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Fourier–Finite-element Approximation of Elliptic Interface Problems in Axisymmetric Domains

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The paper deals with the Fourier-finite-element method (FFEM), which combines the approximate Fourier method with the finite-element method, and its application to Poisson-like equations $-\hat{p}\Delta_3\hat{u} = \hat{f}$ in three-dimensional axisymmetric domains $\hat{\Omega}$. Here, $\hat{p}$ is a piecewise constant coefficient having a jump at some axisymmetric interface. Special emphasis is given to estimates of the Fourier-finite-element error in the Sobolev space $H^1(\hat{\Omega})$, if the interface is smooth or if it meets the boundary of $\hat{\Omega}$ at some edge. In general, the solution $\hat{u}$ contains a singularity at the interface, which is described by a tensor product representation and treated numerically by appropriate mesh grading in the meridian plane of $\hat{\Omega}$. The rate of convergence of the combined approximation in $H^1(\hat{\Omega})$ is proved to be $\mathcal{O}(h + N^{-1})$ (h, N: the parameters of the finite-element- and Fourier-approximation, with $h \to 0$, $N \to \infty$). The theoretical results are confirmed by numerical experiments.

1. Introduction

The aim of this paper is the investigation of the so-called Fourier-finite-element method (briefly, FFEM), which combines the approximate Fourier method with the finite-element method, for the numerical solution of elliptic problems in three-dimensional axisymmetric domains, here especially for problems with discontinuous coefficients.

The FFEM is often applied to three-dimensional boundary value problems (BVPs) in physics and engineering, when the domain is axisymmetric or prismatic and the data as well as the solution of the BVP vary with all co-ordinates. On the one hand, the approximate Fourier method (see e.g. [3, 4, 23]) is employed, which involves trigonometrical polynomials of degree $\leqslant N$ in one space direction, here with respect to the rotational angle; on the other hand, the finite element method is taken for the approximate calculation of Fourier coefficients of the solution on triangulations with mesh size h of the plane meridian domain $\Omega_a \subset \mathbb{R}^2_+$, where Ω_a generates the axisymmetric domain $\hat{\Omega} \subset \mathbb{R}^3$. For algorithmic aspects and some applications of the FFEM, see e.g. [2, 13, 21, 29, 30, 32, 34].

It should be noted that the FFEM provides several advantages. Thus, the approximate solution of the BVP in 3D can be reduced to the approximate solution of a finite set of elliptic problems in 2D yielding approximations of the Fourier coefficients u_k of

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the solution $\hat{u}$ in 3D. Moreover, for basic problems involving Laplace- or Lamé-like operators, the BVPs in 2D are not coupled and can be solved in parallel, cf. [14, 28].

Elliptic problems with discontinuous coefficients, sometimes called interface problems, arise e.g. when engineering problems involving composite materials with different properties are modelled mathematically. Here, we consider Poisson-like equations $-\hat{p}\Delta_3\hat{u} = \hat{f}$ (Δ_3: Laplacian in 3D) in axisymmetric domains $\hat{\Omega}$, where the coefficient $\hat{p}$ is piecewise constant and independent of the rotational angle φ. It is admitted that the axisymmetric face of discontinuity of $\hat{p}$, the interface, is curved and provided with edges or that it intersects the boundary $\hat{\Gamma}$ of $\hat{\Omega}$. It is well-known that such data may cause singularities of the solution.

For studies of the analysis of such problems, see e.g. [16, 18–20, 22, 25, 31]. For aspects of the numerical treatment of two-dimensional interface problems, cf. [1, 8, 10, 27]. Basic results on the error analysis of the FFEM are given in [5, 24]. In [24], only regular solutions $\hat{u} \in H^2(\hat{\Omega})$ are admitted, and in [5] one-dimensional meridian domains $\Omega_a = (0, 1) \subset \mathbb{R}^1$ are considered, where interfaces touching the boundary do not occur.

In this paper, we are concerned with the error analysis of the FFEM under various regularity assumptions on the solution $\hat{u}$. Especially, solutions belonging piecewise to H^2 (we shall write $\hat{u} \in PH^2(\hat{\Omega})$) or containing interface singularities ($\hat{u} \notin PH^2(\hat{\Omega})$) are taken into account. We employ results and methods given in [7, 12, 15, 16, 24, 31] and get extensions of the results of [24] to the more general class of problems introduced above. As a main result, estimates of the error norm $\|\hat{u} - \hat{u}_{hN}\|_{H^1(\hat{\Omega})}$ are proved, where $\hat{u}$ and $\hat{u}_{hN}$ denote the exact solution and its Fourier-finite-element approximation. The estimate

$$\|\hat{u} - \hat{u}_{hN}\|_{H^1(\hat{\Omega})} \leqslant C(h^\lambda + N^{-1}), \quad \tfrac{1}{4} < \lambda < 1, \tag{1.1}$$

is proved for problems with interface singularities, where singularity functions of tensor-product-type containing an R^λ-term are taken into account and where the triangular meshes covering the meridian plane Ω_a are quasi-uniform. Moreover, the estimate

$$\|\hat{u} - \hat{u}_{hN}\|_{H^1(\hat{\Omega}))} \leqslant C(h + N^{-1}) \tag{1.2}$$

is verified for piecewise regular solutions ($\hat{u} \in PH^2(\hat{\Omega})$) and quasi-uniform meshes as well as for solutions containing interface singularities and meshes with appropriate local grading depending on the exponent λ.

The plan of this paper is as follows. In section 2, the interface problem in 3D, some function spaces with power weights and the partial Fourier analysis with respect to the rotational angle φ are described. Moreover, regularity results for problems with C^2-smooth interfaces (cf. [19]) and interfaces touching the boundary (cf. [16, 31]) are given. Especially, singularity functions of tensor product type are taken into account.

In section 3, the triangulation of the plane meridian domain by means of shape-regular triangles and the Fourier-finite-element space are introduced. Due to the curved parts of the interface and of the boundary, the finite-element method is weakly non-conforming. The Fourier-finite-element approximation, an adapted version of Strang's lemma and the basic idea of error estimates are given.

In section 4, problems with curved interfaces and $\hat{u} \in PH^2(\hat{\Omega})$ are considered. We give estimates of the errors of interpolation and of projection-interpolation for the Fourier coefficients of the solution $\hat{u}$. It is proved that for $\hat{u} \in PH^2(\hat{\Omega})$ the error of the

Fourier-finite-element approximation in the energy norm behaves like $\mathcal{O}(h + N^{-1})$, where h and N denote the parameters of the finite-element- and Fourier-approximation. Under the assumption of less regularity of $\hat{u}$, e.g. if the interface touches the boundary, in section 5 it is shown that the error goes with $\mathcal{O}(h^{\lambda} + N^{-1})$, $\frac{1}{4} < \lambda < 1$, where the value λ depends on $\hat{p}$ as well as on the geometry of the domain $\hat{\Omega}$ and on the interface $\hat{\Gamma}^*$. Theorem 5.4 yields the main result of this paper: it is stated that for appropriate local mesh refinement the error behaves now like $\mathcal{O}(h + N^{-1})$.

Finally, in section 6, a special interface problem provided with a tensor product singularity function is given and numerical experiments verifying the rates of convergence as indicated by relations (1.1) and (1.2) are presented.

2. Analytical preliminaries

Let us consider the interface problem for a Poisson-like equation with Dirichlet boundary conditions in a three-dimensional domain, written as variational equation: Find $\hat{u} \in H_0^1(\hat{\Omega})$ such that

$$\hat{b}(\hat{u}, \hat{v}) := \int_{\hat{\Omega}} \hat{p}(x) \sum_{i=1}^{3} \frac{\partial \hat{u}}{\partial x_i} \frac{\overline{\partial \hat{v}}}{\partial x_i} \mathrm{d}x = \int_{\hat{\Omega}} \hat{f} \bar{\hat{v}} \mathrm{d}x =: \hat{f}(\hat{v}) \quad \forall \hat{v} \in H_0^1(\hat{\Omega}), \tag{2.1}$$

where $H_0^1(\hat{\Omega}) := \{\hat{v} \in H^1(\hat{\Omega}): \hat{v}|_{\partial\hat{\Omega}} = 0\}$, $\hat{f} \in L_2(\hat{\Omega})$, and $\bar{\hat{v}}$ denotes the complex conjugate of $\hat{v}$. The coefficient $\hat{p}(x)$ is assumed to be piecewise constant, i.e.

$$\hat{p}(x) = p_j > 0 \quad \text{for } x \in \hat{\Omega}^j \ (j = 1, 2), \quad \bar{\hat{\Omega}} = \bar{\hat{\Omega}}^1 \cup \bar{\hat{\Omega}}^2, \ \hat{\Omega}^1 \cap \hat{\Omega}^2 = \emptyset, \tag{2.2}$$

with $p_1 \neq p_2$, in general. Here, H^1, L_2 and $\cdot|_{\partial\hat{\Omega}}$ are, respectively, the usual Sobolev spaces and the trace operator, and (x_1, x_2, x_3) are the Cartesian co-ordinates of the point $x \in \mathbb{R}^3$. Furthermore, assume that $\hat{\Omega}$ is bounded and that $\hat{\Omega}$, $\hat{\Omega}^j$ $(j = 1, 2)$ are axisymmetric with respect to the x_3-axis, with boundaries $\hat{\Gamma} := \partial\hat{\Omega}$ and $\partial\hat{\Omega}^j$ $(j = 1, 2)$ belonging to $C^{0,1} \cap PC^2$ (Lipschitz-continuous and piecewise twice continuously differentiable).

Clearly, the Lax–Milgram lemma implies that there is a unique solution $\hat{u}$ of (2.1). On the interface $\hat{\Gamma}^* := \bar{\hat{\Omega}}^1 \cap \bar{\hat{\Omega}}^2$, the coefficient $\hat{p}$ has a jump and the solution $\hat{u}$ satisfies the so-called interface conditions (cf. [19, 33]), which are naturally included in formulation (2.1). Let Γ_0 denote the part of the x_3-axis contained in $\hat{\Omega}$. Then, the sets $\hat{\Omega}\backslash\Gamma_0$, $\hat{\Omega}^j\backslash\Gamma_0$ $(j = 1, 2)$ and $\hat{\Gamma}\backslash\bar{\Gamma}_0$, $\hat{\Gamma}^*\backslash\bar{\Gamma}_0$ are generated by rotation of the corresponding plane meridian domains Ω_a, Ω_a^j and curves Γ_a, Γ_a^* about the x_3-axis (cf. Fig. 1). Especially, we have

$$\bar{\Omega}_a = \bar{\Omega}_a^1 \cup \bar{\Omega}_a^2, \quad \Omega_a^1 \cap \Omega_a^2 = \emptyset, \quad \Gamma_a = \partial\Omega_a\backslash\Gamma_0, \quad \Gamma_a^* = \bar{\Omega}_a^1 \cap \bar{\Omega}_a^2, \tag{2.3}$$

where $\partial\Omega_a$, $\partial\Omega_a^j$, Γ_a and Γ_a^* belong to $C^{0,1} \cap PC^2$. Assume that in the neighbourhood of the points of intersection $\bar{\Gamma}_0 \cap \bar{\Gamma}_a$ the boundary Γ_a is straightline and forms right angles with Γ_0. Moreover, if $\Gamma_0 \cap \Gamma_a^* \neq \emptyset$, the same should hold for Γ_a^*. Hence, on $\hat{\Gamma}$ and $\hat{\Gamma}^*$ conical points do not occur.

Introduce cylindrical co-ordinates r, φ, z $(x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, $x_3 = z)$. Then, we get one-to-one mappings $\hat{\Omega}\backslash\Gamma_0 \to \Omega := \Omega_a \times (-\pi, \pi]$, $\hat{\Omega}^j\backslash\Gamma_0 \to \Omega^j := \Omega_a^j \times (-\pi, \pi]$ $(j = 1, 2)$ and $\hat{\Gamma}\backslash\bar{\Gamma}_0 \to \Gamma_a \times (-\pi, \pi]$. Consequently, for each function $\hat{v}(x)$ with $x \in \hat{\Omega}\backslash\Gamma_0$, some function v on Ω is defined by

$$v(r, \varphi, z) := \hat{v}(r\cos\varphi, r\sin\varphi, z). \tag{2.4}$$

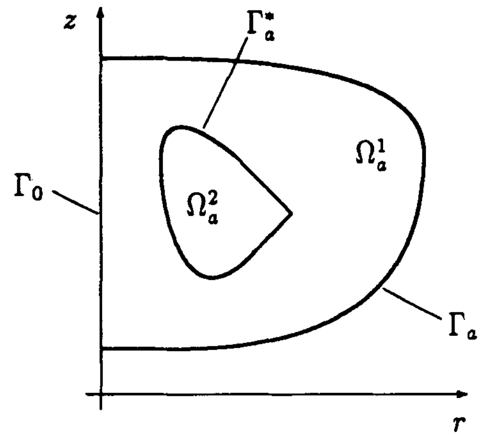

Fig. 1

By (2.4) mappings $H^l(\hat{\Omega}\backslash\Gamma_0) \to X^l_{1/2}(\Omega)$ $(l = 0, 1, 2)$ are defined, with $X^l_{1/2}(\Omega)$ from (2.5). Since Γ_0 is one-dimensional, $H^l(\hat{\Omega}\backslash\Gamma_0)$ and $H^l(\hat{\Omega})$ can be identified.
Let the space $L_2^*(\Omega)$ be defined by

$$L_2^*(\Omega) = \left\{u = u(r, \varphi, z): \int_\Omega |u|^2 \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}z < \infty, 2\pi\text{-periodic with respect to } \varphi\right\},$$

or with Ω^j $(j = 1, 2)$ instead of Ω. (For such spaces, see e.g. [26].) Then, the spaces of Sobolev-type, which correspond to $H^l(\hat{\Omega})$ and $H^1_0(\hat{\Omega})$, are given by

$$X^0_{1/2}(\Omega) := \{u = u(r, \varphi, z): r^{1/2} u \in L_2^*(\Omega)\},$$

$$X^1_{1/2}(\Omega) := \left\{u \in X^0_{1/2}(\Omega): \frac{\partial u}{\partial r}, \frac{\partial u}{\partial z}, \frac{1}{r}\frac{\partial u}{\partial \varphi} \in X^0_{1/2}(\Omega)\right\},$$

$$X^2_{1/2}(\Omega) := \left\{u \in X^1_{1/2}(\Omega): \frac{\partial^2 u}{\partial r^2}, \frac{\partial^2 u}{\partial z^2}, \frac{\partial^2 u}{\partial z \partial r}, \frac{1}{r}\frac{\partial^2 u}{\partial z \partial \varphi}, \frac{1}{r}\frac{\partial^2 u}{\partial \varphi \partial r} - \frac{1}{r^2}\frac{\partial u}{\partial \varphi},\right. \tag{2.5}$$
$$\left. \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{1}{r}\frac{\partial u}{\partial r} \in X^0_{1/2}(\Omega)\right\},$$

$$V_0^*(\Omega) := \{u \in X^1_{1/2}(\Omega): u|_{\Gamma_a \times (-\pi, \pi]} = 0\}.$$

The norms $\|\cdot\|_{X^l_{1/2}(\Omega)}$ are generated from $\|\cdot\|_{H^l(\hat{\Omega})}$ by transformation (2.4), viz.,

$$\|u\|_{X^0_{1/2}(\Omega)} := \left\{\int_\Omega |u|^2 r \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}z\right\}^{1/2}, \quad \|u\|_{V_0^*(\Omega)} := |u|_{X^1_{1/2}(\Omega)} \qquad \text{for } u \in V_0^*(\Omega),$$

$$|u|_{X^1_{1/2}(\Omega)} := \left\{\int_\Omega \left(\left|\frac{\partial u}{\partial r}\right|^2 + \left|\frac{1}{r}\frac{\partial u}{\partial \varphi}\right|^2 + \left|\frac{\partial u}{\partial z}\right|^2\right) r \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}z\right\}^{1/2},$$

$$|u|_{X^2_{1/2}(\Omega)} := \left\{\int_\Omega \left(\left|\frac{\partial^2 u}{\partial r^2}\right|^2 + \left|\frac{\partial^2 u}{\partial z^2}\right|^2 + 2\left|\frac{\partial^2 u}{\partial z \partial r}\right|^2 + 2\left|\frac{1}{r}\frac{\partial^2 u}{\partial z \partial \varphi}\right|^2\right.\right.$$
$$\left.\left. + 2\left|\frac{1}{r}\frac{\partial^2 u}{\partial \varphi \partial r} - \frac{1}{r^2}\frac{\partial u}{\partial \varphi}\right|^2 + \left|\frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right|^2\right) r \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}z\right\}^{1/2}, \tag{2.6}$$

$$\|u\|_{X^l_{1/2}(\Omega)} := \{\|u\|^2_{X^{l-1}_{1/2}(\Omega)} + |u|^2_{X^l_{1/2}(\Omega)}\}^{1/2}, \quad l = 1, 2.$$

The spaces $X^l_{1/2}(\Omega^j)$ and norms $\|\cdot\|_{X^l_{1/2}(\Omega^j)}$ ($j = 1,2$; $l = 0,1,2$) are defined by analogy to (2.5), (2.6). Clearly, for u and $\hat{u}$ connected by (2.4), we get $\|u\|_{X^l_{1/2}(\Omega)} = \|\hat{u}\|_{H^l(\hat{\Omega})}$ for $l = 0,1,2$. For cylindrical co-ordinates, the weak formulation of problem (2.1) can be described as follows. Find $u \in V^*_0(\Omega)$:

$$b(u, v) = f(v) \quad \forall v \in V^*_0(\Omega), \tag{2.7}$$

with

$$b(u, v) := \int_\Omega p(r, \varphi, z)\left\{\frac{\partial u}{\partial r}\frac{\overline{\partial v}}{\partial r} + \frac{1}{r^2}\frac{\partial u}{\partial \varphi}\frac{\overline{\partial v}}{\partial \varphi} + \frac{\partial u}{\partial z}\frac{\overline{\partial v}}{\partial z}\right\} r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z,$$

$$f(v) := \int_\Omega f\bar{v}r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z, \quad f \in X^0_{1/2}(\Omega).$$

The functions p, f, u and v correspond to $\hat{p}, \hat{f}, \hat{u}$ and $\hat{v}$ from (2.1). Especially, p is of the type

$$p(r, \varphi, z) = p(r, z) = p_j \quad \text{for } (r, z) \in \Omega^j_a \quad (j = 1, 2). \tag{2.8}$$

i.e. $p(r, z)$ has a jump along the interface Γ^*_a from (2.3).

For functions $v(r, \varphi, z)$, $v \in X^l_{1/2}(\Omega)$, partial Fourier analysis with respect to the rotational angle φ will be employed. Take the system $\{e^{ik\varphi}\}_{k\in\mathbb{Z}}$ ($i^2 = -1$, $\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$), which is orthogonal and complete in $L_2(-\pi, \pi)$, and expand $v(r, \varphi, z)$ as a Fourier series with Fourier coefficients $v_k(r, z)$, $(r, z) \in \Omega_a$:

$$v(r, \varphi, z) = \sum_{k\in\mathbb{Z}} v_k(r, z)e^{ik\varphi}, \qquad v_k(r, z) := \frac{1}{2\pi}\int_{-\pi}^{\pi} v(r, \varphi, z)e^{-ik\varphi}\,\mathrm{d}\varphi$$
$$\text{for } k \in \mathbb{Z}. \tag{2.9}$$

In [11], completeness relations in the spaces $X^l_{1/2}(\Omega)$ are described, which involve norms of v on Ω and of v_k on the meridian plane Ω_a. Some spaces of functions defined on Ω_a, needed in the following and partially provided with power weights r^α (α real), are given by

$$L_2(\Omega_a) := \left\{w = w(r, z): \int_{\Omega_a} |w|^2\,\mathrm{d}r\,\mathrm{d}z < \infty\right\},$$

$$W^{l,2}_\alpha(\Omega_a) := \{w = w(r, z): r^\alpha D^\beta w \in L_2(\Omega_a), 0 \leqslant |\beta| \leqslant l\} \quad \text{for } l \in \{0, 1, 2\};$$

$$D^\beta w := \frac{\partial^{|\beta|}w}{\partial r^{\beta_1}\partial z^{\beta_2}}, \quad |\beta| = \beta_1 + \beta_2; \quad W^{0,2}_\alpha(\Omega_a) = L_{2,\alpha}(\Omega_a), \tag{2.10}$$

$$V^a_0(\Omega_a) := \{w \in W^{1;2}_{1/2}(\Omega_a): w|_{\Gamma_a} = 0\},$$

$$W^a_0(\Omega_a) := \{w \in V^a_0(\Omega_a): w \in L_{2,-1/2}(\Omega_a)\}.$$

The norms in these spaces are defined as follows

$$\|w\|_{L_2(\Omega_a)} := \left\{\int_{\Omega_a} |w|^2\,\mathrm{d}r\,\mathrm{d}z\right\}^{1/2}, \qquad \|w\|_{L_{2,\alpha}(\Omega_a)} := \left\{\int_{\Omega_a} |r^\alpha w|^2\,\mathrm{d}r\,\mathrm{d}z\right\}^{1/2},$$

$$|w|_{W^{l,2}_\alpha(\Omega_a)} := \left\{\sum_{|\beta|=l} \|r^\alpha D^\beta w\|^2_{L_2(\Omega_a)}\right\}^{1/2} \quad \text{for } l \in \{0, 1, 2\},$$

$$\|w\|_{W_\alpha^{l,2}(\Omega_a)} := \{\|w\|^2_{W_\alpha^{l-1,2}(\Omega_a)} + |w|^2_{W_\alpha^{l,2}(\Omega_a)}\}^{1/2}, \qquad \|w\|_{W_\alpha^{0,2}(\Omega_a)} = \|w\|_{L_{2,\alpha}(\Omega_a)}, \tag{2.11}$$

$$\|w\|_{V_0^a(\Omega_a)} := |w|_{W^{1,2}_{1/2}(\Omega_a)}, \quad \|w\|_{W_0^a(\Omega_a)} := \{\|w\|^2_{L_{2,-1/2}(\Omega_a)} + |w|^2_{W^{1,2}_{1/2}(\Omega_a)}\}^{1/2}.$$

By means of the partial Fourier analysis with respect to the rotational angle φ, the 3D-interface problem (2.7) can be decomposed into an infinite sequence of decoupled 2D-interface problems.

Lemma 2.1. *For $u, v \in X^1_{1/2}(\Omega), f \in X^0_{1/2}(\Omega)$, the functionals $b(u, v)$ and $f(v)$ from (2.7) can be represented by*

$$b(u,v) = 2\pi \sum_{k \in \mathbb{Z}} b_k(u_k, v_k), \qquad f(v) = 2\pi \sum_{k \in \mathbb{Z}} f_k(v_k),$$

$$b_k(u_k, v_k) = \int_{\Omega_a} p \left\{ \frac{\partial u_k}{\partial r} \overline{\frac{\partial v_k}{\partial r}} + \frac{\partial u_k}{\partial z} \overline{\frac{\partial v_k}{\partial z}} + \frac{k^2}{r^2} u_k \bar{v}_k \right\} r \, \mathrm{d}r \, \mathrm{d}z, \tag{2.12}$$

$$f_k(v_k) = \int_{\Omega_a} f_k \bar{v}_k r \, \mathrm{d}r \, \mathrm{d}z \quad \text{for } k \in \mathbb{Z},$$

where u_k, v_k and f_k are the Fourier coefficients of u, v and f, respectively. Moreover, the Fourier coefficients $u_k(r, z)$ of the solution $u(r, \varphi, z)$ of the BVP (2.7) are the solutions of the following BVPs in 2D:

$$k = 0\text{: find } u_0 \in V_0^a(\Omega_a)\text{:} \quad b_0(u_0, w) = f_0(w) \quad \forall w \in V_0^a(\Omega_a),$$

$$k \in \mathbb{Z}\backslash\{0\}\text{: find } u_k \in W_0^a(\Omega_a)\text{:} \quad b_k(u_k, w) = f_k(w) \quad \forall w \in W_0^a(\Omega_a). \tag{2.13}$$

Since the coefficient p does not depend on φ (cf. 2.8)), the proof is analogous to [11, proofs of Lemma 4.1 and Theorem 4.2].

Moreover, the Fourier coefficients of $u \in V_0^*(\Omega)$ satisfy $u_k|_{\Gamma_0} = 0$ for $|k| \geqslant 1$ (see [24, Proposition 4.1]).

For the error analysis of the FFEM, some regularity properties of the solution $\hat{u}$ (or u) of the interface problem (2.1) (or 2.7)) are needed, and the regularity depends essentially on the geometry of $\hat{\Omega}$, $\hat{\Omega}^j$, $j = 1, 2$. First, let the following assumption be satisfied.

Assumption 2.2. *The boundary $\hat{\Gamma}$ and the interface $\hat{\Gamma}^*$ are C^2-smooth and do not touch each other.*

Under this assumption and using the notations

$$PH^2(\hat{\Omega}) := \{\hat{v} \in H^1(\hat{\Omega})\text{: } \hat{v}|_{\hat{\Omega}^j} \in H^2(\hat{\Omega}^j),\ j = 1, 2\}, \quad PX^2_{1/2}(\Omega) \text{ analogously,}$$

$$\|\hat{v}\|^2_{PH^2(\hat{\Omega})} := \{\|\hat{v}\|^2_{H^1(\hat{\Omega})} + \sum_{j=1}^{2} |\hat{v}|^2_{H^2(\hat{\Omega}^j)}\}^{1/2}, \quad \|v\|_{PX^2_{1/2}(\Omega)} \text{ analogously,} \tag{2.14}$$

the solution u from (2.7) satisfies (cf. [19, 20]): $u \in PX^2_{1/2}(\Omega)$ and

$$\|u\|_{PX^2_{1/2}(\Omega)} \leqslant C \|f\|_{X^0_{1/2}(\Omega)}. \tag{2.15}$$

If there are edges on the interface $\hat{\Gamma}^*$, or if the interface $\hat{\Gamma}^*$ meets the boundary $\hat{\Gamma}$ at some edge, then $\hat{u} \notin PH^2(\hat{\Omega})$ may occur. In the following, we shall study the second problem more extensively and assume that Ω_a^1 and Ω_a^2 are such that $\Gamma_a^* \in C^2$ and there

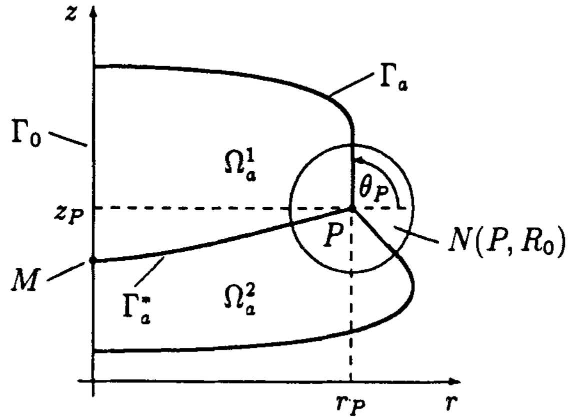

Fig. 2

are only two points of intersection of Γ_a^* with $\partial\Omega_a$, viz., $P \in \Gamma_a \cap \Gamma_a^*$ and $M \in \Gamma_0 \cap \Gamma_a^*$ (see Fig. 2). Introduce local polar co-ordinates (R, θ) at P by

$$r - r_P = R\cos(\theta + \theta_P), \qquad z - z_P = R\sin(\theta + \theta_P) \tag{2.16}$$

(cf. Fig. 2) and denote some neighbourhood, where Γ_a and Γ_a^* are supposed to be straightline, by $N(P, R_0) := \{(r, z) \in \mathbb{R}^2 : (r - r_P)^2 + (z - z_P)^2 < R_0^2\}$. We define subsets $G_a \subset \Omega_a$ and $G \subset \Omega$ by

$$G_a := \Omega_a \cap N(P, R_0), \qquad G := G_a \times (-\pi, \pi]. \tag{2.17}$$

Furthermore, let $\eta = \eta(r, \varphi, z) = \tilde{\eta}(R)$ be a cut-off function, with

$$0 \leqslant \tilde{\eta} \leqslant 1, \quad \tilde{\eta} \in C^\infty[0, \infty), \quad \tilde{\eta} = 1 \quad \text{for } 0 \leqslant R \leqslant R_0/3, \quad \tilde{\eta} = 0$$
$$\text{for } R \geqslant 2R_0/3. \tag{2.18}$$

The regularity of solutions of the BVPs (2.13) can be characterized by means of the eigenvalues and eigenfunctions of some one-dimensional Sturm–Liouville problem associated with the interface problem in 2D, see e.g. [16, p. 12] or [18, p. 360]. By means of the regularity results in 2D and under the additional assumption $\partial^l f/\partial\varphi^l \in X_{1/2}^0(G)$ $(l = 1, 2)$, we get some representation of the solution in 3D involving tensor product singularity functions.

Theorem 2.3 *Assume that* $f \in X_{1/2}^0(\Omega)$ *and* $\partial^l f/\partial\varphi^l \in X_{1/2}^0(G)$ $(l = 1, 2)$ *are satisfied. Then, the solution* u *of the interface problem* (2.7) *can be represented by a sum of singularity functions of tensor product type and a piecewise regular part, viz.,*

$$u(r, \varphi, z) = u_{\mathrm{si}}(r, \varphi, z) + w(r, \varphi, z) \quad \textit{with } u \in V_0^*(\Omega),$$
$$u_{\mathrm{si}}(r, \varphi, z) = \tilde{\eta}(R) \sum_{m \in \mathcal{M}} c^{(m)}(\varphi) R^{\lambda_m} t_m(\theta), \quad u_{\mathrm{si}} \in V_0^*(G),$$
$$c^{(m)}(\varphi) = \sum_{k \in \mathbb{Z}} c_k^{(m)} \mathrm{e}^{\mathrm{i}k\varphi}, \quad c^{(m)} \in H^2(-\pi, \pi), \tag{2.19}$$
$$w(r, \varphi, z) = \sum_{k \in \mathbb{Z}} w_k(r, z) \mathrm{e}^{\mathrm{i}k\varphi}, \quad w \in V_0^*(\Omega) \cap PX_{1/2}^2(\Omega).$$

Here, R, θ and $\tilde{\eta}$ are taken from (2.16), (2.18); *λ_m^2 $(0 < \lambda_1^2 \leqslant \lambda_2^2 \leqslant \cdots)$ and $t_m(\theta)$ denote the eigenvalues and the eigenfunctions of the previously mentioned Sturm–Liouville problem. The set $\mathscr{M}$ is defined by $\mathscr{M} := \{m \in \{1,2, \dots\}\colon 0 < \lambda_m^2 < 1\}$. The Fourier coefficients $c_k^{(m)}$ of $c^{(m)}(\varphi)$ are complex constants, and the Fourier coefficients $w_k(r, z)$ of $w(r, \varphi, z)$ satisfy $w_k \in V_0^a(\Omega_a) \cap W_{1/2}^{2;2}(\Omega_a^j)$ for $j = 1, 2$. Moreover, the following a priori estimate holds*:

$$\sum_{m \in \mathscr{M}} \|c^{(m)}\|_{H^2(-\pi,\pi)} + \|w\|_{PX_{1/2}^2(\Omega)} \leqslant C \left\{ \|f\|_{X_{1/2}^0(\Omega)} + \sum_{l=1}^{2} \left\| \frac{\partial^l f}{\partial \varphi^l} \right\|_{X_{1/2}^0(G)} \right\}. \tag{2.20}$$

The proof of Theorem 2.3 is given in [16].

Remark 2.4. For problems where the boundary $\hat{\Gamma}$ is smooth and the interface $\hat{\Gamma}^*$ is provided with edges within $\hat{\Omega}$, the representation of the solution u from (2.7) is analogous to (2.19); see [16, 31] for more details.

3. The Fourier–finite–element approximation

First we approximate the solution $u(r, \varphi, z)$ of the BVP (2.7) by the truncated Fourier series

$$u_N(r, \varphi, z) = \sum_{|k| \leqslant N} u_k(r, z) e^{ik\varphi} \quad \text{for } N > 0. \tag{3.1}$$

Here, $u_k(r, z)$, $(k = 0, \pm 1, \dots, \pm N)$ are the first $2N + 1$ Fourier coefficients of u, which are the solutions of the 2D-BVPs (2.13) for $|k| \leqslant N$. They shall be approximated by the FEM in 2D (see e.g. [6]).

Approximate Ω_a and Ω_a^j $(j = 1, 2)$ by polygonally bounded domains Ω_{ah} and Ω_{ah}^j, with boundaries $\partial\Omega_{ah}$ and $\partial\Omega_{ah}^j$, respectively, such that $\bar{\Omega}_{ah} = \bar{\Omega}_{ah}^1 \cup \bar{\Omega}_{ah}^2$, $\Omega_{ah}^1 \cap \Omega_{ah}^2 = \emptyset$, and denote the corresponding triangulations by $\mathscr{T}_h$, $\mathscr{T}_h^j$ $(j = 1, 2)$. Let h denote the mesh parameter of the triangulation $\mathscr{T}_h = \{T\}$, with $0 < h \leqslant h_0$ and sufficiently small h_0 (e.g. $h = \max\{h_T, T \in \mathscr{T}_h\}$, h_T from Assumption 3.1(ii)).

Assumption 3.1. (i) *Arbitrary two triangles T, $T' \in \mathscr{T}_h$ are either disjoint or have a common vertex, or a common edge. The points of the sets $\bar{\Gamma}_a \cap \bar{\Gamma}_0$, $\Gamma_a \cap \Gamma_a^*$ and the corner points of Γ_a and Γ_a^* are vertices of triangles $T \in \mathscr{T}_h$. The vertices belonging to $\partial\Omega_{ah}^j$ $(j = 1, 2)$ are, at the same time, located on $\partial\Omega_a^j$.* For $T \in \mathscr{T}_h$, *at most two vertices belong to $\partial\Omega_a^j$.*

(ii) *The mesh is 'shape regular', i.e. for h_T and ρ_T as the diameter of T and of the largest ball contained in T, respectively, we have*

$$\frac{h_T}{\rho_T} \leqslant C \quad \textit{for any } T \in \mathscr{T}_h,\ C \textit{ is independent of } T \textit{ and } h. \tag{3.2}$$

(iii) *The global regularity for some part of the mesh is required,*

$$\max_{T \in \mathscr{T}'_h} h_T / \min_{T \in \mathscr{T}'_h} \rho_T \leqslant C, \tag{3.3}$$

where $\mathscr{T}'_h$ is a subset of $\mathscr{T}_h$, possibly $\mathscr{T}'_h = \mathscr{T}_h$.

Thus, for the angles Θ at the vertices of triangles T and for the length h'_T of any edge of T, the following inequalities hold, where Θ_0 and ε_0 are independent of h and T:

$$0 < \Theta_0 \leqslant \Theta \leqslant \pi - \Theta_0, \quad \varepsilon_0 h_T \leqslant h'_T \leqslant h_T \ (0 < \varepsilon_0 < 1). \tag{3.4}$$

Clearly, for convex Ω_a^j the relations $\bar{\Omega}_{ah}^j \subset \bar{\Omega}_a^j$ $(j = 1, 2)$ hold; otherwise $\bar{\Omega}_{ah}^j \not\subset \bar{\Omega}_a^j$ and $\bar{\Omega}_{ah}^j \not\supset \bar{\Omega}_a^j$ may occur.

Introduce the following finite element spaces V_{0h}^a and W_{0h}^a, $0 < h \leqslant h_0$:

$$\begin{aligned} V_{0h}^a &:= \{w_h = w_h(r, z)\colon w_h \in C(\bar{\Omega}_{ah}),\ w_h \in \mathbb{P}_1(T)\,\forall T \in \mathscr{T}_h,\ w_h = 0 \text{ on } \bar{\Gamma}_{ah}\},\\ W_{0h}^a &:= \{w_h = w_h(r, z)\colon w_h \in V_{0h}^a,\ w_h = 0 \text{ on } \bar{\Gamma}_0\}, \end{aligned} \tag{3.5}$$

where $\mathbb{P}_1(T)$ denotes the space of all polynomials of degree $\leqslant 1$ on T; $\Gamma_{ah} := \partial\Omega_{ah} \backslash \bar{\Gamma}_0$. For $h \in (0, h_0]$ and integer $N > 0$, the Fourier-finite-element space is defined by

$$V_{hN} := \{v\colon v(r, \varphi, z) = \sum_{|k| \leqslant N} v_{kh}(r, z)\mathrm{e}^{\mathrm{i}k\varphi},\ v_{0h} \in V_{0h}^a,\ v_{kh} \in W_{0h}^a,\ k \neq 0\}. \tag{3.6}$$

Lemma 3.2. *The following relations hold*:

$$V_{0h}^a \subset V_0^a(\Omega_{ah}), \quad W_{0h}^a \subset W_0^a(\Omega_{ah}), \quad V_{hN} \subset V_0^*(\Omega_h), \tag{3.7}$$

where $V_0^*(\Omega_h) := \{v\colon v \in X_{1/2}^1(\Omega_h),\ v|_{\Gamma_{ah} \times (-\pi, \pi]} = 0\}$.

The Fourier-finite-element discretization of the BVP (2.7) is given as follows.

$$\text{Find } u_{hN} \in V_{hN}\colon \quad b_h(u_{hN}, v_{hN}) = f_h(v_{hN}) \ \ \forall v_{hN} \in V_{hN}, \tag{3.8}$$

with

$$b_h(u_{hN}, v_{hN}) = \int_{\Omega_h} p_h \left\{ \frac{\partial u_{hN}}{\partial r}\frac{\overline{\partial v_{hN}}}{\partial r} + \frac{1}{r^2}\frac{\partial u_{hN}}{\partial \varphi}\frac{\overline{\partial v_{hN}}}{\partial \varphi} + \frac{\partial u_{hN}}{\partial z}\frac{\overline{\partial v_{hN}}}{\partial z} \right\} r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z,$$

$$f_h(v_{hN}) = \int_{\Omega_h} f_h \overline{v_{hN}}\, r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z.$$

Here, u_{hN} is called the Fourier-finite-element approximation of u. The functions p_h and f_h are defined by

$$p_h(r, \varphi, z) = p_h(r, z) = p_j \quad \text{for } (r, z) \in \Omega_{ah}^j \quad (j = 1, 2),\ \varphi \in (-\pi, \pi], \tag{3.9}$$

$$f_h(r, \varphi, z) = \begin{cases} f(r, \varphi, z) & \text{for } (r, \varphi, z) \in (\Omega \cap \Omega_h),\\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

We conclude from relations (2.8) and (3.9) that $p_h \neq p$ holds if $\Omega_{ah}^1 \neq \Omega_a^1$ or $\Omega_{ah}^2 \neq \Omega_a^2$. Using arguments as in the Lax-Milgram lemma (see e.g. [6]) we can verify that there is a unique solution $u_{hN} \in V_{hN}$ of (3.8), see [31]. By means of partial Fourier analysis applied to $b_h(u_{hN}, v_{hN})$ and $f_h(v_{hN})$ with respect to φ, we get relations analogous to (2.12), but with summation over $|k| \leqslant N$ only. This implies a splitting of the 3D variational equation (3.8) into a decoupled finite system of 2D variational equations. The finite element approximations u_{kh} $(|k| \leqslant N)$ of the Fourier coefficients u_k are obtained as follows.

$$\begin{aligned} &\text{Find } u_{0h} \in V_{0h}^a\colon \quad b_{0h}(u_{0h}, w_h) = f_{0h}(w_h) \ \ \forall w_h \in V_{0h}^a,\\ &u_{kh} \in W_{0h}^a\colon \quad b_{kh}(u_{kh}, w_h) = f_{kh}(w_h) \ \ \forall w_h \in W_{0h}^a,\ 1 \leqslant |k| \leqslant N, \end{aligned} \tag{3.11}$$

with

$$b_{kh}(u_{kh}, w_h) = \int_{\Omega_{ah}} p_h \left\{ \frac{\partial u_{kh}}{\partial r} \frac{\overline{\partial w_h}}{\partial r} + \frac{\partial u_{kh}}{\partial z} \frac{\overline{\partial w_h}}{\partial z} + \frac{k^2}{r^2} u_{kh} \overline{w_h} \right\} r \, dr \, dz,$$

$$f_{kh}(w_h) = \int_{\Omega_{ah}} f_{kh} \overline{w_h} r \, dr \, dz,$$

and f_{kh}, $k \in \mathbb{Z}$, are the Fourier coefficients of f_h from (3.10).

Since $\bar{\Omega}_h \not\subset \bar{\Omega}$ may occur, we define an extension $\tilde{u}$ (according to Calderon) of the function u beyond the boundary $\partial\Omega$ to the domain $\tilde{\Omega} := \tilde{\Omega}_a \times (-\pi, \pi]$. Here, $\tilde{\Omega}_a$ is a bounded domain, with $\Omega_a \cup \Omega_{ah} \subset \tilde{\Omega}_a$ for $0 < h \leqslant h_0$, and with subdomains $\tilde{\Omega}_a^1$, $\tilde{\Omega}_a^2$ satisfying $\bar{\tilde{\Omega}}_a = \bar{\tilde{\Omega}}_a^1 \cup \bar{\tilde{\Omega}}_a^2$, $\tilde{\Omega}_a^1 \cap \tilde{\Omega}_a^2 = \emptyset$ and $\Omega_a^j \subset \tilde{\Omega}_a^j$ $(j = 1, 2)$. Moreover, let $r > 0$ for any $(r, z) \in \tilde{\Omega}_a$ and $\partial\tilde{\Omega}_a, \partial\tilde{\Omega}_a^j \in C^{0,1} \cap PC^2$ be satisfied. Then for $\tilde{\Omega}^j := \tilde{\Omega}_a^j \times (-\pi, \pi]$, the relations $\bar{\tilde{\Omega}} = \bar{\tilde{\Omega}}^1 \cup \bar{\tilde{\Omega}}^2$, $\tilde{\Omega}^1 \cap \tilde{\Omega}^2 = \emptyset$ hold. Furthermore, let the subdomains G_δ and G_δ^j be given by $G_\delta := G_{a,\delta} \times (-\pi, \pi]$, with $G_{a,\delta} := \{(r, z) \in G_a : R < \delta\}$ for some δ (G_a is taken from (2.17)), and $G_\delta^j := G_\delta \cap \Omega^j$, $j = 1, 2$. Starting from two important cases of regularity of the solution u, viz.,

(i) $u \in PX_{1/2}^2(\Omega)$ according to (2.15), or
(ii) $u = u_{si} + w$ according to (2.19), $u \in PX_{1/2}^2(\Omega \backslash G_\delta)$, $0 < \delta_0 \leqslant \delta \leqslant \frac{2}{3}R_0$,

we define two types of extensions $\tilde{u}$ of u, with $\tilde{u} = u$ on Ω, beyond the boundary $\partial\Omega$. For (i), take Calderon's extension (see e.g. [33]) $\tilde{u}$ of u. Thus, we have

$$\tilde{u} \in PX_{1/2}^2(\tilde{\Omega}), \qquad \|\tilde{u}\|_{PX_{1/2}^2(\tilde{\Omega})} \leqslant C \|u\|_{PX_{1/2}^2(\Omega)}. \tag{3.12}$$

In case (ii), for suitable choice of $\tilde{\Omega}_a$ (i.e. if $\partial\tilde{\Omega}_a \cap \partial G_{a,\delta}$ coincides with $\partial\Omega_a \cap \partial G_{a,\delta}$ for $0 < \delta_0 \leqslant \delta \leqslant \frac{2}{3}R_0$), the function $\tilde{u}$ can be extended such that the following relations hold:

$$\tilde{u} \in X_{1/2}^1(\tilde{\Omega}) \cap PX_{1/2}^2(\tilde{\Omega} \backslash G_\delta), \qquad \|\tilde{u}\|_{PX_{1/2}^2(\tilde{\Omega} \backslash G_\delta)} \leqslant C \|u\|_{PX_{1/2}^2(\Omega \backslash G_\delta)}, \tag{3.13}$$

with $0 < \delta_0 \leqslant \delta \leqslant \frac{2}{3}R_0$.

Using these extensions, the error $\tilde{u} - u_{hN}$ can be estimated by (3.14).

Theorem 3.3. *Let $\tilde{u}$ be the extension of the solution u from* (2.7) *and u_{hN} the Fourier-finite-element approximation from* (3.8). *Then, there is a constant C which does not depend on h $(0 < h \leqslant h_0)$ and N $(N > 0)$ such that the following estimate holds:*

$$\|\tilde{u} - u_{hN}\|_{X_{1/2}^1(\Omega_h)} \leqslant C \left\{ \inf_{v_{hN} \in V_{hN}} \|\tilde{u} - v_{hN}\|_{X_{1/2}^1(\Omega_h)} + \sup_{w_{hN} \in V_{hN}} \frac{|b_h(\tilde{u}, w_{hN}) - f_h(w_{hN})|}{\|w_{hN}\|_{X_{1/2}^1(\Omega_h)}} \right\}. \tag{3.14}$$

The proof is given in [31, Proof of Theorem 4.1}; it is similar to the proof of Strang's lemma (see [6, Theorem 4.2.2]).

For further estimates via (3.14) we use

$$\inf_{v_{hN} \in V_{hN}} \|\tilde{u} - v_{hN}\|_{X_{1/2}^1(\Omega_h)} \leqslant \|\tilde{u} - r_{hN} u\|_{X_{1/2}^1(\Omega_h)}, \tag{3.15}$$

and define $r_{hN}u$ by

$$(r_{hN}u)(r, \varphi, z) := \sum_{|k| \leqslant N} u_{kh}(r, z)e^{ik\varphi} \quad \text{where } u_{kh} = \begin{cases} \Pi_h u_k & \text{for } |k| \leqslant 1, \\ P_h u_k & \text{for } 2 \leqslant |k| \leqslant N. \end{cases} \tag{3.16}$$

Here, Π_h and P_h are interpolation and projection-interpolation operators as proposed and investigated for convex geometry of Ω and regular solutions $u \in X^2_{1/2}(\Omega)$ (without interfaces or corners) in [24]. It turns out that this approach is also convenient for problem (2.7), where 'partial regularity' $u \in PX^2_{1/2}(\Omega)$, interface singularities and curved (non-convex) boundaries may occur. For some arguments of choosing Π_h and P_h according to the values $|k|$ specified at (3.16), see [24] or [31], and for the definition of Π_h as well as the extended (in comparison with [24]) definition of P_h, cf. Section 4.

4. The convergence for piecewise regular solutions

In this section, we study the error $\|\tilde{u} - u_{hN}\|_{X^1_{1/2}(\Omega_h)}$ for piecewise regularity of u, i.e., we assume that $\hat{\Gamma}^* \cap \hat{\Gamma} = \emptyset$ and $\hat{\Gamma}, \hat{\Gamma}^* \in C^2$ (cf. Assumption 2.2), or that $u_{si} \equiv 0$ in (2.19). Then, for $f \in X^0_{1/2}(\Omega)$ inequality (2.15) can be shown and the Fourier coefficients u_k are provided with the properties

$$\begin{aligned} &u_k|_{\Omega^j_a} \in W^{2,2}_{1/2}(\Omega^j_a), \quad u_k \in C(\bar{\Omega}_a) \quad \text{for } |k| \geqslant 0; \quad u_k|_{\Omega^j_a} \in W^{1,2}_{-1/2}(\Omega^j_a) \\ &\text{for } |k| \geqslant 2, j = 1, 2. \end{aligned} \tag{4.1}$$

These relations can be concluded from [24, Lemma 3.1, Corollary 3.1, Theorem 4.7]. Of course, for the Fourier coefficients $\tilde{u}_k$ of the extension $\tilde{u}$ from (3.12), we have $u_k = \tilde{u}_k$ on Ω_a and the relations (4.1) are also satisfied for $\tilde{u}_k$, but now with $\tilde{\Omega}_a$, $\tilde{\Omega}^j_a$ instead of Ω_a, Ω^j_a.

Owing to (3.14)–(3.16), for the error analysis of $\tilde{u} - u_{hN}$ it suffices to study the approximation properties of the operators Π_h and P_h. For analysing the approximation errors on triangles touching the curved interface Γ^*_a, we consider extensions (according to Calderon) of the function $u|_{\Omega^j} \in X^2_{1/2}(\Omega^j)$ to u^j defined on the domain Ω such that

$$u^j = u \text{ in } \Omega^j, \quad \|u^j\|_{X^2_{1/2}(\Omega)} \leqslant C\|u\|_{X^2_{1/2}(\Omega^j)} \qquad \text{for } j = 1, 2 \tag{4.2}$$

holds. For the Fourier coefficients u^j_k of u^j ($j = 1, 2$) we have the relations

$$\begin{aligned} &u^j_k = u_k \quad \text{in } \Omega^j_a, \qquad u^j_k \in W^{2,2}_{1/2}(\Omega_a) \quad \text{for } |k| \geqslant 0; \qquad u^j_k \in W^{1,2}_{-1/2}(\Omega_a) \\ &\text{for } |k| \geqslant 2. \end{aligned}$$

Let Assumption 3.1 be satisfied and assume that $\mathscr{T}'_h = \mathscr{T}_h$ holds, with $\mathscr{T}'_h$ from (3.3). We consider the usual Lagrangian interpolation operator Π_h. Let $\psi \in C(\bar{\Omega}_a)$, with $\psi|_{\Gamma_a} = 0$, be satisfied. Then, $\Pi_h\psi \in V^a_{0h}$ is that function (interpolant of ψ) which coincides with ψ at all nodes of the triangulation $\mathscr{T}_h$.

In the following theorem, we give bounds of the error $\tilde{u}_k - \Pi_h u_k$.

Theorem 4.1. *For the interpolation error of the Fourier coefficients u_k, the estimates*

$$|\tilde{u}_k - \Pi_h u_k|_{W^{1;2}_{1/2}(\Omega_{ah})} \leqslant ChS^0(u_k) \quad \text{for } k \in \mathbb{Z}, \tag{4.3}$$

$$\|\tilde{u}_k - \Pi_h u_k\|_{L_{2,-1/2}(\Omega_{ah})} \leqslant ChS^0(u_k) \quad \text{for } k \in \mathbb{Z}\backslash\{0\} \tag{4.4}$$

hold, with $S^0(u_k) := \sum_{j=1}^{2} \{ \|\tilde{u}_k\|_{W^{2;2}_{1/2}(\Omega_a^j)} + \|u_k^j\|_{W^{2;2}_{1/2}(\Omega_a)} \}$.

The proof is given in [31, section 4.2.1].

Additionally, we mention that the following relation holds (for the proof, see [31, section 4.2.2]):

$$\|\tilde{u}_k - \Pi_h u_k\|_{L_{2,1/2}(\Omega_{ah})} \leqslant ChS^0(u_k) \quad \text{for } k \in \mathbb{Z}. \tag{4.5}$$

According to (3.16), for $|k| \geqslant 2$ the approximation error of the projection-interpolation-operator P_h is to be estimated. For this purpose, we introduce P_h given in [24] in a modified version, which takes into account that $u_k \in W^{2;2}_{1/2}((\Omega_a^j)$ $(j = 1,2)$ and $u_k \notin W^{2;2}_{1/2}(\Omega_a)$ hold, in general.

Introduce some subsets of triangles covering Ω_{ah}. For any node Q of $\mathscr{T}_h$, let T_l $(1 \leqslant l \leqslant l_Q)$ be the triangles having Q as vertex. Further, let T_l^j $(j = 1,2; 1 \leqslant l \leqslant l_Q^j)$ be such of them which have all their nodes in $\bar{\Omega}_a^j$. Then, introduce S_Q and S_Q^j as defined by

$$S_Q = \bigcup_{l=1}^{l_Q} T_l \quad \text{and} \quad S_Q^j = \bigcup_{l=1}^{l_Q^j} T_l^j, \quad j = 1,2. \tag{4.6}$$

Clearly, $S_Q = S_Q^1 \cup S_Q^2$ holds, and $S_Q^j = \emptyset$ may occur for $j = 1$ or $j = 2$. If $Q \subset \Gamma_a^*$ and Γ_a^* is straight line near Q, then $S_Q^j = S_Q \cap \Omega_a^j$ $(j = 1,2)$ holds.

Let B_h^0 be the interior of the union of all triangles $T \in \mathscr{T}_h$ with $T \cap \bar{\Gamma}_0 \neq \emptyset$. Furthermore, let Σ_h be the set of all nodes of $\mathscr{T}_h$, Σ_h^0 the set of all nodes $Q \in \Sigma_h$ with $Q \notin \bar{B}_h^0$ and $Q \notin \Gamma_a$, and Σ_h^1 the set of all nodes $Q \in \Sigma_h$ where $S_Q \cap \Omega_a^j = \emptyset$ holds for $j = 1$ or $j = 2$ (i.e. the interface Γ_a^* does not intersect S_Q). Moreover, let Σ_h^2 be given by $\Sigma_h^2 := \Sigma_h \backslash \Sigma_h^1$, i.e. Σ_h^2 is the set of all nodes Q where S_Q is intersected by Γ_a^*. Note that even for $Q \in \Sigma_h^2$, $S_Q^j = \emptyset$ with $j = 1$ or $j = 2$ may occur (e.g. if a curvilinear part of the interface Γ_a^* intersects S_Q, but all nodes of triangles $T \subset S_Q$ belong to $\bar{\Omega}_a^j$ with $j = 1$ or $j = 2$).

Introduce Courant's basis function $\Phi_Q \in C(\bar{\Omega}_{ah})$ associated with the node $Q \in \Sigma_h$:

$$\Phi_Q \in \mathbb{P}_1(T)\,\forall T \in \mathscr{T}_h, \quad \Phi_Q(Q') = \begin{cases} 1 & \text{for } Q' = Q, \\ 0 & \text{for } Q' \neq Q,\ Q' \in \Sigma_h. \end{cases} \tag{4.7}$$

Define orthogonal projection operators $P_Q : L_2(S_Q) \to \mathbb{P}_1(S_Q)$ and $P_Q^j : L_2(S_Q^j) \to \mathbb{P}_1(S_Q^j)$ $(j = 1,2)$ by $v \to P_Q v$, $v \to P_Q^j v$ via the relations

$$\begin{aligned} &(v - P_Q v, p)_{L_2(S_Q)} = 0 \quad \forall p \in \mathbb{P}_1(S_Q),\ Q \in \Sigma_h, \\ &(v - P_Q^j v, p)_{L_2(S_Q^j)} = 0 \quad \forall p \in \mathbb{P}_1(S_Q^j),\ Q \in \Sigma_h^2 \text{ and } S_Q^j \neq \emptyset \text{ for } j = 1,2. \end{aligned} \tag{4.8}$$

Using the Fourier coefficients $\tilde{u}_k$, u_k^j of the extensions $\tilde{u}$, u^j $(j = 1,2)$ of u, the values $(u_k)_Q$ $(Q \in \Sigma_h)$ are defined as follows:

$$(u_k)_Q := \begin{cases} (P_Q \tilde{u}_k)(Q) & \text{for } Q \in \Sigma_h^1, \\ (P_Q^j u_k^j)(Q) & \text{for } Q \in \Sigma_h^2, S_Q = S_Q^j \ (j = 1 \text{ or } j = 2), \\ \frac{1}{2}[(P_Q^1 u_k^1)(Q) + (P_Q^2 u_k^2)(Q)] & \text{for } Q \in \Sigma_h^2,\ S_Q^1 \neq \emptyset,\ S_Q^2 \neq \emptyset. \end{cases} \tag{4.9}$$

Taking $(u_k)_Q$ from (4.9) and Φ_Q from (4.7), the operator P_h is given by

$$P_h u_k = \sum_{Q \in \Sigma_h^0} (u_k)_Q \Phi_Q. \tag{4.10}$$

Obviously, $P_h u_k = 0$ on $\bar{B}_h^0$, $P_h u_k \in W_{0h}^a$ hold for $k \in \mathbb{Z}$. The approximation properties of the operator P_h can be described as follows.

Theorem 4.2. *For P_h from* (4.10) *and the Fourier coefficients u_k of u, with $|k| \geqslant 2$, the following estimates hold:*

$$|\tilde{u}_k - P_h u_k|_{W^{1;2}_{1/2}(\Omega_{ah})} \leqslant Ch(S^1(u_k) + S^2(u_k)), \tag{4.11}$$

$$\|\tilde{u}_k - P_h u_k\|_{L_{2,-1/2}(\Omega_{ah})} \leqslant Ch\, S^1(u_k), \tag{4.12}$$

with

$$S^1(u_k) = \sum_{j=1}^{2} \{|\tilde{u}_k|_{W^{1;2}_{-1/2}(\bar{\Omega}_a^j)} + |u_k^j|_{W^{1;2}_{-1/2}(\Omega_a)}\},$$

$$S^2(u_k) = \sum_{j=1}^{2} \{|\tilde{u}_k|_{W^{2;2}_{1/2}(\bar{\Omega}_a^j)} + |u_k^j|_{W^{2;2}_{1/2}(\Omega_a)}\}.$$

Proof: See [15, Proof of Theorem 4.5] or [31, Section 4.2.2].

Clearly, relation (4.12) also implies

$$\|\tilde{u}_k - P_h u_k\|_{L_{2,1/2}(\Omega_{ah})} \leqslant ChS^1(u_k) \quad \text{for } |k| \geqslant 2. \tag{4.13}$$

For further estimates of the error $\tilde{u} - u_{hN}$, we use the inequalities given in Theorems 4.1 and 4.2. For the first term on the right-hand side of (3.14) we get

Lemma 4.3. *Let $\tilde{u}$ be the extension of u from* (2.7), *cf.* (3.12). *Then, the estimate*

$$\inf_{v_{hN} \in V_{hN}} \|\tilde{u} - v_{hN}\|_{X^1_{1/2}(\Omega_h)} \leqslant C(h + N^{-1})\|f\|_{X^0_{1/2}(\Omega)}. \tag{4.14}$$

is satisfied.

Proof. Obviously, for $r_{hN}u$ from (3.16) we have $r_{hN}u \in X^1_{1/2}(\Omega_h)$. Using the truncated Fourier series

$$\tilde{u}_N(r, \varphi, z) := \sum_{|k| \leqslant N} \tilde{u}_k(r, z) e^{ik\varphi} \tag{4.15}$$

and relation (3.15), we also get

$$\inf_{v_{hN} \in V_{hN}} \|\tilde{u} - v_{hN}\|_{X^1_{1/2}(\Omega_h)} \leqslant \|\tilde{u} - \tilde{u}_N\|_{X^1_{1/2}(\Omega_h)} + \|\tilde{u}_N - r_{hN}u\|_{X^1_{1/2}(\Omega_h)}. \tag{4.16}$$

The error $\tilde{u} - \tilde{u}_N$ can be bounded by

$$\|\tilde{u} - \tilde{u}_N\|_{X^1_{1/2}(\Omega_h)} \leqslant \|\tilde{u} - \tilde{u}_N\|_{X^1_{1/2}(\tilde{\Omega})} \leqslant CN^{-1}\|\tilde{u}\|_{PX^2_{1/2}(\tilde{\Omega})}, \tag{4.17}$$

cf. [31, Proof of Lemma 4.11]. For the estimate of the error $\tilde{u}_N - r_{hN}u$, we apply the relation (3.16) and completeness relations which are analogous to that of

[11, Lemma 3.2], viz.,

$$\begin{aligned}\frac{1}{2\pi}\|\tilde{u}_N - r_{hN}u\|^2_{X^1_{1/2}(\Omega_h)} &= \sum_{|k|\leq 1}\|\tilde{u}_k - \Pi_h u_k\|^2_{W^{1,2}_{1/2}(\Omega_{ah})} + \sum_{|k|=1} k^2\|\tilde{u}_k - \Pi_h u_k\|^2_{L_{2,-1/2}(\Omega_{ah})}\\ &+ \sum_{2\leq|k|\leq N}\|\tilde{u}_k - P_h u_k\|^2_{W^{1,2}_{1/2}(\Omega_{ah})}\\ &+ \sum_{2\leq|k|\leq N} k^2\|\tilde{u}_k - P_h u_k\|^2_{L_{2,-1/2}(\Omega_{ah})}.\end{aligned} \tag{4.18}$$

By means of (4.3)–(4.5) and (4.11)–(4.13) we get

$$\begin{aligned}\|\tilde{u}_N - r_{hN}u\|^2_{X^1_{1/2}(\Omega_h)} \leq Ch^2\Bigg\{&\sum_{|k|\leq 1}\sum_{j=1}^{2}\left(\|\tilde{u}_k\|^2_{W^{2,2}_{1/2}(\tilde{\Omega}^j_a)} + \|u^j_k\|^2_{W^{2,2}_{1/2}(\Omega_a)}\right)\\ &+ \sum_{2\leq|k|\leq N}\sum_{j=1}^{2}\left(k^2|\tilde{u}_k|^2_{W^{1,2}_{-1/2}(\tilde{\Omega}^j_a)} + k^2|u^j_k|^2_{W^{1,2}_{-1/2}(\Omega_a)}\right.\\ &\left.+ |\tilde{u}_k|^2_{W^{2,2}_{1/2}(\tilde{\Omega}^j_a)} + |u^j_k|^2_{W^{2,2}_{1/2}(\Omega_a)}\right)\Bigg\}.\end{aligned} \tag{4.19}$$

Furthermore, we use

$$\begin{aligned}\sum_{|k|\geq 2} k^2\left\|\frac{1}{r}\frac{\partial\tilde{u}_k}{\partial r}\right\|^2_{L_{2,1/2}(\tilde{\Omega}^j_a)} \leq C\sum_{|k|\geq 2}\Bigg\{&k^2\left\|\frac{1}{r}\frac{\partial\tilde{u}_k}{\partial r} - \frac{1}{r^2}\tilde{u}_k\right\|^2_{L_{2,1/2}(\tilde{\Omega}^j_a)}\\ &+\left\|\frac{1}{r}\frac{\partial\tilde{u}_k}{\partial r} - \frac{k^2}{r^2}\tilde{u}_k\right\|^2_{L_{2,1/2}(\tilde{\Omega}^j_a)}\Bigg\}\end{aligned} \tag{4.20}$$

(cf. [24, Proof of Corollary 3.1]) and an inequality analogous to (4.20), where $\tilde{u}_k$ and $\tilde{\Omega}^j_a$ are replaced by $\tilde{u}^j_k$ and $\tilde{\Omega}_a$, respectively. This, combined with (2.11), completeness relations according to [11, Lemma 3.2], and relation (4.19), leads to the estimate

$$\|\tilde{u}_N - r_{hN}u\|^2_{X^1_{1/2}(\Omega_h)} \leq Ch^2\left\{\|\tilde{u}\|^2_{PX^2_{1/2}(\tilde{\Omega})} + \sum_{j=1}^{2}\|u^j\|^2_{X^2_{1/2}(\Omega)}\right\}. \tag{4.21}$$

Finally, (4.16), (4.17) and (4.21), together with (3.12), (4.2) and (2.15), yield (4.14).

We derive now estimates of the second term on the right-hand side of inequality (3.14). First, we associate with $w_{hN} \in V_{hN}$ (see (3.6)) some auxiliary function $\hat{w} \in V^*_0(\Omega)$ given by

$$\hat{w}(r, \varphi, z) = \sum_{|k|\leq N}\hat{w}_k(r, z)\,e^{ik\varphi}, \tag{4.22}$$

where $\hat{w}_k \in C(\bar{\Omega}_a) \cap W^{1,2}_{1/2}(\Omega_a)$ arises from w_{kh} by modification of w_{kh} on triangles T with $T \cap \Omega_a \neq T$ (i.e. T is located at some non-convex curved boundary part). This is done as follows: the linear finite element function on the straight line reference triangle T_{ref} is transformed into a function on $T^{\text{id}} := T \cap \Omega_a$ going with the non-linear mapping $T_{\text{ref}} \to T^{\text{id}}$ (see [9, 35] or [31, section 4.2.4] for more details). Moreover, $\hat{w}_k = 0$ on $\Omega_a \backslash \Omega_{ah}$ holds.

Lemma 4.4. *For $\tilde{u}$ from* (3.12) *and any function $w_{hN} \in V_{hN}$, the following inequality holds:*

$$\frac{|b_h(\tilde{u}, w_{hN}) - f_h(w_{hN})|}{\| w_{hN} \|_{X^1_{1/2}(\Omega_h)}} \leqslant Ch \| f \|_{X^0_{1/2}(\Omega)}. \tag{4.23}$$

Proof. First we see that for $w_{hN} \in V_{hN}$, $\hat{w} \in V^*_0(\Omega)$ from (4.22) and u from (2.7), with extension $\tilde{u}$, the inequality

$$\begin{aligned} |b_h(\tilde{u}, w_{hN}) - f_h(w_{hN})| &= |b_h(\tilde{u}, w_{hN}) - b(u, \hat{w}) + f(\hat{w}) - f_h(w_{hN})| \\ &\leqslant 2\pi \sum_{|k| \leqslant N} |b_{kh}(\tilde{u}_k, w_{kh}) - b_h(u_k, \hat{w}_k) \\ &\quad + f_k(\hat{w}_k) - f_{kh}(w_{kh})| \end{aligned} \tag{4.24}$$

holds, with $b_k(\cdot,\cdot), f_k(\cdot)$ from (2.12) and $b_{kh}(\cdot,\cdot), f_{kh}(\cdot)$ from (3.11). This is a consequence of the generalized completeness relation, cf. Lemma 2.1, and of the triangle inequality. By means of (4.24), Lemma 4.14 from [31], Hölders's inequality and the completeness relations according to [11, Lemma 3.2] we get

$$\begin{aligned} &|b_h(\tilde{u}, w_{hN}) - f_h(w_{hN})| \\ &\leqslant Ch \left\{ (\| \tilde{u} \|^2_{PX^2_{1/2}(\tilde{\Omega})} + \| f \|^2_{X^0_{1/2}(\Omega)}) \sum_{|k| \leqslant N} (|w_{kh}|^2_{W^{1,2}_{1/2}(\Omega_{ah})} + k^2 \| w_{kh} \|^2_{L_{2,-1/2}(\Omega_{ah})}) \right\}^{1/2}. \end{aligned}$$

This, together with

$$\| w_{hN} \|^2_{X^1_{1/2}(\Omega_h)} = 2\pi \sum_{|k| \leqslant N} \{ \| w_{kh} \|^2_{W^{1,2}_{1/2}(\Omega_{ah})} + k^2 \| w_{kh} \|^2_{L_{2,-1/2}(\Omega_{ah})} \}$$

(cf. (3.6) and [11, Lemma 3.2]), relation (3.12) and estimate (2.15), leads to (4.23).

Theorem 4.5. *Let $u_{hN} \in V_{hN}$ be the Fourier-finite-element approximation of $u \in V^*_0(\Omega)$, where u and u_{hN} are given by* (2.7) *and* (3.8). *Moreover, suppose that $u_{\mathrm{si}} \equiv 0$ and $u \in PX^2_{1/2}(\Omega)$ are satisfied (e.g. by Assumption 2.2). Then, the estimate*

$$\| \tilde{u} - u_{hN} \|_{X^1_{1/2}(\Omega_h)} \leqslant C(h + N^{-1}) \| f \|_{X^0_{1/2}(\Omega)} \tag{4.25}$$

holds, where $\tilde{u}$ is the extension of u according to (3.12).

Proof. Combining the estimates of Theorem 3.3, Lemmas 4.3 and 4.4, we get (4.25).

5. Adapted Fourier-FEM for interface singularities

Since interface singularities diminish the rate of convergence $u_{hN} \to u$, at least with respect to the discretization parameter h, we employ triangulations $\mathcal{T}_h$ of the meridian plane Ω_a with local mesh grading near the points associated with singularities of the Fourier coefficients u_k, here e.g. $P \in \Gamma_a \cap \Gamma^*_a$.
Introduce the real grading parameter μ, $0 < \mu \leqslant 1$, the grading function R_j, the step size h_j and regions of mesh grading B_j by (meanwhile take $R'_0 := R_0$, cf. Fig. 2)

$$\begin{aligned} &R_j := \tfrac{1}{3} R'_0 (jh)^{1/\mu}, \quad j = 1, \ldots, J, \\ &h_1 := R_1, \quad h_j := R_j - R_{j-1}, \quad j = 2, \ldots, J, \\ &B_1 := \{(r,z) \in \Omega_a : 0 < R \leqslant R_1\}, \; B_j := \{(r,z) \in \Omega_a : R_{j-1} < R \leqslant R_j\}, \quad j = 2, \ldots, J, \end{aligned} \tag{5.1}$$

with $J = [h^{-1}]$ (integer part of h^{-1}), $0 < h \leqslant h_0$. The co-ordinates R, θ are related with r, z by (2.16).

Assumption 5.1. *For $h \in (0, h_0]$ and fixed $\varepsilon_1 \in (0,1)$, the triangulation $\mathscr{T}_h$ satisfies Assumption* 3.1 *and*

(i) $0 < \varepsilon_1 h \leqslant h_T \leqslant \varepsilon_1^{-1} h$ *for* $T \in \mathscr{T}_h$: $T \cap B_j = \emptyset$ *for* $j = 1, \ldots, J$,

(ii) $0 < \varepsilon_1 h_j \leqslant h_T \leqslant \varepsilon_1^{-1} h_j$ *for* $T \in \mathscr{T}_h$: $T \cap B_j \neq \emptyset$ *at least for one* $j \in \{1, \ldots, J\}$,

i.e. for $\mathscr{T}'_h$ *from* (3.3) *we have* $\mathscr{T}'_h \subset \mathscr{T}_h$, *but* $\mathscr{T}'_h \neq \mathscr{T}_h$ *for* $\mu \neq 1$.

For simplicity and without loss of generality of the essential results on the interpolation and approximation errors assume that each T has a non-empty intersection at most with two consecutive regions B_j. Let n_j denote the number of triangles satisfying $T \cap B_j \neq \emptyset$ for any $j \in \{1, \ldots, J\}$; then by elementary considerations it is obvious that

$$n_j \leqslant Cj, \quad j = 1, \ldots, J \tag{5.2}$$

holds, where C does not depend on h $(0 < h \leqslant h_0)$ and j. Clearly, for $\mu = 1$ the mesh is quasi-uniform, for $0 < \mu < 1$ it is quasi-uniform only outside the region of grading the mesh. But the total number of nodes of $\mathscr{T}_h$ is always of the order $\mathcal{O}(h^{-2})$. In section 6, an example of mesh grading satisfying Assumption 5.1 is given.

The approximation error of the FFEM on meshes with grading can be estimated by Theorem 3.3, with the extension $\tilde{u}$ according to (3.13) and u from (2.19). Define extensions u^j $(j = 1, 2)$ of $u|_{\Omega^j \setminus G^j_\delta}$ to the domain $\Omega \setminus G_\delta$, with G_δ, G^j_δ from section 3 and any fixed δ, $0 < \delta_0 \leqslant \delta \leqslant \frac{2}{3} R_0$. Applying Calderon's extension theorem, we get

$$\begin{aligned} &u^j = u \quad \text{on } \Omega^j, \qquad \| u^j \|_{X^2_{1/2}(\Omega \setminus G_\delta)} \leqslant C \| u \|_{X^2_{1/2}(\Omega^j \setminus G^j_\delta)} \\ &\text{for } j = 1, 2;\ 0 < \delta_0 \leqslant \delta \leqslant \tfrac{2}{3} R_0. \end{aligned} \tag{5.3}$$

Then, the Fourier coefficients u^j_k of u^j $(j = 1, 2)$ satisfy

$$\begin{aligned} &u^j_k = u_k \ \text{on } \Omega^j_a, \qquad u^j_k \in W^{2;2}_{1/2}(\Omega_a \setminus G_{a,\delta}) \quad \text{for } |k| \geqslant 0; \\ &u^j_k \in W^{1;2}_{-1/2}(\Omega_a \setminus G_{a,\delta}) \text{ for } |k| \geqslant 2. \end{aligned}$$

Introduce subsets M^h_1, M^h_2 of $\mathscr{T}_h$ as follows:

$$M^h_1 = \left\{ T \in \mathscr{T}_h \colon \sup_{(r,z) \in T} R \leqslant \frac{R_0}{3} \right\}, \qquad M^h_2 = \left\{ T \in \mathscr{T}_h \colon \sup_{(r,z) \in T} R > \frac{R_0}{3} \right\}. \tag{5.4}$$

Clearly, $M^h_1 \cap M^h_2 = \emptyset$, $M^h_1 \cup M^h_2 = \mathscr{T}_h$, and the mesh comprising the triangles of M^h_1 is graded. For the sake of simplicity and w.l.o.g., assume that the singular part u_{si} of u consists of one function only, i.e. $\lambda = \lambda_1$, with $0 < \lambda^2_1 < 1$, cf. Theorem 2.3. Further we shall omit the index m occurring in (2.19), (2.20).

Lemma 5.2. *For* $u_k = c_k s_k + w_k$ $(k \in \mathbb{Z})$, *where* $s_k = s = \tilde{\eta} R^\lambda t(\theta)$, $0 < \lambda < 1$, *and* $w_k \in V^a_0(\Omega_a) \cap W^{2;2}_{1/2}(\Omega^j_a)$, $j = 1, 2$ (*cf. also Theorem* 2.3), *the following estimates hold*:

$$\sum_{T \in M^h_1} |\tilde{u}_k - \Pi_T u_k|^2_{W^{1;2}_{1/2}(T)} \leqslant C h^{2\alpha} S(c_k, w_k), \tag{5.5}$$

$$\sum_{T \in M^h_1} \| \tilde{u}_k - \Pi_T u_k \|^2_{L_{2,\beta}(T)} \leqslant C h^{2\gamma} S(c_k, w_k), \ \beta = \pm \tfrac{1}{2}, \tag{5.6}$$

with $\alpha = \lambda$, $\gamma = \lambda + 1$ *for quasi-uniform meshes* ($\mu = 1$), *and* $\alpha = 1$, $\gamma = 2$ *for meshes associated with the real grading parameter* μ: $0 < \mu < \lambda$, *and* $S(c_k, w_k)$ *is given by*

$$S(c_k, w_k) := |c_k|^2 + \sum_{T \in M_1^h} |w_k|^2_{W^{2,2}_{1/2}(T)}.$$

Proof. We only sketch the proof and refer to [31] for more details. Since the boundary Γ_a as well as the interface Γ_a^* are straight line in a neighbourhood $N(P, R_0)$ of the singular point P and all triangles $T \in M_1^h$ have a positive distance to the z-axis, the interpolation error analysis can be studied on straight line triangles, and weighting factors r^β are not active. It is well-known that for $v \in H^2(T)$ relation

$$\| v - \Pi_T v \|_{L_2(T)} + h_T |v - \Pi_T v|_{H^1(T)} \leqslant C h_T^2 |v|_{H^2(T)} \tag{5.7}$$

holds. Relation (5.7) can be applied to $v = w_k$ on triangles $T \in M_1^h$, and to $v = s_k$ on $T \in M_1^h$: $T \cap B_1 = \emptyset$, since on such triangles w_k and s_k belong to $H^2(T)$. For $T \cap B_1 \neq \emptyset$, we take advantage of the explicitly known singularity function $s_k = \tilde{\eta} R^\lambda t(\theta)$ (for $t(\theta)$, cf. [16, 31]; $\tilde{\eta} \equiv 1$ in B_1, w.l.o.g.) and find that

$$\| s_k \|_{L_2(T)} + \| \Pi_T s_k \|_{L_2(T)} + h_T \{ |s_k|_{H^1(T)} + |\Pi_T s_k|_{H^1(T)} \} \leqslant C |c_k| h_T^{\lambda+1} \tag{5.8}$$

holds, cf. [31, Proof of Lemma 5.1]. Using relations (5.1), (5.2), Assumption 5.1 and a careful summation of the error bounds assigned to B_j ($j = 1, \ldots, J$), we confirm (5.5), (5.6). This is already demonstrated in [31, 12] in more detail.

For triangles $T \in M_2^h$, the local error estimates given in [31, Lemmas 4.2–4.6] can be applied to $u_k = c_k s_k + w_k$, since here $u_k \in W^{2,2}_{1/2}(T)$, $k \in \mathbb{Z}$, holds. We get

$$\sum_{T \in M_2^h} \{ |\tilde{u}_k - \Pi_T u_k|^2_{W^{1,2}_{1/2}(T)} + \| \tilde{u}_k - \Pi_T u_k \|^2_{L_{2,\beta}(T)} \}$$
$$\leqslant C h^2 \sum_{j=1}^{2} \{ \| \tilde{u}_k \|^2_{W^{2,2}_{1/2}(\tilde{\Omega}_a^j \setminus G_{a,\delta}^j)} + \| u_k^j \|^2_{W^{2,2}_{1/2}(\Omega_a \setminus G_{a,\delta})} \}, \tag{5.9}$$

with $\beta = \pm \frac{1}{2}$ and $\delta = R_0/6$, since for $0 < h \leqslant h_0$ (h_0 sufficiently small) and for all $T \in M_2^h$ we have $T \cap G_{a,\delta} = \emptyset$.

Now we shall study the approximation properties of the operator P_h with respect to $u_k|_T$, $T \in M_1^h$.

Lemma 5.3. *For* u_k *from Lemma* 5.2, *the estimates*

$$\sum_{T \in M_1^h} |\tilde{u}_k - P_h u_k|^2_{W^{1,2}_{1/2}(T)} \leqslant C h^{2\alpha} S(c_k, w_k) \tag{5.10}$$

$$\sum_{T \in M_1^h} \| \tilde{u}_k - P_h u_k \|^2_{L_{2,\beta}(T)} \} \leqslant C h^{2\gamma} S(c_k, w_k), \quad \beta = \pm \tfrac{1}{2} \tag{5.11}$$

hold, with $\alpha = \lambda$, $\gamma = \lambda + 1$ *for quasi-uniform meshes* ($\mu = 1$), *and* $\alpha = 1$, $\gamma = 2$ *for meshes with local grading such that* $0 < \mu < \lambda$.

For the proof we refer to [31, Proof of Lemma 5.2] or [15, Proof of Lemma 6.3].

We consider now triangles $T \in M_2^h$. Here, the local error estimates given in [31, Lemmas 4.8–4.10] are valid, and we get by analogy to (4.11)–(4.13)

$$\sum_{T \in M_2^h} |\tilde{u}_k - P_h u_k|^2_{W^{1,2}_{1/2}(T)} \leqslant C h (S_\delta^1(u_k) + S_\delta^2(u_k)), \tag{5.12}$$

$$\sum_{T \in M_2^h} \| \tilde{u}_k - P_h u_k \|^2_{L_{2,\beta}(T)} \leqslant C h S_\delta^1(u_k), \quad \beta = \pm \tfrac{1}{2}, \tag{5.13}$$

with

$$S_\delta^1(u_k) := \sum_{j=1}^{2} \{|\tilde{u}_k|^2_{W^{1,2}_{-1/2}(\tilde{\Omega}_a^j\setminus G_{a,\delta}^j)} + |u_k^j|^2_{W^{1,2}_{-1/2}(\Omega_a\setminus G_{a,\delta})}\},$$

$$S_\delta^2(u_k) := \sum_{j=1}^{2} \{|\tilde{u}_k|^2_{W^{2,2}_{1/2}(\tilde{\Omega}_a^j\setminus G_{a,\delta}^j)} + |u_k^j|^2_{W^{2,2}_{1/2}(\Omega_a\setminus G_{a,\delta})}\}, \quad \delta = \frac{R_0}{6}.$$

By means of the estimates derived previously, we are in a position to formulate the following theorem about the rate of convergence of the Fourier-finite-element approximation.

Theorem 5.4. *Let u be the solution of the interface problem* (2.7) *and $u_{hN} \in V_{hN}$ the Fourier-finite-element approximation* (3.8). *Suppose that the assumptions of Theorem* 2.3 *are satisfied and that $\tilde{u}$ is the extension of u according to* (3.13). *Then, the error $\tilde{u} - u_{hN}$ satisfies the estimate*

$$\|\tilde{u} - u_{hN}\|_{X^1_{1/2}(\Omega_h)} \leqslant C(h^\alpha + N^{-1})\left\{\|f\|_{X^0_{1/2}(\Omega)} + \left\|\frac{\partial f}{\partial \varphi}\right\|_{X^0_{1/2}(G)}\right\}, \tag{5.14}$$

where

$$\alpha := \begin{cases} \lambda & \text{for quasi-uniform meshes } (\mu = 1), \\ 1 & \text{for meshes with local grading: } 0 < \mu < \lambda. \end{cases}$$

Proof. First, we employ estimate (3.14). For the first term on the right-hand side of (3.14), we get (4.16). The term $\|\tilde{u} - \tilde{u}_N\|_{X^1_{1/2}(\Omega_h)}$ can be bounded by

$$\|\tilde{u} - \tilde{u}_N\|_{X^1_{1/2}(\Omega_h)} \leqslant CN^{-1}\|f\|_{X^0_{1/2}(\Omega)}, \tag{5.15}$$

i.e. the rate of convergence $\tilde{u} \to \tilde{u}_N$ is not affected by the singularities admitted here (for more details, see [31, Proof of Lemma 5.3]).

Next we consider the term $\|\tilde{u}_N - r_{hN}u\|_{X^1_{1/2}(\Omega_h)}$ and employ the completeness relation (4.18). For estimating the terms on the right-hand side of (4.18), inequalities (5.5), (5.6) and (5.9) are added for $|k| \leqslant 1$, and inequalities (5.10)–(5.13) for $2 \leqslant |k| \leqslant N$. For further estimates of the resulting sums we apply relations by analogy to (4.19)–(4.21), where $\tilde{\Omega}_a^j$, Ω_a, $\tilde{\Omega}$ and Ω are replaced by $\tilde{\Omega}_a^j\setminus G_{a,\delta}^j$, $\Omega_a\setminus G_{a,\delta}$, $\tilde{\Omega}\setminus G_\delta$ and $\Omega\setminus G_\delta$, respectively. Moreover, in comparison with the proof of Lemma 4.3 we have to take into account additionally the terms

$$\sum_{|k|\leqslant N} k^{2l}|c_k|^2, \quad \sum_{|k|\leqslant N} k^{2l}|w_k|^2_{W^{2,2}_{1/2}(G_{a,\delta}^j)},$$

where $l = 0, 1; j = 1, 2; \delta = R_0/3$ (cf. the definition of M_1^h in (5.4)). These terms result from (5.5), (5.6), (5.10) and (5.11). Using the *a priori* estimate

$$\sum_{|k|\leqslant N} (1 + k^2)\left\{|c_k|^2 + \sum_{j=1}^{2} |w_k|^2_{W^{2,2}_{1/2}(G_{a,\delta}^j)}\right\} \leqslant C\left\{\|f\|^2_{X^0_{1/2}(\Omega)} + \left\|\frac{\partial f}{\partial \varphi}\right\|^2_{X^0_{1/2}(G)}\right\}, \tag{5.16}$$

which was proved in [16, 31], we are led to the estimate

$$\|\tilde{u}_N - r_{hN}u\|^2_{X^1_{1/2}(\Omega_h)} \leqslant Ch^{2\alpha}\left\{\|\tilde{u}\|^2_{PX^2_{1/2}(\tilde{\Omega}\setminus G_\delta)} + \sum_{j=1}^{2} \|u^j\|^2_{X^2_{1/2}(\Omega\setminus G_\delta)} + \|f\|^2_{X^0_{1/2}(\Omega)} \right.$$
$$\left. + \left\|\frac{\partial f}{\partial \varphi}\right\|^2_{X^0_{1/2}(G)}\right\}, \tag{5.17}$$

with $\delta = R_0/6$, and $\alpha = \lambda$ for quasiuniform meshes or $\alpha = 1$ for meshes with local grading parameter μ: $0 < \mu < \lambda$. By means of relations (3.13), (4.16), (5.3), (5.15), and (5.17) and by the *a priori* estimate

$$\| u \|_{PX^2_{1/2}(\Omega \backslash G_\delta)} \leqslant C \| f \|_{X^0_{1/2}(\Omega)}, \tag{5.18}$$

which can be concluded from [19, Theorem 10.1] and [20, Remark 7.1], we finally get the following bound of the approximation error of u in V_{hN}:

$$\inf_{v_{hN} \in V_{hN}} \| \tilde{u} - v_{hN} \|_{X^1_{1/2}(\Omega_h)} \leqslant C(h^\alpha + N^{-1}) \left\{ \| f \|_{X^0_{1/2}(\Omega)} + \left\| \frac{\partial f}{\partial \varphi} \right\|_{X^0_{1/2}(G)} \right\}. \tag{5.19}$$

We consider now the second term on the right-hand side of (3.14). For functions u of the type (2.19) and its extension $\tilde{u}$ according to (3.13), the estimate

$$\sup_{v_{hN} \in V_{hN}} \frac{|b_h(\tilde{u}, w_{hN}) - f_h(w_{hN})|}{\| w_{hN} \|_{X^1_{1/2}(\Omega_h)}} \leqslant Ch\{ \| u \|_{PX^2_{1/2}(\Omega \backslash G_\delta)} + \| f \|_{X^0_{1/2}(\Omega)} \} \tag{5.20}$$

holds, where $\delta = \frac{2}{3} R_0$ (for the proof, see [31, Proof of Theorem 5.1]). Combining Theorem 3.3 and the relations (5.18)–(5.20), we get the completion of the proof.

Remark 5.5. Owing to Remark 2.4, the Fourier-FEM on locally graded meshes as described in this section can also be applied to BVPs with non-smooth interfaces $\hat{\Gamma}^*$.

6. Numerical example

For verifying the rate of convergence of the FFEM according to Theorem 5.4 and for studying the influence of the mesh grading on the error, we consider some interface problem of type (2.7). The meridian domain Ω_a, which generates $\hat{\Omega}$, is L-shaped and given by Fig. 3. The interface Γ^*_a meets the boundary Γ_a at the corner P, with angle $\theta_0 = \frac{3}{2}\pi$.

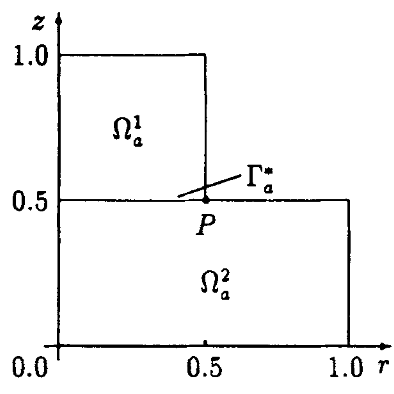

Fig. 3

The right-hand side f is chosen such that the assumptions of Theorem 2.3 prove to be satisfied and that the solution of (2.7) is given by

$$u(r,\varphi,z)=u_1(R,\varphi,\theta)+u_2(r,\varphi,z), \tag{6.1}$$

with

$$u_1(R,\varphi,\theta)=\begin{cases}R^\lambda(\alpha_0+\alpha_1R+\alpha_2R^{2-\lambda}+\alpha_3R^{3-\lambda})c(\varphi)t(\theta) & \text{for} \quad R\leqslant 0.5,\\ 0 & \text{for} \quad R>0.5,\end{cases}$$

$$c(\varphi)=\begin{cases}-[|\varphi|(\pi+\varphi)]^{1.51} & \text{for } \varphi\in(-\pi,0],\\ [\varphi(\pi-\varphi)]^{1.51} & \text{for } \varphi\in(0,\pi],\end{cases} \tag{6.2}$$

$$t(\theta)=\begin{cases}\sin(\lambda\theta) & \text{for } (r,z)\in\Omega_a^1,\\ \frac{1}{2}\sin[\lambda(\frac{3\pi}{2}-\theta)][\cos(\lambda\frac{\pi}{2})]^{-1} & \text{for } (r,z)\in\Omega_a^2,\end{cases}$$

$$u_2(r,\varphi,z)=0.01r^{1.1}c(\varphi), \tag{6.3}$$

and with constants $\alpha_0=-0.5^{3-\lambda}$, $\alpha_1=0.5^{2-\lambda}(\lambda-3)(\lambda-1)^{-1}$, $\alpha_2=0.5(\lambda-3)(1-\lambda)^{-1}$, $\alpha_3=1$. Since $u_1(R,\varphi,\theta)|_{\Gamma_a\times(-\pi,\pi]}=0$ holds, the Dirichlet boundary condition is given by $u(r,\varphi,z)|_{\Gamma_a\times(-\pi,\pi]}=u_2(r,\varphi,z)|_{\Gamma_a\times(-\pi,\pi]}$, with u_2 from (6.3). The variables R,θ,r,φ,z are to be understood as in section 2, especially the centre of the local polar co-ordinates (R,θ) is $P(r,z)=(0.5,\ 0.5)$, cf. (2.16). The parameter λ from (6.2) is determined by the ratio of the coefficients p_1 and p_2 from (2.8) and will be obtained as the smallest positive solution of the equation

$$-p_2\sin(\tfrac{\lambda\pi}{2})\cos(\lambda\pi)=p_1\cos(\tfrac{\lambda\pi}{2})\sin(\lambda\pi), \tag{6.4}$$

see also [31, section 6.2.2]. For the L-shaped domain, the relation $0.5<\lambda<1$ holds. In Table 1, for some pairs of p_1, p_2, the corresponding values of λ are presented. By means of (2.5) and (6.1)–(6.3), relation $u-\alpha_0R^\lambda c(\varphi)t(\theta)\in PX^2_{1/2}(\Omega)$ can be verified, and the term $\alpha_0R^\lambda c(\varphi)t(\theta)$ represents the singularity function u_{si} of u. Obviously, $c\in H^2(-\pi,\pi)$ holds, u_{si} is of tensor product type (cf. Theorem 2.3), and the Fourier series of u requires infinitely many non-zero Fourier coefficients.

For solving the interface problem provided by the solution (6.1), we utilized the package FEFOP (Finite Element Fourier Package), developed as an implementation of the FFEM for basic elliptic equations in axisymmetric domains. Some aspects of the implementation are briefly described in [13, 31], and a version of the algorithm for a MIMD parallel computer is given in [14, 28].

For numerical experiments, quasi-uniform meshes as well as graded meshes around $P(r,z)=(0.5,\ 0.5)$ are employed, especially for levels of consecutive mesh sizes $h_i=2^{-(i+2)}$ $(i=1,2,3,4)$ and grading parameters $\mu=0.7\lambda$. The meshes with local grading are provided with the properties of Assumption 5.1, and for some algorithm of generating such meshes, cf. [17]. Examples of locally graded meshes on the levels $h=h_1$ and $h=h_2$, with grading parameter $\mu=0.7\lambda$ for $\lambda=0.6$, are given in Figs. 4(a) and (b).

For the approximate measuring of the convergence rates stated in (5.14), the hypothesis for the tests is

$$\|u-u_{hN}\|_{X^1_{1/2}(\Omega)}\approx C_1h^\alpha+C_2N^{-\beta}, \tag{6.5}$$

with u from (6.1) and u_{hN} as its Fourier-finite-element approximation. The parameters C_1 and C_2 are assumed to be approximately constant for three consecutive levels of

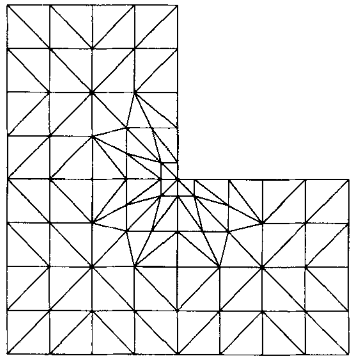

Fig. 4a

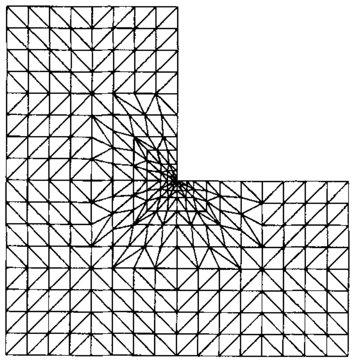

Fig. 4b

Table 1. Solutions λ of equation (6.4) for some pairs of p_1, p_2

p_1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
p_2	30.83623	5.39245	2.23607	1.00000	0.70130	0.23606
λ	0.51	0.55	0.6	0.66667	0.7	0.8

h and N, and h as well as N are varied independently from each other. For more details of calculating α and β occurring in relation (6.5), see [31, section 6.2.1] and [12, section 7].

Table 2 shows the observed values $\alpha = \alpha(\lambda, \mu)$ on the level $N = 50$ for different values of λ from Table 1 and grading parameters $\mu = \mu_1 = 1.0$ (quasi-uniform meshes) as well as $\mu = \mu_2 = 0.7\lambda$. According to Theorem 5.4, $\alpha_{\exp}(\lambda, 1) = \lambda$ and $\alpha_{\exp}(\lambda, 0.7\lambda) = 1$ are the theoretically expected values of α, and the observed values $\alpha(\lambda, \mu)$ are good approximations of $\alpha_{\exp}(\lambda, \mu)$.

In Table 3, for different values of λ and μ, the measured convergence order $\beta = \beta(\lambda, \mu)$ on the level $h = h_4 = 2^{-6}$ is given. The theoretically expected value is $\beta_{\exp}(\lambda, \mu) = 1$ for all λ, μ, and the experiments confirm that β is nearly constant and equal to 1.

In Fig. 5, for $1 \leqslant N \leqslant 30$ the approximate values of the error norms $e = e(N, h, \mu) = \| u - \check{u}_{hN} \|_{X^1_{1/2}(\Omega)}$, with $\check{u}_{hN}$ as the calculated value of u_{hN}, are

Table 2. Convergence order α on the level $N = 50$ for different values of λ and $\mu_1 = 1.0$, $\mu_2 = 0.7\lambda$

λ	0.51	0.55	0.6	0.67	0.7	0.8
$\alpha(\lambda, \mu_1)$	0.520	0.559	0.607	0.671	0.702	0.793
$\alpha_{exp}(\lambda, \mu_1)$	0.510	0.550	0.600	0.670	0.700	0.800
$\alpha(\lambda, \mu_2)$	1.090	1.098	1.006	1.117	1.122	1.049
$\alpha_{exp}(\lambda, \mu_2)$	1.000	1.000	1.000	1.000	1.000	1.000

Table 3. Convergence order β on the level $h = h_4$ for different values of λ and $\mu_1 = 1.0$, $\mu_2 = 0.7\lambda$

λ	0.51	0.55	0.6	0.67	0.7	0.8
$\beta(\lambda, \mu_1)$	0.975	0.975	0.974	0.975	0.975	0.976
$\beta(\lambda, \mu_2)$	0.976	0.975	0.975	0.975	0.975	0.977
$\beta_{exp}(\lambda, \mu)$	1.000	1.000	1.000	1.000	1.000	1.000

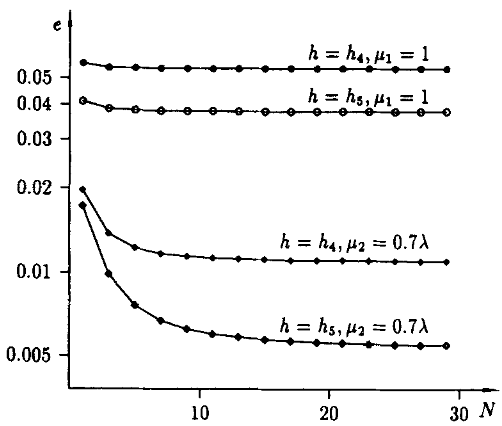

Fig. 5

visualized. Hence, the numerical experiments show that local mesh grading is suited to overcome the loss of accuracy and the diminishing of the order of convergence due to interface singularities.

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