A SPECTRAL THEORETIC PROOF OF PERRON–FROBENIUS

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Abstract

An elementary proof of the famous Perron–Frobenius Theorem for non-negative matrices is given via classical spectral theory. The key to the proof lies in the inherited characteristics of certain spectral projections.

1. Preliminaries

In this paper T will be a linear operator on a complex finite-dimensional vector space. We shall formally denote its matrix relative to a fixed basis by $[T]$ and the actual element in row i, column j by $[T]_{ij}$. Where there is no danger of confusion the square brackets are omitted. The main diagonal vector of $[T]$ is denoted by $diag(T)$. The *spectrum* or set of eigenvalues of T is denoted by $\sigma(T)$ and the largest modulus amongst its eigenvalues, the *spectral radius of* T, is denoted by $r(T)$. We use $P(\lambda; T)$ to denote the *spectral projection* associated with the eigenvalue λ relative to the operator T. The *peripheral spectrum* $\pi(T)$ is the set of eigenvalues of T with modulus $r(T)$, and the number of elements in $\pi(T)$ is called the *index of imprimitivity* of T.

An important attribute of any matrix $T \geq 0$ is its *potency*, namely the smallest positive integer c such that $diag(T^c) > 0$. If no such number exists then T is said to be *impotent*. Note that a potent matrix always has a positive spectral radius.

The matrix $T \geq 0$ is *connected* if for every i, j there exists n such that $[T^n]_{ij} > 0$. Clearly if $T \geq 0$ is connected then T is potent and $r(T) > 0$. For to every i there is an $n(i)$ such that $[T^{n(i)}]_{ii} > 0$, and the product of the $n(i)$ is then always a possible candidate (and therefore an upper bound) for the potency.

The following three technical lemmas lay the foundations of our proof. For background the reader is assumed to have a working knowledge of basic spectral theory such as may be found in [1].

Lemma 1. *If* $T \geq 0$ *and* $diag(T) > 0$ *then* $\pi(T) = \{r(T)\}$.

Proof. Clearly $r(T) > 0$ so we can assume without loss of generality that $r(T) = 1$. Let $C = \{e^{i\theta}; \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\}$. Fix n and let ε be the smallest entry on the diagonal of T^n. Then $\varepsilon > 0$. Since reducing the diagonal elements clearly cannot increase the spectral radius, we have $r(T^n - \varepsilon 1) \leq r(T^n) = 1$ and $\sigma(T^n - \varepsilon 1)$ is just $\sigma(T^n)$ shifted left by ε so T^n cannot have eigenvalues on C. As n is arbitrary, $\pi(T) = \{1\}$ by the spectral mapping theorem. ■

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Lemma 2 (the peripheral projection). *Suppose that $T \geq 0$ is connected with potency c and spectral radius r. The spectral projection (called the peripheral projection of T) given by $P = P(\pi(T); T)$ satisfies the following:*

(i) $TP = PT$ *and* $P = P(r^c; T^c)$*;*
(ii) $P \geq 0$*;*
(iii) $T^c P = r^c P$*;*
(iv) $diag(P) > 0$.

Proof. The results of (i) are standard properties of spectral projections. Since $r > 0$ we may assume, using appropriate scaling, that $r = 1$. Then $\pi(T^c) = \{1\}$ by Lemma 1, and by the usual spectral decomposition $T^c = U \oplus V$ where $U = T^c|PX$ and $V = T^c|(1-P)X$. If v is chosen such that $r(V) < v < 1$ then ultimately the norm of V^n is bounded by v^n and so $V^n \to 0$ as $n \to \infty$. Now suppose that 1 is a pole of T^c of order $q+1$ where $q \geq 1$. Then $U = 1 + N$ where $N^q \neq 0, N^{q+1} = 0$, so if $n \geq q$ then $U^n = \Sigma_0^q \, {}^nC_k N^k$, which is a polynomial in n of degree q, the coefficient of n^q being $N^q/q!$, which is a non-zero operator, W say. Moreover, it follows immediately that $UW = W$ and $W^2 = 0$. Then $n^{-q}U^n \to W$ and writing $E = W \oplus 0$ shows that $n^{-q}T^{nc} = n^{-q}U^n \oplus n^{-q}V^n \to E$. Note that $E \geq 0$ since $T \geq 0$ and also $TE = ET$. Now $E^2 = W^2 \oplus 0 = 0$. It follows that $ET^nE = 0$ for all n. But if $[E]_{ij} \neq 0$ then by connectedness there would exist n such that $[T^n]_{ji} \neq 0$, which would mean that $ET^nE \neq 0$, which is false. Therefore 1 is a simple pole of T^c and $U = 1$. Hence $T^cP = P$, which proves (iii). Note that $T^{nc} = 1 \oplus V^n \to 1 \oplus 0 = P$ so, relative to the original basis, P is real and $P \geq 0$, proving (ii). Finally, to prove (iv) we show $[P]_{kk} > 0$. Since $P \geq 0$ is non-zero choose any $[P]_{ij} > 0$. By connectedness choose m and n such that $[T^m]_{ki} > 0$ and $[T^n]_{jk} > 0$. Then $[T^{m+n}P]_{kk} = [T^mPT^n]_{kk} > 0$ by positivity, and raising this to the cth power gives $[T^{(m+n)c}P]_{kk} > 0$. But $T^{(m+n)c}P = P$, hence $[P]_{kk} > 0$. ■

That the peripheral projection is non-negative is perhaps hardly surprising as one might expect it to inherit at least something from the original matrix. By contrast, the positivity of its diagonal is quite unexpected and indeed is the key that unlocks our proof.

Lemma 3 (the child operator). *Let $T \geq 0$ be connected with spectral radius r, peripheral projection P, and index of imprimitivity h. Then the child operator of T given by $R = TP$ satisfies:*

(i) $\pi(R) = \pi(T) = \sigma(R)\backslash\{0\}$ *and* $P = P(\pi(R); R)$; *also* $P(\lambda; R) = P(\lambda; T)$ *for all* $\lambda \in \pi(R)$*;*
(ii) $R \geq 0$ *is connected;*
(iii) *if (for any n) an entry in $[R^n]$ is zero the corresponding entry in $[T^n]$ is also zero;*
(iv) *if $h|n$ then $diag(R^n) > 0$; otherwise $diag(R^n) = 0$;*
(v) *if $\lambda^h = r^h$ then $P(\lambda; R) = h^{-1}\Sigma_1^h \lambda^{-k}R^k$;*
(vi) $R^h = r^hP$ *and* $R^{h+1} = r^hR$*;*
(vii) *for positive integers m, n if $[R^n]_{ij} > 0$ then $[R^m]_{ji} > 0 \Longleftrightarrow h|(m+n)$;*
(viii) *for positive integers m, n if $[R^n]_{ij} > 0$ then $[R^m]_{ij} > 0 \Longleftrightarrow h|(m-n)$.*

PROOF. (i) is standard. Also R is real ≥ 0 by Lemma 2(ii). As $R^n = T^n P$ we have $[R^n]_{ij} = [T^n P]_{ij} \geq [T^n]_{ij}[P]_{jj}$ and using $diag(P) > 0$ shows that R is connected because T is, whilst if $[R^n]_{ij} = 0$ then $[T^n]_{ij} = 0$, proving (ii) and (iii). Let b be the potency of R. We shall show $b = h$. Applying Lemma 2 to R and using (i) gives $R^b = r^b P$, so $b \geq h$ by the spectral mapping theorem. If $[R^n]_{jj} > 0$ by connectedness choose u, v with $[R^u]_{ij} > 0$ and $[R^v]_{ji} > 0$. Then $[R^{u+v}]_{ii} \geq [R^u]_{ij}[R^v]_{ji} > 0$ so $[R^u]_{ij}[R^n]_{jj}[R^v]_{ji}[R^{(u+v)(b-1)}]_{ii} > 0$; since $R^b = r^b P$ we have $R^{b(u+v)+n} = r^{b(u+v)} P R^n = r^{b(u+v)} R^n$ so $[R^n]_{ii} > 0$. This, combined with the definition of potency, shows $diag(R^n) = 0$ for $n < b$. Let $\lambda^b = r^b$ and write $Q = b^{-1}\Sigma_1^b \lambda^{-k} R^k$. Then Q is a projection as $R^b = r^b P$, and $QR = \lambda Q = RQ$ so Q reduces R. Now $(R-\lambda)|QX$ is zero, and also $(R-\lambda)(1-Q)x = 0 \Longrightarrow Rx = \lambda x \Longrightarrow Qx = x \Longrightarrow (1-Q)x = 0$ so $(R-\lambda)|(1-Q)X$ is one-to-one and hence invertible. Therefore $Q = P(\lambda; R) = P(\lambda; T)$. Also $Q \neq 0$ because $diag(Q) = b^{-1}\lambda^{-b} diag(R^b) > 0$ so $\lambda \in \pi(R)$. Hence all of the bth roots of r^b lie in $\pi(R)$ and so $h \geq b$. Thus $b = h$ and now substituting h for b in what has already been proved above yields (iv), (v) and (vi). Suppose now that $[R^n]_{ij} > 0$. By (iv) if $(m+n)$ is not divisible by h then $0 = [R^{(m+n)}]_{ii} \geq [R^n]_{ij}[R^m]_{ji} \geq 0$, hence $[R^m]_{ji} = 0$. On the other hand, as R is connected we know by (vi) that $[R^m]_{ji}$ is periodically positive so $[R^m]_{ji} > 0$ whenever $h|(m+n)$. This proves (vii) and an application of (vii) to itself establishes (viii). ■

2. The main results

Corollary 1. *If $T \geq 0$ then $r(T) \in \sigma(T)$.*

PROOF. If $\delta > 0$ then $\pi(T + \delta 1) = \{r(T + \delta 1)\}$ by Lemma 1. As $r(T + \delta 1) \geq r(T)$ it follows that $T + \delta 1$ has a real eigenvalue not less than $r(T)$, so T has an eigenvalue in the closed real interval $[r(T) - \delta, r(T)]$. Since δ is arbitrary, $r(T) \in \sigma(T)$. ■

This theme is explored in more detail in [3]. The eigenvalue $r(T)$ of a non-negative matrix T is often referred to as the *Perron root of T.* The spectral projection $P(r(T); T)$ corresponding to the Perron root is called the *Perron projection.*

We shall need the standard result that a positive projection always has rank one. As the proof is easy we include it for completeness. Let $P > 0$ be a projection and let u be its first column. Clearly $Pu = u$ and $u > 0$. Let v be any other column and λ be the largest real number such that $w = v - \lambda u \geq 0$. Then $Pw = w$ and at least one component, say number k, of w is zero so $\Sigma_j [P]_{kj} w_j = 0$, hence, by positivity, $w = 0$ and so every column of P is a multiple of the first one. Thus P has rank one as required.

Corollary 2. *If $T \geq 0$ is connected its Perron projection is positive and has rank one.*

PROOF. By Lemma 3(ii) and (v) we have $P(r; T) \geq h^{-1} r^{-1} R \geq 0$ so $P(r; T)$ is a connected projection, hence it must be positive. From the above it has rank one. ■

Theorem (Perron). *Suppose $T > 0$. Then $r(T)$ is the only point in the peripheral* *spectrum of T; the corresponding eigenspace is 1-dimensional and consists of multiples of a positive eigenvector. Also $r(T)$ is a simple root of the characteristic equation of T.*

Proof. Any $T > 0$ has potency 1 so $TP = rP$ by Lemma 2(iii). As $P \geq 0$ and $diag(P) > 0$ then, by positivity, $TP > 0$, thus $P > 0$ and must therefore have rank one. By Lemma 1, r is the only point in $\pi(T)$, hence the corresponding eigenspace is 1-dimensional and r is a simple root of the characteristic equation of T. ■

So far we have refrained from examining how $[P]$ and $[R]$ behave under permutations largely because there has been no pressing need to do so. However, such an examination is required in order to establish the deeper properties of T.

Let $T \geq 0$ be connected with peripheral projection P and child R. Our immediate objective is to find the best possible form for $[ETE^{-1}]$ where $[E]$ is a permutation matrix. Note that the application of a permutation matrix in this way is equivalent to permuting the basis of the underlying space and in no way affects either positivity or connectivity. Moreover, $[EPE^{-1}]$ is the peripheral projection of the non-negative connected matrix $[ETE^{-1}]$ and the corresponding child is $[ERE^{-1}]$. There are actually two steps involved in finding an optimal $[E]$. The first assigns a block structure to the various matrices; the second is a block-level refinement that reveals their cyclical nature.

Step 1

Define $i \sim j \iff [P]_{ij} > 0$. Then $\sim$ is reflexive by Lemma 2(iv), symmetric by Lemma 3(vi) and (vii), and transitive because if $i \sim j$ and $j \sim k$ then $[P]_{ik} = [P^2]_{ik} \geq [P]_{ij}[P]_{jk} > 0$. Hence $\sim$ is an equivalence relation. Suppose that there are b equivalence classes. Choose any permutation matrix $[E]$ that makes the elements of each equivalence class consecutive. Then the resultant projection $[EPE^{-1}]$ has b positive square blocks on its diagonal and every other block is zero. Moreover, it is the peripheral projection of the non-negative connected matrix $[ETE^{-1}]$ which is controlled by its child $[ERE^{-1}]$ by Lemma 3(iii) in that a zero entry in $[ERE^{-1}]$ forces the corresponding entry in $[ETE^{-1}]$ to be zero. This completes the first step.

Step 2

We now concentrate on the child matrix. For simplicity we assume the original basis to be optimal in the sense that the elements in each equivalence class are already consecutive so that no rearrangement becomes necessary. In other words, we will choose $E = 1$ in the first step. This assumption does not involve any loss of generality provided that we bear in mind that $[T]$ has already been subject to permutation. For $1 \leq k \leq h$ note that $R^k = PR^kP$ so, multiplying out, we see that every block of $[R^k]$ is either zero or positive. By Lemma 3(iv) it follows that $G = \{r^{-k}R^k\}_1^h$ is a cyclic group of order h with identity P. We claim that there are precisely b positive blocks in each $[R^k]$. Certainly there are at least b as $[R^k]$ cannot have any rows or columns of zero blocks. If $[R^k]$ had a row, say row j, containing two positive blocks, then by choosing i such that the (i, j)-th block of $[R]$ is positive we see, by positivity, that $[R^{k+1}]$ would also contain two positive blocks (in row i). But $[R^h]$ has only one positive block per row so our claim must be true. Now by Lemma 3(viii) if a block is positive in one element of G then it must be zero in all the others, so the Perron projection $h^{-1}\Sigma_1^h r^{-k}R^k$ contains hb positive blocks. On the other hand, the Perron projection is positive and therefore contains b^2 positive blocks. Hence $b = h$ and so $[R]$ has an $h \times h$ block form with exactly one positive block in every row and column; all the remaining blocks are zero. Since $r^{-1}R$ generates a h-cycle, by applying a further block-level permutation we can now arrange for all blocks of the permuted $[R]$ to be zero except for those immediately right of the main diagonal and in the bottom left corner, all of which will be positive. The construction is complete.

The argument above shows that $[P]$ is the direct sum of h positive projections. Since a positive projection necessarily has rank one, this implies that $rank(P) = h$. Now $R = TP$ so $rank(T) \geq rank(R) \leq h$ and as $h = rank(R^h) \leq rank(R)$ it follows that $rank(R) = h$.

The most common case is of course $h = 1$. Then there is only the one equivalence class so $P > 0$, and since $R = PRP$ straightforward multiplication shows that $R > 0$ as well. Conversely, if $R > 0$ then $P > 0$ by Lemma 3(vi), hence there is just the one equivalence class and so $h = 1$.

The following lemma summarises the above construction and subsequent observations.

Lemma 4. *Let $T \geq 0$ be connected with index of imprimitivity h, peripheral projection P, and child operator R. Then $h = rank(P) = rank(R) \leq rank(T)$; furthermore:*

(i) $h = 1 \Longleftrightarrow P > 0 \Longleftrightarrow R > 0$;

(ii) *if $h > 1$ then there exists a permutation matrix E such that EPE^{-1} has a $h \times h$ block form in which all diagonal blocks are positive and all off-diagonal blocks are zero; in ERE^{-1} the blocks immediately right of the main diagonal and in the bottom left corner are positive, and every other block is zero; in ETE^{-1} a block is zero if and only if it corresponds to a zero block in ERE^{-1}.*

Perron's Theorem originally appeared in 1907. In 1912 Frobenius extended it to connected non-negative matrices and it was this extension that ultimately became known as the Perron–Frobenius Theorem (see, for example, [2, 124]). The observant reader will notice that statement (IV) below is stronger than the version quoted there, namely that $\sigma(T)$ is invariant under rotation by an angle $2\pi/h$. Our result implies that spectral characteristics such as pole order and algebraic and geometric multiplicities are faithfully preserved under rotation. Used in conjunction with Corollary 2 it also shows that if $\lambda \in \pi(T)$ then $P(\lambda; T)$ has rank one and consequently that all points of the peripheral spectrum are simple roots of the characteristic equation of T.

Theorem (Perron–Frobenius). *Let $T \geq 0$ be connected with spectral radius r. Then:*

(I) *$r > 0$ is an eigenvalue and a simple root of the characteristic equation of T;*

(II) *there exists a positive eigenvector corresponding to r;*

(III) *if T has h eigenvalues of modulus r these are the h distinct roots of $z^h = r^h$;*

(IV) *if $\omega = e^{2\pi i/h}$ then T is similar to ωT;*

(V) *if $h > 1$ there exists a permutation matrix E such that ETE^{-1} has a representation consisting of h square blocks on the main diagonal and all blocks except those directly above the main diagonal and the one in the bottom left corner are zero.*

PROOF. By Corollary 2 the Perron projection is rank one, which proves (I); moreover, its first column is a positive eigenvector corresponding to the Perron root. This proves (II). If $\lambda^h = r^h$ then Lemma 3(iv) and (v) show that $P(\lambda; R) \neq 0$, therefore $\lambda \in \pi(R)$. Conversely, if $\lambda \in \pi(R)$ then $\lambda^{h+1} = r^h \lambda$ by Lemma 3(vi), so $\lambda^h = r^h$. Since $\pi(R) = \pi(T)$ this proves (III). (V) is part of Lemma 4. Finally, let $1 = 1_1 \oplus 1_2 \oplus \cdots \oplus 1_h$ be the block decomposition of the identity matrix. Let $\omega = e^{2\pi i/h}$ and write $D = \oplus_1^h \omega^{k-1} 1_k$. Clearly D is a diagonal matrix and $D^h = 1$. A simple multiplication check shows that $D^{-1}TD = \omega T$, proving (IV). ∎

3. Examples

An instructive and very simple example is given by the matrix

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then $\sigma(T) = \{0, 1\}$ so the peripheral and Perron projections are equal. In fact

$$P(1; T) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix},$$

which has rank 2. Therefore in this situation, in the absence of connectedness, the peripheral and Perron projections are equal, have rank 2 and are not non-negative.

We remark that by Lemma 3(iii) and (iv) the potency c of a connected non-negative matrix is divisible by, but not always equal to, h. For example, if

$$T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

then $\sigma(T) = \{2, -1\}$, so T is invertible, $h = 1$ and $c = 2$. Observe also that $rank(R) < rank(T)$. Moreover, (V) above shows that a positive trace always means $h = 1$. This trace-zero matrix has $h = 1$ and shows that the converse is false.

We conclude with some general observations about a connected non-negative matrix. The peripheral projection is real, non-negative and potent, whereas the Perron projection is real, positive and rank one. Of course the spectral projections of general eigenvalues are not normally real, but by taking non-real eigenvalues in conjugate pairs it is easy to see, using the standard complex integral formula, that the spectral projections of conjugate pairs of eigenvalues are real, and so too are the spectral projections corresponding to real eigenvalues. However, none of these other projections can be non-negative since they must all be orthogonal to the Perron projection.

It is natural to ask whether the positivity of all non-zero blocks of R induces any related property in T. The answer is that, although it implies that T has no zero rows or columns, it yields nothing new because this is a trivial property of any connected matrix. Matrices with this property are interesting objects of study in their own right. For example, it is well known that a non-negative projection with no zero rows or columns is a direct sum of positive projections. This could have largely replaced Step 1 of Lemma 4, but we have preferred to give a self-contained proof. In fact there is an interesting sequence of implications for non-negative matrices which seems worthy of further investigation:

$$\textit{positivity} \Longrightarrow \textit{connectedness} \Longrightarrow \textit{potency} \Longrightarrow \textit{no zero rows or columns.}$$

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