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Nonlinear Spinor Equation and Asymmetric Connection in General Relativity

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In order to take full account of spin in general relativity, it is necessary to consider space–time as a metric space *with* torsion, as was shown elsewhere. We treat a Dirac particle in such a space. The generalized Dirac equation turns out to be of a Heisenberg–Pauli type. The nonlinear terms induced by torsion express a universal spin–spin interaction of range zero.

1. INTRODUCTION

The realization that momentum and spin angular momentum are, in a certain sense, quantities of the same kind suggests a generalization of the general theory of relativity which encompasses spin angular momentum.[1–4] Momentum is a dynamic quantity which corresponds to translation, whereas spin angular momentum corresponds to rotation. If one further observes that in differential geometry the metric tensor field is related in a definite way to translations and the torsion tensor field to rotations, one is led to consider a geometry with curvature *and torsion* (Riemann–Cartan geometry) in place of the Riemannian geometry.[5,6]

The physical model now proposed is as follows: The matter-free space–time continuum has neither curvature nor torsion and therefore possesses Minkowskian structure. (This is merely a model. Strictly speaking, according to general relativity, space–time does not exist in the absence of matter.) Imagine that in this space–time continuum matter with momentum and spin angular momentum is introduced and distributed continuously. Owing to the influence of matter, curvature and torsion are produced. The curvature can be derived from metric and torsion in a well-known way. Hence, there should exist functional relationships between momentum and spin angular momentum on one side and metric and torsion on the other side. These relationships are the field equations (generalized Einstein field equations); the momentum and the spin angular momentum densities appear as the sources of the metric and torsion fields.

The solutions of these equations are of the following type: Given the densities of momentum and spin angular momentum, the metric and torsion fields are to be found. These solutions obviously touch only a part of the physical problem. Therefore, one should also be able to calculate the momentum and the spin angular momentum densities from the physical conditions. These conditions could perhaps be given in terms of the (classical) wave amplitude of matter on a spacelike hypersurface. What one still needs in the theory are the matter equations. (These are of course not entirely independent of the field equations.)

The field equations as well as the matter equations will be derived from a variational principle. In order to obtain the conventional form of general relativity in the case of vanishing spin, one introduces a Lagrangian density which is the sum of the field Lagrangian and the matter Lagrangian. The field Lagrangian density is determined by metric and torsion; the usual arguments of simplicity lead to the curvature density. The material Lagrangian density can be obtained from that of special relativity by minimal coupling to metric and torsion, i.e., the partial derivatives are replaced by the derivatives which are covariant with respect to the curved and contorted Riemann–Cartan space–time.

In this paper, we treat a classical *Dirac field* in the way discussed above; hence the matter field will be represented by a four-component spinor.

Note added in proof: The whole theory for arbitary matter fields is represented in an article by one of the authors (F. W. Hehl, "Spin und Torsion in der Allgemeinen Relativitätstheorie Oder die Riemann-Cartansche Geometrie der Welt," Habilitation thesis TU Clausthal, 1970). An English version has been submitted for publication to Fortsch Physik.

In accord with these preliminary remarks, we set up the theory in the following way.

In Sec. 2 we summarize all the geometrical apparatus necessary for our theory. In Sec. 3 we introduce in a well-known manner orthonormal tetrads as anholonomic coordinates in the space–time under consideration. This allows us to define the covariant differentiation of a spinor in a straightforward way.

In Sec. 4 we introduce the action function of a Dirac particle interacting with a gravitational field. We explicitly compute the additional terms characteristic for our non-Riemannian geometry. Through a variational procedure we deduce the field equations in Sec. 5 and the matter equations which constitute the generalized Dirac equation in Sec. 6. Eliminating the contortion in the Dirac equation, we arrive at a nonlinear spinor equation of the Heisenberg–Pauli type, thereby generalizing somewhat similar results obtained first by Rodichev.[7]

After the general theory, in Sec. 7 we work out and stress the difference between conventional general relativity and our non-Riemannian theory. The torsion terms in the action function and therefore the nonlinear term in the spinor equation are recognized as corresponding to a universal spin–spin contact interaction. We propose to regard this universal spin–spin interaction as a classical model of weak interaction.

2. RIEMANN–CARTAN GEOMETRY

As was shown elsewhere,[3,4] it is reasonable to assume for the affine connection of space–time the expression

$$\Gamma^{k}_{ij} = \{{}_{i}{}^{k}{}_{j}\} + S_{ij}{}^{k} - S_{j}{}^{k}{}_{i} + S^{k}{}_{ij}. \qquad (2.1)$$

In our notation we essentially follow the book of Schouten.[8] The physical conventions are taken from Landau-Lifshitz.[9] In (2.1), $\{{}_{i}{}^{k}{}_{j}\}$ is Christoffel's symbol of the second kind belonging to the metric g_{ij}. Cartan's torsion tensor is defined according to

$$S_{ij}{}^{k} = \tfrac{1}{2}(\Gamma^{k}_{ij} - \Gamma^{k}_{ji}) \equiv \Gamma^{k}_{[ij]}. \qquad (2.2)$$

Latin indices run from 0 to 3. With the contortion tensor

$$K_{ij}{}^{k} = -S_{ij}{}^{k} + S_{j}{}^{k}{}_{i} - S^{k}{}_{ij}, \qquad (2.3)$$

(2.1) can be written as

$$\Gamma^{k}_{ij} = \{{}_{i}{}^{k}{}_{j}\} - K_{ij}{}^{k}. \qquad (2.4)$$

We remark that (2.1) and therefore (2.4) are equivalent to the relation

$$\nabla_{i} g_{jk} = 0. \qquad (2.5)$$

The manifold equipped with a connection of the form (2.1) will be called a U_4 ("Riemann–Cartan space"). It is the most general metric space with a linear affine connection. For vanishing torsion we arrive at a Riemannian space V_4.

The Riemann–Christoffel curvature tensor is defined in the usual way as

$$R_{ijk}{}^{l} = 2\partial_{[i}\Gamma^{l}_{j]k} + 2\Gamma^{l}{}_{[i|m|}\Gamma^{m}_{j]k}. \qquad (2.6)$$

The first two identities of the curvature tensor are valid in each affine space:

$$\tfrac{1}{2}(R_{ijk}{}^{l} + R_{jik}{}^{l}) \equiv R_{(ij)k}{}^{l} = 0, \qquad (2.7)$$

$$R_{[ijk]}{}^{l} = 2\nabla_{[i}S_{jk]}{}^{l} - 4S_{[ij}{}^{m}S_{k]m}{}^{l}. \qquad (2.8)$$

The third identity for a U_4 reads

$$R_{ij(kl)} = 0. \qquad (2.9)$$

For these and other formulas the book of Schouten[8] should be referred to. Bianchi's identity is given by

$$\nabla_{[i}R_{jk]l}{}^{m} = 2S_{[ij}{}^{n}R_{k]nl}{}^{m}. \qquad (2.10)$$

We define the Ricci tensor $R_{ij} = R_{kij}{}^{k}$ and the Einstein tensor as

$$G_{ij} = R_{ij} - \tfrac{1}{2}g_{ij}R_{k}{}^{k}. \qquad (2.11)$$

For a V_4 (2.8) with (2.9) *inter alia* tells us that the antisymmetric part of the Einstein tensor vanishes; but not so for a U_4. We contract (2.8) and get

$$\tfrac{3}{2}R_{[kij]}{}^{k} = \overset{*}{\nabla}_{k}T_{ij}{}^{k}, \qquad (2.12)$$

where we have introduced a modified torsion tensor

$$T_{ij}{}^{k} = S_{ij}{}^{k} + \delta^{k}_{i}S_{jl}{}^{l} - \delta^{k}_{j}S_{il}{}^{l} \qquad (2.13)$$

and used the notation

$$\overset{*}{\nabla}_{i} = \nabla_{i} + 2S_{ik}{}^{k}. \qquad (2.14)$$

(2.12) together with (2.9) and (2.11) yields

$$\overset{*}{\nabla}_{k}T_{ij}{}^{k} - G_{[ij]} = 0. \qquad (2.15)$$

As we have shown,[4] (2.15) is the geometrical image of the angular momentum conservation theorem.

Bianchi's identity can also be written in a contracted form. Using (2.10), (2.11), and (2.13) we get

$$\overset{*}{\nabla}_{j}G_{i}{}^{j} + 2S_{ji}{}^{k}G_{k}{}^{j} = T_{jk}{}^{l}R_{il}{}^{jk}. \qquad (2.16)$$

Of course this relation corresponds to energy–momentum conservation; therefore the right-hand side will represent a certain volume force density apart from a dimensional factor.

3. TETRADS AS ANHOLONOMIC COORDINATES

Equation (2.1) determines the geometry of space–time. There is nothing in our formalism like an independent tetrad connection or similar entities often described in the literature. Thus there is no place for the so-called Palatini formalism in our theory. We prefer the method worked out for instance in Ref. 7. The independent geometrical quantities describing the U_4 are metric and torsion (or contortion). One is then able to introduce anholonomic coordinates. Let us choose at each point a tetrad $e_\alpha{}^i$. $\alpha = 1, 2, 3, 4$, numbers the four different and linearly independent vectors. Because we treat a metric space, it is convenient to use (pseudo-)orthonormal tetrads. This yields the relations ($e^\beta{}_j$ is the reciprocal of $e_\alpha{}^i$):

$$e_\alpha{}^i e^\alpha{}_j = \delta^i_j, \quad e_\alpha{}^i e^\beta{}_i = \delta^\beta_\alpha, \tag{3.1}$$

$$e_\alpha{}^i = g_{\alpha\beta} g^{ij} e^\beta{}_j, \quad g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}. \tag{3.2}$$

The object of anholonomity

$$\Omega^\gamma_{\alpha\beta} = e_\alpha{}^i e_\beta{}^j \partial_{[i} e^\gamma{}_{j]}, \quad \Omega_{\alpha\beta\gamma} = g_{\gamma\delta} \Omega^\delta_{\alpha\beta} \tag{3.3}$$

depends on the coordinates we chose.

The reason for introducing orthonormal tetrads is the following. As is well known, tensors are connected with the group of general coordinate transformations. Accordingly, there exists no necessity for introducing anholonomic coordinates for computations with tensors in the U_4. Spinors, however, are connected with the Lorentz group. We are compelled therefore to define at each point of the U_4 a tangent Minkowski space R_4 via the tetrads $e_\alpha{}^i$. This procedure cannot be circumvented in principle because of the very nature of spinors mentioned above. The details of this discussion will be worked out in a forthcoming publication.

The covariant derivative in anholonomic coordinates reads

$$\nabla_\alpha \psi = \partial_\alpha \psi + \Gamma^\gamma_{\alpha\beta} f_\gamma{}^\beta \psi. \tag{3.4}$$

Because of (3.1) and (3.2) the operator $f_\gamma{}^\beta$ describes the behavior of ψ under an infinitesimal Lorentz transformation δx^γ:

$$\delta\psi = \partial_\beta(\delta x^\gamma) f_\gamma{}^\beta \psi. \tag{3.5}$$

If we remember (3.1) and (3.2), the connection (2.1) expressed in anholonomic coordinates is given by

$$\Gamma_{\alpha\beta\gamma} = g_{\gamma\delta}\Gamma^\delta_{\alpha\beta} = -\Omega_{\alpha\beta\gamma} + \Omega_{\beta\gamma\alpha} - \Omega_{\gamma\alpha\beta} - K_{\alpha\beta\gamma}. \tag{3.6}$$

This yields the formulas

$$\Gamma_{\alpha(\beta\gamma)} = 0, \tag{3.7}$$

$$\Gamma_{[\alpha\beta\gamma]} = -\Omega_{[\alpha\beta\gamma]} - K_{[\alpha\beta\gamma]} \tag{3.8}$$

$$g^{\beta\gamma}\Gamma_{\beta\gamma\alpha} = -2\Omega^\beta_{\alpha\beta} + K_{\beta\alpha}{}^\beta. \tag{3.9}$$

Let us define for later use the scalar density

$$e = (-\det g_{ij})^{\frac{1}{2}} = \det(e^\alpha{}_i). \tag{3.10}$$

The definition of the determinant together with (3.1) leads to

$$(\partial_\alpha e)/e = e_\beta{}^k \partial_\alpha e^\beta{}_k. \tag{3.11}$$

Later we will also be concerned with the ordinary divergence of tetrads. With (3.1), (3.3), and (3.11) we arrive at

$$\partial_k e_\alpha{}^k = 2\Omega^\beta_{\alpha\beta} - e_\beta{}^k \partial_\alpha e^\beta{}_k = 2\Omega^\beta_{\alpha\beta} - (\partial_\alpha e)/e. \tag{3.12}$$

Let us now turn to 4-spinors, because we want to treat a Dirac particle in a U_4. From Corson,[10] for instance, we get

$$f_{\alpha\beta} = \tfrac{1}{4}\gamma_{[\alpha}\gamma_{\beta]} \tag{3.13}$$

with the well-known Dirac matrices. Covariant differentiation of a 4-spinor is hence given by

$$\nabla_\alpha \psi = \partial_\alpha \psi - \tfrac{1}{4}\Gamma_{\alpha\beta\gamma}\gamma^\beta\gamma^\gamma\psi. \tag{3.14}$$

The derivative of the Dirac adjoint $\psi^+ = \psi^*\gamma_4$ can be calculated easily:

$$\nabla_\alpha \psi^+ = \partial_\alpha \psi^+ - \tfrac{1}{4}\Gamma_{\alpha\beta\gamma}\psi^+\gamma^\gamma\gamma^\beta. \tag{3.15}$$

For later purpose we also introduce a covariant derivative with respect to a V_4:

$$\overset{\{\}}{\nabla}_\alpha \psi = \partial_\alpha \psi + \tfrac{1}{4}(\Omega_{\alpha\beta\gamma} - \Omega_{\beta\gamma\alpha} + \Omega_{\gamma\alpha\beta})\gamma^\beta\gamma^\gamma\psi. \tag{3.16}$$

4. ACTION FUNCTION OF A DIRAC PARTICLE INTERACTING WITH A GRAVITATIONAL FIELD

For the special relativistic Lagrangian density of a Dirac particle we use the usual expression in a R_4, given, for example, in Ref. 10. Instead of taking the partial derivatives, we substitute the covariant ones of (3.14) and (3.15) in the sense of minimal coupling to metric and torsion. This results in ($2\pi\hbar$ = Planck's constant, c = velocity of light, $\hbar m/c$ = mass of the electron)

$$\mathfrak{L} = -e(i\hbar c/2)[(\nabla_\alpha\psi^+)\gamma^\alpha\psi - \psi^+\gamma^\alpha\nabla_\alpha\psi + 2im\psi^+\psi]. \tag{4.1}$$

Hence, we assume that Pauli-type terms do not enter (4.1). Substituting (3.14) and (3.15) in (4.1) and noting (3.16), we have

$$\mathfrak{L} = -e(i\hbar c/2)[(\overset{\{\}}{\nabla}_\alpha\psi^+)\gamma^\alpha\psi - \psi^+\gamma^\alpha\overset{\{\}}{\nabla}_\alpha\psi + 2im\psi^+\psi] + e(i\hbar c/4)K_{\alpha\beta\gamma}\psi^+\gamma^{[\alpha|}\gamma^\beta\gamma^{|\gamma]}\psi. \tag{4.2}$$

The first term on the right-hand side is identically the same as one would get in a V_4 or already in a R_4 in curvilinear coordinates. The second term can be simplified using the formula

$$\gamma^\alpha\gamma^\beta\gamma^\gamma = \gamma^{[\alpha}\gamma^\beta\gamma^{\gamma]} + g^{\alpha\beta}\gamma^\gamma + g^{\beta\gamma}\gamma^\alpha - g^{\gamma\alpha}\gamma^\beta \tag{4.3}$$

or more specifically

$$\gamma^{[\alpha|}\gamma^\beta\gamma^{|\gamma]} = \gamma^{[\alpha}\gamma^\beta\gamma^{\gamma]}, \tag{4.4}$$

which follows from the anticommutation relations for the γ's. Thus, we have for (4.2)

$$\mathfrak{L} = \mathfrak{L}(\{\}) - e(i\hbar c/4)\psi^+\gamma^{[\gamma}\gamma^\beta\gamma^{\alpha]}\psi K_{\alpha\beta\gamma}. \quad (4.5)$$

A similar consideration leads to the equivalent expression

$$\mathfrak{L} = \mathfrak{L}(\partial_\alpha) + e(i\hbar c/4)\psi^+\gamma^{[\gamma}\gamma^\beta\gamma^{\alpha]}\psi\Gamma_{\alpha\beta\gamma}, \quad (4.6)$$

where $\mathfrak{L}(\partial_\alpha)$ is the Lagrangian (4.1), but with the difference that partial derivatives replace the covariant ones.

According to the general theory treated in Refs. 3 and 4, spin angular momentum $\tau_{ij}{}^k$ is coupled to the contortion of the U_4:

$$e\tau_k{}^{ji} = \delta\mathfrak{L}/\delta K_{ij}{}^k. \quad (4.7)$$

(4.5) and (4.7) yield

$$\tau^{\alpha\beta\gamma} = e^\alpha{}_i e^\beta{}_j e^\gamma{}_k \tau^{ijk} = \tau^{[\alpha\beta\gamma]} = -\frac{i\hbar c}{4}\psi^+\gamma^{[\alpha}\gamma^\beta\gamma^{\gamma]}\psi. \quad (4.8)$$

This is the well-known canonical spin angular momentum as required by the theory of Refs. 1, 2, and 4. It is totally antisymmetric and hence has only four independent components. Thus (4.5) and (4.6) can be written in the form

$$\mathfrak{L} = \mathfrak{L}(\{\}) + e\tau^{\gamma\beta\alpha}K_{\alpha\beta\gamma} = \mathfrak{L}(\partial_\alpha) - e\tau^{\gamma\beta\alpha}\Gamma_{\alpha\beta\gamma}. \quad (4.9)$$

(In spite of some other remarks in the literature, a material Lagrangian density generally contains a third term depending on the contortion. A formula of the type (4.9) is valid only if the spin is independent of contortion, as in Dirac's case. This remark is not crucial, however, because the mentioned third term represents only a correction.)

Let the field Lagrangian be given by

$$\mathfrak{R} = eR_k{}^k. \quad (4.10)$$

This is the only scalar density which can be derived from the curvature tensor by contraction. Additional torsion-dependent terms[4] destroy the simplicity of the theory. Thus (4.10) is suggested by analogy with conventional relativity. A straightforward but lengthy calculation reveals that (4.10) can be separated into a Riemannian and a torsion part:

$$\mathfrak{R} = \mathfrak{R}(\{\}) + \partial_i(2eK_k{}^{ik}) - eT_k{}^{ji}K_{ij}{}^k. \quad (4.11)$$

Since in what follows we do not vary on the boundary of the integration volume of the action function, we can forget the divergence in (4.11). The last term of (4.11) can be written in anholonomic coordinates as well.

Correspondingly the total action function is given by

$$W = \frac{1}{c}\int d\Omega\Big[\mathfrak{L}(\{\}) + e\tau^{\gamma\beta\alpha}K_{\alpha\beta\gamma} + \frac{1}{2k}\mathfrak{R}(\{\}) - \frac{e}{2k}T^{\gamma\beta\alpha}K_{\alpha\beta\gamma}\Big], \quad (4.12)$$

where $kc^4/8\pi$ is Newton's gravitational constant. Observe that we get two different additional terms characteristic for a U_4: a term coupling spin and contortion and a term quadratic in the contortion. (Using a tetrad formalism and quantizing a Dirac particle interacting with a gravitational field, Kibble[11] and Kannenberg and Arnowitt[12] subtracted out such terms, because they wanted to have the Riemannian result. The same is true for Lemmer,[13] who quantized a general matter field under the same conditions.)

Rodichev[7] was the first who stated an action function similar to (4.12). He treated the case of $\mathfrak{R}(\{\}) = 0$, which leads to a sort of a teleparallelism, and required the torsion to be totally antisymmetric *a priori*, which is not necessary in our theory. See also Braunss[14] for an interpretation of Rodichev's results. For vanishing spin the theory presented here simplifies to ordinary general relativity; furthermore the constant in front of the last term of (4.12) is specified. For related papers with action functions resembling (4.12), in which torsion is considered more as a secondary concept, see Peres,[15] Lenoir,[16] and Wainwright.[17,18]

5. FIELD EQUATIONS

We must now vary (4.12) with respect to $e^\alpha{}_i$ and $K_{ij}{}^k$ and equate the variations to zero as required by Hamilton's principle. This yields

$$\frac{\delta\mathfrak{L}}{\delta e^\alpha{}_i}e^\beta{}_i \overset{\text{def}}{=} e\sigma_\alpha{}^\beta = -\frac{1}{2k}\frac{\delta\mathfrak{R}}{\delta e^\alpha{}_i}e^\beta{}_i, \quad (5.1)$$

$$\frac{\delta\mathfrak{L}}{\delta K_{ij}{}^k} \overset{\text{def}}{=} e\tau_k{}^{ji} = -\frac{1}{2k}\frac{\delta\mathfrak{R}}{\delta K_{ij}{}^k}. \quad (5.2)$$

Here the left-hand side of (5.1) is by definition the metric energy–momentum density $e\sigma_\alpha{}^\beta$ and the left-hand side of (5.2) the spin angular momentum density according to (4.7).

Notice that $\mathfrak{R}$ depends on $e^\alpha{}_i$ only via g_{kl}. Thus we can use the variational method worked out in detail in Ref. 4. If we define the canonical energy–momentum tensor according to

$$\Sigma_{\alpha\beta} \overset{\text{def}}{=} \sigma_{\alpha\beta} + \overset{*}{\nabla}_\gamma(\tau_{\alpha\beta}{}^\gamma - \tau_\beta{}^\gamma{}_\alpha + \tau^\gamma{}_{\alpha\beta}) \quad (5.3)$$

(see Ref. 4), the above-mentioned procedure applied to (5.1) leads to the equations

$$G_{\alpha\beta} = k\Sigma_{\alpha\beta}. \quad (\text{A})$$

Using (4.12) and (5.2) we immediately have

$$T_{\alpha\beta\gamma} = k\tau_{\alpha\beta\gamma}. \tag{5.4}$$

(A) and (5.4) are the field equations which are generally valid in this form.

Using now (4.8) we see that in the Dirac case discussed here the contortion tensor is totally antisymmetric and the second set of field equations can be specialized to

$$T^{\alpha\beta\gamma} = T^{[\alpha\beta\gamma]} = S^{[\alpha\beta\gamma]} = -K^{[\alpha\beta\gamma]} = k\tau^{\alpha\beta\gamma}$$
$$= -(il^2/4)\psi^+\gamma^{[\alpha}\gamma^{\beta}\gamma^{\gamma]}\psi. \tag{B}$$

Here we have introduced $l^2 = \hbar ck$ or $l \approx 10^{-32}$ cm.

This results in a very special U_4: Only four independent components of $K_{\alpha\beta\gamma}$ are "excited" by a Dirac particle, and this indicates the relatively simple nature of such a particle. It is of course possible to introduce the axial vector corresponding to $\tau_{\alpha\beta\gamma}$ making explicit the independence of only four components. We use the well-known relation (see Ref. 9, for instance)

$$i\gamma_5\gamma^\alpha = \frac{1}{3!}\epsilon^{\alpha\beta\gamma\delta}\gamma_\beta\gamma_\gamma\gamma_\delta. \tag{5.5}$$

[$\epsilon^{\alpha\beta\gamma\delta} = \pm 1/e$ if α, β, δ is an even (odd) permutation of 1, 2, 3, 4; otherwise 0.] Inverting (5.5) and substituting it in (B) yields

$$K_{\alpha\beta\gamma} = -k\tau_{\alpha\beta\gamma} = \tfrac{1}{4}l^2\epsilon_{\alpha\beta\gamma\delta}\psi^+\gamma_5\gamma^\delta\psi. \tag{5.6}$$

Squaring (5.5), we get for later use

$$\gamma^{[\alpha}\gamma^{\beta}\gamma^{\gamma]}\gamma_\alpha\gamma_\beta\gamma_\gamma = -6(\gamma_5\gamma^\alpha)(\gamma_5\gamma_\alpha). \tag{5.7}$$

6. GENERALIZED DIRAC EQUATION

We vary (4.12) with respect to ψ and ψ^+, and then the action principle yields the Euler–Lagrange equations

$$\frac{\partial\mathfrak{L}}{\partial\psi^{(+)}} - \partial_k\left(e_\alpha{}^k\frac{\partial\mathfrak{L}}{\partial(\partial_\alpha\psi^{(+)})}\right) = 0. \tag{6.1}$$

In view of (3.12) we get

$$\frac{\partial\mathfrak{L}}{\partial\psi^{(+)}} - \left(\partial_\alpha + 2\Omega^\beta_{\alpha\beta} - \frac{(\partial_\alpha e)}{e}\right)\frac{\partial\mathfrak{L}}{\partial\partial_\alpha\psi^{(+)}} = 0. \tag{6.2}$$

For convenience we now use the Lagrangian in the form (4.6). Computing the necessary partial derivatives, we obtain from (6.2)

$$[\gamma^\alpha\partial_\alpha - \tfrac{1}{4}\Gamma_{\alpha\beta\gamma}\gamma^{[\alpha}\gamma^{\beta}\gamma^{\gamma]} + \Omega^\beta_{\alpha\beta}\gamma^\alpha]\psi = im\gamma. \tag{6.3}$$

The partial derivative can be transformed into a derivative with respect to a V_4 or a U_4. Substituting (3.14) in (6.3) we have

$$[\gamma^\alpha\nabla_\alpha + \tfrac{1}{4}\Gamma_{\alpha\beta\gamma}(\gamma^\alpha\gamma^\beta\gamma^\gamma - \gamma^{[\alpha}\gamma^{\beta}\gamma^{\gamma]}) + \Omega^\beta_{\alpha\beta}\gamma^\alpha]\psi = im\psi. \tag{6.4}$$

With (4.3), (3.7), and (3.9) one is led to

$$\gamma^\alpha[\nabla_\alpha + \tfrac{1}{2}K_{\beta\alpha}{}^{\beta}]\psi = im\psi \tag{6.5}$$

and with (3.14), (3.6), (3.16), and (4.3) to

$$[\gamma^\alpha\overset{\{\}}{\nabla}_\alpha + \tfrac{1}{4}K_{\alpha\beta\gamma}\gamma^{[\alpha}\gamma^{\beta}\gamma^{\gamma]}]\psi = im\psi. \tag{C}$$

The comparison of this equation with the result in a V_4 is especially instructive.

With the help of (B) it is possible to eliminate the contortion everywhere. Substituting (B) in (C), we immediately have

$$[\gamma^\alpha\overset{\{\}}{\nabla}_\alpha + (il^2/16)(\psi^+\gamma^{[\alpha}\gamma^{\beta}\gamma^{\gamma]}\psi)\gamma_\alpha\gamma_\beta\gamma_\gamma]\psi = im\psi \tag{6.6}$$

or together with (5.7)

$$[\gamma^\alpha\overset{\{\}}{\nabla}_\alpha - \tfrac{3}{8}il^2(\psi^+\gamma_5\gamma^\alpha\psi)\gamma_5\gamma_\alpha]\psi = im\psi. \tag{C$'$}$$

This is a classical spinor equation of the Heisenberg–Pauli type and because of (6.5) it can be equivalently written as

$$\gamma^\alpha\nabla_\alpha\psi = im\psi. \tag{6.7}$$

It should be noted that we arrive at this simple equation only in view of (B).

In addition to the above-mentioned authors of Refs. 7 and 14–18, Gürsey[19] and Finkelstein[20] discussed nonlinear spinor equations, more or less resembling (C′), in space–times with torsion. They both worked with a connection allowing teleparallelism, and Finkelstein even used a space with constant torsion.

7. UNIVERSAL SPIN–SPIN INTERACTION OF RANGE ZERO

Let us now reflect on the difference between the theory presented here and usual general relativity. With the help of the field equations (B) we can collect the contortion terms in the Lagrangian scalar of (4.12) in one term

$$\tfrac{1}{2}\tau^{\gamma\beta\alpha}K_{\alpha\beta\gamma} = \tfrac{1}{2}\tau^{[\gamma\beta\alpha]}K_{\alpha\beta\gamma} = \tfrac{1}{2}k\tau^{[\alpha\beta\gamma]}\tau_{[\alpha\beta\gamma]}. \tag{7.1}$$

We recognize the interaction term (7.1), characteristic for a U_4, as a universal spin–spin contact interaction. That is to say there is nothing like a "spin field" which is emitted and which is the carrier of a new interaction; there is rather a very weak classical interaction as soon as any spinning matter is overlapping. This leads also to a certain self-interaction of spinning matter automatically introducing nonlinearities as in (C′). This interaction is very weak, as can be seen from the smallness of 1 in (C′). Hence this theory describes in a unified manner two universal interactions: the far reaching gravitational interaction and a weak spin–spin interaction of vanishing range. It is very tempting to regard such a theory as a *classical model* unifying gravitational and *weak interaction.*

For a Dirac particle according to (5.6) the canonical spin can be represented by an axial vector. A spin–spin interaction thus leads in this special case to an axial vector interaction and (7.1) can be rewritten, using (B) and (5.7), in the form

$$\tfrac{1}{2}\tau^{\gamma\beta\alpha}K_{\alpha\beta\gamma} = \frac{3l^4}{16k}(\psi^+\gamma_5\gamma_\alpha\psi)(\psi^+\gamma_5\gamma^\alpha\psi). \qquad (7.2)$$

(7.2) naturally corresponds to the nonlinear term entering the spinor equation (C′). It is easy to obtain a V–A interaction instead of (7.2) by modifying the matter Lagrangian in a suitable way.

Apart from all speculations, we have formally arrived at the result that the second term of (C′) can be derived from the first term just by using the connection (2.1) of a U_4 instead of the Christoffel symbol of a V_4. This leads to a deeper understanding of such nonlinear spinor equations and their connection with geometry of space–time.

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Phase-Plane Analysis of Nonlinear, Second-Order, Ordinary Differential Equations*

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Group invariance is used to analyze the solutions of several well-known differential equations.

INTRODUCTION

Nonlinear mechanics is the name generally given to the study of nonlinear, second-order, ordinary differential equations. The independent variable is usually interpreted as time. The principal tool in nonlinear mechanics is the phase plane, the plane in which the dependent variable and its first derivative are the coordinates. The phase plane is particularly useful for studying autonomous differential equations, i.e., those equations in which the independent variable does not appear. For them, the paths in the phase plane traced out by the solutions are independent of time and can be obtained from the solution of a first-order differential equation. In the event that the first-order equation cannot be solved in terms of elementary functions, which is frequently the case, its direction field can be sketched in the phase plane and the qualitative nature of the paths determined. From these paths, certain properties of the solutions of the original equation can be inferred: whether they are oscillatory or monotone, whether they are stable or unstable, whether they have any asymptotic limits, whether they have roots or singularities.

When the differential equation is not autonomous, the paths in the phase plane are no longer time-independent nor are they determined by a first-order differential equation. The phase plane loses much of its usefulness. However, if the differential equation is