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Further Remarks on the Localization of the Energy in the General Theory of Relativity

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The question of the localizability of the energy in gravitational fields is discussed once more. It is shown that the expression for the energy density, derived in an earlier paper γ, doet not have all the properties required for a satisfactory description of the energy distribution. An essential correction to one of the statements in the mentioned paper is made. A careful analysis of the desirable properties of the notions in question is carried through, the conclusion being that none of the expressions derived so far satisfies all the requirements necessary for a satisfactory description of the localization of the energy. At the end, it is indicated how the "four-leg" formulation of the gravitational theory might provide a solution of the problem.

About forty years ago, the question of the localizability of the energy in the general theory of relativity was extensively discussed in a series of papers by different authors, and in a concluding paper by Einstein [*1*] it was shown that the question had to be answered in the negative. While the energy-momentum complex established by Einstein furnishes a satisfactory expression for the total energy and momentum of a closed system in the form of integrals over the three-dimensional space, the integrand of the integral representing the total energy does not satisfy the requirements necessary for a physically acceptable definition of the energy distribution. In a recent paper [*2*], the present author suggested a different expression for the energy-momentum complex which seemed to provide a satisfactory *local* description of the energy distribution. In the present paper it will be shown, however, that globally the new expression gives rise to unexpected difficulties which seriously limit its applicability and the question of the localizability of the energy cannot yet be regarded as finally settled.

1. SURVEY OF THE PROBLEM AND OF EARLIER RESULTS

Einstein's energy-momentum complex $\theta_i^{\ k}$, which satisfies the divergence relation

$$\theta_i^{\ k}{}_{,k} \equiv \frac{\partial \theta_i^{\ k}}{\partial x^k} = 0, \tag{1}$$

can be written in different forms. In the first place, it can be given as the sum of a matter part and a gravitational part:

$$\theta_i^{\,k} = \sqrt{-g}(T_i^{\,k} + \vartheta_i^{\,k}). \tag{2}$$

Here $T_i^{\,k}$ is the matter tensor occurring as source in Einstein's field equations

$$R_i^{\,k} - \tfrac{1}{2}\delta_i^{\,k} R = -\kappa T_i^{\,k} \tag{3}$$

and $\vartheta_i^{\,k}$ is defined by the usual Lagrangian $\mathfrak{L}(g^{ik}, g^{ik}{}_{,l})$ as

$$\sqrt{-g}\vartheta_i^{\,k} = \frac{1}{2\kappa}\left\{\frac{\partial \mathfrak{L}}{\partial g^{lm}{}_{,k}} g^{lm}{}_{,i} - \delta_i^{\,k}\mathfrak{L}\right\}. \tag{4}$$

Since $\mathfrak{L}$ is a homogeneous function of degree 2 of the first-order derivatives $g^{ik}{}_{,l}$ of the metric tensor, the same obviously holds for the complex $\vartheta_i^{\,k}$.

Next, the matter tensor can be eliminated from (2) by means of (3), yielding $\theta_i^{\,k}$ as a function of the metric tensor and its derivatives of the first and second orders. In this way, $\theta_i^{\,k}$ turns out to be derivable from a "superpotential" $h_i^{\,kl}$ in the form[1]

$$\theta_i^{\,k} = h_i^{\,kl}{}_{,l} \tag{5}$$

with

$$h_i^{\,kl} = -h_i^{\,lk} = \frac{g_{in}}{2\kappa\sqrt{-g}}\,[(-g)(g^{kn}g^{lm} - g^{ln}g^{km})]_{,m}. \tag{6}$$

Now, consider a closed system in which the matter is confined to a finite part of space. In that case, we may assume that the 4-space is asymptotically flat for large spatial distances r from the system, and we can introduce "asymptotically rectilinear" coordinates for which the g_{ik}'s approach constant values for $r \to \infty$. Using such coordinates only, one can show the quantities

$$P_i = \frac{1}{c}\iiint_{x^4=\text{const.}} \theta_i^{\,4}\, dx^1 dx^2 dx^3 \tag{7}$$

to have the following important properties:

I. *The P_i are constant in time. They transform as the covariant components of a free 4-vector under linear transformations. Further, the P_i are unchanged under arbitrary transformations of the coordinates provided that the new system of coordinates coincides with the original system at large spatial distances.*

The well-known proof of I is based on the fact that $\theta_i^{\,k}$ is an affine tensor density (of weight one) satisfying the divergence relation (1).

For a closed system at rest as a whole in our system of reference the coordi-

[1] See, for instance, the Appendix of Ref. *2*. Further, see Ref. *3*.

nates (x_0^i) may be chosen so that the line element at large spatial distances is of the form

$$\begin{aligned} ds^2 &= g_{ik} dx_0^i dx_0^k \\ &= (1 + \alpha/r)[(dx_0^1)^2 + (dx_0^2)^2 + (dx_0^3)^2] - (1 - \alpha/r)(dx_0^4)^2, \\ r^2 &= (x_0^1)^2 + (x_0^2)^2 + (x_0^3)^2. \end{aligned} \tag{8}$$

Here, α is a constant connected with the total Newtonian gravitational mass M_0 by

$$\alpha = 2kM_0/c^2. \tag{9}$$

In fact, the scalar potential following from (8) is then

$$\chi = \frac{c^2}{2}(-g_{44} - 1) = -\frac{c^2\alpha}{2r} = -\frac{kM_0}{r}. \tag{10}$$

Since the θ_i^4 in (5) are of the form of a usual 3-divergence of $\mathrm{h}_i^{4\lambda}$, the integrals P_i in (7) can be expressed in the form of a surface integral lying at spatial infinity. Using the form (8) of the g_{ik} holding for $r \to \infty$ in this "center-of-gravity system", one easily finds

$$P_i^0 = -\delta_i^4 \frac{4\pi\alpha}{\kappa c} = -\delta_i^4 M_0 c. \tag{11}$$

This, in connection with the transformation properties I, shows that the P_i are the components of the total energy-momentum 4-vector of the closed system. A Lorentz transformation like

$$x^1 = \frac{x_0^1 + \frac{v}{c} x_0^4}{\sqrt{1 - v^2/c^2}}, \qquad x^2 = x_0^2, \qquad x^3 = x_0^3, \qquad x^4 = \frac{x_0^4 + \frac{v}{c} x_0^1}{\sqrt{1 - v^2/c^2}} \tag{12}$$

leads to a new system of reference in which P_i has the values

$$P_i = \left\{ \frac{M_0 v}{\sqrt{1 - v^2/c^2}}, \qquad 0, \qquad 0, \qquad -\frac{M_0 c}{\sqrt{1 - v^2/c^2}} \right\}, \tag{13}$$

i.e., the same values as have the components of the 4-momentum of a particle of proper mass M_0 moving with the velocity v in the direction of the x-axis in a system of inertia.

However, as was recognized at an early stage of the development,[2] Einstein's theory did not supply a physically satisfactory description of the energy distribution or of the energy contained in a limited part of space, as the integrand

[2] See, in particular, the paper by Bauer [*1*].

$(-\theta_4^{\ 4})$ occurring in the expression (7) for the total energy

$$H = -cP_4 = -\iiint \theta_4^{\ 4}\, dx^1 dx^2 dx^3 \tag{14}$$

does not behave like a scalar density under purely spatial transformations

$$\bar{x}^\iota = f^\iota(x^\kappa), \qquad \bar{x}^4 = x^4. \tag{15}$$

This means that the integral (14) extended over a finite part of space will in general depend on the names (coordinates) given to the different points in the system of reference considered.

Now, the expression $\theta_i^{\ k}$ is of course not uniquely determined by relation (1), since we can always add another affine tensor density with an identically vanishing usual divergence. In Ref. *2*, this freedom was utilized to find a new complex $\mathcal{T}_i^{\ k}$ which also satisfies the "local conservation law" (1), and which, moreover, has the desired transformation properties under the group of transformations (15). $\mathcal{T}_i^{\ k}$ can also be written in terms of a superpotential $\psi_i^{\ kl}$ as

$$\mathcal{T}_i^{\ k} = \chi_i^{\ kl}{}_{,l} \tag{16}$$

with

$$\chi_i^{\ kl} = -\chi_i^{\ lk} = \frac{\sqrt{-g}}{\kappa}(g_{in,m} - g_{im,n})g^{km}g^{ln}. \tag{17}$$

The connection between the superpotentials $h_i^{\ kl}$ and $\chi_i^{\ kl}$ is given by

$$h_i^{\ kl} = \tfrac{1}{2}\,[\chi_i^{\ kl} + K_i^{\ kl}], \tag{18}$$

where

$$K_i^{\ kl} = \delta_i^{\ k} h_r^{\ rl} - \delta_i^{\ l} h_r^{\ rk} \tag{19}$$

and

$$h_r^{\ rl} = \frac{1}{\kappa\sqrt{-g}}(-gg^{lm})_{,m}\,. \tag{20}$$

It is easily seen that $-\mathcal{T}_4^{\ 4}$ behaves like a scalar density under the transformation (15). In fact, the quantity $\mathcal{T}^k \equiv \mathcal{T}_4^{\ k}$ is a 4-vector density under these transformations and, since

$$\mathcal{T}^k{}_{,k} = 0 \tag{21}$$

is a special case of the relation

$$\mathcal{T}_i^{\ k}{}_{,k} = 0, \tag{22}$$

the components of $\mathcal{T}^k$ could be interpreted as the densities of energy and energy current.

Actually, $\mathfrak{T}^k$ behaves like a 4-vector density under the somewhat wider group of transformation

$$\bar{x}^\iota = f^\iota(x^\kappa), \qquad \bar{x}^4 = x^4 + g(x^\kappa), \tag{23}$$

where $g(x^\kappa)$ is an arbitrary function of the spatial coordinates, only. In the case of more general transformations of the time-variable, however, where also the rates of the coordinate clocks are changed, the energy distribution will change in a more complicated way, as would indeed be expected in view of the intimate connection between the notions of energy and time.

It can be shown[3] that the expression (16), (17) for the complex $\mathfrak{T}_i^k$ follows uniquely (apart from an arbitrary constant factor) from the following requirements:

II. *$\mathfrak{T}_i^k(x^l)$ at the event-point (x^l) is an affine tensor density depending algebraically on the g^{ik}'s and their derivatives of the first and second orders at the same point (x^l)*

III. *$\mathfrak{T}_i^k$ satisfies the local conservation law*

$$\mathfrak{T}_{i\ ,k}^{\ k} = 0 \tag{24}$$

identically.

This means that $\mathfrak{T}_i^k$ is expressible in terms of a superpotential $\chi_i^{\ kl} = -\chi_i^{\ lk}$ which is an affine tensor density depending algebraically on the metric tensor and its first-order derivatives, only.

IV. *$\mathfrak{T}^k = \mathfrak{T}_4^k$ transforms as a 4-vector density with respect to the group of purely spatial transformations* (15).

The constant factor left arbitrary by the conditions II–IV may then be fixed so that

$$\iiint \mathfrak{T}_4^{\ 4}\, dx_0^{\ 1}dx_0^{\ 2}dx_0^{\ 3} = \iiint \theta_4^{\ 4}\, dx_0^{\ 1}dx_0^{\ 2}dx_0^{\ 3} = -M_0c^2 \tag{25}$$

in the center-of-gravity system, where (8) holds asymptotically.

A further corroboration of the expression (16), (17) for the energy-momentum complex was obtained through application of the "method of infinitesimal transformations" to the curvature scalar density $\sqrt{-g}R$ entering as integrand in the invariant variational principle of the gravitational field equations [5].[4] In fact, if we consider an arbitrary infinitesimal space-time transformation

$$\bar{x}^i = x^i + \xi^i(x), \tag{26}$$

the transformation of a scalar density V is expressed by the equation

$$\delta V + (V\xi^k)_{,k} = 0. \tag{27}$$

[3] See Møller [4] and, for the complete proof, Magnusson [4].

[4] See also Lorentz [1] in which the author derives a complex which, on closer inspection, turns out to be equivalent to the complex $\mathfrak{T}_i^k$.

Then, if we put ξ^i equal to arbitrary constants, ϵ^i, and $V = (\sqrt{-g}/\kappa)R$, Eq. (27) takes the form

$$-\epsilon^i \mathfrak{T}_i{}^k{}_{,k} = 0, \tag{28}$$

with $\mathfrak{T}_i^k$ given by (16), (17). The method also allows us to determine the transformation properties of $\mathfrak{T}_i^k$ under arbitrary space-time transformations. If, in (27), we take $V = (1/2\kappa)\mathfrak{L}$, which is a scalar density with respect to linear transformations only, we have instead of (28)

$$-\epsilon^i \theta_i{}^k{}_{,k} = 0, \tag{29}$$

where θ_i^k is the Einstein expression (2), (4) or (5), (6).

2. GLOBAL ASPECTS

Since $\mathfrak{T}_i^k$ (like θ_i^k) is an affine tensor density (of weight one) satisfying the divergence relation (22) (or (1)), it was concluded in Ref. 2 that the quantities

$$\breve{P}_i = \frac{1}{c} \iiint_{x^4=\text{const.}} \mathfrak{T}_i^4 \, dx^1 dx^2 dx^3 \tag{30}$$

must have all the properties of the Einstein quantities P_i listed in I. Since $\breve{P}_i^0 = P_i^0$ in the center-of-gravity system (8), this would mean that also $\breve{P}_i$ represents the total energy-momentum vector in all systems of coordinates which are asymptotically rectilinear. A direct calculation of $\mathfrak{T}_i^k$ and of P_i in the system of coordinates (x^i) defined by (12) shows, however, that this is *not* the case.

Since $\mathfrak{T}_i^k$ transforms as a tensor under the Lorentz transformation (12), we have

$$\mathfrak{T}_i^4 = \frac{\partial x_0^r}{\partial x^i} \frac{\partial x^4}{\partial x_0^s} \mathfrak{T}_r^{0s} = \frac{\partial x_0^r}{\partial x^i} \frac{\mathfrak{T}_r^{04} + \frac{v}{c} \mathfrak{T}_r^{01}}{\sqrt{1 - v^2/c^2}}. \tag{31}$$

Further, since the time-variable x^4 is kept constant in the integrals (30), we may introduce the variables

$$x_0^1 = \frac{x^1 - \frac{v}{c} x^4}{\sqrt{1 - v^2/c^2}}, \qquad x_0^2 = x^2, \qquad x_0^3 = x^3 \tag{32}$$

as new variables of integration. Hence,

$$\begin{aligned}\breve{P}_i &= \frac{1}{c} \iiint \mathfrak{T}_i^4 \sqrt{1 - v^2/c^2} \, dx_0^1 dx_0^2 dx_0^3 \\ &= \frac{\partial x_0^r}{\partial x^i} \left[\frac{1}{c} \iiint \mathfrak{T}^0{}_r{}^4 \, dx_0^1 dx_0^2 dx_0^3 + \frac{v}{c^2} \iiint \mathfrak{T}^0{}_r{}^1 \, dx_0^1 dx_0^2 dx_0^3 \right],\end{aligned}$$

i.e.,

$$\breve{P}_i = \frac{\partial x_0^{\,r}}{\partial x^i}\left[\breve{P}_r^{\,0} + \frac{v}{c^2}\iiint \mathcal{T}^{0\,\,1}_{\,\,r}\, dx_0^{\,1} dx_0^{\,2} dx_0^{\,3}\right]. \tag{33}$$

We see that the quantities $\breve{P}_i$ will transform as a 4-vector only if the last quantity in (33) is zero. The condition for the $\breve{P}_i$ being a 4-vector with respect to all Lorentz transformations is, therefore, that the quantities

$$\begin{aligned}\breve{X}^{0\,\,\kappa}_{\,\,i} &= \iiint \mathcal{T}^{0\,\,\kappa}_{\,\,i}\, dx_0^{\,1} dx_0^{\,2} dx_0^{\,3} = \iiint \chi^{0\,\,\kappa\lambda}_{\,\,i\,\,\,\,,\lambda}\, dx_0^{\,1} dx_0^{\,2} dx_0^{\,3} \\ &= \lim_{r\to\infty} \iint_{r=\text{const.}} \chi^{0\,\,\kappa\lambda}_{\,\,i} n_\lambda\, dS\end{aligned} \tag{34}$$

are equal to zero in a static center-of-gravity system. The last integral in (34) is an integral over a large "surface of constant radius r", dS is a surface element of this "sphere", and $n_\lambda = n^\lambda = x_0^\lambda/r$ is a unit 3-vector perpendicular to the surface elements dS. Now, for large r, the metric is given by (8) and, neglecting terms of the order $1/r^3$, we have by (17)

$$\chi^{0\,\,\kappa\lambda}_{\,\,i} = \frac{a_{(i)}'}{\kappa}\,(n^\kappa \delta_i^\lambda - n^\lambda \delta_i^\kappa), \tag{35}$$

where

$$\begin{aligned} a_{(i)} &= g_{ii} = \{1 + \alpha/r,\quad 1 + \alpha/r,\quad 1 + \alpha/r,\ -(1 - \alpha/r)\} \\ a_{(i)}' &= da_{(i)}/dr, \qquad n^\kappa = n_\kappa = x_0^{\,\kappa}/r.\end{aligned} \tag{36}$$

Thus,

$$\begin{aligned}\chi^{0\,\,\kappa\lambda}_{\,\,4} &= 0, \quad \text{i.e.,} \\ \breve{X}^{0\,\,\kappa}_{\,\,4} &= 0.\end{aligned} \tag{37}$$

However, for $i = \iota = 1, 2, 3$ we have for $r \to \infty$

$$\begin{aligned}\chi^{0\,\,\kappa\lambda}_{\,\,\iota} &= -\frac{\alpha}{r^2}\,(n^\kappa \delta_\iota^\lambda - n^\lambda \delta_\iota^\kappa), \\ \chi^{0\,\,\kappa\lambda}_{\,\,\iota} n_\lambda &= -\frac{\alpha}{r^2}\,(n_\iota n^\kappa - \delta_\iota^\kappa),\end{aligned} \tag{38}$$

and, since

$$\iint_{r=\text{const.}} n_\iota n^\kappa\, dS = \frac{4\pi r^2}{3}\,\delta_\iota^\kappa, \tag{39}$$

we obtain by (34), (38), and (39),

$$\breve{X}^{0}{}_{\iota}{}^{\kappa} = \frac{2}{3}\frac{4\pi\alpha}{\kappa}\,\delta_{\iota}{}^{\kappa} = \frac{2}{3}\,M_0c^2\delta_{\iota}{}^{\kappa} \neq 0. \tag{40}$$

This clearly shows that $\breve{P}_i$ is not a 4-vector. For the particular Lorentz transformation (12) we have, by (33), (37), (40), and using the fact that

$$\breve{P}_i^{\,0} = P_i^{\,0} = -\delta_i^{\,4}M_0c,$$

$$\breve{P}_i = \frac{\partial x_0^{\,r}}{\partial x^i}\left[-\delta_r^{\,4}M_0c + \frac{v}{c^2}\,\breve{X}^{0}{}_{r}{}^{1}\right] = -\frac{\partial x_0^{\,4}}{\partial x^i}\,M_0c + \frac{2}{3}\,M_0v\,\frac{\partial x_0^{\,1}}{\partial x^i}, \tag{41}$$

i.e.,

$$\breve{P}_i = \left\{\frac{5}{3}\frac{M_0v}{\sqrt{1-v^2/c^2}},\qquad 0,\qquad 0,\qquad -\frac{M_0c}{\sqrt{1-v^2/c^2}}\left(1+\frac{2}{3}\frac{v^2}{c^2}\right)\right\}. \tag{42}$$

On the other hand, similar considerations applied to the Einstein expression P_i show that these quantities really have the properties stated in I, for the corresponding quantities

$$X^{0}{}_{\alpha}{}^{\kappa} = \lim_{r\to\infty}\iint_{r=\text{const.}} h^{0}{}_{i}{}^{\kappa\lambda}n_\lambda\,dS \tag{43}$$

are easily seen to be zero. In fact we have by (6) and (8) for $r \to \infty$

$$h^{0}{}_{i}{}^{\kappa\lambda} = \frac{\alpha^2}{8\kappa r^3}\,(n^\lambda\delta_i^{\,\kappa} - n^\kappa\delta_i^{\,\lambda}) = O(1/r^3), \tag{44}$$

and, since dS is proportional to r^2, the integrals in (43) vanish for $r \to \infty$. This means that P_i is a 4-vector and, instead of (42), we get Eqs. (13).

The question now arises whether $\theta_i^{\,k}$ is uniquely determined by the conditions (1) and I. If we confine ourselves to affine tensors $\theta_i^{\,k}$ which do not depend on derivatives of the metric tensor of orders higher than the second and which only contain powers up to the third of the first-order derivatives, the said question can be answered in the affirmative. For this means that the superpotential $h_i^{\,kl}$, which must exist on account of (1), can only be of the form

$$\begin{aligned} h_i^{\,kl} = {}& \lambda_1\sqrt{-g}\,g_{in,m}(g^{km}g^{ln} - g^{lm}g^{kn}) + \lambda_2\sqrt{-g}(\delta_i^{\,k}g^{lm}{}_{,m} - \delta_i^{\,l}g^{km}{}_{,m}) \\ & + \lambda_3(\delta_i^{\,k}g^{lm} - \delta_i^{\,l}g^{km})(\sqrt{-g})_{,m}, \end{aligned} \tag{45}$$

where λ_1, λ_2, λ_3 are constants.

Necessary conditions for the validity of I are now that

1. the $h_i^{\,k\lambda}$ vanish like $1/r^3$ for $r \to \infty$ in the center-of-gravity system defined by (8),

2. the integral $\iiint h_4^{\,4l}{}_{,l}\,dx^1dx^2dx^3$ has the value $-M_0c^2$ both in the rest system with the asymptotic metric (8) and, for instance, in the rest system obtained

by the transformation

$$r' = (1 + \alpha/4r)^2 r, \qquad x'^{\iota} = \frac{r'}{r} x^{\iota}, \tag{46}$$

in which the metric has the asymptotic form

$$\begin{aligned} ds^2 &= \left(\delta_{\iota\kappa} - \frac{\alpha}{r'} n_\iota' n_\kappa'\right) dx'^{\iota}\, dx'^{\kappa} - (1 - \alpha/r')(dx^4)^2 \\ n_\iota' &= \frac{x'^{\iota}}{r'}. \end{aligned} \tag{47}$$

It is easily seen that these conditions are satisfied only if

$$\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{2\kappa}, \tag{48}$$

in which case (45) is identical with the superpotential (6).

Thus, Einstein's expressions are the only ones which satisfy the conditions I, II, and III, at least if we exclude derivatives of higher order as well as higher powers of the first-order derivatives than those assumed in the preceding considerations.

Let us now have a look at the usual proof of I to see why this proof is valid for the P_i and not for the $\breve{P}_i$. Consider a region Σ in 4-space bounded by a three-dimensional hypersurface V that consists of two space-like surfaces V_1 and V_2 connected by a "cylinder" V_3 lying at spatial infinity

$$V = V_1 + V_2 + V_3. \tag{49}$$

Integrating (1) over Σ, we obtain by Gauss' theorem

$$\begin{aligned} 0 = \int_V \theta_i^{\ k}\, dS_k = \int_{V_1} \theta_i^{\ k}\, dS_k + \int_{V_2} \theta_i^{\ k}\, dS_k \\ + \int_{V_3} \theta_i^{\ k}\, dS_k = \mathfrak{F}_i^{(1)} + \mathfrak{F}_i^{(2)} + \mathfrak{F}_i^{(3)}, \end{aligned} \tag{50}$$

where

$$dS_k = \delta_{klmn}\, dx^l \delta x^m \Delta x^n \tag{51}$$

is a 4-vector normal to the hypersurface V. Since $\theta_i^{\ k}$ is an affine tensor density, each of the integrals $\mathfrak{F}_i^{(\alpha)}$ in (50) behaves like a 4-vector under linear transformations.

The essential point in the proof is now that in any system of asymptotically rectilinear coordinates the $\theta_i^{\ k}$ decrease with the fourth power of $1/r$, where r is the spatial distance. This follows from the fact that $\vartheta_i^{\ k}$ is quadratic in the first-order derivatives which themselves decrease like $1/r^2$ in the systems of coordinates in question. On the other hand, since the "area" of the hypersurface V_3 can at the most increase with the third power of r, the last integral $\mathfrak{F}_i^{(3)}$ in (50) over the cylinder V_3 vanishes. Then, if V_1 and V_2 are two surfaces of constant x^4, the integrals $\mathfrak{F}_i^{(1)}$ and $\mathfrak{F}_i^{(2)}$ are proportional to the quantities P_i taken at two different values of the time, but with opposite signs, and (50) at once leads to the first part of the statement I about the constancy of P_i in time. Further, if (x^i) and (x'^i) are two systems of asymptotically rectilinear coordinates connected by a linear transformation, we can choose V_1 and V_2 to be the surfaces $x^4 = \text{const.}$ and $x'^4 = \text{const.}$, respectively. On account of the transformation properties of the integrals $\mathfrak{F}_i^{(1)}$ and $\mathfrak{F}_i^{(2)}$ under linear transformations, (50) gives in this case

$$P_i' = \frac{\partial x^k}{\partial x'^i} P_k \,, \tag{52}$$

which shows that P_i is a 4-vector.

The reason for the breakdown of the proof in the case of the $\check{P}_i$ is now obviously that the corresponding quantity

$$\check{\mathfrak{F}}_i^{(3)} = \int_{V_3} \mathfrak{T}_i^k \, dS_k \tag{53}$$

is not always equal to zero. This is connected with the fact that $t_i^k = -T_i^k + \mathfrak{T}_i^k/\sqrt{-g}$ depends linearly also on the *second*-order derivatives of the metric tensor, which vanish like $1/r^3$ only. For the proof of the constancy in time of $\check{P}_i$ in all systems of coordinates connected with the center-of-gravity system by linear transformations, this is now sufficient, since the cylinder V_3 connecting the two space-like surfaces $x^4 = \text{const.}$ has an area which increases like r^2, only. In that case, $\check{\mathfrak{F}}_i^{(3)}$ is therefore equal to zero, and the proof runs as before. However, if V_1 and V_2 are the surfaces $x^4 = \text{const.}$ and $x'^4 = \text{const.}$, where (x^i) and (x'^i) are connected by a Lorentz transformation, the area of the surface V_3 will increase like r^3, so that $\mathfrak{F}_i^{(3)}$ is in general different from zero. This is the reason for the occurrence of the extra terms in (33) and (42).

3. CONCLUSION

The result of the preceding investigation is that it is impossible to satisfy all requirements I–IV simultaneously. In fact, the assumptions I, II, III lead uniquely to the Einstein expressions (5), (6), which do not satisfy the localization requirement IV. On the other hand, the assumptions II, III, IV as well as the method of infinitesimal transformations applied to the invariant variational principle lead uniquely (apart from a constant factor) to the complex (16), (17), in which case the global quantities $\check{P}_i$ do not have the properties I.

It is obviously necessary to give up at least one of the conditions I–IV. If we drop IV, we have to put up with the situation that only the total energy, but not the energy contained in a finite region of space, has an exact physical meaning, for the value of the integral

$$\iiint_\Omega \theta_4^{\ 4}\, dx' dx^2 dx^3$$

taken over a finite part of space depends on the type of spatial coordinates used in the evaluation of the integral. This means, for instance, that we cannot give an exact value to the total energy inside our laboratory placed in the gravitational field of the earth, which seems a little queer. It is true that we can avoid these somewhat weird consequences by the prescription that the complex $\theta_i^{\ k}$ has a physical meaning only for certain limited types of space-time coordinates such as, for instance, the "harmonic" coordinates used by Fock. However, this would mean that we should have to give up the idea of general covariance, which was the guiding principle in the development of the general theory of relativity; therefore, this proposition seems hardly acceptable.

Before abandoning the idea of localizability of the energy we shall try to relax the requirements I–IV and formulate a set of minimum requirements. First, we remark that the total momentum and energy must be given by Einstein's expression P_i, for only this expression satisfies the conditions I necessary for such interpretation. Since the transformations involved in I are only asymptotically restricted, this does not imply any serious restriction of the general covariance. Next, it should be noted that we are only concerned with the localization of the energy, a localization of the momentum being physically meaningless for reasons explained elsewhere.[5] Thus, we shall only assume the existence of an energy and energy current complex satisfying the following conditions:

1. $\mathfrak{T}^k$ is a function of the coordinates (x^i) depending in a covariant way on the metric tensor and its space-time derivatives.

We do not any more require algebraic dependence, and $\mathfrak{T}^k$ may even be a *functional* of the metric tensor and its derivatives so long as the dependence is covariant, i.e., so long as the functional is the same in every system of coordinates.

2. $\mathfrak{T}^k$ satisfies the differential conservation law

$$\mathfrak{T}^k{}_{,k} = 0. \tag{54}$$

3. $\mathfrak{T}^k$ is a 4-vector density with respect to the group of purely spatial transformations (15).

4. In any system of asymptotically rectilinear coordinates we have

$$H = \iiint \mathfrak{T}^4\, dx^1 dx^2 dx^3 = -cP_4 = -\iiint \theta_4^{\ 4}\, dx^1 dx^2 dx^3. \tag{55}$$

[5] See the considerations in Ref. *2*, Section 4.

From the preceding considerations it is clear that such quantity $\mathcal{T}^k$, if it exists at all, cannot be closely connected with the components $\theta_4^{\ k}$ of the complex $\theta_i^{\ k}$, and we need not at all assume that $\mathcal{T}^k$ appears as one row of a complex $\mathcal{T}_i^{\ k}$ with two indices. This gives us considerable freedom (in fact too much) in our choice of $\mathcal{T}^k$.

In view of the fact that the different methods, by which the complex $\mathcal{T}_i^k$ defined by (16), (17) was obtained, determined this quantity apart only from an arbitrary constant factor, it is suggestive to assume that $\mathcal{T}_4^{\ 4}$ describes only the *relative* distribution of the energy inside a given system of reference and that the absolute distribution is given by

$$\mathcal{T}^k = -\lambda \mathcal{T}_4^{\ k} = -\lambda \chi_4^{\ kl}{}_{,l}\,. \tag{56}$$

Here, λ is a constant inside a given system of reference, but it need not be an invariant, i.e., the value of λ may depend on the system of coordinates in question. Equation (56) obviously satisfies the conditions 2 and 3, and condition *4* gives for λ the value

$$\lambda = \frac{P_4}{\check{P}_4} = \iiint \theta_4^{\ 4}\, dx^1 dx^2 dx^3 \Big/ \iiint \mathcal{T}_4^{\ 4}\, dx^1 dx^2 dx^3 \tag{57}$$

or

$$\lambda = \lim_{r\to\infty} \iint_{r=\text{const.}} h_4^{\ 4\lambda} n_\lambda \, dS \Big/ \lim_{r\to\infty} \iint_{r=\text{const.}} \chi_4^{\ 4\lambda} n_\lambda \, dS. \tag{58}$$

Since the ratio $P_4/\check{P}_4$ is time-independent, λ is really a constant. The expression (56) also satisfies condition *1*, but now only the relative energy-distribution is an algebraic function of the metric tensor and its derivatives, while the absolute distribution, described by (56)–(58), is a functional of these quantities.

In the center-of-gravity system defined by (8), λ has the value 1 and, on account of (58), λ is invariant under all transformations which do not change the g_{ik}*'s* and their first-order derivatives at spatial infinity. Also under purely spatial transformations, for instance on the introduction of polar coordinates, the constant λ is by definition invariant. However, with respect to more general space-time transformations, λ is not invariant. For instance in the system of coordinates defined by (12), where the system as a whole is moving with the velocity v, λ has, by (13) and (42), the value

$$\lambda = P_4/\check{P}_4 = \frac{1}{1 + \frac{2}{3}\,(v^2/c^2)}\,. \tag{59}$$

Since v cannot exceed the value c, λ is positive in all physically acceptable systems of reference.

The expressions for λ and $\mathcal{T}^k$ given by (56)–(58) satisfy the condition of general covariance, but they are highly nonlocal, since they depend on the values of the metric tensor far away; in fact, by (58), λ is seen to depend effectively only on the metric tensor at spatial infinity. From a physical point of view, this may appear to be even more unacceptable than the nonlocalizability of the energy in Einstein's original theory.

Finally, it should be mentioned that it is possible to define an "energy-momentum complex" S_i^k satisfying the conditions I, III, IV of Section 1, if the condition II is changed into the following condition:

II′. *$S_i^k(x^l)$ at the event-point (x^l) is an affine tensor depending algebraically on the gravitational field variables and their derivatives of the first and second orders at the same point (x^l).*

At first sight, there seems to be no difference between the conditions II and II′ since the gravitational field variables usually are taken to be the components of the metric tensor. However, there are indications that the g_{ik}'s are not the fundamental gravitational variables. It is well known, for instance, that the field equations for a Fermion field in the presence of a gravitational field cannot be expressed in terms of the matter-field variable ψ and the g_{ik}'s, only; but, as shown by various authors [6], it is convenient in this case to describe the gravitational field by four-legs, i.e., a set of four orthogonal normalized vectors, at each point of 4-space. If we label these four 4-vectors by an index, a, running from 1 to 4, the contravariant components of these vectors will be 16 functions $h_a{}^i(x)$ of the space-time coordinates. Introducing further

$$h^{ai}(x) \equiv \epsilon_{(a)} h_a{}^i(x)$$
$$\epsilon_a = \{1, 1, 1, -1\}, \tag{60}$$

the connection between the four-leg variables $h_a{}^i(x)$ and the components of the metric tensor may be written

$$h_a{}^i h^{ak} = g^{ik}. \tag{61}$$

This relation, and in fact any function of the g^{ik}'s and their derivations, is form-invariant under independent "rotations" of the four-legs

$$\bar{h}_a{}^i = h_b{}^i L^b{}_a{}^{(x)}, \tag{62}$$

where the functions $L^b{}_a(x)$ and

$$L_b{}^a = \epsilon_{(b)} \epsilon_{(a)} L^b{}_a \tag{63}$$

satisfy the "orthogonality" relations

$$L^a{}_c L_b{}^c = \delta_b^a . \tag{64}$$

Also the field equations of the Fermion field are invariant under the rotations (62).

By means of (61), any function of the g^{ik}'s and their derivatives can be expressed in terms of the vectors $h_a{}^i$ and their derivatives. In particular, it can be shown that the curvature scalar density can be written in the form

$$\sqrt{-g}R = \hat{\mathfrak{L}} + \hat{h}, \tag{65}$$

where $\hat{\mathfrak{L}}$ is a homogeneous quadratic form in the first-order derivatives of the $h_a{}^i$, and $\hat{h}$ has the form of a usual divergence. In the variational principle of the gravitational field equations, we can therefore omit the term $\hat{h}$.

The explicit expression for $\hat{\mathfrak{L}}$ is

$$\hat{\mathfrak{L}} = \sqrt{-g}(h^{ar}{}_{;s}h_a{}^s{}_{;r} - h^{ar}{}_{;r}h_a{}^s{}_{;s}), \tag{66}$$

where the semicolon, as usual, indicates covariant derivation of the four-leg vectors. Since the Christoffel symbols are linear expressions in the first-order derivations $h_a{}^i{}_{,k}$ of the four-legs, the same holds for the tensors $h_a{}^r{}_{;s}$. From (66) it is seen that $\hat{\mathfrak{L}}$ (in contrast to the usual Langrangian density $\mathfrak{L}$ in (4)) is a scalar density under arbitrary space-time transformation. We may, therefore, apply the method of infinitesimal transformations described in Reference 6 to this quantity or to the quantity $V = \hat{\mathfrak{L}}/2\kappa$. In this way, we find that the complex

$$S_i^k = \sqrt{-g}(T_i^k + \hat{t}_i^k) \tag{67}$$

with

$$\sqrt{-g}\,\hat{t}_i^k = \frac{1}{2\kappa}\left(\frac{\partial\hat{\mathfrak{L}}}{\partial h_a{}^r{}_{,k}}\,h_a{}^r{}_{,i} - \delta_i^k\,\hat{\mathfrak{L}}\right) \tag{68}$$

satisfies the divergence relation

$$S_i^k{}_{,k} = 0. \tag{69}$$

Further, the method shows that S_i^k can be written in terms of a superpotential

$$S_i^k = U_i{}^{kl}{}_{,l} \tag{70}$$

with

$$\begin{aligned} U_i{}^{kl} &= \frac{1}{4\kappa}\left(\frac{\partial\hat{\mathfrak{L}}}{\partial h_a{}^i{}_{,l}}\,h_a{}^k - \frac{\partial\hat{\mathfrak{L}}}{\partial h_a{}^i{}_{,k}}\,h_a{}^l\right) \\ &= \frac{\sqrt{-g}}{2\kappa}\,[h_a{}^k h^{al}{}_{;i} - h_a{}^l h^{ak}{}_{;i} + 2(\delta_i^k h^{al} - \delta_i^l h^{ak})h_a{}^s{}_{;s}]. \end{aligned} \tag{71}$$

In contrast to the superpotentials h_i^{kl} and $\chi_i{}^{kl}$, the quantity U_i^{kl} is a true tensor density of rank 3. Therefore, the quantity U_4^{kl} is an antisymmetric tensor density of rank 2 under the group of purely spatial transformations (15). Consequently,

$$S^k = U_4{}^{kl}{}_{,l} \tag{72}$$

is a vector density with respect to such transformations. Thus, the expressions (67), (68), (70), and (71) for S_i^k satisfy the conditions II′, III, and IV.

Further, in the case of a closed system at rest where (8) holds asymptotically, it is seen that we are in accordance with (61) if we choose a set of four-legs $h_a^{\ i}$ which asymptotically is of the form

$$h_a^{\ i} = \frac{1}{\sqrt{|a_{(i)}|}}\,\delta_a^{\ i} \tag{73}$$

with $a_{(i)}$ given by (36); for this gives

$$h_a^{\ i} h^{ak} = \frac{\epsilon_a}{\sqrt{|a_{(i)}a_{(k)}|}}\,\delta_a^{\ i}\delta_a^{\ k} = \frac{1}{a_{(i)}}\,\delta^{ik}\,. \tag{74}$$

Now, define

$$\hat{P}_i = \frac{1}{c}\iiint S_i^4\,dx^1dx^2dx^3 = \lim_{r\to\infty}\iint_{r=\text{const.}} U_i^{4\lambda} n_\lambda\,dS. \tag{75}$$

Then, a simple calculation shows by means of (71) and (73) that the $\hat{P}_i$ in the center-of-gravity system have the same values

$$\hat{P}_i^0 = -\delta_i^4 M_0 c \tag{76}$$

as the Einstein quantities P_i^0 in (11). Further, since the first-order derivatives $h_{a\ ,k}^{\ i}$, according to (73) and (36), decrease as $1/r^2$ for $r \to \infty$, the quantity (68), which is a quadratic homogeneous function of the $h_{a\ ,k}^{\ i}$ will decrease as $1/r^4$. Therefore, the condition for the validity of the usual proof of I is satisfied exactly as in the case of the Einstein expressions.

Thus, we have found a complex which satisfies all the conditions I, II′, III, and IV. The trouble is only that we have actually found an infinite number of them, for S_i^k is not invariant under all rotations (62) of the four-legs. S_i^k is invariant under a rotation (62) with constant transformation coefficients $\mathfrak{L}^b{}_a$, but, if $\mathfrak{L}^b{}_a(x)$ is an arbitrary function of the coordinates satisfying the conditions (64), the transformed complex $\bar{S}_i^k$ will in general be different from S_i^k. Hence, any $\bar{S}_i^k$ obtained by a rotation with coefficients $L^b{}_a(x)$, which sufficiently rapidly approach constant values $L_0{}^b{}_a$ for $r \to \infty$, will furnish a complex satisfying all the conditions I, II′, III and IV. Obviously, we need certain conditions restricting the relative orientation of the four-legs in the different points of space-time. Investigations along these lines are now in progress in collaboration with Drs. M. Magnusson and C. Pellegrini, and we hope in a subsequent paper to be able to report on the results of these investigations.

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