A COUNTEREXAMPLE TO THE UNIT CONJECTURE FOR GROUP RINGS

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ABSTRACT. The unit conjecture, commonly attributed to Kaplansky, predicts that if K is a field and G is a torsion-free group then the only units of the group ring $K[G]$ are the trivial units, that is, the non-zero scalar multiples of group elements. We give a concrete counterexample to this conjecture; the group is virtually abelian and the field is order two.

1. INTRODUCTION

Three long-standing open problems on the group rings of torsion-free groups are commonly attributed to Kaplansky: the unit conjecture, the zero divisor conjecture and the idempotent conjecture. Let K be a field and G be a torsion-free group and consider the group ring $K[G]$. The unit conjecture states that every unit in $K[G]$ is of the form kg for $k \in K \setminus \{0\}$ and $g \in G$, the zero divisor conjecture states that $K[G]$ has no non-trivial zero divisors, and the idempotent conjecture states that $K[G]$ has no idempotents other than 0 and 1. The unit conjecture implies the zero divisor conjecture [Pas85, Lemma 13.1.2], which in turn implies the idempotent conjecture; these implications hold for each individual group ring $K[G]$.

In this paper we disprove the strongest of these three conjectures, namely the unit conjecture.

Theorem A. *Let P be the torsion-free group $\langle a, b \mid (a^2)^b = a^{-2}, (b^2)^a = b^{-2} \rangle$ and set $x = a^2, y = b^2, z = (ab)^2$. Set*

$$p = (1+x)(1+y)(1+z^{-1})$$
$$q = x^{-1}y^{-1} + x + y^{-1}z + z$$
$$r = 1 + x + y^{-1}z + xyz$$
$$s = 1 + (x + x^{-1} + y + y^{-1})z^{-1}.$$

Then $p + qa + rb + sab$ is a non-trivial unit in the group ring $\mathbb{F}_2 P$.

The unit conjecture and the zero divisor conjecture were formulated by Higman in his unpublished 1940 thesis [Hig40a, p. 77] (see also [San81, p. 112]), in which he proved

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the unit conjecture for locally indicable groups (the corresponding paper [Hig40b] was written earlier and omits the conjectures). The zero divisor conjecture appeared in print in the report of a 1956 talk of Kaplansky [Kap57, Problem 6]; for integral group rings it appears in the original 1965 Kourovka Notebook as a "well-known problem" [KM18, 1.3]. The unit conjecture was later posed alongside the zero divisor conjecture by Kaplansky [Kap70] having also been asked for integral group rings by Smirnov and Bovdi. Further context is provided in Section 2, where our focus on the group P is explained. Theorem A is proved in Section 3.

Remark. After the fact, Theorem A is of course readily verified using computer algebra. Since it admits a short human-readable proof, we present such a proof.

2. Background

The zero divisor conjecture and the idempotent conjecture have turned out to be susceptible to analytic and K-theoretic methods, despite having being posed with very little evidence; for example, the zero divisor conjecture holds for elementary amenable groups [KLM88, Theorem 1.4] and holds over $\mathbb{C}$ for groups satisfying the Atiyah conjecture (see [Lin93], [Lüc02, Lemma 10.39]), and the idempotent conjecture over $\mathbb{C}$ follows from either the Baum–Connes or Farrell–Jones conjecture (see [Val02, §6.3] and [BLR08, Theorem 1.12] respectively). In contrast, the unit conjecture has only been established as a consequence of the stronger combinatorial and purely group-theoretic property of having unique products. A group G is said to *have unique products* if for every choice of finite subsets $A, B \subset G$ the set $A \cdot B = \{ab : a \in A, B \in B\}$ contains some element uniquely expressible as ab for $a \in A, b \in B$. This implies the *a priori* stronger 'two unique products' property [Str80] and thus the unit conjecture.

The first example of a torsion-free group without unique products was constructed by Rips and Segev using small cancellation techniques [RS87]. Shortly thereafter, Promislow provided an elementary example [Pro88] in the (torsion-free) virtually abelian group P variously known as the Hantzsche–Wendt group, Passman group, Promislow group, or Fibonacci group $F(2,6)$. The group P is the unique 3-dimensional crystallographic group with finite abelianization; specifically, it is a non-split extension

$$1 \to \mathbb{Z}^3 \to P \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1.$$

In Promislow's example we have $A = B$ and $|A| = 14$.

Craven and Pappas attacked the question of whether group rings of P have non-trivial units, filtering potential units according to a complexity measure called length L that comes from the word length in the infinite dihedral quotient of P. They established the unit conjecture for P amongst elements of length $L \leq 3$ which as an application shows that Promislow's set does not support a non-trivial unit [CP13, Theorem 12.1]. We have to entertain the possibility that the unit conjecture is equivalent to unique products; if it is, and failure of unique products begets non-trivial units, then the aforementioned theorem shows that this cannot happen "locally" in the corresponding sense. Our counterexample has length $L = 4$ (after conjugation by a).

The zero divisor conjecture is known for P so this counterexample does not directly suggest a line of attack. If the zero divisor conjecture is true, then we have established that it is not true for the perhaps more combinatorial considerations of the unit conjecture. On the other hand, if the zero divisor conjecture is false, then this paper removes a serious psychological impediment to finding a counterexample.

3. The counterexample

3.1. **Setup.** In this paper we will work with the structure of P as an extension of $\mathbb{Z}^3$ by a $\mathbb{Z}/2 \times \mathbb{Z}/2$ quotient. In order to facilitate calculations, we will describe this extension very explicitly, including its defining cocycle (factor set) and action. One could perform this computation in various ways; our approach is flavoured with Bass–Serre theory.

We adopt the convention of conjugation acting on the right: $s^t := t^{-1}st$. We introduce new variables $x = a^2$ and $y = b^2$ into the presentation

$$\langle\, x, b, y, a \mid x^b = x^{-1},\, y^a = y^{-1},\, x = a^2,\, b^2 = y \,\rangle$$

and observe that this expresses P as an amalgam of two Klein bottle groups, namely $\langle\, x, b \mid x^b = x^{-1} \,\rangle$ and $\langle\, y, a \mid y^a = y^{-1} \,\rangle$, along their isomorphic index-2 $\mathbb{Z}^2$ subgroups $\langle x, b^2 \rangle = \langle a^2, y \rangle$. Being normal in each factor, this subgroup is normal in the amalgam, with corresponding quotient $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2$. We pick $z = abab$ as a lift to P of a generator of the infinite cyclic group $[D_\infty, D_\infty]$. As $x^z = (a^2)^{(abab)} = (a^{-2})^{(ab)} = x$ and similarly $y^z = y$, we see that in fact $\langle x, y, z \rangle \cong \mathbb{Z}^3$ is the kernel of $P \to \mathbb{Z}/2 \times \mathbb{Z}/2$.

Write Q for the quotient $\mathbb{Z}/2 \times \mathbb{Z}/2$. The action of P by conjugation on $\langle x, y, z \rangle$ induces an action of Q in which the non-trivial elements act as conjugation by a, b and ab. The action of a and b on $\langle x, y \rangle$ can be read off the presentation; for the action on z note that $z^{ab} = z$ and

$$z^a z = bab(a^2b)ab = bab(ba^{-2})ab = b(ab^2)a^{-1}b = b(b^{-2}a)a^{-1}b = 1.$$

Let us explicitly record the action for all 3 non-trivial elements of Q.

$$\begin{array}{lll} x^a = x & y^a = y^{-1} & z^a = z^{-1} \\ x^b = x^{-1} & y^b = y & z^b = z^{-1} \\ x^{ab} = x^{-1} & y^{ab} = y^{-1} & z^{ab} = z \end{array}$$

The set-theoretic section $\sigma\colon Q \to P$ with image $\{1, a, b, ab\}$ defines the cocycle $f\colon Q \times Q \to \mathbb{Z}^3$ by $f(g,h) = \sigma(g)\sigma(h)\sigma(gh)^{-1}$. In order to compute it we just need to know how to push an a past a b. One of the defining relations tells us that $b^{-1}a^2 = a^{-2}b^{-1}$ and thus

$$bab^{-1}a^{-1} = b^2(b^{-1}a^2)a^{-1}b^{-1}a^{-1} = b^2(a^{-2}b^{-1})a^{-1}b^{-1}a^{-1} = yx^{-1}z^{-1} = x^{-1}yz^{-1}.$$

With this identity in hand we determine

$f(g,h)$	1	a	b	ab
1	1	1	1	1
a	1	x	1	x
b	1	$x^{-1}yz^{-1}$	y	$x^{-1}z^{-1}$
ab	1	$y^{-1}z$	y^{-1}	z

where the table reads left-to-right (the rows give g and the columns give h).

3.2. **Proof of Theorem A.**

Proof. Let $\alpha = p + qa + rb + sab$. Since P is virtually abelian it certainly satisfies the zero divisors conjecture, so it suffices to check that α is left-invertible in order to show it is a unit (as $\alpha'\alpha = 1$ implies that $\alpha'(\alpha\alpha' - 1) = 0$). If $\alpha' = p' + q'a + r'b + s'ab$ for $p', q', r', s' \in \mathbb{F}_2[x^{\pm1}, y^{\pm1}, z^{\pm1}]$ then referring to the table of values of the cocycle f lets us write down the general form of the product $\alpha'\alpha$ as follows, in terms of the 4 unique Laurent polynomials in $\mathbb{F}_2[x^{\pm1}, y^{\pm1}, z^{\pm1}]$ such that $\alpha'\alpha = (\alpha'\alpha)_1 + (\alpha'\alpha)_a a + (\alpha'\alpha)_b b + (\alpha'\alpha)_{ab} ab$:

$$\begin{aligned}
(\alpha'\alpha)_1 &= p'p + xq'q^a + yr'r^b + zs's^{ab} \\
(\alpha'\alpha)_a &= p'q + q'p^a + x^{-1}z^{-1}r's^b + y^{-1}s'r^{ab} \\
(\alpha'\alpha)_b &= p'r + xq's^a + r'p^b + y^{-1}zs'q^{ab} \\
(\alpha'\alpha)_{ab} &= p's + q'r^a + x^{-1}yz^{-1}r'q^b + s'p^{ab}.
\end{aligned}$$

We shall prove that the following choices p', q', r', s' make α' inverse to α. In this table we also record the action of ab and b on the polynomials p, q, r, s. Geometrically, the action of ab corresponds to rotation by π about the z-axis, so the middle column expresses rotational symmetry in $\mathbb{Z}^3 \otimes \mathbb{R}$ about axes parallel to the z-axis; the third column of equations is simply the second column conjugated by a.

$$\begin{aligned}
p' &:= x^{-1}p^a & p^{ab} &= x^{-1}y^{-1}p & p^b &= x^{-1}yp^a \\
q' &:= x^{-1}q & q^{ab} &= yq & q^b &= y^{-1}q^a \\
r' &:= y^{-1}r & r^{ab} &= x^{-1}r & r^b &= x^{-1}r^a \\
s' &:= z^{-1}s^a & s^{ab} &= s & s^b &= s^a
\end{aligned}$$

The symmetry equations satisfied by the polynomials already do half of the computation of $\alpha'\alpha$, without needing to consider the actual polynomials: substituting in immediately gives (over $\mathbb{F}_2$)

$$\begin{aligned}(\alpha'\alpha)_a &= x^{-1}p^a q + x^{-1}qp^a + x^{-1}z^{-1}y^{-1}rs^a + y^{-1}z^{-1}s^a x^{-1}r = 0,\\ (\alpha'\alpha)_b &= x^{-1}p^a r + xx^{-1}qs^a + y^{-1}rx^{-1}yp^a + y^{-1}zz^{-1}s^a yq = 0.\end{aligned}$$

Let us now determine $(\alpha'\alpha)_{ab}$. We see that

$$\begin{aligned}(\alpha'\alpha)_{ab} &= x^{-1}p^a s + x^{-1}qr^a + x^{-1}yz^{-1}y^{-1}ry^{-1}q^a + z^{-1}s^a x^{-1}y^{-1}p\\ &= x^{-1}(p^a s + qr^a + y^{-1}z^{-1}q^a r + y^{-1}z^{-1}ps^a)\\ &= x^{-1}(\xi + y^{-1}z^{-1}\xi^a)\end{aligned}$$

where $\xi := p^a s + qr^a$. Define $\psi\colon \mathbb{F}_2[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}] \to \mathbb{F}_2[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ by $\psi(\gamma) = y^{-1}z^{-1}\gamma^a$. We show that $\psi(\xi) = \xi$, so that $(\alpha'\alpha)_{ab}$ vanishes as claimed, by modifying each of the summands of ξ by $\epsilon := (1+x)(1+y^{-1})(1+z)$. Note that

$$p^a s + \epsilon = (1+x)(1+y^{-1})(1+z)(x + x^{-1} + y + y^{-1})z^{-1}$$

is invariant under ψ as $(1+y^{-1})(1+z^{-1})$ is invariant and the remaining factor $(1+x)(x + x^{-1} + y + y^{-1})$ is centralized by a. For qr^a we have no choice but to expand it out; it is convenient to bracket the terms according to the power of x:

$$\begin{aligned}qr^a &= (x^{-1}y^{-1} + (y^{-1}z + z) + x)(1 + yz^{-1} + x(1 + y^{-1}z^{-1})\\ &= x^{-1}(y^{-1} + z^{-1}) + (y^{-1} + y^{-2}z^{-1} + y^{-1}z + 1 + z + y)\\ &\quad + x(y^{-1}z + y^{-2} + z + y^{-1} + 1 + yz^{-1}) + x^2(1 + y^{-1}z^{-1}).\end{aligned}$$

We now see that

$$qr^a + \epsilon = x^{-1}(y^{-1} + z^{-1}) + (y^{-2}z^{-1} + y) + x(y^{-2} + yz^{-1}) + x^2(1 + y^{-1}z^{-1})$$

is likewise invariant under ψ, completing the proof that $(\alpha'\alpha)_{ab} = 0$, as $\psi(\xi) = \psi(p^a + \epsilon + qr^a + \epsilon) = \xi$.

It remains to show that $(\alpha'\alpha)_1 = 1$. Substituting in, we see that

$$(\alpha'\alpha)_1 = x^{-1}pp^a + qq^a + x^{-1}rr^a + ss^a.$$

For convenience, write $\bar{x} = x + x^{-1}$, $\bar{y} = y + y^{-1}$ and $\bar{z} = z + z^{-1}$. We expand out qq^a and $x^{-1}rr^a$ according to the power of z. Let us write them down in factorized form first:

$$\begin{aligned}qq^a &= \left(x^{-1}y^{-1} + x + (y^{-1}+1)z\right)\left((y+1)z^{-1} + (x^{-1}y + x)\right),\\ (x^{-1}r)r^a &= \left(x^{-1} + 1 + (x^{-1}y^{-1} + y)z\right)\left((y + xy^{-1})z^{-1} + 1 + x\right).\end{aligned}$$

Observe that the coefficient of z^0 in qq^a is

$$(x^{-1}y^{-1}+x)(x^{-1}y+x)+(y^{-1}+1)(y+1)=x^2+x^{-2}=\bar{x}^2.$$

Similarly, the coefficient of z^0 in $x^{-1}rr^a$ is $\bar{y}^2$. Thus, multiplying out and collecting terms:

$$\begin{aligned}
qq^a+x^{-1}rr^a &= ((x^{-1}+x^{-1}y^{-1}+xy+x)+(x^{-1}y+y^{-1}+y+xy^{-1}))z^{-1}+\bar{x}^2+\bar{y}^2\\
&\quad+((x^{-1}+xy^{-1}+x^{-1}y+x)+(x^{-1}y^{-1}+y^{-1}+y+xy))z\\
&=(xy+xy^{-1}+x^{-1}y+x^{-1}y^{-1}+x+x^{-1}+y+y^{-1})(z^{-1}+z)+\bar{x}^2+\bar{y}^2\\
&=\bar{x}\bar{y}\bar{z}+\bar{x}\bar{z}+\bar{y}\bar{z}+\bar{x}^2+\bar{y}^2.
\end{aligned}$$

Exploiting characteristic 2 to get $(1+x)(1+x^{-1})=\bar{x}$ and similarly for y and z, we see that

$$x^{-1}pp^a=x^{-1}(1+x)(1+y)(1+z^{-1})(1+x)(1+y^{-1})(1+z)=\bar{x}\bar{y}\bar{z}.$$

Furthermore

$$\begin{aligned}
ss^a=(1+(\bar{x}+\bar{y})z^{-1})(1+(\bar{x}+\bar{y})z) &= 1+(\bar{x}+\bar{y})(z+z^{-1})+(\bar{x}+\bar{y})^2\\
&=1+\bar{x}\bar{z}+\bar{y}\bar{z}+\bar{x}^2+\bar{y}^2.
\end{aligned}$$

Thus

$$(\alpha'\alpha)_1=\bar{x}\bar{y}\bar{z}+(\bar{x}\bar{y}\bar{z}+\bar{x}\bar{z}+\bar{y}\bar{z}+\bar{x}^2+\bar{y}^2)+(1+\bar{x}\bar{z}+\bar{y}\bar{z}+\bar{x}^2+\bar{y}^2)=1. \qquad \square$$

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