European Journal of Physics

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To cite this article: B Carrascal *et al* 1991 *Eur. J. Phys.* **12** 184

Eur. J. Phys. 12 (1991) 184–191. Printed in the UK

Vector spherical harmonics and their application to classical electrodynamics

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Received 16 March 1990, in final form 6 February 1991

Abstract. A set of vector spherical harmonics which was earlier introduced and shown to be applicable to *magnetostatics* problems, is employed to decompose and solve Maxwell equations. Expressions for the radiation fields for the transverse electric and magnetic modes, the angular distribution of multipole radiation, and the total power radiated *in terms* of the set of vector spherical harmonics are also derived. The interesting and important Rayleigh scattering problem of electromagnetic plane waves scattered by a conducting sphere is discussed in some detail within the present approach. Numerous other problems particularly well adapted for the vector spherical harmonic approach advocated in this paper are suggested.

Zusammenfassung. Ein Satz von Vektorkugelfunktionen, welcher in einer früheren Arbeit eingeführt und dessen Anwendbarkeit auf magnetostatische Probleme gezeigt wurde, wird zur Zerlegung und Lösung der Maxwell Gleichungen verwendet. Es werden Ausdrücke für die Strahlungsfelder transversaler elektrischer und magnetischer Moden, für die Winkelverteilung der Multipolstrahlung und für die gesamte abgestrahlte Leistung mit Hilfe dieses Satzes von Vektorkugelfunktionen abgeleitet. Das interessante und wichtige Problem der Rayleigh-Streuung von ebenen elektromagnetischen Wellen, die an einer leitenden Kugel gestreut werden, wird mittels dieses Zuganges im Detail besprochen. Zahlreiche andere Probleme, die sich besonders gut für den Zugang mit Hilfe der Vektorkugelfunktionen, wie er in dieser Arbeit vorgestellt wurde, eignen, werden vorgeschlagen.

1. Introduction

In an earlier article Barrera *et al* (1985)† introduced a set of vector spherical harmonics (VSH) in analogy with the customarily employed scalar spherical harmonics. It was stipulated in that paper that any everywhere-finite V can be expressed as an appropriate linear combination of the three fundamental vector spherical harmonics: $\mathbf{\Psi}_{lm}(\theta,\phi) = r\nabla Y_{lm}(\theta,\phi)$, $\mathbf{\Phi}_{lm} = \hat{\boldsymbol{e}}_r \times \mathbf{\Psi}_{lm}$, and $\mathbf{Y}_{lm} = \hat{\boldsymbol{e}}_r Y_{lm}$, where $Y_{lm}(\theta,\phi)$ is the scalar spherical harmonic of order (l,m) (see, for example, Jackson 1975). Barrera *et al* stated without proof that the set of VSH constituted an orthogonal basis set. The utility of the VSH was amply demonstrated in the realm of magnetostics; the new set of VSH, however, is also useful in other contexts. An example of the latter is in the study of Rayleigh scattering problems, i.e. scattering of a free-space plane wave solution to Maxwell equations by objects whose dimensions are small compared with the wavelength of the incident plane wave. The basic problem in classical electrodynamics is to solve Maxwell equations for the electric and magnetic fields. The multipole decomposition of Maxwell equations was not attempted by Barrera *et al.* This article extends the same ideas and formulae to the consideration of radiation and scattering problems. In section 2 the multipole decomposition of Maxwell equations is furnished. Section 3 is devoted to the derivation of the multipole expansion of the magnetostatic and electrostatic fields. In section 4 (the core of this article), formulae for the fields of the sources are established and finally, in section 5, we illustrate the usefulness of

† In the caption to table 1 of this paper the word 'quadrupole' should be replaced by 'octupole'. The first term in the first of equations (3.19) is missing a negative sign. Also in the second of equations (3.23) the function $\mathbf{\Psi}_{lm}$ should be starred.

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the VSH with two examples chosen from radiation and scattering. To make the article self-contained and to find the total power radiated by a system when many modes are excited, the orthogonality relations of the set of VSH over all space are demonstrated.

This material has been given in lectures by the authors to final year physics students. As background, the students have been exposed to the contents of the article by Barrera *et al* (1985).

2. Multipole decomposition of Maxwell equations

To decompose Maxwell equations in their Y_{lm}, $\boldsymbol{\Psi}_{lm}$, and $\boldsymbol{\Phi}_{lm}$ components, we start by writing the electric field as

$$\boldsymbol{E} = E^r_{lm} \boldsymbol{Y}_{lm} + E^{(1)}_{lm}\boldsymbol{\Psi}_{lm} + E^{(2)}_{lm}\boldsymbol{\Phi}_{lm}. \tag{2.1}$$

Similar expansions to equation (2.1) can be written down for the magnetic induction field $\boldsymbol{B}$ and the current density $\boldsymbol{J}$. The charge density ρ being a scalar can only be written as $\rho = \rho_{lm} Y_{lm}$.

To simplify the treatment, the magnetic permeability μ and the relative electric permittivity ε will be constant quantities. A dot on top of a quantity will indicate a derivative with respect to time. Also, for simplicity the subscripts l and m will be omitted. In spherical polar coordinates and in Gaussian units, the Maxwell equations then read:

$$\nabla \cdot \boldsymbol{E} = \frac{4\pi\rho}{\varepsilon}$$

or (2.2)

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E^r) - \frac{l(l+1)}{r}E^{(1)} = \frac{4\pi\rho}{\varepsilon}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

or (2.3)

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 B^r) - \frac{l(l+1)}{r}B^{(1)} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}$$

or (2.4)

$$-\frac{E^r}{r}\boldsymbol{\Phi} + \frac{1}{r}\frac{\partial}{\partial r}(rE^{(1)})\boldsymbol{\Phi} - \frac{l(l+1)}{r}E^{(2)}\boldsymbol{Y}$$

$$-\frac{1}{r}\frac{\partial}{\partial r}(rE^{(2)})\boldsymbol{\Psi} = -\frac{1}{c}(\dot{B}^r \boldsymbol{Y} + \dot{B}^{(1)}\boldsymbol{\Psi} + \dot{B}^{(2)}\boldsymbol{\Phi})$$

$$\nabla \times \boldsymbol{B} = \frac{\mu\varepsilon}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi\mu}{c}\boldsymbol{J}$$

or

$$-\frac{B^r}{r}\boldsymbol{\Phi} + \frac{1}{r}\frac{\partial}{\partial r}(rB^{(1)})\boldsymbol{\Phi} - \frac{l(l+1)}{r}B^{(2)}\boldsymbol{Y} - \frac{1}{r}\frac{\partial}{\partial r}(rB^{(2)})\boldsymbol{\Psi}$$

$$= \frac{\mu\varepsilon}{c}(\dot{E}^r \boldsymbol{Y} + \dot{E}^{(1)}\boldsymbol{\Psi} + \dot{E}^{(2)}\boldsymbol{\Phi})$$

$$+ \frac{4\pi\mu}{c}(J^r \boldsymbol{Y} + J^{(1)}\boldsymbol{\Psi} + J^{(2)}\boldsymbol{\Phi}). \tag{2.5}$$

The expressions for the 'electric' or 'even' or 'transverse magnetic' (TM) modes are

$$\frac{\partial}{\partial r}(r^2 E^r) = l(l+1)rE^{(1)} + \frac{4\pi}{\varepsilon}r^2\rho \tag{2.6}$$

$$\frac{1}{c}\frac{\partial B^{(2)}}{\partial t} = \frac{E^r}{r} - \frac{1}{r}\frac{\partial}{\partial r}(rE^{(1)}) \tag{2.7}$$

$$\frac{l(l+1)B^{(2)}}{r} = \frac{-\mu\varepsilon}{c}\frac{\partial E^r}{\partial t} - \frac{4\pi\mu}{c}J^r \tag{2.8}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB^{(2)}) = \frac{-\mu\varepsilon}{c}\frac{\partial E^{(1)}}{\partial t} - \frac{4\pi\mu}{c}J^{(1)}. \tag{2.9}$$

From the last four expressions above, wave equations can be deduced. For example, in regions where $\boldsymbol{J} = \boldsymbol{0}$ and $\rho = 0$ we have:

$$\frac{\partial^2}{\partial r^2}(r^2 E^r) - \frac{\mu\varepsilon}{c^2}r^2\frac{\partial^2 E^r}{\partial t^2} - l(l+1)E^r = 0 \tag{2.10}$$

$$\frac{\partial^2}{\partial r^2}(rB^{(2)}) - \frac{\mu\varepsilon}{c^2}r\frac{\partial^2 B^{(2)}}{\partial t^2} - \frac{l(l+1)}{r}B^{(2)} = 0. \tag{2.11}$$

The expression for the 'magnetic', 'odd' or 'transverse electric' (TE) modes are

$$\frac{\partial}{\partial r}(r^2 B^r) = l(l+1)rB^{(1)} \tag{2.12}$$

$$\frac{\mu\varepsilon}{c}\frac{\partial E^{(2)}}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}(rB^{(1)}) - \frac{B^r}{r} - \frac{4\pi\mu}{c}J^{(2)} \tag{2.13}$$

$$\frac{l(l+1)}{r}E^{(2)} = \frac{1}{c}\frac{\partial B^r}{\partial t} \tag{2.14}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE^{(2)}) = \frac{1}{c}\frac{\partial B^{(1)}}{\partial t}. \tag{2.15}$$

The expressions for the wave equations corresponding to the TE modes are:

$$\frac{\partial^2}{\partial r^2}(r^2 B^r) - \frac{\mu\varepsilon}{c^2}r^2\frac{\partial^2 B^r}{\partial t^2} - l(l+1)B^r = 0 \tag{2.16}$$

$$\frac{\partial^2}{\partial r^2}(rE^{(2)}) - \frac{\mu\varepsilon}{c^2}r\frac{\partial^2 E^{(2)}}{\partial t^2} - \frac{l(l+1)}{r}E^{(2)} = 0. \tag{2.17}$$

3. Multipole expansion of the magnetostatic and electrostatic fields

The magnetic flux density, $\boldsymbol{B}(\boldsymbol{r})$, at points outside a steady localized current density distribution, $\boldsymbol{J}(\boldsymbol{r})$, is calculated in this section. Expressed as an expansion in VHS we have

$$\boldsymbol{B}(\boldsymbol{r}) = \sum\sum(B^r_{lm}\boldsymbol{Y}_{lm} + B^{(1)}_{lm}\boldsymbol{\Psi}_{lm} + B^{(2)}_{lm}\boldsymbol{\Phi}_{lm}) \quad (3.1)$$

The desired result hinges on a theorem due to Gray (1978a, b) that states that the field $\boldsymbol{B}(\boldsymbol{r})$ can be determined from its radial component at a source-free region. This theorem can be readily demonstrated employing the multipole decomposition of the Maxwell equations presented in section 2. Using equations (2.3) and (2.8), we have:

$$B^{(1)}_{lm} = \frac{1}{l(l+1)r}\frac{\partial}{\partial r}(r^2 B^r_{lm}) \quad (3.2)$$

$$B^{(2)}_{lm} = 0 \quad (3.3)$$

The vanishing of the coefficients $B^{(2)}_{lm}$ is to be expected since in a region outside the sources $\nabla \times \boldsymbol{B} = \boldsymbol{0}$ and thus $\nabla \times (B^{(2)}_{lm}\boldsymbol{\Phi}_{lm})$ can be written as a linear combination of $\boldsymbol{Y}_{lm}$ and $\boldsymbol{\Psi}_{lm}$, whereas $\nabla \times (B^r_{lm}\boldsymbol{Y}_{lm})$ and $\nabla \times (B^{(1)}_{lm}\boldsymbol{\Psi}_{lm})$ are proportional to $\boldsymbol{\Phi}_{lm}$ (see equations (3.12*a*), (3.12*b*) and (3.12*c*) of the paper by Barrera *et al* (1985)).

The initial problem has been thus reduced to the computation of the coefficients B^r_{lm}; such a calculation can be easily performed employing equation (5) of Gray (1978a):

$$\nabla^2(\boldsymbol{r}\cdot\boldsymbol{B}) = -(4\pi/c)\boldsymbol{r}\cdot\nabla\times\boldsymbol{J}. \quad (3.4)$$

Equation (3.4) can be readily solved by analogy with $\nabla^2\Phi_E = -4\pi\rho$. The result is:

$$\boldsymbol{r}\cdot\boldsymbol{B} = \sum\sum rB^r_{lm}Y_{lm}(\theta,\phi) = \sum\sum\frac{4\pi}{2l+1}\frac{A_{lm}}{r^{l+1}}Y_{lm}(\theta,\phi). \quad (3.5)$$

The set of numbers A_{lm} in equation (3.5) depends on $J^{(2)}_{lm}$ and are $(l+1)$ times the expression for the magnetic multipole moments, M_{lm}. In fact,

$$A_{lm} = \frac{1}{c}\int\boldsymbol{r}\cdot(\nabla\times\boldsymbol{J})Y^*_{lm}(\theta,\phi)r^l\,\mathrm{d}^3x$$
$$= -\frac{l(l+1)}{c}\int r^{l+2}J^{(2)}_{lm}(r)\,\mathrm{d}r \quad (3.6)$$

(see equations (3.12) and (4.11) of the paper by Barrera *et al* (1985)). The coefficients B^r_{lm} can thus be written

$$B^r_{lm} = \frac{4\pi(l+1)}{2l+1}\frac{M_{lm}}{r^{l+2}} \quad (3.7)$$

From equations (3.7) and (3.2) we get

$$B^{(1)}_{lm} = -\frac{4\pi}{2l+1}\frac{M_{lm}}{r^{l+2}} = -\frac{1}{l+1}B^r_{lm}. \quad (3.8)$$

Substitution of equations (3.3), (3.7) and (3.8) into equation (3.1) yields for $\boldsymbol{B}(\boldsymbol{r})$ the compact expression

$$\boldsymbol{B}(\boldsymbol{r}) = \sum\sum\frac{4\pi M_{lm}}{(2l+1)r^{l+2}}[(l+1)\boldsymbol{Y}_{lm} - \boldsymbol{\Psi}_{lm}]. \quad (3.9)$$

The VSH expansion corresponding to the electric field, $\boldsymbol{E}(\boldsymbol{r})$, due to a localized charge density distribution, $\rho(\boldsymbol{r})$, in a charge free region can be similarly shown to be

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\Phi_E = -\sum\sum\frac{4\pi}{2l+1}q_{lm}\nabla\left(\frac{Y_{lm}}{r^{l+1}}\right)$$
$$= \sum\sum\frac{4\pi q_{lm}}{(2l+1)r^{l+2}}[(l+1)\boldsymbol{Y}_{lm} - \boldsymbol{\Psi}_{lm}] \quad (3.10)$$

where the quantities q_{lm} are the multipole moments. This expression, strikingly similar to equation (3.9), shows that the vector spherical harmonics $\boldsymbol{\Phi}_{lm}$ are characteristic of the multipole expansion of the dynamic fields. We phrase this conclusion in a slightly different manner: the static fields ($\boldsymbol{E}(\boldsymbol{r})$ or $\boldsymbol{B}(\boldsymbol{r})$) can always be written as linear combination of the $\boldsymbol{Y}_{lm}$ and $\boldsymbol{\Psi}_{lm}$. The above expansion of the field allows one to separate the radial component of the fields from their $\hat{\boldsymbol{e}}_\theta$ and $\hat{\boldsymbol{e}}_\phi$ components.

4. Radiation multipoles

This section establishes formulae for the fields outside the sources. Expressions for the radiation fields for the transverse magnetic and transverse electric modes, the time-averaged power radiated per unit solid angle, and the total power radiated *in terms* of the set of VSH will be derived.

The first equation in each set of table 1 serves the purpose of defining the electric and magnetic multipole coefficients α_{lm} and β_{lm} respectively. In the remaining entries are defined the coefficients of the decomposition of the electric field and the magnetic induction for electric (l, m) multipoles and magnetic (l, m) multipoles. The first term within square brackets in the integrand of the expression for α_{lm} in table 1 is the dominant term in the long wavelength limit. In table 1 the quantity $j_l(kr)$ is the spherical Bessel function, regular at the origin; $h^{(1)}_l$ is the spherical Hankel function of the first kind.

For fields outside the sources, the radial functions f_l and g_l will be taken to be $h^{(1)}_l$ in equation (16.46) of the text by Jackson (1975). This equation then becomes:

TM (Electric)

$$\boldsymbol{B}_{lm} = a_{\mathrm{E}}(l,m)h^{(1)}_l\boldsymbol{X}_{lm}$$

$$\boldsymbol{E}_{lm} = \frac{\mathrm{i}}{k}a_{\mathrm{E}}(l,m)\nabla\times h^{(1)}_l\boldsymbol{X}_{lm} \quad (4.1a)$$

Table 1. Definition of all VSH coefficients needed in the study of the TE and TM modes. The quantities α_{lm} and β_{lm} designate the electric and magnetic multipole coefficients, respectively.

TM ('electric') mode	TE ('magnetic') mode
$B_{lm}^{(2)} = \alpha_{lm} h_l^{(1)}(kr)$	$E_{lm}^{(2)} = \beta_{lm} h_l^{(1)}(kr)$
$E_{lm}^{(r)} = \frac{\alpha_{lm}}{\mathrm{i}k}\frac{l(l+1)}{r} h_l^{(1)}(kr)$	$B_{lm}^{(r)} = \frac{\mathrm{i}\beta_{lm}}{k}\frac{l(l+1)}{r} h_l^{(1)}(kr)$
$E_{lm}^{(1)} = \frac{\alpha_{lm}}{\mathrm{i}k}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rh_l(1))$	$B_{lm}^{(1)} = \frac{\mathrm{i}\beta_{lm}}{kr}\frac{\mathrm{d}}{\mathrm{d}r}(rh_l^{(1)})$
$\alpha_{lm} = -\frac{4\pi k^2}{l(l+1)}\int \mathrm{d}r\left(r^2\rho_{lm}\frac{\mathrm{d}}{\mathrm{d}r}(rj_l) + \frac{\mathrm{i}kr^2}{c}J_{lm}^r rj_l\right)$	$\beta_{lm} = -\frac{4\pi k^2}{c}\int \mathrm{d}r J_{lm}^{(2)}(r) r^2 j_l(kr)$

TE (Magnetic)

$$\boldsymbol{E}_{lm} = a_{\mathrm{M}}(l,m) h_l^{(1)} \boldsymbol{X}_{lm}$$

$$\boldsymbol{B}_{lm} = -\frac{\mathrm{i}}{k} a_{\mathrm{M}}(l,m) \nabla \times h_l^{(1)} \boldsymbol{X}_{lm}. \tag{4.1b}$$

In these expressions the quantity $\boldsymbol{X}_{lm}$ is the vector spherical harmonic defined in the text of Jackson (1975, sections 16.1 and 16.2). The definitions of $\boldsymbol{X}_{lm}$ and $\boldsymbol{\Phi}_{lm}$ permit us to write

$$\boldsymbol{X}_{lm} = \frac{\boldsymbol{r} \times \nabla Y_{lm}}{\mathrm{i}\sqrt{l(l+1)}} = \frac{-\mathrm{i}\boldsymbol{\Phi}_{lm}}{\sqrt{l(l+1)}}. \tag{4.2}$$

Employing this relationship together with the definition given in table 1 and equation (3.12c) of Barrera *et al* (1985), i.e.

$$\nabla \times h_l^{(1)}\boldsymbol{\Phi} = -\frac{l(l+1)}{r} h_l^{(1)} Y_{lm} - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rh_l^{(1)})\boldsymbol{\Psi}_{lm} \tag{4.3}$$

it can be concluded that the two sets of equations (4.1) are consistent if the coefficients $a_{\mathrm{E}}(l,m)$ and $a_{\mathrm{M}}(l,m)$ satisfy the conditions,

$$a_{\mathrm{E}}(l,m) = \mathrm{i}\sqrt{l(l+1)}\alpha_{lm}$$

$$a_{\mathrm{M}}(l,m) = \mathrm{i}\sqrt{l(l+1)}\beta_{lm}. \tag{4.4}$$

Taking $kr \gg 1$ so that one can use the large kr behaviour of $h_l^{(1)}$, that is, $h_l^{(1)}(kr \to \infty) \to (-\mathrm{i})^{l+1}\mathrm{e}^{ikr}/(kr)$, and ignoring fields that fall off faster than $1/r$, one arrives at the following expressions for the radiation fields:

TM (Electric)

$$\boldsymbol{B}_{lm} = \alpha_{lm}(-\mathrm{i})^{l+1}\frac{\mathrm{e}^{\mathrm{i}(kr-wt)}}{kr}\boldsymbol{\Phi}_{lm} = \hat{\boldsymbol{e}}_r \times \boldsymbol{E}_{lm}$$

$$\boldsymbol{E}_{lm} = \beta_{lm}(-\mathrm{i})^{l+1}\frac{\mathrm{e}^{\mathrm{i}(kr-wt)}}{kr}\boldsymbol{\Phi}_{lm} = \boldsymbol{B}_{lm} \times \hat{\boldsymbol{e}}_r \tag{4.5a}$$

TE (Magnetic)

$$\boldsymbol{E}_{lm} = \alpha_{lm}(-\mathrm{i})^{l+1}\frac{\mathrm{e}^{\mathrm{i}(kr-wt)}}{kr}\boldsymbol{\Psi}_{lm} = \boldsymbol{B}_{lm} \times \hat{\boldsymbol{e}}_r$$

$$\boldsymbol{B}_{lm} = -\beta_{lm}(-\mathrm{i})^{l+1}\frac{\mathrm{e}^{\mathrm{i}(kr-wt)}}{kr}\boldsymbol{\Psi}_{lm} = \hat{\boldsymbol{e}}_r \times \boldsymbol{E}_{lm}. \tag{4.5b}$$

Note the presence in the expressions of the time-harmonic factor $\mathrm{e}^{-\mathrm{i}wt}$. The power radiated can be readily obtained from the expression for the real part of the complex Poynting vector:

$$\langle \boldsymbol{S} \rangle = \frac{c}{8\pi}\mathrm{Re}(\boldsymbol{E} \times \boldsymbol{B}^*). \tag{4.6}$$

Consider first the TM mode with single (l,m). We have

$$\langle \boldsymbol{S} \rangle = \frac{c}{8\pi}\left|\frac{\alpha_{lm}}{kr}\right|^2 (\boldsymbol{\Psi}_{lm} \times \boldsymbol{\Phi}_{lm}^*) = \frac{c}{8\pi}\left|\frac{\alpha_{lm}}{kr}\right|^2 |\boldsymbol{\Psi}_{lm}|^2 \hat{\boldsymbol{e}}_r \tag{4.7}$$

For the TE mode with single (l,m) we have

$$\langle \boldsymbol{S} \rangle = \frac{c}{8\pi}\left|\frac{\beta_{lm}}{kr}\right|^2 [\boldsymbol{\Phi} \times (-\boldsymbol{\Psi}_{lm}^*)] = \frac{c}{8\pi}\left|\frac{\beta_{lm}}{kr}\right|^2 |\boldsymbol{\Phi}_{lm}|^2 \hat{\boldsymbol{e}}_r. \tag{4.8}$$

The quantities $|\boldsymbol{\Psi}_{lm}|^2$ and $|\boldsymbol{\Phi}_{lm}|^2$ in equations (4.7) and (4.8) indicate the polarization.

For a pure multipole of order (l,m), i.e. for a single (l,m) mode, the time-averaged power radiated per unit solid angle is

$$\frac{\mathrm{d}P(l,m)}{\mathrm{d}\Omega} = r^2\hat{\boldsymbol{e}}_r \cdot \langle \boldsymbol{S} \rangle = \frac{c}{8\pi k^2}\begin{cases} |\alpha_{lm}|^2 |\boldsymbol{\Psi}_{lm}|^2 \\ |\beta_{lm}|^2 |\boldsymbol{\Phi}_{lm}|^2 \end{cases}. \tag{4.9}$$

Since $|\boldsymbol{\Psi}_{lm}|^2 = |\boldsymbol{\Phi}_{lm}|^2$ then the angular distributions of the multipole radiations are given by $|\boldsymbol{\Psi}_{lm}|^2$. Two examples are listed in table 2.

The total power radiated by a multipole of order (l,m) is obtained by integrating over $\mathrm{d}\Omega$ equation

Table 2. Angular distribution $|\mathbf{\Psi}_{lm}(\theta,\phi)|^2$ to quadrupole order.

l	m = 0	± 1	± 2
1 'Dipole'	$\sin^2\theta$	$1+\cos^2\theta$	
2 'Quadrupole'	$\sin^2\theta\cos^2\theta$	$1-3\cos^2\theta+4\cos^4\theta$	$1-\cos^4\theta$

(4.9). Thus when there is a single mode we have:

$$P(l,m)=\frac{c}{8\pi k^2}\begin{Bmatrix}|\alpha_{lm}|^2\\|\beta_{lm}|^2\end{Bmatrix}\int d\Omega|\mathbf{\Psi}_{lm}|^2.$$

$$P(l,m)=\frac{cl(l+1)}{8\pi k^2}\begin{Bmatrix}|\alpha_{lm}|^2\\|\beta_{lm}|^2\end{Bmatrix}\tag{4.10}$$

Since it will be needed in the evaluation of the power we state for the moment without proof that unless $l=l'$ and $m=m'$ then

$$\int d\Omega\,\mathbf{\Psi}_{lm}\cdot\mathbf{\Psi}^*_{l'm'}=0\tag{4.11a}$$

$$\int d\Omega\,\mathbf{\Phi}_{lm}\cdot\mathbf{\Phi}^*_{l'm'}=0\tag{4.11b}$$

Furthermore for all l,l',m,m'

$$\int d\Omega\,\mathbf{\Phi}_{lm}\cdot\mathbf{\Psi}^*_{l'm'}=0.\tag{4.11c}$$

The proofs of equations (4.11*a*) and (4.11*c*) are somewhat obtuse. Outlines of these as well as that of equation (4.11*b*) are given in the appendix. Thus if many modes are excited, the total power radiated, P, is the sum of the power in each mode and we can write

$$P=\frac{cl(l+1)}{8\pi k^2}\sum_l\sum_m\{|\alpha_{lm}|^2+|\beta_{lm}|^2\}.\tag{4.12}$$

5. Illustrative examples

As a first example we will calculate the angular distribution of radiation, $dP/d\Omega$, and the total power P radiated by an almost spherical volume possessing axial symmetry, bounded by a surface whose equation in spherical polar coordinates is

$$R(\theta)=R_0[1+\eta P_2(\cos\theta)]\tag{5.1}$$

and characterized by a constant charge density ρ. The dimensionless parameter η is very small ($\eta^2 << 1$) and varies harmonically in time at frequency w. We designate by Q the total charge of the radiating system. The volume V is

$$V=2\pi\int_0^\pi\sin\theta\,d\theta\left(\int_0^{R_0}r^2\,dr\right)=\frac{4\pi R_0^3}{3}+O(\eta^2).\tag{5.2}$$

Neglecting squares of η the charge density is

$$\rho=\frac{Q}{V}\cong\frac{3Q}{4\pi R_0^3}.\tag{5.3}$$

The angular distribution of radiation is given by equation (4.9). We need to find the electric multipole moments q_{lm}. These radiating moments are given by the expression

$$q_{lm}=\int\rho r^l Y^*_{lm}(\theta,\phi)\,d^3x.\tag{5.4}$$

The volume of integration is broken up into the domain of a sphere of radius R_0 and the domain of a thin surface layer, at $r=R_0$, of radial extent $\Delta R=R_0\eta P_2(\cos\theta)$. Thus

$$q_{lm}=\int_{\text{sphere}}\rho r^l Y^*_{lm}\,d^3x+\rho R_0^l R_0^2\int\Delta R\,Y^*_{lm}\,d\Omega.\tag{5.5}$$

The first integral on the right-hand side of equation (5.5) vanishes except for $l=0$. The second integral yields $R_0\eta(4\pi/5)^{1/2}$ if $(l,m)=(2,0)$, and vanishes otherwise. The only non-zero electric multipole moment (besides monopole) is thus

$$q_{20}=\left(\frac{4\pi}{5}\right)^{1/2}\rho\eta R_0^5=\frac{3\eta Q}{4\pi}\left(\frac{4\pi}{5}\right)^{1/2}R_0^2.\tag{5.6}$$

It can be readily verified that (k is the wavenumber)

$$\alpha_{20}=-\frac{Qk^4}{10}\left(\frac{4\pi}{5}\right)^{1/2}\eta R_0^2.\tag{5.7}$$

Since $l=2$, $m=0$ the angular distribution of radiation is thus

$$\frac{dP}{d\Omega}=\frac{c}{8\pi k^2}|\alpha_{20}|^2|\mathbf{\Psi}_{20}|^2\tag{5.8}$$

where $\mathbf{\Psi}_{20}=r\nabla Y_{20}(\theta,\phi)$. We thus have

$$\frac{dP}{d\Omega}=\frac{9ck^6Q^2R_0^4}{800\pi}|\eta|^2\sin^2\theta\cos^2\theta\tag{5.9}$$

The total power radiated, P, is obtained by integrating the last expression above over the whole range of angles θ,ϕ. The result is

$$P=\frac{3ck^6Q^2|\eta|^2R_0^4}{500}\tag{5.10}$$

We now consider a more complicated situation, namely, the scattering of a circularly polarized plane wave of radiation of frequency $\omega=ck$ by a non-magnetic, conducting sphere of radius R. Assuming that the conductivity of the sphere is infinite we propose to find expressions for the total external electric and magnetic fields in the neighbourhood of the sphere in the long-wavelength limit, $kR \ll 1$.

The problem considered here is cylindrically symmetric therefore $m = 1$. The *general* expressions for the time-independent parts of the incident and scattered field vectors in the case of positive incident helicity are given by the following four expansions:

$$\boldsymbol{E}_{\text{inc}} = \sum_{l=1}^{\infty} \mathrm{i}^l \left(\frac{4\pi(2l+1)}{l(l+1)}\right)^{1/2} \Big[-ij_l(kr)\boldsymbol{\Phi}_{l1} - \frac{1}{ikr}\Big(l(l+1)j_l \boldsymbol{Y}_{l1} + \frac{\mathrm{d}}{\mathrm{d}r}(rj_l)\boldsymbol{\Psi}_{l1}\Big)\Big] = (\hat{\boldsymbol{e}}_x + \mathrm{i}\hat{\boldsymbol{e}}_y)\mathrm{e}^{ikz} \tag{5.11}$$

$$\boldsymbol{B}_{\text{inc}} = \sum_{l=1}^{\infty} \mathrm{i}^l \left(\frac{4\pi(2l+1)}{l(l+1)}\right)^{1/2} \Big[-j_l(kr)\boldsymbol{\Phi}_{l1} + \frac{1}{ikr}\Big(l(l+1)j_l \boldsymbol{Y}_{l1} + \frac{\mathrm{d}}{\mathrm{d}r}(rj_l)\boldsymbol{\Psi}_{l1}\Big)\Big] \tag{5.12}$$

$$\boldsymbol{E}_{\text{scat}} = \sum_{l=1}^{\infty} \mathrm{i}^l \left(\frac{4\pi(2l+1)}{l(l+1)}\right)^{1/2} \Big[\mathrm{e}^{\mathrm{i}\delta_l} h_l^{(1)} \sin\delta_l \boldsymbol{\Phi}_{l1} - \frac{\mathrm{e}^{\mathrm{i}\delta'_l}}{kr}\sin\delta'_l\Big(l(l+1)h_l^{(1)} \boldsymbol{Y}_{l1} + \frac{\mathrm{d}}{\mathrm{d}r}(rh_l^{(1)})\boldsymbol{\Psi}_{l1}\Big)\Big] \tag{5.13}$$

$$\boldsymbol{B}_{\text{scat}} = \sum_{l=1}^{\infty} \mathrm{i}^l \left(\frac{4\pi(2l+1)}{l(l+1)}\right)^{1/2} \Big[-\mathrm{i}\mathrm{e}^{\mathrm{i}\delta'_l} h_l^{(1)} \sin\delta'_l \boldsymbol{\Phi}_{l1} + \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\delta l}}{kr}\sin\delta_l\Big(l(l+1)h_l^{(1)} \boldsymbol{Y}_{l1} + \frac{\mathrm{d}}{\mathrm{d}r}(rh_l^{(1)})\boldsymbol{\Psi}_{l1}\Big)\Big] \tag{5.14}$$

where the phase angles δ_l and δ'_l are defined by the equations:

$$\mathrm{e}^{2\mathrm{i}\delta l} = -\frac{h_l^{(2)}(kR)}{h_l^{(1)}(kR)}$$

$$\mathrm{e}^{2\mathrm{i}\delta' l} = -\frac{\mathrm{d}(xh_l^{(2)})/\mathrm{d}x}{\mathrm{d}(xh_l^{(1)})/\mathrm{d}x}\bigg|_{x=kR} \tag{5.15}$$

When the radius of the sphere is much smaller than the wavelength, i.e. in the long-wavelength limit, $kR \ll 1$, the following approximations may be used for the scattering phaseshifts δ_l and δ'_l due to the conducting sphere:

$$\delta_l \simeq -\frac{(kR)^{2l+1}}{(2l+1)!!(2l-1)!!}$$

$$\delta'_l \simeq -\frac{(l+1)\delta_l}{l} \tag{5.16}$$

where the double factorial is defined by $(2l+1)!! = 1.3.5.7\ldots(2l+1)$.

In the neighbourhood of the sphere we have the particularly simply forms

$$j_l(kr) \simeq \frac{(kr)^l}{(2l+1)!!} \qquad \frac{\mathrm{d}(rj_l)}{dr} \simeq \frac{(l+1)(kr)}{(2l+1)!!} \tag{5.17}$$

$$h_l^{(1)}(kr) \simeq \frac{-i(2l-1)!!}{(kr)^{l+1}} \qquad \frac{\mathrm{d}(rh_l^{(1)})}{\mathrm{d}r} \simeq \frac{\mathrm{i}l(2l-1)!!}{(kr)^{l+1}}. \tag{5.18}$$

Clearly the dipole $(l = 1)$ term dominates. Note that $h_l \sin\delta_l \simeq (kR)^{2l+1}/(kr)^{l+1}$, so that the scattering phaseshifts tend to become quite small with increasing values of l faster than the h_l get large. If we limit ourselves only to the dominant near field $l = 1$ terms we obtain for the spatial dependence the expressions:

$$\boldsymbol{E}_{\text{inc}} = \mathrm{i}\sqrt{6\pi}\Big[\frac{-ikr}{3}\boldsymbol{\Phi}_{11} - \frac{1}{ikr}\Big(\frac{2kr}{3}\boldsymbol{Y}_{11} + \frac{2kr}{3}\boldsymbol{\Psi}_{11}\Big)\Big] \tag{5.19}$$

$$\boldsymbol{B}_{\text{inc}} = \mathrm{i}\sqrt{6\pi}\Big[\frac{-ikr}{3}\boldsymbol{\Phi}_{11} + \frac{1}{ikr}\Big(\frac{2kr}{3}\boldsymbol{Y}_{11} + \frac{2kr}{3}\boldsymbol{Y}_{11}\Big)\Big] \tag{5.20}$$

$$\boldsymbol{E}_{\text{scat}} = \mathrm{i}\sqrt{6\pi}\Big[\delta_1\Big(\frac{-\mathrm{i}}{k^2r^2}\Big)\boldsymbol{\Phi}_{11} - \frac{\delta'_l}{kr}\Big(\frac{-2\mathrm{i}\boldsymbol{Y}_{11}}{k^2r^2} + \frac{\mathrm{i}}{k^2r^2}\boldsymbol{\Psi}_{11}\Big)\Big] \tag{5.21}$$

$$\boldsymbol{B}_{\text{scat}} = \mathrm{i}\sqrt{6\pi}\Big[-\mathrm{i}\delta'_1\Big(-\frac{\mathrm{i}}{k^2r^2}\Big)\boldsymbol{\Phi}_{11} + \frac{\mathrm{i}}{kr}\delta_1\Big(\frac{-2\mathrm{i}}{k^2r^2}\boldsymbol{Y}_{11} + \frac{\mathrm{i}}{k^2r^2}\boldsymbol{\Psi}_{11}\Big)\Big] \tag{5.22}$$

where $\delta_1 = -(kR)^3/3$ and $\delta'_1 = 2(kR)^3/3$.

The total near fields are the sum of the plane wave and the scattered fields:

$$\boldsymbol{E} \simeq -\sqrt{\frac{8\pi}{3}}\Big[\Big(1 - \frac{R^3}{r^3}\Big)\boldsymbol{\Psi}_{11} + \Big(1 + \frac{2R^3}{r^3}\Big)\boldsymbol{Y}_{11} - \frac{kr}{2}\Big(1 - \frac{R^3}{r^3}\Big)\boldsymbol{\Phi}_{11}\Big] \tag{5.23}$$

$$\boldsymbol{B} \simeq \mathrm{i}\sqrt{\frac{8\pi}{3}}\Big[\Big(1 + \frac{R^3}{2r^3}\Big)\boldsymbol{\Psi}_{11} + \Big(1 - \frac{R^3}{r^3}\Big)\boldsymbol{Y}_{11} - \frac{kr}{2}\Big(1 + \frac{2R^2}{3r^2}\Big)\boldsymbol{\Phi}_{11}\Big]. \tag{5.24}$$

The first two terms in the expressions for $\boldsymbol{E}$ and $\boldsymbol{B}$ are the correct fields to the lowest order in the wavenumber k. The first order in k terms of $\boldsymbol{E}$ and $\boldsymbol{B}$ arising from the case $l = 1$ can be dropped because there are other terms of first order in k arising from $l = 2$ in the multipole expansion and these have already been dropped. Note that $\boldsymbol{E}_{\tan}$ and $\boldsymbol{B}_{\perp}$ vanish on the surface of the sphere, as expected for perfect conductivity.

A *numerical* solution of the specific class of problems in which the boundary of a perfectly conducting body is a surface of revolution and the incident wave is a plane wave propagating along the axis of symmetry has recently been reported (Aziz *et al* 1982).

6. Concluding remarks

The second example presented in section 5 dealt with the scattering of a circularly polarized plane wave by a non-magnetic, conducting sphere. This example of Rayleigh scattering is pedagogically important as it paves the way for student comprehension of its quantum mechanical structural analogue, i.e. the scattering of a beam of low-energy monoenergetic particles (represented by a plane wave of infinite extent in space) by a spherically symmetric infinitely repulsive potential (Flügge 1974). Other useful applications with the set of vector spherical harmonics advocated here include solutions to problems dealing with oscillatory dipoles in vacuum and in the study of the radiation problems involving differential and quarter-wave antenna problems. The ideas and formulae considered in this article are also useful to study slowly moving bounded charge distribution problems. Furthermore, the VSH are useful in the deduction of fields set up by localized sources of charge and current.

In conclusion we have demonstrated that expressing the electromagnetic fields in terms of the VSH formulation results in a description that is compact and elegant. Furthermore, when applied in a straightforward manner the VSH formulation usually decreases the difficulty of solving problems of classical electrodynamics. These features should render this article especially useful to undergraduate students of physics and engineering.

Acknowledgements

Dedicated to J David Jackson in admiration and affection on his sixty fifth birthday. We wish to acknowledge discussions, some old, some new, with V C Aguilera-Navarro, R G Barrera, L B Bhuiyan, F M Fernández, J Giraldo, Wilfrido Solano and Galileo Violini. We thank the anonymous referees for their exceptionally helpful comments.

Appendix

This appendix verifies the orthogonality properties (4.11). Since it will be needed in the demonstration of equation (4.11*a*) we show here that

$$\int \mathrm{d}\Omega\, \nabla \cdot (Y_{lm} \nabla Y^*_{l'm'}) = 0.$$

This is equivalent to showing that

$$\int \mathrm{d}\Omega\, r\nabla \cdot (Y_{lm} \nabla Y^*_{l'm'}) = \int \mathrm{d}\Omega\, \nabla \cdot (Y_{lm} r \nabla Y^*_{l'm'}) = 0$$

If we define a tangential vector $\boldsymbol{F}$ as $\boldsymbol{F} = Y_{lm} r \nabla Y^*_{l'm'}$, we then show that $\int \mathrm{d}\Omega\, \nabla \cdot \boldsymbol{F} = 0$. Note that $\boldsymbol{F} \cdot \hat{\boldsymbol{e}}_r = 0$. Furthermore, $\boldsymbol{F}$ has no r dependence. Let us denote by V the volume of a sphere of radius R, and take $a < R$. Denoting by $\Theta(x) = \frac{1}{2}[1 + \mathrm{sgn}(x)]$ the unit step function we have by the divergence theorem

$$\int \mathrm{d}^3x\, \nabla \cdot [\Theta(a - r)\boldsymbol{F}] = 0.$$

This vanishing integral can be rewritten as follows

$$\int_V \mathrm{d}^3x\, \nabla \cdot [\Theta(a - r)\boldsymbol{F}] = \int_V \mathrm{d}^3x\, \Theta(a - r) \nabla \cdot \boldsymbol{F}$$

$$- \int_V \mathrm{d}^3x\, \delta(a - r) \hat{\boldsymbol{e}}_r \cdot \boldsymbol{F}.$$

Since $\mathrm{d}^3x \equiv r^2 \mathrm{d}r\, \mathrm{d}\Omega$, and $\hat{\boldsymbol{e}}_r \cdot \boldsymbol{F} = 0$ then

$$0 = \int_0^R \Theta(a - r) r^2 \mathrm{d}r \int \mathrm{d}\Omega\, \nabla \cdot \boldsymbol{F} = \frac{a^3}{3} \int \mathrm{d}\Omega\, \nabla \cdot \boldsymbol{F}$$

hence $\int \mathrm{d}\Omega\, \nabla \cdot \boldsymbol{F} = 0$.

With these preliminaries, we are ready to show equation (4.11*a*)

$$\int \mathrm{d}\Omega\, \boldsymbol{\Psi}_{lm} \cdot \boldsymbol{\Psi}^*_{l'm'} = r^2 \int \mathrm{d}\Omega\, \nabla Y_{lm} \cdot \nabla Y^*_{l'm'}$$

$$= r^2 \left(\int \mathrm{d}\Omega\, \nabla \cdot (Y_{lm} \nabla Y^*_{l'm'}) - \int \mathrm{d}\Omega\, Y_{lm} \nabla^2 Y^*_{l'm'} \right).$$

Since the first integral has been shown to vanish then

$$\int \mathrm{d}\Omega\, \boldsymbol{\Psi}_{lm} \cdot \boldsymbol{\Psi}^*_{l'm'} = l'(l' + 1) \oint \mathrm{d}\Omega\, Y_{lm} Y^*_{l'm'}$$

$$= l(l + 1) \delta_{ll'} \delta_{mm'}.$$

In the proof of equation (4.11*b*) the following product is needed:

$$\boldsymbol{\Phi}_{lm} \cdot \boldsymbol{\Phi}^*_{l'm'} = (\boldsymbol{r} \times \nabla Y_{lm}) \cdot (\boldsymbol{r} \times \nabla Y^*_{l'm'})$$

$$= \nabla Y^*_{l'm'} \cdot [(\boldsymbol{r} \times \nabla Y_{lm}) \times \boldsymbol{r}]$$

$$= \nabla Y^*_{l'm'} [r^2 \nabla Y_{lm} - \boldsymbol{r}(\boldsymbol{r} \cdot \nabla Y_{lm})].$$

Since $\boldsymbol{r} \cdot \nabla Y_{lm} = 0$ then $\boldsymbol{\Phi}_{lm} \cdot \boldsymbol{\Phi}^*_{l'm'} = \boldsymbol{\Psi}_{lm} \cdot \boldsymbol{\Psi}^*_{l'm'}$. From this result and equation (4.11*a*) equation (4.11*b*) follows at once.

The only orthogonality property that remains to be shown is thus equation (4.11*c*). Stokes' theorem establishes that for any vector field $\boldsymbol{A}$

$$\int_{\mathscr{S}} (\nabla \times \boldsymbol{A}) \cdot \hat{\boldsymbol{n}}\, \mathrm{d}a = \oint_{\mathscr{C}} \boldsymbol{A} \cdot \mathbf{d}\boldsymbol{l}$$

where $\mathscr{C}$ is the boundary to $\mathscr{S}$. If we take $\mathscr{S}$ to be the surface of a sphere of radius R, the boundary of this closed surface has zero length, therefore, assuming that $\boldsymbol{A}$ is non-singular on the surface ($\mathrm{d}a \equiv R^2\,\mathrm{d}\Omega$)

$$\int_{\substack{\text{spherical}\\ \text{surface}}} R^2\,\mathrm{d}\Omega\,(\boldsymbol{\nabla}\times\boldsymbol{A})\cdot\hat{\boldsymbol{e}}_r = 0.$$

For any non-singular field $\boldsymbol{A}$ we thus have

$$\int \mathrm{d}\Omega\,\hat{\boldsymbol{e}}_r\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) = 0.$$

If we now define a vector $\boldsymbol{A}$ as $\boldsymbol{A} = Y_{lm}\boldsymbol{\nabla} Y^*_{l'm'}$, then $\boldsymbol{\nabla}\times\boldsymbol{A} = \boldsymbol{\nabla} Y_{lm}\times\boldsymbol{\nabla} Y^*_{l'm'}$, therefore

$$0 = \int \mathrm{d}\Omega\,\hat{\boldsymbol{e}}_r\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) = \int \mathrm{d}\Omega\,\hat{\boldsymbol{e}}_r(\boldsymbol{\nabla} Y_{lm}\times\boldsymbol{\nabla} Y^*_{l'm'})$$

The proof is concluded by simply observing that this last expression can also be written

$$0 = \int \mathrm{d}\Omega\,\boldsymbol{\nabla} Y^*_{l'm'}(\hat{\boldsymbol{e}}_r\times\boldsymbol{\nabla} Y_{lm}) = \frac{1}{r^2}\int \mathrm{d}\Omega\,\boldsymbol{\Psi}^*_{l'm'}\cdot\boldsymbol{\Phi}_{lm}.$$

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