Finite Element Exterior Calculus

Part IV

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- Chain complexes + Hilbert spaces operators = Hilbert complexes
- The abstract Hodge Laplacian
- Discretization of Hilbert complexes
- Finite element differential forms
- Constructing complexes and their discretizations

Finite element spaces of differential forms, continued

The Koszul differential

- $\kappa u = u \lrcorner \mathrm{id}$
- $\kappa(f\, dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma^k}) = \sum_{i=1}^k (-)^i f\, x^{\sigma_i}\, dx^{\sigma_1} \wedge \cdots \widehat{dx^{\sigma_i}} \cdots \wedge dx^{\sigma^k}$
- In $\mathbb{R}^3$: $\mathcal{P}_r\Lambda^3 \xrightarrow{\vec{x}} \mathcal{P}_{r+1}\Lambda^2 \xrightarrow{\times\vec{x}} \mathcal{P}_{r+2}\Lambda^1 \xrightarrow{\cdot\vec{x}} \mathcal{P}_{r+3}\Lambda^0$

Two basic properties:

- κ is a *differential:* $\kappa \circ \kappa = 0$
- *Homotopy property:* $(d\kappa + \kappa d)u = (r+k)u,\ u \in \mathcal{H}_r\Lambda^k$

e.g., $\mathrm{curl}(\vec{x} \times \vec{v}) + \vec{x}\,(\mathrm{div}\,\vec{v}) = (\deg \vec{v} + 2)\,\vec{v}$

Consequences:

- The *Koszul complex* $0 \to \mathcal{H}_r\Lambda^n \xrightarrow{\kappa} \mathcal{H}_{r+1}\Lambda^{n-1} \xrightarrow{\kappa} \cdots \xrightarrow{\kappa} \mathcal{H}_{r+n}\Lambda^0 \to 0$ is exact. The polynomial de Rham complex is exact.
- $\mathcal{H}_r\Lambda^k = \kappa\, \mathcal{H}_{r-1}\Lambda^{k+1} \oplus d\, \mathcal{H}_{r+1}\Lambda^{k-1}$

Definition: $\mathcal{P}_r^-\Lambda^k = \mathcal{P}_{r-1}\Lambda^k + \kappa\mathcal{H}_{r-1}\Lambda^{k+1}$

$\mathcal{P}_{r-1}\Lambda^k \subsetneq \mathcal{P}_r^-\Lambda^k \subsetneq \mathcal{P}_r\Lambda^k$ except $\mathcal{P}_r^-\Lambda^0 = \mathcal{P}_r\Lambda^0$, $\mathcal{P}_r^-\Lambda^n = \mathcal{P}_{r-1}\Lambda^n$)

The $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ finite element spaces

Elements: A triangulation $\mathcal{T}_h$ consisting of simplices T

Shape functions: $V(T) = \mathcal{P}_r\Lambda^k(T)$ or $\mathcal{P}_r^-\Lambda^k(T)$, some $r \ge 1$

Degrees of freedom for $\mathcal{P}_r\Lambda^k$ (*unisolvent*):

$$u \mapsto \int_f (\mathrm{tr}_f\, u) \wedge q, \quad q \in \mathcal{P}^-_{r-d+k}\Lambda^{d-k}(f),\, f \in \Delta_d(T),\, d \ge k$$

Degrees of freedom for $\mathcal{P}_r^-\Lambda^k$ (*unisolvent*):

$$u \mapsto \int_f (\mathrm{tr}_f\, u) \wedge q, \quad q \in \mathcal{P}_{r-d+k-1}\Lambda^{d-k}(f),\, f \in \Delta_d(T),\, d \ge k$$

Hiptmair '99

The resulting finite element spaces belongs to $H\Lambda^k$. In fact, they equal

$$\{\, u \in H\Lambda^k(\Omega) \;:\; u|_T \in V(T)\ \forall T \in \mathcal{T}_h \,\}$$

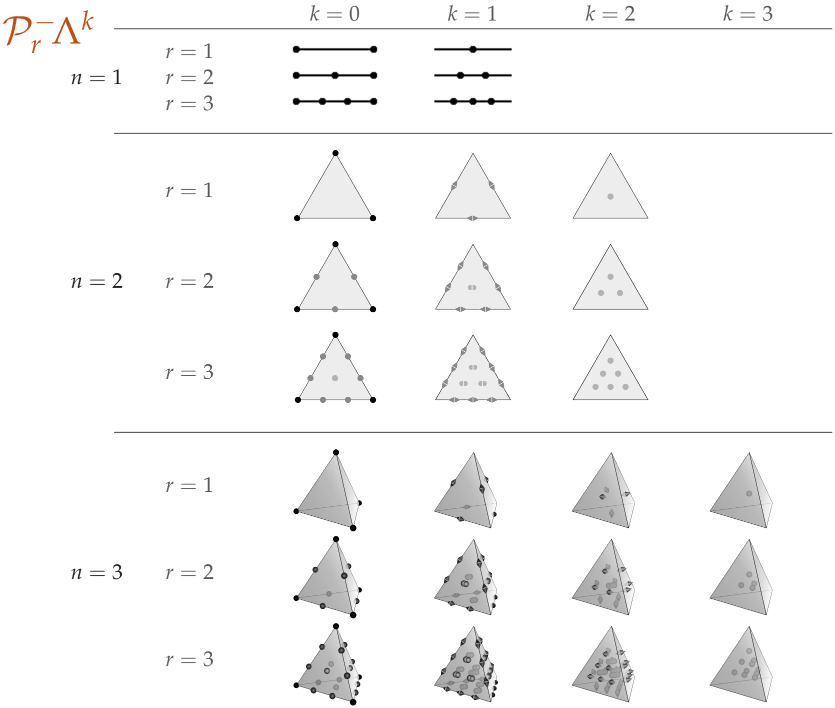
- The DOFs given above for $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$ are *unisolvent*.
- The DOFs define cochain projections for the complexes

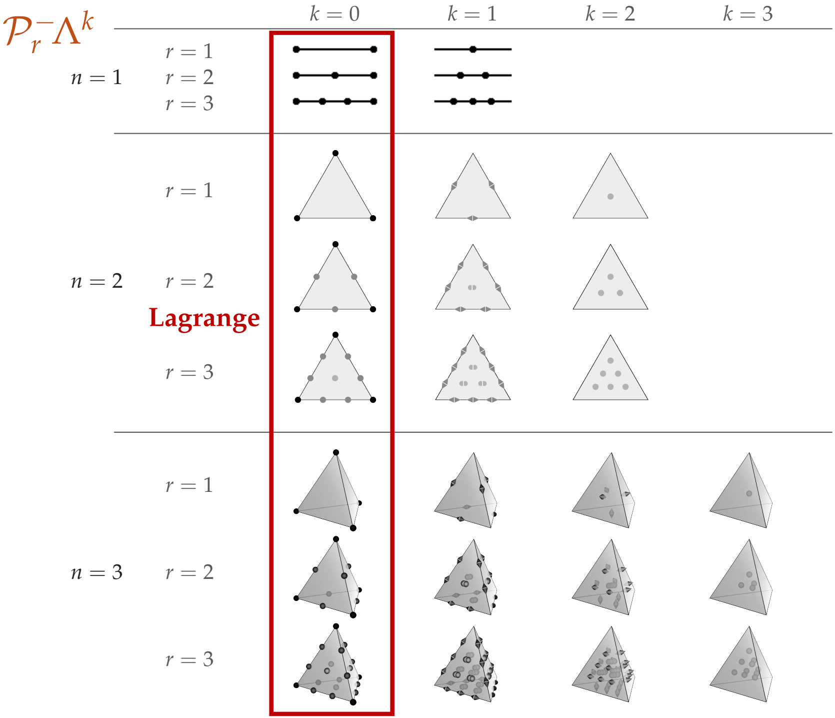
$$
\begin{array}{l}
0 \to \mathcal{P}_r\Lambda^0(\mathcal{T}_h) \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1(\mathcal{T}_h) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n(\mathcal{T}_h) \to 0 \\
0 \to \mathcal{P}_r^-\Lambda^0(\mathcal{T}_h) \xrightarrow{d} \mathcal{P}_r^-\Lambda^1(\mathcal{T}_h) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^-\Lambda^n(\mathcal{T}_h) \to 0
\end{array}
$$

decreasing degree　　　constant degree

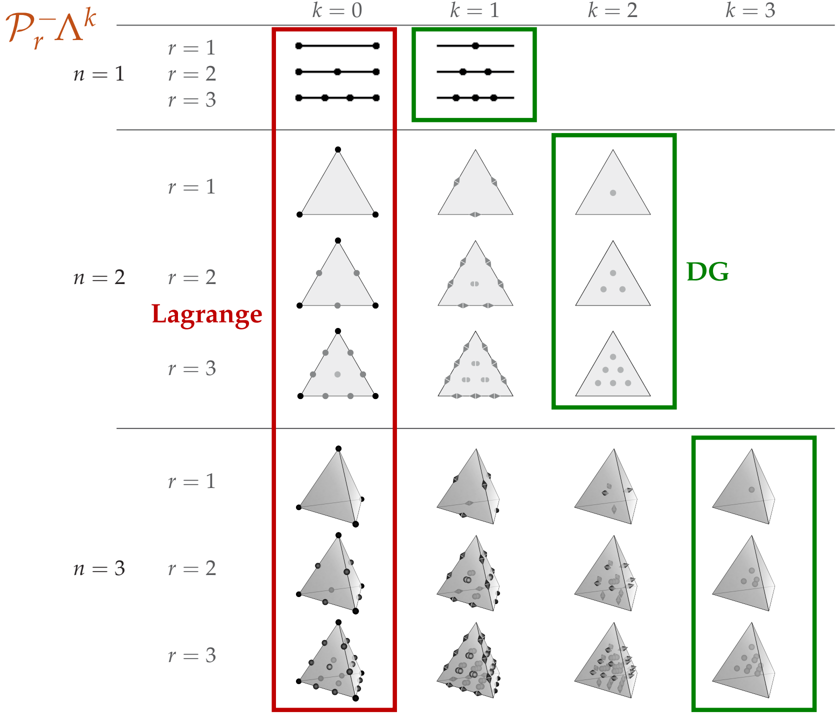
- This leads to four stable discretizations of the mixed k-form Laplacian:

$$
\begin{array}{ll}
\mathcal{P}_r\Lambda^{k-1}(\mathcal{T}_h) \times \mathcal{P}_{r-1}\Lambda^k(\mathcal{T}_h) & \mathcal{P}_r\Lambda^{k-1}(\mathcal{T}_h) \times \mathcal{P}_r^-\Lambda^k(\mathcal{T}_h) \\
\mathcal{P}_r^-\Lambda^{k-1}(\mathcal{T}_h) \times \mathcal{P}_{r-1}\Lambda^k(\mathcal{T}_h) & \mathcal{P}_r^-\Lambda^{k-1}(\mathcal{T}_h) \times \mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)
\end{array}
$$

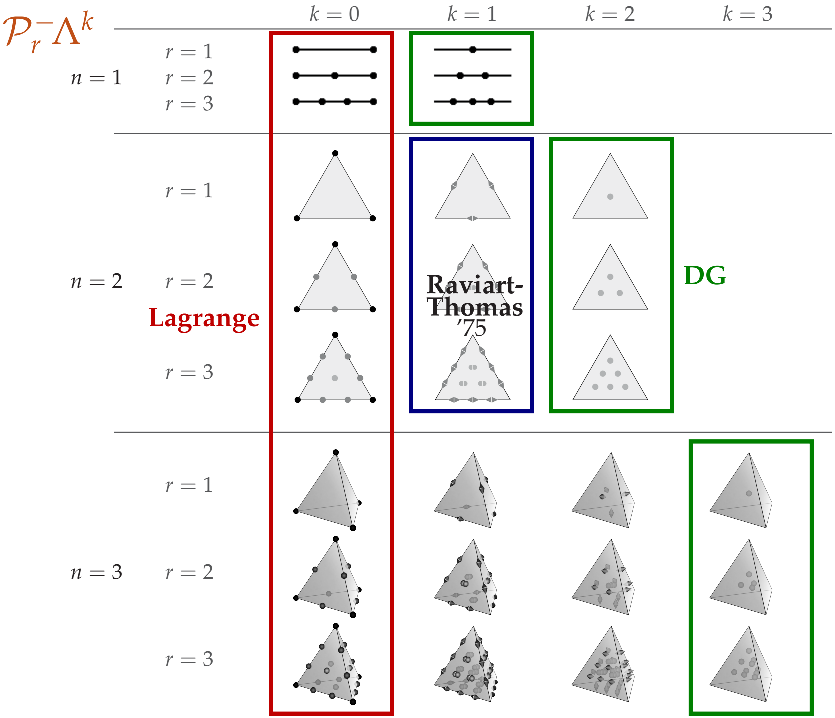

$\mathcal{P}_r^-\Lambda^k$
$k = 0$
$k = 1$
$k = 2$
$k = 3$
$n = 1$
$n = 2$
$n = 3$
$r = 1$
$r = 2$
$r = 3$

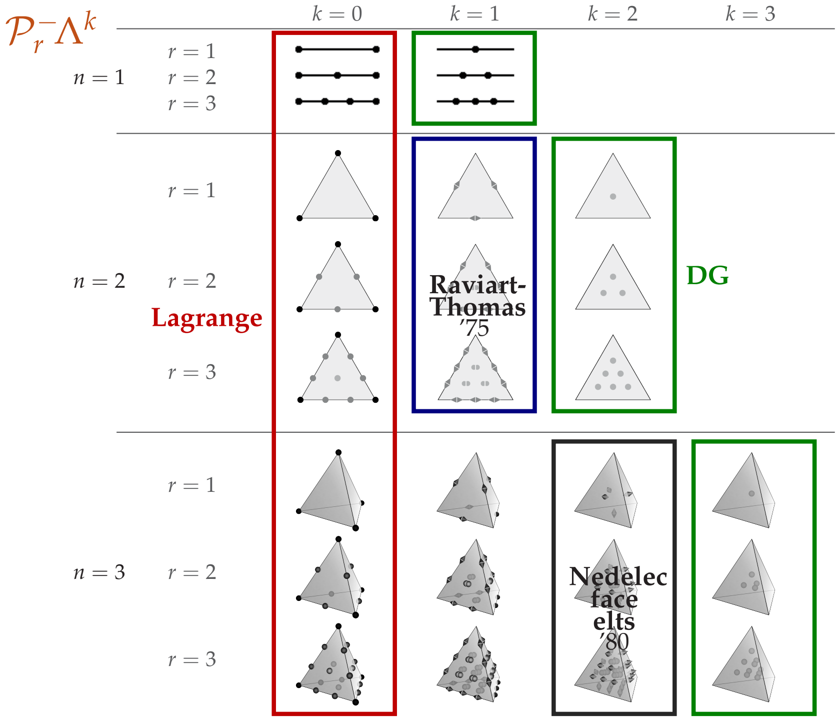


$\mathcal{P}_r^-\Lambda^k$
$k = 0$
$k = 1$
$k = 2$
$k = 3$
$n = 1$
$r = 1$
$r = 2$
$r = 3$
$n = 2$
$r = 1$
$r = 2$
Lagrange
$r = 3$
$n = 3$
$r = 1$
$r = 2$
$r = 3$



$\mathcal{P}_r^-\Lambda^k$
$k=0$
$k=1$
$k=2$
$k=3$
$n=1$
$n=2$
$n=3$
$r=1$
$r=2$
$r=3$
Lagrange
DG


$\mathcal{P}_r^-\Lambda^k$
$k=0$
$k=1$
$k=2$
$k=3$
$n=1$
$n=2$
$n=3$
$r=1$
$r=2$
$r=3$
Lagrange
Raviart-Thomas '75
DG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$k=2$
$k=3$
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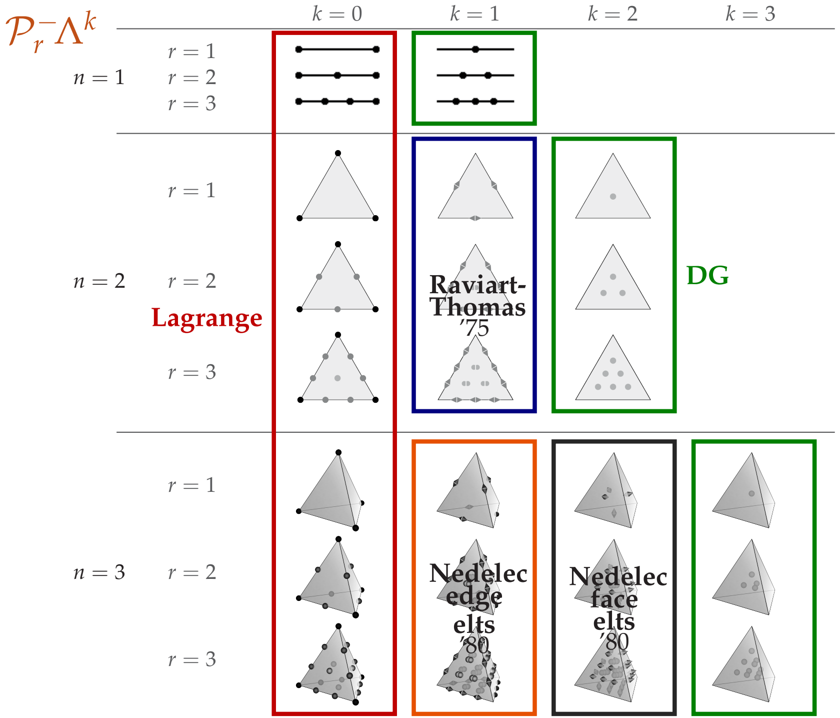
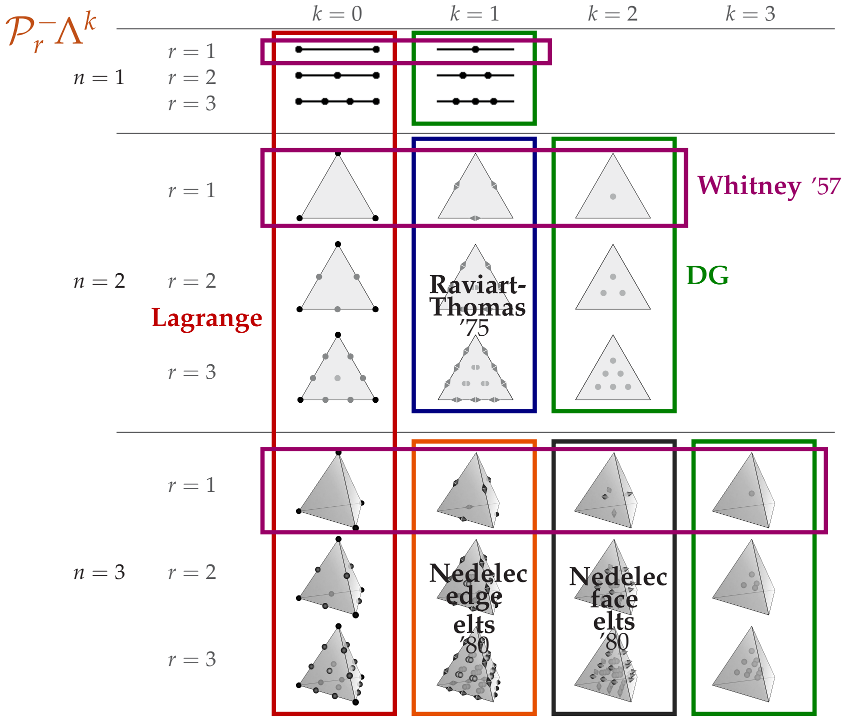
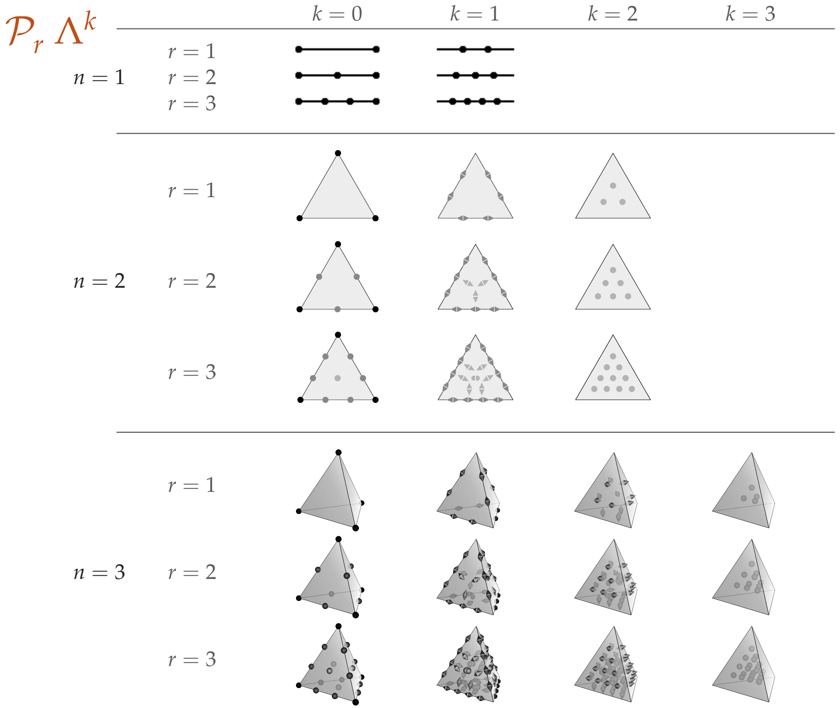
$r=1$
$r=2$
$r=3$
$n=2$
$r=1$
$r=2$
$r=3$
Lagrange
Raviart-Thomas '75
DG
$n=3$
$r=1$
$r=2$
$r=3$
Nedelec face elts '80

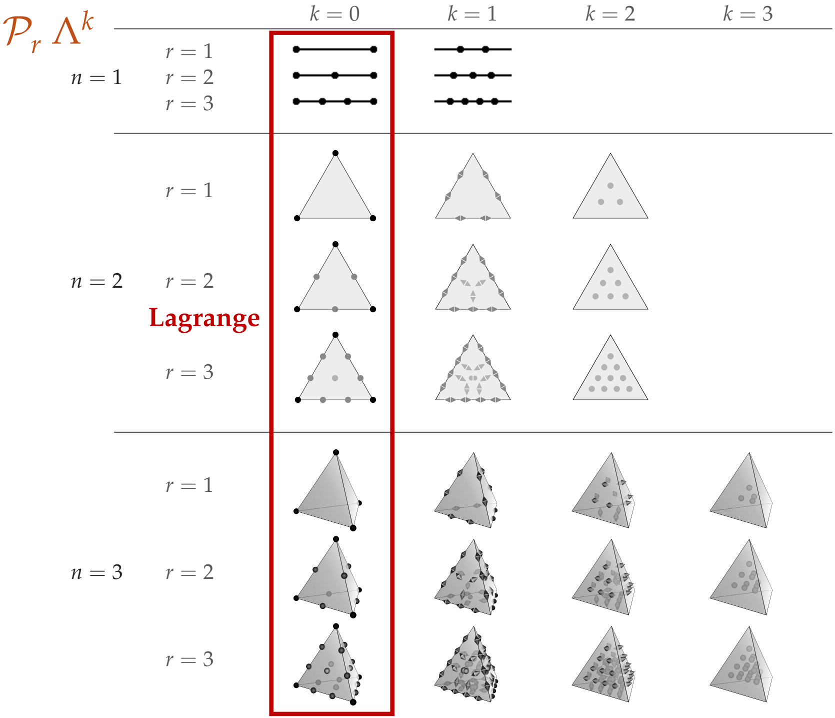


$\mathcal{P}_r^-\Lambda^k$
$k=0$
$k=1$
$k=2$
$k=3$
$n=1$
$n=2$
$n=3$
$r=1$
$r=2$
$r=3$
Lagrange
Raviart-Thomas '75
DG
Nedelec edge elts '80
Nedelec face elts '80

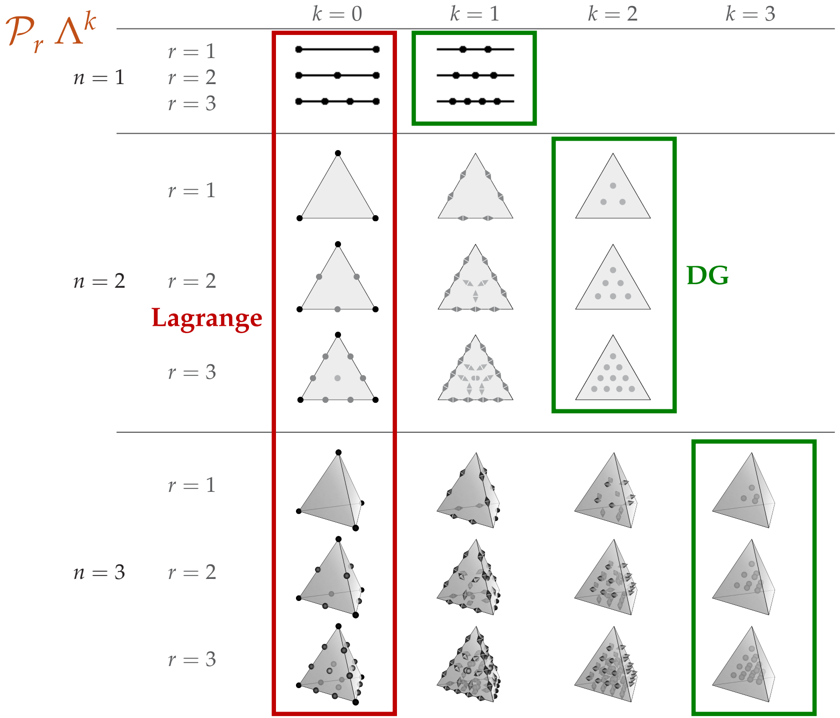


$\mathcal{P}_r^-\Lambda^k$
$k=0$
$k=1$
$k=2$
$k=3$
$n=1$
$n=2$
$n=3$
$r=1$
$r=2$
$r=3$
Whitney '57
DG
Lagrange
Raviart-Thomas '75
Nedelec edge elts '80
Nedelec face elts '80

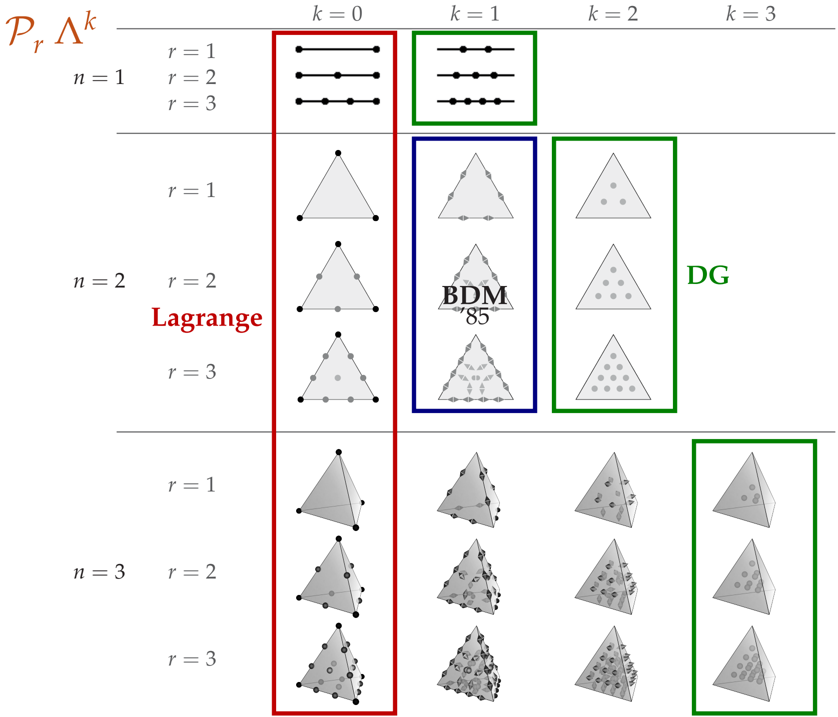

$\mathcal{P}_r\Lambda^k$
$k=0$
$k=1$
$k=2$
$k=3$
$n=1$
$n=2$
$n=3$
$r=1$
$r=2$
$r=3$



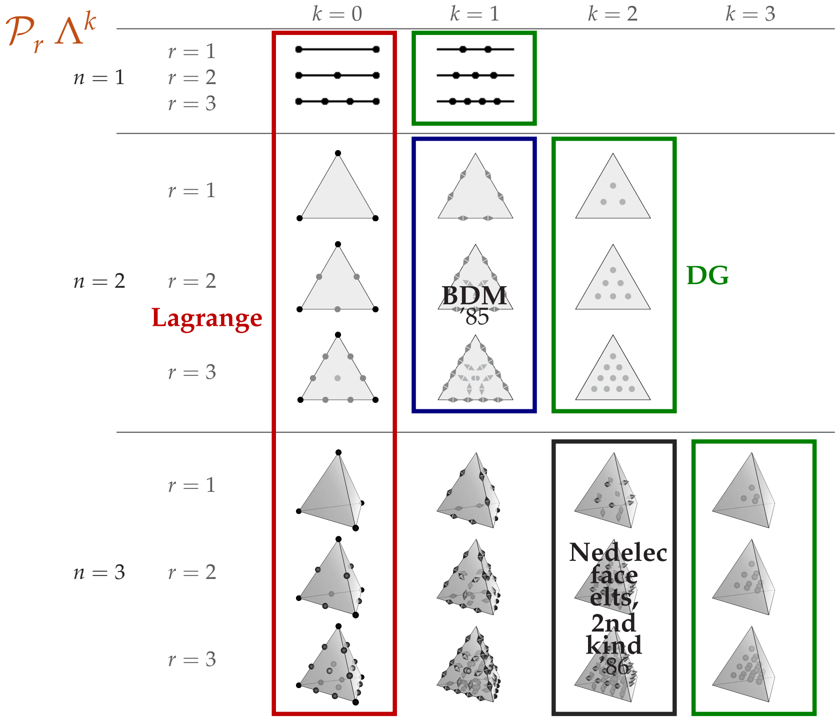
$\mathcal{P}_r \Lambda^k$
$k = 0$
$k = 1$
$k = 2$
$k = 3$
$n = 1$
$r = 1$
$r = 2$
$r = 3$
$n = 2$
$r = 1$
$r = 2$
Lagrange
$r = 3$
$n = 3$
$r = 1$
$r = 2$
$r = 3$



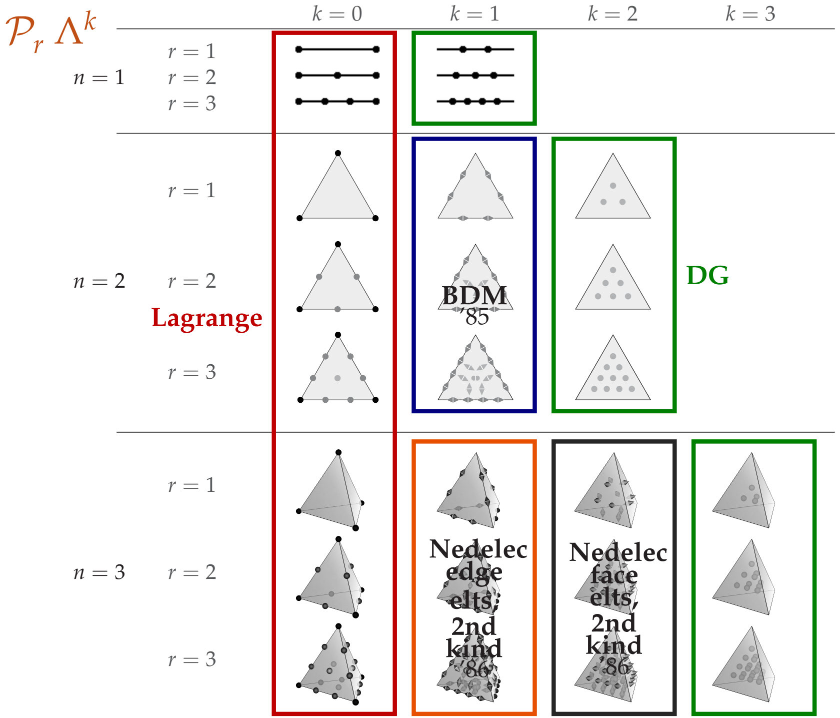
$\mathcal{P}_r \Lambda^k$
$k = 0$
$k = 1$
$k = 2$
$k = 3$
$n = 1$
$n = 2$
$n = 3$
$r = 1$
$r = 2$
$r = 3$
Lagrange
DG



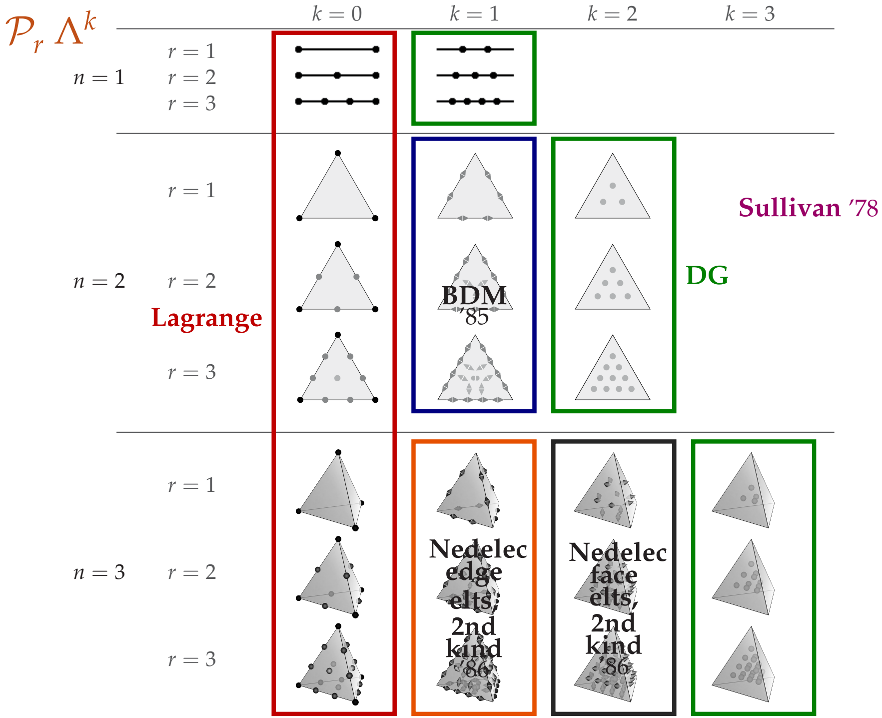
$\mathcal{P}_r \Lambda^k$
$k=0$
$k=1$
$k=2$
$k=3$
$n=1$
$n=2$
$n=3$
$r=1$
$r=2$
$r=3$
Lagrange
BDM '85
DG



$\mathcal{P}_r\Lambda^k$
$k=0$
$k=1$
$k=2$
$k=3$
$n=1$
$n=2$
$n=3$
$r=1$
$r=2$
$r=3$
Lagrange
BDM '85
DG
Nedelec face elts, 2nd kind '86



$\mathcal{P}_r \Lambda^k$
$k=0$
$k=1$
$k=2$
$k=3$
$n=1$
$n=2$
$n=3$
$r=1$
$r=2$
$r=3$
Lagrange
BDM '85
DG
Nedelec edge elts, 2nd kind '86
Nedelec face elts, 2nd kind '86



$\mathcal{P}_r \Lambda^k$
$k = 0$
$k = 1$
$k = 2$
$k = 3$
$n = 1$
$n = 2$
$n = 3$
$r = 1$
$r = 2$
$r = 3$
Lagrange
BDM '85
DG
Sullivan '78
Nedelec edge elts, 2nd kind '86
Nedelec face elts, 2nd kind '86

Periodic Table of the Finite Elements

$\mathcal{P}_r^-\Lambda^k$

The shape function space for $\mathcal{P}_r^-\Lambda^k$ is $\mathcal{P}_{r-1}\Lambda^k + \kappa\mathcal{P}_{r-1}\Lambda^{k+1}$, where κ is the Koszul differential.[7] It includes the full polynomial space $\mathcal{P}_{r-1}\Lambda^k$, is included in $\mathcal{P}_r\Lambda^k$, and has dimension

$$\dim \mathcal{P}_r^-\Lambda^k(\Delta_n) = \binom{r+n}{r+k}\binom{r+k-1}{k}.$$

The degrees of freedom are given on faces f of dimension $d \ge k$ by moments of the trace weighted by a full polynomial space:

$$u \mapsto \int_f (\mathrm{tr}_f\, u)\wedge q, \quad q \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f).$$

The spaces with constant degree r form a complex:

$$\mathcal{P}_r^-\Lambda^0 \xrightarrow{d} \mathcal{P}_r^-\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^-\Lambda^n.$$

		$k=0$	$k=1$	$k=2$	$k=3$
$n=1$	$r=1$	$\mathbf{P_1}$ (2)	$\mathbf{dP_0}$ (1)		
	$r=2$	$\mathbf{P_2}$ (3)	$\mathbf{dP_1}$ (2)		
	$r=3$	$\mathbf{P_3}$ (4)	$\mathbf{dP_2}$ (3)		
$n=2$	$r=1$	$\mathbf{P_1}$ (3)	$\mathbf{RT_1^{[e/f]}}$ (3)	$\mathbf{dP_0}$ (1)	
	$r=2$	$\mathbf{P_2}$ (6)	$\mathbf{RT_2^{[e/f]}}$ (8)	$\mathbf{dP_1}$ (3)	
	$r=3$	$\mathbf{P_3}$ (10)	$\mathbf{RT_3^{[e/f]}}$ (15)	$\mathbf{dP_2}$ (6)	
$n=3$	$r=1$	$\mathbf{P_1}$ (4)	$\mathbf{N1_1^e}$ (6)	$\mathbf{N1_1^f}$ (4)	$\mathbf{dP_0}$ (1)
	$r=2$	$\mathbf{P_2}$ (10)	$\mathbf{N1_2^e}$ (20)	$\mathbf{N1_2^f}$ (15)	$\mathbf{dP_1}$ (4)
	$r=3$	$\mathbf{P_3}$ (20)	$\mathbf{N1_3^e}$ (45)	$\mathbf{N1_3^f}$ (36)	$\mathbf{dP_2}$ (10)

$\mathcal{P}_r\Lambda^k$

The shape function space for $\mathcal{P}_r\Lambda^k$ consists of all differential k-forms with polynomial coefficients of degree at most r, and has dimension

$$\dim \mathcal{P}_r\Lambda^k(\Delta_n) = \binom{r+n}{r+k}\binom{r+k}{k}.$$

The degrees of freedom are given on faces f of dimension $d \ge k$ by moments of the trace weighted by a $\mathcal{P}_r^-$ space:

$$u \mapsto \int_f (\mathrm{tr}_f\, u)\wedge q, \quad q \in \mathcal{P}^-_{r+k-d}\Lambda^{d-k}(f).$$

The spaces with decreasing degree r form a complex:

$$\mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n.$$

		$k=0$	$k=1$	$k=2$	$k=3$
$n=1$	$r=1$	$\mathbf{P_1}$ (2)	$\mathbf{dP_1}$ (2)		
	$r=2$	$\mathbf{P_2}$ (3)	$\mathbf{dP_2}$ (3)		
	$r=3$	$\mathbf{P_3}$ (4)	$\mathbf{dP_3}$ (4)		
$n=2$	$r=1$	$\mathbf{P_1}$ (3)	$\mathbf{BDM_1^{[e/f]}}$ (6)	$\mathbf{dP_1}$ (3)	
	$r=2$	$\mathbf{P_2}$ (6)	$\mathbf{BDM_2^{[e/f]}}$ (12)	$\mathbf{dP_2}$ (6)	
	$r=3$	$\mathbf{P_3}$ (10)	$\mathbf{BDM_3^{[e/f]}}$ (20)	$\mathbf{dP_3}$ (10)	
$n=3$	$r=1$	$\mathbf{P_1}$ (4)	$\mathbf{N2_1^e}$ (12)	$\mathbf{N2_1^f}$ (12)	$\mathbf{dP_1}$ (4)
	$r=2$	$\mathbf{P_2}$ (10)	$\mathbf{N2_2^e}$ (30)	$\mathbf{N2_2^f}$ (30)	$\mathbf{dP_2}$ (10)
	$r=3$	$\mathbf{P_3}$ (20)	$\mathbf{N2_3^e}$ (60)	$\mathbf{N2_3^f}$ (60)	$\mathbf{dP_3}$ (20)

$\mathcal{Q}_r^-\Lambda^k$

This family is constructed from the complex of 1-dimensional finite elements using a tensor product construction.[10] The shape function space on the unit cube $\Box_n = I^n$ is given by

$$\bigoplus_{\sigma\in\Sigma(k,n)}\left[\bigotimes_{i=1}^{n}\mathcal{P}_{r-\delta_{i,\sigma}}(I)\right] dx^{\sigma_1}\wedge\cdots\wedge dx^{\sigma_k},$$

where $\Sigma(k,n)$ denotes the increasing maps $\{1,\ldots,k\}\to\{1,\ldots,n\}$. Its dimension is $\dim \mathcal{Q}_r^-\Lambda^k(\Box_n) = \binom{n}{k}r^k(r+1)^{n-k}$. The degrees of freedom are

$$u \mapsto \int_f (\mathrm{tr}_f\, u)\wedge q, \quad q \in \mathcal{Q}^-_{r-1}\Lambda^{d-k}(f).$$

The spaces with constant degree r form a complex:

$$\mathcal{Q}_r^-\Lambda^0 \xrightarrow{d} \mathcal{Q}_r^-\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{Q}_r^-\Lambda^n.$$

		$k=0$	$k=1$	$k=2$	$k=3$
$n=1$	$r=1$	$\mathbf{Q_1}$ (2)	$\mathbf{dQ_0}$ (1)		
	$r=2$	$\mathbf{Q_2}$ (3)	$\mathbf{dQ_1}$ (2)		
	$r=3$	$\mathbf{Q_3}$ (4)	$\mathbf{dQ_2}$ (3)		
$n=2$	$r=1$	$\mathbf{Q_1}$ (4)	$\mathbf{RTc_1^{[e/f]}}$ (4)	$\mathbf{dQ_0}$ (1)	
	$r=2$	$\mathbf{Q_2}$ (9)	$\mathbf{RTc_2^{[e/f]}}$ (12)	$\mathbf{dQ_1}$ (4)	
	$r=3$	$\mathbf{Q_3}$ (16)	$\mathbf{RTc_3^{[e/f]}}$ (24)	$\mathbf{dQ_2}$ (9)	
$n=3$	$r=1$	$\mathbf{Q_1}$ (8)	$\mathbf{Nc_1^e}$ (12)	$\mathbf{Nc_1^f}$ (6)	$\mathbf{dQ_0}$ (1)
	$r=2$	$\mathbf{Q_2}$ (27)	$\mathbf{Nc_2^e}$ (54)	$\mathbf{Nc_2^f}$ (36)	$\mathbf{dQ_1}$ (8)
	$r=3$	$\mathbf{Q_3}$ (64)	$\mathbf{Nc_3^e}$ (144)	$\mathbf{Nc_3^f}$ (108)	$\mathbf{dQ_2}$ (27)

$\mathcal{S}_r\Lambda^k$

The shape function space for $\mathcal{S}_r\Lambda^k$ is given by

$$\mathcal{P}_r\Lambda^k \oplus \bigoplus_{\ell\ge 1}\left[\kappa\mathcal{H}_{r+\ell-1,\ell}\Lambda^{k+1}\oplus d\kappa\mathcal{H}_{r+\ell,\ell}\Lambda^k\right],$$

where $\mathcal{H}_{r,\ell}\Lambda^k$ consists of homogeneous polynomial k-forms of degree r which are linear and undifferentiated in at least ℓ variables.[11] Its dimension is $\dim \mathcal{S}_r\Lambda^k(\Box_n) = \sum_{d\ge k} 2^{n-d}\binom{n}{d}\binom{r-d+2k}{d}\binom{d}{k}$. The degrees of freedom are

$$u \mapsto \int_f (\mathrm{tr}_f\, u)\wedge q, \quad q \in \mathcal{P}_{r-2(d-k)}\Lambda^{d-k}(f).$$

The spaces with decreasing degree r form a complex:

$$\mathcal{S}_r\Lambda^0 \xrightarrow{d} \mathcal{S}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{S}_{r-n}\Lambda^n.$$

		$k=0$	$k=1$	$k=2$	$k=3$
$n=1$	$r=1$	$\mathbf{S_1}$ (2)	$\mathbf{dPc_1}$ (2)		
	$r=2$	$\mathbf{S_2}$ (3)	$\mathbf{dPc_2}$ (3)		
	$r=3$	$\mathbf{S_3}$ (4)	$\mathbf{dPc_3}$ (4)		
$n=2$	$r=1$	$\mathbf{S_1}$ (4)	$\mathbf{BDMc_1^{[e/f]}}$ (8)	$\mathbf{dPc_1}$ (3)	
	$r=2$	$\mathbf{S_2}$ (8)	$\mathbf{BDMc_2^{[e/f]}}$ (14)	$\mathbf{dPc_2}$ (6)	
	$r=3$	$\mathbf{S_3}$ (12)	$\mathbf{BDMc_3^{[e/f]}}$ (22)	$\mathbf{dPc_3}$ (10)	
$n=3$	$r=1$	$\mathbf{S_1}$ (8)	$\mathbf{AA_1^e}$ (24)	$\mathbf{AA_1^f}$ (18)	$\mathbf{dPc_1}$ (4)
	$r=2$	$\mathbf{S_2}$ (20)	$\mathbf{AA_2^e}$ (48)	$\mathbf{AA_2^f}$ (39)	$\mathbf{dPc_2}$ (10)
	$r=3$	$\mathbf{S_3}$ (32)	$\mathbf{AA_3^e}$ (84)	$\mathbf{AA_3^f}$ (72)	$\mathbf{dPc_3}$ (20)

Legend

Element with degrees of freedom (DOFs)

Dimension of element function space

Symbol of element

Finite element exterior calculus notation

Weight functions for DOFs

Element specification in FEniCS

$\dim \mathcal{P}_r^-\Lambda^k(\Delta_n) = \binom{r+n}{r+k}\binom{r+k-1}{k}$

$\dim \mathcal{P}_r\Lambda^k(\Delta_n) = \binom{r+n}{r+k}\binom{r+k}{k}$

$\dim \mathcal{Q}_r^-\Lambda^k(\Box_n) = \binom{n}{k}r^k(r+1)^{n-k}$

$\dim \mathcal{S}_r\Lambda^k(\Box_n) = \sum_{d=k}^{\min(n,\lfloor r/2\rfloor+k)} 2^{n-d}\binom{n}{d}\binom{r-d+2k}{d}\binom{d}{k}$

Finite elements

The table presents the primary spaces of finite elements for the discretization of the fundamental operators of vector calculus: the gradient, curl, and divergence. A finite element space is a space of piecewise polynomial functions on a domain determined by: (1) a mesh of the domain into polyhedral cells called *elements*, (2) a finite dimensional space of polynomial functions on each element called the *shape functions*, and (3) a unisolvent set of functionals on the shape functions of each element called *degrees of freedom* (DOFs), each DOF being associated to a (generalized) face of the element, and specifying a quantity which takes a single value for all elements sharing the face. The element diagrams depict the DOFs and their association to faces.

The spaces $\mathcal{P}_r^-\Lambda^k$ and $\mathcal{P}_r\Lambda^k$ depicted on the left half of the table are the two primary families of finite element spaces for meshes of simplices, and the spaces $\mathcal{Q}_r^-\Lambda^k$ and $\mathcal{S}_r\Lambda^k$ on the right side are for meshes of cubes or boxes. Each is defined in any dimension $n \ge 1$ for each value of the polynomial degree $r \ge 1$, and each value of $0 \le k \le n$. The parameter k refers to the operator: the spaces consist of differential k-forms which belong to the domain of the kth exterior derivative. Thus for $k=0$, the spaces discretize the Sobolev space H^1, the domain of the gradient operator; for $k=1$, they discretize $H(\mathrm{curl})$, the domain of the curl; for $k=n-1$ they discretize $H(\mathrm{div})$, the domain of the divergence; and for $k=n$, they discretize L^2.

The spaces $\mathcal{P}_r^-\Lambda^0$ and $\mathcal{P}_r\Lambda^0$, which coincide, are the earliest finite elements, going back in the case $r=1$ of linear elements to *Courant*,[1] and collectively referred to as *the Lagrange elements*. The spaces $\mathcal{P}_{r+1}^-\Lambda^n$ and $\mathcal{P}_r\Lambda^n$, which also coincide, are the discontinuous *Galerkin* elements, consisting of piecewise polynomials with no interelement continuity imposed, first introduced by *Reed and Hill*.[2] The space $\mathcal{P}_r^-\Lambda^1$ in 2 dimensions was introduced by *Raviart and Thomas*[3] and generalized to the 3-dimensional spaces $\mathcal{P}_r^-\Lambda^1$ and $\mathcal{P}_r^-\Lambda^2$ by *Nédélec*,[4] while $\mathcal{P}_r\Lambda^1$ is due to *Brezzi, Douglas and Marini*[5] in 2 dimensions, its generalization to 3 dimensions again due to *Nédélec*.[6] The unified treatment and notation of the $\mathcal{P}_r^-\Lambda^k$ and $\mathcal{P}_r\Lambda^k$ families is due to *Arnold, Falk and Winther* as part of *finite element exterior calculus*,[7] extending earlier work of *Hiptmair* for the $\mathcal{P}_r^-\Lambda^k$ family.[8] The space $\mathcal{P}_1^-\Lambda^k$ is the span of the *elementary forms* introduced by *Whitney*.[9]

The family $\mathcal{Q}_r^-\Lambda^k$ of cubical elements can be derived from the 1-dimensional Lagrange and discontinuous Galerkin elements by a tensor product construction detailed by *Arnold, Boffi and Bonizzoni*,[10] but for the most part were presented individually along with the corresponding simplicial elements in the papers mentioned. The second cubical family $\mathcal{S}_r\Lambda^k$ is due to *Arnold and Awanou*.[11]

The finite elements in this table have been implemented as part of the *FEniCS Project*.[12,13,14] Each may be referenced in the *Unified Form Language* (UFL)[15] by giving its family, shape, and degree, with the family as shown on the table. For example, the space $\mathcal{P}_3\Lambda^1(\Delta_3)$ may be referred to in UFL as:

```
FiniteElement("N2E", tetrahedron, 3)
```

Alternatively, the elements may be accessed in a uniform fashion as:

```
FiniteElement("P-", shape, r, k)
FiniteElement("P", shape, r, k)
FiniteElement("Q-", shape, r, k)
FiniteElement("S", shape, r, k)
```

for $\mathcal{P}_r^-\Lambda^k$, $\mathcal{P}_r\Lambda^k$, $\mathcal{Q}_r^-\Lambda^k$, and $\mathcal{S}_r\Lambda^k$, respectively.

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[simula . research laboratory] - by thinking constantly about it

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Periodic Table of the Finite Elements

http://femtable.org

printed copies available upon request

Constructing complexes and their discretizations

The Sobolev–de Rham complex

For each $s \in \mathbb{R}$:

$$0 \longrightarrow H^s\Lambda^0 \xrightarrow{d} H^{s-1}\Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} H^{s-n}\Lambda^n \longrightarrow 0$$

THEOREM (COSTABEL–MCINTOSH 2010)

$\exists$ finite dimensional spaces $\mathfrak{H}^k_\infty \subset \Lambda^k(\Omega)$, $k = 0, \ldots, n$, independent of s which represents the cohomology of the Sobolev–de Rham complex:

$$\mathcal{N}(d, H^s\Lambda^k) = \mathcal{R}(d, H^{s+1}\Lambda^{k-1}) \oplus \mathfrak{H}^k_\infty.$$

Consequences of Costabel–McIntosh theorem

- The cohomology spaces are independent of the smoothness.
- The Sobolev–de Rham complex has closed ranges for all s.
- Regular potentials: If $dH\Lambda^k = dH^1\Lambda^k$.
- Regular decomposition: $H\Lambda^k = dH^1\Lambda^{k-1} + H^1\Lambda^k$
- The L^2 Hilbert complex has closed ranges.
  ($\Longrightarrow$ Hodge decomposition, Poincaré inequality, well-posed Hodge Laplacian).
- Compactness property: $H\Lambda^k \cap \mathring{H}^*\Lambda^k \hookrightarrow L^2\Lambda^k$ compact

Other complexes

The de Rham complex is suitable for problems built from the operators grad, curl, div: Laplacian, Darcy flow, vector Laplacian, Maxwell's equations, div-curl problems, etc. Other PDEs bring in other complexes.

The 3D elasticity complex (or Kröner complex or Calabi complex):

$$0 \longrightarrow C^\infty \otimes \mathbb{V} \xrightarrow{\text{def}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{V} \longrightarrow 0$$

- $\operatorname{def} u := \operatorname{sym}\operatorname{grad} u$, the strain tensor associated to a displacement vector field u
- $\operatorname{inc} = \operatorname{curl} T \operatorname{curl}$, the St. Venant operator, measures the incompatibility of a symmetric matrix field to be a strain. ($\operatorname{inc}\operatorname{def} u$ is the density of dislocations (Kröner))
- div, the vector-valued divergence, gives the force density generated by a stress field; primary operator in mixed formulation of elasticity

Some other complexes

- Hessian complex

$$0 \longrightarrow C^\infty \xrightarrow{\text{hess}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{curl}} C^\infty \otimes \mathbb{T} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{V} \longrightarrow 0$$

- div div complex

$$0 \longrightarrow C^\infty \otimes \mathbb{V} \xrightarrow{\text{dev grad}} C^\infty \otimes \mathbb{T} \xrightarrow{\text{sym curl}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{div div}} C^\infty \longrightarrow 0$$

both with applications to plates (Pauly–Zulehner) and to the Einstein equations (Einstein–Bianchi system, Quenneville–Belair)

- grad curl, curl div complex, and many more…

Bringing order to the chaos

- We will give a systematic derivation of all these complexes and more.
- The construction starts with two understood complexes, and derives a new complex which inherits their properties.
- From the derivation we obtain the cohomology of the new complex. The dimension is the sum of the dimensions from the original complexes.
- We obtain the H^s Sobolev versions of the complexes for all s, with cohomology representatives independent of s. From this follows all the important properties such as closed ranges, regular decomposition, compactness property etc.

Complexes from complexes in four easy steps

1. Start with two abstract complexes of Hilbert spaces and bounded linear operators:

$$0 \longrightarrow Z^0 \xrightarrow{D^0} Z^1 \xrightarrow{D^1} \cdots \xrightarrow{D^{n-1}} Z^n \longrightarrow 0$$

$$0 \longrightarrow \tilde{Z}^0 \xrightarrow{\tilde{D}^0} \tilde{Z}^1 \xrightarrow{\tilde{D}^1} \cdots \xrightarrow{\tilde{D}^{n-1}} \tilde{Z}^n \longrightarrow 0$$

where

$$Z^k = V^k \otimes \mathbb{E}^k, \quad \tilde{Z}^k = \tilde{V}^k \otimes \tilde{\mathbb{E}}^k$$

for Hilbert spaces V^k, $\tilde{V}^k$ and *finite dimensional* Hilbert spaces $\mathbb{E}^k$, $\tilde{\mathbb{E}}^k$.

For example, if the Z complex is the Sobolev–de Rham complex, then $Z^k = H^{s-k}\Lambda^k,$ so $V^k = H^{s-k}$ and $\mathbb{E}^k = \mathrm{Alt}^k$.

2. We require a relation between the complexes:
   - $\tilde{V}^k = V^{k+1}$
   - There exist linear operators $s^k : \tilde{\mathbb{E}}^k \to \mathbb{E}^{k+1}$

Then we can define connecting maps $\quad S^k = \mathrm{id} \otimes s^k : \tilde{Z}^k \to Z^{k+1}$.

$$
\begin{tikzcd}
0 \arrow[r] & V^0 \otimes \mathbb{E}^0 \arrow[r, "D^0"] & V^1 \otimes \mathbb{E}^1 \arrow[r, "D^1"] & \cdots \arrow[r, "D^{n-1}"] & V^n \otimes \mathbb{E}^n \arrow[r] & 0 \\
0 \arrow[r] & \tilde{V}^0 \otimes \tilde{\mathbb{E}}^0 \arrow[r, "\tilde{D}^0"'] \arrow[ur, "S^0"] & \tilde{V}^1 \otimes \tilde{\mathbb{E}}^1 \arrow[r, "\tilde{D}^1"'] \arrow[ur, "S^1"] & \cdots \arrow[r, "\tilde{D}^{n-1}"'] \arrow[ur, "S^{n-1}"] & \tilde{V}^n \otimes \tilde{\mathbb{E}}^n \arrow[r] & 0
\end{tikzcd}
$$

We further require

- Anticommutivity: $S^{k+1}\tilde{D}^k = -D^{k+1}S^k$
- Injectivity/surjectivity: $\exists\, J < n$ s.t. s^k is $\begin{cases} \text{injective}, & 0 \le k \le J, \\ \text{surjective} & k \ge J \end{cases}$

3. Set $\mathbb{F}^k = \mathcal{R}(s^{k-1})^\perp \subset \mathbb{E}^k$. Reduce the first $J+1$ spaces of the first complex from $Z^k = V^k \otimes \mathbb{E}^k$ to $V^k \otimes \mathbb{F}^k$ and replace D^k with $P_{\mathbb{F}^k} D^k$.

$$0 \longrightarrow V^0 \otimes \mathbb{F}^0 \xrightarrow{P_{\mathbb{F}^1} D^0} V^1 \otimes \mathbb{F}^1 \xrightarrow{P_{\mathbb{F}^2} D^1} \cdots \xrightarrow{P_{\mathbb{F}^J} D^{J-1}} V^J \otimes \mathbb{F}^J$$

Similarly, set $\tilde{\mathbb{F}}^k = \mathcal{N}(s^k) \subset \tilde{\mathbb{E}}^k$. Reduce the final $n-J$ spaces of the second complex from $\tilde{Z}^k = V^{k+1} \otimes \tilde{\mathbb{E}}^k$ to $V^{k+1} \otimes \tilde{\mathbb{F}}^k$.

$$\tilde{V}^{J+1} \otimes \tilde{\mathbb{F}}^{J+1} \xrightarrow{\tilde{D}^{J+1}} \tilde{V}^{J+2} \otimes \tilde{\mathbb{F}}^{J+2} \xrightarrow{\tilde{D}^{J+2}} \cdots \xrightarrow{\tilde{D}^{J}} \tilde{V}^n \otimes \tilde{\mathbb{F}}^n \longrightarrow 0$$

4. Finally, splice together the two sequences through this diagram:

$$\begin{tikzcd} V^J \otimes \mathbb{F}^J \arrow[r, "D^J"] & V^{J+1} \otimes \mathbb{E}^{J+1} \\ \tilde{V}^J \otimes \tilde{\mathbb{E}}^J \arrow[r, "\tilde{D}^J"] \arrow[ur, "S^J"] & \tilde{V}^{J+1} \otimes \tilde{\mathbb{F}}^{J+1} \end{tikzcd} \implies V^J \otimes \mathbb{F}^J \xrightarrow{\tilde{D}^J (S^J)^{-1} D^J} \tilde{V}^{J+1} \otimes \tilde{\mathbb{F}}^{J+1}$$

Summary of the construction

$$0 \to V^0 \otimes \mathbb{E}^0 \longrightarrow \cdots \longrightarrow V^J \otimes \mathbb{E}^J \longrightarrow V^{J+1} \otimes \mathbb{E}^{J+1} \longrightarrow \cdots \longrightarrow V^n \otimes \mathbb{E}^n \to 0$$

$$0 \to \tilde{V}^0 \otimes \tilde{\mathbb{E}}^0 \longrightarrow \cdots \longrightarrow \tilde{V}^J \otimes \tilde{\mathbb{E}}^J \longrightarrow \tilde{V}^{J+1} \otimes \tilde{\mathbb{E}}^{J+1} \longrightarrow \cdots \longrightarrow \tilde{V}^n \otimes \tilde{\mathbb{E}}^n \to 0$$

1. Input complexes (of tensor product form)

Summary of the construction

$$
\begin{tikzcd}
0 \arrow[r] & V^0 \otimes \mathbb{E}^0 \arrow[r] & \cdots \arrow[r] & V^J \otimes \mathbb{E}^J \arrow[r] & V^{J+1} \otimes \mathbb{E}^{J+1} \arrow[r] & \cdots \arrow[r] & V^n \otimes \mathbb{E}^n \arrow[r] & 0 \\
0 \arrow[r] & \tilde{V}^0 \otimes \tilde{\mathbb{E}}^0 \arrow[r] \arrow[ur, "S"] & \cdots \arrow[r] \arrow[ur, "S"] & \tilde{V}^J \otimes \tilde{\mathbb{E}}^J \arrow[r] \arrow[ur, "S", "\cong"'] & \tilde{V}^{J+1} \otimes \tilde{\mathbb{E}}^{J+1} \arrow[r] \arrow[ur, "S"] & \cdots \arrow[r] \arrow[ur, "S"] & \tilde{V}^n \otimes \tilde{\mathbb{E}}^n \arrow[r] & 0
\end{tikzcd}
$$

1. Input complexes (of tensor product form)
2. Connecting maps (anticommuting and J-surjective/injective)

Summary of the construction

$$
\begin{tikzcd}
0 \arrow[r] & V^0 \otimes \mathbb{F}^0 \arrow[r] & \cdots \arrow[r] & V^J \otimes \mathbb{F}^J \arrow[r] & V^{J+1} \otimes \mathbb{E}^{J+1} \arrow[r] & \cdots \arrow[r] & V^n \otimes \mathbb{E}^n \arrow[r] & 0 \\
0 \arrow[r] & \tilde{V}^0 \otimes \tilde{\mathbb{E}}^0 \arrow[r] \arrow[ur, "S"] & \cdots \arrow[r] \arrow[ur, "S"] & \tilde{V}^J \otimes \tilde{\mathbb{E}}^J \arrow[r] \arrow[ur, "S", "\cong"'] & \tilde{V}^{J+1} \otimes \tilde{\mathbb{F}}^{J+1} \arrow[r] \arrow[ur, "S"] & \cdots \arrow[r] \arrow[ur, "S"] & \tilde{V}^n \otimes \tilde{\mathbb{F}}^n \arrow[r] & 0
\end{tikzcd}
$$

1. Input complexes (of tensor product form)
2. Connecting maps (anticommuting and J-surjective/injective)
3. Space reduction

Summary of the construction

$$
\begin{tikzcd}
0 \arrow[r] & V^0 \otimes \mathbb{F}^0 \arrow[r] & \cdots \arrow[r] & V^J \otimes \mathbb{F}^J \arrow[r] & V^{J+1} \otimes \mathbb{E}^{J+1} \arrow[r] & \cdots \arrow[r] & V^n \otimes \mathbb{E}^n \arrow[r] & 0 \\
0 \arrow[r] & \tilde{V}^0 \otimes \tilde{\mathbb{E}}^0 \arrow[r] \arrow[ur, "S"] & \cdots \arrow[r] \arrow[ur, "S"] & \tilde{V}^J \otimes \tilde{\mathbb{E}}^J \arrow[r] \arrow[ur, "S", "\cong"'] & \tilde{V}^{J+1} \otimes \tilde{\mathbb{F}}^{J+1} \arrow[r] \arrow[ur, "S"] & \cdots \arrow[r] \arrow[ur, "S"] & \tilde{V}^n \otimes \tilde{\mathbb{F}}^n \arrow[r] & 0
\end{tikzcd}
$$

1. Input complexes (of tensor product form)
2. Connecting maps (anticommuting and J-surjective/injective)
3. Space reduction
4. Splicing

Cohomology of the derived complex

Theorem

The dimension of the kth cohomology space of the derived complex is bounded by the sum of the dimensions of the kth cohomology spaces of the two input complexes. Equality holds if the connecting maps induce the zero map on cohomology, i.e., $S^k(\mathcal{N}(\tilde{D}^k)) \subset \mathcal{R}(D^k)$.

Corollary

If the input complexes have finite dimensional cohomology, so does the derived complex. Consequently, it has closed ranges.

Fix $0 \le J < n$. For the two input complexes take Sobolev–de Rham complexes tensored with Alt^J and Alt^{J+1}, respectively:

$$0 \to H^s\Lambda^0 \otimes \mathrm{Alt}^J \xrightarrow{d} H^{s-1}\Lambda^1 \otimes \mathrm{Alt}^J \xrightarrow{d} \cdots \xrightarrow{d} H^{s-n}\Lambda^n \otimes \mathrm{Alt}^J \to 0$$

$$0 \to H^{s-1}\Lambda^0 \otimes \mathrm{Alt}^{J+1} \xrightarrow{d} H^{s-2}\Lambda^1 \otimes \mathrm{Alt}^{J+1} \xrightarrow{d} \cdots \xrightarrow{d} H^{s-n-1}\Lambda^n \otimes \mathrm{Alt}^{J+1} \to 0$$

$$V^i = \tilde{V}^{i-1} = H^{s-i}, \quad \mathbb{E}^i = \mathrm{Alt}^i \otimes \mathrm{Alt}^J, \quad \tilde{\mathbb{E}}^i = \mathrm{Alt}^i \otimes \mathrm{Alt}^{J+1}$$

The connecting maps $s^i : \mathrm{Alt}^i \otimes \mathrm{Alt}^{J+1} \to \mathrm{Alt}^{i+1} \otimes \mathrm{Alt}^J$ are naturally defined in terms of components:

$$M_{\underbrace{k_1 \cdots k_i}_{\text{antisym}} \underbrace{l_1 \cdots l_{J+1}}_{\text{antisym}}} \quad \xmapsto{s^i} \quad M_{\underbrace{[l_1 k_1 \cdots k_i]}_{\text{antisymmetrize}} l_2 \cdots l_{J+1}}$$

More on the connecting maps

THEOREM

The algebraic maps $s^i : Alt^i \otimes Alt^{J+1} \to Alt^{i+1} \otimes Alt^J$ satisfy the anticommutativity and J-injectivity/surjectivity assumptions.

The s^i in terms of vector proxies. In 3D, Alt^k, $k = 0, 1, 2, 3$, identifies with $\mathbb{R}$, $\mathbb{V}$, $\mathbb{V}$, $\mathbb{R}$.

$$\begin{tikzcd}
i \backslash J & 0 & 1 & 2 & 3 \\
0 & \mathbb{R} & \mathbb{V} & \mathbb{V} & \mathbb{R} \\
1 & \mathbb{V} \arrow[ur, "\mathrm{id}"] & \mathbb{M} \arrow[ur, "\mathrm{vskw}"] & \mathbb{M} \arrow[ur, "\mathrm{tr}"] & \mathbb{V} \\
2 & \mathbb{V} \arrow[ur, "\mathrm{mskw}"] & \mathbb{M} \arrow[ur, "S^1"] & \mathbb{M} \arrow[ur, "\mathrm{vskw}"] & \mathbb{V} \\
3 & \mathbb{R} \arrow[ur, "\iota"] & \mathbb{V} \arrow[ur, "\mathrm{mskw}"] & \mathbb{V} \arrow[ur, "\mathrm{id}"] & \mathbb{R}
\end{tikzcd}$$

vskw : $\mathbb{M} \to \mathbb{V}$, axial vector of the skew part

tr : $\mathbb{M} \to \mathbb{R}$, trace

mskw : $\mathbb{V} \to \mathbb{M}$, axial vector to matrix

$\iota : \mathbb{R} \to \mathbb{M}$, scalar to scalar matrix: $c \mapsto c\,I$

$S^1 : \mathbb{M} \to \mathbb{M}$, $S^1\tau = \tau^T - \mathrm{tr}(\tau)I$ (bijection)

The derived complexes in 3D

$$
\begin{tikzcd}
H^s \otimes \mathbb{R} \arrow[r] & H^{s-1} \otimes \mathbb{V} \arrow[r] & H^{s-2} \otimes \mathbb{V} \arrow[r] & H^{s-3} \otimes \mathbb{R} \\
H^{s-1} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-2} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \\
H^{s-2} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \\
H^{s-3} \otimes \mathbb{R} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-6} \otimes \mathbb{R}
\end{tikzcd}
$$

The derived complexes in 3D

$$
\begin{tikzcd}
H^s \otimes \mathbb{R} \arrow[r] & H^{s-1} \otimes \mathbb{V} \arrow[r] & H^{s-2} \otimes \mathbb{V} \arrow[r] & H^{s-3} \otimes \mathbb{R} \\
H^{s-1} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-2} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \\
H^{s-2} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \\
H^{s-3} \otimes \mathbb{R} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-6} \otimes \mathbb{R}
\end{tikzcd}
$$

The Hessian complex:

$$0 \to H^s \xrightarrow{\text{hess}} H^{s-2} \otimes \mathbb{S} \xrightarrow{\text{curl}} H^{s-3} \otimes \mathbb{T} \xrightarrow{\text{div}} H^{s-4} \otimes \mathbb{V} \to 0$$

The derived complexes in 3D

$$\begin{tikzcd}
H^s \otimes \mathbb{R} \arrow[r] & H^{s-1} \otimes \mathbb{V} \arrow[r] & H^{s-2} \otimes \mathbb{V} \arrow[r] & H^{s-3} \otimes \mathbb{R} \\
H^{s-1} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-2} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \\
H^{s-2} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \\
H^{s-3} \otimes \mathbb{R} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-6} \otimes \mathbb{R}
\end{tikzcd}$$

The elasticity complex:

$$0 \to H^{s-1} \otimes \mathbb{V} \xrightarrow{\text{def}} H^{s-2} \otimes \mathbb{S} \xrightarrow{\text{inc}} H^{s-4} \otimes \mathbb{S} \xrightarrow{\text{div}} H^{s-5} \otimes \mathbb{V} \to 0$$

The derived complexes in 3D

$$\begin{tikzcd}
H^s \otimes \mathbb{R} \arrow[r] & H^{s-1} \otimes \mathbb{V} \arrow[r] & H^{s-2} \otimes \mathbb{V} \arrow[r] & H^{s-3} \otimes \mathbb{R} \\
H^{s-1} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-2} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \\
H^{s-2} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \\
H^{s-3} \otimes \mathbb{R} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-6} \otimes \mathbb{R}
\end{tikzcd}$$

The div-div complex:

$$0 \to H^{s-2} \otimes \mathbb{V} \xrightarrow{\operatorname{dev}\operatorname{grad}} H^{s-3} \otimes \mathbb{T} \xrightarrow{\operatorname{sym}\operatorname{curl}} H^{s-4} \otimes \mathbb{S} \xrightarrow{\operatorname{div}\operatorname{div}} H^{s-6} \to 0$$

The derived complexes in 3D

$$
\begin{tikzcd}
H^s \otimes \mathbb{R} \arrow[r] & H^{s-1} \otimes \mathbb{V} \arrow[r] & H^{s-2} \otimes \mathbb{V} \arrow[r] & H^{s-3} \otimes \mathbb{R} \\
H^{s-1} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-2} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \\
H^{s-2} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \\
H^{s-3} \otimes \mathbb{R} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-6} \otimes \mathbb{R}
\end{tikzcd}
$$

The grad-curl complex:

$$
0 \to H^s \xrightarrow{\text{grad}} H^{s-1} \otimes \mathbb{V} \xrightarrow{\text{grad curl}} H^{s-3} \otimes \mathbb{T} \xrightarrow{\text{curl}} H^{s-4} \otimes \mathbb{M} \xrightarrow{\text{div}} H^{s-5} \otimes \mathbb{V} \to 0
$$

The derived complexes in 3D

$$\begin{tikzcd}
H^s \otimes \mathbb{R} \arrow[r] & H^{s-1} \otimes \mathbb{V} \arrow[r] & H^{s-2} \otimes \mathbb{V} \arrow[r] & H^{s-3} \otimes \mathbb{R} \\
H^{s-1} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-2} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \\
H^{s-2} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \\
H^{s-3} \otimes \mathbb{R} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-6} \otimes \mathbb{R}
\end{tikzcd}$$

The curl-div complex:

$$0 \to H^{s-1} \otimes \mathbb{V} \xrightarrow{\text{grad}} H^{s-2} \otimes \mathbb{M} \xrightarrow{\text{dev curl}} H^{s-3} \otimes \mathbb{T} \xrightarrow{\text{curl div}} H^{s-5} \otimes \mathbb{M} \xrightarrow{\text{div}} H^{s-6} \otimes \mathbb{V} \to 0$$

$$
\begin{tikzcd}
H^s \otimes \mathbb{R} \arrow[r] & H^{s-1} \otimes \mathbb{V} \arrow[r] & H^{s-2} \otimes \mathbb{V} \arrow[r] & H^{s-3} \otimes \mathbb{R} \\
H^{s-1} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-2} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \\
H^{s-2} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-3} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{M} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \\
H^{s-3} \otimes \mathbb{R} \arrow[r] \arrow[ur] & H^{s-4} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-5} \otimes \mathbb{V} \arrow[r] \arrow[ur] & H^{s-6} \otimes \mathbb{R}
\end{tikzcd}
$$

The grad-div complex:

$$
0 \to H^s \xrightarrow{\mathrm{grad}} H^{s-1} \otimes \mathbb{V} \xrightarrow{\mathrm{curl}} H^{s-2} \otimes \mathbb{V} \xrightarrow{\mathrm{grad\,div}} H^{s-4} \otimes \mathbb{V} \xrightarrow{\mathrm{curl}} H^{s-5} \otimes \mathbb{V} \xrightarrow{\mathrm{div}} H^{s-6} \to 0
$$

Complexes from complexes from complexes

We can iterate this procedure, using the derived complexes as inputs to derive more complexes. For example,

elasticity complex + Hessian complex $\implies$ conformal elasticity complex

which involves the deviatoric strain tensor $\operatorname{dev}\operatorname{def} u$, and a third order differential operator known as the Cotton tensor and arising in relativity.

$$\begin{tikzcd}
0 \arrow[r] & H^s \otimes \mathbb{V} \arrow[r, "\operatorname{def}"] & H^{s-1} \otimes \mathbb{S} \arrow[r, "\operatorname{inc}"] & H^{s-3} \otimes \mathbb{S} \arrow[r, "\operatorname{div}"] & H^{s-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{s-1} \arrow[r, "\operatorname{hess}"] \arrow[ur, "\iota"] & H^{s-3} \otimes \mathbb{S} \arrow[r, "\operatorname{curl}"] \arrow[ur, "S^1"] & H^{s-4} \otimes \mathbb{T} \arrow[r, "\operatorname{div}"] \arrow[ur, "\operatorname{vskw}"] & H^{s-5} \otimes \mathbb{V} \arrow[r] & 0
\end{tikzcd}$$

$$\Downarrow$$

$$0 \to H^s \otimes \mathbb{V} \xrightarrow{\operatorname{dev}\operatorname{def}} H^{s-1} \otimes (\mathbb{S} \cap \mathbb{T}) \xrightarrow{\operatorname{Cott}} H^{s-4} \otimes (\mathbb{S} \cap \mathbb{T}) \xrightarrow{\operatorname{div}} H^{s-5} \otimes \mathbb{V} \to 0$$

The first Poincaré inequality for this complex is a strengthened Korn's inequality:

$$\|u\|_{H^1} \le c(\|u\|_{L^2} + \|\operatorname{dev}\operatorname{def} u\|_{L^2})$$

A 2D example, continuous and discrete

Beginning with an H^2 de Rham complex and an H^1 vector-valued de Rham complex, we get a 2D elasticity complex:

$$\begin{tikzcd} 0 \arrow[r] & H^2 \arrow[r, "\mathrm{curl}"] & H^1 \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & L^2 \arrow[r] & 0 \\ 0 \arrow[r] & H^1 \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{id}"] & L^2 \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{skw}"] & H^{-1} \otimes \mathbb{V} \arrow[r] & 0 \end{tikzcd}$$

$$\Downarrow$$

$$0 \longrightarrow H^2 \xrightarrow{\mathrm{curl}\,\mathrm{curl}} L^2 \otimes \mathbb{S} \xrightarrow{\mathrm{div}} H^{-1} \otimes \mathbb{V} \longrightarrow 0$$

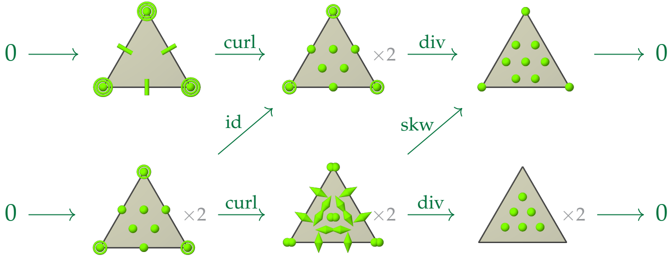
The corresponding L^2 Hilbert complex is what we need to discretize mixed elasticity finite elements.

$$0 \longrightarrow H^2 \xrightarrow{\mathrm{curl}\,\mathrm{curl}} H(\mathrm{div}, \mathbb{S}) \xrightarrow{\mathrm{div}} L^2 \otimes \mathbb{V} \longrightarrow 0$$

Finite element discretization

Discretize with finite elements on a triangulation:

- H^2: Argyris quintic 1968 ($\mathcal{P}_5$)
- $H^1 \otimes \mathbb{V}$: Hermite quartic elements ($\mathcal{P}_4$)
- L^2: piecewise cubics with vertex continuity ($\mathcal{P}_3$)
- $L^2 \otimes \mathbb{M}$: two copies of "nonstandard" variant of BDM, Stenberg 2010 ($\mathcal{P}_3$)
- $H^{-1} \otimes \mathbb{V}$: DG2 ($\mathcal{P}_2$)

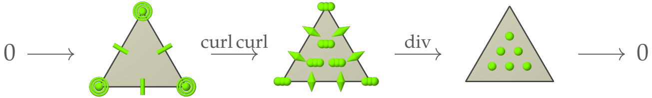


The top sequence was studied by Falk and Neilan 2013 as a discretization for the Stokes complex.

The resulting complex

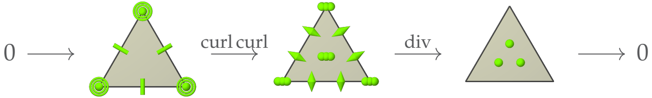
The resulting derived complex is a discretization of the 2D elasticity complex.

- The scalar space (Airy potential) is Argyris.
- Stress: symmetric matrices with rows in Stenberg $\mathcal{P}_3$ space
- Displacement: DG2 vectors



mixed elasticity elements proposed by J. Hu–S. Zhang 2015

A similar construction may be given for the original mixed elasticity elements of Arnold–Winther 2002 (cf. Arnold–Falk–Winter, IMA vol. 142).



Elasticity with weak symmetry

The mixed formulation of elasticity with *weak symmetry* is more amenable to discretization than the standard mixed formulation.

Fraeijs de Veubeke '75

$$p = \operatorname{skw} \operatorname{grad} u, \quad A\sigma = \operatorname{grad} u - p$$

Find $\quad \sigma \in L^2(\Omega; \mathbb{R}^{n\times n}), u \in L^2(\Omega; \mathbb{R}^n), p \in L^2(\Omega; \mathbb{R}^{n\times n}_{\text{skw}}) \quad$ s.t.

$$\begin{aligned} \langle A\sigma, \tau\rangle + \langle u, \operatorname{div}\tau\rangle + \langle p, \tau\rangle &= 0, && \tau \in L^2(\Omega; \mathbb{R}^{n\times n}) \\ -\langle \operatorname{div}\sigma, v\rangle &= \langle f, v\rangle, && v \in L^2(\Omega; \mathbb{R}^n) \\ -\langle \sigma, q\rangle &= 0, && q \in L^2(\Omega; \mathbb{R}^{n\times n}_{\text{skw}}) \end{aligned}$$

This is exactly the mixed Hodge Laplacian for the complex

$$L^2_A(\Omega; \mathbb{R}^{n\times n}) \xrightarrow{(-\operatorname{div}, -\operatorname{skw})} L^2(\Omega; \mathbb{R}^n) \oplus L^2(\Omega; \mathbb{R}^{n\times n}_{\text{skw}}) \longrightarrow 0$$

supposing that it is exact.

Well-posedness

$$L^2_A(\Omega;\mathbb{R}^{n\times n}) \xrightarrow{(-\operatorname{div},-\operatorname{skw})} L^2(\Omega;\mathbb{R}^n) \oplus L^2(\Omega;\mathbb{R}^{n\times n}_{\operatorname{skw}}) \longrightarrow 0$$

To show the complex is exact, and so the system is well-posed, we relate it to two de Rham complexes with commuting connecting maps:

$$\begin{tikzcd}
 & L^2(\Omega;\mathbb{R}^n \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}}) \arrow[rr, "\operatorname{div}"] & & L^2(\Omega;\mathbb{R}^{n\times n}_{\operatorname{skw}}) \arrow[r] & 0 \\
L^2(\Omega;\mathbb{R}^{n\times n}) \arrow[ur, "S"] \arrow[rr, "\operatorname{curl}"] & & L^2(\Omega;\mathbb{R}^{n\times n}) \arrow[ur, "-\operatorname{skw}"] \arrow[rr, "-\operatorname{div}"] & & L^2(\Omega;\mathbb{R}^n) \arrow[r] & 0
\end{tikzcd}$$

$S\tau = \tau^T - \operatorname{tr}(\tau)I$ (invertible)

Well-posedness

$$L^2_A(\Omega;\mathbb{R}^{n\times n}) \xrightarrow{(-\operatorname{div},-\operatorname{skw})} L^2(\Omega;\mathbb{R}^n)\oplus L^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{skw}}) \longrightarrow 0$$

To show the complex is exact, and so the system is well-posed, we relate it to two de Rham complexes with commuting connecting maps:

$$\begin{tikzcd}
 & L^2(\Omega;\mathbb{R}^n\otimes\mathbb{R}^{n\times n}_{\mathrm{skw}}) \arrow[r, "\operatorname{div}"] & \overset{q}{L^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{skw}})} \arrow[r] & 0 \\
L^2(\Omega;\mathbb{R}^{n\times n}) \arrow[ur, "S"] \arrow[r, "\operatorname{curl}"] & L^2(\Omega;\mathbb{R}^{n\times n}) \arrow[ur, "-\operatorname{skw}"] \arrow[r, "-\operatorname{div}"] & \underset{v}{L^2(\Omega;\mathbb{R}^n)} \arrow[r] & 0
\end{tikzcd}$$

$S\tau = \tau^T - \operatorname{tr}(\tau)I$ (invertible)

Well-posedness

$$L^2_A(\Omega;\mathbb{R}^{n\times n}) \xrightarrow{(-\operatorname{div}, -\operatorname{skw})} L^2(\Omega;\mathbb{R}^n) \oplus L^2(\Omega;\mathbb{R}^{n\times n}_{\operatorname{skw}}) \longrightarrow 0$$

To show the complex is exact, and so the system is well-posed, we relate it to two de Rham complexes with commuting connecting maps:

$$\begin{tikzcd}
 & & q & \\
 & L^2(\Omega;\mathbb{R}^n \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}}) \arrow[r, "\operatorname{div}"] & L^2(\Omega;\mathbb{R}^{n\times n}_{\operatorname{skw}}) \arrow[r] & 0 \\
L^2(\Omega;\mathbb{R}^{n\times n}) \arrow[ur, "S"] \arrow[r, "\operatorname{curl}"] & L^2(\Omega;\mathbb{R}^{n\times n}) \arrow[ur, "-\operatorname{skw}"] \arrow[r, "-\operatorname{div}"] & L^2(\Omega;\mathbb{R}^n) \arrow[r] & 0 \\
 & \rho & v \arrow[l] &
\end{tikzcd}$$

$S\tau = \tau^T - \operatorname{tr}(\tau)I \quad$ (invertible)

Well-posedness

$$L^2_A(\Omega;\mathbb{R}^{n\times n}) \xrightarrow{(-\operatorname{div},-\operatorname{skw})} L^2(\Omega;\mathbb{R}^n)\oplus L^2(\Omega;\mathbb{R}^{n\times n}_{\operatorname{skw}}) \longrightarrow 0$$

To show the complex is exact, and so the system is well-posed, we relate it to two de Rham complexes with commuting connecting maps:

$$\begin{tikzcd}
 & & & q-\operatorname{skw}\rho & & \\
 & L^2(\Omega;\mathbb{R}^n\otimes\mathbb{R}^{n\times n}_{\operatorname{skw}}) \arrow[rr, "\operatorname{div}"] & & L^2(\Omega;\mathbb{R}^{n\times n}_{\operatorname{skw}}) \arrow[rr] & & 0 \\
L^2(\Omega;\mathbb{R}^{n\times n}) \arrow[ur, "S"] \arrow[rr, "\operatorname{curl}"] & & L^2(\Omega;\mathbb{R}^{n\times n}) \arrow[ur, "-\operatorname{skw}"] \arrow[rr, "-\operatorname{div}"] & & L^2(\Omega;\mathbb{R}^n) \arrow[r] & 0 \\
 & & \rho \arrow[uuur] & & v \arrow[ll] &
\end{tikzcd}$$

$S\tau = \tau^T - \operatorname{tr}(\tau)I$ (invertible)

Well-posedness

$$L^2_A(\Omega;\mathbb{R}^{n\times n}) \xrightarrow{(-\operatorname{div},-\operatorname{skw})} L^2(\Omega;\mathbb{R}^n)\oplus L^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{skw}}) \longrightarrow 0$$

To show the complex is exact, and so the system is well-posed, we relate it to two de Rham complexes with commuting connecting maps:

$$\begin{tikzcd}
 & \psi & q-\operatorname{skw}\rho \arrow[l] & \\
 & L^2(\Omega;\mathbb{R}^n\otimes\mathbb{R}^{n\times n}_{\mathrm{skw}}) \arrow[r, "\operatorname{div}"] & L^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{skw}}) \arrow[r] & 0 \\
L^2(\Omega;\mathbb{R}^{n\times n}) \arrow[ur, "S"] \arrow[r, "\operatorname{curl}"] & L^2(\Omega;\mathbb{R}^{n\times n}) \arrow[ur, "-\operatorname{skw}"] \arrow[r, "-\operatorname{div}"] & L^2(\Omega;\mathbb{R}^n) \arrow[r] & 0 \\
 & \rho \arrow[uuur] & v \arrow[l] &
\end{tikzcd}$$

$S\tau = \tau^T - \operatorname{tr}(\tau)I \quad$ (invertible)

Well-posedness

$$L^2_A(\Omega;\mathbb{R}^{n\times n}) \xrightarrow{(-\operatorname{div},-\operatorname{skw})} L^2(\Omega;\mathbb{R}^n)\oplus L^2(\Omega;\mathbb{R}^{n\times n}_{\operatorname{skw}}) \longrightarrow 0$$

To show the complex is exact, and so the system is well-posed, we relate it to two de Rham complexes with commuting connecting maps:

$$\begin{tikzcd}
 & \psi \arrow[dddl] & q-\operatorname{skw}\rho \arrow[l] & \\
 & L^2(\Omega;\mathbb{R}^n\otimes\mathbb{R}^{n\times n}_{\operatorname{skw}}) \arrow[r, "\operatorname{div}"] & L^2(\Omega;\mathbb{R}^{n\times n}_{\operatorname{skw}}) \arrow[r] & 0 \\
L^2(\Omega;\mathbb{R}^{n\times n}) \arrow[r, "\operatorname{curl}"] \arrow[ur, "S"] & L^2(\Omega;\mathbb{R}^{n\times n}) \arrow[r, "-\operatorname{div}"] \arrow[ur, "-\operatorname{skw}"] & L^2(\Omega;\mathbb{R}^n) \arrow[r] & 0 \\
\phi & \rho \arrow[uuur] & v \arrow[l] &
\end{tikzcd}$$

$S\tau = \tau^T - \operatorname{tr}(\tau)I$ (invertible)

Well-posedness

$$L^2_A(\Omega;\mathbb{R}^{n\times n}) \xrightarrow{(-\operatorname{div},-\operatorname{skw})} L^2(\Omega;\mathbb{R}^n)\oplus L^2(\Omega;\mathbb{R}^{n\times n}_{\text{skw}}) \longrightarrow 0$$

To show the complex is exact, and so the system is well-posed, we relate it to two de Rham complexes with commuting connecting maps:

$$\begin{tikzcd}
 & \psi \arrow[dddl] & & q-\operatorname{skw}\rho \arrow[ll] & & \\
 & L^2(\Omega;\mathbb{R}^n\otimes\mathbb{R}^{n\times n}_{\text{skw}}) \arrow[rr, "\operatorname{div}"] & & L^2(\Omega;\mathbb{R}^{n\times n}_{\text{skw}}) \arrow[rr] & & 0 \\
L^2(\Omega;\mathbb{R}^{n\times n}) \arrow[ur, "S"] \arrow[rr, "\operatorname{curl}"] & & L^2(\Omega;\mathbb{R}^{n\times n}) \arrow[ur, "-\operatorname{skw}"] \arrow[rr, "-\operatorname{div}"] & & L^2(\Omega;\mathbb{R}^n) \arrow[r] & 0 \\
\phi \arrow[rr] & & \operatorname{curl}\phi+\rho \arrow[uuur] & & v \arrow[ll] &
\end{tikzcd}$$

$S\tau = \tau^T - \operatorname{tr}(\tau)I$ (invertible)

To discretize we select discrete de Rham subcomplexes with commuting projs

$$\bar{V}_h^1 \xrightarrow{\operatorname{div}} \bar{V}_h^2 \to 0, \qquad \tilde{V}_h^0 \xrightarrow{\operatorname{curl}} \tilde{V}_h^1 \xrightarrow{-\operatorname{div}} \tilde{V}_h^2 \to 0$$

to get the discrete complex

$$\tilde{V}_h^1 \otimes \mathbb{R}^n \xrightarrow{(-\operatorname{div}, -\bar{\pi}_h^2 \operatorname{skw})} (\tilde{V}_h^2 \otimes \mathbb{R}^n) \times (\bar{V}_h^2 \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}}) \to 0$$

We get stability if we can carry out the diagram chase on:

$$\begin{tikzcd}
 & & \bar{V}_h^1 \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}} \arrow[rr, "\operatorname{div}"] & & \bar{V}_h^2 \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}} \arrow[r] & 0 \\
\tilde{V}_h^0 \otimes \mathbb{R}^n \arrow[rr, "\operatorname{curl}"] \arrow[urr, "\bar{\pi}_h^1 S"] & & \tilde{V}_h^1 \otimes \mathbb{R}^n \arrow[rr, "-\operatorname{div}"] \arrow[urr, "-\bar{\pi}_h^2 \operatorname{skw}"] & & \tilde{V}_h^2 \otimes \mathbb{R}^n \arrow[r] & 0
\end{tikzcd}$$

This requires that $\bar{\pi}_h^1 S : \tilde{V}_h^0 \otimes \mathbb{R}^n \to \bar{V}_h^1 \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}}$ is *surjective*.

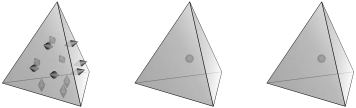
Stable elements

The requirement that $\bar{\pi}_h^1 S : \tilde{V}_h^0 \otimes \mathbb{R}^n \to \bar{V}_h^1 \otimes \mathbb{R}^{n\times n}_{\mathrm{skw}}$ is surjective can be checked by looking at DOFs.

The simplest choice is

$$\mathcal{P}_r^- \Lambda^{n-1} \xrightarrow{\mathrm{div}} \mathcal{P}_r^- \Lambda^n \to 0, \quad \mathcal{P}_{r+1}^- \Lambda^{n-2} \xrightarrow{\mathrm{curl}} \mathcal{P}_r \Lambda^{n-1} \xrightarrow{-\mathrm{div}} \mathcal{P}_{r-1}\Lambda^n \to 0$$

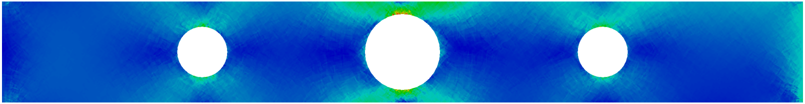
This gives the elements of DNA–Falk–Winther ’07

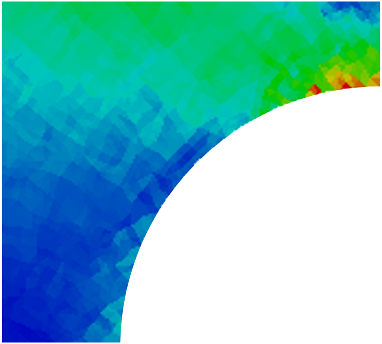


σ u p

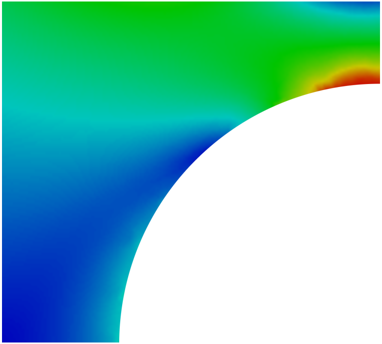
Other elements:
Cockburn–Gopalakrishnan–Guzmán,
Gopalakrishnan–Guzmán, Stenberg, …

Nearly incompressible material





displacement



mixed