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Edth – a differential operator on the sphere

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Introduction. In (9) Newman and Penrose introduced a differential operator which they denoted ð, the phonetic symbol *edth*. This operator acts on *spin weighted*, or *spin* and *conformally weighted* functions on the two-sphere. It turns out to be very useful in the theory of relativity via the isomorphism of the conformal group of the sphere and the proper inhomogeneous Lorentz group (11, 4). In particular, it can be viewed (2) as an *angular momentum lowering operator* for a suitable representation of SO(3) and can be used to investigate the representations of the Lorentz group (4). More recently, edth has appeared in the *good cut equation* describing Newman's $\mathscr{H}$*-space* for an *asymptotically flat* space–time (10). This development is closely related to Penrose's theory of *twistors* and, in particular, to *asymptotic twistors* (14).

The identification of spin and conformally weighted functions as sections of certain complex line-bundles over the sphere is a familiar one (cf. (1)). Our original contribution is to associate ð with the ∂-operator of complex analysis and to show how the properties of ð arise from well-known properties of ∂ by this association. We discuss spin and conformally weighted functions on the *future null infinity* $\mathscr{I}^+$ of an asymptotically flat space–time from this point of view, and then describe the good cut equation which determines the $\mathscr{H}$-space of such a space–time.

In an appendix, following the work of G. A. J. Sparling, we show how the good cut equation arises in the context of complex deformations of *twistor space*

It should be noted that, for notational convenience, the conventions of this paper are slightly changed from the usual. Normally the primed spinor $\pi_{A'}$ is regarded as holomorphic and the stereographic coordinate ζ is $\pi_{0'}/\pi_{1'}$. Then the unprimed spinor appears as $\bar{\pi}_A$ and is antiholomorphic. Also ð is normally defined in terms of $\partial/\partial\pi_{A'}$ so that $\bar{ð}$ is associated with $\partial/\partial\pi_A$. Thus agreement with the usual conventions may be regained if $\pi_{A'}$ is written for π_A and $\bar{\pi}_A$ for $\bar{\pi}_{A'}$ throughout the main body of the paper (with corresponding interchange of primed and unprimed indices where appropriate). The usual conventions are followed in the appendix.

1. *Spin weighted functions.* The two-sphere may be regarded as a complex manifold, the Riemann sphere, obtained as the one point compactification of the complex plane. Let ζ denote the coordinate on the complex plane and η the coordinate near 'infinity' so that $\zeta\eta = 1$. A *spin weighted function* f of *spin* s is usually defined as a pair of smooth functions $(v(\zeta,\bar{\zeta}), \hat{v}(\eta,\bar{\eta}))$ such that

$$\hat{v}(\eta,\bar{\eta}) = (\zeta/\bar{\zeta})^s v(\zeta,\bar{\zeta}). \tag{1·1}$$

We use the notation $v(\zeta,\bar{\zeta})$ for a smooth function reserving the notation $v(\zeta), v(\bar{\zeta})$ for holomorphic, antiholomorphic functions respectively. This notation is not meant to imply that v is in any way composed of a holomorphic and antiholomorphic part. Note that $(\zeta/\bar{\zeta})^s = \zeta^{2s}/(\zeta\bar{\zeta})^s$ makes good sense provided $2s$ is an integer (and otherwise not).

Alternatively the Riemann sphere can be regarded as the projective space $\mathbb{P}_1(\mathbb{C})$. This is preferable since it is possible to avoid the use of coordinates necessary in formulae such as (1·1). Suppose $\mathbb{C}_A$ is a two-dimensional complex vector-space. Elements of this vector-space will be denoted π_A or just π. The reason for this notation is that in applications $\mathbb{C}_A$ will be the space of unprimed spinors at a point in a space–time. The two-sphere is the sphere of null-directions at that point (i.e. the celestial sphere) and spin-weighted functions arise as the spinor description of some tensor field. The *abstract index* A acts only as a label which serves to indicate what would happen if we were to write out our variations constructions in terms of some basis for $\mathbb{C}_A$. For example, if we choose a basis for $\mathbb{C}_A$ then the components of π_A with respect to this basis may be called π_0 and π_1 so that the abstract index A may be replaced by a numerical index ranging over $(0, 1)$. For more details concerning spinors and this notation, see (12). The projective space $\mathbb{P}(\mathbb{C}_A)$ (which we shall denote just by $\mathbb{P}$ from now on) may be identified with the two-sphere above in the usual way by choosing a basis for $\mathbb{C}_A$ and setting $\zeta = \pi_1/\pi_0$, $\eta = \pi_0/\pi_1$. Equation (1·1) may now be rewritten

$$\hat{v}(\eta,\bar{\eta})\,(\bar{\pi}_1/\pi_1)^s = v(\zeta,\bar{\zeta})\,(\bar{\pi}_0/\pi_0)^s. \tag{1·2}$$

The left-hand side of (1·2) makes sense where π_1 does not vanish, the right-hand side where π_0 does not vanish. Thus, (1·2) gives rise to a function $f(\pi,\bar{\pi})$ defined on $\mathbb{C}_A$ save for the origin. It is clear from (1·2) that f satisfies

$$f(\lambda\pi,\bar{\lambda}\bar{\pi}) = (\bar{\lambda}/\lambda)^s f(\pi,\bar{\pi}). \tag{1·3}$$

This equation is independent of any coordinate system, so we take this as our *definition* of a function of spin weight s. Equation (1·3) is clearly a local condition on $\mathbb{P}$ and therefore gives rise to a sheaf $\mathscr{B}(s)$, the *sheaf* of *germs* of *spin weight s* functions. Note that complex conjugation gives rise to an isomorphism $\overline{\mathscr{B}(s)} \simeq \mathscr{B}(-s)$. The sheaf $\mathscr{B}(s)$ is locally free of rank 1 over $\mathscr{E}$, the sheaf of germs of smooth functions, and so corresponds to a smooth line-bundle on $\mathbb{P}$. The smooth line-bundles on $\mathbb{P}$ are classified by their Chern class which in this case can be identified as an integer. With the usual normalization it turns out that the Chern class of $\mathscr{B}(s)$ is $-2s$. In other words, we claim that $\mathscr{B}(s)$ is isomorphic to $\mathscr{E}(-2s)$, the smooth line-bundle underlying the holomorphic line-bundle $\mathcal{O}(-2s)$. To exhibit such an isomorphism choose a positive definite Hermitian form P on $\mathbb{C}_A$. Note that if $\mathbb{C}_A$ is the spin space at a point q in space–time, then choosing P is the same thing as choosing a time-like future pointing vector $t^{AA'}$ at q via $P(\pi,\bar{\pi}) = t^{AA'}\pi_A\bar{\pi}_{A'}$ (repeated indices indicate contraction). For $\pi \neq 0$, $P(\pi,\bar{\pi})$ is real and positive, so we may consider $g = P^{-s}f$. From (1·3) we see that g satisfies

$$g(\lambda\pi,\bar{\lambda}\bar{\pi}) = \lambda^{-2s}g(\pi,\bar{\pi}), \tag{1·4}$$

i.e. g is homogeneous of degree $-2s$ or, equivalently, g is a section of $\mathscr{E}(-2s)$, as required. Similarly, we may consider $h = P^s f$. This function satisfies

$$h(\lambda\pi,\bar{\lambda}\bar{\pi}) = \bar{\lambda}^{2s}h(\pi,\bar{\pi}), \tag{1·5}$$

i.e. h is a section of $\bar{\mathscr{E}}(2s)$. Thus, P induces isomorphisms

$$\mathscr{E}(-2s) \underset{\times P^s}{\overset{\simeq}{\rightarrow}} \mathscr{B}(s) \underset{\times P^s}{\overset{\simeq}{\rightarrow}} \bar{\mathscr{E}}(2s).$$

Let $\mathscr{E}^{p,q}$ denote the sheaf of germs of forms of type (p, q). Then there are differential operators

$$\partial : \bar{\mathscr{E}}(k) \to \bar{\mathscr{E}}(k) \otimes \mathscr{E}^{1,0}$$

$$\bar{\partial} : \mathscr{E}(k) \to \mathscr{E}(k) \otimes \mathscr{E}^{0,1}.$$

In the local holomorphic coordinate ζ, $f \in \bar{\mathscr{E}}(k)$ is expressed as a function of ζ and $\bar{\zeta}$ and the operator ∂ is defined by (cf. (16))

$$\partial f = (\partial f / \partial \zeta)\, d\zeta.$$

It is readily apparent that this definition is then independent of the choice of holomorphic coordinate. Also, ∂f vanishes for anti-holomorphic f, so that ∂ respects the structure of $\bar{\mathscr{E}}(k)$ as a module over $\bar{\mathscr{O}}(k)$. Analogously, $\bar{\partial}$ is defined by

$$\bar{\partial} g = (\partial g / \partial \bar{\zeta})\, d\bar{\zeta}.$$

Suppose ϵ^{AB} is a skew form on $\mathbb{C}_A$. A spin-space comes equipped with such a form. Then $\epsilon^{AB} \pi_A d\pi_B$ is a nowhere vanishing form of type $(1, 0)$ on $\mathbb{C}_A$ homogeneous of degree 2 which therefore trivializes $\mathscr{E}(2) \otimes \mathscr{E}^{1,0}$. In other words, we may identify

$$\mathscr{E}^{1,0} = \mathscr{E}(-2),$$

and similarly,

$$\mathscr{E}^{0,1} = \bar{\mathscr{E}}(-2).$$

We define the differential operator $\eth$ as the composition

$$\begin{tikzcd} \bar{\mathscr{E}}(2s) \arrow[r, "\partial"] & \bar{\mathscr{E}}(2s) \otimes \mathscr{E}^{1,0} = \bar{\mathscr{E}}(2s) \otimes \mathscr{E}(-2) \arrow[d, "\wp \times P^{-s+1}"] \\ \mathscr{B}(s) \arrow[u, "\wp \times P^{s}"] \arrow[r, "\eth"] & \mathscr{B}(s+1) \end{tikzcd} \tag{1·6}$$

and the operator $\bar{\eth}$ by means of the diagram

$$\begin{tikzcd} \mathscr{B}(s-1) & \mathscr{B}(s) \arrow[l, "\bar{\eth}"'] \arrow[d] \\ \mathscr{E}(-2s) \otimes \bar{\mathscr{E}}(-2) = \mathscr{E}(-2s) \otimes \mathscr{E}^{0,1} \arrow[u] & \mathscr{E}(-2s). \arrow[l, "\bar{\partial}"'] \end{tikzcd} \tag{1·7}$$

Thus, $\eth$ is just ∂ in disguise. More specifically, what we have done is write down explicit isomorphisms between the usual bundles $\mathscr{E}(k)$, $\mathscr{E}^{1,0}$, … and a series of standard bundles $\mathscr{B}(s)$. Many properties of $\eth$ follow immediately from the corresponding properties of ∂. For example, it is clear that $\eth$ satisfies a Leibnitz rule

$$\eth(fg) = f\eth g + g\eth f \quad \text{for} \quad f \in \mathscr{B}(s), g \in \mathscr{B}(t). \tag{1·8}$$

Also $\eth$ is elliptic since the same is true for ∂. The usual definition (9) of $\eth$ is in terms of a choice of coordinate ζ on $\mathbb{P}$ and a choice of metric $(1 + \zeta\bar{\zeta})$. It is straightforward to check that this definition agrees with our coordinate-free method.

It is useful to write out our definition in terms of homogeneous functions. A section h of $\bar{\mathscr{E}}(2s)$ can be regarded as a homogeneous function (1·5) or, equivalently, as satisfying the Euler equations

$$\begin{aligned} \bar{\pi}_{A'}(\partial f / \partial \bar{\pi}_{A'}) &= 2sf \\ \pi_A(\partial f / \partial \pi_A) &= 0. \end{aligned} \tag{1·9}$$

Our identification $\mathscr{E}^{1,0} = \mathscr{E}(-2)$ means that we can define ∂h via the equation

$$\partial h/\partial \pi_A + \pi^A \partial h = 0, \tag{1·10}$$

where $\pi^A = \epsilon^{AB}\pi_B$. Note that the second Euler equation shows that (1·10) is consistent. Now introduce $\lambda^A = t^{AA'}\pi_{A'}$ or, equivalently, $\lambda^A = \partial P/\partial \pi_A$. We may contract (1·10) with λ^A to obtain

$$\partial h = P^{-1}\lambda_A(\partial h/\partial \pi_A). \tag{1·11}$$

This formula may be used to obtain a formula for $\eth$. So suppose $f \in \mathscr{B}(s)$. Then

$$\begin{aligned} \eth f &= P^{-s+1}\partial(P^s f), \quad \text{by definition,} \\ &= P^{-s}\lambda_A(\partial(P^s f)/\partial \pi_A), \quad \text{by} \quad (1·11). \end{aligned}$$

But $\lambda_A(\partial P/\partial \pi_A) = \lambda_A \lambda^A = 0$, so

$$\eth f = \lambda_A \partial f/\partial \pi_A. \tag{1·12}$$

Note that s has apparently disappeared from this definition but may be recovered from $\partial f/\partial \pi_A$ by $\pi_A \partial f/\partial \pi_A = -sf$.

Edth may be investigated using *spin weighted spherical harmonics* (cf. (9, 2)). Using the notation developed above, we introduce the subspace ${}_s\mathscr{Y}_l$ of $\Gamma(\mathbb{P}, \mathscr{B}(s))$ for $l = |s|, |s|+1, |s|+2, \ldots$ by

$${}_s\mathscr{Y}_l = \{\alpha^{A\ldots BC\ldots D}\pi_A \ldots \pi_B \lambda_C \ldots \lambda_D \quad \text{for} \quad \alpha^{A\ldots D} = \alpha^{(A\ldots D)}\}$$

with $l-s$ π's and $l+s$ λ's. Here we are using the homogeneous representation (1·3) for sections of $\mathscr{B}(s)$ and round brackets around spinor indices to indicate symmetrization. From (1·12) and its complex conjugate it follows easily that

$$\begin{aligned} \eth \pi_A &= \lambda_A, & \bar{\eth}\pi_A &= 0 \\ \eth \lambda_A &= 0, & \bar{\eth}\lambda_A &= -\|P\|\pi_{A'} \end{aligned} \tag{1·13}$$

where $\|P\| = \frac{1}{2}t^{AA'}t_{AA'} = \det(t^{AA'})$. Note that for the usual choice of coordinate and metric $P = \pi_0\bar{\pi}_{0'} + \pi_1\bar{\pi}_{1'}$, so $t^{AA'} = 1$, and $\|P\| = 1$. From (1·13) we see that

$$\begin{aligned} \eth^t &: {}_s\mathscr{Y}_l \xrightarrow{\simeq} {}_{s+t}\mathscr{Y}_l \quad \text{for} \quad 0 \leqslant t \leqslant l-s, \\ \bar{\eth}^t &: {}_s\mathscr{Y}_l \xrightarrow{\simeq} {}_{s-t}\mathscr{Y}_l \quad \text{for} \quad 0 \leqslant t \leqslant l+s, \end{aligned} \tag{1.14}$$

whereas $\eth^t$ (respectively $\bar{\eth}^t$) annihilates everything in ${}_s\mathscr{Y}_l$ for $t > l-s$ (respectively $l+s$). It is also immediate from (1·13) that ${}_s\mathscr{Y}_l$ is an eigenspace for the operator $\bar{\eth}\eth$, namely,

$$\bar{\eth}\eth Y_l = -\|P\|(l-s)(l+s+1)Y_l \quad \text{for} \quad Y_l \in {}_s\mathscr{Y}_l. \tag{1·15}$$

This fact allows us to establish a certain orthogonality property. We introduce an inner product on $\Gamma(\mathbb{P}, \mathscr{B}(s))$ by

$$\langle f, g \rangle = \int_{\mathbb{P}} f\bar{g}.$$

The measure on $\mathbb{P}$ is determined by the Hermitian form P. More precisely, we use P and the skew form ϵ^{AB} to identify $\mathscr{B}(0)$ with $\mathscr{E}^{1,1}$, i.e. to specify a volume form on the two-sphere. This is explained further in Section 3. The details are unimportant here.

In Section 3 we also observe how $\eth : \mathscr{B}(-1) \to \mathscr{B}(0)$ may be regarded as $\partial : \mathscr{E}^{0,1} \to \mathscr{E}^{1,1}$. It then follows from Stokes' theorem that

$$\int \eth f = 0 \quad \text{for} \quad \mathscr{B}(-1).$$

Similarly,

$$\int \bar{\eth} g = 0 \quad \text{for} \quad \mathscr{B}(1).$$

Hence, integration by parts shows that $\bar{\eth}\eth$ is self-adjoint on $\Gamma(\mathbb{P}, \mathscr{B}(s))$. Therefore, (1·15) implies that ${}_s\mathscr{Y}_k$ and ${}_s\mathscr{Y}_l$ are orthogonal for $k \neq l$.

These spaces ${}_s\mathscr{Y}_l$ are finite-dimensional. More precisely, $\dim_{\mathbb{C}} {}_s\mathscr{Y}_l = 2l+1$. It is usual (cf. (9)) to choose an orthonormal basis ${}_sY_{l,m}$ of ${}_s\mathscr{Y}_l$ for $m = -l, -l+1, \ldots, l-1, l$. This choice depends upon choosing a basis for $\mathbb{C}_A$ and, for the present discussion, is not important. These functions ${}_sY_{l,m}$ are called *spin-s spherical harmonics*. For $s = 0$ they are just the usual spherical harmonics and so are complete in $L^2(0)$, the L^2 completion of $\Gamma(\mathbb{P}, \mathscr{B}(0))$; see (5), for example. More generally, if we denote by $L^2(s)$ the completion of $\Gamma(\mathbb{P}, \mathscr{B}(s))$, then a modification of the proof in case $s = 0$ yields

PROPOSITION 1·1. *The Hilbert space $L^2(s)$ admits the decomposition*

$$L^2(s) = \hat{\bigotimes}\, {}_s\mathscr{Y}_l,$$

or equivalently, ${}_sY_{l,m}$ are complete in $L^2(s)$.

We cannot use this proposition directly to investigate $\eth$ since it is unbounded as an operator on L^2. Indeed, if $f, g \in {}_s\mathscr{Y}_l$, then

$$\begin{aligned} \langle \eth f, \eth g \rangle &= \int (\eth f)(\bar{\eth}\bar{g}) \\ &= -\int (\bar{\eth}\eth f)\bar{g}, \quad \text{by parts}, \\ &= \|P\|(l-s)(l+s+1)\langle f, g\rangle, \quad \text{by (1·15)}. \end{aligned}$$

By the Sobolev lemma, however, if $f \in L^2(s)$ is written according to Proposition 1·1 as

$$f = \Sigma f_l \quad \text{for} \quad f_l \in {}_s\mathscr{Y}_l,$$

then $f \in \Gamma(\mathbb{P}, \mathscr{B}(s))$ if and only if $\|f\|_l$ decreases faster than any polynomial as $l \to \infty$. Hence we can calculate the kernel and cokernel of $\eth_s : \Gamma(\mathbb{P}, \mathscr{B}(s)) \to \Gamma(\mathbb{P}, \mathscr{B}(s+1))$ immediately from (1·14):

$$\ker \eth_s = \begin{cases} {}_s\mathscr{Y}_s & \text{for} \quad s \geqslant 0 \\ 0 & \text{else}, \end{cases}$$

$$\operatorname{coker} \eth_s = \begin{cases} {}_{s+1}\mathscr{Y}_{-s-1} & \text{for} \quad s \leqslant -1 \\ 0 & \text{else}. \end{cases} \tag{1·16}$$

Note that under the pairing

$$\Gamma(\mathbb{P}, \mathscr{B}(s)) \otimes \Gamma(\mathbb{P}, \mathscr{B}(-s)) \to \mathbb{C}$$
$$f \otimes g \mapsto \int fg$$

${}_{-s}\mathscr{Y}_s$ may be regarded as the dual of ${}_s\mathscr{Y}_s$. Thus we may rewrite (1·16) as

$$\ker \eth_s = \begin{cases} \{\alpha^{A'B'\ldots D'}\bar{\pi}_{A'}\ldots\bar{\pi}_{D'}P^{-s}\} & \text{for} \quad s \geqslant 0 \\ 0 & \text{else}, \end{cases} \tag{1·17}$$

$$\operatorname{coker} \eth_s = (\ker \eth_{-s-1})^* \quad \text{for all} \quad s.$$

This result is familiar from complex analysis, for if we regard $\eth:\mathscr{B}(s)\to\mathscr{B}(s+1)$ as providing a fine resolution of $\bar{\mathcal{O}}(2s)$,

$$0 \to \bar{\mathcal{O}}(2s) \xrightarrow{\times P^{-s}} \mathscr{B}(s) \xrightarrow{\eth} \mathscr{B}(s+1) \to 0, \tag{1·18}$$

isomorphic to the Dolbeault resolution, then we see that (1·17) may be interpreted as

$$H^0(\mathbb{P},\bar{\mathcal{O}}(k)) = \begin{cases} \mathbb{C}^{(A'B'\dots D')} & \text{for} \quad k \geqslant 0, \quad k \quad \text{primed indices} \\ 0 & \text{else,} \end{cases} \tag{1·19}$$
$$H^1(\mathbb{P},\bar{\mathcal{O}}(k)) = H^0(\mathbb{P},\bar{\mathcal{O}}(-k-1))^* \quad \text{for all} \quad k.$$

These cohomology groups can be computed in many other ways of course. For example, one can use the Čech method with the Leray cover $\mathbb{P}=\{\pi_0\neq 0\}\cup\{\pi_1\neq 0\}$ by expanding functions in Laurent series and comparing coefficients. Alternatively, one can obtain the first statement of (1·19) directly from the definition. The second statement of (1·19) is then Serre duality (15). The fact that $\eth^s:\Gamma(\mathbb{P},\mathscr{B}(0))\to\Gamma(\mathbb{P},\mathscr{B}(s))$ is surjective is crucial in (9). We see that the standard machinery of complex analysis provides a familiar proof of this.

2. *Conformally weighted functions.* A spin space $\mathbb{C}_A$ comes equipped with a skew form ϵ^{AB}, but the other ingredient of edth, namely, the positive Hermitian form $P=t^{AA'}\pi_A\bar{\pi}_{A'}$, introduces a choice into the definition. This choice can be investigated by the introduction of *conformal weight*. If we perform a linear automorphism Λ of $\mathbb{C}_A$, then P is altered, but ϵ^{AB} is preserved provided $\det\Lambda=1$. Such Λ induce holomorphic or *conformal* automorphisms of $\mathbb{P}$. Indeed, $\mathrm{SL}(2,\mathbb{C})$ is a 2–1 covering of the conformal group of the sphere. Conformal weight is introduced in order to compensate for these possible changes in P. A function $f(\pi,\bar{\pi})$ on $\mathbb{C}_A$-{origin} is said to have spin weight s and conformal weight w if and only if

$$f(\lambda\pi,\bar{\lambda}\bar{\pi}) = (\lambda\bar{\lambda})^w(\bar{\lambda}/\lambda)^s f(\pi,\bar{\pi}). \tag{2·1}$$

This equation reduces to (1·3) when $w=0$. Also, as with (1·3), it is clearly local on $\mathbb{P}$ and so gives rise to a sheaf (or line-bundle) $\mathscr{B}(s,w)$ such that $\mathscr{B}(s)=\mathscr{B}(s,0)$. As before, $2s$ must be an integer. The conformal weight w, however, can be an arbitrary real number. Note that

$$\mathscr{B}(s,s)=\bar{\mathscr{E}}(2s) \quad \text{and} \quad \mathscr{B}(s,-s)=\mathscr{E}(-2s).$$

For a fixed positive Hermitian form P, we see that conformal weight can be varied. Specifically, P defines isomorphisms

$$\times P^{x-w}:\mathscr{B}(s,w)\to\mathscr{B}(s,x). \tag{2·2}$$

Equivalently, P provides a nowhere-vanishing section, i.e. a trivialization, of $\mathscr{B}(0,1)$. Note that there are natural identifications

$$\mathscr{B}(s,w)\otimes\mathscr{B}(t,x)=\mathscr{B}(s+t,w+x)$$
$$\overline{\mathscr{B}(s,w)}=\mathscr{B}(-s,w).$$

From now on we will concentrate on $\eth$ since $\bar{\eth}$ can be recovered as its complex conjugate, $\bar{\eth}f=\overline{\eth\bar{f}}$. For a function with equal spin and conformal weight, we define $\eth$ without reference to an Hermitian form by

$$\begin{tikzcd}
\bar{\mathscr{E}}(2s) \arrow[r, "\partial"] \arrow[dd, equal] & \bar{\mathscr{E}}(2s)\otimes\mathscr{E}^{1,0} = \bar{\mathscr{E}}(2s)\otimes\mathscr{E}(-2) \arrow[d, equal] \\
 & \mathscr{B}(s,s)\otimes\mathscr{B}(1,-1) \arrow[d, equal] \\
\mathscr{B}(s,s) \arrow[r, "\eth"] & \mathscr{B}(s+1,s-1).
\end{tikzcd} \tag{2·3}$$

From this point of view ð is ∂ with less disguise than before. Since P has been eliminated, we say that ð is *conformally invariant* when acting on $\mathscr{B}(s,s)$. The edth operator of Section 1 is defined simply by using (2·2) to vary the conformal weight where necessary. With functions which have a conformal weight as well as a spin weight, it is convenient to insist that edth lower the conformal weight by one as well as increase the spin weight by one

$$\mathscr{B}(s,w) \xrightarrow{\simeq \times P^{w-s}} \mathscr{B}(s,s) \xrightarrow{ð} \mathscr{B}(s+1,s-1) \xrightarrow{\simeq \times P^{w-s}} \mathscr{B}(s+1,w-1).$$

This at least eliminates change of scale $P \mapsto \lambda P$ from the definition of ð.

As in Section 1 we can derive a formula for ð considered as acting on homogeneous functions (cr. (1·12))

$$ðf = P^{-1}\lambda_A \partial f/\partial \pi_A. \tag{2·4}$$

Note that this formula is independent of weights.

In many applications the operator $ð^2: \mathscr{B}(s,w) \to \mathscr{B}(s+2, w-2)$ appears (for example, in the good cut equation and Newman equation of Section 4). This operator is not usually considered in complex analysis. We can use (2·4) to calculate $ð^2$.

$$\begin{aligned} ð^2 f &= P^{-1}\lambda_A \frac{\partial}{\partial \pi_A}(P^{-1}\lambda_B)\frac{\partial f}{\partial \pi_B} \\ &= P^{-2}\lambda_A \lambda_B \partial^2 f/\partial \pi_A \partial \pi_B, \end{aligned}$$

since $\lambda_A \partial P/\partial \pi_A = 0 = \partial \lambda_B/\partial \pi_A$. More generally, $ð^k: \mathscr{B}(s,w) \to \mathscr{B}(s+k, w-k)$ is given by

$$ð^k f = P^{-k}\lambda_A \lambda_B \ldots \lambda_D \partial^k f/\partial \pi_A \ldots \partial \pi_D. \tag{2·5}$$

For suitable conformal weight, $ð^k$ is conformally invariant.

PROPOSITION 2·1. *If $k = w - s + 1$ is a positive integer, then $ð^k: \mathscr{B}(s,w) \to \mathscr{B}(w+1, s-1)$ is conformally invariant.*

Proof. Indeed, we claim that $ð^k$ is given by (cf. (1·10))

$$\pi^A \pi^B \ldots \pi^D ð^k f = (-1)^k \partial^k f/\partial \pi_A \partial \pi_B \ldots \partial \pi_D. \tag{2·6}$$

By the homogeneity of f, $\pi_A \partial^k f/\partial \pi_A \partial \pi_B \ldots \partial \pi_D = 0$, so this equation is consistent. If (2·6) is satisfied, then, contracting both sides with $\lambda_A \lambda_B \ldots \lambda_D$,

$$(-P)^k ð^k f = (-1)^k \lambda_A \lambda_B \ldots \lambda_D \partial^k f/\partial \pi_A \ldots \partial \pi_D.$$

This is equation (2·5). ∎

It is straightforward to use (2·4) and its complex conjugate to calculate the commutator $[ð, \bar{ð}]$:

PROPOSITION 2·2. *If $f \in \mathscr{B}(s,w)$, then*

$$(ð\bar{ð} - \bar{ð}ð) f = -2s\|P\| P^{-2} f. \tag{2·7}$$

Proof. According to (2·4) and its complex conjugate

$$\begin{aligned} (ð\bar{ð} - \bar{ð}ð) f &= P^{-2}[(\lambda_A \partial \bar{\lambda}_{A'}/\partial \pi_A)\,\partial/\partial \bar{\pi}_{A'} - (\bar{\lambda}_{A'} \partial \lambda_A/\partial \bar{\pi}_{A'})\,\partial/\partial \pi_A] \\ &= P^{-2}[\lambda_A t^A_{A'} \partial/\partial \bar{\pi}_{A'} - \bar{\lambda}_{A'} t^{A'}_A \partial/\partial \pi_A] \\ &= \|P\| P^{-2}[\pi_A \partial/\partial_A - \bar{\pi}_{A'} \partial/\partial \bar{\pi}_{A'}] \\ &= \|P\| P^{-2}[\times (w-s) - \times (w+s)] = \times -2s\|P\| P^{-2}. \blacksquare \end{aligned}$$

In the next section we will see how (2·7) may be interpreted as a curvature. Note that $\eth$ and $\bar{\eth}$ commute on functions of spin weight zero. More generally, we have

PROPOSITION 2·3. *If $f \in \mathscr{B}(s, w)$, then*

$$\eth^p \bar{\eth}^q f = \bar{\eth}^q \eth^p f \quad \textit{when} \quad q-p = 2s.$$

Proof. Since multiplication by P commutes with $\eth$ and $\bar{\eth}$, we can assume that $w = p+s-1$ so that, according to Proposition 2·1, $\eth^p$ is conformally invariant and is given by (2·6). Since $q-p = 2s$, it follows that $\bar{\eth}^q$ is also conformally invariant and is given by

$$\bar{\pi}^{E'} \bar{\pi}^{F'} \ldots \bar{\pi}^{H} \bar{\eth}'^{\,q} f = (-1)^q \, \partial^q f / \partial \bar{\pi}_{E'} \partial \bar{\pi}_{F'} \ldots \partial \bar{\pi}_{H'}.$$

The result follows immediately since $\partial^p / \partial \pi_A \partial \pi_B \ldots \partial \pi_D$ and $\partial^q / \partial \bar{\pi}_{E'} \partial \bar{\pi}_{F'} \ldots \partial \bar{\pi}_{H'}$ commute. ∎

3. *Connections*. Suppose $\mathscr{B}$ is the sheaf of germs of smooth sections of a smooth complex line-bundle and $\nabla : \mathscr{B} \to \mathscr{B} \otimes \mathscr{E}^1$, where $\mathscr{E}^1$ is the sheaf of smooth complex-valued 1-forms, is a connection. Recall that ∇ is said to be *compatible* with an Hermitian structure $\langle\, , \rangle$ on $\mathscr{B}$ if and only if

$$d\langle f, g \rangle = \langle \nabla f, g \rangle + \langle f, \nabla g \rangle \quad \text{for all} \quad f, g \in \mathscr{B}. \tag{3·1}$$

PROPOSITION 3·1. *Suppose $\mathscr{B}$ has an Hermitian structure and is based on a complex manifold. Suppose that*

$$\nabla^{1,0} : \mathscr{B} \to \mathscr{B} \otimes \mathscr{E}^{1,0}$$

satisfies a Leibnitz rule

$$\nabla^{1,0}(\alpha f) = \alpha \nabla^{1,0} f + f \otimes \partial \alpha$$

for all $f \in \mathscr{B}$ and $\alpha \in \mathscr{E}$, the sheaf of germs of smooth functions. Then there is a unique connection $\nabla : \mathscr{B} \to \mathscr{B} \otimes \mathscr{E}^1$ compatible with the metric whose $(1, 0)$ *component is* $\nabla^{1,0}$.

Proof. Choose $s \in \mathscr{B}$ a local unitary frame, i.e. $\langle s, s \rangle = 1$. Then $\nabla^{1,0} s = s \otimes \tau$ for some $\tau \in \mathscr{E}^{1,0}$. Any choice of $\omega \in \mathscr{E}^1$ gives rise to a local connection via $\nabla(\alpha s) = \alpha s \otimes \omega + s \otimes d\alpha$, and it is easy to check that (3·1) is equivalent to insisting that $\omega + \bar{\omega} = 0$. Thus, we are forced to take $\omega = \tau - \bar{\tau}$. Since ∇ is uniquely determined locally, these definitions automatically patch together to give a global definition. ∎

Note that we could equally well specify $\nabla^{0,1}$ and the Hermitian form. In particular, if $\mathscr{B}$ is a holomorphic line-bundle, then we could insist that $\nabla^{0,1} = \bar{\partial}$ whence Proposition 3·1 would be the usual result that there is always a unique connection compatible with both the holomorphic structure and the Hermitian structure on a line-bundle. The same proof applies more generally to a vector-bundle instead of a line-bundle.

We claim that $\eth$ and $\bar{\eth}$ are linked as in Proposition 3·1 when $\mathscr{B}(s, w)$ is given the metric induced from P, namely, the composition

$$\begin{array}{ccccc}
\mathscr{B}(s, w) & \otimes & \mathscr{B}(-s, w) & = & \mathscr{B}(0, 2w) \\
\| & & & & \big\downarrow{\scriptstyle \wr\, \times P^{-2w}} \\
\mathscr{B}(s, w) & \otimes & \overline{\mathscr{B}(s, w)} & \xrightarrow{\langle , \rangle} & \mathscr{E}.
\end{array}$$

PROPOSITION 3·2. *Let $\nabla = \eth + \bar{\eth}$, i.e.*

$$\begin{array}{ccccc}
\eth \oplus \bar{\eth} : & \mathscr{B}(s, w) & \to & \mathscr{B}(s, w) \otimes (\mathscr{E}^{1,0} \oplus \mathscr{E}^{0,1}) \\
& & {\scriptstyle \nabla} \searrow & \| \\
& & & \mathscr{B}(s, w) \otimes \mathscr{E}^1.
\end{array}$$

Then ∇ is a connection compatible with the metric.

Proof. Since $\eth$ and $\bar{\eth}$ satisfy Leibnitz rules, so does ∇, i.e. it is a connection. If $f, g \in \mathscr{B}(w, s)$,

$$\begin{aligned} d\langle f, g\rangle &= (\partial + \bar{\partial})(P^{-2w} f\bar{g}) = (\eth + \bar{\eth})(P^{-2w} f\bar{g}) \\ &= P^{-2w}((\eth + \bar{\eth})f)\,\bar{g} + P^{-2w} f((\eth + \bar{\eth})\,\bar{g}) \\ &= P^{-2w}(\nabla f)\,\bar{g} + P^{-2w} f\,\overline{\nabla g} \\ &= \langle \nabla f, g\rangle + \langle f, \nabla g\rangle. \end{aligned}$$

So (3·1) is satisfied. ∎

Thus, Propositions 3·1 and 3·2 show that $\eth$ and $\bar{\eth}$ determine each other via the metric on $\mathscr{B}(s, w)$.

In general, a connection ∇ on a smooth line-bundle $\mathscr{B}$ determines a differential operator for which we shall use the same notation

$$\nabla : \mathscr{B} \otimes \mathscr{E}^1 \to \mathscr{B} \otimes \mathscr{E}^2$$

characterized by the requirement that

$$\nabla(f \otimes \omega) = f \otimes d\omega + \nabla f \wedge \omega$$

for $f \in \mathscr{B}$ and $\omega \in \mathscr{E}^1$. If the base is a 1-dimensional complex manifold as in our case, then $\mathscr{E}^2 = \mathscr{E}^{1,1} = \mathscr{E}^{1,0} \otimes \mathscr{E}^{0,1}$. Not surprisingly, a short calculation shows that ∇: $\mathscr{B}(s, w) \otimes \mathscr{E}^1 \to \mathscr{B}(s, w) \otimes \mathscr{E}^2$ can be expressed in terms of $\eth$ and $\bar{\eth}$. More specifically,

$$\begin{tikzcd} \mathscr{B}(s,w) \arrow[r, "\nabla"] \arrow[d, equal] & \mathscr{B}(s,w) \otimes \mathscr{E}^1 \arrow[r, "\nabla"] \arrow[d, equal] & \mathscr{B}(s,w) \otimes \mathscr{E}^2 \arrow[d, equal] \\ \mathscr{B}(s,w) \arrow[r, "\eth", "\bar{\eth}"'] & \begin{matrix} \mathscr{B}(s+1, w-1) \\ \oplus \\ \mathscr{B}(s-1, w-1) \end{matrix} \arrow[r, "-\bar{\eth}", "\eth"'] & \mathscr{B}(s, w-2) \end{tikzcd}$$

where we have identified $\mathscr{E}^{1,0} = \mathscr{B}(1, -1)$, $\mathscr{E}^{0,1} = \mathscr{B}(-1, -1)$ and $\mathscr{E}^2 = \mathscr{E}^{1,0} \otimes \mathscr{E}^{0,1} = \mathscr{B}(0, -2)$. In general, $\nabla^2 : \mathscr{B} \to \mathscr{B} \otimes \mathscr{E}^2$ does not involve differentiation. It is simply given by $f \mapsto f \otimes \Omega$ for a 2-form Ω called the *curvature* of ∇. Thus we see that Proposition 2·2 $((\eth\bar{\eth} - \bar{\eth}\eth)f = -2s\|P\|P^{-2}f)$ may be interpreted as giving the curvature of $\eth + \bar{\eth}$. It is easy to check that if we identify P^{-2} as a 2-form on $\mathbb{P}$, then

$$\int_{\mathbb{P}} P^{-2} = -2\pi i/\|P\|.$$

Hence,

$$\frac{i}{2\pi}\int_{\mathbb{P}} \text{curvature of} \quad (\eth + \bar{\eth}) = -2s.$$

This is precisely the integral formula for the Chern class of a complex line-bundle (cf. ((7), Appendix C)).

4. *The good cut equation and the Newman equation.* A space–time M which is *asymptotically flat* according to Penrose's criterion of *weak asymptotic simplicity* (12) admits two *conformal infinities*: $\mathscr{I}^+$, the remote future in null directions, and $\mathscr{I}^-$, the remote past in null directions. The conformal structure of M, determined by the space–time metric, extends to $\mathscr{I}^+$ and $\mathscr{I}^-$. These become null hypersurfaces, topologically $S^2 \times \mathbb{R}$, in $\hat{M} = M \cup \mathscr{I}^+ \cup \mathscr{I}^-$, the conformally extended manifold. If M is Minkowski space, $\mathscr{I}^+$ and $\mathscr{I}^-$ fit together to form a single null cone in *compactified Minkowski space*,

but in a space–time with curvature they must be regarded as separate. The outgoing radiation parts of the gravitational field of M are determined by a single piece of data on $\mathscr{I}^+$; likewise the incoming radiation is determined by data on $\mathscr{I}^-$. Newman's $\mathscr{H}$-space construction (10) arose from attempts to reconstruct the space–time M from quantities solely on $\mathscr{I}^+$.

In general, the shear σ of an outgoing null hypersurface in an asymptotically flat space–time has leading terms of order r^{-2} in an affine parameter r along the null generators, i.e.

$$\sigma = \sigma^0/r^2 + \text{higher order terms},$$

where σ^0 is the *asymptotic shear* (and is a function of the remaining three coordinates).

In Minkowski space, the null cones of points, which have identically vanishing shear, can be characterized as outgoing null hypersurfaces with vanishing σ^0, since the propagation equation for σ along the generators then ensures that it is identically zero. An outgoing null hypersurface Σ intersects $\mathscr{I}^+$ in a *cut* S, a section of $\mathscr{I}^+$ topologically S^2. Σ can be recovered from S by taking the null geodesics from the interior which meet S orthogonally. Thus, the points of Minkowski space can be characterized as giving rise to cuts of $\mathscr{I}^+$ which give rise to asymptotically shear-free null hypersurfaces. These are known as *good cuts*. The metric separation of two points in the interior of Minkowski space can then be defined as an integral involving the corresponding good cuts. In this way, Minkowski space is obtained entirely from structures and operations at $\mathscr{I}^+$.

In attempting to parallel this construction for a general asymptotically flat space–time M, one is led to a differential equation for the good cuts, the *good cut equation* (4·3), which involves the radiation data of the space–time as a source term. In general, this has no solutions, but if complex-valued solutions are allowed, then for sufficiently small radiation data there is a four complex parameter family of solutions (3). This family forms a complex manifold called the $\mathscr{H}$-space of M. The $\mathscr{H}$-space has a natural holomorphic metric which satisfies the Einstein equations and has self-dual Weyl tensor.

In this section we give a precise formulation of the good cut equation, and in the appendix, following the work of G. A. J. Sparling, we show how the good cut equation may be interpreted as defining a complex deformation of flat twistor space.

Consider the bundle $\mathscr{B}(0,1)$ over $\mathbb{P}$. Recall that its sections may be identified as complex-valued functions satisfying

$$f(\lambda\pi, \bar{\lambda}\bar{\pi}) = \lambda\bar{\lambda} f(\pi, \bar{\pi}).$$

Since $\lambda\bar{\lambda}$ is always real, it makes sense to consider the subsheaf $\mathscr{B}_{\mathbb{R}}(0,1)$ of real-valued f. Then $\mathscr{B}(0,1)$ is the complexification of the real line-bundle $\mathscr{B}_{\mathbb{R}}(0,1)$. We define $\mathscr{I}^+$ as the total space of this real line bundle. More concretely, $\mathscr{I}^+ = (\mathbb{C}_A - \{0\}) \times \mathbb{R}/\sim$ where the equivalence relation $\sim$ is given by

$$(\pi, \bar{\pi}, u) \sim (\lambda\pi, \bar{\lambda}\bar{\pi}, \lambda\bar{\lambda}u) \quad \text{for} \quad \lambda \in \mathbb{C} - \{0\}. \tag{4·1}$$

Actually, the natural structure of $\mathscr{I}^+$ is as an affine bundle rather than a line-bundle. To regard it as a line-bundle we have chosen a section which we designate as the origin. A different choice corresponds to effecting a *supertranslation* in the Bondi–Metzner–Sachs group (cf. (9)). Of course, $\mathscr{B}_{\mathbb{R}}(0,1)$ may be trivialized just like $\mathscr{B}(0,1)$, but it is important to preserve the distinction. In the case of a complex space–time, the corresponding complex line-bundle, i.e. the complex null cone of some point considered as a bundle over the null directions, is a topologically nontrivial bundle (1). We will denote by $\mathbb{C}\mathscr{I}^+$ the total space of the bundle $\mathscr{B}(0,1)$ over $\mathbb{P}$. In our concrete description (4·1) this just means that we allow u to be complex. We remark that we are breaking from the standard notation here (14). Usually $\mathbb{C}\mathscr{I}^+$ means to complexify the base $\mathbb{P}$ as well, i.e. to regard $\mathbb{P}$ as a real analytic 2-dimensional manifold which is thickened into the germ of a complex 2-dimensional manifold. Concretely this means that we embed $\mathbb{P} \hookrightarrow \mathbb{P}\times\mathbb{P}$ by $\pi \to (\pi, \bar\pi)$, i.e. the antiholomorphic diagonal, and then take the germ of this embedding. Then a real-analytic complex-valued function on $\mathbb{P}$ is the same as the germ of a complex-analytic function of π and $\bar\pi$. Although it is necessary to indulge in some complexification in order to obtain a reasonable good cut equation, we will keep this to a minimum with the above definition of $\mathbb{C}\mathscr{I}^+$.

The bundles $\mathscr{B}(s,w)$ on $\mathbb{P}$ give rise via pullback to bundles on $\mathbb{C}\mathscr{I}^+$. We will denote by $\tilde{\mathscr{B}}(s,w)$ the sheaf of germs of sections of the appropriate pullback bundle which are smooth in the $\mathbb{P}$ variable but holomorphic in the fibre direction. If we use the concrete description (4·1) of $\mathbb{C}\mathscr{I}^+$ (i.e. with $u\in\mathbb{C}$), then a section of $\tilde{\mathscr{B}}(s,w)$ may be represented by a function $f(\pi,\bar\pi,u)$ smooth in $\pi,\bar\pi$ yet holomorphic in u which is homogeneous in the sense of satisfying the transformation law:

$$f(\lambda\pi,\bar\lambda\bar\pi,\lambda\bar\lambda u) = \lambda^{w-s}\bar\lambda^{w+s}f(\pi,\bar\pi,u) \quad \text{for} \quad \lambda\in\mathbb{C}-\{0\}. \tag{4·2}$$

A *cut* of $\mathbb{C}\mathscr{I}^+$ is a section $Z\in\Gamma(\mathbb{P},\mathscr{B}(0,1))$. Such a section is usually regarded by means of its image, an embedded sphere in $\mathbb{C}\mathscr{I}^+$.

To define the good cut equation, we need to say what the radiation data is at $\mathbb{C}\mathscr{I}^+$. We first define a *Bondi u-coordinate* U by choosing a time-like future pointing vector $t^{AA'}$. The cuts of constant U, given a choice of origin for u, are then defined by

$$u = Ut^{AA'}\pi_A\bar\pi_{A'}.$$

These cuts determine outgoing null hypersurfaces in the space–time M, and the radiation data is the asymptotic shear σ^0 of these hypersurfaces. This is a spin weight 2, conformal weight -1 function on $\mathbb{C}\mathscr{I}^+$, i.e. a section $\sigma^0\in\Gamma(\mathbb{C}\mathscr{I}^+,\tilde{\mathscr{B}}(2,-1))$. The cut $u = Z(\pi,\bar\pi)$ is now a *good cut* if and only if

$$\eth^2 Z(\pi,\bar\pi) = \sigma^0(Z(\pi,\bar\pi),\pi,\bar\pi). \tag{4·3}$$

This is derived in (6). Note that the equation makes good sense, both sides being elements of $\Gamma(\mathbb{P},\mathscr{B}(2,-1))$. We can check the homogeneity of the right-hand side:

$$\begin{aligned}\sigma^0(Z(\lambda\pi,\bar\lambda\bar\pi),\lambda\pi,\bar\lambda\bar\pi) &= \sigma^0(\lambda\bar\lambda Z(\pi,\bar\pi),\lambda\pi,\bar\lambda\bar\pi)\\ &= \lambda^{-3}\bar\lambda\sigma^0(Z(\pi,\bar\pi),\pi,\bar\pi),\end{aligned}$$

by (2·1) and (4·2). By Proposition 2·1 the good cut equation (4·3) is conformally invariant and may be written (by (2·6))

$$\partial^2 Z/\partial\pi_A\partial\pi_B = \pi^A\pi^B\sigma^0. \tag{4·4}$$

In (3) it is shown that (4·4) has a four complex parameter family of solutions for sufficiently 'calm' σ^0. This family is Newman's $\mathscr{H}$*-space*.

In the analysis of (4·4) it is necessary to consider the linearized good cut equation, or *Newman equation*, (4·5) which determines the tangent space of $\mathscr{H}$-space. Suppose

$Z_0(\pi,\bar\pi)$ is a solution, and replace $Z(\pi,\bar\pi)$ in (4·4) by $Z_0(\pi,\bar\pi)+V(\pi,\bar\pi)$. Since σ^0 is analytic in the u variable, we may expand it in a power series

$$\sigma^0(Z_0(\pi,\bar\pi)+V(\pi,\bar\pi),\pi,\bar\pi) = \sum_{j=0}^{\infty}\frac{1}{j!}\frac{\partial^j\sigma^0}{\partial u^j}(Z_0(\pi,\bar\pi),\pi,\bar\pi)\,(V(\pi,\bar\pi))^j.$$

Note that $\partial^j\sigma/\partial u^j\in\Gamma(\mathbb{C}\mathscr{I}^+,\tilde{\mathscr{B}}(2,-j-1))$. If $V(\pi,\bar\pi)$ is small, then we neglect higher than first order terms

$$\sigma^0(Z_0(\pi,\bar\pi)+V(\pi,\bar\pi),\pi,\bar\pi)\simeq\sigma^0(Z_0(\pi,\bar\pi),\pi,\bar\pi)+\dot\sigma^0(Z_0(\pi,\bar\pi),\pi,\bar\pi)\,V(\pi,\bar\pi)$$

where $\dot\sigma^0=\partial\sigma^0/\partial u$ and substitute into (4·4) to obtain the Newman equation

$$\partial^2V/\partial\pi_A\,\partial\pi_B=\dot\sigma^0(Z_0(\pi,\bar\pi),\pi,\bar\pi)\,V(\pi,\bar\pi)\,\pi^A\pi^B. \tag{4·5}$$

Appendix. In this appendix, we begin with a brief description of *flat twistor space* and say what is meant by a *deformed twistor space*. It is known (13) that the general anti-self-dual solution to Einstein's equations may be obtained locally from a deformed twistor space. Our purpose here is not to describe this construction but to point out, following unpublished work of G. A. J. Sparling, how the good cut equation arises in the construction of a deformed twistor space.

Flat twistor space $\mathbb{T}$ (see (14), for example) is a four-dimensional complex vector space. It is usual to choose coordinates $(\omega^A,\pi_{A'})=(\omega^0,\omega^1,\pi_{0'},\pi_{1'})$ on $\mathbb{T}$, and regard it as fibered over $\mathbb{C}_{A'}$ by the projection $p(\omega^A,\pi_{A'})=\pi_{A'}$. Now forget that $\mathbb{T}$ is a vector space. We regard it merely as a complex manifold, yet endowed with the following structures:

(i) a projection $p:\mathbb{T}\to\mathbb{C}_{A'}$,

(ii) an *Euler vector field* $\Upsilon=\omega^A\partial/\partial\omega^A+\pi_{A'}\partial/\partial\pi_{A'}$,

(iii) a *Poisson bracket* $\hat\mu=\epsilon^{AB}\partial/\partial\omega^A\otimes\partial/\partial\omega^B$,

or, equivalently, a 2-form $\mu=\epsilon_{AB}d\omega^A\wedge d\omega^B$ on the fibres of p.

Holomorphic sections of p invariant under the action of Υ form a complex four-parameter family which may be identified as (complexified) Minkowski space, and the space–time metric may be recovered from μ, and a skew form $\epsilon^{A'B'}$ on $\mathbb{C}_{A'}$ (see (13)). In the usual notation these holomorphic sections take the form $\omega^A=ix^{AA'}\pi_{A'}$.

As it stands, this structure is rigid; however, if we remove the origin from $\mathbb{C}_{A'}$ and the fibre above it from $\mathbb{T}$, then the resulting space admits *complex deformations* (see (8)) which preserve (i), (ii), and (iii), and it is a consequence of theorems of Kodaira (13) that for sufficiently small deformations there is still a complex four-parameter family of sections of p invariant under Υ. In (13) Penrose shows how this family may be regarded as a complex Einstein manifold with anti-self-dual Weyl tensor. More generally, we could just deform a neighbourhood N of a holomorphic Υ-invariant section of p, and this produces the general local solution of the anti-self-dual Einstein equations.

One way of constructing a deformation is to change the $\bar\partial$ operator, i.e. change the Cauchy–Riemann equations. Let θ denote the sheaf of holomorphic vector fields, and suppose $\Phi\in\Gamma(N,\mathscr{E}^{0,1}(\theta))$. Then we can try $\bar\partial+\Phi\partial$ as our new Cauchy–Riemann operator. The Newlander–Nirenberg theorem asserts that this defines a bona fide complex structure provided Φ is small and satisfies the integrability condition

$$\bar\partial\Phi+[\Phi,\Phi]=0. \tag{A 1}$$

See (8) for details of this discussion. Sparling chooses

$$\Phi = \sigma\bar{\pi}^A\bar{\pi}^B d\bar{\pi}_A \partial/\partial\omega^B$$

for some smooth function σ on N. We compute,

$$[\Phi, \Phi] = \sigma\bar{\pi}^A\bar{\pi}^B\bar{\pi}^C\bar{\pi}^D d\bar{\pi}_A \wedge d\bar{\pi}_C(\partial\sigma/\partial\omega^B)\,\partial/\partial\omega^D = 0,$$

so (A 1) reduces to $\bar{\partial}\Phi = 0$. The corresponding cohomology class in $H^1(N, \theta)$ represents an *infinitesimal deformation*. A short computation shows that $\bar{\partial}\Phi = 0$ is equivalent to the following restrictions on σ,

$$\partial\sigma/\partial\bar{\omega}^{A'} = 0 \tag{A 2}$$

and

$$\bar{\pi}_A\partial\sigma/\partial\bar{\pi}_A + 3\sigma = 0.$$

Clearly (i) is preserved by this choice of Φ. To preserve (ii) we require

$$\Upsilon\sigma = \sigma, \tag{A 3}$$

and for (iii) we need

$$\bar{\pi}^A\,\partial\sigma/\partial\omega^A = 0. \tag{A 4}$$

Introducing $u = \omega^A\bar{\pi}_A$ we see that (A 2–4) imply that

$$\sigma = \sigma^0(u, \pi_{A'}, \bar{\pi}_A)$$

for some function σ^0 holomorphic in u and homogeneous in the sense of satisfying

$$\sigma^0(\lambda\bar{\lambda}u, \lambda\pi, \bar{\lambda}\bar{\pi}) = \lambda\bar{\lambda}^{-3}\sigma^0(u, \pi, \bar{\pi}).$$

But this means, by (4·2), that we may regard σ^0 as a section of $\tilde{\mathscr{B}}(-2, -1)$ over $\mathbb{C}\mathscr{I}^+$. The good cut equation arises when we look for the holomorphic sections of p. These sections will be given by equations

$$\omega^A = \nu^A(\pi, \bar{\pi}).$$

To be invariant under Υ we need $\nu^A(\lambda\pi, \bar{\lambda}\bar{\pi}) = \lambda\nu^A(\pi, \bar{\pi})$, and now this section is holomorphic in the deformed space provided

$$(\partial/\partial\bar{\pi}_A + \sigma\bar{\pi}^A\bar{\pi}^B\,\partial/\partial\omega^B)\,(\omega^C - \nu^C(\pi, \bar{\pi})) = 0,$$

which becomes

$$\partial\nu^C/\partial\bar{\pi}_A = \sigma^0(u, \pi, \bar{\pi})\,\bar{\pi}^A\bar{\pi}^C, \tag{A 5}$$

where

$$u = \omega^C\bar{\pi}_C = \nu^C(\pi, \bar{\pi})\,\bar{\pi}_C.$$

Now define $Z(\pi, \bar{\pi}) = \nu^C(\pi, \bar{\pi})\,\bar{\pi}_C$. According to (A 5), we may recover ν^C as $\partial Z/\partial\bar{\pi}_C$, and (A 5) may be rewritten

$$\partial^2 Z/\partial\bar{\pi}_A\partial\bar{\pi}_B = \bar{\pi}^A\bar{\pi}^B\sigma^0(Z, \pi, \bar{\pi}),$$

with

$$Z(\lambda\pi, \lambda\bar{\pi}) = \lambda\bar{\lambda}Z(\pi, \bar{\pi}).$$

This is precisely the good cut equation (4·4), or rather its complex conjugate. Thus, from this point of view, the good cut equation is precisely the condition that a section of a deformed twistor space be holomorphic. A deformed twistor space obtained in this way is referred to as an *asymptotic twistor space*.

In a later paper, the Newman equation is used in this context to discuss possible singularities of the $\mathscr{H}$-space construction associated with jumps in the normal bundle of a section of the deformed twistor space.

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