Letters in Mathematical Physics **28**: 123–137, 1993.

Stochastic Differential Calculus, the Moyal $*$-Product, and Noncommutative Geometry

ARISTOPHANES DIMAKIS
Department of Mathematics, University of Crete, Iraklion, Greece

and

FOLKERT MÜLLER-HOISSEN
Institut für Theoretische Physik, Bunsenstr. 9, Göttingen, Germany

(Received: 10 March 1993)

Abstract. A reformulation of the Itô calculus of stochastic differentials is presented in terms of a differential calculus in the sense of noncommutative geometry (with an exterior derivative operator d satisfying $d^2 = 0$ and the Leibniz rule). In this calculus, differentials do not commute with functions. The relation between both types of differential calculi is mediated by a generalized Moyal $*$-product. In contrast to the Itô calculus, the new framework naturally incorporates analogues of higher-order differential forms. A first step is made towards an understanding of their stochastic meaning.

Mathematics Subject Classifications (1991). 17B37, 58A10, 58G32, 81R50, 81S20.

1. Introduction

The basic structure underlying noncommutative geometry in its various forms (see [1], for example) is a differential calculus on an associative algebra $\mathscr{A}$. It generalizes the calculus of differential forms on a manifold (respectively its algebra of C^∞-functions) to, in general, noncommutative algebras. This structure has been employed during recent years in different fields of mathematical physics, either as a tool (e.g., in the context of quantum field theory [2], quantum Hall effect [3], quantum groups [4]), or as an essential part of the mathematical framework for new physical models (e.g., for particle physics [5]).

In a recent work [6], it has also been demonstrated that even the commutative algebra of functions on a manifold admits nonstandard differential calculi which provide us with a unified framework for dealing with continuum, lattice and certain q-deformed theories and which allows us to explore their relations. In [7], we developed a kind of 'second-order' differential geometry on the basis of a deformation of the standard differential calculus on a manifold. Later, we realized that some formulae which arose in that work are very much alike those appearing in the Itô calculus of stochastic differentials (see [8] for example) which is not a differential calculus in the ordinary sense. The calculus of stochastic differentials plays a role in stochastic mechanics and stochastic quantization (see [9, 10] for reviews). An attempt to reveal a more direct correspondence between our noncommutative differential calculus and the Itô calculus, despite their apparent dissimilarity, led to the present Letter.

The following short section recalls some notions from stochastic calculus. The essential step from the calculus of stochastic differentials to noncommutative differential calculus is done in Section 3. It involves a product between functions and differentials which is related to the Moyal product [11]. This relation is established in Section 4 (and Appendix A). Section 5 generalizes the results of Section 3 to stochastic calculus on manifolds where we have to address questions of general covariance. Of particular importance, then is, Section 6 where we take our noncommutative differential calculus as the basic structure and derive the Itô formula from it. Exterior products of stochastic differential forms have not been discussed in the literature according to our knowledge. In a differential calculus, such forms of higher grade are constructed using $\mathrm{d}^2 = 0$ and the derivation property (Leibniz rule) for the exterior derivative d. In this way we are led beyond the usual stochastic framework. (A kind of differential geometric formalism has been developed for the Itô calculus (see [12], for example). It also includes the notion of higher-order forms. But in that case differentials commute (instead of anticommute) and we do not have $\mathrm{d}^2 = 0$.) Section 7 is a step towards an interpretation of these forms of a higher grade in terms of stochastic differentials. Some conclusions and further remarks are collected in Section 8. In Appendix B, we briefly discuss a generalization of our noncommutative differential calculus involving 'generalized Galilei structures' and establish a relation with the classical limit of bicovariant differential calculus on quantum groups.

2. Calculus of Stochastic Differentials

The differences between the calculus of stochastic differentials and ordinary differential calculus can be traced back to a kind of 'weak nonlocality' inherent in stochastic processes. For a Wiener process W_t, this manifests itself in $(\mathrm{d}W_t)^2$ being proportional to $\mathrm{d}t$. Wiener processes are of unbounded variation so that an expression like $\int W_t\,\mathrm{d}W_t$ cannot be interpreted as an ordinary Riemann–Stieltjes integral [8]. It is defined as

$$\int_{t_0}^{t} W_s\,\mathrm{d}W_s = \text{qm-lim}_{\delta_N \to 0} \sum_{k=1}^{N} W_{\tau_k}(W_{t_k} - W_{t_{k-1}}), \tag{2.1}$$

where $t_{k-1} \leqslant \tau_k \leqslant t_k$ and $t_N = t$. (The t_k and τ_k should carry an index N which we omit in order to simplify the notation.) qm-lim means 'quadratic mean limit' and $\delta_N = \max_k(t_k - t_{k-1})$. Evaluation of this expression shows that the result depends on the choice of the intermediate points τ_k. In fact, choosing $\tau_k = (1 - \alpha)t_{k-1} + \alpha t_k$ with a real parameter α, where $0 \leqslant \alpha \leqslant 1$, one obtains

$$\int_{t_0}^{t} W_s\,\mathrm{d}W_s = \tfrac{1}{2}(W_t^2 - W_{t_0}^2) + (\alpha - \tfrac{1}{2})(t - t_0) \tag{2.2}$$

(cf. [8], p. 60, for example). This is usually expressed in terms of differentials as follows

$$W_t\,\mathrm{d}W_t = \tfrac{1}{2}\,\mathrm{d}(W_t^2) + (\alpha - \tfrac{1}{2})\,\mathrm{d}t. \tag{2.3}$$

Special cases of this formula appear in the Stratonovich calculus (where $\alpha = \frac{1}{2}$) and the Itô calculus (where $\alpha = 0$). The calculus of stochastic differentials encodes in a convenient (although formal) way the basic formulae which appeared in Itô's theory of stochastic integration. In the remainder of this section, we will concentrate on the case of the Itô calculus, i.e. $\alpha = 0$.

If W_t^i are n Wiener processes with covariance matrix $\gamma\delta^{ij}$, where γ is a positive constant, then one has the following simple rules for products of the differentials $\mathrm{d}t$ and $\mathrm{d}W_t^i$

$$\mathrm{d}t\cdot\mathrm{d}t = 0, \qquad \mathrm{d}t\cdot\mathrm{d}W_t^i = 0, \qquad \mathrm{d}W_t^i\cdot\mathrm{d}W_t^j = \gamma\delta^{ij}\,\mathrm{d}t. \tag{2.4}$$

The product is commutative and associative. For a smooth function $f(t,x)$ let $f_t := f(t, W_t)$. The differentials $\mathrm{d}W_j^i$ and $\mathrm{d}t$ are assumed to commute with f_t. Then one obtains the Itô formula

$$\mathrm{d}f_t = \mathrm{d}t\left(\partial_t f_t + \frac{\gamma}{2}(\Delta f)_t\right) + \mathrm{d}W_t^i(\partial_i f)_t \tag{2.5}$$

by Taylor-expanding $f(t + \mathrm{d}t, W_t + \mathrm{d}W_t)$ up to second order in $\mathrm{d}t$ and $\mathrm{d}W_t^i$ and using (2.4). We have set

$$\partial_i := \frac{\partial}{\partial x^i}, \qquad \partial_t := \frac{\partial}{\partial t} \quad \text{and} \quad \Delta := \delta^{ij}\partial_i\partial_j.$$

The Itô formula is very helpful for evaluating Itô integrals. The differential operator d defined above does not have the properties of an exterior derivative (like the d in the calculus of differential forms on a manifold). Instead of the Leibniz rule (derivation property), it satisfies

$$\mathrm{d}(f_t h_t) = \mathrm{d}f_t\,h_t + f_t\,\mathrm{d}h_t + \mathrm{d}f_t\cdot\mathrm{d}h_t. \tag{2.6}$$

3. Making the Stochastic d a Derivation

Let us define a new product which takes account of the nonlocality by encoding it into its noncommutativity. For any function f, we set

$$\begin{aligned} \mathrm{d}W_t^i * f_t &:= f_t\,\mathrm{d}W_t^i - \alpha\gamma\delta^{ij}(\partial_j f)_t\,\mathrm{d}t, \\ f_t * \mathrm{d}W_t^i &:= f_t\,\mathrm{d}W_t^i + (1-\alpha)\gamma\delta^{ij}(\partial_j f)_t\,\mathrm{d}t. \end{aligned} \tag{3.1}$$

These relations have to be supplemented by

$$f_t * \mathrm{d}t = \mathrm{d}t * f_t = f_t\mathrm{d}t \quad \text{and} \quad f_t * g_t = f_t g_t.$$

The $*$-product is obviously not commutative. Adding the rules

$$(f_t \mathrm{d}W_t^i) * h_t := f_t(\mathrm{d}W_t^i * h_t), \qquad f_t * (\mathrm{d}W_t^i h_t) := (f_t * \mathrm{d}W_t^i) h_t \tag{3.2}$$

one can show that $*$ is associative. As a consequence of the definition of $*$ we obtain

$$[f_t, \mathrm{d}W_t^i]_* := f_t * \mathrm{d}W_t^i - \mathrm{d}W_t^i * f_t = \gamma \delta^{ij} (\partial_j f)_t \, \mathrm{d}t, \tag{3.3}$$

which does not explicitly depend on α. When $\alpha = 0$, the Itô formula (2.5) can now be written as

$$\mathrm{d}f_t = \mathrm{d}t \left(\partial_t f_t + \frac{\gamma}{2} (\Delta f)_t \right) + \mathrm{d}W_t^i * (\partial_i f)_t.$$

$$= \mathrm{d}t \left(\partial_t f_t - \frac{\gamma}{2} (\Delta f)_t \right) + (\partial_i f)_t * \mathrm{d}W_t^i. \tag{3.4}$$

Our $*$-product obviously depends on the parameter α. The corresponding algebras for different values of α are isomorphic, however. Applying the isomorphism which transforms the algebra with $\alpha = 0$ into the algebra with $\alpha \neq 0$ to (3.4), we obtain the generalized Itô formula

$$\mathrm{d}f_t = \left(\partial_t f_t + \frac{1 - 2\alpha}{2} \gamma (\Delta f)_t \right) \mathrm{d}t + (\partial_i f)_t \, \mathrm{d}W_t^i \tag{3.5}$$

which for $\alpha = \frac{1}{2}$ yields the ordinary formula for the differential of a function and thus corresponds to the case of the Stratonovich calculus. Using this formula and the properties of the $*$-product, it is easily verified that

$$\mathrm{d}(f_t * h_t) = \mathrm{d}f_t * h_t + f_t * \mathrm{d}h_t, \tag{3.6}$$

i.e. the introduction of the new product achieves that d becomes a *derivation*.

Expressing (3.5) in terms of the $*$-product we find that (3.4) is valid for all values of α. (3.4) is then 'universal' in the sense that it does not make explicit reference to the parameter α. This raises the question of whether (3.4) can be derived directly from (3.3) (cf. Section 6).

4. Relation with the Moyal Product

The $*$-product introduced in Section 3 is a generalization of the Moyal product [13, 11]. This is, however, not evident. Let us consider Equations (3.1) for one Wiener process only (for simplicity). Using the map

$$W_t \mapsto x, \qquad \mathrm{d}W_t \mapsto p \, \mathrm{d}t \tag{4.1}$$

to phase-space coordinates x and p, we obtain

$$p * x = xp + a, \qquad x * p = xp + b, \tag{4.2}$$

where $a = -\alpha\gamma$ and $b = (1-\alpha)\gamma$. More generally, we will allow a and b to be arbitrary complex numbers. We think it is even of interest to consider this structure independent of our previous discussion. Using the rules

$$\begin{aligned}
&\lambda * f(x,p) = \lambda f(x,p) = f(x,p) * \lambda,\\
&g(x) * [f(x,p)k(x)] = [g(x) * f(x,p)]k(x),\\
&g(p) * [f(x,p)k(p)] = [g(p) * f(x,p)]k(p),\\
&[g(x)f(x,p)] * k(x) = g(x)[f(x,p) * k(x)],\\
&[g(p)f(x,p)] * k(p) = g(p)[f(x,p) * k(p)],
\end{aligned} \tag{4.3}$$

(where $\lambda \in \mathbb{C}$) and requiring associativity of $*$, one can show by induction that

$$f * h = \sum_{r=0}^{\infty} \sum_{i+j=r} \frac{1}{i!j!} a^j b^i (\partial_x^i \partial_p^j f)(\partial_p^i \partial_x^j h), \tag{4.4}$$

where f and h are arbitrary functions which can be expanded as power series in x and p. A proof of the associativity of the $*$-product defined in (4.4) is given in Appendix A. With the choice★ $b = -a = i\hbar/2$, we recover the *Moyal product* as defined in [11], i.e.

$$f *_M h = f \exp\left(\frac{i\hbar}{2} \overleftrightarrow{P}\right) h, \tag{4.5}$$

where $\overleftrightarrow{P}$ is the (Poisson bracket) bidifferential operator acting on the couple (f, h) as follows

$$f \overleftrightarrow{P} h = \partial_x f \partial_p h - \partial_p f \partial_x h = \{f, h\}. \tag{4.6}$$

Other choices appeared in [14] where they were related to operator ordering prescriptions for the transition from classical to quantum mechanics and, furthermore, to the discretisation ambiguity in the definition of a path integral (which is, of course, related to the problem of defining a stochastic integral mentioned in Section 2).

With $a = \lambda - \mu$, $b = \lambda + \mu$ and

$$Q := \partial_x \otimes \partial_p + \partial_p \otimes \partial_x, \qquad P := \partial_x \otimes \partial_p - \partial_p \otimes \partial_x, \tag{4.7}$$

our $*$-product can be expressed as

$$f * h = \mathcal{M} e^{\lambda Q} e^{\mu P} (f \otimes h), \tag{4.8}$$

where $\mathcal{M}$ is the multiplication map $f \otimes h \mapsto fh$. This implies

$$[f, h]_* = 2\mathcal{M} \sinh(\mu P) e^{\lambda Q} (f \otimes h). \tag{4.9}$$

★This corresponds to $\alpha = \frac{1}{2}$ and $\gamma = i\hbar$ which is imaginary whereas in Sections 2 and 3 the constant γ was taken to be real and positive. This converts the diffusion operator which appears in (3.4) into the corresponding Schrödinger operator.

With $\lambda = 0$ (and $\mu = i\hbar/2$), we recover the Moyal bracket [13] which, divided by 2μ, yields the Poisson bracket in the limit $\mu \to 0$. This is no longer true when $\lambda \neq 0$. (In this way our bracket escapes the uniqueness result in [15] for the Moyal bracket.) Then we still have a deformation of the Poisson bracket in the limit $\mu \to 0$

$$\lim_{\mu \to 0} \frac{1}{2\mu}[f, h]_* = (\partial_x f) \circ (\partial_p h) - (\partial_p f) \circ (\partial_x h), \tag{4.10}$$

where

$$f \circ h := \mathcal{M} e^{\lambda Q}(f \otimes h). \tag{4.11}$$

5. Stochastic Differential Calculus on Manifolds

Let us now associate stochastic processes x_t^μ with the coordinate functions on an n-dimensional manifold M [10, 16]. Instead of (2.4), we consider

$$\mathrm{d}t \cdot \mathrm{d}t = 0, \qquad \mathrm{d}t \cdot \mathrm{d}x_t^\mu = 0, \qquad \mathrm{d}x_t^\mu \cdot \mathrm{d}x_t^\nu = \gamma g_t^{\mu\nu}\, \mathrm{d}t \tag{5.1}$$

with a symmetric tensor field $g^{\mu\nu}$. The $*$-product can be defined as before, but with x_t^μ and $g_t^{\mu\nu}$ replacing W_t^i and δ^{ij}, respectively. We then have the following commutation relations

$$[t, \mathrm{d}t]_* = 0, \qquad [x_t^\mu, \mathrm{d}t]_* = 0, \qquad [t, \mathrm{d}x_t^\mu]_* = 0, \qquad [x_t^\mu, \mathrm{d}x_t^\nu]_* = \gamma g_t^{\mu\nu} * \mathrm{d}t. \tag{5.2}$$

In the same way as before, we obtain the Itô formula

$$\mathrm{d}f_t = \mathrm{d}t\left(\partial_t f_t + \frac{\gamma}{2}(\partial^2 f)_t\right) + \mathrm{d}x_t^\mu * (\partial_\mu f)_t, \tag{5.3}$$

where we suppressed $*$ in the first term on the right-hand side because in this case the two products are identical. We have introduced the abbreviation $\partial^2 := g^{\mu\nu}\partial_\mu\partial_\nu$.

Let us consider a coordinate transformation $x'^\mu = x'^\mu(t, x^\nu)$. The Itô formula yields

$$\mathrm{d}x_t'^\mu = \mathrm{d}t\left(\partial_t x_t'^\mu + \frac{\gamma}{2}(\partial^2 x'^\mu)_t\right) + \mathrm{d}x_t^\nu * (\partial_\nu x'^\mu)_t. \tag{5.4}$$

It is now easily verified that the relations (5.1) do not depend on the choice of coordinates. From the defining relations of the $*$-product, we find

$$\mathrm{d}x_t'^\mu * f_t = (\partial_\nu x'^\mu)_t(\mathrm{d}x_t^\nu * f_t) + \mathrm{d}t\left(\partial_t x_t'^\mu + \frac{\gamma}{2}(1 - 2\alpha)(\partial^2 x'^\mu)_t\right) f_t,$$

$$f_t * \mathrm{d}x_t'^\mu = (\partial_\nu x'^\mu)_t(f_t * \mathrm{d}x_t^\nu) + \mathrm{d}t\left(\partial_t x_t'^\mu + \frac{\gamma}{2}(1 - 2\alpha)(\partial^2 x'^\mu)_t\right) f_t. \tag{5.5}$$

It follows that despite of the noncovariance of the coordinate differential the commutator $[f_t, \mathrm{d}x_t^\mu]_*$ is covariant (for all values of α). Hence,

$$[f_t, \mathrm{d}x_t^\mu]_* = \gamma g_t^{\mu\nu}(\partial_\nu f)_t\, \mathrm{d}t \tag{5.6}$$

(of which the last relation in (5.2) is a special case) is coordinate invariant. In the following section, we will therefore take this commutator as the basic formula and develop a noncommutative differential calculus.

6. Noncommutative Differential Calculus

The rules to deal with stochastic differentials are very different from those of the calculus of differential forms on a manifold. However, in Section 3 we have seen that there is a kind of translation to a *noncommutative* differential calculus.

In Sections 3 and 5, a deformed commutation relation between functions and differentials emerged as a structure not explicitly depending on the parameter α. This 'universality' suggests to regard these commutation relations as the basic structure and not refer to the definition of $*$. (In order to stress this we will suppress the symbol $*$ in this section.) It turns out that all relevant formulae, in particular the Itô formula (see below), are consequences of this stucture.

Let us recall what is meant by a *differential calculus* on an associative algebra $\mathscr{A}$. This involves the following structures. There is an associative $\mathbb{Z}$-graded algebra

$$\bigwedge(\mathscr{A}) = \bigoplus_{r=0}^{\infty} \overset{r}{\bigwedge}(\mathscr{A}), \qquad \overset{0}{\bigwedge}(\mathscr{A}) = \mathscr{A}.$$

$\bigwedge^r(\mathscr{A})$ is a vector space over $\mathbb{C}$. The elements of $\bigwedge^r(\mathscr{A})$ are called *r-forms*. There is a $\mathbb{C}$-linear *exterior derivative* operator d: $\bigwedge^r(\mathscr{A}) \to \bigwedge^{r+1}(\mathscr{A})$ which satisfied $\mathrm{d}^2 = 0$ and the (graded) Leibniz rule

$$\mathrm{d}(\omega\omega') = (\mathrm{d}\omega)\omega' + (-1)^r \omega\, \mathrm{d}\omega', \tag{6.1}$$

where ω and ω' are r- and r'-forms, respectively. $(\bigwedge(\mathscr{A}), \mathrm{d})$ is called a *differential algebra.*

In the case under consideration, the algebra $\mathscr{A}$ is simply the algebra of smooth functions $f(t, x)$ on $\mathbb{R} \times M$, where M is a manifold.

We take (5.2) as the basic formulae for a noncommutative analogue of the ordinary calculus of differential forms on the algebra $\mathscr{A}$, i.e.★

$$[t, \mathrm{d}t] = 0, \qquad [x^\mu, \mathrm{d}t] = 0, \qquad [t, \mathrm{d}x^\mu] = 0, \qquad [x^\mu, \mathrm{d}x^\nu] = \gamma g^{\mu\nu}\, \mathrm{d}t. \tag{6.2}$$

This is indeed compatible with the requirements of a differential calculus. (This calculus has been studied in [7] with different motivations. If we replace $i\hbar$ in that paper by γ, then we an take all the formulae over to the case under consideration.) *Right-partial derivatives* of a function f are introduced by

$$\mathrm{d}f = \mathrm{d}t\, \vec{\partial}_t f + \mathrm{d}x^\mu\, \vec{\partial}_\mu f. \tag{6.3}$$

★Here and in the rest of this section we omit the indices t.

$\vec{\partial}_t$ and $\vec{\partial}_\mu$ are operators $\mathscr{A} \to \mathscr{A}$. They are determined by $\vec{\partial}_t t = 1$, $\vec{\partial}_\mu x^\nu = \delta^\nu_\mu$, and their action on a product of two functions. The latter can be derived from the derivation property of the exterior derivative d. One finds (see also [7])

$$\vec{\partial}_\mu = \partial_\mu, \qquad \vec{\partial}_t = \partial_t + \frac{\gamma}{2} g^{\mu\nu} \partial_\mu \partial_\nu. \tag{6.4}$$

Now (6.3) becomes

$$\mathrm{d}f = \mathrm{d}t \left(\partial_t + \frac{\gamma}{2} g^{\mu\nu} \partial_\mu \partial_\nu \right) f + \mathrm{d}x^\mu \, \partial_\mu f. \tag{6.5}$$

We see that the Itô formula is simply a consequence of the differential calculus with (6.2). The deformation of the ordinary differential calculus expressed by the last relation in (6.2) may therefore be regarded as replacing coordinates (and functions) on a manifold by stochastic processes.

The coordinate invariance of the differential calculus can be demonstrated directly as follows

$$\begin{aligned}
[x'^\mu, \mathrm{d}x'^\nu] &= [x'^\mu, \mathrm{d}x^\lambda \, \partial_\lambda x'^\nu + \mathrm{d}t \, \vec{\partial}_t x'^\nu] \\
&= [x'^\mu, \mathrm{d}x^\lambda] \, \partial_\lambda x'^\nu \\
&= [x^\lambda, \mathrm{d}x'^\mu] \, \partial_\lambda x'^\nu \\
&= [x^\lambda, \mathrm{d}x^\kappa] (\partial_\kappa x'^\mu)(\partial_\lambda x'^\nu) \\
&= \gamma \, \mathrm{d}t \, g^{\kappa\lambda} (\partial_\kappa x'^\mu)(\partial_\lambda x'^\nu).
\end{aligned} \tag{6.6}$$

The relations (6.2) are thus indeed coordinate invariant if g transforms like a tensor.

Acting with d on the last relation of (6.2), using the Leibniz rule and $\mathrm{d}^2 = 0$, we obtain

$$\mathrm{d}x^\mu \, \mathrm{d}x^\nu + \mathrm{d}x^\nu \, \mathrm{d}x^\mu = -\gamma \, \mathrm{d}g^{\mu\nu} \, \mathrm{d}t. \tag{6.7}$$

This shows that the (*-) product of 1-forms is in general not given by the usual wedge product (of the ordinary differential calculus on a manifold).

7. *-Product for Stochastic Differential Forms

After the reformulation of the calculus of stochastic differentials into a (noncommutative) differential calculus, the resulting framework naturally incorporates higher grade forms. This raises the question of what corresponds to them in the Itô calculus and whether these forms have a stochastic interpretation. We do not yet have a complete answer to these questions. But we believe that the following discussion contributes to clarifying these points.

It is natural to consider 'stochastic r-forms' ($r > 1$) simply as products of stochastic 1-forms using the ordinary wedge product (we will suppress the wedge sign, however). In this section we construct a realization of the $*$-product of differential forms in terms of the exterior algebra of stochastic forms.

Let Φ_t denote the 'stochastization map' $f(x^\mu, t) \mapsto f(x_t^\mu, t) =: f_t$. We would like to extend it as a linear map to the algebra of ordinary differential forms on a manifold.

For an ordinary differential form ϕ we set

$$\mathrm{d}x^\mu \cdot \phi := \gamma\, \mathrm{d}t\, g^{\mu\nu}\, \partial_\nu \lrcorner\, \phi \tag{7.1}$$

where $\lrcorner$ is the interior product operator.

Furthermore, we introduce a connection which is compatible with $g^{\mu\nu}$, i.e. $\nabla_\rho g^{\mu\nu} = 0$. It satisfies $\nabla_\rho(f\phi) = (\partial_\rho f)\phi + f\nabla_\rho\phi$. We will assume that the connection has vanishing torsion. Then $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$, where $\nabla_\nu\, \mathrm{d}x^\mu =: -\Gamma^\mu_{\rho\nu}\, \mathrm{d}x^\rho$. As a consequence, $\mathrm{d}\phi = \mathrm{d}x^\mu \nabla_\mu \phi$.

Now we define the action of Φ_t on differential forms recursively by

$$\begin{aligned}\Phi_t(\psi\phi) := (\Phi_t\psi)(\Phi_t\phi) - (\beta + \tfrac{1}{2})\Phi_t(\nabla_\rho\psi)\Phi_t(\mathrm{d}x^\rho\cdot\phi) - \\ - (\alpha - \tfrac{1}{2})\Phi_t(\mathrm{d}x^\rho\cdot\psi)\Phi_t(\nabla_\rho\phi),\end{aligned} \tag{7.2}$$

where $\beta = \alpha - 1$. This is defined in such a way that we have consistency with the algebraic structure of the algebra of differential forms. Indeed,★

$$\Phi_t(\psi\phi - (-1)^{rs}\phi\psi) = \Phi_t(\psi\phi) - (-1)^{rs}\Phi_t(\phi\psi) = 0, \tag{7.3}$$

$$\Phi_t(\psi(\phi\omega) - (\psi\phi)\omega) = \Phi_t(\psi(\phi\omega)) - \Phi_t((\psi\phi)\omega) = 0, \tag{7.4}$$

where ψ and ϕ are r- and s-forms, respectively. In (7.4) one has to use the identities

$$\nabla_\rho(\psi\phi) = (\nabla_\rho\psi)\phi + \psi(\nabla_\rho\phi), \tag{7.5}$$

$$\mathrm{d}x^\rho\cdot(\psi\phi) = (\mathrm{d}x^\rho\cdot\psi)\phi + \psi(\mathrm{d}x^\rho\cdot\phi) \tag{7.6}$$

and the fact that terms with more than one '$\cdot$' vanish identically (since $\mathrm{d}t\,\mathrm{d}t = 0$).

For an exterior derivative d acting on stochastic forms, we require★★

$$\mathrm{d}\Phi_t\psi := \Phi_t\, \mathrm{d}\psi. \tag{7.7}$$

Applying Φ_t to (7.1), where $\mathrm{d}x^\nu$ is taken for ϕ, we recover formula (5.1) for the product of two stochastic differentials.

There is an additional consistency condition,

$$\begin{aligned} 0 &= \Phi_t(\mathrm{d}(\psi\phi) - (\mathrm{d}\psi)\phi - (-1)^r\psi\, \mathrm{d}\phi) \\ &= \mathrm{d}\Phi_t(\psi\phi) - \Phi_t((\mathrm{d}\psi)\phi) - (-1)^r\Phi_t(\psi\, \mathrm{d}\phi).\end{aligned} \tag{7.8}$$

★We need $\beta = \alpha - 1$ for the first of these equations to hold.

★★Note that this equation involves two different d's, the ordinary exterior derivative on the right-hand side and a stochastic d on the left-hand side.

This further determines the action of the stochastic d. For a product of two functions, this together with (7.2) implies

$$\begin{aligned} \mathrm{d}(f_t h_t) &= \mathrm{d}\Phi_t(fh) = \Phi_t((\mathrm{d}f)h) + \Phi_t(f\,\mathrm{d}h) \\ &= (\mathrm{d}f_t)h_t + f_t\,\mathrm{d}h_t - (\alpha+\beta)\gamma\,\mathrm{d}t\, g_t^{\mu\nu}(\partial_\mu f)_t(\partial_\nu h)_t \end{aligned} \tag{7.9}$$

which is the modified Leibniz rule of stochastic calculus. More generally, we obtain the following result from (7.2) and (7.8). It extends (7.9) for the stochastic d acting on functions to arbitrary differential forms. To simplify the notation, we will write ψ_t instead of $\Phi_t\psi$.

LEMMA

$$\begin{aligned} \mathrm{d}(\psi_t\phi_t) = (\mathrm{d}\psi_t)\phi_t + (-1)^r\psi_t\,\mathrm{d}\phi_t + \gamma(\alpha+\beta)\,\mathrm{d}t\,\times \\ \times\,[(-1)^r\Omega^{\mu\nu}(\partial_\mu \lrcorner\, \psi)(\partial_\nu \lrcorner\, \phi) - g^{\mu\nu}(\nabla_\mu\psi)(\nabla_\nu\phi)]_t \end{aligned} \tag{7.10}$$

where $\Omega^\mu_\nu = \frac{1}{2}R^\mu_{\nu\rho\sigma}\,\mathrm{d}x^\rho\,\mathrm{d}x^\sigma$ *is the curvature 2-form of the connection* Γ.

Proof. First we note that $(\psi\phi)_t = \psi_t\phi_t$ if either ψ or ϕ contains $\mathrm{d}t$. As a consequence of (7.7), the Leibniz rule then holds for d acting on such an expression. Applied to (7.2) this yields

$$\begin{aligned} \mathrm{d}(\psi\phi)_t = \mathrm{d}(\psi_t\phi_t) - (\beta+\tfrac{1}{2})[\mathrm{d}(\nabla_\rho\psi)(\mathrm{d}x^\rho\cdot\phi) + (-1)^r(\nabla_\rho\psi)\,\mathrm{d}(\mathrm{d}x^\rho\cdot\phi)]_t - \\ - (\alpha-\tfrac{1}{2})[\mathrm{d}(\mathrm{d}x^\rho\cdot\psi)(\nabla_\rho\phi) + (-1)^r(\mathrm{d}x^\rho\cdot\psi)\,\mathrm{d}(\nabla_\rho\phi)]_t. \end{aligned} \tag{7.11}$$

On the other hand, we have

$$\begin{aligned} \mathrm{d}(\psi\phi)_t = \mathrm{d}(\psi_t)\phi_t - (\beta+\tfrac{1}{2})[(\nabla_\rho\,\mathrm{d}\psi)(\mathrm{d}x^\rho\cdot\phi) + (-1)^r(\nabla_\rho\psi)(\mathrm{d}x^\rho\cdot\mathrm{d}\phi)]_t + \\ + (-1)^r\psi_t\,\mathrm{d}\phi_t - (\alpha-\tfrac{1}{2})[(\mathrm{d}x^\rho\cdot\mathrm{d}\psi)(\nabla_\rho\phi) + \\ + (-1)^r(\mathrm{d}x^\rho\cdot\psi)(\nabla_\rho\,\mathrm{d}\phi)]_t \end{aligned} \tag{7.12}$$

using (7.8) and (7.2). Hence,

$$\begin{aligned} &\mathrm{d}(\psi_t\phi_t) - (\mathrm{d}\psi_t)\phi_t - (-1)^r\psi_t\,\mathrm{d}\phi_t \\ &\quad = (\beta+\tfrac{1}{2})[(\mathrm{d}\nabla_\rho\psi - \nabla_\rho\,\mathrm{d}\psi)(\mathrm{d}x^\rho\cdot\phi) + (-1)^r(\nabla_\rho\psi)(\mathrm{d}(\mathrm{d}x^\rho\cdot\phi) - \mathrm{d}x^\rho\cdot\mathrm{d}\phi)]_t + \\ &\qquad + (\alpha-\tfrac{1}{2})[(\mathrm{d}(\mathrm{d}x^\rho\cdot\psi) - \mathrm{d}x^\rho\cdot\mathrm{d}\psi)(\nabla_\rho\phi) + (-1)^r(\mathrm{d}x^\rho\cdot\psi)(\mathrm{d}\nabla_\rho\phi - \nabla_\rho\mathrm{d}\phi)]_t. \end{aligned} \tag{7.13}$$

In order to further evaluate this formula, we recall some identities,

$$\mathrm{d}\nabla_\rho\phi = \mathrm{d}x^\sigma(\nabla_\sigma\nabla_\rho\phi + \Gamma^\lambda_{\rho\sigma}\nabla_\lambda\phi), \tag{7.14}$$

$$\nabla_\rho\,\mathrm{d}\phi = \mathrm{d}x^\sigma\nabla_\rho\nabla_\sigma\phi, \tag{7.15}$$

$$(\nabla_\sigma\nabla_\rho - \nabla_\rho\nabla_\sigma)\phi = -R^\mu_{\nu\sigma\rho}\,\mathrm{d}x^\nu(\partial_\mu \lrcorner\, \phi). \tag{7.16}$$

Using the first Bianchi identity, they imply

$$\mathrm{d}\nabla_\rho\phi - \nabla_\rho\,\mathrm{d}\phi = -\Omega^\mu_\rho(\partial_\mu \lrcorner\, \phi) + \mathrm{d}x^\nu\,\Gamma^\mu_{\nu\rho}\nabla_\mu\phi. \tag{7.17}$$

Furthermore, using the metric compatibility of the connection, we have

$$d(A \cdot \phi) = dx^\sigma(\nabla_\sigma A \cdot \phi + A \cdot \nabla_\sigma \phi) \tag{7.18}$$

for a 1-form A. With this identity, we find

$$d(dx^\rho \cdot \phi) - dx^\rho \cdot d\phi = -dx^\sigma \Gamma^\rho_{\kappa\sigma}(dx^\kappa \cdot \phi) - \gamma\, dt\, \nabla^\rho \phi. \tag{7.19}$$

Inserting (7.17) and (7.19) in (7.13) leads to (7.10). □

As a consequence of (7.4), the map Φ_t defines an associative product on the algebra of stochastic forms: $\psi_t \tilde{*} \phi_t := (\psi\phi)_t$. But this is not the $*$-product we are looking for. In particular, it does not reproduce our formulae for the $*$-product between functions and differentials. We have to construct the $*$-product of differential forms in such a way that d becomes a derivation, i.e. we have to convert (7.10) into

$$d(\psi_t * \phi_t) = (d\psi_t) * \phi_t + (-1)^r \psi_t * d\phi_t. \tag{7.20}$$

Indeed, a deformation of the $\tilde{*}$-product (which corresponds to the shifts $\alpha \mapsto \alpha + \frac{1}{2}$ and $\beta \mapsto \beta - \frac{1}{2}$ in (7.2)) leads to an associative product with the required properties

$$\psi_t * \phi_t := \psi_t \phi_t - \beta(\nabla_\rho \psi)_t (dx^\rho \cdot \phi)_t - \alpha(dx^\rho \cdot \psi)_t (\nabla_\rho \phi)_t. \tag{7.21}$$

In the special cases where ϕ_t and ψ_t are 0- or 1-forms we recover our previous definition of the $*$-product and also the coordinate invariant equation (6.7).

8. Concluding Remarks

In this Letter, we have established a transformation of the calculus of stochastic differentials into a (noncommutative) differential calculus. The essential step was the introduction of a product between functions and differentials which is closely related to the Moyal product. In view of Moyal's original work [13], the appearance of the Moyal product should not come too much as a surprise in the context of stochastic processes.

The differential calculus which we obtained in this way allows a generalization of gauge theory and differential geometry in a straightforward way [7]. The essential formulae of gauge theory expressed in terms of differential forms can be taken over to the noncommutative calculus without changes. There are thus analogues of the common notions of Yang–Mills field strength, spacetime curvature, etc. [7]. It seems that these notions do not have a known counterpart in the usual stochastic framework (see Section 7, however). Whereas there is a clear stochastic interpretation of 1-forms, a clear interpretation for differential r-forms with $r > 1$ is still lacking. Are these related to (r-dimensional) random surfaces?

We should also mention that our noncommutative differential calculus appears as the 'classical limit' of bicovariant differential calculi on some quantum groups. This is briefly discussed in Appendix B.

There exists a generalization of the Itô calculus to noncommutative C^*-algebras replacing the algebra of smooth functions on a manifold (see [18], for example). Also, our noncommutative differential calculus remains consistent on some noncommutative algebras like Heisenberg or Weyl algebras.

The prospects of our reformulation of the calculus of stochastic differentials as a noncommutative differential calculus in the usual sense are thus manifold. It remains to be seen whether the new framework is helpful to understand and solve physical problems.

Appendix A: Associativity of the Generalized Moyal Product

Let

$$A := a\,\partial_p \otimes \partial_x + b\,\partial_x \otimes \partial_p \tag{A.1}$$

and let $\mathcal{M}$ denote the multiplication map $f \otimes h \mapsto fh$. Then we have the expression

$$f * h = \mathcal{M} e^A (f \otimes h) \tag{A.2}$$

for the $*$-product of Section 4. It is easily verified that

$$A\mathcal{M}_{23} = \mathcal{M}_{23}(A_{12} + A_{13}), \qquad A\mathcal{M}_{12} = \mathcal{M}_{12}(A_{13} + A_{23}). \tag{A.3}$$

(This acts on a threefold tensor product and the indices refer to the respective factors.) As a consequence, we have

$$e^A \mathcal{M}_{23} = \mathcal{M}_{23}\, e^{A_{12}+A_{13}} = \mathcal{M}_{23}(e^A)_{12}(e^A)_{13}, \tag{A.4}$$

$$e^A \mathcal{M}_{12} = \mathcal{M}_{12}\,(e^A)_{13}(e^A)_{23}. \tag{A.5}$$

Hence

$f * (g * h) = \mathcal{M} e^A$ [illegible]

[illegible]

$$\begin{aligned}
&= \mathcal{M}\mathcal{M}_{23}(e^A)_{12}(e^A)_{13}(e^A)_{23}(f \otimes g \otimes h)\\
&= \mathcal{M}\mathcal{M}_{12}(e^A)_{12}(e^A)_{13}(e^A)_{23}(f \otimes g \otimes h)\\
&= \mathcal{M}\mathcal{M}_{12}(e^A)_{13}(e^A)_{23}(e^A)_{12}(f \otimes g \otimes h)\\
&= \mathcal{M}\, e^A\, \mathcal{M}_{12}(e^A)_{12}(f \otimes g \otimes h)\\
&= (f * g) * h,
\end{aligned} \tag{A.6}$$

which proves the associativity of the $*$-product. The further terms $c\partial_p \otimes \partial_p + d\partial_x \otimes \partial_x$ with constants c and d can be added to A without spoiling associativity.

Appendix B: Generalized Galilei Structures and Differential Calculus on

on an n-dimensional manifold M a *generalized Galilei structure* on M. This generalizes the four-dimensional Galilean spacetime which is described by a 'space metric' tensor field $\gamma^{\mu\nu}$ with rank 3 and $\tau = \mathrm{d}t$ where t is the absolute time.

Let us consider the deformation of the ordinary differential calculus on M with

$$[x^\mu, \mathrm{d}x^\nu] = \tau\gamma^{\mu\nu}, \tag{B.2}$$

where $\tau = \mathrm{d}x^\mu \tau_\mu$. In particular, the commutation relations of the differential calculus of Section 6 can be expressed in this form. As a consequence of (B.1) and (B.2), the 1-form τ commutes with all functions. Introducing *right-partial derivatives* by

$$\mathrm{d}f = \mathrm{d}x^\mu \vec{\partial}_\mu f \tag{B.3}$$

the derivation rule for d implies

$$\vec{\partial}_\mu(fg) = (\vec{\partial}_\mu f)g + f(\vec{\partial}_\mu g) - \tau_\mu \gamma^{\kappa\lambda}(\vec{\partial}_\kappa f)(\vec{\partial}_\lambda g). \tag{B.4}$$

On C^2-functions, $\vec{\partial}_\mu$ is represented by

$$\vec{\partial}_\mu = \partial_\mu + \tfrac{1}{2}\tau_\mu \gamma^{\kappa\lambda}\partial_\kappa \partial_\lambda, \tag{B.5}$$

where ∂_μ are the ordinary partial derivatives. This expression is uniquely determined by (B.4) and $\vec{\partial}_\mu x^\nu = \delta^\nu_\mu$ which follows from the definition of the right-partial derivatives.

For a coordinate transformation we obtain

$$[x'^\mu, \mathrm{d}x'^\nu] = [x^\kappa, \mathrm{d}x^\lambda](\vec{\partial}_\kappa x'^\mu)(\vec{\partial}_\lambda x'^\nu). \tag{B.6}$$

If τ is invariant, this implies

$$\gamma'^{\mu\nu} = \gamma^{\kappa\lambda}(\vec{\partial}_\kappa x'^\mu)(\vec{\partial}_\lambda x'^\nu) = \gamma^{\kappa\lambda}\partial_\kappa x'^\mu \partial_\kappa x'^\mu \partial_\lambda x'^\nu \tag{B.7}$$

and

$$\tau_\mu = \tau'_\nu \vec{\partial}_\mu x'^\nu. \tag{B.8}$$

The last two equations ensure that (B.1) is coordinate invariant.

Differentiation of $[f, \tau] = 0$ leads to

$$[f, \mathrm{d}\tau] + [\mathrm{d}f, \tau]_+ = 0 \tag{B.9}$$

where $[\ ,\ ]_+$ denotes an anti-commutator. Furthermore, acting with d on (B.2) gives

$$\mathrm{d}x^\mu\, \mathrm{d}x^\nu + \mathrm{d}x^\nu\, \mathrm{d}x^\mu = \mathrm{d}\tau\, \gamma^{\mu\nu} - \tau\, \mathrm{d}\gamma^{\mu\nu}. \tag{B.10}$$

Commuting x^κ through (B.10) using (B.2) and (B.9) yields

$$[\tau, \mathrm{d}x^{(\kappa}]_+ \gamma^{\mu\nu)} + \tau^2 \gamma^{\lambda(\kappa}\partial_\lambda \gamma^{\mu\nu)} = 0. \tag{B.11}$$

Contracting (B.10) with $\tau_\mu \tau_\nu$ from the right and using (B.1) implies

$$\tau^2[\gamma^{\kappa\lambda}\gamma^{\mu\nu}(\partial_\kappa \tau_\mu)(\partial_\lambda \tau_\nu) + 2] = 0. \tag{B.12}$$

In general, (B.2) need not be consistent with the rules of differential calculus. The last two equations are corresponding consistency conditions. There are additional conditions, but we do not attempt at a general analysis of the consistency problem here. Rather, we will add another example (for which the consistency is already known) to the one of Section 6.

The structure (B.2) appears in the classical ($q \to 1$) limit of bicovariant differential calculus [4] on the quantum group $GL_q(2)$ [17]. In that case, the x^μ are the four entries of a $GL(2, \mathbb{C})$-matrix, regarded as (complex) functions on the group, and the differential calculus depends on a complex parameter s★

$$\gamma^{\mu\nu} = -\mathscr{D}^{-1} x^\mu x^\nu + 4(\delta_1^{(\mu}\delta_4^{\nu)} - \delta_2^{(\mu}\delta_3^{\nu)}), \tag{B.13}$$

$$\tau = s(x^4\,\mathrm{d}x^1 - x^3\,\mathrm{d}x^2 - x^2\,\mathrm{d}x^3 + x^1\,\mathrm{d}x^4). \tag{B.14}$$

Here $\mathscr{D} := x^1 x^4 - x^2 x^3$ denotes the determinant of the $GL(2, \mathbb{C})$-matrix. Acting with d on the expression for τ and using (B.10), one finds $\mathrm{d}\tau = 0$ if $s \neq \frac{1}{3}$. Also for $s = \frac{1}{3}$ the bicovariant calculus on $GL_q(2)$ leads to $\mathrm{d}\tau = 0$ in the classical limit [17]. Formula (B.9) then implies that τ anticommutes with all 1-forms and thus $\tau^2 = 0$ in particular. The consistency conditions (B.11) and (B.12) are therefore satisfied.

For a specific value of the parameter s, the bicovariant differential calculus on $GL_q(2)$ can be consistently restricted to $SL_q(2)$. This happens when $s = 1/(1 + q + q^2)$. For the classical limit this means $s = \frac{1}{3}$. If we solve the constraint $\mathscr{D} = 1$ for one of the functions, say x^4, and try to eliminate it in τ, we find that τ remains as a nonvanishing 1-form which can *not* be written as a linear combination of the 1-forms $\mathrm{d}x^i$. For $SL(2, \mathbb{R})$ we then obtain

$$[x^i, \mathrm{d}x^j] = \tau g^{ij} \quad (i, j = 1, 2, 3), \tag{B.15}$$

where g^{ij} turns out to be the natural group metric [17]. As a consequence, we have the Itô formula

$$\mathrm{d}f = \mathrm{d}x^i\,\partial_i f + \tfrac{1}{2}\tau g^{ij}\,\partial_i\,\partial_j f \tag{B.16}$$

for a function $f(x^i)$ on $SL(2, \mathbb{R})$.

All this establishes a relation between (the classical limit of) bicovariant differential calculus on quantum groups and stochastic geometry.

Acknowledgements

A. D. would like to thank C. Tzanakis for pointing out the similarity between the calculus in [7] and the Itô stochastic calculus. He also acknowledges discussions with E. Floratos.

★There are actually *two* branches of 1-parameter families of differential calculi. Here we only consider one of them.

References

1. Connes, A., Non-commutative differential geometry, *Publ. IHES* **62**, 257 (1986);
   Dubois-Violette, M., Dérivations et calcul différentiel non commutatif, *CR Acad. Sci. Paris Série I* **307** 403 (1988);
   Coquereaux, R., Noncommutative geometry and theoretical physics, *J. Geom. Phys.* **6**, 425 (1989);
   Manin, Yu. I., *Topics in Noncommutative Geometry*, Princeton University Press, Princeton, 1991.
2. Jaffe, A., Quantum physics as non-commutative geometry, in K. Schmüdgen (ed), *Mathematical Physics X*, Springer, Berlin, 1992, p. 281.
3. Bellissard, J., Ordinary quantum Hall effect and non-commutative cohomology, in W. Weller and P. Ziesche (eds), *Localization of Disordered Systems*, Teubner, Leipzig, 1988.
4. Woronowicz, S. L., Differential calculus on compact matrix pseudogroups (quantum groups), *Comm. Math. Phys.* **122**, 125 (1989).
5. Connes, A. and Lott, J., Particle models and noncommutative geometry, *Nuclear Phys. B* (*Proc. Suppl.*) **18**, 29 (1990);
   Chamseddine, A. H., Felder, G., and Fröhlich, J., Grand unification in noncommutative geometry, Preprint ETH/TH/92–41;
   Dubois-Violette, M., Kerner, R., and Madore, J., Noncommutative differential geometry and new models of gauge theory, *J. Math. Phys.* **31**, 323 (1990);
   Coquereaux, R., Esposito-Farèse, G., and Scheck, F., Noncommutative geometry and graded algebras in electroweak interactions, *J. Mod. Phys. A* **7**, 6555 (1992).
6. Dimakis, A. and Müller-Hoissen, F., Quantum mechanics on a lattice and q-deformations, *Phys. Lett. B* **295**, 242 (1992);
   Dimakis, A., Müller-Hoissen, F., and Striker, T., From continuum to lattice theory via deformation of the differential calculus, *Phys. Lett. B* **300**, 141 (1993); Noncommutative differential calculus and lattice gauge theory, to appear in *J. Phys. A.*
7. Dimakis, A. and Müller-Hoissen, F., Noncommutative differential calculus, gauge theory and gravitation, Preprint GOET-TP 33/92.
8. Arnold, L., *Stochastic Differential Equations*, Wiley, New York, 1974.
9. Guerra, F., Structural aspects of stochastic mechanics and stochastic field theory, *Phys. Rep.* **77**, 263 (1981);
   Damgaard, P. H. and Hüffel, H., Stochastic quantization, *Phys. Rep.* **152**, 227 (1987); Namiki, M., *Stochastic Quantization*, Lect. Notes in Phys. m9, Springer, Berlin, 1992.
10. Nelson, E., Derivation of the Schrödinger equation from Newtonian mechanics, *Phys. Rev.* **150**, 1079 (1966); *Dynamical Theories of Brownian Motion*, Princeton University Press, Princeton, 1967; *Quantum Fluctuations*, Princeton University Press, Princeton, 1985.
11. Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., and Sternheimer, D., Deformation theory and quantization, *Ann. Phys.* **111**, 61 and 111 (1978).
12. Meyer, P. A., A differential geometric formalism for the Ito calculus, in D. Williams (ed), *Stochastic Integrals*, Lecture Notes in Mathematics 851, Springer, Berlin, 1981, p. 256; Meyer, P.-A., Qu'est ce qu'une différentielle d'ordre n? *Exposition Math.* **7**, 249 (1989).
13. Moyal, J. E., Quantum mechanics as a statistical theory, *Camb. Phil. Soc.* **45**, 99 (1949).
14. Dunne, G. V., Quantum canonical invariance - A Moyal approach, *J. Phys. A.* **21**, 2321 (1988).
15. Fletcher, P., The uniqueness of the Moyal algebra, *Phys. Lett. B* **248**, 323 (1990).
16. Emery, M., *Stochastic Calculus in Manifolds*, Springer, Berlin, 1989.
17. Müller-Hoissen, F., Differential calculi on the quantum group $\mathrm{GL}_{p,q}(2)$, *J. Phys. A* **25**, 1703 (1992);
    Müller-Hoissen, F., and Reuten, C., Bicovariant differential calculus on $\mathrm{GL}_{p,q}(2)$ and quantum subgroups, to appear in *J. Phys. A.*
18. Hudson, R. L. and Robinson, P., Quantum diffusions and the noncommutative torus, *Lett. Math. Phys.* **15**, 47 (1988).