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Noether-type theorems for difference equations

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Abstract

The Noether's type constructions for difference functionals, difference equations and meshes (lattices) are reviewed. It is shown in Dorodnitsyn [J. Soviet Math. 55 (1999) 1490]; [Dokl. Akad. Nauk SSSR 328 (1993) 678] that the invariance of a functional (together with a mesh) does not mean the invariance of the corresponding difference Euler's equation. The stationary value of invariant difference functional is reached on the new difference equations (quasi-extremal equations) which are different for different subgroups. In the present paper the properties of quasi-extremals are considered. Any quasi-extremal equation is invariant under the corresponding subgroup and possesses its own conservation law. Every group operator which commutes with discrete differentiation transforms one quasi-extremal equation into another one, since there exists the group basis of quasi-extremal equations, which corresponds to the basis of conservation laws. It is shown that the intersection of quasi-extremals is invariant with respect to the whole group admitted by difference functional. This intersection has got the full set of difference conservation laws. The last proposition could be viewed as a discrete analog of Noether's theorem; it sufficiently differs from the result early obtained in Dorodnitsyn [Dokl. Akad. Nauk SSSR 328 (1993) 678].

1. Introduction

Symmetries are intrinsic and fundamental features of the differential equations of mathematical physics. Consequently, they should be retained when discrete analogs of such equations are constructed.

We recall that Lie point symmetries yield a number of useful properties of differential equations [10, 14,15]:

- A group action transforms the complete set of solutions into itself; so it is possible to obtain new solutions from a given one.
- There exists a standard procedure to obtain the whole set of invariants and differential invariants of a symmetry group of transformations; it yields the invariant representation of the differential equations and the forms of invariant solutions in which they could be found (symmetry reduction of PDEs).

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- For ODEs the known symmetry yields the reduction of the order; if the dimension of symmetry is equal to (or greater than) the order of ODE, then there exists a complete integrability of ODE.
- The invariance of PDEs is a necessary condition for the application of Noether's theorem to variational problems to obtain conservation laws (first integrals for ODEs).

It should be noticed that Lie point transformations have a clear geometrical interpretation and one can construct the orbits of a group in a finite dimensional space of independent and dependent variables.

The structure of the admitted group essentially effects the construction of difference equations and meshes. Group transformations can break the geometric structure of a mesh that influences the approximation and algebraic properties of a difference equation. Early contributions to the construction of the difference meshes based on the symmetries of the initial difference model were done in [6,8]. There were separated out the classes of transformations that conserve the uniformity, the orthogonality, and other properties of meshes. It formed the background for the construction of series of difference schemes and meshes which conserve all group properties of original differential equations (see, e.g., [3, 5,6]).

In the present report we will consider conservative properties of difference equations, based on Lie-point symmetry.

Difference (discrete) representation of Euler's operator depends on type of difference mesh (lattice) see [5–7]. In the case of one independent variable x and one or more dependent variables $(u^1, u^2, u^3, \dots, u^m)$ the Euler operator on *regular lattice* is the following:

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - \sum_{s=1}^{\infty} (-h)^{s-1} D^s \left(\frac{\partial}{\partial \underset{h}{u}_x} \right) = \frac{\partial}{\partial u} - \underset{-h}{D} \left(\frac{\partial}{\partial \underset{h}{u}_x} \right), \tag{1}$$

where

$$\underset{-h}{D} = \sum_{s \geqslant 1} \frac{(-h)^{s-1}}{s!} D^s$$

is a left difference operator, which has as a coupled counterpart a right difference operator

$$\underset{+h}{D} = \sum_{s \geqslant 1} \frac{h^{s-1}}{s!} D^s;$$

$$D = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + \cdots$$

is a total differentiation by x; $\underset{h}{u}_{\bar{x}}$ is a left difference derivative, $\underset{h}{u}_x$ is a right difference derivative. The Euler operator (1) is supposed to be applied to a function $\mathcal{L} = \mathcal{L}(x, u, \underset{h}{u}_x)$. Notice that in (1) firstly "continuous" differentiation by difference derivative $\underset{h}{u}_x$ is applied, then "discrete" (difference left) differentiation is used.

We will mention a finite-difference equation

$$\frac{\delta \mathcal{L}}{\delta u} = \frac{\partial \mathcal{L}}{\partial u} - \underset{-h}{D} \left(\frac{\partial \mathcal{L}}{\partial \underset{h}{u}_x} \right) = 0, \tag{2}$$

as a *difference Euler's equation* on regular mesh, a function $\mathcal{L} = \mathcal{L}(x, u, \underset{h}{u}_x)$ as a *difference Lagrange function*, and any solution of (2) as an *extremal*.

For a nonregular mesh the difference operators are "local", i.e., they are connected with local steps (spacings) h^- and h^+ (left and right) on a given point of x.

The Euler equation on nonregular mesh is [1,6,7]

$$\frac{\partial \mathcal{L}}{\partial u} - \left(\frac{h^-}{h^+}\right) \underset{-h}{D} \left(\frac{\partial \mathcal{L}}{\partial \underset{h}{u}_x}\right) = 0. \tag{3}$$

The multiplication factor (h^-/h^+) in (3) characterizing a proportion of *difference stencil* on a given point:

$$\frac{h^-}{h^+} = \varphi(x).$$

The Euler equation on nonregular mesh could be written in the other form:

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{1}{h^+}\left(\frac{\partial \mathcal{L}}{\partial \underset{h}{u}_x} - \frac{\partial \check{\mathcal{L}}}{\partial \underset{h}{u}_{\bar{x}}}\right) = 0, \tag{3*}$$

where $\check{\mathcal{L}} = \underset{-h}{S}(\mathcal{L}) = \mathcal{L}(x^-, u^-, \underset{h}{u}_{\bar{x}})$, and $\underset{-h}{S}$ is a left shift operator.

Let us consider the Euler operator in the *two-dimension case*. The 1+1 dimension mesh is considered to be quadrilateral with regular steps in both directions. Let

$$\mathcal{L} = \mathcal{L}\big(x^1, x^2, u, \underset{h}{u}_1, \underset{h}{u}_2, \big),$$

where $\underset{h}{u}_1$ and $\underset{h}{u}_2$ are first right difference derivatives, be a Lagrangian function. Then the Euler equation will be the following (see [1,7,8]):

$$\frac{\delta \mathcal{L}}{\delta u} = \frac{\partial \mathcal{L}}{\partial u} - \underset{-h}{D}_1\left(\frac{\partial \mathcal{L}}{\partial \underset{h}{u}_1}\right) - \underset{+h}{D}_2\left(\frac{\partial \mathcal{L}}{\partial \underset{h}{u}_2}\right) = 0. \tag{4}$$

If the Lagrangian depends on the left difference derivatives $\underset{h}{u}_{\bar{i}}$

$$\mathcal{L} = \mathcal{L}\big(x^1, x^2, u, \underset{h}{u}_1, \underset{h}{u}_{\bar{1}}, \underset{h}{u}_2, \underset{h}{u}_{\bar{2}}\big),$$

then the Euler equation on nonregular quadrilateral $1+1$ D mesh is [1,7,8]

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{h_i^-}{h_i^+} \underset{-h}{D}_i\left(\frac{\partial \mathcal{L}}{\partial \underset{h}{u}_i}\right) - \frac{h_i^+}{h_i^-} \underset{+h}{D}_i\left(\frac{\partial \mathcal{L}}{\partial \underset{h}{u}_{\bar{i}}}\right) = 0, \quad i = 1, 2. \tag{5}$$

Now we consider the invariance of the difference functionals. Let the finite-difference functional

$$L = \sum_{\underset{h}{\Omega}} \mathcal{L}(x, u, \underset{h}{u}_x) h^+, \tag{6}$$

(where $\underset{h}{u}_x$ is the first finite-difference derivative) be defined on a domain $\underset{h}{\Omega}$ of a one-dimensional mesh $\underset{h}{\omega}$, while the step h^+ at a given point explicitly depends on solution:

$$h^+ = \varphi(x, u). \tag{7}$$

Let one-parameter Lie group G_1 of point transformations be defined in $(x, u, \underset{h}{u}_x, \underset{h}{u}_{xx}, \ldots, h^+)$ by the generator

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \cdots + h^+ \underset{+h}{D}(\xi)\frac{\partial}{\partial h^+} + h^- \underset{-h}{D}(\xi)\frac{\partial}{\partial h^-}. \tag{8}$$

Then the action of a group G_1 with the operator (8) changes both the difference functional (6) and the difference mesh $\underset{h}{\omega}$ together with a domain $\underset{h}{\Omega} \subset \underset{h}{\omega}$. Thus *the transformation of a mesh* $\underset{h}{\omega}$ *should be included in the definition of the transformed functional.*

We will call *the transformed value of difference functional* (6) on a mesh (7) the following expression:

$$L^* = \sum_{\underset{h}{\Omega^*}} \mathcal{L}^*\big(x^*, u^*, \underset{h}{u^*_x}\big)h^{+*}, \quad h^{+^*} = \varphi^*\big(x^*, u^*\big),$$

where $\varphi \neq \varphi^*$ and $\mathcal{L} \neq \mathcal{L}^*$ in general.

Notice that the transformed domain $\underset{h}{\Omega^*}$ can depend on a solution u if a transformed x^* depends on u.

We will say that a difference functional L is *invariant* with respect to a group G_1 , if for any transform of G_1 and for any domain $\underset{h}{\Omega}$ the following is true:

$$\sum_{\underset{h}{\Omega}} \mathcal{L}(x, u, \underset{h}{u}_x)h^+ = \sum_{\underset{h}{\Omega^*}} \mathcal{L}\big(x^*, u^*, \underset{h}{u^*_x}\big)h^{+*}, \quad h^{+^*} = \varphi\big(x^*, u^*\big). \tag{9}$$

The invariance condition of (6) on the mesh (7) is the following [1,6,7]:

$$\begin{aligned} &\xi\frac{\partial\mathcal{L}}{\partial x} + \eta\frac{\partial\mathcal{L}}{\partial u} + \big(\underset{+h}{D}(\eta) - \underset{h}{u}_x \underset{+h}{D}(\xi)\big)\frac{\partial\mathcal{L}}{\partial \underset{h}{u}_x} + \mathcal{L}\underset{+h}{D}(\xi) = 0, \\ &\underset{+h}{S}(\xi) - \xi(1 + \varphi_x) - \eta\varphi_u = 0, \end{aligned} \tag{10}$$

where $\underset{+h}{S}$ is the right shift operator, connected with difference differentiation by $h^+\underset{+h}{D} = \underset{+h}{S} - 1$.

It was developed in [6,7] that a difference functional

$$L = \sum_{\underset{h}{\Omega}} \mathcal{L}\big(x^1, x^2, u, \underset{h}{u}_1, \underset{h}{u}_2\big)h_1^+ h_2^+ \tag{11}$$

is invariant with respect to G_1 on two-dimensional regular orthogonal mesh $\underset{h_1}{\omega} \times \underset{h_2}{\omega}$, $h_i^+ = h_i^-$, with two constant steps h_1 and h_2, if and only if

$$\begin{aligned} &X(\mathcal{L}) + \mathcal{L}\big(\underset{+h}{D}_1(\xi^1) + \underset{+h}{D}_2(\xi^2)\big) = 0, \\ &\underset{+h}{D}_1 \underset{-h}{D}_1(\xi^1) = 0, \qquad \underset{+h}{D}_2 \underset{-h}{D}_2(\xi^2) = 0, \\ &\underset{\pm h}{D}_1(\xi^2) = -\underset{\pm h}{D}_2(\xi^1), \end{aligned} \tag{12}$$

where $\underset{\pm h}{D}_i$ is the difference differentiation in i-direction, X is an operator of a group G_1:

$$X = \xi^1 \frac{\partial}{\partial x^1} + \xi^2 \frac{\partial}{\partial x^2} + \eta \frac{\partial}{\partial u} + \underset{h}{\zeta_1} \frac{\partial}{\partial \underset{h}{u}_1} + \underset{h}{\zeta_2} \frac{\partial}{\partial \underset{h}{u}_2} + \cdots$$
$$+ \big(\underset{+h}{S}_1(\xi^1) - \xi^1\big) \frac{\partial}{\partial h_1^+} + \big(\underset{+h}{S}_2(\xi^2) - \xi^2\big) \frac{\partial}{\partial h_2^+}, \tag{13}$$

where $\underset{h}{\zeta_i}$ are linear forms of ξ^1, ξ^2, η (see [1,5]).

It is well known [13] (see also [10,14,15]) that in continuous case the invariance of the Euler equation is a consequence of the invariance (or divergent invariance, see [4]) of the corresponding functional. This situation is changing while considering discrete case.

For simplicity we will follow one-dimensional case. For the invariance of a difference Euler's equation (1) of the invariant difference functional (6) on a regular mesh $\underset{h}{\omega}$ it is necessary and sufficient that for every solution of (2) the following is true (see [7,8]):

$$\xi_u \left(\frac{\partial \mathcal{L}}{\partial x} + \underset{h}{u}_{\bar{x}} \frac{\partial \mathcal{L}}{\partial u} + \underset{h}{u}_{x\bar{x}} \frac{\partial \mathcal{L}}{\partial \underset{h}{u}_x} - \underset{-h}{D}(\mathcal{L}) \right) = 0, \qquad \underset{-h}{D}\,\underset{+h}{D}(\xi) = 0. \tag{14}$$

This condition "disappears" in continuous limit since the operator in brackets tends to zero as $h \to 0$. Thus, any Euler's differential equation admits every symmetry group of a corresponding functional (the inverse proposition is not true!).

The conditions (14) are true for degenerate functionals which are linear and for the special transformation groups:

$$\xi_u = 0, \qquad \underset{-h}{D}\,\underset{+h}{D}(\xi) = 0. \tag{15}$$

So, the only case (14) is similar to the continuous situation, where invariance of a functional leads to the invariance of the Euler equation. The transformations which do not change independent variables x (means $\xi = 0$) satisfy the conditions (14) for sure. This special case was considered in pioneering papers [11,12]. Meanwhile it is clear that a lot of transformations do not satisfy the condition (14). It means that those transformations could change the Euler equation into another one without changing of difference functional.

As it was developed in [6,7] the stationary value of a difference functional is reached on the new difference equations, named quasi-extremal equations, *which are different for different subgroups*:

$$\xi_\alpha \left(\frac{\partial \mathcal{L}}{\partial x} + \underset{-h}{D} \left(\underset{h}{u}_x \frac{\partial \mathcal{L}}{\partial \underset{h}{u}_x} - \mathcal{L} \right) \right) + \eta_\alpha \left(\frac{\partial \mathcal{L}}{\partial u} - \underset{-h}{D} \left(\frac{\partial \mathcal{L}}{\partial \underset{h}{u}_x} \right) \right) = 0. \tag{16}$$

The quasi-extremal equations could be rewritten as

$$\xi_\alpha \frac{\delta \mathcal{L}}{\delta x} + \eta_\alpha \frac{\delta \mathcal{L}}{\delta u} = 0, \quad \alpha = 1, 2, 3, \ldots, \tag{17}$$

where

$$\frac{\delta \mathcal{L}}{\delta x} = \frac{\partial \mathcal{L}}{\partial x} h^+ + \frac{\partial \mathcal{L}^-}{\partial x} h^- + \mathcal{L}^- - \mathcal{L}, \quad \mathcal{L}^- = \underset{-h}{S}(\mathcal{L}), \tag{18}$$

$$\frac{\delta \mathcal{L}}{\delta u} = h^+ \frac{\partial \mathcal{L}}{\partial u} + h^- \frac{\partial \mathcal{L}^-}{\partial u}. \tag{19}$$

It is supposed in (17)–(19) that a Lagrangian has the form $\mathcal{L} = \mathcal{L}(x, x^+, u, u^+)$.

It is easy to show that in the continuous limit the difference between quasi-extremal equations disappears; all of them tend to one and the same Euler's differential equation.

Example. Let us illustrate the above by example. We consider a difference model for the following ODE:

$$u_{xx} = u^{-3}. \tag{20}$$

This equation could be viewed as the Euler equation for the Lagrange function $(1/u^2 - u_x^2)$.

In accordance with Noether's theorem one can find the following first integrals:

$$J_1 = u_x^2 + \frac{1}{u^2} = A^0, \qquad J_2 = 2\frac{x}{u^2} - 2(u - u_x x)u_x = 2B^0, \qquad J_3 = \frac{x^2}{u^2} + (u - xu_x)^2 = C^0.$$

The structure of action of the adjoint Lie algebra (see [10,14]) permits us to find all integrals from the basic integral J_3:

$$X_1(J_3) = J_2, \qquad X_1(J_2) = 2J_1.$$

Let us choose a difference analog of the Lagrangian $(1/u^2 - u_x^2)$:

$$\mathcal{L} = \frac{1}{uu^+} - \left(\frac{u^+ - u}{h^+}\right)^2, \tag{21}$$

which is defined on two points of some difference lattice $\underset{h}{\omega}$ (see Fig. 1).

Then we could check the variational invariance of (21) with regard to the symmetry of original ODE (20):

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = 2x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \qquad X_3 = x^2\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u}. \tag{22}$$

The difference functional

$$L = \sum_{\Omega}\left(\frac{1}{uu^+} - \left(\frac{u^+ - u}{h^+}\right)^2\right)h^+ = \sum_{\Omega}\left(\frac{h^+}{uu^+} - \frac{(u^+ - u)^2}{h^+}\right), \tag{23}$$

where $h^+ = x^+ - x$, admits translation X_1 for sure. The invariance of $\mathcal{L}h^+$:

$$X\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi) = 0 \tag{24}$$

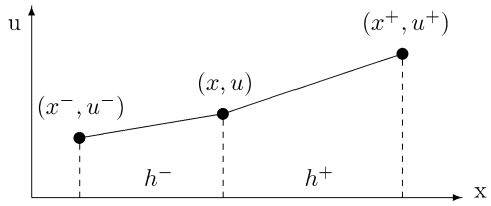

Fig. 1.

could easily be checked for the dilatation X_2. The action of the operator X_3 yields the divergent invariance of L, i.e., the action gives finite difference of some function:

$$X_3\mathcal{L} + \mathcal{L} \underset{+h}{D}(x^2) = \frac{u^{+2} - u^2}{h^+} \equiv \underset{+h}{D}(u^2). \tag{25}$$

The extremal Euler's equation of (21) is

$$2 \underset{h}{u}_x - 2 \underset{h}{u}_{\bar{x}} - \frac{1}{u^2}\left(\frac{h^+}{u^+} + \frac{h^-}{u^-}\right) = 0, \tag{26}$$

where

$$\underset{h}{u}_x = \frac{u^+ - u}{h^+}, \qquad \underset{h}{u}_{\bar{x}} = \frac{u - u^-}{h^-}.$$

Notice that a mesh $\underset{h}{\omega}$ is not still defined:

$$h^+ = \varphi(x, u, u^+, u^-). \tag{27}$$

Due to the fact that the group G_3 satisfies the condition (14) the Euler equation (26) admits symmetries (22) *provided a mesh* (27) *is invariant*.

Let us write down the quasi-extremal equations of the functional (23). The symmetry operator X_1 yields the following quasi-extremal equation:

$$\frac{1}{u}\left(\frac{1}{u^-} - \frac{1}{u^+}\right) + \underset{h}{u}{}_x^2 - \underset{h}{u}{}_{\bar{x}}^2 = 0. \tag{28}$$

The quasi-extremal for the operator X_2 is

$$\frac{x}{u}\left(\frac{1}{u^-} - \frac{1}{u^+}\right) + \frac{1}{u}\left(\frac{x - h^-}{u^-} - \frac{x + h^+}{u^+}\right) + 2x\left(\underset{h}{u}{}_{\bar{x}}^2 - \underset{h}{u}{}_x^2\right) + 2u\left(\underset{h}{u}_x - \underset{h}{u}_{\bar{x}}\right) = 0. \tag{29}$$

The third quasi-extremal equation springs from symmetry X_3:

$$x^2\left(\frac{1}{uu^-} - \frac{1}{uu^+} - \underset{h}{u}{}_x^2 + \underset{h}{u}{}_{\bar{x}}^2\right) + xu\left(2\underset{h}{u}_x - 2\underset{h}{u}_{\bar{x}} - \frac{h^+}{u^2u^+} - \frac{h^-}{u^2u^-}\right) = 0. \tag{30}$$

We could rewrite all quasi-extremal equations in the uniform form:

$$\begin{aligned} &(1) \quad \frac{\delta\mathcal{L}}{\delta x} = 0, \\ &(2) \quad 2x\left(\frac{\delta\mathcal{L}}{\delta x}\right) + u\left(\frac{\delta\mathcal{L}}{\delta u}\right) = 0, \\ &(3) \quad x^2\left(\frac{\delta\mathcal{L}}{\delta x}\right) + xu\left(\frac{\delta\mathcal{L}}{\delta u}\right) = 0, \end{aligned} \tag{31}$$

where

$$\frac{\delta\mathcal{L}}{\delta x} = \frac{\partial\mathcal{L}}{\partial x}h^+ + \frac{\partial\mathcal{L}^-}{\partial x}h^- + \mathcal{L}^- - \mathcal{L}, \quad \mathcal{L}^- = \underset{-h}{S}(\mathcal{L}), \tag{32}$$

$$\frac{\delta\mathcal{L}}{\delta u} = h^+\frac{\partial\mathcal{L}}{\partial u} + h^-\frac{\partial\mathcal{L}^-}{\partial u}. \tag{33}$$

It is evident that the quasi-extremal equations (31) depend on corresponding coordinates of the subgroup operators.

2. The properties of quasi-extremal equations

Now we will develop some common properties of quasi-extremal equations, which are true for any invariant variational functional.

Notice, that quasi-extremal equations

$$\xi_\alpha \frac{\delta \mathcal{L}}{\delta x} + \eta_\alpha \frac{\delta \mathcal{L}}{\delta u} = 0, \quad \alpha = 1, 2, 3, \ldots \tag{34}$$

have the intersection domain:

$$\frac{\delta \mathcal{L}}{\delta x} = 0, \qquad \frac{\delta \mathcal{L}}{\delta u} = 0. \tag{35}$$

More exactly, any solution of (35) yields the solution of the system (34).

Remarkable that the system (35) admits the entire set of symmetries of the original difference functional (see [2,8]).

Theorem 1. *Let a difference functional*

$$L = \sum \mathcal{L}(x, x^+, u, u^+) h^+, \tag{36}$$

be invariant with respect to a point transformation group G_1 with an operator

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \xi^+ \frac{\partial}{\partial x^+} + \eta^+ \frac{\partial}{\partial u^+}. \tag{37}$$

Then the system (35) *admits the same symmetry group G_1.*

Proof. From the invariance of the functional (36) we have the following relation (fragment) on some point x:

$$\begin{aligned} &\cdots + \mathcal{L}(x, x^+, u, u^+) h^+ + \mathcal{L}^-(x^-, x, u^-, u) h^- \\ &\quad = \mathcal{L}(x^*, x^{+*}, u^*, u^{+*}) h^{+*} + \mathcal{L}^-(x^{-*}, x^*, u^{-*}, u^*) h^{-*} + \cdots. \end{aligned} \tag{38}$$

Differentiations by x and by u yield:

$$\begin{aligned} &\frac{\partial \mathcal{L}}{\partial x} h^+ + \frac{\partial \mathcal{L}^-}{\partial x} h^- + \mathcal{L}^- - \mathcal{L} \\ &\quad = \frac{\partial x^*}{\partial x}\left(\mathcal{L}^- - \mathcal{L} + \frac{\partial \mathcal{L}}{\partial x^*} h^{+*} + \frac{\partial \mathcal{L}^-}{\partial x^*} h^{-*}\right) + \frac{\partial u^*}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial u^*} h^{+*} + \frac{\partial \mathcal{L}^-}{\partial u^*} h^{-*}\right), \end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial x} h^+ + \frac{\partial \mathcal{L}^-}{\partial x} h^- = \frac{\partial x^*}{\partial u}\left(\frac{\partial \mathcal{L}}{\partial x^*} h^{+*} + \frac{\partial \mathcal{L}^-}{\partial x^*} h^{-*} + \mathcal{L}^- - \mathcal{L}\right) + \frac{\partial u^*}{\partial u}\left(\frac{\partial \mathcal{L}}{\partial u^*} h^{+*} + \frac{\partial \mathcal{L}^-}{\partial u^*} h^{-*}\right).$$

Applying the procedure $(\partial/\partial a)|_{a=0}$ we get:

$$\frac{\partial}{\partial a}\left(\frac{\partial \mathcal{L}}{\partial x^*} h^{+*} + \frac{\partial \mathcal{L}^-}{\partial x^*} h^{-*} + \mathcal{L}^- - \mathcal{L}\right)\bigg|_{a=0} + \xi_x \frac{\delta \mathcal{L}}{\delta x} + \eta_x \frac{\delta \mathcal{L}}{\delta u} = 0,$$

$$\frac{\partial}{\partial a}\left(\frac{\partial \mathcal{L}}{\partial u^*} h^{+*} + \frac{\partial \mathcal{L}^-}{\partial u^*} h^{-*}\right)\bigg|_{a=0} + \xi_u \frac{\delta \mathcal{L}}{\delta x} + \eta_u \frac{\delta \mathcal{L}}{\delta u} = 0.$$

Substituting Eq. (35) completes the proof:

$$X\left(\frac{\delta\mathcal{L}}{\delta x}\right)\Big|_{(35)} = 0, \qquad X\left(\frac{\delta\mathcal{L}}{\delta u}\right)\Big|_{(35)} = 0. \qquad \square$$

Corollary. *Let intersection* (35) *correspond to the functional* (36), *which is invariant under the action of r-parameter group G_r, then the system* (35) *is invariant with respect to G_r.*

Consequence. *Being invariant the manifold* (35) *can be represented by means of difference invariants of a group G_r.*

One more property of quasi-extremal equations is the following: every quasi-extremal equation is invariant with respect to its "own" subgroup.

Theorem 2. *Let a quasi-extremal equation*

$$\xi\frac{\delta\mathcal{L}}{\delta x} + \eta\frac{\delta\mathcal{L}}{\delta u} = 0, \tag{39}$$

correspond to the invariant functional (36) *and a group G_1 with an operator* (37). *Then Eq.* (39) *admits the same group G_1.*

Let us consider an opposite situation: Eq. (39) does not admit the operator X with coordinates ξ, η. Then under the action of corresponding group G_1 this equation would be transformed into some other equation (39)* while functional is constant. But (39) was obtained as the *entire set* of solutions where (36) has stationary value, since the equation (39)* cannot be different from (39). Thus, it proves the invariance of quasi-extremal equation (39) with respect to the group G_1 with the operator X.

So, the quasi-extremal equations are invariant with respect to their "own" subgroups and the intersection domain possesses the *entire set* of symmetries. It makes rise for one more question: how does "foreign" subgroup act on a given quasi-extremal equation?

The particular answer on this question springs from the following identity:

$$\begin{aligned}&\xi\frac{\partial\mathcal{L}}{\partial x} + \xi^+\frac{\partial\mathcal{L}}{\partial x^+} + \eta\frac{\partial\mathcal{L}}{\partial u} + \eta^+\frac{\partial\mathcal{L}}{\partial u^+} + \mathcal{L}\underset{+h}{D}(\xi)\\ &\quad\equiv \xi\left(\frac{\partial\mathcal{L}}{\partial x} + \frac{h^-}{h^+}\frac{\partial\mathcal{L}^-}{\partial x} - \underset{+h}{D}(\mathcal{L}^-)\right) + \eta\left(\frac{\partial\mathcal{L}}{\partial u} + \frac{h^-}{h^+}\frac{\partial\mathcal{L}^-}{\partial u}\right) + \underset{+h}{D}\left(h^-\eta\frac{\partial\mathcal{L}^-}{\partial u} + h^-\xi\frac{\partial\mathcal{L}^-}{\partial x} + \xi\mathcal{L}^-\right).\end{aligned} \tag{40}$$

The identity (40) yields the following proposition.

Theorem 3. *Let quasi-extremal equations* (39) *of invariant functional* (36) *correspond to a group G_1 with an operator X. Let there exists an operator $\overline{X}$ which commutes with the difference differentiation on every solution of* (39):

$$\left[\overline{X}, \underset{+h}{D}\right] = 0. \tag{41}$$

Then the action of $\overline{X}$ transforms a quasi-extremal equation (39) *into some other quasi-extremal equation of the same functional.*

Corollary. *This property is valid for the operators of adjoint Lie algebra of operators, admitted by* (36). *So, we are defining a* new action *on the manifold of all quasi-extremal equations of invariant functional, that is action of adjoint Lie algebra. Due to this action it is possible to introduce a new definition*: a group basis of quasi-extremal equations, *means a minimum set of quasi-extremal equations which yields all others by actions of some operators* $\overline{X}$.

Notice, that all last properties are sufficiently "difference", and they "disappear" in the continuous limit, as all quasi-extremal equations tend to one and the same differential Euler's equation, which is invariant with respect to all symmetries of corresponding functional.

Example. Let us come back to our example ODE for illustrating the developed properties of quasi-extremal equations. All the above quasi-extremal equations

$$\begin{aligned}
&(1)\quad \frac{\delta\mathcal{L}}{\delta x}=0,\\
&(2)\quad 2x\left(\frac{\delta\mathcal{L}}{\delta x}\right)+u\left(\frac{\delta\mathcal{L}}{\delta u}\right)=0,\\
&(3)\quad x^2\left(\frac{\delta\mathcal{L}}{\delta x}\right)+xu\left(\frac{\delta\mathcal{L}}{\delta u}\right)=0,
\end{aligned}\tag{42}$$

have the following intersection:

$$\begin{cases}
\dfrac{1}{uu^-}-\dfrac{1}{uu^+}+\underset{h}{u}{}_x^2-\underset{h}{u}{}_{\bar{x}}^2=0,\\
2\underset{h}{u}{}_x-2\underset{h}{u}{}_{\bar{x}}-\dfrac{h^+}{u^2u^+}-\dfrac{h^-}{u^2u^-}=0.
\end{cases}\tag{43}$$

It is easy to check that the system (43) admits all the symmetries (22), but the system (42) does not. The first equation in (42) admits the operator X_1, the second admits X_2, the third admits X_3.

One can easily see that the operator $\partial/\partial x$ transforms the third equation into the second, and the operator $\frac{1}{2}(\partial/\partial x)$ changes the second equation into the first. So, the third equation is the basis among (42). This relation is true for conservation laws as well.

Notice that a difference mesh is not still determined. From invariance property of intersection system we know the possibility of constructing a mesh from difference invariants of the group G_3. The standard calculating procedure yields the following set of difference invariants:

$$J_1=\frac{h^+}{uu^+},\qquad J_2=\frac{h^-}{uu^-},\qquad J_3=u^2u^-\frac{\underset{h}{u}{}_x-\underset{h}{u}{}_{\bar{x}}}{h^-}.\tag{44}$$

Thus, the general equation of invariant lattices is the following:

$$F_\alpha\left(\frac{h^+}{uu^+},\frac{h^-}{uu^-},u^2u^-\frac{\underset{h}{u}{}_x-\underset{h}{u}{}_{\bar{x}}}{h^-}\right)=0,\tag{45}$$

where F_α are arbitrary functions of its arguments.

We choose the simplest case of invariant lattice:

$$\frac{h^+}{uu^+}=\varepsilon,\qquad \frac{h^-}{uu^-}=\varepsilon,\qquad \varepsilon=\text{const},\quad 0<\varepsilon\ll 1,\tag{46}$$

or $J_1=J_2=\varepsilon$.

Substituting the mesh (46) into intersection system (43) yields the the following mapping:

$$u^+u^-(2-\varepsilon^2) = u(u^+ + u^-).$$

This mapping together with the mesh (46) is equivalent the following difference equation:

$$\frac{\underset{h}{u}_x - \underset{h}{u}_{\bar{x}}}{h^-} = \frac{1}{u^2u^-}, \tag{47}$$

or $J_3 = 1$.

It is easy to show that Eq. (47) on the mesh (46) approximates Eq. (20) with the second order.

Notice that developed intersection equation (47) is different from Euler's equation (26):

$$\underset{h}{u}_x - \underset{h}{u}_{\bar{x}} = \frac{1}{2u^2}\left(\frac{h^+}{u^+} + \frac{h^-}{u^-}\right).$$

3. Difference analog of the Noether theorem

The difference analog of the Noether-type identity

$$\begin{aligned}
&\xi\frac{\partial \mathcal{L}}{\partial x} + \xi^+\frac{\partial \mathcal{L}}{\partial x^+} + \eta\frac{\partial \mathcal{L}}{\partial u} + \eta^+\frac{\partial \mathcal{L}}{\partial u^+} + \mathcal{L}\underset{+h}{D}(\xi) \\
&\quad\equiv \xi\left(\frac{\partial \mathcal{L}}{\partial x} + \frac{h^-}{h^+}\frac{\partial \mathcal{L}^-}{\partial x} - \underset{+h}{D}(\mathcal{L}^-)\right) + \eta\left(\frac{\partial \mathcal{L}}{\partial u} + \frac{h^-}{h^+}\frac{\partial \mathcal{L}^-}{\partial u}\right) + \underset{+h}{D}\left(h^-\eta\frac{\partial \mathcal{L}^-}{\partial u} + h^-\xi\frac{\partial \mathcal{L}^-}{\partial x} + \xi\mathcal{L}^-\right)
\end{aligned}$$

gives possibility to find conservation laws for difference equations.

Theorem 4. *Let the difference equation* (*or system*)

$$F_\alpha(x, u, u^+, u^-, \ldots, h^+, h^-) = 0 \tag{48}$$

be given on a mesh $\underset{h}{\omega}$:

$$\Omega_\beta(x, u, u^+, u^-, \ldots, h^+, h^-) = 0, \tag{49}$$

where F_α *and* Ω_β *are some smooth functions. Let the solutions of* (48) *on the mesh* (49) *be quasi-extremals of the functional* (36) *corresponding to the coordinates* ξ, η *of the group operator* (37).

Then the invariance of (36) *is necessary and sufficient condition of conservativeness of the system* (48)–(49):

$$\underset{+h}{D}\left(h^-\eta\frac{\partial \mathcal{L}^-}{\partial u} + h^-\xi\frac{\partial \mathcal{L}^-}{\partial x} + \xi\mathcal{L}^-\right)\bigg|_{(48)-(49)} = 0. \tag{50}$$

Remark. In classical continuous case the generalization of the invariance condition of functional is that the action of a group on the last one could give a divergence of some function [4]; those functionals were called divergent invariant [14]. The same generalization is true for a difference functionals as well. If

$$\xi\frac{\partial \mathcal{L}}{\partial x} + \xi^+\frac{\partial \mathcal{L}}{\partial x^+} + \eta\frac{\partial \mathcal{L}}{\partial u} + \eta^+\frac{\partial \mathcal{L}}{\partial u^+} + \mathcal{L}\underset{+h}{D}(\xi) = \underset{+h}{D}\big(B(x, u, u^+, \ldots)\big),$$

where B is some smooth function, then the same function B could be added to a first integral (50).

Theorems 1 and 4 have the following corollary, which could be viewed as *the difference analog of Noether's theorem*.

Corollary. *Let the functional* (36) *admit r-parameter group G_r, and there are corresponding r quasi-extremal equations*

$$\xi^{\alpha}\frac{\delta\mathcal{L}}{\delta x}+\eta^{\alpha}\frac{\delta\mathcal{L}}{\delta u}=0,\quad \alpha=1,2,\ldots,r. \tag{51}$$

Then the intersection system

$$\begin{cases}\dfrac{\delta\mathcal{L}}{\delta x}=0,\\[2mm] \dfrac{\delta\mathcal{L}}{\delta u}=0,\end{cases} \tag{52}$$

is invariant with respect to G_r and possesses r conservation laws of type (50).

Notice that this version of difference analog of the Noether theorem is sufficiently different from such early obtained in [7].

Example. An application of the above corollary to the difference scheme (46), (47) yields the following first integrals:

$$\begin{array}{ll}(1) & \underset{h}{u}{}_x^2+\dfrac{1}{uu^+}=A=\text{const},\\[2mm] (2) & \dfrac{2x+h^+}{2}\underset{h}{u}{}_x^2+\dfrac{2x+h^+}{2uu^+}-\dfrac{u+u^+}{2}\underset{h}{u}{}_x=B=\text{const},\\[2mm] (3) & \dfrac{x(x+h^+)}{uu^+}+\left(\dfrac{u+u^+}{2}-\dfrac{2x+h^+}{2}\underset{h}{u}{}_x\right)^2=C=\text{const}.\end{array} \tag{53}$$

The structure of an action of the adjoint Lie algebra permits, as in the continuous case, to find all integrals from the basic integral J_3:

$$X_1(J_3)=J_2,\qquad X_1(J_2)=2J_1.$$

Due to these integrals it is easy to find the general solution of the difference scheme (46), (47):

$$Au^2=(Ax+B)^2+1-\frac{\varepsilon^2}{4}, \tag{54}$$

where $A=\text{const}$, $B=\text{const}$.

So, the developed difference model (46), (47):

- is invariant under the same Lie point symmetry group G_3 as the original differential equation,
- possesses the difference first integrals with one basic integral and the same action of adjoint Lie algebra,
- is completely integrable and its general solution differs by $\varepsilon^2/4$ from the general solution of Eq. (20).

4. Discrete representation of ODE (exact scheme)

The exact solution (54) is very close to the exact solution of the original ODE (20):

$$A_0u^2 = (A_0x + B_0)^2 + 1. \tag{55}$$

The question arises; is it possible to construct the *exact* difference scheme (and lattice) for (20)?

If YES, then such scheme should be invariant with respect to the same group G_3, since it could be expressed by means of the same difference invariants:

$$J_1 = \frac{h^+}{uu^+}, \qquad J_2 = \frac{h^-}{uu^-}, \qquad J_3 = u^2u^- \frac{\underset{h}{u}_x - \underset{h}{u}_{\bar{x}}}{h^-}.$$

Let us choose the difference Lagrangian:

$$\mathcal{L} = \frac{\delta}{uu^+} - \left(\frac{u^+ - u}{h^+}\right)^2, \tag{56}$$

where $\delta = \text{const}$ is not determined.

Let us choose the same invariant lattice:

$$\frac{h^+}{uu^+} = \varepsilon, \quad \frac{h^-}{uu^-} = \varepsilon, \qquad \varepsilon = \text{const}, \quad 0 < \varepsilon \ll 1. \tag{57}$$

Then the same variational procedure yields the intersection system for quasi-extremal equations of the Lagrangian (56):

$$\frac{\underset{h}{u}_x - \underset{h}{u}_{\bar{x}}}{h^-} = \frac{\delta}{u^2u^-}, \tag{58}$$

Substituting the exact solution (55) into the scheme (57), (58) yields the constant δ:

$$\delta = 2\frac{1 - \sqrt{1 - \varepsilon^2}}{\varepsilon^2}. \tag{59}$$

Thus, the developed scheme (57), (58) with the constant (59) being the exact scheme for ODE (20) means the family of solutions (55) of ODE (20) satisfies this scheme identically. Of course, the exact scheme (57), (58) gives numbers of points instead of a curve; the density of points depends on the parameter ε.

Notice that the developed intersection equation (58) is sufficiently different from Euler's equation for (56):

$$\underset{h}{u}_x - \underset{h}{u}_{\bar{x}} = \frac{\delta}{2u^2}\left(\frac{h^+}{u^+} + \frac{h^-}{u^-}\right).$$

An application of the above corollary to the exact scheme (57), (58) yields the following first integrals:

$$\begin{aligned}
&(1) \quad \underset{h}{u}_x^2 + \frac{\delta}{uu^+} = A^0 = \text{const},\\
&(2) \quad \frac{2x + h^+}{2}\underset{h}{u}_x^2 + \delta\frac{2x + h^+}{2uu^+} - \frac{u + u^+}{2}\underset{h}{u}_x = B^0 = \text{const},\\
&(3) \quad \delta\frac{x(x + h^+)}{uu^+} + \left(\frac{u + u^+}{2} - \frac{2x + h^+}{2}\underset{h}{u}_x\right)^2 = C^0 = \text{const}.
\end{aligned} \tag{60}$$

Notice that conservation laws (60) are nonlocal, and they cannot be obtained from classical Noether's theorem.

Remarks.

(1) The external transformation

$$u(x)=\frac{v(t)}{t},\quad x=-\frac{1}{t}, \tag{61}$$

do not change Eq. (20), transforming the family of solutions into itself. The same situation is true for any invariant difference scheme. Indeed, the exchange (61) transforms the group G_3 with the operators (22) into itself, so it conserves the finite-difference invariants:

$$\begin{aligned} J_1&=\frac{h^+}{uu^+}\quad\Longrightarrow\quad \widetilde{J}_1=\frac{\tau^+}{vv^+},\\ J_2&=\frac{h^-}{uu^-}\quad\Longrightarrow\quad \widetilde{J}_2=\frac{\tau^-}{vv^-},\\ J_3&=u^2\left(\frac{1}{h^+}+\frac{1}{h^-}\right)\quad\Longrightarrow\quad \widetilde{J}_3=v^2\left(\frac{1}{\tau^+}+\frac{1}{\tau^-}\right), \end{aligned} \tag{62}$$

where τ^+, τ^- are steps in t-direction.

Thus, the exchange (61) conserves any invariant difference equation as well. In particularly, the exact scheme (57), (58) is invariant with respect to the transformation (61).

(2) The invariant scheme of (46), (47) is very close to the exact scheme (57), (58). Moreover, the difference between corresponding families of exact solutions is homogeneous.

It turned out that the difference between the exact scheme and the invariant second-order approximation scheme could be expressed by scaling. Namely, the scaling of x or u, or their combination

$$\widetilde{x}=x\cdot\alpha^2\sqrt{1-\frac{\varepsilon^2}{4}},\qquad \widetilde{u}=u\cdot\alpha, \tag{63}$$

where $\alpha\neq 0$ is an arbitrary constant, transforms the scheme (46), (47) into the exact scheme (57), (58).

(3) Transforming the initial equation (20) we can find *the differential equation that has approximate scheme* (46), (47) *as the difference representation or the exact scheme*. For any fixed ε this equation is the following:

$$u_{xx}=\frac{1}{u^3}\left(1-\frac{\varepsilon^2}{4}\right). \tag{64}$$

Thus, the approximate invariant scheme (46), (47) is the exact scheme of the approximate differential equation (64).

(4) The exact scheme (57), (58) has “an infinite order of approximation”, i.e., an accuracy does not depend on steps h^+, h^- or on a number of points on a given domain. So, the minimum number of points in calculating procedure is 3.

The entire set of second-order ordinary difference equations with nontrivial symmetry where the difference analog of Noether-type theorem can be applied is published elsewhere [9].

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References

[1] W.F. Ames, R.L. Anderson, V.A. Dorodnitsyn, E.V. Ferapontov, R.K. Gazizov, N.H. Ibragimov, S.R. Svirshchevskii, CRC Handbook of Lie Group Analysis of Differential Equations, Vol. I: Symmetries, Exact Solutions and Conservation Laws, CRC Press, Boca Raton, FL, 1994.

[2] Yu. Artamonov, Master's Thesis, Physical Department of Moscow State University, 1995 (in Russian).

[3] M.I. Bakirova, V.A. Dorodnitsyn, R.V. Kozlov, Symmetry preserving difference schemes for some heat transfer equations, J. Phys. A: Math. Gen. 30 (1997) 8139–8155.

[4] E. Bessel-Hagen, Uber die Erhaltungssatze der Elektrodynamik, Math. Ann. 84 (1921) 258–276.

[5] V.A. Dorodnitsyn, Transformation groups in mesh spaces, J. Soviet Math. 55 (1) (1991) 1490.

[6] V.A. Dorodnitsyn, Finite-difference models entirely inheriting symmetry of original differential equations, in: Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics, Kluwer Academic, Dordrecht, 1993, p. 191.

[7] V.A. Dorodnitsyn, Finite-difference analog of Noether's theorem, Dokl. Akad. Nauk SSSR 328 (6) (1993) 678 (in Russian).

[8] V. Dorodnitsyn, Conservation laws for difference equation, in: N. Ibragimov, K. Razi Nagvi, E. Straume (Eds.), Modern Group Analysis VII, Developments in Theory, Computations and Application, MARS Publishers, Symmetry Foundation, 1999, pp. 91–97.

[9] V. Dorodnitsyn, R. Kozlov, P. Winternitz, Lie group classification of second-order ordinary difference, J. Math. Phys. 41 (1) (2000) 480–504.

[10] N.H. Ibragimov, Transformation Groups Applied to Mathematical Physics, D. Reidel, Dordrecht, 1985.

[11] S. Maeda, Extension of discrete Noether theorem, Math. Japon. 26 (1) (1981) 85–90.

[12] S. Maeda, The similarity method for difference equations, J. Inst. Math. Appl. 38 (1987) 129–134.

[13] E. Noether, Invariante Variathionsprobleme, Konig. Gesell. Wissen., Gottingen, Math.-Phys. Kl., Heft 2, 235.

[14] P.J. Olver, Application of Lie Groups to the Differential Equations, Springer, New York, 1986.

[15] L.V. Ovsiannikov, Group Analysis of Differential Equations, Academic Press, New York, 1982.