullet$.

Exercise 1.52. Repeat Exercise 1.50 for $\mathrm{sCAlg}_{R_\bullet}$. Namely, prove that if $S_\bullet$ and $T_\bullet$ are simplicial commutative $R_\bullet$-algebras, then $S_\bullet \otimes_{R_\bullet} T_\bullet$ computes the pushout of $S_\bullet$ and $T_\bullet$ in the 1-category $\mathrm{sCAlg}_{R_\bullet}$.

The proof of the next proposition gives a taste of a common device used later, especially in the study of finiteness results for the cotangent complex (and de Rham cohomology and Hochschild homology). We can give most of the proof now, but one point has to remain a bit of a black box. We will cover the needed connectivity growth at the end of the proof using décalage in Week 3. But, if you are impatient, see [37, Prop. 3.13] or [18, Satz 12.1].

Proposition 1.53. *Let k be a noetherian commutative ring and let R be a finitely presented commutative k-algebra. There is a cofibrant simplicial commutative k-algebra $S_\bullet$ and a weak equivalence $S_\bullet \to R$ such that each S_n is a polynomial ring over k in finitely many generators over k. Here, we view R as a constant simplicial commutative ring.*[3]

Proof. Let $k[x_1, \ldots, x_n] \to R$ be a surjective map presenting R. Let $S_\bullet^{(0)}$ be the constant simplicial commutative k-algebra on $k[x_1, \ldots, x_n]$. We have a map $S_\bullet^{(0)} \to R$ which is surjective in homotopy, but is typically not a quasi-isomorphism. Suppose that we have constructed cofibrations $S_\bullet^{(0)} \hookrightarrow S_\bullet^{(1)} \hookrightarrow \cdots \hookrightarrow S_\bullet^{(n)} \to R$ such that

(a) each $S_\bullet^{(i)}$ is free and finitely generated over k in each degree,

[3] I found this presentation in Akhil Mathew's helpful notes available at http://math.uchicago.edu/~amathew/SCR.pdf.

(b) $S_j^{(i)} \to S_j^{(i+1)}$ is an isomorphism for $0 \leqslant j < i$ and all $i \geqslant 0$,

(c) each $\pi_j S_\bullet^{(i)}$ is a finitely presented $\pi_0 S_\bullet^{(i)}$-module, and

(d) $S_\bullet^{(i)} \to R$ is an i-equivalence, meaning that $\pi_j S_\bullet^{(i)} \to \pi_j R$ is an isomorphism for $j < i$ and a surjection in degree i.

We will construct $S_\bullet^{(n)} \hookrightarrow S_\bullet^{(n+1)} \to R$. Then, $S_\bullet \simeq \operatorname{colim}_n S_\bullet^{(n)}$ works.

It follows from Exercise 1.44 that $\pi_0 S_\bullet^{(n)}$ is noetherian since it is a quotient of $S_0^{(n)}$ which is a finitely generated polynomial ring over k. Since $\pi_n S_\bullet^{(n)}$ is a finitely presented $\pi_0 S_\bullet^{(n)}$-module, it follows that $\ker\left(\pi_n S_\bullet^{(n)} \to \pi_n R\right)$ is also a finitely presented $\pi_0 S_\bullet^{(n)}$-module. Fix generators, $x_1, \ldots, x_r$ of the kernel. Since $\partial\Delta^{n+1}$ is cofibrant (as is every simplicial set) and since $S_\bullet^{(n)}$ is fibrant as a simplicial set (as is every simplicial abelian group by Exercise 1.39), the x_j are classified by maps $\partial\Delta^{n+1} \to S_\bullet^{(n)}$ where we view $S_\bullet^{(n)}$ as a simplicial set.[4] By adjunction, we get maps of simplicial commutative k-algebras

$$k[\partial\Delta^{n+1}] \xrightarrow{x_j} S_\bullet^{(n)}.$$

We also have inclusion $k[\partial\Delta^{n+1}] \to k[\Delta^{n+1}]$.

Since x_i is in the kernel of $\pi_n S_\bullet^{(n)} \to \pi_n R$, we have a commutative diagram

$$\begin{tikzcd} \partial\Delta^{n+1} \arrow[r] \arrow[d] & S_\bullet^{(n)} \arrow[d] \\ \Delta^{n+1} \arrow[r] & R. \end{tikzcd}$$

It follows that we get an induced map

$$S_\bullet^{(n)} \otimes_{\bigotimes_{j=1}^r k[\partial\Delta^{n+1}]} \bigotimes_{j=1}^r k[\Delta^{n+1}] \to R.$$

We let $S_\bullet^{(n+1)}$ denote the tensor product on the left. The tensor product results in 'killing' the x_i. Note however that $k[\partial\Delta^{n+1}]_j \to k[\Delta^{n+1}]_j$ is a bijection for $j \leqslant n$. This implies that the map $\pi_n S_\bullet^{(n)} \to \pi_n S_\bullet^{(n+1)}$ is a surjection, so we do not get any new homotopy classes in low degrees.

This gives the inductive step; in particular parts (a), (b), and (d) follow from the construction. The only remaining subtlety is to see that $\pi_j S_\bullet^{(n+1)}$ is is finitely presented over $\pi_0 S_\bullet^{(n+1)}$. This comes down to proving that $\pi_j k[\partial\Delta^{n+1}]$ is a finitely presented $\pi_0 k[\partial\Delta^{n+1}]$-module for $j \geqslant 0$. If $n = 0$, this is trivial. If $n > 0$, then $\pi_0 k[\partial\Delta^{n+1}] \cong k$. The only way I really know how to do this is by using the LSym functors we will encounter later.

To be specific, note that our Quillen adjunctions $\mathrm{sSet} \leftrightarrows \mathrm{sMod}_k \leftrightarrows \mathrm{sCAlg}_k$ imply that $k[\partial\Delta^{n+1}]$ is equivalent to $\mathrm{Sym}_k K(k[n])$, where K is the inverse in the Dold–Kan correspondence and where $k[n]$ denotes the simplicial k-module associated to the chain complex $k[n]$ consisting of k sitting in (homological) degree n and 0 elsewhere.

Note that $\mathrm{Sym}_k K(k[n]) \cong \bigoplus \mathrm{Sym}_k^r K(k[n])$. I claim that $\mathrm{Sym}_k^r K(k[n])$ is $(n + 2r - 2)$-connective:

$$\pi_j \mathrm{Sym}_k^r K(k[n]) = 0$$

[4] When $n = 0$, send one of the boundary components of $\partial\Delta^1$ to 0.

for $j < n + 2r - 2$. Now, the homotopy of $\mathrm{Sym}^r_k K(k[n])$ is computed by the homology of the normalized (or unnormalized) chain complex $N(\mathrm{Sym}^r_k K(k[n]))$, which is by inspection a finitely generated k-module in each degree. Since k is noetherian, the homotopy of $\mathrm{Sym}^r_k K(k[n])$ is finitely presented in each degree. Since the connectivity of $\mathrm{Sym}^r_k K(k[n])$ increases as $r \to \infty$, it follows that each $\pi_j \mathrm{Sym}_k K(k[n])$ is finitely presented, as desired.

The proof of the claim relies on décalage and will be given in Week 3. □

Remark 1.54. The simplicial commutative k-algebra $S_\bullet$ is a particularly nice cofibrant replacement of R.

2 Week 2: Animated commutative rings (23–29 Sept.)

We describe in this section a process of animation which constructs ∞-categories from certain 1-categories. The main examples for us are the following

- $\mathrm{Set} \rightsquigarrow \mathfrak{a}\mathrm{Set} \simeq \mathcal{S}$,
- $\mathcal{A}\mathrm{b} \rightsquigarrow \mathfrak{a}\mathcal{A}\mathrm{b} \simeq \mathcal{D}(\mathbb{Z})_{\geqslant 0}$,
- $\mathrm{Alg} \rightsquigarrow \mathfrak{a}\mathrm{Alg} \simeq \mathrm{dga}[\mathrm{qiso}^{-1}]$,
- $\mathrm{CAlg} \rightsquigarrow \mathfrak{a}\mathrm{CAlg}$

where $\rightsquigarrow$ denotes the process of animation and where $\mathcal{S}$ is the ∞-category of spaces, which we will call the ∞-category of animae. The ∞-category $\mathfrak{a}\mathrm{CAlg}$ is our ∞-category of **animated commutative rings**.

Anima is a feminine noun in Latin, meaning **air, breeze, breath, soul, life**. The plural form is animae.

The basic idea here is that one is very rarely interested in a simplicial set $X_\bullet$ viewed as an object of the 1-category sSet. Rather, it is the image of $X_\bullet$ in $\mathfrak{a}\mathrm{Set} \simeq \mathcal{S}$, which is the 'homotopy type' of $X_\bullet$ and reveals its true nature. I am reminded of a phrase from the song 'Love, Love, Love' by the Mountain Goats:[5]

> Now we see things
> As in a mirror dimly
> Then we shall see each other
> Face to face

The process of animation serves not only an introspective role, but an extrospective one as well: revealing the true nature of the ∞-categories of animae, animated commutative rings, etc by identifying the universal properties of these objects among all ∞-categories.

2.1 Cocompletion of 1-categories

References 2.1. Adámek–Rosický–Vitale [1], Lurie [38, Sec. 5.5.8-9], and Česnavičius–Scholze [15, Sec. 5.1].

To make this process precise, we define three notions of free cocompletion, answering the question of how, given a small category $\mathcal{C}$, we can freely adjoin colimits to $\mathcal{C}$.

[5] Also, 1 Corinthians 13:12 (King James Version): For now we see through a glass, darkly; but then face to face: now I know in part; but then shall I know even as also I am known.

Definition 2.2. Let $\mathcal{C}$ be a small category. The functor category $\mathcal{P}(\mathcal{C})^{\heartsuit} \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ is the category of **presheaves of sets** on $\mathcal{C}$.[6] There is a Yoneda embedding $j\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ which sends an object $Y \in \mathcal{C}$ to the representable functor $\mathrm{Hom}_{\mathcal{C}}(-, Y)$.

Lemma 2.3. *Let $\mathcal{C}$ be a small category.*

(a) *The presheaf category $\mathcal{P}(\mathcal{C})^{\heartsuit}$ has all small colimits.*

(b) *For any cocomplete category $\mathcal{D}$, restriction along Yoneda induces an equivalence $\mathrm{Fun}'(\mathcal{P}(\mathcal{C})^{\heartsuit}, \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, where $\mathrm{Fun}'(\mathcal{P}(\mathcal{C})^{\heartsuit}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{P}(\mathcal{C})^{\heartsuit}, \mathcal{D})$ is the full subcategory of colimit preserving functors.*

Proof. Part (a) follows from the 1-categorical version of [38, Cor. 5.1.2.4] and part (b) follows from the 1-categorical version of [38, Thm. 5.1.5.6]. These 1-categorical versions can themselves be deduced from the ∞-categorical versions by using [38, Cor. 5.5.6.22]. □

Remark 2.4. In Lemma 2.3(b), if $\mathcal{D}$ is presentable, then $\mathrm{Fun}'(\mathcal{P}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{P}(\mathcal{C}), \mathcal{D})$ by the adjoint functor theorem, where $\mathrm{Fun}^{\mathrm{L}}(\mathcal{P}(\mathcal{C}), \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{P}(\mathcal{C}), \mathcal{D})$ is the full subcategory consisting of those functors which admit a right adjoint (i.e., the full subcategory of left adjoint functors).

Remark 2.5. We see from Lemma 2.3 that $\mathcal{P}(\mathcal{C})^{\heartsuit}$ deserves to be called the **free 1-categorical cocompletion** of $\mathcal{C}$: it has all colimits by (a) and (b) implies that there are no unnecessary relations between these colimits. Given $F\colon \mathcal{C} \to \mathcal{D}$, the associated functor $\mathcal{P}(\mathcal{C})^{\heartsuit} \to \mathcal{D}$ is called the **left Kan extension** of F along Yoneda. Because the Yoneda functor $j\colon \mathcal{C} \to \mathcal{P}(\mathcal{C})^{\heartsuit}$ is fully faithful, one obtains a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "j"'] & \mathcal{D} \\ \mathcal{P}(\mathcal{C})^{\heartsuit} \arrow[ur, "\mathrm{Lan}_j F"'] & \end{tikzcd}$$

of functors. To compute the value of the left Kan extension $\mathrm{Lan}_j F$ at a presheaf $\mathcal{G}$, we write $\mathcal{G}$ as a colimit in $\mathcal{P}(\mathcal{C})$ of representables, $\mathrm{colim}_{j(c)\to\mathcal{G}}\, j(c)$ and we define $\mathrm{Lan}_j F(\mathcal{G})$ as $\mathrm{colim}_{j(c)\to\mathcal{G}}\, F(c)$.

Exercise 2.6. Show that the Yoneda functor $j\colon \mathcal{C} \to \mathcal{P}(\mathcal{C})^{\heartsuit}$ preserves any limits that exist in $\mathcal{C}$.

Exercise 2.7. Show that $j\colon \mathcal{C} \to \mathcal{P}(\mathcal{C})^{\heartsuit}$ typically does not preserve colimits. For example, consider the functor $\mathrm{Set}^{\omega} \to \mathcal{P}(\mathrm{Set}^{\omega})^{\heartsuit}$, where $\mathrm{Set}^{\omega} \subseteq \mathrm{Set}$ is the full subcategory of finite sets.

If one wants to adjoin colimits in a way so that Yoneda also preserves certain colimits, there are two additional constructions.

Definition 2.8. Let $\mathcal{C}$ be a category.

(i) We say that $\mathcal{C}$ is **1-sifted** if $\mathcal{C}$-indexed colimits in Set commute with finite products.

(ii) We say that $\mathcal{C}$ is **filtered** if $\mathcal{C}$-indexed colimits in Set commute with finite limits.

[6] We will shortly define $\mathcal{P}(\mathcal{C})$, which will be an ∞-category even if $\mathcal{C}$ is a 1-category. Often, we will use the notation $(-)^{\heartsuit}$ to denote the 1-categorical version of an ∞-categorical construction; this conflicts with the standard notation used for t-structures, but we will not study t-structures much in this course. To be precise, we adopt the following convention. Unless otherwise stated, for an ∞-category $\mathcal{D}$, we let $\mathcal{D}^{\heartsuit}$ denote the full subcategory of 0-truncated objects, where an object $Y \in \mathcal{D}$ is 0-truncated if $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is 0-truncated (i.e., equivalent to a set) for all $X \in \mathcal{D}$. In the case of a 1-category $\mathcal{C}$, $\mathcal{P}(\mathcal{C})^{\heartsuit}$ is equivalent as a 1-category to $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$. We will see other examples. **Beware:** $\mathcal{D}^{\heartsuit}$ is typically not the homotopy category of $\mathcal{D}$. For example, $\mathcal{S}^{\heartsuit} \simeq \mathrm{Set}$ while the homotopy category is equivalent to the ordinary weak homotopy category of topological spaces.

Remark 2.9. A 1-category $\mathcal{C}$ is 1-sifted if and only if it is non-empty and the diagonal functor $\mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is cofinal. Recall that a functor $F \colon \mathcal{C} \to \mathcal{D}$ is cofinal if for every $d \in \mathcal{D}$ the comma category d/F consisting of objects $c \in \mathcal{C}$ with a map $d \to F(c)$ is non-empty and connected. Equivalently, one demands that for every functor $X \colon \mathcal{D} \to \mathrm{Set}$, the induced map $\operatorname{colim} X \circ F \to \operatorname{colim} X$ is an isomorphism.

A 1-category $\mathcal{C}$ is filtered if and only if it satisfies the following conditions: (1) for every finite collection of objects $\{X_i\}_{i \in I}$ of $\mathcal{C}$ there is an object $X \in \mathcal{C}$ and morphisms $X_i \to X$ for each $i \in I$ and (2) for every pair of morphisms $f, g \colon X \to Y$ there is a morphism $h \colon Y \to Z$ that coequalizes f and g in the sense that $h \circ f = h \circ g$. This notion should be familiar from filtered or directed colimits.

Exercise 2.10. Let P be a poset. Find necessary and sufficient conditions for P to be filtered. When these hold, we say that P is a **filtered poset**. Other names for this concept are directed set, directed preorder, or filtered set.

Remark 2.11. For every filtered category $\mathcal{C}$ there is a filtered poset P and a cofinal map $P \to \mathcal{C}$. See [59, Tag 0032] or [38, Prop. 5.3.16]. This means that filtered colimits are no more general than colimits over filtered posets.

Exercise 2.12. Let P be a poset. Find necessary and sufficient conditions for P to be 1-sifted.

Exercise 2.13. Prove that every filtered category is 1-sifted.

Exercise 2.14. Prove that every category with finite coproducts is 1-sifted.

Exercise 2.15. Prove that the **coequalizer category** $\{[1] \rightrightarrows [0]\}$ is *not* 1-sifted.

Exercise 2.16. Prove that the **reflexive coequalizer category** $\Delta^{\mathrm{op}}_{\leqslant 1} \simeq \{[1] \overset{\rightrightarrows}{\leftarrow} [0]\}$ is 1-sifted. Here, there are two maps $\partial_0, \partial_1 \colon [1] \to [0]$ and one map $\sigma_0 \colon [0] \to [1]$ with the composition $\partial_i \circ \sigma_0 = \mathrm{id}_{[0]}$ for $i = 0, 1$. In particular, note that $[0]$ is a retract of $[1]$.

Exercise 2.17. Prove that the reflexive coequalizer category $\Delta^{\mathrm{op}}_{\leqslant 1}$ of Exercise 2.16 is not filtered.

Definition 2.18. Let $\mathcal{C}$ be a small category.

(i) If $\mathcal{C}$ has finite coproducts, then we let $\mathcal{P}_\Sigma(\mathcal{C})^\heartsuit \subseteq \mathcal{P}(\mathcal{C})^\heartsuit$ be the full subcategory of those functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ that preserve finite products. We will call $\mathcal{P}_\Sigma(\mathcal{C})$ the **1-sifted completion** of $\mathcal{C}$.[7]

(ii) If $\mathcal{C}$ has finite colimits, then we let $\mathrm{Ind}(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C})^\heartsuit$ be the full subcategory of those functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ that preserve finite limits. The 1-category $\mathrm{Ind}(\mathcal{C})$ is called the **ind-completion** of $\mathcal{C}$.

Remark 2.19. Note that if $\mathcal{C}$ has finite colimits, then $\mathrm{Ind}(\mathcal{C}) \subseteq \mathcal{P}_\Sigma(\mathcal{C})^\heartsuit$.

Both $\mathcal{P}_\Sigma(\mathcal{C})^\heartsuit$ and $\mathrm{Ind}(\mathcal{C})$ admit universal properties and they can be defined as freely adjoining either 1-sifted or filtered colimits to $\mathcal{C}$, depending on the context.

Proposition 2.20. *Let $\mathcal{C}$ be a small 1-category.*

(i) *Suppose that $\mathcal{C}$ has finite coproducts.*

  (a) *The inclusion $\mathcal{P}_\Sigma(\mathcal{C})^\heartsuit \subseteq \mathcal{P}(\mathcal{C})^\heartsuit$ preserves sifted colimits and admits a left adjoint (and hence $\mathcal{P}_\Sigma(\mathcal{C})^\heartsuit$ is presentable).*

[7]It seems like this should be called the 1-sifted cocompletion, but no one calls $\mathrm{Ind}(\mathcal{C})$ the ind-*co*completion. The `nLab` page for ind-objects claims that the terminology arises from inductive colimits. I do not know of an analogous term for objects which are 1-sifted colimits.

(b) *The Yoneda functor* $j\colon \mathcal{C} \to \mathcal{P}(\mathcal{C})$ *lands in* $\mathcal{P}_\Sigma(\mathcal{C})^\heartsuit$.

(c) *An object* $X \in \mathcal{P}(\mathcal{C})^\heartsuit$ *is in* $\mathcal{P}_\Sigma(\mathcal{C})^\heartsuit$ *if and only if it can be written as the colimit of a reflexive coequalizer* $X_1 \overset{\rightarrow}{\underset{\rightarrow}{\leftarrow}} X_0$ *where* X_1 *and* X_0 *are filtered colimits of objects in* $\mathcal{C} \subseteq \mathcal{P}(\mathcal{C})^\heartsuit$.

(d) *If* $\mathcal{D}$ *is a* 1*-category with* 1*-sifted colimits, then restriction along Yoneda induces an equivalence*

$$\mathrm{Fun}''(\mathcal{P}_\Sigma(\mathcal{C})^\heartsuit, \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D}),$$

where $\mathrm{Fun}''(\mathcal{P}_\Sigma(\mathcal{C})^\heartsuit, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{P}_\Sigma(\mathcal{C})^\heartsuit, \mathcal{D})$ *is the full subcategory of functors which preserve* 1*-sifted colimits.*

(e) *If* $\mathcal{D}$ *is a* 1*-category with all colimits, then restriction along Yoneda induces an equivalence*

$$\mathrm{Fun}'(\mathcal{P}_\Sigma(\mathcal{C})^\heartsuit, \mathcal{D}) \simeq \mathrm{Fun}^\sqcup(\mathcal{C}, \mathcal{D}),$$

where $\mathrm{Fun}^\sqcup(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ *is the full subcategory of functors which preserve finite coproducts.*

(ii) *Suppose that* $\mathcal{C}$ *has finite colimits.*

(a) *The inclusion* $\mathrm{Ind}(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C})^\heartsuit$ *preserves filtered colimits and admits a left adjoint (and hence* $\mathrm{Ind}(\mathcal{C})$ *is presentable).*

(b) *The Yoneda functor* $j\colon \mathcal{C} \to \mathcal{P}(\mathcal{C})^\heartsuit$ *lands in* $\mathrm{Ind}(\mathcal{C})$.

(c) *An object* $X \in \mathcal{P}(\mathcal{C})^\heartsuit$ *is in* $\mathrm{Ind}(\mathcal{C})$ *if and only if it can be written as the filtered colimit objects in* $\mathcal{C} \subseteq \mathcal{P}(\mathcal{C})^\heartsuit$.

(d) *If* $\mathcal{D}$ *is a* 1*-category with filtered colimits, then restriction along Yoneda induces an equivalence*

$$\mathrm{Fun}'''(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D}),$$

where $\mathrm{Fun}'''(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) \subseteq \mathrm{Fun}(\mathrm{Ind}(\mathcal{C}), \mathcal{D})$ *is the full subcategory of functors which preserve filtered colimits.*

(e) *If* $\mathcal{D}$ *is a* 1*-category with all colimits, then restriction along Yoneda induces an equivalence*

$$\mathrm{Fun}'(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{rex}}(\mathcal{C}, \mathcal{D}),$$

where $\mathrm{Fun}^{\mathrm{rex}}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ *is the full subcategory of functors which preserve finite colimits.*

Proof. This is extracted from [38, Chap. 5], especially [38, Sec. 5.5.8] in the 1-sifted case and [38, Sec. 5.3.5] in the filtered case. □

Remark 2.21. The equivalences of parts (d) and (e) in Proposition 2.20 give rise to new Kan extensions. In particular, if $\mathcal{C}$ has finite coproducts and $\mathcal{D}$ has 1-sifted colimits, we will say that $F\colon \mathcal{P}_\Sigma(\mathcal{C})^\heartsuit \to \mathcal{D}$ is **left Kan extended** from a functor $F'\colon \mathcal{C} \to \mathcal{D}$ if F and F' correspond via the equivalence in part (i.d). Note that *every* functor $\mathcal{P}_\Sigma(\mathcal{C})^\heartsuit \to \mathcal{D}$ which preserves 1-sifted colimits is left Kan extended from a unique F'. That is the crucial universal property of these constructions, and there is a similar story for the ind-completion.

Exercise 2.22. Prove that if $\mathcal{C}$ is a 1-category with finite coproducts, the Yoneda embedding $j\colon \mathcal{C} \to \mathcal{P}_\Sigma(\mathcal{C})^\heartsuit$ preserves finite coproducts.

Exercise 2.23. Prove that if $\mathcal{C}$ is a 1-category with finite colimits, the Yoneda embedding $j\colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ preserves finite colimits.

Motto 2.24. Filtered colimits plus reflexive coequalizers give 1-sifted colimits.

This motto is discussed in detail in [1]. For example, it means that to check that a functor $F\colon \mathcal{P}_\Sigma(\mathcal{C})^\heartsuit \to \mathcal{D}$ preserves 1-sifted colimits (and hence is left Kan extended from a functor $\mathcal{C} \to \mathcal{D}$), it is enough to check that F preserves sifted colimits and reflexive coequalizers.

2.2 Cocompletion of ∞-categories

Here is the ∞-categorical version.

Definition 2.25. Let $\mathcal{C}$ be a small ∞-category (or any simplicial set). The functor category $\mathcal{P}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is the ∞-category of **presheaves of animae** on $\mathcal{C}$. There is a Yoneda embedding $j\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ which sends an object $Y \in \mathcal{C}$ to the representable functor $\mathrm{Map}_{\mathcal{C}}(-, Y)$.

Lemma 2.26. *Let $\mathcal{C}$ be a small ∞-category.*

(a) *The ∞-category $\mathcal{P}(\mathcal{C})$ has all small colimits.*

(b) *For any cocomplete ∞-category $\mathcal{D}$, restriction along Yoneda induces an equivalence $\mathrm{Fun}'(\mathcal{P}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, where $\mathrm{Fun}'(\mathcal{P}(\mathcal{C}), \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{P}(\mathcal{C}), \mathcal{D})$ is the full subcategory of colimit preserving functors.*

Proof. Part (a) is [38, Cor. 5.1.2.4] and (b) follows from [38, Thm. 5.1.5.6]. □

Remark 2.27. We see from Lemma 2.3 that $\mathcal{P}(\mathcal{C})$ deserves to be called the **free cocompletion** of $\mathcal{C}$: it has all colimits by (a) and (b) implies that there are no unnecessary relations between these colimits. Given $F\colon \mathcal{C} \to \mathcal{D}$, the associated functor $\mathcal{P}(\mathcal{C}) \to \mathcal{D}$ is called the **left Kan extension** of F along Yoneda, although this makes sense in greater generality which we will not discuss here. Because the Yoneda functor $j\colon \mathcal{C} \to \mathcal{P}(\mathcal{C})$ is fully faithful, one obtains a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "j"'] & \mathcal{D} \\ \mathcal{P}(\mathcal{C}) \arrow[ur, "\mathrm{Lan}_j F"'] & \end{tikzcd}$$

of functors.

Remark 2.28. Let $\mathcal{C}$ be a 1-category. It turns out that $\mathcal{C}$ has all 1-categorical colimits if and only if it has all ∞-categorical colimits. So, the notions of cocomplete coincide in this case. The reason for this is that if $\mathcal{D} \to \mathcal{C}$ is any functor where $\mathcal{D}$ is a simplicial set, then there is a canonical factorization $\mathcal{D} \to \mathrm{Ho}(\mathcal{D}) \to \mathcal{C}$ and the colimits in $\mathcal{C}$ computed over $\mathcal{D}$ or over $\mathrm{Ho}(\mathcal{D})$ are the same. The factorization comes from an adjunction

$$\mathrm{Ho}(-)\colon \mathrm{Cat}_\infty \rightleftarrows \mathrm{Cat}_1,$$

where the right adjoint is the inclusion of 1-categories into ∞-categories.

Remark 2.29. The free cocompletion of a 1-category is typically an ∞-category, not a 1-category. For example, suppose that we let $*$ denote the final 1-category. Then, $\mathcal{P}(*) \simeq \mathrm{Fun}(*^{\mathrm{op}}, \mathcal{S}) \simeq \mathcal{S}$ is the ∞-category of animae.

As in the 1-categorical situation, we can look at slightly different constructions in order to force Yoneda to preserve certain colimits.

Definition 2.30. Let $\mathcal{C}$ be an ∞-category.

(i) We say that $\mathcal{C}$ is **sifted** if $\mathcal{C}$-indexed colimits in $\mathcal{S}$ commute with finite products.

(ii) We say that $\mathcal{C}$ is **filtered** if $\mathcal{C}$-index colimits in $\mathcal{S}$ commute with finite limits.

Remark 2.31. For every filtered ∞-category $\mathcal{C}$ there is a filtered poset P and a cofinal map $P \to \mathcal{C}$. See [38, Prop. 5.3.16]. This means that filtered colimits, even ∞-categorical, are no more general than colimits over filtered posets.

Proposition 2.32. *The opposite of the simplex category Δ^{op} is sifted.*

Exercise 2.33. Read the proof of Proposition 2.32 at [38, Lem. 5.5.8.5].

Remark 2.34. This is at heart the reason why given a bisimplicial anima $X_{\bullet\bullet}$ the colimit over $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ can be computed by restriction to the diagonal simplicial anima. Indeed, $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ is cofinal.

Remark 2.35. An ∞-category $\mathcal{C}$ is sifted if and only if it is non-empty and the diagonal functor $\mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is cofinal. Note that a sifted 1-category is 1-sifted, but a 1-sifted 1-category is not necessarily sifted.

An ∞-category $\mathcal{C}$ is filtered if and only if for every finite simplicial set K and every morphism $i\colon K \to \mathcal{C}$, there is an extension of i to a morphism $K^{\triangleright} \to \mathcal{C}$.

Exercise 2.36. Prove that the reflexive coequalizer category $\Delta^{\mathrm{op}}_{\leqslant 1}$ is not sifted.

Definition 2.37. Let $\mathcal{C}$ be a small ∞-category.

(i) If $\mathcal{C}$ has finite coproducts, then we let $\mathcal{P}_\Sigma(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C})$ be the full subcategory of those functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ that preserve finite products.

(ii) If $\mathcal{C}$ has finite colimits, then we let $\mathrm{Ind}(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C})$ be the full subcategory of those functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ that preserve finite limits.

Remark 2.38. Note that if $\mathcal{C}$ has finite colimits, then $\mathrm{Ind}(\mathcal{C}) \subseteq \mathcal{P}_\Sigma(\mathcal{C})$.

As in the 1-categorical case, these constructions admit universal characterizations.

Proposition 2.39. *Let $\mathcal{C}$ be a small ∞-category.*

(i) *Suppose that $\mathcal{C}$ has finite coproducts.*

 (a) *The inclusion $\mathcal{P}_\Sigma(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C})$ preserves sifted colimits and admits a left adjoint (and hence $\mathcal{P}_\Sigma(\mathcal{C})$ is presentable).*

 (b) *The Yoneda functor $j\colon \mathcal{C} \to \mathcal{P}(\mathcal{C})$ lands in $\mathcal{P}_\Sigma(\mathcal{C})$.*

 (c) *An object $X \in \mathcal{P}(\mathcal{C})$ is in $\mathcal{P}_\Sigma(\mathcal{C})$ if and only if it can be written as the geometric realization of a simplicial object $X_\bullet$ where each X_n is a filtered colimit of objects in $\mathcal{C} \subseteq \mathcal{P}(\mathcal{C})$.*

 (d) *If $\mathcal{D}$ is an ∞-category with sifted colimits, then restriction along Yoneda induces an equivalence*

$$\mathrm{Fun}''(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D}),$$

 where $\mathrm{Fun}''(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D})$ is the full subcategory of functors which preserve sifted colimits.

 (e) *If $\mathcal{D}$ is an ∞-category with all colimits, then restriction along Yoneda induces an equivalence*

$$\mathrm{Fun}'(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}^{\sqcup}(\mathcal{C}, \mathcal{D}),$$

 where $\mathrm{Fun}^{\sqcup}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is the full subcategory of functors which preserve finite coproducts.

(ii) *Suppose that $\mathcal{C}$ has finite colimits.*

 (a) *The inclusion $\mathrm{Ind}(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C})$ preserves filtered colimits and admits a left adjoint (and hence $\mathrm{Ind}(\mathcal{C})$ is presentable).*

 (b) *The Yoneda functor $j\colon \mathcal{C} \to \mathcal{P}(\mathcal{C})$ lands in $\mathrm{Ind}(\mathcal{C})$.*

(c) *An object $X \in \mathcal{P}(\mathcal{C})$ is in* $\mathrm{Ind}(\mathcal{C})$ *if and only if it can be written as the filtered colimit objects in* $\mathcal{C} \subseteq \mathcal{P}(\mathcal{C})$.

(d) *If $\mathcal{D}$ is an ∞-category with filtered colimits, then restriction along Yoneda induces an equivalence*

$$\mathrm{Fun}'''(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D}),$$

where $\mathrm{Fun}'''(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) \subseteq \mathrm{Fun}(\mathrm{Ind}(\mathcal{C}), \mathcal{D})$ *is the full subcategory of functors which preserve filtered colimits.*

(e) *If $\mathcal{D}$ is an ∞-category with all colimits, then restriction along Yoneda induces an equivalence*

$$\mathrm{Fun}'(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{rex}}(\mathcal{C}, \mathcal{D}),$$

where $\mathrm{Fun}^{\mathrm{rex}}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ *is the full subcategory of functors which preserve finite colimits.*

Proof. This is extracted from [38, Chap. 5], especially [38, Sec. 5.5.8] in the 1-sifted case and [38, Sec. 5.3.5] in the filtered case. □

Remark 2.40. It is important to note that if $\mathcal{C}$ is a 1-category with finite colimits, then $\mathrm{Ind}(\mathcal{C})$ is still a 1-category, whereas $\mathcal{P}(\mathcal{C})$ and $\mathcal{P}_\Sigma(\mathcal{C})$ are generally ∞-categories. The reason for this is as follows. Suppose that $X, Y \in \mathrm{Ind}(\mathcal{C})$ and assume that $X \simeq \mathrm{colim}_I X_i$ and $Y \simeq \mathrm{colim}_K Y_k$ where I and J are appropriately chosen posets and $X_i, Y_k \in \mathcal{C} \subseteq \mathrm{Ind}(\mathcal{C})$. Then,

$$\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(X, Y) \simeq \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\operatorname*{colim}_I X_i, \operatorname*{colim}_K Y_k) \simeq \lim_I \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(X_i, \operatorname*{colim}_K Y_k) \simeq \lim_I \operatorname*{colim}_K \mathrm{Map}_{\mathcal{C}}(X_i, Y_k),$$

where the final equivalence is because X_i is **compact** in $\mathrm{Ind}(\mathcal{C})$, which means that maps out of X_i commutes with filtered colimits. Since $\mathcal{C}$ is a 1-category, $\mathrm{Map}_{\mathcal{C}}(X_i, Y_k)$ is a set. But, $\mathrm{Set} \subseteq \mathcal{S}$ is closed under filtered colimits (as you will prove below in Exercise 2.41) and all limits (since the inclusion $\mathrm{Set} \subseteq \mathcal{S}$ has a left adjoint given by π_0). Thus, we see that $\lim_I \mathrm{colim}_K \mathrm{Map}_{\mathcal{C}}(X_i, Y_k)$ is a set. Much of this argument works for $\mathcal{P}_\Sigma(\mathcal{C})$. However, at the end, one sees that $\mathrm{Set} \subseteq \mathcal{S}$ is *not* closed under sifted colimits.

Exercise 2.41. Prove that $\mathrm{Set} \subseteq \mathcal{S}$ is closed under filtered colimits.

Exercise 2.42. Prove that $\mathrm{Set} \subseteq \mathcal{S}$ is not closed under sifted colimits.

Exercise 2.43. Prove that if $\mathcal{C}$ is an ∞-category with finite coproducts, the Yoneda embedding $j\colon \mathcal{C} \to \mathcal{P}_\Sigma(\mathcal{C})$ preserves finite coproducts.

Exercise 2.44. Prove that if $\mathcal{C}$ is an ∞-category with finite colimits, the Yoneda embedding $j\colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ preserves finite colimits.

Motto 2.45. Filtered colimits plus geometric realizations give sifted colimits.

2.3 Animation

To each notion of cocompletion from the previous sections, there is a notion of a 'small object'.

Definition 2.46 (Small objects)**.** Let $\mathcal{C}$ be a presentable ∞-category and let $X \in \mathcal{C}$ be an object. We let $\mathrm{Map}_{\mathcal{C}}(X, -)\colon \mathcal{C} \to \mathcal{S}$ denote the functor corepresented by X.

(a) We say that X is an **absolute object** if $\mathrm{Map}_{\mathcal{C}}(X, -)$ commutes with all colimits.

(b) We say that X is a **compact projective object** if $\mathrm{Map}_{\mathcal{C}}(X, -)$ commutes with all sifted colimits.

(c) We say that X is a **compact object** if $\mathrm{Map}_{\mathcal{C}}(X, -)$ commutes with all filtered colimits.

We also consider a presentable 1-category $\mathcal{D}$ and say that $Y \in \mathcal{D}$ is **compact 1-projective** if $\mathrm{Hom}_{\mathcal{D}}(Y, -)\colon \mathcal{D} \to \mathrm{Set}$ commutes with 1-sifted colimits.

Notation 2.47. If $\mathcal{C}$ is a presentable ∞-category, we let $\mathcal{C}^{\mathrm{abs}}$, $\mathcal{C}^{\omega\mathrm{proj}}$, and $\mathcal{C}^{\omega}$ denote the full subcategories of $\mathcal{C}$ consisting of the absolute, compact projective, and compact objects, respectively. If $\mathcal{C}$ is a presentable 1-category, we let $\mathcal{C}^{\omega 1\mathrm{proj}}$ denote the full subcategory of compact 1-projective objects.

Remark 2.48. Absolute objects are compact projective and compact projective objects are compact. If $\mathcal{C}$ is a presentable 1-category, and $X \in \mathcal{C}$ is compact projective, then it is compact 1-projective.

Question 2.49. Is the converse true? That is, if $\mathcal{C}$ is a presentable 1-category and $X \in \mathcal{C}$ is compact 1-projective, does it follow that X is compact projective? What is at stake here is again the fact that Set is not closed under sifted colimits in $\mathcal{S}$.

Example 2.50. Consider the three objects

$$\emptyset, \quad *, \quad * \sqcup *$$

in $\mathcal{S}$. Which are absolute, which compact projective, and which compact?

Exercise 2.51. Prove the following three statements. Absolute objects are closed under retracts. Compact projective objects are closed under finite coproducts. Compact objects are closed under finite colimits.

Exercise 2.52. Let $\mathcal{C}$ be a small ∞-category. Show that every object of $\mathcal{C}$ is absolute when viewed in $\mathcal{P}(\mathcal{C})$ via the Yoneda embedding.

Exercise 2.53. Let $\mathcal{C}$ be a small ∞-category with finite coproducts. Show that every object of $\mathcal{C}$ is compact projective when viewed in $\mathcal{P}_{\Sigma}(\mathcal{C})$ via the Yoneda embedding.

Exercise 2.54. Let $\mathcal{C}$ be a small ∞-category with finite colimits. Show that every object of $\mathcal{C}$ is compact when viewed in $\mathrm{Ind}(\mathcal{C})$ via the Yoneda embedding.

Definition 2.55. Let $\mathcal{C}$ be a presentable ∞-category.

(a) We say that $\mathcal{C}$ is **absolutely generated** if the natural left adjoint functor $\mathcal{P}(\mathcal{C}^{\mathrm{abs}}) \to \mathcal{C}$ is an equivalence.

(b) We say that $\mathcal{C}$ is **projectively generated** if the natural left adjoint functor $\mathcal{P}_{\Sigma}(\mathcal{C}^{\omega\mathrm{proj}}) \to \mathcal{C}$ is an equivalence.

(c) We say that $\mathcal{C}$ is **compactly generated** if the natural left adjoint functor $\mathrm{Ind}(\mathcal{C}^{\omega}) \to \mathcal{C}$ is an equivalence.

Finally, if $\mathcal{C}$ is a presentable 1-category, we say that $\mathcal{C}$ is 1-projectively generated if $\mathcal{P}_{\Sigma}(\mathcal{C}^{\omega 1\mathrm{proj}})^{\heartsuit} \to \mathcal{C}$ is an equivalence.

Exercise 2.56. Prove that if $\mathcal{C}$ is absolutely generated, then it is projectively generated. Similarly, prove that if $\mathcal{C}$ is projectively generated, then it is compactly generated.

Definition 2.57. Let $\mathcal{C}$ be a 1-projectively generated category. The **animation** of $\mathcal{C}$ is

$$\mathfrak{a}\mathcal{C} \simeq \mathcal{P}_{\Sigma}(\mathcal{C}^{\omega\mathrm{proj}}).$$

The objects of $\mathfrak{a}\mathcal{C}$ are called **animated objects** of $\mathcal{C}$.

Lemma 2.58. *If $\mathcal{C}$ is a 1-projectively generated category, then $\mathfrak{a}\mathcal{C}$ is a projectively generated ∞-category and $\mathfrak{a}\mathcal{C}^{\omega\mathrm{proj}} \simeq \mathcal{C}^{\omega 1\mathrm{proj}}$.*

Proof. The 1-category $\mathcal{C}^{\omega 1\mathrm{proj}}$ has finite coproducts and is idempotent-complete. The result now follows from [38, Prop. 5.5.8.25]. In slightly more detail, the main thing to prove is that every compact projective object p of $\mathfrak{a}\mathcal{C}$ is a retract of an object in the image of the Yoneda embedding $j\colon \mathcal{C}^{\omega 1\mathrm{proj}} \to \mathfrak{a}\mathcal{C}$. By Prop. 2.39(i.c), we see that there is an equivalence $|c_\bullet| \simeq p$, where each $c_\bullet$ is a filtered colimit of objects of $\mathcal{C}^{\omega 1\mathrm{proj}}$. Now, since p is compact projective, it follows that the identity map $p \xrightarrow{\mathrm{id}} p$ factors through $c_0 \to p$ via a map $f\colon p \to c_0$. Now, writing c_0 as $\operatorname{colim}_{i\in I} b_i$ for objects b_i in the image of Yoneda, and using that p is in particular compact, we see that f factors through some $b_i \to c_0$. In particular, we have a map $p \to b_i \to p$, which is equivalent to the identity and where $b_i \in \mathcal{C}^{\omega 1\mathrm{proj}}$. This is what we had to show. □

Remark 2.59. We see that if $\mathcal{D}$ has all sifted colimits, then composition with Yoneda induces an equivalence

$$\mathrm{Fun}^{\mathrm{sifted}}(\mathfrak{a}\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}^{\omega\mathrm{proj}}, \mathcal{D}),$$

where $\mathrm{Fun}^{\mathrm{sifted}}(\mathfrak{a}\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathfrak{a}\mathcal{C}, \mathcal{D})$ is the full subcategory of functors that preserve sifted colimits. If $\mathcal{D}$ has all colimits, then we get an equivalence

$$\mathrm{Fun}'(\mathfrak{a}\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}^{\sqcup}(\mathcal{C}^{\omega\mathrm{proj}}, \mathcal{D}),$$

where as above $\mathrm{Fun}'(\mathfrak{a}\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathfrak{a}\mathcal{C}, \mathcal{D})$ is the full subcategory of colimit preserving functors and where $\mathrm{Fun}^{\sqcup}(\mathcal{C}^{\omega\mathrm{proj}}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}^{\omega\mathrm{proj}}, \mathcal{D})$ is the full subcategory of functors that preserve finite coproducts.

This is the real point of animation: it is to allow us to build functors. It is often possible to construct a functor out of $\mathcal{C}^{\omega\mathrm{proj}}$, and left Kan extension allows us to extend such a functor to every animated object of $\mathcal{C}$.

Now, let us discuss some examples.

Example 2.60. The category Set of sets is compact 1-projectively generated. Indeed, the final object $* \in \mathrm{Set}$ is a compact 1-projective object, and every set is built as a colimit of some diagram involving only $*$ (in fact a coproduct). It is not hard to check that the only compact 1-projective objects of Set are the finite coproducts of $*$, i.e., the finite sets. So, we see that $\mathrm{Set} \simeq \mathcal{P}_\Sigma(\mathrm{Set}^{\omega\mathrm{proj}})^{\heartsuit}$. In particular, we can animate Set to obtain $\mathfrak{a}\mathrm{Set} \simeq \mathcal{P}_\Sigma(\mathrm{Set}^{\omega\mathrm{proj}})$, which is equivalent to $\mathcal{S}$.

Example 2.61. Let R be an associative ring. The category Mod_R of right R-modules is compact 1-projectively generated by the finitely presented projective right R-modules. Thus, the ∞-category $\mathfrak{a}\mathrm{Mod}_R$ of animated right R-modules makes sense and is given by $\mathcal{P}_\Sigma(\mathrm{Mod}_R^{\omega\mathrm{proj}})$. One shows that this is equivalent to the ∞-category $\mathcal{D}(R)_{\geqslant 0}$ obtained from $\mathrm{Ch}(R)_{\geqslant 0}$ by inverting quasi-isomorphisms.

Exercise 2.62. Show that if R is an associative ring and P is a finitely presented projective right R-module, then P is a compact 1-projective object of Mod_R. Prove the converse too: if P is a compact 1-projective right R-module, then P is finitely presented projective in the usual sense.

Remark 2.63. Based on Example 2.61, one finds $\mathcal{P}_\Sigma(\mathcal{C})$ called the **nonabelian derived category** of $\mathcal{C}$ in the literature.

Example 2.64. As far as I know, the (lax symmetric monoidal) functor

$$N\colon \mathrm{sAlg} \to \mathrm{dga}_{\geqslant 0}$$

is not an equivalence of 1-categories. But it becomes an equivalence after inverting weak equivalences on the right and quasi-isomorphisms on the left by a theorem of Schwede and Shipley [55, Thm. 1.1(3)]. This means we get equivalences

$$\mathfrak{a}\mathrm{Alg} \simeq \mathrm{sAlg}[W^{-1}] \simeq \mathrm{dga}_{\geqslant 0}[\mathrm{qiso}^{-1}]$$

of ∞-categories. Moreover, Shipley proves in [57] that these are all equivalent additionally to the ∞-category of $\mathbb{E}_1$-algebras (or A_∞-algebras) in $\mathcal{D}(\mathbb{Z})_{\geqslant 0}$.

2.4 A tiny model categorical digression

Lurie proves in [38, Sec. 5.5.9] that when $\mathcal{C}$ is a 1-category with finite coproducts, then the sifted-cocompletion (or nonabelian derived category) $\mathcal{P}_\Sigma(\mathcal{C})$ admits a nice model categorical description. The model category is due to Quillen. One takes $\mathrm{Fun}^\sqcap(\mathcal{C}^{\mathrm{op}}, \mathrm{sSet})$, the 1-category of finite product preserving functors from $\mathcal{C}^{\mathrm{op}}$ to the 1-category of simplicial sets. If $\eta\colon F \to G$ is a morphism in $\mathrm{Fun}^\sqcap(\mathcal{C}^{\mathrm{op}}, \mathrm{sSet})$, we say that η is a weak equivalence (resp. fibration) if $\eta(c)\colon F(c) \to G(c)$ is a weak equivalence (resp. fibration) in sSet for each object $c \in \mathcal{C}$. It is a theorem that these classes of morphisms are the weak equivalences and fibrations for a model category structure on the functor category, called the **projective model category structure**. Lurie proves that the ∞-categorical nerve of this model category structure naturally recovers $\mathcal{P}_\Sigma(\mathcal{C})$.

2.5 Animated commutative rings

The ∞-category of derived commutative rings we will use for this course is

$$\mathfrak{a}\mathrm{CAlg} \simeq \mathfrak{a}\mathrm{CAlg}_{\mathbb{Z}},$$

the ∞-category of **animated commutative $\mathbb{Z}$-algebras**, or **animated commutative rings**. If k is a base commutative ring, then we similarly will consider $\mathfrak{a}\mathrm{CAlg}_k$, the ∞-category of animated commutative k-algebras when we are working over $\mathrm{Spec}\, k$. In fact, if $k \in \mathfrak{a}\mathrm{CAlg}_{\mathbb{Z}}$ is any animated commutative ring, we will abuse notation and let $\mathfrak{a}\mathrm{CAlg}_k$ denote the ∞-category $(\mathfrak{a}\mathrm{CAlg})_{k/}$ of animated commutative rings equipped with a map from k. This will allow us to work not only over $\mathrm{Spec}\, k$ when k is an ordinary commutative ring, but also over "$\mathrm{Spec}\, k$" when k is 'derived'.

We give some small examples of how to work in $\mathfrak{a}\mathrm{CAlg}$ in this section. To begin, we identify the objects of $\mathrm{CAlg}_k^{\omega 1\mathrm{proj}} \simeq \mathfrak{a}\mathrm{CAlg}_k^{\omega\mathrm{proj}}$.

Lemma 2.65. *Let k be an ordinary commutative ring. An object $R \in \mathrm{CAlg}_k$ is compact projective if and only if R is a retract of a finitely presented polynomial algebra over k.*

Proof. Consider the adjunction $k[-]\colon \mathrm{Set} \rightleftarrows \mathrm{CAlg}_k$. Using the standard simplicial object associated to the associated comonad on CAlg_k, we get a functor $S_{\bullet\leqslant 1}\colon \Delta^{\mathrm{op}}_{\leqslant 1} \to \mathrm{CAlg}_k$ which looks like $S_1 \cong k[k[R]]$ and $S_0 \cong k[R]$. There is an augmentation $S_0 \to R$ which coequalizes $\partial_0, \partial_1\colon S_1 \rightrightarrows S_0$. Moreover, $\mathrm{colim}(S_1 \overset{\leftleftarrows}{\rightrightarrows} S_0) \cong R$. Since R is compact projective, we have

$$\mathrm{colim}(\mathrm{Hom}_{\mathrm{CAlg}_k}(R, S_1) \overset{\leftleftarrows}{\rightrightarrows} \mathrm{Hom}_{\mathrm{CAlg}_k}(R, S_0)) \cong \mathrm{Hom}_{\mathrm{CAlg}_k}(R, \mathrm{colim}(S_1 \overset{\leftleftarrows}{\rightrightarrows} S_0)) \cong \mathrm{Hom}_{\mathrm{CAlg}_k}(R, R).$$

In particular, the identity map $\mathrm{id}_R\colon R \to R$ factors through the surjection $S_0 \twoheadrightarrow R$. Now, since R is additionally compact, the map $R \to S_0$ factors through $R \to k[F]$ for some finite subset $F \subseteq R$. But, this means that $R \to k[F] \to R$ is a retract of a finitely presented polynomial ring over k. $\square$

Question 2.66. For which k is it the case that every retract of a finitely presented polynomial algebra over k is itself a finitely presented polynomial ring? I honestly have no idea. I guess it should be necessary for every finitely presented projective k-module to be free.

Definition 2.67. Let $\mathrm{LSym}_k\colon \mathfrak{a}\mathrm{Mod}_k \to \mathfrak{a}\mathrm{CAlg}_k$ be the left adjoint to the forgetful functor. Note that this is simply described as the left Kan extension (or animation) of the composite functor

$$\mathrm{Mod}_k^{\omega 1\mathrm{proj}} \to \mathrm{CAlg}_k^{\omega 1\mathrm{proj}} \xrightarrow{j} \mathfrak{a}\mathrm{CAlg}_k$$

which sends a finitely presented projective k-module P to the commutative k-algebra $\mathrm{Sym}_k P$.

Lemma 2.68. *The forgetful functor* $\mathfrak{a}\mathrm{CAlg}_k \to \mathfrak{a}\mathrm{Mod}_k$ *preserves sifted colimits.*

Proof. This follows from the fact that $\mathrm{LSym}_k\colon \mathfrak{a}\mathrm{Mod}_k \to \mathfrak{a}\mathrm{CAlg}_k$ preserves compact projective objects. □

Remark 2.69. It follows that the monad $\mathrm{LSym}_k\colon \mathfrak{a}\mathrm{Mod}_k \to \mathfrak{a}\mathrm{Mod}_k$ preserves sifted colimits and is thus determined by its value on finitely presented projective k-modules. This means that to compute $\mathrm{LSym}(M)$ we can write $M \simeq |P_\bullet|$, where $P_\bullet$ is a simplicial object with each P_t a flat k-module. Then, we let $\mathrm{LSym}_k(M) \simeq |\mathrm{LSym}_k(P_\bullet)|$. Here, for a flat k-module Q, we have $\mathrm{LSym}_k(Q) \simeq \bigoplus_{n\geqslant 0} \mathrm{H}_0((Q^{\otimes n})_{hS_n})$ is the usual polynomial algebra on Q.

Example 2.70 (The coproduct)**.** Let k be an ordinary commutative ring and let $R, S \in \mathfrak{a}\mathrm{CAlg}_k$ be animated commutative k-algebras. The pushout $R \coprod S \in \mathfrak{a}\mathrm{CAlg}_k$ has underlying complex of k-modules given by $R \otimes_k S$ in $\mathcal{D}(k)_{\geqslant 0}$, so we will typically write the pushout as $R \otimes_k S$. To see this, note that it is true for free objects. That is, let P and Q be finitely presented projective k-modules. Then, $\mathrm{LSym}_k(P) \otimes_k \mathrm{LSym}_k(Q) \simeq \mathrm{LSym}(P \oplus Q)$ is the pushout. Indeed, the left adjoint functor

$$\mathrm{LSym}_k\colon \mathcal{D}(k)_{\geqslant 0} \to \mathfrak{a}\mathrm{CAlg}_k$$

preserves colimits. Now, it is also true for $\mathrm{LSym}_k(F) \otimes_k \mathrm{LSym}_k(I)$ where F and I are filtered colimits of finitely presented projective k-modules (i.e., flat k-modules by Lazard's theorem). We can now write R and S as geometric realizations $|R_\bullet|$ and $|S_\bullet|$ where each R_i and S_j is free on a flat k-module.

Exercise 2.71 (The pushout)**.** Prove that more generally, $R \otimes_T S$ computes the coproduct of R and S over T in $\mathfrak{a}\mathrm{CAlg}_k$.

Remark 2.72 (Sym versus LSym)**.** There is a different notion of a commutative algebra object in homotopy theory, which is that of an $\mathbb{E}_\infty$-algebra. If $\mathcal{C}$ is a symmetric monoidal ∞-category with small colimits, then there is a monad on $\mathcal{C}$ called the Sym monad. Given $X \in \mathcal{C}$, we have $\mathrm{Sym}(X) \simeq \coprod_{n\geqslant 0} (X^{\otimes n})_{hS_n}$, where we take homotopy orbits with respect to the canonical symmetric group action on $X^{\otimes n}$ obtained by permuting the factors. Then, $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C})$ is the ∞-category of Sym-algebras in $\mathcal{C}$. In the case of $\mathcal{D}(k)_{\geqslant 0} \simeq \mathfrak{a}\mathrm{Mod}_k$, we have a natural map $\mathrm{Sym}(P) \to \mathrm{LSym}(P)$ obtained from the natural map

$$(P^{\otimes n})_{hS_n} \to \mathrm{H}_0((P^{\otimes n})_{hS_n}) \cong \mathrm{LSym}^n(P)$$

in the case of a finitely presented projective k-module P and then left Kan extending. In particular, we obtain a forgetful functor from LSym_k-algebras to Sym-algebras and hence a forgetful functor

$$\mathfrak{a}\mathrm{CAlg}_k \to \mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{D}(k)_{\geqslant 0}).$$

Construction 2.73 (The derived category of an animated commutative ring)**.** Given an animated commutative ring $R \in \mathfrak{a}\mathrm{CAlg}_{\mathbb{Z}}$, we can consider $\mathrm{Mod}_R(\mathfrak{a}\mathrm{Mod}_{\mathbb{Z}}) \simeq \mathrm{Mod}_R(\mathcal{D}(\mathbb{Z})_{\geqslant 0})$. This is the ∞-category $\mathcal{D}(R)_{\geqslant 0}$ of connective R-modules. By formally inverting the suspension functor $M \mapsto M[1]$ in Pr^{L} we obtain the stable ∞-category $\mathcal{D}(R)$. That is, we let $\mathcal{D}(R)$ be the colimit

$$\mathcal{D}(R)_{\geqslant 0} \xrightarrow{(-)[1]} \mathcal{D}(R)_{\geqslant 0} \xrightarrow{(-)[1]} \mathcal{D}(R)_{\geqslant 0} \to \cdots,$$

where we compute the colimit in Pr^{L}. Colimits in Pr^{L} can be computed as limits of the associated diagram of right adjoint functors by [38, Thm. 5.5.3.18]. So, we can instead write

$$\mathcal{D}(R) \simeq \lim\left(\cdots \to \mathcal{D}(R)_{\geqslant 0} \xrightarrow{\Omega} \mathcal{D}(R)_{\geqslant 0} \xrightarrow{\Omega} \mathcal{D}(R)_{\geqslant 0}\right),$$

where the limit is computed either in Pr^{R} or in the ∞-category of large ∞-categories. We see that $\mathcal{D}(R)$ consists of Ω-spectrum objects of $\mathcal{D}(R)_{\geqslant 0}$, which is to say sequences $(X_n)_{n\geqslant 0}$ of objects $X_n \in \mathcal{D}(R)_{\geqslant 0}$ together with equivalences $\Omega X_n \simeq X_{n-1}$ for $n \geqslant 1$.

Remark 2.74. Here is an alternative construction. We let $\mathrm{CAlg}_{\mathcal{D}(\mathbb{Z})}(\mathrm{Pr}^{\mathrm{L}})$ be the ∞-category of $\mathbb{E}_\infty$-$\mathcal{D}(\mathbb{Z})$-algebras in Pr^{L}. This is a subcategory of the ∞-category of symmetric monoidal stable presentable ∞-categories and left adjoint functors symmetric monoidal functors. Although $\mathrm{CAlg}_{\mathcal{D}(\mathbb{Z})}(\mathrm{Pr}^{\mathrm{L}})$ is not presentable (because it is very large), it does have all small colimits. The functor $R \mapsto \mathcal{D}(R)$ gives a functor $\mathrm{CAlg}_{\mathbb{Z}}^{\omega 1\mathrm{proj}} \to \mathrm{Mod}_{\mathcal{D}(\mathbb{Z})}(\mathrm{Pr}^{\mathrm{L}})$ which preserves finite coproducts, since $\mathcal{D}(R) \otimes_{\mathcal{D}(\mathbb{Z})} \mathcal{D}(S) \simeq \mathcal{D}(R \otimes_{\mathbb{Z}} S)$ for R and S finitely generated polynomial commutative $\mathbb{Z}$-algebras. Thus, we can left Kan extend and use the universal property of $\mathfrak{a}\mathrm{CAlg}_{\mathbb{Z}}$ to get a colimit-preserving functor $\mathfrak{a}\mathrm{CAlg}_{\mathbb{Z}} \to \mathrm{CAlg}_{\mathcal{D}(\mathbb{Z})}(\mathrm{Pr}^{L})$ which gives another definition of $\mathcal{D}(R)$ for $R \in \mathfrak{a}\mathrm{CAlg}_{\mathbb{Z}}$.

3 Week 3: The cotangent complex (30 Sept.–6 Oct.)

Everything comes down to the cotangent complex. This is true not only of deformation theory and the study of moduli problems, but also Hochschild homology, de Rham cohomology, and, in the end, prismatic cohomology. We will discuss the theory of Kähler differentials, the classical construction of the cotangent complex, and the universal property of the cotangent complex.

3.1 Kähler differentials

References 3.1. Any standard algebraic geometry textbook will do. For example [26, Sec. II.8] or [59, Tag 00RM].

We give several ways of thinking about Kähler differentials: via derivations, via an explicit elementwise construction, via a universal property, and via an interpretation in terms of the diagonal $X \hookrightarrow X \times X$. Everything globalizes, so we will focus on the affine case.

Definition 3.2. Let $k \to R$ be a map of commutative rings and let M be an R-module. A k-**derivation** of R in M is a function $d\colon R \to M$ such that

(a) $d(r+s) = d(r) + d(s)$ for $r, s \in R$,

(b) $d(rs) = rd(s) + sd(r)$ for $r, s \in R$, and

(c) $d(t) = 0$ for $t \in k$.

Let $\mathrm{Der}_k(R, M)$ be the set of all k-derivations of R in M. It is easy to see that $\mathrm{Der}_k(R, M)$ is an R-module.

Definition 3.3. Let $k \to R$ be a map of commutative rings. The R-module of **Kähler differentials** $\Omega^1_{R/k}$ is a quotient of the free module $\bigoplus_{r\in R} R \cdot \mathrm{d}r$ on symbols $\mathrm{d}r$ subject to the relations

(a) $\mathrm{d}(r+s) = \mathrm{d}r + \mathrm{d}s$ for $r, s \in R$,

(b) $\mathrm{d}(rs) = r\mathrm{d}s + s\mathrm{d}r$ for $r, s \in R$, and

(c) $\mathrm{d}t = 0$ for $t \in k$.

Example 3.4. The function $\mathrm{d}\colon R \to \Omega^1_{R/k}$ with $r \mapsto \mathrm{d}r$ is a k-derivation of R in $\Omega^1_{R/k}$ called the **universal derivation**.[8]

Example 3.5. If $k \to R$ is a surjection, then $\Omega^1_{R/k} = 0$ by the second relation.

[8] We will later see an even more universal derivation, which we will probably also call the universal derivation.

Exercise 3.6. Prove that there is a natural isomorphism $\operatorname{Hom}_R(\Omega^1_{R/k}, M) \cong \operatorname{Der}_k(R, M)$ from the R-module of R-module homomorphisms from $\Omega^1_{R/k}$ to M to the R-module of k-derivations of R in M.

The best way to think about differentials is via a different universal property. After all, the elementwise definition of a derivation will be tricky to formulate precisely once there is homotopy coherence to observe.

Definition 3.7. Let k be a commutative ring. A **square-zero extension** of commutative k-algebras is a surjective map $\tilde{S} \twoheadrightarrow S$ in CAlg_k such that the kernel J satisfies $J^2 = 0$.

Remark 3.8. Note that if $\tilde{S} \twoheadrightarrow S$ is a square-zero extension with kernel J, then J is naturally a S-module. Indeed, given $j \in J$ and $s \in S$, lift s to an element $\tilde{s} \in \tilde{S}$ and define $s \cdot j = \tilde{s}j$. This is well-defined since J is square-zero.

Definition 3.9. Let k be a commutative ring, let S be a commutative k-algebra, and let J be an S-module. The **trivial square-zero extension** of S by J is the ring $S \oplus J$ with multiplication defined by $(r, m)(s, n) = (rs, rn + sm)$. Note that $(0, m)(0, n) = (0, 0) = 0$, which is the square-zero condition. In particular, trivial square-zero extensions are square-zero extensions.

Exercise 3.10. Show that if $\tilde{S} \twoheadrightarrow S$ is a square-zero extension in CAlg_k with kernel J then $\tilde{S}$ is the trivial square-zero extension of S by J if and only if there is a commutative k-algebra section, i.e., a commutative k-algebra map $S \to \tilde{S}$ such that the composition $S \to \tilde{S} \to S$ is the identity.

Example 3.11. The map $\mathbb{Z}/p^2 \to \mathbb{F}_p$ is a square-zero extension of commutative rings (commutative $\mathbb{Z}$-algebras). It is not trivial, since $\mathbb{Z}/p^2$ is not a commutative $\mathbb{F}_p$-algebra.

Example 3.12. The map $k[t]/(t^2) \to k$ sending t to 0 is a trivial square-zero extension of k by itself when viewed as a map of commutative k-algebras.

Example 3.13. The map $k[t]/(t^2) \to k$ is a square-zero extension, but is *not* trivial, when viewed as a map of commutative $k[t]$-algebras.

Exercise 3.14. Let $R \in \mathrm{CAlg}_k$ and let $\tilde{S} \twoheadrightarrow S$ be a square-zero extension in CAlg_k with kernel J. Let $g\colon R \to S$ be a map and suppose that $f_0, f_1\colon R \to \tilde{S}$ are two lifts of g, i.e., the diagram

$$\begin{tikzcd} & \tilde{S} \arrow[d] \\ R \arrow[ur, "f_i"] \arrow[r, "g"'] & S \end{tikzcd}$$

commutes for $i = 0, 1$. Prove that the difference $f_0 - f_1\colon R \to J$ is a k-derivation of R in J, where J is viewed as an R-module via restriction of scalars along g.

Exercise 3.15. Let S be a commutative k-algebra. We let $\mathrm{CAlg}_{k//S} \simeq (\mathrm{CAlg}_k)_{/S}$ denote the category of k-algebras equipped with a map to S. Let $g\colon R \to S$ be an object of $\mathrm{CAlg}_{k//S}$ and let $f\colon \tilde{S} \twoheadrightarrow S$ be a square-zero extension of S in CAlg_k with kernel J, viewed as an object of $\mathrm{CAlg}_{k//S}$. Prove that $\operatorname{Hom}_{\mathrm{CAlg}_{k//S}}((R, g), (\tilde{S}, f))$ is naturally a $\operatorname{Hom}_R(\Omega^1_{R/k}, J) \cong \operatorname{Der}_k(R, J)$-torsor.

Exercise 3.16. Suppose in the situation of Exercise 3.15 that $f\colon S \oplus J \twoheadrightarrow S$ is the trivial square-zero extension of S by J. Show that in this case there is a natural isomorphism

$$\operatorname{Hom}_{\mathrm{CAlg}_{k//S}}((R, g), (S \oplus J, f)) \cong \operatorname{Hom}_R(\Omega^1_{R/k}, J) \cong \operatorname{Der}_k(R, J).$$

This interpretation of the Kähler differentials will be the one which extends well to animated commutative rings.

Example 3.17. Using the universal property established in the exercises, we see that if $R = k[x]$, then $\Omega^1_{R/k}$ is a free module on $\mathrm{d}x$. Indeed, one checks that that

$$\mathrm{Hom}_{\mathrm{CAlg}_{k//S}}(k[x], (\tilde{S}, f)) \cong J,$$

since we can send x to any point of the fiber of $g(x)$. This isomorphism is not canonical for general square-zero extensions, but it is for trivial square-zero extensions.

Example 3.18. More generally, if $R = k[x_i : i \in I]$, then $\Omega^1_{R/k}$ is a free R-module on $\{\mathrm{d}x_i : i \in I\}$.

Theorem 3.19 (Conormal sequence I)**.** *Let $k \to R \to S$ be maps in CAlg_k. Then, there is a natural exact sequence*

$$S \otimes_R \Omega^1_{R/k} \to \Omega^1_{S/k} \to \Omega^1_{S/R} \to 0$$

of S-modules.

Proof. By Yoneda, it is enough to consider the functors corresponding to the objects in question. Specifically, we consider the map

$$\mathrm{Hom}_{\mathrm{CAlg}_{k//T}}(S, \tilde{T}) \to \mathrm{Hom}_{\mathrm{CAlg}_{k//T}}(R, \tilde{T})$$

where $\tilde{T} \to T$ is a trivial square-zero extension with kernel J. The fiber over $g\colon R \to \tilde{T}$ is by definition $\mathrm{Hom}_{\mathrm{CAlg}_{R//T}}(S, \tilde{T})$, where we view $\tilde{T}$ as a commutative R-algebra via g. Now, we see that there is a natural exact sequence

$$0 \to \mathrm{Hom}_S(\Omega^1_{S/R}, J) \to \mathrm{Hom}_S(\Omega^1_{S/k}, J) \to \mathrm{Hom}_R(\Omega^1_{R/k}, J).$$

But, by adjunction, since $R \to T$ factors through R, we can write the rightmost term as $\mathrm{Hom}_S(S \otimes_R \Omega^1_{R/k}, J)$. This completes the proof. ☐

When $R \to S$ is surjective, $\Omega^1_{S/R} = 0$ and we can extend the sequence above one term to the left.

Theorem 3.20 (Conormal sequence II)**.** *If $R \to S$ is a surjection with kernel I, then there is a natural exact sequence*

$$I/I^2 \to S \otimes_R \Omega^1_{R/k} \to \Omega^1_{S/k} \to 0$$

of S-modules, where $f \in I$ maps to $1 \otimes \mathrm{d}f$.

Exercise 3.21. Prove Theorem 3.20 along the lines of the proof of Theorem 3.19.

Exercise 3.22. If $R \to S$ is a surjection with kernel I and there is a commutative k-algebra map $S \to R$ splitting the surjection, then there is a split exact sequence

$$0 \to I/I^2 \to S \otimes_R \Omega^1_{R/k} \to \Omega^1_{S/k} \to 0.$$

For a hint, see Tag 02HP of [59].

Exercise 3.23. Prove that if R and S are commutative k-algebras, then $\Omega^1_{R \otimes_k S/k} \cong \left(\Omega^1_R \otimes_k S\right) \oplus \left(R \otimes_k \Omega^1_S\right)$ as an $(R \otimes_k S)$-module.

Exercise 3.24. Suppose that S is a multiplicative subset of R. Show that $\Omega^1_{S^{-1}R/k} \cong S^{-1}R \otimes_R \Omega^1_{R/k}$.

Exercise 3.25. Show that $k' \otimes_k \Omega^1_{R/k} \cong \Omega^1_{k' \otimes_k R/k'}$.

We follow Tag 00RW of [59] in the proof of the next proposition.

Proposition 3.26. *Let $k \to R$ be a map of commutative rings and let I be the kernel of the (surjective) multiplication map $R \otimes_k R \to R$. There is a natural isomorphism*

$$\Omega^1_{R/k} \cong I/I^2.$$

The map sends adb to $a \otimes b - ab \otimes 1$.

Proof. The base change of $k \to R$ along $k \to R$ is $R \to R \otimes_k R$ which sends r to $1 \otimes r$. The composition with multiplication $R \otimes_k R \to R$ is the identity on R. It follows that $R \otimes_{R \otimes_k R} \Omega^1_{R \otimes_k R/R} \cong I/I^2$. But, $\Omega^1_{R \otimes_k R/R} \cong R \otimes_k \Omega^1_{R/k} \cong (R \otimes_k R) \otimes_R \Omega^1_{R/k}$ by Exercise 3.25. So, we have

$$I/I^2 \cong R \otimes_{R \otimes_k R} \Omega^1_{R \otimes_k R/R} \cong R \otimes_{R \otimes_k R} (R \otimes_k R) \otimes_R \Omega^1_{R/k} \cong \Omega^1_{R/k}.$$

□

Remark 3.27. The isomorphism of Proposition 3.26 is often taken as the definition of the module of Kähler differentials.

3.2 The cotangent complex

Construction 3.28. Let $k_\bullet$ be a simplicial commutative ring and let $R_\bullet \in \mathrm{sCAlg}_{k_\bullet}$ be a simplicial commutative $k_\bullet$-algebra. Let $\widetilde{R}_\bullet \to R_\bullet$ be a weak equivalence where $\widetilde{R}_\bullet$ is a cofibrant simplicial commutative $k_\bullet$-algebra. Let $\Omega^1_{\widetilde{R}_\bullet/k_\bullet}$ be the simplicial $\widetilde{R}_\bullet$-module obtained by taking the Kähler differentials in each degree. We extend scalars to obtain

$$\mathrm{L}_{R_\bullet/k_\bullet} \simeq R_\bullet \otimes_{\widetilde{R}_\bullet} \Omega^1_{\widetilde{R}_\bullet/k_\bullet},$$

which is the **cotangent complex** of $R_\bullet$ over $k_\bullet$.

Remark 3.29. In order to obtain a strictly functorial construction, we can use the canonical resolution of $R_\bullet$ as a simplicial commutative $k_\bullet$-algebra, using the simplicial object associated to the comonad on $\mathrm{sCAlg}_{k_\bullet}$ arising from the adjunction $\mathrm{sSet} \rightleftarrows \mathrm{sCAlg}_{k_\bullet}$.

The construction given above of the cotangent complex does not make it immediately obvious that it is independent of the chosen resolution, or independent (up to weak equivalence) of weak equivalences in $k_\bullet$ or $R_\bullet$. But, it is. This will be made clear by an alternative definition.

Let $k \in \mathfrak{a}\mathrm{CAlg}$ be an animated commutative ring which will serve as our base for the following discussion. There is a natural coCartesian fibration

$$\mathfrak{a}\mathrm{Mod} \to \mathfrak{a}\mathrm{CAlg}_k,$$

where the fiber over an animated commutative ring R is the ∞-category of animated R-modules. In other words, the fiber is $\mathcal{D}(R)_{\geqslant 0}$. The objects of $\mathfrak{a}\mathrm{Mod}$ are pairs (R, M) where $R \in \mathfrak{a}\mathrm{CAlg}_k$ and M is an animated R-module. The morphisms $(R, M) \to (S, N)$ are pairs of maps $f\colon R \to S$ and $f^*M \to N$ (or equivalently $M \to f_*N$).

We would like to construct a colimit-preserving section

$$\mathrm{L}_{-/k}\colon \mathfrak{a}\mathrm{CAlg}_k \to \mathfrak{a}\mathrm{Mod}$$

whose value at R will be the cotangent complex $\mathrm{L}_{R/k}$. By the universal property of $\mathfrak{a}\mathrm{CAlg}_k$, which applies because $\mathfrak{a}\mathrm{Mod}$ is presentable, we know that it is enough to construct a functor

$$\mathfrak{a}\mathrm{CAlg}_k^{\omega\mathrm{proj}} \to \mathfrak{a}\mathrm{Mod}$$

which preserves finite coproducts, where $\mathfrak{a}\mathrm{CAlg}_k^{\omega\mathrm{proj}} = (\mathfrak{a}\mathrm{CAlg}_k)^{\omega\mathrm{proj}} \simeq \mathrm{CAlg}_k^{\omega 1\mathrm{proj}}$ is the 1-category of finitely generated polynomial rings over k.

Remark 3.30. We proved the desired universal property of $\mathfrak{a}\mathrm{CAlg}_k$ when k is discrete. However, it is also true when k is animated. In this case, $\mathfrak{a}\mathrm{CAlg}_k^{\omega\mathrm{proj}}$ is not a 1-category but an ∞-category in general. Nevertheless, we still have $\mathfrak{a}\mathrm{Mod}_k \simeq \mathcal{P}_\Sigma(\mathfrak{a}\mathrm{CAlg}_k^{\omega\mathrm{proj}})$. The easiest way to see this to check that the forgetful functor $\mathfrak{a}\mathrm{CAlg}_k \to \mathfrak{a}\mathrm{CAlg}_\mathbb{Z}$ preserves sifted colimits and hence that the objects $\mathbb{Z}[x_1, \ldots, x_r] \otimes_\mathbb{Z} k$ give compact projective generators of $\mathfrak{a}\mathrm{CAlg}_k$.

In general, it is not so easy to construct functors between ∞-categories. But, in the case where k is discrete, we want the functor to send R to $(R, \Omega^1_{R/k})$ when R is a finitely generated polynomial ring over k.

Exercise 3.31. The ∞-category $\mathfrak{a}\mathrm{Mod}$ is presentable and the 0-truncation $(\mathfrak{a}\mathrm{Mod})_{\leqslant 0}$ consists of the pairs (R, M) where R and M are both discrete.

Exercise 3.32. Suppose that (R, M) and (S, N) are objects of $\mathfrak{a}\mathrm{Mod}$. Then, $(R, M) \coprod (S, N)$ is equivalent to $(R \otimes_k S, (M \otimes_k S) \oplus (R \otimes_k N))$.

Suppose that k is a discrete animated commutative ring, i.e., a commutative ring. Exercise 3.31 implies that $R \mapsto (R, \Omega^1_{R/k})$ defines a functor $\mathfrak{a}\mathrm{CAlg}_k^{\omega\mathrm{proj}} \to \mathfrak{a}\mathrm{Mod}_{\leqslant 0}$. Moreover, this functor preserves finite coproducts by Exercises 3.23 and 3.32. In general, the forgetful functor $(\mathfrak{a}\mathrm{Mod})_{\leqslant 0} \to \mathfrak{a}\mathrm{Mod}$ does *not* preserve finite coproducts. However, since polynomial rings are flat, the composition

$$\mathfrak{a}\mathrm{CAlg}^{\omega\mathrm{proj}} \to \mathfrak{a}\mathrm{Mod}_{\leqslant 0} \to \mathfrak{a}\mathrm{Mod}$$

does preserve finite coproducts.

Definition 3.33. If k is discrete, the cotangent complex functor $\mathrm{L}_{-/k} \colon \mathfrak{a}\mathrm{CAlg}_k \to \mathfrak{a}\mathrm{Mod}$ is the unique functor obtained via left Kan extension along the inclusion of finitely presented polynomial rings.

The definition above is still not quite satisfactory: it does not tell us what the cotangent complex really is. We will settle that in the next section.

Exercise 3.34. Suppose that $M \in \mathfrak{a}\mathrm{Mod}_k$ and $R = \mathrm{LSym}_k M$ is the free animated commutative ring on M. Prove that $\mathrm{L}_{R/k} \simeq M \otimes_k R$.

Remark 3.35. We will construct the cotangent complex in a different way in the next section. However, the reader might be interested in extending the approach above to the case where the base k is animated commutative, justifying Construction 3.28. For this, I can imagine doing the following. We can take the conormal sequence as a definition. Given $k \in \mathfrak{a}\mathrm{CAlg}$ and $R \in \mathfrak{a}\mathrm{CAlg}_k$, we have a natural map $R \otimes_k \mathrm{L}_{k/\mathbb{Z}} \to \mathrm{L}_{R/\mathbb{Z}}$ coming from the cotangent complex functor $\mathrm{L}_{-/\mathbb{Z}} \colon \mathfrak{a}\mathrm{CAlg} \to \mathfrak{a}\mathrm{Mod}$. We can define $\mathrm{L}_{R/k}$ to be the cofiber, so that we have a natural cofiber sequence

$$R \otimes_k \mathrm{L}_{k/\mathbb{Z}} \to \mathrm{L}_{R/\mathbb{Z}} \to \mathrm{L}_{R/k}$$

for any $R \in \mathfrak{a}\mathrm{CAlg}_k$. The functor $R \mapsto R \otimes_k \mathrm{L}_{k/\mathbb{Z}}$ preserves sifted colimits, so it follows that $R \mapsto \mathrm{L}_{R/k}$ preserves sifted colimits as well. Using that $\mathrm{L}_{-/\mathbb{Z}}$ takes weak equivalences to weak equivalences (since it is after all a functor), we see that Construction 3.28 is independent of the choices made there and descends to a functor on $\mathfrak{a}\mathrm{CAlg}_k$.

3.3 The universal property

Let k be a base animated commutative ring and let $k \to R$ be a morphism of animated commutative rings. We want to give a definition of $\mathrm{L}_{k/R}$ which is somewhat more functorial and enlightening for the purposes of proving basic properties and also deformation theory.

Definition 3.36. Let $R \in \mathfrak{a}\mathrm{CAlg}_k$ and let $M \in \mathcal{D}(R)_{\geqslant 0}$. In $\mathfrak{a}\mathrm{CAlg}_{k//R}$, we can consider maps from R to $R \oplus M$. Based on what we saw in Section 3.1, it makes sense to call these derivations:

$$\mathrm{Der}_k(R, M) \simeq \mathrm{Map}_{\mathfrak{a}\mathrm{CAlg}_{k//R}}(R, R \oplus M).$$

Remark 3.37. Note that there is a natural fiber sequence

$$\mathrm{Der}_k(R, M) \to \mathrm{Map}_{\mathfrak{a}\mathrm{CAlg}_k}(R, R \oplus M) \to \mathrm{Map}_{\mathfrak{a}\mathrm{CAlg}_k}(R, R),$$

where $\mathrm{Der}_k(R, M)$ is the fiber over the identity.

Definition 3.38. We say that $\mathrm{L}_{R/k} \in \mathfrak{a}\mathrm{Mod}_R$ is a **cotangent complex** for R relative to k if there is a natural equivalence of functors $\mathrm{Map}_R(\mathrm{L}_{R/k}, -) \simeq \mathrm{Der}_k(R, -)$.

Remark 3.39. Note that $R \oplus M$ is a loopspace object of $\mathfrak{a}\mathrm{CAlg}_{k//R}$. Indeed, it fits into a pullback square

$$\begin{tikzcd}
R \oplus M \arrow[r] \arrow[d] & R \arrow[d] \\
R \arrow[r] & R \oplus \Sigma M,
\end{tikzcd}$$

where both maps from R to $R \oplus \Sigma M$ are given by the inclusion of R (which corresponds to the trivial derivation of R in ΣM). It follows that $\mathrm{Der}_k(R, M)$ is in fact an infinite loopspace.

Existence and uniqueness is now just categorical nonsense.

Proposition 3.40. *The cotangent complex $\mathrm{L}_{R/k}$ exists and is unique (up to homotopy).*

Proof. The functor $\mathfrak{a}\mathrm{Mod}_R \to \mathcal{S}$ given by sending M to $\mathrm{Der}_k(R, M)$ is accessible and preserves limits. It is easy to see accessibility since the forgetful functor $\mathfrak{a}\mathrm{CAlg}_{k//R} \to \mathfrak{a}\mathrm{CAlg}_k$ preserves colimits and R is κ-compact in $\mathfrak{a}\mathrm{CAlg}_k$ for some κ. But, κ-filtered colimits preserve κ-small limits and we can compute the mapping spaces in $\mathfrak{a}\mathrm{CAlg}_{k//R}$ via the fiber sequence

$$\mathrm{Map}_{\mathfrak{a}\mathrm{CAlg}_{k//R}}(S, T) \to \mathrm{Map}_{\mathfrak{a}\mathrm{CAlg}_k}(S, T) \to \mathrm{Map}_{\mathfrak{a}\mathrm{CAlg}_k}(S, R),$$

where we take the fiber over the structure map $S \to T$. This gives accessibility. The forgetful functor $\mathfrak{a}\mathrm{CAlg}_{k//R} \to \mathfrak{a}\mathrm{CAlg}_k$ does not preserve limits (for example, it does not preserve final objects), but it does detect limits: if $T \colon I^{\triangleleft} \to \mathfrak{a}\mathrm{CAlg}_{k//R}$ is a diagram which is a limit in $\mathfrak{a}\mathrm{CAlg}_k$, then it is a limit in $\mathfrak{a}\mathrm{CAlg}_{k//R}$. This follows again from the fiber description of the mapping spaces in $\mathfrak{a}\mathrm{CAlg}_{k//R}$. Now, we see that $M \mapsto R \oplus M$ preserves limits as a functor $\mathfrak{a}\mathrm{Mod}_R \to \mathfrak{a}\mathrm{CAlg}_{k//R}$ and thus $M \mapsto \mathrm{Der}_k(R, M) \simeq \mathrm{Map}_{\mathfrak{a}\mathrm{CAlg}_{k//R}}(R, R \oplus M)$ preserves limits. Thus, there is a left adjoint $F \colon \mathcal{S} \to \mathfrak{a}\mathrm{Mod}_R$ and we have

$$\mathrm{Der}_k(R, M) \simeq \mathrm{Map}_{\mathcal{S}}(*, \mathrm{Der}_k(R, M)) \simeq \mathrm{Map}_{\mathfrak{a}\mathrm{Mod}_R}(F(*), M).$$

This proves existence and uniqueness: we see that $F(*)$ is a cotangent complex for R over k. $\square$

Exercise 3.41. Prove that this definition coincides with the definition from the previous section when k is discrete.

Exercise 3.42 (Universal property)**.** Prove that $\mathrm{L}_{R/k}$ also has the following universal property. If $S \in \mathfrak{a}\mathrm{CAlg}_k$ and $M \in \mathfrak{a}\mathrm{Mod}_S$ and we fix a map $R \to S$, then

$$\mathrm{Map}_{\mathfrak{a}\mathrm{Mod}_R}(\mathrm{L}_{R/k}, M) \simeq \mathrm{Map}_{\mathfrak{a}\mathrm{CAlg}_{k//S}}(R, S \oplus M).$$

3.4 A little deformation theory

Now, we want to understand what it means for an extension $\tilde{S} \to S$ to be a square-zero extension in $\mathfrak{a}\mathrm{CAlg}_k$.

Definition 3.43. We say that $\tilde{S} \to S$ is a square-zero extension of S by M if there exists a derivation $d\colon S \to \Sigma M$ such that $\tilde{S}$ fits into a pullback square

$$\begin{tikzcd} \tilde{S} \arrow[r] \arrow[d] & S \arrow[d] \\ S \arrow[r, "d"'] & S \oplus \Sigma M, \end{tikzcd}$$

where the bottom horizontal arrow is the map corresponding to the derivation d and the right vertical arrow is the map corresponding to the 0 derivation.

Example 3.44. The base commutative ring is important. For example, the only square-zero extension of $\mathbb{F}_p$ by itself *as an* $\mathbb{F}_p$*-algebra* is $\mathbb{F}_p[x]/(x^2)$. However, since $\mathrm{L}_{\mathbb{F}_p/\mathbb{Z}}$ is more interesting, we see that there is a non-trivial derivation $d\colon \mathbb{F}_p \to \mathbb{F}_p[1]$ corresponding to the square-zero extension $\mathbb{Z}/p^2$.

Proposition 3.45. *Let M be an animated S-module concentrated in degree $k \geqslant 0$. If $\tilde{S}$ is a square-zero extension of S by M and $R \to S$ is a map in $\mathfrak{a}\mathrm{CAlg}_k$, then $\mathrm{Map}_{\mathfrak{a}\mathrm{CAlg}_{k//S}}(R, \tilde{S})$ is a $\mathrm{Der}_k(R, M)$-torsor. That is, the mapping space might be empty, but if it is not, then the choice of a basepoint induces an equivalence to $\mathrm{Der}_k(R, M)$.*

Proof. See Lurie [37, Prop 3.3.5]. □

Remark 3.46 (The obstruction to lifting)**.** In the situation of the proposition, to construct a lift $R \to \tilde{S}$ it is necessary and sufficient to prove that the composite derivation $R \to S \xrightarrow{d} S \oplus \Sigma M$ is trivial. In many cases, this can be checked by hand. For example, if R is smooth over k, then $\pi_0 \mathrm{Der}_k(R, \Sigma M) \simeq \pi_0 \mathrm{Map}_{\mathfrak{a}\mathrm{Mod}_R}(\mathrm{L}_{R/k}, \Sigma M) \cong 0$.

Example 3.47 (Postnikov extensions)**.** Postnikov extensions of animated commutative rings are square-zero. This is not obvious, but it follows from [37, Proposition 3.3.5]. In other words, we find $\tau_{\leqslant n} R$ for $n \geqslant 1$ as a pullback

$$\begin{tikzcd} \tau_{\leqslant n} R \arrow[r] \arrow[d] & \tau_{\leqslant n-1} R \arrow[d] \\ \tau_{\leqslant n-1} R \arrow[r, "d"'] & \tau_{\leqslant n-1} R \oplus \pi_n R[n+1], \end{tikzcd}$$

for some derivation d. Therefore, every animated commutative ring R can be built up from $\pi_0 R$ by a series of square-zero extensions.

Here is a sketch of the argument. Consider the two maps $\tau_{\leqslant n-1} \to \tau_{\leqslant n-1} R \otimes_{\tau_{\leqslant n} R} \tau_{\leqslant n-1 R} S$; these are equalized by the canonical map $\tau_{\leqslant n} R \to \tau_{\leqslant n-1} R$. Applying $\tau_{\leqslant n+1}$, we see that $\tau_{\leqslant n} R$ also equalizes

$$\tau_{\leqslant n-1} R \rightrightarrows \tau_{\leqslant n+1} \left(\tau_{\leqslant n-1} R \otimes_{\tau_{\leqslant n} R} \tau_{\leqslant n-1} R \right).$$

It is not difficult to check that the $(n+1)$-truncation on the right is equivalent to $\tau_{\leqslant n-1} R \oplus \pi_n R[n+1]$. Thus, we see that there is a commutative square

$$\begin{tikzcd} \tau_{\leqslant n} R \arrow[r] \arrow[d] & \tau_{\leqslant n-1} R \arrow[d] \\ \tau_{\leqslant n-1} R \arrow[r] & \tau_{\leqslant n-1} R \oplus \pi_n R[n+1]. \end{tikzcd}$$

One checks that the induced map from $\tau_{\leqslant n}R$ to the pullback is an equivalence. Certainly they have the same homotopy groups, and the isomorphism on π_n can be check by mapping out of $\mathrm{LSym}_k(\pi_n R[n])$ which maps to everything in sight.

Definition 3.48 (Formally étale and formally smooth)**.** Let $k \to R$ be a map of animated commutative rings. We say that R is **formally smooth** over k if for every square-zero extension $\tilde{S} \to S$ and every map $R \to S$, there exists a dotted arrow

$$\begin{tikzcd} & \tilde{S} \arrow[d] \\ R \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

making the diagram commute. If the space of such lifts is contractible, we say that $k \to R$ is **formally étale**.

Exercise 3.49. Prove that $k \to R$ is formally smooth if and only if $\pi_0 \mathrm{Map}_{\mathfrak{a}\mathrm{Mod}_R}(\mathrm{L}_{R/k}, \Sigma M) \cong 0$ for every $M \in \mathfrak{a}\mathrm{Mod}_R$. This occurs if $\mathrm{L}_{R/k}$ is in $\mathfrak{a}\mathrm{Mod}_R^{\mathrm{proj}}$, i.e., if $\mathrm{L}_{R/k}$ is a retract of $R^{\oplus N}$ for some set N, possibly infinite. (In fact, this condition is also necessary.)

Exercise 3.50. Prove that $k \to R$ is formally étale if and only if $\mathrm{L}_{R/k} \simeq 0$.

3.5 Properties of the cotangent complex

By far the most important computations are the following.

Exercise 3.51. Show that if $k \to R$ is a map of animated commutative rings and k' is another animated commutative k-algebra, then the natural map $k' \otimes_k \mathrm{L}_{R/k} \to \mathrm{L}_{k' \otimes_k R/k}$ is an equivalence.

Exercise 3.52. Prove that $\mathrm{L}_{R \otimes_k S/k} \simeq (\mathrm{L}_{R/k} \otimes_k S) \oplus (R \otimes_k \mathrm{L}_{S/k})$.

Exercise 3.53. If $R \in \mathfrak{a}\mathrm{CAlg}_k$ and $S \subseteq \pi_0 R$ is a multiplicative subset, then $S^{-1}R \otimes_R \mathrm{L}_{R/k} \simeq \mathrm{L}_{S^{-1}R/k}$.

Exercise 3.54 (The general conormal sequence)**.** Let $k \to R \to S$ be animated commutative rings. Prove that $S \otimes_R \mathrm{L}_{R/k} \to \mathrm{L}_{S/k} \to \mathrm{L}_{S/R}$.

Exercise 3.55. Prove that $\mathrm{L}_{\mathbb{Z}/\mathbb{Z}[x]} \simeq (x)/(x^2)[1]$.

Exercise 3.56. More generally, prove that if R is a commutative ring and $I \subseteq R$ is an ideal generated by a regular sequence $x_1, \ldots, x_c$, then $\mathrm{L}_{(R/I)/R} \cong I/I^2[1]$, where I/I^2 is a projective R/I module of rank c (the **conormal bundle**, dual to the normal bundle of $\operatorname{Spec} R/I$ in $\operatorname{Spec} R$).

Exercise 3.57. Even more generally, let R be an animated commutative ring and let $x_1, \ldots, x_c$ be a sequence of elements in $\pi_0 R$ generating an ideal $I \subseteq \pi_0 R$. Let $R//(x_1, \ldots, x_c)$ be the pushout

$$\begin{tikzcd} R[y_1, \ldots, y_c] \arrow[r, "y_i \mapsto x_i"] \arrow[d, "y_i \mapsto 0"'] & R \arrow[d] \\ R \arrow[r] & R//(x_1, \ldots, x_c). \end{tikzcd}$$

If R is discrete and each x_i is a non-zero divisor, then $R//(x_1, \ldots, x_c)$ is equivalent to $R/(x_1) \otimes_R R/(x_2) \otimes \cdots \otimes R/(x_c)$, where the tensor product is derived. If R is animated, then in any case $\pi_0 R//(x_1, \ldots, x_c) \cong (\pi_0 R)/I$. Prove that

$$\mathrm{L}_{(R//(x_1,\ldots,x_c))/R} \simeq (R//(x_1, \ldots, x_c))^{\oplus c}[1].$$

More invariantly, if J denotes the fiber of $R \to R//(x_1, \ldots, x_c)$, then the cotangent complex is equivalent to $J \otimes_R (R//(x_1, \ldots, x_c))[1]$, a derived version of $J/J^2[1]$.

Exercise 3.58. Let $R = k[x]/(x^2)$. Compute $\mathrm{L}_{R/k}$ and $\mathrm{L}_{k/R}$.

There is a major structural result about the cotangent complex, which says in good cases that if $\mathrm{L}_{R/k}$ has bounded Tor-amplitude, then in fact $k \to R$ is a locally complete intersection morphism and $\mathrm{L}_{R/k}$ has Tor-amplitude in $[0, 1]$.

Definition 3.59. Let R be an animated commutative ring and let $M \in \mathfrak{a}\mathrm{Mod}_R$. We say that M has **Tor-amplitude** in $[a, b]$ if for any $\pi_0 R$-module N, we have $\mathrm{H}_i(M \otimes_k N) = 0$ for $i \notin [a, b]$. If M has Tor-amplitude in $[a, b]$ for some $[a, b]$, then we say that M has **bounded Tor-amplitude**.

Example 3.60. If R is discrete and M is quasi-isomorphic to a complex of the form

$$\cdots \leftarrow 0 \leftarrow F_a \leftarrow F_{a+1} \leftarrow \cdots \leftarrow F_b \leftarrow 0 \leftarrow \cdots,$$

where each F_i is flat, then M has Tor-amplitude in $[a, b]$.

Example 3.61. If R is discrete and M is an R-module, then M has Tor-amplitude in $[0, b]$ if and only if the flat dimension of M is at most b.

The following theorem is deep and we will not prove it. See [7].

Theorem 3.62 (Avramov)**.** *Suppose that $f: k \to R$ is a map of noetherian commutative rings. The map f is a local complete intersection if and only if R is locally of finite flat dimension over k and $\mathrm{L}_{R/k}$ has bounded Tor-amplitude.*

Remark 3.63. Note that if we let $R = k[x]/(x^2)$, then $\mathrm{L}_{k/R}$ has Tor-amplitude in $[1, 2]$ by Exercise 3.58. However, k is certainly not lci over R. This does not contradict Avramov's theorem because k does not have finite flat dimension over R.

3.6 Finiteness

We have the following fundamental results.

Proposition 3.64. *Let k be an animated commutative ring. If R is a compact object of $\mathfrak{a}\mathrm{CAlg}_k$, then $\mathrm{L}_{R/k}$ is a compact object of $\mathfrak{a}\mathrm{Mod}_R$. In particular, $\mathrm{L}_{R/k}$ has bounded Tor-amplitude.*

Proof. Since R is compact in $\mathfrak{a}\mathrm{CAlg}_k$, it is also compact in $\mathfrak{a}\mathrm{CAlg}_{k//R}$. (Check this!) In particular, $\mathrm{Map}_{\mathfrak{a}\mathrm{CAlg}_{k//R}}(R, R \oplus M)$ preserves filtered colimits in M. It follows that

$$\mathrm{Map}_{\mathfrak{a}\mathrm{Mod}_R}(\mathrm{L}_{R/k}, M) \simeq \mathrm{Map}_{\mathfrak{a}\mathrm{CAlg}_{k//R}}(R, R \oplus M)$$

preserves filtered colimits in M. Thus, $\mathrm{L}_{R/k}$ is compact in $\mathfrak{a}\mathrm{Mod}_R$. Now, it turns out that every compact object of $\mathfrak{a}\mathrm{Mod}_R$ has bounded Tor-amplitude. See for example [60, Proposition 2.22]. $\square$

Proposition 3.65. *Let k be an animated commutative ring and let $\mathrm{colim}_{i\in I} R_i \simeq R$ be a colimit in $\mathfrak{a}\mathrm{CAlg}_k$. The natural map $\mathrm{colim}_{i\in I} R \otimes_{R_i} \mathrm{L}_{R_i/k} \to \mathrm{L}_{R/k}$ is an equivalence.*

Proof. It is enough to note that if $S \oplus J \to S$ is a trivial square-zero extension, with $S \in \mathfrak{a}\mathrm{CAlg}_k$ and $J \in \mathfrak{a}\mathrm{Mod}_S$, and $R \to S$ is a fixed map in $\mathfrak{a}\mathrm{CAlg}_k$, then $\mathrm{colim}_{i\in I} S_i \to R$ is also an equivalence in $\mathfrak{a}\mathrm{CAlg}_{k//S}$, where each R_i is viewed as an object of $\mathfrak{a}\mathrm{CAlg}_{k//S}$ via $R_i \to R \to S$. We conclude by using the universal property of Exercise 3.42. $\square$

Remark 3.66. In particular, if $R_\bullet$ is a simplicial object of $\mathfrak{a}\mathrm{CAlg}_k$ (for example a simplicial commutative ring), then $\mathrm{L}_{|R_\bullet|/k} \simeq ||R_\bullet| \otimes_{R_\bullet} \mathrm{L}_{R_\bullet/k}|$.

4 Week 4: Derived de Rham cohomology (10 Oct.– 16 Oct.)

The search for cohomology theories goes back to the foundations of algebraic topology and algebraic geometry. We have Čech cohomology, sheaf cohomology, computed via Čech complexes, and singular cohomology defined using the simplicial set $\mathrm{Sing}_\bullet(X)$ if X is a topological space. In algebraic geometry, there is coherent sheaf cohomology, but this is not really fine enough for most problems. For example, it does not have something like Poincaré duality. Good algebraic theories include étale cohomology, algebraic de Rham cohomology, and de Rham cohomology's family: its sibling Hodge–Tate cohomology, its parent crystalline cohomology, and its ancestor prismatic cohomology.[9] Many of the more sophisticated theories are covered in Tsygan's course this quarter, so I will focus on the de Rham theory. For the time being, we focus on the affine case.

4.1 The discrete de Rham complex

References 4.1. The discrete de Rham complex as studied below originates as far as I can tell with Grothendieck [23] and Hartshorne (see [25] and the survey article [24]), although Hartshorne goes much farther and studies a variant which as we will see is suitable for working with singular schemes.[10] It was also heavily used in early days by Katz–Oda [34], Katz [31, 32, 33], and Deligne [16].

Definition 4.2. Recall that if R is a commutative ring and M is an R-module, we define the graded ring ΛM to be the quotient of the tensor algebra $\bigoplus_{i\geqslant 0} M^{\otimes i}$ by the 2-sided graded ideal generated by $m\otimes m$ for $m\in M$. In particular, $m^2=0$ in ΛM for $m\in M$. The ith exterior power of M is defined to be $\Lambda^i M$, the ith graded piece of $\Lambda^i M$.

Remark 4.3. The exterior algebra is graded-commutative, meaning that $xy=(-1)^{|x||y|}yx$. Indeed, $0=(m+n)^2=m^2+n^2+mn+nm=mn+nm$, so that $mn=-nm$ if $m,n\in M$.

Construction 4.4. Let R be a commutative k-algebra, where k is discrete. We can build a commutative differential graded k-algebra

$$0\to R\xrightarrow{\mathrm{d}}\Omega^1_{R/k}\xrightarrow{\mathrm{d}}\Omega^2_{R/k}\to\cdots,$$

where $\Omega^i_{R/k}=\Lambda^i\Omega^1_{R/k}$ and d is the exterior, or de Rham, derivative. This is exactly the same construction as in differential calculus on manifolds, but we are restricting our attention to algebraic functions on $X=\operatorname{Spec} R$. We will write $\Omega^\bullet_{R/k}$ for the **discrete de Rham complex** of R over k. Since $\omega^2=0$ if ω has odd degree in $\Omega^\bullet_{R/k}$, this cdga is **strict**.[11]

Example 4.5. Suppose that $R=k[x]$. Then $\Omega^1_{R/k}\cong R\cdot\mathrm{d}x$ and the differential $R\xrightarrow{\mathrm{d}}\Omega^1_{R/k}$ can be written in this basis as $f\mapsto\frac{\partial f}{\partial x}\mathrm{d}x$. If k is a commutative $\mathbb{Q}$-algebra, then every term $x^n\mathrm{d}x$ has an antiderivative: $\mathrm{d}(\frac{x^{n+1}}{n+1})=x^n\mathrm{d}x$, so we see in this case that $k\simeq\Omega^\bullet_{R/k}$.

Exercise 4.6. Compute $\mathrm{H}^*(\Omega^\bullet_{\mathbb{Z}[x]/\mathbb{Z}})$.

Exercise 4.7. Compute $\mathrm{H}^*(\Omega^\bullet_{k[x^{\pm1}]/k})$ when k is a commutative $\mathbb{Q}$-algebra.

Exercise 4.8. Compute the discrete de Rham complex of $\mathbb{Z}[x]/(x^2)$ over $\mathbb{Z}$.

Exercise 4.9. Suppose that $R\to R/I$ is a surjection. Compute the discrete de Rham complex of R/I over R.

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[9] Some of these are better than others: for example, de Rham cohomology in characteristic p is not enough to prove, or even state, the Weil conjectures.

[10] While published in 1975, Hartshorne places the birth of [25] in 1967.

[11] The term 'discrete de Rham complex' is my own. It will be justified below.

Construction 4.10. If X is a k-scheme, then we can define a complex of sheaves

$$0 \to \mathcal{O}_X \to \Omega^1_{X/k} \to \Lambda^2\Omega^2_{X/k} \to \cdots$$

by globalizing the discrete de Rham complex on open affine subschemes. The discrete de Rham cohomology of X is defined to be $\mathrm{R}\Gamma(\Omega_{X/k})$, the (hyper)cohomology of the complex, where $\Omega_{X/k}$ denotes the underlying object in $\mathcal{D}(\mathcal{O}_X)$, the derived ∞-category of sheaves of $\mathcal{O}_X$-modules on X. Note that this complex is not quasicoherent: even though each $\Omega^i_{X/k}$ is a quasicoherent sheaf, the differentials are not $\mathcal{O}_X$-linear. Nevertheless, it has a complete filtration (the Hodge filtration) with quasicoherent graded pieces. The **Hodge filtration** on discrete de Rham cohomology is given by

$$\mathrm{F}^\star_{\mathrm{H}}\mathrm{R}\Gamma(\Omega_{X/k}) \simeq \mathrm{R}\Gamma(\Omega^{\geqslant\star}_{X/k})$$

with graded pieces

$$\mathrm{gr}^s_{\mathrm{H}}\mathrm{R}\Gamma(\Omega_{X/k}) \simeq \mathrm{R}\Gamma(\Omega^s_{X/k})[-s].$$

The associated **Hodge–de Rham spectral sequence** is

$$\mathrm{E}^{s,t}_1 = \mathrm{H}^t(X, \Omega^s_{X/k}) \Rightarrow \mathrm{H}^{s+t}(\Omega^\bullet_{X/k}).$$

The starting point for this entire story is the following theorem of Grothendieck [23].

Theorem 4.11 (Grothendieck's comparison theorem)**.** *If X is a smooth $\mathbb{C}$-scheme, then there is a natural quasi-isomorphism*

$$\mathrm{R}\Gamma(\Omega^\bullet_{X/\mathbb{C}}) \simeq \mathrm{R}\Gamma(X(\mathbb{C}), \mathbb{C})$$

induced by inclusion of algebraic differential forms into smooth $\mathbb{C}$-valued differential forms.

The theorem says we can compute the singular cohomology of the topological space $X(\mathbb{C})$ with $\mathbb{C}$-coefficients via this purely algebraic object, the discrete de Rham complex.

Example 4.12 (Badness in the singular case)**.** The singular situation is not so good, but it is not easy to find examples in the literature. For example the cases of the dual numbers $\mathbb{C}[x]/(x^2)$, the coordinate axes $\mathbb{C}[x,y]/(xy)$, and the cusp $\mathbb{C}[x,y]/(y^2-x^3)$ all have the property that the discrete de Rham complex computes the correct $\mathbb{C}$-valued Betti cohomology. A MathOverflow question led me to an example of Arapura–Kang [5, Ex. 4.4], the affine curve $\mathbb{C}[x,y]/(x^5+y^5+x^2y^2)$.

Challenge 4.13. Compute the singular cohomology and the discrete de Rham cohomology of the affine curve

$$\mathbb{C}[x,y]/(x^5+y^5+x^2y^2).$$

Prove that they are different. I do not know the solution to this challenge at the moment of writing.

Here is a nice universal property of the discrete de Rham complex.

Theorem 4.14. *Let k be a commutative ring. The functor $\mathrm{scdga}^{\geqslant 0}_k \to \mathrm{CAlg}_k$ which sends a (cohomological) strict cdga over k*

$$0 \to R^0 \to R^1 \to \cdots$$

to R^0 admits a left adjoint given by $S \mapsto \Omega^\bullet_{S/k}$.

Proof. We have to show that the forgetful map $\mathrm{Hom}_{\mathrm{scdga}_k}(\Omega^\bullet_{S/k}, R^\bullet) \to \mathrm{Hom}_{\mathrm{CAlg}_k}(S, R^0)$ is a bijection. If $f^\bullet, g^\bullet$ are two maps $\Omega^\bullet_{S/k} \to R^\bullet$ of strict cdgas over k such that $f^0 = g^0$, then $f^1 = g^1$. Indeed, the elements $\mathrm{d}x$ for $x \in S$ give an S-basis for $\Omega^1_{S/k}$ and we must have $f^1(\mathrm{d}x) = \mathrm{d}(f^0(x)) = \mathrm{d}(g^0(x)) = g^1(\mathrm{d}x)$ since $f^\bullet$ and $g^\bullet$ are cdga maps. Since $\Omega^\bullet_{S/k}$ is generated in degree 1, it follows that $f^\bullet = g^\bullet$. This proves injectivity. On the other hand, if we have $S \to R^0$, then the composition $S \to R^0 \to R^1$ is a k-derivation so there is a map $\Omega^1_{R/k} \to R^1$. We leave it to the reader that this extends to a map $\Omega^\bullet_{S/k} \to R^\bullet$. of cdgas over k. □

Corollary 4.15. *It follows that* $\Omega^\bullet_{-/k} \colon \mathrm{CAlg}_k \to \mathrm{scdga}_k$ *is fully faithful.*

Proof. For any $R \in \mathrm{CAlg}_k$, the counit map $R \to \Omega^0_{R/k} \cong R$ is an isomorphism. □

4.2 Derived commutative versus $\mathbb{E}_\infty$-k-algebras

References 4.16. See Raksit's paper [53, Sec. 4] for the Bhatt–Mathew theory of derived commutative rings. See also [14, Sec. 3] for more details on extending monads. For ∞-operads and $\mathbb{E}_\infty$-rings, see [39].

We will define derived de Rham cohomology as the result of left Kan extending the discrete de Rham complex functor from finitely presented polynomial rings to all animated commutative k-algebras. We have to think however about the target category and there are two natural choices. Indeed, there is an issue because even though we are working with animated commutative k-algebras, which are connective, the discrete de Rham complex of a commutative k-algebra is coconnective: it has *cohomology* concentrated in non-negative degrees. So, it does not typically admit the structure of an animated commutative k-algebra.

There are two remedies for this. We can either use $\mathbb{E}_\infty$-rings or we can use **derived commutative rings**, which is a notion due to Bhatt and Mathew. They prove that the LSym_k monad on $\mathcal{D}(k)_{\geqslant 0}$, which has the property that $\mathrm{Mod}_{\mathrm{LSym}_k}(\mathcal{D}(k)_{\geqslant 0}) \simeq \mathfrak{a}\mathrm{CAlg}_k$, extends to a monad on the entire derived ∞-category $\mathcal{D}(k)$, which might as well also be called LSym_k. The resulting ∞-category of modules[12] $\mathrm{Mod}_{\mathrm{LSym}_k}(\mathcal{D}(k))$ for the LSym_k-monad acting on $\mathcal{D}(k)$ is the ∞-category $\mathfrak{d}\mathrm{CAlg}_k$ of **derived commutative k-algebras**. It is an analogue of the theory of animated commutative rings, but it includes nonconnective objects, meaning that there objects $R \in \mathfrak{d}\mathrm{CAlg}_k$ with $\pi_i R \neq 0$ for $i < 0$.

Bhatt and Mathew prove that $\mathfrak{d}\mathrm{CAlg}_k$ is closed under all limits and colimits in $\mathrm{Alg}_k^{\mathbb{E}_\infty}$, the ∞-category of $\mathbb{E}_\infty$-k-algebras. In particular, this means that if X is a scheme, then $\mathrm{R}\Gamma(X, \mathcal{O}_X)$ naturally admits the structure of a derived commutative k-algebra, which is nice because it clearly should have more structure than simply that of an $\mathbb{E}_\infty$-ring.

For simplicity, we will work with $\mathbb{E}_\infty$-algebras over k, as this is something that makes sense in any symmetric monoidal ∞-category, whereas $\mathfrak{d}\mathrm{CAlg}_k$ requires some work to translate into filtered or graded settings. But, it is worth working out the theory in those situations as additional structure and universal properties are revealed. This is what Raksit does in [53].

4.3 Derived de Rham cohomology

References 4.17. Illusie [30, Sec. VIII.2.1], Beilinson [8], Bhatt [12]. See also Szamuely–Zábrádi and Raksit [53].

Let k be a commutative ring. Consider the functor $\mathrm{CAlg}_k \to \mathrm{scdga}_k$ defined by taking R to the discrete de Rham complex $\Omega^\bullet_{R/k}$. While there is not a good homotopy theory of (strict) commutative differential graded k-algebras unless $\mathbb{Q} \subseteq k$, there is nevertheless a functor $\mathrm{scdga}_k \to \mathrm{Alg}_k^{\mathbb{E}_\infty}$ which inverts quasi-isomorphisms. Thus, we obtain a functor

$$\mathrm{CAlg}_k \to \mathrm{Alg}_k^{\mathbb{E}_\infty}$$

[12] Modules for a monad are often also called algebras for that monad.

which takes R to $\Omega_{R/k}$, the $\mathbb{E}_\infty$-ring underlying $\Omega^\bullet_{R/k}$.

Definition 4.18. The **derived de Rham complex** functor

$$\mathrm{L}\Omega_{-/k} \colon \mathfrak{a}\mathrm{CAlg} \to \mathrm{Alg}_k^{\mathbb{E}_\infty}$$

is obtained by left Kan extension from the restriction of $\Omega_{-/k}$ to $\mathrm{CAlg}_k^{\omega 1\mathrm{proj}}$. For an animated commutative k-algebra R, we call $\mathrm{L}\Omega_{R/k}$ the derived de Rham complex of R over k.

Remark 4.19. In practice, one can compute $\mathrm{L}\Omega_{R/k}$ by first extending $\Omega_{-/k}$ to all polynomial rings (not just the finitely presented polynomial rings). Then, choosing a resolution $|S_\bullet| \simeq R$, one lets $\mathrm{L}\Omega_{R/k} \simeq |\Omega_{S_\bullet/k}|$, where the geometric realization (colimit) is computed in $\mathrm{Alg}_k^{\mathbb{E}_\infty}$.

Remark 4.20. So far, the 'complex' in derived de Rham complex is a bit of a misnomer, but it will come into play when we consider the Hodge filtration.

Proposition 4.21. *The derived de Rham complex functor* $\mathrm{L}\Omega_{-/k} \colon \mathfrak{a}\mathrm{CAlg}_k \to \mathrm{Alg}_k^{\mathbb{E}_\infty}$ *preserves colimits.*

Proof. As a left Kan extension of the category of $\mathrm{CAlg}_k^{\omega 1\mathrm{proj}}$, the functor $\mathrm{L}\Omega_{-/k}$ automatically preserves sifted colimits by Proposition 3.39, Week 2. By the same proposition, it is enough to see that $\mathrm{L}\Omega_{-/k}$ commutes with finite coproducts when restricted to $\mathrm{CAlg}_k^{\omega 1\mathrm{proj}}$. In other words, we must show that if X and Y are finite sets, the natural map $\Omega^\bullet_{k[X]/k} \otimes_k \Omega^\bullet_{k[Y]/k} \to \Omega^\bullet_{k[X \coprod Y]/k}$ is a quasi-isomorphism. This is easy to see because in fact the map is an *isomorphism* of strict commutative dgas:

$$\mathrm{gr}^n \left(\Omega^\bullet_{k[X]/k} \otimes_k \Omega^\bullet_{k[Y]/k} \right) \cong \bigoplus_{i+j=n} \Omega^i_{k[X]/k} \otimes_k \Omega^j_{k[Y]/k} \cong \Omega^n_{k[X \coprod Y]/k}.$$

□

Corollary 4.22. *Given animated commutative rings* $R, S \in \mathfrak{a}\mathrm{CAlg}_k$*, we have a natural equivalence of* $\mathbb{E}_\infty$*-*k*-algebras* $\mathrm{L}\Omega_{R/k} \otimes_k \mathrm{L}\Omega_{S/k} \simeq \mathrm{L}\Omega_{R \otimes_k S/k}$.

We will show in the remainder of this section that derived de Rham cohomology is not interesting in characteristic 0.

Example 4.23. Suppose that k is a commutative $\mathbb{Q}$-algebra. If $R \cong k[X]$ is a polynomial ring on a set X, then the natural map $k \to \Omega_{R/k}$ is a quasi-isomorphism. Indeed, this reduces (by tensor products and filtered colimits) to the case of $R \cong k[x]$, where it is an easy computation, the algebraic Poincaré lemma.

Lemma 4.24. *If* k *is a commutative* $\mathbb{Q}$*-algebra, then* $\mathrm{L}\Omega_{-/k} \colon \mathfrak{a}\mathrm{CAlg}_k \to \mathrm{Alg}_k^{\mathbb{E}_\infty}$ *is the constant functor sending* R *to* k*.*

Proof. Indeed, by Example 4.23, we are left Kan extending the constant functor. □

Thus, this theory is not interesting in characteristic 0. It is much more interesting in mixed or positive characteristic.

4.4 The conjugate filtration

References 4.25. For a good reference on the conjugate filtration (also known as the Cartier isomorphism) on the discrete de Rham cohomology of smooth schemes in characteristic p, see [32, Thm. 7.2]. The filtration is also used heavily in [17]. In characteristic p, derived de Rham cohomology is very interesting by virtue of the conjugate filtration, a deep observation of Bhatt [12].

Lemma 4.26. *Let k be a commutative $\mathbb{F}_p$-algebra and let R be a flat commutative k-algebra. Write $R^{(p)} = R \otimes_k (\mathrm{F}_k)_* k$, where $\mathrm{F}_k \colon k \xrightarrow{x \mapsto x^p} k$ is the absolute Frobenius, so that $(\mathrm{F}_k)_* k$ denotes k as a k-module via restriction of scalars $\mathrm{F}_k \colon k \xrightarrow{x \mapsto x^p} k$. This is a sub-$k$-algebra of R. The discrete de Rham complex $\Omega^\bullet_{R/k}$ is a complex of $R^{(p)}$-modules, which is to say that the differential is $R^{(p)}$-linear.*

Proof. This follows from the fact that $\mathrm{d}x^p = px^{p-1}\mathrm{d}x = 0$ in $\Omega^\bullet_{R/k}$ for any $x^p \in R^{(p)}$. □

Theorem 4.27 (Cartier)**.** *Suppose that k is a commutative $\mathbb{F}_p$-algebra and let R be a smooth commutative k-algebra. There are natural isomorphisms*[13]

$$\mathrm{C}^{-1} \colon \Omega^i_{R^{(p)}/k} \cong \mathrm{H}^i(\Omega_{R/k})$$

of $R^{(p)}$-modules for all $i \geqslant 0$ induced by $\mathrm{d}x^p \mapsto x^{p-1}\mathrm{d}x$ when $x^p \in R^{(p)}$.

Exercise 4.28. Prove Cartier's theorem when $R = \mathbb{F}_p[x]$ with $k = \mathbb{F}_p$.

Construction 4.29. Let k be a commutative $\mathbb{F}_p$-algebra and let R be a smooth commutative k-algebra. The **conjugate filtration** on $\Omega_{R/k}$ is the increasing Whitehead filtration $\mathrm{F}^{\mathrm{conj}}_i \Omega_{R/k} \simeq \tau_{\geqslant -i}\Omega_{R/k}$. This looks like $0 \to \mathrm{F}^{\mathrm{conj}}_0 \Omega_{R/k} \to \mathrm{F}^{\mathrm{conj}}_1 \Omega_{R/k} \to \cdots$. The ith graded piece is given by

$$\mathrm{gr}^{\mathrm{conj}}_i \Omega_{R/k} \simeq \frac{\mathrm{F}_i \Omega_{R/k}}{\mathrm{F}_{i-1}\Omega_{R/k}} \simeq \mathrm{H}^i(\Omega_{R/k})[-i] \cong \Omega^i_{R^{(p)}/k}[-i],$$

where the final equivalence is via the Cartier isomorphism.

Remark 4.30. This filtration is complete, exhaustive, and multiplicative. An increasing filtration $\mathrm{F}_\star S$ on an $\mathbb{E}_\infty$-ring is **complete** if $\lim_{i \mapsto -\infty} \mathrm{F}_i R \simeq 0$, **exhaustive** if $\mathrm{colim}_{i \mapsto \infty} \mathrm{F}_i R \simeq R$, and **multiplicative** if roughly speaking $\mathrm{F}_i R \otimes \mathrm{F}_j R \to R \otimes R \to R$ factors through $\mathrm{F}_{i+j} R \to R$.

Definition 4.31. Let k be a commutative $\mathbb{F}_p$-algebra. We define the complete, exhaustive, multiplicative **conjugate filtration** on the derived de Rham complex functor relative to k by left Kan extending the conjugate filtration from finitely presented polynomial k-algebras. Given $R \in \mathfrak{a}\mathrm{CAlg}_k$, the resulting filtration $\mathrm{F}^{\mathrm{conj}}_\star \mathrm{L}\Omega_{R/k}$ has graded pieces

$$\mathrm{gr}^{\mathrm{conj}}_i \mathrm{L}\Omega_{R/k} \simeq \mathrm{L}\Lambda^i \mathrm{L}_{R^{(p)}/k}[-i],$$

for each i. Note in particular that $\mathrm{gr}^{\mathrm{conj}}_0 \mathrm{L}\Omega_{R/k} \simeq R^{(p)}$, so that $\mathrm{L}\Omega_{R/k}$ is an $\mathbb{E}_\infty$-algebra over $R^{(p)}$. However, if R is not flat over k, $R^{(p)}$ must be interpreted as the pushout $R \otimes_k (F_k)_* k$ in $\mathfrak{a}\mathrm{CAlg}_k$; in particular it might be a non-discrete animated commutative k-algebra.

Remark 4.32. The term $\mathrm{L}\Lambda^i \mathrm{L}_{R^{(p)}/k}$ denotes the ith derived exterior power, meaning the nonabelian derived functor of $P \mapsto \Lambda^i P$ for P a finitely presented projective $R^{(p)}$-module. It is computed on a general $M \in \mathcal{D}(R^{(p)})_{\geqslant 0}$ by writing $M \simeq P_\bullet$ were $P_\bullet$ is a simplicial resolution of M which is term-wise projective, and then $|\Lambda^i P_\bullet| \simeq \mathrm{L}\Lambda^i M$.

Exercise 4.33. Show that if k is a perfect $\mathbb{F}_p$-algebra, then $\mathrm{L}_{k/\mathbb{F}_p} \simeq 0$. Thus, it follows using the conjugate filtration that $\mathrm{L}\Omega_{k/\mathbb{F}_p} \simeq k$.

[13] These are called the **inverse Cartier transforms**.

4.5 The Hodge filtration

Definition 4.34. We let $\mathcal{DF}(k)$ denote the ∞-category of decreasing filtered complexes, i.e., $\mathcal{DF}(k) \simeq \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{D}(k))$, where $\mathbb{Z}^{\mathrm{op}}$ is viewed as the category corresponding to the totally ordered set $\{\cdots \to n+1 \to n \to n-1 \to \cdots\}$. An object $X(\star) \in \mathcal{DF}(k)$ is called a **filtration**. A filtration on an object $M \in \mathcal{D}(k)$ is a filtration $X(\star)$ together with a map $\mathrm{colim}_{n\to-\infty} X(n) \to M$. We will write $\mathrm{F}^{\star}M$ for these filtrations. We say that such a filtration is exhaustive if $\mathrm{colim}_{n\to-\infty} \mathrm{F}^n M \to M$ is an equivalence. We say that a filtration $X(\star)$ or $\mathrm{F}^{\star}M$ is complete if $\lim_{n\to\infty} \mathrm{F}^n \simeq 0$.

We picture a (decreasing) filtration as a sequence of maps

$$\cdots \to \mathrm{F}^{n+1}M \to \mathrm{F}^n M \to \mathrm{F}^{n-1}M \to \cdots .$$

There is a tendency to view a filtration as being a filtration of some underlying object we are trying to understand. This is not strictly necessary as we can always view $X(\star)$ as being a filtration on $\mathrm{colim}_{n\to-\infty} X(n)$.

Definition 4.35. The ∞-category $\mathcal{DF}(k)$ admits a symmetric monoidal structure under **Day convolution**, which means that if $X(\star), Y(\star) \in \mathcal{DF}(k)$, then

$$(X \otimes Y)(n) \cong \operatorname*{colim}_{i+j\geqslant n} X(i) \otimes_k Y(j).$$

Thus, it makes sense to think about $\mathbb{E}_\infty$-algebra objects of $\mathcal{DF}(k)$.

Example 4.36. Any cdga $R^\bullet$ over k defines an $\mathbb{E}_\infty$-algebra object of $\mathcal{DF}(k)$ where the underlying object comes equipped with the stupid or naive filtration with $\mathrm{gr}^n R^\bullet \simeq R^n$.

Warning 4.37. There is a big difference between $\mathbb{E}_\infty$-algebras of $\mathcal{DF}(k)$ and filtrations of $\mathbb{E}_\infty$-algebras, which would be given by $\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{Alg}_k^{\mathbb{E}_\infty})$.

Definition 4.38. We defined **Hodge-filtered derived de Rham cohomology** to be the left Kan extension $\mathrm{F}^{\star}_{\mathrm{H}}\mathrm{L}\Omega_{R/k} \colon \mathfrak{a}\mathrm{CAlg}_k \to \mathrm{Alg}^{\mathbb{E}_\infty}(\mathcal{DF}(k))$ of $\mathrm{F}^{\star}_{\mathrm{H}}\Omega^\bullet_{-/k} \colon \mathfrak{a}\mathrm{CAlg}_k^{\omega 1\mathrm{proj}} \to \mathrm{scdga}_k \to \mathrm{Alg}^{\mathbb{E}_\infty}(\mathcal{DF}(k))$.

Definition 4.39. Hodge-complete derived de Rham cohomology of R over k is defined to be $\widehat{\mathrm{L}\Omega}_{R/k}$, the completion of $\mathrm{L}\Omega_{R/k}$ with respect to the (derived) Hodge filtration. Alternatively, we could define this as the left Kan extension of $\mathrm{F}^{\star}_{\mathrm{H}}\Omega^\bullet_{-/k} \colon \mathfrak{a}\mathrm{CAlg}_k^{\omega 1\mathrm{proj}} \to \mathrm{Alg}^{\mathbb{E}_\infty}\left(\widehat{\mathcal{DF}}(k)\right)$ as a functor to $\mathbb{E}_\infty$-algebras in *complete* filtered complexes.

Remark 4.40 (Graded pieces)**.** There are natural equivalences

$$\mathrm{gr}^s_{\mathrm{H}}\mathrm{L}\Omega_{R/k} \simeq \mathrm{gr}^s_{\mathrm{H}}\widehat{\mathrm{L}\Omega}_{R/k} \simeq \mathrm{L}\Lambda^s \mathrm{L}_{R/k}[-s],$$

where $\mathrm{L}\Lambda^s\mathrm{L}_{R/k}$ denotes the derived sth exterior power functor.

Exercise 4.41. Prove that for any smooth k-algebra R, we have an equivalence $\widehat{\mathrm{L}\Omega}_{R/k} \simeq \Omega_{R/k}$.

Remark 4.42. This works even for commutative $\mathbb{Q}$-algebras, so we see that Hodge-complete derived de Rham cohomology can be more interesting than the non-complete variant.

Remark 4.43 (Homotopy-coherent cochain complexes)**.** Let $\mathrm{F}^{\star}M$ be a filtration. For each $s \in \mathbb{Z}$, there is a fiber sequence

$$\mathrm{gr}^{s+1}M \to \frac{\mathrm{F}^s M}{\mathrm{F}^{s+2}M} \to \mathrm{gr}^s M$$

and hence a map $\mathrm{gr}^s M \to \mathrm{gr}^{s+1} M[1]$. We can assemble these maps into a kind of cochain complex:

$$\cdots \to \mathrm{gr}^{-1} M[-1] \to \mathrm{gr}^0 M \to \mathrm{gr}^1 M[1] \to \mathrm{gr}^2 M[2] \to \cdots,$$

having the property that any composition d^2 is nullhomotopic. The ∞-category of complete filtered complexes is equivalent to the ∞-category of homotopy-coherent cochain complexes, in the sense above, in $\mathcal{D}(k)$. This perspective is made precise in the forthcoming PhD thesis [6] of Stefano Ariotta and is also explained in detail in [53, Sec. 5].

Remark 4.44. If we look at $\mathrm{F}^\star_{\mathrm{H}}\widehat{\mathrm{L}\Omega}_{R/k}$ as a homotopy-coherent chain complex, we obtain the pleasing picture

$$0 \to R \to \mathrm{L}_{R/k} \to \mathrm{L}\Lambda^2 \mathrm{L}_{R/k} \to \cdots, \tag{1}$$

which looks exactly like a derived version of the discrete de Rham complex as a cochain complex. In fact, applying π_0 to (1) we obtain

$$0 \to R \to \Omega^1_{R/k} \to \Omega^2_{R/k} \to \cdots,$$

the discrete de Rham complex, which justifies the terminology.

Remark 4.45. We will see in Week 7 that there is a t-structure on complete filtered complexes, the Beilinson t-structure, which has heart the abelian category of cochain complexes of k-modules. With respect to this t-structure $\mathrm{F}^\star\widehat{\mathrm{L}\Omega}_{R/k}$ is connective and $\pi_0^{\mathrm{B}}\mathrm{F}^\star\widehat{\mathrm{L}\Omega}_{R/k} \cong \Omega^\bullet_{R/k}$, the discrete de Rham complex.

4.6 Hartshorne's algebraic de Rham cohomology

Hartshorne constructs in [25] an alternative theory of de Rham cohomology for finitely presented morphisms. If R is a finitely presented commutative k-algebra, we consider a presentation $S \twoheadrightarrow R$ where S is a finitely presented polynomial ring over k (and is in particular smooth over k). The differentials d in the discrete de Rham complex $\Omega^\bullet_{S/k}$ are continuous with respect to the I-adic topology, where $I = \ker(S \to R)$. Thus, we can complete to obtain a complex

$$\widehat{S} \xrightarrow{\mathrm{d}} \Omega^1_{S/k} \otimes_S \widehat{S} \xrightarrow{\mathrm{d}} \cdots.$$

Hartshorne proved that if X is of finite presentation over $\mathbb{C}$, then the corresponding theory recovers complex singular cohomology.

Then, Bhatt proved in [11] that $\widehat{\mathrm{L}\Omega}_{R/k} \simeq \Omega^H_{R/k}$, so that Hodge-complete derived de Rham cohomology recovers complex singular cohomology of finite type $\mathbb{C}$-schemes.

Exercise 4.46. Show that if S is a commutative k-algebra and $I \subseteq S$ is a finitely generated ideal, then the differentials in the discrete de Rham complex $\Omega^\bullet_{S/k}$ are continuous with respect to the I-adic topology.

4.7 Décalage

References 4.47. I have not found a source I love for the divided power functors. Berthelot's book contains a nice overview [9, I.1.4] and cites Roby's paper [54, Chapitre III].

Consider the three derived functors

$$\mathrm{LSym}, \quad \mathrm{L}\Lambda, \quad \mathrm{L}\Gamma,$$

which define derived commutative, derived exterior, and derived divided power algebras. These are defined by left Kan extending the appropriate monad from flat k-modules to all animated k-modules. Moreover, they are graded and have graded parts LSym^s, $\mathrm{L}\Lambda^s$, and $\mathrm{L}\Gamma^s$, respectively. If k is a commutative ring and M is a flat k-module, then

$$\Lambda^s(M) \simeq (M^{\otimes s})^{\Sigma_s} \subseteq M^{\otimes s},$$

the submodule of elements fixed by the Σ_s-action.

Here are two results of Illusie.

Proposition 4.48. *Let k be an animated commutative ring and let let M be an animated k-module.*

(a) *There is a natural equivalence* $\mathrm{LSym}^s(M[1]) \simeq \mathrm{L}\Lambda^s(M)[s]$.

(b) *There is a natural equivalence* $\mathrm{L}\Lambda^s(M[1]) \simeq \mathrm{L}\Gamma^s(M)[s]$.

Proof. We follow the proofs in [37, Lem. 3.1.1] and [37, Lem. 3.1.2]. The functors

$$\mathrm{LSym}^s(M[1]), \quad \mathrm{L}\Lambda^s(M)[s], \quad \mathrm{L}\Lambda^s(M[1]), \quad \mathrm{L}\Gamma^s(M)[2]$$

all preserve sifted colimits in M, which lets us reduce to the case where k and M are both discrete and M is a free k-module.

If R is an animated commutative ring and $x \in \pi_1 R$, then $x^2 = 0$ in $\pi_2 R$. It is clear that $2x^2 = 0$ from graded-commutativity. However, we can reduce this question to the case of the free object $\mathrm{LSym}(\mathbb{Z}[1]) \simeq \mathbb{Z} \otimes_{\mathbb{Z}[y]} \mathbb{Z}$ (where the equivalence follows because $\mathrm{LSym}\colon \mathfrak{a}\mathrm{Mod}_k \to \mathfrak{a}\mathrm{CAlg}_k$ commutes with colimits) and we have $\mathrm{Tor}_t^{\mathbb{Z}[y]}(\mathbb{Z},\mathbb{Z}) = 0$ for $t > 1$. It follows that there is a natural map $\Lambda^s(\pi_1 R) \to \pi_s R$ for any s; in particular, there is a natural map $\Lambda^s(M) \to \pi_s \mathrm{LSym}(M[1]) \to \pi_s \mathrm{LSym}^s(M[1])$. We want to show that this is an isomorphism for all $s \geqslant 0$ when M is a free k-module and that in this case $\pi_r \mathrm{LSym}^s(M[1]) = 0$ for $r \neq s$. We know both claims are true when $s = 0, 1$ and in general we can assume that M is finite rank and we can induct on the rank using that

$$\mathrm{Sym}^s((M \oplus N)[1]) \simeq \bigoplus_{i+j=s} \mathrm{Sym}^i(M[1]) \otimes \mathrm{Sym}^j(N[1]).$$

This reduces us to the case where $M \simeq k$ and even $k \simeq \mathbb{Z}$. But, we also saw the claim because $\mathrm{LSym}(\mathbb{Z}[1]) \simeq \mathbb{Z} \otimes_{\mathbb{Z}[y]} \mathbb{Z}$ so that $\mathbb{Z} \oplus \mathrm{LSym}^1(\mathbb{Z}[1]) \simeq \mathrm{LSym}(\mathbb{Z}[1])$. This proves (a).

To prove part (b), we can use part (a) to reduce to a calculation of $\mathrm{L}\Lambda^s(M[1])$ for M free over a commutative ring. We can use the Dold–Kan correspondence to replace $M[1]$ by a simplicial k-module, which is precisely $\mathrm{B}_\bullet M$, where M is viewed as a group, as in Example 1.25. In particular, $\mathrm{B}_n M$ is the direct sum of n copies of M. Thus, by definition

$$\mathrm{L}\Lambda^s(M[1]) \simeq |\Lambda^s \mathrm{B}_\bullet M|.$$

We have

$$\Lambda^s \mathrm{B}_n M \cong \Lambda^s(\oplus_{i=1}^n M) \cong \bigoplus_{a_1+\cdots+a_n=s} \Lambda^{a_1}(M) \otimes_k \cdots \otimes_k \Lambda^{a_n}(M).$$

We have $M^{\otimes s} \subseteq \Lambda^s \mathrm{B}_s M$, corresponding to the summand $a_1 = \cdots a_n = 1$.

Exercise 4.49. Each face map $\partial_i\colon \Lambda^s \mathrm{B}_s M \to \Lambda^s \mathrm{B}_{s-1} M$ vanishes on $\Gamma^s M \subseteq M^{\otimes s} \subseteq \Lambda^s \mathrm{B}_s M$. Hint: use the definition of the face maps on $\mathrm{B}_\bullet M$ from Example 1.25 as well as the definition of the exterior algebra $\Lambda(M)$.

Thus, we obtain natural maps $\Gamma^s M \to \pi_s \Lambda^s(M[1])$. We must prove that this is an isomorphism and that $\pi_r \Lambda^s(M[1]) = 0$ for $r \neq s$. As above, we can reduce to the case where M is fact free of rank 1. In this case, $\Lambda^a M \cong 0$ if $a > 1$.

Thus, we have

$$\Lambda^s \mathrm{B}_n M \cong \bigoplus_{a_1+\cdots+a_n=s, 0\leqslant a_i \leqslant 1} \Lambda^{a_1}(M) \otimes_k \cdots \otimes_k \Lambda^{a_i}(M) \cong \bigoplus_{a_1+\cdots+a_n, 0 \leqslant a_i \leqslant 1} M^{\otimes s}$$

when M is free of rank 1. It follows that $\Lambda^s \mathrm{B}_\bullet M \simeq M^{\otimes s} \otimes Q_\bullet$, where $Q_\bullet$ is some simplicial abelian group. Specifically, Q_n is the free abelian group on $a_1 + \cdots + a_n = s$ with $0 \leqslant a_i \leqslant 1$. Thus, Q_n is the free abelian group on surjections $[n] \twoheadrightarrow [s]$.[14] There is an exact sequence

$$0 \to \mathbb{Z}[\partial\Delta^s_\bullet] \to \mathbb{Z}[\Delta^s_\bullet] \to Q_\bullet \to 0,$$

since the simplices of $\partial\Delta^s_\bullet$ correspond to the non-surjective maps $[n] \to [s]$. It follows that

$$\pi_n(Q_\bullet) \cong \mathrm{H}_n(\Delta^s_{\mathrm{top}}, \partial\Delta^s_{\mathrm{top}}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } n = s \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

It follows that when M is free of rank 1 we have

$$\pi_s \mathrm{L}\Lambda^s(M[1]) \cong M \cong \Gamma^s(M)$$

and $\pi_r \mathrm{L}\Lambda^s(M[1]) = 0$ when $r \neq s$. One checks that this unwinds to agree with the map $\Gamma^s(M) \to \pi_s\Lambda^s(M[1])$ constructed above, and this completes the proof. $\square$

Corollary 4.50. *For an animated k-module M, we have* $\mathrm{LSym}^s(M[2]) \simeq \mathrm{L}\Gamma^s(M)[2s]$.

At long last we can give a proof of the claim at the end of the proof of Proposition 1.53.

Corollary 4.51. *If k is an animated commutative ring and M is an animated k-module which is such that M is n-connected for some $n > 0$ (meaning that $\pi_i M = 0$ for $i \leqslant n$), then* $\mathrm{LSym}^s(M)$ *is $(n + 2s - 2)$-connected.*

Proof. Since M is 1-connected, $M[-2]$ remains an animated k-module and we have $\mathrm{LSym}^s(M)[-2s] \simeq \mathrm{L}\Gamma^s(M[-2])$. We can reduce as usual to the case where k is discrete and then assume (using the Dold–Kan correspondence) that $M[-2]$ is represented by a simplicial k-module $N_\bullet$ with each N_i projective and $N_i = 0$ for $i \leqslant n-2$. Thus, certainly, $\pi_i\Lambda^s(N_\bullet) = 0$ for $i \leqslant n-2$. It follows that $\mathrm{L}\Lambda^s(M[-2])$ is $(n-2)$-connected and hence that $\mathrm{LSym}^s(M) \simeq \mathrm{L}\Gamma^s(M[-2])[2s]$ is $(n+2s-2)$-connected, as desired. $\square$

Example 4.52. Let k be a commutative $\mathbb{Q}$-algebra. Then, $\widehat{\mathrm{L}\Omega}_{k/k[x]} \simeq k[\![x]\!]$ and the Hodge filtration on the left corresponds to the x-adic filtration on the right. Indeed, $\mathrm{L}_{k/k[x]} \simeq k[1]$, so we have $\mathrm{gr}^t_{\mathrm{H}}\widehat{\mathrm{L}\Omega}_{k/k[x]} \simeq \mathrm{L}\Lambda^t(k[1])[-t] \simeq \mathrm{L}\Gamma^t(k) \simeq \mathrm{LSym}^t(k) \simeq k$, where the second-to-last equivalence follows because k is a commutative $\mathbb{Q}$-algebra. We see that the associated graded of $\mathrm{F}^\star_{\mathrm{H}}\widehat{\mathrm{L}\Omega}_{k/k[x]}$ is $k[\![x]\!]$. As there is a natural map $k[x] \to \widehat{\mathrm{L}\Omega}_{k/k[x]}$, it suffices to check it is compatible with the x-adic filtration on the right and the Hodge filtration on the right and that it induces an equivalence on graded pieces. But, since the associated graded of the x-adic filtration $x^\star k[x]$ is generated in degree 1, it suffices to check that the induced map $(x)/(x^2) \to \mathrm{L}_{k/k[x]}$ is an equivalence, which follows from the conormal sequence.

Exercise 4.53. Let $R = \mathbb{F}_p[x^{1/p^\infty}] \cong \mathbb{F}_p[x]_{\mathrm{perf}}$ and let $S = R/x$. Prove that $\mathrm{L}\Omega_{S/\mathbb{F}_p}$ and $\widehat{\mathrm{L}\Omega}_{S/\mathbb{F}_p}$ are discrete, i.e., $\mathrm{H}^t(\mathrm{L}\Omega_{S/\mathbb{F}_p}) = 0$ and $\mathrm{H}^t(\mathrm{L}\Omega_{S/\mathbb{F}_p}) = 0$ for $t \neq 0$. This ring is an example of a **quasiregular semiperfect** ring: absolute Frobenius is surjective on S and the cotangent complex $\mathrm{L}_{S/\mathbb{F}_p}$ has Tor-amplitude in $[1,1]$.

Remark 4.54. Life is easier in characteristic 0. Indeed, if k is a commutative $\mathbb{Q}$-algebra and P is a finitely presented projective k-module, the natural composition

$$\Gamma^s(P) \cong (P^{\otimes s})^{\Sigma_s} \to P^{\otimes s} \to (P^{\otimes s})_{\Sigma_s} \cong \mathrm{Sym}^s(P)$$

[14]For example, $1 + 0 + 1 + 1 = 3$ with $n = 4$ and $s = 3$ corresponds to $\sigma^0 \colon [4] \to [3]$.

is an isomorphism. It follows that

$$\mathrm{LSym}^s(M[2n]) \simeq \mathrm{LSym}^s(M)[2ns]$$

and

$$\mathrm{LSym}^s(M[2n+1]) \simeq \mathrm{L}\Lambda^s(M[1])[2ns].$$

This tells us exactly how to compute the homotopy groups of free animated commutative rings in characteristic 0. They are polynomial on even generators and exterior on odd generators. Specifically, if k is a commutative $\mathbb{Q}$-algebra and if $M \simeq \bigoplus i \in Ik[a_i]$ is a shifted free complex, with $a_i \geqslant 0$, then

$$\pi_*\mathrm{LSym}(M) \cong k[x_i : i \in I]/(x_ix_j = -x_jx_i \text{ if } a_i \text{ and } a_j \text{ are odd}),$$

where $|x_i| = a_i$. In other words, $\pi_*\mathrm{LSym}(M)$ is the free graded commutative k-algebra on classes x_i of degrees a_i.

Example 4.55. This example was explained to us by Shubhodip Mondal who credited Bhatt. One consequence of Bhatt's comparison theorem to Hartshorne's de Rham cohomology is that for finite type maps $R \to S$ of noetherian $\mathbb{Q}$-algebras, one has that $\widehat{\mathrm{L}\Omega}_{S/R}$ is coconnective. I asked Mondal if this is true for arbitrary maps of commutative $\mathbb{Q}$-algebras. The answer is 'no'. The example is to take $R = \mathbb{Q}[x^{1/2^\infty}]/(x)$. One checks that $\mathrm{L}_{R/\mathbb{Q}}$ has Tor-amplitude in $[1,2]$ but that $\mathrm{L}_{\mathbb{Q}/R}$ has Tor-amplitude in $[2,2]$ and is in fact equivalent to $\mathbb{Q}[2]$. Now, it is an easy exercise to prove using the Hodge–de Rham spectral sequence that if $R \to S$ is any map of commutative rings where $\mathrm{L}_{S/R}$ is 2-connective and non-zero, then $\widehat{\mathrm{L}\Omega}_{S/R}$ is connective (not coconnective!) and one has that $\pi_n\widehat{\mathrm{L}\Omega}_{S/R} \neq 0$ for some $n > 0$.

4.8 Hacks

In this chapter, we have defined derived de Rham cohomology $\mathrm{L}\Omega_{R/k}$ and various completed versions for $R \in \mathfrak{a}\mathrm{CAlg}_k$, but we require k to be discrete. This is because in order to use the universal property of animated commutative rings, we must write down the restriction of the functor to polynomial k-algebras. Here are three ways to extend the definition to the case when k is an animated commutative rings.

First, we can realize k as the geometric realization (in animated commutative rings) of a simplicial commutative ring $k_\bullet$ and we can write R as a cofibrant simplicial commutative $k_\bullet$-algebra $R_\bullet$. Then, applying the de Rham complex in each degree we obtain, $\Omega^\bullet_{R_\bullet/k_\bullet}$, a simplicial strict cdga over $k_\bullet$. The geometric realization of $\Omega^\bullet_{R_\bullet/k_\bullet}$ in $\mathrm{Alg}_{\mathbb{Z}}^{\mathbb{E}_\infty}$ is naturally an $\mathbb{E}_\infty$-k-algebra, and we can take this to be $\mathrm{L}\Omega_{R/k}$. Similar approaches apply to the Hodge filtration and the Hodge-complete version.

Second, just as for the cotangent complex, we can realize that there should be base-change formula. Specifically, we define

$$\mathrm{L}\Omega_{R/k} \simeq \mathrm{L}\Omega_{R/\mathbb{Z}} \otimes_{\mathrm{L}\Omega_{k/\mathbb{Z}}} k.$$

Or, for the Hodge-complete version, we can let

$$\widehat{\mathrm{L}\Omega}_{R/k} \simeq \widehat{\mathrm{L}\Omega}_{R/\mathbb{Z}} \widehat{\otimes}_{\widehat{\mathrm{L}\Omega}_{k/\mathbb{Z}}} k,$$

where we use the completed tensor product.

Third, there is a conceptual approach due to Raksit [53], which has the benefit of isolating the universal property of Hodge-completed derived de Rham cohomology. Specifically, there is an ∞-category $\mathfrak{d}\mathfrak{f}\mathrm{CAlg}_k$ of derived complete non-negatively graded filtered commutative k-algebras[15] and there is a functor

$$\mathrm{gr}^0 \colon \mathfrak{d}\mathfrak{f}\mathrm{CAlg}_k \to \mathfrak{d}\mathrm{CAlg}_k.$$

[15] In fact, there are several different possibilities of what this might mean. It is important to take a particular choice in [53], corresponding to what Raksit calls h_+-cochain complexes.

Raksit proves that this admits a left adjoint

$$\mathrm{F}^\star_{\mathrm{H}}\widehat{\mathrm{L}\Omega}_{-/k}\colon \mathfrak{d}\mathrm{CAlg}_k \to \mathfrak{d}\mathfrak{f}\mathrm{CAlg}_k.$$

The universal property of the adjunction says that to give a map $R \to \mathrm{gr}^0 S$ is equivalent to give a map $\mathrm{F}^\star_{\mathrm{H}}\widehat{\mathrm{L}\Omega}_{R/k} \to \mathrm{F}^\star S$ when S is a derived complete non-negatively graded filtered commutative k-algebra. This is a precise, derived version of Theorem 4.14.

5 Week 5: circle actions (17–23 Oct.)

Circle actions play a surprisingly important role in derived algebraic geometry via their connection to bundles with flat connection, Hochschild homology, and, via periodic cyclic homology, de Rham cohomology.

5.1 Stable ∞-categories and t-structures

References 5.1. See [39, Chap. 1] for stable ∞-categories and t-structures; see [40, App. C] for prestable ∞-categories. The original source of Beilinson–Bernstein–Deligne [10] remains a must-read on t-structures.

An ∞-category is **pointed** if it admits an object $*$ which is both initial and terminal. An object which is initial and terminal is often called a **0-object**. If $\mathcal{C}$ is pointed and admits pushouts, then there is a natural **suspension functor** $\Sigma\colon \mathcal{C} \to \mathcal{C}$ which is defined by letting ΣX be the pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X. \end{tikzcd}$$

Example 5.2. The ∞-category $\mathcal{S}_*$ of **pointed animae** is pointed. The suspension ΣS^n is equivalent to S^{n+1} for $n \geqslant 0$ (and for any choice of basepoint of S^n).

Example 5.3. If $\mathcal{C}$ is pointed, then for any $X, Y \in \mathcal{C}$, the mapping space

$$\mathrm{Map}_{\mathcal{C}}(X, Y)$$

is naturally pointed by the map $X \to * \to Y$.

Definition 5.4. Let $\mathcal{C}$ be an ∞-category. We say that $\mathcal{C}$ is **stable** if it satisfies the following equivalent conditions:

(a) $\mathcal{C}$ is pointed, $\mathcal{C}$ admits all finite limits and colimits, and a commutative square

$$\begin{tikzcd} w \arrow[r] \arrow[d] & x \arrow[d] \\ y \arrow[r] & z \end{tikzcd}$$

is a pullback square in $\mathcal{C}$ if and only if it is a pushout square;

(b) $\mathcal{C}$ is pointed, has finite colimits, and the suspension functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$ is an equivalence;

(c) $\mathcal{C}$ is pointed, has pushouts, and the suspension functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$ is an equivalence.

Remark 5.5. For further details on stable ∞-categories, see [39, Section 1.1.1].

Remark 5.6. The inverse to the suspension functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$ is the **loops functor** $\Omega\colon \mathcal{C} \to \mathcal{C}$. If $\mathcal{C}$ is a pointed ∞-category with finite limits, Ωx is defined to be the fiber product

$$\begin{tikzcd} \Omega x \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & x. \end{tikzcd}$$

We will also write $x[n] = \Sigma^n x$ for $n \in \mathbb{Z}$, where, if $n \leqslant 0$, this means $\Omega^{-n}x$.

Remark 5.7. If we weaken the conditions somewhat, then we obtain Lurie's definition of a **prestable ∞-category**: a pointed ∞-category $\mathcal{C}$ is prestable if it admits finite colimits, the suspension functor Σ is fully faithful, and there is a limit for every diagram of the form

$$\begin{tikzcd} & x \arrow[d] \\ 0 \arrow[r] & \Sigma w. \end{tikzcd}$$

The platonic example of a prestable ∞-category is $\mathfrak{a}\mathrm{Mod}_k^{\omega}$ when k is a commutative ring. This is the homotopy theory of complexes of finitely presented projective k-modules in non-negative degrees and quasi-isomorphisms (or homotopy equivalences of chain complexes). Indeed, there is a 0 object given by the 0 animated k-module, suspension is clearly fully faithful, and certain morphisms have fibers, namely the morphisms which do not create negative homology.

Exercise 5.8. Let $\mathcal{A}$ be an abelian category. Is $\mathcal{A}$ stable?

Example 5.9. Every pointed ∞-category $\mathcal{C}$ with finite colimits admits a **stabilization**, obtained by taking the **Spanier–Whitehead category** of $\mathcal{C}$:

$$\mathrm{SW}(\mathcal{C}) = \mathrm{colim}(\mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \cdots),$$

where the colimit is computed in Cat_∞. In the case of finite pointed spaces $\mathcal{S}_*^{\omega}$, we obtain $\mathcal{S}\mathrm{p}^{\omega} \simeq \mathrm{SW}(\mathcal{S}_*^{\omega})$, the stable ∞-category of finite spectra.

Remark 5.10. Lurie shows that if $\mathcal{C}$ is a stable ∞-category, then the homotopy category $\mathrm{Ho}(\mathcal{C})$ admits a canonical triangulated category structure. The shift functor $(-)[1]$ is induced by the suspension functor. A triangle is distinguished if and only if it arises from a pushout square

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d] \\ 0 \arrow[r] & z \end{tikzcd}$$

in $\mathcal{C}$. The map $z \to x[1]$ comes by fitting the previous square into a larger pushout square

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & z \arrow[r] & \Sigma x. \end{tikzcd}$$

Construction 5.11. For us, the most important stable ∞-category is the **derived ∞-category** $\mathcal{D}(k)$ of an animated commutative ring. This is obtained from $\mathfrak{a}\mathrm{Mod}_k \simeq \mathcal{D}(k)_{\geqslant 0}$ by inverting the suspension operation in Pr^{L}.

Remark 5.12. Colimits in Pr^{L} are not computed in $\widehat{\mathrm{Cat}}_\infty$. Rather, one replaces a diagram in Pr^{L} by the corresponding diagram of right adjoint functors and computes the limit of that diagram in $\widehat{\mathrm{Cat}}_\infty$. For example, if we compute

$$\mathrm{colim}(\mathcal{D}(k)_{\geqslant 0} \xrightarrow{\Sigma} \mathcal{D}(k)_{\geqslant 0} \xrightarrow{\Sigma} \cdots)$$

in $\widehat{\mathrm{Cat}}_\infty$, then we obtain $\mathcal{D}(k)^-$, the bounded below derived ∞-category of k. Because every object of the colimit (computed in $\widehat{\mathrm{Cat}}_\infty$) must come from a finite stage, we see it must be bounded below. On the other hand, computing the limit of the right adjoint tower, we get

$$\mathcal{D}(k) \simeq \lim(\cdots \xrightarrow{\Omega} \mathcal{D}(k)_{\geqslant 0} \xrightarrow{\Omega} \mathcal{D}(k)_{\geqslant 0}),$$

which is the ∞-category of Ω-spectrum objects of $\mathcal{D}(k)_{\geqslant 0}$.

The theory of t-structures axiomatizes the Postnikov and Whitehead towers in homotopy theory and the good truncations of chain complexes.

Definition 5.13. Let $\mathcal{C}$ be a stable ∞-category. A ***t*-structure** on $\mathcal{C}$ is a pair $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ of full subcategories of $\mathcal{C}$ satisfying the following conditions:

(i) $\Sigma\mathcal{C}_{\geqslant 0} \subseteq \mathcal{C}_{\geqslant 0}$, $\Omega\mathcal{C}_{\leqslant 0} \subseteq \mathcal{C}_{\leqslant 0}$,

(ii) if $x \in \mathcal{C}_{\geqslant 0}$ and $y \in \mathcal{C}_{\leqslant 0}$, then $\mathrm{Map}_{\mathcal{C}}(x, \Omega y) \simeq 0$, and

(iii) every object $y \in \mathcal{C}$ fits into a fiber sequence

$$x \to y \to z$$

where $x \in \mathcal{C}_{\geqslant 0}$ and $\Sigma z \in \mathcal{C}_{\leqslant 0}$.

Remark 5.14. If $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ is a t-structure on $\mathcal{C}$, then $\mathcal{C}_{\geqslant 0}$ is prestable. Moreover, the t-structure is determined by $\mathcal{C}_{\geqslant 0} \subseteq \mathcal{C}$: the subcategory $\mathcal{C}_{\leqslant 0}$ is the subcategory of objects y satisfying the orthogonality condition (ii) $\mathrm{Map}_{\mathcal{C}}(x, \Omega y) \simeq 0$ for all $x \in \mathcal{C}$.

Remark 5.15. We let $\mathcal{C}_{\geqslant n} \simeq \Sigma^n\mathcal{C}$ and $\mathcal{C}_{\leqslant n} \simeq \Sigma^n\mathcal{C}_{\leqslant 0}$ for $n \in \mathbb{Z}$. The inclusion $\mathcal{C}_{\geqslant n} \subseteq \mathcal{C}$ admits a right adjoint $\tau_{\geqslant n} \colon \mathcal{C} \to \mathcal{C}_{\geqslant n}$, while the inclusion $\mathcal{C}_{\leqslant n} \subseteq \mathcal{C}$ admits a left adjoint $\tau_{\leqslant n} \colon \mathcal{C} \to \mathcal{C}_{\leqslant n}$. Given $x \in \mathcal{C}$ we obtain the **Whitehead tower**

$$\cdots \to \tau_{\geqslant n+1}x \to \tau_{\geqslant n}x \to \tau_{\geqslant n-1}x \to \cdots \to x$$

and the **Postnikov tower**

$$x \to \cdots \to \tau_{\leqslant n+1}x \to \tau_{\leqslant n}x \to \tau_{\leqslant n-1}x \to \cdots .$$

Remark 5.16. Given an interval $[a, b]$ where $a < b \in \mathbb{Z}$, we let $\mathcal{C}_{[a,b]} \simeq \mathcal{C}_{\geqslant a} \cap \mathcal{C}_{\leqslant b}$. The **heart** of the t-structure is $\mathcal{C}^{\heartsuit} \simeq \mathcal{C}_{[0,0]} \simeq \mathcal{C}_{\geqslant 0} \cap \mathcal{C}_{\leqslant 0}$.

The most important aspect of a t-structure is that the heart is an abelian category. This is due to [10].

Theorem 5.17. *The heart of a t-structure is an abelian category.*

Construction 5.18. There are functors $\pi_n : \mathcal{C} \to \mathcal{C}^{\heartsuit}$ obtained by letting $\pi_n x \simeq \tau_{\leqslant 0}\tau_{\geqslant 0}(x[-n])$. (Think of this as shifting down so that degree n sits in degree 0 and then taking the homotopy objects in degree 0.) The $\pi_n x \in \mathcal{C}^{\heartsuit}$ are called the **homotopy objects** of $x \in \mathcal{C}$.

Proposition 5.19. *Given a fiber sequence $x \to y \to z$ in $\mathcal{C}$, there is a natural long exact sequence*

$$\cdots \to \pi_{n+1}z \to \pi_n x \to \pi_n y \to \pi_n z \to \pi_{n-1}x \to \cdots$$

in $\mathcal{C}^{\heartsuit}$.

Example 5.20. The ∞-category $\mathcal{S}\mathrm{p}_{\geqslant 0} \subseteq \mathcal{S}\mathrm{p}$ of connective spectra (i.e., those spectra X with $\pi_i X \cong 0$ for $i < 0$) is the connective part of a unique t-structure on $\mathcal{S}\mathrm{p}$. The heart $\mathcal{S}\mathrm{p}^{\heartsuit} \simeq \mathrm{Mod}_{\mathbb{Z}}$ is the abelian category of abelian groups. The homotopy objects are the usual homotopy groups.

Example 5.21. If k is an animated commutative ring, then $\mathfrak{a}\mathrm{Mod}_k \simeq \mathcal{D}(k)_{\geqslant 0} \subseteq \mathcal{D}(k)$ is the connective part of a unique t-structure on $\mathcal{D}(k)$. There is a natural equivalence $\mathcal{D}(k)^{\heartsuit} \simeq \mathrm{Mod}_{\pi_0 k}$. Given $X \in \mathcal{D}(k)$, the homotopy objects are the homology groups $\mathrm{H}_*(X)$, which are naturally $\pi_0 k$-modules.

Exercise 5.22. Let k be a commutative ring. Suppose that $M_\bullet$ is a chain complex of k-modules. Construct natural (in maps of chain complexes, not in the ∞-category) maps $\tau_{\geqslant 0}M_\bullet \to M_\bullet$ and $M_\bullet \to \tau_{\leqslant 0}M_\bullet$. You should write down the truncations as explicit chain complexes in terms of $M_\bullet$.

Exercise 5.23. Show that the functors $M_\bullet \mapsto \tau_{\geqslant 0}M_\bullet$ and $M_\bullet \mapsto \tau_{\leqslant 0}M_\bullet$ of Exercise 5.22, as well as the natural transformations $\tau_{\geqslant 0} \to \mathrm{id}$ and $\mathrm{id} \to \tau_{\leqslant 0}$, descend to the ∞-category $\mathcal{D}(k)$.

For the next example, recall that if k is a commutative ring, $\mathrm{Mod}_k^{\omega} \subseteq \mathrm{Mod}_k$ is the full subcategory of finitely presented k-modules.[16]

Example 5.24. For a small example, we consider a noetherian commutative ring k and let $\mathcal{D}^b(k) \subseteq \mathcal{D}(k)$ be the full subcategory of objects $X \in \mathcal{D}(k)$ such that $\mathrm{H}_i(X) \in \mathrm{Mod}_k^{\omega}$ for all i and $\mathrm{H}_i(X)$ vanishes for $|i|$ very large. The t-structure on $\mathcal{D}(k)$ restricts to a t-structure on $\mathcal{D}^b(k)$ where $\mathcal{D}^b(X)_{\geqslant 0}$ consists of the objects X with $\mathrm{H}_i(X) \simeq 0$ for $i < 0$. There is an equivalence $\mathcal{D}^b(k)^{\heartsuit} \simeq \mathrm{Mod}_k^{\omega}$.

Warning 5.25. It would be better to write $\mathcal{D}^b(\mathrm{Mod}_k^{\omega})$ instead of $\mathcal{D}^b(k)$ because the latter could alternatively mean the full subcategory of $\mathcal{D}(k)$ on the bounded objects. However, this is the standard convention in the literature.

Definition 5.26. A commutative ring k is **coherent** if every finitely generated ideal $I \subseteq k$ is finitely presented.

Exercise 5.27. Suppose that k is coherent. Show that every finitely generated submodule N of a finitely presented module M is itself finitely presented.

Exercise 5.28. Show that k is coherent if and only if Mod_k^{ω} is abelian.

Remark 5.29. If k is coherent, then $\mathcal{D}^b(k) \simeq \mathcal{D}^b(\mathrm{Mod}_k^{\omega}) \subseteq \mathcal{D}(k)$ makes sense and admits a t-structure with heart Mod_k^{ω}.

Exercise 5.30. Let k be a coherent commutative ring. Let $\mathcal{P}\mathrm{erf}(k)_{\geqslant 0} \subseteq \mathcal{P}\mathrm{erf}(k)$ denote the full subcategory of connective perfect complexes. Show that $\mathcal{P}\mathrm{erf}(k)_{\geqslant 0}$ is the connective part of a t-structure on $\mathcal{P}\mathrm{erf}(k)$ if and only if k is **weakly regular** in the sense that every finitely presented k-module his finite projective dimension. If this is the case, the heart is Mod_k^{ω}.

[16] These are the compact objects of Mod_k.

5.2 Parametrized spectra

Now, we will look at an anima X and study objects (for examples, animae, spectra, or complexes) parametrized over X.

Definition 5.31. Let $\mathcal{C}$ be an ∞-category and let X be an anima. The functor category $\mathrm{Fun}(X, \mathcal{C}) \simeq \mathcal{C}^X$ will be called the ∞-category of objects of $\mathcal{C}$ parametrized over X.

Remark 5.32. Since X is an anima, we can view X as an ∞-category and it is in fact an ∞-groupoid.[17] This means in particular that every morphism in X (i.e., a path) is invertible. To give an object $F \in \mathrm{Fun}(X, \mathcal{C})$ is to give an object $F_x \in \mathcal{C}$ for each point $x \in X$, to give an equivalence $F_x \simeq F_y$ for each path $x \simeq y$ in X, to give 2-simplices

$$\begin{tikzcd} & F_y \arrow[dr, "\simeq"] & \\ F_x \arrow[ur, "\simeq"] \arrow[rr, "\simeq"] & & F_z \end{tikzcd}$$

for each 2-simplex $\Delta^2 \to X$, and on and on.[18]

Example 5.33. If $X \simeq *$ is a final object of $\mathcal{S}$, then $\mathcal{C}^* \simeq \mathrm{Fun}(*, \mathcal{C}) \simeq \mathcal{C}$.

Example 5.34. Suppose that $\mathcal{C}$ is a 1-category and that X is an anima. Any functor $X \to \mathcal{C}$ factors through $\tau_{\leqslant 1}X$, the fundamental groupoid of X. To see this, it is enough to see that in the adjunction $\mathrm{Cat}_\infty \rightleftarrows \mathrm{Cat}_1$, the left adjoint $\mathrm{Cat}_\infty \to \mathrm{Cat}_1$, which is given by taking the homotopy category, takes a space X to its fundamental groupoid $\tau_{\leqslant 1}X$. In fact, there is a commutative square

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\tau_{\leqslant 1}"] \arrow[d] & \mathcal{S}_{\leqslant 1} \arrow[d] \\ \mathrm{Cat}_\infty \arrow[r] & (\mathrm{Cat}_\infty)_{\leqslant 1} \end{tikzcd}$$

and every 1-category is 1-truncated when viewed as an object of Cat_∞. It follows in this case that the natural map

$$\mathcal{C}^{\tau_{\leqslant 1}X} \to \mathcal{C}^X$$

is an equivalence.

Remark 5.35. It is not super clear to me what the meaning of $(\mathrm{Cat}_\infty)_{\leqslant n}$. It is tempting to guess that these are the n-categories. But, there is an argument on MathOverflow that suggests this is not the case. A 0-category is an ∞-category in which every mapping space is a (-1)-type, meaning it is empty or contractible. A (-1)-category is an ∞-category in which every mapping space is a (-2)-type, meaning contractible. Any (-1)-category is equivalent to the final category. A 0-category $\mathcal{C}$ however is equivalent to a poset: if $x, y \in \mathcal{C}$, we say that $x < y$ if $\mathrm{Map}_{\mathcal{C}}(x, y)$ is non-empty. However, the Kronecker quiver $\bullet \rightrightarrows \bullet$ defines a 1-category $\mathcal{K}$ which is not a poset, but which is 0-truncated in Cat_∞. Indeed, it suffices to check that $\mathrm{Map}_{\mathrm{Cat}_\infty}(\Delta^0, \mathcal{K})$ and

[17] In fact, this justifies using $\mathrm{Fun}(X, \mathcal{C})$ instead of the possibly more natural presheaf perspective $\mathrm{Fun}(X^{\mathrm{op}}, \mathcal{C})$ in our definition of parametrized objects. Indeed, there is a natural equivalence $X \simeq X^{\mathrm{op}}$ when we view an anima as an ∞-category. The equivalence is induced by the identity on objects and sending u to u^{-1} on morphisms.

[18] In fact, if $X_\bullet$ is in fact a Kan complex model of X and $\mathcal{C}$ is an ∞-category, then

$$\mathrm{Fun}(X, \mathcal{C}) \simeq \lim_{\Delta} \mathrm{Fun}(X_\bullet, \mathcal{C}),$$

which expresses this idea precisely.

$\operatorname{Map}_{\mathrm{Cat}_\infty}(\Delta^1, \mathcal{K})$ are 0-truncated (i.e., sets). For Δ^0, this is clear as $\operatorname{Map}_{\mathrm{Cat}_\infty}(\Delta^0, \mathcal{K}) \simeq \iota_0\mathcal{K} \simeq \partial S^0$. We can compute $\operatorname{Fun}(\Delta^1, \mathcal{K})$, which consists of 4 functors and some natural transformations. However, $\iota_0 \operatorname{Fun}(\Delta^1, \mathcal{K})$ is discrete as none of the natural transformations is invertible.

Exercise 5.36. Let G be a group and let $\mathrm{B}G$ denote the animated classifying space of G. This is an Eilenberg–Mac Lane space which is connected and has $\pi_1 \mathrm{B}G \cong G$ and $\pi_i \mathrm{B}G \cong 0$ for $i > 1$. Suppose that $\mathcal{C}$ is a 1-category. Show that $\mathcal{C}^{\mathrm{B}G}$ is equivalent to the category $\operatorname{Rep}_G(\mathcal{C})$ of G-representations on objects of $\mathcal{C}$, i.e., the category of objects of $\mathcal{C}$ equipped with a G-action.

Proposition 5.37. *Let X be an anima. There is a natural equivalence $\mathcal{S}^X \simeq \mathcal{S}_{/X}$.*

Proof. The assignment $X \mapsto \mathcal{S}^X$ defines a limit preserving functor $\mathcal{S}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}$. Similarly, since $\mathcal{S}_{/X}$ is an ∞-topos, the slice construction $X \mapsto \mathcal{S}_{/X}$ defines a limit preserving functor $\mathcal{S}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}$.[19] Since $\mathcal{S}^{\mathrm{op}}$ is generated under limits by the initial object $*$ (since we are in the *opposite* of the ∞-category of animae), it is enough to note that there is a natural equivalence $\mathcal{S}^* \simeq \mathcal{S} \simeq \mathcal{S}_{/*}$. Right Kan extending this equivalence gives the natural equivalence $\mathcal{S}^{(-)} \simeq \mathcal{S}_{/(-)}$. □

Remark 5.38. The equivalence implements a kind of Grothendieck construction, so that the fibers of a map $Y \to X$ become the values of a functor $X \to \mathcal{S}$. Thus, we often think of parametrized objects as living 'over' X as the total space of a kind of fibration.

Definition 5.39 (Parametrized complexes)**.** Let X be an anima and let k be an animated commutative ring. The ∞-category of **parametrized complexes** over X is $\mathcal{D}(k)^X \simeq \operatorname{Fun}(X, \mathcal{D}(k))$. We will also call these **local systems**.

Challenge 5.40. Let M be a real n-manifold and let $\widetilde{\mathbb{Z}} \to M$ be the local system of generators of $\mathrm{H}_n(M, M - \{x\}; \mathbb{Z})$. This is the canonical local system and gives a theory of twisted coefficients with twisted cohomology groups $\mathrm{H}^*(X, \widetilde{\mathbb{Z}})$. This should also define a local system over the underlying anima of M. How can we make this precise?

This theory is especially rich because of the large number of adjoints.

Construction 5.41. Let $f \colon X \to Y$ be a map of animae. Composition with f gives a pullback map

$$f^* \colon \mathcal{D}(k)^Y \to \mathcal{D}(k)^X.$$

Since limits and colimits of presheaf categories are computed pointwise,

$$f^*$$

commutes with limits and colimits, so that there are left and right adjoints, $f_!, f_* \colon \mathcal{D}(k)^X \to \mathcal{D}(k)^Y$.

Exercise 5.42. If $p \colon X \to *$ is the canonical map, prove that $p_! k$ computes k-chains on X and $p_* k$ computes k-cochains on X, where k denotes the constant local system on X with coefficients in k. In other words, prove that $p_! k \simeq \mathrm{C}_\bullet(X, k)$ in $\mathcal{D}(k)$ and that $p_* k \simeq \mathrm{C}^\bullet(X, k)$.

Exercise 5.43. Let $f \colon X \to Y$ be a map of animae. Prove that $y^*(f_! k) \simeq (f_y)_! k$, where $y \in Y$ and $f_y \colon X \times_Y \{y\} \to y$.

[19] This is the universality of colimits in the definition of an ∞-topos. See [38, Chap. 6].

The next theorem is an important summary of some facts in derived Morita theory. See [56] for the original model categorical version and [39, Thm. 7.1.2.1] for the ∞-categorical version.

An important facet of the theory is that a stable ∞-category $\mathcal{C}$ is enriched not only over anima but also over spectra. That is, given $X, Y \in \mathcal{C}$, there is naturally a mapping *spectrum* $\mathbf{Map}_{\mathcal{C}}(X,Y)$ and $\Omega^\infty \mathbf{Map}_{\mathcal{C}}(X,Y) \simeq \mathrm{Map}_{\mathcal{C}}(X,Y)$. In particular, $\mathbf{Map}_{\mathcal{C}}(X,X)$ is an $\mathbb{E}_1$-ring spectrum for any $X \in \mathcal{C}$.

Theorem 5.44 (Some Morita theory)**.** *Suppose that $\mathcal{C}$ is a stable presentable ∞-category and that $X \in \mathcal{C}$ is a compact generator: X is compact and if the mapping spectrum $\mathbf{Map}_{\mathcal{C}}(X,Y) \simeq 0$, then $Y \simeq 0$. Setting $A \simeq \mathbf{Map}_{\mathcal{C}}(X,X)$, we obtain an equivalence $\mathcal{C} \simeq \mathcal{D}(A) \simeq \mathrm{Mod}_A(\mathcal{S}\mathrm{p})$.*

Corollary 5.45 (Chains on the loopspace)**.** *Suppose that X is a connected anima (so that $\pi_0 X \cong *$). If k is an animated commutative ring (or more generally and $\mathbb{E}_\infty$-ring), then*

$$\mathcal{D}(k)^X \simeq \mathcal{D}(k[\Omega X]),$$

where $k[\Omega X]$ is the group algebra of the loopspace ΩX.

Proof. Pick a point $x \in X$. We have $x_! \colon \mathcal{D}(k) \to \mathcal{D}(k)^X$ which has a right adjoint x^*. The right adjoint preserves all colimits and in particular filtered colimits. Thus, $x_!$ preserves compact objects. (See [38, Prop. 5.5.7.2] for the simple argument.) In particular, $x_! k$ is compact. We claim it is a compact generator. Indeed, if $\mathbf{Map}_{\mathcal{D}(k)^X}(x_! k, F) \simeq 0$, then $\mathbf{Map}_{\mathcal{D}(k)}(k, x^* F) \simeq 0$, so that $x^* F \simeq 0$. But, this means that $x^* F$ vanishes at every point of X since X is connected. So, $F \simeq 0$. It follows from Theorem 5.44 that $\mathcal{D}(k)^X \simeq \mathcal{D}(A)$ where $A \simeq \mathbf{Map}_{\mathcal{D}(k)^X}(x_! k, x_! k) \simeq \mathbf{Map}_{\mathcal{D}(k)}(k, x^* x_! k) \simeq x^* x_! k$. By definition, $x_! k$ is the free parametrized complex on k, which corresponds to inducing up from k to the group algebra $k[\Omega X]$. ☐

Remark 5.46. It may be helpful to think of Corollary 5.45 as follows. If X is connected with a basepoint x, then a parametrized complex $M \colon X \to \mathcal{D}(k)$ is determined first by where the basepoint x goes, i.e., to some complex M, then then the rest of the structure is determined by the induced map $\mathrm{Map}_X(x,x) \to \mathrm{Map}_{\mathcal{D}(k)}(M,M)$. However, $\mathrm{Map}_X(x,x) \simeq \Omega_x X$ is a grouplike $\mathbb{E}_1$-anima and $\mathrm{Map}_{\mathcal{D}(k)}(M,M) \simeq \Omega^\infty \mathbf{Map}_{\mathcal{D}(k)}(M,M)$, where $\mathbf{Map}_{\mathcal{D}(k)}(M,M)$ is an $\mathbb{E}_1$-algebra in $\mathcal{D}(k)$. Now, there is an adjunction

$$k[-] \colon \mathrm{Gr}(\mathcal{S}) \rightleftarrows \mathrm{Alg}_k^{\mathbb{E}_1} \colon \Omega^\infty,$$

where the right adjoint takes the underlying grouplike $\mathbb{E}_1$-anima and the left adjoint gives the group algebra. It follows by adjunction that there is a map $k[\Omega_x X] \to \mathbf{Map}_{\mathcal{D}(k)}(M,M)$ of $\mathbb{E}_1$-algebras. But, giving such a map is precisely the data of giving a $k[\Omega_x X]$-module structure on M.

Remark 5.47 (Symmteric monoidal structure)**.** The ∞-category $\mathcal{D}(k)^X$ admits a natural, **pointwise symmetric monoidal structure** induced by the composition

$$\mathrm{Fun}(X \times X, \mathcal{D}(k) \times \mathcal{D}(k)) \to \mathrm{Fun}(X \times X, \mathcal{D}(k)) \xrightarrow{\Delta^*} \mathrm{Fun}(X, \mathcal{D}(k)).$$

Given two parametrized complexes F and G, the tensor product $F \otimes_k G$ is a parametrized complex with $x^*(F \otimes_k G) \simeq (x^* F) \otimes_k (x^* G)$ for $x \in X$. The unit is the constant parametrized complex given by k.

5.3 Group actions

Proposition 5.48 (The canonical t-structure)**.** *Let X be an anima and let k be an animated commutative ring. There is a t-structure on $\mathcal{D}(k)^X$ defined by*

$$(\mathcal{D}(k))_{\geqslant 0} = \{F \in \mathcal{D}(k)^X \colon x^* F \in \mathcal{D}(k)_{\geqslant 0} \text{ for } x \in X\}$$

and

$$(\mathcal{D}(k))_{\leqslant 0} = \{F \in \mathcal{D}(k)^X : x^*F \in \mathcal{D}(k)_{\leqslant 0} \text{ for } x \in X\}.$$

If X is connected, then the choice of a basepoint induces an equivalence

$$\mathcal{D}(k)^{X,\heartsuit} \simeq \mathrm{Mod}_{\pi_0 k[\pi_1 X]}$$

of an abelian categories, where $\pi_0 k[\pi_1 X]$ is the group algebra of the fundamental group $\pi_1 X$ over $\pi_0 k$.

Proof. It is easy to see that this pair satisfies (i) of Definition 5.13. Given any $F \in \mathcal{D}(k)$, there is a natural fiber sequence

$$\tau_{\geqslant 0} F \to F \to \tau_{\leqslant -1} F.$$

We can apply this construction pointwise to obtain such a sequence for $F \in \mathcal{D}(k)^X$. This gives (ii) of Definition 5.13. Finally, to check orthogonality, we have to computing the mapping complexes in $\mathcal{D}(k)^X$. Representing the anima X as a simplicial set $X_\bullet$, given $F \in \mathcal{D}(k)^X_{\geqslant 0}$ and $Y \in \mathcal{D}(k)^X_{\leqslant -1}$, we have

$$\mathrm{Map}_{\mathcal{D}(k)^X}(X, Y) \simeq \lim_{\Delta} \left(\prod_{x \in X_0} \mathrm{Map}_{\mathcal{D}(k)}(F_x, Y_x) \rightrightarrows \prod_{(x \xrightarrow{\sigma} y) \in X_1} \mathrm{Map}_{\mathcal{D}(k)}(F_x, Y_y) \cdots \right).$$

Each term of the limit vanishes if $X \in \mathcal{D}(k)_{\geqslant 0}$ and $Y \in \mathcal{D}(k)_{\leqslant -1}$. This verifies (ii).

To see the identification of the heart, recall that $\mathcal{D}(k)^X \simeq \mathcal{D}(k[\Omega X])$. Under this equivalence, an object of $\mathcal{D}(k)^X$ is connective if and only if it is connective as a $k[\Omega X]$-module. In particular, the heart is equivalent to $\mathrm{Mod}_{\pi_0(k[\Omega X])}\mathcal{D}(k)^\heartsuit \simeq \mathrm{Mod}_{\pi_0 k[\pi_1 X]}$. ☐

Remark 5.49. Returning to Challenge 5.40, we see that if M is connected, we can directly construct $\tilde{\mathbb{Z}}$ as an object of $\mathcal{D}(\mathbb{Z})^M$ by using the heart. We have $\tilde{\mathbb{Z}}$ as a $\pi_1 M$-module, and to this there corresponds a natural object of the heart of the canonical t-structure on $\mathcal{D}(\mathbb{Z})^M$; we can view this as an object of $\mathcal{D}(\mathbb{Z})^M$, which is what we wanted to do.

Remark 5.50. If X is a pointed connected anima, It is natural to view ΩX as a kind of generalized group. In fact, ΩX is a loopspace, or equivalently a grouplike $\mathbb{E}_1$-space. Thus, we view $\mathcal{D}(k)^X$ as being a good model for the ∞-category of $G = \Omega X$-representations: $\mathcal{D}(k)^X = \mathrm{Rep}_{\Omega X}(\mathcal{D}(k))$. This is justified by what happens on the heart of the canonical t-structure:

$$\mathcal{D}(k)^{X,\heartsuit} \simeq \mathrm{Mod}_{\pi_0 k[\pi_1 X]} \simeq \mathrm{Rep}_{\pi_1 X}(\mathrm{Mod}_{\pi_0 k}).$$

5.4 Circle actions

Definition 5.51. We will write S^1 for the **animated circle** and $\mathrm{B}S^1$ for its **classifying anima**. The classifying anima “is” $\mathbb{CP}^\infty$. These can be modeled by simplicial set $\mathrm{B}_\bullet\mathbb{Z}$ and the bisimplicial set $\mathrm{B}_{\bullet\bullet}\mathbb{Z}$, respectively.

Definition 5.52. Let $\mathcal{C}$ be an ∞-category. The ∞-category of objects of $\mathcal{C}$ equipped with a **circle action** is $\mathcal{C}^{\mathrm{B}S^1}$. If $X \in \mathcal{C}$, a circle action on X is obtained by lifting X along the forgetful functor $\mathcal{C}^{\mathrm{B}S^1} \to \mathcal{C}$.

Example 5.53. In the crucial case, $\mathcal{C} \simeq \mathcal{D}(k)$. In this case, we can identify the ∞-category of complexes with circle action as

$$\mathcal{D}(k)^{\mathrm{B}S^1} \simeq \mathcal{D}(k[\Omega \mathrm{B}S^1]) \simeq \mathcal{D}(k[S^1]),$$

where $k[S^1]$ is the k-linear group algebra of S^1. On homotopy, we have $\pi_* k[S^1]$ is a graded-commutative ring with the Pontryagin product arising from the multiplication map $S^1 \times S^1 \to S^1$. In particular, we $k = \mathbb{Z}$, we have $\pi_*\mathbb{Z}[S^1] \cong \mathbb{Z}[x]/(x^2)$ where $|x| = 1$. To give a module over $k[S^1]$ is thus roughly speaking to give a complex $F \in \mathcal{D}(k)$ together with a map

$$x\colon F \to F[-1]$$

such that the composition $F \xrightarrow{x} F[-1] \xrightarrow{x} F[-2]$ is nullhomotopic. This makes the homology groups of F into a cochain complex

$$\cdots \to \mathrm{H}_{-1}(X) \to \mathrm{H}_0(X) \to \mathrm{H}_1(X) \to \cdots .$$

Remark 5.54. There is a more combinatorial way to model objects with circle action, via representations of the 1-category Λ of Connes, also called Connes' **cyclic category**. For details, see [36]. Or, ask Tamarkin or Tsygan.

6 Week 6: The HKR theorem (24–30 Oct.)

At the heart of the topological approach to p-adic cohomology theories is the central role of Hochschild homology and the Hochschild–Kostant–Rosenberg theorem.

The material in Section 6.4 is optional, but is included for those curious to see how the signs work out.

6.1 Hochschild homology

References 6.1. Loday's book [36] remains a crucial source of information.

We limit ourselves to the Hochschild homology of animated commutative rings in this course. This is for the simple reason that Hochschild homology has a natural universal property in this setting (or in any commutative setting such as $\mathbb{E}_\infty$). For noncommutative algebras, Hochschild homology is important, but its meaning is a bit more opaque.

Fix an animated commutative ring k. Let X be an anima with a point $x \in X$. Evaluation at x gives us a pullback map

$$\mathfrak{a}\mathrm{CAlg}_k^X \to \mathfrak{a}\mathrm{CAlg}_k,$$

where $\mathfrak{a}\mathrm{CAlg}_k^X = \mathrm{Fun}(X, \mathfrak{a}\mathrm{CAlg}_k)$. Since limits in $\mathfrak{a}\mathrm{CAlg}_k^X$ are computed pointwise, x^* commutes with limits and hence admits a left adjoint

$$x_!\colon \mathfrak{a}\mathrm{CAlg}_k \to \mathfrak{a}\mathrm{CAlg}_k^X,$$

which might be called the free animated k-algebra with $\Omega_x X$-action functor.[20]

Definition 6.2. Fix a point x in $\mathrm{B}S^1$. We let

$$\mathrm{HH}(-/k)\colon \mathfrak{a}\mathrm{CAlg}_k \to \mathfrak{a}\mathrm{CAlg}_k^{\mathrm{B}S^1}$$

denote the left adjoint to the forgetful functor x^*. The value $\mathrm{HH}(R/k)$ at $R \in \mathfrak{a}\mathrm{CAlg}_k$ is the **Hochschild homology** of R over k; it is an animated commutative k-algebra with S^1-action.

[20] Note that x^* commutes with animated commutative k-algebra structures in the sense that we get a natural commutative diagram

$$\begin{tikzcd} \mathfrak{a}\mathrm{CAlg}_k^X \arrow[r] \arrow[d, "x^*"] & \mathfrak{a}\mathrm{Mod}_k^X \arrow[d, "x^*"] \\ \mathfrak{a}\mathrm{CAlg}_k \arrow[r] & \mathfrak{a}\mathrm{Mod}_k, \end{tikzcd}$$

where the horizontal maps forget the commutative algebra structure. The analogous statement is not true for $x_!$.

Remark 6.3 (Universal property)**.** By construction, Hochschild homology has a universal property: if R is an animated commutative k-algebra and S is an animated commutative k-algebra with S^1-action, then there is a natural equivalence

$$\mathrm{Map}_{\mathfrak{a}\mathrm{CAlg}_k}(R, S) \simeq \mathrm{Map}_{\mathfrak{a}\mathrm{CAlg}_k^{\mathrm{B}S^1}}(\mathrm{HH}(R/k), S).$$

Put another way, given a map $R \to S$ of animated commutative k-algebras, there is a unique dotted arrow filling in the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \\ \mathrm{HH}(R/k) \arrow[ur, dotted] & \end{tikzcd}$$

where the vertical map is the counit map $R \to \mathrm{HH}(R/k)$ of the adjunction.

Remark 6.4 (Non-equivariant unit map)**.** The unit map of the adjunction

$$\mathrm{HH}(-/k)\colon \mathfrak{a}\mathrm{CAlg}_k \rightleftarrows \mathfrak{a}\mathrm{CAlg}_k^{\mathrm{B}S^1} : x^*$$

gives a map $R \to x^*\mathrm{HH}(R/k)$ for an animated commutative ring R, but it is important to note that this map is *not* S^1-equivariant. Nevertheless, we will typically write $R \to \mathrm{HH}(R/k)$ for this morphism and leave implicit the abandonment of the S^1-action.

Remark 6.5 (Collapse map)**.** We can compose

$$* \xrightarrow{x} \mathrm{B}S^1 \xrightarrow{p} *$$

and get that if we project

$$p_!\mathrm{HH}(R/k) \simeq p_!x_!R \simeq (p \circ x)_!R \simeq R$$

and use the adjunction between $p_!$ and p^* we get a canonical morphism

$$\mathrm{HH}(R/k) \to p^*R$$

in $\mathfrak{a}\mathrm{CAlg}_k^{\mathrm{B}S^1}$, where p^*R denotes R viewed as an animated commutative k-algebra with the trivial S^1-action. We will call this the **collapse map**.

Definition 6.6 (Copowers/tensors)**.** Let $\mathcal{C}$ be an ∞-category. If $X \in \mathcal{S}$ is an anima and $C \in \mathcal{C}$ is an object, we say that an object ${}^X C \in \mathcal{C}$ is a **copower** or **tensor** of C by X if there is a natural equivalence $\mathrm{Map}_{\mathcal{C}}({}^X C, D) \simeq \mathrm{Map}_{\mathcal{S}}(X, \mathrm{Map}_{\mathcal{C}}(C, D))$ of animae for $D \in \mathcal{C}$.[21] If a copower ${}^X C$ exists for every pair of objects $X \in \mathcal{S}$ and $C \in \mathcal{C}$, we say that $\mathcal{C}$ is **copowered** (or **tensored**) over $\mathcal{S}$.[22]

Lemma 6.7. *Suppose that $\mathcal{C}$ is a presentable ∞-category. Then, $\mathcal{C}$ is copowered over animae. Moreover, copowers are functorial in the sense that there is a functor $T\colon \mathcal{S} \times \mathcal{C} \to \mathcal{C}$ with $T(X, C) \simeq {}^X C$.*

Proof. Fix $X \in \mathcal{S}$ and $C \in \mathcal{C}$ and consider the functor

$$\mathrm{Map}_{\mathcal{S}}(X, \mathrm{Map}_{\mathcal{C}}(C, -))\colon \mathcal{C} \to \mathcal{S}.$$

This functor preserves limits. Moreover, since X and C are κ-compact for some cardinal κ, it follows that the functor is κ-accessible. Thus, by the adjoint functor theorem, there is a left adjoint $F\colon \mathcal{S} \to \mathcal{C}$. The left adjoint has the property that

$$\mathrm{Map}_{\mathcal{C}}(F(*), Y) \simeq \mathrm{Map}_{\mathcal{S}}(*, \mathrm{Map}_{\mathcal{S}}(X, \mathrm{Map}_{\mathcal{C}}(C, Y))) \simeq \mathrm{Map}_{\mathcal{S}}(X, \mathrm{Map}_{\mathcal{C}}(C, Y)).$$

In other words, $F(*)$ is a copower of C by X. We leave functoriality as an exercise to the reader. $\square$

[21] This makes sense more generally for $\mathcal{V}$-enriched ∞-categories when $\mathcal{V}$ is presentably symmetric monoidal.

[22] Frequently, one will see the notation $X \otimes C$ for the copower of X by C.

Lemma 6.8. *The copower functor $\mathcal{S} \to \mathcal{C}$ given by $X \mapsto {}^X C$ preserves colimits.*

Proof. This is follows from the universal property of the copower. $\square$

Example 6.9. For any ∞-category $\mathcal{C}$ and any object $C \in \mathcal{C}$ the copower ${}^*C$ exists and is equivalent to C.

Exercise 6.10. Let $R \in \mathfrak{a}\mathrm{CAlg}_k$ be an animated commutative k-algebra. Compute ${}^{S^0}R$ where S^0 is the 0-sphere.

Lemma 6.11. *If $\mathcal{C}$ is a presentable ∞-category and $C \in \mathcal{C}$, the copower functor ${}^{(-)}C \colon \mathcal{S} \to \mathcal{C}$ is the left Kan extension*

$$\begin{tikzcd} * \arrow[r, "C"] \arrow[d] & \mathcal{C} \\ \mathcal{S}. \arrow[ur, "(-)^C"'] & \end{tikzcd}$$

Proof. This follows from Lemma 6.8 and the fact that the ∞-category of anima is freely generated under colimits by the point. $\square$

Proposition 6.12. *The underlying animated commutative ring of Hochschild homology $\mathrm{HH}(R/k)$ is computed as the copower ${}^{S^1}R$ in $\mathfrak{a}\mathrm{CAlg}_k$.*[23]

Proof. By definition, $\mathrm{HH}(R/k)$ is the left Kan extension

$$\begin{tikzcd} * \arrow[r, "R"] \arrow[d, "x"'] & \mathfrak{a}\mathrm{CAlg}_k \\ \mathrm{B}S^1, \arrow[ur, "\mathrm{HH}(R/k)"'] & \end{tikzcd}$$

and the slice ∞-category

$$* \downarrow \mathrm{B}S^1$$

is equivalent to $\Omega_x \mathrm{B}S^1 \simeq S^1$. Therefore, we can evaluate $\mathrm{HH}(R/k) \to \mathrm{B}S^1 \to \mathfrak{a}\mathrm{CAlg}_k$ at $x \in \mathrm{B}S^1$ by the formula for the left Kan extension

$$\operatorname*{colim}_{x \to x \text{ in } \mathrm{B}S^1} R \simeq \operatorname*{colim}_{\Omega_x \mathrm{B}S^1} R \simeq \operatorname*{colim}_{S^1} R \simeq {}^{S^1}R.$$

Here, the colimit diagram is the constant diagram at R. $\square$

Corollary 6.13. *As an animated commutative ring, we have*

$$\mathrm{HH}(R/k) \simeq R \otimes_{R \otimes_k R} R,$$

where R is viewed as an animated commutative $R \otimes_k R$-algebra via the multiplication map $R \otimes_k R \to R$.[24]

[23] This lemma is often stated as $\mathrm{HH}(R/k) \simeq S^1 \otimes R$ and was first proved, in the setting of $\mathbb{E}_\infty$-ring spectra, by McClure, Schwänzl, and Vogt in [44].

[24] So there is no possibility for confusion, this means $R \otimes^{\mathrm{L}}_{R \otimes^{\mathrm{L}}_k R} R$: both tensor products are derived. Of course, since we are working in the ∞-category of animated commutative rings, there is no other possible definition. See Warning 6.16.

Proof. Since the copower functor preserves colimits, it takes the pushout square

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & S^1 \end{tikzcd}$$

to a pushout square

$$\begin{tikzcd} {}^{S^0}R \arrow[r] \arrow[d] & {}^{*}R \arrow[d] \\ {}^{*}R \arrow[r] & {}^{S^1}R, \end{tikzcd}$$

which is equivalent to a pushout square

$$\begin{tikzcd} R \otimes_k R \arrow[r] \arrow[d] & R \arrow[d] \\ R \arrow[r] & \mathrm{HH}(R/k), \end{tikzcd}$$

which completes the proof. □

Remark 6.14 (Unit and collapse maps via copowers)**.** The unit map $R \to \mathrm{HH}(R/k)$ is (equivalent to) the copower functor applied to the choice of a basepoint $* \xrightarrow{y} S^1$. The collapse map $\mathrm{HH}(R/k) \to R$ is the copower functor applied to $S^1 \to *$.

Remark 6.15. Suppose that A is merely an $\mathbb{E}_1$-k-algebra (for example a dg algebra over k if k is discrete). Then, A is naturally an $A^{\mathrm{op}} \otimes_k A$-module and $\mathrm{HH}(A/k)$ is defined to be $A \otimes_{A^{\mathrm{op}} \otimes_k A} A$. This agrees with the definition above for commutative k-algebras, for example $\mathbb{E}_\infty$-k-algebras or animated commutative k-algebras. However, there is no nice universal property known of $\mathrm{HH}(A/k)$ in the noncommutative case.

Warning 6.16. What we have defined here as $\mathrm{HH}(R/k)$ is sometimes called Shukla homology in the literature. The classical definition of Hochschild homology over a commutative ring k would have been written

$$R \otimes^{\mathrm{L}}_{R \otimes_k R} R$$

where the outer tensor product is derived but the base $R \otimes_k R$ is not derived. These agree when R is flat over k, but in general it is the twice-derived version, which we have given above, which has good properties.

Exercise 6.17. Prove that the functor $\mathrm{HH}(-/k)\colon \mathfrak{a}\mathrm{CAlg}_k \to \mathfrak{a}\mathrm{CAlg}_k^{\mathrm{B}S^1}$ preserves colimits.

Exercise 6.18. Prove that the functor $\mathrm{HH}(-/k)\colon \mathfrak{a}\mathrm{CAlg}_k \to \mathfrak{a}\mathrm{Mod}_k^{\mathrm{B}S^1}$, obtained by forgetting the animated commutative k-algebra structure on $\mathrm{HH}(-/k)$, preserves sifted colimits.

Now, we discuss some important properties of Hochschild homology.

Künneth formula. This is the statement that $\mathrm{HH}(-/k)\colon \mathfrak{a}\mathrm{CAlg}_k \to \mathfrak{a}\mathrm{CAlg}_k^{\mathrm{B}S^1}$ and in particular that it preserves pushouts:

$$\mathrm{HH}(R \otimes_k S/k) \simeq \mathrm{HH}(R/k) \otimes_k \mathrm{HH}(S/k).$$

Base change formula. If $R, S \in \mathfrak{a}\mathrm{CAlg}_k$, then

$$\mathrm{HH}(R/k) \otimes_k S \simeq \mathrm{HH}(R \otimes_k S/S).$$

Collapse reduction formula. If $k \to R \to S$ is a sequence of maps of animated commutative rings, then

$$\mathrm{HH}(S/k) \otimes_{\mathrm{HH}(R/k)} R \simeq \mathrm{HH}(S/R).$$

Exercise 6.19. Prove the base change and collapse reduction formulae.

6.2 The HKR theorem

References 6.20. The original source is [28]. Another good place to look is [65] and [36, Sec. 3.4]. The paper of Bhatt–Morrow–Scholze [13] also has a lot of information and strongly influenced our perspective here.

Example 6.21 (Free animated commutative rings)**.** We compute $\mathrm{HH}(R/k)$ when $R \simeq \mathrm{LSym}_k(M)$ for an animated k-module M. Note that since $\mathrm{LSym}_k \colon \mathfrak{a}\mathrm{Mod}_k \to \mathfrak{a}\mathrm{CAlg}_k$ commutes with colimits, Lemma 6.11 implies that it commutes with copowers: $\mathrm{LSym}_k({}^X M) \simeq {}^X \mathrm{LSym}_k(M)$ for any anima X and any $M \in \mathfrak{a}\mathrm{Mod}_k$. Thus,

$$\mathrm{HH}(R/k) \simeq {}^{S^1} R \simeq {}^{S^1} \mathrm{LSym}_k(M) \simeq \mathrm{LSym}_k({}^{S^1} M).$$

Arguing as in 6.13, we have a pushout square

$$\begin{tikzcd} M \oplus M \arrow[r] \arrow[d] & M \arrow[d] \\ M \arrow[r] & {}^{S^1} M, \end{tikzcd}$$

which implies that ${}^{S^1} M \simeq M \oplus \Sigma M \simeq M \oplus M[1]$. Indeed, the fiber of the addition map $M \oplus M \to M$ is M, so the cofiber of $M \to {}^{S^1} M$ is $M[1]$. But, M splits off of ${}^{S^1} M$ using the maps $* \to S^1 \to *$. Thus, we see that

$$\mathrm{HH}(R/k) \simeq \mathrm{LSym}_k(M \oplus M[1]) \simeq \mathrm{LSym}_k(M) \otimes_k \mathrm{LSym}_k(M[1]) \simeq R \otimes_k \mathrm{LSym}_k(M[1]) \simeq \mathrm{LSym}_R(R \otimes_k M[1]).$$

Using that $\mathrm{L}_{R/k} \simeq R \otimes_k M$ in this case, we see that

$$\mathrm{HH}(R/k) \simeq \mathrm{LSym}_R(\mathrm{L}_{R/k}[1]).$$

Exercise 6.22. Use Example 6.21 to compute $\pi_* \mathrm{HH}(k[x]/k)$.

The main purpose of this section is to prove the following theorem.

Theorem 6.23. *Let k be an animated commutative ring and let R be an animated commutative k-algebra. There is a natural S^1-equivariant multiplicative decreasing $\mathbb{Z}_{\geqslant 0}$-indexed filtration $\mathrm{F}^\star_{\mathrm{HKR}} \mathrm{HH}(R/k)$ with*

$$\mathrm{gr}^s_{\mathrm{HKR}} \mathrm{HH}(R/k) \simeq \Lambda^s \mathrm{L}_{R/k}[s]$$

with trivial S^1-action.

Proof. First, we assume that k and R are discrete commutative rings. Since $\mathrm{HH}(R/k)$ is in particular an animated commutative ring, $\mathrm{HH}_*(R/k)$ is a graded-commutative ring. Moreover, the S^1-action induces an operator $d\colon \mathrm{HH}_t(R/k) \to \mathrm{HH}_{t+1}(R/k)$. We claim that this makes $\mathrm{HH}_*(R/k)$ into a k-linear strict cdga. To prove this, we consider the meaning of the S^1-equivariant multiplication map $\mathrm{HH}(R/k) \otimes_k \mathrm{HH}(R/k) \to \mathrm{HH}(R/k)$. We have

$$\begin{tikzcd}
\bigoplus_{i+j=s} \pi_i\mathrm{HH}(R/k) \otimes_k^{\heartsuit} \pi_j\mathrm{HH}(R/k) \arrow[r] \arrow[d] & \pi_s\left(\mathrm{HH}(R/k) \otimes_k \mathrm{HH}(R/k)\right) \arrow[r] \arrow[d] & \pi_s\mathrm{HH}(R/k) \arrow[d] \\
\bigoplus_{a+b=s+1} \pi_a\mathrm{HH}(R/k) \otimes_k^{\heartsuit} \pi_b\mathrm{HH}(R/k) \arrow[r] & \pi_{s+1}\left(\mathrm{HH}(R/k) \otimes_k \mathrm{HH}(R/k)\right) \arrow[r] & \pi_{s+1}\mathrm{HH}(R/k),
\end{tikzcd}$$

where $\otimes_k^{\heartsuit}$ denotes the ordinary, non-derived tensor product. The S^1-action restricts to give a commutative diagram

$$\begin{tikzcd}
\pi_i\mathrm{HH}(R/k) \otimes_k^{\heartsuit} \pi_j\mathrm{HH}(R/k) \arrow[r] \arrow[d] & \pi_{i+j}(\mathrm{HH}(R/k) \otimes_k \mathrm{HH}(R/k)) \arrow[d] \\
\pi_{i+1}\mathrm{HH}(R/k) \otimes_k^{\heartsuit} \pi_j\mathrm{HH}(R/k) \oplus \pi_i\mathrm{HH}(R/k) \otimes_k^{\heartsuit} \pi_{j+1}\mathrm{HH}(R/k) \arrow[r] & \pi_{i+j+1}\mathrm{HH}(R/k).
\end{tikzcd}$$

Going around clockwise from the top left corner, we get $d(xy)$. Going around counterclockwise from the top left corner, we get $d(x)y + (-1)^i x d(y)$. The sign here confused me for a little while. The reason is that there is an asymmetry in the signs involved in the identifications

$$\mathrm{HH}(R/k)[-1] \otimes_k \mathrm{HH}(R/k) \simeq (\mathrm{HH}(R/k) \otimes_k \mathrm{HH}(R/k))[-1] \simeq \mathrm{HH}(R) \otimes_k (\mathrm{HH}(R/k)[-1]).$$

The left-hand map induces the usual multiplication map

$$\pi_{i+1}\mathrm{HH}(R/k) \otimes_k \pi_j\mathrm{HH}(R/k) \to \pi_{i+j+1}(\mathrm{HH}(R/k) \otimes_k \mathrm{HH}(R/k)),$$

but the right-hand map induces $(-1)^i$ times the multiplication map

$$\pi_i\mathrm{HH}(R/k) \otimes_k \pi_{j+1}\mathrm{HH}(R/k) \to \pi_{i+j+1}(\mathrm{HH}(R/k) \otimes_k \mathrm{HH}(R/k)).$$

For details, see Definition 6.38. For strictness, we use that any odd degree class in the homotopy ring of animated commutative ring squares to zero. We already proved the case where the degree is 1 in the proof of Proposition 4.47. For the general case, one uses the shuffle product on simplicial commutative rings.

Using the universal property of the discrete de Rham complex, we get a map $\Omega^*_{R/k} \to \mathrm{HH}_*(R/k)$ of strict k-linear cdgas. Example 6.21 implies that this map is an isomorphism when R is a polynomial ring over k. It follows that the Whitehead tower

$$\tau_{\geqslant \star}\mathrm{HH}(R/k)$$

gives a multiplicative S^1-equivariant decreasing filtration with $\mathrm{gr}^s\mathrm{HH}(R/k) \simeq \Omega^s_{R/k}[s]$ when R is a polynomial ring over k. Since $R \mapsto \mathrm{HH}(R/k)$ preserves sifted colimits by Exercise 6.18, we can left Kan extend this filtration to obtain the HKR filtration as in the statement of the theorem. Now, the action of S^1 on $\Omega^s_{R/k}[s]$ is trivial when R is a polynomial k-algebra, so the action on $\Lambda^s\mathrm{L}_{R/k}[s]$ is trivial in general.

The general animated commutative case can be proved in a similar way by resolving k. Indeed, we start by writing $k \simeq |k_\bullet|$, where each $k_\bullet$ is a polynomial $\mathbb{Z}$-algebra. Then,

$$|\mathrm{HH}(R/k_\bullet)| \simeq \mathrm{HH}(R/k).$$

Using the naturality of the HKR filtration constructed above in the discrete case, we obtain a filtration

$$\mathrm{F}^{\star}_{\mathrm{HKR}} \simeq |\mathrm{F}^{\star}_{\mathrm{HKR}}\mathrm{HH}(R/k_{\bullet})|$$

with graded pieces

$$\mathrm{gr}^{s}_{\mathrm{HKR}} \simeq |\mathrm{gr}^{s}_{\mathrm{HKR}}\mathrm{HH}(R/k_{\bullet})| \simeq |\Lambda^{s}\mathrm{L}_{R/k_{\bullet}}| \simeq \Lambda^{s}\mathrm{L}_{R/k},$$

as desired. □

Remark 6.24. The natural of S^1-equivariance of the HKR filtration means that each $\mathrm{F}^{s}_{\mathrm{HKR}}\mathrm{HH}(R/k)$ admits the structure of a complex with S^1-action and each map

$$\cdots \to \mathrm{F}^{s+1}_{\mathrm{HKR}}\mathrm{HH}(R/k) \to \mathrm{F}^{s}_{\mathrm{HKR}}\mathrm{HH}(R/k) \to \cdots$$

in the tower is S^1-equivariant. From this perspective, however, we have the trivial S^1-action on the graded pieces. Recall from Example 5.49 that the S^1-action is roughly speaking a map $\mathrm{B} : \mathrm{HH}(R/k) \to \mathrm{HH}(R/k)[-1]$ which squares to (something nullhomotopic to) 0. The triviality of the S^1-action on the graded pieces means that

$$\mathrm{B}\colon \mathrm{gr}^{s}_{\mathrm{HKR}}\mathrm{HH}(R/k) \to \mathrm{gr}^{s}_{\mathrm{HKR}}\mathrm{HH}(R/k)[-1]$$

is nullhomotopic and thus the action map

$$\mathrm{B}\colon \mathrm{gr}^{\star}_{\mathrm{HKR}}\mathrm{HH}(R/k) \to \mathrm{gr}^{\star}_{\mathrm{HKR}}\mathrm{HH}(R/k)[-1]$$

is trivial. However, it is natural to want to view B as having its own weight of 1, so that we would get a map

$$\mathrm{B}\colon \mathrm{gr}^{\star}_{\mathrm{HKR}}\mathrm{HH}(R/k) \to \mathrm{gr}^{\star+1}_{\mathrm{HKR}}\mathrm{HH}(R/k)[-1]$$

which on graded pieces would be the de Rham differential

$$\Lambda^{s}\mathrm{L}_{R/k}[s] \to \Lambda^{s+1}\mathrm{L}_{R/k}[s+1][-1] \simeq \Lambda^{s+1}\mathrm{L}_{R/k}[s].$$

This is possible by using the filtered circle of Raksit [53] or Moulinos–Robalo–Toën [46]. We will see more about this in the next chapter when we discuss the de Rham filtration on negative cyclic homology and periodic cyclic homology.

Remark 6.25. If k is a commutative $\mathbb{Q}$-algebra, then there is a formality theorem due to Yekutieli [65] and Toën–Vezzosi [63] which says that the HKR filtration $\mathrm{HH}(R/k)$ splits canonically, so that

$$\mathrm{HH}(R/k) \simeq \bigoplus_{s\geqslant 0} \Lambda^{s}\mathrm{L}_{R/k}[s],$$

although it is important to note that this splitting is *not* S^1-equivariant. In characteristic p, the filtration is split for smooth affine schemes over k. However, the splitting cannot be chosen naturally as shown in [3]: the HKR spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{H}^{-s}(X, \Omega^{t}_{X/k}) \Rightarrow \mathrm{HH}_{s+t}(X/k)$$

does not always degenerate for smooth proper schemes over $\mathbb{F}_p$.[25]

[25] We define Hochschild homology of a scheme or a stack by right Kan extension from the case of affine schemes, so

$$\mathrm{HH}(X/k) \simeq \lim_{\mathrm{Spec}\, R \to X} \mathrm{HH}(R/k).$$

As it turns out, Hochschild homology is an fppf (hyper)sheaf, so we can compute the limit via the Čech complex of an fppf hypercover if desired.

This non-degeneration might be surprising in light of Example 6.21. However, the equivalence $\mathrm{HH}(R/k) \simeq \mathrm{LSym}_R(\mathrm{L}_{R/k}[1])$ for $R = \mathrm{LSym}_k M$ free over k is only functorial in maps of M, not in all maps between free animated commutative k-algebras. For example, it is functorial in maps $k[x_1, \ldots, x_m] \to k[y_1, \ldots, y_n]$ that send each x_i to a homogeneous degree 1 polynomial in the y_j, but it is not functorial in general polynomial maps.

Remark 6.26. Another way of constructing Hochschild homology of R over k is as the geometric realization of the cyclic bar complex $\mathrm{B}^{\mathrm{cyc}}_\bullet(R/k)$ which is a simplicial animated commutative k-algebra

$$|\cdots \substack{\rightarrow\\ \rightarrow\\ \rightarrow} R \otimes_k R \rightrightarrows R|$$

with $\mathrm{B}^{\mathrm{cyc}}_n(R/k) \simeq R^{\otimes n+1}$. For details on the differentials, in a slightly different context, see Definition 6.48.

If k and R are discrete and R is flat over k, we can construct explicit maps

$$\Omega^*_{R/k} \to \mathrm{H}_*(\mathrm{B}^{\mathrm{cyc}}_\bullet(R/k))$$

by

$$f\mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_n \mapsto \sum_{\sigma \in \Sigma_n} f \otimes g_{\sigma(1)} \otimes \cdots g_{\sigma(n)}.$$

This is well-defined only as a map to the homology of the cyclic bar complex.

On the other hand, we also have a map

$$R^{\otimes n+1} \to \Omega^n_{R/k}$$

obtained by

$$f_0 \otimes g_1 \otimes \cdots g_n \mapsto f_0 \mathrm{d}g_1 \wedge \cdots \mathrm{d}g_n.$$

The composition

$$\Omega^n_{R/k} \to \mathrm{HH}_n(R/k) \to \Omega^n_{R/k}$$

is multiplication by $n!$, which shows that the discrete de Rham complex splits off of Hochschild homology in characteristic 0. Moreover, this explicitly gives the formality of Remark 6.25 in characteristic 0, because it proves that $\mathrm{HH}(R/k)$ is formal as a complex. Richer formality statements are explored in [63].

The Künneth, base change, and collapse reduction formulae can all be upgraded to HKR-filtered statements.

Lemma 6.27 (HKR-filtered Künneth formula)**.** *If $R, S \in \mathfrak{a}\mathrm{CAlg}_k$, then*

$$\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HH}(R/k) \otimes_k \mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HH}(S/k) \simeq \mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HH}(R \otimes_k S/k).$$

Lemma 6.28 (HKR-filtered base change formula)**.** *If $R, S \in \mathfrak{a}\mathrm{CAlg}_k$, then*

$$\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HH}(R/k) \otimes_k S \simeq \mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HH}(R \otimes_k S/S).$$

Lemma 6.29 (HKR-filtered collpase reduction formula)**.** *If $k \to R \to S$ is a sequence of maps of animated commutative ring, then*

$$\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HH}(S/k) \otimes_{\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HH}(R/k)} R \simeq \mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HH}(S/R).$$

Exercise 6.30. Prove Lemma 6.29 when R is free over k and S is free ove R.

Remark 6.31. Taking gr^1 in Lemma 6.27, we get $\mathrm{L}_{R/k}[1] \otimes_k S \oplus R \otimes_k \mathrm{L}_{S/k}[1] \simeq \mathrm{L}_{S \otimes_k R/k}[1]$, which follows from Exercise 3.52. Similarly, taking gr^1 in Lemma 6.28 yields

$$\mathrm{L}_{R/k}[1] \otimes_k S \simeq \mathrm{L}_{R \otimes_k S/S}[1], .$$

which is Exercise 3.51. The statement of Lemma 6.29 is trickier because it involves taking the tensor product over a non-trivially filtered animated commutative ring, $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HH}(R/k)$. The lemmas can be proved by reducing to the case where R is polynomial over k and S is polynomial over R. Alternatively, they can be derived from Raksit's paper [53].

6.3 Descent

References 6.32. See Mathew's paper [41] or Bhatt–Morrow–Scholze [13].

We will use the following two theorems, but will not prove them. The first theorem was proved (in greater generality) in [41].

Theorem 6.33 (Étale base change)**.** *If $R \to S$ is an étale map of animated commutative k-algebras, then the natural (non-S^1-equivariant) map*

$$S \otimes_R \mathrm{HH}(R/k) \to \mathrm{HH}(S/k)$$

is an equivalence.

The second theorem was proved in [13].

Theorem 6.34 (Descent)**.** *The functors*

$$R \mapsto \mathrm{HH}(R/k), \quad R \mapsto \mathrm{L}_{R/k}, \quad R \mapsto \mathrm{L}\Lambda^s \mathrm{L}_{R/k}$$

satisfy flat descent (for $s \geqslant 0$ in the third case).

6.4 Appendix: signs

Lacking a standard reference with quite the detail we want, we include an appendix with various technicalities on the sign choices used in the proof. A good source is Loday's book [36, Sec. 5.4], although note that Loday only works with non-negatively graded differential graded algebras. This material is entirely well-known, but there are some signed surprises for the reader who, like us, initially attempts to naïvely extend the usual formulas as found, for example, in Weibel's book [64] to the dg world.

6.4.1 The symmetric monoidal category of chain complexes

Definition 6.35. Let k be a commutative ring and Ch_k the category of chain complexes of k-modules. This is a tensor category with

$$(M \otimes_k N)_n = \bigoplus_{p+q=n} M_p \otimes_k N_q,$$

where the differential is

$$d(m \otimes n) = d(m) \otimes n + (-1)^p m \otimes d(n)$$

for $m \in M_p$. The tensor is symmetrically braided with

$$\Phi : M \otimes_k N \to N \otimes_k M$$

given by the following convention:

$$\Phi(m \otimes n) = (-1)^{pq} n \otimes m$$

for $m \in M_p$ and $n \in N_q$. To see that Φ is indeed a morphism of chain complexes, one checks that

$$d(\Phi(m \otimes n)) = (-1)^{pq} d(n \otimes m) = (-1)^{pq} \left(d(n) \otimes m + (-1)^q n \otimes d(m) \right),$$

while

$$\Phi(d(m \otimes n)) = \Phi(d(m) \otimes n + (-1)^p m \otimes d(n)) = (-1)^{pq-q} n \otimes d(m) + (-1)^p (-1)^{pq-p} d(n) \otimes m,$$

and these two terms are the same. The associativity constraints are the identities (or rather, come directly from the associativity constraint on the underlying k-modules).

Definition 6.36. The suspension of a chain complex M is defined as ΣM with p part

$$(\Sigma M)_p = M_{p-1}.$$

The differential $d_{\Sigma M}(m) = -d_M(m)$. More generally,

$$\left(\Sigma^k M\right)_p = M_{p-k},$$

with $d_{\Sigma^k M}(m) = (-1)^k d_M(m)$. The suspension becomes an endofunctor of Ch_k by decreeing that

$$\Sigma(f)(m) = f(m)$$

for all homogeneous m.

Remark 6.37. For clarity we will often write $\Sigma^k m$ for m viewed as a homogeneous element of $\Sigma^k M$. Thus, for example, the differential formula can then be viewed as

$$d_{\Sigma^k M}\left(\Sigma^k m\right) = (-1)^k \Sigma^k d_M(m).$$

This will be an especially important bookkeeping method when we define the map from the double complex computing Hochschild homology to the symmetric algebra on the Kähler differentials.

Definition 6.38. There is an additional canonical structure we need, which says how the suspension interacts with the tensor product. Namely, we fix isomorphisms

$$S^R_{M,N} : M \otimes_k \Sigma N \to \Sigma\left(M \otimes_k N\right)$$

by

$$S^R_{M,N}\left(m \otimes \Sigma n\right) = (-1)^p \Sigma\left(m \otimes n\right)$$

for $m \in M_p$ and $n \in N_q$, while

$$S^L_{M,N} : \Sigma M \otimes_k N \to \Sigma\left(M \otimes_k N\right)$$

is defined as the identity on each component. Note that $S^R_{M,N} = \Sigma(\Phi_{N,M}) \circ S^L_{N,M} \circ \Phi_{M,\Sigma N}$.

Lemma 6.39. *For any M and N, the functions $S^R_{M,N}$ and $S^L_{M,N}$ are maps of chain complexes.*

Proof. Suppose that $m \in M_p$ and $n \in N_q$. Write $m \otimes \Sigma n$ for the element $m \otimes n$ viewed in $M \otimes_k \Sigma N$. Then,

$$\begin{aligned} d_{\Sigma(M\otimes_k N)}(S^R(m \otimes \Sigma n)) &= (-1)^p d_{\Sigma(M\otimes_k N)}(\Sigma(m \otimes n)) \\ &= -(-1)^p \Sigma d_{M\otimes_k N}(m \otimes n) \\ &= -(-1)^p \Sigma(d(m) \otimes n + (-1)^p m \otimes d_N(n)), \end{aligned}$$

while

$$\begin{aligned} S^R(d_{M\otimes_k \Sigma N}(m \otimes \Sigma n)) &= S^R(d(m) \otimes \Sigma n + (-1)^p m \otimes d_{\Sigma N}(\Sigma n)) \\ &= S^R(d(m) \otimes \Sigma n - (-1)^p m \otimes \Sigma d_N(n)) \\ &= (-1)^{p-1} \Sigma\left(d(m) \otimes n\right) - (-1)^p(-1)^p \Sigma(m \otimes d_N(n)), \end{aligned}$$

which shows that S^R is a map of chain complexes. The verification of the other side is similar, or follows from the relationship between S^R and S^L noted above. $\square$

Definition 6.40. A *differential graded algebra* (k-dga) is a monoid in Ch_k; a *commutative differential graded algebra* (k-cdga) is a commutative monoid in Ch_k. If A is a k-dga, then the fact that $A \otimes_k A \to A$ is a map of chain complexes implies that

$$d(ab) = d(a)b + (-1)^p ad(b),$$

where $a \in A_p$. A k-cdga satisfies additionally the property that

$$ab = (-1)^{pq} ba,$$

if $a \in A_p$ and $b \in A_q$.

Now, all the usual properties of rings and modules can be formulated in the symmetric monoidal category Ch_k.

Definition 6.41. A left A-module is a chain complex M together with a map of chain complexes $A \otimes_k M \to M$ satisfying the usual unit and associativity axioms. In particular,

$$d(am) = d(a)m + (-1)^p ad(m)$$

for $a \in A_p$. Similarly, a right A-module has a multiplication $M \otimes_k A \to M$ satisfying

$$d(ma) = d(m)a + (-1)^p md(a)$$

for $m \in M_p$. Note the asymmetry here: we take the exponent from the degree of m for the right module structure.

The following point, that one has to twist suspensions of left A-modules, confused me for quite a while.

Definition 6.42. If $m^L : A \otimes_k M \to M$ is a left A-module, we can naturally put a left A-module structure on ΣM via

$$A \otimes_k \Sigma M \xrightarrow{S^R} \Sigma(A \otimes_k M) \xrightarrow{\Sigma(m^L)} \Sigma M.$$

In particular,

$$a \otimes \Sigma m = (-1)^p \Sigma(am)$$

for $a \in A_p$. Similarly, if $m^R : M \otimes_k A \to M$ is a right A-module, then ΣM becomes a right A-module via

$$\Sigma M \otimes_k A \xrightarrow{S^L} \Sigma(M \otimes_K A) \xrightarrow{\Sigma(m^R)} \Sigma M.$$

This time, we have $(\Sigma m)a = \Sigma(ma)$.

Definition 6.43. If R is a k-cdga, then a symmetric R-bimodule is a chain complex M with left and right R-module structures satisfying

$$am = (-1)^{pq} ma$$

for $a \in A_p$ and $m \in M_q$.

6.4.2 Differential graded derivations

Definition 6.44. A square-zero extension of k-cdgas is a map $S' \to S$ of k-cdgas with kernel M a square-zero ideal of S'. Concretely, this means that

$$mn = 0$$

for m, n homogeneous elements of M. This makes M into a symmetric R bimodule.

Now, we outline the considerations that motivated our definition of the sign rule we adopt for a (degree zero) derivation of a k-cdga. Let $f, g : R \to S'$ be two lifts to S' of a k-cdga map $h : R \to S$. In the case of discrete (i.e., classical) commutative algebras, the difference $D = f - g$ is supposed to be the defining example of a k-derivation of R in M. Note that via h (or equivalently f or g), M inherits the structure of a symmetric R-bimodule. Here is what we can say about D. Let $r \in R_p$ and $s \in S_q$. Then,

$$\begin{aligned} D(rs) &= f(rs) - g(rs) \\ &= f(r)f(s) - g(r)g(s) \\ &= (f(r) - g(r))f(s) + g(r)f(s) + g(r)(f(s) - g(s)) - g(r)f(s) \\ &= D(r)s + rD(s), \end{aligned}$$

and we use the symmetric bimodule structure to write this as

$$D(rs) = rD(s) + (-1)^{pq} sD(r).$$

This motivates our next definition.

Definition 6.45. Let R be a k-cdga, and let M be a (left) R-module. Then, a k-derivation of R in M is a map of chain complexes $D : R \to M$ such that $D(a) = 0$ for $a \in k$ and

$$D(rs) = rD(s) + (-1)^{pq} sD(r)$$

for all homogeneous elements $r \in R_p$ and $s \in R_q$.

Definition 6.46. Let R be a k-cdga, and make $R \otimes_k R$ a k-cdga by the rule

$$(a \otimes b) \otimes (x \otimes y) = (-1)^{|b||x|}(ax \otimes by).$$

Multiplication $R \otimes_k R \to R$ is a ring map, and hence it has an ideal I as kernel. We define

$$\Omega_{R/k} = I/I^2,$$

a symmetric R-bimodule. There is a canonical map $D : R \to \Omega_{R/k}$ defined by

$$D(r) = r \otimes 1 - 1 \otimes r.$$

We see that this is a k-derivation by checking

$$\begin{aligned} D(rs) &= rs \otimes 1 - 1 \otimes rs \\ &= r(s \otimes 1 - 1 \otimes s) + r \otimes s + (r \otimes 1 - 1 \otimes r)s - r \otimes s \\ &= rD(s) + D(r)s \\ &= rD(s) + (-1)^{pq} sD(r), \end{aligned}$$

if $r \in R_p$ and $s \in R_q$.

Example 6.47. The key example for us is when $R = \mathrm{Sym}_k M$, where M is a differential graded k-module with zero differential and free in each degree. Then, $\Omega_{R/k} \simeq R \otimes_k M$.

6.4.3 The cyclic bar complex

The cyclic bar complex makes sense for any algebra object in a braided monoidal category. We specialize to the case of Ch_k and outline carefully what the various maps are.

Definition 6.48. Let A be a k-dga, and let M be an A-bimodule. The cyclic bar construction

$$\mathrm{B}_\bullet(A/k, M)$$

is the simplicial object of Ch_k with

$$\mathrm{B}_p(A/k, M) = M \otimes A^{\otimes p},$$

where the face maps are

$$\partial_i(m \otimes a_1 \otimes \cdots \otimes a_p) = \begin{cases} ma_1 \otimes a_2 \otimes \cdots \otimes a_p & \text{if } i = 0, \\ m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_p & \text{if } 1 \leqslant i \leqslant p-1, \text{ and} \\ (-1)^{|a_p|(|m|+|a_1|+\cdots+|a_{p-1}|)}\, a_p m \otimes a_1 \otimes \cdots \otimes a_{p-1} & \text{if } i = p, \end{cases}$$

and the degeneracy maps are

$$\sigma_i(m \otimes a_1 \otimes \cdots \otimes a_p) = m \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_p$$

for $0 \leqslant i \leqslant p$. When $M = A$ there is a cyclic structure on $\mathrm{B}_\bullet A = \mathrm{B}_\bullet(A/k, A)$ given by

$$t(a_0 \otimes \cdots a_p) = (-1)^{|a_p|(|a_0|+\cdots+|a_{p-1}|)}\, a_p \otimes a_0 \otimes \cdots \otimes a_{p-1}.$$

The signs in ∂_p and t are not mysterious. They arise because in order to perform the wrap-around multiplication a_p needs to commute all the way past the first p terms, and this is done via the braiding rule given in Definition 6.35.

We will leave the verification that these maps do indeed make $\mathrm{B}_\bullet(A/k, M)$ (resp. $\mathrm{B}_\bullet A$) into a simplicial (resp. cyclic) chain complex to the reader.

Definition 6.49. We can associate to $\mathrm{B}_\bullet A$ a double complex $\mathrm{B}_{\bullet,\bullet} A$ in the standard way, using the (vertical) sign trick. Namely, we let

$$\mathrm{B}_{p,q} = \left(A^{\otimes p+1}\right)_q,$$

and we equip the resulting lattice with horizontal differentials

$$d^h = b = \partial_0 - \partial_1 + \cdots + (-1)^p \partial_p$$

coming from the simplicial structure, and with vertical differentials

$$d^v = (-1)^p d_{A^{\otimes p+1}}.$$

The function $\overline{d} = d^h + d^v$ is a differential on

$$\mathrm{Tot}^{\oplus} \mathrm{B}_{\bullet,\bullet} A,$$

and the homology of the resulting complex is the Hochschild homology $\mathrm{HH}^k_\bullet(A)$ of A.

Remark 6.50. Since we take the direct sum total complex, and as this is a double-complex concentrated in the right half-plane, filtration by columns leads to a *convergent* spectral sequence

$${}^I\mathrm{E}^2_{p,q} = \mathrm{H}^h_p \mathrm{H}^v_q \mathrm{B}_{\bullet,\bullet} A \Rightarrow \mathrm{HH}^k_{p+q}(A).$$

Often, A will in fact be a formal k-cdga, which means that the spectral sequence will degenerate at the E^2-page.

6.4.4 From Hochschild homology to differential forms

The purpose of this section is to construct a map from the Hochschild homology double complex $\mathrm{B}_{\bullet,\bullet}R$ to a double complex that computes the homology of the symmetric algebra $\mathrm{Sym}_R(\Sigma\Omega_{R/k})$. The importance of having this concrete map is that we can compute exactly what happens in the trace map of K-theory. This is the point where the sign conventions get serious, and we use both the commutativity sign constraint, the compatibility with suspension sign constraints S^R and S^L, as well as the signs appearing in ∂_p and the definition of a k-derivation. That it all turns out to work is something of a miracle, but one that *a posteriori* validates all of the choices we have made.

Let R be a k-cdga. Let

$$S_{p,q} = \left(\Sigma^{-p}\left(\Sigma\Omega_{R/k}^{\otimes p}\right)_{S_p}\right)_q,$$

where the subscript S_p denotes as usual the S_p-coinvariants. We view $S_{p,q}$ as a double complex by applying the (vertical) sign trick to the vertical differentials, and equipping it with zero horizontal differential. Our goal is to define a map

$$\mathrm{B}_{\bullet,\bullet}R \to S_{\bullet,\bullet}$$

of double complexes.

Definition 6.51. Consider the map $D^{\wedge p}$ of complexes

$$\Sigma^p R^{\otimes p+1} \to R\otimes \Sigma R\otimes\cdots\otimes\Sigma R \xrightarrow{id\otimes\Sigma D\otimes\cdots\otimes\Sigma D} R\otimes\Sigma\Omega_R\otimes\cdots\otimes\Sigma\Omega_R \to \Sigma\Omega_R\otimes\cdots\otimes\Sigma\Omega_R \to \left(\Sigma\Omega_R^{\otimes p}\right)_{S_p},$$

where the first map is an appropriate combination of S^R and S^L maps, and where the third map is induced from the left R-module structure on the left-most copy of $\Sigma\Omega_R$. On a typical homogeneous $r_0\otimes\cdots r_p$ element of $\Sigma^p R^{\otimes p+1}$, we find by tracing through the definitions that

$$D^{\wedge p}(r_0\cdots r_p) = (-1)^{\sum_{i=0}^{p-1}|r_i|}(-1)^{|r_0|}\,\Sigma(r_0D(r_1))\otimes\Sigma D(r_1)\otimes\cdots\otimes\Sigma D(r_p),$$

where the first sign term comes from the p-fold S^Rs followed by a single S^L, and where the second sign term comes from the shift in $R\otimes\Sigma\Omega_R\to\Omega_R$. We write $D^{\wedge p}$ also for the same map but desuspended p times.

Proposition 6.52. *The identity $D^{\wedge p}\circ b = 0$ holds on $R^{\otimes p+2}$.*

Proof. Fix a homogeneous element $r_0\otimes\cdots\otimes r_{p+1}$ of $R^{\otimes p+2}$. Then,

$$\begin{aligned} D^{\wedge p}(\partial_0(r_0\otimes\cdots\otimes r_{p+1})) &= D^{\wedge p}(r_0r_1\otimes r_2\otimes\cdots\otimes r_{p+1})\\ &= (-1)^{p(|r_0|+|r_1|)+\sum_{i=1}^{p-1}(p-i)||r_{i+1}|}(-1)^{|r_0|+|r_1|}\,\Sigma(r_0r_1D(r_2))\otimes\Sigma D(r_3)\otimes\cdots\otimes\Sigma D(r_{p+1}),\end{aligned}$$

$$\begin{aligned} D^{\wedge p}(\partial_j(r_0\otimes\cdots\otimes r_{p+1})) &= D^{\wedge p}(r_0\otimes\cdots r_jr_{j+1}\otimes\cdots\otimes r_{p+1})\\ &= (-1)^{\sum_{i=0}^{j-1}(p-i)|r_i|+(p-j)(|r_j|+|r_{j+1}|)+\sum_{i=j+1}^{p-1}(p-i)|r_{i+1}|}(-1)^{|r_0|}\\ &\cdot\Sigma(r_0D(r_1))\otimes\cdots\otimes\Sigma\left(r_jD(r_{j+1})+(-1)^{|r_j||r_{j+1}|}r_{j+1}D(r_j)\right)\otimes\cdots\otimes\Sigma D(r_{p+1})\end{aligned}$$

for $1\leqslant j\leqslant p$, and

$$\begin{aligned} D^{\wedge p}(\partial_{p+1}(r_0\otimes\cdots\otimes r_{p+1})) &= (-1)^{|r_{p+1}|(|r_0|+\cdots+|r_p|)}D^{\wedge p}(r_{p+1}r_0\otimes r_1\otimes\cdots\otimes r_p)\\ &= (-1)^{p(|r_0|+|r_{p+1}|)+\sum_{i=1}^{p-1}(p-i)|r_i|}(-1)^{|r_0|+|r_{p+1}|}(-1)^{|r_{p+1}|(|r_0|+\cdots+|r_p|)}\\ &\cdot\Sigma(r_{p+1}r_0D(r_1))\otimes\Sigma D(r_2)\otimes\cdots\otimes\Sigma D(r_p).\end{aligned}$$

Taken together, for $1 \leqslant j \leqslant p+1$, we see that up to a sign each term of the form

$$X_j = \Sigma(r_0 r_j D(r_1)) \otimes \cdots \otimes \widehat{\Sigma D(r_j)} \otimes \cdots \otimes \Sigma D(r_{p+1}),$$

where we omit $\Sigma D(r_j)$, appears exactly twice, once in $D^{\wedge p}(\partial_{j-1}(r_0 \otimes \cdots \otimes r_{p+1}))$, and once in $D^{\wedge p}(\partial_j(r_0 \otimes \cdots \otimes r_{p+1}))$. Since these occur with opposite signs in the expansion of b, it is enough to show that they occur with the same sign in the equations above.

In order to transform all of the terms into the common for X_j, we use that $\Sigma(r_j D(r_{j+1})) = (-1)^{|r_j|} r_j \Sigma D(r_{j+1})$ together with the braiding, and so on. The equations above then become

$$\begin{aligned}
D^{\wedge p}(\partial_0(r_0 \otimes \cdots \otimes r_{p+1})) &= (-1)^{p(|r_0|+|r_1|)+\sum_{i=1}^{p-1}(p-i)||r_{i+1}|} \, (-1)^{|r_0|+|r_1|} \, X_1, \\
D^{\wedge p}(\partial_j(r_0 \otimes \cdots \otimes r_{p+1})) &= (-1)^{\sum_{i=0}^{j-1}(p-i)|r_i|+(p-j)(|r_j|+|r_{j+1}|)+\sum_{i=j+1}^{p-1}(p-i)|r_{i+1}|} \, (-1)^{|r_0|} \\
&\quad \cdot \Big((-1)^{|r_j|+|r_j|(j-1+|r_0|+\cdots+|r_{j-1}|)+|r_j|+|r_0||r_j|} \, X_j + (-1)^{|r_j||r_{j+1}|+|r_{j+1}|+|r_{j+1}|(j-1+|r_0|+\cdots+|r_{j-1}|)+|r_{j-}} \\
D^{\wedge p}(\partial_{p+1}(r_0 \otimes \cdots \otimes r_{p+1})) &= (-1)^{p(|r_0|+|r_{p+1}|)+\sum_{i=1}^{p-1}(p-i)|r_i|} \, (-1)^{|r_0|+|r_{p+1}|} \, (-1)^{|r_{p+1}|(|r_0|+\cdots+|r_p|)} \, (-1)^{|r_0||r_{p+1}|} \\
&\quad \cdot X_{p+1},
\end{aligned}$$

where the second equation is for $1 \leqslant j \leqslant p$.

The exponent of X_1 in $D^{\wedge p}(\partial_0)$ is

$$p(|r_0| + |r_1|) + \sum_{i=1}^{p-1}(p-i)||r_{i+1}| + |r_0| + |r_1|,$$

while it appears in $D^{\wedge p}(\partial_1)$ with exponent

$$\begin{aligned}
&\sum_{i=0}^{j-1}(p-i)|r_i| + (p-j)(|r_j| + |r_{j+1}|) + \sum_{i=j+1}^{p-1}(p-i)|r_{i+1}| + |r_0| + |r_j| + |r_j|(j-1+|r_0|+\cdots+|r_{j-1}|) + |r_j| + |r_0||r_j| \\
&\qquad = p|r_0| + (p-1)(|r_1| + |r_2|) + \sum_{i=2}^{p-1}(p-i)|r_{i+1}| + |r_0| + 2|r_1| + 2|r_0||r_1|;
\end{aligned}$$

these are equivalent modulo 2, as desired.

For $2 \leqslant j \leqslant p$, we have that the exponent of (-1) appearing in front of X_j in $D^{\wedge p}(\partial_{j-1})$ is

$$\begin{aligned}
&\sum_{i=0}^{j-2}(p-i)|r_i| + (p-j+1)(|r_{j-1}| + |r_j|) + \sum_{i=j}^{p-1}(p-i)|r_{i+1}| + |r_0| \\
&+ |r_{j-1}||r_j| + |r_j| + |r_j|(j-2+|r_0|+\cdots+|r_{j-2}|) + |r_j| + |r_0||r_j|,
\end{aligned}$$

while that of (-1) appearing in front of X_j in $D^{\wedge p}(\partial_j)$ is

$$\begin{aligned}
&\sum_{i=0}^{j-1}(p-i)|r_i| + (p-j)(|r_j| + |r_{j+1}|) + \sum_{i=j+1}^{p-1}(p-i)|r_{i+1}| + |r_0| + |r_j| \\
&+ |r_j|(j-1+|r_0|+\cdots+|r_{j-1}|) + |r_j| + |r_0||r_j|.
\end{aligned}$$

Inspection shows that these are the same.

Finally, for X_{p+1} we have

$$\sum_{i=0}^{p-1}(p-i)|r_i| + |r_0| + |r_p||r_{p+1}| + |r_{p+1}| + |r_{p+1}|(p-1+|r_0|+\cdots+|r_{p-1}|) + |r_{p+1}| + |r_0||r_{p+1}|$$

from $D^{\wedge p}(\partial_p)$ and

$$p(|r_0|+|r_{p+1}|) + \sum_{i=1}^{p-1}(p-i)|r_i| + |r_0| + |r_{p+1}| + |r_{p+1}|(|r_0|+\cdots+|r_p|) + |r_0||r_{p+1}|$$

from $D^{\wedge p}(\partial_{p+1})$. These agree, so the proof is complete. $\square$

7 Week 7: The de Rham filtration on periodic cyclic homology (31 Oct.–6 Nov.)

The purpose of this section is to construct a kind of motivic or Atiyah–Hirzebruch spectral sequence relating de Rham cohomology to periodic cyclic homology. To motivate this, consider the double-speed Postnikov tower

$$\tau_{\leqslant 2\star}\mathrm{KU}$$

of the complex K-theory spectrum. Now, let X be an anima. We can form the decreasing filtered spectrum

$$\mathrm{F}^{\star}X^{\mathrm{KU}} = X^{\tau_{\leqslant -2\star}\mathrm{KU}}.$$

In other words, we have

$$\cdots \to X^{\tau_{\leqslant -2}\mathrm{KU}} \to X^{\tau_{\leqslant 0}\mathrm{KU}} \to X^{\tau_{\leqslant 2}\mathrm{KU}} \to \cdots$$

where the graded pieces are

$$\mathrm{gr}^s X \simeq X^{\mathbb{Z}[-2s]} \simeq X^{\mathbb{Z}}[2s],$$

in other words shifts of the integral singular cohomology of X. We rewrite the associated E_1-spectral sequence as an E_2-spectral sequence via

$$\mathrm{E}_2^{s,t} = \mathrm{H}^{s+t}(\mathrm{gr}^{-t}X^{\mathrm{KU}}) \Rightarrow \mathrm{KU}^{s+t}(X)$$

or equivalently as

$$\mathrm{E}_2^{s,t} = \mathrm{H}^{s-t}(X,\mathbb{Z}) \Rightarrow \mathrm{KU}^{s+t}(X).$$

This is slightly different from the standard indexing of the Atiyah–Hirzebruch spectral sequence because we used the double-speed Postnikov filtration. It has the advantage that it agrees more closely with motivic-style spectral sequences and the spectral sequences of [13].

The goal of this section is to construct another spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{H}^{s-t}_{\widehat{\mathrm{dR}}}(X/k) \Rightarrow \mathrm{H}^{s+t}(\mathrm{HP}(X/k)),$$

which will have good convergence properties when X is quasi-lci over k. This was done first in [13] in the p-complete case. I extended it integrally via a different construction in [2] and then it has been studied by Raksit in [53] and Moulinos–Robalo–Toën in [46] (for negative cyclic homology).

7.1 Cyclic homology and negative cyclic homology

References 7.1. Loday's book [36] remains a great reference. See also [48] and [13] for modern approaches.

Let X be an anima with $p\colon X \to *$ the structure map. To p we associate the functor $p^*\colon \mathcal{D}(k) \to \mathcal{D}(k)^X$, which has left adjoint $p_!$ and right adjoint p_*. The functor $p_!$ is called **homology** and p_* is called **cohomology**. Indeed, $p_!k$ and p_*k compute the singular homology and cohomology of X with coefficients in k.

Example 7.2. When $X \simeq \mathrm{B}G$ for some group (or grouplike $\mathbb{E}_1$-space), these functors are computing the group homology and cohomology of G. As such, if $M \in \mathcal{D}(k)^{\mathrm{B}G}$, we set $M_{\mathrm{h}G} = p_!M$ and $M^{\mathrm{h}G} = p_*M$. These are the complexes of **homotopy G-orbits** and **homotopy G-fixed points**, respectively.

Since p^* is symmetric monoidal, the right adjoint p_* is lax symmetric monoidal.

Exercise 7.3. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal 1-categories and that $f^*\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon f_*$ is an adjunction where f^* is symmetric monoidal. Show that f_* is lax symmetric monoidal.

Exercise 7.4. Show that if $f\colon \mathcal{C} \to \mathcal{D}$ is a lax symmetric monoidal functor between symmetric monoidal 1-categories, then f preserves algebra objects in the sense that it induces a functor $\mathrm{Alg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{D})$. A similar result holds for commutative algebra objects.

Remark 7.5. Of course, Exercises 7.3 and 7.4 have analogues in the world of ∞-categories, but it is slightly outside the scope of this course to develop the machinery to give those arguments. But, we will nevertheless implicitly use those two facts in the context of ∞-categories too.

Definition 7.6. Let R be an animated commutative k-algebra and let $\mathrm{HH}(R/k)$ be its Hochschild homology. Forgetting the animated commutative structure, we view $\mathrm{HH}(R/k)$ as a complex with S^1-action, i.e., as an object of $\mathcal{D}(k)^{\mathrm{B}S^1}$. The **cyclic homology** of R is defined to be the complex of homotopy S^1-orbits

$$\mathrm{HC}(R/k) \simeq \mathrm{HH}(R/k)_{\mathrm{h}S^1}.$$

The **negative cyclic homology** of R is defined to be the complex of homotopy S^1-fixed points:

$$\mathrm{HC}^-(R/k) \simeq \mathrm{HH}(R/k)^{\mathrm{h}S^1}.$$

Remark 7.7. Since $\mathrm{HH}(R/k)$ admits the structure of an $\mathbb{E}_\infty$-ring (by forgetting the derived commutative structure), $\mathrm{HC}^-(R/k)$ naturally admits the structure of an $\mathbb{E}_\infty$-ring too, and we typically view it as such. In particular, since $\mathrm{HH}(R/k)$ is an $\mathbb{E}_\infty$-k-algebra, $\mathrm{HC}^-(R/k)$ is an $\mathbb{E}_\infty$-algebra over $k^{\mathrm{h}S^1}$, the complex computing $\mathrm{H}^*(\mathrm{B}S^1, k) \simeq \mathrm{H}^*(\mathbb{CP}^\infty, k) \simeq k[v]$, where $|v| = 2$.[26] Alternatively, $\mathrm{HC}^-(R/k)$ is a derived commutative ring over the derived commutative ring $k^{\mathrm{h}S^1}$; these admit structures of derived commutative k-algebras because $\mathfrak{d}\mathrm{CAlg}_k$ is closed under all limits (and colimits) in $\mathrm{Alg}_k^{\mathbb{E}_\infty}$.

Exercise 7.8. Cyclic homology is defined by applying homotopy S^1-orbits to the underlying parametrized complex of $\mathrm{HH}(R/k)$, thus forgetting the commutative algebra structure and applying $p_!\colon \mathcal{D}(k)^{\mathrm{B}S^1} \to \mathcal{D}(k)$. What happens if we compute a version of cyclic homology using $p_!\colon \mathfrak{a}\mathrm{CAlg}_k^{\mathrm{B}S^1} \to \mathfrak{a}\mathrm{CAlg}_k$. What is $p_!(\mathrm{HH}(R/k))$?

Remark 7.9. The original impetus to study these theories is the observation that there is a natural map $\mathrm{K}(R) \to \mathrm{HH}(R/\mathbb{Z})$ which factors through a map $\mathrm{K}(R) \to \mathrm{HC}^-(R/k)$. Here, $\mathrm{K}(R)$ denotes algebraic K-theory,

[26] We are using here the fact that we are working over $\mathbb{Z}$. The computation of $\mathrm{H}^*(\mathrm{B}S^1, R)$ with coefficients in general $\mathbb{E}_\infty$-rings R is more complicated.

an important but ultimately extremely difficult invariant of (animated) commutative rings R. Goodwillie proved in [22] that if R is an associative $\mathbb{Q}$-algebra and $I \subseteq R$ is a two-sided ideal, then the natural square

$$\begin{tikzcd} \mathrm{K}(R) \arrow[r] \arrow[d] & \mathrm{K}(R/I) \arrow[d] \\ \mathrm{HC}^-(R/\mathbb{Q}) \arrow[r] & \mathrm{HC}^-((R/I)/\mathbb{Q}) \end{tikzcd}$$

is a pullback square. In particular, as we we will see, the negative cyclic homology groups are rather accessible, so one can hope to compute $\mathrm{K}(R)$ from $\mathrm{K}(R/I)$.

7.2 Periodic cyclic homology

References 7.10. See Loday's book [36] for the classical approach. The Nikolaus–Scholze paper [48] clearly explains the Tate construction, extending previous work of Farrell [20] and Klein [35]. Morrow's Arizona Winter School lectures notes [45] are also nice.

We first give a 'hack' definition of periodic cyclic homology.

Definition 7.11 (Periodic cyclic homology I)**.** The **periodic cyclic homology** of R over k is

$$\mathrm{HP}(R/k) \simeq \mathrm{HC}^-(R/k)[v^{-1}].$$

This turns out to admit the structure of an $\mathbb{E}_\infty$-ring, which is an algebra over $k^{\mathrm{h}S^1}[v^{-1}]$. In practice, one can compute the v-localization as

$$\operatorname{colim}\left(\mathrm{HC}^-(R/k) \xrightarrow{v} \mathrm{HC}^-(R/k)[2] \xrightarrow{v} \mathrm{HC}^-(R/k)[4] \to \cdots\right).$$

This definition is fine and one can work effectively with it when necessary. However, it ignores one of the crucial features of group homology and cohomology over S^1 which is *Tate cohomology*.

Construction 7.12. Let X be an anima with structure map $p\colon X \to *$. Let $p'_!\colon \mathcal{D}(k)^X \to \mathcal{D}(k)$ denote the unique colimit-preserving approximation to p_*. In other words, we define $p'_!$ be defining it first on the compact objects M of $\mathcal{D}(k)^X$ as p_*M and then left Kan extending. In particular, the natural transformation $p'_! \to p_*$ induces an equivalence on compact objects. Given a parametrized complex $M \in \mathcal{D}(k)^X$, we call the cofiber $\mathrm{cofib}(p'_!(M) \to p_*(M))$ the **Tate cohomology** of X with coefficients in M.

Lemma 7.13. *The functor $p'_!$ is equivalent to $p_!(\mathrm{D}_X \otimes_k (-))$ for a unique parametrized complex $\mathrm{D}_X \in \mathcal{D}(k)^X$, the* ***Spivak–Klein dualizing complex****.*

Proof. See [48, Thm. I.4.1]. We explain the case of $k = \mathbb{S}$, so that $\mathcal{D}(k) \simeq \mathcal{S}\mathrm{p}$, the ∞-category of spectra. The adjunction $\mathrm{Pr}^{\mathrm{L}} \rightleftarrows \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, where the left adjoint is stabilization, shows that giving a colimit preserving functor $\mathcal{S}\mathrm{p}^X \to \mathcal{S}\mathrm{p}$ is equivalent to giving a colimit preserving functor $\mathcal{S}^X \to \mathcal{S}\mathrm{p}$. By Yoneda and left Kan extension, using that $\mathcal{S}^X \simeq \mathrm{Fun}(X, \mathcal{S})$, this in turn is equivalent to giving a functor $X \simeq X^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ which is a parametrized spectrum, D_X. □

Remark 7.14. Lemma 7.13 says that one computes $p'_!$ of $M \in \mathcal{D}(k)^X$ by first forming the tensor product $\mathrm{D}_X \otimes_k M \in \mathcal{D}(k)^X$ and then taking homology.

Remark 7.15. We suppress the dependence on k in the notation for the dualizing complex D_X. This is not a problem because everything commutes with base change. If $k \to R$ is a map of animated commutative rings (or simply $\mathbb{E}_\infty$-rings), then the dualizing complex D_X^R of X relative to R is equivalent to $R \otimes_k \mathrm{D}_X^k$, where D_X^k denotes for this moment the dualizing complex of X relative to k.

Example 7.16 (Finite groups)**.** If G is a finite group, then $\mathrm{D}_{\mathrm{B}G} \simeq k$. In particular, this says that the Tate cohomology of $M \in \mathcal{D}(k)^{\mathrm{B}G}$ is given by

$$\mathrm{cofib}(p_!(M) \to p^*(M)).$$

In this case, we write the cofiber as $M^{\mathrm{t}G}$ and use the equivalences $p_!(M) \simeq M_{\mathrm{h}G}$ and $p_*(M) \simeq M^{\mathrm{h}G}$ to rewrite the cofiber sequence as

$$M_{\mathrm{h}G} \to M^{\mathrm{h}G} \to M^{\mathrm{t}G}.$$

This is in fact the original context of Tate cohomology. To give a bit more detail, suppose that k is discrete and that M is a discrete parametrized complex, i.e., a G-representation. Let M_G denote the k-module of G-orbits and M^G the k-module of G-fixed points. There is a natural **norm map**

$$M_G \to M^G$$

obtained by first defining $M \to M^G$ via $m \mapsto \sum_{g \in G} gm$ and observing that this factors through the G-orbits. Now, we can use the norm map to glue together the group homology and cohomology of M. Indeed, since $M_{\mathrm{h}G}$ is connective (and computes $\mathrm{H}_*(G, M)$) and $M^{\mathrm{h}G}$ is coconnective (and computes $\mathrm{H}^*(G, M)$), any map $M_{\mathrm{h}G} \to M^{\mathrm{h}G}$ factors through

$$M_{\mathrm{h}G} \to \pi_0(M_{\mathrm{h}G}) \to \pi_0(M^{\mathrm{h}G}) \to M^{\mathrm{h}G}.$$

In particular, we get

$$M_{\mathrm{h}G} \to M^{\mathrm{h}G}$$

by the factorization

$$M_{\mathrm{h}G} \to M_G \xrightarrow{\mathrm{Nm}_G} M^G \to M^{hG}.$$

Exercise 7.17. Show that if G is a finite group of order g and if $M \in \mathcal{D}(\mathbb{Z}[1/g])$, then $M^{\mathrm{t}G} \simeq 0$.

Example 7.18 (Finite anima)**.** If X is a compact anima, then $p_* \colon \mathcal{D}(k)^X \to \mathcal{D}(k)$ preserves filtered colimits and hence (because of stability) all colimits. It follows that the colimit preserving approximation of p_* is p_* itself. In particular, the Tate cohomology vanishes.

Example 7.19 (Manifolds)**.** Let $k = \mathbb{Z}$. If M is a real n-manifold with orientation bundle $\tilde{\mathbb{Z}}$, then $\mathrm{D}_M \simeq \tilde{\mathbb{Z}}[-n]$. This gives us the map

$$\mathrm{H}_*(M, \tilde{\mathbb{Z}} \otimes_{\mathbb{Z}} F) \to \mathrm{H}^{n-*}(M, F) \tag{2}$$

for any parametrized complex F. If M is reasonably finite (for example if it is closed), then the Tate cohomology vanishes by Example 7.18. This says that the maps of (2) are isomorphisms, which is **Poincaré duality**.[27]

Example 7.20 (Classifying spaces)**.** Klein proves in [35, Thm. 10.1] that if G is a compact Lie group, then $\mathrm{D}_{\mathrm{B}G} \simeq k[\widetilde{S^{\mathrm{Ad}G}}]$, the reduced k-chains of a certain representation sphere. Here, $\mathrm{Ad}G$ is the adjoint representation of G, i.e., the tangent space to the identity in G with the action coming from conjugation. The space $S^{\mathrm{Ad}G}$ is defined to be the 1-point compactification of $\mathrm{Ad}G$ as a G-space, or equivalent it is the space of norm 1 elements in $\mathbb{R} \oplus \mathrm{Ad}G$. In particular, if $\dim G = g$, then $S^{\mathrm{Ad}G}$ is a g-sphere with a possibly non-trivial G-action. Thus, if $x \in \mathrm{B}G$ is a point, we have $x^*\mathrm{D}_{\mathrm{B}G} \simeq k[g]$. For any classifying space, we let $M^{\mathrm{t}G}$ denote the Tate cohomology of M over $\mathrm{B}G$.

[27] A more general statement, **Atiyah duality**, is true over the sphere spectrum.

Example 7.21 ($\mathrm{B}S^1$)**.** Since S^1 is abelian, the adjoint representation is a trivial 1-dimensional representation. Hence, $\mathrm{D}_{\mathrm{B}S^1} \simeq k[1]$ equivariantly. Thus, the cofiber sequence defining Tate cohomology takes the form

$$p_!(M[1]) \to p_*(M) \to M^{\mathrm{t}S^1},$$

or equivalently

$$M_{\mathrm{h}S^1}[1] \to M^{\mathrm{h}S^1} \to M^{\mathrm{t}S^1}.$$

We return to periodic cyclic homology.

Definition 7.22 (Periodic cyclic homology II)**.** Let R be an animated commutative k-algebra. We define the **periodic cyclic homology** of R over k to be

$$\mathrm{HP}(R/k) \simeq \mathrm{HH}(R/k)^{\mathrm{t}S^1}.$$

Remark 7.23 (Norm sequence)**.** The defining **norm sequence**

$$\mathrm{HC}(R/k)[1] \to \mathrm{HC}^-(R/k) \to \mathrm{HP}(R/k)$$

is one of the main tools for understanding these theories.

Remark 7.24. For compact Lie groups, there is a natural lax symmetric monoidal structure on Tate cohomology by [48, Thm. I.4.1]. Applied to the circle, this makes $\mathrm{HP}(R/k)$ into an $\mathbb{E}_\infty$-algebra over k and the map $\mathrm{HC}^-(R/k) \to \mathrm{HP}(R/k)$ corresponding to $\mathrm{HH}(R/k)^{\mathrm{h}S^1} \to \mathrm{HH}(R/k)^{\mathrm{t}S^1}$ into a map of $\mathbb{E}_\infty$-algebras over k. Moreover, $\mathrm{HC}(R/k)$ is naturally an $\mathrm{HC}^-(R/k)$-module.

Now, we check that the two definitions of periodic cyclic homology given in this section agree.

Lemma 7.25. *The complex* $\mathrm{HP}(R/k)$ *is* v*-local and the natural map* $\mathrm{HC}^-(R/k)[v^{-1}] \to \mathrm{HP}(R/k)$ *is an equivalence.*

Proof. For the first part, note that $\mathrm{HP}(R/k)$ is an $\mathbb{E}_\infty$-ring over $k^{\mathrm{t}S^1}$ by commutativity and that $\pi_* k^{\mathrm{t}S^1} \simeq k[v^{\pm 1}]$, where $|v| = -2$. For the second, it is enough now to check that $\mathrm{HH}(R/k)_{\mathrm{h}S^1}[1][v^{-1}] \simeq 0$ using the Norm sequence defining Tate cohomology. But, $\mathrm{HH}(R/k)[1]_{\mathrm{h}S^1}$ is connective, so v acts nilpotently and hence the v-localization vanishes, as desired. □

Remark 7.26. In fact, the lemma works for any $M \in \mathcal{D}(k)^{\mathrm{B}S^1}$ in the sense that $M^{\mathrm{t}G} \simeq M^{\mathrm{h}S^1}[v^{-1}]$.

Remark 7.27. We can also compute $\mathrm{HP}(R/k)$ as

$$\mathrm{HP}(R/k) \simeq \lim\Big(\cdots \to \mathrm{HC}(R/k)[-2] \xrightarrow{v} \mathrm{HC}(R/k) \xrightarrow{v} \mathrm{HC}(R/k)[2].\Big)$$

Remark 7.28. As $(-)^{\mathrm{h}S^1}$ commutes with limits, but not colimits, and $(-)_{\mathrm{h}S^1}$ commutes with colimits, but not limits, $(-)^{\mathrm{t}S^1}$ commutes in general with neither limits nor colimits, although there are notable exceptions.

Summarizing the relations between the cyclic, negative cyclic, and periodic cyclic theories, we have the following proposition.

Proposition 7.29. *Let $R \in \mathfrak{a}\mathrm{CAlg}_k$. There is a natural commutative diagram*

$$\begin{tikzcd}
\mathrm{HC}^-(R/k)[-2] \arrow[r, equal] \arrow[d, "v"'] & \mathrm{HC}^-(R/k)[-2] \arrow[d] & \\
\mathrm{HC}^-(R/k) \arrow[r] \arrow[d] & \mathrm{HP}(R/k) \arrow[r] \arrow[d] & \mathrm{HC}(R/k)[2] \arrow[d, equal] \\
\mathrm{HH}(R/k) \arrow[r] & \mathrm{HC}(R/k) \arrow[r, "v"'] & \mathrm{HC}(R/k)[2],
\end{tikzcd}$$

where both rows and both columns are cofiber sequences.

Proof. We will not prove this at the moment. □

Remark 7.30. The best approach to much of this material is to show that $\mathcal{D}(k)^{\mathrm{B}S^1}$ is naturally equivalent to the ∞-category of v-complete objects in $\mathcal{D}(k^{\mathrm{h}S^1})$, where $\pi_* k^{\mathrm{h}S^1} \cong k[v]$ with $|v| = -2$. See [42, Thm. 7.35].

7.3 The idea of the motivic filtration on $\mathrm{HC}^-(R/k)$ and $\mathrm{HP}(R/k)$

We will explain spectral sequences in more detail in the next section. But, It suffices to say now that they arise from filtrations. In particular, consider $\mathrm{HH}(R/k)$ with its HKR filtration $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HH}(R/k)$, which has graded pieces

$$\mathrm{gr}^s_{\mathrm{HKR}}\mathrm{HH}(R/k) \simeq \Lambda^s \mathrm{L}_{R/k}[s].$$

Since this filtration is S^1-equivariant, there is an induced HKR filtration $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HC}^-(R/k)$ on $\mathrm{HC}^-(R/k)$ with graded peices

$$\mathrm{gr}^s_{\mathrm{HKR}}\mathrm{HC}^-(R/k) \simeq (\Lambda^s \mathrm{L}_{R/k}[s])^{hS^1}.$$

Assume that R is smooth over k, so that $\Lambda^s \mathrm{L}_{R/k}[s] \simeq \Omega^s_{R/k}[s]$ is concentrated in degree s. In this case,

$$(\Omega^s_{R/k})^{\mathrm{h}S^1} \simeq (\Omega^s_{R/k})^{\mathrm{h}S^1}[s] \simeq p_*(\Omega^s_{R/k})[s],$$

where $p\colon \mathrm{B}S^1 \to *$. In other words, we are taking the cohomology of $\mathrm{B}S^1 \simeq \mathbb{CP}^\infty$ with coefficients in $\Omega^s_{R/k}$ and then shifting up into degree s. Thus, we have

$$\pi_t(\Omega^s_{R/k})^{\mathrm{h}S^1}[s] \cong \mathrm{H}^{-t-s}(\mathrm{B}S^1, \Omega^s_{R/k}) \cong \mathrm{H}^{-t-s}(\mathbb{CP}^\infty, \Omega^s_{R/k}) \cong \begin{cases} \Omega^s_{R/k} & \text{if } -t-s \geqslant 0 \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

We will use cohomological grading conventions below to keep as close as contact as possible with the motivic spectral sequence. The filtration above leads (in the smooth case) to the **homotopy S^1-fixed points spectral sequence**

$$\mathrm{E}_2^{s,t} = \mathrm{H}^s(\mathrm{B}S^1, \Omega^{-t}_{R/k}) \Rightarrow \mathrm{H}^{s+t}(\mathrm{HC}^-(R/k)) = \mathrm{HC}^-_{-s-t}(R/k),$$

a so-called fourth quadrant spectral sequence.

Remark 7.31. This is the result of applying a standard transformation to turn an E_1-spectral sequence into an E_2-spectral sequence.

Figures 1 and 2 show the E_2 and E_4-pages of the homotopy S^1-fixed points spectral sequence in the smooth case. Observe the beautiful fact that something like the de Rham complex appears in the differentials on E_2-page! In fact, the differentials shown *is* are the de Rham differentials. In particular, the E_4-page[28] of the spectral sequence has as its terms the de Rham cohomology of R over k, except at the left-hand boundary where some groups Z^* of cycles because there is nothing non-zero further left to hit the boundaries.

$$\begin{tikzcd}
\Omega^0_{R/k} \arrow[drr] & 0 & \Omega^0_{R/k} \arrow[drr] & 0 & \Omega^0_{R/k} & 0 \\
\Omega^1_{R/k} \arrow[drr] & 0 & \Omega^1_{R/k} \arrow[drr] & 0 & \Omega^1_{R/k} & 0 \\
\Omega^2_{R/k} & 0 & \Omega^1_{R/k} & 0 & \Omega^1_{R/k} & 0
\end{tikzcd}$$

Figure 1: The E_2-page of the homotopy S^1-fixed points spectral sequence for a smooth k-algebra R in the square $[0,5]\times[-2,0]$. The pattern repeats to the right and is 0 to the left of the displayed patch.

$$\begin{tikzcd}
\mathrm{H}^0_{\mathrm{dR}}(R/k) \arrow[ddrrrr] & 0 & \mathrm{H}^0_{\mathrm{dR}}(R/k) & 0 & \mathrm{H}^0_{\mathrm{dR}}(R/k) & 0 \\
\mathrm{Z}^1_{\mathrm{dR}}(R/k) & 0 & \mathrm{H}^1_{\mathrm{dR}}(R/k) & 0 & \mathrm{H}^1_{\mathrm{dR}}(R/k) & 0 \\
\mathrm{Z}^2_{\mathrm{dR}}(R/k) & 0 & \mathrm{H}^2_{\mathrm{dR}}(R/k) & 0 & \mathrm{H}^2_{\mathrm{dR}}(R/k) & 0
\end{tikzcd}$$

Figure 2: The E_4-page of the homotopy S^1-fixed points spectral sequence for a smooth k-algebra R in the square $[0,5]\times[-2,0]$.

A similar phenomenon appears in Tate cohomology. We obtain an HKR filtration

$$\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HP}(R/k)$$

with graded pieces

$$\mathrm{gr}^s_{\mathrm{HKR}}\mathrm{HP}(R/k) \simeq (\Lambda^s \mathrm{L}_{R/k}[s])^{\mathrm{t}S^1}.$$

The Tate cohomology of any parametrized complex is 2-periodic as it is a module over $k^{\mathrm{t}S^1}$, which has $\pi_* k^{\mathrm{t}S^1} \cong k[v^{\pm 1}]$ where $|v| = -2$. In particular, if R is smooth over k, we have

$$\pi_t(\Omega^s_{R/k})^{\mathrm{t}S^1}[s] \cong \mathrm{H}^{-t-s}(\mathrm{B}S^1, \Omega^s_{R/k}) \cong \widehat{\mathrm{H}}^{-t-s}(\mathbb{CP}^\infty, \Omega^s_{R/k}) \cong \begin{cases} \Omega^s_{R/k} & \text{if } -t-s \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

[28]The same is true of the E_3-page but there are no non-zero differentials on the E_3-page because of the 0 columns.

Here, $\widehat{\mathrm{H}}^*(\mathbb{CP}^\infty, -)$ is common notation for the Tate cohomology of S^1.

Inverting v has the effect of making the **S^1-Tate spectral sequence**

$$\mathrm{E}_2^{s,t} = \widehat{\mathrm{H}}^s(\mathrm{B}S^1, \Omega^{-t}_{R/k}) \Rightarrow \mathrm{H}^{s+t}(\mathrm{HP}(R/k)) \cong \mathrm{HP}_{-s-t}(R/k)$$

2-periodic in the columns. In particular, the cycle groups $\mathrm{Z}^*_{\mathrm{dR}}(R/k)$ appearing in the E_4-page of the homotopy S^1-fixed points spectral sequence now have terms to the left of them to see the boundaries. See Figure 3.

$$\begin{tikzcd}
\mathrm{H}^0_{\mathrm{dR}}(R/k) \arrow[ddrrrr] & 0 & \mathrm{H}^0_{\mathrm{dR}}(R/k) & 0 & \mathrm{H}^0_{\mathrm{dR}}(R/k) & 0 \\
\mathrm{H}^1_{\mathrm{dR}}(R/k) & 0 & \mathrm{H}^1_{\mathrm{dR}}(R/k) & 0 & \mathrm{H}^1_{\mathrm{dR}}(R/k) & 0 \\
\mathrm{H}^2_{\mathrm{dR}}(R/k) & 0 & \mathrm{H}^2_{\mathrm{dR}}(R/k) & 0 & \mathrm{H}^2_{\mathrm{dR}}(R/k) & 0
\end{tikzcd}$$

Figure 3: The E_4-page of the S^1-Tate spectral sequence for a smooth k-algebra R in the square $[0,5] \times [-2,0]$. This patch repeats infinitely to the right *and* to the left.

The spectral sequences make it clear that de Rham cohomology is somehow related to negative cyclic and periodic cyclic homology. But how? We would like to say that there are filtrations on these theories having de Rham cohomology as a graded pieces. But, we have only obtained these objects on the E_3 or E_4 pages of the spectral sequences. Moreover, there is a problem as the analysis above really holds only for R smooth over k. If R is not smooth over k, the terms $(\Lambda^s \mathrm{L}_{R/k}[s])^{\mathrm{t}S^1}$ will be a mess, incorporating both the Tate cohomology of S^1 and the homology groups of the sth wedge powers of the cotangent complex. However, we know that the strength of the cotangent complex is as a complex, not its particular homotopy groups.

The answer to these problems is (1) to use the Beilinson t-structure on filtered complexes to implement décalage[29] and (2) left Kan extension from the smooth case.

7.4 Spectral sequences

References 7.32. See [13, Sec. 5] or [53] or [2].

Definition 7.33. Let k be an animated commutative ring (or connective $\mathbb{E}_\infty$-ring spectrum). Let $\mathcal{DF}(k) = \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{D}(k))$ denote the ∞-category of decreasing filtrations of k-modules.

The next definition gives our answer to the question: what is a spectral sequence? We will attempt to justify it in the rest of this section.

Definition 7.34. A **k-linear spectral sequence** is an object of $\mathcal{DF}(k)$.

Theorem 7.35 (Beilinson, Bhatt–Morrow–Scholze)**.** *Consider the two subcategories*

$$\mathcal{DF}(k)_{\geqslant 0} = \{\mathrm{F}^\star M \in \mathcal{DF}(k) \colon \mathrm{gr}^s M \in \mathcal{D}(k)_{\geqslant -s}\}$$

[29] Décalage means shift in French. In earlier sections, this was used to describe the derived symmetric powers of the shift $M[1]$. Here, shift might be taken to mean 'page-turning' in the spectral sequence.

and

$$\mathcal{DF}(k)_{\leqslant 0} = \{F^\star M \in \mathcal{DF} \colon \mathrm{F}^s M \in \mathcal{D}(k)_{\leqslant -s}\}.$$

*These form the connective and coconnective parts of a t-structure on $\mathcal{DF}(k)$, the **Beilinson t-structure**, and the heart is naturally equivalent to*

$$\mathcal{DF}(k)^{\mathrm{B}\heartsuit} \simeq \mathrm{Ch}^\bullet(\pi_0 k),$$

the abelian category of cochain complexes over $\pi_0 k$.

Remark 7.36. Note the asymmetry in the definition. Connectivity in the Beilinson t-structure is about the graded pieces, whereas coconnectivity is about the filtered pieces.

Remark 7.37. In general, $\mathrm{gr}^s \tau^{\mathrm{B}}_{\geqslant n}(\mathrm{F}^\star M) \simeq \tau_{\geqslant -s+n}\mathrm{gr}^s M$. An easy calculation then implies that

$$\mathrm{gr}^s \pi^{\mathrm{B}}_n(\mathrm{F}^\star M) \simeq (\pi_{-s+n}\mathrm{gr}^s M)[-s].$$

Remark 7.38. A cochain complex $C^\bullet$ is viewed as a filtered complex via the stupid filtration which has $\mathrm{F}^n C^\bullet \cong C^{\bullet \geqslant n}$. Note that $\mathrm{gr}^n C^\bullet \simeq C^n[-n]$ in $\mathcal{D}(k)$, which implies that this filtration is indeed in the heart of the Beilinson t-structure.

Remark 7.39. We let $\pi^{\mathrm{B}}_*(\mathrm{F}^\star M)$ denote the homotopy objects of a filtered complex $\mathrm{F}^\star M$ with respect to the Beilinson t-structure. By the theorem, these are naturally cochain complexes. Which ones? They are the cochain complexes appearing in the E_1-spectral sequence of the filtration!

Remark 7.40 (Homotopy-coherent chain complexes)**.** Consider a filtered object $\mathrm{F}^\star M$, which we view as a sequence

$$\cdots \to \mathrm{F}^{s+2}M \to \mathrm{F}^{s+1}M \to \mathrm{F}^s M \to \mathrm{F}^{s-1}M \to \cdots$$

and take associated graded pieces. But, these graded pieces themselves fit into fiber sequences

$$\mathrm{gr}^{s+1}M \to \frac{\mathrm{F}^s M}{\mathrm{F}^{s+2}M} \to \mathrm{gr}^s M$$

with associated boundary map $\mathrm{gr}^s M \to \mathrm{gr}^{s+1}M[1]$. Now, we claim that the composition $\mathrm{gr}^s M \to \mathrm{gr}^{s+1}M[1] \to \mathrm{gr}^{s+2}M[2]$ is canonically nullhomotopic. Indeed, considering the commutative diagram

$$\begin{tikzcd}
\frac{\mathrm{F}^1 M}{\mathrm{F}^3 M} \arrow[r] \arrow[d] & \frac{\mathrm{F}^0 M}{\mathrm{F}^3 M} \arrow[r] \arrow[d] & \mathrm{gr}^0 M \arrow[r] \arrow[d, equal] & \frac{\mathrm{F}^1 M}{\mathrm{F}^3 M}[1] \arrow[d] \\
\mathrm{gr}^1 M \arrow[r] & \frac{\mathrm{F}^0 M}{\mathrm{F}^2 M} \arrow[r] & \mathrm{gr}^0 M \arrow[r] & \mathrm{gr}^1 M[1] \arrow[d] \\
 & & & \mathrm{gr}^2 M[2],
\end{tikzcd}$$

we see the desired nullhomotopy. Now, we can string these together to obtain

$$\cdots \to \mathrm{gr}^{-2}M[-2] \to \mathrm{gr}^{-1}M[-1] \to \mathrm{gr}^0 M \to \mathrm{gr}^1 M[1] \to \mathrm{gr}^2 M[2] \to \cdots,$$

which, since $d^2 \simeq 0$, is a kind of cochain complex internal to $\mathcal{D}(k)$. In fact, it is possible to make this notation of a homotopy coherent chain complex precise[30] and to show that $\widehat{\mathcal{DF}}(k)$, the ∞-category of complete filtered complexes, is equivalent to homotopy-coherent chain complexes.[31]

[30] See Ariotta's thesis [6] as well as [53, Sec. 5].

[31] Here, homotopy-coherent means that we specify compatible nullhomotopies $d^2 \simeq 0$, $d^3 \simeq 0$, and so on.

Remark 7.41. Given a complete filtered complex $\mathrm{F}^\star M$ we obtain the Beilinson homotopy objects $\pi_*^{\mathrm{B}}(\mathrm{F}^\star M)$ by applying π_* to the associated homotopy coherent cochain complex

$$\cdots \to \mathrm{gr}^{-2}M[-2] \to \mathrm{gr}^{-1}M[-1] \to \mathrm{gr}^0 M \to \mathrm{gr}^1 M[1] \to \mathrm{gr}^2 M[2] \to \cdots .$$

Since $d^2 \simeq 0$ in the homotopy-coherent chain complex, the associated sequence of maps

$$\cdots \to \pi_n(\mathrm{gr}^{-2}M[-2]) \to \pi_n(\mathrm{gr}^{-1}M[-1]) \to \pi_n(\mathrm{gr}^0 M) \to \pi_n(\mathrm{gr}^1 M[1]) \to \pi_n(\mathrm{gr}^2 M[2]) \to \cdots$$

forms a cochain complex, which is $\pi_n^{\mathrm{B}}(\mathrm{F}^\star M)$. here, $\pi_n \mathrm{gr}^0 M$ occurs in cohomological degree 0 of $\pi_n^{\mathrm{B}}(\mathrm{F}^\star M)$.

7.5 The motivic filtration

References 7.42. The ideas here go back to Connes, Quillen–Loday, and Feigin–Tsygan. The p-adic quasisyntomic case is taken up by Bhatt–Morrow–Scholze in [13] and the integral case is given in [2].

Definition 7.43. We say that $k \to R$ in $\mathfrak{a}\mathrm{CAlg}$ is **quasi-lci** if $\mathrm{L}_{R/k}$ has Tor-amplitude in $[0,1]$.

We can finally state the main theorem.

Theorem 7.44. *Let k be an animated commutative ring and let $R \in \mathfrak{a}\mathrm{CAlg}_k$. There are complete multiplicative decreasing filtrations*

$$\mathrm{F}_{\mathrm{B}}^\star \mathrm{HC}^-(R/k) \to \mathrm{F}_{\mathrm{B}}^\star \mathrm{HP}(R/k)$$

with

$$\mathrm{gr}_{\mathrm{B}}^s \mathrm{HC}^-(R/k) \to \mathrm{gr}_{\mathrm{B}}^s \mathrm{HP}(R/k)$$

equivalent to

$$\widehat{\mathrm{L}\Omega}^{\geqslant s}_{R/k}[2s] \to \widehat{\mathrm{L}\Omega}_{R/k}[2s].$$

If R is quasi-lci over k, the filtrations are exhaustive. In this case, we obtain a motivic spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{H}^{s-t}(\widehat{\mathrm{L}\Omega}_{R/k}) \Rightarrow \mathrm{H}^{s+t}(\mathrm{HP}(R/k)).$$

Proof. I will give the proof in the case when k is discrete and R is a smooth commutative k-algebra. The general case follows by a careful left Kan extension argument. I will also focus on the case of $\mathrm{HP}(R/k)$ for simplicity. Consider $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HP}(R/k)$. This is a complete exhaustive multiplicative *finite* decreasing filtration with $\mathrm{gr}^s_{\mathrm{HKR}}\mathrm{HP}(R/k) \simeq (\Omega^s_{R/k}[s])^{\mathrm{t}S^1}$. Now, consider

$$\tau^{\mathrm{B}}_{\geqslant\star}(\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HP}(R/k)).$$

This is getting a little mind-bending, but it is a decreasing multiplicative filtration in $\mathcal{DF}(k)$! In fact, $\pi^{\mathrm{B}}_{2s+1}(\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HP}(R/k)) \cong 0$ as one can easily check using the descriptions of the previous section. Thus, we will consider the double-speed Whitehead filtration $\tau^{\mathrm{B}}_{\geqslant 2\star}(\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HP}(R/k))$. Each $\tau^{\mathrm{B}}_{\geqslant 2s}(\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HP}(R/k))$ is a filtered complex, which we write as $\tau^{\mathrm{B},\star}_{\geqslant s}(\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HP}(R/k))$ but we want to forget this internal filtration complex. Let us write $\mathrm{F}^s_{\mathrm{B}}\mathrm{HP}(R/k)$ for $\mathrm{colim}_{i\mapsto-\infty}\, \tau^{\mathrm{B},i}_{\geqslant s}(\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HP}(R/k))$. It is the complex underlying the filtration. In this case, $\mathrm{gr}^s_{\mathrm{B}}\mathrm{HP}(R/k)$ is by definition the complex underlying the filtered object

$$\pi^{\mathrm{B}}_{2s}(\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HP}(R/k))[2s].$$

Now, we have

$$\mathrm{gr}^n \pi^{\mathrm{B}}_{2s}(\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HP}(R/k)) \simeq \pi_{-n+2s}\mathrm{gr}^n_{\mathrm{HKR}}\mathrm{HP}(R/k)[-n] \simeq \pi_{-n+2s}(\Omega^n_{R/k}[n])^{\mathrm{t}S^1}[-n] \simeq \Omega^n_{R/k}[-n].$$

Thus, we see that $\mathrm{gr}^s_{\mathrm{B}}\mathrm{HP}(R/k)[-2s]$ is the object of $\mathcal{D}(k)$ underlying the complex $\Omega^0_{R/k} \to \Omega^1_{R/k} \to \cdots$. That is, we have

$$\mathrm{gr}^s_{\mathrm{B}}\mathrm{HP}(R/k) \simeq \Omega_{R/k}[2s].$$

This is what we wanted to prove. □

Remark 7.45. Taking the cofiber of $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}^-(R/k) \to \mathrm{F}^\star_{\mathrm{B}}\mathrm{HP}(R/k)$ we get a complete decreasing filtration $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}(R/k)[2]$ on a shift of cyclic homology with graded pieces $\mathrm{gr}^s_{\mathrm{B}}\mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}(R/k)[2] \simeq \mathrm{L}\Omega^{\leqslant s-1}_{R/k}[2s]$. Or, shifting back down, we get a filtration $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}(R/k)$ with $\mathrm{gr}^s_{\mathrm{B}}\mathrm{HC}(R/k) \simeq \mathrm{L}\Omega^{\leqslant s}[2s]$. In any case, the cofiber sequence

$$\mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}^-(R/k) \to \mathrm{F}^\star_{\mathrm{B}}\mathrm{HP}(R/k) \to \mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}(R/k)$$

induces

$$\widehat{\mathrm{L}\Omega}^{\geqslant s}_{R/k}[2s] \to \widehat{\mathrm{L}\Omega}_{R/k}[2s] \to \mathrm{L}\Omega^{\leqslant s-1}_{R/k}[2s]$$

on graded pieces. This filtration on cyclic homology can also be constructed directly and is in fact used in the proof of the theorem.

Letting $\mathrm{H}^*_{\widehat{\mathrm{dR}}}(R/k)$ denote $\mathrm{H}^*(\widehat{\mathrm{L}\Omega}_{R/k})$, we can rewrite the spectral sequence as

$$\mathrm{E}_2^{s,t} = \mathrm{H}^{s-t}_{\widehat{\mathrm{dR}}}(R/k) \Rightarrow \mathrm{H}^{s+t}(\mathrm{HP}(R/k)),$$

just like the Atiyah–Hirzebruch spectral sequence. We call this the **motivic spectral sequence** for HP or the **de Rham**–HP **spectral sequence**.

Exercise 7.46. The theorem globalizes to the case of k-schemes. Explain how.

Exercise 7.47. Compute $\mathrm{HP}_*(\mathbb{Z}[x]/\mathbb{Z})$.

Exercise 7.48. Let k be an animated commutative $\mathbb{Q}$-algebra, and let $R = k[\epsilon]/(\epsilon^2)$, where $|\epsilon| = 0$. Compute $\mathrm{HP}(R/k)$.

7.6 Degeneration in characteristic 0

References 7.49. I like Toën–Vezzosi [63].

Theorem 7.50. *The motivic filtration on* $\mathrm{HP}(R/k)$ *is naturally split if* k *is an animated commutative* $\mathbb{Q}$*-algebra and* $k \to R$ *is is quasi-lci. In fact, there is an equivalence* $\mathrm{HP}(R/k) \simeq (\widehat{\mathrm{L}\Omega}_{R/k})^{\mathrm{t}S^1}$, *where* $\widehat{\mathrm{L}\Omega_{R/k}}$ *is given the trivial* S^1*-action, and the motivic filtration on the left corresponds to the Tate filtration on the right.*

Remark 7.51. The last comment runs through many people's work, especially Ben-Zvi–Nadler and Preygel. I have discussed it with Devalapurkar.

8 Week 8: Derived stacks (7–13 Nov.)

In this section we finally begin to globalize our animated commutative rings to introduce derived schemes and stacks.

8.1 The classical functor of points picture

References 8.1. I have always liked the explanation of the functor of points in [19, Chap. VI]. The original reference for n-geometric higher stacks is [58].

Definition 8.2. A map $f\colon R \to S$ of commutative rings is **étale** if it satisfies one of the following equivalent conditions:

(a) f is formally étale and of finite presentation;

(b) f is of finite presentation and $\mathrm{L}_f \simeq 0$;

(c) f is smooth and $\mathrm{L}_f \simeq 0$.

Exercise 8.3. Show that if $f \in R$ is non-nilpotent, then $R \to R[\frac{1}{f}]$ is étale.

Definition 8.4. Let k be a commutative ring. We let $\mathrm{Aff}_k = \mathrm{CAlg}_k^{\mathrm{op}}$ be the **category of affine schemes** over k. Given, $R \in \mathrm{CAlg}_k$ we let $\operatorname{Spec} R \in \mathrm{Aff}_k$ denote the corresponding object in the opposite category. We make Aff_k into a site by declaring

$$\{\operatorname{Spec} S_i \to \operatorname{Spec} R\}_{i\in I}$$

to be a cover if each $R \to S_i$ is étale and if the family is jointly surjective (in the scheme theoretic sense). This is the **large étale topology**.

Remark 8.5. Étale morphisms are open, so the quasicompactness of $\operatorname{Spec} R$ implies that for any étale cover $\{\operatorname{Spec} S_i \to \operatorname{Spec} R\}_{i\in I}$ there is a finite subcover $\{\operatorname{Spec} S_f \to \operatorname{Spec} R\}_{f\in F}$ for some finite subset $F \subseteq I$.

Definition 8.6. The **category of prestacks of sets** is $\mathcal{P}(\mathrm{Aff}_k)^{\heartsuit} \simeq \mathrm{Fun}(\mathrm{Aff}_k^{\mathrm{op}}, \mathrm{Set}) \simeq \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$. The **category of stacks of sets** is the full subcategory $\mathrm{St}_k^{\heartsuit}$ of $\mathcal{P}(\mathrm{Aff}_k)^{\heartsuit}$ on the prestacks which are étale sheaves of sets.[32]

Definition 8.7. Let X be a k-scheme. The **functor of points** of X is $X\colon \mathrm{Aff}_k^{\mathrm{op}} \to \mathrm{Set}$ given by

$$X(\operatorname{Spec} R) = \mathrm{Hom}_{\mathrm{Sch}_k}(\operatorname{Spec} R, X).$$

The functor of points gives a functor

$$\mathrm{Sch}_k \to \mathcal{P}(\mathrm{Aff}_k)^{\heartsuit}.$$

Proposition 8.8. *The functor of points*

$$\mathrm{Sch}_k \to \mathrm{St}_k^{\heartsuit}$$

is fully faithful.

Remark 8.9. We will not prove the important Proposition 8.8 in this course. It is important to note however that it is a subtle and important fact. Indeed, on the one side we are considering a class of locally ringed spaces while on the other side we forget the internal topological spaces and remember just sheaves of sets on the large étale site.

[32] Here and below we perform a common abuse of terminology and notation. One should really fix a large cardinal κ and consider only κ-small affine schemes, namely the opposite of CAlg_k^{κ}, the category of κ-compact commutative k-algebras. Then, the full subcategory of étale sheaves of sets on $(\mathrm{CAlg}_k^{\kappa,\mathrm{op}})$ is in fact a topos. It is in particular presentable, and the inclusion of stacks into all presheaves admits a finite limit preserving left adjoint. The issue in general is that the sheafification of an object of $\mathrm{Fun}(\mathrm{Aff}_k^{\mathrm{op}}, \mathrm{Set})$ might not be defined, or might only be defined in a strictly larger universe. However, this is not a problem in practice because all constructions will happen locally in some sense, i.e., at the level of κ-small objects for some κ.

Now, we identify the essential image. To give our sheaf-theoretic description of schemes, it will be convenient to identify the quasi-affine schemes among the étale sheaves.

Definition 8.10. A scheme U is **quasi-affine** if it is an open subscheme of $\operatorname{Spec} R$ for some commutative ring R.

Exercise 8.11. Let k be a field and let $U = \mathbb{A}^2_k - \{(0,0)\}$. Prove that U is quasi-affine but not affine.

Lemma 8.12. *A scheme U is quasi-affine if and only if its functor of points, also denoted U, admits a monomorphism $U \hookrightarrow \operatorname{Spec} R$ where U is the union*

$$\bigcup_{i \in I} \operatorname{Spec} R[\tfrac{1}{f_i}] \simeq U$$

in the sheaf theoretic sense.

Remark 8.13. If U is quasicompact, then only finitely many $\operatorname{Spec} R[\frac{1}{f_i}]$ are needed.

Remark 8.14. Note that it is important to take the union in the sheaf theoretic sense. For example, $\operatorname{Spec} \mathbb{Z}[\frac{1}{2}] \cup \operatorname{Spec} \mathbb{Z}[\frac{1}{3}] \to \operatorname{Spec} \mathbb{Z}$ is not an isomorphism when computed in prestacks of sets.

Definition 8.15. Let $f\colon X \to Y$ be a morphism in $\mathrm{St}_k^{\heartsuit}$. We say that f is **representable and open** if for every $\operatorname{Spec} R \to Y$ the pullback

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is isomorphic to a quasi-affine open subscheme U of $\operatorname{Spec} R$.

Theorem 8.16 (Functor of points)**.** *The Yoneda functor $\mathrm{Sch}_k \to \mathrm{St}_k^{\heartsuit}$ is fully faithful. The essential image consists of those stacks X which admit an open cover by affine schemes, i.e., a sheaf theoretically surjective morphism $\coprod \operatorname{Spec} R_i \to X$ such that each $\operatorname{Spec} R_i \to X$ is representable and open.*

Motto 8.17. Schemes are stacks which admit an open cover by affine schemes.

Remark 8.18. It is important to note that the notion of representability is crucial to making precise the notion of 'openness'. It is a kind of inductive condition, where we assume that the fibers of the morphism between stacks of sets are given by an identifiable class of geometric objects, in this case quasi-affine schemes.

There are now three important generalizations of schemes going back to the foundation of modern algebraic geometry.

Definition 8.19 (Algebraic spaces)**.** A stack of sets $X \in \mathrm{St}_k^{\heartsuit}$ is an **algebraic space with affine diagonal** if there exists a surjection $\coprod_i \operatorname{Spec} S_i \to X$ such that for any $\operatorname{Spec} R \to X$ and any i the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ \operatorname{Spec} S_i \arrow[r] & X \end{tikzcd}$$

is isomorphic to $\operatorname{Spec} T$ for an étale commutative R-algebra T. A map $X \to Y$ is a **relative algebraic space with affine diagonal** if for each $\operatorname{Spec} R \to Y$ the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

is an algebraic space with affine diagonal (in $\mathrm{St}_R^{\heartsuit}$). A morphism $X \to \operatorname{Spec} R$ is an **étale algebraic space** over $\operatorname{Spec} R$ if X admits a cover $\coprod_\alpha \operatorname{Spec} R_\alpha \to X$ in $\mathrm{St}_R^{\heartsuit}$ such that each R_α is étale over R and such that each morphism $\operatorname{Spec} R_\alpha \to X$ is a relative algebraic space with affine diagonal. A morphism of stacks of sets $X \to Y$ is **representable and étale** if for each $\operatorname{Spec} R \to Y$ the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

is an étale algebraic space over $\operatorname{Spec} R$. An **algebraic space** is a stack of sets $X \in \mathrm{St}_k^{\heartsuit}$ such that X admits a representable and étale cover by affine schemes. Finally, $X \to Y$ is a **relative algebraic space** if for each $\operatorname{Spec} R \to Y$ the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

is an algebraic space (in $\mathrm{St}_R^{\heartsuit}$).

Motto 8.20. Algebraic spaces are stacks which admit an étale cover by affine schemes. The meaning of "étale cover" is subtle in this generality and is why we need the long definition above.

Example 8.21. To see a bunch of random examples, look at [59, Tag 02Z0]. The general approach to constructing algebraic spaces is to take the quotient of a scheme X by an étale equivalence relation.[33] For example, we can take the quotient of $\mathbb{A}^1$ by the étale equivalence relation generated by $x \sim (-x)$ for every $x \in \mathbb{A}^1$. This is an algebraic space which is not a scheme. See [59, Tag 02Z1] for details.

Exercise 8.22. Prove that the example above is an algebraic space and that it is not a scheme if the characteristic of k is not 2.

To formulate the next two generalizations, we enlarge the category of stacks to obtain the 2-category of étale sheaves of groupoids, $\mathrm{St}_{k,\leqslant 1}$.

Definition 8.23 (Deligne–Mumford stacks)**.** A stack of groupoids $X \in \mathrm{St}_{k,\leqslant 1}$ is a **Deligne–Mumford stack** (or DM stack) if it admits a representable and étale surjection $\coprod_i \operatorname{Spec} S_i \to X$ by affine schemes. Equivalently, we could ask for a surjective étale relative algebraic space $\coprod_i \operatorname{Spec} S_i \to X$.

Remark 8.24. We see that algebraic spaces are DM stacks which happen to take discrete values on Aff_k.

Definition 8.25 (Smooth algebraic spaces)**.** An algebraic space $X \in \mathrm{St}_k$ is **smooth** if it admits a representable and étale surjection $\coprod_i \operatorname{Spec} S_i \to X$ where each S_i is smooth over k.

[33] An **étale equivalence relation** on a k-scheme X is a monomorphism of schemes $R \to X \times_k X$ such that $R(T) \to X(T) \times X(T)$ is an equivalence relation for each k-scheme T and such that each projection $R \to X$ is étale.

Definition 8.26 (Artin stacks)**.** A stack of groupoids $X \in \mathrm{St}_{k,\leqslant 1}$ is an **Artin stack** if it admits a representable and smooth surjection $\coprod_i \operatorname{Spec} S_i \to X$, which means that for any $\operatorname{Spec} R \to X$ and each i the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ \operatorname{Spec} S_i \arrow[r] & X \end{tikzcd}$$

is a smooth algebraic space over $\operatorname{Spec} R$.

Example 8.27. Let $\mathcal{M}_g$ denote the functor which takes an affine k-scheme $\operatorname{Spec} R$ to the groupoid of genus g-curves over $\operatorname{Spec} R$. Here, a genus g curve is a map $C \to \operatorname{Spec} R$ which is smooth, proper, and has genus g curves as fibers. If $g \geqslant 2$, then $\mathcal{M}_g$ is a Deligne–Mumford stack. This expresses the fact that higher genus curves have only finite automorphism groups. If $g = 0, 1$, then $\mathcal{M}_g$ is an Artin stack. This expresses the fact that the automorphism groups of genus 0 or 1 curves are large: for genus 0 one obtains geometrically the affine algebraic group scheme PGL_2. For genus 1 one gets extensions of finite group schemes by elliptic curves.

Exercise 8.28. Prove that $\mathcal{M}_0$ is equivalent to the Artin stack $\mathbf{BPGL}_2$.

Exercise 8.29. The moduli stack $\mathcal{M}_1$ is more complicated. Let $\mathcal{M}_{1,1}$ be the moduli stack of elliptic curves. This is a DM stack. There is a natural functor $\mathcal{M}_1 \to \mathcal{M}_{1,1}$ obtained by replacing a genus 1 fibration $C \to \operatorname{Spec} C$ with the associated Jacobian fibration (given by taking relative $\mathrm{Pic}^{\circ}_{C/\operatorname{Spec} R}$). Let $\mathcal{E}$ be the universal elliptic curve on $\mathcal{M}_{1,1}$. Then, $\mathbf{B}\mathcal{E} \to \mathcal{M}_{1,1}$ is a relative Artin stack so that $\mathbf{B}\mathcal{E}$ is an Artin stack over $\operatorname{Spec} \mathbb{Z}$. Prove that $\mathcal{M}_1$ is an Artin stack with the help of $\mathcal{M}_{1,1}$ and $\mathbf{B}\mathcal{E}$.

Example 8.30. Let X be a stack of groupoids. The stabilizer of a point $x\colon \operatorname{Spec} R \to X$ is the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d, "x"] \\ \operatorname{Spec} R \arrow[r, "x"'] & X. \end{tikzcd}$$

The stack P is a stack of groups (it is like a loopspace). For example, the stabilizer of a point $\operatorname{Spec} R \to \mathcal{M}_g$ corresponding to a smooth proper genus g curve $C \to \operatorname{Spec} R$ is the group scheme of automorphisms of C over $\operatorname{Spec} R$.

Example 8.31. If G is a smooth group scheme over $\operatorname{Spec} k$, then $\mathrm{B}G$, which classifies the groupoid of G-torsors, is an Artin stack. Indeed, since smooth surjections admit étale local sections, $\operatorname{Spec} k \to \mathrm{B}G$ is a smooth cover.

Finally, we have a sweeping generalization of Artin stacks due to Simpson [58].

Definition 8.32 (Higher stacks)**.** We let $\mathrm{St}_k \subseteq \mathcal{P}(\mathrm{Aff}_k)$ be the full subcategory of étale sheaves of animae on affine k-schemes. We call the objects of St_k **higher stacks**.

The higher stacks are a natural higher categorical analogue of an arbitrary étale sheaf of sets. We also want a higher categorical analogue of the kinds of geometric phenomena highlighted by algebraic spaces, Deligne–Mumford stacks, or Artin stacks.

Definition 8.33. A 0-geometric higher stack is a disjoint union of representables (affines)

$$\coprod_{i\in I}\operatorname{Spec} S_i.$$

A 0-geometric higher stack is smooth if each $\operatorname{Spec} S_i$ is smooth over $\operatorname{Spec} k$. Suppose we have defined $(n-1)$-geometric higher stacks and smooth $(n-1)$-geometric higher stacks for some $n \geqslant 1$. We say that a map $f\colon U \to X$ is $(n-1)$**-geometric** if for each $\operatorname{Spec} R \to Y$ the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

is $(n-1)$-geometric. An $(n-1)$-geometric morphism is smooth if the fibers P are smooth $(n-1)$-geometric higher stacks. A higher stack X is n**-geometric** if there is a smooth $(n-1)$-geometric morphism $U \to X$ where U is 0-geometric (i.e., a disjoint union of affines).

We will treat this notion in greater detail in the context of derived stacks in the next section.

Remark 8.34. The higher stacks X which arise in practice typically satisfy some finiteness hypotheses which guarantee that one can choose $U \to X$ where $U \simeq \operatorname{Spec} S$ is in fact representable.

Remark 8.35. It is not hard to prove inductively that an n-geometric stack takes n-truncated values, i.e., belongs to $\mathrm{St}_{k,\leqslant n}$.

Example 8.36. The Eilenberg–Mac Lane space

$$\mathrm{B}^n\mathbb{G}_m \simeq K(\mathbb{G}_m, n)$$

is the platonic n-geometric higher stack. Here, $\operatorname{Spec} k \to \mathrm{B}^n\mathbb{G}_m$ is an $(n-1)$-geometric atlas. The higher stack $\mathrm{B}^n\mathbb{G}_m$ classifies the n-groupoid of $(n-1)$-$\mathbb{G}_m$-gerbes, or $\mathrm{B}^{n-1}\mathbb{G}_m$-torsors. Given a k-scheme X we have

$$\pi_i(\mathrm{B}^n\mathbb{G}_m(X)) \cong \mathrm{H}^{n-i}_{\text{ét}}(X, \mathbb{G}_m).$$

8.2 n-geometric derived stacks

References 8.37. See [37, Sec. 5], Toën–Vezzosi [61, 62], or [4, Sec. 4.3].

Definition 8.38. Let $f\colon R \to S$ be a map of animated commutative rings. We say that f is **flat** if $\pi_0 R \to \pi_0 S$ is flat and the induced map

$$\pi_i R \otimes_{\pi_0 R} \pi_0 S \to \pi_i S$$

is an isomorphism for each $i \geqslant 1$. We say that f is **étale** if it is flat and if additionally $\pi_0 R \to \pi_0 S$ is étale. We say that f is an **étale cover** if it is étale and faithful, meaning that $S \otimes_R (-)\colon \mathcal{D}(R) \to \mathcal{D}(S)$ is conservative.

Theorem 8.39 (Lurie)**.** *Let R be an animated commutative ring and let $\mathfrak{a}\mathrm{CAlg}_R^{\text{ét}} \subseteq \mathfrak{a}\mathrm{CAlg}_R$ be the full subcategory of étale morphisms $R \to S$. Then,*

$$\pi_0\colon \mathfrak{a}\mathrm{CAlg}_R^{\text{ét}} \to \mathfrak{a}\mathrm{CAlg}_{\pi_0 R}^{\text{ét}} \simeq \mathrm{CAlg}_{\pi_0 R}^{\text{ét}}$$

is an equivalence of ∞-categories. In particular, $\mathfrak{a}\mathrm{CAlg}_R^{\text{ét}}$ is a 1-category.

Remark 8.40. The theorem says in particular that every étale map $\pi_0 R \to S$ admits a unique deformation to an étale map $R \to \widetilde{S}$ where $\pi_0 \widetilde{S} \cong S$. In the end, this is proved using the cotangent complex.

Definition 8.41. Let k be an animated commutative ring. A **derived affine k-scheme** is an object $\operatorname{Spec} R$ of $\mathfrak{d}\mathrm{Aff}_k \simeq \mathfrak{a}\mathrm{CAlg}_k^{\mathrm{op}}$.

Definition 8.42. Let k be an animated commutative ring. A **derived prestack** over k (or over $\operatorname{Spec} k$) is an object of $\mathcal{P}(\mathfrak{d}\mathrm{Aff}_k)$, i.e. it is a presheaf of animae on derived affine schemes.

Definition 8.43. Let $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$ be a map of derived affine k-schemes. The Čech complex $\check{\mathrm{C}}_\bullet(f)$ of f is the simplicial derived affine scheme

$$\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} \operatorname{Spec} S \times_{\operatorname{Spec} R} \operatorname{Spec} S \rightrightarrows \operatorname{Spec} S.$$

Remark 8.44. The Čech complex of f is opposite to the Amitsur complex

$$S \rightrightarrows S \otimes_R S \substack{\rightarrow\\\rightarrow\\\rightarrow} S \otimes_R S \otimes_R S \cdots .$$

The natural map

$$R \to \lim_{\Delta} (S^{\otimes \bullet + 1})$$

is an equivalence if f is an étale cover.

Definition 8.45. A **derived stack** over k is an object X of $\mathcal{P}(\mathfrak{d}\mathrm{Aff}_k)$ which satisfies **étale descent**, meaning that if $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$ is an étale cover, the natural map

$$X(\operatorname{Spec} R) \to \lim_{\Delta} X(\check{\mathrm{C}}_\bullet(f))$$

is an equivalence. We let $\mathfrak{d}\mathrm{St}_k \subseteq \mathcal{P}(\mathfrak{d}\mathrm{Aff}_k)$ be the full subcategory of derived stacks.

Exercise 8.46. Use Remark 8.44 to show that every derived affine scheme $X = \operatorname{Spec} R$ is a derived stack.

Definition 8.47 (n-geometric derived stacks)**.** Let k be an animated commutative ring.

(i) We say that a derived stack is 0-**geometric** if it is equivalent to $\coprod_{i \in I} \operatorname{Spec} S_i$ for a collection of animated commutative k-algebras $\{S_i\}$.

(ii) A 0-geometric derived stack $X \simeq \coprod_{i \in I} \operatorname{Spec} S_i$ is smooth if each S_i is smooth over k, meaning that S_i is compact in $\mathfrak{a}\mathrm{CAlg}_k$ and $\mathrm{L}_{S_i/k}$ has Tor-amplitude in $[0,0]$ (so it is a finitely presented projective S_i-module).

(iii) Now, suppose we have defined $(n-1)$ geometric derived stacks and smooth $(n-1)$ geometric derived stacks. We say that a morphism $U \to X$ is $(n-1)$-geometric if for every animated commutative k-algebra R and every $\operatorname{Spec} R \to X$ in $\mathfrak{d}\mathrm{St}_k$ the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

is $(n-1)$-geometric.

(iv) We say an $(n-1)$-geometric morphism $U \to X$ is smooth if the fibers P are smooth $(n-1)$-geometric stacks.

(v) A derived stack X is **n-geometric** if there is a smooth $(n-1)$-geometric surjection $U \to X$ where U is 0-geometric (i.e., a disjoint union of affines).

Remark 8.48. The conventions above give just one choice of notion of "n-geometricity". Any other choice would be found by replacing the class of 0-geometric derived stacks. We chose the disjoint unions of representables. Another reasonable choice would be to let the 0-geometric derived stacks be the class of derived algebraic spaces, i.e., those derived stacks admitting an étale cover by affine derived schemes and which additionally take on 0-truncated values on $\operatorname{Spec} R$ when R is discrete. This choice leads to what Lurie calls in [37] the relative n-stacks and we give the definition in the next section. Every n-geometric derived stack is a relative n-stack.

Remark 8.49. The choice here has the slightly strange property that non-affine schemes are not 0-geometric, but rather 1-geometric.

Example 8.50. Suppose that X is a separated scheme. Then, X is 1-geometric. Indeed, if $\{U_i\}_{i\in I}$ is an open affine cover of X, then $\coprod_{i\in I} U_i \to X$ is smooth and 0-geometric by separatedness.

Example 8.51. Let $\widetilde{\mathbb{A}}^2$ be the affine plane with the origin doubled. Let U_0 and U_1 be the two affine opens, each isomorphic to $\mathbb{A}^2$. Then, $U_0 \times_{\widetilde{\mathbb{A}}^2} U_1 \cong \mathbb{A}^2 - \{0\}$. We see that $U_0 \coprod U_1 \to \mathbb{A}^2$ is a smooth 1-geometric surjection. Thus, $\widetilde{\mathbb{A}}^2$ is 2-geometric. It is not 1-geometric. (Note that $\mathbb{A}^2 - \{0\}$ is not affine.)

Exercise 8.52. Prove that if $X \to Y$ is n-geometric and $W \to Y$ is any morphism of derived stacks, then the pullback $X \times_Y W \to W$ is n-geometric.

Exercise 8.53. Prove that $X \to Y$ is n-geometric if and only if for every map $\operatorname{Spec} S \to Y$, the pullback $X \times_Y \operatorname{Spec} S \to \operatorname{Spec} S$ is n-geometric.

Exercise 8.54. Prove that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ are maps of stacks and if f and g are n-geometric, then $g \circ f$ is n-geometric.

Motto 8.55. A stack on a stack is a stack.

There are many, many more permanence properties, which we will only study as needed. I would strongly suggest skimming [4, Sec. 4.3].

8.3 Relative n-stacks

To state the precise version of Artin–Lurie representability in Week 9, we will need the following generalization of n-geometric stacks.

Definition 8.56. A **relative derived algebraic space** is defined starting with derived affine schemes exactly as in Definition 8.19.

Definition 8.57 (Relative n-stacks)**.** Let k be an animated commutative ring.

(i) A morphism $X \to Y$ of derived stacks is a **relative 0-stack** if the fibers are derived algebraic spaces. A relative 0-stack $X \to Y$ is smooth if the fibers are smooth algebraic spaces.

(ii) A morphism $X \to Y$ is a **relative n-stack** if for each $\operatorname{Spec} R \to Y$ the fiber

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

admits a surjection $U \to P$ which is a smooth relative $(n-1)$-stack and where U is a disjoint union of affine schemes. If U can be chosen to be smooth over $\operatorname{Spec} R$, we say that $X \to Y$ is a smooth relative n-stack.

8.4 The classical locus

There are several important constructions relating higher stacks and derived stacks. To begin, we have the adjoint functors

$$\pi_0 \colon \mathfrak{a}\mathrm{CAlg}_k \rightleftarrows \mathrm{CAlg}_{\pi_0 k} \colon \iota,$$

which induce adjoint functors

$$\mathrm{Aff}_{\pi_0 k} \rightleftarrows \mathfrak{d}\mathrm{Aff}_k.$$

Definition 8.58. Given a derived stack $X \colon \mathfrak{d}\mathrm{Aff}_k \to \mathcal{S}$, we let X^{cl} denote the classical locus, i.e., the composition $\mathrm{Aff}_{\pi_0 k} \to \mathfrak{d}\mathrm{Aff}_k \to \mathcal{S}$. This is a higher stack.

Example 8.59. If $X = \operatorname{Spec} R$, then $X^{\mathrm{cl}} \simeq \operatorname{Spec} \pi_0 R$.

Definition 8.60. Given a higher stack Y, there are two constructions of a derived stack from Y. Specifically, the classical locus functor

$$\mathrm{St}_{\pi_0 k} \leftarrow \mathfrak{d}\mathrm{St}_k \colon (-)^{\mathrm{cl}}$$

admits left and right adjoints, which we will call $\mathfrak{d}_!$ and $\mathfrak{d}_*$. Given a higher stack Y, we have $(\mathfrak{d}_* Y)(\operatorname{Spec} R) \simeq Y(\operatorname{Spec} \pi_0 R)$, while the defining property of $\mathfrak{d}_!$ is that $\mathfrak{d}_! \operatorname{Spec} R \simeq \operatorname{Spec} R$ for R a commutative $\pi_0 k$-algebra, where on the left we view $\operatorname{Spec} R$ as a higher stack and on the right it denotes a derived stack. A general higher stack is a colimit of affines and $\mathfrak{d}_!$ commutes with colimits, so we obtain $\mathfrak{d}_!$ in general by left Kan extension. We will also write $\mathfrak{d}^*$ for the classical locus functor.

Exercise 8.61. Prove that $\mathfrak{d}^*$ takes n-geometric derived stacks to n-geometric higher stacks.

Exercise 8.62. Show that if k is discrete, then $\mathfrak{d}_!$ takes n-geometric higher stacks to n-geometric derived stacks.

Example 8.63. To get a feeling for the difference between $\mathfrak{d}_!$ and $\mathfrak{d}_*$, we consider the case of the group of multiplicative units functor, $\mathbb{G}_m$. Specifically, we let $\mathbb{G}_m$ denote the ordinary affine scheme $\operatorname{Spec} \mathbb{Z}[u^{\pm 1}]$. We have $\mathbb{G}_m(R) = R^\times$ if R is a commutative ring. Now, $(\mathfrak{d}_* \mathbb{G}_m)(\operatorname{Spec} S) \simeq (\pi_0 S)^\times$ if S is a derived commutative ring, while

$$(\mathfrak{d}_! \mathbb{G}_m)(\operatorname{Spec} S) \simeq \mathrm{Map}_{\mathfrak{a}\mathrm{CAlg}_k}(\mathbb{Z}[u^{\pm 1}], S) \simeq \mathrm{GL}_1(S),$$

where $\mathrm{GL}_1(S)$ denotes the space of units, obtained as the pullback

$$\begin{tikzcd} \mathrm{GL}_1(S) \arrow[r] \arrow[d] & \Omega^\infty S \arrow[d] \\ (\pi_0 S)^\times \arrow[r] & \pi_0 S. \end{tikzcd}$$

In particular, $\mathfrak{d}_! \mathbb{G}_m$ is 0-geometric and we can describe the functor it represents. On the other hand, $\mathfrak{d}_* \mathbb{G}_m$ is not geometric, although this is somewhat tricky to prove at the moment.

Exercise 8.64. Derive the description of $\mathrm{Map}_{\mathfrak{a}\mathrm{CAlg}_k}(\mathbb{Z}[u^{\pm 1}], S)$ given in Example 8.63.

Remark 8.65. For any affine scheme, the unit map $\operatorname{Spec} R \to \mathfrak{d}^*\mathfrak{d}_! \operatorname{Spec} R$ is an equivalence. Since $\mathfrak{d}^*$ and $\mathfrak{d}_!$ preserve colimits and $\mathrm{St}_{\mathbb{Z}}$ is generated by affines under colimits, we see that the unit map $X \to \mathfrak{d}^*\mathfrak{d}_! X$ is an equivalence for any higher stack X. This implies that $\mathfrak{d}_! \colon \mathrm{St}_k \to \mathfrak{d}\mathrm{St}_k$ is fully faithful.

8.5 Quasicoherent sheaves on derived stacks

Definition 8.66. Let X be a derived stack. We let

$$\mathcal{D}(X) \simeq \lim_{\operatorname{Spec} R \to X} \mathcal{D}(R)$$

be the ∞-category of quasicoherent sheaves on X.

Remark 8.67. By definition, to give a quasicoherent sheaf on X is to give for each point $x\colon \operatorname{Spec} R \to X$ a complex $\mathcal{F}_x \in \mathcal{D}(R)$ which is functorial in the point. This means for example that if $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$, then there is a fixed equivalence $S \otimes_R \mathcal{F}_x \simeq \mathcal{F}_{x \circ f}$ and these satisfy the cocycle condition and higher coherent analogues of the cocycle condition.

Example 8.68 (The structure sheaf)**.** Given a derived stack X, the structure sheaf $\mathcal{O} \in \mathcal{D}(X)$ is the quasicoherent sheaf such that $x * \mathcal{O} \simeq R$ for all points $x\colon \operatorname{Spec} R \to X$. Since the transition maps in the limit diagram defining $\mathcal{D}(X)$ are all exact and symmetric monoidal, $\mathcal{D}(X)$ is a symmetric monoidal stable ∞-category and the unit is $\mathcal{O}$.

Example 8.69 (Perfect complexes)**.** A perfect complex on a derived stack is a quasicoherent sheaf $\mathcal{F} \in \mathcal{D}(X)$ such that $x^*\mathcal{F}$ is perfect for all points $x\colon \operatorname{Spec} R \to X$. The perfect complexes are precisely the (fully) dualizable objects of $\mathcal{D}(X)$.

Example 8.70. If $f\colon U \to X$ is a cover with Čech complex

$$U_\bullet \simeq \check{\mathrm{C}}(f),$$

for example if X is n-geometric and f is a smooth $(n-1)$-geometric surjection, then we can compute

$$\mathcal{D}(X) \simeq \lim_{\Delta} \mathcal{D}(U_\bullet).$$

Example 8.71. When X is presented as a colimit of a simple diagram of affine schemes, then we can compute $\mathcal{D}(X)$ as a simple limit. For example, $\mathbb{P}^1_k$ is given as the pushout

$$\begin{tikzcd} \mathrm{GL}_1 \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

so $\mathcal{D}(\mathbb{P}^1)$ is the pullback

$$\begin{tikzcd} \mathcal{D}(\mathbb{P}^1) \arrow[r] \arrow[d] & \mathcal{D}(\mathbb{A}^1) \arrow[d] \\ \mathcal{D}(\mathbb{A}^1) \arrow[r] & \mathcal{D}(\mathrm{GL}_1). \end{tikzcd}$$

We recover the usual 'clutching function' definition of quasicoherent sheaves on $\mathbb{P}^1$. To give a quasicoherent sheaf on $\mathbb{P}^1$, we must give $\mathcal{F}_0$ on the $\mathbb{A}^1$ around 0 and $\mathcal{F}_\infty$ on the $\mathbb{A}^1$ around ∞ together with an equivalence $(\mathcal{F}_0)|_{\mathrm{GL}_1} \simeq (\mathcal{F}_1)|_{\mathrm{GL}_1}$ of their restrictions to GL_1. It is crucial here that the equivalence is part of the data.

Example 8.72. If G is an affine algebraic group, then

$$\mathcal{D}(\mathrm{B}G)$$

is equivalent to the (left completion of the)[34] ∞-category of G-representations (often called rational G-representations in the literature).

8.6 The cotangent complex of a derived stack

Definition 8.73. Let $f\colon X \to Y$ be a morphism of derived stacks and let $x\colon \operatorname{Spec} R \to X$ be a point of X. Fix a connective R-module M. Let

$$\mathrm{Der}_f(x, M)$$

denote the fiber of

$$X(R \oplus M) \to X(R) \times_{Y(R)} Y(R \oplus M)$$

over the point corresponding to $x \in X(R)$ and the composition $\operatorname{Spec} R \oplus M \to \operatorname{Spec} R \xrightarrow{x} X \to Y$ in $Y(R \oplus M)$.

Definition 8.74. Let $f\colon X \to Y$ be a morphism of derived stacks. Let $L \in \mathcal{D}(X)$ be a complex. We say that L is a **cotangent complex** for f if for every $x\colon \operatorname{Spec} R \to X$ and connective R-module M, there is a natural equivalence

$$\mathrm{Der}_f(x, M) \simeq \mathrm{Map}_R(x^* L, M).$$

Remark 8.75. If L is a bounded below cotangent complex for f, then any two cotangent complexes for f are naturally equivalence. In this case, we write L_f or $\mathrm{L}_{X/Y}$ for the cotangent complex. If $X \to \operatorname{Spec} k$ is the structure morphism, we will write L_X for the cotangent complex.

Remark 8.76. The possession of a cotangent complex is one of the most important things we can ask about a derived stack. If we think of derived stacks as representing abstract moduli problems, having a cotangent complex means there is a well-defined deformation theory for the moduli problem.

Theorem 8.77. *If $f\colon X \to Y$ is an n-geometric morphism, then L_f exists and is $(-n)$-connective. If $f\colon X \to Y$ is an n-geometric morphism of finite presentation, then f is smooth if and only if L_f has Tor-amplitude in $[-n, 0]$.*

Proof. See [4, Prop. 4.45]. □

Remark 8.78. For example, while μ_n is not a smooth group scheme over $\operatorname{Spec} \mathbb{Z}$, the classifying stack $\mathrm{B}_{\mathrm{fppf}}\mu_n$ of fppf-locally trivial μ_n-torsors is smooth.

Definition 8.79. Let k be an animated commutative $\mathbb{Q}$-algebra and let X be a derived stack over k. We define a new derived stack X_{dR} by

$$X_{\mathrm{dR}}(\operatorname{Spec} R) \simeq X(\operatorname{Spec}(\pi_0 R)_{\mathrm{red}}).$$

In other words, the space of R-points of X_{dR} is the space of $(\pi_0 R)_{\mathrm{red}}$-points of X, where $(\pi_0 R)_{\mathrm{red}}$ denotes the quotient of $\pi_0 R$ by its nilradical. There is a natural map $X \to X_{\mathrm{dR}}$. This is the **de Rham stack** of X and was defined by Simpson. It has some remarkable properties. For example, if X is a scheme, then $\mathrm{R}\Gamma(X_{\mathrm{dR}}, \mathcal{O})$ computes the derived de Rham cohomology of X.

Exercise 8.80. Compute $\mathrm{L}_{X_{\mathrm{dR}}}$ for any derived stack X.

[34] Ignore this technicality, which is just saying that $\mathcal{D}(\mathrm{B}G)$ and $\mathcal{D}(\mathrm{Rep}_G)$ do not quite agree at $+\infty$ homologically. For an example, see [47]. Rather, the bounded above derived categories $\mathcal{D}(\mathrm{B}G)^+$ and $\mathcal{D}(\mathrm{Rep}_G)^+$ agree and the left completion of $\mathcal{D}(\mathrm{Rep}_G)$ is $\mathcal{D}(\mathrm{B}G)$. We know $\mathcal{D}(\mathrm{B}G)$ must be left complete for general reasons, namely since we can write it as a limit of left complete t-structures along t-exact functors using Example 8.70 and flatness of G.

8.7 Properties of geometric derived stacks

Since these notes have already gotten long enough, we keep this section very, very short.

Definition 8.81. If **P** is a finiteness condition (such as finite presentation, almost finite presentation, noetherianity, regularity), then we say that an n-geometric derived stack $X \to \operatorname{Spec} k$ has property **P** there exists a smooth $(n-1)$-geometric surjection $U \to X$ where $U \simeq \coprod_{i \in I} \operatorname{Spec} S_i$ and each $\operatorname{Spec} S_i$ has property **P**.

Definition 8.82. We say that a 0-geometric derived stack is quasicompact if it is affine (representable). A 0-geometric morphism $X \to Y$ is quasicompact if the fibers are quasicompact. An n-geometric derived stack if quasicompact if it admits a smooth cover by a representable. An n-geometric morphism $X \to Y$ is quasicompact if the fibers are quasicompact n-geometric stacks.

9 Week 9: Artin–Lurie representability (14–24 Nov.)

In this section we present the two major theorems of derived algebraic geometry, the Artin–Lurie representability theorem, which gives necessary and sufficient conditions for derived stacks of finite presentation to be geometric, and the Toën–Vaquié representability theorem, which proves that a certain *moduli stack of objects* is representable. We will then give applications to Picard stacks.

9.1 Artin–Lurie representability: statement of result

References 9.1. See [37].

Theorem 9.2. *Let k be an animated commutative G-ring and let $\mathcal{F} \in \mathfrak{d}\mathrm{St}_k$ be a derived stack. Then, $\mathcal{F}$ is representable by a relative n-stack almost of finite presentation over k if and only if the following conditions hold:*

(a) *$\mathcal{F}$ is almost of finite presentation;*

(b) *formal deformations of points of $\mathcal{F}$ are algebraizable;*

(c) *$\mathcal{F}$ admits a cotangent complex $\mathrm{L}_{\mathcal{F}}$;*

(d) *$\mathcal{F}$ is nilcomplete;*

(e) *$\mathcal{F}$ is infinitesimally cohesive;*

(f) *$\mathcal{F}(R)$ is n-truncated if $R \in \mathfrak{a}\mathrm{CAlg}_k$ is discrete.*

In the rest of this section, we explain hypotheses (a)-(f).

Definition 9.3. A commutative ring k is a ***G*-ring** if k is noetherian and if for every prime ideal $\mathfrak{p}$ of k the completion map $k \to \hat{k}_{\mathfrak{p}}$ is geometrically regular. An animated commutative ring k is an animated commutative G-ring if $\pi_0 k$ is a G-ring and if each $\pi_n k$ is a finitely presented $\pi_0 k$-module.

Example 9.4. Every localization of a finitely presented commutative $\mathbb{Z}$-ring is a G-ring. Complete noetherian local rings are G-rings as are Dedekind domains in characteristic 0.

Definition 9.5. A derived stack $\mathcal{F} \in \mathfrak{d}\mathrm{St}_k$ is of **finite presentation** if the functor $\mathcal{F} \colon \mathfrak{a}\mathrm{CAlg}_k \to \mathcal{S}$ commutes with filtered colimits. It is **almost of finite presentation** if $\mathcal{F} \colon \mathfrak{a}\mathrm{CAlg}_{k,\leqslant n} \to \mathcal{S}$ commutes with filtered colimits for each $n \geqslant 0$.

Remark 9.6. We saw in the last section that a geometric derived stack is of finite presentation if and only if it admits a cover by affine schemes $\operatorname{Spec} S$ where each S is almost compact in $\mathfrak{a}\mathrm{CAlg}_k$.

Definition 9.7. We say that formal deformations of points of $\mathcal{F}$ are **algebraizable** if for every complete local commutative $\pi_0 k$-algebra R with maximal ideal $\mathfrak{m}$ the natural map

$$\mathcal{F}(R) \to \lim \mathcal{F}(R/\mathfrak{m}^n)$$

is an equivalence.

Definition 9.8. A derived stack $\mathcal{F} \in \mathfrak{d}\mathrm{St}_k$ is **nilcomplete** if for every $R \in \mathfrak{a}\mathrm{CAlg}_k$ the natural map

$$\mathcal{F}(R) \to \lim \mathcal{F}(\tau_{\leqslant n} R)$$

is an equivalence.

Definition 9.9. A derived stack $\mathcal{F} \in \mathfrak{d}\mathrm{St}_k$ is **infinitesimally cohesive** if for every square-zero extension

$$\begin{tikzcd} \tilde{R} \arrow[r] \arrow[d] & R \arrow[d] \\ R \arrow[r, "d"'] & R \oplus \Sigma M \end{tikzcd}$$

of R by a connective R-module M, the associated square

$$\begin{tikzcd} \mathcal{F}(\tilde{R}) \arrow[r] \arrow[d] & \mathcal{F}(R) \arrow[d] \\ \mathcal{F}(R) \arrow[r] & \mathcal{F}(R \oplus \Sigma M) \end{tikzcd}$$

is cartesian.

We will not prove the theorem. However, the following exercises and lemmas establish one direction in the special case of n-geometric derived stacks. For hints, see the proof of [4, Prop. 4.45].

Exercise 9.10. Let $\mathcal{F}$ be an n-geometric derived stack. Prove that $\mathcal{F}$ is almost of finite presentation if and only if it admits a smooth surjective $(n-1)$-geometric morphism $U \to \mathcal{F}$ such that $U \simeq \coprod_i \operatorname{Spec} S_i$ and each S_i is almost compact in $\mathfrak{a}\mathrm{CAlg}_k$. Here, almost compact means that $\tau_{\leqslant n} S_i$ is compact in $(\mathfrak{a}\mathrm{CAlg}_k)_{\leqslant n}$ for all $n \geqslant 0$, where $(\mathfrak{a}\mathrm{CAlg}_k)_{\leqslant n} \subseteq \mathfrak{a}\mathrm{CAlg}_k$ is the full subcategory of animated commutative k-algebras R with $\pi_i R = 0$ for $i > n$. Similarly, prove that $\mathcal{F}$ is locally of finite presentation if and only if it admits a smooth surjective cover as above where each S_i is compact in $\mathfrak{a}\mathrm{CAlg}_k$.

Lemma 9.11. *Show that if $\mathcal{F}$ is an n-geometric derived stack, then formal deformations of points of $\mathcal{F}$ are algebraizable.*

Proof. If $\mathcal{F}$ is representable by a derived affine scheme, the statement is clear since $\mathrm{Map}_{\mathfrak{a}\mathrm{CAlg}_k}(R, S) \simeq \mathrm{Map}_{\mathfrak{a}\mathrm{CAlg}_k}(R, \lim S/\mathfrak{m}^n)$ if S is a complete local noetherian $\pi_0 k$-algebra with maximal ideal $\mathfrak{m}$. Suppose then that the statement is true for $(n-1)$-geometric derived stacks for some $n \geqslant 1$. Choose a smooth surjective $(n-1)$-geometric morphism $U \to \mathcal{F}$ where $U \simeq \coprod \operatorname{Spec} R_i$. The map $\mathcal{F}(S) \to \lim_n \mathcal{F}(S/\mathfrak{m}^n)$ is an inclusion of base points (i.e., it is fully faithful). To see this, one can choose $x \in \mathcal{F}(S)$ and pull back $U \to \mathcal{F}$ over the corresponding map $\operatorname{Spec} S \to \mathcal{F}$. The main thing is to see that $\pi_0 \mathcal{F}(S) \to \pi_0 \left(\lim_n \mathcal{F}(S/\mathfrak{m}^n)\right)$

is surjective. However, since $U \to \mathcal{F}$ is smooth, it is formally smooth. Fix a compatible system of points $x_n \colon \operatorname{Spec} S/\mathfrak{m}^n \to \mathcal{F}$. Since $U \to \mathcal{F}$ is surjective, there is a faithful étale map $S/\mathfrak{m} \to T_1$ such that the point x_1 lifts through a map $\operatorname{Spec} T_1 \to U$. The étale map deforms to an étale map $S \to T$ such that $T/\mathfrak{m}T \cong T_1$. We now use formal smoothness of $U \to \mathcal{F}$ to lift each $T/\mathfrak{m}^n T$ to a map to U. But, U is 0-representable, so we get a corresponding lift to T by the base case of this argument. This map does not descend to a map $\operatorname{Spec} S \to U$ but it does descend to a map $\operatorname{Spec} S \to \mathcal{F}$, which by construction is the algebraization we wanted. $\square$

Exercise 9.12. Show that if $\mathcal{F}$ is an n-geometric derived stack, it admits a cotangent complex which is $(-n)$-connective.

Exercise 9.13. Show that an n-geometric derived stack $\mathcal{F}$ is nilcomplete.

Exercise 9.14. Show that an n-geometric derived stack $\mathcal{F}$ is infinitesimally cohesive.

Exercise 9.15. Show that an n-geometric derived stack $\mathcal{F}$ takes n-truncated values on $\operatorname{Spec} R$ when R is discrete.

Remark 9.16. The idea of the proof of the sufficiency of the conditions in the theorem is to start with a given K-point $\mathcal{F}(K)$ where K is a field and deform this eventually to a point $\operatorname{Spec} R \to \mathcal{F}$ which is smooth, where R is an animated commutative ring with $\pi_0 R$ having a maximal ideal with residue field K. Taking enough of these to cover $\mathcal{F}$, we get a candidate atlas which must be shown to be $(n-1)$-geometric.

9.2 Toën-Vaquié representability

References 9.17. Of course, the main reference here is [60], but I will follow the proof of [4], which was given in the $\mathbb{E}_\infty$-setting.

The Artin–Lurie representability theorem can be used to prove the following incredibly general representability theorem. It in some sense contains the representability of Picard schemes, Hilbert schemes, and so much more in it statement. There are two proofs. One, due to Pandit [49] does indeed use Artin–Lurie. However, the original proof, due to Toën and Vaquié uses the simpler notion of n-geometric derived stacks, and we will present that here. The main advantage of Pandit's result is that it applies to a broader class of representation categories in Section 9.4.

Definition 9.18. We say that a derived stack $\mathcal{F}$ is **locally geometric** if it can be written as a filtered colimit $\mathcal{F} \simeq \operatorname{colim}_{i \in I} \mathcal{F}_i$ where each $\mathcal{F}_i$ is n_i-geometric for some integer n_i and where the transition maps $\mathcal{F}_i \to \mathcal{F}_j$ in the colimit diagram are monomorphisms for $i \to j$.

Definition 9.19. Fix an animated commutative ring k. The **moduli of objects** is the derived stack over k with R-points

$$\mathbf{Perf}(R) \simeq \iota_0 \mathcal{P}\mathrm{erf}(R)$$

for an animated commutative ring R, where we recall that $\iota_0 \mathcal{P}\mathrm{erf}(R)$ is the maximal subgroupoid (or underlying ∞-groupoid) of the ∞-category $\mathcal{P}\mathrm{erf}(R)$.

Theorem 9.20 (Toën–Vaquié representability)**.** *The moduli of objects derived stack* $\mathbf{Perf}$ *is locally geometric. More precisely, if* $\mathbf{Perf}_{[a,b]} \subseteq \mathbf{Perf}$ *denotes the substack classifying perfect complexes of Tor-amplitude contained in* $[a, b]$*, then* $\mathbf{Perf}_{[a,b]}$ *is* $(b - a + 1)$*-geometric and locally of finite presentation.*

Remark 9.21. Note that the conclusion is stronger than what we would obtain from the statement of Artin–Lurie representability: n-geometric stacks are relative n-stacks, but the converse is not true.

Question 9.22. Is there a relative n-stack which is not m-geometric for any m?

Before giving the proof of the theorem, we give some facts about Tor-amplitude. Recall that if k is an animated commutative ring and $M \in \mathcal{D}(k)$, then M has Tor-amplitude in $[a, b]$ if for every $\pi_0 k$-module N the tensor product $M \otimes_k N \in \mathcal{D}(\pi_0 k)_{[a,b]}$; i.e., $\mathrm{H}_i(M \otimes_k N) = 0$ for $i \notin [a, b]$. We will mostly be interested in Tor-amplitude of perfect complexes.

Proposition 9.23 (Tor-amplitude omnibus)**.** *Let k be an animated commutative ring.*

(a) *There is an equivalence* $\mathrm{Ho}(\mathrm{Proj}_k) \simeq \mathrm{Proj}_{\pi_0 k}$.

(b) *If $P \in \mathcal{P}\mathrm{erf}(k)$, then P has Tor-amplitude in $[a, b]$ for some interval $[a, b]$; moreover, a can be taken to be the smallest integer such that $\mathrm{H}_a(P) \neq 0$.*

(c) *If $P \in \mathcal{P}\mathrm{erf}(k)$ and a is the smallest integer such that $\mathrm{H}_a(P) \neq 0$, then $\mathrm{H}_a(P)$ is a finitely presented $\pi_0 k$-module.*

(d) *If P has Tor-amplitude in $[a, b]$, then there is a cofiber sequence $F[a] \to P \to Q$ where $F \simeq k^{\oplus N}$ for some integer N and Q has Tor-amplitude in $[a + 1, b]$.*

(e) *If P has Tor-amplitude in $[a, b]$ then the dual perfect complex $P^\vee$ has Tor-amplitude in $[-b, -a]$.*

(f) Every perfect complex $P \in \mathcal{P}\mathrm{erf}(k)$ has Tor-amplitude in $[a, b]$ for some interval $[a, b]$ (where a, b are finite).

Theorem 9.24. *The derived prestack* **Perf** *is a derived stack.*

Proof. For example, this follows from three facts: (1) the assignment $R \mapsto \mathcal{D}(R)$ satisfies étale descent[35]; (2) the perfect complexes $\mathcal{P}\mathrm{erf}(R)$ are precisely the fully dualizable of $\mathcal{D}(R)$; (3) taking dualizable objects preserves limits of presentably symmetric monoidal stable ∞-categories, for example by the cobordism hypothesis. The full proof is outside the scope of this course. □

Proof. See [4, Prop. 2.13]. □

Remark 9.25. In case k is discrete, one should think of a perfect complex P of Tor-amplitude in $[a, b]$ as being modeled by a complex

$$\cdots \to 0 \to P_b \to \cdots \to P_a \to 0 \to \cdots$$

where P_a in (homological) degree a is a finitely presented projective k-module.

Now we can prove the main theorem.

Proof of Theorem 9.20. By Theorem 9.24, **Perf** is a derived stack. It is also not hard to see that each $\mathbf{Perf}_{[a,b]} \subseteq \mathbf{Perf}$ is a derived stack.

The proof of the geometricity of $\mathbf{Perf}_{[a,b]}$ will be by induction on the length of the interval $(b - a)$. Note that suspension gives an equivalence $\mathbf{Perf}_{[a,b]} \simeq \mathbf{Perf}_{[a+n,b+n]}$ for any a, b, n. So, we can and will work with $\mathbf{Perf}_{[0,n]}$ and prove $(n + 1)$-geometricity by induction on n.

The base case, when $n = 0$, is relatively straightforward. To prove it, we first claim that $\mathbf{Perf}_{[0,0]}$ is the moduli of finitely presented projective modules, which in turn is equivalent to

$$\mathbf{Perf}_{[0,0]} \simeq \coprod_{n \geqslant 0} \mathbf{BGL}_n,$$

[35]This is a generalization to derived categories of Grothendieck's faithfully flat descent for quasicoherent sheaves.

where GL_n is the sheaf of automorphisms of the free R-module of rank n. So, suppose that $M \in \mathcal{P}\mathrm{erf}(R)$ has Tor-amplitude in $[0,0]$. We want to show that M is a retract of a finitely presented free R-module. Now, by base-change, it is clear that $N = M \otimes_R \pi_0 R$ has Tor-amplitude in $[0,0]$ as a $\pi_0 R$-module. In particular, N is a finitely presented $\pi_0 R$-module with flat dimension 0. That is N is flat and finitely presented. Thus, it is finitely presented projective. We also have $\pi_0 M \cong N$. Thus, by adjunction and by Proposition 9.23(a) there is a map $P \to M$ which is an isomorphism on π_0 and where P is a finitely presented projective R-module. Let C be the cofiber of $P \to M$. We see that C is bounded below and, since $P \otimes_R \pi_0 R \to M \otimes_R \pi_0 R$ is an equivalence, that $C \otimes_R \pi_0 R \simeq 0$. But, an easy Tor spectral sequence argument now implies that $C \simeq 0$ and hence that $P \to M$ is an equivalence.

Now, there is an obvious map $\coprod_{n \geqslant 0} \mathbf{BGL}_n \to \mathbf{Perf}_{[0,0]}$ induced by including rank n projective modules into all perfect complexes of Tor-amplitude in $[0,0]$. This is an equivalence up to retracts by the paragraph above. But, it is also an equivalence Zariski locally and hence it is an equivalence of derived stacks.

To finish the proof that $\mathbf{Perf}_{[0,0]}$ is 1-geometric, we want to show that $\mathbf{GL}_n$ is smooth for each $n \geqslant 0$ and hence that $\mathbf{BGL}_n$ is 1-geometric. By definition, $\mathbf{GL}_n$ is the derived stack of automorphisms of the rank n free module. Thus, there is an inclusion of $\mathbf{GL}_n \subseteq \mathbf{M}_{n\times n}$, where $\mathbf{M}_{n\times n}$ is the derived stack of endomorphisms of the rank n-free module. That is, $\mathbf{M}_{n\times n}(R) \simeq \mathrm{Map}_R(R^{\oplus n}, R^{\oplus n})$ for any animated commutative ring R. But, we have $\mathrm{Map}_R(R^{\oplus n}, R^{\oplus n}) \simeq R^{\oplus n^2}$. It follows that

$$\mathbf{M}_{n\times n} \simeq \mathbb{A}^{n^2} \simeq \operatorname{Spec} k[x_{11}, \ldots, x_{nn}].$$

In particular, we see that $\mathbf{M}_{n\times n}$ is smooth and locally of finite presentation over $\operatorname{Spec} k$. An R-linear endomorphism $T\colon R^{\oplus n} \to R^{\oplus n}$ is an equivalence if and only if $\pi_0 T\colon \pi_0 R^{\oplus n} \to \pi_0 R^{\oplus}$ is an equivalence which occurs if and only if $\det(\pi_0 T)$ is invertible. Thus, we see that $\mathbf{GL}_n \subseteq \mathbf{M}_{n\times n}$ is an open substack obtained by inverting a single function so it is affine. In particular, since $\mathbf{M}_{n\times n}$ is smooth, so is $\mathbf{GL}_n$. This proves the base case of our induction, that $\mathbf{Perf}_{[0,0]}$ is 1-geometric and locally of finite presentation.

Now, suppose we have proved that $\mathbf{Perf}_{[0,n-1]}$ is n-geometric and locally of finite presentation for some $n \geqslant 1$. Recall from Proposition 9.23 that we can write every perfect P of Tor-amplitude in $[0,n]$ as an extension $F \to P \to Q$, where F is free of finite rank and Q has Tor-amplitude in $[1,n]$. We will make this observation into a cover of $\mathbf{Perf}_{[0,n]}$.

Specifically, we form the pullback

$$\begin{tikzcd} U \arrow[r] \arrow[d, "p"] & \iota_0(\mathrm{Fun}(\Delta^1, \mathcal{P}\mathrm{erf})) \arrow[d] \\ \mathbf{Perf}_{[1,n]} \times_{\operatorname{Spec} k} \mathbf{Perf}_{[1,1]} \arrow[r] & \mathbf{Perf}^{\partial\Delta^1}, \end{tikzcd}$$

where the right-hand vertical map takes an arrow to its source and target. Thus, U is the derived stack of triples $(Q, F[1], \tau)$ where Q is perfect with Tor-amplitude in $[1,n]$, F is finitely presented projective, and τ is a map $Q \to F[1]$. In particular, there is a map $q\colon U \to \mathbf{Perf}_{[0,n]}$ obtained by taking the fiber of τ. We will show (1) that U is n-geometric and locally of finite presentation and (2) that $q\colon U \to \mathbf{Perf}_{[0,n]}$ is a smooth n-geometric surjection. This is enough to prove that $\mathbf{Perf}_{[0,n]}$ is $(n+1)$-geometric and locally of finite presentation, completing the inductive step.

The fiber of $p\colon U \to \mathbf{Perf}_{[1,n]} \times_{\operatorname{Spec} k} \mathbf{Perf}_{[1,1]}$ over an R-point $(Q, F[1])$ consists of the mapping space $\mathrm{Map}_R(Q, F[1])$. It follows that the fiber is the R-scheme $\operatorname{Spec} \mathrm{LSym}_R(Q \otimes_R F[-1])$. Note that since Q is 1-connective the tensor product $Q \otimes_R F[-1]$ is still connective, so that $\mathrm{LSym}_R(Q \otimes_R F[-1])$ is animated commutative (and not a more general derived commutative ring). It follows, using that each $\mathrm{LSym}_R(Q \otimes_R F[-1])$ is a compact animated commutative R-algebra, that $p\colon U \to \mathbf{Perf}_{[1,n]} \times_{\operatorname{Spec} k} \mathbf{Perf}_{[1,1]}$ is 0-geometric and locally of finite presentation. Since $\mathbf{Perf}_{[1,n]} \times_{\operatorname{Spec} k} \mathbf{Perf}_{[1,1]}$ is n-geometric and locally of finite presentation, it follows that U is n-geometric and locally of finite presentation.

Now, consider the fiber map $q\colon U \to \mathbf{Perf}_{[0,n]}$. This is a surjective map of derived stacks by Proposition 9.23(d). The fiber of q over a perfect complex P consists of all presentations of P as a fiber of a map $Q \to F[1]$ where Q has Tor-amplitude in $[1,n]$ and F is finitely presented projective. This fiber is equivalent to the space of maps $F \to P$ where F is finitely presented projective and where $\pi_0 F \to \pi_0 P$ is surjective, as this condition is equivalent to the cofiber Q have Tor-amplitude in $[1,n]$. The fiber decomposes into pieces corresponding to the rank of the finitely presented projective module F. Now, the rank n piece is the substack of $\operatorname{Spec}\operatorname{LSym}_R(R^{\oplus n}\otimes_R P^\vee)$ corresponding to those morphisms $R^{\oplus n} \to P$ which are surjective on π_0. Now, $\operatorname{LSym}_R(R^{\oplus n}\otimes_R P^\vee)$ is not an animated commutative ring in general, since $P^\vee$ will typically have negative homology groups. Nevertheless, it is possible to show that $\operatorname{Spec}\operatorname{LSym}_R(R^{\oplus n}\otimes_R P^\vee)$ is smooth and n-geometric. See Theorem 9.29. Again, the condition of surjectivity on π_0 is an open condition (since $\pi_0 P$ is finitely presented), so it follows that q is smooth, surjective, and n-geometric, as desired. □

Next, we will need a little bit of information about the cotangent complex of $\mathbf{Perf}$.

Proposition 9.26. *If $P \in \mathbf{Perf}(R)$ is an R-point of the moduli stack of objects, then there is a natural equivalence*

$$\mathrm{L}_{\mathbf{Perf}(R),P} \simeq \mathbf{End}_R(P)^\vee[-1]$$

in $\mathcal{D}(R)$, where $\mathbf{End}_R(P)$ is the derived endomorphism algebra of P over R.[36]

Proof. We consider the loopstack

$$\begin{tikzcd} \Omega_P\mathbf{Perf} \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d, "P"] \\ \operatorname{Spec} R \arrow[r, "P"'] & \mathbf{Perf}. \end{tikzcd}$$

Then, using the compatibility of the cotangent complex and limits of derived stacks, we have that $\mathrm{L}_{\Omega_P\mathbf{Perf},*} \simeq \Sigma\mathrm{L}_{\mathbf{Perf},P}$. On the other hand, we see that $\Omega_P\mathbf{Perf}$ is the derived stack of automorphisms of P. Thus, it is an open substack of the derived stack of endomorphisms of P, which is represented by $\operatorname{Spec}\operatorname{LSym}_R(\operatorname{End}_R(P)^\vee)$. While $\operatorname{LSym}_R(\operatorname{End}_R(P)^\vee)$ will often not be connective, the derived stack $\operatorname{Spec}\operatorname{LSym}_R(\operatorname{End}_R(P)^\vee)$ has cotangent complex given by $S\otimes_R \operatorname{End}_R(P)^\vee$, where $S \simeq \operatorname{LSym}_R(\operatorname{End}_R(P))^\vee$. See Theorem 9.29 again. It follows that $\mathrm{L}_{\Omega_R\mathbf{Perf},*} \simeq \operatorname{End}_R(P)^\vee$. and hence that $\mathrm{L}_{\mathbf{Perf},P} \simeq \operatorname{End}_R(P)^\vee[-1]$. □

Question 9.27. Is it true that in fact $\mathrm{L}_{\mathbf{Perf}} \simeq \mathbf{End}_R(\mathcal{P})^\vee[-1]$ where $\mathcal{P}$ is the universal perfect complex on $\mathbf{Perf}$? Ask Chris Brav.

Example 9.28. If we look at $\mathbf{BGL}_n$ in $\mathbf{Perf}$, the cotangent complex at a basepoint $\operatorname{Spec} k \xrightarrow{*} \mathbf{BGL}_n$ is given by $\mathrm{L}_{\mathbf{BGL}_n,*} \simeq \mathbf{End}_k(k^{\oplus n})^\vee[-1]$. This is precisely the adjoint action of $\mathbf{BGL}_n$ on the dual of the Lie algebra $\mathfrak{g}$ of GL_n.

9.3 Local moduli

We will use the following theorem.

Theorem 9.29 (Toën–Vaquié)**.** *Let k be an animated commutative ring and let $P \in \mathcal{D}(k)$. Let $R = \operatorname{LSym}_k(P)$ be the free derived commutative ring on P. We let $\operatorname{Spec} R$ be the derived prestack with*

$$(\operatorname{Spec} R)(S) \simeq \operatorname{Map}_{\mathfrak{d}\mathrm{CAlg}_k}(R,S).$$

[36] In other words, $\mathbf{End}_R(P)$ is the k-linear mapping spectrum from P to itself. We have $\mathrm{H}_i(\mathbf{End}_R(P)) \cong \operatorname{Ext}_R^{-i}(P,P)$.

Then, $\operatorname{Spec} R$ *is a derived stack and if* P *is perfect and has Tor-amplitude in* $[a,b]$ *with* $a \leqslant 0$*, then* $\operatorname{Spec} R$ *is a quasicompact, quasiseparated,* $(-a)$*-geometric derived stack which is locally of finite presentation. Moreover, the cotangent complex at an* R*-point corresponding to* $\operatorname{Spec} S \xrightarrow{x} \operatorname{Spec} R$ *is*

$$\mathrm{L}_{\operatorname{Spec} R,x} \simeq S \otimes_k P.$$

In particular, if $b \leqslant 0$*, then* $\operatorname{Spec} R$ *is smooth.*

Proof. See for example [4, Thm. 5.2]. The dedicated reader could prove this fairly directly, using induction on $(-a)$ and the cofiber sequences of Proposition 9.23. ☐

Definition 9.30. We call the objects $\operatorname{Spec} R$ of the theorem **nonconnective derived affine schemes**.

Exercise 9.31. Let $\mathbb{A}^1$ be the smooth derived affine group scheme with $\mathbb{A}^1(R) \simeq R$, viewed as a grouplike $\mathbb{E}_\infty$-space under addition. The classifying stack $\mathbf{B}\mathbb{A}^1$ is 1-geometric. Prove that $\mathbf{B}\mathbb{A}^1$ is representable by the nonconnective derived affine scheme $\operatorname{Spec} \operatorname{Sym}_{\mathbb{Z}}(\mathbb{Z}[-1])$.

9.4 Generalized Toën–Vaquié representability

The theorem of the previous section is the main representability theorem of [60], but it can be used to prove a useful extension which is often used in practice. The exact setting of [60] is of representations of homotopically finitely presented dg categories, but in order to avoid technicalities for the moment, we will assume that k is an animated commutative ring and that $A \in \mathrm{Alg}_k^{\mathbb{E}_1}$ is an $\mathbb{E}_1$-algebra over k. For example, if k is discrete, then $\mathrm{Alg}_k^{\mathbb{E}_1}$ is the homotopy theory of dg algebras by a theorem of Shipley [57].

Definition 9.32. If k is an animated commutative ring and A is an $\mathbb{E}_1$-algebra over k, then we define the stable ∞-category

$$\mathcal{R}\mathrm{ep}_A(R) \simeq \mathrm{Mod}_{A\otimes_k R}(\mathcal{P}\mathrm{erf}(R)),$$

the ∞-category of **representations of** A **in perfect complexes of** R**-modules**. We let $\mathbf{Rep}_A(R) \simeq \iota_0\mathcal{R}\mathrm{ep}_A(R)$ and take $\mathbf{Rep}_A$ to be the derived (pre)stack of A-representations.

Lemma 9.33. *If* A *is an* $\mathbb{E}_1$*-algebra over the animated commutative ring* k*, then* $\mathbf{Rep}_A$ *is a derived stack.*

Proof. Consider the forgetful map $p\colon \mathbf{Rep}_A \to \mathbf{Perf}$, which forgets the A-representation and remembers the underlying perfect complex. We have already seen that $\mathbf{Perf}$ is a derived stack (and is even locally geometric). Thus, it suffices to check that the fibers of $\mathbf{Rep}_A \to \mathbf{Perf}$ are derived stacks. However, given $P \in \mathbf{Perf}(R)$, the fiber of p over P is given by the derived R-prestack of A-module structures on P, which is given by

$$(R \to S) \mapsto \mathrm{Map}_{\mathrm{Alg}_k^{\mathbb{E}_1}}(A, \mathbf{End}_S(P \otimes_R S)).$$

Since $\mathcal{P}\mathrm{erf}$ satisfies étale descent descent, the assignment $(R \to S) \mapsto \mathbf{End}_S(P \otimes_R S)$ satisfies étale descent as a functor from $\mathfrak{a}\mathrm{CAlg}_R$ to $\mathrm{Alg}_k^{\mathbb{E}_1}$. Thus, since we are mapping out of A into a limit, we see that the fiber over P is a derived stack. Since $\mathbf{Perf}$ is a derived stack and the fibers of $\mathbf{Rep}_A \to \mathbf{Perf}$ are derived stacks, it follows that $\mathbf{Rep}_A$ is a derived stack. ☐

Exercise 9.34. Justify the logic used in the last sentence of the proof of Lemma 9.33. Specifically, suppose that $p\colon \mathcal{F} \to \mathcal{G}$ is a morphism in $\mathcal{P}(\mathfrak{d}\mathrm{Aff}_k)$ where $\mathcal{G} \in \mathfrak{d}\mathrm{St}_k \subseteq \mathcal{P}(\mathfrak{d}\mathrm{Aff}_k)$ and where for each point $x\colon \operatorname{Spec} R \to \mathcal{G}$ the pullback

$$\mathrm{fib}_x(f) \simeq \mathcal{F} \times_{\mathcal{G}} \operatorname{Spec} R$$

is in $\mathfrak{d}\mathrm{St}_R \subseteq \mathcal{P}(\mathfrak{d}\mathrm{Aff}_R)$. Show that $\mathcal{F} \in \mathfrak{d}\mathrm{St}_k$, i.e., that $\mathcal{F}$ satisfies étale descent.

Theorem 9.35. *If $A \in \mathrm{Alg}_k^{\mathbb{E}_1}$ is compact as an $\mathbb{E}_1$-algebra, then $\mathbf{Rep}_A$ is locally geometric and $\mathbf{Rep}_A \times_{\mathbf{Perf}} \mathbf{Perf}_{[a,b]}$ is locally of finite presentation.*

Proof. Since n-geometric morphisms are closed under finite limits and retracts, it is enough to consider the case where $A \simeq \mathrm{T}(P)$ is the free $\mathbb{E}_1$-algebra on a perfect complex P. In this case, the fiber of the map $\mathbf{Rep}_A \to \mathbf{Perf}$ over a perfect R-module Q is given by

$$\mathrm{Map}_{\mathrm{Alg}_k^{\mathbb{E}_1}}(\mathrm{T}(P), \mathbf{End}_R(Q)) \simeq \mathrm{Map}_k(P, \mathbf{End}_R(Q)) \simeq \mathrm{Map}_{\mathfrak{d}\mathrm{CAlg}_R}(\mathrm{Sym}_R(P \otimes_k R \otimes_R \mathbf{End}_R(Q)^\vee), R).$$

We see that the fiber is represented by $\mathrm{Spec}\,\mathrm{Sym}_R(P \otimes_k R \otimes_R \mathbf{End}_R(Q))$. If P has Tor-amplitude in $[a, b]$ and Q has Tor-amplitude in $[c, d]$, then $P \otimes_k R \otimes_R \mathbf{End}_R(Q)$ has Tor-amplitude in $[a+c-d, b+d-c]$ and in particular the nonconnective derived affine scheme $\mathrm{Spec}\,\mathrm{Sym}_R(P \otimes_k R \otimes_R \mathbf{End}_R(Q))$ is $\max(0, (d-a-c))$-geometric (and locally of finite presentation). ☐

9.5 Example: $\mathbf{Bun}_G$

One of the most important stacks studied in algebraic geometry, especially in the context of the geometric Langlands program, is $\mathbf{Bun}_{G \to C}$ when C is a curve. This is the stack of principal G-bundles on C where G is a reductive algebraic group. We will study this in somewhat greater generality here to illustrate the advantage of derived stacks.

Definition 9.36. Suppose that k is a commutative ring and that $p\colon X \to \mathrm{Spec}\,k$ is a separated k-scheme. Let $G \to X$ be a smooth affine algebraic group. Thus, $X \to \mathrm{Spec}\,k$ is 1-geometric and $G \to X$ is 0-geometric, while $\mathbf{B}G \to X$ is smooth and 1-geometric. It follows that the composition

$$\mathbf{B}G \to X \to \mathrm{Spec}\,k$$

is 1-geometric; it is smooth if X is smooth over $\mathrm{Spec}\,k$. We set $\mathbf{Bun}_{G \to X} = p_*\mathbf{B}G$. This is $\mathbf{Bun}_{G \to X}$, the **moduli stack of G-bundles on X**. By definition, for $R \in \mathfrak{a}\mathrm{CAlg}_k$, we have

$$\mathrm{Bun}_{G \to X}(T) \simeq (\mathbf{B}G)(X \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,T).$$

Example 9.37. The **classical Picard stack** is

$$\mathbf{Pic}^{\heartsuit}_{X/S} \simeq \mathbf{Bun}_{\mathbf{G}_{m,X} \to X}.$$

Warning: this is a derived stack whose *classical locus* is the ordinary Picard stack. However, there is a more general Picard stack $\mathbf{Pic}_{X/S}$ which we will meet in the next section.

9.6 Example: the Picard stack

Let $p\colon X \to S$ be a smooth proper morphism of ordinary schemes viewed as derived schemes over $\mathrm{Spec}\,\mathbb{Z}$. It is an important fact that $p_*\mathcal{P}\mathrm{erf} \simeq \mathcal{P}\mathrm{erf}_A$ for a sheaf of compact $\mathbb{E}_1$-algebras A. In particular, Theorem 9.35 implies that $p_*\mathbf{Perf}$ is locally geometric.

Definition 9.38. We let $\mathbf{Pic}_{X/S} \subseteq p_*\mathbf{Perf}$ be the open substack consisting of $\otimes$-invertible perfect complexes.

Exercise 9.39. Prove that $\mathbf{Pic}_{X/S} \subseteq p_*\mathbf{Perf}$ is indeed open.

Proposition 9.40. *We see that $\mathbf{Pic}_{X/S}$ takes 1-truncated values in discrete rings, so that it is a relative 1-stack over S by Artin–Lurie. It will also be 1-geometric. The cotangent complex $\mathrm{L}_{\mathbf{Pic}_{X/S}, \mathcal{L}}$ at a line bundle $\mathcal{L}\colon \mathrm{Spec}\,R \to \mathbf{Pic}_{X/S}$ is*

$$\mathrm{R}\Gamma(X_R, \mathcal{O})^\vee[-1].$$

Proof. This comes down to the fact that $\mathbf{End}(\mathcal{L}) \simeq \mathcal{O}$ and the computation of the cotangent complex of $\mathbf{Perf}$. $\square$

Now, we can ask the following question: when is $\mathbf{Pic}_{X/S}$ representable by an ordinary Artin stack? By Avramov's theorem, this will turn out to be the case if and only if $\mathrm{L}_{\mathbf{Pic}_{X/S}/S}$ has Tor-amplitude in $[-1,1]$ which happens if and only if $\mathrm{R}\Gamma(X,\mathcal{O})$ is a perfect complex on S with Tor-amplitude in $[-2,0]$, meaning that $\mathrm{H}^i(X,\mathcal{O}) = 0$ universally for $i > 2$. So, surfaces are OK, but, for example, abelian 3-folds are out.

We can quotient out $\mathbf{Pic}_{X/S}$ by the action of $\mathbf{GL}_{1,S}$ (in the case where the morphism $X \to S$ is geometrically connected) to obtain to obtain a derived scheme $\mathrm{Pic}_{X/S}$. In the case where $\mathrm{H}^i(X,\mathcal{O}) \neq 0$ for some $i > 2$, this is a derived scheme which is not a classical scheme (so not of the form $\mathfrak{d}_!Y$ for some ordinary scheme Y). Nevertheless, $\mathfrak{d}^*\,\mathrm{Pic}_{X/S}$ is the classical Picard scheme (with a few more connected components corresponding to shifts $\mathcal{L}[n]$ in the derived category). Thus, we see that every Picard scheme comes equipped with a canonical derived structure. This seems interesting and unstudied.

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Index

$(n-1)$-geometric, **80**
$(n-1)$-geometric morphism, 81
0-geometric, **81**
0-object, **42**
1-sifted, **13**
1-sifted completion, **14**
G-ring, **86**
S^1-Tate spectral sequence, **72**
$\Omega_x X$-action, 50
∞-category of animae, **8**
∞-category of spaces, **8**
∞-category
    pointed, **42**
    prestable, **43**
    stable, **42**
k-linear spectral sequence, **72**
n-geometric, **80**, **82**
t-structure, **44**
étale, **76**, **80**
étale algebraic space, **78**
étale cover, **80**
étale descent, **81**
étale equivalence relation, **78**

absolute object, **18**
absolutely generated, **19**
algebraic space, **78**
algebraic space with affine diagonal, **77**
algebraizable, **87**
almost of finite presentation, **86**
anima, 12
animated circle, **49**
animated commutative $\mathbb{Z}$-algebras, **21**
animated commutative rings, **12**, **21**
animated objects, **19**
animation, 12, **19**
Artin stack, **79**
Atiyah duality, **68**

Beilinson t-structure, **73**
bounded Tor-amplitude, **31**

category of affine schemes, **76**
category of prestacks of sets, **76**
category of stacks of sets, **76**
Čech complex, **6**
circle action, **49**
classical Picard stack, **93**
classifying anima, **49**
classifying space, **8**
coequalizer category, **14**
coherent, **45**
cohomology, **66**
collapse map, **51**, 53
comonad, **6**
compact, **18**
compact 1-projective, **19**
compact object, **19**
compact projective object, **18**
compactly generated, **19**
connective, **3**
conormal bundle, **30**
copower, **51**
copowered, **51**
cosimplicial object, **4**
cotangent complex, **26**, **28**, **85**
cyclic category, **50**
cyclic homology, **66**

Day convolution, **37**
de Rham stack, **85**
de Rham–HP spectral sequence, **75**
degeneracy maps, **3**
Deligne–Mumford stack, **78**
derivation, **23**
derived ∞-category, **44**
derived affine k-scheme, **81**
derived commutative k-algebras, **34**
derived commutative rings, **34**
derived de Rham complex, **35**
derived prestack, **81**
derived stack, **81**
directed preorder, 14
directed set, 14
discrete de Rham complex, **32**

face maps, **3**
filtered, **13**, **16**
filtered poset, **14**
filtered set, 14
filtration, **37**
    complete, **36**, 37
    conjugate, **36**
    exhaustive, **36**, 37
    multiplicative, **36**
finite presentation, **86**
flat, **80**
formally étale, **30**
formally smooth, **30**
free 1-categorical cocompletion, **13**

free cocompletion, 16
functor of points, 76

geometric realization, 7

heart of a t-structure, 44
higher stacks, 79
Hochschild homology, 50
Hodge filtration, 33
Hodge–de Rham spectral sequence, 33
Hodge-complete derived de Rham cohomology, 37
Hodge-filtered derived de Rham cohomology, 37
homology, 66
homotopy G-fixed points, 66
homotopy G-orbits, 66
homotopy S^1-fixed points spectral sequence, 70
homotopy objects, 45

ind-completion, 14
infinitesimally cohesive, 87
inverse Cartier transforms, 36

Kähler differentials, 23
Kan fibration, 8
Kan complex, 8

large étale topology, 76
left Kan extended, 15
left Kan extension, 13, 16, 20
local systems, 47
locally geometric, 88
loops functor, 43

moduli of objects, 88
moduli stack of G-bundles on X, 93
monad, 6
motivic spectral sequence, 75

negative cyclic homology, 66
nilcomplete, 87
nonabelian derived category, 20
nonconnective derived affine schemes, 92
norm map, 68
norm sequence, 69
normalized chain complex, 4

parametrized complexes, 47
periodic cyclic homology, 67, 69
Poincaré duality, 68
pointed animae, 42
pointwise symmetric monoidal structure, 48
Postnikov tower, 44
presheaves of animae, 16
presheaves of sets, 13
projective model category structure, 21
projectively generated, 19

quasi-affine, 77
quasi-lci, 74
quasiregular semiperfect, 40

reflexive coequalizer category, 14
relative 0-stack, 82
relative n-stack, 83
relative algebraic space, 78
relative algebraic space with affine diagonal, 78
relative derived algebraic space, 82
representable and étale, 78
representable and open, 77
representations of A in perfect complexes of R-modules, 92

set of n-simplices, 6
sifted, 16
simplex category, 3
simplicial object, 4
singular homology groups, 6
smooth $(n-1)$-geometric morphism, 82
smooth 0-geometric derived stack, 81
smooth algebraic space, 78
Spanier–Whitehead category, 43
Spivak–Klein dualizing complex, 67
square-zero extension, 24
stabilization, 43
strict, 32
suspension functor, 42

Tate cohomology, 67
tensor, 51
tensored, 51
Tor-amplitude, 31
trivial square-zero extension, 24

unit map, 53
universal derivation, 23
unnormalized chain complex, 4

weak equivalence, 8
weak homotopy equivalence, 7
weakly regular, 45
Whitehead tower, 44

Yoneda embedding, 13, 16