Notes on modeling the London equation using FEM
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1 The model system

We'd like to model a superconductor subject to a magnetic field. The superconductor can have various geometries; currently, we're mostly interested in a slab geometry (a rectangular parallelepiped with sides ℓ and w, and height d), or various ring geometries. The latter include a circular disk with a hole in the center, or a square plate with thickness d, also with a centered hole. The typical length scales of these objects is from a few to a few tens of microns, with hole sizes ranging from slightly submicron up to a few microns.

The so-called London model (see below) is appropriate for these superconductors. The London model applies when the coherence length ξ (the distance over which the magnitude of the superconductivity varies) is very short; in the materials of interest, ξ is only a few nanometers. The only parameter in the London equations is then the penetration length λ, the typical distance over which magnetic fields and currents can vary. This value can vary from perhaps 0.1 µm in conventional superconductors up to several microns in various exotic superconductors. In addition, the penetration length is temperature dependent, diverging at the critical temperature of the superconductor. This means that magnetic fields can penetrate appreciably into small superconductors of the kind we are studying.

Figure 1 shows the idea for a bulk sample that is much larger than λ. For the sphere shown, the external magnetic field falls off exponentially with characteristic length λ. Thus the interior of the superconductor is *screened* from the field. This expulsion of an external field by a superconductor is called the *Meissner effect.* In order to effect this screening there must be currents, also falling off as λ, flowing around the surface of the sample. So a bulk superconductor exhibiting the Meissner effect has zero magnetic fields and currents inside; the fields and currents are only non-zero in a thin layer at the surface.

Note, however, that if the sample is comparable in size to λ, like ours are, then fields and currents can penetrate appreciably into the interior.

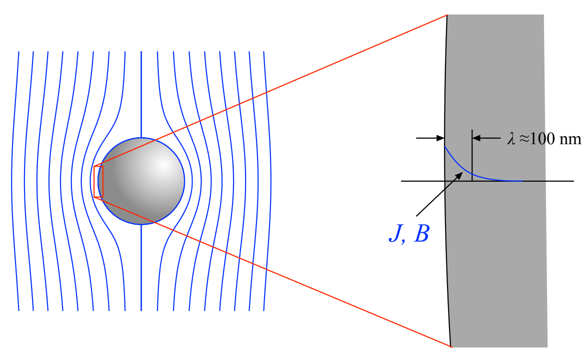

Figure 1: In the Meissner state, the magnetic field inside a bulk superconductor is zero; the fields and currents exist only in a thin surface layer.

1.1 Vortices

Superconductors can support magnetic structures called *vortices*, which are thin threads or filaments of magnetic flux that penetrate the superconductor. We can think of a vortex as a straight flux tube that penetrates the sample; magnetic field lines enter at one end of the tube (where the tube meets the sample surface), thread through the vortex, and leave at the other surface. One interesting aspect of vortices is that every vortex, independent of the particular superconducting material, contains the same amount of flux, namely the *flux quantum*

$$\Phi_0 = \frac{h}{2e} \approx 2.07\times10^{-15}\,\mathrm{T}\cdot\mathrm{m}^2 = 20.7\,\mathrm{G}\cdot\mu\mathrm{m}^2.$$

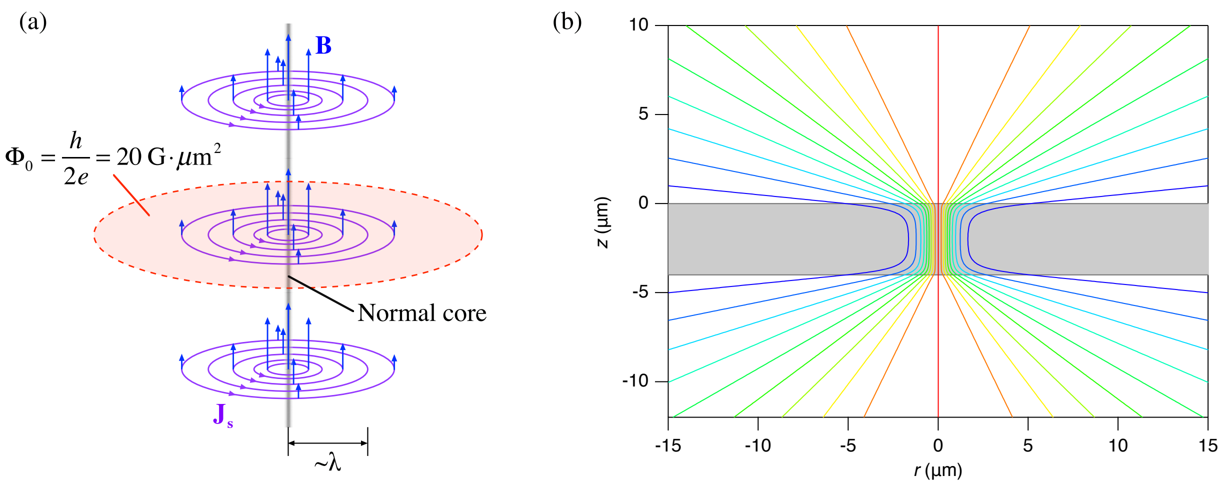

Figure 2: (a) Schematic of a long, straight vortex. (b) Magnetic field lines of a vortex in a film of thickness 4 µm.

Figure 2a is a schematic of a long, straight vortex as would exist in a bulk sample. There is a narrow, non-superconducting core of radius $\sim\xi$ surrounded by a circulating supercurrent and a magnetic field that fall off exponentially with distance from the core. Again, the characteristic length for this falloff is λ. The flux through a surface large enough such that J_s and B are ~ 0 is Φ_0. In a thin film geometry, as in Fig. 2b, the fields and currents near the core still fall of rapidly, but further from the core there is a power-law $\sim 1/r$ falloff. For not-too-thin films, the magnetic field above the surface is approximately that of a magnetic dipole located a distance λ below the surface.

2 The London Equation

The London equation is a phenomenological equation that describes the response of a superconductor to a magnetic field. As mentioned above, it is applicable in the case where the coherence length is very short compared to both the penetration length and the size of the sample. Both of these conditions hold well for our materials. (There are actually two London equations, but the other one is only needed in the case where the current and fields are varying in time.) The London equation can be written in a number of ways, because there are connections (from Ampère's law) between currents and fields.

2.1 The London Equation in J and B

The most basic version of London's equation is

$$\boxed{\mu_0\lambda^2\nabla\times\mathbf{J_s}+\mathbf{B}=\Phi_0\delta(\mathbf{r})\hat{\mathbf{z}}.} \tag{1}$$

Here, μ_0 is the permittivity of free space, $\mathbf{J_s}$ the supercurrent density, $\mathbf{B}$ the magnetic field, $\delta(\mathbf{r})$ the two-dimensional delta function, and $\hat{\mathbf{z}}$ the unit vector in the z-direction (or whichever direction the vortex is directed). The right-hand side of this equation represents the contribution of a vortex in the sample; if there is no vortex in the sample, then the right-hand side is zero.

We can interpret this equation as follows. Recall Ampère's law,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J_s}. \tag{2}$$

This equation is generally interpreted as "currents cause magnetic fields." We note that London's equation has a similar mathematical form, but with the roles of $\mathbf{J_s}$ and $\mathbf{B}$ reversed. Thus we can interpret London's equation as "in superconductors, magnetic fields cause currents." The application of even a static field to a superconductor causes persistent currents to flow in the superconductor.

2.2 The Fluxoid

London's equations (including the first one, which deals with time-varying fields) imply an important relationship between the current and magnetic flux. This relationship will be useful in studying non-simply-connected superconductors (i.e., superconductors with a hole). Consider an arbitrary closed integration path Γ entirely within the superconductor (but possibly encircling a hole). Then London defined the *fluxoid* Φ' as

$$\Phi' = \mu_0 \lambda^2 \oint_\Gamma (\mathbf{J_s}) \cdot d\mathbf{s} + \Phi = n\Phi_0, \tag{3}$$

where $\Phi = \int \mathbf{B} \cdot d\mathbf{a}$ is the magnetic flux through Γ, and n is an integer.

For a simply connected superconductor, without a hole, the fluxoid equals zero (i.e., $n = 0$). But if the superconductor contains a hole, then a circulating supercurrent can be trapped in the hole. The fluxoid Φ' in this case is equal to $n\Phi_0$, with $n > 0$.

Note that for a thick superconductor, the current inside the material far from the surface is zero. For such a superconductor with a hole, a path Γ can be chosen such that $\mathbf{J_s} = 0$ everywhere on Γ. In this case, $\Phi' = \Phi = n\Phi_0$. In this case it is the *flux* that is quantized. A vortex, with its non-superconducting core, can also be thought of as a supercurrent encircling a "hole." Thus the total flux of a vortex is quantized as $n\Phi_0$. In general, vortices with $n = 1$ are favored energetically.

2.3 The London Equation in B

Equation 1 can be written in terms of the magnetic field by using Ampère's law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J_s}. \tag{4}$$

Inserting $\mathbf{J_s}$ from Eq. 4 into Eq. 1 gives

$$\mu_0 \lambda^2 \nabla \times (\nabla \times \mathbf{B}/\mu_0) + \mathbf{B} = \Phi_0 \delta(\mathbf{r})\hat{\mathbf{z}} \tag{5}$$

or

$$\nabla \times \nabla \times \mathbf{B} + \frac{\mathbf{B}}{\lambda^2} = \frac{\Phi_0}{\lambda^2}\delta(\mathbf{r})\hat{\mathbf{z}}. \tag{6}$$

We can now apply the vector identity

$$\nabla \times \nabla \times \mathbf{a} = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}. \tag{7}$$

Because $\nabla \cdot \mathbf{B} = 0$ for any magnetic field, Eq. 6 reduces to

$$\boxed{\nabla^2 \mathbf{B} - \frac{\mathbf{B}}{\lambda^2} = -\frac{\Phi_0}{\lambda^2}\delta(\mathbf{r})\hat{\mathbf{z}}} \tag{8}$$

when a vortex is present or

$$\boxed{\nabla^2 \mathbf{B} - \frac{\mathbf{B}}{\lambda^2} = 0} \tag{9}$$

without a vortex.

2.4 The London Equation in A

The London equation can also be written in terms of the vector potential $\mathbf{A}$, defined by $\nabla \times \mathbf{A} = \mathbf{B}$. We can do so by rewriting Eq. 1 (without the source term for now) as

$$\mu_0 \lambda^2 \nabla \times \mathbf{J_s} + \nabla \times \mathbf{A} = 0 \tag{10}$$

or

$$\nabla \times \left(\mathbf{J_s} + \frac{\mathbf{A}}{\mu_0 \lambda^2} \right) = 0. \tag{11}$$

If the curl of a vector function is zero, that function must be expressible as the gradient of a scalar function, so we can write

$$\mathbf{J_s} + \frac{\mathbf{A}}{\mu_0 \lambda^2} = \nabla \chi(\mathbf{r}), \tag{12}$$

where $\chi(\mathbf{r})$ is a scalar function of position [2].

The London gauge

Equation 12 shows that, because of the arbitrary function ξ, there are an infinite number of potentials $\mathbf{A}$ In Eq. 12, $\mathbf{J_s}$ is a physical quantity that is independent of the choice of the function ξ.

We know that the vector potential $\mathbf{A}$ is defined only up to the gradient of a scalar potential, while $\mathbf{J_s}$ is a physical quantity independent of the gauge chosen. Thus choosing the function χ is equivalent to making the choice of the gauge of $\mathbf{A}$. One convenient choice of gauge is equivalent to choosing $\nabla \cdot \mathbf{A} = 0$, the so-called *London gauge*. With this choice we have

$$\boxed{\mathbf{J_s} + \frac{\mathbf{A}}{\mu_0 \lambda^2} = 0.} \tag{13}$$

In this gauge, $\mathbf{J_s}$ and $\mathbf{A}$ are *proportional* to each other. Thus $\nabla \cdot \mathbf{A} = 0$ implies $\nabla \cdot \mathbf{J_s} = 0$, a result we anticipate from the continuity equation.

This choice also imposes a boundary condition for $\mathbf{A}$ at the surface of the superconductor. We know that no current passes through the surface, so that $\mathbf{n} \cdot \mathbf{J_s} = 0$. The equivalent boundary condition on $\mathbf{A}$ is then $\mathbf{n} \cdot \mathbf{A} = 0$.

Equation 13 is not convenient for calculation because it contains two fields, $\mathbf{J_s}$ and $\mathbf{A}$. We can rewrite Eq. 13 in terms of the vector potential alone by using Ampère's law. We have

$$\frac{1}{\mu_0} \nabla \times \mathbf{B} + \frac{\mathbf{A}}{\mu_0 \lambda^2} = \frac{1}{\mu_0} \nabla \times \nabla \times \mathbf{A} + \frac{\mathbf{A}}{\mu_0 \lambda^2} = 0, \tag{14}$$

or

$$\nabla \times \nabla \times \mathbf{A} + \frac{\mathbf{A}}{\lambda^2} = 0 \tag{15}$$

If there is a vortex present, then we need a source term. This is [3]

$$\boxed{\nabla \times \nabla \times \mathbf{A} + \frac{\mathbf{A}}{\lambda^2} = \frac{\Phi_0}{2\pi \lambda^2 r} \hat{\phi}.} \tag{16}$$

Here, we assume a straight vortex oriented in the z-direction. Then r is the cylindrical-coordinate distance from the vortex, and $\hat{\phi}$ the unit vector in the tangential direction.

Equation 16 can be written in another form that is more useful for numerical work. We use the vector identity of Eq. 7 to write Eq. 16 as

$$\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}+\frac{\mathbf{A}}{\lambda^2}=\frac{\Phi_0}{2\pi\lambda^2 r}\hat{\phi}, \tag{17}$$

or, because we have chosen $\nabla\cdot\mathbf{A}=0$,

$$-\nabla^2\mathbf{A}+\frac{\mathbf{A}}{\lambda^2}=\frac{\Phi_0}{2\pi\lambda^2 r}\hat{\phi}, \tag{18}$$

or

$$\boxed{\nabla^2\mathbf{A}-\frac{\mathbf{A}}{\lambda^2}=-\frac{\Phi_0}{2\pi\lambda^2 r}\hat{\phi},} \tag{19}$$

again with the auxiliary condition $\nabla\cdot\mathbf{A}=0$.

3 Boundary Conditions

Finding a solution the equations just devised requires the specification of appropriate boundary conditions. There are actually two boundaries to be considered: The surface of the sample itself, and the outer boundary of the larger space (a vacuum) in which the sample is embedded. Let's consider these independently.

3.1 Boundary Conditions at the Sample Surface

Whether we are using the field $\mathbf{B}$ or $\mathbf{A}$, the boundary condition at the surface is that $\mathbf{J_s}\mathbf{A}\cdot\mathbf{n}=0$, where $\mathbf{n}$ is the unit vector normal to the surface. This boundary condition is somewhat complicated in the $\mathbf{B}$ representation, as it implies (from Ampère's law) that

$$(\nabla\times\mathbf{B})\cdot\mathbf{n}=0. \tag{20}$$

The situation is simpler for the $\mathbf{A}$ representation, where we have

$$\mathbf{A}\cdot\mathbf{n}=0, \tag{21}$$

but again we need to somehow enforce the condition $\nabla\cdot\mathbf{A}=0$.

3.2 Boundary Conditions at the Outer Boundary

The boundary condition on the distant outer boundary will depend on what kind of system we're trying to model. Let's look at a few examples.

3.2.1 Uniform magnetic field

Consider a sample in a uniform magnetic field $\mathbf{B}_0$, with an outer boundary that consists of the faces of a cube. Then we can use the boundary condition $B_z=B_0,\ B_x=0,\ B_y=0$ on all faces. Similarly, in the $\mathbf{A}$-representation we have $A_x=-B_0y,\ A_y=0,\ A_z=0$. Because of gauge freedom, there are other reasonable choices for $\mathbf{A}$ in this case.

3.2.2 Sample with a hole

A superconductor with a hole can support a trapped flux with no external field applied. How to enforce a fixed amount of flux in such a sample? As noted previous, a current circulating around a hole will create a dipole field at large distances. Thus we can enforce that the field (or vector potential) on the distant outer boundary has the correct dipolar form. The magnetic field and vector potential of a dipole can be written most conveniently in spherical coordinates. The magnetic field components are

$$B_r = \frac{2m\cos\theta}{r^3} \tag{22}$$

$$B_\theta = \frac{m\sin\theta}{r^3}, \tag{23}$$

where $m = IA$ is the magnetic moment of a current loop with current I and area$A$.

The vector potential only has a component in the ϕ-direction, given by

$$A_\phi(r,\theta) = \frac{m\sin\theta}{r^2}. \tag{24}$$

What value should be chosen for m? The idea is to choose some arbitrary value (say, $m = 1$) and calculate the current in the ring. Then calculate the fluxoid from Eq. 3. Then scale m so that the fluxoid equals $n\Phi_0$.

References

[1] F. London, *Superfluids* Vol./ 1, pp.70–73 (Wiley, New York, 1950).

[2] Philippe Mangin and Rémi Kahn, *Superconductivity, an Introduction*, Springer (2017).

[3] G. Carneiro and E. Brandt, Phys. Rev. B **61**, 6370 (2000).