



# All-in at the River

## Standard Code Library

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Desprado2 fstqwq AntiLeaf

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# 1 数学

## 1.1 多项式

### 1.1.1 FFT

```

1 // 使用时一定要注意double的精度是否足够(极限大概是10 ^ ↪ 14)
2
3 const double pi = acos((double)-1.);
4
5 // 手写复数类
6 // 支持加减乘三种运算
7 // += 运算符如果用的不多可以不重载
8 struct Complex {
9     double a, b; // 由于long double精度和double几乎相同,
    → 通常没有必要用long double
10
11     Complex(double a = 0., double b = 0.) : a(a), b(b)
    → {}
12
13     Complex operator + (const Complex &x) const {
14         return Complex(a + x.a, b + x.b);
15     }
16
17     Complex operator - (const Complex &x) const {
18         return Complex(a - x.a, b - x.b);
19     }
20
21     Complex operator * (const Complex &x) const {
22         return Complex(a * x.a - b * x.b, a * x.b + b * ↪ x.a);
23     }
24
25     Complex operator * (double x) const {
26         return Complex(a * x, b * x);
27     }
28
29     Complex &operator += (const Complex &x) {
30         return *this = *this + x;
31     }
32
33     Complex conj() const { // 共轭, 一般只有MTT需要用
34         return Complex(a, -b);
35     }
36 } omega[maxn], omega_inv[maxn];
37 const Complex ima = Complex(0, 1); // i = sqrt(-1)
38
39 int fft_n; // 要在主函数里初始化
40
41 // FFT初始化
42 void FFT_init(int n) {
43     fft_n = n;
44
45     for (int i = 0; i < n; i++) // 根据单位根的旋转性质
        → 可以节省计算单位根逆元的时间
        omega[i] = Complex(cos(2 * pi / n * i), sin(2 * ↪ pi / n * i));
46
47     omega_inv[0] = omega[0];
48     for (int i = 1; i < n; i++)
        omega_inv[i] = omega[n - i];
49
50     // 当然不存单位根也可以, 只不过在FFT次数较多时很可能
        → 会增大常数
51 }
52
53 // FFT主过程
54 void FFT(Complex *a, int n, int tp) {
55     for (int i = 1, j = 0, k; i < n - 1; i++) {

```

```

57         k = n;
58         do
59             j = (k >= 1);
60         while (j < k);
61
62         if (i < j)
63             swap(a[i], a[j]);
64     }
65
66     for (int k = 2, m = fft_n / 2; k <= n; k *= 2, m /= ↪ 2)
67         for (int i = 0; i < n; i += k)
68             for (int j = 0; j < k / 2; j++) {
69                 Complex u = a[i + j], v = (tp > 0 ?
                    → omega : omega_inv)[m * j] * a[i + j
                    → + k / 2];
70
71                 a[i + j] = u + v;
72                 a[i + j + k / 2] = u - v;
73             }
74
75         if (tp < 0)
76             for (int i = 0; i < n; i++) {
77                 a[i].a /= n;
78                 a[i].b /= n; // 一般情况下是不需要的, 只
                    → 有MTT时才需要
79         }
80     }

```

### 1.1.2 NTT

```

1 constexpr int p = 998244353; // p为模数
2
3 int ntt_n, omega[maxn], omega_inv[maxn]; // ntt_n要在主
    → 函数里初始化
4
5 void NTT_init(int n) {
6     ntt_n = n;
7
8     int wn = qpow(3, (p - 1) / n); // 这里的3代表模数的
    → 任意一个原根
9
10    omega[0] = omega_inv[0] = 1;
11
12    for (int i = 1; i < n; i++)
13        omega_inv[n - i] = omega[i] = (long
            → long)omega[i - 1] * wn % p;
14
15
16 void NTT(int *a, int n, int tp) { // n为变换长度,
    → tp为1或-1, 表示正/逆变换
17
18     for (int i = 1, j = 0, k; i < n - 1; i++) { // → O(n)旋转算法, 原理是模拟加1
19         k = n;
20         do
21             j = (k >= 1);
22         while (j < k);
23
24         if (i < j)
25             swap(a[i], a[j]);
26     }
27
28     for (int k = 2, m = ntt_n / 2; k <= n; k *= 2, m /= ↪ 2)
29         for (int i = 0; i < n; i += k)
30             for (int j = 0; j < k / 2; j++) {
31                 int w = (tp > 0 ? omega : omega_inv)[m
                    → * j];

```

```

32     int u = a[i + j], v = (long long)w *
33         ↪ a[i + j + k / 2] % p;
34     a[i + j] = u + v;
35     if (a[i + j] ≥ p)
36         a[i + j] -= p;
37
38     a[i + j + k / 2] = u - v;
39     if (a[i + j + k / 2] < 0)
40         a[i + j + k / 2] += p;
41 }
42
43 if (tp < 0) {
44     int inv = qpow(n, p - 2);
45     for (int i = 0; i < n; i++)
46         a[i] = (long long)a[i] * inv % p;
47 }

```

### 1.1.3 任意模数卷积(MTT, 毛梯梯)

三模数NTT和直接拆系数FFT都太慢了, 不要用.

MTT的原理就是拆系数FFT, 只不过优化了做变换的次数.

考虑要对 $A(x), B(x)$ 两个多项式做DFT, 可以构造两个复多项式

$$P(x) = A(x) + iB(x) \quad Q(x) = A(x) - iB(x)$$

只需要DFT一个, 另一个DFT实际上就是前者反转再取共轭, 再利用

$$A(x) = \frac{P(x) + Q(x)}{2} \quad B(x) = \frac{P(x) - Q(x)}{2i}$$

即可还原出 $A(x), B(x)$ .

IDFT的道理更简单, 如果要对 $A(x)$ 和 $B(x)$ 做IDFT, 只需要对 $A(x) + iB(x)$ 做IDFT即可, 因为IDFT的结果必定为实数, 所以结果的实部和虚部就分别是 $A(x)$ 和 $B(x)$ .

实际上任何同时对两个实序列进行DFT, 或者同时对结果为实序列的DFT进行逆变换时都可以按照上面的方法优化, 可以减少一半的DFT次数.

```

30     FFT(c, n, -1);
31
32     for (int i = 0; i < n; i++) {
33         a[i].a = c[i].a;
34         b[i].a = c[i].b;
35     }
36 }
37
38 Complex a[2][maxn], b[2][maxn], c[3][maxn];
39 int ans[maxn];
40
41 int main() {
42     int n, m;
43     scanf("%d%d%d", &n, &m, &p);
44     n++;
45     m++;
46
47     base = (int)(sqrt(p) + 0.5);
48
49     for (int i = 0; i < n; i++) {
50         int x;
51         scanf("%d", &x);
52         x %= p;
53
54         a[1][i].a = x / base;
55         a[0][i].a = x % base;
56     }
57
58     for (int i = 0; i < m; i++) {
59         int x;
60         scanf("%d", &x);
61         x %= p;
62
63         b[1][i].a = x / base;
64         b[0][i].a = x % base;
65     }
66
67     int N = 1;
68     while (N < n + m - 1)
69         N <<= 1;
70
71     FFT_init(N);
72
73     DFT(a[0], a[1], N);
74     DFT(b[0], b[1], N);
75
76     for (int i = 0; i < N; i++)
77         c[0][i] = a[0][i] * b[0][i];
78
79     for (int i = 0; i < N; i++)
80         c[1][i] = a[0][i] * b[1][i] + a[1][i] * b[0]
81             ↪ [i];
82
83     for (int i = 0; i < N; i++)
84         c[2][i] = a[1][i] * b[1][i];
85
86     FFT(c[1], N, -1);
87     IDFT(c[0], c[2], N);
88
89     for (int j = 2; ~j; j--)
90         for (int i = 0; i < n + m - 1; i++)
91             ans[i] = ((long long)ans[i] * base + (long
92                 ↪ long)(c[j][i].a + 0.5)) % p;
93
94     // 实际上就是c[2] * base ^ 2 + c[1] * base + c[0],
95     // 这样写可以改善地址访问连续性
96
97     for (int i = 0; i < n + m - 1; i++) {
98         if (i)
99

```

```

1 // 常量和复数类略
2
3 const Complex ima = Complex(0, 1);
4
5 int p, base;
6
7 // FFT略
8
9 void DFT(Complex *a, Complex *b, int n) {
10     static Complex c[maxn];
11
12     for (int i = 0; i < n; i++)
13         c[i] = Complex(a[i].a, b[i].a);
14
15     FFT(c, n, 1);
16
17     for (int i = 0; i < n; i++) {
18         int j = (n - i) & (n - 1);
19
20         a[i] = (c[i] + c[j].conj() * 0.5;
21         b[i] = (c[i] - c[j].conj() * -0.5 * ima;
22     }
23
24 void IDFT(Complex *a, Complex *b, int n) {
25     static Complex c[maxn];
26
27     for (int i = 0; i < n; i++)
28         c[i] = a[i] + ima * b[i];

```

```

96     printf(" ");
97
98     printf("%d", ans[i]);
99
100    return 0;
101}

```

### 1.1.4 多项式操作

```

1 // A为输入, C为输出, n为所需长度且必须是2^k
2 // 多项式求逆, 要求A常数项不为0
3 void get_inv(int *A, int *C, int n) {
4     static int B[maxn];
5
6     memset(C, 0, sizeof(int) * (n * 2));
7     C[0] = qpow(A[0], p - 2); // 一般常数项都是1, 直接赋
→ 值为1就可以
8
9     for (int k = 2; k <= n; k *= 2) {
10        memcpy(B, A, sizeof(int) * k);
11        memset(B + k, 0, sizeof(int) * k);
12
13        NTT(B, k * 2, 1);
14        NTT(C, k * 2, 1);
15
16        for (int i = 0; i < k * 2; i++) {
17            C[i] = (2 - (long long)B[i] * C[i]) % p *
→ C[i] % p;
18            if (C[i] < 0)
19                C[i] += p;
20        }
21
22        NTT(C, k * 2, -1);
23
24        memset(C + k, 0, sizeof(int) * k);
25    }
26
27 // 开根
28 void get_sqrt(int *A, int *C, int n) {
29     static int B[maxn], D[maxn];
30
31     memset(C, 0, sizeof(int) * (n * 2));
32     C[0] = 1; // 如果不是1就要考虑二次剩余
33
34     for (int k = 2; k <= n; k *= 2) {
35        memcpy(B, A, sizeof(int) * k);
36        memset(B + k, 0, sizeof(int) * k);
37
38        get_inv(C, D, k);
39
40        NTT(B, k * 2, 1);
41        NTT(D, k * 2, 1);
42
43        for (int i = 0; i < k * 2; i++)
44            B[i] = (long long)B[i] * D[i] % p;
45
46        NTT(B, k * 2, -1);
47
48        for (int i = 0; i < k; i++)
49            C[i] = (long long)(C[i] + B[i]) * inv_2 %
→ p; // inv_2是2的逆元
50    }
51
52 // 求导
53 void get_derivative(int *A, int *C, int n) {
54     for (int i = 1; i < n; i++)

```

```

57         C[i - 1] = (long long)A[i] * i % p;
58
59     C[n - 1] = 0;
60 }
61
62 // 不定积分, 最好预处理逆元
63 void get_integrate(int *A, int *C, int n) {
64     for (int i = 1; i < n; i++)
65         C[i] = (long long)A[i - 1] * qpow(i, p - 2) %
→ p;
66
67     C[0] = 0; // 不定积分没有常数项
68 }
69
70 // 多项式ln, 要求A常数项不为0
71 void get_ln(int *A, int *C, int n) { // 通常情况下A常数
→ 项都是1
72     static int B[maxn];
73
74     get_derivative(A, B, n);
75     memset(B + n, 0, sizeof(int) * n);
76
77     get_inv(A, C, n);
78
79     NTT(B, n * 2, 1);
80     NTT(C, n * 2, 1);
81
82     for (int i = 0; i < n * 2; i++)
83         B[i] = (long long)B[i] * C[i] % p;
84
85     NTT(B, n * 2, -1);
86
87     get_integrate(B, C, n);
88
89     memset(C + n, 0, sizeof(int) * n);
90 }
91
92 // 多项式exp, 要求A没有常数项
93 // 常数很大且总代码较长, 一般来说最好替换为分治FFT
94 // 分治FFT依据: 设 $G(x) = \exp F(x)$ , 则有
95 //  $\rightarrow g_i = \sum_{k=1}^{i-1} f_{i-k} * k * g_k$ 
96 void get_exp(int *A, int *C, int n) {
97     static int B[maxn];
98
99     memset(C, 0, sizeof(int) * (n * 2));
100    C[0] = 1;
101
102    for (int k = 2; k <= n; k *= 2)
103        get_ln(C, B, k);
104
105    for (int i = 0; i < k; i++) {
106        B[i] = A[i] - B[i];
107        if (B[i] < 0)
108            B[i] += p;
109    }
110    (++B[0]) %= p;
111
112    NTT(B, k * 2, 1);
113    NTT(C, k * 2, 1);
114
115    for (int i = 0; i < k * 2; i++)
116        C[i] = (long long)C[i] * B[i] % p;
117
118    NTT(C, k * 2, -1);
119
120    memset(C + k, 0, sizeof(int) * k);
121 }
122
123 // 多项式k次幂, 在A常数项不为1时需要转化

```

```

124 // 常数较大且总代码较长，在时间要求不高时最好替换为暴力快
125 // →速幂
126 void get_pow(int *A, int *C, int n, int k) {
127     static int B[maxn];
128
129     get_ln(A, B, n);
130
131     for (int i = 0; i < n; i++)
132         B[i] = (long long)B[i] * k % p;
133
134     get_exp(B, C, n);
135 }
136
137 // 多项式除法，A / B，结果输出在C
138 // A的次数为n, B的次数为m
139 void get_div(int *A, int *B, int *C, int n, int m) {
140     static int f[maxn], g[maxn], gi[maxn];
141
142     if (n < m) {
143         memset(C, 0, sizeof(int) * m);
144         return;
145     }
146
147     int N = 1;
148     while (N < (n - m + 1))
149         N *= 2;
150
151     memset(f, 0, sizeof(int) * N * 2);
152     memset(g, 0, sizeof(int) * N * 2);
153     // memset(gi, 0, sizeof(int) * N);
154
155     for (int i = 0; i < n - m + 1; i++)
156         f[i] = A[n - i - 1];
157     for (int i = 0; i < m && i < n - m + 1; i++)
158         g[i] = B[m - i - 1];
159
160     get_inv(g, gi, N);
161
162     for (int i = n - m + 1; i < N; i++)
163         gi[i] = 0;
164
165     NTT(f, N * 2, 1);
166     NTT(gi, N * 2, 1);
167
168     for (int i = 0; i < N * 2; i++)
169         f[i] = (long long)f[i] * gi[i] % p;
170
171     NTT(f, N * 2, -1);
172
173     for (int i = 0; i < n - m + 1; i++)
174         C[i] = f[n - m - i];
175
176 // 多项式取模，余数输出到C，商输出到D
177 void get_mod(int *A, int *B, int *C, int *D, int n, int
178             m) {
179     static int b[maxn], d[maxn];
180
181     if (n < m) {
182         memcpy(C, A, sizeof(int) * n);
183
184         if (D)
185             memset(D, 0, sizeof(int) * m);
186
187         return;
188     }
189
190     get_div(A, B, d, n, m);
191
192     for (int i = 0; i < n - m + 1; i++)
193         D[i] = d[i];
194
195     int N = 1;
196     while (N < n)
197         N *= 2;
198
199     memcpy(b, B, sizeof(int) * m);
200
201     NTT(b, N, 1);
202     NTT(d, N, 1);
203
204     for (int i = 0; i < N; i++)
205         b[i] = (long long)d[i] * b[i] % p;
206
207     NTT(b, N, -1);
208
209     for (int i = 0; i < m - 1; i++)
210         C[i] = (A[i] - b[i] + p) % p;
211
212     memset(b, 0, sizeof(int) * N);
213     memset(d, 0, sizeof(int) * N);
214
215 }
216
217 // 多点求值要用的数组
218 int q[maxn], ans[maxn]; // q是要代入的各个系数, ans是求
219 // →出的值
220 int tg[25][maxn * 2], tf[25][maxn]; // 辅助数组, tg是预
221 // →处理乘积
222 // tf是项数越来越少的f, tf[0]就是原来的函数
223 void pretreat(int l, int r, int k) { // 多点求值预处理
224     static int A[maxn], B[maxn];
225
226     int *g = tg[k] + l * 2;
227
228     if (r - l + 1 ≤ 200) { // 小范围暴力, 能跑得快点
229         g[0] = 1;
230
231         for (int i = l; i ≤ r; i++) {
232             for (int j = i - l + 1; j; j--) {
233                 g[j] = (g[j - 1] - (long long)g[j] *
234                     → q[i]) % p;
235                 if (g[j] < 0)
236                     g[j] += p;
237             }
238             g[0] = (long long)g[0] * (p - q[i]) % p;
239
240         }
241
242         return;
243     }
244
245     int mid = (l + r) / 2;
246
247     pretreat(l, mid, k + 1);
248     pretreat(mid + 1, r, k + 1);
249
250     if (!k)
251         return;
252
253     int N = 1;
254     while (N ≤ r - l + 1)
255         N *= 2;
256
257     int *gl = tg[k + 1] + l * 2, *gr = tg[k + 1] + (mid
258         → + 1) * 2;
259
260     memset(A, 0, sizeof(int) * N);

```

```

257     memset(B, 0, sizeof(int) * N);
258
259     memcpy(A, gl, sizeof(int) * (mid - l + 2));
260     memcpy(B, gr, sizeof(int) * (r - mid + 1));
261
262     NTT(A, N, 1);
263     NTT(B, N, 1);
264
265     for (int i = 0; i < N; i++)
266         A[i] = (long long)A[i] * B[i] % p;
267
268     NTT(A, N, -1);
269
270     for (int i = 0; i <= r - l + 1; i++)
271         g[i] = A[i];
272 }
273
274 void solve(int l, int r, int k) { // 多项式多点求值主过
→ 程
275     int *f = tf[k];
276
277     if (r - l + 1 <= 200) {
278         for (int i = l; i <= r; i++) {
279             int x = q[i];
280
281             for (int j = r - l; ~j; j--) {
282                 ans[i] = ((long long)ans[i] * x + f[j])
283                     → % p;
284             }
285
286             return;
287         }
288
289         int mid = (l + r) / 2;
290         int *ff = tf[k + 1], *gl = tg[k + 1] + l * 2, *gr =
→ tg[k + 1] + (mid + 1) * 2;
291
292         get_mod(f, gl, ff, nullptr, r - l + 1, mid - l +
→ 2);
293         solve(l, mid, k + 1);
294
295         memset(gl, 0, sizeof(int) * (mid - l + 2));
296         memset(ff, 0, sizeof(int) * (mid - l + 1));
297
298         get_mod(f, gr, ff, nullptr, r - l + 1, r - mid +
→ 1);
299         solve(mid + 1, r, k + 1);
300
301         memset(gr, 0, sizeof(int) * (r - mid + 1));
302         memset(ff, 0, sizeof(int) * (r - mid));
303     }
304 // f < x^n, m个询问, 询问是0-based, 当然改成1-based也很
→ 简单
305 void get_value(int *f, int *x, int *a, int n, int m) {
306     if (m <= n)
307         m = n + 1;
308     if (n < m - 1)
309         n = m - 1; // 补零方便处理
310
311     memcpy(tf[0], f, sizeof(int) * n);
312     memcpy(q, x, sizeof(int) * m);
313
314     pretreat(0, m - 1, 0);
315     solve(0, m - 1, 0);
316
317     if (a) // 如果a是nullptr, 代表不复制答案, 直接
→ 用ans数组
318     memcpy(a, ans, sizeof(int) * m);

```

319 }

### 1.1.5 更优秀的多项式多点求值

这个做法不需要写取模, 求逆也只有一次, 但是神乎其技, 完全搞不懂原理

清空和复制之类的地方容易抄错, 抄的时候要注意

```

1 int q[maxn], ans[maxn]; // q是要代入的各个系数, ans是求
→ 出的值
2 int tg[25][maxn * 2], tf[25][maxn]; // 辅助数组, tg是预
→ 处理乘积,
3 // tf是项数越来越少的f, tf[0]就是原来的函数
4
5 void pretreat(int l, int r, int k) { // 预处理
6     static int A[maxn], B[maxn];
7
8     int *g = tg[k] + l * 2;
9
10    if (r - l + 1 <= 1) { // 小范围暴力
11        g[0] = 1;
12
13        for (int i = l; i <= r; i++) {
14            for (int j = i - l + 1; j; j--) {
15                g[j] = (g[j - 1] - (long long)g[j] *
→ q[i]) % p;
16                if (g[j] < 0)
17                    g[j] += p;
18            }
19            g[0] = (long long)g[0] * (p - q[i]) % p;
20        }
21
22        reverse(g, g + r - l + 2);
23
24        return;
25    }
26
27    int mid = (l + r) / 2;
28
29    pretreat(l, mid, k + 1);
30    pretreat(mid + 1, r, k + 1);
31
32    int N = 1;
33    while (N <= r - l + 1)
34        N *= 2;
35
36    int *gl = tg[k + 1] + l * 2, *gr = tg[k + 1] + (mid
→ + 1) * 2;
37
38    memset(A, 0, sizeof(int) * N);
39    memset(B, 0, sizeof(int) * N);
40
41    memcpy(A, gl, sizeof(int) * (mid - l + 2));
42    memcpy(B, gr, sizeof(int) * (r - mid + 1));
43
44    NTT(A, N, 1);
45    NTT(B, N, 1);
46
47    for (int i = 0; i < N; i++)
48        A[i] = (long long)A[i] * B[i] % p;
49
50    NTT(A, N, -1);
51
52    for (int i = 0; i <= r - l + 1; i++)
53        g[i] = A[i];
54
55
56 void solve(int l, int r, int k) { // 主过程
57     static int a[maxn], b[maxn];
58

```

```

59     int *f = tf[k];
60
61     if (l == r) {
62         ans[l] = f[0];
63         return;
64     }
65
66     int mid = (l + r) / 2;
67     int *ff = tf[k + 1], *gl = tg[k + 1] + l * 2, *gr =
68     ↪ tg[k + 1] + (mid + 1) * 2;
69
70     int N = 1;
71     while (N < r - l + 2)
72         N *= 2;
73
74     memcpy(a, f, sizeof(int) * (r - l + 2));
75     memcpy(b, gr, sizeof(int) * (r - mid + 1));
76     reverse(b, b + r - mid + 1);
77
78     NTT(a, N, 1);
79     NTT(b, N, 1);
80     for (int i = 0; i < N; i++)
81         b[i] = (long long)a[i] * b[i] % p;
82
83     reverse(b + 1, b + N);
84     NTT(b, N, 1);
85     int n_inv = qpow(N, p - 2);
86     for (int i = 0; i < N; i++)
87         b[i] = (long long)b[i] * n_inv % p;
88
89     for (int i = 0; i < mid - l + 2; i++)
90         ff[i] = b[i + r - mid];
91
92     memset(a, 0, sizeof(int) * N);
93     memset(b, 0, sizeof(int) * N);
94
95     solve(l, mid, k + 1);
96
97     memset(ff, 0, sizeof(int) * (mid - l + 2));
98
99     memcpy(a, f, sizeof(int) * (r - l + 2));
100    memcpy(b, gl, sizeof(int) * (mid - l + 2));
101    reverse(b, b + mid - l + 2);
102
103    NTT(a, N, 1);
104    NTT(b, N, 1);
105    for (int i = 0; i < N; i++)
106        b[i] = (long long)a[i] * b[i] % p;
107
108    reverse(b + 1, b + N);
109    NTT(b, N, 1);
110    for (int i = 0; i < N; i++)
111        b[i] = (long long)b[i] * n_inv % p;
112
113    for (int i = 0; i < r - mid + 1; i++)
114        ff[i] = b[i + mid - l + 1];
115
116    memset(a, 0, sizeof(int) * N);
117    memset(b, 0, sizeof(int) * N);
118
119    solve(mid + 1, r, k + 1);
120
121    memset(gl, 0, sizeof(int) * (mid - l + 2));
122    memset(gr, 0, sizeof(int) * (r - mid + 1));
123    memset(ff, 0, sizeof(int) * (r - mid + 1));
124
125 // f < x^n, m个询问, θ-based
126 void get_value(int *f, int *x, int *a, int n, int m) {

```

```

127     static int c[maxn], d[maxn];
128
129     if (m ≤ n)
130         m = n + 1;
131     if (n < m - 1)
132         n = m - 1; // 补零
133
134     memcpy(q, x, sizeof(int) * m);
135     pretreat(0, m - 1, 0);
136
137     int N = 1;
138     while (N < m)
139         N *= 2;
140
141     get_inv(tg[0], c, N);
142
143     fill(c + m, c + N, 0);
144     reverse(c, c + m);
145
146     memcpy(d, f, sizeof(int) * m);
147
148     NTT(c, N * 2, 1);
149     NTT(d, N * 2, 1);
150     for (int i = 0; i < N * 2; i++)
151         c[i] = (long long)c[i] * d[i] % p;
152     NTT(c, N * 2, -1);
153
154     for (int i = 0; i < m; i++)
155         tf[0][i] = c[i + n];
156
157     solve(0, m - 1, 0);
158
159     if (a) // 如果a是nullptr, 代表不复制答案, 直接
160     ↪ 用ans数组
161         memcpy(a, ans, sizeof(int) * m);
162

```

### 1.1.6 多项式快速插值

问题: 给出 $n$ 个 $x_i$ 与 $y_i$ , 求一个 $n-1$ 次多项式满足 $F(x_i) = y_i$ . 考虑拉格朗日插值:

$$F(x) = \sum_{i=1}^n \frac{\prod_{i \neq j} (x - x_j)}{\prod_{i \neq j} (x_i - x_j)} y_i$$

第一步要先对每个 $i$ 求出

$$\prod_{i \neq j} (x_i - x_j)$$

设

$$M(x) = \prod_{i=1}^n (x - x_i)$$

那么想要的是

$$\frac{M(x)}{x - x_i}$$

取 $x = x_i$ 时, 上下都为0, 使用洛必达法则, 则原式化为 $M'(x)$ . 使用分治算出 $M(x)$ , 使用多点求值即可算出每个

$$\prod_{i \neq j} (x_i - x_j) = M'(x_i)$$

设

$$v_i = \frac{y_i}{\prod_{i \neq j} (x_i - x_j)}$$

第二步要求出

$$\sum_{i=1}^n v_i \prod_{i \neq j} (x - x_j)$$

使用分治. 设

$$L(x) = \prod_{i=1}^{\lfloor n/2 \rfloor} (x - x_i), \quad R(x) = \prod_{i=\lfloor n/2 \rfloor + 1}^n (x - x_i)$$

则原式化为

$$\left( \sum_{i=1}^{\lfloor n/2 \rfloor} v_i \prod_{i \neq j, j \leq \lfloor n/2 \rfloor} (x - x_j) \right) R(x) +$$

$$\left( \sum_{i=\lfloor n/2 \rfloor + 1}^n v_i \prod_{i \neq j, j > \lfloor n/2 \rfloor} (x - x_j) \right) L(x)$$

递归计算, 复杂度 $O(n \log^2 n)$

注意由于整体和局部的 $M(x)$ 都要用到，要预处理一下

```

1 int qx[maxn], qy[maxn];
2 int th[25][maxn * 2], ansf[maxn]; // th存的是各阶段
3   ← 的M(x)
4 void pretreat2(int l, int r, int k) { // 预处理
5   static int A[maxn], B[maxn];
6   int *h = th[k] + l * 2;
7
8   if (l == r) {
9     h[0] = p - qx[l];
10    h[1] = 1;
11    return;
12  }
13
14  int mid = (l + r) / 2;
15
16  pretreat2(l, mid, k + 1);
17  pretreat2(mid + 1, r, k + 1);
18
19  int N = 1;
20  while (N <= r - l + 1)
21    N *= 2;
22
23  int *hl = th[k + 1] + l * 2, *hr = th[k + 1] +
24    → + 1) * 2;
25
26  memset(A, 0, sizeof(int) * N);
27  memset(B, 0, sizeof(int) * N);
28
29  memcpy(A, hl, sizeof(int) * (mid - l + 2));
30  memcpy(B, hr, sizeof(int) * (r - mid + 1));
31
32  NTT(A, N, 1);
33  NTT(B, N, 1);
34
35  for (int i = 0; i < N; i++)
36    A[i] = (long long)A[i] * B[i] % p;
37
38  NTT(A, N, -1);
39
40  for (int i = 0; i <= r - l + 1; i++)
41    h[i] = A[i];
42
43 void solve2(int l, int r, int k) { // 分治
44   static int A[maxn], B[maxn], t[maxn];

```

```

46     if (l == r)
47         return;
48
49     int mid = (l + r) / 2;
50
51     solve2(l, mid, k + 1);
52     solve2(mid + 1, r, k + 1);
53
54     int *hl = th[k + 1] + l * 2, *hr = th[k + 1] + (mid
55     ↪ + 1) * 2;
56
57     int N = 1;
58
59     while (N < r - l + 1)
60         N *= 2;
61
62     memset(A, 0, sizeof(int) * N);
63     memset(B, 0, sizeof(int) * N);
64
65     memcpy(A, ansf + l, sizeof(int) * (mid - l + 1));
66     memcpy(B, hr, sizeof(int) * (r - mid + 1));
67
68     NTT(A, N, 1);
69     NTT(B, N, 1);
70
71     for (int i = 0; i < N; i++)
72         t[i] = (long long)A[i] * B[i] % p;
73
74     memset(A, 0, sizeof(int) * N);
75     memset(B, 0, sizeof(int) * N);
76
77     memcpy(A, ansf + mid + 1, sizeof(int) * (r - mid));
78     memcpy(B, hl, sizeof(int) * (mid - l + 2));
79
80     NTT(A, N, 1);
81     NTT(B, N, 1);
82
83     for (int i = 0; i < N; i++)
84         t[i] = (t[i] + (long long)A[i] * B[i]) % p;
85
86     NTT(t, N, -1);
87
88     memcpy(ansf + l, t, sizeof(int) * (r - l + 1));
89 }
90
91 // 主过程
92 // 如果x, y传nullptr表示询问已经存在了qx, qy里
93 void interpolation(int *x, int *y, int n, int *f =
94     ↪ nullptr) {
95     static int d[maxn];
96
97     if (x)
98         memcpy(qx, x, sizeof(int) * n);
99     if (y)
100        memcpy(qy, y, sizeof(int) * n);
101
102    pretreat2(0, n - 1, 0);
103
104    get_derivative(th[0], d, n + 1);
105
106    multipoint_eval(d, qx, nullptr, n, n);
107
108    for (int i = 0; i < n; i++)
109        ansf[i] = (long long)qy[i] * qpow(ans[i], p -
110        ↪ 2) % p;
111
112    solve2(0, n - 1, 0);
113
114    if (f)
115        memcpy(f, ansf, sizeof(int) * n);

```

113 }

### 1.1.7 拉格朗日反演(多项式复合逆)

如果 $f(x)$ 与 $g(x)$ 互为复合逆, 则有

$$[x^n] g(x) = \frac{1}{n} [x^{n-1}] \left( \frac{x}{f(x)} \right)^n$$

$$[x^n] h(g(x)) = \frac{1}{n} [x^{n-1}] h'(x) \left( \frac{x}{f(x)} \right)^n$$

### 1.1.8 分治FFT

```

1 void solve(int l, int r) {
2     if (l == r)
3         return;
4
5     int mid = (l + r) / 2;
6
7     solve(l, mid);
8
9     int N = 1;
10    while (N <= r - l + 1)
11        N *= 2;
12
13    for (int i = l; i <= mid; i++)
14        B[i - l] = (long long)A[i] * fac_inv[i] % p;
15    fill(B + mid - l + 1, B + N, 0);
16    for (int i = 0; i < N; i++)
17        C[i] = fac_inv[i];
18
19    NTT(B, N, 1);
20    NTT(C, N, 1);
21
22    for (int i = 0; i < N; i++)
23        B[i] = (long long)B[i] * C[i] % p;
24
25    NTT(B, N, -1);
26
27    for (int i = mid + 1; i <= r; i++)
28        A[i] = (A[i] + B[i - l] * 2 % p * (long
29            → long)fac[i] % p) % p;
30
31    solve(mid + 1, r);
32 }
```

### 1.1.9 半在线卷积

```

1 void solve(int l, int r) {
2     if (r <= m)
3         return;
4
5     if (r - l == 1) {
6         if (l == m)
7             f[l] = a[m];
8         else
9             f[l] = (long long)f[l] * inv[l - m] % p;
10
11     for (int i = l, t = (long long)l * f[l] % p; i
12         → ≤ n; i += l)
13         g[i] = (g[i] + t) % p;
14
15     return;
16 }
17
18     int mid = (l + r) / 2;
19
20     solve(l, mid);
21 }
```

```

21
22     if (l == 0) {
23         for (int i = 1; i < mid; i++) {
24             A[i] = f[i];
25             B[i] = (c[i] + g[i]) % p;
26         }
27         NTT(A, r, 1);
28         NTT(B, r, 1);
29         for (int i = 0; i < r; i++)
30             A[i] = (long long)A[i] * B[i] % p;
31         NTT(A, r, -1);
32
33         for (int i = mid; i < r; i++)
34             f[i] = (f[i] + A[i]) % p;
35     }
36     else {
37         for (int i = 0; i < r - l; i++) {
38             A[i] = f[i];
39             for (int i = l; i < mid; i++)
40                 B[i - l] = (c[i] + g[i]) % p;
41         NTT(A, r - l, 1);
42         NTT(B, r - l, 1);
43         for (int i = 0; i < r - l; i++)
44             A[i] = (long long)A[i] * B[i] % p;
45         NTT(A, r - l, -1);
46
47         for (int i = mid; i < r; i++)
48             f[i] = (f[i] + A[i - l]) % p;
49
50         memset(A, 0, sizeof(int) * (r - l));
51         memset(B, 0, sizeof(int) * (r - l));
52
53         for (int i = l; i < mid; i++) {
54             A[i - l] = f[i];
55             for (int i = 0; i < r - l; i++)
56                 B[i] = (c[i] + g[i]) % p;
57         NTT(A, r - l, 1);
58         NTT(B, r - l, 1);
59         for (int i = 0; i < r - l; i++)
60             A[i] = (long long)A[i] * B[i] % p;
61         NTT(A, r - l, -1);
62
63         for (int i = mid; i < r; i++)
64             f[i] = (f[i] + A[i - l]) % p;
65     }
66
67     memset(A, 0, sizeof(int) * (r - l));
68     memset(B, 0, sizeof(int) * (r - l));
69
70     solve(mid, r);
71 }
```

### 1.1.10 常系数齐次线性递推 $O(k \log k \log n)$

如果只有一次这个操作可以照抄, 否则就开一个全局flag.

```

1 // 多项式取模, 余数输出到C, 商输出到D
2 void get_mod(int *A, int *B, int *C, int *D, int n, int
3 → m) {
4     static int b[maxn], d[maxn];
5     static bool flag = false;
6
7     if (n < m) {
8         memcpy(C, A, sizeof(int) * n);
9
10     if (D)
11         memset(D, 0, sizeof(int) * m);
12
13     return;
14 }
```

```

14
15     get_div(A, B, d, n, m);
16
17     if (D) { // D是商, 可以选择不要
18         for (int i = 0; i < n - m + 1; i++)
19             D[i] = d[i];
20     }
21
22     int N = 1;
23     while (N < n)
24         N *= 2;
25
26     if (!flag) {
27         memcpy(b, B, sizeof(int) * m);
28         NTT(b, N, 1);
29
30         flag = true;
31     }
32
33     NTT(d, N, 1);
34
35     for (int i = 0; i < N; i++)
36         d[i] = (long long)d[i] * b[i] % p;
37
38     NTT(d, N, -1);
39
40     for (int i = 0; i < m - 1; i++)
41         C[i] = (A[i] - d[i] + p) % p;
42
43     // memset(b, 0, sizeof(int) * N);
44     memset(d, 0, sizeof(int) * N);
45 }

// g < x^n, f是输出答案的数组
46 void pow_mod(long long k, int *g, int n, int *f) {
47     static int a[maxn], t[maxn];
48
49     memset(f, 0, sizeof(int) * (n * 2));
50
51     f[0] = a[1] = 1;
52
53     int N = 1;
54     while (N < n * 2 - 1)
55         N *= 2;
56
57     while (k) {
58         NTT(a, N, 1);
59
60         if (k & 1) {
61             memcpy(t, f, sizeof(int) * N);
62
63             NTT(t, N, 1);
64             for (int i = 0; i < N; i++)
65                 t[i] = (long long)t[i] * a[i] % p;
66             NTT(t, N, -1);
67
68             get_mod(t, g, f, NULL, n * 2 - 1, n);
69         }
70
71         for (int i = 0; i < N; i++)
72             a[i] = (long long)a[i] * a[i] % p;
73         NTT(a, N, -1);
74
75         memcpy(t, a, sizeof(int) * (n * 2 - 1));
76         get_mod(t, g, a, NULL, n * 2 - 1, n);
77         fill(a + n - 1, a + N, 0);
78
79         k >>= 1;
80     }
81
82 }


```

```

84     memset(a, 0, sizeof(int) * (n * 2));
85 }
86
87 // f_n = \sum_{i=1}^m f_{n-i} a_i
88 // f是0~m-1项的初值
89 int linear_recurrence(long long n, int m, int *f, int
90     *a) {
91     static int g[maxn], c[maxn];
92
93     memset(g, 0, sizeof(int) * (m * 2 + 1));
94
95     for (int i = 0; i < m; i++)
96         g[i] = (p - a[m - i]) % p;
97     g[m] = 1;
98
99     pow_mod(n, g, m + 1, c);
100
101    int ans = 0;
102    for (int i = 0; i < m; i++)
103        ans = (ans + (long long)c[i] * f[i]) % p;
104
105    return ans;
106 }


```

### 1.1.11 应用: $O(\sqrt{n} \log^2 n)$ 快速求阶乘

问题: 求  $n! \pmod{p}$ ,  $n < p$ ,  $p$  是 NTT 模数.

考虑令  $m = \lfloor \sqrt{n} \rfloor$ , 那么我们可以写出连续  $m$  个数相乘的多项式:

$$f(x) = \prod_{i=1}^m (x + i)$$

那么显然就有

$$n! = \left( \prod_{k=0}^{m-1} f(km) \right) \prod_{i=m^2+1}^n i$$

$f(x)$  的系数可以用倍增求(或者懒一点直接分治 FFT), 然后  $f(km)$  可以用多项式多点求值求出, 所以总复杂度就是  $O(\sqrt{n} \log^2 n)$ .

当然如果  $p$  不变并且多次询问的话我们只需要取一个  $m$ , 也就是预处理  $O(\sqrt{p} \log^2 p)$ , 询问  $O(\sqrt{p})$ .

## 1.2 插值

### 1.2.1 牛顿插值

牛顿插值的原理是二项式反演.

二项式反演:

$$f(n) = \sum_{k=0}^n \binom{n}{k} g(k) \Leftrightarrow g(n) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} f(k)$$

可以用  $e^x$  和  $e^{-x}$  的麦克劳林展开式证明.

套用二项式反演的结论即可得到牛顿插值:

$$f(n) = \sum_{i=0}^k \binom{n}{i} r_i$$

$$r_i = \sum_{j=0}^i (-1)^{i-j} \binom{i}{j} f(j)$$

其中  $k$  表示  $f(n)$  的最高次项系数.

实现时可以用  $k$  次差分替代右边的式子:

```

1 for (int i = 0; i <= k; i++)
2     r[i] = f(i);
3 for (int j = 0; j < k; j++)
4     for (int i = k; i > j; i--)
5         r[i] -= r[i - 1];

```

注意到预处理 $r_i$ 的式子满足卷积形式,必要时可以用FFT优化至 $O(k \log k)$ 预处理.

### 1.2.2 拉格朗日(Lagrange)插值

$$f(x) = \sum_i f(x_i) \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

## 1.3 FWT快速沃尔什变换

```

// 注意FWT常数比较小, 这点与FFT/NTT不同
// 以下代码均以模质数情况为例, 其中n为变换长度, tp表示
// → 正/逆变换

// 按位或版本
void FWT_or(int *A, int n, int tp) {
    for (int k = 2; k <= n; k *= 2)
        for (int i = 0; i < n; i += k)
            for (int j = 0; j < k / 2; j++) {
                if (tp > 0)
                    A[i + j + k / 2] = (A[i + j + k / 2] + A[i + j]) % p;
                else
                    A[i + j + k / 2] = (A[i + j] - A[i + j] + p) % p;
            }
}

// 按位与版本
void FWT_and(int *A, int n, int tp) {
    for (int k = 2; k <= n; k *= 2)
        for (int i = 0; i < n; i += k)
            for (int j = 0; j < k / 2; j++) {
                if (tp > 0)
                    A[i + j] = (A[i + j] + A[i + j + k / 2]) % p;
                else
                    A[i + j] = (A[i + j] - A[i + j + k / 2] + p) % p;
            }
}

// 按位异或版本
void FWT_xor(int *A, int n, int tp) {
    for (int k = 2; k <= n; k *= 2)
        for (int i = 0; i < n; i += k)
            for (int j = 0; j < k / 2; j++) {
                int a = A[i + j], b = A[i + j + k / 2];
                A[i + j] = (a + b) % p;
                A[i + j + k / 2] = (a - b + p) % p;
            }

    if (tp < 0) {
        int inv = qpow(n % p, p - 2); // n的逆元, 在不取
        // 模时需要用每层除以2代替
        for (int i = 0; i < n; i++)
            A[i] = A[i] * inv % p;
    }
}

```

### 1.3.1 三行FWT

```

1 void fwt_or(int *a, int n, int tp) {
2     for (int j = 0; (1 << j) < n; j++)
3         for (int i = 0; i < n; i++)
4             if (i >> j & 1) {
5                 if (tp > 0)
6                     a[i] += a[i ^ (1 << j)];
7                 else
8                     a[i] -= a[i ^ (1 << j)];
9             }
10
11 // and自然就是or反过来
12 void fwt_and(int *a, int n, int tp) {
13     for (int j = 0; (1 << j) < n; j++)
14         for (int i = 0; i < n; i++)
15             if (!(i >> j & 1)) {
16                 if (tp > 0)
17                     a[i] += a[i | (1 << j)];
18                 else
19                     a[i] -= a[i | (1 << j)];
20             }
21
22 }
23 // xor同理

```

## 1.4 线性代数

稀疏矩阵操作参见Berlekamp-Massey算法的应用(12页).

### 1.4.1 矩阵乘法

```

1 for (int i = 1; i <= n; i++)
2     for (int k = 1; k <= n; k++)
3         for (int j = 1; j <= n; j++)
4             a[i][j] += b[i][k] * c[k][j];
5 // 通过改善内存访问连续性, 显著提升速度

```

### 1.4.2 高斯消元

**高斯-约当消元法 Gauss-Jordan** 每次选取当前行绝对值最大的数作为代表元, 在做浮点数消元时可以很好地保证精度.

```

1 void Gauss_Jordan(int A[][maxn], int n) {
2     for (int i = 1; i <= n; i++) {
3         int ii = i;
4         for (int j = i + 1; j <= n; j++)
5             if (fabs(A[j][i]) > fabs(A[ii][i]))
6                 ii = j;
7
8         if (ii != i) // 这里没有判是否无解, 如果有可能无
9             // 解的话要判一下
10            for (int j = i; j <= n + 1; j++)
11                swap(A[i][j], A[ii][j]);
12
13         for (int j = 1; j <= n; j++)
14             if (j != i) // 消成对角
15                 for (int k = n + 1; k >= i; k--)
16                     A[j][k] -= A[j][i] / A[i][i] * A[i]
17                         ^ [k];
18     }
19 }

```

**解线性方程组** 在矩阵的右边加上一列表示系数即可, 如果消成上三角的话最后要倒序回代.

**求逆矩阵** 维护一个矩阵 $B$ , 初始设为 $n$ 阶单位矩阵, 在消元的同时对 $B$ 进行一样的操作, 当把 $A$ 消成单位矩阵时 $B$ 就是逆矩阵.

**行列式** 消成对角之后把代表元乘起来. 如果是任意模数, 要注意消元时每交换一次行列要取反一次.

### 1.4.3 行列式取模

```

1 int p;
2
3 int Gauss(int A[maxn][maxn], int n) {
4     int det = 1;
5
6     for (int i = 1; i <= n; i++) {
7         for (int j = i + 1; j <= n; j++)
8             while (A[j][i]) {
9                 int t = (p - A[i][i] / A[j][i]) % p;
10                for (int k = i; k <= n; k++)
11                    A[i][k] = (A[i][k] + (long
12                        ↪ long)A[j][k] * t) % p;
13
14                swap(A[i], A[j]);
15                det = (p - det) % p; // 交换一次之后行列
16                ↪ 式取负
17            }
18
19            if (!A[i][i])
20                return 0;
21
22            det = (long long)det * A[i][i] % p;
23        }
24
25    return det;
26 }
```

### 1.4.4 线性基(消成对角)

```

1 void add(unsigned long long x) {
2     for (int i = 63; i >= 0; i--)
3         if (x >> i & 1) {
4             if (b[i])
5                 x ^= b[i];
6             else {
7                 b[i] = x;
8
9                 for (int j = i - 1; j >= 0; j--)
10                    if (b[j] && (b[i] >> j & 1))
11                        b[i] ^= b[j];
12
13                 for (int j = i + 1; j < 64; j++)
14                     if (b[j] >> i & 1)
15                         b[j] ^= b[i];
16
17                 break;
18             }
19         }
20 }
```

### 1.4.5 线性代数知识

**行列式:**

$$\det A = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_i a_{i,\sigma_i}$$

**逆矩阵:**

$$B = A^{-1} \iff AB = 1$$

**代数余子式:**

$$C_{i,j} = (-1)^{i+j} M_{i,j} = (-1)^{i+j} |A^{i,j}|$$

也就是 $A$ 去掉一行一列之后的行列式.

伴随矩阵:

$$A^* = C^T$$

即代数余子式矩阵的转置.

同时我们有

$$A^* = |A|A^{-1}$$

**特征多项式:**

$$P_A(x) = \det(Ix - A)$$

**特征根:** 特征多项式的所有 $n$ 个根(可能有重根).

### 1.4.6 矩阵树定理, BEST定理

**无向图** 设图 $G$ 的基尔霍夫矩阵 $L(G)$ 等于度数矩阵减去邻接矩阵, 则 $G$ 的生成树个数等于 $L(G)$ 的任意一个代数余子式的值.

**有向图** 类似地定义 $L_{in}(G)$ 等于入度矩阵减去邻接矩阵( $i$ 指向 $j$ 有边, 则 $A_{i,j} = 1$ ),  $L_{out}(G)$ 等于出度矩阵减去邻接矩阵.

则以 $i$ 为根的内向树个数即为 $L_{out}$ 的第 $i$ 个主子式(即关于第 $i$ 行第 $i$ 列的余子式), 外向树个数即为 $L_{in}$ 的第 $i$ 个主子式.

(可以看出, 只有无向图才满足 $L(G)$ 的所有代数余子式都相等.)

**BEST定理(有向图欧拉回路计数)** 如果 $G$ 是有向欧拉图, 则 $G$ 的欧拉回路的个数等于以一个任意点为根的内/外向树个数乘以 $\prod_v (\deg(v) - 1)!$ .

并且在欧拉图里, 无论以哪个结点为根, 无论内向树还是外向树, 个数都是一样的.

另外无向图欧拉回路计数是NP问题.

## 1.5 Berlekamp-Massey最小递推式

如果要求出一个次数为 $k$ 的递推式, 则输入的数列需要至少有 $2k$ 项. 返回的内容满足 $\sum_{j=0}^{m-1} a_{i-j} c_j = 0$ , 并且 $c_0 = 1$ . 称为最小递推式. 如果不加最后的处理的话, 代码返回的结果会变成 $a_i = \sum_{j=0}^{m-1} c_{j-1} a_{i-j}$ , 有时候这样会方便接着跑递推, 需要的话就删掉最后的处理.

(实际上Berlekamp-Massey是对每个前缀都求出了递推式, 但一般没啥用.)

```

1 vector<int> berlekamp_massey(const vector<int> &a) {
2     vector<int> v, last; // v is the answer, 0-based
3     int k = -1, delta = 0;
4
5     for (int i = 0; i < (int)a.size(); i++) {
6
7         int tmp = 0;
8         for (int j = 0; j < (int)v.size(); j++)
9             tmp = (tmp + (long long)a[i - j - 1] *
10                  ↪ v[j]) % p;
11
12         if (a[i] == tmp)
13             continue;
14
15         if (k < 0) {
16             k = i;
17             delta = (a[i] - tmp + p) % p;
18             v = vector<int>(i + 1);
19
20             continue;
21         }
22
23         vector<int> u = v;
24         int val = (long long)(a[i] - tmp + p) *
25             ↪ qpow(delta, p - 2) % p;
26
27         if (v.size() < last.size() + i - k)
28             v.resize(last.size() + i - k);
```

```

27     v[i - k - 1] += val) %= p;
28
29     for (int j = 0; j < (int)last.size(); j++) {
30         v[i - k + j] = (v[i - k + j] - (long)
31             → long)val * last[j]) % p;
32         if (v[i - k + j] < 0)
33             v[i - k + j] += p;
34     }
35
36     if ((int)u.size() - i < (int)last.size() - k) {
37         last = u;
38         k = i;
39         delta = a[i] - tmp;
40         if (delta < 0)
41             delta += p;
42     }
43
44
45     for (auto &x : v) // 一般是需要最小递推式的, 所以处理
46         ← 一下
47         x = (p - x) % p;
48     v.insert(v.begin(), 1);
49
50     return v; // ∀i, ∑_{j=0}^m a_{i-j}v_j = 0
}

```

如果要求向量序列的递推式, 就把每位乘一个随机权值(或者说是乘一个随机行向量 $v^T$ )变成求数列递推式即可.

如果是矩阵序列的话就随机一个行向量 $u^T$ 和列向量 $v$ , 然后把矩阵变成 $u^T A v$ 的数列就行了.

### 1.5.1 优化矩阵快速幂DP

如果 $f_i$ 是一个向量, 并且转移是一个矩阵, 那显然 $\{f_i\}$ 是一个线性递推序列.

假设 $f_i$ 有 $n$ 维, 先暴力求出 $f_{0-2n-1}$ , 然后跑Berlekamp-Massey, 最后调用前面的快速齐次线性递推(8页)即可. (快速齐次线性递推的结果是一个序列, 某个给定初值的结果就是点乘, 所以只需要跑一次.)

如果要求 $f_m$ , 并且矩阵有 $k$ 个非零项的话, 复杂度就是 $O(nk + n \log m \log n)$ . (因为暴力求前 $2n - 1$ 个 $f_i$ 需要 $O(nk)$ 时间.)

### 1.5.2 求矩阵最小多项式

矩阵 $A$ 的最小多项式是次数最小的并且 $f(A) = 0$ 的多项式 $f$ .

实际上最小多项式就是 $\{A^i\}$ 的最小递推式, 所以直接调用Berlekamp-Massey就好了, 并且显然它的次数不超过 $n$ .

瓶颈在于求出 $A^i$ , 实际上我们只要处理 $A^i v$ 就行了, 每次对向量做递推.

假设 $A$ 有 $k$ 个非零项, 则复杂度为 $O(kn + n^2)$ .

### 1.5.3 求稀疏矩阵的行列式

如果能求出特征多项式, 则常数项乘上 $(-1)^n$ 就是行列式, 但是最小多项式不一定就是特征多项式.

把 $A$ 乘上一个随机对角阵 $B$ (实际上就是每行分别乘一个随机数), 则 $AB$ 的最小多项式有很大概率就是特征多项式, 最后再除掉 $\det B$ 就行了.

设 $A$ 有 $k$ 个非零项, 则复杂度为 $O(kn + n^2)$ .

### 1.5.4 求稀疏矩阵的秩

设 $A$ 是一个 $n \times m$ 的矩阵, 首先随机一个 $n \times n$ 的对角阵 $P$ 和一个 $m \times m$ 的对角阵 $Q$ , 然后计算 $QAPAT^Q$ 的最小多项式即可.

实际上不用计算这个矩阵, 因为求最小多项式时要用它乘一个向量, 我们依次把这几个矩阵乘到向量里就行了. 答案就是最小多项式除掉所有 $x$ 因子后剩下的次数.

设 $A$ 有 $k$ 个非零项, 复杂度为 $O(kn + n^2)$ .

### 1.5.5 解稀疏方程组

**问题**  $Ax = b$ , 其中 $A$ 是一个 $n \times n$ 的满秩稀疏矩阵,  $b$ 和 $x$ 是 $1 \times n$ 的列向量,  $A, b$ 已知, 需要解出 $x$ .

**做法** 显然 $x = A^{-1}b$ . 如果我们能求出 $\{A^i b\}(i \geq 0)$ 的最小递推式 $\{r_{0-m-1}\}(m \leq n)$ , 那么就有结论

$$A^{-1}b = -\frac{1}{r_{m-1}} \sum_{i=0}^{m-2} A^i b r_{m-2-i}$$

因为 $A$ 是稀疏矩阵, 直接按定义递推出 $b \sim A^{2n-1}b$ 即可. 设 $A$ 中有 $k$ 个非零项, 则复杂度为 $O(kn + n^2)$ .

```

1 vector<int> solve_sparse_equations(const
2     → vector<tuple<int, int, int>> &A, const vector<int>
3     → &b) {
4         int n = (int)b.size(); // 0-based
5
6         vector<vector<int>> f({b});
7
8         for (int i = 1; i < 2 * n; i++) {
9             vector<int> v(n);
10            auto &u = f.back();
11
12            for (auto [x, y, z] : A) // [x, y, value]
13                v[x] = (v[x] + (long long)u[y] * z) % p;
14
15            f.push_back(v);
16        }
17
18        vector<int> w(n);
19        mt19937 gen;
20        for (auto &x : w)
21            x = uniform_int_distribution<int>(1, p - 1)
22                → (gen);
23
24        vector<int> a(2 * n);
25        for (int i = 0; i < 2 * n; i++)
26            for (int j = 0; j < n; j++)
27                a[i] = (a[i] + (long long)f[i][j] * w[j]) %
28                    → p;
29
30        auto c = berlekamp_massey(a);
31        int m = (int)c.size();
32
33        vector<int> ans(n);
34
35        for (int i = 0; i < m - 1; i++)
36            for (int j = 0; j < n; j++)
37                ans[j] = (ans[j] + (long long)c[m - 2 - i]
38                    → * f[i][j]) % p;
39
40        int inv = qpow(p - c[m - 1], p - 2);
41
42        for (int i = 0; i < n; i++)
43            ans[i] = (long long)ans[i] * inv % p;
44
45        return ans;
46    }

```

## 1.6 单纯形

```

1 const double eps = 1e-10;
2
3 double A[maxn][maxn], x[maxn];
4 int n, m, t, id[maxn * 2];
5
6 // 方便起见, 这里附上主函数
7 int main() {
8     scanf("%d%d%d", &n, &m, &t);
9
10    for (int i = 1; i ≤ n; i++) {
11        scanf("%lf", &A[0][i]);
12        id[i] = i;
13    }
14
15    for (int i = 1; i ≤ m; i++) {
16        for (int j = 1; j ≤ n; j++) {
17            scanf("%lf", &A[i][j]);
18
19            scanf("%lf", &A[i][0]);
20        }
21
22    if (!initialize())
23        printf("Infeasible"); // 无解
24    else if (!simplex())
25        printf("Unbounded"); // 最优解无限大
26
27    else {
28        printf("%.15lf\n", -A[0][0]);
29        if (t) {
30            for (int i = 1; i ≤ m; i++)
31                x[id[i + n]] = A[i][0];
32            for (int i = 1; i ≤ n; i++)
33                printf("%.15lf ", x[i]);
34        }
35    }
36    return 0;
37}

```

// 初始化

// 对于初始解可行的问题, 可以把初始化省略掉

```

41 bool initialize() {
42     while (true) {
43         double t = 0.0;
44         int l = 0, e = 0;
45
46         for (int i = 1; i ≤ m; i++) {
47             if (A[i][0] + eps < t) {
48                 t = A[i][0];
49                 l = i;
50             }
51
52         if (!l)
53             return true;
54
55         for (int i = 1; i ≤ n; i++) {
56             if (A[l][i] < -eps && (!e || id[i] <
57             → id[e])) {
58                 e = i;
59
60             if (!e)
61                 return false;
62
63             pivot(l, e);
64         }
65
66 // 求解
67 bool simplex() {

```

```

68     while (true) {
69         int l = 0, e = 0;
70         for (int i = 1; i ≤ n; i++)
71             if (A[0][i] > eps && (!e || id[i] < id[e]))
72                 e = i;
73
74         if (!e)
75             return true;
76
77         double t = 1e50;
78         for (int i = 1; i ≤ m; i++)
79             if (A[i][e] > eps && A[i][0] / A[i][e] < t)
80                 → {
81                     l = i;
82                     t = A[i][0] / A[i][e];
83                 }
84
85         if (!l)
86             return false;
87
88         pivot(l, e);
89     }
90
91 // 转轴操作, 本质是在凸包上沿着一条棱移动
92 void pivot(int l, int e) {
93     swap(id[e], id[n + l]);
94     double t = A[l][e];
95     A[l][e] = 1.0;
96
97     for (int i = 0; i ≤ n; i++)
98         A[l][i] /= t;
99
100    for (int i = 0; i ≤ m; i++) {
101        if (i != l) {
102            t = A[i][e];
103            A[i][e] = 0.0;
104            for (int j = 0; j ≤ n; j++)
105                A[i][j] -= t * A[l][j];
106        }
107    }

```

### 1.6.1 线性规划对偶原理

给定一个原始线性规划:

$$\text{Minimize} \quad \sum_{j=1}^n c_j x_j$$

$$\text{Subject to} \quad \sum_{j=1}^n a_{ij} x_j \geq b_i,$$

$$x_j \geq 0$$

定义它的对偶线性规划为:

$$\text{Maximize} \quad \sum_{i=1}^m b_i y_i$$

$$\text{Subject to} \quad \sum_{i=1}^m a_{ij} y_i \leq c_j,$$

$$y_i \geq 0$$

用矩阵可以更形象地表示为:

$$\begin{array}{ll} \text{Minimize} & \mathbf{c}^T \mathbf{x} \\ \text{Subject to} & \mathbf{Ax} \geq \mathbf{b}, \iff \text{Subject to} & \mathbf{A}^T \mathbf{y} \leq \mathbf{c}, \\ & \mathbf{x} \geq 0 & \mathbf{y} \geq 0 \end{array}$$

## 1.7 博弈论

### 1.7.1 SG定理

对于一个平等游戏，可以为每个状态定义一个SG函数。

一个状态的SG函数等于所有它能一步到达的状态的SG函数的mex，也就是最小的没有出现过的自然数。

那么所有先手必败态的SG函数为0，先手必胜态的SG函数非0。

如果有一个游戏，它由若干个独立的子游戏组成，且每次行动时只能选一个子游戏进行操作，则这个游戏的SG函数就是所有子游戏的SG函数的异或和。（比如最经典的Nim游戏，每次只能选一堆取若干个石子。）

同时操作多个子游戏的结论参见1.7.3.经典博弈(14页)。

### 1.7.2 纳什均衡

**纯策略，混合策略** 纯策略是指你一定会选择某个选项，混合策略是指你对每个选项都有一个概率分布 $p_i$ ，你会以相应的概率选择这个选项。

考虑这样的游戏：有几个人（当然也可以是两个）各自独立地做决定，然后同时公布每个人的决定，而每个人的收益和所有人的选择有关。

那么纳什均衡就是每个人都决定一个混合策略，使得在其他人都是纯策略的情况下，这个人最坏情况下（也就是说其他人的纯策略最针对他的时候）的收益是最大的。也就是说，收益函数对这个人的混合策略求一个偏导，结果是0（因为是极大值）。

纳什均衡点可能存在多个，不过在一个双人零和游戏中，纳什均衡点一定唯一存在。

### 1.7.3 经典博弈

#### 1. 阶梯博弈

台阶的每层都有一些石子，每次可以选一层（但不能是第0层），把任意一个石子移到低一层。

**结论** 奇数层的石子数量进行异或和即可。

实际上只要路径长度唯一就可以，比如在树上博弈，然后石子向根节点方向移动，那么就是奇数深度的石子数量进行异或和。

#### 2. 可以同时操作多个子游戏

如果某个游戏由若干个独立的子游戏组成，并且每次可以任意选几个（当然至少一个）子游戏进行操作，那么结论是：所有子游戏都必败时先手才会必败，否则先手必胜。

#### 3. 每次最多操作 $k$ 个子游戏(Nim-K)

如果每次最多操作 $k$ 个子游戏，结论是：把所有子游戏的SG函数写成二进制表示，如果每一位上的1个数都是 $(k+1)$ 的倍数，则先手必败，否则先手必胜。

（实际上上面一条可以看做 $k = \infty$ 的情况，也就是所有SG值都是0时才会先手必败。）

如果要求整个游戏的SG函数，就按照上面的方法每个二进制位相加后 $\text{mod}(k+1)$ ，视为 $(k+1)$ 进制数后求值即可。（未验证）

#### 4. 反Nim游戏(Anti-Nim)

和Nim游戏差不多，唯一的不同是取走最后一个石子的输。

分两种情况：

- 所有堆石子个数都是1：有偶数堆时先手必胜，否则先手必败。
- 存在某个堆石子数多于1：异或和不为0则先手必胜，否则先手必败。

当然石子个数实际上就是SG函数，所以判别条件全都改成SG函数也是一样的。

#### 5. 威佐夫博弈

有两堆石子，每次要么从一堆中取任意个，要么从两堆中都取走相同数量。也等价于两个人移动一个只能向左上方走的皇后，不能动的输。

**结论** 设两堆石子分别有 $a$ 个和 $b$ 个，且 $a < b$ ，则先手必败当且仅当 $a = \left\lfloor (b-a)\frac{1+\sqrt{5}}{2} \right\rfloor$ 。

#### 6. 删子树博弈

有一棵有根树，两个人轮流操作，每次可以选一个点（除了根节点）然后把它的子树都删掉，不能操作的输。

#### 结论

$$SG(u) = \text{XOR}_{v \in \text{son}_u} (SG(v) + 1)$$

#### 7. 无向图游戏

在一个无向图上的某个点上摆一个棋子，两个人轮流把棋子移动到相邻的点，并且每个点只能走一次，不能操作的输。

**结论** 如果某个点一定在最大匹配中，则先手必胜，否则先手必败。

#### 1.7.4 例题

##### 1. 黑白棋游戏

一些棋子排成一列，棋子两面分别是黑色和白色。两个人轮流动，每次可以选择一个白色朝上的棋子，把它和它左边的所有棋子都翻转，不能行动的输。

**结论** 只需要看最左边的棋子即可，因为每次操作最左边的棋子都一定会被翻转。

二维情况同理，如果每次是把左上角的棋子全部翻转，那么就只需要看左上角的那个棋子。

## 1.8 自适应Simpson积分

Forked from fstqwq's template.

```

1 // Adaptive Simpson's method : double simpson::solve
2   ↪ (double (*f)(double), double l, double r, double
3     ↪ eps) : integrates f over (l, r) with error eps.
4
5 double area (double (*f)(double), double l, double r)
6   ↪ {
7     double m = l + (r - l) / 2;
8     return (f(l) + 4 * f(m) + f(r)) * (r - l) / 6;
9   }
10
11 double solve (double (*f)(double), double l, double r,
12   ↪ double eps, double a) {
13   double m = l + (r - l) / 2;
14   double left = area(f, l, m), right = area(f, m, r);
15   if (fabs(left + right - a) ≤ 15 * eps)
16     return left + right + (left + right - a) /
17       ↪ 15.0;
18   return solve(f, l, m, eps / 2, left) + solve(f, m,
19             ↪ r, eps / 2, right);
20 }
21
22 double solve (double (*f)(double), double l, double r,
23   ↪ double eps) {
24   return solve(f, l, r, eps, area(f, l, r));
25 }
```

## 1.9 常见数列

查表参见8.15.OEIS(84页)。

### 1.9.1 斐波那契数 卢卡斯数

**斐波那契数**  $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$   
 $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$

**卢卡斯数**  $L_0 = 2, L_1 = 1$

$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, \dots$

通项公式  $\phi = \frac{1+\sqrt{5}}{2}$ ,  $\hat{\phi} = \frac{1-\sqrt{5}}{2}$

$F_n = \frac{\phi^n - \hat{\phi}^n}{\sqrt{5}}$ ,  $L_n = \phi^n + \hat{\phi}^n$

实际上有  $\frac{L_n + F_n \sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^n$ , 所以求通项的话写一个类然后快速幂就可以同时得到两者.

快速倍增法  $F_{2k} = F_k(2F_{k+1} - F_k)$ ,  $F_{2k+1} = F_{k+1}^2 + F_k^2$

```

1 pair<int, int> fib(int n) { // 返回F(n)和F(n + 1)
2     if (n == 0)
3         return {0, 1};
4     auto p = fib(n >> 1);
5     int c = p.first * (2 * p.second - p.first);
6     int d = p.first * p.first + p.second * p.second;
7     if (n & 1)
8         return {d, c + d};
9     else
10        return {c, d};
11 }
```

### 1.9.2 伯努利数, 自然数幂次和

指数生成函数:  $B(x) = \sum_{i \geq 0} \frac{B_i x^i}{i!} = \frac{x}{e^x - 1}$

$$B_n = [n=0] - \sum_{i=0}^{n-1} \binom{n}{i} \frac{B_i}{n-k+1}$$

$$\sum_{i=0}^n \binom{n+1}{i} B_i = 0$$

$$S_n(m) = \sum_{i=0}^{m-1} i^n = \sum_{i=0}^n \binom{n}{i} B_{n-i} \frac{m^{i+1}}{i+1}$$

$B_0 = 1$ ,  $B_1 = -\frac{1}{2}$ ,  $B_4 = -\frac{1}{30}$ ,  $B_6 = \frac{1}{42}$ ,  $B_8 = -\frac{1}{30}$ , ...

(除了  $B_1 = -\frac{1}{2}$  以外, 伯努利数的奇数项都是0.)

自然数幂次和关于次数的EGF:

$$\begin{aligned} F(x) &= \sum_{k=0}^{\infty} \frac{\sum_{i=0}^n i^k}{k!} x^k \\ &= \sum_{i=0}^n e^{ix} \\ &= \frac{e^{(n+1)x}-1}{e^x-1} \end{aligned}$$

### 1.9.3 分拆数

```

1 int b = sqrt(n);
2 ans[0] = tmp[0] = 1;
3
4 for (int i = 1; i <= b; ++i) {
5     for (int rep = 0; rep < 2; ++rep)
6         for (int j = i; j <= n - i * i; ++j)
7             add(tmp[j], tmp[j - i]);
8
9     for (int j = i * i; j <= n; ++j)
10        add(ans[j], tmp[j - i * i]);
11 }
12 // —
13
14 long long a[100010];
15 long long p[50005]; // 欧拉五边形数定理
16
17 int main() {
```

```

19     p[0] = 1;
20     p[1] = 1;
21     p[2] = 2;
22     int i;
23     for (i = 1; i < 50005; i++) { // 递推式系
24         a[2 * i] = i * (i * 3 - 1) / 2; // 五边形数
25         // 为1, 5, 12, 22 ... i*(3*i-1)/2
26         a[2 * i + 1] = i * (i * 3 + 1) / 2;
27     }
28     for (i = 3; i < 50005; i++) { //
29         p[n] = p[n-1] + p[n-2] - p[n-5] -
30         p[n-7] + p[12] + p[15] - ... + p[n-i*[3i-1]/2] + p[n-
31         i*[3i+1]/2]
32         p[i] = 0;
33         int j;
34         for (j = 2; a[j] <= i; j++) { // 可能为负数, 式
35             // 中加1000007
36             if (j & 2)
37                 p[i] = (p[i] + p[i - a[j]] + 1000007) %
38             // 1000007;
39             else
40                 p[i] = (p[i] - p[i - a[j]] + 1000007) %
41             // 1000007;
42     }
43     int n;
44     while (~scanf("%d", &n))
45         printf("%lld\n", p[n]);
46 }
```

### 1.9.4 斯特林数

#### 1. 第一类斯特林数

$[n]_k$  表示  $n$  个元素划分成  $k$  个轮换的方案数.

递推式:  $[n]_k = [n-1]_{k-1} + (n-1)[n-1]_k$ .

求同一行: 分治FFT  $O(n \log^2 n)$ , 或者倍增  $O(n \log n)$  (每次都是  $f(x) = g(x)g(x+d)$  的形式, 可以用  $g(x)$  反转之后做一个卷积求出后者).

$$\sum_{k=0}^n [n]_k x^k = \prod_{i=0}^{n-1} (x+i)$$

求同一列: 用一个轮换的指数生成函数做  $k$  次幂

$$\sum_{n=0}^{\infty} [n]_k \frac{x^n}{n!} = \frac{(\ln(1-x))^k}{k!} = \frac{x^k}{k!} \left( \frac{\ln(1-x)}{x} \right)^k$$

#### 2. 第二类斯特林数

$\{n\}_k$  表示  $n$  个元素划分成  $k$  个子集的方案数.

递推式:  $\{n\}_k = \{n-1\}_{k-1} + k \{n-1\}_k$ .

求一个: 容斥, 狗都会做

$$\{n\}_k = \frac{1}{k!} \sum_{i=0}^k (-1)^i \binom{k}{i} (k-i)^n = \sum_{i=0}^k \frac{(-1)^i}{i!} \frac{(k-i)^n}{(k-i)!}$$

求同一行: FFT, 狗都会做

求同一列: 指数生成函数

$$\sum_{n=0}^{\infty} \{n\}_k \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!} = \frac{x^k}{k!} \left( \frac{e^x - 1}{x} \right)^k$$

普通生成函数

$$\sum_{n=0}^{\infty} \{n\}_k x^n = x^k \left( \prod_{i=1}^k (1 - ix) \right)^{-1}$$

## 3. 斯特林反演

$$f(n) = \sum_{k=0}^n \begin{Bmatrix} n \\ k \end{Bmatrix} g(k) \iff g(n) = \sum_{k=0}^n (-1)^{n-k} \begin{Bmatrix} n \\ k \end{Bmatrix} f(k)$$

## 4. 幂的转换

上升幂与普通幂的转换

$$x^{\bar{n}} = \sum_k \begin{Bmatrix} n \\ k \end{Bmatrix} x^k$$

$$x^n = \sum_k \begin{Bmatrix} n \\ k \end{Bmatrix} (-1)^{n-k} x^{\bar{k}}$$

下降幂与普通幂的转换

$$x^n = \sum_k \begin{Bmatrix} n \\ k \end{Bmatrix} x^k = \sum_k \binom{x}{k} \begin{Bmatrix} n \\ k \end{Bmatrix} k!$$

$$x^n = \sum_k \begin{Bmatrix} n \\ k \end{Bmatrix} (-1)^{n-k} x^k$$

另外, 多项式的点值表示的每项除以阶乘之后卷上  $e^{-x}$  乘上阶乘之后是牛顿插值表示, 或者不乘阶乘就是下降幂系数表示. 反过来的转换当然卷上  $e^x$  就行了. 原理是每次差分等价于乘以  $(1-x)$ , 展开之后用一次卷积取代多次差分.

## 5. 斯特林多项式(斯特林数关于斜线的性质)

定义:

$$\sigma_n(x) = \frac{\begin{bmatrix} x \\ n \end{bmatrix}}{x(x-1)\dots(x-n)}$$

$\sigma_n(x)$  的最高次数是  $x^{n-1}$ . (所以作为唯一的特例,  $\sigma_0(x) = \frac{1}{x}$  不是多项式.)

斯特林多项式实际上非常神奇, 它与两类斯特林数都有关系.

$$\begin{Bmatrix} n \\ n-k \end{Bmatrix} = n^{k+1} \sigma_k(n)$$

$$\begin{Bmatrix} n \\ n-k \end{Bmatrix} = (-1)^{k+1} n^{k+1} \sigma_k(-(n-k))$$

不过它并不好求. 可以  $O(k^2)$  直接计算前几个点值然后插值, 或者如果要推式子的话可以用后面提到的二阶欧拉数.

## 1.9.5 贝尔数

$$B_0 = 1, B_1 = 1, B_2 = 2, B_3 = 5,$$

$$B_4 = 15, B_5 = 52, B_6 = 203, \dots$$

$$B_n = \sum_{k=0}^n \begin{Bmatrix} n \\ k \end{Bmatrix}$$

递推式:

$$B_{n+1} = \sum_{k=0}^n \binom{n}{k} B_k$$

指数生成函数:

$$B(x) = e^{e^x - 1}$$

Touchard同余:

$$B_{n+p} \equiv (B_n + B_{n+1}) \pmod{p}, p \text{ is a prime}$$

## 1.9.6 欧拉数(Eulerian Number)

## 1. 欧拉数

$\begin{Bmatrix} n \\ k \end{Bmatrix}$ :  $n$ 个数的排列, 有  $k$ 个上升的方案数.

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = (n-k) \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} + (k+1) \begin{Bmatrix} n-1 \\ k \end{Bmatrix}$$

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \sum_{i=0}^{k+1} (-1)^i \binom{n+1}{i} (k+1-i)^n$$

$$\sum_{k=0}^{n-1} \begin{Bmatrix} n \\ k \end{Bmatrix} = n!$$

$$x^n = \sum_{k=0}^{n-1} \begin{Bmatrix} n \\ k \end{Bmatrix} \binom{x+k}{n}$$

$$k! \begin{Bmatrix} n \\ k \end{Bmatrix} = \sum_{i=0}^{n-1} \begin{Bmatrix} n \\ i \end{Bmatrix} \binom{i}{n-k}$$

## 2. 二阶欧拉数

$\begin{Bmatrix} n \\ k \end{Bmatrix}$ : 每个数都出现两次的多重排列, 并且每个数两次出现之间的数都比它要大. 在此前提下有  $k$ 个上升的方案数.

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = (2n-k-1) \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} + (k+1) \begin{Bmatrix} n-1 \\ k \end{Bmatrix}$$

$$\sum_{k=0}^{n-1} \begin{Bmatrix} n \\ k \end{Bmatrix} = (2n-1)!! = \frac{(2n)^n}{2^n}$$

## 3. 二阶欧拉数与斯特林数的关系

$$\begin{Bmatrix} x \\ x-n \end{Bmatrix} = \sum_{k=0}^{n-1} \begin{Bmatrix} n \\ k \end{Bmatrix} \binom{x+n-k-1}{2n}$$

$$\begin{Bmatrix} x \\ x-n \end{Bmatrix} = \sum_{k=0}^{n-1} \begin{Bmatrix} n \\ k \end{Bmatrix} \binom{x+k}{2n}$$

## 1.9.7 卡特兰数, 施罗德数, 默慈金数

## 1. 卡特兰数

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}$$

- $n$ 个元素按顺序入栈, 出栈序列方案数
- 长为  $2n$  的合法括号序列数
- $n+1$  个叶子的满二叉树个数

递推式:

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}$$

$$C_n = C_{n-1} \frac{4n-2}{n+1}$$

普通生成函数:

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$$

扩展: 如果有  $n$  个左括号和  $m$  个右括号, 方案数为

$$\binom{n+m}{n} - \binom{n+m}{m-1}$$

## 2. 施罗德数

$$S_n = S_{n-1} + \sum_{i=0}^{n-1} S_i S_{n-i-1}$$

$$(n+1)s_n = (6n-3)s_{n-1} - (n-2)s_{n-2}$$

其中  $S_n$  是(大)施罗德数,  $s_n$  是小施罗德数(也叫超级卡特兰数).

除了  $S_0 = s_0 = 1$  以外, 都有  $S_i = 2s_i$ .

施罗德数的组合意义:

- 从  $(0, 0)$  走到  $(n, n)$ , 每次可以走右, 上, 或者右上一步, 并且不能超过  $y = x$  这条线的方案数
  - 长为  $n$  的括号序列, 每个位置也可以为空, 并且括号对数和空位置数加起来等于  $n$  的方案数
  - 凸  $n$  边形的任意剖分方案数
- (有些人会把大(而不是小)施罗德数叫做超级卡特兰数.)

## 3. 默慈金数

$$M_{n+1} = M_n + \sum_{i=0}^{n-1} M_i M_{n-1-i} = \frac{(2n+3)M_n + 3nM_{n-1}}{n+3}$$

$$M_n = \sum_{i=0}^{\frac{n}{2}} \binom{n}{2i} C_i$$

在圆上的  $n$  个不同的点之间画任意条不相交(包括端点)的弦的方案数.

也等价于在网格图上, 每次可以走右上, 右下, 正右方一步, 且不能走到  $y < 0$  的位置, 在此前提下从  $(0, 0)$  走到  $(n, 0)$  的方案数.

扩展: 默慈金数画的弦不可以共享端点. 如果可以共享端点的话是 A054726, 后面的表里可以查到.

## 1.10 常用公式及结论

## 1.10.1 方差

$m$  个数的方差:

$$s^2 = \frac{\sum_{i=1}^m x_i^2}{m} - \bar{x}^2$$

随机变量的方差:  $D^2(x) = E(x^2) - E^2(x)$

## 1.10.2 min-max 反演

$$\max(S) = \sum_{T \subset S} (-1)^{|T|+1} \min(T)$$

$$\min(S) = \sum_{T \subset S} (-1)^{|T|+1} \max(T)$$

推广: 求第  $k$  大

$$k\text{-}\max(S) = \sum_{T \subset S} (-1)^{|T|-k} \binom{|T|-1}{k-1} \min(T)$$

显然只有大小至少为  $k$  的子集才是有用的.

1.10.3 单位根反演(展开整除条件  $[n|k]$ )

$$[n|k] = \frac{1}{n} \sum_{i=0}^{n-1} \omega_n^{ik}$$

$$\sum_{i \geq 0} [x^{ik}] f(x) = \frac{1}{k} \sum_{j=0}^{k-1} f(\omega_k^j)$$

## 1.10.4 康托展开(排列的排名)

求排列的排名: 先对每个数都求出它后面有几个数比它小(可以用树状数组预处理), 记为  $c_i$ , 则排列的排名就是

$$\sum_{i=1}^n c_i (n-i)!$$

已知排名构造排列: 从前到后先分别求出  $c_i$ , 有了  $c_i$  之后再用一个平衡树(需要维护排名)倒序处理即可.

## 1.10.5 连通图计数

设大小为  $n$  的满足一个限制  $P$  的简单无向图数量为  $g_n$ , 满足限制  $P$  且连通的简单无向图数量为  $f_n$ , 如果已知  $g_{1 \dots n}$  求  $f_n$ , 可以得到递推式

$$f_n = g_n - \sum_{k=1}^{n-1} \binom{n-1}{k-1} f_k g_{n-k}$$

这个递推式的意义就是用任意图的数量减掉不连通的数量, 而不连通的数量可以通过枚举 1 号点所在连通块大小来计算.

注意, 由于  $f_0 = 0$ , 因此递推式的枚举下界取 0 和 1 都是可以的. 推一推式子会发现得到一个多项式求逆, 再仔细看看, 其实就是一个多项式  $\ln$ .

## 1.10.6 线性齐次线性常系数递推求通项

- 定理 3.1: 设数列  $\{u_n : n \geq 0\}$  满足  $r$  阶齐次线性常系数递推关系  $u_n = \sum_{j=1}^r c_j u_{n-j}$  ( $n \geq r$ ). 则

$$(i). \quad U(x) = \sum_{n \geq 0} u_n x^n = \frac{h(x)}{1 - \sum_{j=1}^r c_j x^j}, \quad \deg(h(x)) < r.$$

(ii). 若特征多项式

$$c(x) = x^r - \sum_{j=1}^r c_j x^{r-j} = (x - \alpha_1)^{e_1} \cdots (x - \alpha_s)^{e_s},$$

其中  $\alpha_1, \dots, \alpha_s$  互异,  $e_1 + \dots + e_s = r$  则  $u_n$  有表达式

$$u_n = p_1(n)\alpha_1^n + \dots + p_s(n)\alpha_s^n, \quad \deg(p_i) < e_i, i = 1, \dots, s.$$

多项式  $p_1, \dots, p_s$  的共  $e_1 + \dots + e_s = r$  个系数可由初始值  $u_0, \dots, u_{r-1}$  唯一确定.



## 1.11 常用生成函数变换

$$\frac{x}{(1-x)^2} = \sum_{i \geq 0} i x^i$$

$$\frac{1}{(1-x)^k} = \sum_{i \geq 0} \binom{i+k-1}{i} x^i = \sum_{i \geq 0} \binom{i+k-1}{k-1} x^i, \quad k > 0$$

$$\begin{aligned} \sum_{i=0}^{\infty} i^n x^i &= \sum_{k=0}^n \binom{n}{k} k! \frac{x^k}{(1-x)^{k+1}} = \sum_{k=0}^n \binom{n}{k} k! \frac{x^k (1-x)^{n-k}}{(1-x)^{n+1}} \\ &= \frac{1}{(1-x)^{n+1}} \sum_{i=0}^n \frac{x^i}{(n-i)!} \sum_{k=0}^i \binom{n}{k} k! (n-k)! \frac{(-1)^{i-k}}{(i-k)!} \end{aligned}$$

(用上面的方法可以把分子化成一个  $n$  次以内的多项式, 并且可以用一次卷积求出来.)

如果把  $i^n$  换成任意的一个  $n$  次多项式, 那么我们可以求出它的下降幂表示形式(或者说是牛顿插值)的系数  $r_i$ , 发现用  $r_k$  替换掉上面的  $\binom{n}{k} k!$  之后其余过程完全相同.

2 数论

## 2.1 $O(n)$ 预处理逆元

```
// 要求p为质数
1 inv[0] = inv[1] = 1;
2 for (int i = 2; i <= n; i++)
3     inv[i] = (long long)(p - (p / i)) * inv[p % i] % p;
4     → // p为模数
5 // i ^ -1 = -(p / i) * (p % i) ^ -1
```

## 2.2 线性筛

```

1 // 此代码以计算约数之和函数\sigma_1(对10^9+7取模)为例
2 // 适用于任何f(p^k)便于计算的积性函数
3 constexpr int p = 1000000007;
4
5 int prime[maxn / 10], sigma_one[maxn], f[maxn],
6    → g[maxn];
7 // f: 除掉最小质因子后剩下的部分
8 // g: 最小质因子的幂次, 在f(p^k)比较复杂时很有用,
9 // → 但f(p^k)可以递推时就可以省略了
10 // 这里没有记录最小质因子, 但根据线性筛的性质, 每个合数只
11 // → 会被它最小的质因子筛掉
12 bool notp[maxn]; // 顾名思义
13
14 void get_table(int n) {
15     sigma_one[1] = 1; // 积性函数必有f(1) = 1
16     for (int i = 2; i ≤ n; i++) {
17         if (!notp[i]) { // 质数情况
18             prime[++prime[0]] = i;
19             sigma_one[i] = i + 1;
20             f[i] = g[i] = 1;
21         }
22
23         for (int j = 1; j ≤ prime[0] && i * prime[j]
24             → ≤ n; j++) {
25             notp[i * prime[j]] = true;
26
27             if (i % prime[j]) { // 加入一个新的质因子,
28                 → 这种情况很简单
29                 sigma_one[i * prime[j]] = (long
30                     → long)sigma_one[i] * (prime[j] + 1)
31                     → % p;
32                 f[i * prime[j]] = i;
33                 g[i * prime[j]] = 1;
34             }
35             else { // 再加入一次最小质因子, 需要再进行分类
36                 → 讨论
37                 f[i * prime[j]] = f[i];
38                 g[i * prime[j]] = g[i] + 1;
39                 // 对于f(p^k)可以直接递推的函数, 这里的判
40                 → 断可以改成
41                 // i / prime[j] % prime[j] != 0, 这样可
42                 → 以省下f[]的空间,
43                 // 但常数很可能会稍大一些
44
45                 if (f[i] == 1) // 质数的幂次, 这
46                 → 里\sigma_1可以递推
47                 sigma_one[i * prime[j]] =
48                     → (sigma_one[i] + i * prime[j]) %
49                     → p;
50                 // 对于更一般的情况, 可以借助g[]计
51                 → 算f(p^k)
52             }
53             else sigma_one[i * prime[j]] = // 否则直
54                 → 接利用积性, 两半乘起来
55                 (long long)sigma_one[i * prime[j]] /
56                     → f[i] * sigma_one[f[i]] % p;
57         }
58     }
59 }

```

```
40 |  
41 |  
42 |  
43 |  
44 | } break
```

## 2.3 杜教筛

$$S_\varphi(n) = \frac{n(n+1)}{2} - \sum_{d=2}^n S_\varphi\left(\left\lfloor \frac{n}{d} \right\rfloor\right)$$

$$S_\mu(n) = 1 - \sum_{d=2}^n S_\mu\left(\left\lfloor \frac{n}{d} \right\rfloor\right)$$

```

1 // 用于求可以用狄利克雷卷积构造出好求和的东西的函数的前缀
2 // → 和(有点绕)
3 // 有些题只要求  $n \leq 10^9$ , 这时就没必要开long long了, 但
4 // → 记得乘法时强转
5 // 常量/全局变量/数组定义
6 const int maxn = 5000000, table_size = 5000000, p =
7 // → 1000000007, inv_2 = (p + 1) / 2;
8 bool notp[maxn];
9 int prime[maxn / 20], phi[maxn], tbl[100005];
10 // tbl用来顶替哈希表, 其实开到  $n^{1/3}$  就够了, 不过保险
11 // → 起见开成  $\sqrt{n}$  比较好
12 long long N;
13
14 // 主函数前面加上这么一句
15 memset(tbl, -1, sizeof(tbl));
16
17 // 线性筛预处理部分略去
18
19 // 杜教筛主过程 总计  $O(n^{2/3})$ 
20 // 递归调用自身
21 // 递推式还需具体情况具体分析, 这里以求欧拉函数前缀和(mod
22 // →  $10^9 + 7$ )为例
23 int S(long long n) {
24     if (n <= table_size)
25         return phi[n];
26     else if (~tbl[N / n])
27         return tbl[N / n];
28     // 原理: n除以所有可能的数的结果一定互不相同
29
30     int ans = 0;
31     for (long long i = 2, last; i <= n; i = last + 1) {
32         last = n / (n / i);
33         ans = (ans + (last - i + 1) % p * S(n / i)) %
34             → p; // 如果n是int范围的话记得强转
35     }
36
37     ans = (n % p * ((n + 1) % p) % p * inv_2 - ans + p)
38     → % p; // 同上
39     return tbl[N / n] = ans;
40 }

```

## 2.4 Powerful Number筛

注意 Powerful Number 筛只能求积性函数的前缀和。  
本质上就是构造一个方便求前缀和的函数，然后做类似杜教筛的操作。

定义Powerful Number表示每个质因子幂次都大于1的数，显然最多有 $\sqrt{n}$ 个。

设我们要求和的函数是 $f(n)$ , 构造一个方便求前缀和的积性函数 $g(n)$ 使得 $g(p) = f(p)$ .

那么就存在一个积性函数  $h = f * g^{-1}$ , 也就是  $f = g * h$ . 可以证明  $h(p) = 0$ , 所以只有 Powerful Number 的  $h$  值不为 0.

$$S_f(i) = \sum_{d=1}^n h(d) S_g\left(\left\lfloor \frac{n}{d} \right\rfloor\right)$$

只需要枚举每个 Powerful Number 作为  $d$ , 然后用杜教筛计算  $g$  的前缀和.

求  $h(d)$  时要先预处理  $h(p^k)$ , 显然有

$$h(p^k) = f(p^k) - \sum_{i=1}^k g(p^i) h(p^{k-i})$$

处理完之后 DFS 就行了. (显然只需要筛  $\sqrt{n}$  以内的质数.)

复杂度取决于杜教筛的复杂度, 特殊题目构造的好也可以做到  $O(\sqrt{n})$ .

例题:

- $f(p^k) = p^k(p^k - 1)$  :  $g(n) = \text{id}(n)\varphi(n)$ .
- $f(p^k) = p \text{xor } k$  :  $n$  为偶数时  $g(n) = 3\varphi(n)$ , 否则  $g(n) = \varphi(n)$ .

## 2.5 洲阁筛

计算积性函数  $f(n)$  的前  $n$  项之和时, 我们可以把所有项按照是否有  $> \sqrt{n}$  的质因子分两类讨论, 最后将两部分的贡献加起来即可.

### 1. 有 $> \sqrt{n}$ 的质因子

显然  $> \sqrt{n}$  的质因子幂次最多为 1, 所以这一部分的贡献就是

$$\sum_{i=1}^{\sqrt{n}} f(i) \sum_{d=\sqrt{n}+1}^{\lfloor \frac{n}{i} \rfloor} [d \in \mathbb{P}] f(d)$$

我们可以 DP 后面的和式. 由于  $f(p)$  是一个关于  $p$  的低次多项式, 我们可以对每个次幂分别 DP: 设  $g_{i,j}$  表示  $[1, j]$  中和前  $i$  个质数都互质的数的  $k$  次方之和. 设  $\sqrt{n}$  以内的质数总共有  $m$  个, 显然贡献就转换成了

$$\sum_{i=1}^{\sqrt{n}} i^k g_{m, \lfloor \frac{n}{i} \rfloor}$$

边界显然就是自然数幂次和, 转移是

$$g_{i,j} = g_{i-1,j} - p_i^k g_{i-1, \lfloor \frac{j}{p_i} \rfloor}$$

也就是减掉和第  $i$  个质数不互质的贡献.

在滚动数组的基础上再优化一下: 首先如果  $j < p_i$  那肯定就只有一个数; 如果  $p_i \leq j < p_i^2$ , 显然就有  $g_{i,j} = g_{i-1,j} - p_i^k$ , 那么对每个  $j$  记下最大的  $i$  使得  $p_i^2 \leq j$ , 比这个还大的情况就不需要递推了, 用到的时候再加上一个前缀和解决.

### 2. 所有质因子都 $\leq \sqrt{n}$

类似的道理, 我们继续 DP:  $h_{i,j}$  表示只含有第  $i$  到  $m$  个质数作为质因子的所有数的  $f(i)$  之和. (这里不需要对每个次幂单独 DP 了; 另外倒着 DP 是为了方便卡上限.)

边界显然是  $h_{m+1,j} = 1$ , 转移是

$$h_{i,j} = h_{i+1,j} + \sum_c f(p_i^c) h_{i+1, \lfloor \frac{j}{p_i^c} \rfloor}$$

跟上面一样的道理优化, 分成三段:  $j < p_i$  时  $h_{i,j} = 1$ ,  $j < p_i^2$  时  $h_{i,j} = h_{i+1,j} + f(p_i)$  (同样用前缀和解决), 再小的部分就老实递推.

预处理  $\sqrt{n}$  以内的部分之后跑两次 DP, 最后把两部分的贡献加起来就行了.

两部分的复杂度都是  $\Theta\left(\frac{n^{\frac{3}{4}}}{\log n}\right)$  的.

以下代码以洛谷 P5325 ( $f(p^k) = p^k(p^k - 1)$ ) 为例.

```

1 constexpr int maxn = 200005, p = 1000000007;
2
3 long long N, val[maxn]; // 询问的n和存储所有整除结果的表
4 int sqrtn;
5
6 inline int getid(long long x) {
7     if (x <= sqrtn)
8         return x;
9
10    return val[0] - N / x + 1;
11}
12
13 bool notp[maxn];
14 int prime[maxn], prime_cnt, rem[maxn]; // 线性筛用数组
15
16 int f[maxn], pr[maxn], g[2][maxn], dp[maxn];
17 int l[maxn], r[maxn];
18
19 // 线性筛省略
20
21 inline int get_sum(long long n, int k) {
22     n %= p;
23
24     if (k == 1)
25         return n * (n + 1) % p * ((p + 1) / 2) % p;
26
27     else
28         return n * (n + 1) % p * (2 * n + 1) % p * ((p
29             - 1) / 6) % p;
30 }
31
32 void get_dp(long long n, int k, int *dp) {
33     for (int j = 1; j <= val[0]; j++)
34         dp[j] = get_sum(val[j], k);
35
36     for (int i = 1; i <= prime_cnt; i++) {
37         long long lb = (long long)prime[i] * prime[i];
38         int pw = (k == 1 ? prime[i] : (int)(lb % p));
39
40         pr[i] = (pr[i - 1] + pw) % p;
41
42         for (int j = val[0]; j && val[j] >= lb; j--) {
43             int t = getid(val[j] / prime[i]);
44
45             int tmp = dp[t];
46             if (l[t] < i)
47                 tmp = (tmp - pr[min(i - 1, r[t])] +
48                     pr[l[t]]) % p;
49
50             dp[j] = (dp[j] - (long long)pw * tmp) % p;
51             if (dp[j] < 0)
52                 dp[j] += p;
53         }
54
55         for (int j = 1; j <= val[0]; j++)
56             dp[j] = (dp[j] - pr[r[j]] + pr[l[j]]) % p;
57
58         dp[j] = (dp[j] + p - 1) % p; // 因为DP数组是
59             // 有1的, 但后面计算不应该有1
60     }
61
62     int calc1(long long n) {
63         get_dp(n, 1, g[0]);
64         get_dp(n, 2, g[1]);
65
66         int ans = 0;

```

```

67     for (int i = 1; i <= sqrtN; i++)
68         ans = (ans + (long long)f[i] * (g[1][getid(N /
69             → i)] - g[0][getid(N / i)])) % p;
70
71     if (ans < 0)
72         ans += p;
73
74     return ans;
75 }
76
77 int calc2(long long n) {
78     for (int j = 1; j <= val[0]; j++)
79         dp[j] = 1;
80
81     for (int i = 1; i <= prime_cnt; i++)
82         pr[i] = (pr[i - 1] + f[prime[i]]) % p;
83
84     for (int i = prime_cnt; i; i--) {
85         long long lb = (long long)prime[i] * prime[i];
86
87         for (int j = val[0]; j && val[j] ≥ lb; j--)
88             for (long long pc = prime[i]; pc ≤ val[j];
89                 → pc *= prime[i]) {
90                 int t = getid(val[j] / pc);
91
92                 int tmp = dp[t];
93                 if (r[t] > i)
94                     tmp = (tmp + pr[r[t]] - pr[max(i,
95                         → l[t])]) % p;
96
97                 dp[j] = (dp[j] + pc % p * ((pc - 1) %
98                     → p) % p * tmp) % p;
99             }
100
101     }
102
103     return (long long)(dp[val[0]] + pr[r[val[0]]] -
104         → pr[l[val[0]]] + p) % p;
105 }
106
107 int main() {
108
109     // ios::sync_with_stdio(false);
110
111     cin >> N;
112
113     sqrtN = (int)sqrt(N);
114
115     get_table(sqrtN);
116
117     for (int i = 1; i <= sqrtN; i++)
118         val[++val[0]] = i;
119
120     for (int i = 1; i <= sqrtN; i++)
121         val[++val[0]] = N / i;
122
123     sort(val + 1, val + val[0] + 1);
124
125     val[0] = unique(val + 1, val + val[0] + 1) - val -
126         → 1;
127
128     int li = 0, ri = 0;
129     for (int j = 1; j <= val[0]; j++) {
130         while (ri < prime_cnt && prime[ri + 1] ≤
131             → val[j])
132             ri++;
133
134         while (li < prime_cnt && (long long)prime[li]
135             → * prime[li] ≤ val[j])
136             li++;
137     }
138 }
```

```

129     l[j] = li - 1;
130     r[j] = ri;
131 }
132
133 cout << (calc1(N) + calc2(N)) % p << endl;
134
135 }
136 }
```

## 2.6 Miller-Rabin

```

1 // 复杂度可以认为是常数
2
3 // 封装好的函数体
4 // 需要调用check
5 bool Miller_Rabin(long long n) {
6     if (n == 1)
7         return false;
8     if (n == 2)
9         return true;
10    if (n % 2 == 0)
11        return false;
12
13    for (int i : {2, 3, 5, 7, 11, 13, 17, 19, 23, 29,
14        → 31, 37}) {
15        if (i ≥ n)
16            break;
17        if (!check(n, i))
18            return false;
19    }
20
21    return true;
22}
23
24 // 用一个数检测
25 // 需要调用long long快速幂和O(1)快速乘
26 bool check(long long n, long long b) { // b: base
27     long long a = n - 1;
28     int k = 0;
29
30     while (a % 2 == 0) {
31         a /= 2;
32         k++;
33     }
34
35     long long t = qpow(b, a, n); // 这里的快速幂函数需要
36         → 写O(1)快速乘
37     if (t == 1 || t == n - 1)
38         return true;
39
40     while (k--) {
41         t = mul(t, t, n); // mul是O(1)快速乘函数
42         if (t == n - 1)
43             return true;
44     }
45 }
```

## 2.7 Pollard's Rho

```

1 // 注意, 虽然Pollard's Rho的理论复杂度是O( $n^{1/4}$ )的,
2 // 但实际跑起来比较慢, 一般用于做long long范围内的质因数
3 // 分解
4
5 // 封装好的函数体
6 // 需要调用solve
```

```

7 void factorize(long long n, vector<long long> &v) { // 
8     → v用于存分解出来的质因子，重复的会放多个
9     for (int i : {2, 3, 5, 7, 11, 13, 17, 19}) {
10        while (n % i == 0) {
11            v.push_back(i);
12            n /= i;
13        }
14    }
15    solve(n, v);
16    sort(v.begin(), v.end()); // 从小到大排序后返回
17}
18
19 // 递归过程
20 // 需要调用Pollard's Rho主过程，同时递归调用自身
21 void solve(long long n, vector<long long> &v) {
22     if (n == 1)
23         return;
24
25     long long p;
26     do
27         p = Pollards_Rho(n);
28     while (!p); // p是任意一个非平凡因子
29
30     if (p == n) {
31         v.push_back(p); // 说明n本身就是质数
32         return;
33     }
34
35     solve(p, v); // 递归分解两半
36     solve(n / p, v);
37
38 // Pollard's Rho主过程
39 // 需要使用Miller-Rabin作为子算法
40 // 同时需要调用O(1)快速乘和gcd函数
41 long long Pollards_Rho(long long n) {
42     // assert(n > 1);
43
44     if (Miller_Rabin(n))
45         return n;
46
47     long long c = rand() % (n - 2) + 1, i = 1, k = 2, x
48     → = rand() % (n - 3) + 2, u = 2; // 注意这里rand函
49     → 数需要重定义一下
50     while (true) {
51         i++;
52         x = (mul(x, x, n) + c) % n; // mul是O(1)快速乘函
53         → 数
54
55         long long g = gcd((u - x + n) % n, n);
56         if (g > 1 && g < n)
57             return g;
58
59         if (u == x)
60             return 0; // 失败，需要重新调用
61
62         if (i == k) {
63             u = x;
64             k *= 2;
65         }
66     }
67 }

```

## 2.8 快速阶乘算法

参见1.1.11.应用:  $O(\sqrt{n} \log^2 n)$  快速求阶乘(9页).

## 2.9 扩展欧几里德

```

1 void exgcd(LL a, LL b, LL &c, LL &x, LL &y) {
2     if (b == 0) {
3         c = a;
4         x = 1;
5         y = 0;
6         return;
7     }
8
9     exgcd(b, a % b, c, x, y);
10
11    LL tmp = x;
12    x = y;
13    y = tmp - (a / b) * y;

```

### 2.9.1 求通解的方法

假设我们已经找到了一组解  $(p_0, q_0)$  满足  $ap_0 + bq_0 = \gcd(a, b)$ , 那么其他的解都满足

$$p = p_0 + \frac{b}{\gcd(p, q)} \times t \quad q = q_0 - \frac{a}{\gcd(p, q)} \times t$$

其中  $t$  为任意整数.

### 2.9.2 类欧几里德算法(直线下整点个数)

$a, b \geq 0, m > 0$ , 计算  $\sum_{i=0}^{n-1} \lfloor \frac{a+bi}{m} \rfloor$ .

```

1 int solve(int n, int a, int b, int m) {
2     if (!b)
3         return n * (a / m);
4     if (a ≥ m)
5         return n * (a / m) + solve(n, a % m, b, m);
6     if (b ≥ m)
7         return (n - 1) * n / 2 * (b / m) + solve(n, a,
8             → b % m, m);
9
10    return solve((a + b * n) / m, (a + b * n) % m, m,
11        → b);

```

## 2.10 中国剩余定理

$$x \equiv a_i \pmod{m_i}$$

$$M = \prod_i m_i, M_i = \frac{M}{m_i}$$

$$M'_i \equiv M_i^{-1} \pmod{m_i}$$

$$x \equiv \sum_i a_i M_i M'_i \pmod{M}$$

### 2.10.1 ex-CRT

设两个方程分别是  $x \equiv a_1 \pmod{m_1}$  和  $x \equiv a_2 \pmod{m_2}$ .

将它们转化为不定方程  $x = m_1 p + a_1 = m_2 q + a_2$ , 其中  $p, q$  是整数, 则有  $m_1 p - m_2 q = a_2 - a_1$ .

当  $a_2 - a_1$  不能被  $\gcd(m_1, m_2)$  整除时无解, 否则可以通过扩展欧几里德解出来一组可行解  $(p, q)$ .

则原来的两方程组成的模方程组的解为  $x \equiv b \pmod{M}$ , 其中  $b = m_1 p + a_1$ ,  $M = \text{lcm}(m_1, m_2)$ .

## 2.11 原根阶

**阶** 最小的整数  $k$  使得  $a^k \equiv 1 \pmod{p}$ , 记为  $\delta_p(a)$ .

显然  $a$  在阶以下的幂次是两两不同的.

一个性质: 如果  $a, b$  均与  $p$  互质, 则  $\delta_p(ab) = \delta_p(a)\delta_p(b)$  的充分必要条件是  $\gcd(\delta_p(a), \delta_p(b)) = 1$ .

另外, 如果  $a$  与  $p$  互质, 则有  $\delta_p(a^k) = \frac{\delta_p(a)}{\gcd(\delta_p(a), k)}$ . (也就是环上一次跳  $k$  步的周期.)

**原根** 阶等于  $\varphi(p)$  的数.

只有形如  $2, 4, p^k, 2p^k$  ( $p$  是奇素数) 的数才有原根, 并且如果一个数  $n$  有原根, 那么原根的个数是  $\varphi(\varphi(n))$  个.

暴力找原根代码:

```

1 def split(n): # 分解质因数
2     i = 2
3     a = []
4     while i * i <= n:
5         if n % i == 0:
6             a.append(i)
7
8             while n % i == 0:
9                 n /= i
10
11            i += 1
12
13        if n > 1:
14            a.append(n)
15
16    return a
17
18 def getg(p): # 找原根
19     def judge(g):
20         for i in d:
21             if pow(g, (p - 1) / i, p) == 1:
22                 return False
23         return True
24
25     d = split(p - 1)
26     g = 2
27
28     while not judge(g):
29         g += 1
30
31     return g
32
33 print(getg(int(input())))

```

## 2.12 常用数论公式

### 2.12.1 莫比乌斯反演

$$f(n) = \sum_{d|n} g(d) \Leftrightarrow g(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d)$$

$$f(d) = \sum_{d|k} g(k) \Leftrightarrow g(d) = \sum_{d|k} \mu\left(\frac{k}{d}\right) f(k)$$

### 2.12.2 降幂公式

$$a^k \equiv a^{k \bmod \varphi(p)+\varphi(p)}, k \geq \varphi(p)$$

### 2.12.3 其他常用公式

$$\mu * I = e \quad (e(n) = [n = 1])$$

$$\mu * id = \varphi$$

$$\sigma_0 = I * I, \sigma_1 = id * I, \sigma_k = id^{k-1} * I$$

$$\sum_{i=1}^n [(i, n) = 1] i = n \frac{\varphi(n) + e(n)}{2}$$

$$\sum_{i=1}^n \sum_{j=1}^i [(i, j) = d] = S_\varphi\left(\left\lfloor \frac{n}{d} \right\rfloor\right)$$

$$\sum_{i=1}^n \sum_{j=1}^m [(i, j) = d] = \sum_{d|k} \mu\left(\frac{k}{d}\right) \left\lfloor \frac{n}{k} \right\rfloor \left\lfloor \frac{m}{k} \right\rfloor$$

$$\sum_{i=1}^n f(i) \sum_{j=1}^{\lfloor \frac{n}{i} \rfloor} g(j) = \sum_{i=1}^n g(i) \sum_{j=1}^{\lfloor \frac{n}{i} \rfloor} f(j)$$

# 3 图论

## 3.1 最小生成树

### 3.1.1 Boruvka算法

思想: 每次选择连接每个连通块的最小边, 把连通块缩起来.

每次连通块个数至少减半, 所以迭代 $O(\log n)$ 次即可得到最小生成树.

一种比较简单的实现方法: 每次迭代遍历所有边, 用并查集维护连通性和每个连通块的最小边权.

应用: 最小异或生成树

### 3.1.2 动态最小生成树

动态最小生成树的离线算法比较容易, 而在线算法通常极为复杂.

一个跑得比较快的离线做法是对时间分治, 在每层分治时找出一定在不在MST上的边, 只带着不确定边继续递归.

简单起见, 找确定边的过程用Kruskal算法实现, 过程中的两种重要操作如下:

- Reduction: 待修改边标为 $+\infty$ , 跑MST后把非树边删掉, 减少无用边
- Contraction: 待修改边标为 $-\infty$ , 跑MST后缩除待修改边之外的所有MST边, 计算必须边

每轮分治需要Reduction-Contraction, 借此减少不确定边, 从而保证复杂度.

复杂度证明: 假设当前区间有 $k$ 条待修改边,  $n$ 和 $m$ 表示点数和边数, 那么最坏情况下R-C的效果为 $(n, m) \rightarrow (n, n + k - 1) \rightarrow (k + 1, 2k)$ .

```

1 // 全局结构体与数组定义
2 struct edge { // 边的定义
3     int u, v, w, id; // id表示边在原图中的编号
4     bool vis; // 在Kruskal时用, 记录这条边是否是树边
5     bool operator < (const edge &e) const { return w <
6         → e.w; }
7 } e[20][maxn], t[maxn]; // 为了便于回滚, 在每层分治存一个
8 → 副本
9
10 // 用于存储修改的结构体, 表示第id条边的权值从u修改为v
11 struct A {
12     int id, u, v;
13 } a[maxn];
14
15 int id[20][maxn]; // 每条边在当前图中的编号
16 int p[maxn], size[maxn], stk[maxn], top; // p和size是并
17 → 查集数组, stk是用来撤销的栈
18 int n, m, q; // 点数, 边数, 修改数
19
20 // 方便起见, 附上可能需要用到的预处理代码
21 int main() {
22     for (int i = 1; i ≤ n; i++) { // 并查集初始化
23         p[i] = i;
24         size[i] = 1;
25     }
26
27     for (int i = 1; i ≤ m; i++) { // 读入与预标号
28         scanf("%d%d%d", &e[0][i].u, &e[0][i].v, &e[0]
29             → [i].w);
30         e[0][i].id = i;
31         id[0][i] = i;
32     }
33
34     for (int i = 1; i ≤ q; i++) { // 预处理出调用数组
35         scanf("%d%d", &a[i].id, &a[i].v);
36     }
37 }
```

```

35     a[i].u = e[0][a[i].id].w;
36     e[0][a[i].id].w = a[i].v;
37 }
38
39 for(int i = q; i; i--)
40     e[0][a[i].id].w = a[i].u;
41
42 CDQ(1, q, 0, m, 0); // 这是调用方法
43 }

44 // 分治主过程 O(nlog^2n)
45 // 需要调用Reduction和Contraction
46 void CDQ(int l, int r, int d, int m, long long ans) {
47     ← // CDQ分治
48     if (l == r) { // 区间长度已减小到1, 输出答案, 退出
49         e[d][id[d][a[l].id]].w = a[l].v;
50         printf("%lld\n", ans + Kruskal(m, e[d]));
51         e[d][id[d][a[l].id]].w = a[l].u;
52         return;
53     }
54
55     int tmp = top;
56
57     Reduction(l, r, d, m);
58     ans += Contraction(l, r, d, m); // R-C
59
60     int mid = (l + r) / 2;
61
62     copy(e[d] + 1, e[d] + m + 1, e[d + 1] + 1);
63     for (int i = 1; i ≤ m; i++)
64         id[d + 1][e[d][i].id] = i; // 准备好下一层要用的
65 → 数组
66
67     CDQ(l, mid, d + 1, m, ans);
68
69     for (int i = l; i ≤ mid; i++)
70         e[d][id[d][a[i].id]].w = a[i].v; // 进行左边的修
71 → 改
72
73     copy(e[d] + 1, e[d] + m + 1, e[d + 1] + 1);
74     for (int i = 1; i ≤ m; i++)
75         id[d + 1][e[d][i].id] = i; // 重新准备下一层要用
76 → 的数组
77
78     CDQ(mid + 1, r, d + 1, m, ans);
79
80     for (int i = top; i > tmp; i--)
81         cut(stk[i]); // 撤销所有操作
82     top = tmp;
83
84 // Reduction(减少无用边): 待修改边标为 $+\infty$ , 跑MST后把非树边
85 → 删掉, 减少无用边
86 // 需要调用Kruskal
87 void Reduction(int l, int r, int d, int &m) {
88     for (int i = l; i ≤ r; i++)
89         e[d][id[d][a[i].id]].w = INF; // 待修改的边标为INF
90
91     Kruskal(m, e[d]);
92
93     copy(e[d] + 1, e[d] + m + 1, t + 1);
94
95     int cnt = 0;
96     for (int i = 1; i ≤ m; i++)
97         if (t[i].w == INF || t[i].vis) { // 非树边扔掉
98             id[d][t[i].id] = ++cnt; // 给边重新编号
99             e[d][cnt] = t[i];
99     }
99 }
```

```

100
101    for (int i = r; i >= l; i--) {
102        e[d][id[d][a[i].id]].w = a[i].u; // 把待修改的边
103        ↪ 改回去
104    }
105
106
107
108 // Contraction(缩必须边):待修改边标为- $\text{INF}$ ,跑MST后缩除待修
109 → 改边之外的所有树边
110 // 返回缩掉的边的总权值
111 // 需要调用Kruskal
112 long long Contraction(int l, int r, int d, int &m) {
113     long long ans = 0;
114
115     for (int i = l; i <= r; i++) {
116         e[d][id[d][a[i].id]].w = - $\text{INF}$ ; // 待修改边标
117         ↪ 为- $\text{INF}$ 
118
119         Kruskal(m, e[d]);
120         copy(e[d] + 1, e[d] + m + 1, t + 1);
121
122         int cnt = 0;
123         for (int i = 1; i <= m; i++) {
124
125             if (t[i].w != - $\text{INF}$  && t[i].vis) { // 必须边
126                 ans += t[i].w;
127                 mergeset(t[i].u, t[i].v);
128             }
129             else { // 不确定边
130                 id[d][t[i].id] = ++cnt;
131                 e[d][cnt] = t[i];
132             }
133
134         for (int i = r; i >= l; i--) {
135             e[d][id[d][a[i].id]].w = a[i].u; // 把待修改的边
136             ↪ 改回去
137             e[d][id[d][a[i].id]].vis = false;
138
139         m = cnt;
140
141         return ans;
142
143
144 // Kruskal算法  $O(m \log n)$ 
145 // 方便起见,这里直接沿用进行过缩点的并查集,在过程结束后撤
146 → 销即可
147 long long Kruskal(int m, edge *e) {
148     int tmp = top;
149     long long ans = 0;
150
151     sort(e + 1, e + m + 1); // 比较函数在结构体中定义过
152     ↪ 了
153
154     for (int i = 1; i <= m; i++) {
155         if (findroot(e[i].u) != findroot(e[i].v)) {
156             e[i].vis = true;
157             ans += e[i].w;
158             mergeset(e[i].u, e[i].v);
159         }
160         else
161             e[i].vis = false;
162
163     for (int i = top; i > tmp; i--)
164         cut(stk[i]); // 撤销所有操作

```

```

164     top = tmp;
165
166     return ans;
167 }
168
169 // 以下是并查集相关函数
170 int findroot(int x) { // 因为需要撤销,不写路径压缩
171     while (p[x] != x)
172         x = p[x];
173
174     return x;
175 }
176
177 void mergeset(int x, int y) { // 按size合并,如果想跑得更
178 → 快就写一个按秩合并
179     x = findroot(x); // 但是按秩合并要再开一个栈记录合并
180     ↪ 之前的秩
181     y = findroot(y);
182
183     if (x == y)
184         return;
185
186     if (size[x] > size[y])
187         swap(x, y);
188
189     p[x] = y;
190     size[y] += size[x];
191     stk[++top] = x;
192
193 void cut(int x) { // 并查集撤销
194     int y = x;
195
196     do
197         size[y = p[y]] -= size[x];
198     while (p[y] != y);
199
200     p[x] = x;
201 }

```

### 3.1.3 最小树形图

对每个点找出最小的入边, 如果是一个DAG那么就已经结束了。否则把环都缩起来, 每个点的边权减去环上的边权之后再跑一遍, 直到没有环为止。

可以用可并堆优化到  $O(m \log n)$ , 需要写一个带懒标记的左偏树。  
 $O(nm)$  版本

```

1 constexpr int maxn = 105, maxe = 10005, inf =
2   ↪ 0x3f3f3f3f;
3
4 struct edge {
5     int u, v, w;
6 } e[maxe];
7
8 int mn[maxn], pr[maxn], ufs[maxn], vis[maxn];
9 bool alive[maxn];
10
11 int edmonds(int n, int m, int rt) {
12     for (int i = 1; i <= n; i++)
13         alive[i] = true;
14
15     int ans = 0;
16
17     while (true) {
18         memset(mn, 63, sizeof(int) * (n + 1));
19         memset(pr, 0, sizeof(int) * (n + 1));
20         memset(ufs, 0, sizeof(int) * (n + 1));
21
22         for (int i = 1; i <= m; i++) {
23             if (mn[e[i].u] < mn[e[i].v]) {
24                 mn[e[i].v] = mn[e[i].u];
25                 pr[e[i].v] = e[i].u;
26             }
27         }
28
29         for (int i = 1; i <= n; i++) {
30             if (!alive[i]) continue;
31
32             int cur = i;
33             int sum = 0;
34
35             while (cur != pr[cur]) {
36                 sum += e[pr[cur]].w;
37                 cur = pr[cur];
38             }
39
40             if (sum > mn[i]) {
41                 mn[i] = sum;
42                 pr[i] = rt;
43             }
44         }
45
46         for (int i = 1; i <= n; i++) {
47             if (!alive[i]) continue;
48
49             int cur = i;
50             int sum = 0;
51
52             while (cur != pr[cur]) {
53                 sum += e[pr[cur]].w;
54                 cur = pr[cur];
55             }
56
57             if (sum < mn[i]) {
58                 mn[i] = sum;
59                 pr[i] = rt;
60             }
61         }
62
63         for (int i = 1; i <= m; i++) {
64             if (mn[e[i].u] < mn[e[i].v]) {
65                 mn[e[i].v] = mn[e[i].u];
66                 pr[e[i].v] = e[i].u;
67             }
68         }
69
70         for (int i = 1; i <= n; i++) {
71             if (!alive[i]) continue;
72
73             int cur = i;
74             int sum = 0;
75
76             while (cur != pr[cur]) {
77                 sum += e[pr[cur]].w;
78                 cur = pr[cur];
79             }
80
81             if (sum > mn[i]) {
82                 mn[i] = sum;
83                 pr[i] = rt;
84             }
85         }
86
87         for (int i = 1; i <= m; i++) {
88             if (mn[e[i].u] < mn[e[i].v]) {
89                 mn[e[i].v] = mn[e[i].u];
90                 pr[e[i].v] = e[i].u;
91             }
92         }
93
94         for (int i = 1; i <= n; i++) {
95             if (!alive[i]) continue;
96
97             int cur = i;
98             int sum = 0;
99
100            while (cur != pr[cur]) {
101                sum += e[pr[cur]].w;
102                cur = pr[cur];
103            }
104
105            if (sum < mn[i]) {
106                mn[i] = sum;
107                pr[i] = rt;
108            }
109        }
110
111        for (int i = 1; i <= m; i++) {
112            if (mn[e[i].u] < mn[e[i].v]) {
113                mn[e[i].v] = mn[e[i].u];
114                pr[e[i].v] = e[i].u;
115            }
116        }
117
118        for (int i = 1; i <= n; i++) {
119            if (!alive[i]) continue;
120
121            int cur = i;
122            int sum = 0;
123
124            while (cur != pr[cur]) {
125                sum += e[pr[cur]].w;
126                cur = pr[cur];
127            }
128
129            if (sum > mn[i]) {
130                mn[i] = sum;
131                pr[i] = rt;
132            }
133        }
134
135        for (int i = 1; i <= m; i++) {
136            if (mn[e[i].u] < mn[e[i].v]) {
137                mn[e[i].v] = mn[e[i].u];
138                pr[e[i].v] = e[i].u;
139            }
140        }
141
142        for (int i = 1; i <= n; i++) {
143            if (!alive[i]) continue;
144
145            int cur = i;
146            int sum = 0;
147
148            while (cur != pr[cur]) {
149                sum += e[pr[cur]].w;
150                cur = pr[cur];
151            }
152
153            if (sum < mn[i]) {
154                mn[i] = sum;
155                pr[i] = rt;
156            }
157        }
158
159        for (int i = 1; i <= m; i++) {
160            if (mn[e[i].u] < mn[e[i].v]) {
161                mn[e[i].v] = mn[e[i].u];
162                pr[e[i].v] = e[i].u;
163            }
164        }
165
166        for (int i = 1; i <= n; i++) {
167            if (!alive[i]) continue;
168
169            int cur = i;
170            int sum = 0;
171
172            while (cur != pr[cur]) {
173                sum += e[pr[cur]].w;
174                cur = pr[cur];
175            }
176
177            if (sum > mn[i]) {
178                mn[i] = sum;
179                pr[i] = rt;
180            }
181        }
182
183        for (int i = 1; i <= m; i++) {
184            if (mn[e[i].u] < mn[e[i].v]) {
185                mn[e[i].v] = mn[e[i].u];
186                pr[e[i].v] = e[i].u;
187            }
188        }
189
190        for (int i = 1; i <= n; i++) {
191            if (!alive[i]) continue;
192
193            int cur = i;
194            int sum = 0;
195
196            while (cur != pr[cur]) {
197                sum += e[pr[cur]].w;
198                cur = pr[cur];
199            }
200
201            if (sum < mn[i]) {
202                mn[i] = sum;
203                pr[i] = rt;
204            }
205        }
206
207        for (int i = 1; i <= m; i++) {
208            if (mn[e[i].u] < mn[e[i].v]) {
209                mn[e[i].v] = mn[e[i].u];
210                pr[e[i].v] = e[i].u;
211            }
212        }
213
214        for (int i = 1; i <= n; i++) {
215            if (!alive[i]) continue;
216
217            int cur = i;
218            int sum = 0;
219
220            while (cur != pr[cur]) {
221                sum += e[pr[cur]].w;
222                cur = pr[cur];
223            }
224
225            if (sum > mn[i]) {
226                mn[i] = sum;
227                pr[i] = rt;
228            }
229        }
230
231        for (int i = 1; i <= m; i++) {
232            if (mn[e[i].u] < mn[e[i].v]) {
233                mn[e[i].v] = mn[e[i].u];
234                pr[e[i].v] = e[i].u;
235            }
236        }
237
238        for (int i = 1; i <= n; i++) {
239            if (!alive[i]) continue;
240
241            int cur = i;
242            int sum = 0;
243
244            while (cur != pr[cur]) {
245                sum += e[pr[cur]].w;
246                cur = pr[cur];
247            }
248
249            if (sum < mn[i]) {
250                mn[i] = sum;
251                pr[i] = rt;
252            }
253        }
254
255        for (int i = 1; i <= m; i++) {
256            if (mn[e[i].u] < mn[e[i].v]) {
257                mn[e[i].v] = mn[e[i].u];
258                pr[e[i].v] = e[i].u;
259            }
260        }
261
262        for (int i = 1; i <= n; i++) {
263            if (!alive[i]) continue;
264
265            int cur = i;
266            int sum = 0;
267
268            while (cur != pr[cur]) {
269                sum += e[pr[cur]].w;
270                cur = pr[cur];
271            }
272
273            if (sum > mn[i]) {
274                mn[i] = sum;
275                pr[i] = rt;
276            }
277        }
278
279        for (int i = 1; i <= m; i++) {
280            if (mn[e[i].u] < mn[e[i].v]) {
281                mn[e[i].v] = mn[e[i].u];
282                pr[e[i].v] = e[i].u;
283            }
284        }
285
286        for (int i = 1; i <= n; i++) {
287            if (!alive[i]) continue;
288
289            int cur = i;
290            int sum = 0;
291
292            while (cur != pr[cur]) {
293                sum += e[pr[cur]].w;
294                cur = pr[cur];
295            }
296
297            if (sum < mn[i]) {
298                mn[i] = sum;
299                pr[i] = rt;
300            }
301        }
302
303        for (int i = 1; i <= m; i++) {
304            if (mn[e[i].u] < mn[e[i].v]) {
305                mn[e[i].v] = mn[e[i].u];
306                pr[e[i].v] = e[i].u;
307            }
308        }
309
310        for (int i = 1; i <= n; i++) {
311            if (!alive[i]) continue;
312
313            int cur = i;
314            int sum = 0;
315
316            while (cur != pr[cur]) {
317                sum += e[pr[cur]].w;
318                cur = pr[cur];
319            }
320
321            if (sum > mn[i]) {
322                mn[i] = sum;
323                pr[i] = rt;
324            }
325        }
326
327        for (int i = 1; i <= m; i++) {
328            if (mn[e[i].u] < mn[e[i].v]) {
329                mn[e[i].v] = mn[e[i].u];
330                pr[e[i].v] = e[i].u;
331            }
332        }
333
334        for (int i = 1; i <= n; i++) {
335            if (!alive[i]) continue;
336
337            int cur = i;
338            int sum = 0;
339
340            while (cur != pr[cur]) {
341                sum += e[pr[cur]].w;
342                cur = pr[cur];
343            }
344
345            if (sum < mn[i]) {
346                mn[i] = sum;
347                pr[i] = rt;
348            }
349        }
350
351        for (int i = 1; i <= m; i++) {
352            if (mn[e[i].u] < mn[e[i].v]) {
353                mn[e[i].v] = mn[e[i].u];
354                pr[e[i].v] = e[i].u;
355            }
356        }
357
358        for (int i = 1; i <= n; i++) {
359            if (!alive[i]) continue;
360
361            int cur = i;
362            int sum = 0;
363
364            while (cur != pr[cur]) {
365                sum += e[pr[cur]].w;
366                cur = pr[cur];
367            }
368
369            if (sum > mn[i]) {
370                mn[i] = sum;
371                pr[i] = rt;
372            }
373        }
374
375        for (int i = 1; i <= m; i++) {
376            if (mn[e[i].u] < mn[e[i].v]) {
377                mn[e[i].v] = mn[e[i].u];
378                pr[e[i].v] = e[i].u;
379            }
380        }
381
382        for (int i = 1; i <= n; i++) {
383            if (!alive[i]) continue;
384
385            int cur = i;
386            int sum = 0;
387
388            while (cur != pr[cur]) {
389                sum += e[pr[cur]].w;
390                cur = pr[cur];
391            }
392
393            if (sum < mn[i]) {
394                mn[i] = sum;
395                pr[i] = rt;
396            }
397        }
398
399        for (int i = 1; i <= m; i++) {
400            if (mn[e[i].u] < mn[e[i].v]) {
401                mn[e[i].v] = mn[e[i].u];
402                pr[e[i].v] = e[i].u;
403            }
404        }
405
406        for (int i = 1; i <= n; i++) {
407            if (!alive[i]) continue;
408
409            int cur = i;
410            int sum = 0;
411
412            while (cur != pr[cur]) {
413                sum += e[pr[cur]].w;
414                cur = pr[cur];
415            }
416
417            if (sum > mn[i]) {
418                mn[i] = sum;
419                pr[i] = rt;
420            }
421        }
422
423        for (int i = 1; i <= m; i++) {
424            if (mn[e[i].u] < mn[e[i].v]) {
425                mn[e[i].v] = mn[e[i].u];
426                pr[e[i].v] = e[i].u;
427            }
428        }
429
430        for (int i = 1; i <= n; i++) {
431            if (!alive[i]) continue;
432
433            int cur = i;
434            int sum = 0;
435
436            while (cur != pr[cur]) {
437                sum += e[pr[cur]].w;
438                cur = pr[cur];
439            }
440
441            if (sum < mn[i]) {
442                mn[i] = sum;
443                pr[i] = rt;
444            }
445        }
446
447        for (int i = 1; i <= m; i++) {
448            if (mn[e[i].u] < mn[e[i].v]) {
449                mn[e[i].v] = mn[e[i].u];
450                pr[e[i].v] = e[i].u;
451            }
452        }
453
454        for (int i = 1; i <= n; i++) {
455            if (!alive[i]) continue;
456
457            int cur = i;
458            int sum = 0;
459
460            while (cur != pr[cur]) {
461                sum += e[pr[cur]].w;
462                cur = pr[cur];
463            }
464
465            if (sum > mn[i]) {
466                mn[i] = sum;
467                pr[i] = rt;
468            }
469        }
470
471        for (int i = 1; i <= m; i++) {
472            if (mn[e[i].u] < mn[e[i].v]) {
473                mn[e[i].v] = mn[e[i].u];
474                pr[e[i].v] = e[i].u;
475            }
476        }
477
478        for (int i = 1; i <= n; i++) {
479            if (!alive[i]) continue;
480
481            int cur = i;
482            int sum = 0;
483
484            while (cur != pr[cur]) {
485                sum += e[pr[cur]].w;
486                cur = pr[cur];
487            }
488
489            if (sum < mn[i]) {
490                mn[i] = sum;
491                pr[i] = rt;
492            }
493        }
494
495        for (int i = 1; i <= m; i++) {
496            if (mn[e[i].u] < mn[e[i].v]) {
497                mn[e[i].v] = mn[e[i].u];
498                pr[e[i].v] = e[i].u;
499            }
500        }
501
502        for (int i = 1; i <= n; i++) {
503            if (!alive[i]) continue;
504
505            int cur = i;
506            int sum = 0;
507
508            while (cur != pr[cur]) {
509                sum += e[pr[cur]].w;
510                cur = pr[cur];
511            }
512
513            if (sum > mn[i]) {
514                mn[i] = sum;
515                pr[i] = rt;
516            }
517        }
518
519        for (int i = 1; i <= m; i++) {
520            if (mn[e[i].u] < mn[e[i].v]) {
521                mn[e[i].v] = mn[e[i].u];
522                pr[e[i].v] = e[i].u;
523            }
524        }
525
526        for (int i = 1; i <= n; i++) {
527            if (!alive[i]) continue;
528
529            int cur = i;
530            int sum = 0;
531
532            while (cur != pr[cur]) {
533                sum += e[pr[cur]].w;
534                cur = pr[cur];
535            }
536
537            if (sum < mn[i]) {
538                mn[i] = sum;
539                pr[i] = rt;
540            }
541        }
542
543        for (int i = 1; i <= m; i++) {
544            if (mn[e[i].u] < mn[e[i].v]) {
545                mn[e[i].v] = mn[e[i].u];
546                pr[e[i].v] = e[i].u;
547            }
548        }
549
550        for (int i = 1; i <= n; i++) {
551            if (!alive[i]) continue;
552
553            int cur = i;
554            int sum = 0;
555
556            while (cur != pr[cur]) {
557                sum += e[pr[cur]].w;
558                cur = pr[cur];
559            }
560
561            if (sum > mn[i]) {
562                mn[i] = sum;
563                pr[i] = rt;
564            }
565        }
566
567        for (int i = 1; i <= m; i++) {
568            if (mn[e[i].u] < mn[e[i].v]) {
569                mn[e[i].v] = mn[e[i].u];
570                pr[e[i].v] = e[i].u;
571            }
572        }
573
574        for (int i = 1; i <= n; i++) {
575            if (!alive[i]) continue;
576
577            int cur = i;
578            int sum = 0;
579
580            while (cur != pr[cur]) {
581                sum += e[pr[cur]].w;
582                cur = pr[cur];
583            }
584
585            if (sum < mn[i]) {
586                mn[i] = sum;
587                pr[i] = rt;
588            }
589        }
590
591        for (int i = 1; i <= m; i++) {
592            if (mn[e[i].u] < mn[e[i].v]) {
593                mn[e[i].v] = mn[e[i].u];
594                pr[e[i].v] = e[i].u;
595            }
596        }
597
598        for (int i = 1; i <= n; i++) {
599            if (!alive[i]) continue;
600
601            int cur = i;
602            int sum = 0;
603
604            while (cur != pr[cur]) {
605                sum += e[pr[cur]].w;
606                cur = pr[cur];
607            }
608
609            if (sum > mn[i]) {
610                mn[i] = sum;
611                pr[i] = rt;
612            }
613        }
614
615        for (int i = 1; i <= m; i++) {
616            if (mn[e[i].u] < mn[e[i].v]) {
617                mn[e[i].v] = mn[e[i].u];
618                pr[e[i].v] = e[i].u;
619            }
620        }
621
622        for (int i = 1; i <= n; i++) {
623            if (!alive[i]) continue;
624
625            int cur = i;
626            int sum = 0;
627
628            while (cur != pr[cur]) {
629                sum += e[pr[cur]].w;
630                cur = pr[cur];
631            }
632
633            if (sum < mn[i]) {
634                mn[i] = sum;
635                pr[i] = rt;
636            }
637        }
638
639        for (int i = 1; i <= m; i++) {
640            if (mn[e[i].u] < mn[e[i].v]) {
641                mn[e[i].v] = mn[e[i].u];
642                pr[e[i].v] = e[i].u;
643            }
644        }
645
646        for (int i = 1; i <= n; i++) {
647            if (!alive[i]) continue;
648
649            int cur = i;
650            int sum = 0;
651
652            while (cur != pr[cur]) {
653                sum += e[pr[cur]].w;
654                cur = pr[cur];
655            }
656
657            if (sum > mn[i]) {
658                mn[i] = sum;
659                pr[i] = rt;
660            }
661        }
662
663        for (int i = 1; i <= m; i++) {
664            if (mn[e[i].u] < mn[e[i].v]) {
665                mn[e[i].v] = mn[e[i].u];
666                pr[e[i].v] = e[i].u;
667            }
668        }
669
670        for (int i = 1; i <= n; i++) {
671            if (!alive[i]) continue;
672
673            int cur = i;
674            int sum = 0;
675
676            while (cur != pr[cur]) {
677                sum += e[pr[cur]].w;
678                cur = pr[cur];
679            }
680
681            if (sum < mn[i]) {
682                mn[i] = sum;
683                pr[i] = rt;
684            }
685        }
686
687        for (int i = 1; i <= m; i++) {
688            if (mn[e[i].u] < mn[e[i].v]) {
689                mn[e[i].v] = mn[e[i].u];
690                pr[e[i].v] = e[i].u;
691            }
692        }
693
694        for (int i = 1; i <= n; i++) {
695            if (!alive[i]) continue;
696
697            int cur = i;
698            int sum = 0;
699
700            while (cur != pr[cur]) {
701                sum += e[pr[cur]].w;
702                cur = pr[cur];
703            }
704
705            if (sum > mn[i]) {
706                mn[i] = sum;
707                pr[i] = rt;
708            }
709        }
710
711        for (int i = 1; i <= m; i++) {
712            if (mn[e[i].u] < mn[e[i].v]) {
713                mn[e[i].v] = mn[e[i].u];
714                pr[e[i].v] = e[i].u;
715            }
716        }
717
718        for (int i = 1; i <= n; i++) {
719            if (!alive[i]) continue;
720
721            int cur = i;
722            int sum = 0;
723
724            while (cur != pr[cur]) {
725                sum += e[pr[cur]].w;
726                cur = pr[cur];
727            }
728
729            if (sum < mn[i]) {
730                mn[i] = sum;
731                pr[i] = rt;
732            }
733        }
734
735        for (int i = 1; i <= m; i++) {
736            if (mn[e[i].u] < mn[e[i].v]) {
737                mn[e[i].v] = mn[e[i].u];
738                pr[e[i].v] = e[i].u;
739            }
740        }
741
742        for (int i = 1; i <= n; i++) {
743            if (!alive[i]) continue;
744
745            int cur = i;
746            int sum = 0;
747
748            while (cur != pr[cur]) {
749                sum += e[pr[cur]].w;
750                cur = pr[cur];
751            }
752
753            if (sum > mn[i]) {
754                mn[i] = sum;
755                pr[i] = rt;
756            }
757        }
758
759        for (int i = 1; i <= m; i++) {
760            if (mn[e[i].u] < mn[e[i].v]) {
761                mn[e[i].v] = mn[e[i].u];
762                pr[e[i].v] = e[i].u;
763            }
764        }
765
766        for (int i = 1; i <= n; i++) {
767            if (!alive[i]) continue;
768
769            int cur = i;
770            int sum = 0;
771
772            while (cur != pr[cur]) {
773                sum += e[pr[cur]].w;
774                cur = pr[cur];
775            }
776
777            if (sum < mn[i]) {
778                mn[i] = sum;
779                pr[i] = rt;
780            }
781        }
782
783        for (int i = 1; i <= m; i++) {
784            if (mn[e[i].u] < mn[e[i].v]) {
785                mn[e[i].v] = mn[e[i].u];
786                pr[e[i].v] = e[i].u;
787            }
788        }
789
790        for (int i = 1; i <= n; i++) {
791            if (!alive[i]) continue;
792
793            int cur = i;
794            int sum = 0;
795
796            while (cur != pr[cur]) {
797                sum += e[pr[cur]].w;
798                cur = pr[cur];
799            }
800
801            if (sum > mn[i]) {
802                mn[i] = sum;
803                pr[i] = rt;
804            }
805        }
806
807        for (int i = 1; i <= m; i++) {
808            if (mn[e[i].u] < mn[e[i].v]) {
809                mn[e[i].v] = mn[e[i].u];
810                pr[e[i].v] = e[i].u;
811            }
812        }
813
814        for (int i = 1; i <= n; i++) {
815            if (!alive[i]) continue;
816
817            int cur = i;
818            int sum = 0;
819
820            while (cur != pr[cur]) {
821                sum += e[pr[cur]].w;
822                cur = pr[cur];
823            }
824
825            if (sum < mn[i]) {
826                mn[i] = sum;
827                pr[i] = rt;
828            }
829        }
830
831        for (int i = 1; i <= m; i++) {
832            if (mn[e[i].u] < mn[e[i].v]) {
833                mn[e[i].v] = mn[e[i].u];
834                pr[e[i].v] = e[i].u;
835            }
836        }
837
838        for (int i = 1; i <= n; i++) {
839            if (!alive[i]) continue;
840
841            int cur = i;
842            int sum = 0;
843
844            while (cur != pr[cur]) {
845                sum += e[pr[cur]].w;
846                cur = pr[cur];
847            }
848
849            if (sum > mn[i]) {
850                mn[i] = sum;
851                pr[i] = rt;
852            }
853        }
854
855        for (int i = 1; i <= m; i++) {
856            if (mn[e[i].u] < mn[e[i].v]) {
857                mn[e[i].v] = mn[e[i].u];
858                pr[e[i].v] = e[i].u;
859            }
860        }
861
862        for (int i = 1; i <= n; i++) {
863            if (!alive[i]) continue;
864
865            int cur = i;
866            int sum = 0;
867
868            while (cur != pr[cur]) {
869                sum += e[pr[cur]].w;
870                cur = pr[cur];
871            }
872
873            if (sum < mn[i]) {
874                mn[i] = sum;
875                pr[i] = rt;
876            }
877        }
878
879        for (int i = 1; i <= m; i++) {
880            if (mn[e[i].u] < mn[e[i].v]) {
881                mn[e[i].v] = mn[e[i].u];
882                pr[e[i].v] = e[i].u;
883            }
884        }
885
886        for (int i = 1; i <= n; i++) {
887            if (!alive[i]) continue;
888
889            int cur = i;
890            int sum = 0;
891
892            while (cur != pr[cur]) {
893                sum += e[pr[cur]].w;
894                cur = pr[cur];
895            }
896
897            if (sum > mn[i]) {
898                mn[i] = sum;
899                pr[i] = rt;
900            }
901        }
902
903        for (int i = 1; i <= m; i++) {
904            if (mn[e[i].u] < mn[e[i].v]) {
905                mn[e[i].v] = mn[e[i].u];
906                pr[e[i].v] = e[i].u;
907            }
908        }
909
910        for (int i = 1; i <= n; i++) {
911            if (!alive[i]) continue;
912
913            int cur = i;
914            int sum = 0;
915
916            while (cur != pr[cur]) {
917                sum += e[pr[cur]].w;
918                cur = pr[cur];
919            }
920
921            if (sum < mn[i]) {
922                mn[i] = sum;
923                pr[i] = rt;
924            }
925        }
926
927        for (int i = 1; i <= m; i++) {
928            if (mn[e[i].u] < mn[e[i].v]) {
929                mn[e[i].v] = mn[e[i].u];
930                pr[e[i].v] = e[i].u;
931            }
932        }
933
934        for (int i = 1; i <= n; i++) {
935            if (!alive[i]) continue;
936
937            int cur = i;
938            int sum = 0;
939
940            while (cur != pr[cur]) {
941                sum += e[pr[cur]].w;
942                cur = pr[cur];
943            }
944
945            if (sum > mn[i]) {
946                mn[i] = sum;
947                pr[i] = rt;
948            }
949        }
950
951        for (int i = 1; i <= m; i++) {
952            if (mn[e[i].u] < mn[e[i].v]) {
953                mn[e[i].v] = mn[e[i].u];
954                pr[e[i].v] = e[i].u;
955            }
956        }
9
```

```

20     memset(vis, 0, sizeof(int) * (n + 1));
21
22     mn[rt] = 0;
23
24     for (int i = 1; i ≤ m; i++) {
25         if (e[i].u != e[i].v && e[i].w <
26             → mn[e[i].v]) {
27             mn[e[i].v] = e[i].w;
28             pr[e[i].v] = e[i].u;
29         }
30
31         for (int i = 1; i ≤ n; i++) {
32             if (!alive[i]) {
33                 if (mn[i] ≥ inf)
34                     return -1; // 不存在最小树形图
35
36             ans += mn[i];
37         }
38
39         bool flag = false;
40
41         for (int i = 1; i ≤ n; i++) {
42             if (!alive[i])
43                 continue;
44
45             int x = i;
46             while (x && !vis[x]) {
47                 vis[x] = i;
48                 x = pr[x];
49             }
50
51             if (x && vis[x] == i) {
52                 flag = true;
53                 for (int u = x; !ufs[u]; u = pr[u])
54                     ufs[u] = x;
55             }
56
57             for (int i = 1; i ≤ m; i++) {
58                 e[i].w -= mn[e[i].v];
59
60                 if (ufs[e[i].u])
61                     e[i].u = ufs[e[i].u];
62                 if (ufs[e[i].v])
63                     e[i].v = ufs[e[i].v];
64             }
65
66             if (!flag)
67                 return ans;
68
69             for (int i = 1; i ≤ n; i++)
70                 if (ufs[i] && i != ufs[i])
71                     alive[i] = false;
72     }
73 }
```

$O(m \log n)$  版本

(堆优化版本可以参考fstqwq的模板，在最后没有目录的部分。)

### 3.1.4 Steiner Tree 斯坦纳树

问题：一张图上有  $k$  个关键点，求让关键点两两连通的最小生成树

做法：状压 DP， $f_{i,S}$  表示以  $i$  号点为树根， $i$  与  $S$  中的点连通的最小边权和

转移有两种：

1. 枚举子集：

$$f_{i,S} = \min_{T \subset S} \{f_{i,T} + f_{i,S \setminus T}\}$$

2. 新加一条边：

$$f_{i,S} = \min_{(i,j) \in E} \{f_{j,S} + w_{i,j}\}$$

第一种直接枚举子集 DP 就行了，第二种可以用 SPFA 或者 Dijkstra 松弛（显然负边一开始全选就行了，所以只需要处理非负边）。

复杂度  $O(n3^k + 2^k \text{SSSP}(n, m))$ ，其中  $\text{SSSP}(n, m)$  可以是  $nm$  或者  $n^2 + m$  或者  $m \log n$ 。

```

1 constexpr int maxn = 105, inf = 0x3f3f3f3f;
2
3 int dp[maxn][(1 << 10) + 1];
4 int g[maxn][maxn], a[15];
5 bool inq[maxn];
6
7 int main() {
8
9     int n, m, k;
10    scanf("%d%d%d", &n, &m, &k);
11
12    memset(g, 63, sizeof(g));
13
14    while (m--) {
15        int u, v, c;
16        scanf("%d%d%d", &u, &v, &c);
17
18        g[u][v] = g[v][u] = min(g[u][v], c); // 不要忘了
19        → 是双向边
20    }
21
22    memset(dp, 63, sizeof(dp));
23
24    for (int i = 0; i < k; i++) {
25        scanf("%d", &a[i]);
26
27        dp[a[i]][1 << i] = 0;
28    }
29
30    for (int s = 1; s < (1 << k); s++) {
31        for (int i = 1; i ≤ n; i++)
32            for (int t = (s - 1) & s; t; (~t) &= s)
33                dp[i][s] = min(dp[i][s], dp[i][t] +
34                    → dp[i][s ^ t]);
35
36    // SPFA
37    queue<int> q;
38    for (int i = 1; i ≤ n; i++)
39        if (dp[i][s] < inf) {
40            q.push(i);
41            inq[i] = true;
42        }
43
44    while (!q.empty()) {
45        int i = q.front();
46        q.pop();
47        inq[i] = false; // 最终结束时 inq 一定全 0，所
48        → 以不用清空
49
50        for (int j = 1; j ≤ n; j++)
51            if (dp[i][s] + g[i][j] < dp[j][s]) {
52                dp[j][s] = dp[i][s] + g[i][j];
53                if (!inq[j]) {
54                    q.push(j);
55                    inq[j] = true;
56                }
57            }
58        }
59    }
60 }
```

```

57
58     int ans = inf;
59     for (int i = 1; i <= n; i++)
60         ans = min(ans, dp[i][(1 << k) - 1]);
61
62     printf("%d\n", ans);
63
64     return 0;
65 }
```

### 3.1.5 最小直径生成树

首先要找到图的绝对中心(可能在点上, 也可能在某条边上), 然后以绝对中心为起点建最短路树就是最小直径生成树.

```

1 #include <bits/stdc++.h>
2
3 using namespace std;
4
5 typedef long long ll;
6 typedef pair<int, int> pii;
7
8 const int MAXN = 502;
9 const ll INF = 1e17;
10
11 ll d[MAXN][MAXN], dd[MAXN][MAXN], rk[MAXN][MAXN],
12     → val[MAXN];
13 int n, m;
14
15 bool cmp(int a, int b) { return val[a] < val[b]; }
16
17 void floyd() {
18     for (int k = 1; k <= n; k++)
19         for (int i = 1; i <= n; i++)
20             for (int j = 1; j <= n; j++) d[i][j] =
21                 → min(d[i][j], d[i][k] + d[k][j]);
22
23 struct node {
24     ll u, v, w;
25 } a[MAXN * (MAXN - 1) / 2];
26
27 void solve() {
28     // 求图的绝对中心
29     floyd();
30     for (int i = 1; i <= n; i++) {
31         for (int j = 1; j <= n; j++) {
32             rk[i][j] = j;
33             val[j] = d[i][j];
34         }
35         sort(rk[i] + 1, rk[i] + 1 + n, cmp);
36     }
37     ll P = 0, ansP = INF;
38     // 在点上
39     for (int i = 1; i <= n; i++) {
40         if (d[i][rk[i][n]] * 2 < ansP) {
41             ansP = d[i][rk[i][n]] * 2;
42             P = i;
43         }
44     }
45     // 在边上
46     int f1 = 0, f2 = 0;
47     ll disu = INT_MIN, disv = INT_MIN, ansL = INF;
48     for (int i = 1; i <= m; i++) {
49         ll u = a[i].u, v = a[i].v, w = a[i].w;
50         for (int p = n, i = n - 1; i >= 1; i--) {
51             if (d[v][rk[u][i]] > d[v][rk[u][p]]) {
52                 if (d[u][rk[u][i]] + d[v][rk[u][p]] + w
53                     → < ansL) {
54                     ansL = d[u][rk[u][i]] + d[v][rk[u]
55                         → [p]] + w;
56                     f1 = u, f2 = v;
57                     disu = (d[u][rk[u][i]] + d[v][rk[u]
58                         → [p]] + w) / 2 - d[u][rk[u][i]];
59                     disv = w - disu;
60                 }
61             }
62         }
63     }
64     cout << min(ansP, ansL) / 2 << '\n';
65 }
```

```

52
53         ansL = d[u][rk[u][i]] + d[v][rk[u]
54             → [p]] + w;
55         f1 = u, f2 = v;
56         disu = (d[u][rk[u][i]] + d[v][rk[u]
57             → [p]] + w) / 2 - d[u][rk[u][i]];
58         disv = w - disu;
59     }
60
61     p = i;
62 }
63
64 cout << min(ansP, ansL) / 2 << '\n';
65
66 // 最小路径生成树
67 vector<pii> pp;
68 for (int i = 1; i <= 501; ++i)
69     for (int j = 1; j <= 501; ++j) dd[i][j] = INF;
70 for (int i = 1; i <= 501; ++i) dd[i][i] = 0;
71 if (ansP <= ansL) {
72     for (int j = 1; j <= n; j++) {
73         for (int i = 1; i <= m; ++i) {
74             ll u = a[i].u, v = a[i].v, w = a[i].w;
75             if (dd[P][u] + w == d[P][v] && dd[P][u]
76                 → + w < dd[P][v]) {
77                 dd[P][v] = dd[P][u] + w;
78                 pp.push_back({u, v});
79             }
80             u = a[i].v, v = a[i].u, w = a[i].w;
81             if (dd[P][u] + w == d[P][v] && dd[P][u]
82                 → + w < dd[P][v]) {
83                 dd[P][v] = dd[P][u] + w;
84                 pp.push_back({u, v});
85             }
86         }
87     }
88     for (auto [x, y] : pp)
89         cout << x << ' ' << y << '\n';
90 }
91 else {
92     d[n + 1][f1] = disu;
93     d[f1][n + 1] = disu;
94     d[n + 1][f2] = disv;
95     d[f2][n + 1] = disv;
96     a[m + 1].u = n + 1, a[m + 1].v = f1, a[m + 1].w
97         → = disu;
98     a[m + 2].u = n + 1, a[m + 2].v = f2, a[m + 2].w
99         → = disv;
100    n += 1;
101    m += 2;
102    floyd();
103    P = n;
104    for (int j = 1; j <= n; j++) {
105        for (int i = 1; i <= m; ++i) {
106            ll u = a[i].u, v = a[i].v, w = a[i].w;
107            if (dd[P][u] + w == d[P][v] && dd[P][u]
108                → + w < dd[P][v]) {
109                dd[P][v] = dd[P][u] + w;
110                pp.push_back({u, v});
111            }
112        }
113    }
114    cout << f1 << ' ' << f2 << '\n';
115    for (auto [x, y] : pp)
```

```

113     |     |     if (x != n && y != n) cout << x << ' ' << y
114     |     |
115     |
116 void init() {
117     for (int i = 1; i ≤ 501; ++i)
118         for (int j = 1; j ≤ 501; ++j) d[i][j] = INF;
119     for (int i = 1; i ≤ 501; ++i) d[i][i] = 0;
120 }
121
122 int main() {
123     init();
124     cin >> n >> m;
125     for (int i = 1; i ≤ m; ++i) {
126         ll u, v, w;
127         cin >> u >> v >> w;
128         w *= 2;
129         d[u][v] = w, d[v][u] = w;
130         a[i].u = u, a[i].v = v, a[i].w = w;
131     }
132     solve();
133     return 0;
134 }
135

```

```

25     void refresh(){
26         d = rc → d + 1;
27     }
28 } null[maxm], *ptr = null, *root[maxn];
29
30 struct B { // 维护答案用
31     int x, w; // x是结点编号, w表示之前已经产生的权值
32     node *rt; // 这个答案对应的堆顶,注意可能不等于任何一个结点的堆
33
34     B(int x, node *rt, int w) : x(x), w(w), rt(rt) {}
35
36     bool operator < (const B &a) const {
37         return w + rt → w > a.w + a.rt → w;
38     }
39 };
40
41 // 全局变量和数组定义
42 vector<int> G[maxn], W[maxn], id[maxn]; // 最开始要存反
43 // 向图,然后把G清空作为儿子列表
44 bool vis[maxn], used[maxe]; // used表示边是否在最短路树
45 // 上
46 int u[maxe], v[maxe], w[maxe]; // 存下每条边,注意是有向
47 // 边
48 int d[maxn], p[maxn]; // p表示最短路树上每个点的父边
49 int n, m, k, s, t; // s, t分别表示起点和终点
50
51 // 以下是主函数中较关键的部分
52 for (int i = 0; i ≤ n; i++)
53     root[i] = null; // 一定要加上!!!
54
55 // (读入&建反向图)
56 Dijkstra();
57
58 // (清空G, W, id)
59
60 for (int i = 1; i ≤ n; i++) {
61     if (p[i]) {
62         used[p[i]] = true; // 在最短路树上
63         G[v[p[i]]].push_back(i);
64     }
65
66 for (int i = 1; i ≤ m; i++) {
67     w[i] -= d[u[i]] - d[v[i]]; // 现在的w[i]表示这条边能
68     // 使路径长度增加多少
69     if (!used[i])
70         root[u[i]] = merge(root[u[i]], newnode(w[i],
71                             → i));
72
73 dfs(t);
74
75 priority_queue<B> heap;
76 heap.push(B(s, root[s], 0)); // 初始状态是找贡献最小的边
77 // 加进去
78
79 printf("%d\n", d[s]); // 第1短路需要特判
80 while (--k) { // 其余k - 1短路径用二叉堆维护
81     if (heap.empty())
82         printf("-1\n");
83     else {
84         int x = heap.top().x, w = heap.top().w;
85         node *rt = heap.top().rt;
86         heap.pop();
87
88         printf("%d\n", d[s] + w + rt → w);
89     }
90 }
91
92
93
94
95
96
97
98
99
100
101
102
103
104
105
106
107
108
109
110
111
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132
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134
135

```

## 3.2 最短路

### 3.2.1 Dijkstra

参见3.2.3.k短路(27页), 注意那边是求到 $t$ 的最短路.

### 3.2.2 Johnson算法(负权图多源最短路)

首先前提是图没有负环.

先任选一个起点 $s$ , 跑一遍SPFA, 计算每个点的势 $h_u = d_{s,u}$ , 然后将每条边 $u \rightarrow v$ 的权值 $w$ 修改为 $w + h[u] - h[v]$ 即可, 由最短路的性质显然修改后边权非负.

然后对每个起点跑Dijkstra, 再修正距离 $d_{u,v} = d'_{u,v} - h_u + h_v$ 即可, 复杂度 $O(nm \log n)$ , 在稀疏图上是要优于Floyd的.

### 3.2.3 k短路

```

1 // 注意这是个多项式算法, 在k比较大时很有优势, 但k比较小时
2 // → 最好还是用A*
3 // DAG和有环的情况都可以, 有重边或自环也无所谓, 但不能有
4 // → 零环
5 // 以下代码以Dijkstra + 可持久化左偏树为例
6
7 constexpr int maxn = 1005, maxe = 10005, maxm = maxe *
8     → 30; // 点数,边数,左偏树结点数
9
10 // 结构体定义
11 struct A { // 用来求最短路
12     int x, d;
13
14     A(int x, int d) : x(x), d(d) {}
15
16     bool operator < (const A &a) const {
17         return d > a.d;
18     }
19
20 struct node { // 左偏树结点
21     int w, i, d; // i: 最后一条边的编号 d: 左偏树附加信息
22     node *lc, *rc;
23
24     node() {}
25
26     node(int w, int i) : w(w), i(i), d(0) {}
27
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131
132
133
134
135

```

```

88     if (rt → lc != null || rt → rc != null)
89         heap.push(B(x, merge(rt → lc, rt → rc),
90             → w)); // pop掉当前边，换成另一条贡献大一
91             → 点的边
92         if (root[v[rt → i]] != null)
93             heap.push(B(v[rt → i], root[v[rt → i]], w
94                 → + rt → w)); // 保留当前边，往后面再接上
95                 → 另一条边
96
97     }
98
99 // 主函数到此结束
100
101
102
103
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107
108
109
110
111
112
113
114
115
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148
149
150
151
152

```

```

153     if (z → lc → d < z → rc → d)
154         swap(z → lc, z → rc);
155         z → refresh();
156
157     return z;
158 }
```

### 3.3 Tarjan算法

#### 3.3.1 强连通分量

```

1 int dfn[maxn], low[maxn], tim = 0;
2 vector<int> G[maxn], scc[maxn];
3 int sccid[maxn], scc_cnt = 0, stk[maxn];
4 bool instk[maxn];
5
6 void dfs(int x) {
7     dfn[x] = low[x] = ++tim;
8
9     stk[++stk[0]] = x;
10    instk[x] = true;
11
12    for (int y : G[x]) {
13        if (!dfn[y]) {
14            dfs(y);
15            low[x] = min(low[x], low[y]);
16        } else if (instk[y])
17            low[x] = min(low[x], dfn[y]);
18    }
19
20    if (dfn[x] == low[x]) {
21        scc_cnt++;
22
23        int u;
24        do {
25            u = stk[stk[0]--];
26            instk[u] = false;
27            sccid[u] = scc_cnt;
28            scc[scc_cnt].push_back(u);
29        } while (u != x);
30    }
31
32 }
33
34 void tarjan(int n) {
35     for (int i = 1; i ≤ n; i++)
36         if (!dfn[i])
37             dfs(i);
38 }
```

#### 3.3.2 割点 点双

```

1 vector<int> G[maxn], bcc[maxn];
2 int dfn[maxn], low[maxn], tim = 0, bccid[maxn], bcc_cnt
3     → = 0;
4 bool iscut[maxn];
5
6 pair<int, int> stk[maxn];
7 int stk_cnt = 0;
8
9 void dfs(int x, int pr) {
10     int child = 0;
11     dfn[x] = low[x] = ++tim;
12
13     for (int y : G[x]) {
14         if (!dfn[y]) {
15             stk[++stk_cnt] = make_pair(x, y);
16             child++;
17         }
18     }
19 }
```

```

16     dfs(y, x);
17     low[x] = min(low[x], low[y]);
18
19     if (low[y] >= dfn[x]) {
20         iscut[x] = true;
21         bcc_cnt++;
22
23         while (true) {
24             auto pi = stk[stk_cnt--];
25
26             if (bccid[pi.first] != bcc_cnt) {
27                 bcc[bcc_cnt].push_back(pi.first);
28                 bccid[pi.first] = bcc_cnt;
29             }
30             if (bccid[pi.second] != bcc_cnt) {
31                 bcc[bcc_cnt].push_back(pi.second);
32                 bccid[pi.second] = bcc_cnt;
33             }
34
35             if (pi.first == x && pi.second ==
36                 y)
37                 break;
38         }
39     }
40     else if (dfn[y] < dfn[x] && y != pr) {
41         stk[++stk_cnt] = make_pair(x, y);
42         low[x] = min(low[x], dfn[y]);
43     }
44
45     if (!pr && child == 1)
46         iscut[x] = false;
47 }
48
49 void Tarjan(int n) {
50     for (int i = 1; i <= n; i++)
51         if (!dfn[i])
52             dfs(i, 0);
53 }
```

```

27     if (dfn[x] == low[x]) {
28         bcc_cnt++;
29         int y;
30         do {
31             y = stk[top--];
32             bcc[bcc_cnt].push_back(y);
33         } while (y != x);
34     }
35 }
```

### 3.3.3 桥 边双

```

1 int u[maxe], v[maxe];
2 vector<int> G[maxn]; // 存的是边的编号
3
4 int stk[maxn], top, dfn[maxn], low[maxn], tim, bcc_cnt;
5 vector<int> bcc[maxn];
6
7 bool isbridge[maxe];
8
9 void dfs(int x, int pr) { // 这里pr是入边的编号
10     dfn[x] = low[x] = ++tim;
11     stk[++top] = x;
12
13     for (int i : G[x]) {
14         int y = (u[i] == x ? v[i] : u[i]);
15
16         if (!dfn[y]) {
17             dfs(y, i);
18             low[x] = min(low[x], low[y]);
19
20             if (low[y] > dfn[x])
21                 bridge[i] = true;
22         }
23         else if (i != pr)
24             low[x] = min(low[x], dfn[y]);
25     }
26 }
```

## 3.4 仙人掌

一般来说仙人掌问题都可以通过圆方树转成有两种点的树上问题来做。

### 3.4.1 仙人掌DP

```

1 struct edge {
2     int to, w, prev;
3 } e[maxn * 2];
4
5 vector<pair<int, int>> v[maxn];
6 vector<long long> d[maxn];
7 stack<int> stk;
8
9 int p[maxn];
10 bool vis[maxn], vise[maxn * 2];
11 int last[maxn], cnt;
12
13 long long f[maxn], g[maxn], sum[maxn];
14 int n, m, cnt;
15
16 void addedge(int x, int y, int w) {
17     v[x].push_back(make_pair(y, w));
18 }
19
20 void dfs(int x) {
21
22     vis[x] = true;
23
24     for (int i = last[x]; ~i; i = e[i].prev) {
25         if (vise[i ^ 1])
26             continue;
27
28         int y = e[i].to, w = e[i].w;
29
30         vise[i] = true;
31
32         if (!vis[y]) {
33             stk.push(i);
34             p[y] = x;
35             dfs(y);
36
37             if (!stk.empty() && stk.top() == i) {
38                 stk.pop();
39                 addedge(x, y, w);
40             }
41         }
42
43         else {
44             cnt++;
45
46             long long tmp = w;
47             while (!stk.empty()) {
48                 int i = stk.top();
49                 stk.pop();
50
51                 int yy = e[i].to, ww = e[i].w;
```

```

52
53     |     |     |     addedge(cnt, yy, 0);
54
55     |     |     |     d[cnt].push_back(tmp);
56
57     |     |     |     tmp += ww;
58
59     |     |     |     if (e[i ^ 1].to == y)
60     |     |     |     break;
61     |
62
63     |     |     addedge(y, cnt, 0);
64
65     |     |     sum[cnt] = tmp;
66
67     |
68 }
69
70 void dp(int x) {
71
72     for (auto o : v[x]) {
73         int y = o.first, w = o.second;
74         dp(y);
75     }
76
77     if (x <= n) {
78         for (auto o : v[x]) {
79             int y = o.first, w = o.second;
80
81             f[x] += 2 * w + f[y];
82         }
83
84         g[x] = f[x];
85
86         for (auto o : v[x]) {
87             int y = o.first, w = o.second;
88
89             g[x] = min(g[x], f[x] - f[y] - 2 * w + g[y]
90                         ↵ + w);
91         }
92     } else {
93         f[x] = sum[x];
94         for (auto o : v[x]) {
95             int y = o.first;
96
97             f[x] += f[y];
98         }
99
100        g[x] = f[x];
101
102        for (int i = 0; i < (int)v[x].size(); i++) {
103            int y = v[x][i].first;
104
105            g[x] = min(g[x], f[x] - f[y] + g[y] +
106                         ↵ min(d[x][i], sum[x] - d[x][i]));
107        }
108    }

```

```
6  bool dfs(int x) {
7      for (int y : G[x])
8          if (!vis[y]) {
9              vis[y] = true;
10
11             if (!boy[y] || dfs(boy[y])) {
12                 girl[x] = y;
13                 boy[y] = x;
14
15                 return true;
16             }
17         }
18
19     return false;
20 }
21
22 int hungary() {
23     int ans = 0;
24
25     for (int i = 1; i ≤ n; i++)
26         if (!girl[i]) {
27             memset(vis, 0, sizeof(vis));
28             ans += dfs(i);
29         }
30
31     return ans;
32 }
```

### 3.5.2 Hopcroft-Karp二分图匹配

其实长得和Dinic差不多，或者说像匈牙利和Dinic的缝合怪。

```
1 vector<int> G[maxn];
2
3 int girl[maxn], boy[maxn]; // girl: 左边匹配右边 boy:
4     → 右边匹配左边
5
6 bool vis[maxn]; // 右半的点是否已被访问
7 int dx[maxn], dy[maxn];
8 int q[maxn];
9
10 bool bfs(int n) {
11     memset(dx, -1, sizeof(int) * (n + 1));
12     memset(dy, -1, sizeof(int) * (n + 1));
13
14     int head = 0, tail = 0;
15     for (int i = 1; i <= n; i++) {
16         if (!girl[i]) {
17             q[tail++] = i;
18             dx[i] = 0;
19         }
20     }
21
22     bool flag = false;
23
24     while (head != tail) {
25         int x = q[head++];
26
27         for (auto y : G[x])
28             if (dy[y] == -1) {
29                 dy[y] = dx[x] + 1;
30
31                 if (boy[y]) {
32                     if (dx[boy[y]] == -1) {
33                         dx[boy[y]] = dy[y] + 1;
34                         q[tail++] = boy[y];
35                     }
36                 } else
37                     flag = true;
38             }
39     }
40
41     return flag;
42 }
```

### 3.5 二分图

### 3.5.1 匈牙利

```
1 vector<int> G[maxn];
2
3 int girl[maxn], boy[maxn]; // 男孩在左边, 女孩在右边
4 bool vis[maxn];
5
```

```

37     }
38 }
39
40     return flag;
41 }
42
43 bool dfs(int x) {
44     for (int y : G[x])
45         if (!vis[y] && dy[y] == dx[x] + 1) {
46             vis[y] = true;
47
48             if (boy[y] && !dfs(boy[y]))
49                 continue;
50
51             girl[x] = y;
52             boy[y] = x;
53             return true;
54         }
55
56     return false;
57 }
58
59 int hopcroft_karp(int n) {
60     int ans = 0;
61
62     for (int x = 1; x ≤ n; x++) // 先贪心求出一组初始匹
→ 配, 当然不写贪心也行
63         for (int y : G[x])
64             if (!boy[y]) {
65                 girl[x] = y;
66                 boy[y] = x;
67                 ans++;
68                 break;
69             }
70
71     while (bfs(n)) {
72         memset(vis, 0, sizeof(bool) * (n + 1));
73
74         for (int x = 1; x ≤ n; x++)
75             if (!girl[x])
76                 ans += dfs(x);
77     }
78
79     return ans;
80 }

```

```

20     return false;
21 }
22
23     while (y) {
24         boy[y] = p[y];
25         swap(y, girl[p[y]]);
26     }
27
28     return true;
29 }
30
31 // bfs每个点
32 void bfs(int x) {
33     memset(q, 0, sizeof(q));
34     head = tail = 0;
35
36     q[tail++] = x;
37     visx[x] = true;
38
39     while (true) {
40         while (head != tail) {
41             int x = q[head++];
42
43             for (int y = 1; y ≤ N; y++)
44                 if (!visy[y]) {
45                     long long d = lx[x] + ly[y] - w[x]
→ [y];
46
47                     if (d < slack[y]) {
48                         p[y] = x;
49                         slack[y] = d;
50
51                         if (!slack[y] && check(y))
52                             return;
53                     }
54                 }
55         }
56
57         long long d = INF;
58         for (int i = 1; i ≤ N; i++)
59             if (!visy[i])
60                 d = min(d, slack[i]);
61
62         for (int i = 1; i ≤ N; i++) {
63             if (visx[i])
64                 lx[i] -= d;
65
66             if (visy[i])
67                 ly[i] += d;
68             else
69                 slack[i] -= d;
70         }
71
72         for (int i = 1; i ≤ N; i++)
73             if (!visy[i] && !slack[i] && check(i))
74                 return;
75     }
76
77 // 主过程
78 long long KM() {
79     for (int i = 1; i ≤ N; i++) {
80         // lx[i] = 0;
81         ly[i] = -INF;
82         // boy[i] = girl[i] = -1;
83
84         for (int j = 1; j ≤ N; j++)
85             ly[i] = max(ly[i], w[j][i]);

```

### 3.5.3 KM二分图最大权匹配

```

1 const long long INF = 0x3f3f3f3f3f3f3f3f;
2
3 long long w[maxn][maxn], lx[maxn], ly[maxn],
→ slack[maxn];
4 // 边权 顶标 slack
5 // 如果要求最大权完美匹配就把不存在的边设为-INF, 否则所有
→ 边对0取max
6
7 bool visx[maxn], visy[maxn];
8
9 int boy[maxn], girl[maxn], p[maxn], q[maxn], head,
→ tail; // p : pre
10
11 int n, m, N, e;
12
13 // 增广
14 bool check(int y) {
15     visy[y] = true;
16
17     if (boy[y]) {
18         visx[boy[y]] = true;
19         q[tail++] = boy[y];

```

```

87 }
88
89 for (int i = 1; i ≤ N; i++) {
90     memset(slack, 0x3f, sizeof(slack));
91     memset(visx, 0, sizeof(visx));
92     memset(visy, 0, sizeof(visy));
93     bfs(i);
94 }
95
96 long long ans = 0;
97 for (int i = 1; i ≤ N; i++)
98     ans += w[i][girl[i]];
99 return ans;
}
100
101 // 为了方便贴上主函数
102 int main() {
103
104     scanf("%d%d%d", &n, &m, &e);
105     N = max(n, m);
106
107     while (e--) {
108         int x, y, c;
109         scanf("%d%d%d", &x, &y, &c);
110         w[x][y] = max(c, 0);
111     }
112 }
113
114 printf("%lld\n", KM());
115
116 for (int i = 1; i ≤ n; i++) {
117     if (i > 1)
118         printf(" ");
119     printf("%d", w[i][girl[i]] > 0 ? girl[i] : 0);
120 }
121 printf("\n");
122
123 return 0;
}

```

### 3.5.4 二分图原理

- **最大匹配的可行边与必须边, 关键点**

以下的“残量网络”指网络流图的残量网络.

- 可行边: 一条边的两个端点在残量网络中处于同一个SCC, 不论是正向边还是反向边.
- 必须边: 一条属于当前最大匹配的边, 且残量网络中两个端点不在同一个SCC中.
- 关键点(必须点): 这里不考虑网络流图而只考虑原始的图, 将匹配边改成从右到左之后从左边的每个未匹配点进行floodfill, 左边没有被标记的点即为关键点. 右边同理.

- **独立集**

二分图独立集可以看成最小割问题, 割掉最少的点使得S和T不连通, 则剩下的点自然都在独立集中.

所以独立集输出方案就是求出不在最小割中的点, 独立集的必须点/可行点就是最小割的不可行点/非必须点.

割点等价于割掉它与源点或汇点相连的边, 可以通过设置中间的边权为无穷以保证不能割掉中间的边, 然后按照上面的方法判断即可. (由于一个点最多流出一个流量, 所以中间的边权其实是可以任取的.)

- **二分图最大权匹配**

二分图最大权匹配的对偶问题是最小顶标和问题, 即: 为图中的每个顶点赋予一个非负顶标, 使得对于任意一条边, 两端点的顶标和都要不小于边权, 最小化顶标之和.

显然KM算法的原理实际上就是求最小顶标和.

## 3.6 一般图匹配

### 3.6.1 高斯消元

```

1 // 这个算法基于Tutte定理和高斯消元, 思维难度相对小一些,
2 // →也更方便进行可行边的判定
3 // 注意这个算法复杂度是满的, 并且常数有点大, 而带花树通常
4 // →是跑不满的
5 // 以及, 根据Tutte定理, 如果求最大匹配的大小的话直接输出
6 // →出Tutte矩阵的秩/2即可
7 // 需要输出方案时才需要再写后面那些乱七八糟的东西
8
9 // 复杂度和常数所限, 1s之内500已经是这个算法的极限了
10 const int maxn = 505, p = 1000000007; // p可以是任意10^9以内的质数
11
12 // 全局数组和变量定义
13 int A[maxn][maxn], B[maxn][maxn], t[maxn][maxn],
14 // →id[maxn], a[maxn];
15 bool row[maxn] = {false}, col[maxn] = {false};
16 int n, m, girl[maxn]; // girl是匹配点, 用来输出方案
17
18 // 为了方便使用, 贴上主函数
19 // 需要调用高斯消元和eliminate
20 int main() {
21     srand(19260817);
22
23     scanf("%d%d", &n, &m); // 点数和边数
24     while (m--) {
25         int x, y;
26         scanf("%d%d", &x, &y);
27         A[x][y] = rand() % p;
28         A[y][x] = -A[x][y]; // Tutte矩阵是反对称矩阵
29     }
30
31     for (int i = 1; i ≤ n; i++)
32         id[i] = i; // 输出方案用的, 因为高斯消元的时候会
33         →交换列
34     memcpy(t, A, sizeof(t));
35     Gauss(A, NULL, n);
36
37     m = n;
38     n = 0; // 这里变量复用纯属个人习惯
39
40     for (int i = 1; i ≤ m; i++) {
41         if (A[id[i]][id[i]])
42             a[++n] = i; // 找出一个极大满秩子矩阵
43
44     for (int i = 1; i ≤ n; i++) {
45         for (int j = 1; j ≤ n; j++)
46             A[i][j] = t[a[i]][a[j]];
47
48     Gauss(A, B, n);
49
50     for (int i = 1; i ≤ n; i++) {
51         if (!girl[a[i]])
52             for (int j = i + 1; j ≤ n; j++)
53                 if (!girl[a[j]] && t[a[i]][a[j]] &&
54                     →B[j][i]) {
55                     // 注意上面那句if的写法, 现在t是邻接
56                     →矩阵的备份,
57                     // 逆矩阵j行i列不为0当且仅当这条边可
58                     →行
59                     girl[a[i]] = a[j];
60                     girl[a[j]] = a[i];
61
62                     eliminate(i, j);
63                     eliminate(j, i);
64                     break;
65                 }
66             }
67         }
68     }
69 }

```

```

59     printf("%d\n", n / 2);
60     for (int i = 1; i <= m; i++)
61         printf("%d ", girl[i]);
62
63     return 0;
64 }
65
66 // 高斯消元 O(n^3)
67 // 在传入B时表示计算逆矩阵，传入NULL则只需计算矩阵的秩
68 void Gauss(int A[][maxn], int B[][maxn], int n) {
69     if(B) {
70         memset(B, 0, sizeof(t));
71         for (int i = 1; i <= n; i++)
72             B[i][i] = 1;
73     }
74
75
76     for (int i = 1; i <= n; i++) {
77         if (!A[i][i]) {
78             for (int j = i + 1; j <= n; j++)
79                 if (A[j][i])
80                     swap(id[i], id[j]);
81                 for (int k = i; k <= n; k++)
82                     swap(A[i][k], A[j][k]);
83
84             if (B)
85                 for (int k = 1; k <= n; k++)
86                     swap(B[i][k], B[j][k]);
87             break;
88         }
89
90         if (!A[i][i])
91             continue;
92     }
93
94     int inv = qpow(A[i][i], p - 2);
95
96     for (int j = 1; j <= n; j++)
97         if (i != j && A[j][i]) {
98             int t = (long long)A[j][i] * inv % p;
99
100            for (int k = i; k <= n; k++)
101                if (A[i][k])
102                    A[j][k] = (A[j][k] - (long
103                        long)t * A[i][k]) % p;
104
105            if (B)
106                for (int k = 1; k <= n; k++)
107                    if (B[i][k])
108                        B[j][k] = (B[j][k] - (long
109                            long)t * B[i][k]) % p;
110
111     if (B)
112         for (int i = 1; i <= n; i++) {
113             int inv = qpow(A[i][i], p - 2);
114
115             for (int j = 1; j <= n; j++)
116                 if (B[i][j])
117                     B[i][j] = (long long)B[i][j] * inv
118                         % p;
119     }
120
121 // 消去一行一列 O(n^2)
122 void eliminate(int r, int c) {
123     row[r] = col[c] = true; // 已经被消掉
124
125     int inv = qpow(B[r][c], p - 2);

```

```

126
127     for (int i = 1; i <= n; i++)
128         if (!row[i] && B[i][c]) {
129             int t = (long long)B[i][c] * inv % p;
130
131             for (int j = 1; j <= n; j++)
132                 if (!col[j] && B[r][j])
133                     B[i][j] = (B[i][j] - (long long)t *
134                         → B[r][j]) % p;
135     }

```

### 3.6.2 带花树

```

1 // 带花树通常比高斯消元快很多，但在只需要求最大匹配大小的
2 → 时候并没有高斯消元好写
3 // 当然输出方案要方便很多
4
5 // 全局数组与变量定义
6 vector<int> G[maxn];
7 int girl[maxn], f[maxn], t[maxn], p[maxn], vis[maxn],
8 → tim, q[maxn], head, tail;
9 int n, m;
10
11 // 封装好的主过程 O(nm)
12 int blossom() {
13     int ans = 0;
14
15     for (int i = 1; i <= n; i++)
16         if (!girl[i])
17             ans += bfs(i);
18
19     return ans;
20 }
21
22 // bfs找增广路 O(m)
23 bool bfs(int s) {
24     memset(t, 0, sizeof(t));
25     memset(p, 0, sizeof(p));
26
27     for (int i = 1; i <= n; i++)
28         f[i] = i; // 并查集
29
30     head = tail = 0;
31     q[tail++] = s;
32     t[s] = 1;
33
34     while (head != tail) {
35         int x = q[head++];
36         for (int y : G[x]) {
37             if (findroot(y) == findroot(x) || t[y] ==
38                 → 2)
39                 continue;
40
41             if (!t[y]) {
42                 t[y] = 2;
43                 p[y] = x;
44
45                 if (!girl[y]) {
46                     for (int u = y, t; u; u = t) {
47                         t = girl[p[u]];
48                         girl[p[u]] = u;
49                         girl[u] = p[u];
50
51                     }
52                 }
53             }
54         }
55     }
56
57     return true;
58 }

```

```

52         t[girl[y]] = 1;
53         q[tail++] = girl[y];
54     }
55     else if (t[y] == 1) {
56         int z = LCA(x, y);
57
58         shrink(x, y, z);
59         shrink(y, x, z);
60     }
61 }
62 }
63
64 return false;
65 }

//缩奇环 O(n)
66 void shrink(int x, int y, int z) {
67     while (findroot(x) != z) {
68         p[x] = y;
69         y = girl[x];
70
71         if (t[y] == 2) {
72             t[y] = 1;
73             q[tail++] = y;
74         }
75
76         if (findroot(x) == x)
77             f[x] = z;
78         if (findroot(y) == y)
79             f[y] = z;
80
81         x = p[y];
82     }
83 }

//暴力找LCA O(n)
84 int LCA(int x, int y) {
85     tim++;
86     while (true) {
87         if (x) {
88             x = findroot(x);
89
90             if (vis[x] == tim)
91                 return x;
92             else {
93                 vis[x] = tim;
94                 x = p[girl[x]];
95             }
96         }
97
98         swap(x, y);
99     }
100 }

//并查集的查找 O(1)
101 int findroot(int x) {
102     return x == f[x] ? x : (f[x] = findroot(f[x]));
103 }


```

### 3.6.3 带权带花树

Forked from the template of Imperisble Night.

(有一说一这玩意实在太难写了，抄之前建议先想想算法是不是假的或者有SB做法)

```

1 //maximum weight blossom, change g[u][v].w to INF -
2 //→ g[u][v].w when minimum weight blossom is needed
3 //type of ans is long long
4 //replace all int to long long if weight of edge is
5 //→ long long

```

```

60 }
61 inline void augment(int u, int v){
62     for(; ;){
63         int xnv=st[match[u]];
64         set_match(u, v);
65         if(!xnv) return;
66         set_match(xnv, st[pa[xnv]]);
67         u=st[pa[xnv]], v=xnv;
68     }
69 }
70 inline int get_lca(int u, int v){
71     static int t=0;
72     for(++; u || v; swap(u, v)){
73         if(u == 0) continue;
74         if(vis[u] == t) return u;
75         vis[u] = t;
76         u = st[match[u]];
77         if(u) u = st[pa[u]];
78     }
79     return 0;
80 }
81 inline void add_blossom(int u, int lca, int v){
82     int b = n + 1;
83     while(b <= n_x && st[b]) ++b;
84     if(b > n_x) ++n_x;
85     lab[b] = 0, S[b] = 0;
86     match[b] = match[lca];
87     flower[b].clear();
88     flower[b].push_back(lca);
89     for(int x = u, y; x != lca; x = st[pa[y]]) {
90         flower[b].push_back(x),
91         flower[b].push_back(y = st[match[x]]),
92         q_push(y);
93     }
94     reverse(flower[b].begin() + 1,
95             flower[b].end());
96     for(int x = v, y; x != lca; x = st[pa[y]]) {
97         flower[b].push_back(x),
98         flower[b].push_back(y = st[match[x]]),
99         q_push(y);
100    }
101    set_st(b, b);
102    for(int x = 1; x <= n_x; ++x) g[b][x].w = g[x]
103        .w = 0;
104    for(int x = 1; x <= n; ++x) flower_from[b][x] =
105        0;
106    for(size_t i = 0 ; i < flower[b].size(); ++i){
107        int xs = flower[b][i];
108        for(int x = 1; x <= n_x; ++x)
109            if(g[b][x].w == 0 || e_delta(g[xs][x])
110                < e_delta(g[b][x]))
111                g[b][x] = g[xs][x], g[x][b] = g[x]
112                    .w = xs;
113        for(int x = 1; x <= n; ++x)
114            if(flower_from[xs][x]) flower_from[b]
115                [x] = xs;
116    }
117    set_slack(b);
118 }
119 inline void expand_blossom(int b){ // S[b] == 1
120     for(size_t i = 0; i < flower[b].size(); ++i)
121         set_st(flower[b][i], flower[b][i]);
122     int xr = flower_from[b][g[b][pa[b]].u], pr =
123         get_pr(b, xr);
124     for(int i = 0; i < pr; i += 2){
125         int xs = flower[b][i], xns = flower[b][i +
126             1];
127         pa[xs] = g[xns][xs].u;
128         S[xs] = 1, S[xns] = 0;
129     }
130 }
131 inline void set_slack(int xns){
132     q_push(xns);
133 }
134 S[xr] = 1, pa[xr] = pa[b];
135 for(size_t i = pr + 1; i < flower[b].size(); +
136     i){
137     int xs = flower[b][i];
138     S(xs) = -1, set_slack(xs);
139 }
140 st[b] = 0;
141 }
142 inline bool on_found_edge(const edge &e){
143     int u = st[e.u], v = st[e.v];
144     if(S[v] == -1){
145         pa[v] = e.u, S[v] = 1;
146         int nu = st[match[v]];
147         slack[v] = slack[nu] = 0;
148         S[nu] = 0, q_push(nu);
149     }else if(S[v] == 0){
150         int lca = get_lca(u, v);
151         if(!lca) return augment(u, v), augment(v,
152             u), true;
153         else add_blossom(u, lca, v);
154     }
155     return false;
156 }
157 inline bool matching(){
158     memset(S + 1, -1, sizeof(int) * n_x);
159     memset(slack + 1, 0, sizeof(int) * n_x);
160     q = queue<int>();
161     for(int x = 1; x <= n_x; ++x)
162         if(st[x] == x && !match[x]) pa[x] = 0,
163             S[x] = 0, q.push(x);
164     if(q.empty()) return false;
165     for(; ;){
166         while(q.size()){
167             int u = q.front(); q.pop();
168             if(S[st[u]] == 1) continue;
169             for(int v = 1; v <= n; ++v)
170                 if(g[u][v].w > 0 && st[u] != st[v])
171                     if(e_delta(g[u][v]) == 0){
172                         if(on_found_edge(g[u]
173                             [v])) return true;
174                         }else update_slack(u, st[v]);
175         }
176         int d = INF;
177         for(int b = n + 1; b <= n_x; ++b)
178             if(st[b] == b && S[b] == 1)d = min(d,
179                 lab[b]/2);
180         for(int x = 1; x <= n_x; ++x)
181             if(st[x] == x && slack[x]){
182                 if(S[x] == -1)d = min(d,
183                     e_delta(g[slack[x]][x]));
184                 else if(S[x] == 0)d = min(d,
185                     e_delta(g[slack[x]][x])/2);
186             }
187         for(int u = 1; u <= n; ++u){
188             if(S[st[u]] == 0){
189                 if(lab[u] <= d) return 0;
190                 lab[u] -= d;
191             }else if(S[st[u]] == 1)lab[u] += d;
192         }
193         for(int b = n+1; b <= n_x; ++b)
194             if(st[b] == b){
195                 if(S[st[b]] == 0) lab[b] += d * 2;
196             }
197     }
198 }

```

```

180     |     |     |     |     else if(S[st[b]] == 1) lab[b] -= d
181     |     |     |     |     ↪ * 2;
182     |     |     |     |
183     |     |     |     q=queue<int>();
184     |     |     |     for(int x = 1; x ≤ n_x; ++x)
185     |     |     |     if(st[x] == x && slack[x] &&
186     |     |     |     ↪ st[slack[x]] != x &&
187     |     |     |     ↪ e_delta(g[slack[x]][x]) == 0)
188     |     |     |     if(on_found_edge(g[slack[x]])
189     |     |     |     ↪ [x]))return true;
190     |     |     |
191     |     |     for(int b = n + 1; b ≤ n_x; ++b)
192     |     |     if(st[b] == b && S[b] == 1 && lab[b] ==
193     |     |     ↪ 0)expand_blossom(b);
194     |
195     |     |     return false;
196     |
197     |     |     inline pair<long long, int> solve(){
198     |     |     memset(match + 1, 0, sizeof(int) * n);
199     |     |     n_x = n;
200     |     |     int n_matches = 0;
201     |     |     long long tot_weight = 0;
202     |     |     for(int u = 0; u ≤ n; ++u) st[u] = u,
203     |     |     ↪ flower[u].clear();
204     |     |     int w_max = 0;
205     |     |     for(int u = 1; u ≤ n; ++u)
206     |     |     for(int v = 1; v ≤ n; ++v){
207     |     |     |     flower_from[u][v] = (u == v ? u : 0);
208     |     |     |     w_max = max(w_max, g[u][v].w);
209     |     |     }
210     |     |     for(int u = 1; u ≤ n; ++u) lab[u] = w_max;
211     |     |     while(matching()) ++n_matches;
212     |     |     for(int u = 1; u ≤ n; ++u)
213     |     |     if(match[u] && match[u] < u)
214     |     |     tot_weight += g[u][match[u]].w;
215     |     |     return make_pair(tot_weight, n_matches);
216     |
217     |     |     inline void init(){
218     |     |     for(int u = 1; u ≤ n; ++u)
219     |     |     for(int v = 1; v ≤ n; ++v)
220     |     |     |     g[u][v]=edge(u, v, 0);
221     |
222     |     |     }
223     |
224     |     |     }
225     |

```

### 3.6.4 原理

设图 $G$ 的Tutte矩阵是 $\tilde{A}$ , 首先是最基础的引理:

- $G$ 的最大匹配大小是 $\frac{1}{2}\text{rank}\tilde{A}$ .
- $(\tilde{A}^{-1})_{i,j} \neq 0$ 当且仅当 $G - \{v_i, v_j\}$ 有完美匹配.  
(考虑到逆矩阵与伴随矩阵的关系, 这是显然的.)

构造最大匹配的方法见板子. 对于更一般的问题, 可以借助构造方法转化为完美匹配问题.

设最大匹配的大小为 $k$ , 新建 $n - 2k$ 个辅助点, 让它们和其他所有点连边, 那么如果一个点匹配了一个辅助点, 就说明它在原图的匹配中不匹配任何点.

- 最大匹配的可行边: 对原图中的任意一条边 $(u, v)$ , 如果删掉 $u, v$ 后新图仍然有完美匹配(也就是 $\tilde{A}_{u,v}^{-1} \neq 0$ ), 则它是一条可行边.
- 最大匹配的必须边: 待补充
- 最大匹配的必须点: 可以删掉这个点和一个辅助点, 然后判断剩下的图是否还有完美匹配, 如果有则说明它不是必须的, 否则是必须的. 只需要用到逆矩阵即可.
- 最大匹配的可行点: 显然对于任意一个点, 只要它不是孤立点, 就是可行点.

## 3.7 支配树

记得建反图!

```

1 vector<int> G[maxn], R[maxn], son[maxn]; // R是反图,
2     ↪ son存的是sdom树上的儿子
3
4     int ufs[maxn];
5
6     int idom[maxn], sdom[maxn], anc[maxn]; // anc:
7     ↪ sdom的dfn最小的祖先
8
9     int p[maxn], dfn[maxn], id[maxn], tim;
10
11    int findufs(int x) {
12        if (ufs[x] == x)
13            return x;
14
15        int t = ufs[x];
16        ufs[x] = findufs(ufs[x]);
17
18        if (dfn[sdom[anc[x]]] > dfn[sdom[anc[t]]])
19            anc[x] = anc[t];
20
21    return ufs[x];
22}
23
24    void dfs(int x) {
25        dfn[x] = ++tim;
26        id[tim] = x;
27        sdom[x] = x;
28
29        for (int y : G[x])
30            if (!dfn[y]) {
31                p[y] = x;
32                dfs(y);
33            }
34
35    void get_dominator(int n) {
36        for (int i = 1; i ≤ n; i++)
37            ufs[i] = anc[i] = i;
38
39        dfs(1);
40
41        for (int i = n; i > 1; i--) {
42            int x = id[i];
43
44            for (int y : R[x])
45                if (dfn[y]) {
46                    findufs(y);
47                    if (dfn[sdom[x]] > dfn[sdom[anc[y]]])
48                        sdom[x] = sdom[anc[y]];
49
50
51            son[sdom[x]].push_back(x);
52            ufs[x] = p[x];
53
54            for (int u : son[p[x]]) {
55                findufs(u);
56                idom[u] = (sdom[u] == sdom[anc[u]] ? p[x] :
57                           ↪ anc[u]);
58            }
59
60            son[p[x]].clear();
61
62            for (int i = 2; i ≤ n; i++) {
63                int x = id[i];
64
65                if (idom[x] != sdom[x])

```

```

65     |     idom[x] = idom[idom[x]];
66
67     |     son[idom[x]].push_back(x);
68 }

```

## 3.8 2-SAT

如果限制满足对称性(每个命题的逆否命题对应的边也存在), 那么可以使用Tarjan算法求SCC搞定.

具体来说就是, 如果某个变量的两个点在同一SCC中则显然无解, 否则按拓扑序倒序尝试选择每个SCC即可.

由于Tarjan算法的特性, 找到SCC的顺序就是拓扑序倒序, 所以判断完是否有解之后, 每个变量只需要取SCC编号较小的那个.

```

1 if (!ok)
2     printf("IMPOSSIBLE\n");
3 else {
4     printf("POSSIBLE\n");
5
6     for (int i = 1; i <= n; i++)
7         printf("%d%c", sccid[i * 2 - 1] > sccid[i * 2],
8                i < n ? ' ' : '\n');
9 }

```

如果要字典序最小就用DFS, 注意可以压位优化. 另外代码是0-base的.

```

1 bool vis[maxn];
2 int stk[maxn], top;
3
4 // 主函数
5 for (int i = 0; i < n; i += 2)
6     if (!vis[i] && !vis[i ^ 1]) {
7         top = 0;
8         if (!dfs(i)) {
9             while (top)
10                 vis[stk[top--]] = false;
11
12             if (!dfs(i + 1)) {
13                 bad = true;
14                 break;
15             }
16         }
17     }
18 // 最后stk中的所有元素就是选中的值
19
20 // dfs
21 bool dfs(int x) {
22     if (vis[x ^ 1])
23         return false;
24
25     if (vis[x])
26         return true;
27
28     vis[x] = true;
29     stk[++top] = x;
30
31     for (int i = 0; i < (int)G[x].size(); i++)
32         if (!dfs(G[x][i]))
33             return false;
34
35     return true;
36 }

```

## 3.9 最大流

### 3.9.1 Dinic

```

1 // 注意Dinic适用于二分图或分层图, 对于一般稀疏图ISAP更
2 // →优, 稠密图则HPP更优
3
4 struct edge {
5     int to, cap, prev;
6 } e[maxe * 2];
7
8 int last[maxn], len, d[maxn], cur[maxn], q[maxn];
9
10 // main函数里要初始化
11 memset(last, -1, sizeof(last));
12
13 void AddEdge(int x, int y, int z) {
14     e[len].to = y;
15     e[len].cap = z;
16     e[len].prev = last[x];
17     last[x] = len++;
18 }
19
20 void addedge(int x, int y, int z) {
21     AddEdge(x, y, z);
22     AddEdge(y, x, 0);
23 }
24
25 void bfs() {
26     int head = 0, tail = 0;
27     memset(d, -1, sizeof(int) * (t + 5));
28     q[tail++] = s;
29     d[s] = 0;
30
31     while (head != tail) {
32         int x = q[head++];
33         for (int i = last[x]; ~i; i = e[i].prev)
34             if (e[i].cap > 0 && d[e[i].to] == -1)
35                 d[e[i].to] = d[x] + 1;
36                 q[tail++] = e[i].to;
37     }
38 }
39
40 int dfs(int x, int a) {
41     if (x == t || !a)
42         return a;
43
44     int flow = 0, f;
45     for (int &i = cur[x]; ~i; i = e[i].prev)
46         if (e[i].cap > 0 && d[e[i].to] == d[x] + 1 &&
47             (f = dfs(e[i].to, min(e[i].cap, a))) > 0) {
48             e[i].cap -= f;
49             e[i ^ 1].cap += f;
50             flow += f;
51             a -= f;
52
53             if (!a)
54                 break;
55         }
56
57     return flow;
58 }
59
60 int Dinic() {
61     int flow = 0;
62     while (bfs(), ~d[t]) {
63         memcpy(cur, last, sizeof(int) * (t + 5));
64         flow += dfs(s, inf);
65     }
66 }

```

```

65     }
66     return flow;
67 }
```

### 3.9.2 ISAP

可能有毒，慎用。

```

1 // 注意ISAP适用于一般稀疏图，对于二分图或分层图情
2 // → 况Dinic比较优，稠密图则HLPP更优
3
4 // 边的定义
5 // 这里没有记录起点和反向边，因为反向边即为正向边xor 1,
6 // → 起点即为反向边的终点
7 struct edge {
8     int to, cap, prev;
9 } e[maxe * 2];
10
11 // 全局变量和数组定义
12 int last[maxn], cnte = 0, d[maxn], p[maxn], c[maxn],
13 // → cur[maxn], q[maxn];
14 int n, m, s, t; // s, t一定要开成全局变量
15
16 void AddEdge(int x, int y, int z) {
17     e[cnte].to = y;
18     e[cnte].cap = z;
19     e[cnte].prev = last[x];
20     last[x] = cnte++;
21 }
22
23 void addedge(int x, int y, int z) {
24     AddEdge(x, y, z);
25     AddEdge(y, x, 0);
26 }
27
28 // 预处理到t的距离标号
29 // 在测试数据组数较少时可以省略，把所有距离标号初始化为0
30 void bfs() {
31     memset(d, -1, sizeof(d));
32
33     int head = 0, tail = 0;
34     d[t] = 0;
35     q[tail++] = t;
36
37     while (head != tail) {
38         int x = q[head++];
39         c[d[x]]++;
40
41         for (int i = last[x]; ~i; i = e[i].prev)
42             if (e[i ^ 1].cap && d[e[i].to] == -1) {
43                 d[e[i].to] = d[x] + 1;
44                 q[tail++] = e[i].to;
45             }
46     }
47
48 // augment函数 O(n) 沿增广路增广一次，返回增广的流量
49 int augment() {
50     int a = (~0u) >> 1; // INT_MAX
51
52     for (int x = t; x != s; x = e[p[x] ^ 1].to)
53         a = min(a, e[p[x]].cap);
54
55     for (int x = t; x != s; x = e[p[x] ^ 1].to) {
56         e[p[x]].cap -= a;
57         e[p[x] ^ 1].cap += a;
58     }
59
60     return a;
61 }
```

```

60 // 主过程 O(n^2 m)，返回最大流的流量
61 // 注意这里的n是编号最大值，在这个值不为n的时候一定要开个
62 // → 变量记录下来并修改代码
63 int ISAP() {
64     bfs();
65
66     memcpy(cur, last, sizeof(cur));
67
68     int x = s, flow = 0;
69
70     while (d[s] < n) {
71         if (x == t) { // 如果走到了t就增广一次，并返
72             // → 回s重新找增广路
73             flow += augment();
74             x = s;
75         }
76
77         bool ok = false;
78         for (int &i = cur[x]; ~i; i = e[i].prev)
79             if (e[i].cap && d[x] == d[e[i].to] + 1) {
80                 p[e[i].to] = i;
81                 x = e[i].to;
82
83                 ok = true;
84                 break;
85             }
86
87         if (!ok) { // 修改距离标号
88             int tmp = n - 1;
89             for (int i = last[x]; ~i; i = e[i].prev)
90                 if (e[i].cap)
91                     tmp = min(tmp, d[e[i].to] + 1);
92
93             if (!--c[d[x]])
94                 break; // gap优化，一定要加上
95
96             c[d[x]] = tmp++;
97             cur[x] = last[x];
98
99             if (x != s)
100                 x = e[p[x] ^ 1].to;
101         }
102     }
103
104     // 重要！main函数最前面一定要加上如下初始化
105     memset(last, -1, sizeof(last));
106 }
```

### 3.9.3 HLPP 最高标号预流推进

```

1 constexpr int maxn = 1205, maxe = 120005;
2
3 struct edge {
4     int to, cap, prev;
5 } e[maxe * 2];
6
7 int n, m, s, t;
8 int last[maxn], cnte;
9 int h[maxn], gap[maxn * 2];
10 long long ex[maxn]; // 多余流量
11 bool inq[maxn];
12
13 struct cmp {
14     bool operator()(int x, int y) const {
15         return h[x] < h[y];
16     }
17 };
```

```

18 priority_queue<int, vector<int>, cmp> heap;
19
20 void adde(int x, int y, int z) {
21     e[cnte].to = y;
22     e[cnte].cap = z;
23     e[cnte].prev = last[x];
24     last[x] = cnte++;
25 }
26
27 void addedge(int x, int y, int z) {
28     adde(x, y, z);
29     adde(y, x, 0);
30 }
31
32 bool bfs() {
33     static int q[maxn];
34
35     fill(h, h + n + 1, 2 * n); // 如果没有全局的n, 记得
36     → 改这里
37     int head = 0, tail = 0;
38     q[tail++] = t;
39     h[t] = 0;
40
41     while (head < tail) {
42         int x = q[head++];
43         for (int i = last[x]; ~i; i = e[i].prev)
44             if (e[i ^ 1].cap && h[e[i].to] > h[x] + 1)
45                 → {
46                     h[e[i].to] = h[x] + 1;
47                     q[tail++] = e[i].to;
48                 }
49
50     return h[s] < 2 * n;
51 }
52
53 void push(int x) {
54     for (int i = last[x]; ~i; i = e[i].prev)
55         if (e[i].cap && h[x] == h[e[i].to] + 1) {
56             int d = min(ex[x], (long long)e[i].cap);
57
58             e[i].cap -= d;
59             e[i ^ 1].cap += d;
60             ex[x] -= d;
61             ex[e[i].to] += d;
62
63             if (e[i].to != s && e[i].to != t &&
64                 → !inq[e[i].to]) {
65                 heap.push(e[i].to);
66                 inq[e[i].to] = true;
67             }
68
69             if (!ex[x])
70                 break;
71 }
72
73 void relabel(int x) {
74     h[x] = 2 * n;
75
76     for (int i = last[x]; ~i; i = e[i].prev)
77         if (e[i].cap)
78             h[x] = min(h[x], h[e[i].to] + 1);
79 }
80
81 long long hlpp() {
82     if (!bfs())
83         return 0;
84
85     // memset(gap, 0, sizeof(int) * 2 * n);
86     h[s] = n;
87
88     for (int i = 1; i ≤ n; i++)
89         gap[h[i]]++;
90
91     for (int i = last[s]; ~i; i = e[i].prev)
92         if (e[i].cap) {
93             int d = e[i].cap;
94
95             e[i].cap -= d;
96             e[i ^ 1].cap += d;
97             ex[s] -= d;
98             ex[e[i].to] += d;
99
100            if (e[i].to != s && e[i].to != t &&
101                → !inq[e[i].to]) {
102                 heap.push(e[i].to);
103                 inq[e[i].to] = true;
104             }
105
106         while (!heap.empty()) {
107             int x = heap.top();
108             heap.pop();
109             inq[x] = false;
110
111             push(x);
112             if (ex[x]) {
113                 if (!--gap[h[x]]) { // gap
114                     for (int i = 1; i ≤ n; i++)
115                         if (i != s && i != t && h[i] >
116                             → h[x])
117                             h[i] = n + 1;
118
119             relabel(x);
120             ++gap[h[x]];
121             heap.push(x);
122             inq[x] = true;
123         }
124
125         return ex[t];
126     }
127
128 //记得初始化
129 memset(last, -1, sizeof(last));
130

```

## 3.10 费用流

### 3.10.1 SPFA费用流

```

1 constexpr int maxn = 20005, maxm = 200005;
2
3 struct edge {
4     int to, prev, cap, w;
5 } e[maxm * 2];
6
7 int last[maxn], cnte, d[maxn], p[maxn]; // 记得把last初
8     → 始化成-1, 不然会死循环
9     bool inq[maxn];
10
11 void spfa(int s) {
12
13     memset(d, -63, sizeof(d));
14     memset(p, -1, sizeof(p));
15     queue<int> q;

```

```

16
17     q.push(s);
18     d[s] = 0;
19
20     while (!q.empty()) {
21         int x = q.front();
22         q.pop();
23         inq[x] = false;
24
25         for (int i = last[x]; ~i; i = e[i].prev)
26             if (e[i].cap) {
27                 int y = e[i].to;
28
29                 if (d[x] + e[i].w > d[y]) {
30                     p[y] = i;
31                     d[y] = d[x] + e[i].w;
32                     if (!inq[y])
33                         q.push(y);
34                     inq[y] = true;
35                 }
36             }
37     }
38 }
39
40 int mcmf(int s, int t) {
41     int ans = 0;
42
43     while (spfa(s), d[t] > 0) {
44         int flow = 0x3f3f3f3f;
45         for (int x = t; x != s; x = e[p[x] ^ 1].to)
46             flow = min(flow, e[p[x]].cap);
47
48         ans += flow * d[t];
49
50         for (int x = t; x != s; x = e[p[x] ^ 1].to) {
51             e[p[x]].cap -= flow;
52             e[p[x] ^ 1].cap += flow;
53         }
54     }
55
56     return ans;
57 }
58
59 void add(int x, int y, int c, int w) {
60     e[cnte].to = y;
61     e[cnte].cap = c;
62     e[cnte].w = w;
63
64     e[cnte].prev = last[x];
65     last[x] = cnte++;
66 }
67
68 void addedge(int x, int y, int c, int w) {
69     add(x, y, c, w);
70     add(y, x, 0, -w);
71 }
72

```

### 3.10.2 Dijkstra费用流

有的地方也叫原始-对偶费用流。

原理和求多源最短路的Johnson算法是一样的，都是给每个点维护一个势 $h_u$ ，使得对任何有向边 $u \rightarrow v$ 都满足 $w + h_u - h_v \geq 0$ 。

如果有负费用则从 $s$ 开始跑一遍SPFA初始化，否则可以直接初始化 $h_u = 0$ 。

每次增广时得到的路径长度就是 $d_{s,t} + h_t$ ，增广之后让所有 $h_u = h'_u + d'_{s,u}$ ，直到 $d_{s,t} = \infty$ (最小费用最大流)或 $d_{s,t} \geq 0$ (最小费用流)为止。

注意最大费用流要转成取负之后的最小费用流，因为Dijkstra求的是最短路。

```

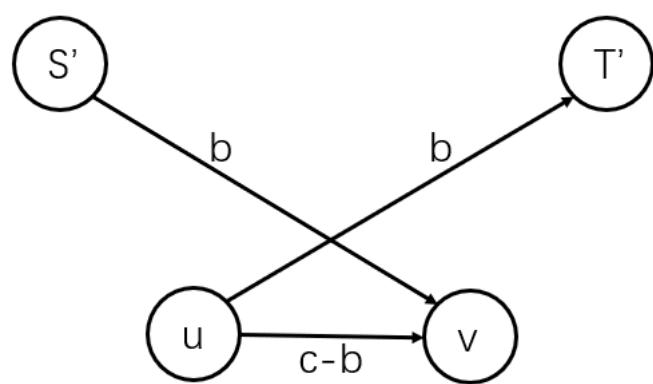
1 struct edge {
2     int to, cap, prev, w;
3 } e[maxe * 2];
4
5 int last[maxn], cnte;
6
7 long long d[maxn], h[maxn];
8 int p[maxn];
9
10 bool vis[maxn];
11 int s, t;
12
13 void Adde(int x, int y, int z, int w) {
14     e[cnte].to = y;
15     e[cnte].cap = z;
16     e[cnte].w = w;
17     e[cnte].prev = last[x];
18     last[x] = cnte++;
19 }
20
21 void addedge(int x, int y, int z, int w) {
22     Adde(x, y, z, w);
23     Adde(y, x, 0, -w);
24 }
25
26 void dijkstra() {
27     memset(d, 63, sizeof(d));
28     memset(vis, 0, sizeof(vis));
29
30     priority_queue<pair<long long, int>> heap;
31
32     d[s] = 0;
33     heap.push(make_pair(0ll, s));
34
35     while (!heap.empty()) {
36         int x = heap.top().second;
37         heap.pop();
38
39         if (vis[x])
40             continue;
41
42         vis[x] = true;
43         for (int i = last[x]; ~i; i = e[i].prev)
44             if (e[i].cap > 0 && d[e[i].to] > d[x] +
45                 → e[i].w + h[x] - h[e[i].to]) {
46                 d[e[i].to] = d[x] + e[i].w + h[x] -
47                 → h[e[i].to];
48                 p[e[i].to] = i;
49                 heap.push(make_pair(-d[e[i].to],
50                                     → e[i].to));
51             }
52     }
53
54 pair<long long, long long> mcmf() {
55     /*
56     spfa();
57     for (int i = 1; i <= t; i++)
58         h[i] = d[i];
59     // 如果初始有负权就像这样跑一遍SPFA预处理
60     */
61
62     long long flow = 0, cost = 0;
63
64     while (dijkstra(), d[t] < 0x3f3f3f3f) {
65         for (int i = 1; i <= t; i++)
66             h[i] += d[i];
67
68         int a = 0x3f3f3f3f;
69     }
70 }
71
72

```

```

67
68     for (int x = t; x != s; x = e[p[x] ^ 1].to)
69         a = min(a, e[p[x]].cap);
70
71     flow += a;
72     cost += (long long)a * h[t];
73
74     for (int x = t; x != s; x = e[p[x] ^ 1].to) {
75         e[p[x]].cap -= a;
76         e[p[x] ^ 1].cap += a;
77     }
78
79 }
80
81     return make_pair(flow, cost);
82 }
83
84 // 记得初始化
85 memset(last, -1, sizeof(last));

```



然后从 $S'$ 到 $S$ , 从 $T$ 到 $T'$ 分别连容量为正无穷的边即可.

因为附加边实际上算了两次流量, 所以最终答案应该减掉所有下界之和.

#### 有源汇上下界最小流

按照上面的方法转换后先跑一遍最大流, 然后撤掉超级源汇和附加边, 反过来跑一次最大流退流, 最大流减去退掉的流量就是最小流.

#### 无源汇上下界可行流

转化方法和上面的图是一样的, 只不过不需要考虑原有的源汇了.

在新图跑一遍最大流之后检查一遍辅助边, 如果有辅助边没满流则无解, 否则把每条边的流量加上 $b$ 就是一组可行方案.

## 3.11 网络流原理

### 3.11.1 最大流

- 判断一条边是否必定满流

在残量网络中跑一遍Tarjan, 如果某条满流边的两端处于同一SCC中则说明它不一定满流. (因为可以找出包含反向边的环, 增广之后就不满流了.)

### 3.11.2 最小割

首先牢记最小割的定义: 选权值和尽量小的一些边, 使得删除这些边之后 $s$ 无法到达 $t$ .

- 最小割输出一种方案

在残量网络上从 $S$ 开始floodfill, 源点可达的记为 $S$ 集, 不可达的记为 $T$ , 如果一条边的起点在 $S$ 集而终点在 $T$ 集, 就将其加入最小割中.

- 最小割的可行边与必须边

- 可行边: 满流, 且残量网络上不存在 $u$ 到 $v$ 的路径, 也就是 $u$ 和 $v$ 不在同一SCC中. (实际上也就是最大流必定满流的边.)

- 必须边: 满流, 且残量网络上 $S$ 可达 $u$ ,  $v$ 可达 $T$ .

- 字典序最小的最小割

直接按字典序从小到大的顺序依次判断每条边能否在最小割中即可.

如果一条边是可行边, 我们就需要把它删掉, 同时进行退流,  $u \rightarrow s$ 和 $t \rightarrow v$ 都退掉等同于这条边容量的流量.

退流用Dinic实现即可.

### 3.11.3 费用流

### 3.11.4 上下界网络流

#### 有源汇上下界最大流

新建超级源汇 $S', T'$ , 然后如图所示转化每一条边.

### 3.11.5 常见建图方法

- 最大流/费用流

流量不是很多的时候可以理解成很多条路径, 并且每条边可以经过的次数有限.

- 最小割

常用的模型是**最大权闭合子图**. 当然它并不是万能的, 因为限制条件可以带权值.

- 如果某些点全部在 $S$ 集或者 $T$ 集则获得一个正的收益

把这个条件建成一个点, 向要求的点连 $\infty$ 边, 然后 $s$ 向它连 $\infty$ 边. (如果是 $T$ 集就都反过来)

那么如果它在 $S$ 集就一定满足它要求的点都在 $S$ 集, 反之如果是 $T$ 集亦然.

- 如果两个点不在同一集合中则需要付出代价

建双向边, 那显然如果它们不在同一集合中就需要割掉中间的边, 付出对应的代价.

- 二分图, 如果相邻的两个点在同一集合则需要付出代价

染色后给一半的点反转源汇, 就转换成上面的问题了.

### 3.11.6 例题

- 费用流

- 序列上选和尽量大的数, 但连续 $k$ 个数中最多选 $p$ 个.

费用流建图, 先建一条 $n + 1$ 个点的无限容量的链表示不选, 然后每个点往后面 $k$ 个位置连边, 答案是流量为 $p$ 的最大费用流. 因为条件等价于选 $p$ 次并且每次选的所有数间隔都至少是 $k$ .

- 还要求连续 $k$ 个数中最少选 $q$ 个.

任选一个位置把图前后切开就会发现通过截面的流量总和恰为 $p$ . 注意到如果走了最开始的链就代表不选, 因此要限制至少有 $q$ 的流量不走链, 那么只需要把链的容量改成 $p - q$ 就行了.

## 3.12 Prufer序列

对一棵有 $n \geq 2$ 个结点的树, 它的Prufer编码是一个长为 $n - 2$ , 且每个数都在 $[1, n]$ 内的序列.

构造方法: 每次选取编号最小的叶子结点, 记录它的父亲, 然后把它删掉, 直到只剩两个点为止. (并且最后剩的两个点一定有一个是 $n$ 号点.)

相应的，由Prufer编码重构树的方法：按顺序读入序列，每次选取编号最小的且度数为1的结点，把这个点和序列当前点连上，然后两个点剩余度数同时-1。

### Prufer编码的性质

- 每个至少2个结点的树都唯一对应一个Prufer编码. (当然也就可  
以做无根树哈希.)
  - 每个点在Prufer序列中出现的次数恰好是度数-1. 所以如果给定  
某些点的度数然后求方案数, 就可以用简单的组合数解决.

最后，构造和重构直接写都是 $O(n \log n)$ 的，想优化成线性需要一些技巧。

线性求Prufer序列代码:

```

// 0-based
vector<vector<int>> adj;
vector<int> parent;

void dfs(int v) {
    for (int u : adj[v]) {
        if (u != parent[v]) parent[u] = v, dfs(u);
    }
}

vector<int> pruefer_code() { // pruefer是德语
    int n = adj.size();
    parent.resize(n), parent[n - 1] = -1;
    dfs(n - 1);

    int ptr = -1;
    vector<int> degree(n);
    for (int i = 0; i < n; i++) {
        degree[i] = adj[i].size();
        if (degree[i] == 1 && ptr == -1) ptr = i;
    }

    vector<int> code(n - 2);
    int leaf = ptr;
    for (int i = 0; i < n - 2; i++) {
        int next = parent[leaf];
        code[i] = next;
        if (--degree[next] == 1 && next < ptr)
            leaf = next;
        else {
            ptr++;
            while (degree[ptr] != 1)
                ptr++;
            leaf = ptr;
        }
    }
    return code;
}

```

```
        while (degree[ptr] != 1)
            ptr++;
        leaf = ptr;
    }
edges.emplace_back(leaf, n - 1);
return edges;
```

### 3.13 弦图相关

Forked from the template of NEW CODE!!!.

1. 团数  $\leq$  色数，弦图团数 = 色数
  2. 设  $next(v)$  表示  $N(v)$  中最前的点。令  $w^*$  表示所有满足  $A \in B$  的  $w$  中最后的一个点，判断  $v \cup N(v)$  是否为极大团，只需判断是否存在一个  $w$ ，满足  $Next(w) = v$  且  $|N(v)| + 1 \leq |N(w)|$  即可。
  3. 最小染色：完美消除序列从后往前依次给每个点染色，给每个点染上可以染的最小的颜色
  4. 最大独立集：完美消除序列从前往后能选就选
  5. 弦图最大独立集数 = 最小团覆盖数，最小团覆盖：设最大独立集为  $\{p_1, p_2, \dots, p_t\}$ ，则  $\{p_1 \cup N(p_1), \dots, p_t \cup N(p_t)\}$  为最小团覆盖

线性重构树代码:

```
// 0-based
vector<pair<int, int>> pruefer_decode(vector<int> const
→ &code) {
    int n = code.size() + 2;
    vector<int> degree(n, 1);
    for (int i : code) degree[i]++;
    int ptr = 0;
    while (degree[ptr] != 1) ptr++;
    int leaf = ptr;
    vector<pair<int, int>> edges;
    for (int v : code) {
        edges.emplace_back(leaf, v);
        if (--degree[v] == 1 && v < ptr)
            leaf = v;
        else {
            ptr++;
        }
    }
}
```

# 4 数据结构

## 4.1 线段树

### 4.1.1 非递归线段树

- 如果  $M = 2^k$ , 则只能维护  $[1, M - 2]$  范围
- 找叶子:  $i$  对应的叶子就是  $i + M$
- 单点修改: 找到叶子然后向上跳
- 区间查询: 左右区间各扩展一位, 转换成开区间查询

```

1 int query(int l, int r) {
2     l += M - 1;
3     r += M + 1;
4
5     int ans = 0;
6     while (l ^ r != 1) {
7         ans += sum[l ^ 1] + sum[r ^ 1];
8
9         l >>= 1;
10        r >>= 1;
11    }
12
13    return ans;
14 }
```

区间修改要标记永久化, 并且求区间和和求最值的代码不太一样.

### 区间加, 区间求和

```

1 void update(int l, int r, int d) {
2     int len = 1, cntl = 0, cntr = 0; // cntl, cntr 是左右
3     → 两边分别实际修改的区间长度
4     for (l += n - 1, r += n + 1; l ^ r ^ 1; l >>= 1, r
5     → >>= 1, len <= 1) {
6         tree[l] += cntl * d, tree[r] += cntr * d;
7         if (~l & 1) tree[l ^ 1] += d * len, mark[l ^ 1]
8         → += d, cntl += len;
9         if (r & 1) tree[r ^ 1] += d * len, mark[r ^ 1]
10        → += d, cntr += len;
11    }
12
13    for (; l; l >>= 1, r >>= 1)
14        tree[l] += cntl * d, tree[r] += cntr * d;
15
16    int query(int l, int r) {
17        int ans = 0, len = 1, cntl = 0, cntr = 0;
18        for (l += n - 1, r += n + 1; l ^ r ^ 1; l >>= 1, r
19        → >>= 1, len <= 1) {
20            ans += cntl * mark[l] + cntr * mark[r];
21            if (~l & 1) ans += tree[l ^ 1], cntl += len;
22            if (r & 1) ans += tree[r ^ 1], cntr += len;
23        }
24
25        for (; l; l >>= 1, r >>= 1)
26            ans += cntl * mark[l] + cntr * mark[r];
27
28        return ans;
29    }
30 }
```

### 区间加, 区间求最大值

```

1 void update(int l, int r, int d) {
2     for (l += N - 1, r += N + 1; l ^ r ^ 1; l >>= 1, r
3     → >>= 1) {
4         if (l < N) {
5             tree[l] = max(tree[l < 1], tree[l < 1 |
6             → 1]) + mark[l];
7
8         }
9     }
10 }
```

```

5         tree[r] = max(tree[r < 1], tree[r < 1 |
6             → 1]) + mark[r];
7     }
8
9     if (~l & 1) {
10         tree[l ^ 1] += d;
11         mark[l ^ 1] += d;
12     }
13     if (r & 1) {
14         tree[r ^ 1] += d;
15         mark[r ^ 1] += d;
16     }
17
18     for (; l; l >= 1, r >= 1)
19         if (l < N) tree[l] = max(tree[l < 1], tree[l
20         → < 1 | 1]) + mark[l],
21             tree[r] = max(tree[r < 1], tree[r
22             → < 1 | 1]) + mark[r];
23
24 void query(int l, int r) {
25     int maxl = -INF, maxr = -INF;
26
27     for (l += N - 1, r += N + 1; l ^ r ^ 1; l >= 1, r
28     → >= 1) {
29         maxl += mark[l];
30         maxr += mark[r];
31
32         if (~l & 1)
33             maxl = max(maxl, tree[l ^ 1]);
34         if (r & 1)
35             maxr = max(maxr, tree[r ^ 1]);
36
37         while (l) {
38             maxl += mark[l];
39             maxr += mark[r];
40
41             l >= 1;
42             r >= 1;
43         }
44
45     }
46
47     return max(maxl, maxr);
48 }
```

### 4.1.2 线段树维护矩形并

为线段树的每个结点维护  $cover_i$  表示这个区间被完全覆盖的次数. 更新时分情况讨论, 如果当前区间已被完全覆盖则长度就是区间长度, 否则长度是左右儿子相加.

```

1 constexpr int maxn = 100005, maxm = maxn * 70;
2
3 int lc[maxm], rc[maxm], cover[maxm], sum[maxm], root,
4     → seg_cnt;
5 int s, t, d;
6
7 void refresh(int l, int r, int o) {
8     if (cover[o])
9         sum[o] = r - l + 1;
10    else
11        sum[o] = sum[lc[o]] + sum[rc[o]];
12
13 void modify(int l, int r, int &o) {
14     if (!o)
15         o = ++seg_cnt;
16
17     if (s <= l && t >= r) {
```

```

18     cover[o] += d;
19     refresh(l, r, o);
20
21     return;
22 }
23
24     int mid = (l + r) / 2;
25
26     if (s <= mid)
27         modify(l, mid, lc[o]);
28     if (t > mid)
29         modify(mid + 1, r, rc[o]);
30
31     refresh(l, r, o);
32 }
33
34 struct modi {
35     int x, l, r, d;
36
37     bool operator < (const modi &o) {
38         return x < o.x;
39     }
40 } a[maxn * 2];
41
42 int main() {
43
44     int n;
45     scanf("%d", &n);
46
47     for (int i = 1; i <= n; i++) {
48         int lx, ly, rx, ry;
49         scanf("%d%d%d%d", &lx, &ly, &rx, &ry);
50
51         a[i * 2 - 1] = {lx, ly + 1, rx, 1};
52         a[i * 2] = {rx, ly + 1, ry, -1};
53     }
54
55     sort(a + 1, a + n * 2 + 1);
56
57     int last = -1;
58     long long ans = 0;
59
60     for (int i = 1; i <= n * 2; i++) {
61         if (last != -1)
62             ans += (long long)(a[i].x - last) * sum[1];
63         last = a[i].x;
64
65         s = a[i].l;
66         t = a[i].r;
67         d = a[i].d;
68
69         modify(1, 1e9, root);
70     }
71
72     printf("%lld\n", ans);
73
74     return 0;
75 }
```

## 4.2 陈丹琦分治

### 4.2.1 动态图连通性(分治并查集)

```

1 vector<pair<int, int>> seg[(1 << 22) + 5];
2
3 int s, t;
4 pair<int, int> d;
5
6 void add(int l, int r, int o) {

```

```

7     if (s > t)
8         return;
9
10    if (s <= l && t >= r) {
11        seg[o].push_back(d);
12        return;
13    }
14
15    int mid = (l + r) / 2;
16
17    if (s <= mid)
18        add(l, mid, o * 2);
19    if (t > mid)
20        add(mid + 1, r, o * 2 + 1);
21 }
22
23 int ufs[maxn], sz[maxn], stk[maxn], top;
24
25 int findufs(int x) {
26     while (ufs[x] != x)
27         x = ufs[x];
28
29     return ufs[x];
30 }
31
32 void link(int x, int y) {
33     x = findufs(x);
34     y = findufs(y);
35
36     if (x == y)
37         return;
38
39     if (sz[x] < sz[y])
40         swap(x, y);
41
42     ufs[y] = x;
43     sz[x] += sz[y];
44     stk[++top] = y;
45 }
46
47 int ans[maxm];
48
49 void solve(int l, int r, int o) {
50     int tmp = top;
51
52     for (auto pi : seg[o])
53         link(pi.first, pi.second);
54
55     if (l == r)
56         ans[l] = top;
57     else {
58         int mid = (l + r) / 2;
59
60         solve(l, mid, o * 2);
61         solve(mid + 1, r, o * 2 + 1);
62     }
63
64     for (int i = top; i > tmp; i--) {
65         int x = stk[i];
66
67         sz[ufs[x]] -= sz[x];
68         ufs[x] = x;
69     }
70
71     top = tmp;
72 }
73
74 map<pair<int, int>, int> mp;
```

### 4.2.2 四维偏序

```

// 四维偏序
void CDQ1(int l, int r) {
    if (l >= r)
        return;

    int mid = (l + r) / 2;

    CDQ1(l, mid);
    CDQ1(mid + 1, r);

    int i = l, j = mid + 1, k = l;
    while (i <= mid && j <= r) {
        if (a[i].x < a[j].x) {
            a[i].ins = true;
            b[k++] = a[i++];
        }
        else {
            a[j].ins = false;
            b[k++] = a[j++];
        }
    }

    while (i <= mid) {
        a[i].ins = true;
        b[k++] = a[i++];
    }

    while (j <= r) {
        a[j].ins = false;
        b[k++] = a[j++];
    }

    copy(b + l, b + r + 1, a + l); // 后面的分治会破坏排序，所以要复制一份
}

CDQ2(l, r);
}

void CDQ2(int l, int r) {
    if (l >= r)
        return;

    int mid = (l + r) / 2;

    CDQ2(l, mid);
    CDQ2(mid + 1, r);

    int i = l, j = mid + 1, k = l;
    while (i <= mid && j <= r) {
        if (b[i].y < b[j].y) {
            if (b[i].ins)
                add(b[i].z, 1); // 树状数组

            t[k++] = b[i++];
        }
        else{
            if (!b[j].ins)
                ans += query(b[j].z - 1);

            t[k++] = b[j++];
        }
    }

    while (i <= mid) {
        if (b[i].ins)

```

```
68 |         add(b[i].z, 1);
69 |
70 |         t[k++] = b[i++];
71 |     }
72 |
73 |     while (j <= r) {
74 |         if (!b[j].ins)
75 |             ans += query(b[j].z - 1);
76 |
77 |         t[k++] = b[j++];
78 |     }
79 |
80 |     for (i = l; i <= mid; i++)
81 |         if (b[i].ins)
82 |             add(b[i].z, -1);
83 |
84 |     copy(t + l, t + r + 1, b + l);
85 }
```

### 4.3 整体二分

修改和询问都要划分，备份一下，递归之前copy回去。

如果是满足可减性的问题(例如查询区间 $k$ 小数)可以直接在划分的时候把询问的 $k$ 修改一下。否则需要维护一个全局的数据结构,一般来说可以先递归右边再递归左边,具体维护方法视情况而定。

以下代码以ZJOI K大数查询为例(区间都添加一个数, 询问区间k大数).

```

37     qk[i] -= tmp;
38     vl.push_back(i);
39 }
40 }
41
42 for (int i : vr)
43     if (op[i] == 1)
44         update(ql[i], qr[i], -1);
45
46 v.clear();
47
48 solve(l, mid, vl);
49 solve(mid + 1, r, vr);
50 }
51
52 int main() {
53     int n, m;
54     scanf("%d%d", &n, &m);
55
56     M = 1;
57     while (M < n + 2)
58         M *= 2;
59
60     for (int i = 1; i ≤ m; i++)
61         scanf("%d%d%d%lld", &op[i], &ql[i], &qr[i],
62               &qk[i]);
63
64     vector<int> v;
65     for (int i = 1; i ≤ m; i++)
66         v.push_back(i);
67
68     solve(1, 1e9, v);
69
70     for (int i = 1; i ≤ m; i++)
71         if (op[i] == 2)
72             printf("%d\n", ans[i]);
73
74     return 0;
75 }
```

```

20 // 伪构造函数 O(1)
21 // 为了方便，在结点类外面再定义一个伪构造函数
22 node *newnode(int x) { // 键值为x
23     ***ptr = node(x);
24     ptr → ch[0] = ptr → ch[1] = null;
25     return ptr;
26 }
27
28 // 插入键值 期望O(log n)
29 // 需要调用旋转
30 void insert(int x, node *&rt) { // rt为当前结点，建议调
31     ← 用时传入root, 下同
32     if (rt == null) {
33         rt = newnode(x);
34         return;
35     }
36     int d = x > rt → key;
37     insert(x, rt → ch[d]);
38     rt → refresh();
39
40     if (rt → ch[d] → p < rt → p)
41         rot(rt, d ^ 1);
42 }
43
44 // 删除一个键值 期望O(log n)
45 // 要求键值必须存在至少一个，否则会导致RE
46 // 需要调用旋转
47 void erase(int x, node *&rt) {
48     if (x == rt → key) {
49         if (rt → ch[0] != null && rt → ch[1] != null)
50             ← {
51                 int d = rt → ch[0] → p < rt → ch[1] →
52                     ← p;
53                 rot(rt, d);
54                 erase(x, rt → ch[d]);
55             }
56         else
57             rt = rt → ch[rt → ch[0] == null];
58     }
59     else
60         erase(x, rt → ch[x > rt → key]);
61
62     if (rt != null)
63         rt → refresh();
64
65 // 求元素的排名(严格小于键值的个数 + 1) 期望O(log n)
66 // 非递归
67 int rank(int x, node *rt) {
68     int ans = 1, d;
69     while (rt != null) {
70         if ((d = x > rt → key))
71             ans += rt → ch[0] → size + 1;
72         rt = rt → ch[d];
73     }
74
75     return ans;
76 }
77
78 // 返回排名第k(从1开始)的键值对应的指针 期望O(log n)
79 // 非递归
80 node *kth(int x, node *rt) {
81     int d;
82     while (rt != null) {
83         if (x == rt → ch[0] → size + 1)
84             return rt;
85         if ((d = x > rt → ch[0] → size))
86             ←
87 }
```

## 4.4 平衡树

pb\_ds平衡树参见8.11.Public Based DataStructure(PB\_DS)  
(83页).

### 4.4.1 Treap

```

1 // 注意：相同键值可以共存
2
3 struct node { // 结点类定义
4     int key, size, p; // 分别为键值，子树大小，优先度
5     node *ch[2]; // 0表示左儿子，1表示右儿子
6
7     node(int key = 0) : key(key), size(1), p(rand()) {}
8
9     void refresh() {
10         size = ch[0] → size + ch[1] → size + 1;
11     } // 更新子树大小(和附加信息，如果有的话)
12 } null[maxn], *root = null, *ptr = null; // 数组名叫
13 ← 做null是为了方便开哨兵节点
14 // 如果需要删除而空间不能直接开下所有结点，则需要再写一个
15 ← 垃圾回收
16 // 注意：数组里的元素一定不能delete，否则会导致RE
17 null → ch[0] = null → ch[1] = null;
18 null → size = 0;
19 }
```

```

87     |     |     x -= rt → ch[0] → size + 1;
88     |     |
89     |     rt = rt → ch[d];
90     |
91     |
92     return rt;
93 }
94
95 // 返回前驱(最大的比给定键值小的键值)对应的指针 期望O(log n)
96 // 非递归
97 node *pred(int x, node *rt) {
98     node *y = null;
99     int d;
100
101    while (rt != null) {
102        if ((d = x > rt → key))
103            y = rt;
104
105        rt = rt → ch[d];
106    }
107
108    return y;
109 }
110
111 // 返回后继(最小的比给定键值大的键值)对应的指针 期望O(log n)
112 // 非递归
113 node *succ(int x, node *rt) {
114     node *y = null;
115     int d;
116
117     while (rt != null) {
118         if ((d = x < rt → key))
119             y = rt;
120
121         rt = rt → ch[d ^ 1];
122     }
123
124     return y;
125 }
126
127 // 旋转(Treap版本) O(1)
128 // 平衡树基础操作
129 // 要求对应儿子必须存在, 否则会导致后续各种莫名其妙的问题
130 void rot(node *&x, int d) { // x为被转下去的结点, 会被修改以维护树结构
131     node *y = x → ch[d ^ 1];
132
133     x → ch[d ^ 1] = y → ch[d];
134     y → ch[d] = x;
135
136     x → refresh();
137     (x = y) → refresh();
138 }

```

```

13
14 node *copied(node *x) { // 如果不用可持久化的话, 直接用
15     →就行了
16     return new node(*x);
17 }
18
19 node *merge(node *x, node *y) {
20     if (x == null)
21         return y;
22     if (y == null)
23         return x;
24
25     node *z;
26     if (rand() % (x → size + y → size) < x → size)
27         z = copied(y);
28         z → ch[0] = merge(x, y → ch[0]);
29     else {
30         z = copied(x);
31         z → ch[1] = merge(x → ch[1], y);
32     }
33
34     z → refresh(); // 因为每次只有一边会递归到儿子, 所以z不可能取到null
35     return z;
36 }
37
38 pair<node*, node*> split(node *x, int k) { // 左边大小为k
39     if (x == null)
40         return make_pair(null, null);
41
42     pair<node*, node*> pi(null, null);
43
44     if (k ≤ x → ch[0] → size) {
45         pi = split(x → ch[0], k);
46
47         node *z = copied(x);
48         z → ch[0] = pi.second;
49         z → refresh();
50         pi.second = z;
51     }
52     else {
53         pi = split(x → ch[1], k - x → ch[0] → size - 1);
54
55         node *y = copied(x);
56         y → ch[1] = pi.first;
57         y → refresh();
58         pi.first = y;
59     }
60
61     return pi;
62 }
63
64 // 记得初始化null
65 int main() {
66     for (int i = 0; i ≤ n; i++)
67         null[i].ch[0] = null[i].ch[1] = null;
68         null → size = 0;
69
70     // do something
71
72     return 0;
73 }

```

#### 4.4.2 无旋Treap/可持久化Treap

```

1 struct node {
2     int val, size;
3     node *ch[2];
4
5     node(int val) : val(val), size(1) {}
6
7     inline void refresh() {
8         size = ch[0] → size + ch[1] → size;
9     }
10
11 } null[maxn];

```

### 4.4.3 Splay

如果插入的话可以直接找到底然后splay一下，也可以直接splay前驱后继。

```

1 #define dir(x) ((x) == (x) → p → ch[1])
2
3 struct node {
4     int size;
5     bool rev;
6     node *ch[2], *p;
7
8     node() : size(1), rev(false) {}
9
10    void pushdown() {
11        if (!rev)
12            return;
13
14        ch[0] → rev ≈ true;
15        ch[1] → rev ≈ true;
16        swap(ch[0], ch[1]);
17
18        rev = false;
19    }
20
21    void refresh() {
22        size = ch[0] → size + ch[1] → size + 1;
23    }
24 } null[maxn], *root = null;
25
26 void rot(node *x, int d) {
27     node *y = x → ch[d ^ 1];
28
29     if ((x → ch[d ^ 1] = y → ch[d]) != null)
30         y → ch[d] → p = x;
31     ((y → p = x → p) != null ? x → p → ch[dir(x)] :
32      → root) = y;
33     (y → ch[d] = x) → p = y;
34
35     x → refresh();
36     y → refresh();
37 }
38
39 void splay(node *x, node *t) {
40     while (x → p != t) {
41         if (x → p → p == t) {
42             rot(x → p, dir(x) ^ 1);
43             break;
44         }
45
46         if (dir(x) == dir(x → p))
47             rot(x → p → p, dir(x → p) ^ 1);
48         else
49             rot(x → p, dir(x) ^ 1);
50         rot(x → p, dir(x) ^ 1);
51     }
52
53 node *kth(int k, node *o) {
54     int d;
55     k++; // 因为最左边有一个哨兵
56
57     while (o != null) {
58         o → pushdown();
59
60         if (k == o → ch[0] → size + 1)
61             return o;
62
63         if ((d = k > o → ch[0] → size))
64             k -= o → ch[0] → size + 1;

```

```

65         o = o → ch[d];
66     }
67
68     return null;
69 }
70
71 void reverse(int l, int r) {
72     splay(kth(l - 1));
73     splay(kth(r + 1), root);
74
75     root → ch[1] → ch[0] → rev ≈ true;
76 }
77
78 int n, m;
79
80 int main() {
81     null → size = 0;
82     null → ch[0] = null → ch[1] = null → p = null;
83
84     scanf("%d%d", &n, &m);
85     root = null + n + 1;
86     root → ch[0] = root → ch[1] = root → p = null;
87
88     for (int i = 1; i ≤ n; i++) {
89         null[i].ch[1] = null[i].p = null;
90         null[i].ch[0] = root;
91         root → p = null + i;
92         (root = null + i) → refresh();
93     }
94
95     null[n + 2].ch[1] = null[n + 2].p = null;
96     null[n + 2].ch[0] = root; // 这里直接建成一条链的,
97     // 如果想减少常数也可以递归建一个平衡的树
98     root → p = null + n + 2; // 总之记得建两个哨兵, 这
99     // 样splay起来不需要特判
100    (root = null + n + 2) → refresh();
101
102    // Do something
103
104    return 0;
105 }

```

## 4.5 树链剖分

### 4.5.1 动态树形DP(最大权独立集)

```

1 #include <bits/stdc++.h>
2
3 using namespace std;
4
5 constexpr int maxn = 100005, maxm = 262155, inf =
6     → 0x3f3f3f3f;
7
8 struct binary_heap {
9     priority_queue<int> q, t;
10
11     binary_heap() {}
12
13     void push(int x) {
14         q.push(x);
15     }
16
17     void erase(int x) {
18         t.push(x);
19     }
20
21     int top() {
22         while (!t.empty() && q.top() == t.top()) {
23             q.pop();
24         }
25         return q.top();
26     }
27
28     void clear() {
29         while (!q.empty())
30             q.pop();
31         while (!t.empty())
32             t.pop();
33     }
34
35     int size() {
36         return q.size();
37     }
38
39     bool empty() {
40         return q.empty();
41     }
42
43     int front() {
44         return q.front();
45     }
46
47     void pop() {
48         q.pop();
49     }
50
51     void swap(binary_heap &other) {
52         swap(q, other.q);
53         swap(t, other.t);
54     }
55
56     void print() {
57         cout << "q: ";
58         for (int x : q)
59             cout << x << " ";
60         cout << endl;
61         cout << "t: ";
62         for (int x : t)
63             cout << x << " ";
64         cout << endl;
65     }
66
67     void print() {
68         cout << "q: ";
69         for (int x : q)
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71         cout << endl;
72         cout << "t: ";
73         for (int x : t)
74             cout << x << " ";
75         cout << endl;
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108        cout << endl;
109    }
110
111    void print() {
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113        for (int x : q)
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115        cout << endl;
116        cout << "t: ";
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119        cout << endl;
120    }
121
122    void print() {
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126        cout << endl;
127        cout << "t: ";
128        for (int x : t)
129            cout << x << " ";
130        cout << endl;
131    }
132
133    void print() {
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240        cout << endl;
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262        cout << endl;
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867        cout << endl;
868    }
869
870    void print() {
871        cout << "q: ";
872        for (int x : q)
873            cout << x << " ";
874        cout
```

```

23         t.pop();
24     }
25
26     return q.top();
27 }
28 } heap;
29
30 int pool[maxm][2][2], (*pt)[2][2] = pool;
31
32 void merge(int a[2][2], int b[2][2]) {
33     static int c[2][2];
34     memset(c, 0, sizeof(c));
35
36     for (int i = 0; i < 2; i++)
37         for (int j = 0; j < 2; j++)
38             for (int k = 0; k < 2; k++)
39                 if (!(j && k))
40                     for (int t = 0; t < 2; t++)
41                         c[i][t] = max(c[i][t], a[i][j]
42                             ↪ + b[k][t]);
43
44     memcpy(a, c, sizeof(c));
45 }
46
47 vector<pair<int, int>> tw;
48
49 struct seg_tree {
50     int (*tr)[2][2], n;
51
52     int s, d[2];
53
54     seg_tree() {}
55
56     void update(int o) {
57         memcpy(tr[o], tr[o * 2], sizeof(int) * 4);
58         merge(tr[o], tr[o * 2 + 1]);
59     }
60
61     void build(int l, int r, int o) {
62         if (l == r) {
63             tr[o][0][0] = tw[l - 1].first;
64             tr[o][0][1] = tr[o][1][0] = -inf;
65             tr[o][1][1] = tw[l - 1].second;
66
67             return;
68         }
69
70         int mid = (l + r) / 2;
71
72         build(l, mid, o * 2);
73         build(mid + 1, r, o * 2 + 1);
74
75         update(o);
76     }
77
78     void modify(int l, int r, int o) {
79         if (l == r) {
80             tr[o][0][0] = d[0];
81             tr[o][0][1] = tr[o][1][0] = -inf;
82             tr[o][1][1] = d[1];
83
84             return;
85         }
86
87         int mid = (l + r) / 2;
88
89         if (s <= mid)
90             modify(l, mid, o * 2);
91         else
92             modify(mid + 1, r, o * 2 + 1);
93
94         update(o);
95     }
96
97     int getval() {
98         int ans = 0;
99         for (int i = 0; i < 2; i++)
100             for (int j = 0; j < 2; j++)
101                 ans = max(ans, tr[1][i][j]);
102
103         return ans;
104     }
105
106     pair<int, int> getpair() {
107         int ans[2] = {0};
108         for (int i = 0; i < 2; i++)
109             for (int j = 0; j < 2; j++)
110                 ans[i] = max(ans[i], tr[1][i][j]);
111
112         return make_pair(ans[0], ans[1]);
113     }
114
115     void build(int len) {
116         n = len;
117         int N = 1;
118         while (N < n * 2)
119             N *= 2;
120
121         tr = pt;
122         pt += N;
123
124         build(1, n, 1);
125     }
126
127     void modify(int x, int dat[2]) {
128         s = x;
129         for (int i = 0; i < 2; i++)
130             d[i] = dat[i];
131         modify(1, n, 1);
132     }
133 } seg[maxn];
134
135 vector<int> G[maxn];
136
137 int p[maxn], d[maxn], sz[maxn], son[maxn], top[maxn];
138 int dp[maxn][2], dptr[maxn][2], w[maxn];
139
140 void dfs1(int x) {
141     d[x] = d[p[x]] + 1;
142     sz[x] = 1;
143
144     for (int y : G[x])
145         if (y != p[x]) {
146             p[y] = x;
147             dfs1(y);
148
149             if (sz[y] > sz[son[x]])
150                 son[x] = y;
151
152             sz[x] += sz[y];
153         }
154
155 void dfs2(int x) {
156     if (x == son[p[x]])
157         top[x] = top[p[x]];
158     else
159         top[x] = x;

```

```

161     for (int y : G[x])
162         if (y != p[x])
163             dfs2(y);
164
165     dp[x][1] = w[x];
166     for (int y : G[x])
167         if (y != p[x] && y != son[x]) {
168             dp[x][1] += dptr[y][0];
169             dp[x][0] += max(dptr[y][0], dptr[y][1]);
170         }
171
172     if (top[x] == x) {
173         tw.clear();
174
175         for (int u = x; u; u = son[u])
176             tw.push_back(make_pair(dp[u][0], dp[u]
177                                     ↪ [1]));
178
179         seg[x].build((int)tw.size());
180
181         tie(dptr[x][0], dptr[x][1]) = seg[x].getpair();
182
183     }
184 }
185
186 void modify(int x, int dat) {
187     dp[x][1] -= w[x];
188     dp[x][1] += (w[x] = dat);
189
190     while (x) {
191         if (p[top[x]]) {
192             dp[p[top[x]]][0] -= max(dptr[top[x]][0],
193                                     ↪ dptr[top[x]][1]);
194             dp[p[top[x]]][1] -= dptr[top[x]][0];
195         }
196
197         heap.erase(seg[top[x]].getval());
198         seg[top[x]].modify(d[x] - d[top[x]] + 1,
199                           ↪ dp[x]);
200         heap.push(seg[top[x]].getval());
201
202         tie(dptr[top[x]][0], dptr[top[x]][1]) =
203             ↪ seg[top[x]].getpair();
204
205         if (p[top[x]]) {
206             dp[p[top[x]]][0] += max(dptr[top[x]][0],
207                                     ↪ dptr[top[x]][1]);
208             dp[p[top[x]]][1] += dptr[top[x]][0];
209         }
210
211     x = p[top[x]];
212 }
213
214 int main() {
215     int n, m;
216     scanf("%d%d", &n, &m);
217
218     for (int i = 1; i ≤ n; i++)
219         scanf("%d", &w[i]);
220
221     for (int i = 1; i < n; i++) {
222         int x, y;
223         scanf("%d%d", &x, &y);
224
225         G[x].push_back(y);
226         G[y].push_back(x);
227     }
228 }
```

```

226
227     dfs1(1);
228     dfs2(1);
229
230     while (m--) {
231         int x, dat;
232         scanf("%d%d", &x, &dat);
233
234         modify(x, dat);
235
236         printf("%d\n", heap.top());
237     }
238
239     return 0;
240 }
```

## 4.6 树分治

### 4.6.1 动态树分治

```

1 // 为了减小常数，这里采用bfs写法，实测预处理比dfs快将近一
2 ↪ 半
3 // 以下以维护一个点到每个黑点的距离之和为例
4
5 // 全局数组定义
6 vector<int> G[maxn], W[maxn];
7 int size[maxn], son[maxn], q[maxn];
8 int p[maxn], depth[maxn], id[maxn][20], d[maxn][20]; // ↪ id是对应层所在子树的根
9 int a[maxn], ca[maxn], b[maxn][20], cb[maxn][20]; // 维
10 ↪ 护距离和用的
11 bool vis[maxn], col[maxn];
12
13 // 建树 总计O(n log n)
14 // 需要调用找重心和预处理距离，同时递归调用自身
15 void build(int x, int k, int s, int pr) { // 结点，深度,
16     ↪ 连通块大小，点分树上的父亲
17     x = getcenter(x, s);
18     vis[x] = true;
19     depth[x] = k;
20     p[x] = pr;
21
22     for (int i = 0; i < (int)G[x].size(); i++)
23         if (!vis[G[x][i]]) {
24             d[G[x][i]][k] = W[x][i];
25             p[G[x][i]] = x;
26
27             getdis(G[x][i], k, G[x][i]); // bfs每个子树,
28             ↪ 预处理距离
29         }
30
31     for (int i = 0; i < (int)G[x].size(); i++)
32         if (!vis[G[x][i]])
33             build(G[x][i], k + 1, size[G[x][i]], x); // ↪ 递归建树
34
35
36 // 找重心 O(n)
37 int getcenter(int x, int s) {
38     int head = 0, tail = 0;
39     q[tail++] = x;
40
41     while (head != tail) {
42         x = q[head++];
43         size[x] = 1; // 这里不需要清空，因为以后要用的话
44         ↪ 一定会重新赋值
45         son[x] = 0;
46
47         for (int i = 0; i < (int)G[x].size(); i++)
48             if (!vis[G[x][i]])
```

```

43     if (!vis[G[x][i]] && G[x][i] != p[x]) {
44         p[G[x][i]] = x;
45         q[tail++] = G[x][i];
46     }
47 }
48
49 for (int i = tail - 1; i; i--) {
50     x = q[i];
51     size[p[x]] += size[x];
52
53     if (size[x] > size[son[p[x]]])
54         son[p[x]] = x;
55 }
56
57 x = q[0];
58 while (son[x] && size[son[x]] * 2 ≥ s)
59     x = son[x];
60
61 return x;
62 }

// 预处理距离 O(n)
// 方便起见，这里直接用了笨一点的方法，O(n log n)全存下来
64 void getdis(int x, int k, int rt) {
65     int head = 0, tail = 0;
66     q[tail++] = x;
67
68     while (head != tail) {
69         x = q[head++];
70         size[x] = 1;
71         id[x][k] = rt;
72
73         for (int i = 0; i < (int)G[x].size(); i++)
74             if (!vis[G[x][i]] && G[x][i] != p[x]) {
75                 p[G[x][i]] = x;
76                 d[G[x][i]][k] = d[x][k] + W[x][i];
77
78                 q[tail++] = G[x][i];
79             }
80     }
81
82     for (int i = tail - 1; i; i--)
83         size[p[q[i]]] += size[q[i]]; // 后面递归建树要用
84         ↪ 到子问题大小
85 }
86
87 // 修改 O(log n)
88 void modify(int x) {
89     if (col[x])
90         ca[x]--;
91     else
92         ca[x]++; // 记得先特判自己作为重心的那层
93
94     for (int u = p[x], k = depth[x] - 1; u; u = p[u],
95          ↪ k--) {
96         if (col[x]) {
97             a[u] -= d[x][k];
98             ca[u]--;
99
100            b[id[x][k]][k] -= d[x][k];
101            cb[id[x][k]][k]--;
102        }
103        else {
104            a[u] += d[x][k];
105            ca[u]++;
106
107            b[id[x][k]][k] += d[x][k];
108            cb[id[x][k]][k]++;
109        }
110    }
111
112    col[x] = true;
113 }
114
115 // 询问 O(log n)
116 int query(int x) {
117     int ans = a[x]; // 特判自己是重心的那层
118
119     for (int u = p[x], k = depth[x] - 1; u; u = p[u],
120          ↪ k--)
121         ans += a[u] - b[id[x][k]][k] + d[x][k] * (ca[u]
122             ↪ - cb[id[x][k]][k]);
123
124     return ans;
125 }

```

#### 4.6.2 紫荆花之恋

稍微重构了一下，修改了代码风格。

另外这个是BFS版本，跑得比DFS要快不少。（虽然主要复杂度并不在重构上）

```

1 #include <bits/stdc++.h>
2
3 using namespace std;
4
5 constexpr int maxn = 100005, maxk = 49;
6 constexpr double alpha = .75;
7
8 mt19937 rnd(23333333);
9
10 struct node {
11     int key, size, p;
12     node *ch[2];
13
14     node() {}
15
16     node(int key) : key(key), size(1), p(rnd()) {}
17
18     inline void update() {
19         size = ch[0] → size + ch[1] → size + 1;
20     }
21     ~node() { delete ch[0]; delete ch[1]; }
22     node *null[maxn * maxk], *pt = null;
23
24     vector<node*> pool;
25
26     node *newnode(int val) {
27         node *x;
28
29         if (!pool.empty())
30             x = pool.back();
31         pool.pop_back();
32         else
33             x = ++pt;
34
35         *x = node(val);
36         x → ch[0] = x → ch[1] = null;
37
38         return x;
39     }
40
41     void rot(node *&x, int d) {
42         node *y = x → ch[d ^ 1];
43         x → ch[d ^ 1] = y → ch[d];
44         y → ch[d] = x;
45
46         x → update();
47         (x = y) → update();
48     }
49
50     void print() {
51         cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
52         for (int i = 0; i < 2; i++)
53             ch[i] → print();
54     }
55
56     void print() {
57         cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
58         for (int i = 0; i < 2; i++)
59             ch[i] → print();
60     }
61
62     void print() {
63         cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
64         for (int i = 0; i < 2; i++)
65             ch[i] → print();
66     }
67
68     void print() {
69         cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
70         for (int i = 0; i < 2; i++)
71             ch[i] → print();
72     }
73
74     void print() {
75         cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
76         for (int i = 0; i < 2; i++)
77             ch[i] → print();
78     }
79
80     void print() {
81         cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
82         for (int i = 0; i < 2; i++)
83             ch[i] → print();
84     }
85
86     void print() {
87         cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
88         for (int i = 0; i < 2; i++)
89             ch[i] → print();
90     }
91
92     void print() {
93         cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
94         for (int i = 0; i < 2; i++)
95             ch[i] → print();
96     }
97
98     void print() {
99         cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
100        for (int i = 0; i < 2; i++)
101            ch[i] → print();
102    }
103
104    void print() {
105        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
106        for (int i = 0; i < 2; i++)
107            ch[i] → print();
108    }
109
110    void print() {
111        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
112        for (int i = 0; i < 2; i++)
113            ch[i] → print();
114    }
115
116    void print() {
117        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
118        for (int i = 0; i < 2; i++)
119            ch[i] → print();
120    }
121
122    void print() {
123        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
124        for (int i = 0; i < 2; i++)
125            ch[i] → print();
126    }
127
128    void print() {
129        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
130        for (int i = 0; i < 2; i++)
131            ch[i] → print();
132    }
133
134    void print() {
135        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
136        for (int i = 0; i < 2; i++)
137            ch[i] → print();
138    }
139
140    void print() {
141        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
142        for (int i = 0; i < 2; i++)
143            ch[i] → print();
144    }
145
146    void print() {
147        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
148        for (int i = 0; i < 2; i++)
149            ch[i] → print();
150    }
151
152    void print() {
153        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
154        for (int i = 0; i < 2; i++)
155            ch[i] → print();
156    }
157
158    void print() {
159        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
160        for (int i = 0; i < 2; i++)
161            ch[i] → print();
162    }
163
164    void print() {
165        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
166        for (int i = 0; i < 2; i++)
167            ch[i] → print();
168    }
169
170    void print() {
171        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
172        for (int i = 0; i < 2; i++)
173            ch[i] → print();
174    }
175
176    void print() {
177        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
178        for (int i = 0; i < 2; i++)
179            ch[i] → print();
180    }
181
182    void print() {
183        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
184        for (int i = 0; i < 2; i++)
185            ch[i] → print();
186    }
187
188    void print() {
189        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
190        for (int i = 0; i < 2; i++)
191            ch[i] → print();
192    }
193
194    void print() {
195        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
196        for (int i = 0; i < 2; i++)
197            ch[i] → print();
198    }
199
200    void print() {
201        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
202        for (int i = 0; i < 2; i++)
203            ch[i] → print();
204    }
205
206    void print() {
207        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
208        for (int i = 0; i < 2; i++)
209            ch[i] → print();
210    }
211
212    void print() {
213        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
214        for (int i = 0; i < 2; i++)
215            ch[i] → print();
216    }
217
218    void print() {
219        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
220        for (int i = 0; i < 2; i++)
221            ch[i] → print();
222    }
223
224    void print() {
225        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
226        for (int i = 0; i < 2; i++)
227            ch[i] → print();
228    }
229
230    void print() {
231        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
232        for (int i = 0; i < 2; i++)
233            ch[i] → print();
234    }
235
236    void print() {
237        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
238        for (int i = 0; i < 2; i++)
239            ch[i] → print();
240    }
241
242    void print() {
243        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
244        for (int i = 0; i < 2; i++)
245            ch[i] → print();
246    }
247
248    void print() {
249        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
250        for (int i = 0; i < 2; i++)
251            ch[i] → print();
252    }
253
254    void print() {
255        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
256        for (int i = 0; i < 2; i++)
257            ch[i] → print();
258    }
259
260    void print() {
261        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
262        for (int i = 0; i < 2; i++)
263            ch[i] → print();
264    }
265
266    void print() {
267        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
268        for (int i = 0; i < 2; i++)
269            ch[i] → print();
270    }
271
272    void print() {
273        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
274        for (int i = 0; i < 2; i++)
275            ch[i] → print();
276    }
277
278    void print() {
279        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
280        for (int i = 0; i < 2; i++)
281            ch[i] → print();
282    }
283
284    void print() {
285        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
286        for (int i = 0; i < 2; i++)
287            ch[i] → print();
288    }
289
290    void print() {
291        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
292        for (int i = 0; i < 2; i++)
293            ch[i] → print();
294    }
295
296    void print() {
297        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
298        for (int i = 0; i < 2; i++)
299            ch[i] → print();
300    }
301
302    void print() {
303        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
304        for (int i = 0; i < 2; i++)
305            ch[i] → print();
306    }
307
308    void print() {
309        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
310        for (int i = 0; i < 2; i++)
311            ch[i] → print();
312    }
313
314    void print() {
315        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
316        for (int i = 0; i < 2; i++)
317            ch[i] → print();
318    }
319
320    void print() {
321        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
322        for (int i = 0; i < 2; i++)
323            ch[i] → print();
324    }
325
326    void print() {
327        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
328        for (int i = 0; i < 2; i++)
329            ch[i] → print();
330    }
331
332    void print() {
333        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
334        for (int i = 0; i < 2; i++)
335            ch[i] → print();
336    }
337
338    void print() {
339        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
340        for (int i = 0; i < 2; i++)
341            ch[i] → print();
342    }
343
344    void print() {
345        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
346        for (int i = 0; i < 2; i++)
347            ch[i] → print();
348    }
349
350    void print() {
351        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
352        for (int i = 0; i < 2; i++)
353            ch[i] → print();
354    }
355
356    void print() {
357        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
358        for (int i = 0; i < 2; i++)
359            ch[i] → print();
360    }
361
362    void print() {
363        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
364        for (int i = 0; i < 2; i++)
365            ch[i] → print();
366    }
367
368    void print() {
369        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
370        for (int i = 0; i < 2; i++)
371            ch[i] → print();
372    }
373
374    void print() {
375        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
376        for (int i = 0; i < 2; i++)
377            ch[i] → print();
378    }
379
380    void print() {
381        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
382        for (int i = 0; i < 2; i++)
383            ch[i] → print();
384    }
385
386    void print() {
387        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
388        for (int i = 0; i < 2; i++)
389            ch[i] → print();
390    }
391
392    void print() {
393        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
394        for (int i = 0; i < 2; i++)
395            ch[i] → print();
396    }
397
398    void print() {
399        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
400        for (int i = 0; i < 2; i++)
401            ch[i] → print();
402    }
403
404    void print() {
405        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
406        for (int i = 0; i < 2; i++)
407            ch[i] → print();
408    }
409
410    void print() {
411        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
412        for (int i = 0; i < 2; i++)
413            ch[i] → print();
414    }
415
416    void print() {
417        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
418        for (int i = 0; i < 2; i++)
419            ch[i] → print();
420    }
421
422    void print() {
423        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
424        for (int i = 0; i < 2; i++)
425            ch[i] → print();
426    }
427
428    void print() {
429        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
430        for (int i = 0; i < 2; i++)
431            ch[i] → print();
432    }
433
434    void print() {
435        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
436        for (int i = 0; i < 2; i++)
437            ch[i] → print();
438    }
439
440    void print() {
441        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
442        for (int i = 0; i < 2; i++)
443            ch[i] → print();
444    }
445
446    void print() {
447        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
448        for (int i = 0; i < 2; i++)
449            ch[i] → print();
450    }
451
452    void print() {
453        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
454        for (int i = 0; i < 2; i++)
455            ch[i] → print();
456    }
457
458    void print() {
459        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
460        for (int i = 0; i < 2; i++)
461            ch[i] → print();
462    }
463
464    void print() {
465        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
466        for (int i = 0; i < 2; i++)
467            ch[i] → print();
468    }
469
470    void print() {
471        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
472        for (int i = 0; i < 2; i++)
473            ch[i] → print();
474    }
475
476    void print() {
477        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
478        for (int i = 0; i < 2; i++)
479            ch[i] → print();
480    }
481
482    void print() {
483        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
484        for (int i = 0; i < 2; i++)
485            ch[i] → print();
486    }
487
488    void print() {
489        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
490        for (int i = 0; i < 2; i++)
491            ch[i] → print();
492    }
493
494    void print() {
495        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
496        for (int i = 0; i < 2; i++)
497            ch[i] → print();
498    }
499
500    void print() {
501        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
502        for (int i = 0; i < 2; i++)
503            ch[i] → print();
504    }
505
506    void print() {
507        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
508        for (int i = 0; i < 2; i++)
509            ch[i] → print();
510    }
511
512    void print() {
513        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
514        for (int i = 0; i < 2; i++)
515            ch[i] → print();
516    }
517
518    void print() {
519        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
520        for (int i = 0; i < 2; i++)
521            ch[i] → print();
522    }
523
524    void print() {
525        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
526        for (int i = 0; i < 2; i++)
527            ch[i] → print();
528    }
529
530    void print() {
531        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
532        for (int i = 0; i < 2; i++)
533            ch[i] → print();
534    }
535
536    void print() {
537        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
538        for (int i = 0; i < 2; i++)
539            ch[i] → print();
540    }
541
542    void print() {
543        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
544        for (int i = 0; i < 2; i++)
545            ch[i] → print();
546    }
547
548    void print() {
549        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
550        for (int i = 0; i < 2; i++)
551            ch[i] → print();
552    }
553
554    void print() {
555        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
556        for (int i = 0; i < 2; i++)
557            ch[i] → print();
558    }
559
560    void print() {
561        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
562        for (int i = 0; i < 2; i++)
563            ch[i] → print();
564    }
565
566    void print() {
567        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
568        for (int i = 0; i < 2; i++)
569            ch[i] → print();
570    }
571
572    void print() {
573        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
574        for (int i = 0; i < 2; i++)
575            ch[i] → print();
576    }
577
578    void print() {
579        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
580        for (int i = 0; i < 2; i++)
581            ch[i] → print();
582    }
583
584    void print() {
585        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
586        for (int i = 0; i < 2; i++)
587            ch[i] → print();
588    }
589
590    void print() {
591        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
592        for (int i = 0; i < 2; i++)
593            ch[i] → print();
594    }
595
596    void print() {
597        cout << "key: " << key << ", size: " << size << ", p: " << p << endl;
598        for (int i = 0; i < 2; i++)
599            ch[i] → print();
60
```

```

48 }
49
50 void insert(node *&o, int x) {
51     if (o == null) {
52         o = newnode(x);
53         return;
54     }
55
56     int d = (x > o->key);
57
58     insert(o->ch[d], x);
59     o->update();
60
61     if (o->ch[d]->p < o->p)
62         rot(o, d ^ 1);
63 }
64
65 int get_order(node *o, int x) {
66     int ans = 0;
67
68     while (o != null) {
69         int d = (x > o->key);
70
71         if (d)
72             ans += o->ch[0]->size + 1;
73
74         o = o->ch[d];
75     }
76
77     return ans;
78 }
79
80 void destroy(node *x) {
81     if (x == null)
82         return;
83
84     pool.push_back(x);
85     destroy(x->ch[0]);
86     destroy(x->ch[1]);
87 }
88
89 struct my_tree { // 封装了一下，如果不卡内存直接换
90     // 成PBDS就好了
91     node *rt;
92
93     my_tree() : rt(null) {}
94
95     void clear() {
96         ::destroy(rt);
97         rt = null;
98     }
99
100    void insert(int x) {
101        ::insert(rt, x);
102    }
103
104    int order_of_key(int x) { // less than x
105        return ::get_order(rt, x);
106    }
107 } tr[maxn], tre[maxn][maxk];
108
109 vector<pair<int, int>> G[maxn];
110
111 int p[maxn], depth[maxn], d[maxn][maxk], rid[maxn]
112     -> [maxk];
113 int sz[maxn], siz[maxn][maxk], q[maxn];
114 bool vis[maxn];
115
116 int w[maxn];

```

```

116 void destroy(int o) {
117     int head = 0, tail = 0;
118     q[tail++] = o;
119     vis[o] = false;
120
121     while (head != tail) {
122         int x = q[head++];
123         tr[x].clear();
124
125         for (int i = depth[o]; i <= depth[x]; i++) {
126             tre[x][i].clear();
127             d[x][i] = rid[x][i] = siz[x][i] = 0;
128         }
129
130         for (auto pi : G[x]) {
131             int y = pi.first;
132
133             if (vis[y] && depth[y] >= depth[o]) {
134                 vis[y] = false;
135                 q[tail++] = y;
136             }
137         }
138     }
139 }
140
141 int getcenter(int o, int s) {
142     int head = 0, tail = 0;
143     q[tail++] = o;
144
145     while (head != tail) {
146         int x = q[head++];
147         sz[x] = 1;
148
149         for (auto pi : G[x]) {
150             int y = pi.first;
151
152             if (!vis[y] && y != p[x]) {
153                 p[y] = x;
154                 q[tail++] = y;
155             }
156         }
157     }
158
159     for (int i = s - 1; i; i--)
160         sz[p[q[i]]] += sz[q[i]];
161
162     int x = o;
163     while (true) {
164         bool ok = false;
165
166         for (auto pi : G[x]) {
167             int y = pi.first;
168             if (!vis[y] && y != p[x] && sz[y] * 2 > s)
169                 -> {
170                     x = y;
171                     ok = true;
172                     break;
173                 }
174
175         if (!ok)
176             break;
177     }
178
179     return x;
180 }
181
182 void getdis(int st, int o, int k) {
183     int head = 0, tail = 0;

```

```

184     q[tail++] = st;
185
186     while (head != tail) {
187         int x = q[head++];
188         sz[x] = 1;
189         rid[x][k] = st;
190
191         tr[o].insert(d[x][k] - w[x]);
192         tre[st][k].insert(d[x][k] - w[x]);
193
194         for (auto pi : G[x]) {
195             int y = pi.first, val = pi.second;
196
197             if (!vis[y] && y != p[x]) {
198                 p[y] = x;
199                 d[y][k] = d[x][k] + val;
200                 q[tail++] = y;
201             }
202         }
203     }
204
205     for (int i = tail - 1; i; i--)
206         sz[p[q[i]]] += sz[q[i]];
207
208     siz[st][k] = sz[st];
209 }
210
211 void rebuild(int x, int s, int pr) {
212     x = getcenter(x, s);
213     vis[x] = true;
214     p[x] = pr;
215     depth[x] = depth[pr] + 1;
216     sz[x] = s;
217
218     tr[x].insert(-w[x]);
219
220     for (auto pi : G[x]) {
221         int y = pi.first, val = pi.second;
222
223         if (!vis[y]) {
224             p[y] = x;
225             d[y][depth[x]] = val;
226             getdis(y, x, depth[x]);
227         }
228     }
229
230     for (auto pi : G[x]) {
231         int y = pi.first;
232
233         if (!vis[y])
234             rebuild(y, sz[y], x);
235     }
236 }
237
238 long long add_node(int x, int nw) { // nw是边权
239     depth[x] = depth[p[x]] + 1;
240     sz[x] = 1;
241     vis[x] = true;
242
243     tr[x].insert(-w[x]);
244
245     long long tmp = 0;
246     int goat = 0; // 替罪羊
247
248     for (int u = p[x], k = depth[x] - 1; u; u = p[u],
249          k--) {
250         d[x][k] = d[p[x]][k] + nw;
251         rid[x][k] = (rid[p[x]][k] ? rid[p[x]][k] : x);

```

```

252         tmp += tr[u].order_of_key(w[x] - d[x][k] + 1);
253         tmp -= tre[rid[x][k]][k].order_of_key(w[x] -
254                                     → d[x][k] + 1);
255
256         tr[u].insert(d[x][k] - w[x]);
257         tre[rid[x][k]][k].insert(d[x][k] - w[x]);
258
259         sz[u]++;
260         siz[rid[x][k]][k]++;
261
262         if (siz[rid[x][k]][k] > sz[u] * alpha + 5)
263             goat = u;
264     }
265
266     if (goat) {
267         destroy(goat);
268         rebuild(goat, sz[goat], p[goat]);
269     }
270
271     return tmp;
272 }
273
274 int main() {
275     null → ch[0] = null → ch[1] = null;
276     null → size = 0;
277
278     int n;
279     scanf("%*d%d", &n);
280
281     scanf("%*d%*d%d", &w[1]);
282     vis[1] = true;
283     sz[1] = 1;
284     tr[1].insert(-w[1]);
285
286     printf("0\n");
287
288     long long ans = 0;
289
290     for (int i = 2; i ≤ n; i++) {
291         int nw;
292         scanf("%d%d%d", &p[i], &nw, &w[i]);
293
294         p[i] ≈ (ans % 1000000000);
295
296         G[i].push_back(make_pair(p[i], nw));
297         G[p[i]].push_back(make_pair(i, nw));
298
299         ans += add_node(i, nw);
300
301         printf("%lld\n", ans);
302     }
303
304     return 0;
305 }

```

## 4.7 LCT动态树

### 4.7.1 不换根(弹飞绵羊)

```

1 #define isroot(x) ((x) != (x) → p → ch[0] && (x) !=
2 → (x) → p → ch[1]) // 判断是不是Splay的根
3
4 #define dir(x) ((x) == (x) → p → ch[1]) // 判断它是它
→ 父亲的左 / 右儿子
5
6 struct node { // 结点类定义
7     int size; // Splay的子树大小
8     node *ch[2], *p;
9 }

```

```

8  node(): size(1) {}
9  void refresh() {
10   |   size = ch[0] → size + ch[1] → size + 1;
11 } // 附加信息维护
12 } null[maxn];
13
14 // 在主函数开头加上这句初始化
15 null → size = 0;
16
17 // 初始化结点
18 void initialize(node *x) {
19   |   x → ch[0] = x → ch[1] = x → p = null;
20 }
21
22 // Access 均摊O(log n)
23 // LCT核心操作，把结点到根的路径打通，顺便把与重儿子的连
24 // 边变成轻边
25 // 需要调用splay
26 node *access(node *x) {
27   |   node *y = null;
28
29   while (x != null) {
30     |     splay(x);
31
32     |     x → ch[1] = y;
33     |     (y = x) → refresh();
34
35     |     x = x → p;
36   }
37
38   return y;
39 }
40
41 // Link 均摊O(log n)
42 // 把x的父亲设为y
43 // 要求x必须为所在树的根节点否则会导致后续各种莫名其妙的
44 // → 问题
45 // 需要调用splay
46 void link(node *x, node *y) {
47   |   splay(x);
48   |   x → p = y;
49 }
50
51 // Cut 均摊O(log n)
52 // 把x与其父亲的连边断掉
53 // x可以是所在树的根节点，这时此操作没有任何实质效果
54 // 需要调用access和splay
55 void cut(node *x) {
56   |   access(x);
57   |   splay(x);
58
59   |   x → ch[0] → p = null;
60   |   x → ch[0] = null;
61
62   |   x → refresh();
63 }
64
65 // Splay 均摊O(log n)
66 // 需要调用旋转
67 void splay(node *x) {
68   |   while (!isroot(x)) {
69   |     |   if (isroot(x → p)) {
70   |       |       rot(x → p, dir(x) ^ 1);
71   |       |       break;
72   |     }
73
74   |     |   if (dir(x) == dir(x → p))
75   |       |       rot(x → p → p, dir(x → p) ^ 1);
76   |     else
77   |       |       rot(x → p, dir(x) ^ 1);
78   |     rot(x → p, dir(x) ^ 1);
79 }
80
81 // 旋转(LCT版本) O(1)
82 // 平衡树基本操作
83 // 要求对应儿子必须存在，否则会导致后续各种莫名其妙的问题
84 void rot(node *x, int d) {
85   |   node *y = x → ch[d ^ 1];
86
87   |   y → p = x → p;
88   |   if (!isroot(x))
89   |     |     x → p → ch[dir(x)] = y;
90
91   |   if ((x → ch[d ^ 1] = y → ch[d]) != null)
92   |     |     y → ch[d] → p = x;
93   |     |     (y → ch[d] = x) → p = y;
94
95   |   x → refresh();
96   |   y → refresh();
97 }
```

```

77   }
78 }
79
80 // 旋转(LCT版本) O(1)
81 // 平衡树基本操作
82 // 要求对应儿子必须存在，否则会导致后续各种莫名其妙的问题
83 void rot(node *x, int d) {
84   |   node *y = x → ch[d ^ 1];
85
86   |   y → p = x → p;
87   |   if (!isroot(x))
88   |     |     x → p → ch[dir(x)] = y;
89
90   |   if ((x → ch[d ^ 1] = y → ch[d]) != null)
91   |     |     y → ch[d] → p = x;
92   |     |     (y → ch[d] = x) → p = y;
93
94   |   x → refresh();
95   |   y → refresh();
96 }
```

#### 4.7.2 换根/维护生成树

```

1 #define isroot(x) ((x) → p == null || ((x) → p →
2 → ch[0] != (x) && (x) → p → ch[1] != (x)))
3 #define dir(x) ((x) == (x) → p → ch[1])
4
5 using namespace std;
6
7 const int maxn = 200005;
8
9 struct node{
10   |   int key, mx, pos;
11   |   bool rev;
12   |   node *ch[2], *p;
13
14   |   node(int key = 0): key(key), mx(key), pos(-1),
15   |   |   rev(false) {}
16
17   |   void pushdown() {
18   |     |   if (!rev)
19   |     |     return;
20
21   |     |   ch[0] → rev ≈ true;
22   |     |   ch[1] → rev ≈ true;
23   |     |   swap(ch[0], ch[1]);
24
25   |     |   if (pos != -1)
26   |     |     pos ≈ 1;
27
28   |     |   rev = false;
29   }
30
31   |   void refresh() {
32   |     |   mx = key;
33   |     |   pos = -1;
34   |     |   if (ch[0] → mx > mx) {
35   |     |     mx = ch[0] → mx;
36   |     |     pos = 0;
37   |   }
38   |     |   if (ch[1] → mx > mx) {
39   |     |     mx = ch[1] → mx;
40   |     |     pos = 1;
41   |   }
42   |   null[maxn * 2];
43   void init(node *x, int k) {
```

```

44     x → ch[0] = x → ch[1] = x → p = null;
45     x → key = x → mx = k;
46 }
47
48 void rot(node *x, int d) {
49     node *y = x → ch[d ^ 1];
50     if ((x → ch[d ^ 1] = y → ch[d]) != null)
51         y → ch[d] → p = x;
52
53     y → p = x → p;
54     if (!isroot(x))
55         x → p → ch[dir(x)] = y;
56
57     (y → ch[d] = x) → p = y;
58
59     x → refresh();
60     y → refresh();
61 }
62
63 void splay(node *x) {
64     x → pushdown();
65
66     while (!isroot(x)) {
67         if (!isroot(x → p))
68             x → p → p → pushdown();
69         x → p → pushdown();
70         x → pushdown();
71
72         if (isroot(x → p)) {
73             rot(x → p, dir(x) ^ 1);
74             break;
75         }
76
77         if (dir(x) == dir(x → p))
78             rot(x → p → p, dir(x → p) ^ 1);
79         else
80             rot(x → p, dir(x) ^ 1);
81
82         rot(x → p, dir(x) ^ 1);
83     }
84 }
85
86 node *access(node *x) {
87     node *y = null;
88
89     while (x != null) {
90         splay(x);
91
92         x → ch[1] = y;
93         (y = x) → refresh();
94
95         x = x → p;
96     }
97
98     return y;
99 }
100
101 void makeroott(node *x) {
102     access(x);
103     splay(x);
104     x → rev = true;
105 }
106
107 void link(node *x, node *y) {
108     makeroott(x);
109     x → p = y;
110 }
111
112 void cut(node *x, node *y) {
113     makeroott(x);
114     access(y);
115     splay(y);
116
117     y → ch[0] → p = null;
118     y → ch[0] = null;
119     y → refresh();
120 }
121
122 node *getroot(node *x) {
123     x = access(x);
124     while (x → pushdown(), x → ch[0] != null)
125         x = x → ch[0];
126     splay(x);
127     return x;
128 }
129
130 node *getmax(node *x, node *y) {
131     makeroott(x);
132     x = access(y);
133
134     while (x → pushdown(), x → pos != -1)
135         x = x → ch[x → pos];
136     splay(x);
137
138     return x;
139 }
140
141 // 以下为主函数示例
142 for (int i = 1; i ≤ m; i++) {
143     init(null + n + i, w[i]);
144     if (getroot(null + u[i]) != getroot(null + v[i])) {
145         ans[q + 1] -= k;
146         ans[q + 1] += w[i];
147
148         link(null + u[i], null + n + i);
149         link(null + v[i], null + n + i);
150         vis[i] = true;
151     }
152     else {
153         int ii = getmax(null + u[i], null + v[i]) -
154             → null - n;
155         if (w[i] ≥ w[ii])
156             continue;
157
158         cut(null + u[ii], null + n + ii);
159         cut(null + v[ii], null + n + ii);
160
161         link(null + u[i], null + n + i);
162         link(null + v[i], null + n + i);
163
164         ans[q + 1] -= w[ii];
165         ans[q + 1] += w[i];
166     }
167 }

```

### 4.7.3 维护子树信息

```

1 // 这个东西虽然只需要抄板子但还是极其难写，常数极其巨大，
2 // 没必要的时候就不要用
3 // 如果维护子树最小值就需要套一个可删除的堆来维护，复杂度
4 // 会变成 $O(n \log^2 n)$ 
5 // 注意由于这道题与边权有关，需要边权拆点变点权
6
7 // 宏定义
8 #define isroot(x) ((x) → p == null || ((x) != (x) → p
9 // → ch[0] && (x) != (x) → p → ch[1]))
10 #define dir(x) ((x) == (x) → p → ch[1])
11
12 // 节点类定义
13 struct node { // 以维护子树中黑点到根距离和为例

```

```

11 int w, chain_cnt, tree_cnt;
12 long long sum, suml, sumr, tree_sum; // 由于换根需要
13   ↪ 子树反转，需要维护两个方向的信息
14 bool rev, col;
15 node *ch[2], *p;
16
17 node() : w(0), chain_cnt(0),
18   ↪ tree_cnt(0), sum(0), suml(0), sumr(0),
19   ↪ tree_sum(0), rev(false), col(false) {}
20
21 inline void pushdown() {
22   if(!rev)
23     return;
24
25   ch[0]→rev ≈ true;
26   ch[1]→rev ≈ true;
27   swap(ch[0], ch[1]);
28   swap(suml, sumr);
29
30   rev = false;
31 }
32
33 inline void refresh() { // 如果不想这样特判
34   ↪ 就pushdown一下
35   // pushdown();
36
37   sum = ch[0] → sum + ch[1] → sum + w;
38   suml = (ch[0] → rev ? ch[0] → sumr : ch[0] →
39     ↪ suml) + (ch[1] → rev ? ch[1] → sumr :
40     ↪ ch[1] → suml) + (tree_cnt + ch[1] →
41     ↪ chain_cnt) * (ch[0] → sum + w) + tree_sum;
42   sumr = (ch[0] → rev ? ch[0] → suml : ch[0] →
43     ↪ sumr) + (ch[1] → rev ? ch[1] → suml :
44     ↪ ch[1] → sumr) + (tree_cnt + ch[0] →
45     ↪ chain_cnt) * (ch[1] → sum + w) + tree_sum;
46   chain_cnt = ch[0] → chain_cnt + ch[1] →
47     ↪ chain_cnt + tree_cnt;
48 }
49 null[maxn * 2]; // 如果没有边权变点权就不用乘2了
50
51 // 封装构造函数
52 node *newnode(int w) {
53   node *x = nodes.front(); // 因为有删边加边，可以用一
54   ↪ 个队列维护可用结点
55   nodes.pop();
56   initialize(x);
57   x → w = w;
58   x → refresh();
59   return x;
60 }
61
62 // 封装初始化函数
63 // 记得在进行操作之前对所有结点调用一遍
64 inline void initialize(node *x) {
65   *x = node();
66   x → ch[0] = x → ch[1] = x → p = null;
67 }
68
69 // 注意一下在Access的同时更新子树信息的方法
70 node *access(node *x) {
71   node *y = null;
72
73   while (x != null) {
74     splay(x);
75
76     x → tree_cnt += x → ch[1] → chain_cnt - y →
77       ↪ chain_cnt;
78     x → tree_sum += (x → ch[1] → rev ? x →
79       ↪ ch[1] → sumr : x → ch[1] → suml) - y →
80       ↪ suml;
81   }
82
83   x → ch[1] = y;
84   (y = x) → refresh();
85   x = x → p;
86 }
87
88 // 找到一个点所在连通块的根
89 // 对比原版没有变化
90 node *getroot(node *x) {
91   x = access(x);
92
93   while (x → pushdown(), x → ch[0] != null)
94     x = x → ch[0];
95   splay(x);
96
97   return x;
98 }
99
100 // 换根，同样没有变化
101 void makeroott(node *x) {
102   access(x);
103   splay(x);
104   x → rev ≈ true;
105   x → pushdown();
106 }
107
108 // 连接两个点
109 // !!! 注意这里必须把两者都变成根，因为只能修改根结点
110 void link(node *x, node *y) {
111   makeroott(x);
112   makeroott(y);
113
114   x → p = y;
115   y → tree_cnt += x → chain_cnt;
116   y → tree_sum += x → suml;
117   y → refresh();
118 }
119
120 // 删除一条边
121 // 对比原版没有变化
122 void cut(node *x, node *y) {
123   makeroott(x);
124   access(y);
125   splay(y);
126
127   y → ch[0] → p = null;
128   y → ch[0] = null;
129   y → refresh();
130 }
131
132 // 修改/询问一个点，这里以询问为例
133 // 如果是修改就在换根之后搞一些操作
134 long long query(node *x) {
135   makeroott(x);
136   return x → suml;
137 }
138
139 // Splay函数
140 // 对比原版没有变化
141 void splay(node *x) {
142   x → pushdown();
143
144   while (!isroot(x)) {
145     if (!isroot(x → p))
146       x → p → p → pushdown();
147     x → p → pushdown();
148   }
149   x → pushdown();
150 }

```

```

137
138     if (isroot(x → p)) {
139         rot(x → p, dir(x) ^ 1);
140         break;
141     }
142
143     if (dir(x) == dir(x → p))
144         rot(x → p → p, dir(x → p) ^ 1);
145     else
146         rot(x → p, dir(x) ^ 1);
147
148     rot(x → p, dir(x) ^ 1);
149 }
150
151 // 旋转函数
152 // 对比原版没有变化
153 void rot(node *x, int d) {
154     node *y = x → ch[d ^ 1];
155
156     if ((x → ch[d ^ 1] = y → ch[d]) != null)
157         y → ch[d] → p = x;
158
159     y → p = x → p;
160     if (!isroot(x))
161         x → p → ch[dir(x)] = y;
162
163     (y → ch[d] = x) → p = y;
164
165     x → refresh();
166     y → refresh();
167 }
168

34
35     q1.pop();
36     q2.pop();
37 }
38
39     return q1.top();
40 }

41 long long top2() {
42     if (size() < 2)
43         return -INF;
44
45     long long a = top();
46     erase(a);
47     long long b = top();
48     push(a);
49     return a + b;
50 }

51 int size() {
52     return q1.size() - q2.size();
53 }

54 bool empty() {
55     return q1.size() == q2.size();
56 }

57 } heap; // 全局堆维护每条链的最大子段和

58 struct node {
59     long long sum, maxsum, prefix, suffix;
60     int key;
61     binary_heap heap; // 每个点的堆存的是它的子树中到它的
62     // 最远距离, 如果它是黑点的话还会包括自己
63     node *ch[2], *p;
64 }
65

```

#### 4.7.4 模板题:动态QTREE4(询问树上相距最远点)

```

1 #include <bits/stdc++.h>
2 #include <ext/pb_ds/assoc_container.hpp>
3 #include <ext/pb_ds/tree_policy.hpp>
4 #include <ext/pb_ds/priority_queue.hpp>
5
6 #define isroot(x) ((x) → p == null || ((x) != (x) → p
7     ↪ → ch[0] && (x) != (x) → p → ch[1]))
8 #define dir(x) ((x) == (x) → p → ch[1])
9
10 using namespace std;
11 using namespace __gnu_pbds;
12
13 constexpr int maxn = 100005;
14 constexpr long long INF = 1000000000000000000l;
15
16 struct binary_heap {
17     __gnu_pbds::priority_queue<long long, less<long
18         ↪ long>, binary_heap_tag> q1, q2;
19     binary_heap() {}
20
21     void push(long long x) {
22         if (x > (-INF) >> 2)
23             q1.push(x);
24     }
25
26     void erase(long long x) {
27         if (x > (-INF) >> 2)
28             q2.push(x);
29     }
30
31     long long top() {
32         if (empty())
33             return -INF;
34
35         while (!q2.empty() && q1.top() == q2.top()) {
36
37             if (!rev)
38                 return;
39
40             ch[0] → rev ≈ true;
41             ch[1] → rev ≈ true;
42             swap(ch[0], ch[1]);
43             swap(prefix, suffix);
44             rev = false;
45         }
46
47         inline void pushdown() {
48             if (!rev)
49                 return;
50
51             ch[0] → rev ≈ true;
52             ch[1] → rev ≈ true;
53             swap(ch[0], ch[1]);
54             swap(prefix, suffix);
55             rev = false;
56         }
57
58         inline void refresh() {
59             pushdown();
60             ch[0] → pushdown();
61             ch[1] → pushdown();
62             sum = ch[0] → sum + ch[1] → sum + key;
63             prefix = max(ch[0] → prefix,
64                         ch[0] → sum + key + ch[1] →
65                             ↪ prefix);
66             suffix = max(ch[1] → suffix,
67                         ch[1] → sum + key + ch[0] →
68                             ↪ suffix);
69             maxsum = max(max(ch[0] → maxsum, ch[1] →
70                             ↪ maxsum),
71                         ch[0] → suffix + key + ch[1] →
72                             ↪ prefix);
73
74             if (!heap.empty()) {
75                 prefix = max(prefix,
76                               ch[0] → sum + key +
77                                   ↪ heap.top());
78                 suffix = max(suffix,
79                               ch[1] → sum + key +
80                                   ↪ heap.top());
81                 maxsum = max(maxsum, max(ch[0] → suffix,
82
83
84
85
86
87
88
89
90
91
92
93
94
95
96

```

```

97         ch[1] → prefix) +
98             ↪ key +
99             ↪ heap.top());
100
101     if (heap.size() > 1) {
102         maxsum = max(maxsum, heap.top2() +
103             ↪ key);
104     }
105 }
106 void addedge(int, int, int);
107 void deledge(int, int);
108 void modify(int, int, int);
109 void modify_color(int);
110 node *newnode(int);
111 node *access(node *);
112 void makeroott(node *);
113 void link(node *, node *);
114 void cut(node *, node *);
115 void splay(node *);
116 void rot(node *, int);
117
118 queue<node *> freenodes;
119 tree<pair<int, int>, node *> mp;
120
121 bool col[maxn] = {false};
122 char c;
123 int n, m, k, x, y, z;
124
125 int main() {
126     null → ch[0] = null → ch[1] = null → p = null;
127     scanf("%d%d%d", &n, &m, &k);
128
129     for (int i = 1; i ≤ n; i++)
130         newnode(0);
131
132     heap.push(0);
133
134     while (k--) {
135         scanf("%d", &x);
136
137         col[x] = true;
138         null[x].heap.push(0);
139     }
140
141     for (int i = 1; i < n; i++) {
142         scanf("%d%d%d", &x, &y, &z);
143
144         if (x > y)
145             swap(x, y);
146         addedge(x, y, z);
147     }
148
149     while (m--) {
150         scanf(" %c%d", &c, &x);
151
152         if (c == 'A') {
153             scanf("%d", &y);
154
155             if (x > y)
156                 swap(x, y);
157             deledge(x, y);
158         }
159         else if (c == 'B') {
160             scanf("%d%d", &y, &z);
161
162             if (x > y)
163                 swap(x, y);
164             addedge(x, y, z);
165         }
166     }
167
168     else if (c == 'C') {
169         scanf("%d%d", &y, &z);
170
171         if (x > y)
172             swap(x, y);
173         modify(x, y, z);
174     }
175     else
176         modify_color(x);
177
178     printf("%lld\n", (heap.top() > 0 ? heap.top() :
179         ↪ -1));
180 }
181
182 void addedge(int x, int y, int z) {
183     node *tmp;
184     if (freenodes.empty())
185         tmp = newnode(z);
186     else {
187         tmp = freenodes.front();
188         freenodes.pop();
189         *tmp = node(z);
190     }
191
192     tmp → ch[0] = tmp → ch[1] = tmp → p = null;
193
194     heap.push(tmp → maxsum);
195     link(tmp, null + x);
196     link(tmp, null + y);
197     mp[make_pair(x, y)] = tmp;
198 }
199
200 void deledge(int x, int y) {
201     node *tmp = mp[make_pair(x, y)];
202
203     cut(tmp, null + x);
204     cut(tmp, null + y);
205
206     freenodes.push(tmp);
207     heap.erase(tmp → maxsum);
208     mp.erase(make_pair(x, y));
209 }
210
211 void modify(int x, int y, int z) {
212     node *tmp = mp[make_pair(x, y)];
213     makeroott(tmp);
214     tmp → pushdown();
215
216     heap.erase(tmp → maxsum);
217     tmp → key = z;
218     tmp → refresh();
219     heap.push(tmp → maxsum);
220 }
221
222 void modify_color(int x) {
223     makeroott(null + x);
224     col[x] ≈ true;
225
226     if (col[x])
227         null[x].heap.push(0);
228     else
229         null[x].heap.erase(0);
230
231     heap.erase(null[x].maxsum);
232     null[x].refresh();
233     heap.push(null[x].maxsum);
234 }
235
236 node *newnode(int k) {

```

```

237     *(++ptr) = node(k);
238     ptr → ch[0] = ptr → ch[1] = ptr → p = null;
239     return ptr;
240 }
241
242 node *access(node *x) {
243     splay(x);
244     heap.erase(x → maxsum);
245     x → refresh();
246
247     if (x → ch[1] != null) {
248         x → ch[1] → pushdown();
249         x → heap.push(x → ch[1] → prefix);
250         x → refresh();
251         heap.push(x → ch[1] → maxsum);
252     }
253
254     x → ch[1] = null;
255     x → refresh();
256     node *y = x;
257     x = x → p;
258
259     while (x != null) {
260         splay(x);
261         heap.erase(x → maxsum);
262
263         if (x → ch[1] != null) {
264             x → ch[1] → pushdown();
265             x → heap.push(x → ch[1] → prefix);
266             heap.push(x → ch[1] → maxsum);
267         }
268
269         x → heap.erase(y → prefix);
270         x → ch[1] = y;
271         (y = x) → refresh();
272         x = x → p;
273     }
274
275     heap.push(y → maxsum);
276     return y;
277 }
278
279 void makeroott(node *x) {
280     access(x);
281     splay(x);
282     x → rev ~ true;
283 }
284
285 void link(node *x, node *y) { // 新添一条虚边, 维护y对应的堆
286     makeroott(x);
287     makeroott(y);
288
289     x → pushdown();
290     x → p = y;
291     heap.erase(y → maxsum);
292     y → heap.push(x → prefix);
293     y → refresh();
294     heap.push(y → maxsum);
295 }
296
297 void cut(node *x, node *y) { // 断开一条实边, 一条链变成
298     // 两条链, 需要维护全局堆
299     makeroott(x);
300     access(y);
301     splay(y);
302
303     heap.erase(y → maxsum);
304     heap.push(y → ch[0] → maxsum);
305     y → ch[0] → p = null;
306     y → ch[0] = null;
307     y → refresh();
308 }
309
310 void splay(node *x) {
311     x → pushdown();
312
313     while (!isroot(x)) {
314         if (!isroot(x → p))
315             x → p → p → pushdown();
316
317         x → p → pushdown();
318         x → pushdown();
319
320         if (isroot(x → p)) {
321             rot(x → p, dir(x) ^ 1);
322             break;
323         }
324
325         if (dir(x) == dir(x → p))
326             rot(x → p → p, dir(x → p) ^ 1);
327         else
328             rot(x → p, dir(x) ^ 1);
329
330         rot(x → p, dir(x) ^ 1);
331     }
332 }
333
334 void rot(node *x, int d) {
335     node *y = x → ch[d ^ 1];
336
337     if ((x → ch[d ^ 1] = y → ch[d]) != null)
338         y → ch[d] → p = x;
339
340     y → p = x → p;
341
342     if (!isroot(x))
343         x → p → ch[dir(x)] = y;
344
345     (y → ch[d] = x) → p = y;
346
347     x → refresh();
348     y → refresh();
349 }

```

## 4.8 K-D树

### 4.8.1 动态K-D树(定期重构)

```

1 int l[2], r[2], x[B + 10][2], w[B + 10];
2 int n, op, ans = 0, cnt = 0, tmp = 0;
3 int d;
4
5 struct node {
6     int x[2], l[2], r[2], w, sum;
7     node *ch[2];
8
9     bool operator < (const node &a) const {
10         return x[d] < a.x[d];
11     }
12
13     void refresh() {
14         sum = ch[0] → sum + ch[1] → sum + w;
15         l[0] = min(x[0], min(ch[0] → l[0], ch[1] →
16                     → l[0]));
17         l[1] = min(x[1], min(ch[0] → l[1], ch[1] →
18                     → l[1]));
19         r[0] = max(x[0], max(ch[0] → r[0], ch[1] →
20                     → r[0]));
21         r[1] = max(x[1], max(ch[0] → r[1], ch[1] →
22                     → r[1]));
23     }
24 }
25
26
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30
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```

```

19     }
20 } null[maxn], *root = null;
21
22 void build(int l, int r, int k, node *&rt) {
23     if (l > r) {
24         rt = null;
25         return;
26     }
27
28     int mid = (l + r) / 2;
29
30     d = k;
31     nth_element(null + l, null + mid, null + r + 1);
32
33     rt = null + mid;
34     build(l, mid - 1, k ^ 1, rt → ch[0]);
35     build(mid + 1, r, k ^ 1, rt → ch[1]);
36
37     rt → refresh();
38 }
39
40 void query(node *rt) {
41     if (l[0] ≤ rt → l[0] && l[1] ≤ rt → l[1] && rt
42         → r[0] ≤ r[0] && rt → r[1] ≤ r[1]) {
43         ans += rt → sum;
44         return;
45     }
46     else if (l[0] > rt → r[0] || l[1] > rt → r[1] ||
47             → r[0] < rt → l[0] || r[1] < rt → l[1])
48         return;
49
50     if (l[0] ≤ rt → x[0] && l[1] ≤ rt → x[1] && rt
51         → x[0] ≤ r[0] && rt → x[1] ≤ r[1])
52         ans += rt → w;
53
54     query(rt → ch[0]);
55     query(rt → ch[1]);
56 }
57
58 int main() {
59
60     null → l[0] = null → l[1] = 100000000;
61     null → r[0] = null → r[1] = -100000000;
62     null → sum = 0;
63     null → ch[0] = null → ch[1] = null;
64     scanf("%*d");
65
66     while (scanf("%d", &op) == 1 && op != 3) {
67         if (op == 1) {
68             tmp++;
69             scanf("%d%d%d", &x[tmp][0], &x[tmp][1],
70                   → &w[tmp]);
71             x[tmp][0] ≈ ans;
72             x[tmp][1] ≈ ans;
73             w[tmp] ≈ ans;
74
75             if (tmp == B) {
76                 for (int i = 1; i ≤ tmp; i++) {
77                     null[cnt + i].x[0] = x[i][0];
78                     null[cnt + i].x[1] = x[i][1];
79                     null[cnt + i].w = w[i];
80                 }
81
82                 build(1, cnt + tmp, 0, root);
83                 tmp = 0;
84             }
85         }
86         else {
87             scanf("%d%d%d%d", &l[0], &l[1], &r[0],
88                   → &r[1]);
89         }
90     }
91 }
92
93 }
```

```

84     l[0] ≈ ans;
85     l[1] ≈ ans;
86     r[0] ≈ ans;
87     r[1] ≈ ans;
88     ans = 0;
89
90     for (int i = 1; i ≤ tmp; i++)
91         if (l[0] ≤ x[i][0] && l[1] ≤ x[i][1]
92             → && x[i][0] ≤ r[0] && x[i][1] ≤
93                 → r[1])
94             ans += w[i];
95
96     query(root);
97     printf("%d\n", ans);
98 }
99
100 }
```

## 4.9 LCA最近公共祖先

### 4.9.1 Tarjan LCA $O(n + m)$

```

1 vector<pair<int, int>> q[maxn];
2 int lca[maxn];
3
4 void dfs(int x) {
5     dfn[x] = ++tim; // 其实求LCA是用不到DFS序的，但是一般其他步骤要用
6     ufs[x] = x;
7
8     for (auto pi : q[x]) {
9         int y = pi.first, i = pi.second;
10        if (dfn[y])
11            lca[i] = findufs(y);
12    }
13
14    for (int y : G[x])
15        if (y != p[x]) {
16            p[y] = x;
17            dfs(y);
18        }
19
20    ufs[x] = p[x];
21 }
```

## 4.10 虚树

```

1 struct Tree {
2     vector<int> G[maxn], W[maxn];
3     int p[maxn], d[maxn], size[maxn], mn[maxn],
4         → mx[maxn];
5     bool col[maxn];
6     long long ans_sum;
7     int ans_min, ans_max;
8
9     void add(int x, int y, int z) {
10         G[x].push_back(y);
11         W[x].push_back(z);
12     }
13
14     void dfs(int x) {
15         size[x] = col[x];
16         mx[x] = (col[x] ? d[x] : -inf);
17         mn[x] = (col[x] ? d[x] : inf);
18
19         for (int i = 0; i < (int)G[x].size(); i++) {
```

```

19     d[G[x][i]] = d[x] + W[x][i];
20     dfs(G[x][i]);
21     ans_sum += (long long) size[x] * size[G[x]
22         ↪ [i]] * d[x];
23     ans_max = max(ans_max, mx[x] + mx[G[x][i]]
24         ↪ - (d[x] << 1));
25     ans_min = min(ans_min, mn[x] + mn[G[x][i]]
26         ↪ - (d[x] << 1));
27     size[x] += size[G[x][i]];
28     mx[x] = max(mx[x], mx[G[x][i]]);
29     mn[x] = min(mn[x], mn[G[x][i]]);
30 }
31
32 void clear(int x) {
33     G[x].clear();
34     W[x].clear();
35     col[x] = false;
36 }
37
38 void solve(int rt) {
39     ans_sum = 0;
40     ans_max = -inf;
41     ans_min = inf;
42     dfs(rt);
43     ans_sum <= 1;
44 }
45 void dfs(int);
46 int LCA(int, int);
47
48 vector<int> G[maxn];
49 int f[maxn][20], d[maxn], dfn[maxn], tim = 0;
50
51 bool cmp(int x, int y) {
52     return dfn[x] < dfn[y];
53 }
54
55 int n, m, lgn = 0, a[maxn], s[maxn], v[maxn];
56
57 int main() {
58     scanf("%d", &n);
59
60     for (int i = 1, x, y; i < n; i++) {
61         scanf("%d%d", &x, &y);
62         G[x].push_back(y);
63         G[y].push_back(x);
64     }
65
66     G[n + 1].push_back(1);
67     dfs(n + 1);
68
69     for (int i = 1; i ≤ n + 1; i++)
70         G[i].clear();
71
72     lgn--;
73
74     for (int j = 1; j ≤ lgn; j++)
75         for (int i = 1; i ≤ n; i++)
76             f[i][j] = f[f[i][j - 1]][j - 1];
77
78     scanf("%d", &m);
79
80     while (m--) {
81         int k;
82         scanf("%d", &k);
83
84         for (int i = 1; i ≤ k; i++)
85             scanf("%d", &a[i]);
86
87         sort(a + 1, a + k + 1, cmp);
88
89         int top = 0, cnt = 0;
90         s[++top] = v[++cnt] = n + 1;
91         long long ans = 0;
92
93         for (int i = 1; i ≤ k; i++) {
94             virtree.col[a[i]] = true;
95             ans += d[a[i]] - 1;
96             int u = LCA(a[i], s[top]);
97
98             if (s[top] != u) {
99                 while (top > 1 && d[s[top - 1]] ≥
100                     ↪ d[u]) {
101                     virtree.add(s[top - 1], s[top],
102                         ↪ d[s[top]] - d[s[top - 1]]);
103                     top--;
104                 }
105
106                 if (s[top] != u) {
107                     virtree.add(u, s[top], d[s[top]] -
108                         ↪ d[u]);
109                     s[top] = v[++cnt] = u;
110                 }
111
112                 s[++top] = a[i];
113
114             for (int i = top - 1; i; i--)
115                 virtree.add(s[i], s[i + 1], d[s[i + 1]] -
116                     ↪ d[s[i]]);
117
118             virtree.solve(n + 1);
119             ans *= k - 1;
120             printf("%lld %d %d\n", ans - virtree.ans_sum,
121                   ↪ virtree.ans_min, virtree.ans_max);
122
123             for (int i = 1; i ≤ k; i++)
124                 virtree.clear(a[i]);
125             for (int i = 1; i ≤ cnt; i++)
126                 virtree.clear(v[i]);
127
128             }
129
130             return 0;
131         }
132
133         void dfs(int x) {
134             dfn[x] = ++tim;
135             d[x] = d[f[x][0]] + 1;
136
137             while ((1 << lgn) < d[x])
138                 lgn++;
139
140             for (int i = 0; i < (int)G[x].size(); i++) {
141                 if (G[x][i] != f[x][0]) {
142                     f[G[x][i]][0] = x;
143                     dfs(G[x][i]);
144                 }
145             }
146
147             int LCA(int x, int y) {
148                 if (d[x] != d[y]) {
149                     if (d[x] < d[y])
150                         swap(x, y);
151
152                     for (int i = lgn; i ≥ 0; i--)
153                         if (((d[x] - d[y]) >> i) & 1)
154                             x = f[x][i];
155
156                     if (x == y)
157                         return x;
158                 }
159             }
160         }
161     }
162 }

```

```

155
156     for (int i = lgn; i >= 0; i--) {
157         if (f[x][i] != f[y][i]) {
158             x = f[x][i];
159             y = f[y][i];
160         }
161
162     return f[x][0];
163 }
```

## 4.11 长链剖分

```

1 // 顾名思义，长链剖分是取最深的儿子作为重儿子
2
3 // O(n)维护以深度为下标的子树信息
4 vector<int> G[maxn], v[maxn];
5 int n, p[maxn], h[maxn], son[maxn], ans[maxn];
6
7 // 原题题意：求每个点的子树中与它距离是几的点最多，相同的
8 // → 取最大深度
9 // 由于vector只能在后面加入元素，为了写代码方便，这里反过来
10 // → 来存
11 // 或者开一个结构体维护倒过来的vector
12 void dfs(int x) {
13     h[x] = 1;
14
15     for (int y : G[x])
16         if (y != p[x]){
17             p[y] = x;
18             dfs(y);
19
20             if (h[y] > h[son[x]])
21                 son[x] = y;
22
23         if (!son[x]) {
24             v[x].push_back(1);
25             ans[x] = 0;
26             return;
27         }
28
29     h[x] = h[son[x]] + 1;
30     swap(v[x], v[son[x]]);
31
32     if (v[x][ans[son[x]]] == 1)
33         ans[x] = h[x] - 1;
34     else
35         ans[x] = ans[son[x]];
36
37     v[x].push_back(1);
38
39     int mx = v[x][ans[x]];
40     for (int y : G[x])
41         if (y != p[x] && y != son[x]) {
42             for (int j = 1; j <= h[y]; j++) {
43                 v[x][h[x] - j - 1] += v[y][h[y] - j];
44
45                 int t = v[x][h[x] - j - 1];
46                 if (t > mx || (t == mx && h[x] - j - 1
47                     > ans[x])) {
48                     mx = t;
49                     ans[x] = h[x] - j - 1;
50
51                 }
52
53             v[y].clear();
54         }
55 }
```

### 4.11.1 梯子剖分

```

1 // 在线求一个点的第k祖先 O(n log n)-O(1)
2 // 理论基础：任意一个点x的k级祖先所在长链长度一定 ≥ k
3
4 // 全局数组定义
5 vector<int> G[maxn], v[maxn];
6 int d[maxn], mxd[maxn], son[maxn], top[maxn],
7     ↪ len[maxn];
8 int f[19][maxn], log_tbl[maxn];
9
10 // 在主函数中两遍dfs之后加上如下预处理
11 log_tbl[0] = -1;
12 for (int i = 1; i ≤ n; i++)
13     log_tbl[i] = log_tbl[i / 2] + 1;
14 for (int j = 1; (1 << j) < n; j++)
15     for (int i = 1; i ≤ n; i++)
16         f[j][i] = f[j - 1][f[j - 1][i]];
17
18 // 第一遍dfs，用于计算深度和找出重儿子
19 void dfs1(int x) {
20     mxd[x] = d[x];
21
22     for (int y : G[x])
23         if (y != f[0][x]){
24             f[0][y] = x;
25             d[y] = d[x] + 1;
26
27             dfs1(y);
28
29             mxd[x] = max(mxd[x], mxd[y]);
30             if (mxd[y] > mxd[son[x]])
31                 son[x] = y;
32         }
33
34 // 第二遍dfs，用于进行剖分和预处理梯子剖分(每条链向上延伸
35 // → 一倍)数组
36 void dfs2(int x) {
37     top[x] = (x == son[f[0][x]] ? top[f[0][x]] : x);
38
39     for (int y : G[x])
40         if (y != f[0][x])
41             dfs2(y);
42
43     if (top[x] == x) {
44         int u = x;
45         while (top[son[u]] == x)
46             u = son[u];
47
48         len[x] = d[u] - d[x];
49         for (int i = 0; i < len[x]; i++, u = f[0][u])
50             v[x].push_back(u);
51
52         u = x;
53         for (int i = 0; i < len[x] && u; i++, u = f[0]
54             ↪ [u])
55             v[x].push_back(u);
56
57 // 在线询问x的k级祖先 O(1)
58 // 不存在时返回0
59 int query(int x, int k) {
60     if (!k)
61         return x;
62     if (k > d[x])
63         return 0;
```

```
65     x = f[log_tbl[k]][x];
66     k ≈ 1 << log_tbl[k];
67     return v[top[x]][d[top[x]] + len[top[x]] - d[x] +
68     ↵ k];
```

## 4.12 堆

### 4.12.1 左偏树

参见3.2.3.k短路(27页).

### 4.12.2 二叉堆

```
1 struct my_binary_heap {
2     static constexpr int maxn = 100000;
3
4     int a[maxn], size;
5
6     my_binary_heap() : size(0) {}
7
8     void push(int val) {
9         a[++size] = val;
10
11        for (int x = size; x > 1; x /= 2)
12            if (a[x] < a[x / 2])
13                swap(a[x], a[x / 2]);
14            else
15                break;
16    }
17
18    int &top() {
19        return a[1];
20    }
21
22    int pop() {
23        int res = a[1];
24        a[1] = a[size--];
25
26        for (int x = 1, son; ; x = son)
27            if (x * 2 == size)
28                son = x * 2;
29            else if (x * 2 > size)
30                break;
31            else if (a[x * 2] < a[x * 2 + 1])
32                son = x * 2;
33            else
34                son = x * 2 + 1;
35
36            if (a[son] < a[x])
37                swap(a[x], a[son]);
38            else
39                break;
40    }
41
42    return res;
43
44 }
```

#### 4.13.1 莫队二次离线

适用范围：询问的是点对相关(或者其它可以枚举每个点和区间算贡献)的信息，并且可以离线；更新时可以使用一些牺牲修改复杂度来改善询问复杂度的数据结构(如单点修改询问区间和)。

先按照普通的莫队将区间排序。考虑区间移动的情况，以 $(l, r)$ 向右移动右端点到 $(l, t)$ 为例。

对于每个  $i \in (r, t]$  来说，它都要对区间  $[l, i]$  算贡献。可以拆成  $[1, i)$  和  $[1, l]$  两部分，那么前一部分因为都是  $i$  和  $[1, i)$  做贡献的形式所以可以直接预处理。

考虑后一部分,  $i$ 和 $(1, l]$ 做贡献, 因为莫队的性质我们可以保证这样的询问次数不超过 $O((n+m)\sqrt{n})$ , 因此我们可以对每个 $l$ 记录下来哪些 $i$ 要和它询问. 并且每次移动时询问的 $i$ 都是连续的, 所以对每个 $l$ 开一个vector记录下对应的区间和编号就行了.

剩余的三种情况(右端点左移或者移动左端点)都是类似的, 具体可以看代码.

例: Yuno loves sqrt technology II (询问区间逆序对数)

4.13 莫队

注意如果  $n$  和  $q$  不平衡，块大小应该设为  $\frac{n}{\sqrt{q}}$ .

另外如果裸的莫队要卡常可以按块编号奇偶性分别对右端点正序或者倒序排序，期望可以减少一半的移动次数。

```

51     return 0; // 为了防止越界就判一下
52     return sa[x] + sb[bid[x] + 1];
53 }

54 vector<Q> vp[maxn], vs[maxn]; // prefix, suffix
55 long long fp[maxn], fs[maxn]; // prefix, suffix
56
57 int a[maxn], b[maxn];
58
59 long long ta[maxn], ans[maxn];
60
61 int main() {
62
63     int n, m;
64     scanf("%d%d", &n, &m);
65
66     global_n = n;
67
68     for (int i = 1; i ≤ n; i++)
69         scanf("%d", &a[i]);
70
71     memcpy(b, a, sizeof(int) * (n + 1));
72     sort(b + 1, b + n + 1);
73
74     for (int i = 1; i ≤ n; i++)
75         a[i] = lower_bound(b + 1, b + n + 1, a[i]) - b;
76
77     for (int i = 1; i ≤ n; i++) {
78         bid[i] = (i - 1) / B + 1;
79
80         if (!L[bid[i]])
81             L[bid[i]] = i;
82
83         R[bid[i]] = i;
84         cntb = bid[i];
85     }
86
87     for (int i = 1; i ≤ m; i++) {
88         scanf("%d%d", &q[i].l, &q[i].r);
89
90         q[i].d = bid[q[i].l];
91         q[i].id = i;
92     }
93
94     sort(q + 1, q + m + 1);
95
96     int l = 2, r = 1; // l, r是上一个询问的端点
97
98     for (int i = 1; i ≤ m; i++) {
99         int s = q[i].l, t = q[i].r; // s, t是当前要调整
100        → 到的端点
101
102        if (s < l)
103            vs[r + 1].push_back(Q(s, l - 1, 1, i));
104        else if (s > l)
105            vs[r + 1].push_back(Q(l, s - 1, -1, i));
106
107        l = s;
108
109        if (t > r)
110            vp[l - 1].push_back(Q(r + 1, t, 1, i));
111        else if (t < r)
112            vp[l - 1].push_back(Q(t + 1, r, -1, i));
113
114        r = t;
115    }
116
117    for (int i = 1; i ≤ n; i++) { // 第一遍正着处理, 解
118        → 决关于前缀的询问

```

```

119     fp[i] = fp[i - 1] + querys(a[i] + 1);
120
121     adds(a[i]);
122
123     for (auto q : vp[i]) {
124         long long tmp = 0;
125         for (int k = q.l; k ≤ q.r; k++)
126             tmp += querys(a[k] + 1);
127
128         ta[q.id] -= q.d * tmp;
129     }
130
131     memset(sa, 0, sizeof(sa));
132     memset(sb, 0, sizeof(sb));
133
134     for (int i = n; i; i--) { // 第二遍倒着处理, 解决关
135        → 于后缀的询问
136         fs[i] = fs[i + 1] + queryp(a[i] - 1);
137
138         addp(a[i]);
139
140         for (auto q : vs[i]) {
141             long long tmp = 0;
142             for (int k = q.l; k ≤ q.r; k++)
143                 tmp += queryp(a[k] - 1);
144
145             ta[q.id] -= q.d * tmp;
146         }
147     }
148
149     l = 2;
150     r = 1;
151
152     for (int i = 1; i ≤ m; i++) { // 求出fs和fp之后再加
153        → 上这部分的贡献
154         int s = q[i].l, t = q[i].r;
155
156         ta[i] += fs[s] - fs[l];
157         ta[i] += fp[t] - fp[r];
158
159         l = s;
160         r = t;
161
162         ta[i] += ta[i - 1]; // 因为算出来的是相邻两个询
163        → 问之间的贡献, 所以要前缀和
164         ans[q[i].id] = ta[i];
165
166     for (int i = 1; i ≤ m; i++)
167         printf("%lld\n", ans[i]);
168
169 }

```

### 4.13.2 带修莫队在线化 $O(n^{\frac{5}{3}})$

最简单的带修莫队: 块大小设成 $n^{\frac{2}{3}}$ , 排序时第一关键字是左端点块编号, 第二关键字是右端点块编号, 第三关键字是时间. (记得把时间压缩成只有修改的时间.)

现在要求在线的同时支持修改, 仍然以 $B = n^{\frac{2}{3}}$ 分一块, 预处理出两块之间的贡献, 那么预处理复杂度就是 $O(n^{\frac{5}{3}})$ .

修改时最简单的方法是直接把 $n^{\frac{2}{3}}$ 个区间全更新一遍. 嫌慢的话可以给每个区间打一个懒标记, 询问的时候如果解了再更新区间的信息.

注意如果附加信息是可减的(比如每个数的出现次数), 那么就只需要存 $O(n^{\frac{1}{3}})$ 个.

总复杂度仍然是 $O(n^{\frac{5}{3}})$ , 如果打懒标记的话是跑不太满的. 如果附加

信息可减则空间复杂度是 $O(n^{\frac{4}{3}})$ , 否则和时间复杂度同阶

### 4.13.3 莫队二次离线 在线化 $O((n + m)\sqrt{n})$

和之前的道理是一样的， $i$ 和 $[1, i]$ 的贡献这部分仍然可以预处理掉，而前后缀对区间的贡献那部分只保存块端点处的信息。  
按照莫队二次离线的转移方法操作之后发现只剩两边散块的贡献没有解决。这时可以具体问题具体解决，例如求逆序对的话直接预处理出排序后的数组然后归并即可。

由排序后的数组累加归并  
时空复杂度均为 $O(n\sqrt{n})$

以下代码以强制在线求区间逆序对为例(洛谷上被卡常了, 正常情况下极限数据应该在1.5s内.)

```

1 constexpr int maxn = 100005, B = 315, maxb = maxn / B +
2   ↪ 5;
3
4 int n, bid[maxn], L[maxb], R[maxb], cntb;
5
6 struct DS { // O(sqrt(n))修改 O(1)查询
7     int total;
8     int sa[maxn], sb[maxb];
9
10 void init(const DS &o) {
11     total = o.total;
12     memcpy(sa, o.sa, sizeof(int) * (n + 1));
13     memcpy(sb, o.sb, sizeof(int) * (cntb + 1));
14 }
15
16 void add(int x) {
17     total++;
18     for (int k = 1; k ≤ bid[x]; k++)
19         sb[k]++;
20     for (int i = L[bid[x]]; i ≤ x; i++)
21         sa[i]++;
22 }
23
24 int querys(int x) {
25     if (x > n)
26         return 0;
27
28     return sb[bid[x] + 1] + sa[x];
29 }
30
31 int queryp(int x) {
32     return total - querys(x + 1);
33 }
34 pr[maxb];
35
36 int c[maxn]; // 树状数组
37
38 void addc(int x, int d) {
39     while (x) {
40         c[x] += d;
41         x -= x & -x;
42     }
43 }
44
45 int queryc(int x) {
46     int ans = 0;
47     while (x ≤ n) {
48         ans += c[x];
49         x += x & -x;
50     }
51     return ans;
52 }
53
54 long long fp[maxn], fs[maxn];
55
56 int rk[maxn], val[maxn][B + 5];

```

```

57 long long dat[maxb][maxb];
58
59 int a[maxn];
60
61 int main() {
62
63     int m;
64     cin >> n >> m;
65
66     for (int i = 1; i ≤ n; i++) {
67         cin >> a[i];
68
69         bid[i] = (i - 1) / B + 1;
70         if (!L[bid[i]])
71             L[bid[i]] = i;
72         R[bid[i]] = i;
73         cntb = bid[i];
74
75         rnk[i] = i;
76     }
77
78     for (int k = 1; k ≤ cntb; k++)
79         sort(rnk + L[k], rnk + R[k] + 1, [](int x, int
80                                     → y) {return a[x] < a[y];}); // 每个块排序
81
82     for (int i = n; i; i--)
83         for (int j = 2; i + j - 1 ≤ R[bid[i]]; j++) {
84             val[i][j] = val[i + 1][j - 1] + val[i][j -
85                                         → 1] - val[i + 1][j - 2];
86             if (a[i] > a[i + j - 1])
87                 val[i][j]++; // 块内用二维前缀和预处理
88
89     for (int k = 1; k ≤ cntb; k++) {
90         for (int i = L[k]; i ≤ R[k]; i++) {
91             dat[k][k] += queryc(a[i] + 1); // 单块内的逆
92                                         → 序对直接用树状数组预处理
93             addc(a[i], 1);
94
95             for (int i = L[k]; i ≤ R[k]; i++)
96                 addc(a[i], -1);
97     }
98
99     for (int i = 1; i ≤ n; i++) {
100         if (i > 1 && i == L[bid[i]])
101             pr[bid[i]].init(pr[bid[i] - 1]);
102
103         fp[i] = fp[i - 1] + pr[bid[i]].query(a[i] +
104                                         → 1);
105
106         pr[bid[i]].add(a[i]);
107     }
108
109     for (int i = n; i; i--) {
110         fs[i] = fs[i + 1] + (n - i - queryc(a[i] + 1));
111         addc(a[i], 1);
112     }
113
114     for (int s = 1; s ≤ cntb; s++)
115         for (int t = s + 1; t ≤ cntb; t++) {
116             dat[s][t] = dat[s][t - 1] + dat[t][t];
117
118             for (int i = L[t]; i ≤ R[t]; i++) // 块间的
119                                         → 逆序对用刚才处理的分块求出
120             dat[s][t] += pr[t - 1].query(a[i] + 1)
121                                         → - pr[s - 1].query(a[i] + 1);
122         }
123
124     long long ans = 0;

```

```

121
122     while (m--) {
123         long long s, t;
124         cin >> s >> t;
125
126         int l = s ^ ans, r = t ^ ans;
127
128         if (bid[l] == bid[r])
129             ans = val[l][r - l + 1];
130         else {
131             ans = dat[bid[l] + 1][bid[r] - 1];
132
133             ans += fp[r] - fp[L[bid[r]] - 1];
134             for (int i = L[bid[r]]; i <= r; i++)
135                 ans -= pr[bid[l]].query(a[i] + 1);
136
137             ans += fs[l] - fs[R[bid[l]] + 1];
138             for (int i = l; i <= R[bid[l]]; i++)
139                 ans -= (a[i] - 1) - pr[bid[r] -
140                               1].query(a[i] - 1);
141
142             int i = L[bid[l]], j = L[bid[r]], w = 0; // ← 手写归并
143
144             while (true) {
145                 while (i <= R[bid[l]] && rnk[i] < l)
146                     i++;
147                 while (j <= R[bid[r]] && rnk[j] > r)
148                     j++;
149
150                 if (i > R[bid[l]] && j > R[bid[r]])
151                     break;
152
153                 int x = (i <= R[bid[l]] ? a[rnk[i]] : (int)1e9), y = (j <= R[bid[r]] ?
154                               a[rnk[j]] : (int)1e9);
155
156                 if (x < y) {
157                     ans += w;
158                     i++;
159                 } else {
160                     j++;
161                     w++;
162                 }
163             }
164
165             cout << ans << '\n';
166         }
167
168     }
169 }
```

- 如果输入规模之和不大于 $n$ (例如给定多个小字符串与大字符串进行询问), 那么规模超过 $\sqrt{n}$ 的问题最多只有 $\sqrt{n}$ 个

## 2. 序列

- 某些维护序列的问题可以用分块/块状链表维护
- 对于静态区间询问问题, 如果可以快速将左/右端点移动一位, 可以考虑莫队
  - 如果强制在线可以分块预处理, 但是一般空间需要 $n\sqrt{n}$
  - \* 例题: 询问区间中有几种数出现次数恰好为 $k$ , 强制在线
  - 如果带修改可以试着想一想带修莫队, 但是复杂度高达 $n^{\frac{5}{3}}$
- 线段树可以解决的问题也可以用分块来做到 $O(1)$ 询问或是 $O(1)$ 修改, 具体要看哪种操作更多

## 3. 树

- 与序列类似, 树上也有树分块和树上莫队
  - 树上带修莫队很麻烦, 常数也大, 最好不要先考虑
  - 树分块不要想当然
- 树分治也可以套根号分治, 道理是一样的

## 4. 字符串

- 循环节长度大于 $\sqrt{n}$ 的子串最多只有 $O(n)$ 个, 如果是极长子串则只有 $O(\sqrt{n})$ 个

## 4.14 常见根号思路

### 1. 通用

- 出现次数大于 $\sqrt{n}$ 的数不会超过 $\sqrt{n}$ 个
- 对于带修改问题, 如果不方便分治或者二进制分组, 可以考虑对操作分块, 每次查询时暴力最后的 $\sqrt{n}$ 个修改并更正答案
- 根号分治:** 如果分治时每个子问题需要 $O(N)$ ( $N$ 是全局问题的大小)的时间, 而规模较小的子问题可以 $O(n^2)$ 解决, 则可以使用根号分治
  - 规模大于 $\sqrt{n}$ 的子问题用 $O(N)$ 的方法解决, 规模小于 $\sqrt{n}$ 的子问题用 $O(n^2)$ 暴力
  - 规模大于 $\sqrt{n}$ 的子问题最多只有 $\sqrt{n}$ 个
  - 规模不大于 $\sqrt{n}$ 的子问题大小的平方和也必定不会超过 $n\sqrt{n}$

# 5 字符串

## 5.1 KMP

```

1 char s[maxn], t[maxn];
2 int fail[maxn];
3 int n, m;
4
5 void init() { // 注意字符串是0-based, 但是fail是1-based
6     // memset(fail, 0, sizeof(fail));
7
8     for (int i = 1; i < m; i++) {
9         int j = fail[i];
10        while (j && t[i] != t[j])
11            j = fail[j];
12
13        if (t[i] == t[j])
14            fail[i + 1] = j + 1;
15        else
16            fail[i + 1] = 0;
17    }
18}
19
20 int KMP() {
21     int cnt = 0, j = 0;
22
23     for (int i = 0; i < n; i++) {
24         while (j && s[i] != t[j])
25             j = fail[j];
26
27         if (s[i] == t[j])
28             j++;
29         if (j == m)
30             cnt++;
31     }
32
33     return cnt;
34}

```

### 5.1.1 ex-KMP

```

1 //全局变量与数组定义
2 char s[maxn], t[maxn];
3 int n, m, a[maxn];
4
5 // 主过程 O(n + m)
6 // 把t的每个后缀与s的LCP输出到a中, s的后缀和自己的LCP存
7 // 在nx中
8 // 0-based, s的长度是m, t的长度是n
9 void exKMP(const char *s, const char *t, int *a) {
10     static int nx[maxn];
11
12     memset(nx, 0, sizeof(nx));
13
14     int j = 0;
15     while (j + 1 < m && s[j] == s[j + 1])
16         j++;
17     nx[1] = j;
18
19     for (int i = 2, k = 1; i < m; i++) {
20         int pos = k + nx[k], len = nx[i - k];
21
22         if (i + len < pos)
23             nx[i] = len;
24         else {
25             j = max(pos - i, 0);
26             while (i + j < m && s[j] == s[i + j])
27                 j++;
28             nx[i] = j;
29         }
30     }
31 }

```

```

27
28         nx[i] = j;
29         k = i;
30     }
31 }
32
33 j = 0;
34 while (j < n && j < m && s[j] == t[j])
35     j++;
36 a[0] = j;
37
38 for (int i = 1, k = 0; i < n; i++) {
39     int pos = k + a[k], len = nx[i - k];
40     if (i + len < pos)
41         a[i] = len;
42     else {
43         j = max(pos - i, 0);
44         while (j < m && i + j < n && s[j] == t[i + j])
45             j++;
46         a[i] = j;
47         k = i;
48     }
49 }
50
51

```

## 5.2 AC自动机

```

1 int ch[maxm][26], f[maxm][26], q[maxm], sum[maxm], cnt
2     = 0;
3
4 // 在字典树中插入一个字符串 O(n)
5 int insert(const char *c) {
6     int x = 0;
7     while (*c) {
8         if (!ch[x][*c - 'a'])
9             ch[x][*c - 'a'] = ++cnt;
10        x = ch[x][*c++ - 'a'];
11    }
12    return x;
13}
14
15 // 建AC自动机 O(n * sigma)
16 void getfail() {
17     int x, head = 0, tail = 0;
18
19     for (int c = 0; c < 26; c++)
20         if (ch[0][c])
21             q[tail++] = ch[0][c]; // 把根节点的儿子加入
22                             // 队列
23
24     while (head != tail) {
25         x = q[head++];
26
27         G[f[x][0]].push_back(x);
28         fill(f[x] + 1, f[x] + 26, cnt + 1);
29
30         for (int c = 0; c < 26; c++) {
31             if (ch[x][c]) {
32                 int y = f[x][0];
33
34                 f[ch[x][c]][0] = ch[y][c];
35                 q[tail++] = ch[x][c];
36             }
37             else
38                 ch[x][c] = ch[f[x][0]][c];
39         }
40     }
41     fill(f[0], f[0] + 26, cnt + 1);
42 }

```

40

58

### 5.3 后缀数组

### 5.3.1 倍增

```
58 }
59
60 char s[maxn];
61 int sa[maxn], rnk[maxn], height[maxn];
62
63 int main() {
64     cin >> (s + 1);
65
66     int n = strlen(s + 1);
67
68     get_sa(s, n, sa, rnk, height);
69
70     for (int i = 1; i ≤ n; i++)
71         cout << sa[i] << (i < n ? ' ' : '\n');
72
73     for (int i = 2; i ≤ n; i++)
74         cout << height[i] << (i < n ? ' ' : '\n');
75
76     return 0;
77 }
```

### 5.3.2 SA-IS

```

1 // SA-IS求完的SA有效位只有1~n, 但它是0-based, 如果其他部
2 → 分是1-based就抄一下封装
3
4
5 // 判断一个字符是否为LMS字符
6 bool is_lms(int *tp, int x) {
7     return x > 0 && tp[x] == s_type && tp[x - 1] ==
8         l_type;
9
10 // 判断两个LMS子串是否相同
11 bool equal_substr(int *s, int x, int y, int *tp) {
12     do {
13         if (s[x] != s[y])
14             return false;
15         x++;
16         y++;
17     } while (!is_lms(tp, x) && !is_lms(tp, y));
18
19     return s[x] == s[y];
20 }
21
22 // 诱导排序(从*型诱导到L型, 从L型诱导到S型)
23 // 调用之前应将*型按要求放入SA中
24 void induced_sort(int *s, int *sa, int *tp, int *buc,
25 → int *lbuc, int *sbuc, int n, int m) {
26     for (int i = 0; i ≤ n; i++)
27         if (sa[i] > 0 && tp[sa[i] - 1] == l_type)
28             sa[lbuc[s[sa[i] - 1]]++] = sa[i] - 1;
29
30     for (int i = 1; i ≤ m; i++)
31         sbuc[i] = buc[i] - 1;
32
33     for (int i = n; ~i; i--)
34         if (sa[i] > 0 && tp[sa[i] - 1] == s_type)
35             sa[sbuc[s[sa[i] - 1]]--] = sa[i] - 1;
36
37 // s是输入字符串, n是字符串的长度, m是字符集的大小
38 int *sa_is(int *s, int len, int m) {
39     int n = len - 1;
40
41     int *tp = new int[n + 1];
42     int *pos = new int[n + 1];
43     int *name = new int[n + 1];

```

```

44 int *sa = new int[n + 1];
45 int *buc = new int[m + 1];
46 int *lbuc = new int[m + 1];
47 int *sbuc = new int[m + 1];
48
49 memset(buc, 0, sizeof(int) * (m + 1));
50 memset(lbuc, 0, sizeof(int) * (m + 1));
51 memset(sbuc, 0, sizeof(int) * (m + 1));
52
53 for (int i = 0; i ≤ n; i++)
| buc[s[i]]++;
54
55
56 for (int i = 1; i ≤ m; i++) {
| buc[i] += buc[i - 1];
57
| lbuc[i] = buc[i - 1];
58 | sbuc[i] = buc[i] - 1;
59 }
60
61
62 tp[n] = s_type;
63 for (int i = n - 1; ~i; i--) {
| if (s[i] < s[i + 1])
| | tp[i] = s_type;
| else if (s[i] > s[i + 1])
| | tp[i] = l_type;
| else
| | tp[i] = tp[i + 1];
64 }
65
66 int cnt = 0;
67 for (int i = 1; i ≤ n; i++)
| if (tp[i] == s_type && tp[i - 1] == l_type)
| | pos[cnt++] = i;
68
69
70 memset(sa, -1, sizeof(int) * (n + 1));
71 for (int i = 0; i < cnt; i++)
| | sa[sbuc[s[pos[i]]]--] = pos[i];
72 induced_sort(s, sa, tp, buc, lbuc, sbuc, n, m);
73
74 memset(name, -1, sizeof(int) * (n + 1));
75 int lastx = -1, namecnt = 1;
76 bool flag = false;
77
78 for (int i = 1; i ≤ n; i++) {
79 | int x = sa[i];
80
81 | if (is_lms(tp, x)) {
82 | | if (lastx ≥ 0 && !equal_substr(s, x,
| | | → lastx, tp))
| | | namecnt++;
83
84 | | if (lastx ≥ 0 && namecnt == name[lastx])
| | | flag = true;
85
86 | | name[x] = namecnt;
87 | | lastx = x;
88 }
89
90 name[n] = 0;
91
92 int *t = new int[cnt];
93 int p = 0;
94 for (int i = 0; i ≤ n; i++)
| if (name[i] ≥ 0)
| | t[p++] = name[i];
95
96 int *tsa;
97 if (!flag) {
98 | tsa = new int[cnt];
99
100
101
102
103
104
105
106
107
108
109
110
111
112
113 | for (int i = 0; i < cnt; i++)
114 | | tsa[t[i]] = i;
115 }
116 else
117 | | tsa = sais(t, cnt, namecnt);
118
119 | lbuc[0] = sbuc[0] = 0;
120 for (int i = 1; i ≤ m; i++) {
121 | | lbuc[i] = buc[i - 1];
122 | | sbuc[i] = buc[i] - 1;
123 }
124
125 memset(sa, -1, sizeof(int) * (n + 1));
126 for (int i = cnt - 1; ~i; i--)
127 | | sa[sbuc[s[pos[tsa[i]]]]--] = pos[tsa[i]];
128 induced_sort(s, sa, tp, buc, lbuc, sbuc, n, m);
129
130 // 多组数据的时候最好 delete 掉
131 delete[] tp;
132 delete[] pos;
133 delete[] name;
134 delete[] buc;
135 delete[] lbuc;
136 delete[] sbuc;
137 delete[] t;
138 delete[] tsa;
139
140 return sa;
141 }
142
143 // 封装好的函数, 1-based
144 void get_sa(char *s, int n, int *sa, int *rnk, int
145 → *height) {
146 static int a[maxn];
147
148 for (int i = 1; i ≤ n; i++)
| | a[i - 1] = s[i];
149
150 a[n] = '$';
151
152 int *t = sais(a, n + 1, 256);
153 memcpy(sa, t, sizeof(int) * (n + 1));
154 delete[] t;
155
156 sa[0] = 0;
157 for (int i = 1; i ≤ n; i++)
| | rnk[+sa[i]] = i;
158
159 for (int i = 1, k = 0; i ≤ n; i++) { // 求 height
160 | if (k)
161 | | k--;
162
163 while (s[i + k] == s[sa[rnk[i] - 1] + k])
| | k++;
164
165 height[rnk[i]] = k; // height[i] = lcp(sa[i],
166 → sa[i - 1])
167 }
168 }
169

```

### 5.3.3 SAMSA

```

1 bool vis[maxn * 2];
2 char s[maxn];
3 int n, id[maxn * 2], ch[maxn * 2][26], height[maxn],
4 → tim = 0;
5 void dfs(int x) {

```

```

6   if (id[x]) {
7     height[tim++] = val[last];
8     sa[tim] = id[x];
9
10    last = x;
11  }
12
13  for (int c = 0; c < 26; c++)
14    if (ch[x][c])
15      dfs(ch[x][c]);
16
17  last = par[x];
18}
19
20 int main() {
21  last = ++cnt;
22
23  scanf("%s", s + 1);
24  n = strlen(s + 1);
25
26  for (int i = n; i; i--) {
27    expand(s[i] - 'a');
28    id[last] = i;
29  }
30
31  vis[1] = true;
32  for (int i = 1; i <= cnt; i++) {
33    if (id[i])
34      for (int x = i, pos = n; x && !vis[x]; x =
35        → par[x]) {
36      vis[x] = true;
37      pos -= val[x] - val[par[x]];
38      ch[par[x]][s[pos + 1] - 'a'] = x;
39    }
40
41  dfs(1);
42
43  for (int i = 1; i <= n; i++) {
44    if (i > 1)
45      printf(" ");
46    printf("%d", sa[i]); // 1-based
47  }
48  printf("\n");
49
50  for (int i = 1; i < n; i++) {
51    if (i > 1)
52      printf(" ");
53    printf("%d", height[i]);
54  }
55  printf("\n");
56
57  return 0;
}

```

## 5.4 后缀平衡树

如果不需要查询排名，只需要维护前驱后继关系的题目，可以直接用二分哈希+set去做。

一般的题目需要查询排名，这时候就需要写替罪羊树或者Treap维护tag。插入后缀时如果首字母相同只需比较各自删除首字母后的tag大小即可。

(Treap也具有重量平衡树的性质，每次插入后影响到的子树大小期望是 $O(\log n)$ 的，所以每次做完插入操作之后直接暴力重构子树内tag就行了。)

## 5.5 后缀自动机

```

1 // 在字符集比较小的时候可以直接开go数组，否则需要用map或
2 // →者哈希表替换
3
4 // 全局变量与数组定义
5 int last, val[maxn], par[maxn], go[maxn][26], sam_cnt;
6 int c[maxn], q[maxn]; // 用来桶排序
7
8 // 在主函数开头加上这句初始化
9 last = sam_cnt = 1;
10
11 // 以下是按val进行桶排序的代码
12 for (int i = 1; i <= sam_cnt; i++) {
13   c[val[i] + 1]++;
14 for (int i = 1; i <= n; i++)
15   c[i] += c[i - 1]; // 这里n是串长
16 for (int i = 1; i <= sam_cnt; i++)
17   q[++c[val[i]]] = i;
18
19 // 加入一个字符 均摊O(1)
20 void extend(int c) {
21   int p = last, np = ++sam_cnt;
22   val[np] = val[p] + 1;
23
24   while (p && !go[p][c]) {
25     go[p][c] = np;
26     p = par[p];
27   }
28
29   if (!p)
30     par[np] = 1;
31   else {
32     int q = go[p][c];
33
34     if (val[q] == val[p] + 1)
35       par[np] = q;
36     else {
37       int nq = ++sam_cnt;
38       val[nq] = val[p] + 1;
39       memcpy(go[nq], go[q], sizeof(go[q]));
40
41       par[nq] = par[q];
42       par[np] = par[q] = nq;
43
44       while (p && go[p][c] == q) {
45         go[p][c] = nq;
46         p = par[p];
47       }
48     }
49   }
50
51   last = np;
52 }

```

### 5.5.1 广义后缀自动机

下面的写法复杂度是 $\Sigma$ 串长的，但是胜在简单。  
如果建字典树然后BFS建自动机可以做到 $O(n|\Sigma|)$ ( $n$ 是字典树结点数)，但是后者写起来比较麻烦。

```

1 int extend(int p, int c) {
2   int np = 0;
3
4   if (!go[p][c]) {
5     np = ++sam_cnt;
6     val[np] = val[p] + 1;
7     while (p && !go[p][c]) {
8       go[p][c] = np;
9     }
10   }
11
12   last = np;
13 }

```

```

9      p = par[p];
10     }
11 }
12
13 if (!p)
14     par[np] = 1;
15 else {
16     int q = go[p][c];
17
18     if (val[q] == val[p] + 1) {
19         if (np)
20             par[np] = q;
21         else
22             return q;
23     }
24     else {
25         int nq = ++sam_cnt;
26         val[nq] = val[p] + 1;
27         memcpy(go[nq], go[q], sizeof(go[q]));
28
29         par[nq] = par[q];
30         par[q] = nq;
31         if (np)
32             par[np] = nq;
33
34         while (p && go[p][c] == q){
35             go[p][c] = nq;
36             p = par[p];
37         }
38
39         if (!np)
40             return nq;
41     }
42 }
43
44 return np;
45 }
// 调用的时候直接last = 1然后一路调用last = extend(last,
46 ← c)就行了

```

```

x → ch[1] = null;
x → refresh();

if (x → val) // val记录的是上次访问时间，也就
    ↪是right集合最大值
    update(x → val - val[x → r] + 1, x → val
    ↪ - val[par[x → l]], -1);

x → val = tim;
x → lazy = true;

update(x → val - val[x → r] + 1, x → val -
    ↪ val[par[x → l]], 1);

x → ch[1] = y;

(y = x) → refresh();

x = x → p;
}

return y;
}

// 以下是main函数中的用法
for (int i = 1; i ≤ n; i++) {
    tim++;
    access(null + id[i]);

    if (i ≥ m) // 例题询问长度是固定的，如果不固定的话就
        ↪按照右端点离线即可
        ans[i - m + 1] = query(i - m + 1, i);
}

```

## 5.6 回文树

```

1 // 定理：一个字符串本质不同的回文子串个数是O(n)的
2 // 注意回文树只需要开一倍结点，另外结点编号也是一个可用
   ↳ 的bfs序
3
4 // 全局数组定义
5 int val[maxn], par[maxn], go[maxn][26], last, cnt;
6 char s[maxn];
7
8 // 重要！在主函数最前面一定要加上以下初始化
9 par[0] = cnt = 1;
10 val[1] = -1;
11 // 这个初始化和广义回文树不一样，写普通题可以用，广义回文
   ↳ 树就不要乱搞了
12
13 // extend函数 均摊O(1)
14 // 向后扩展一个字符
15 // 传入对应下标
16 void extend(int n) {
17     int p = last, c = s[n] - 'a';
18     while (s[n - val[p] - 1] != s[n])
19         p = par[p];
20
21     if (!go[p][c]) {
22         int q = ++cnt, now = p;
23         val[q] = val[p] + 2;
24
25         do
26             p = par[p];
27         while (s[n - val[p] - 1] != s[n]);
28
29         par[q] = go[p][c];
30         last = go[now][c] = q;
31     }
32 }
```

```
1 int tim; // tim实际上就是当前的右端点
2
3 node *access(node *x) {
4     node *y = null;
5
6     while (x != null) {
7         splay(x);
```

```

32     else
33         last = go[p][c];
34
35     // a[last]++;
36 }

```

### 5.6.1 广义回文树

(代码是梯子剖分的版本, 压力不大的题目换成直接倍增就好了, 常数只差不到一倍)

```

1 #include <bits/stdc++.h>
2
3 using namespace std;
4
5 constexpr int maxn = 1000005, mod = 1000000007;
6
7 int val[maxn], par[maxn], go[maxn][26], fail[maxn][26],
8     → pam_last[maxn], pam_cnt;
9 int weight[maxn], pow_26[maxn];
10
11 int trie[maxn][26], trie_cnt, d[maxn], mxd[maxn],
12     → son[maxn], top[maxn], len[maxn], sum[maxn];
13 char chr[maxn];
14 int f[25][maxn], log_tbl[maxn];
15 vector<int> v[maxn];
16
17 vector<int> queries[maxn];
18
19 char str[maxn];
20 int n, m, ans[maxn];
21
22 int add(int x, int c) {
23     if (!trie[x][c]) {
24         trie[x][c] = ++trie_cnt;
25         f[0][trie[x][c]] = x;
26         chr[trie[x][c]] = c + 'a';
27     }
28
29     return trie[x][c];
30 }
31
32 int del(int x) {
33     return f[0][x];
34 }
35
36 void dfs1(int x) {
37     mxd[x] = d[x] = d[f[0][x]] + 1;
38
39     for (int i = 0; i < 26; i++)
40         if (trie[x][i])
41             int y = trie[x][i];
42
43             dfs1(y);
44
45             mxd[x] = max(mxd[x], mxd[y]);
46             if (mxd[y] > mxd[son[x]])
47                 son[x] = y;
48 }
49
50 void dfs2(int x) {
51     if (x == son[f[0][x]])
52         top[x] = top[f[0][x]];
53     else
54         top[x] = x;
55
56     for (int i = 0; i < 26; i++)
57         if (trie[x][i])
58             int y = trie[x][i];
59             dfs2(y);

```

```

60     if (top[x] == x) {
61         int u = x;
62         while (top[son[u]] == x)
63             u = son[u];
64
65         len[x] = d[u] - d[x];
66
67         for (int i = 0; i < len[x]; i++) {
68             v[x].push_back(u);
69             u = f[0][u];
70         }
71
72         u = x;
73         for (int i = 0; i < len[x]; i++) { // 梯子剖
74             → 分, 要延长一倍
75             v[x].push_back(u);
76             u = f[0][u];
77         }
78     }
79
80     int get_anc(int x, int k) {
81         if (!k)
82             return x;
83         if (k > d[x])
84             return 0;
85
86         x = f[log_tbl[k]][x];
87         k ≈ 1 << log_tbl[k];
88
89         return v[top[x]][d[top[x]] + len[top[x]] - d[x] +
90             → k];
91     }
92
93     char get_char(int x, int k) { // 查询x前面k个的字符是哪
94         → 个
95         return chr[get_anc(x, k)];
96     }
97
98     int getfail(int x, int p) {
99         if (get_char(x, val[p] + 1) == chr[x])
100             return p;
101         return fail[p][chr[x] - 'a'];
102     }
103
104     int extend(int x) {
105         int p = pam_last[f[0][x]], c = chr[x] - 'a';
106
107         p = getfail(x, p);
108
109         int new_last;
110
111         if (!go[p][c]) {
112             int q = ++pam_cnt, now = p;
113             val[q] = val[p] + 2;
114
115             p = getfail(x, par[p]);
116
117             par[q] = go[p][c];
118             new_last = go[now][c] = q;
119
120             for (int i = 0; i < 26; i++)
121                 fail[q][i] = fail[par[q]][i];
122
123             if (get_char(x, val[par[q]]) ≥ 'a')
124                 fail[q][get_char(x, val[par[q]]) - 'a'] =
125                     → par[q];
126
127             if (val[q] ≤ n)
128                 →
129         }
130     }

```

```

127     weight[q] = (weight[par[q]] + (long long)(n
128         - val[q] + 1) * pow_26[n - val[q]]) %
129         mod;
130     else
131         weight[q] = weight[par[q]];
132     }
133     else
134         new_last = go[p][c];
135
136     pam_last[x] = new_last;
137 }
138
139 void bfs() {
140
141     queue<int> q;
142
143     q.push(1);
144
145     while (!q.empty()) {
146         int x = q.front();
147         q.pop();
148
149         sum[x] = sum[f[0][x]];
150         if (x > 1)
151             sum[x] = (sum[x] + extend(x)) % mod;
152
153         for (int i : queries[x])
154             ans[i] = sum[x];
155
156         for (int i = 0; i < 26; i++)
157             if (trie[x][i])
158                 q.push(trie[x][i]);
159     }
160 }
161
162 int main() {
163
164     pow_26[0] = 1;
165     log_tbl[0] = -1;
166
167     for (int i = 1; i <= 1000000; i++) {
168         pow_26[i] = 26ll * pow_26[i - 1] % mod;
169         log_tbl[i] = log_tbl[i / 2] + 1;
170     }
171
172     int T;
173     scanf("%d", &T);
174
175     while (T--) {
176         scanf("%d%d%s", &n, &m, str);
177
178         trie_cnt = 1;
179         chr[1] = '#';
180
181         int last = 1;
182         for (char *c = str; *c; c++)
183             last = add(last, *c - 'a');
184
185         queries[last].push_back(0);
186
187         for (int i = 1; i <= m; i++) {
188             int op;
189             scanf("%d", &op);
190
191             if (op == 1) {
192                 char c;
193                 scanf(" %c", &c);
194
195                 last = add(last, c - 'a');
196             }
197         }
198     }
199
200     else
201         last = del(last);
202
203     queries[last].push_back(i);
204
205     dfs1(1);
206     dfs2(1);
207
208     for (int j = 1; j <= log_tbl[trie_cnt]; j++)
209         for (int i = 1; i <= trie_cnt; i++)
210             f[j][i] = f[j - 1][f[j - 1][i]];
211
212     par[0] = pam_cnt = 1;
213
214     for (int i = 0; i < 26; i++)
215         fail[0][i] = fail[1][i] = 1;
216
217     val[1] = -1;
218     pam_last[1] = 1;
219
220     bfs();
221
222     for (int i = 0; i <= m; i++)
223         printf("%d\n", ans[i]);
224
225     for (int j = 0; j <= log_tbl[trie_cnt]; j++)
226         memset(f[j], 0, sizeof(f[j]));
227
228     for (int i = 1; i <= trie_cnt; i++) {
229         chr[i] = 0;
230         d[i] = mxd[i] = son[i] = top[i] = len[i] =
231             → pam_last[i] = sum[i] = 0;
232         v[i].clear();
233         queries[i].clear();
234
235         memset(trie[i], 0, sizeof(trie[i]));
236     }
237     trie_cnt = 0;
238
239     for (int i = 0; i <= pam_cnt; i++) {
240         val[i] = par[i] = weight[i];
241
242         memset(go[i], 0, sizeof(go[i]));
243         memset(fail[i], 0, sizeof(fail[i]));
244     }
245     pam_cnt = 0;
246
247     return 0;
248 }
249

```

## 5.7 Manacher马拉车

```

1 // n为串长, 回文半径输出到p数组中
2 // 数组要开串长的两倍
3 void manacher(const char *t, int n) {
4     static char s[maxn * 2];
5
6     for (int i = n; i; i--)
7         s[i * 2] = t[i];
8     for (int i = 0; i <= n; i++)
9         s[i * 2 + 1] = '#';
10
11    s[0] = '$';
12    s[(n + 1) * 2] = '\0';
13    n = n * 2 + 1;
14

```

```
15 int mx = 0, j = 0;
16
17 for (int i = 1; i <= n; i++) {
18     p[i] = (mx > i ? min(p[j * 2 - i], mx - i) :
19             1);
20     while (s[i - p[i]] == s[i + p[i]])
21         p[i]++;
22
23     if (i + p[i] > mx) {
24         mx = i + p[i];
25         j = i;
26     }
27 }
```

## 5.8 字符串原理

KMP和AC自动机的fail指针存储的都是它在串或者字典树上的最长后缀，因此要判断两个前缀是否互为后缀时可以直接用fail指针判断。当然它不能做子串问题，也不能做最长公共后缀。

后缀数组利用的主要是LCP长度可以按照字典序做RMQ的性质，与某个串的LCP长度 $\geq$ 某个值的后缀形成一个区间。另外一个比较好用的性质是本质不同的子串个数 = 所有子串数 - 字典序相邻的串的height。

后缀自动机实际上可以接受的是所有后缀，如果把中间状态也算上的话就是所有子串。它的fail指针代表的也是当前串的后缀，不过注意每个状态可以代表很多状态，只要右端点在right集合中且长度处在 $(val_{par_p}, val_p]$ 中的串都被它代表。

后缀自动机的fail树也就是反串的后缀树。每个结点代表的串和后缀自动机同理，两个串的LCP长度也就是他们在后缀树上的LCA。

# 6 动态规划

## 6.1 决策单调性 $O(n \log n)$

```

1 int a[maxn], q[maxn], p[maxn], g[maxn]; // 存左端点, 右端
2   点就是下一个左端点 - 1
3
4 long long f[maxn], s[maxn];
5
6 int n, m;
7
8 long long calc(int l, int r) {
9     if (r < l)
10        return 0;
11
12     int mid = (l + r) / 2;
13     if ((r - l + 1) % 2 == 0)
14         return (s[r] - s[mid]) - (s[mid] - s[l - 1]);
15     else
16         return (s[r] - s[mid]) - (s[mid - 1] - s[l - 1]);
17
18 int solve(long long tmp) {
19     memset(f, 63, sizeof(f));
20     f[0] = 0;
21
22     int head = 1, tail = 0;
23
24     for (int i = 1; i ≤ n; i++) {
25         f[i] = calc(1, i);
26         g[i] = 1;
27
28         while (head < tail && p[head + 1] ≤ i)
29             head++;
30         if (head ≤ tail) {
31             if (f[q[head]] + calc(q[head] + 1, i) <
32                 f[i]) {
33                 f[i] = f[q[head]] + calc(q[head] + 1,
34                                           i);
35                 g[i] = g[q[head]] + 1;
36             }
37             while (head < tail && p[head + 1] ≤ i + 1)
38                 head++;
39             if (head ≤ tail)
40                 p[head] = i + 1;
41         }
42         f[i] += tmp;
43
44         int r = n;
45
46         while (head ≤ tail) {
47             if (f[q[tail]] + calc(q[tail] + 1, p[tail]) >
48                 f[i] + calc(i + 1, p[tail])) {
49                 r = p[tail] - 1;
50                 tail--;
51             }
52             else if (f[q[tail]] + calc(q[tail] + 1, r) ≤
53                     f[i] + calc(i + 1, r)) {
54                 if (r < n) {
55                     q[++tail] = i;
56                     p[tail] = r + 1;
57                 }
58                 break;
59             }
60             else {
61                 int L = p[tail], R = r;
62                 while (L < R) {
63                     int M = (L + R) / 2;
64                     if (f[q[tail]] + calc(q[tail] + 1, M) ≤
65                         f[i] + calc(i + 1, M))
66                         L = M + 1;
67                     else
68                         R = M;
69                 }
70                 q[++tail] = i;
71                 p[tail] = L;
72             }
73             if (head > tail) {
74                 q[++tail] = i;
75                 p[tail] = i + 1;
76             }
77         }
78
79         return g[n];
80     }
81
82     return 0;
83 }

```

## 7 计算几何

## 7.1 Delaunay三角剖分

只要两个点同在某个三角形上，它们就互为一对最近点。注意返回的三角形似乎不保证顺序，所以要加边的话还是要加双向边。

如果要建V图的话求出每个三角形的外心就行了，每个点控制的区域就是所在三角形的外心连起来。

```
1 #include <bits/stdc++.h>
2
3 using namespace std;
4
5 constexpr int maxn = 500005;
6
7 using ll = long long;
8
9 constexpr int INF = 0x3f3f3f3f;
10 constexpr ll LINF = 0x3f3f3f3f3f3f3f3f;
11 constexpr double eps = 1e-8;
12
13 template <class T>
14 int sgn(T x) {
15     return x > 0 ? 1 : x < 0 ? -1 : 0
16 }
17
18 struct point {
19     ll x, y;
20
21     point() = default;
22
23     point(ll x, ll y) : x(x), y(y) {}
24
25     point operator - (const point &p)
26         return point(x - p.x, y - p.y)
27 }
28
29     ll cross(const point &p) const {
30         return x * p.y - y * p.x;
31     }
32
33     ll cross(const point &a, const point &b) const {
34         return (a - *this).cross(b - *this);
35     }
36
37     ll dot(const point &p) const {
38         return x * p.x + y * p.y;
39     }
40
41     ll dot(const point &a, const point &b) const {
42         return (a - *this).dot(b - *this);
43     }
44
45     ll abs2() const {
46         return this -> dot(*this);
47     }
48
49     bool operator == (const point &p) const {
50         return x == p.x && y == p.y;
51     }
52
53     bool operator < (const point &p) const {
54         if (x != p.x) return x < p.x;
55         return y < p.y;
56     }
57 };
58
59
60 const point inf_point = point(1e18,
```

```

62 struct quad_edge {
63     point origin;
64     quad_edge *rot = nullptr;
65     quad_edge *onext = nullptr;
66     bool used = false;
67
68     quad_edge *rev() const {
69         return rot → rot;
70     }
71
72     quad_edge *lnext() const {
73         return rot → rev() → onext → rot;
74     }
75
76     quad_edge *oprev() const {
77         return rot → onext → rot;
78     }
79
80     point dest() const {
81         return rev() → origin;
82     }
83 };
84
85 quad_edge *make_edge(point from, point to) {
86     quad_edge *e1 = new quad_edge;
87     quad_edge *e2 = new quad_edge;
88     quad_edge *e3 = new quad_edge;
89     quad_edge *e4 = new quad_edge;
90
91     e1 → origin = from;
92     e2 → origin = to;
93     e3 → origin = e4 → origin = inf_point;
94
95     e1 → rot = e3;
96     e2 → rot = e4;
97     e3 → rot = e2;
98     e4 → rot = e1;
99
100    e1 → onext = e1;
101    e2 → onext = e2;
102    e3 → onext = e4;
103    e4 → onext = e3;
104
105    return e1;
106}
107
108 void splice(quad_edge *a, quad_edge *b) { // 拼接
109     swap(a → onext → rot → onext, b → onext → rot
110           → → onext);
111     swap(a → onext, b → onext);
112 }
113
114 void delete_edge(quad_edge *e) {
115     splice(e, e → oprev());
116     splice(e → rev(), e → rev() → oprev());
117
118     delete e → rev() → rot;
119     delete e → rev();
120     delete e → rot;
121     delete e;
122 }
123
124 quad_edge *connect(quad_edge *a, quad_edge *b) {
125     quad_edge *e = make_edge(a → dest(), b → origin);
126
127     splice(e, a → lnext());
128     splice(e → rev(), b);
129
130     return e;
131 }
132
133 bool left_of(point p, quad_edge *e) {
134     return p.x < e → origin.x && p.y < e → origin.y;
135 }

```

```

131     return p.cross(e → origin, e → dest()) > 0;
132 }
133
134 bool right_of(point p, quad_edge *e) {
135     return p.cross(e → origin, e → dest()) < 0;
136 }
137
138 template <class T>
139 T det3(T a1, T a2, T a3, T b1, T b2, T b3, T c1, T c2,
140        → T c3) {
141     return a1 * (b2 * c3 - c2 * b3) - a2 * (b1 * c3 -
142           → c1 * b3) +
143           a3 * (b1 * c2 - c1 * b2);
144 }
145
146 bool in_circle(point a, point b, point c, point d) { // ← 如果有__int128就直接计算行列式, 否则算角度
147 #if defined(_LP64_) || defined(_WIN64_)
148     __int128 det = -det3<__int128>(b.x, b.y, b.abs2(),
149           → c.x, c.y, c.abs2(), d.x, d.y, d.abs2());
150     det += det3<__int128>(a.x, a.y, a.abs2(), c.x, c.y,
151           → c.abs2(), d.x, d.y, d.abs2());
152     det -= det3<__int128>(a.x, a.y, a.abs2(), b.x, b.y,
153           → b.abs2(), d.x, d.y, d.abs2());
154     det += det3<__int128>(a.x, a.y, a.abs2(), b.x, b.y,
155           → b.abs2(), c.x, c.y, c.abs2());
156
157     return det > 0;
158 #else
159     auto ang = [] (point l, point mid, point r) {
160         ll x = mid.dot(l, r);
161         ll y = mid.cross(l, r);
162         long double res = atan2((long double)x, (long
163             → double)y);
164         return res;
165     };
166
167     long double kek = ang(a, b, c) + ang(c, d, a) -
168           → ang(b, c, d) - ang(d, a, b);
169
170     return kek > 1e-10;
171 #endif
172 }
173
174 pair<quad_edge*, quad_edge*> divide_and_conquer(int l,
175           → int r, vector<point> &p) {
176     if (r - l + 1 == 2) {
177         quad_edge *res = make_edge(p[l], p[r]);
178         return make_pair(res, res → rev());
179     }
180
181     if (r - l + 1 == 3) {
182         quad_edge *a = make_edge(p[l], p[l + 1]), *b =
183             → make_edge(p[l + 1], p[r]);
184         splice(a → rev(), b);
185
186         int sg = sgn(p[l].cross(p[l + 1], p[r]));
187
188         if (sg == 0)
189             return make_pair(a, b → rev());
190
191         quad_edge *c = connect(b, a);
192
193         if (sg == 1)
194             return make_pair(a, b → rev());
195         else
196             return make_pair(c → rev(), c);
197
198         int mid = (l + r) / 2;
199
200         quad_edge *ldo, *ldi, *rdo, *rdi;
201         tie(ldo, ldi) = divide_and_conquer(l, mid, p);
202         tie(rdi, rdo) = divide_and_conquer(mid + 1, r, p);
203
204         while (true) {
205             if (left_of(rdi → origin, ldi))
206                 ldi = ldi → lnext();
207             else if (right_of(ldi → origin, rdi))
208                 rdi = rdi → rev() → onext();
209             else
210                 break;
211
212         quad_edge *basel = connect(rdi → rev(), ldi);
213         auto is_valid = [&basel] (quad_edge *e) {
214             return right_of(e → dest(), basel);
215         };
216
217         if (ldi → origin == ldo → origin)
218             ldo = basel → rev();
219         if (rdi → origin == rdo → origin)
220             rdo = basel;
221
222         while (true) {
223             quad_edge *lcand = basel → rev() → onext();
224             if (is_valid(lcand)) {
225                 while (in_circle(basel → dest(), basel →
226                     → origin, lcand → dest(), lcand → onext(
227                         → dest())))
228                     quad_edge *t = lcand → onext;
229                     delete_edge(lcand);
230                     lcand = t;
231             }
232
233             quad_edge *rcand = basel → oprev();
234             if (is_valid(rcand)) {
235                 while (in_circle(basel → dest(), basel →
236                     → origin, rcand → dest(), rcand →
237                         → oprev() → dest()))
238                     quad_edge *t = rcand → oprev();
239                     delete_edge(rcand);
240                     rcand = t;
241
242             if (!is_valid(lcand) && !is_valid(rcand))
243                 break;
244
245             if (!is_valid(lcand) || (is_valid(rcand) &&
246                 in_circle(lcand → dest(), lcand → origin,
247                     → rcand → origin, rcand → dest())))
248                 basel = connect(rcand, basel → rev());
249             else
250                 basel = connect(basel → rev(), lcand →
251                     → rev());
252
253         return make_pair(ldo, rdo);
254     }
255
256     vector<tuple<point, point, point>>
257     delaunay(vector<point> p) { // Delaunay 三角剖分
258         sort(p.begin(), p.end(), [] (const point &a, const
259             → point &b) {

```

```

249     |     return a.x < b.x || (a.x == b.x && a.y < b.y);
250     |     // 实际上已经重载小于了，只是为了清晰
251     | };
252     |
253     auto res = divide_and_conquer(0, (int)p.size() - 1,
254                                   p);
255
256     quad_edge *e = res.first;
257     vector<quad_edge*> edges = {e};
258
259     while (e->onext->dest().cross(e->dest(), e->
260                                         ->origin) < 0)
261         e = e->onext;
262
263     auto add = [&p, &e, &edges] () { // 修改 p, e, edges
264         quad_edge *cur = e;
265         do {
266             cur->used = true;
267             p.push_back(cur->origin);
268             edges.push_back(cur->rev());
269
270             cur = cur->lnext();
271         } while (cur != e);
272     };
273
274     add();
275     p.clear();
276
277     int kek = 0;
278     while (kek < (int)edges.size())
279         if (!(e = edges[kek++])->used)
280             add();
281
282
283     vector<tuple<point, point, point>> ans;
284     for (int i = 0; i < (int)p.size(); i += 3)
285         ans.push_back(make_tuple(p[i], p[i + 1], p[i +
286                                   ->2]));
287
288     return ans; // 返回每个三角形
289 }
290
291 #define sq(x) ((x) * (ll)(x)) // 平方
292
293 ll dist(point p, point q) { // 两点间距离的平方
294     return (p - q).abs2();
295 }
296
297 ll sarea2(point p, point q, point r) { // 三角形面积的两
298     // 倍(叉积)
299     return (q - p).cross(r - q);
300 }
301
302 point v[maxn];
303
304 int main() {
305     int n;
306     cin >> n;
307
308     // read the points, v[1 ~ n]
309
310     bool col_linear = true; // 如果给出的所有点都共线则
311     // 需要特判
312     for (int i = 3; i <= n; i++)
313         if (sarea2(v[1], v[2], v[i]))
314             col_linear = false;
315
316     if (col_linear) {
317         // do something
318
319     return 0;
320 }
321
322 }
```

# 8 杂项

## 8.1 $O(1)$ 快速乘

如果对速度要求很高并且不能用指令集，可以去看fstqwq的模板.

```

1 // long double 快速乘
2 // 在两数直接相乘会爆long long时才有必要使用
3 // 常数比直接long long乘法 + 取模大很多，非必要时不建议使用
4 // → 用
5 long long mul(long long a, long long b, long long p) {
6     a %= p;
7     b %= p;
8     return ((a * b - p * (long long)((long double)a / p
9         → * b + 0.5)) % p + p) % p;
10 }
11
12 // 指令集快速乘
13 // 试机记得测试能不能过编译
14 inline long long mul(const long long a, const long long
15     → b, const long long p) {
16     long long ans;
17     __asm__ __volatile__ ("\\tmulq %%rbx\\n\\tdivq %
18     → %rcx\\n" : "=d"(ans) : "a"(a), "b"(b), "c"(p));
19     return ans;
20 }
21
22 // int乘法取模，大概比直接做快一倍
23 inline int mul_mod(int a, int b, int p) {
24     int ans;
25     __asm__ __volatile__ ("\\tmull %%ebx\\n\\tdivl %
26     → %ecx\\n" : "=d"(ans) : "a"(a), "b"(b), "c"(p));
27     return ans;
28 }
```

## 8.2 Kahan求和算法(减少浮点数累加的误差)

当然一般来说是用不到的，累加被卡精度了才有必要考虑.

```

1 double kahanSum(vector<double> vec) {
2     double sum = 0, c = 0;
3     for (auto x : vec) {
4         double y = x - c;
5         double t = sum + y;
6         c = (t - sum) - y;
7         sum = t;
8     }
9     return sum;
10 }
```

## 8.3 Python Decimal

```

1 import decimal
2
3 decimal.getcontext().prec = 1234 # 有效数字位数
4
5 x = decimal.Decimal(2)
6 x = decimal.Decimal('50.5679') # 不要用float，因为
7     → float本身就不准确
8
9 x = decimal.Decimal('50.5679'). \
10    quantize(decimal.Decimal('0.00')) # 保留两位小数,
11    → 50.57
12 x = decimal.Decimal('50.5679'). \
13    quantize(decimal.Decimal('0.00'),
14    → decimal.ROUND_HALF_UP) # 四舍五入
15
16 # 第二个参数可选如下:
17 # ROUND_HALF_UP 四舍五入
```

```

14 # ROUND_HALF_DOWN 五舍六入
15 # ROUND_HALF_EVEN 银行家舍入法，舍入到最近的偶数
16 # ROUND_UP 向绝对值大的取整
17 # ROUND_DOWN 向绝对值小的取整
18 # ROUND_CEILING 向正无穷取整
19 # ROUND_FLOOR 向负无穷取整
20 # ROUND_05UP (away from zero if last digit after
21     → rounding towards zero would have been 0 or 5;
22     → otherwise towards zero)
23
24 print('%f' % x) # 这样做只有float的精度
25 s = str(x)
26
27 decimal.is_finite(x) # x是否有穷(NaN也算)
28 decimal.is_infinite(x)
29 decimal.is_nan(x)
30 decimal.is_normal(x) # x是否正常
31 decimal.is_signed(x) # 是否为负数
32
33 x.exp(), x.ln(), x.sqrt(), x.log10()
34
35 # 可以转复数，前提是import complex
```

## 8.4 $O(n^2)$ 高精度

```

1 // 注意如果只需要正数运算的话
2 // 可以只抄英文名的运算函数
3 // 按需自取
4 // 乘法O(n ^ 2), 除法O(10 * n ^ 2)
5
6 constexpr int maxn = 1005;
7
8 struct big_decimal {
9     int a[maxn];
10    bool negative;
11
12    big_decimal() {
13        memset(a, 0, sizeof(a));
14        negative = false;
15    }
16
17    big_decimal(long long x) {
18        memset(a, 0, sizeof(a));
19        negative = false;
20
21        if (x < 0) {
22            negative = true;
23            x = -x;
24        }
25
26        while (x) {
27            a[++a[0]] = x % 10;
28            x /= 10;
29        }
30
31    }
32
33    big_decimal(string s) {
34        memset(a, 0, sizeof(a));
35        negative = false;
36
37        if (s == "") {
38            return;
39        }
40        if (s[0] == '-') {
41            negative = true;
42            s = s.substr(1);
43        }
44        a[0] = s.size();
```

```

44     for (int i = 1; i <= a[0]; i++)
45         a[i] = s[a[0] - i] - '0';
46
47     while (a[0] && !a[a[0]])
48         a[0]--;
49 }
50
51 void input() {
52     string s;
53     cin >> s;
54     *this = s;
55 }
56
57 string str() const {
58     if (!a[0])
59         return "0";
60
61     string s;
62     if (negative)
63         s = "-";
64
65     for (int i = a[0]; i; i--)
66         s.push_back('0' + a[i]);
67
68     return s;
69 }
70
71 operator string() const {
72     return str();
73 }
74
75 big_decimal operator -() const {
76     big_decimal o = *this;
77     if (a[0])
78         o.negative = true;
79
80     return o;
81 }
82
83 friend big_decimal abs(const big_decimal &u) {
84     big_decimal o = u;
85     o.negative = false;
86     return o;
87 }
88
89 big_decimal &operator <= (int k) {
90     a[0] += k;
91
92     for (int i = a[0]; i > k; i--)
93         a[i] = a[i - k];
94
95     for (int i = k; i; i--)
96         a[i] = 0;
97
98     return *this;
99 }
100
101 friend big_decimal operator < (const big_decimal
102     &u, int k) {
103     big_decimal o = u;
104     return o <= k;
105 }
106
107 friend big_decimal &operator >= (int k) {
108     if (a[0] < k)
109         return *this = big_decimal(0);
110
111     a[0] -= k;
112     for (int i = 1; i <= a[0]; i++)

```

```

112         a[i] = a[i + k];
113
114     for (int i = a[0] + 1; i <= a[0] + k; i++)
115         a[i] = 0;
116
117     return *this;
118 }
119
120 friend big_decimal operator > (const big_decimal
121     &u, int k) {
122     big_decimal o = u;
123     return o >= k;
124 }
125
126 friend int cmp(const big_decimal &u, const
127     big_decimal &v) {
128     if (u.negative || v.negative) {
129         if (u.negative && v.negative)
130             return -cmp(-u, -v);
131
132         if (u.negative)
133             return -1;
134
135         if (v.negative)
136             return 1;
137     }
138
139     if (u.a[0] != v.a[0])
140         return u.a[0] < v.a[0] ? -1 : 1;
141
142     for (int i = u.a[0]; i; i--)
143         if (u.a[i] != v.a[i])
144             return u.a[i] < v.a[i] ? -1 : 1;
145
146     return 0;
147 }
148
149 friend bool operator < (const big_decimal &u, const
150     big_decimal &v) {
151     return cmp(u, v) == -1;
152 }
153
154 friend bool operator > (const big_decimal &u, const
155     big_decimal &v) {
156     return cmp(u, v) == 1;
157 }
158
159 friend bool operator == (const big_decimal &u,
160     const big_decimal &v) {
161     return cmp(u, v) == 0;
162 }
163
164 friend bool operator ≤ (const big_decimal &u,
165     const big_decimal &v) {
166     return cmp(u, v) ≤ 0;
167 }
168
169 friend bool operator ≥ (const big_decimal &u,
170     const big_decimal &v) {
171     return cmp(u, v) ≥ 0;
172 }
173
174 friend big_decimal decimal_plus(const big_decimal
175     &u, const big_decimal &v) { // 保证u, v均为正数
176     big_decimal o;
177
178     o.a[0] = max(u.a[0], v.a[0]);
179
180     for (int i = 1; i <= a[0]; i++)

```

```

172     for (int i = 1; i <= u.a[0] || i <= v.a[0]; i++)
173     {
174         o.a[i] += u.a[i] + v.a[i];
175
176         if (o.a[i] >= 10) {
177             o.a[i + 1]++;
178             o.a[i] -= 10;
179         }
180
181         if (o.a[o.a[0] + 1])
182             o.a[0]++;
183
184         return o;
185     }
186
187     friend big_decimal decimal_minus(const big_decimal
188     &u, const big_decimal &v) { // 保证u, v均为正数
189     // 的话可以直接调用
190     int k = cmp(u, v);
191
192     if (k == -1)
193         return -decimal_minus(v, u);
194     else if (k == 0)
195         return big_decimal(0);
196
197     big_decimal o;
198
199     o.a[0] = u.a[0];
200
201     for (int i = 1; i <= u.a[0]; i++) {
202         o.a[i] += u.a[i] - v.a[i];
203
204         if (o.a[i] < 0) {
205             o.a[i] += 10;
206             o.a[i + 1]--;
207         }
208
209         while (o.a[0] && !o.a[o.a[0]])
210             o.a[0]--;
211
212     return o;
213 }
214
215     friend big_decimal decimal_multi(const big_decimal
216     &u, const big_decimal &v) {
217     big_decimal o;
218
219     o.a[0] = u.a[0] + v.a[0] - 1;
220
221     for (int i = 1; i <= u.a[0]; i++)
222         for (int j = 1; j <= v.a[0]; j++)
223             o.a[i + j - 1] += u.a[i] * v.a[j];
224
225     for (int i = 1; i <= o.a[0]; i++)
226         if (o.a[i] >= 10) {
227             o.a[i + 1] += o.a[i] / 10;
228             o.a[i] %= 10;
229         }
230
231         if (o.a[o.a[0] + 1])
232             o.a[0]++;
233
234     return o;
235 }
236
237     if (v > u)
238         return make_pair(big_decimal(0), u);
239
240     big_decimal o;
241     o.a[0] = u.a[0] - v.a[0] + 1;
242
243     int m = v.a[0];
244     v <<= u.a[0] - m;
245
246     for (int i = u.a[0]; i >= m; i--) {
247         while (u >= v) {
248             u = u - v;
249             o.a[i - m + 1]++;
250         }
251
252         v >>= 1;
253     }
254
255     while (o.a[0] && !o.a[o.a[0]])
256         o.a[0]--;
257
258     return make_pair(o, u);
259 }
260
261     friend big_decimal operator + (const big_decimal
262     &u, const big_decimal &v) {
263     if (u.negative || v.negative) {
264         if (u.negative && v.negative)
265             return -decimal_plus(-u, -v);
266
267         if (u.negative)
268             return v - (-u);
269
270         if (v.negative)
271             return u - (-v);
272     }
273
274     return decimal_plus(u, v);
275 }
276
277     friend big_decimal operator - (const big_decimal
278     &u, const big_decimal &v) {
279     if (u.negative || v.negative) {
280         if (u.negative && v.negative)
281             return -decimal_minus(-u, -v);
282
283         if (u.negative)
284             return -decimal_plus(-u, v);
285
286         if (v.negative)
287             return decimal_plus(u, -v);
288     }
289
290     return decimal_minus(u, v);
291 }
292
293     friend big_decimal operator * (const big_decimal
294     &u, const big_decimal &v) {
295     if (u.negative || v.negative) {
296         big_decimal o = decimal_multi(abs(u),
297                                     abs(v));
298
299         if (u.negative ^ v.negative)
300             return -o;
301         return o;
302     }
303
304     return decimal_multi(u, v);
305 }

```

```

300 }
301
302 big_decimal operator * (long long x) const {
303     if (x >= 10)
304         return *this * big_decimal(x);
305
306     if (negative)
307         return -(*this * x);
308
309     big_decimal o;
310
311     o.a[0] = a[0];
312
313     for (int i = 1; i <= a[0]; i++) {
314         o.a[i] += a[i] * x;
315
316         if (o.a[i] >= 10) {
317             o.a[i + 1] += o.a[i] / 10;
318             o.a[i] %= 10;
319         }
320     }
321
322     if (o.a[a[0] + 1])
323         o.a[0]++;
324
325     return o;
326 }
327
328 friend pair<big_decimal, big_decimal>
329     decimal_div(const big_decimal &u, const
330     big_decimal &v) {
331     if (u.negative || v.negative) {
332         pair<big_decimal, big_decimal> o =
333             decimal_div(abs(u), abs(v));
334
335         if (u.negative ^ v.negative)
336             return make_pair(-o.first, -o.second);
337         return o;
338     }
339
340     return decimal_divide(u, v);
341
342 friend big_decimal operator / (const big_decimal
343     &u, const big_decimal &v) { // v不能是0
344     if (u.negative || v.negative) {
345         big_decimal o = abs(u) / abs(v);
346
347         if (u.negative ^ v.negative)
348             return -o;
349         return o;
350     }
351
352     return decimal_divide(u, v).first;
353
354 friend big_decimal operator % (const big_decimal
355     &u, const big_decimal &v) {
356     if (u.negative || v.negative) {
357         big_decimal o = abs(u) % abs(v);
358
359         if (u.negative ^ v.negative)
360             return -o;
361         return o;
362     }
363
364     return decimal_divide(u, v).second;
365 }

```

## 8.5 笛卡尔树

```

1 int s[maxn], root, lc[maxn], rc[maxn];
2
3 int top = 0;
4 s[++top] = root = 1;
5 for (int i = 2; i <= n; i++) {
6     s[top + 1] = 0;
7     while (top && a[i] < a[s[top]]) // 小根笛卡尔树
8         top--;
9
10    if (top)
11        rc[s[top]] = i;
12    else
13        root = i;
14
15    lc[i] = s[top + 1];
16    s[++top] = i;
17 }

```

## 8.6 GarsiaWachs算法( $O(n \log n)$ 合并石子)

设序列是 $\{a_i\}$ , 从左往右, 找到一个最小的且满足 $a_{k-1} \leq a_{k+1}$ 的 $k$ , 找到后合并 $a_k$ 和 $a_{k-1}$ , 再从当前位置开始向左找最大的 $j$ 满足 $a_j \geq a_k + a_{k-1}$ (当然是指合并前的), 然后把 $a_k + a_{k-1}$ 插到 $j$ 的后面就行. 一直重复, 直到只剩下一堆石子就可以了.

另外在这个过程中, 可以假设 $a_{-1}$ 和 $a_n$ 是正无穷的, 可省略边界的判别. 把 $a_0$ 设为INF,  $a_{n+1}$ 设为INF-1, 可实现剩余一堆石子时自动结束.

## 8.7 常用NTT素数及原根

$p = r \times 2^k + 1$	$r$	$k$	最小原根
104857601	25	22	3
167772161	5	25	3
469762049	7	26	3
985661441	235	22	3
<b>998244353</b>	119	23	3
<b>1004535809</b>	479	21	3
1005060097*	1917	19	5
2013265921	15	27	31
2281701377	17	27	3
31525197391593473	7	52	3
180143985094819841	5	55	6
194555039024054273	27	56	5
4179340454199820289	29	57	3

\*注: 1005060097有点危险, 在变化长度大于 $524288 = 2^{19}$ 时不可用.

## 8.8 xorshift

```

1 ull k1, k2;
2 const int mod = 100000000;
3 ull xorShift128Plus() {
4     ull k3 = k1, k4 = k2;
5     k1 = k4;
6     k3 ^= (k3 << 23);
7     k2 = k3 ^ k4 ^ (k3 >> 17) ^ (k4 >> 26);
8     return k2 + k4;
9 }
10 void gen(ull _k1, ull _k2) {
11     k1 = _k1, k2 = _k2;
12     int x = xorShift128Plus() % threshold + 1;
13     // do sth
14 }

```

```

15
16
17 uint32_t xor128(void) {
18     static uint32_t x = 123456789;
19     static uint32_t y = 362436069;
20     static uint32_t z = 521288629;
21     static uint32_t w = 88675123;
22     uint32_t t;
23
24     t = x ^ (x << 11);
25     x = y; y = z; z = w;
26     return w = w ^ (w >> 19) ^ (t ^ (t >> 8));
27 }

```

## 8.9 枚举子集

(注意这是 $t \neq 0$ 的写法, 如果可以等于0需要在循环里手动break)

```

1 for (int t = s; t; (--t) &= s) {
2     // do something
3 }

```

## 8.10 STL

### 8.10.1 vector

- `vector(int nSize)`: 创建一个vector, 元素个数为nSize
- `vector(int nSize, const T &value)`: 创建一个vector, 元素个数为nSize, 且值均为value
- `vector(begin, end)`: 复制[begin, end)区间内另一个数组的元素到vector中
- `void assign(int n, const T &x)`: 设置向量中前n个元素的值为x
- `void assign(const_iterator first, const_iterator last)`: 向量中[first, last)中元素设置成当前向量元素
- `void emplace_back(Args&& ... args)`: 自动构造并push\_back一个元素, 例如对一个存储pair的vector可以`v.emplace_back(x, y)`

### 8.10.2 list

- `assign()` 给list赋值
- `back()` 返回最后一个元素
- `begin()` 返回指向第一个元素的迭代器
- `clear()` 删除所有元素
- `empty()` 如果list是空的则返回true
- `end()` 返回末尾的迭代器
- `erase()` 删除一个元素
- `front()` 返回第一个元素
- `insert()` 插入一个元素到list中
- `max_size()` 返回list能容纳的最大元素数量
- `merge()` 合并两个list
- `pop_back()` 删除最后一个元素
- `pop_front()` 删除第一个元素
- `push_back()` 在list的末尾添加一个元素
- `push_front()` 在list的头部添加一个元素
- `rbegin()` 返回指向第一个元素的逆向迭代器
- `remove()` 从list删除元素
- `remove_if()` 按指定条件删除元素
- `rend()` 指向list末尾的逆向迭代器
- `resize()` 改变list的大小
- `reverse()` 把list的元素倒转
- `size()` 返回list中的元素个数
- `sort()` 给list排序
- `splice()` 合并两个list
- `swap()` 交换两个list
- `unique()` 删除list中重复的元

### 8.10.3 unordered\_set/map

- `unordered_map<int, int, hash>`: 自定义哈希函数, 其中hash是一个带重载括号的类.

## 8.11 Public Based DataStructure(PB\_DS)

### 8.11.1 哈希表

```

1 #include<ext/pb_ds/assoc_container.hpp>
2 #include<ext/pb_ds/hash_policy.hpp>
3 using namespace __gnu_pbds;
4
5 cc_hash_table<string, int> mp1; // 拉链法
6 gp_hash_table<string, int> mp2; // 查探法(快一些)

```

### 8.11.2 堆

默认也是大根堆, 和`std::priority_queue`保持一致.

```

1 #include<ext/pb_ds/priority_queue.hpp>
2 using namespace __gnu_pbds;
3
4 __gnu_pbds::priority_queue<int> q;
5 __gnu_pbds::priority_queue<int, greater<int>,
   → pairing_heap_tag> pq;

```

效率参考:

- \* 共有五种操作: push、pop、modify、erase、join
- \* pairing\_heap\_tag: push和join为 $O(1)$ , 其余为均摊 $\Theta(\log n)$
- \* binary\_heap\_tag: 只支持push和pop, 均为均摊 $\Theta(\log n)$
- \* binomial\_heap\_tag: push为均摊 $O(1)$ , 其余为 $\Theta(\log n)$
- \* rc\_binomial\_heap\_tag: push为 $O(1)$ , 其余为 $\Theta(\log n)$
- \* thin\_heap\_tag: push为 $O(1)$ , 不支持join, 其余为 $\Theta(\log n)$ ; 果只有increase\_key, 那么modify为均摊 $O(1)$
- \* “不支持”不是不能用, 而是用起来很慢. csdn.net/TRiddle

常用操作:

- `push()`: 向堆中压入一个元素, 返回迭代器
- `pop()`: 将堆顶元素弹出
- `top()`: 返回堆顶元素
- `size()`: 返回元素个数
- `empty()`: 返回是否非空
- `modify(point_iterator, const key)`: 把迭代器位置的key修改为传入的key
- `erase(point_iterator)`: 把迭代器位置的键值从堆中删除
- `join(__gnu_pbds::priority_queue &other)`: 把other合并到`*this`, 并把other清空

### 8.11.3 平衡树

```

1 #include <ext/pb_ds/tree_policy.hpp>
2 #include <ext/pb_ds/assoc_container.hpp>
3 using namespace __gnu_pbds;
4
5 tree<int, null_type, less<int>, rb_tree_tag,
   → tree_order_statistics_node_update> t;
6
7 // rb_tree_tag 红黑树(还有splay_tree_tag和ov_tree_tag,
   → 后者不知道是什么)

```

注意第五个参数要填`tree_order_statistics_node_update`才能使用排名操作.

- `insert(x)`: 向树中插入一个元素x, 返回`pair<point_iterator, bool>`
- `erase(x)`: 从树中删除一个元素/迭代器x, 返回一个`bool`表明是否删除成功
- `order_of_key(x)`: 返回x的排名, 0-based
- `find_by_order(x)`: 返回排名(0-based)所对应元素的迭代器
- `lower_bound(x) / upper_bound(x)`: 返回第一个 $\geq$ 或者 $>x$ 的元素的迭代器
- `join(x)`: 将x树并入当前树, 前提是两棵树的类型一样, 并且二者值域不能重叠, x树会被删除
- `split(x, b)`: 分裂成两部分, 小于等于x的属于当前树, 其余的属于b树
- `empty()`: 返回是否为空
- `size()`: 返回大小

(注意平衡树不支持多重值, 如果需要多重值, 可以再开一个`unordered_map`来记录值出现的次数, 将`x << 32`后加上出现的次数后插入. 注意此时应该为long long类型.)

## 8.12 rope

```

1 #include <ext/rope>
2 using namespace __gnu_cxx;
3
4 push_back(x); // 在末尾添加x
5 insert(pos, x); // 在pos插入x, 自然支持整个char数组的一
   → 次插入
6 erase(pos, x); // 从pos开始删除x个, 不要只传一个参数, 有
   → 毒
7 copy(pos, len, x); // 从pos开始到pos + len为止的部分, 赋
   → 值给x
8 replace(pos, x); // 从pos开始换成x
9 substr(pos, x); // 提取pos开始x个
10 at(x) / [x]; // 访问第x个元素

```

## 8.13 其他C++相关

### 8.13.1 <cmath>

- `std::log1p(x)`: (注意是数字1)返回 $\ln(1 + x)$ 的值,  $x$ 非常接近0时比直接`exp`精确得多.
- `std::hypot(x, y[, z])`: 返回平方和的平方根, 或者说到原点的欧几里得距离.

### 8.13.2 <algorithm>

- `std::all_of(begin, end, f)`: 检查范围内元素调用函数f后是否全返回真. 类似地还有`std::any_of`和`std::none_of`.
- `std::for_each(begin, end, f)`: 对范围内所有元素调用一次f. 如果传入的是引用, 也可以用f修改. (例如`for_each(a, a + n, [](int &x){cout << ++x << "\n";})`)
- `std::for_each_n(begin, n, f)`: 同上, 只不过范围改成了从begin开始的n个元素.
- `std::copy()`, `std::copy_n()`: 用法谁都会, 但标准里说如果元素是可平凡复制的(比如int), 那么它会避免批量赋值, 并且调用`std::memmove()`之类的快速复制函数. (一句话总结: 它跑得快)
- `std::rotate(begin, mid, end)`: 循环移动, 移动后mid位置的元素会跑到first位置. C++11起会返回begin位置的元素移动后的位置.
- `std::unique(begin, end)`: 去重, 返回去重后的end.
- `std::partition(begin, end, f)`: 把f为true的放在前面, false的放在后面, 返回值是第二部分的开头, 不保持相对顺序. 如果要保留相对顺序可以用`std::stable_partition()`, 比如写整体二分.

- `std::partition_copy(begin, end, begin_t, begin_f, f)`: 不修改原数组, 把true的扔到begin\_t, false的扔到begin\_f. 返回值是两部分结尾的迭代器的pair.
- `std::equal_range(begin, end, x)`: 在已经排好序的数组里找到等于x的范围.
- `std::minmax(a, b)`: 返回`pair(min(a, b), max(a, b))`. 比如`tie(l, r) = minmax(l, r)`.

### 8.13.3 std::tuple

- `std::make_tuple(...)`: 返回构造好的tuple
- `std::get<i>(tup)`: 返回tuple的第i项
- `std::tuple_cat(...)`: 传入几个tuple, 返回按顺序连起来的tuple
- `std::tie(x, y, z, ...)`: 把传入的变量的左值引用绑起来作为tuple返回, 例如可以`std::tie(x, y, z) = std::make_tuple(a, b, c)`.

### 8.13.4 <complex>

- `complex<double> imaginary = 1i, x = 2 + 3i`: 可以这样直接构造复数.
- `real/imag(x)`: 返回实部/虚部.
- `conj(x)`: 返回共轭复数.
- `arg(x)`: 返回辐角.
- `norm(x)`: 返回模的平方. (直接求模用`abs(x)`.)
- `polar(len, theta)`: 用绝对值和辐角构造复数.

## 8.14 一些游戏

### 8.14.1 炉石传说

两个随从 $(a_i, h_i)$ 和 $(a_j, h_j)$ 皇城PK, 最后只有 $a_i \times h_i$ 较大的一方才有可能活下来, 当然也有可能一起死.

## 8.15 OEIS

如果没有特殊说明, 那么以下数列都从第0项开始, 除非没有定义也没有好的办法解释第0项的意义.

### 8.15.1 计数相关

1. 卡特兰数(**A000108**)  
1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, ...  
性质见1.9.7.卡特兰数, 施罗德数, 默慈金数(16页).
2. (大)施罗德数(**A006318**)  
1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098, 1037718, 5293446, 27297738, 142078746, 745387038, ... (0-based)  
性质同样见1.9.7.卡特兰数, 施罗德数, 默慈金数(16页).
3. 小施罗德数(**A001003**)  
1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859, 2646723, 13648869, 71039373, 372693519, ... (0-based)  
性质位置同上.  
小施罗德数除了第0项以外都是施罗德数的一半.
4. 默慈金数(**Motzkin numbers, A001006**)  
1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, 113634, 310572, 853467, 2356779, ... (0-based)  
性质位置同上.
5. 将点按顺序排成一圈后不自交的树的个数(**A001764**)  
1, 1, 3, 12, 55, 273, 1428, 7752, 43263, 246675, 1430715, 8414640, 50067108, 300830572, 1822766520, ... (0-based)  
 $a_n = \frac{\binom{3n}{n}}{2n+1}$   
也就是说, 在圆上按顺序排列的n个点之间连 $n - 1$ 条不相交(除端点外)的弦, 组成一棵树的方案数.

也等于每次只能向右或向上，并且不能高于 $y = 2x$ 这条直线，从 $(0,0)$ 走到 $(n, 2n)$ 的方案数。

扩展：如果改成不能高于 $y = kx$ 这条直线，走到 $(n, kn)$ 的方案数，那么答案就是 $\frac{\binom{(k+1)n}{n}}{kn+1}$ 。

#### 6. $n$ 个点的圆上画不相交的弦的方案数(A054726)

1, 1, 2, 8, 48, 352, 2880, 25216, 231168, 2190848, 21292032, 211044352, 2125246464, 21681954816, ... (0-based)

$a_n = 2^n s_{n-2}$  ( $n > 2$ )， $s_n$ 是上面的小施罗德数。

和上面的区别在于，这里可以不连满 $n - 1$ 条边。另外默慈金数画的弦不能共享端点，但是这里可以。

#### 7. Wedderburn-Etherington numbers(A001190)

0, 1, 1, 1, 2, 3, 6, 11, 23, 46, 98, 207, 451, 983, 2179, 4850, 10905, 24631, 56011, 127912, 293547, ... (0-based)

每个结点都有0或者2个儿子，且总共有 $n$ 个叶子结点的二叉树方案数。(无标号)

同时也是 $n - 1$ 个结点的无标号二叉树个数。

$$A(x) = x + \frac{A(x^2)^2 + A(x^2)}{2} = 1 - \sqrt{1 - 2x - A(x^2)}$$

#### 8. 划分数(A000041)

1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, ... (0-based)

#### 9. 贝尔数(A000110)

1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213597, 27644437, 190899322, 1382958545, ... (0-based)

#### 10. 错位排列数(A0000166)

1, 0, 1, 2, 9, 44, 265, 1854, 14833, 133496, 1334961, 14684570, 176214841, 2290792932, 32071101049, ... (0-based)

#### 11. 交替阶乘(A005165)

0, 1, 1, 5, 19, 101, 619, 4421, 35899, 326981, 3301819, 36614981, 442386619, 5784634181, 81393657019, ...

$$n! - (n-1)! + (n-2)! - \dots 1! = \sum_{i=0}^{n-1} (-1)^i (n-i)!$$

$a_0 = 0, a_n = n! - a_{n-1}$ .

### 8.15.2 线性递推数列

#### 1. Lucas数(A000032)

2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571, 5778, 9349, 15127, ...

#### 2. 斐波那契数(A000045)

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, ...

#### 3. 泰波那契数(Tribonacci, A000071)

0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, 5768, 10609, 19513, 35890, ...

$a_0 = a_1 = 0, a_2 = 1, a_n = a_{n-1} + a_{n-2} + a_{n-3}$ .

#### 4. Pell数(A0000129)

0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782, 195025, 470832, 1136689, ...

$a_0 = 0, a_1 = 1, a_n = 2a_{n-1} + a_{n-2}$ .

#### 5. 帕多万(Padovan)数(A0000931)

1, 0, 0, 1, 0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, 265, 351, 465, 616, 816, 1081, 1432, 1897, 2513, 3329, 4410, 5842, 7739, 10252, 13581, 17991, 23833, 31572, ...

$a_0 = 1, a_1 = a_2 = 0, a_n = a_{n-2} + a_{n-3}$ .

#### 6. Jacobsthal numbers(A001045)

0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, 1365, 2731, 5461, 10923, 21845, 43691, 87381, 174763, ...

$a_0 = 0, a_1 = 1, a_n = a_{n-1} + 2a_{n-2}$

同时也是最接近 $\frac{2^n}{3}$ 的整数。

#### 7. 佩林数(A001608)

3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, 90, 119, 158, 209, 277, 367, 486, 644, 853, ...

$a_0 = 3, a_1 = 0, a_2 = 2, a_n = a_{n-2} + a_{n-3}$

### 8.15.3 数论相关

#### 1. Carmichael数, 伪质数(A002997)

561, 1105, 1729, 2465, 2821, 6601, 8911, 10585, 15841, 29341, 41041, 46657, 52633, 62745, 63973, 75361, 101101, 115921, 126217, 162401, 172081, 188461, 252601, 278545, 294409, 314821, 334153, 340561, 399001, 410041, 449065, 488881, 512461, ...

满足 $\forall$ 与 $n$ 互质的 $a$ ，都有 $a^{n-1} \equiv 1 \pmod{n}$ 的所有合数 $n$ 被称为Carmichael数。

Carmichael数在 $10^8$ 以内只有255个。

#### 2. 反质数(A002182)

1, 2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360, 720, 840, 1260, 1680, 2520, 5040, 7560, 10080, 15120, 20160, 25200, 27720, 45360, 50400, 55440, 83160, 110880, 166320, 221760, 277200, 332640, 498960, 554400, 665280, 720720, 1081080, 1441440, 2162160, ...

比所有更小的数的约数数量都更多的数。

#### 3. 前 $n$ 个质数的乘积(A002110)

1, 2, 6, 30, 210, 2310, 30030, 510510, 9699690, 223092870, 6469693230, 200560490130, 7420738134810, ...

#### 4. 梅森质数(A000668)

3, 7, 31, 127, 8191, 131071, 524287, 2147483647, 2305843009213693951, 618970019642690137449562111, 162259276829213363391578010288127, 170141183460469231731687303715884105727

$p$ 是质数，同时 $2^p - 1$ 也是质数。

### 8.15.4 其他

#### 1. 伯努利数(A027641)

见1.9.2.伯努利数，自然数幂次和(15页)。

#### 2. 四个柱子的汉诺塔(A007664)

0, 1, 3, 5, 9, 13, 17, 25, 33, 41, 49, 65, 81, 97, 113, 129, 161, 193, 225, 257, 289, 321, 385, 449, ...

差分之后可以发现其实就是1次+1, 2次+2, 3次+4, 4次+8...的规律。

#### 3. 乌拉姆数(Ulam numbers, A002858)

1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, 36, 38, 47, 48, 53, 57, 62, 69, 72, 77, 82, 87, 97, 99, 102, 106, 114, 126, 131, 138, 145, 148, 155, 175, 177, 180, 182, 189, 197, 206, 209, 219, 221, 236, 238, 241, 243, 253, 258, 260, 273, 282, 309, 316, 319, 324, 339 ...

$a_1 = 1, a_2 = 2, a_n$ 表示在所有 $> a_{n-1}$ 的数中，最小的，能被表示成(前面的两个不同的元素的和)的数。

## 8.16 编译选项

- `-O2 -g -std=c++17`: 狗都知道
- `-Wall -Wextra -Wshadow -Wconversion`: 更多警告
  - `-Werror`: 强制将所有Warning变成Error
- `-fsanitize=(address/undefined)`: 检查有符号整数溢出(算ub)/数组越界
  - 注意无符号类型溢出不算ub.
- `-fno-ms-extensions`: 关闭一些和msvc保持一致的特性，例如，不标返回值类型的函数会报CE而不是默认为`int`.
  - 但是不写return的话它还是管不了.

## 8.17 注意事项

### 8.17.1 常见下毒手法

- 0/1base是不是搞混了
- 高精度高低位搞反了吗
- 线性筛抄对了吗
- 快速乘抄对了吗
- $i \leq n, j \leq m$
- sort比较函数是不是比了个寂寞
- 该取模的地方都取模了吗
- 边界情况(+1-1之类的)有没有想清楚
- 特判是否有必要, 确定写对了吗

### 8.17.2 场外相关

- 安顿好之后查一下附近的咖啡店, 打印店, 便利店之类的位置, 以备不时之需
- 热身赛记得检查一下编译注意事项中的代码能否过编译, 还有熟悉比赛场地, 清楚洗手间在哪儿, 测试打印机(如果可以)
- 比赛前至少要翻一遍板子, 尤其要看原理与例题
- 比赛前一两天不要摸鱼, 要早睡, 有条件最好洗个澡; 比赛当天不要起太晚, 维持好的状态
- 赛前记得买咖啡, 最好直接安排三人份, 记得要咖啡因比较足的; 如果主办方允许, 就带些巧克力之类的高热量零食
- 入场之后记得检查机器, 尤其要逐个检查键盘按键有没有坏的; 如果可以的话, 调一下gedit设置
- 开赛之前调整好心态, 比赛而已, 不必心急.

### 8.17.3 做题策略与心态调节

- 拿到题后立刻按照商量好的顺序读题, 前半小时最好跳过题意太复杂的题(除非被过穿了)
- 签到题写完不要激动, 稍微检查一下最可能的下毒点再交, 避免无谓的罚时

一两行的那种傻逼题就算了

- 读完题及时输出题意, 一方面避免重复读题, 一方面也可以让队友有一个初步印象, 方便之后决定开题顺序
- 如果不能确定题意就不要贸然输出甚至上机, 尤其是签到题, 因为样例一般都很弱
- 一个题如果卡了很久又有其他题可以写, 那不妨先放掉写更容易的题, 不要在一棵树上吊死

不要被一两道题搞得心态爆炸, 一方面急也没有意义, 一方面你很可能真的离AC就差一步

- 榜是不会骗人的, 一个题如果被不少人过了就说明这个题很可能并没有那么难; 如果不是有十足的把握就不要轻易开没什么人交的题; 另外不要忘记最后个小时会封榜
- 想不出题/找不出毒自然容易犯困, 一定不要放任自己昏昏欲睡, 最好去洗手间冷静一下, 没有条件的站起来踱步
- 思考的时候不要挂机, 一定要在草稿纸上画一画, 最好说出声来最容易断掉思路
- 出完算法一定要check一下样例和一些trivial的情况, 不然容易写了半天发现写了个假算法
- 上机前有时间就提前给需要思考怎么写的地方打草稿, 不要浪费机时
- 查毒时如果最难的地方反复check也没有问题, 就从头到脚仔仔细查一遍, 不要放过任何细节, 即使是并查集和sort这种东西也不能想当然
- 后半场如果时间不充裕就不要冒险开难题, 除非真的无事可做

如果是没写过的东西也不要轻举妄动, 在有其他好写的题的时候就等一会再说

- 大多数时候都要听队长安排, 虽然不一定最正确但可以保持组织性
- 任何时候都不要着急, 着急不能解决问题, 不要当皓国王
- 输了游戏, 还有人生; 赢了游戏, 还有人生.

## 8.18 附录: VScode相关

### 8.18.1 插件

- Chinese (Simplified) (简体中文语言包)
- C/C++
- C++ Intellisense (前提是让用)
- Better C++ Syntax
- Python
- Pylance (前提是让用)
- Rainbow Brackets (前提是让用)

### 8.18.2 设置选项

- Editor: Insert Spaces (取消勾选, 改为tab缩进)
- Editor: Line Warp (开启折行)
- 改配色, “深色+：默认深色”
- 自动保存(F1 → “auto”)
- Terminal → Integrated: Cursor Style (修改终端光标形状)
- Terminal → Integrated: Cursor Blinking (终端光标闪烁)
- 字体改为Cascadia Code/Mono SemiLight (Windows可用)

### 8.18.3 快捷键

- F1 / Ctrl+Shift+P: 万能键, 打开命令面板
- F8: 下一个Error Shift+F8: 上一个Error
- Ctrl+\: 水平分栏, 最多3栏
- Ctrl+1/2/3: 切到对应栏
- Ctrl+[/: 当前行向左/右缩进
- Alt+F12: 查看定义的缩略图(显示小窗, 不跳过去)
- Ctrl+H: 查找替换
- Ctrl+D: 下一个匹配的也被选中(用于配合Ctrl+F)
- Ctrl+U: 回退上一个光标操作(防止光标飞了找不回去)
- Ctrl+/: 切换行注释
- Ctrl+'(键盘左上角的倒引号): 显示终端

## 8.19 附录：骂人的艺术—梁实秋

古今中外没有一个不骂人的人。骂人就是有道德观念的意思，因为在骂人的时候，至少在骂人者自己总觉得那人有该骂的地方。何者该骂，何者不该骂，这个抉择的标准，是极道德的。所以根本不骂人，大可不必。骂人是一种发泄感情的方法，尤其是那一种怨怒的感情。想骂人的时候而不骂，时常在身体上弄出毛病，所以想骂人时，骂骂何妨？

但是，骂人是一种高深的学问，不是人人都可以随便试的。有因为骂人挨嘴巴的，有因为骂人吃官司的，有因为骂人反被人骂的，这都是不会骂人的原故。今以研究所得，公诸同好，或可为骂人时之一助乎？

### 1. 知己知彼

骂人是和动手打架一样的，你如其敢打人一拳，你先要自己忖度下，你吃得起别人的一拳否。这叫做知己知彼。骂人也是一样。譬如你骂他是“屈死”，你先要反省，自己和“屈死”有无分别。你骂别人荒唐，你自己想想曾否吃喝嫖赌。否则别人回敬你一二句，你就受不了。所以别人有着某种短处，而足下也正有同病，那么你在骂他的时候只得割爱。

### 2. 无骂不如己者

要骂人须要挑比你大一点的人物，比你漂亮一点的或者比你坏得万倍而比你得势的人物，总之，你要骂人，那人无论在好的方面或坏的一方面都要能胜过你，你才不吃亏。你骂大人物，就怕他不理你，他一回骂，你就算骂着了。因为身份相同的人才肯对骂。在坏的一方面胜过你的，你骂他就如教训一般，他既便回骂，一般人仍不会理会他的。假如你骂一个无关痛痒的人，你越骂他他越得意，时常可以把一个无名小卒骂出名了，你看冤与不冤？

### 3. 适可而止

骂大人物骂到他回骂的时候，便不可再骂；再骂则一般人对你必无同情，以为你是无理取闹。骂小人物骂到他不能回骂的时候，便不可再骂；再骂下去则一般人对你也必无同情，以为你是欺负弱者。

### 4. 旁敲侧击

他偷东西，你骂他是贼；他抢东西，你骂他是盗，这是笨伯。骂人必须先明虚实掩映之法，须要烘托旁衬，旁敲侧击，于要紧处只一语便得，所谓杀于咽喉处着刀。越要骂他你越要原谅他，即便说些恭维话亦不为过，这样的骂法才能显得你所骂的句句是真实确凿，让旁人看起来也可见得你的度量。

### 5. 态度镇定

骂人最忌浮躁。一语不合，面红筋跳，暴躁如雷，此灌夫骂座，泼妇骂街之术，不足以言骂人。善骂者必须态度镇静，行若无事。普通一般骂人，谁的声音高便算谁占理，谁的来势猛便算谁骂赢，惟真善骂人者，乃能避其锋而击其懈。你等他骂得疲倦的时候，你只消轻轻的回敬他一句，让他再狂吼一阵。在他暴躁不堪的时候，你不妨对他冷笑几声，包管你不费力气，把他气得死去活来，骂得他针针见血。

### 6. 出言典雅

骂人要骂得微妙含蓄，你骂他一句要使他不甚觉得是骂，等到想

过一遍才慢慢觉悟这句话不是好话，让他笑着的面孔由白而红，由红而紫，由紫而灰，这才是骂人的上乘。欲达到此种目的，深刻之用意固不可少，而典雅之言词则尤为重要。言词典雅可使听者不致刺耳。如要骂人骂得典雅，则首先要在骂时万勿提起女人身上的某一部分，万勿不要涉及生理学范围。骂人一骂到生理学范围以内，底下再有什么话都不好说了。譬如你骂某甲，千万别提起他的令堂令妹。因为那样一来，便无是非可言，并且你自己也不免有令堂令妹，他若回敬起来，岂非势均力敌，半斤八两？再者骂人的时候，最好不要加入以种种难堪的名词，称呼起来总要客气，即使他是极卑鄙的小人，你也不妨称他先生，越客气，越骂得有力量。骂得时节最好引用他自己的词句，这不但可以使他难堪，还可以减轻他对你骂的力量。俗语少用，因为俗语一览无遗，不若典雅古文曲折含蓄。

### 7. 以退为进

两人对骂，而自己亦有理屈之处，则处于开骂伊始，特宜注意，最好是毅然将自己理屈之处完全承认下来，即使道歉认错均不妨事。先把自己理屈之处轻轻遮掩过去，然后你再重整旗鼓，着着逼人，方可无后顾之忧。即使自己没有理屈的地方，也绝不可自行夸张，务必要谦逊不遑，把自己的位置降到一个不可再降的位置，然后骂起人来，自有一种公正光明的态度。否则你骂他一两句，他便以你个人的事反唇相讥，一场对骂，会变成两人私下口角，是非曲直，无从判断。所以骂人者自己要低声下气，此所谓以退为进。

### 8. 预设埋伏

你把这句话骂过去，你便要想想看，他将用什么话骂回来。有眼光的骂人者，便处处留神，或是先将他要骂你的话替他说出来，或是预先安设埋伏，令他骂回来的话失去效力。他骂你的话，你替他说出来，这便等于缴了他的械一般。预设埋伏，便是在要攻击你的地方，你先轻轻的安下话根，然后他骂过来就等于枪弹打在沙包上，不能中伤。

### 9. 小题大做

如对方有该骂之处，而题目身小，不值一骂，或你所知不多，不足一骂，那时节你便可用小题大做的方法，来扩大题目。先用诚恳而怀疑的态度引申对方的意思，由不紧要之点引到大题目上去，处处用严谨的逻辑逼他说出不逻辑的话来，或是逼他说出合于逻辑但不合乎理的话来，然后你再大举骂他，骂到体无完肤为止，而原来惹动你的小题目，轻轻一提便了。

### 10. 远交近攻

一个时候，只能骂一个人，或一种人，或一派。决不宜多树敌。所以骂人的时候，万勿连累旁人，即使必须牵涉多人，你也要表示好意，否则回骂之声纷至沓来，使你无从应付。

骂人的艺术，一时所能想起来的有上面十条，信手拈来，并无条理。我做此文的用意，是助人骂人。同时也是想把骂人的技术揭破一点，供爱骂人者参考。挨骂的人看看，骂人的心理原来是这样的，也算是揭破一张黑幕给你瞧瞧！

## 8.20 附录：Cheat Sheet

见最后几页。

## Theoretical Computer Science Cheat Sheet

Definitions		Series
$f(n) = O(g(n))$	iff $\exists$ positive $c, n_0$ such that $0 \leq f(n) \leq cg(n) \forall n \geq n_0$ .	$\sum_{i=1}^n i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}.$
$f(n) = \Omega(g(n))$	iff $\exists$ positive $c, n_0$ such that $f(n) \geq cg(n) \geq 0 \forall n \geq n_0$ .	In general: $\sum_{i=1}^n i^m = \frac{1}{m+1} \left[ (n+1)^{m+1} - 1 - \sum_{i=1}^n ((i+1)^{m+1} - i^{m+1} - (m+1)i^m) \right]$ $\sum_{i=1}^{n-1} i^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k n^{m+1-k}.$
$f(n) = \Theta(g(n))$	iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ .	Geometric series: $\sum_{i=0}^n c^i = \frac{c^{n+1} - 1}{c - 1}, \quad c \neq 1, \quad \sum_{i=0}^{\infty} c^i = \frac{1}{1 - c}, \quad \sum_{i=1}^{\infty} c^i = \frac{c}{1 - c}, \quad  c  < 1,$ $\sum_{i=0}^n i c^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \neq 1, \quad \sum_{i=0}^{\infty} i c^i = \frac{c}{(1-c)^2}, \quad  c  < 1.$
$\lim_{n \rightarrow \infty} a_n = a$	iff $\forall \epsilon > 0, \exists n_0$ such that $ a_n - a  < \epsilon, \forall n \geq n_0$ .	Harmonic series: $H_n = \sum_{i=1}^n \frac{1}{i}, \quad \sum_{i=1}^n i H_i = \frac{n(n+1)}{2} H_n - \frac{n(n-1)}{4}.$ $\sum_{i=1}^n H_i = (n+1)H_n - n, \quad \sum_{i=1}^n \binom{i}{m} H_i = \binom{n+1}{m+1} \left( H_{n+1} - \frac{1}{m+1} \right).$
$\sup S$	least $b \in \mathbb{R}$ such that $b \geq s, \forall s \in S$ .	
$\inf S$	greatest $b \in \mathbb{R}$ such that $b \leq s, \forall s \in S$ .	
$\liminf_{n \rightarrow \infty} a_n$	$\lim_{n \rightarrow \infty} \inf \{a_i \mid i \geq n, i \in \mathbb{N}\}$ .	
$\limsup_{n \rightarrow \infty} a_n$	$\lim_{n \rightarrow \infty} \sup \{a_i \mid i \geq n, i \in \mathbb{N}\}$ .	
$\binom{n}{k}$	Combinations: Size $k$ subsets of a size $n$ set.	
$\begin{bmatrix} n \\ k \end{bmatrix}$	Stirling numbers (1st kind): Arrangements of an $n$ element set into $k$ cycles.	1. $\binom{n}{k} = \frac{n!}{(n-k)!k!}, \quad 2. \sum_{k=0}^n \binom{n}{k} = 2^n, \quad 3. \binom{n}{k} = \binom{n}{n-k},$
$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$	Stirling numbers (2nd kind): Partitions of an $n$ element set into $k$ non-empty sets.	4. $\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}, \quad 5. \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$
$\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle$	1st order Eulerian numbers: Permutations $\pi_1 \pi_2 \dots \pi_n$ on $\{1, 2, \dots, n\}$ with $k$ ascents.	6. $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}, \quad 7. \sum_{k=0}^n \binom{r+k}{k} = \binom{r+n+1}{n},$
$\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle$	2nd order Eulerian numbers.	8. $\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}, \quad 9. \sum_{k=0}^n \binom{r}{k} \binom{s}{n-k} = \binom{r+s}{n},$
$C_n$	Catalan Numbers: Binary trees with $n+1$ vertices.	10. $\binom{n}{k} = (-1)^k \binom{k-n-1}{k}, \quad 11. \left\{ \begin{smallmatrix} n \\ 1 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\} = 1,$
14. $\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!$	15. $\begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)!H_{n-1},$	12. $\left\{ \begin{smallmatrix} n \\ 2 \end{smallmatrix} \right\} = 2^{n-1} - 1, \quad 13. \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = k \left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\},$
18. $\begin{bmatrix} n \\ k \end{bmatrix} = (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix},$	19. $\left\{ \begin{smallmatrix} n \\ n-1 \end{smallmatrix} \right\} = \left[ \begin{smallmatrix} n \\ n-1 \end{smallmatrix} \right] = \binom{n}{2},$	16. $\begin{bmatrix} n \\ n \end{bmatrix} = 1, \quad 17. \begin{bmatrix} n \\ k \end{bmatrix} \geq \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\},$
22. $\left\langle \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\rangle = \left\langle \begin{smallmatrix} n \\ n-1 \end{smallmatrix} \right\rangle = 1,$	23. $\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle = \left\langle \begin{smallmatrix} n \\ n-1-k \end{smallmatrix} \right\rangle,$	20. $\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} = n!, \quad 21. C_n = \frac{1}{n+1} \binom{2n}{n},$
25. $\left\langle \begin{smallmatrix} 0 \\ k \end{smallmatrix} \right\rangle = \begin{cases} 1 & \text{if } k=0, \\ 0 & \text{otherwise} \end{cases}$	26. $\left\langle \begin{smallmatrix} n \\ 1 \end{smallmatrix} \right\rangle = 2^n - n - 1,$	24. $\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle = (k+1) \left\langle \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\rangle + (n-k) \left\langle \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\rangle,$
28. $x^n = \sum_{k=0}^n \left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle \binom{x+k}{n},$	29. $\left\langle \begin{smallmatrix} n \\ m \end{smallmatrix} \right\rangle = \sum_{k=0}^m \binom{n+1}{k} (m+1-k)^n (-1)^k,$	27. $\left\langle \begin{smallmatrix} n \\ 2 \end{smallmatrix} \right\rangle = 3^n - (n+1)2^n + \binom{n+1}{2},$
31. $\left\langle \begin{smallmatrix} n \\ m \end{smallmatrix} \right\rangle = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} \binom{n-k}{m} (-1)^{n-k-m} k!,$	32. $\left\langle \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\rangle = 1,$	30. $m! \left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\} = \sum_{k=0}^n \left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle \binom{k}{n-m},$
34. $\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle = (k+1) \left\langle \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\rangle + (2n-1-k) \left\langle \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\rangle,$		33. $\left\langle \begin{smallmatrix} n \\ n \end{smallmatrix} \right\rangle = 0 \quad \text{for } n \neq 0,$
36. $\left\{ \begin{smallmatrix} x \\ x-n \end{smallmatrix} \right\} = \sum_{k=0}^n \left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle \binom{x+n-1-k}{2n},$	37. $\left\{ \begin{smallmatrix} n+1 \\ m+1 \end{smallmatrix} \right\} = \sum_k \binom{n}{k} \left\{ \begin{smallmatrix} k \\ m \end{smallmatrix} \right\} = \sum_{k=0}^n \left\{ \begin{smallmatrix} k \\ m \end{smallmatrix} \right\} (m+1)^{n-k},$	35. $\sum_{k=0}^n \left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle = \frac{(2n)^n}{2^n},$

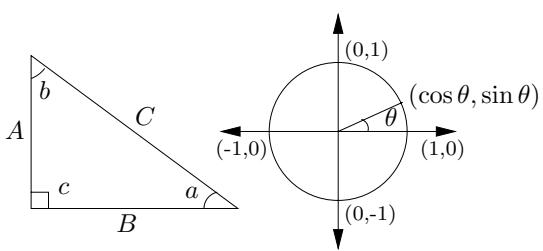
Theoretical Computer Science Cheat Sheet		
Identities Cont.		Trees
38. $\begin{bmatrix} n+1 \\ m+1 \end{bmatrix} = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} \binom{k}{m} = \sum_{k=0}^n \begin{bmatrix} k \\ m \end{bmatrix} n^{n-k} = n! \sum_{k=0}^n \frac{1}{k!} \begin{bmatrix} k \\ m \end{bmatrix}$ ,	39. $\begin{bmatrix} x \\ x-n \end{bmatrix} = \sum_{k=0}^n \begin{Bmatrix} n \\ k \end{Bmatrix} \binom{x+k}{2n}$ ,	Every tree with $n$ vertices has $n-1$ edges.
40. $\begin{Bmatrix} n \\ m \end{Bmatrix} = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} \begin{Bmatrix} k+1 \\ m+1 \end{Bmatrix} (-1)^{n-k}$ ,	41. $\begin{bmatrix} n \\ m \end{bmatrix} = \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \binom{k}{m} (-1)^{m-k}$ ,	Kraft inequality: If the depths of the leaves of a binary tree are $d_1, \dots, d_n$ : $\sum_{i=1}^n 2^{-d_i} \leq 1,$
42. $\begin{Bmatrix} m+n+1 \\ m \end{Bmatrix} = \sum_{k=0}^m k \begin{Bmatrix} n+k \\ k \end{Bmatrix}$ ,	43. $\begin{bmatrix} m+n+1 \\ m \end{bmatrix} = \sum_{k=0}^m k(n+k) \begin{bmatrix} n+k \\ k \end{bmatrix}$ ,	and equality holds only if every internal node has 2 sons.
44. $\begin{bmatrix} n \\ m \end{bmatrix} = \sum_k \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} \binom{k}{m} (-1)^{m-k}$ ,	45. $(n-m)! \begin{bmatrix} n \\ m \end{bmatrix} = \sum_k \begin{bmatrix} n+1 \\ k+1 \end{Bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} (-1)^{m-k}$ , for $n \geq m$ ,	
46. $\begin{Bmatrix} n \\ n-m \end{Bmatrix} = \sum_k \begin{bmatrix} m-n \\ m+k \end{bmatrix} \binom{m+n}{n+k} \begin{bmatrix} m+k \\ k \end{bmatrix}$ ,	47. $\begin{bmatrix} n \\ n-m \end{bmatrix} = \sum_k \begin{bmatrix} m-n \\ m+k \end{bmatrix} \binom{m+n}{n+k} \begin{Bmatrix} m+k \\ k \end{Bmatrix}$ ,	
48. $\begin{Bmatrix} n \\ \ell+m \end{Bmatrix} \binom{\ell+m}{\ell} = \sum_k \begin{Bmatrix} k \\ \ell \end{Bmatrix} \begin{Bmatrix} n-k \\ m \end{Bmatrix} \binom{n}{k}$ ,	49. $\begin{bmatrix} n \\ \ell+m \end{bmatrix} \binom{\ell+m}{\ell} = \sum_k \begin{bmatrix} k \\ \ell \end{bmatrix} \begin{bmatrix} n-k \\ m \end{bmatrix} \binom{n}{k}$ .	
Recurrences		
<p>Master method:  <math>T(n) = aT(n/b) + f(n)</math>, <math>a \geq 1, b &gt; 1</math></p> <p>If <math>\exists \epsilon &gt; 0</math> such that <math>f(n) = O(n^{\log_b a - \epsilon})</math> then  <math>T(n) = \Theta(n^{\log_b a})</math>.</p> <p>If <math>f(n) = \Theta(n^{\log_b a})</math> then  <math>T(n) = \Theta(n^{\log_b a} \log_2 n)</math>.</p> <p>If <math>\exists \epsilon &gt; 0</math> such that <math>f(n) = \Omega(n^{\log_b a + \epsilon})</math>, and <math>\exists c &lt; 1</math> such that <math>af(n/b) \leq cf(n)</math> for large <math>n</math>, then  <math>T(n) = \Theta(f(n))</math>.</p> <p>Substitution (example): Consider the following recurrence  <math>T_{i+1} = 2^{2^i} \cdot T_i^2</math>, <math>T_1 = 2</math>.</p> <p>Note that <math>T_i</math> is always a power of two. Let <math>t_i = \log_2 T_i</math>. Then we have  <math>t_{i+1} = 2^i + 2t_i</math>, <math>t_1 = 1</math>.</p> <p>Let <math>u_i = t_i/2^i</math>. Dividing both sides of the previous equation by <math>2^{i+1}</math> we get  <math>\frac{t_{i+1}}{2^{i+1}} = \frac{2^i}{2^{i+1}} + \frac{t_i}{2^i}</math>.</p> <p>Substituting we find  <math>u_{i+1} = \frac{1}{2} + u_i</math>, <math>u_1 = \frac{1}{2}</math>,</p> <p>which is simply <math>u_i = i/2</math>. So we find that <math>T_i</math> has the closed form <math>T_i = 2^{i2^{i-1}}</math>.</p> <p>Summing factors (example): Consider the following recurrence  <math>T(n) = 3T(n/2) + n</math>, <math>T(1) = 1</math>.</p> <p>Rewrite so that all terms involving <math>T</math> are on the left side  <math>T(n) - 3T(n/2) = n</math>.</p> <p>Now expand the recurrence, and choose a factor which makes the left side “telescope”</p>	$1(T(n) - 3T(n/2) = n)$ $3(T(n/2) - 3T(n/4) = n/2)$ $\vdots \quad \vdots \quad \vdots$ $3^{\log_2 n-1}(T(2) - 3T(1) = 2)$ <p>Let <math>m = \log_2 n</math>. Summing the left side we get <math>T(n) - 3^m T(1) = T(n) - 3^m = T(n) - n^k</math> where <math>k = \log_2 3 \approx 1.58496</math>. Summing the right side we get</p> $\sum_{i=0}^{m-1} \frac{n}{2^i} 3^i = n \sum_{i=0}^{m-1} \left(\frac{3}{2}\right)^i.$ <p>Let <math>c = \frac{3}{2}</math>. Then we have</p> $\begin{aligned} n \sum_{i=0}^{m-1} c^i &= n \left( \frac{c^m - 1}{c - 1} \right) \\ &= 2n(c^{\log_2 n} - 1) \\ &= 2n(c^{(k-1)\log_c n} - 1) \\ &= 2n^k - 2n, \end{aligned}$ <p>and so <math>T(n) = 3n^k - 2n</math>. Full history recurrences can often be changed to limited history ones (example): Consider</p> $T_i = 1 + \sum_{j=0}^{i-1} T_j, \quad T_0 = 1.$ <p>Note that</p> $T_{i+1} = 1 + \sum_{j=0}^i T_j.$ <p>Subtracting we find</p> $\begin{aligned} T_{i+1} - T_i &= 1 + \sum_{j=0}^i T_j - 1 - \sum_{j=0}^{i-1} T_j \\ &= T_i. \end{aligned}$ <p>And so <math>T_{i+1} = 2T_i = 2^{i+1}</math>.</p>	<p>Generating functions:</p> <ol style="list-style-type: none"> <li>Multiply both sides of the equation by <math>x^i</math>.</li> <li>Sum both sides over all <math>i</math> for which the equation is valid.</li> <li>Choose a generating function <math>G(x)</math>. Usually <math>G(x) = \sum_{i=0}^{\infty} x^i g_i</math>.</li> <li>Rewrite the equation in terms of the generating function <math>G(x)</math>.</li> <li>Solve for <math>G(x)</math>.</li> <li>The coefficient of <math>x^i</math> in <math>G(x)</math> is <math>g_i</math>.</li> </ol> <p>Example:  <math>g_{i+1} = 2g_i + 1</math>, <math>g_0 = 0</math>.</p> <p>Multiply and sum:  <math display="block">\sum_{i \geq 0} g_{i+1} x^i = \sum_{i \geq 0} 2g_i x^i + \sum_{i \geq 0} x^i.</math></p> <p>We choose <math>G(x) = \sum_{i \geq 0} x^i g_i</math>. Rewrite in terms of <math>G(x)</math>:</p> $\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i \geq 0} x^i.$ <p>Simplify:  <math display="block">\frac{G(x)}{x} = 2G(x) + \frac{1}{1-x}.</math></p> <p>Solve for <math>G(x)</math>:</p> $G(x) = \frac{x}{(1-x)(1-2x)}.$ <p>Expand this using partial fractions:</p> $\begin{aligned} G(x) &= x \left( \frac{2}{1-2x} - \frac{1}{1-x} \right) \\ &= x \left( 2 \sum_{i \geq 0} 2^i x^i - \sum_{i \geq 0} x^i \right) \\ &= \sum_{i \geq 0} (2^{i+1} - 1)x^{i+1}. \end{aligned}$ <p>So <math>g_i = 2^i - 1</math>.</p>

## Theoretical Computer Science Cheat Sheet

$\pi \approx 3.14159, e \approx 2.71828, \gamma \approx 0.57721, \phi = \frac{1+\sqrt{5}}{2} \approx 1.61803, \hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -.61803$				
$i$	$2^i$	$p_i$	General	Probability
1	2	2	Bernoulli Numbers ( $B_i = 0$ , odd $i \neq 1$ ): $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}.$	Continuous distributions: If $\Pr[a < X < b] = \int_a^b p(x) dx,$ then $p$ is the probability density function of $X$ . If $\Pr[X < a] = P(a),$ then $P$ is the distribution function of $X$ . If $P$ and $p$ both exist then $P(a) = \int_{-\infty}^a p(x) dx.$
2	4	3		
3	8	5		
4	16	7	Change of base, quadratic formula: $\log_b x = \frac{\log_a x}{\log_a b}, \quad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$	
5	32	11		
6	64	13		
7	128	17	Euler's number $e$ : $e = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \dots$ $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$	Expectation: If $X$ is discrete $E[g(X)] = \sum_x g(x) \Pr[X = x].$
8	256	19		If $X$ continuous then $E[g(X)] = \int_{-\infty}^{\infty} g(x)p(x) dx = \int_{-\infty}^{\infty} g(x) dP(x).$
9	512	23		Variance, standard deviation: $\text{VAR}[X] = E[X^2] - E[X]^2,$ $\sigma = \sqrt{\text{VAR}[X]}.$
10	1,024	29		For events $A$ and $B$ : $\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]$
11	2,048	31		$\Pr[A \wedge B] = \Pr[A] \cdot \Pr[B],$ iff $A$ and $B$ are independent.
12	4,096	37		$\Pr[A B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$
13	8,192	41	Harmonic numbers: $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \frac{49}{20}, \frac{363}{140}, \frac{761}{280}, \frac{7129}{2520}, \dots$	For random variables $X$ and $Y$ : $E[X \cdot Y] = E[X] \cdot E[Y],$ if $X$ and $Y$ are independent.
14	16,384	43		$E[X + Y] = E[X] + E[Y],$ $E[cX] = cE[X].$
15	32,768	47		Bayes' theorem:
16	65,536	53		$\Pr[A_i B] = \frac{\Pr[B A_i] \Pr[A_i]}{\sum_{j=1}^n \Pr[A_j] \Pr[B A_j]}.$
17	131,072	59		Inclusion-exclusion:
18	262,144	61		$\Pr\left[\bigvee_{i=1}^n X_i\right] = \sum_{i=1}^n \Pr[X_i] + \sum_{k=2}^n (-1)^{k+1} \sum_{i_1 < \dots < i_k} \Pr\left[\bigwedge_{j=1}^k X_{i_j}\right].$
19	524,288	67	Factorial, Stirling's approximation: $1, 2, 6, 24, 120, 720, 5040, 40320, 362880, \dots$	Moment inequalities:
20	1,048,576	71		$\Pr[ X  \geq \lambda E[X]] \leq \frac{1}{\lambda},$ $\Pr[ X - E[X]  \geq \lambda \cdot \sigma] \leq \frac{1}{\lambda^2}.$
21	2,097,152	73		Geometric distribution:
22	4,194,304	79		$\Pr[X = k] = pq^{k-1}, \quad q = 1 - p,$ $E[X] = \sum_{k=1}^{\infty} k \Pr[X = k] = \frac{p}{1-q}.$
23	8,388,608	83	Ackermann's function and inverse: $a(i, j) = \begin{cases} 2^j & i = 1 \\ a(i-1, 2) & j = 1 \\ a(i-1, a(i, j-1)) & i, j \geq 2 \end{cases}$ $\alpha(i) = \min\{j \mid a(j, j) \geq i\}.$	
24	16,777,216	89		
25	33,554,432	97		
26	67,108,864	101		
27	134,217,728	103		
28	268,435,456	107	Binomial distribution: $\Pr[X = k] = \binom{n}{k} p^k q^{n-k}, \quad q = 1 - p,$ $E[X] = \sum_{k=1}^n k \binom{n}{k} p^k q^{n-k} = np.$	
29	536,870,912	109		
30	1,073,741,824	113	Poisson distribution: $\Pr[X = k] = \frac{e^{-\lambda} \lambda^k}{k!}, \quad E[X] = \lambda.$	
31	2,147,483,648	127	Normal (Gaussian) distribution: $p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad E[X] = \mu.$	
32	4,294,967,296	131	The "coupon collector": We are given a random coupon each day, and there are $n$ different types of coupons. The distribution of coupons is uniform. The expected number of days to pass before we collect all $n$ types is $nH_n.$	
Pascal's Triangle				
	1			
	1 1			
	1 2 1			
	1 3 3 1			
	1 4 6 4 1			
	1 5 10 10 5 1			
	1 6 15 20 15 6 1			
	1 7 21 35 35 21 7 1			
	1 8 28 56 70 56 28 8 1			
	1 9 36 84 126 126 84 36 9 1			
	1 10 45 120 210 252 210 120 45 10 1			

## Theoretical Computer Science Cheat Sheet

### Trigonometry



Pythagorean theorem:

$$C^2 = A^2 + B^2.$$

Definitions:

$$\sin a = A/C, \quad \cos a = B/C,$$

$$\csc a = C/A, \quad \sec a = C/B,$$

$$\tan a = \frac{\sin a}{\cos a} = \frac{A}{B}, \quad \cot a = \frac{\cos a}{\sin a} = \frac{B}{A}.$$

Area, radius of inscribed circle:

$$\frac{1}{2}AB, \quad \frac{AB}{A+B+C}.$$

Identities:

$$\sin x = \frac{1}{\csc x},$$

$$\cos x = \frac{1}{\sec x},$$

$$\tan x = \frac{1}{\cot x},$$

$$\sin^2 x + \cos^2 x = 1,$$

$$1 + \tan^2 x = \sec^2 x,$$

$$1 + \cot^2 x = \csc^2 x,$$

$$\sin x = \cos(\frac{\pi}{2} - x),$$

$$\sin x = \sin(\pi - x),$$

$$\cos x = -\cos(\pi - x),$$

$$\tan x = \cot(\frac{\pi}{2} - x),$$

$$\cot x = -\cot(\pi - x),$$

$$\csc x = \cot \frac{x}{2} - \cot x,$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$

$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y},$$

$$\sin 2x = 2 \sin x \cos x, \quad \sin 2x = \frac{2 \tan x}{1 + \tan^2 x},$$

$$\cos 2x = \cos^2 x - \sin^2 x, \quad \cos 2x = 2 \cos^2 x - 1,$$

$$\cos 2x = 1 - 2 \sin^2 x, \quad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}, \quad \cot 2x = \frac{\cot^2 x - 1}{2 \cot x},$$

$$\sin(x+y) \sin(x-y) = \sin^2 x - \sin^2 y,$$

$$\cos(x+y) \cos(x-y) = \cos^2 x - \sin^2 y.$$

Euler's equation:

$$e^{ix} = \cos x + i \sin x, \quad e^{i\pi} = -1.$$

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### Matrices

Multiplication:

$$C = A \cdot B, \quad c_{i,j} = \sum_{k=1}^n a_{i,k} b_{k,j}.$$

Determinants:  $\det A \neq 0$  iff  $A$  is non-singular.

$$\det A \cdot B = \det A \cdot \det B,$$

$$\det A = \sum_{\pi} \prod_{i=1}^n \text{sign}(\pi) a_{i,\pi(i)}.$$

$2 \times 2$  and  $3 \times 3$  determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = g \begin{vmatrix} b & c \\ e & f \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix} + i \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$= aei + bfg + cdh - ceg - fha - ibd.$$

Permanents:

$$\text{perm } A = \sum_{\pi} \prod_{i=1}^n a_{i,\pi(i)}.$$

### Hyperbolic Functions

Definitions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2},$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \operatorname{csch} x = \frac{1}{\sinh x},$$

$$\operatorname{sech} x = \frac{1}{\cosh x}, \quad \coth x = \frac{1}{\tanh x}.$$

Identities:

$$\cosh^2 x - \sinh^2 x = 1, \quad \tanh^2 x + \operatorname{sech}^2 x = 1,$$

$$\coth^2 x - \operatorname{csch}^2 x = 1, \quad \sinh(-x) = -\sinh x,$$

$$\cosh(-x) = \cosh x, \quad \tanh(-x) = -\tanh x,$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$$

$$\sinh 2x = 2 \sinh x \cosh x,$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x,$$

$$\cosh x + \sinh x = e^x, \quad \cosh x - \sinh x = e^{-x},$$

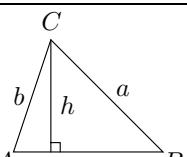
$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx, \quad n \in \mathbb{Z},$$

$$2 \sinh^2 \frac{x}{2} = \cosh x - 1, \quad 2 \cosh^2 \frac{x}{2} = \cosh x + 1.$$

$\theta$	$\sin \theta$	$\cos \theta$	$\tan \theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	$\infty$

... in mathematics you don't understand things, you just get used to them.  
– J. von Neumann

### More Trig.



Law of cosines:

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

Area:

$$A = \frac{1}{2}hc,$$

$$= \frac{1}{2}ab \sin C,$$

$$= \frac{c^2 \sin A \sin B}{2 \sin C}.$$

Heron's formula:

$$A = \sqrt{s \cdot s_a \cdot s_b \cdot s_c},$$

$$s = \frac{1}{2}(a+b+c),$$

$$s_a = s - a,$$

$$s_b = s - b,$$

$$s_c = s - c.$$

More identities:

$$\sin \frac{x}{2} = \sqrt{\frac{1 - \cos x}{2}},$$

$$\cos \frac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$$

$$\tan \frac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}},$$

$$= \frac{1 - \cos x}{\sin x},$$

$$= \frac{\sin x}{1 + \cos x},$$

$$= \frac{\sin x}{\sin x},$$

$$\cot \frac{x}{2} = \sqrt{\frac{1 + \cos x}{1 - \cos x}},$$

$$= \frac{1 + \cos x}{\sin x},$$

$$= \frac{\sin x}{1 - \cos x},$$

$$= \frac{\sin x}{\sin x},$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i},$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$

$$\tan x = -i \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}},$$

$$= -i \frac{e^{2ix} - 1}{e^{2ix} + 1},$$

$$= -i \frac{e^{2ix} - 1}{e^{2ix} + 1},$$

$$\sinh ix = \frac{e^{ix} - e^{-ix}}{i},$$

$$\cos x = \cosh ix,$$

$$\tanh ix = \frac{\sinh ix}{\cosh ix}.$$

## Theoretical Computer Science Cheat Sheet

Theoretical Computer Science Cheat Sheet								
Number Theory	Graph Theory							
<p>The Chinese remainder theorem: There exists a number <math>C</math> such that:</p> $C \equiv r_1 \pmod{m_1}$ $\vdots \quad \vdots \quad \vdots$ $C \equiv r_n \pmod{m_n}$ <p>if <math>m_i</math> and <math>m_j</math> are relatively prime for <math>i \neq j</math>.</p> <p>Euler's function: <math>\phi(x)</math> is the number of positive integers less than <math>x</math> relatively prime to <math>x</math>. If <math>\prod_{i=1}^n p_i^{e_i}</math> is the prime factorization of <math>x</math> then</p> $\phi(x) = \prod_{i=1}^n p_i^{e_i-1} (p_i - 1).$ <p>Euler's theorem: If <math>a</math> and <math>b</math> are relatively prime then</p> $1 \equiv a^{\phi(b)} \pmod{b}.$ <p>Fermat's theorem:</p> $1 \equiv a^{p-1} \pmod{p}.$ <p>The Euclidean algorithm: if <math>a &gt; b</math> are integers then</p> $\gcd(a, b) = \gcd(a \bmod b, b).$ <p>If <math>\prod_{i=1}^n p_i^{e_i}</math> is the prime factorization of <math>x</math> then</p> $S(x) = \sum_{d x} d = \prod_{i=1}^n \frac{p_i^{e_i+1} - 1}{p_i - 1}.$ <p>Perfect Numbers: <math>x</math> is an even perfect number iff <math>x = 2^{n-1}(2^n - 1)</math> and <math>2^n - 1</math> is prime.</p> <p>Wilson's theorem: <math>n</math> is a prime iff</p> $(n-1)! \equiv -1 \pmod{n}.$ <p>Möbius inversion:</p> $\mu(i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i \text{ is not square-free.} \\ (-1)^r & \text{if } i \text{ is the product of } r \text{ distinct primes.} \end{cases}$ <p>If</p> $G(a) = \sum_{d a} F(d),$ <p>then</p> $F(a) = \sum_{d a} \mu(d) G\left(\frac{a}{d}\right).$ <p>Prime numbers:</p> $p_n = n \ln n + n \ln \ln n - n + n \frac{\ln \ln n}{\ln n} + O\left(\frac{n}{\ln n}\right),$ $\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3} + O\left(\frac{n}{(\ln n)^4}\right).$	<p>Definitions:</p> <ul style="list-style-type: none"> <li><i>Loop</i>: An edge connecting a vertex to itself.</li> <li><i>Directed</i>: Each edge has a direction.</li> <li><i>Simple</i>: Graph with no loops or multi-edges.</li> <li><i>Walk</i>: A sequence <math>v_0 e_1 v_1 \dots e_\ell v_\ell</math>.</li> <li><i>Trail</i>: A walk with distinct edges.</li> <li><i>Path</i>: A trail with distinct vertices.</li> <li><i>Connected</i>: A graph where there exists a path between any two vertices.</li> <li><i>Component</i>: A maximal connected subgraph.</li> <li><i>Tree</i>: A connected acyclic graph.</li> <li><i>Free tree</i>: A tree with no root.</li> <li><i>DAG</i>: Directed acyclic graph.</li> <li><i>Eulerian</i>: Graph with a trail visiting each edge exactly once.</li> <li><i>Hamiltonian</i>: Graph with a cycle visiting each vertex exactly once.</li> <li><i>Cut</i>: A set of edges whose removal increases the number of components.</li> <li><i>Cut-set</i>: A minimal cut.</li> <li><i>Cut edge</i>: A size 1 cut.</li> <li><i>k-Connected</i>: A graph connected with the removal of any <math>k-1</math> vertices.</li> <li><i>k-Tough</i>: <math>\forall S \subseteq V, S \neq \emptyset</math> we have <math>k \cdot c(G-S) \leq  S </math>.</li> <li><i>k-Regular</i>: A graph where all vertices have degree <math>k</math>.</li> <li><i>k-Factor</i>: A <math>k</math>-regular spanning subgraph.</li> <li><i>Matching</i>: A set of edges, no two of which are adjacent.</li> <li><i>Clique</i>: A set of vertices, all of which are adjacent.</li> <li><i>Ind. set</i>: A set of vertices, none of which are adjacent.</li> <li><i>Vertex cover</i>: A set of vertices which cover all edges.</li> <li><i>Planar graph</i>: A graph which can be embedded in the plane.</li> <li><i>Plane graph</i>: An embedding of a planar graph.</li> </ul>	<p>Notation:</p> <ul style="list-style-type: none"> <li><math>E(G)</math>: Edge set</li> <li><math>V(G)</math>: Vertex set</li> <li><math>c(G)</math>: Number of components</li> <li><math>G[S]</math>: Induced subgraph</li> <li><math>\deg(v)</math>: Degree of <math>v</math></li> <li><math>\Delta(G)</math>: Maximum degree</li> <li><math>\delta(G)</math>: Minimum degree</li> <li><math>\chi(G)</math>: Chromatic number</li> <li><math>\chi_E(G)</math>: Edge chromatic number</li> <li><math>G^c</math>: Complement graph</li> <li><math>K_n</math>: Complete graph</li> <li><math>K_{n_1, n_2}</math>: Complete bipartite graph</li> <li><math>r(k, \ell)</math>: Ramsey number</li> </ul>						
		Geometry						
<p>Projective coordinates: triples <math>(x, y, z)</math>, not all <math>x, y</math> and <math>z</math> zero.</p> $(x, y, z) = (cx, cy, cz) \quad \forall c \neq 0.$ <table style="width: 100%; border-collapse: collapse;"> <tr> <td style="width: 50%;">Cartesian</td><td style="width: 50%;">Projective</td></tr> <tr> <td><math>(x, y)</math></td><td><math>(x, y, 1)</math></td></tr> <tr> <td><math>y = mx + b</math></td><td><math>(m, -1, b)</math></td></tr> <tr> <td><math>x = c</math></td><td><math>(1, 0, -c)</math></td></tr> </table> <p>Distance formula, <math>L_p</math> and <math>L_\infty</math> metric:</p> $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$ $[ x_1 - x_0 ^p +  y_1 - y_0 ^p]^{1/p},$ $\lim_{p \rightarrow \infty} [ x_1 - x_0 ^p +  y_1 - y_0 ^p]^{1/p}.$ <p>Area of triangle <math>(x_0, y_0)</math>, <math>(x_1, y_1)</math> and <math>(x_2, y_2)</math>:</p> $\frac{1}{2} \operatorname{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}.$ <p>Angle formed by three points:</p> $\cos \theta = \frac{(x_1 - x_0)(x_2 - x_0) + (y_1 - y_0)(y_2 - y_0)}{\ell_1 \ell_2}.$ <p>Line through two points <math>(x_0, y_0)</math> and <math>(x_1, y_1)</math>:</p> $\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$ <p>Area of circle, volume of sphere:</p> $A = \pi r^2, \quad V = \frac{4}{3} \pi r^3.$	Cartesian	Projective	$(x, y)$	$(x, y, 1)$	$y = mx + b$	$(m, -1, b)$	$x = c$	$(1, 0, -c)$
Cartesian	Projective							
$(x, y)$	$(x, y, 1)$							
$y = mx + b$	$(m, -1, b)$							
$x = c$	$(1, 0, -c)$							
<p>If I have seen farther than others, it is because I have stood on the shoulders of giants. – Issac Newton</p>								

## Theoretical Computer Science Cheat Sheet

$\pi$	Calculus
<p>Wallis' identity:</p> $\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$ <p>Brouncker's continued fraction expansion:</p> $\frac{\pi}{4} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{\cdots}}}}$ <p>Gregory's series:</p> $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$ <p>Newton's series:</p> $\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots$ <p>Sharp's series:</p> $\frac{\pi}{6} = \frac{1}{\sqrt{3}} \left( 1 - \frac{1}{3^1 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \cdots \right)$ <p>Euler's series:</p> $\begin{aligned}\frac{\pi^2}{6} &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots \\ \frac{\pi^2}{8} &= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots \\ \frac{\pi^2}{12} &= \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots\end{aligned}$	<p>Derivatives:</p> <ol style="list-style-type: none"> <li>1. <math>\frac{d(cu)}{dx} = c \frac{du}{dx},</math></li> <li>2. <math>\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx},</math></li> <li>3. <math>\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx},</math></li> <li>4. <math>\frac{d(u^n)}{dx} = nu^{n-1} \frac{du}{dx},</math></li> <li>5. <math>\frac{d(u/v)}{dx} = \frac{v(\frac{du}{dx}) - u(\frac{dv}{dx})}{v^2},</math></li> <li>6. <math>\frac{d(e^{cu})}{dx} = ce^{cu} \frac{du}{dx},</math></li> <li>7. <math>\frac{d(c^u)}{dx} = (\ln c)c^u \frac{du}{dx},</math></li> <li>8. <math>\frac{d(\ln u)}{dx} = \frac{1}{u} \frac{du}{dx},</math></li> <li>10. <math>\frac{d(\cos u)}{dx} = -\sin u \frac{du}{dx},</math></li> <li>12. <math>\frac{d(\cot u)}{dx} = \csc^2 u \frac{du}{dx},</math></li> <li>14. <math>\frac{d(\csc u)}{dx} = -\cot u \csc u \frac{du}{dx},</math></li> <li>16. <math>\frac{d(\arccos u)}{dx} = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx},</math></li> <li>18. <math>\frac{d(\operatorname{arccot} u)}{dx} = \frac{-1}{1+u^2} \frac{du}{dx},</math></li> <li>20. <math>\frac{d(\operatorname{arccsc} u)}{dx} = \frac{-1}{u\sqrt{1-u^2}} \frac{du}{dx},</math></li> <li>22. <math>\frac{d(\cosh u)}{dx} = \sinh u \frac{du}{dx},</math></li> <li>24. <math>\frac{d(\coth u)}{dx} = -\operatorname{csch}^2 u \frac{du}{dx},</math></li> <li>26. <math>\frac{d(\operatorname{csch} u)}{dx} = -\operatorname{csch} u \coth u \frac{du}{dx},</math></li> <li>28. <math>\frac{d(\operatorname{arccosh} u)}{dx} = \frac{1}{\sqrt{u^2-1}} \frac{du}{dx},</math></li> <li>30. <math>\frac{d(\operatorname{arccoth} u)}{dx} = \frac{1}{u^2-1} \frac{du}{dx},</math></li> <li>32. <math>\frac{d(\operatorname{arccsch} u)}{dx} = \frac{-1}{ u \sqrt{1+u^2}} \frac{du}{dx}.</math></li> </ol> <p>Integrals:</p> <ol style="list-style-type: none"> <li>1. <math>\int cu \, dx = c \int u \, dx,</math></li> <li>2. <math>\int (u+v) \, dx = \int u \, dx + \int v \, dx,</math></li> <li>3. <math>\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1,</math></li> <li>4. <math>\int \frac{1}{x} \, dx = \ln x,</math></li> <li>5. <math>\int e^x \, dx = e^x,</math></li> <li>6. <math>\int \frac{dx}{1+x^2} = \arctan x,</math></li> <li>7. <math>\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx,</math></li> <li>8. <math>\int \sin x \, dx = -\cos x,</math></li> <li>9. <math>\int \cos x \, dx = \sin x,</math></li> <li>10. <math>\int \tan x \, dx = -\ln  \cos x ,</math></li> <li>11. <math>\int \cot x \, dx = \ln  \cos x ,</math></li> <li>12. <math>\int \sec x \, dx = \ln  \sec x + \tan x ,</math></li> <li>13. <math>\int \csc x \, dx = \ln  \csc x + \cot x ,</math></li> <li>14. <math>\int \arcsin \frac{x}{a} \, dx = \arcsin \frac{x}{a} + \sqrt{a^2 - x^2}, \quad a &gt; 0,</math></li> </ol>
<p>The reasonable man adapts himself to the world; the unreasonable persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable. – George Bernard Shaw</p>	

## Theoretical Computer Science Cheat Sheet

Calculus Cont.

15.  $\int \arccos \frac{x}{a} dx = \arccos \frac{x}{a} - \sqrt{a^2 - x^2}, \quad a > 0,$
16.  $\int \arctan \frac{x}{a} dx = x \arctan \frac{x}{a} - \frac{a}{2} \ln(a^2 + x^2), \quad a > 0,$
17.  $\int \sin^2(ax) dx = \frac{1}{2a} (ax - \sin(ax) \cos(ax)),$
18.  $\int \cos^2(ax) dx = \frac{1}{2a} (ax + \sin(ax) \cos(ax)),$
19.  $\int \sec^2 x dx = \tan x,$
20.  $\int \csc^2 x dx = -\cot x,$
21.  $\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx,$
22.  $\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx,$
23.  $\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx, \quad n \neq 1,$
24.  $\int \cot^n x dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x dx, \quad n \neq 1,$
25.  $\int \sec^n x dx = \frac{\tan x \sec^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx, \quad n \neq 1,$
26.  $\int \csc^n x dx = -\frac{\cot x \csc^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} x dx, \quad n \neq 1,$
27.  $\int \sinh x dx = \cosh x, \quad 28. \int \cosh x dx = \sinh x,$
29.  $\int \tanh x dx = \ln |\cosh x|, \quad 30. \int \coth x dx = \ln |\sinh x|, \quad 31. \int \operatorname{sech} x dx = \arctan \sinh x, \quad 32. \int \operatorname{csch} x dx = \ln |\tanh \frac{x}{2}|,$
33.  $\int \sinh^2 x dx = \frac{1}{4} \sinh(2x) - \frac{1}{2}x, \quad 34. \int \cosh^2 x dx = \frac{1}{4} \sinh(2x) + \frac{1}{2}x, \quad 35. \int \operatorname{sech}^2 x dx = \tanh x,$
36.  $\int \operatorname{arcsinh} \frac{x}{a} dx = x \operatorname{arcsinh} \frac{x}{a} - \sqrt{x^2 + a^2}, \quad a > 0,$
37.  $\int \operatorname{arctanh} \frac{x}{a} dx = x \operatorname{arctanh} \frac{x}{a} + \frac{a}{2} \ln |a^2 - x^2|,$
38.  $\int \operatorname{arccosh} \frac{x}{a} dx = \begin{cases} x \operatorname{arccosh} \frac{x}{a} - \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} > 0 \text{ and } a > 0, \\ x \operatorname{arccosh} \frac{x}{a} + \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} < 0 \text{ and } a > 0, \end{cases}$
39.  $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln \left( x + \sqrt{a^2 + x^2} \right), \quad a > 0,$
40.  $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a}, \quad a > 0,$
41.  $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$
42.  $\int (a^2 - x^2)^{3/2} dx = \frac{x}{8} (5a^2 - 2x^2) \sqrt{a^2 - x^2} + \frac{3a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$
43.  $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a}, \quad a > 0,$
44.  $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right|, \quad 45. \int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2 \sqrt{a^2 - x^2}},$
46.  $\int \sqrt{a^2 \pm x^2} dx = \frac{x}{2} \sqrt{a^2 \pm x^2} \pm \frac{a^2}{2} \ln \left| x + \sqrt{a^2 \pm x^2} \right|,$
47.  $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right|, \quad a > 0,$
48.  $\int \frac{dx}{ax^2 + bx} = \frac{1}{a} \ln \left| \frac{x}{a+bx} \right|,$
49.  $\int x \sqrt{a+bx} dx = \frac{2(3bx - 2a)(a+bx)^{3/2}}{15b^2},$
50.  $\int \frac{\sqrt{a+bx}}{x} dx = 2\sqrt{a+bx} + a \int \frac{1}{x\sqrt{a+bx}} dx,$
51.  $\int \frac{x}{\sqrt{a+bx}} dx = \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}} \right|, \quad a > 0,$
52.  $\int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} - a \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right|,$
53.  $\int x \sqrt{a^2 - x^2} dx = -\frac{1}{3} (a^2 - x^2)^{3/2},$
54.  $\int x^2 \sqrt{a^2 - x^2} dx = \frac{x}{8} (2x^2 - a^2) \sqrt{a^2 - x^2} + \frac{a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$
55.  $\int \frac{dx}{\sqrt{a^2 - x^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right|,$
56.  $\int \frac{x dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2},$
57.  $\int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$
58.  $\int \frac{\sqrt{a^2 + x^2}}{x} dx = \sqrt{a^2 + x^2} - a \ln \left| \frac{a + \sqrt{a^2 + x^2}}{x} \right|,$
59.  $\int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \arccos \frac{a}{|x|}, \quad a > 0,$
60.  $\int x \sqrt{x^2 \pm a^2} dx = \frac{1}{3} (x^2 \pm a^2)^{3/2},$
61.  $\int \frac{dx}{x\sqrt{x^2 + a^2}} = \frac{1}{a} \ln \left| \frac{x}{a + \sqrt{a^2 + x^2}} \right|,$

## Theoretical Computer Science Cheat Sheet

### Calculus Cont.

<p><b>62.</b> <math>\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \arccos \frac{a}{ x }, \quad a &gt; 0,</math></p> <p><b>64.</b> <math>\int \frac{x dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2},</math></p> <p><b>66.</b> <math>\int \frac{dx}{ax^2 + bx + c} = \begin{cases} \frac{1}{\sqrt{b^2 - 4ac}} \ln \left  \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}} \right , &amp; \text{if } b^2 &gt; 4ac, \\ \frac{2}{\sqrt{4ac - b^2}} \arctan \frac{2ax + b}{\sqrt{4ac - b^2}}, &amp; \text{if } b^2 &lt; 4ac, \end{cases}</math></p> <p><b>67.</b> <math>\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \begin{cases} \frac{1}{\sqrt{a}} \ln \left  2ax + b + 2\sqrt{a}\sqrt{ax^2 + bx + c} \right , &amp; \text{if } a &gt; 0, \\ \frac{1}{\sqrt{-a}} \arcsin \frac{-2ax - b}{\sqrt{b^2 - 4ac}}, &amp; \text{if } a &lt; 0, \end{cases}</math></p> <p><b>68.</b> <math>\int \sqrt{ax^2 + bx + c} dx = \frac{2ax + b}{4a} \sqrt{ax^2 + bx + c} + \frac{4ax - b^2}{8a} \int \frac{dx}{\sqrt{ax^2 + bx + c}},</math></p> <p><b>69.</b> <math>\int \frac{x dx}{\sqrt{ax^2 + bx + c}} = \frac{\sqrt{ax^2 + bx + c}}{a} - \frac{b}{2a} \int \frac{dx}{\sqrt{ax^2 + bx + c}},</math></p> <p><b>70.</b> <math>\int \frac{dx}{x\sqrt{ax^2 + bx + c}} = \begin{cases} \frac{-1}{\sqrt{c}} \ln \left  \frac{2\sqrt{c}\sqrt{ax^2 + bx + c} + bx + 2c}{x} \right , &amp; \text{if } c &gt; 0, \\ \frac{1}{\sqrt{-c}} \arcsin \frac{bx + 2c}{ x \sqrt{b^2 - 4ac}}, &amp; \text{if } c &lt; 0, \end{cases}</math></p> <p><b>71.</b> <math>\int x^3 \sqrt{x^2 + a^2} dx = (\frac{1}{3}x^2 - \frac{2}{15}a^2)(x^2 + a^2)^{3/2},</math></p> <p><b>72.</b> <math>\int x^n \sin(ax) dx = -\frac{1}{a} x^n \cos(ax) + \frac{n}{a} \int x^{n-1} \cos(ax) dx,</math></p> <p><b>73.</b> <math>\int x^n \cos(ax) dx = \frac{1}{a} x^n \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax) dx,</math></p> <p><b>74.</b> <math>\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx,</math></p> <p><b>75.</b> <math>\int x^n \ln(ax) dx = x^{n+1} \left( \frac{\ln(ax)}{n+1} - \frac{1}{(n+1)^2} \right),</math></p> <p><b>76.</b> <math>\int x^n (\ln ax)^m dx = \frac{x^{n+1}}{n+1} (\ln ax)^m - \frac{m}{n+1} \int x^n (\ln ax)^{m-1} dx.</math></p>	<p><b>63.</b> <math>\int \frac{dx}{x^2 \sqrt{x^2 \pm a^2}} = \mp \frac{\sqrt{x^2 \pm a^2}}{a^2 x},</math></p> <p><b>65.</b> <math>\int \frac{\sqrt{x^2 \pm a^2}}{x^4} dx = \mp \frac{(x^2 + a^2)^{3/2}}{3a^2 x^3},</math></p>
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### Finite Calculus

Difference, shift operators:

$$\Delta f(x) = f(x+1) - f(x),$$

$$\mathrm{E} f(x) = f(x+1).$$

Fundamental Theorem:

$$f(x) = \Delta F(x) \Leftrightarrow \sum f(x) \delta x = F(x) + C.$$

$$\sum_a^b f(x) \delta x = \sum_{i=a}^{b-1} f(i).$$

Differences:

$$\Delta(cu) = c\Delta u, \quad \Delta(u+v) = \Delta u + \Delta v,$$

$$\Delta(uv) = u\Delta v + \mathrm{E} v \Delta u,$$

$$\Delta(x^n) = nx^{n-1},$$

$$\Delta(H_x) = x^{-1}, \quad \Delta(2^x) = 2^x,$$

$$\Delta(c^x) = (c-1)c^x, \quad \Delta(\binom{x}{m}) = \binom{x}{m-1}.$$

Sums:

$$\sum cu \delta x = c \sum u \delta x,$$

$$\sum(u+v) \delta x = \sum u \delta x + \sum v \delta x,$$

$$\sum u \Delta v \delta x = uv - \sum \mathrm{E} v \Delta u \delta x,$$

$$\sum x^n \delta x = \frac{x^{n+1}}{m+1}, \quad \sum x^{-1} \delta x = H_x,$$

$$\sum c^x \delta x = \frac{c^x}{c-1}, \quad \sum \binom{x}{m} \delta x = \binom{x}{m+1}.$$

Falling Factorial Powers:

$$x^n = x(x-1) \cdots (x-n+1), \quad n > 0,$$

$$x^0 = 1,$$

$$x^n = \frac{1}{(x+1) \cdots (x+|n|)}, \quad n < 0,$$

$$x^{n+m} = x^m (x-m)^n.$$

Rising Factorial Powers:

$$x^{\overline{n}} = x(x+1) \cdots (x+n-1), \quad n > 0,$$

$$x^{\overline{0}} = 1,$$

$$x^{\overline{n}} = \frac{1}{(x-1) \cdots (x-|n|)}, \quad n < 0,$$

$$x^{\overline{n+m}} = x^{\overline{m}} (x+m)^{\overline{n}}.$$

Conversion:

$$x^n = (-1)^n (-x)^{\overline{n}} = (x-n+1)^{\overline{n}} = 1/(x+1)^{\overline{-n}},$$

$$x^{\overline{n}} = (-1)^n (-x)^n = (x+n-1)^n = 1/(x-1)^{\overline{-n}},$$

$$x^n = \sum_{k=1}^n \binom{n}{k} x^k = \sum_{k=1}^n \binom{n}{k} (-1)^{n-k} x^{\overline{k}},$$

$$x^{\overline{n}} = \sum_{k=1}^n \binom{n}{k} (-1)^{n-k} x^k,$$

$$x^{\overline{n}} = \sum_{k=1}^n \binom{n}{k} x^k.$$

$x^1 =$	$x^1$	$=$	$x^{\overline{1}}$
$x^2 =$	$x^2 + x^1$	$=$	$x^{\overline{2}} - x^{\overline{1}}$
$x^3 =$	$x^3 + 3x^2 + x^1$	$=$	$x^{\overline{3}} - 3x^{\overline{2}} + x^{\overline{1}}$
$x^4 =$	$x^4 + 6x^3 + 7x^2 + x^1$	$=$	$x^{\overline{4}} - 6x^{\overline{3}} + 7x^{\overline{2}} - x^{\overline{1}}$
$x^5 =$	$x^5 + 15x^4 + 25x^3 + 10x^2 + x^1$	$=$	$x^{\overline{5}} - 15x^{\overline{4}} + 25x^{\overline{3}} - 10x^{\overline{2}} + x^{\overline{1}}$
$x^{\overline{1}} =$	$x^1$	$=$	$x^1$
$x^{\overline{2}} =$	$x^2 + x^1$	$=$	$x^2 - x^1$
$x^{\overline{3}} =$	$x^3 + 3x^2 + 2x^1$	$=$	$x^3 - 3x^2 + 2x^1$
$x^{\overline{4}} =$	$x^4 + 6x^3 + 11x^2 + 6x^1$	$=$	$x^4 - 6x^3 + 11x^2 - 6x^1$
$x^{\overline{5}} =$	$x^5 + 10x^4 + 35x^3 + 50x^2 + 24x^1$	$=$	$x^5 - 10x^4 + 35x^3 - 50x^2 + 24x^1$

## Theoretical Computer Science Cheat Sheet

### Series

Taylor's series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \dots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a).$$

Expansions:

$\frac{1}{1-x}$	$= 1 + x + x^2 + x^3 + x^4 + \dots$	$= \sum_{i=0}^{\infty} x^i,$
$\frac{1}{1-cx}$	$= 1 + cx + c^2x^2 + c^3x^3 + \dots$	$= \sum_{i=0}^{\infty} c^i x^i,$
$\frac{1}{1-x^n}$	$= 1 + x^n + x^{2n} + x^{3n} + \dots$	$= \sum_{i=0}^{\infty} x^{ni},$
$\frac{x}{(1-x)^2}$	$= x + 2x^2 + 3x^3 + 4x^4 + \dots$	$= \sum_{i=0}^{\infty} ix^i,$
$x^k \frac{d^n}{dx^n} \left( \frac{1}{1-x} \right)$	$= x + 2^nx^2 + 3^nx^3 + 4^nx^4 + \dots$	$= \sum_{i=0}^{\infty} i^n x^i,$
$e^x$	$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$	$= \sum_{i=0}^{\infty} \frac{x^i}{i!},$
$\ln(1+x)$	$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 - \dots$	$= \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^i}{i},$
$\ln \frac{1}{1-x}$	$= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \dots$	$= \sum_{i=1}^{\infty} \frac{x^i}{i},$
$\sin x$	$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$	$= \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!},$
$\cos x$	$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$	$= \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!},$
$\tan^{-1} x$	$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$	$= \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)},$
$(1+x)^n$	$= 1 + nx + \frac{n(n-1)}{2}x^2 + \dots$	$= \sum_{i=0}^{\infty} \binom{n}{i} x^i,$
$\frac{1}{(1-x)^{n+1}}$	$= 1 + (n+1)x + \binom{n+2}{2}x^2 + \dots$	$= \sum_{i=0}^{\infty} \binom{i+n}{i} x^i,$
$\frac{x}{e^x - 1}$	$= 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \dots$	$= \sum_{i=0}^{\infty} \frac{B_i x^i}{i!},$
$\frac{1}{2x}(1 - \sqrt{1-4x})$	$= 1 + x + 2x^2 + 5x^3 + \dots$	$= \sum_{i=0}^{\infty} \frac{1}{i+1} \binom{2i}{i} x^i,$
$\frac{1}{\sqrt{1-4x}}$	$= 1 + x + 2x^2 + 6x^3 + \dots$	$= \sum_{i=0}^{\infty} \binom{2i}{i} x^i,$
$\frac{1}{\sqrt{1-4x}} \left( \frac{1 - \sqrt{1-4x}}{2x} \right)^n$	$= 1 + (2+n)x + \binom{4+n}{2}x^2 + \dots$	$= \sum_{i=0}^{\infty} \binom{2i+n}{i} x^i,$
$\frac{1}{1-x} \ln \frac{1}{1-x}$	$= x + \frac{3}{2}x^2 + \frac{11}{6}x^3 + \frac{25}{12}x^4 + \dots$	$= \sum_{i=1}^{\infty} H_i x^i,$
$\frac{1}{2} \left( \ln \frac{1}{1-x} \right)^2$	$= \frac{1}{2}x^2 + \frac{3}{4}x^3 + \frac{11}{24}x^4 + \dots$	$= \sum_{i=2}^{\infty} \frac{H_{i-1} x^i}{i},$
$\frac{x}{1-x-x^2}$	$= x + x^2 + 2x^3 + 3x^4 + \dots$	$= \sum_{i=0}^{\infty} F_i x^i,$
$\frac{F_n x}{1 - (F_{n-1} + F_{n+1})x - (-1)^n x^2}$	$= F_n x + F_{2n} x^2 + F_{3n} x^3 + \dots$	$= \sum_{i=0}^{\infty} F_{ni} x^i.$

Ordinary power series:

$$A(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Exponential power series:

$$A(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}.$$

Dirichlet power series:

$$A(x) = \sum_{i=1}^{\infty} \frac{a_i}{i^x}.$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k.$$

Difference of like powers:

$$x^n - y^n = (x-y) \sum_{k=0}^{n-1} x^{n-1-k} y^k.$$

For ordinary power series:

$$\alpha A(x) + \beta B(x) = \sum_{i=0}^{\infty} (\alpha a_i + \beta b_i) x^i,$$

$$x^k A(x) = \sum_{i=k}^{\infty} a_{i-k} x^i,$$

$$\frac{A(x) - \sum_{i=0}^{k-1} a_i x^i}{x^k} = \sum_{i=0}^{\infty} a_{i+k} x^i,$$

$$A(cx) = \sum_{i=0}^{\infty} c^i a_i x^i,$$

$$A'(x) = \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i,$$

$$xA'(x) = \sum_{i=1}^{\infty} i a_i x^i,$$

$$\int A(x) dx = \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} x^i,$$

$$\frac{A(x) + A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i} x^{2i},$$

$$\frac{A(x) - A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i+1} x^{2i+1}.$$

Summation: If  $b_i = \sum_{j=0}^i a_j$  then

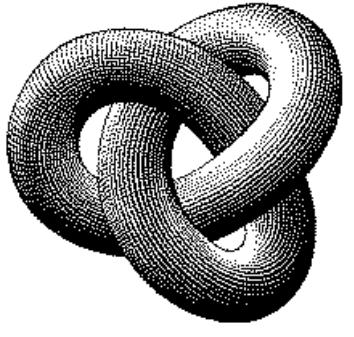
$$B(x) = \frac{1}{1-x} A(x).$$

Convolution:

$$A(x)B(x) = \sum_{i=0}^{\infty} \left( \sum_{j=0}^i a_j b_{i-j} \right) x^i.$$

God made the natural numbers;  
all the rest is the work of man.  
— Leopold Kronecker

## Theoretical Computer Science Cheat Sheet

Series	Escher's Knot																																																																																																				
<p>Expansions:</p> $\frac{1}{(1-x)^{n+1}} \ln \frac{1}{1-x} = \sum_{i=0}^{\infty} (H_{n+i} - H_n) \binom{n+i}{i} x^i,$ $x^{\frac{n}{k}} = \sum_{i=0}^{\infty} \binom{n}{i} x^i,$ $\left(\ln \frac{1}{1-x}\right)^n = \sum_{i=0}^{\infty} \binom{i}{n} \frac{n! x^i}{i!},$ $\tan x = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{2^{2i}(2^{2i}-1)B_{2i}x^{2i-1}}{(2i)!},$ $\frac{1}{\zeta(x)} = \sum_{i=1}^{\infty} \frac{\mu(i)}{i^x},$ $\zeta(x) = \prod_p \frac{1}{1-p^{-x}},$ $\zeta^2(x) = \sum_{i=1}^{\infty} \frac{d(i)}{i^x} \quad \text{where } d(n) = \sum_{d n} 1,$ $\zeta(x)\zeta(x-1) = \sum_{i=1}^{\infty} \frac{S(i)}{i^x} \quad \text{where } S(n) = \sum_{d n} d,$ $\zeta(2n) = \frac{2^{2n-1} B_{2n} }{(2n)!} \pi^{2n}, \quad n \in \mathbb{N},$ $\frac{x}{\sin x} = \sum_{i=0}^{\infty} (-1)^{i-1} \frac{(4^i - 2)B_{2i}x^{2i}}{(2i)!},$ $\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = \sum_{i=0}^{\infty} \frac{n(2i+n-1)!}{i!(n+i)!} x^i,$ $e^x \sin x = \sum_{i=1}^{\infty} \frac{2^{i/2} \sin \frac{i\pi}{4}}{i!} x^i,$ $\sqrt{\frac{1-\sqrt{1-x}}{x}} = \sum_{i=0}^{\infty} \frac{(4i)!}{16^i \sqrt{2}(2i)!(2i+1)!} x^i,$ $\left(\frac{\arcsin x}{x}\right)^2 = \sum_{i=0}^{\infty} \frac{4^i i!^2}{(i+1)(2i+1)!} x^{2i}.$																																																																																																					
	Stieltjes Integration																																																																																																				
	<p>If <math>G</math> is continuous in the interval <math>[a, b]</math> and <math>F</math> is nondecreasing then</p> $\int_a^b G(x) dF(x)$ <p>exists. If <math>a \leq b \leq c</math> then</p> $\int_a^c G(x) dF(x) = \int_a^b G(x) dF(x) + \int_b^c G(x) dF(x).$ <p>If the integrals involved exist</p> $\int_a^b (G(x) + H(x)) dF(x) = \int_a^b G(x) dF(x) + \int_a^b H(x) dF(x),$ $\int_a^b G(x) d(F(x) + H(x)) = \int_a^b G(x) dF(x) + \int_a^b G(x) dH(x),$ $\int_a^b c \cdot G(x) dF(x) = \int_a^b G(x) d(c \cdot F(x)) = c \int_a^b G(x) dF(x),$ $\int_a^b G(x) dF(x) = G(b)F(b) - G(a)F(a) - \int_a^b F(x) dG(x).$																																																																																																				
<p>Cramer's Rule</p> <p>If we have equations:</p> $a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1$ $a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2$ $\vdots \quad \vdots \quad \vdots$ $a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n = b_n$ <p>Let <math>A = (a_{i,j})</math> and <math>B</math> be the column matrix <math>(b_i)</math>. Then there is a unique solution iff <math>\det A \neq 0</math>. Let <math>A_i</math> be <math>A</math> with column <math>i</math> replaced by <math>B</math>. Then</p> $x_i = \frac{\det A_i}{\det A}.$	<p>Fibonacci Numbers</p> <table border="1" style="margin-left: auto; margin-right: auto;"> <tr><td>00</td><td>47</td><td>18</td><td>76</td><td>29</td><td>93</td><td>85</td><td>34</td><td>61</td><td>52</td></tr> <tr><td>86</td><td>11</td><td>57</td><td>28</td><td>70</td><td>39</td><td>94</td><td>45</td><td>02</td><td>63</td></tr> <tr><td>95</td><td>80</td><td>22</td><td>67</td><td>38</td><td>71</td><td>49</td><td>56</td><td>13</td><td>04</td></tr> <tr><td>59</td><td>96</td><td>81</td><td>33</td><td>07</td><td>48</td><td>72</td><td>60</td><td>24</td><td>15</td></tr> <tr><td>73</td><td>69</td><td>90</td><td>82</td><td>44</td><td>17</td><td>58</td><td>01</td><td>35</td><td>26</td></tr> <tr><td>68</td><td>74</td><td>09</td><td>91</td><td>83</td><td>55</td><td>27</td><td>12</td><td>46</td><td>30</td></tr> <tr><td>37</td><td>08</td><td>75</td><td>19</td><td>92</td><td>84</td><td>66</td><td>23</td><td>50</td><td>41</td></tr> <tr><td>14</td><td>25</td><td>36</td><td>40</td><td>51</td><td>62</td><td>03</td><td>77</td><td>88</td><td>99</td></tr> <tr><td>21</td><td>32</td><td>43</td><td>54</td><td>65</td><td>06</td><td>10</td><td>89</td><td>97</td><td>78</td></tr> <tr><td>42</td><td>53</td><td>64</td><td>05</td><td>16</td><td>20</td><td>31</td><td>98</td><td>79</td><td>87</td></tr> </table> <p>Definitions:</p> $F_i = F_{i-1} + F_{i-2}, \quad F_0 = F_1 = 1,$ $F_{-i} = (-1)^{i-1} F_i,$ $F_i = \frac{1}{\sqrt{5}} (\phi^i - \hat{\phi}^i),$ <p>Cassini's identity: for <math>i &gt; 0</math>:</p> $F_{i+1}F_{i-1} - F_i^2 = (-1)^i.$ <p>Additive rule:</p> $F_{n+k} = F_k F_{n+1} + F_{k-1} F_n,$ $F_{2n} = F_n F_{n+1} + F_{n-1} F_n.$ <p>Calculation by matrices:</p> $\begin{pmatrix} F_{n-2} & F_{n-1} \\ F_{n-1} & F_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n.$	00	47	18	76	29	93	85	34	61	52	86	11	57	28	70	39	94	45	02	63	95	80	22	67	38	71	49	56	13	04	59	96	81	33	07	48	72	60	24	15	73	69	90	82	44	17	58	01	35	26	68	74	09	91	83	55	27	12	46	30	37	08	75	19	92	84	66	23	50	41	14	25	36	40	51	62	03	77	88	99	21	32	43	54	65	06	10	89	97	78	42	53	64	05	16	20	31	98	79	87
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<p>Improvement makes strait roads, but the crooked roads without Improvement, are roads of Genius. – William Blake (The Marriage of Heaven and Hell)</p>	<p>The Fibonacci number system: Every integer <math>n</math> has a unique representation</p> $n = F_{k_1} + F_{k_2} + \cdots + F_{k_m},$ <p>where <math>k_i \geq k_{i+1} + 2</math> for all <math>i</math>, <math>1 \leq i &lt; m</math> and <math>k_m \geq 2</math>.</p>																																																																																																				



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