Beam Params.c : GSM equations explained

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1 Introduction

This file explains the GSM equations used in the BeamParams.c file. The equations in the paper by McMorran and Croni (2008) are not presented in the same form as they are in the code. The reason for this is the introduction of a new parameter z_p (see equation 1). This document shows

2 Nomenclature

- $z =$: z position.
- z_p : function to simplify some calculations.
- $v =$: radius of wavefront curvature.
- λ : De Brogilie wavelength of the particle.
- l (el) : coherence width.
- l_0 (el0) : initial coherence width.
- E (sp.energy) :
- w : beamwidth.
- w_0 : initial beamwidth.

3 Equations as in the BeamParams.c file

$$z_p(z, v) = \frac{v \times z}{v + z} \tag{1}$$

$$\lambda = \sqrt{\frac{1.5 \times 10^{-18}}{E}} \tag{2}$$

$$w(z) = w_0 \left|\frac{z}{z_p}\right| \sqrt{1 + \left(\frac{\lambda z_p}{l_0 w_0}\right)^2} \tag{3}$$

$$l(z) = l_0 \left|\frac{z}{z_p}\right| \sqrt{1 + \left(\frac{\lambda z_p}{l_0 w_0}\right)^2} \tag{4}$$

$$v(z) = \frac{z}{1 - \frac{z_p}{z\left(1+\left(\frac{\lambda z_p}{l_0 w_0}\right)^2\right)}} \tag{5}$$

4 Rewriting paper equations to the code form.

This sections takes the equations from the papers and shows how to obtain the form used in the code.

Equation number 3 is the beam's width as it moves along the z-axis. From the article by McMorran and Cronin (2008) we have the following equation for GSM beam's width:

$$w = w_0 \sqrt{\left(1 + \frac{z}{r_0}\right)^2 + \left(\frac{\lambda z}{w_0 l_0}\right)^2} \tag{6}$$

Writing 1 as $\frac{r_0}{r_0}$:

$$w = w_0 \sqrt{\left(\frac{r_0 + z}{r_0}\right)^2 + \left(\frac{\lambda z}{w_0 l_0}\right)^2}$$

Multiplying z^2 in the denominator and numerator in the first term inside the square root then pulling z^2 out of it as $|z|$:

$$w = w_0 \, |z| \sqrt{\left(\frac{r_0 + z}{r_0 z}\right)^2 + \left(\frac{\lambda}{w_0 l_0}\right)^2}$$

Introducing a new variable $z_p = \frac{r_0 \times z}{r_0 + z}$:

$$w = w_0 \, |z| \sqrt{\frac{1}{z_p^2} + \left(\frac{\lambda}{w_0 l_0}\right)^2}$$

$$w = w_0 \left|\frac{z}{z_p}\right| \sqrt{1 + \left(\frac{\lambda z_p}{l_0 w_0}\right)^2}$$

Then we have the simplified equation 3 as it is used in the code.

Similarly, one can simplify equation 4 from:

$$l = l_0 \sqrt{\left(1 + \frac{z}{r_0}\right)^2 + \left(\frac{\lambda z}{w_0 l_0}\right)^2} \tag{7}$$

For equation 5, which is the radius of the wavefront, it is a bit more complicated. From the reference McMorran and Cronin, 2008 we have the following (note that in the code the name of the function is v, whereas in the article it is r):

$$r = z\left(\frac{\left(1+\frac{z}{r_0}\right)^2+\left(\frac{\lambda z}{w_0 l_0}\right)^2}{\left(\frac{z}{r_0}\right)\left(1+\frac{z}{r_0}\right)+\left(\frac{\lambda z}{w_0 l_0}\right)^2}\right) \tag{8}$$

For simplicity let's look first to the numerator of the fraction between brackets.

$$\left(1+\frac{z}{r_0}\right)^2+\left(\frac{\lambda z}{w_0 l_0}\right)^2 = z^2\left(\frac{z+r_0}{zr_0}\right)^2+\left(\frac{\lambda z}{w_0 l_0}\right)^2$$

$$\left(\frac{z}{z_p}\right)^2+\left(\frac{\lambda z}{w_0 l_0}\right)^2 = \frac{z^2}{z_p^2}\left(1+\left(\frac{\lambda z_p}{w_0 l_0}\right)^2\right)$$

Now working in the denominator of the same fraction:

$$\left(\frac{z}{r_0}\right)\left(1+\frac{z}{r_0}\right)+\left(\frac{\lambda z}{w_0 l_0}\right)^2 = \left(\frac{z}{r_0}\right)^2+\frac{2z}{r_0}+1-\frac{z}{r_0}-1+\left(\frac{\lambda z}{w_0 l_0}\right)^2$$

$$\left(\frac{z}{r_0}+1\right)^2-\frac{z}{r_0}-1+\left(\frac{\lambda z}{w_0 l_0}\right)^2 = \left(\frac{z+r_0}{r_0}\right)^2-\frac{z+r_0}{r_0}+\left(\frac{\lambda z}{w_0 l_0}\right)^2$$

$$\frac{z^2}{z_p^2}+\left(\frac{\lambda z}{w_0 l_0}\right)^2-\frac{z}{z_p} = \frac{z}{z_p^2}\left(z\left(1+\left(\frac{\lambda z_p}{w_0 l_0}\right)^2\right)-z_p\right)$$

Now putting everything into the fraction:

$$\frac{\frac{z^2}{z_p^2}\left(1+\left(\frac{\lambda z_p}{w_0 l_0}\right)^2\right)}{\frac{z}{z_p^2}\left(z\left(1+\left(\frac{\lambda z_p}{w_0 l_0}\right)^2\right)-z_p\right)} = \frac{z\left(1+\left(\frac{\lambda z_p}{w_0 l_0}\right)^2\right)}{z\left(1+\left(\frac{\lambda z_p}{w_0 l_0}\right)^2\right)-z_p}$$

$$\frac{1}{\frac{z\left(1+\left(\frac{\lambda z_p}{w_0 l_0}\right)^2\right)-z_p}{z\left(1+\left(\frac{\lambda z_p}{w_0 l_0}\right)^2\right)}} = \frac{1}{1-\frac{z_p}{z\left(1+\left(\frac{\lambda z_p}{l_0 w_0}\right)^2\right)}}$$

Therefore equation 5 equals to 8:

$$\frac{z}{1-\frac{z_p}{z\left(1+\left(\frac{\lambda z_p}{l_0 w_0}\right)^2\right)}} = z\left(\frac{\left(1+\frac{z}{r_0}\right)^2+\left(\frac{\lambda z}{w_0 l_0}\right)^2}{\left(\frac{z}{r_0}\right)\left(1+\frac{z}{r_0}\right)+\left(\frac{\lambda z}{w_0 l_0}\right)}\right)$$

That is a result that is easier to compute in the computer.