Hermitian Adjacency Matrix of Digraphs and Mixed Graphs

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Abstract: The article gives a thorough introduction to spectra of digraphs via its Hermitian adjacency matrix. This matrix is indexed by the vertices of the digraph, and the entry corresponding to an arc from x to y is equal to the complex unity i (and its symmetric entry is $-i$) if the reverse arc yx is not present. We also allow arcs in both directions and unoriented edges, in which case we use 1 as the entry. This allows to use the definition also for mixed graphs. This matrix has many nice properties; it has real eigenvalues and the interlacing theorem holds for a digraph and its induced subdigraphs. Besides covering the basic properties, we discuss many differences from the properties of eigenvalues of undirected graphs and develop basic theory. The main novel results include the following. Several surprising facts are discovered about the spectral radius; some consequences of the interlacing property are obtained; operations that preserve the spectrum are

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discussed—they give rise to a large number of cospectral digraphs; for every $0 \leq \alpha \leq \sqrt{3}$, all digraphs whose spectrum is contained in the interval $(-\alpha, \alpha)$ are determined.

Keywords: *algebraic graph theory; eigenvalue; mixed graph; directed graph; spectral radius; cospectral*

1. INTRODUCTION

Investigation of eigenvalues of graphs has a long history. From early applications in mathematical chemistry (see [4] or chapter 8 of [5]), graph eigenvalues found use in combinatorics (see [1, 2, 5, 7]), combinatorial optimization (see [14, 17]), and most notably in theoretical computer science (see [19] and references therein), where it became one of the standard tools.

On the other hand, results about eigenvalues of digraphs are sparse. One reason is that it is not clear which matrix associated to a digraph D would best reflect interesting combinatorial properties in its spectrum. One candidate is the *adjacency matrix* $A = A(D)$ whose (u, v)-entry is 1 if there is an arc from the vertex u to v, and 0 otherwise. A well-known theorem of Wilf [21] bounding the chromatic number in terms of its largest eigenvalue extends to this setting as shown in [15]. However, the disadvantage of this matrix is that it is not symmetric and we lose the property that eigenvalues are real. Moreover, the algebraic and geometric multiplicities of eigenvalues may be different. Another candidate is the *skew-symmetric adjacency matrix* $S(D)$, where the (u, v)-entry is 1 if there is an arc from u to v, and -1 if there is an arc from v to u (and 0 otherwise). This choice is quite natural but it only works for *oriented graphs* (i.e., when we have no digons). We refer to a survey by Cavers et al. [3].

For a purpose in the Ph.D. thesis of one of the authors [9], the second author suggested to use the *Hermitian adjacency matrix* $H(D)$ of a digraph instead. This matrix is Hermitian and has many of the properties that are most useful for dealing with undirected graphs. For example, there is the eigenvalue interlacing property for eigenvalues of a digraph and its induced subdigraphs (see Section 4). In the meantime, Liu and Li independently introduced the same matrix in [13] and used it to define and study energy of mixed graphs.

In the Hermitian adjacency matrix, the (u, v)-entry is the imaginary unit i if there is an arc from u to v, $-i$ if there is an arc from v to u, 1 if both arcs exist, and 0 otherwise. This notion extends to the setting of partially oriented graphs or *mixed graphs* (see Section 2 for details).

In this article, we investigate basic properties of the Hermitian adjacency matrix, complementary to the results of Liu and Li [13]. We observe that the largest eigenvalue of the Hermitian adjacency matrix is upper bounded by the maximum degree of the underlying graph and we characterize the case when equality is attained (see Theorem 5.1). This is very similar to what happens with undirected graphs. On the other hand, several properties that follow from Perron–Frobenius theory are lost, and the differences that occur are discussed. For example, it may happen that the spectral radius $\rho(X)$ of a digraph X is not equal to the largest eigenvalue $\lambda_1(X)$ but is equal to the absolute value of its smallest eigenvalue. To some surprise, things cannot go arbitrarily "bad," as we are able to prove that

$$\lambda_1(X) \leq \rho(X) \leq 3\lambda_1(X).$$

Both of these inequalities are tight. See Theorem 5.6 and examples preceding it.

Similarly, in another contrast to the case of the adjacency matrix, there does not appear to be a bound on the diameter of the digraph in terms of the number of distinct eigenvalues of the Hermitian adjacency matrix. In fact, there is an infinite family of strongly connected digraphs whose number of distinct eigenvalues is constant, but whose diameter goes to infinity.

Despite the many unintuitive properties that the Hermitian adjacency matrix exhibits, it is still possible to extract combinatorial structure of the digraph from its eigenvalues. In Section 9, we find all digraphs whose H-eigenvalues lie in the range $(-\alpha, \alpha)$ for any $0 \leq \alpha \leq \sqrt{3}$. Using interlacing, one can find spectral bounds for the maximum independent set and maximum acyclic subgraphs for oriented graphs (see Section 4).

In Section 8 we discuss cospectral digraphs. As expected, cospectral pairs are not rare. Moreover, there are several kinds of "switching" operations that change the digraph but preserve its spectrum. In particular, the set of all digraphs obtained by orienting the edges of an arbitrary undirected graph of order n usually contains up to 2^n nonisomorphic digraphs cospectral to each other. This kind of questions occupies Section 8.

2. DEFINITIONS

A *directed graph* (or *digraph*) X consists of a finite set $V(X)$ of vertices together with a subset $E(X) \subseteq V(X) \times V(X)$ of ordered pairs called *arcs* or *directed edges*. Instead of $(x, y) \in E(X)$, we write xy for short. If $xy \in E$ and $yx \in E$, we say that the unordered pair $\{x, y\}$ is a *digon* of X.

A mixed graph is a graph where both directed and undirected edges may exist. More formally, a *mixed graph* is an ordered triple (V, E, A), where V is the vertex set, E is a set of undirected edges, and A is set of arcs, or directed edges. In this article, the Hermitian adjacency matrix is defined in such a way that the undirected edges may be thought of as digons and, from this perspective, mixed graphs are equivalent to the class of digraphs that we consider here.

If a digraph X has every arc contained in a digon, then we say that X is *undirected* or, more simply, that X is a *graph.* The *underlying graph of a digraph* X, denoted $\Gamma(X)$, is the graph with vertex set $V(X)$ and edge set

$$E = \{\{x, y\} \mid xy \in E(X) \text{ or } yx \in E(X)\}.$$

If a digraph X has no digons, we say that X is an *oriented graph.* Given a graph G, the *digraph* of G is the digraph on the same vertex set with every undirected edge replaced by a digon. We will denote the digraph of a graph G by $\vec{D}(G)$.

The *converse* of a digraph X is the digraph X^C with the same vertex set and arc set $E(X^C) = \{xy \mid yx \in E\}$. Observe that every digon of X is unchanged under the operation of taking the converse.

For a vertex $x \in V(X)$, we define the set of *in-neighbors* of x as $N_X^-(x) = \{u \in V(X) \mid ux \in E(X)\}$ and the set of *out-neighbors* as $N_X^+(x) = \{u \in V(X) \mid xu \in E(X)\}$. The *in-degree* of x, denoted $d^-(x)$, is the number of in-neighbors of x. The *out-degree* of x, denoted $d^+(x)$, is the number of out-neighbors of x. The maximum in-degree (resp. out-degree) of X will be denoted $\Delta^-(X)$ (resp. $\Delta^+(X)$).

For a digraph X with vertex set $V = V(X)$ and arc set $E = E(X)$, we consider the *Hermitian adjacency matrix* $H = H(X) \in \mathbb{C}^{V\times V}$, whose entries $H_{uv} = H(u, v)$ are given by

$$H_{uv} = \begin{cases} 1 & \text{if } uv \text{ and } vu \in E; \\ i & \text{if } uv \in E \text{ and } vu \notin E; \\ -i & \text{if } uv \notin E \text{ and } vu \in E; \\ 0 & \text{otherwise.} \end{cases}$$

If every edge of X lies in a digon, then $H(X) = A(X)$, which reflects that X is, essentially, equivalent to an undirected graph. This definition also covers mixed graphs, for which we replace each undirected edge by a digon.

In general, we define the Hermitian adjacency matrix of a mixed multigraph $X = (V, E, A)$ without digons to be the matrix $H = H(X)$ with entries given by $H_{uv} = a + bi - ci$, where a is the multiplicity of uv as an undirected edge, b is the multiplicity of uv as an arc in A, and c is the multiplicity of vu as an arc of A. (Note that either $b = 0$ or $c = 0$ since there are no digons.) In this article, we will mainly restrict our attention to *simple* digraphs and our results will hold for simple mixed graphs when they are considered as simple digraphs obtained by replacing each undirected edge by a digon. Having said this, we will from now on speak only of digraphs.

Observe that H is a Hermitian matrix and so is diagonalizable with real eigenvalues. The following proposition contains properties that are true for adjacency matrices that also carry over to the Hermitian case.

Proposition 2.1. *For a digraph X on n vertices and $H = H(X)$ its Hermitian adjacency matrix, the following are true:*

(i) *All eigenvalues of H are real numbers.*
(ii) *The matrix H has n pairwise orthogonal eigenvectors in $\mathbb{C}^n$ and so H is unitarily similar to a diagonal matrix.*

The eigenvalues of $H(X)$ are the *H-eigenvalues of X* and the spectrum of $H(X)$ (i.e., the multiset of eigenvalues, counting their multiplicities) is the *H-spectrum of X*. We will denote the H-spectrum by $\sigma_H(X)$ and we will express it as either a multiset of H-eigenvalues or a list of distinct H-eigenvalues with multiplicities in superscripts.

The H-eigenvalues of a digraph X will be ordered in the decreasing order, the jth largest eigenvalue will be denoted by $\lambda_j(X) = \lambda_j(H)$, so that $\lambda_1(X) \ge \lambda_2(X) \ge \cdots \ge \lambda_n(X)$. The characteristic polynomial of $H(X)$ will be denoted by $\phi(H(X), t)$ or simply by $\phi(X, t)$.

A direct consequence of Proposition 2.1 is the min–max formula for $\lambda_j(X)$:

$$\lambda_j(X) = \max_{\dim U=j} \min_{\mathbf{z}\in U} \mathbf{z}^* H\mathbf{z} = \min_{\dim U=n-j+1} \max_{\mathbf{z}\in U} \mathbf{z}^* H\mathbf{z}, \tag{1}$$

where $H = H(X)$ and the outer maximum (minimum) is taken over all subspaces U of dimension j $(n - j + 1)$ and the inner minimum (maximum) is taken over all $\mathbf{z}$ of norm 1.

We say that digraphs X and Y are *H-cospectral* if matrices $H(X)$ and $H(Y)$ are cospectral, that is, they have the same characteristic polynomials, $\phi(X, t) = \phi(Y, t)$.

Recall that digraphs X and Y are cospectral (or A-*cospectral*, if we wish to distinguish between the two matrices) if $A(X)$ and $A(Y)$ have the same characteristic polynomial. To avoid ambiguity, we refer to eigenvalues and spectrum of X with respect to its adjacency matrix as the A-*eigenvalues* and A-*spectrum,* respectively.

3. BASIC PROPERTIES

We first examine the coefficients of the characteristic polynomial of $H(X)$.

Lemma 3.1. *For X a digraph and $H = H(X)$ its Hermitian adjacency matrix,*

$$(H^2)_{uu} = d(u),$$

where $d(u)$ is the degree of u in the underlying graph of X.

Proof. Since H is Hermitian and has only entries 0, 1, and $\pm i$, we have

$$H_{uv}H_{vu} = H_{uv}\overline{H_{uv}} = 1$$

whenever $H_{uv} \neq 0$. This implies that the (u, u) diagonal entry in H^2 is the degree of u in the underlying graph of X. ■

Note that the degree of a vertex x of digraph X in the underlying graph of X is equal to $|N_X^-(x) \cup N_X^+(x)|$. Lemma 3.1 gives the following information about the coefficient of t^{n-2} in the characteristic polynomial of a digraph on n vertices.

Corollary 3.2. *Let X be a digraph on n vertices. The coefficient of t^{n-2} in the characteristic polynomial $\phi(H(X), t)$ is $-e$, where e is the number of edges of the underlying graph of X.*

Proof. Let Γ be the underlying graph of X and let $H = H(X)$. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of H, the characteristic polynomial of H can be written as

$$\phi(H, t) = (t - \lambda_1) \cdots (t - \lambda_n).$$

Thus, the coefficient of t^{n-2} is

$$c_2 = \sum_{1 \leq j < k \leq n} \lambda_j \lambda_k.$$

Observe that

$$\left(\sum_{j=1}^{n} \lambda_j \right)^2 = \sum_{j=1}^{n} \lambda_j^2 + 2 \sum_{1 \leq j < k \leq n} \lambda_j \lambda_k = \operatorname{tr}(H^2) + 2c_2.$$

The matrix H has all zeros on the diagonal, and so $\sum_{j=1}^{n} \lambda_j = 0$. Then, we obtain that

$$\operatorname{tr}(H^2) + 2c_2 = 0.$$

By Lemma 3.1, we have that $\operatorname{tr}(H^2) = \sum_{u \in V(X)} d_\Gamma(u) = 2e$, where e is the number of edges of Γ. Thus $c_2 = -e$. ■

Using the definition of matrix determinant as the sum of contributions over all permutations, we can write the characteristic polynomial of $H(X)$ in terms of cycles in the underlying graph as follows. Let X be a digraph on n vertices. A *basic subgraph* *of order j* of X is a subgraph of the underlying graph $\Gamma(X)$ with j vertices, each of whose components is either a single edge or a cycle in $\Gamma(X)$, and when it is a cycle, the corresponding digraph in X has an even number of directed (nondigon) edges. For each basic subgraph B, let $c(B)$ denote the number of cycles in B, and let $r(B)$ be $\frac{1}{2}|f-b|$, where f is the number of forward arcs and b is the number of backward arcs in B (for some orientation of the cycles in B, we are only interested in the value of $r(B)$ modulo 2, which is independent of the chosen orientations). Finally, let $s(B)$ be the number of components of B with even number of vertices (i.e., edges and even cycles).

The following result, which corresponds to a well-known result of Sachs [18] (see also [5, Section 1.4]), appears in [13, Theorem 2.8].

Theorem 3.3 ([13]). *Let X be a digraph of order n. Then the characteristic polynomial $\phi(X,t)=\sum_{j=0}^{n} c_j t^j$ has coefficients equal to*

$$c_j = \sum_B (-1)^{r(B)+s(B)} 2^{c(B)},$$

where the sum runs over all basic subgraphs B of order $n-j$ in $\Gamma(X)$.

In particular, the expression for c_{n-2} from Theorem 3.3 yields the result in Corollary 3.2 since all basic graphs of order 2 are single edges. Using this theorem, we also obtain the following corollary, which has also been used in [13].

Corollary 3.4 ([13]). *Suppose that $\{u,v\}$ is a digon in X. If uv is a cut-edge of $\Gamma(X)$, then the spectrum of $H(X)$ is unchanged when the digon is replaced with a single arc uv or vu.*

Analogous to the results for the adjacency matrix found in standard texts like [1], we may write the (u,v)-entry of $H(X)^k$ as a weighted sum of the walks in $\Gamma(X)$ of length k from u and v.

Proposition 3.5. *Let X be a digraph and $H=H(X)$. For vertices $u,v\in V(X)$ and any positive integer k, the (u,v)-entry of the kth power of H can be expressed as follows:*

$$\left(H^k\right)_{uv} = \sum_{W\in\mathcal{W}} \mathrm{wt}(W),$$

where $\mathcal{W}$ is the set of all walks of length k from u to v in $\Gamma(X)$ and for $W=(v_0,\ldots,v_k)\in\mathcal{W}$, where $v_0=u$ and $v_k=v$, the weight is

$$\mathrm{wt}(W) = \prod_{j=0}^{k-1} H(v_j, v_{j+1}).$$

We will use Proposition 3.5 to find an expression for $\mathrm{tr}(H(X)^3)$ in terms of numbers of subdigraphs isomorphic to certain types of triangles.

Proposition 3.6. *Let X be a digraph of order n and $\lambda_1,\ldots,\lambda_n$ be its H-eigenvalues. Then*

(i) $\sum_{j=1}^{n} \lambda_j = 0$.

(ii) $\sum_{j=1}^{n} \lambda_j^2 = 2e$, *where e is the number of edges of $\Gamma(X)$.*

(iii) $\sum_{j=1}^{n} \lambda_j^3 = 6(x_2+x_3+x_4-x_1)$, *where x_j is the number of copies of X_j as an induced subdigraph of X and X_j $(1\le j\le 4)$ are the digraphs shown in Figure 1.*

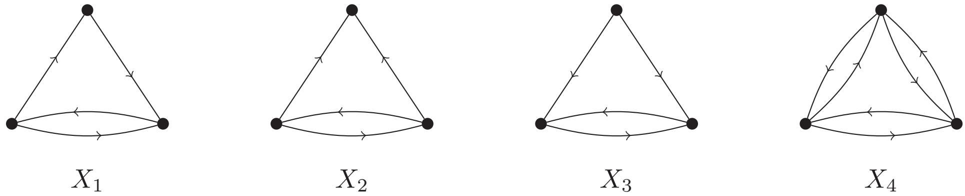


FIGURE 1. All nonisomorphic digraphs with even number of arcs whose underlying graph is the 3-cycle.

Proof. The first two statements are easy consequences of the facts that $\mathrm{tr}(H) = 0$ and $\mathrm{tr}(H^2) = 2e$. Below we provide details for (iii).

Let X be a digraph with $V = V(X)$ and $E = E(X)$. We apply Proposition 3.5 to obtain

$$\mathrm{tr}(H^3) = \sum_{v\in V}(H^3)_{v,v} = \sum_{v\in V}\sum_{W\in\mathcal{W}_v} \mathrm{wt}(W),$$

where W_v is the set of closed walks at v of length 3 in $\Gamma(X)$ and wt as defined in Proposition 3.5. Every closed walk of length 3 has C_3 as its underlying graph. For simplicity, we will refer to any digraph with C_3 as the underlying graph as a *triangle* and we will call the corresponding subdigraph in X a *triangle of* X. In Figure 1, we list all nonisomorphic triangles with an even number of arcs (nondigon edges). Those with an odd number make contributions to $\mathrm{tr}(H)$ that cancel each other out. Finally, note that the weight of X_1 is -1, while the weight of the others is 1. ■

We define $G(X)$, the *symmetric subgraph of a digraph* X, to be the graph with vertex set $V(X)$ and the edge set being the set consisting of all digons of X. Similarly, we define $D(X)$, the *asymmetric subdigraph of* X, to be the digraph with vertex set $V(X)$ and the arc set being the set of arcs of X that are not contained in any digons of X.

Lemma 3.7. *Let X be a digraph. Then $H(X)$ has the all-ones vector $\mathbb{1}$ as an eigenvector if and only if $G(X)$ is regular and $D(X)$ is Eulerian.*

Proof. Suppose $D(X)$ is Eulerian, then the sum along any row of $H(D(X))$ is equal to 0 and so $\mathbb{1}$ is an eigenvector for $H(D(X))$ of eigenvalue 0. If $G(X)$ is regular, then $\mathbb{1}$ is an eigenvector of $H(G(X)) = A(G(X))$ with eigenvalue equal to the valency of $G(X)$, say k. Then

$$H(X)\mathbb{1} = H(D(X))\mathbb{1} + H(G(X))\mathbb{1} = k\mathbb{1}.$$

For the other direction, suppose that $H(X)\mathbb{1} = \gamma\mathbb{1}$ for some $\gamma \in \mathbb{R}$. The row sum along the row indexed by x is equal to $d + (s - t)i$, where $d = d(x)$, $s = d^-(x)$, and $t = d^+(x)$. Then, since γ is a real number, we must have that $s = t$ and $d = \gamma$. ■

4. INTERLACING

Suppose that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_{n-t}$ (where $t \geq 1$ is an integer) be two sequences of real numbers. We say that the sequences λ_l $(1 \leq l \leq n)$ and κ_j $(1 \leq j \leq n - t)$ *interlace* if for every $s = 1, \ldots, n - t$, we have

$$\lambda_s \geq \kappa_s \geq \lambda_{s+t}.$$

The usual version of the eigenvalue interlacing property states that the eigenvalues of any principal submatrix of a Hermitian matrix interlace those of the whole matrix (see [11, Theorems 4.3.8 and 4.3.15]).

Theorem 4.1. *If H is a Hermitian matrix and B is a principal submatrix of H, then the eigenvalues of B interlace those of H.*

Interlacing of eigenvalues is a powerful tool in algebraic graph theory. Theorem 4.1 implies that the eigenvalues of any induced subdigraph interlace those of the digraph itself.

Corollary 4.2. *The eigenvalues of an induced subdigraph interlace the eigenvalues of the digraph.*

To see a simple example how useful the interlacing theorem is, let us consider the following notion. Let $\eta^+(X)$ denote the number of nonnegative H-eigenvalues of a digraph X and $\eta^-(X)$ denote the number of nonpositive H-eigenvalues of X.

Theorem 4.3. *If a digraph X contains a subset of m vertices, no two of which form a digon, then $\eta^+(X) \geq \lceil \frac{m}{2} \rceil$ and $\eta^-(X) \geq \lceil \frac{m}{2} \rceil$.*

Proof. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the H-eigenvalues of X. Let $U \subseteq V(X)$, $|U| = m$, be a subset without digons, and let B be the principal submatrix of $H(X)$ with rows and columns corresponding to U. As shown in [13] (see also Theorem 6.2 in this article), the H-eigenvalues μ_j $(1 \leq j \leq m)$ of B are symmetric about 0. Therefore, $\mu_{\lceil \frac{m}{2} \rceil} \geq 0$ and $\mu_{\lfloor \frac{m}{2} \rfloor + 1} \leq 0$. By interlacing, we see that $\lambda_{\lceil \frac{m}{2} \rceil} \geq \mu_{\lceil \frac{m}{2} \rceil} \geq 0$ and $\lambda_{n - \lceil \frac{m}{2} \rceil + 1} \leq \mu_{\lfloor \frac{m}{2} \rfloor + 1} \leq 0$. This implies the result. ■

Similarly, the Cvetković bound (see [5]) for the largest independent set of a graph extends to digraphs and their Hermitian adjacency matrix.

Proposition 4.4. *If X has an independent set of size α, then $\eta^+(X) \geq \alpha$ and $\eta^-(X) \geq \alpha$.*

Proof. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of $H(X)$. By interlacing, we see that $\lambda_\alpha \geq 0$ and so $H(X)$ has at least α nonnegative eigenvalues. Applying the same argument to $-H(X)$ shows that there are at least α nonpositive eigenvalues as well. ■

We can also obtain a spectral bound on the maximum induced transitive tournament of a digraph. In [8], Gregory et al. found the tight upper bound on the spectral radius of a skew-symmetric matrix and classified the matrices that attain the bound. We will state a special case of their theorem restricted to the Hermitian adjacency matrices of oriented graphs. Let us recall that two tournaments are *switching equivalent* if one can be obtained from the other by reversing all arcs across an edge-cut of the underlying graph; see also Section 10.

Theorem 4.5 ([8]). *If X is an oriented graph of order n, then*

$$\lambda_1(H(X)) \leq \cot\left(\frac{\pi}{2n}\right).$$

Equality holds if and only if X is switching equivalent to T_n, the transitive tournament of order n.

The following lemma is another immediate consequence of interlacing.

Corollary 4.6. *If X is a digraph with an induced subdigraph that is switching equivalent to T_m, then $\lambda_1(H(X)) \geq \cot(\frac{\pi}{2m})$.*

In other words, if $m > \frac{\pi}{2\cot^{-1}(\lambda_1(H(X)))}$ then X cannot contain an induced subdigraph that would be switching equivalent to T_m.

A more general version of the interlacing theorem (see, e.g., [2] or [10]) involves more general orthogonal projections.

Theorem 4.7. *If H is a Hermitian matrix of order n and S is a matrix of order $k \times n$ such that $SS^* = I_k$, then the eigenvalues of H and the eigenvalues of SHS^* interlace.*

For a digraph X, let $\Pi = V_1 \cup \cdots \cup V_m$ be a partition of the vertex set of X and order the vertices of X such that Π induces the following partition of $H(X)$ into block matrices:

$$H(X) = \begin{pmatrix} H_{11} & \cdots & H_{1m} \\ \vdots & \ddots & \vdots \\ H_{m1} & \cdots & H_{mm} \end{pmatrix}.$$

The *quotient* matrix of $H(X)$ with respect to Π is the matrix $B = [b_{jk}]$, where b_{jk} is the average row sum of the block matrix H_{jk}, for $j, k \in [m]$. The following corollary of Theorem 4.7 is proved in the same way as the corresponding result for undirected graphs found in [10].

Theorem 4.8. *Eigenvalues of the quotient matrix of $H(X)$ with respect to a partition of vertices interlace those of $H(X)$.*

The partition Π is said to be *equitable* if each block H_{jk} has constant row sums for $j, k \in [m]$. In this case, more can be said.

Corollary 4.9. *Let Π be an equitable partition of the vertices of X and B is the quotient matrix of $H(X)$ with respect to Π. If λ is an eigenvalue of B with multiplicity μ, then λ is an eigenvalue of $H(X)$ with multiplicity at least μ.*

These results remain true in the setting of digraphs with multiple edges.

5. SPECTRAL RADIUS

For a digraph X, let the eigenvalues of $H = H(X)$ be $\lambda_1 \geq \cdots \geq \lambda_n$. Note that since H is not a matrix with nonnegative entries, there is no analogue of the Perron value of the adjacency matrix and the properties of λ_1 may be highly unintuitive. Figure 3 shows a strongly connected digraph K_3' on three vertices with H-eigenvalues $\{1^{(2)}, -2\}$. This shows that, in general, λ_1 is not necessarily simple or largest in magnitude.

Instead of considering the largest eigenvalue, we may consider the largest eigenvalue in absolute value, for which we find a bound that is analogous to the adjacency matrix. The *spectral radius* $\rho(M)$ of a matrix M is defined as

$$\rho(M) = \max\{|\lambda| \mid \lambda \text{ an eigenvalue of } M\}$$

and we also define $\rho(X) = \rho(H(X))$ as the *spectral radius* of the digraph X.

Theorem 5.1. *If X is a digraph (multiple edges allowed), then $\rho(X) \leq \Delta(\Gamma(X))$. When X is weakly connected, the equality holds if and only if $\Gamma(X)$ is a $\Delta(\Gamma(X))$-regular*

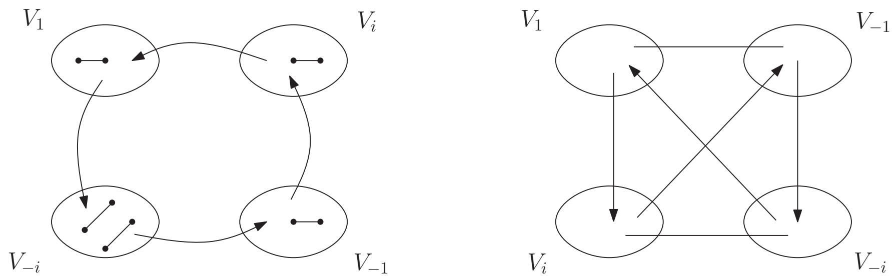


FIGURE 2. Cases (i) and (ii) of Theorem 5.1.

graph and there exists a partition of $V(X)$ into four (possibly empty) parts V_1, V_{-1}, V_i, and V_{-i} such that one of the following holds:

(i) *For $j \in \{\pm 1, \pm i\}$, the digraph induced by V_j in X contains only digons. Every other arc uv of X is such that $u \in V_j$ and $v \in V_{(-i)j}$ for some $j \in \{\pm 1, \pm i\}$. See Figure 2.*

(ii) *For $j \in \{\pm 1, \pm i\}$, the digraph induced by V_j in X is an independent set. For each $j \in \{\pm 1, \pm i\}$, every arc with one end in V_j and one end in V_{-j} is contained in a digon. Every other arc uv of X is such that $u \in V_j$ and $v \in V_{ij}$ for some $j \in \{\pm 1, \pm i\}$. See Figure 2.*

Proof. Let $H = H(X)$ and let λ be an eigenvalue of H with eigenvector $\mathbf{x}$. Choose $v \in V(X)$ such that $|\mathbf{x}(v)|$ is maximum. Now we consider the v-entry of $H\mathbf{x}$. For simplicity of notation, we will write $N(v) := N_X^-(v) \cap N_X^+(v)$. We obtain

$$(H\mathbf{x})(v) = \sum_{u \in N(v)} \mathbf{x}(u) + i \sum_{w \in N_X^+(v) \setminus N(v)} \mathbf{x}(w) - i \sum_{y \in N_X^-(v) \setminus N(v)} \mathbf{x}(y).$$

On the other hand, $(H\mathbf{x})(v) = \lambda \mathbf{x}(v)$. Then

$$\begin{aligned}
|\lambda \mathbf{x}(v)| &= |(H\mathbf{x})(v)| \\
&\le \sum_{u \in N(v)} |\mathbf{x}(u)| + \sum_{w \in N_X^+(v) \setminus N(v)} |\mathbf{x}(w)| + \sum_{y \in N_X^-(v) \setminus N(v)} |\mathbf{x}(y)| \\
&\le \sum_{u \in N(v)} |\mathbf{x}(v)| + \sum_{w \in N_X^+(v) \setminus N(v)} |\mathbf{x}(v)| + \sum_{y \in N_X^-(v) \setminus N(v)} |\mathbf{x}(v)| \\
&= \deg_{\Gamma(X)}(v)\, |\mathbf{x}(v)| \qquad (2)\\
&\le \Delta(\Gamma(X))\, |\mathbf{x}(v)|.
\end{aligned}$$

From this, we obtain that $|\lambda| \le \Delta(\Gamma(X))$.

If $\rho(H(X)) = \Delta(\Gamma(X))$, then all of the inequalities in (2) must hold with equality. From the last inequality of (2), we see that v has degree $\Delta(\Gamma(X))$ in $\Gamma(X)$. If the second inequality of (2) holds with equality, we obtain that $|\mathbf{x}(z)| = |\mathbf{x}(v)|$ for all $z \in N_X^-(v) \cup N_X^+(v)$. Since the choice of v was arbitrary amongst all vertices attaining the maximum absolute value in $\mathbf{x}$, we may apply this same argument to any vertex adjacent to v in $\Gamma(X)$. Since X is weakly connected, we obtain that $|\mathbf{x}(z)| = |\mathbf{x}(v)|$ for any $z \in V(X)$.

The first inequality of (2) follows from the triangle inequality for sums of complex numbers, and so equality holds if and only if every complex number in Z has the same argument as $\lambda\mathbf{x}(v)$, where

$$\begin{aligned} Z = &\{\mathbf{x}(u) \mid u \in N(v)\} \cup \\ &\{i\mathbf{x}(w) \mid w \in N_X^+(v) \setminus N(v)\} \cup \\ &\{-i\mathbf{x}(y) \mid y \in N_X^-(v) \setminus N(v)\}. \end{aligned}$$

We may normalize $\mathbf{x}$ such that $\mathbf{x}(v) = 1$. There are three cases: $\lambda = 0$, λ is positive, or λ is negative. Since we are bounding the spectral radius, we need not consider the $\lambda = 0$ case; the only digraph with $\rho(X) = 0$ is the empty graph and the statement of the theorem holds there.

Suppose that λ is positive. Every complex number in Z has the same norm and same argument as $\mathbf{x}(v) = 1$ and is thus equal to 1. We conclude that

$$\mathbf{x}(z) = \begin{cases} 1, & \text{if } z \in N(v); \\ -i, & \text{if } z \in N_X^+(v) \setminus N(v); \text{ and} \\ i, & \text{if } z \in N_X^-(v) \setminus N(v). \end{cases}$$

Repeating the argument at a vertex w such that $\mathbf{x}(w) = -i$, we see that

$$\mathbf{x}(z) = \begin{cases} -i, & \text{if } z \in N(w); \\ -1, & \text{if } z \in N_X^+(w) \setminus N(w); \text{ and} \\ +1, & \text{if } z \in N_X^-(w) \setminus N(w). \end{cases}$$

Similar argument can be used when $\mathbf{x}(z) = -1$ or $\mathbf{x}(z) = i$. From this we conclude that $V(X)$ has a partition into sets V_1, V_{-1}, V_i, and V_{-i} such that condition (i) of the theorem holds.

Suppose now that λ is negative. Every complex number in Z has the same norm and same argument as -1 and is hence equal to -1. Thus we obtain that

$$\mathbf{x}(z) = \begin{cases} -1, & \text{if } z \in N(v); \\ i, & \text{if } z \in N_X^+(v) \setminus N(v); \text{ and} \\ -i, & \text{if } z \in N_X^-(v) \setminus N(v). \end{cases}$$

Repeating the argument at vertices w such that $\mathbf{x}(w) = -1$ or $\pm i$, we see that $V(X)$ has a partition into V_1, V_{-1}, V_i, and V_{-i} such that condition (ii) of the theorem holds.

We now consider the converse for the two cases of the theorem. Let X be a digraph such that $\Gamma(X)$ is k-regular. Suppose that $V(X)$ has a partition V_1, V_{-1}, V_i, and V_{-i} such that condition (i) or (ii) holds. Let $\mathbf{x}$ be the vector indexed by the vertices of X such that $\mathbf{x}(z) = j$ if $z \in V_j$. Then it is easy to see that for every vertex v we have $(H\mathbf{x})(v) = k\mathbf{x}$ (in case (i)) or $(H\mathbf{x})(v) = -k\mathbf{x}$ (in case (ii)). Thus $\mathbf{x}$ is an eigenvector for H with eigenvalue $\pm k$, and the bound is tight as claimed. ■

For undirected graphs, $\rho(X)$ is always larger or equal to the average degree. However, for digraphs, $\rho(X)$ can be smaller than the minimum degree in $\Gamma(X)$. An example is the digraph $\widetilde{C}_3$ shown in Figure 5, which has eigenvalues $\pm\sqrt{3}$ and 0, while its underlying graph has minimum degree 2. Of course, this anomaly is also justified by Theorem 5.1 since $\widetilde{C}_3$ does not have the structure as in Figure 2.

Next we shall discuss digraphs whose H-spectral radius $\rho(X)$ is larger than the largest H-eigenvalue $\lambda_1(X)$. In that case, $\rho(X)$ is attained by the absolutely largest negative

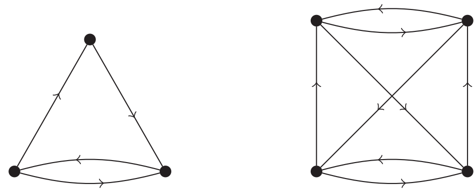


FIGURE 3. K_3' and K_4'.

eigenvalue. First we treat an extremal case in which the eigenvalues are precisely the opposite of those for complete graphs.

A. Digraphs with Spectrum $\{-(n-1), 1^{(n-1)}\}$

The tightness case of Theorem 5.1 shows when $\lambda_1(X)$ is large in terms of vertex degrees. By trying to do the converse—make $\lambda_1(X)$ small—we arrive to digraphs K_3' and K_4' (see Figure 3) with H-spectra $\{-2, 1^{(2)}\}$ and $\{-3, 1^{(3)}\}$, respectively. It is worth mentioning that each of K_3' and K_4' has its H-spectrum that is the negative of the spectrum of their underlying complete graphs K_3 and K_4, respectively. Naturally, we may ask if there are other digraphs on n vertices whose H-spectrum is $\{-(n-1), 1^{(n-1)}\}$. Such digraphs would have a large negative eigenvalue and small positive eigenvalues and thus exhibit extreme spectral behavior, opposite to the behavior of undirected graphs, whose spectral radius always equals the largest (positive) eigenvalue. We answer this in the negative and show that K_3' and K_4' are the only nontrivial digraphs with this property.

Proposition 5.2. *If X is a digraph such that $\sigma_H(X) = \{-(n-1), 1^{(n-1)}\}$, then $X \cong Y$ where $Y \in \{K_1, K_2, T_2, K_3', K_4'\}$, where T_2 is the oriented K_2.*

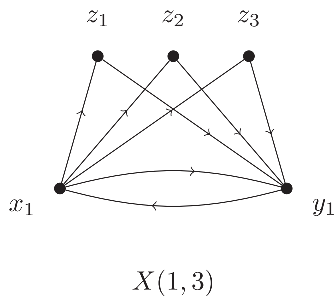
Proof. Let X be a digraph of order n such that $H = H(X)$ has spectrum $\{-(n-1), (-1)^{(n-1)}\}$. If $n = 1$, then $X \cong K_1$ and if $n = 2$, then $X \cong K_2$ or T_2. Suppose now that $n \geq 3$. The characteristic polynomial of H is $\phi(H, t) = (t + (n-1))(t-1)^{n-1}$. Observe that $\phi(H(K_n)) = (t - (n-1))(t+1)^{n-1}$ and so its coefficients are

$$[t^k]\phi(H,t) = \begin{cases} [t^k]\phi(H(K_n), t), & \text{if } n-k \text{ is even;} \\ -[t^k]\phi(H(K_n), t) & \text{if } n-k \text{ is odd.} \end{cases}$$

In particular, $[t^{n-2}]\phi(H,t) = [t^{n-2}]\phi(H(K_n), t)$. Thus, by Corollary 3.2, $\Gamma(X)$ has the same number of edges as K_n, and so $\Gamma(X) \cong K_n$.

Also, $\mathrm{tr}(H^3) = -\mathrm{tr}(H(K_n)^3)$. By Proposition 3.6, we see that $\mathrm{tr}(H(K_n)^3) = 6\binom{n}{3}$. Therefore, $\mathrm{tr}(H^3) = -6\binom{n}{3}$ and so every three vertices of X must induce a digraph isomorphic to the digraph K_3', the only possible triangle with negative weight.

Consider $G(X)$, the symmetric subgraph of X. If there is a path uvw of length 2 in $G(X)$, then $\{u, v, w\}$ induce a triangle of X with more than one digon and hence not isomorphic to K_3', which is a contradiction. Thus, each connected component of $G(X)$ is a copy of either K_1 or K_2. If $G(X)$ has three or more components, then choosing three vertices from different components will give a triangle with no digons and hence not isomorphic to K_3'. Thus, $G(X)$ has at most two components. Since $n \geq 3$, we see that



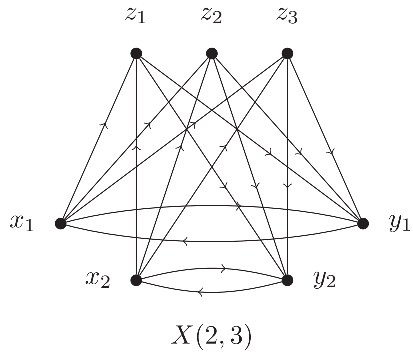


FIGURE 4. Digraphs $X(1,3)$ and $X(2,3)$, constructed as examples of digraphs with a large negative H-eigenvalue.

$G(X)$ has exactly two components and $n \in \{3,4\}$. If $n = 3$, then $X \cong K_3'$ since K_3' is an induced subdigraph.

If $n = 4$, then we may assume that $\{x_1, x_2\}$ and $\{x_3, x_4\}$ are digons in X. The deletion of any vertex of X results in a digraph isomorphic to K_3'. When deleting x_4, we may assume that x_1x_3 and $x_3x_2 \in E(X)$. Now it is easy to see that $x_2x_4 \in E(X)$ (consider deleting x_1) and $x_4x_1 \in E(X)$, giving us K_4'. ■

B. Digraphs with a Large Negative *H*-Eigenvalue

We define a digraph $X(a,b)$ on $2a + b$ vertices, where $a \geq 1$ and $b \geq 1$. The vertices of $X(a,b)$ consist of $X \cup Y \cup Z$, where $X = \{x_1, \ldots, x_a\}$, $Y = \{y_1, \ldots, y_a\}$, and $Z = \{z_1, \ldots, z_b\}$. The arcs are

$$\{x_j y_j, y_j x_j \mid j = 1, \ldots a\}$$

and

$$\{x_j z_\ell, z_\ell y_j \mid j = 1, \ldots, a \text{ and } \ell = 1, \ldots b\}.$$

We see that K_3' from the previous section is isomorphic to $X(1,1)$. Figure 4 shows $X(1,3)$ and $X(2,3)$.

Lemma 5.3. *Digraph $X(a,b)$ defined above has H-spectrum*

$$\left\{ \frac{-1 + \sqrt{1 + 8ab}}{2},\ 1^{(a)},\ 0^{(b-1)},\ -1^{(a-1)},\ \frac{-1 - \sqrt{1 + 8ab}}{2} \right\}.$$

Proof. Let $H = H(X(a,b))$. We may write H in the following form:

$$H = \begin{pmatrix} \mathbf{0} & I_a & iJ_{a,b} \\ I_a & \mathbf{0} & -iJ_{a,b} \\ -iJ_{b,a} & iJ_{b,a} & \mathbf{0} \end{pmatrix},$$

where we recall that I_n denotes the $n \times n$ identity matrix and $J_{m,n}$ denotes the $m \times n$ all-ones matrix.

Observe that the last b rows are all identical and hence linearly dependent. This shows that $\mathrm{rk}(H) \leq 2a + b - (b-1)$, which implies that H has 0 as an eigenvalue with multiplicity at least $b - 1$.

For $j = 1, \ldots, a$, let $\mathbf{v}_j = (\mathbf{e}_j \quad \mathbf{e}_j \quad \mathbf{0})^T$, where $\mathbf{e}_j$ is the a-dimensional jth elementary vector. We see that $H\mathbf{v}_j = \mathbf{v}_j$ for $j = 1, \ldots, a$ and so 1 is an eigenvalue of H with multiplicity at least a.

Similarly, for $j = 1, \ldots, a-1$, let

$$\mathbf{w}_j = \begin{pmatrix} \mathbf{e}_j - \mathbf{e}_a \\ -(\mathbf{e}_j - \mathbf{e}_a) \\ \mathbf{0} \end{pmatrix},$$

where $\mathbf{e}_n$ is defined as above. Then $H\mathbf{w}_j = -\mathbf{w}_j$ and so -1 is an eigenvalue of H with multiplicity at least $a - 1$.

We have found $2a + b - 2$ eigenvalues of H. To find the remaining two eigenvalues, we will use the interlacing theorem. Partition the vertices of $X(a, b)$ (and consequently the rows and columns of H) into the sets X, Y, and Z. Each block of H, under this partition, has constant row sums and so this is an equitable partition of H. We obtain B, the quotient matrix corresponding to this partition as follows:

$$B = \begin{pmatrix} 0 & 1 & ib \\ 1 & 0 & -ib \\ -ia & ia & 0 \end{pmatrix}.$$

We find that the characteristic polynomial of B is

$$\phi(B, t) = t^3 - (2ab + 1)t + 2ab = (t - 1)(t^2 + t - 2ab)$$

whose roots are 1, $\tau = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 8ab}$ and $\sigma = -\frac{1}{2} - \frac{1}{2}\sqrt{1 + 8ab}$. The partition is equitable and so τ and σ are also eigenvalues of H, by Corollary 4.9. Since for $a, b \geq 1$, τ and σ are not equal to any of the eigenvalues of H that we have already found. The trace formula gives that the last eigenvalue is another 1. Thus, H has spectrum $\{\tau, 1^{(a)}, 0^{(b-1)}, -1^{(a-1)}, \sigma\}$. ∎

In the sequel we will need a formula for eigenvalues of Cartesian products of digraphs. Let X and Y be digraphs. The *Cartesian product* of X and Y, denoted by $X \Box Y$, is the graph with vertex set $V(X) \times V(Y)$ such that there is an arc from (x_1, y_1) to (x_2, y_2) when either x_1x_2 is an arc of X and $y_1 = y_2$ or y_1y_2 is an arc of Y and $x_1 = x_2$. The Hermitian adjacency matrix of $X \Box Y$ is

$$H(X \Box Y) = H(X) \otimes I_{|V(Y)|} + I_{|V(X)|} \otimes H(Y),$$

where I_k is the $k \times k$ identity matrix. For definitions of Kronecker products of matrices and vectors, see [5]. The following result is mentioned in [13] without proofs.

Proposition 5.4. *If X and Y are digraphs with H-eigenvalues $\{\lambda_j\}_{j=1}^n$ and $\{\mu_k\}_{k=1}^m$, respectively, then $X \Box Y$ has H-eigenvalues $\lambda_j + \mu_k$ for $j = 1, \ldots, n$ and $k = 1, \ldots, m$.*

Proof. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be an orthonormal eigenbasis of $H(X)$ such that $H(X)\mathbf{v}_j = \lambda_j \mathbf{v}_j$, for $j = 1, \ldots, n$. Let $\mathbf{w}_1, \ldots, \mathbf{w}_n$ be an orthonormal eigenbasis of $H(Y)$ such that $H(Y)\mathbf{w}_k = \mu_k \mathbf{w}_k$, for $k = 1, \ldots, m$. Observe that the vectors $\{\mathbf{v}_j \otimes \mathbf{w}_k \mid j \in$

$\{1, \ldots, n\}$, $k \in \{1, \ldots, m\}\}$ form an orthonormal basis of $\mathbb{C}^{nm}$. We see that

$$\begin{aligned} H(X\Box Y)\mathbf{v}_j \otimes \mathbf{w}_k &= (H(X) \otimes I_m)\,(\mathbf{v}_j \otimes \mathbf{w}_k) + (I_n \otimes H(Y))\,(\mathbf{v}_j \otimes \mathbf{w}_k) \\ &= H(X)\mathbf{v}_j \otimes I_m\mathbf{w}_k + I_n\mathbf{v}_j \otimes H(Y)\mathbf{w}_k \\ &= \lambda_j \left(\mathbf{v}_j \otimes \mathbf{w}_k\right) + \mu_k \left(\mathbf{v}_j \otimes \mathbf{w}_k\right) \\ &= (\lambda_j + \mu_k) \left(\mathbf{v}_j \otimes \mathbf{w}_k\right) \end{aligned}$$

for every $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$. ■

Note that $X(a, b)$ has $\rho(X(a, b)) - \lambda_1(X(a, b)) = 1$. We now use the Cartesian product to construct digraphs where this difference is much larger. We let $X^{\Box n}$ denote the n-fold Cartesian product of X with itself; that is, $X^{\Box n} = X\Box \cdots \Box X$, where there are n terms in the product.

Proposition 5.5. *The digraph $X_n = K_4'^{\Box n}$ has $\rho(X_n) = 3n$ and $\lambda_1(X_n) = n$.*

Proof. By applying Proposition 5.4 n times, the H-eigenvalues of X_n are

$$\left\{ \sum_{j=1}^{n} \beta_j \mid \beta_j \in \{-3, 1\} \right\} = \{-3n, -3n + 2, -3n + 4, \ldots, n - 4, n - 2, n\}.$$

Thus $\rho(X_n) = 3n$ and $\lambda_1(X_n) = n$. ■

The importance of the above examples is that they exhibit the extreme behavior as evidenced by our next result.

Theorem 5.6. *For every digraph X we have*

$$\lambda_1(X) \le \rho(X) \le 3\lambda_1(X).$$

Both inequalities are tight.

Proof. The first inequality is clear by the definition of the spectral radius. Tightness is also clear by Proposition 5.5. To prove the second inequality, let A be the 01-matrix corresponding to all digons in X, and let L be the matrix corresponding to nondigons, so that $H = H(X) = A + L$.

It is easy to see that for every $\mathbf{x} \in \mathbb{R}^V$ $(V = V(X))$, we have $\mathbf{x}^T L\mathbf{x} = 0$. Therefore, $\mathbf{x}^* H\mathbf{x} = \mathbf{x}^T A\mathbf{x}$. By using the min–max formula (1) for the largest and smallest eigenvalues of H and L and noting that the same formula applies to the matrix A, in which case we need to consider only real vectors, we obtain the following inequality:

$$\lambda_1(A) = \max_{\mathbf{x}\in\mathbb{R}^V,\, \|\mathbf{x}\|=1} \mathbf{x}^T A\mathbf{x} = \max_{\mathbf{x}\in\mathbb{R}^V} \mathbf{x}^* H\mathbf{x} \le \max_{\mathbf{z}\in\mathbb{C}^V,\, \|\mathbf{z}\|=1} \mathbf{z}^* H\mathbf{z} = \lambda_1(H).$$

We are done if $\lambda_1(H) \ge \frac{1}{3}\rho(X)$. Thus, we may assume that $\lambda_1(A) \le \lambda_1(H) \le \frac{1}{3}\rho(X)$. Now,

$$\rho(H) = \lambda_n(H) = |\min_{\mathbf{z}\in\mathbb{C}^V,\, \|\mathbf{z}\|=1} \mathbf{z}^* H\mathbf{z}|.$$

Suppose that the minimum (which is negative) is attained by a vector $\mathbf{w} \in \mathbb{C}^V$ with $\|\mathbf{w}\| = 1$. Then

$$\rho(H) = |\mathbf{w}^* H\mathbf{w}| \le |\mathbf{w}^* L\mathbf{w}| + |\mathbf{w}^* A\mathbf{w}| \le |\mathbf{w}^* L\mathbf{w}| + \rho(A) \le |\mathbf{w}^* L\mathbf{w}| + \frac{1}{3}\rho(H).$$

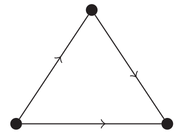


FIGURE 5. Digraph $\widetilde{C}_3$ with H-eigenvalues symmetric about 0 whose underlying graph is not bipartite.

This implies that $|\mathbf{w}^*L\mathbf{w}| \geq \frac{2}{3}\rho(H)$. By [13], Corollary 2.13], see also Theorem 6.2, the spectrum of L is symmetric about 0. Therefore there exists $\mathbf{y} \in \mathbb{C}^V$ with $\|\mathbf{y}\| = 1$ such that $\mathbf{y}^*L\mathbf{y} = |\mathbf{w}^*L\mathbf{w}|$. Now,

$$\lambda_1(H) = \max_{\mathbf{z}\in\mathbb{C}^V,\,\|\mathbf{z}\|=1} \mathbf{z}^*H\mathbf{z} \geq \mathbf{y}^*H\mathbf{y} = \mathbf{y}^*L\mathbf{y} + \mathbf{y}^*A\mathbf{y} \geq \frac{2}{3}\rho(H) - \rho(A) \geq \frac{1}{3}\rho(H).$$

This completes the proof. ■

It follows from the above proof that in the case of equality in the upper bound, the graph corresponding to the digons of X is bipartite, its spectral radius is equal to $\frac{1}{3}\rho(X)$, and $\rho(L) = \frac{2}{3}\rho(X)$.

To end up this section we show that the spectral radius of $\Gamma(X)$ always majorizes $\rho(X)$.

Theorem 5.7. *For every digraph X (with multiple edges allowed), $\rho(X) \leq \rho(\Gamma(X))$.*

Proof. Let $q = \pm\rho(X)$ be an eigenvalue of X, and let $\mathbf{x}$ be a unit eigenvector of $H(X)$ for its eigenvalue q. Define $\mathbf{y}$ by setting $\mathbf{y}(v) = |\mathbf{x}(v)|$, $v \in V(X)$. Then it is easy to see by using the triangular inequality that $\mathbf{y}^*A(\Gamma(X))\mathbf{y} \geq |\mathbf{x}^*H(X)\mathbf{x}| = \rho(X)$. Since $\mathbf{y}$ has norm 1, this implies that $\rho(\Gamma(X)) \geq \mathbf{y}^*A(\Gamma(X))\mathbf{y} \geq \rho(X)$, which we were to prove. ■

6. *H*-EIGENVALUES SYMMETRIC ABOUT 0

For a digraph X, its A-eigenvalues are symmetric about 0 if and only if $\Gamma(X)$ is bipartite. We may also consider digraphs X whose H-eigenvalues are symmetric about 0. Note that the eigenvalues of H are real and can be ordered as $\lambda_1 \geq \cdots \geq \lambda_n$, they are symmetric about 0 if and only if $\lambda_j = -\lambda_{n-j+1}$ for $j = 1, \ldots, n$. The following proposition appears in [13].

Proposition 6.1 ([13]). *For a digraph X, if $\Gamma(X)$ is bipartite, then the H-eigenvalues are symmetric about* 0.

The converse to Proposition 6.1 is not true. For example, the digraph $\widetilde{C}_3$ in Figure 5 has eigenvalues $\pm\sqrt{3}, 0$. In fact, every oriented graph has H-eigenvalues symmetric about 0. This was proved in [13], Corollary 2.13]; see also [3]. Here we give a different and simpler proof.

Theorem 6.2 ([13]). *If X is an oriented graph, then the H-spectrum of X is symmetric about* 0.

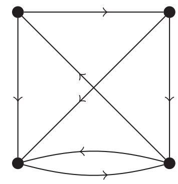


FIGURE 6. An example of a digraph on four vertices, having H-eigenvalues symmetric about 0, but not oriented and not bipartite.

Proof. Let X be an oriented graph on n vertices and $H = H(X)$. Let the eigenvalues of H be $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The matrix iH is skew-symmetric with purely imaginary eigenvalues $i\lambda_1, \ldots, i\lambda_n$. Since iH has entries ± 1, its characteristic polynomial has real coefficients. Thus, every eigenvalue μ of iH occurs with the same multiplicity as its complex conjugate. This implies that the spectrum of H is symmetric about 0.

There are digraphs with H-eigenvalues symmetric about 0, which are neither oriented nor have bipartite underlying graphs. Computationally, we verified that there are no such digraphs on fewer than four vertices. For digraphs of order 4, we found out, using computer, that there are exactly seven H-cospectral classes with H-spectrum symmetric about 0. They contain digraphs that are not oriented and their underlying graph needs not be bipartite. One of these classes contains exclusively such digraphs; this class contains 15 nonisomorphic digraphs all of which have underlying graphs isomorphic to K_4, and each contains at least one digon. One graph from this class, D, is shown in Figure 6. The characteristic polynomial of D is

$$\phi(H(D), t) = t^4 - 6t^2 + 5.$$

A common generalization of Proposition 6.1 and Theorem 6.2 follows from Theorem 3.3 using the fact that the spectrum is symmetric if and only if every even coefficient of the characteristic polynomial is zero. Thus, the following condition is certainly sufficient.

Theorem 6.3 ([13])**.** *If every odd cycle of* $\Gamma(X)$ *contains an even number of digons, then the* H*-spectrum of* X *is symmetric about* 0.

The digraph in Figure 6 shows that the above condition is only sufficient but not necessary.

A simple combinatorial characterization of digraphs with H-eigenvalues symmetric about 0 is not known.

7. *C**-ALGEBRA OF A DIGRAPH

The diameter of undirected graphs is bounded above by the number of distinct eigenvalues of its adjacency matrix. In this section we consider similar question for the Hermitian adjacency matrix.

Let M be a Hermitian matrix (with at least one nonzero off-diagonal element to exclude trivialities). Let $\mathcal{M}$ be the matrix algebra generated by $I, M, M^2, M^3, \ldots$ and let $\psi(M, t)$ be the minimal polynomial of M. Then, $\dim(\mathcal{M}) = \deg(\psi(M, t)) - 1$. On the other hand, the degree of $\psi(M, t)$ is equal to the number of distinct eigenvalues of M. If

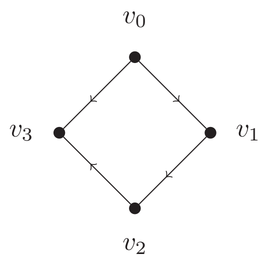


FIGURE 7. Digraph $\widetilde{C}_4$, obtained from C_4 by reversing one arc.

$M = A(X)$, the adjacency matrix of a digraph, it is easy to see that the dimension of $\mathcal{M}$ is at least the diameter of the graph. This implies that the diameter of X is smaller than the number of distinct eigenvalues of $A(X)$.

The case of the Hermitian adjacency matrix is quite different. For example, consider the modified directed cycle $\widetilde{C}_n$, obtained from a directed cycle by changing the orientation on one arc to the opposite direction. Consider, in particular, the digraph $\widetilde{C}_4$ shown in Figure 7. We can compute that

$$\phi(H(\widetilde{C}_4), t) = t^4 - 4t^2 + 4 = (t^2 - 2)^2$$

and we see that $\widetilde{C}_4$ has exactly two distinct H-eigenvalues, but the diameter of $\Gamma(\widetilde{C}_4)$ is 2.

For $n \geq 3$, the nth *necklace digraph,* denoted N_n, is an oriented graph on $3n$ vertices,

$$V(N_n) = \{v_j \mid j \in \mathbb{Z}_{2n}\} \cup \{w_k \mid k \in \mathbb{Z}_n\}$$

and with arcs

$$E(N_n) = \{v_j v_{j+1} \mid j \in \mathbb{Z}_{2n}\} \cup \{v_{2k}w_k, v_{2k+2}w_k \mid k \in \mathbb{Z}_n\}.$$

Let C_j be the cycle $(v_{2j}, v_{2j+1}, v_{2j+2}, w_j, v_{2j})$ in $\Gamma(N_n)$. Each v_{2j} lies on two of these cycles, C_j and C_{j-1}. Every other vertex lies on a unique C_j. Figure 8 shows N_4 with C_0 highlighted.

To find the eigenvalues of N_n, we will use the following lemma.

Lemma 7.1. *For every $n \geq 3$ and $H = H(N_n)$, we have $H^3 = 4H$.*

Proof. We will show that $H^3(u, v) = 4H(u, v)$. Observe that since the underlying graph $\Gamma := \Gamma(N_n)$ is a bipartite graph of girth 4, we have that $H^3(u, u) = 0$ and $H^3(u, v) = 0$ if $\mathrm{dist}_\Gamma(u, v)$ is even or $\mathrm{dist}_\Gamma(u, v) > 3$. We only need to consider the two cases where $\mathrm{dist}_\Gamma(u, v) \in \{1, 3\}$.

Let us first consider the case when u, v are adjacent in Γ. Since H and H^3 are Hermitian, we need only check u, v such that $uv \in E(N_n)$. Let $\mathcal{W}$ be the set of all walks of length 3 from u to v in Γ. The following are all possible types of walks of length 3 in Γ, starting at u and ending at v:

(a) $W_1 = (u, v, u, v)$;
(b) $W_2 = (u, v, x, v)$, where $x \in N_\Gamma(v) \setminus \{u\}$;
(c) $W_3 = (u, w, u, v)$, where $w \in N_\Gamma(u) \setminus \{v\}$; and
(d) $W_4 = (u, w, x, v)$, where $w \in N_\Gamma(u) \setminus \{v\}$, $x \in N_\Gamma(w) \setminus \{u\}$, and $xv \in E(\Gamma)$.

Every edge of Γ that is traversed once in each direction in any W_j contributes a factor of 1 to the weight $\mathrm{wt}(W_j)$. Thus $\mathrm{wt}(W_1) = \mathrm{wt}(W_2) = \mathrm{wt}(W_3) = i$. To find the weight of

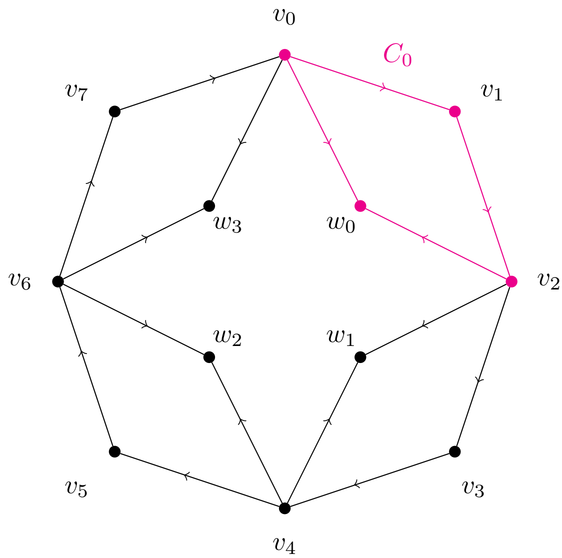


FIGURE 8. N_4 with C_0 in a lighter color.

W_4, we observe that every arc uv lies on the unique cycle C_j, for some j, and that W_4 together with the arc uv gives a 4-cycle in Γ, which must then correspond to C_j. From this we immediately see that, for each such uv, there is exactly one walk from u to v isomorphic to W_4 and that W_4 is a path of length 3 on C_j. We may observe that all such paths either traverse two arcs in the backward direction and one in the forward direction, or all three arcs in the forward direction. In either case, $\mathrm{wt}(W_4) = -i$.

For every arc uv, one of u or v has degree 4 in Γ and the other has degree 2. Then $\mathcal{W}$ contains one walk isomorphic to W_1 and either three walks isomorphic to W_2 and one isomorphic to W_3, or three walks isomorphic W_3 and one isomorphic to W_2. Then, since W_j for $j = 1, 2, 3$ have the same weight, we get

$$\begin{aligned} H^3(u, v) &= \sum_{W \in \mathcal{W}} \mathrm{wt}(W) \\ &= \mathrm{wt}(W_1) + \sum_{N_\Gamma(v)\setminus\{u\}} \mathrm{wt}(W_2) + \sum_{N_\Gamma(u)\setminus\{v\}} \mathrm{wt}(W_3) + \mathrm{wt}(W_4) \\ &= i + (3i + i) - i = 4i = 4H(u, v) \end{aligned}$$

as claimed.

Finally, suppose that u, v are at distance 3 in Γ. In this case, since vertices lying on C_j for any given j can be at distance at most 2, we have that $u \in C_j$ and $v \in C_{j\pm1}$. Also note that either u or v is of degree 4 in $\Gamma(X)$, thus equal to v_{2k} for some $k \in \mathbb{Z}_n$.

Suppose that $u = v_{2k}$ for some k. Then $v \in C_{k+1}$ or $v \in C_{k-2}$. In either case, there are two walks from u to v of opposite weight, and so $H^3(u, v) = 0 = H(u, v)$.

Corollary 7.2. *The H-spectrum of N_n is $\sigma_H(N_n) = \{0^{(n)}, 2^{(n)}, -2^{(n)}\}$.*

Proof. Let $H = H(N_n)$. From Lemma 7.1, we see that the minimal polynomial of H is $t^3 - 4t$. Since every eigenvalue of a matrix is a root of its minimal polynomial,

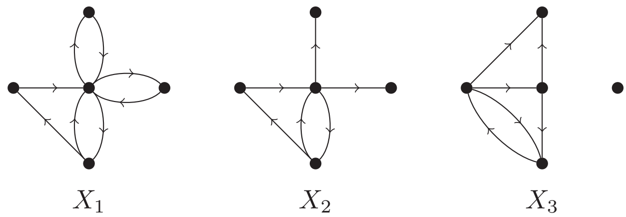


FIGURE 9. *H*-cospectral digraphs on five vertices with different connectivity properties.

the distinct eigenvalues of H are 0, 2, and -2. Let q, r, and s be the multiplicities of $0, 2$, and -2, respectively. Since $\mathrm{tr}(H) = 0$, we see that $r = s$. By Proposition 3.6(ii), $\mathrm{tr}(H^2) = 2|E(\Gamma(N_n))| = 8n$. Then $r(2^2 + (-2)^2) = 8n$, and so $r = n$. Since $q + 2r = 3n$, we have that $q = n$. ■

Corollary 7.3. *There exists an infinite family of digraphs* $\{X_j\}_{j=1}^{\infty}$ *such that the diameter of* $\Gamma(X_j)$ *goes to* ∞ *as* $j \to \infty$*, while each* X_j *has only three distinct* H*-eigenvalues.*

8. COSPECTRALITY

In this section, we study properties of digraphs that are H-cospectral and describe some operations on digraphs that preserve the H-spectrum. In particular, we are motivated to consider digraph operations that preserve the H-spectrum and preserve the underlying graph.

The H-spectrum of a digraph X does not determine if X is strongly connected, weakly connected, or disconnected. In Figure 9, we give an example of three digraphs, X_1, X_2, and X_3. A routine calculation shows that their characteristic polynomials are the same,

$$\phi(H, t) = t^5 - 5t^3 + 2t^2 + 2t.$$

Therefore, X_1, X_2, and X_3 are cospectral to each other. Observe that X_1 is strongly connected, X_2 is weakly connected but not strongly connected, and X_3 is not even weakly connected.

By the computation, as recorded in Tables I and II in Section 8, we see that, for small digraphs, the number of H-cospectral classes is smaller than the number of A-cospectral classes on the same number of vertices. (However, we expect that the opposite may hold when n is large.)

By contrast, any two acyclic digraphs on the same number of vertices are A-cospectral (all their eigenvalues are 0); in particular, there are A-cospectral digraphs on two vertices with nonisomorphic underlying graphs, whereas the smallest pair of H-cospectral digraphs with nonisomorphic underlying graphs have four vertices (shown in Figure 10).

There are many cases of H-cospectral digraphs having isomorphic underlying graphs. In Section 10, we will see that all orientations of an odd cycle are H-cospectral to each other and all digraphs whose underlying graph is an n-star are H-cospectral. From Table III, we see that every pair of H-cospectral digraphs on three vertices have the same underlying digraph.

We will try to explain the spectral information about the underlying graph by looking at some H-spectrum preserving operations that do not change the underlying graph.

TABLE I. The H-spectra of small digraphs

Order	2	3	4	5	6
Number of digraphs	3	16	218	9,608	1,540,944
Number of distinct characteristic polynomials	2	6	27	275	10,920
Number of H-cospectral classes such that					
(a) Characteristic polynomial is irreducible over $\mathbb{Q}$	0	0	0	0	6
(b) Characteristic polynomial is square-free	1	3	14	214	9,980
Maximum size of a H-cospectral class	2	6	21	158	1,338
Number of digraphs determined by H-spectrum	1	2	3	5	16
Number of H-cospectral classes containing:					
(a) No graphs	0	2	16	242	10,769
(b) Only graphs	1	1	1	1	1
(c) At least one graph and a digraph	1	3	10	32	150

TABLE II. The adjacency matrix spectra small digraphs

Order	2	3	4	5	6
Number of digraphs	3	16	218	9,608	1,540,944
Number of distinct characteristic polynomials	2	7	46	718	35,239
Number of A-cospectral classes such that					
(a) Characteristic polynomial is irreducible over $\mathbb{Q}$	0	1	12	277	19,392
(b) Characteristic polynomial is square-free	1	5	36	625	33,146
Maximum size of a A-cospectral class	2	6	42	592	15,842
Number of digraphs determined by spectrum	1	5	23	166	2,317
Number of A-cospectral classes containing:					
(a) No graphs	0	3	35	685	35,088
(b) Only graphs	1	2	5	15	69
(c) At least one graph and a digraph	1	2	6	18	82

FIGURE 10. The smallest pair of H-cospectral digraphs with nonisomorphic underlying graphs.

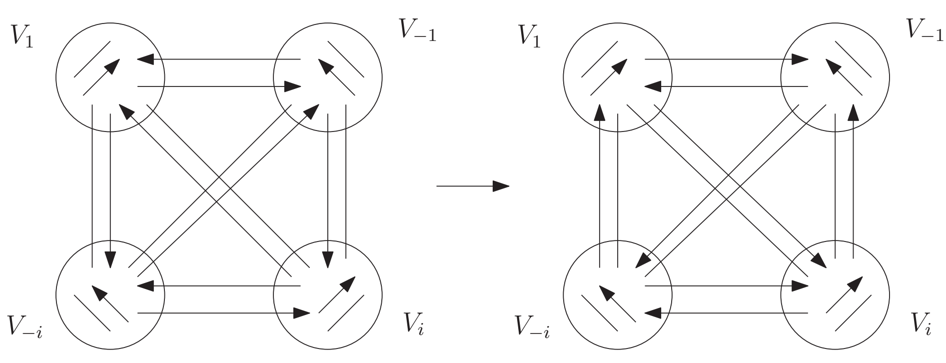
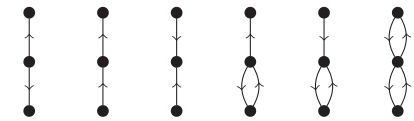
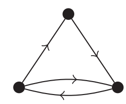
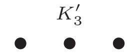
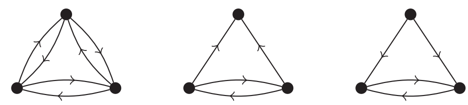
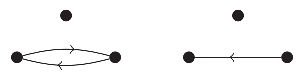
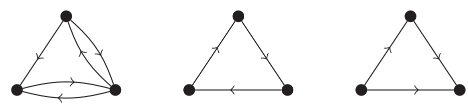


FIGURE 11. Four-way switching on the admissible edges.

TABLE III. H-cospectral classes of all digraphs on three vertices

Polynomial $p(t)$	Roots of $p(t)$	All digraphs X such that $\phi(H(X), t) = p(t)$
$t^3 - 2t$	$\sqrt{2}, 0, -\sqrt{2}$	 Z_1 $\vec{P}_3$ Z_2 Z_3 Z_4 P_3
$t^3 - 3t + 2$	$1, 1, -2$	 K_3'
t^3	$0, 0, 0$	 E_3
$t^3 - 3t - 2$	$2, -1, -1$	 K_3 $Y_{2,1}$ $Y_{1,2}$
$t^3 - t$	$1, 0, -1$	 Z_5 Z_6
$t^3 - 3t$	$\sqrt{3}, 0, -\sqrt{3}$	 Z_7 D_3 $\widetilde{C}_3$

It is immediate from the definition of the Hermitian adjacency matrix that if X is a digraph and X^C is its converse, then $H(X^C) = H(X)^T = \overline{H(X)}$. This implies the following result.

Proposition 8.1. *A digraph X and its converse are H-cospectral.*

Inspired by this, we now define a local operation on a digraph that will also preserve the spectrum with respect to the Hermitian adjacency matrix. For a digraph X and a vertex v of X, the *local reversal of X at v* is the operation of replacing every arc xy incident with v by its converse yx. We can extend this to the *local reversal of X at $S \subset V$* by taking the local reversal at v for each $v \in S$. Observe that the order of reversals does not matter. If an arc xy is incident to two vertices of S, then it is unchanged in the local reversal at S. We denote by $\delta(S)$ the arcs with exactly one end in S. Note that this operation generalizes the concept of switching equivalence, defined earlier for tournaments.

Proposition 8.2. *If X is a digraph and $S \subset V(X)$ such that $\delta(S)$ contains no digons, then X and the digraph obtained by the local reversal of X at S are H-cospectral.*

Proof. Let X' be the digraph obtained by the local reversal of X at S. Let M be the diagonal matrix indexed by the vertices of X given by

$$M_{uu} = \begin{cases} -1, & \text{if } u \in S; \\ +1, & \text{if } u \notin S. \end{cases}$$

Consider $M^{-1}H(X)M$. Applying $M^{-1} = M$ on the left of $H(X)$ changes the sign for all rows indexed by vertices of S and applying M on the right changes the sign of all columns indexed by vertices of S. Since $\delta(S)$ contains no digons, this implies that $M^{-1}H(X)M = H(X')$. Consequently, the matrices $H(X)$ and $H(X')$ are similar and hence cospectral. ■

Proposition 8.2 cannot be generalized to the case when $\delta(S)$ contains digons. However, there is one exceptional case that is described next.

Proposition 8.3. *If X is a digraph and $S \subset V(X)$ such that $\delta(S)$ contains only digons, then X and the digraph obtained by replacing each digon $\{x, y\}$ $(x \notin S, y \in S)$ in the cut by the arc xy are H-cospectral.*

Proposition 8.3 follows directly from Theorem 3.3, and the details are left to the reader. Equivalently, the proof of Proposition 8.2 works, where the only difference is that we take $M_{uu} = i$ if $u \in S$.

In a special case when each edge is a cut-edge, we obtain the following corollary, which was proved earlier in [13, Corollary 2.21].

Corollary 8.4 ([13]). *If X is a digraph whose underlying graph is a forest, then $H(X)$ is cospectral with $A(\Gamma(X))$.*

All of the above operations that preserve the H-spectrum can be generalized as discussed next. It all amounts to a simple similarity transformation that is based on the structure of Theorem 5.1(i). Suppose that the vertex set of X is partitioned in four (possibly empty) sets, $V(X) = V_1 \cup V_{-1} \cup V_i \cup V_{-i}$. An arc xy or a digon $\{x, y\}$ is said to be of *type* (j, k) for $j, k \in \{\pm 1, \pm i\}$ if $x \in V_j$ and $y \in V_k$. The partition is said to be *admissible* if the following conditions hold:

(a) There are no digons of types $(1, -1)$ or $(i, -i)$.
(b) All edges of types $(1, i)$, $(i, -1)$, $(-1, -i)$, $(-i, 1)$ are contained in digons. See Figure 11.

A *four-way switching* with respect to a partition $V(X) = V_1 \cup V_{-1} \cup V_i \cup V_{-i}$ is the operation of changing X into the digraph X' by making the following changes:

(a) reversing the direction of all arcs of types $(1, -1)$, $(-1, 1)$, $(i, -i)$, $(-i, i)$;
(b) replacing each digon of type $(1, i)$ with a single arc directed from V_1 to V_i and replacing each digon of type $(-1, -i)$ with a single arc directed from V_{-1} to V_{-i};
(c) replacing each digon of type $(1, -i)$ with a single arc directed from V_{-i} to V_1 and replacing each digon of type $(-1, i)$ with a single arc directed from V_i to V_{-1};
(d) replacing each nondigon of type $(1, -i)$, $(-1, i)$, $(i, 1)$ or $(-i, -1)$ with the digon.

Theorem 8.5. *If a partition $V(X) = V_1 \cup V_{-1} \cup V_i \cup V_{-i}$ is admissible, then the digraph obtained from X by the four-way switching is cospectral with X.*

Proof. We use a similarity transformation with the diagonal matrix S whose (v, v)-entry is equal to $j \in \{\pm 1, \pm i\}$ if $v \in V_j$. The entries of the matrix $H' = S^{-1}HS$ are given by the formula

$$H'(u, v) = H(u, v)S(v)/S(u).$$

It is clear that H' is Hermitian and that its nonzero elements are in $\{\pm 1, \pm i\}$. Note that the entries within the parts of the partition remain unchanged. Digons of type $(1, -1)$ would give rise to the entries -1, but since these digons are excluded for admissible partitions, this does not happen. On the other hand, any other entry in this part is multiplied by -1, and thus directed edges of types $(1, -1)$ or $(-1, 1)$ just reverse their orientation. Similar conclusions are made for other types of edges. Admissibility is needed in order that H' has no entries equal to -1. It turns out that H' is the Hermitian adjacency matrix of X'. The details are easily read off from the following similarity using the diagonal 4×4 matrix $S = \mathrm{diag}(1, -1, i, -i)$. The first one shows how digons are transformed:

$$S^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} S = \begin{bmatrix} 1 & -1 & i & -i \\ -1 & 1 & -i & i \\ -i & i & 1 & -1 \\ i & -i & -1 & 1 \end{bmatrix}.$$

The second one shows the result for nondigon arcs:

$$S^{-1} \begin{bmatrix} i & i & i & i \\ i & i & i & i \\ i & i & i & i \\ i & i & i & i \end{bmatrix} S = \begin{bmatrix} i & -i & -1 & 1 \\ -i & i & 1 & -1 \\ 1 & -1 & i & -i \\ -1 & 1 & -i & i \end{bmatrix}.$$

Note the resemblance of Theorem 8.5 with Theorem 5.1(i). ■

The partition used in Theorem 8.5 is explored further in [16].

A. Digraphs that are *H*-Cospectral with K_n

In the undirected case, complete graphs have the property of being determined by their spectrum. In the case of digraphs with the Hermitian adjacency matrix it turns out that for each n, there are exactly n nonisomorphic digraphs with the same H-spectrum as K_n. They can all be obtained by applying a single transformation of Proposition 8.3.

For $a = 1, \ldots, n-1$, let $Y_{a,n-a}$ be the digraph of order n that is obtained from the complete digraph $\vec{D}(K_n)$ by replacing the digons between any of the first a vertices (forming the set A) and any of the remaining $n-1$ vertices (forming the set B) by the single arc from A to B joining the same two vertices. In other words, $Y_{a,n-a}$ consists of a copy of $\vec{D}(K_a)$ and a copy of $\vec{D}(K_{n-a})$, with all possible arcs from $\vec{D}(K_a)$ to $\vec{D}(K_{n-a})$. Figure 12 shows $Y_{2,3}$.

Proposition 8.3 implies that the digraph $Y_{a,n-a}$ as defined above is H-cospectral with K_n.

Proposition 8.6. *For each n, there are precisely n nonisomorphic digraphs that have the same H-spectrum as K_n. These are the digraphs K_n and $Y_{a,n-a}$, where $a = 1, \ldots, n-1$.*

Proof. For $a \geq 1$ and $b = n - a$, we observe that the out-degree of a vertex of $Y_{a,b}$ is either $n-1$ or $b-1$. Then, for $c, d \geq 1$, the digraphs $Y_{a,b}$ and $Y_{c,d}$ have different sets of degrees unless $a = c$ and $b = d$. Thus, $Y_{a,b}$ is isomorphic to $Y_{c,d}$ if and only if $a = c$

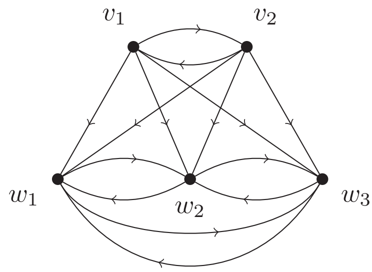


FIGURE 12. Digraph $Y_{2,3}$ is H-cospectral to K_5.

and $b = d$. From Proposition 8.6, we see that the digraphs $Y_{a,n-a}$, $a \in \{1, \dots n-1\}$, are cospectral with K_n. Together with K_n itself, there are at least n nonisomorphic digraphs in the H-cospectral class containing K_n.

Let X be a digraph that is H-cospectral with K_n. We will show that X is isomorphic to one of K_n and $Y_{a,b}$, where $a \in \{1, \dots, n-1\}$ and $b = n - a$. Note that $\rho(X) = n - 1$, thus by Theorem 5.1, $\Gamma(X) = K_n$ and X has the structure as in part (i) of that theorem. Since $\gamma(X)$ is complete, this can only be when at most two of the sets V_j ($j \in \{\pm 1, \pm i\}$) are nonempty. Clearly, this gives either the digraph $\vec{D}(K_n)$ or one of the digraphs $Y_{a,n-a}$. ■

B. Cospectral Classes of Digraphs Whose Underlying Graph is a Cycle

At the end of this section, we determine the cospectral classes of all digraphs whose underlying graph is a cycle. The directed cycle on n vertices is denoted D_n and $\widetilde{C}_n$ is the oriented cycle obtained from D_n by reversing one arc. Further, we let $\widetilde{C}'_n$ be the digraph obtained from D_n by replacing one arc by a digon, and $\widetilde{C}''_n$ be the digraph obtained from D_n by replacing two consecutive arcs, the first one by reversing the arc, and the second one by adding the reverse arc, that is, making it the digon.

We start with oriented cycles. We consider two digraphs X and Y to be *equivalent under taking local reversals* if Y can be obtained from X by taking a series of local reversals.

Lemma 8.7. *All orientations of an odd cycle C_{2m+1} are equivalent under taking local reversals. Each orientation of an even cycle C_{2m} is equivalent either to D_{2m} or to $\widetilde{C}_{2m}$ under taking local reversals.*

Proof. Let X be any orientation of C_n on vertices $V = \{x_1, \dots, x_n\}$ joined in the cyclic order. For $j = 2, 3, \dots, n$ we consecutively make the local reversal at x_j if the current graph obtained from X has the arc $x_j x_{j-1}$, thus changing this arc to $x_{j-1}x_j$. This way we obtain either D_n or $\widetilde{C}_n$. If n is odd and we have $\widetilde{C}_n$, then we make additional reversals at $x_2, x_4, \dots, x_{n-1}$, thus transforming $\widetilde{C}_n$ into D_n. This completes the proof. ■

Lemma 8.7 implies that every orientation of C_n is H-cospectral with D_n or with $\widetilde{C}_n$. We note that for even n, the spectra of D_n and $\widetilde{C}_n$ are not equal (see Section 10).

Proposition 8.8. *Every digraph whose underlying graph is isomorphic to the n-cycle C_n is H-cospectral to one of the following: D_n, $\widetilde{C}_n$, or $\widetilde{C}'_n$ (when n is even), and to D_n, $\widetilde{C}'_n$, or $\widetilde{C}''_n$ (when n is odd).*

Proof. Any two edges of C_n form an edge-cut. By Proposition 8.3, any edge-cut consisting of two digons can be changed to a directed cut without changing its H-spectrum, and any cut consisting of two arcs can be changed to its reverse by Proposition 8.2. Thus, we may assume that we have at most one digon. If there are no digons, Lemma 8.7 completes the proof. On the other hand, having one digon, we can make local reversals in the same way as in the proof of Lemma 8.7 to obtain either $\widetilde{C}'_n$ or $\widetilde{C}''_n$. If n is odd, then $\widetilde{C}''_n$ is cospectral with $\widetilde{C}'_n$. (To see this, make local reversals at $\frac{n-1}{2}$ independent vertices with in-degree and out-degree 1.) This gives the three cases of the proposition.

The H-eigenvalues of oriented cycles are treated fully in Section 10.

9. DIGRAPHS WITH SMALL SPECTRAL RADIUS

In the previous sections, we have seen that the H-eigenvalues of digraphs behave differently and somewhat strangely compared with the A-eigenvalues of graphs. It appears that graph invariants such as diameter, minimum degree, and number of connected components cannot be bounded by the H-spectrum. However, since H is Hermitian, we may use the interlacing theorem. Using interlacing, we can characterize all digraph with all H-eigenvalues small in absolute value. First we will look at a special case where all H-eigenvalues are equal to 1 or -1, then we will consider the general case. Note that mX where X is a digraph denotes the union of m disjoint copies of X.

Theorem 9.1. *A digraph X has the property that $\lambda \in \{-1, 1\}$ for each H-eigenvalue λ of X if and only if $\Gamma(X) \cong mK_2$ for some m.*

Proof. Let X be a digraph on n vertices having the property as in the statement of the theorem. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of $H(X)$. Then $\lambda_i \in \{-1, 1\}$ for $i = 1, \ldots, n$ by assumption. Since the H-eigenvalues of X must sum to $\mathrm{tr}(H) = 0$, the multiplicity of 1 and -1 are equal and X has an even number vertices, say $n = 2m$. Observe that

$$\mathrm{tr}(H(X)^2) = \sum_{i=1}^{n} \lambda_i^2 = n = 2m.$$

Lemma 3.1 gives that $\Gamma(X)$ has m edges. If X has an isolated vertex, then X will have 0 as an H-eigenvalue. Thus, every vertex must have degree at least 1 in $\Gamma(X)$. Then $d_{\Gamma(X)}(v) = 1$ for every vertex v and so $\Gamma(X) \cong mK_2$. ■

Theorem 9.2. *For a digraph X the following are equivalent:*

(a) $\sigma_H(X) \subseteq (-\sqrt{2}, \sqrt{2}\,)$;
(b) $\sigma_H(X) \subseteq [-1, 1]$; *and*
(c) *every component of X is either a single arc, a digon, or an isolated vertex.*

Proof. Let X be a digraph on n vertices with H-eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Suppose that $\lambda_1 < \sqrt{2}$ and $\lambda_n > -\sqrt{2}$. Let Y be an induced subdigraph of X on three vertices

TABLE IV. H-eigenvalues of digraphs with C_4 as the underlying graph

Digraph	Characteristic polynomial	Eigenvalues
C_4	$t^4 - 4t^2$	$\pm 2, 0^{(2)}$
$\widetilde{C}_4$	$t^4 - 4t^2 + 4$	$\pm\sqrt{2}^{(2)}$
$\widetilde{C}'_4$	$t^4 - 4t^2 + 2$	$\pm\sqrt{2 \pm \sqrt{2}}$

and let $\mu_1 \geq \mu_2 \geq \mu_3$ be the H-eigenvalues of Y. Applying the interlacing theorem, we obtain that

$$-\sqrt{2} < \mu_i < \sqrt{2}$$

for $i = 1, 2, 3$. There are exactly 16 digraphs on three vertices and so we may compute their H-eigenvalues and determine which digraphs on three vertices have all eigenvalues strictly between $-\sqrt{2}$ and $\sqrt{2}$. The digraphs on three vertices grouped by H-cospectral classes are given in Table III. Following the naming of the digraphs in Table III, we see that Y is isomorphic to one of E_3, Z_5 and Z_6. In other words, Y is either the empty graph on three vertices or a graph consisting of an isolated vertex and either one arc or one digon.

Since the choice of Y was arbitrary, the above holds for every induced subdigraph of X on three vertices. Then, $\Gamma(X)$ does not have vertices of degree 2 or more. Thus $\Delta(\Gamma(X)) \leq 1$ and so $\Gamma(X)$ consists of the union of disjoint copies of K_2 and isolated vertices. This shows that (a) implies (b).

The implications (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a) are trivial, so this completes the proof. ∎

Note that Theorem 9.2 implies that X has its H-spectrum equal to

$$\{1^{(m)}, 0^{(k)}, -1^{(m)}\},$$

where k is the number of isolated vertices and m is the number of components consisting of a single arc or a digon.

With some more work, we can characterize all digraphs with H-spectrum in $(-\sqrt{3}, \sqrt{3}\,)$. In order to do this, we need following corollaries of Proposition 8.8, where we compute the spectra of all digraphs whose underlying graph is isomorphic to C_4 or C_5. Using Sage, we found the characteristic polynomials and the H-eigenvalues of these digraphs. For C_4, they are collected in Table IV. We can summarize this in the following corollary to Proposition 8.8.

Corollary 9.3. *If the underlying graph of a digraph is isomorphic to the 4-cycle C_4, then its H-spectrum is one of the following: $\{\pm 2, 0^{(2)}\}$, $\{\pm\sqrt{2}^{(2)}\}$, or $\{\pm\sqrt{2 \pm \sqrt{2}}\,\}$.*

We repeat the same process for digraphs whose underlying graph is C_5. Using Sage, we found the H-eigenvalues of these graphs. They are given in Table V. We can summarize this in the following corollary to Proposition 8.8.

Corollary 9.4. *If Y is a digraph with $\Gamma(Y) \cong C_5$, then the H-spectrum of Y is one of the following: $\left\{2, (\frac{-1\pm\sqrt{5}}{2})^{(2)}\right\}$, $\left\{0, \pm\sqrt{\frac{5\pm\sqrt{5}}{2}}\right\}$, or $\left\{2, (\frac{1\pm\sqrt{5}}{2})^{(2)}\right\}$.*

We can now return to graphs with small spectral radius.

TABLE V. H-eigenvalues of digraphs with C_5 as the underlying graph

Digraph	Characteristic polynomial	Eigenvalues
C_5	$t^5 - 5t^3 + 5t - 2$	$2, (\frac{-1\pm\sqrt{5}}{2})^{(2)}$
D_5	$t^5 - 5t^3 + 5t$	$0, \pm\sqrt{\frac{5\pm\sqrt{5}}{2}}$
$\widetilde{C}_5''$	$t^5 - 5t^3 + 5t + 2$	$2, (\frac{1\pm\sqrt{5}}{2})^{(2)}$

Theorem 9.5. *A digraph X has $\sigma_H(X) \subseteq (-\sqrt{3}, \sqrt{3}\,)$ if and only if every component Y of X has $\Gamma(Y)$ isomorphic to a path of length at most 3 or to C_4, where in the latter case Y is isomorphic to $\widetilde{C}_4$, or to one of the two strongly connected digraphs with two digons.*

Proof. Let the H-eigenvalues of X be $\lambda_1 \geq \cdots \geq \lambda_n$ and let $\Gamma = \Gamma(X)$. Let us assume that $\lambda_1 < \sqrt{3}$ and $\lambda_n > -\sqrt{3}$. Again, we consider Y, an induced subdigraph of X on three vertices and let $\mu_1 \geq \mu_2 \geq \mu_3$ be the H-eigenvalues of Y. Applying the interlacing theorem, we obtain that

$$-\sqrt{3} < \mu_i < \sqrt{3}$$

for $i = 1, 2, 3$. Again, we consult Table III to see that $\Gamma(Y)$ is isomorphic to one of P_3, $K_2 \cup K_1$, or E_3. In other words, $\Gamma(Y)$ is acyclic. Since the choice of Y was arbitrary, the above holds for every induced subdigraph of X on three vertices. Then, Γ does not contain a triangle.

If Γ contains a vertex of degree at least 3, then Γ contains either an induced star on four vertices or a triangle. We have already shown that there are no triangles, so Γ must contain a star on four vertices. It is easy to see that every digraph whose underlying graph is a 4-star has $\sqrt{3}$ as an eigenvalue (see also Section 8). By interlacing, this cannot happen in X. Thus, every vertex in Γ has degree at most 2.

The conclusion of the above is that the components of Γ are paths and cycles. Suppose $W \subseteq V(X)$ induced a path of length 4 in Γ. The spectral radius of P_5 is $\sqrt{3}$ and, since all digraphs with P_5 as its underlying graph are H-cospectral by Corollary 3.4, W induces a subgraph of X with maximum H-eigenvalue equal to $\sqrt{3}$, which is again impossible. Every cycle of length at least 6 contains an induced path of length 4 and every path of length at least 4 contains an induced path of length 4, so Γ does not have any cycles of length greater than 5 or paths of length greater than 3.

We see from Table V that every digraph Y such that $\Gamma(Y) \cong C_5$ has an eigenvalue strictly greater than $\sqrt{3}$. Also, we see from Table III that $\Gamma(Y)$ cannot be C_3. Thus we must have that each component of X is either a path on at most four vertices or $\Gamma(Z) \cong C_4$. From Table IV, we can see that Z must be cospectral to $\widetilde{C}_4$. It is easy to see that there are three such digraphs. One is $\widetilde{C}_4$, the other two have two digons and they are strongly connected.

Conversely, a digraph all of whose components are as stated has spectral radius strictly smaller than $\sqrt{3}$. This completes the proof. ■

It is interesting to consider for which values of α the number of weakly connected digraphs whose H-eigenvalues will all lie in the interval $(-\alpha, \alpha)$ or $[-\alpha, \alpha]$ will be finite.

We see from Theorem 9.5 that there are only finitely many weakly connected digraphs with all H-eigenvalues in the interval $(-\sqrt{3}, \sqrt{3})$. It may be true that the same holds for every α with $0 \leq \alpha < 2$. The directed paths show that there are infinitely many weakly connected digraphs whose H-eigenvalues will all lie in the interval $(-2, 2)$, thus $\alpha = 2$ will not give the same conclusion.

10. EXAMPLES

Computation on all isomorphism classes of digraphs of orders 2, 3, 4, 5, and 6 were carried out using Sage open-source mathematical software system [20], for both the adjacency matrix and the Hermitian adjacency matrix. We include some data here to give an idea of how the Hermitian adjacency matrix behaves as compared to the adjacency matrix on small digraphs. We refer to the set of all digraphs that are H-cospectral to a given digraph as an *H-cospectral class.* A digraph is *determined by its H-spectrum* if every digraph that is H-cospectral with it is also isomorphic to it.

Recall that D_n denotes the directed cycle on n vertices. The following is well known.

Lemma 10.1. *The H-eigenvalues of D_n are $2\sin\frac{2\pi k}{n}$, $k = 0, \ldots, n-1$.*

Next we determine the eigenvalues of $\widetilde{C}_n$. Recall that this digraph is obtained from D_n by reversing one arc and that it is cospectral with D_n if n is odd. In order to find the H-eigenvalues of $\widetilde{C}_n$, we will use known theorems about certain types of circulant matrices, which we will use again when discussing the H-eigenvalues of the transitive tournament.

Circulant matrices have been studied extensively, see [11] for more information. A *skew circulant matrix* is a circulant with a change in sign to all entries below the main diagonal. We will follow the notation of [6] and define for $\mathbf{a} = (a_0, \ldots, a_{n-1})$ with real entries, the skew circulant matrix of $\mathbf{a}$ as:

$$\mathrm{S}(\mathbf{a}) = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ -a_{n-1} & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_1 \\ -a_1 & \cdots & -a_{n-1} & a_0 \end{pmatrix}.$$

The eigenvalues of a skew circulant matrix $\mathrm{S}(\mathbf{a})$ are easy to find, see, for example, [6, Section 3.2.1].

Theorem 10.2. *The eigenvalues of $\mathrm{S}(\mathbf{a})$ are $\mu_j(\mathbf{a}) = \sum_{k=0}^{n-1} a_k \sigma^{(2j+1)k}$, where $\sigma = e^{\frac{\pi i}{n}}$, for $j = 0, 1, \ldots, n-1$.*

In particular, it will be useful to simplify this statement for skew-symmetric, skew circulant matrices.

Corollary 10.3. *Suppose that $a_k = a_{n-k}$ for $k = 1, \ldots, n-1$. If n is odd, then $\mathrm{S}(\mathbf{a})$ has eigenvalues*

$$\nu_j(\mathbf{a}) = a_0 + 2i \sum_{k=1}^{\frac{n-1}{2}} a_k \sin\frac{k(2j+1)\pi}{n}, \quad j = 0, 1, \ldots, n-1.$$

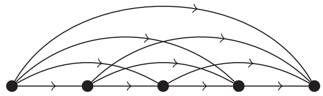


FIGURE 13. Transitive tournament T_5 on five vertices.

If n is even, then $\mathrm{S}(\mathbf{a})$ *has eigenvalues*

$$\nu_j(\mathbf{a}) = a_0 + (-1)^j a_{\frac{n}{2}} i + 2i \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} a_k \sin \tfrac{k(2j+1)\pi}{n}, \qquad j = 0, 1, \ldots, n-1.$$

Proof. Let $\sigma = e^{\frac{\pi i}{n}}$. If $a_j = a_{n-j}$ for $j = 1, \ldots, n-1$, it is clear from the definition that $\mathrm{S}(\mathbf{a})$ is skew-symmetric. Observe that $\sigma^n = -1$. For any $j \in \{0, \ldots, n-1\}$ and $k \in \{1, \ldots, n-1\}$, consider the contribution of terms with a_k and a_{n-k} in $\mu_j(\mathbf{a})$ of Theorem 10.2:

$$\begin{aligned} a_k\sigma^{(2j+1)k} + a_{n-k}\sigma^{(2j+1)(n-k)} &= a_k\left(\sigma^{(2j+1)k} + \sigma^{(2j+1)n}\sigma^{(2j+1)(-k)}\right) \\ &= a_k\left(\sigma^{(2j+1)k} - (\sigma^{-1})^{(2j+1)k}\right) \\ &= a_k\left(2i\mathrm{Im}(\sigma^{(2j+1)k})\right) \\ &= 2ia_k \sin \tfrac{(2j+1)k\pi}{n}. \end{aligned}$$

If n is odd, then we are done. If $n = 2m$, then we also have the contribution $a_m\sigma^{(2j+1)m} = a_m(-1)^j i$ that gives the result for the even case. ■

We will use Corollary 10.3 to find the eigenvalues of $\widetilde{C}_n$.

Proposition 10.4. *The eigenvalues of $\widetilde{C}_n$ ($n \geq 3$) are $2\sin\frac{(2j+1)\pi}{n}$, $j = 0, \ldots, n-1$.*

Proof. Observe that $H(\widetilde{C}_n) = i\mathrm{S}(\mathbf{a})$, where $\mathbf{a} = (a_0, \ldots, a_{n-1})$ and $a_1 = a_{n-1} = 1$ and $a_j = 0$ for $j \notin \{1, n-1\}$. Then, by Corollary 10.3, the eigenvalues of $H(\widetilde{C}_n)$ are $i\nu_j(\mathbf{a})$ for $j = 0, \ldots, n-1$. Let $\sigma = e^{\frac{\pi i}{n}}$ as before. Since $a_{\frac{n}{2}} = 0$, we obtain that $\nu_j(\mathbf{a}) = 2i\sin\frac{(2j+1)\pi}{n}$ for $j \in 0, \ldots, n-1$. Then, the eigenvalues of $H(\widetilde{C}_n)$ are $-2\sin\frac{(2j+1)\pi}{n}$ for $j = 0, \ldots, n-1$, which is easily seen to be the same as claimed. ■

A. Transitive Tournaments

The transitive tournaments are an important class of digraphs to study and the spectra of their skew-symmetric adjacency matrices have been studied as skew circulants and Toeplitz matrices in [8, 12].

Let T_n denote the transitive tournament on n vertices. See Figure 13, which shows T_5. Let $H_n = H(T_n)$ denote its Hermitian adjacency matrix.

Theorem 10.5. *The characteristic polynomial of the transitive tournament T_n is*

$$\phi(T_n, t) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n}{2j} t^{n-2j} = \frac{1}{2}(t+i)^n + \frac{1}{2}(t-i)^n.$$

The H-eigenvalues of T_n are

$$\sum_{k=1}^{\frac{n-1}{2}} 2\sin\left(\frac{k(2j+1)\pi}{n}\right)$$

for $j = 0, \ldots, n-1$, when n is odd, and

$$(-1)^j + \sum_{k=1}^{\frac{n-2}{2}} 2\sin\left(\frac{k(2j+1)\pi}{n}\right)$$

for $j = 0, \ldots, n-1$, when n is even.

Proof. Observe that $H_n = i\mathrm{S}(\mathbf{a})$, where $\mathbf{a} = (a_0, \ldots, a_{n-1})$ and $a_1 = \cdots = a_{n-1} = 1$ and $a_0 = 0$. We see that $\mathrm{S}(\mathbf{a})$ is a skew-symmetric skew circulant and so we may apply Corollary 10.3 to obtain that eigenvalues of $\mathrm{S}(\mathbf{a})$ are

$$\nu_j(\mathbf{a}) = \begin{cases} a_0 + \sum_{k=1}^{\frac{n-1}{2}} a_k 2i\sin\left(\frac{k(2j+1)\pi}{n}\right) & \text{if } n \text{ is odd} \\ a_0 + (-1)^j a_{\frac{n}{2}} i + \sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} a_k 2i\sin\left(\frac{k(2j+1)\pi}{n}\right) & \text{if } n \text{ is even} \end{cases}$$

for $j = 0, \ldots, n-1$. As the eigenvalues of H_n are $i\nu_j$ for $j = 0, \ldots, n-1$, we obtain the expressions for the H-eigenvalues as in the theorem. ■

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