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Twistor Algebra

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A new type of algebra for Minkowski space–time is described, in terms of which it is possible to express any conformally covariant or Poincaré covariant operation. The elements of the algebra (twistors) are combined according to tensor-type rules, but they differ from tensors or spinors in that they describe locational properties in addition to directional ones. The representation of a null line by a pair of two-component spinors, one of which defines the direction of the line and the other, its moment about the origin, gives the simplest type of twistor, with four complex components. The rules for generating other types of twistor are then determined by the geometry. One-index twistors define a four-dimensional, four-valued ("spinor") representation of the (restricted) conformal group. For the Poincaré group a skew-symmetric metric twistor is introduced. Twistor space defines a complex projective three-space C, which gives an alternative picture equivalent to the Minkowski space–time M (which must be completed by a null cone at infinity). Points in C represent null lines or "complexified" null lines in M; lines in C represent real or complex points in M (so M, when complexified, is the Klein representation of C. Conformal transformations of M, including space and time reversals (and complex conjugation) are discussed in detail in twistor terms. A theorem of Kerr is described which shows that the complex analytic surfaces in C define the shear-free null congruences in the real space M. Twistors are used to derive new theorems about the real geometry of M. The general twistor description of physical fields is left to a later paper.

I. INTRODUCTION

In the study of Lorentz covariant local field theories, field quantities (with spin) are normally described by vectors, tensors, or, more generally, by spinors, all finite-dimensional representations of the Lorentz group[1] being expressible in terms of spinors (vectors and tensors being regarded as effectively special cases). A local (restricted) Lorentz transformation then takes the form

$$\psi_{E\cdots J'\cdots}^{A\cdots G'\cdots} \longrightarrow \psi_{Q\cdots V'\cdots}^{M\cdots S'\cdots}\, t^{A}_{M} \cdots \bar{t}^{G'}_{S'} t^{(-1)Q}_{E} \cdots \bar{t}^{(-1)V'}_{J'}, \tag{1.1}$$

(t^A_M) being a complex unimodular (2×2) matrix, with inverse $(t^{(-1)A}_M)$. Except where general relativity is involved, normally it is further required that physical quantities be suitably covariant under the full Poincaré group.[1] However, this covariance is expressed (locally) in a totally different way from the local Lorentz covariance. Whereas the dependence on space–time *direction* is expressed *algebraically* [cf., (1.1)], the dependence on *position* is described by *differential* equations (so that the analog of (1.1) involves integrals). The invariance of all expressions under the translation

$$x^i \rightarrow x^i + a^i \tag{1.2}$$

is generally ensured by the fact that the (Minkowskian) coordinates x^i must, themselves, never enter into the field equations explicitly, only the operators $\partial/\partial x^i$ being permitted to occur.

It is curious that the invariance under two types of transformation (1.1) and (1.2), each of which simply refers to symmetries of the Minkowski space–time, should find mathematical expression in so different a way. This is, moreover, *not* just a property of a difference in group structure between the translation and rotation elements of the Poincaré group, but rather a consequence of *what is meant by a local field theory*. In fact, mathematical formalisms do exist[2] in which *all* the Poincaré transformations are represented according to an algebraic "tensor type" law similar to (1.1). It is the purpose of the present paper to exhibit and discuss in some detail one such formalism, the basic elements of which are re-

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[1] Throughout this paper, "Lorentz group" always refers to to the six-parameter *homogeneous* group and "Poincaré group" to the ten-parameter *inhomogeneous* group. "Conformal group" refers to the fifteen-parameter group of transformations which preserve the local conformal structure of Minkowski space–time. If in any particular context it is important to exclude the space reversing and time reversing transformations, then this is made explicit, for example, by the use of the term "restricted".

[2] The use of homogeneous coordinates in space–time would afford a simple example of such a formalism (but apparently one of limited physical interest). More significant, of course, is the representation of physical quantities in terms of Hilbert space, which has the advantage that the (infinite demensional) analogs of $(t_M{}^A)$ are unitary [cf., particularly E. P. Wigner, Ann. Math. **40**, 149 (1939) V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. U. S. **34**, 211 (1948)]. However, it will be essential here to preserve the *finite* dimensionality of the operations at *this* stage, so that geometrical questions can be kept in the forefront.

ferred to here as *twistors*. It will turn out that the twistor algebra will have the same type of universality, in relation to the *conformal* group,[1] that the well-known and highly effective two-component spinor algebra of van der Waerden[3] has, in relation to the Lorentz group. Twistors are, in fact, the "spinors" which are relevant to the six-dimensional space whose (pseudo-) rotation group is isomorphic with the conformal group of ordinary Minkowski space–time.[4] The simplest (non-scalar) twistors constitute a *four-dimensional, four-valued representation of the restricted conformal group*[5] (eight-dimensional if reflections are included). The general twistor is then a many-index quantity constructible from the above basic twistors by means of the usual "tensor type" rules. Any finite-dimensional representation of the conformal group is thus expressible as a direct sum of twistor representations.[4]

The emphasis here will be on the *geometrical* aspects of twistors. It will, in fact, be possible to give a fairly complete geometrical picture of twistors and of their basic operations. Ordinary space–time concepts can then be translated into twistor terms. However, the geometrical expressions of the most immediate twistor concepts have a somewhat nonlocal character. Thus, the primary geometrical object will not be a point in Minkowski space–time, but rather a null straight line or, more generally, a twisting congruence of null lines. Points do, in fact, emerge, but only at a secondary stage. (It also turns out that a natural description of physical fields in twistor terms is given by quantities having a nonlocal space–time interpretation.) However, any vector, tensor, or spinor operation *can* be translated into twistor terms, if desired, and *vice versa*.

All the basic operations of the formalism are conformally invariant. This is of relevance in the study of zero rest-mass fields, since, for each spin, the free-field equations are all effectively conformally invariant.[6,7] This latter fact has particular importance in the asymptotic analysis of such fields,[7] and it would appear that a "manifestly conformally invariant" formalism should be valuable to this type of analysis. Furthermore, although the presence of *mass* breaks conformal invariance, the conformal group appears to have relevance as an approximate symmetry in very high energy physics.[8] When the energies of the particles are high enough, their rest masses can be neglected.

However, the applicability of the twistor formalism is not restricted to situations in which the conformal group is relevant. Since the Poincaré group is a subgroup of the conformal group, it follows that Poincaré covariant operations can be expressed in twistor terms. It merely becomes necessary to introduce a fixed (skew-symmetric) "metric twistor" $I^{\alpha\beta}$ which singles out a particular Minkowski structure (consistent with the given conformal structure). By including operations involving $I^{\alpha\beta}$ with the basic twistor operations, a formalism invariant under the Poincaré group is obtained, in terms of which, any Poincaré covariant operation is, in principle, expressible. Furthermore, by making slightly different alternative choices for $I^{\alpha\beta}$, the corresponding formalisms for a class of space–times including the de Sitter and Einstein cosmologies are obtained.

Perhaps the most obvious drawback of the twistor formalism, however, from the point of view of a possible *fundamental* applicability in physics, is that it is so tied to the idea of a *conformally flat* background space–time, that it is difficult to conceive of how to incorporate the formalism (as it stands) completely into general relativity. It is not impossible that some modification of twistor algebra might be applicable to general curved background space–times. Indeed, some of the basic ideas of the formalism, namely those concerning the null line congruences (cf., the results of Robinson and Kerr referred to in Sec. VIII) have, as their origins, some researches into general relativity. However, any such modification would have to be of a different order from the comparatively straightforward adaption of spinor algebra into general relativity.[9] The point of view one must apparently take, is that whereas the vector–tensor–spinor algebra refers to the im-

[3] B. L. van der Waerden, Nachr. Ges. Wiss. Gottingen **100**, 1 (1929).

[4] F. Klein, *Gesammelte Mathem. Abhandlungen* (J. Springer, Berlin, 1921); cf. also H. Weyl, *The Classical Groups* (Princeton University Press, Princeton, New Jersey, 1939); R. Brauer and H. Weyl, Am. J. Math. **57**, 425 (1935).

[5] Quantities which are essentially twistors have been described by W. A. Hepner, Nuovo Cimento **26**, 351 (1962). See also Y. Murai, Nucl. Phys. **6**, 489 (1958); Progr. Theoret. Phys. (Kyoto) **9**, 147 (1953); **11**, 441 (1954). Similar quantities have also recently gained prominence in, for example, the work of A. Salam, R. Delbourgo, and J. Strathdee, Proc. Roy. Soc. (London) **A284**, 147 (1965).

[6] E. Cunningham, Proc. London Math. Soc. **8**, 77 (1910); H. Bateman, *ibid.* **8**, 223 (1910); P. A. M. Dirac, Ann. Math. **37**, 429 (1936); J. A. McLennan, Jr., Nuovo Cimento **10**, 1360 (1956); H. A. Buchdahl, *ibid.* **11**, 496 (1959).

[7] R. Penrose, Proc. Roy. Soc. (London) **A284**, 159 (1965).

[8] H. A. Kastrupp, Phys. Letters (Amsterdam) **3**, 78 (1962); Phys. Rev. **142**, 1060 (1966). A recent additional suggestion is that the mass splittings of strong interaction physics may be derivable from conformal group symmetry: D. Bohm, M. Flato, D. Sternheimer, and J. P. Vigier, Nuovo Cimento **38**, 1941 (1965). For a discussion of the relevance of the conformal group in physics, see T. Fulton, F. Rohrlich, and L. Witten, Rev. Mod. Phys. **34**, 442 (1962).

[9] L. Infeld and B. L. van der Waerden, S. B. Preuss. Akad. 9, 380 (1933).

mediate neighborhood of a "point" in space–time, the twistor algebra refers to a more extended "nonlocal" region. It seems to be mathematically more difficult to piece together such regions into a curved space–time, than to build a space–time out of points.

Finally, one of the most important initial motivations for considering a formalism of this kind should be briefly mentioned, although implications of this idea are not discussed in this paper. It concerns possible applications to quantum field theory. One of the most significant quantum mechanical operations is the splitting of field amplitudes into their positive and negative frequency parts. Although this is generally expressed in terms of Fourier transforms, there is the alternative description in terms of singularity-free analytic extensions into the upper or lower complex half-planes.[10] In the present formalism, the idea is to use this second approach and to combine it with global properties of complexified Minkowski space–time. Similarly to the way that the real axis divides the complex plane into two disconnected halves, the space of "null" twistors (describing real Minkowskian null lines) divides twistor space into two disconnected halves, namely the "right-handed" and the "left-handed" twistors (which may be thought of as describing "complexified" null lines).

Free fields of zero rest mass and arbitrary spin can be described particularly conveniently in twistor terms. Moreover, it turns out that they can be generated in a remarkably simple way as contour integrals of arbitrary analytic functions defined in twistor space. (A *null* field is generated when the contour surrounds a simple pole of the function. This leads to a certain generalization of a theorem of Robinson.[11]) If the relevant singularities of this function are associated with one half, rather than the other, of the twistor space, this ensures that the field is of (say) positive frequency. Other properties of zero rest-mass fields also find a natural expression in twistor terms, notably the interrelation with certain types of potential fields[7] and the structure of their total energy-momentum–angular-momentum (which give ten of the fifteen components of a trace-free "Hermitian" twistor E^{α}_{β}). The discussion of these matters is left to a later paper.

Care has been taken here to illustrate that twistor algebra is not merely an abstract formalism, but that the algebraic operations have well-defined meanings in terms of space–time geometry. For whether or not twistors have a significant role to play in future physical theory, the results so far obtained suggest strongly that the formalism ought at least to be thoroughly explored from both geometric and analytic points of view.

II. TWISTORS AND NULL LINES

As a starting point for the geometrical description of a twistor, consider a null straight line L in Minkowski space–time M. Choose a set of Minkowski coordinates[12] (x^i) for M with origin O. Let l^i be the position vector of some point P on L and let n^i be a future-pointing tangent vector to L (see Fig. 1). If we wish to assign a set of coordinates to the line L, we may do this (Plücker–Grassmann coordinates) by selecting n^i and the moment

$$m^{ij} = l^i n^j - n^i l^j \tag{2.1}$$

of the vector n^i (acting at P) about O. Then, the ratios of the ten quantitites (n^i, m^{ij}) will uniquely define L. (Note that this is independent of the choice of P on L.) This is, however, a highly redundant representation. In addition to the requirement here that n^i be null

$$n^i n_i = 0, \tag{2.2}$$

there are the consistency relations for (2.1)

$$\epsilon_{ijkl} m^{ij} n^k = 0 \tag{2.3}$$

[and also $\epsilon_{ijkl} m^{ij} m^{kl} = 0$, which is implied by (2.3)]. Equation (2.3) in fact represents just three independent conditions which, together with (2.2), reduce the set of nine ratios in (n^i, m^{ij}) to just *five* independent real numbers. This is consistent with

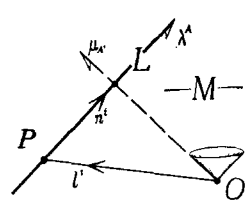

FIG. 1. The representation of a null line.

[10] See any standard work on quantum field theory. The essential matters referred to here can be found in R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That* (W. A. Benjamin, Inc., New York, 1964).

[11] I. Robinson, J. Math. Phys. **2**, 290 (1961).

[12] World vectors and tensors are labeled by lower case Latin indices running over 0,1,2,3, the Minkowski metric being given by $(g^{ij}) = \mathrm{diag}(1,-1,-1,-1)$. Capital Latin index letters, primed or unprimed, denote spinor indices and run over 0,1 or 0′,1′. Greek indices are twistor indices and run over 0,1,2,3. The summation convention applies to each of these four types of index separately. Thus, in particular, no summation takes place between primed and unprimed spinor indices even when the same letter is used, e.g., J and J' are regarded as distinct letters in $x_J{}^{J'}$. This allows us to write the tensor–spinor correspondence in a definite way by simply using the two capital versions of a tensor index as its spinor translation: $x^j \leftrightarrow x^{JJ'}$, etc. (Rindler's convention).

the fact that the null lies in M form an ∞^5 system (for, choosing P as the intersection of L with a fixed spacelike hyperplane, we have ∞^3 choices for P and ∞^2 choices for the null direction at P).

So far, the fact that L is null has not played an essential part. But null directions are very conveniently represented in terms of *two-component spinors.* It will emerge that, by using spinors, we can greatly simplify the description of the null line L; the set of four conditions (2.2), (2.3) being effectively replaced by a *single* condition. Since the following discussion depends essentially on the use of spinors, a very brief review of the ideas required will now be given.[13] The translation from world tensors to spinors is achieved using a quantity[12] $\sigma_i^{JJ'}$ [a Hermitian (2 × 2) matrix for each j] and its inverse $\sigma^i_{JJ'}$ subject to $\sigma_i^{JJ'}\sigma_k^{KK'}\epsilon_{JK}\epsilon_{J'K'} = g_{ik}$, $\sigma_j^{JJ'}\sigma^i_{JJ'} = \delta^i_j$. The ϵ_{JK}, $\epsilon_{J'K'}$ are skew-symmetric Levi-Civita symbols and are used for raising and lowering spinor indices: $\xi^A\epsilon_{AB} = \xi_B$, $\epsilon^{AB}\xi_B = \xi^A$, i.e., $\xi^0 = \xi_1$, $\xi^1 = -\xi_0$, and similarly for primed indices. Any tensor (e.g., $X^{ij}{}_k$) has a spinor translation, which is written using the same base symbol, but with each tensor index replaced by the corresponding pair of spinor indices, e.g.,

$$X^{ij}{}_k \leftrightarrow X^{II'JJ'}{}_{KK'} = X^{ij}{}_k\sigma_i{}^{II'}\sigma_j{}^{JJ'}\sigma^k{}_{KK'}.$$

Under complex conjugation, the roles of primed and unprimed indices are interchanged, so that reality of tensors is expressed as Hermiticity of spinors. Since M is flat, we can here choose $\sigma_i^{JJ'}$ constant and equal to $2^{-\frac{1}{2}}$ times the unit matrix and Pauli matrices. Then, say,

$$(x^0, x^1, x^2, x^3) \leftrightarrow \begin{bmatrix} x^{00'} & x^{01'} \\ x^{10'} & x^{11'} \end{bmatrix} = 2^{-\frac{1}{2}}\begin{bmatrix} x^0 + x^1 & x^2 + ix^3 \\ x^2 - ix^3 & x^0 - x^1 \end{bmatrix}. \tag{2.4}$$

The spinor translation of a complex null vector α^i (i.e., $\alpha^i\alpha_i = 0$) has the form

$$\alpha^i \leftrightarrow \alpha^{JJ'} = \beta^J\gamma^{J'}. \tag{2.5}$$

If α^i is real and future pointing, then we can take $\gamma^{J'}$ to be the complex conjugate of β^J, i.e.,

$$\alpha^i \leftrightarrow \beta^J\bar{\beta}^{J'}. \tag{2.6}$$

[13] For more complete accounts see Refs. 3 and 9 and, for example, W. L. Bade and H. Jehle, Rev. Mod. Phys. **25**, 714 (1953); E, M. Corson, *An Introduction to Tensors, Spinors and Relativistic Wave Equations* (Blackie & Son Ltd., London, 1953); F. A. E. Pirani, in *Brandeis Summer Institute in Theoretical Physics,* 1964, *Lectures on General Relativity* (Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1965), Vol. 1.

Equation (2.6) implies a geometrical realization of a spinor β^J, up to a phase multiplier, namely as a future-pointing null vector. Moreover, β^J can be completely realized geometrically[14] up to *sign,* in terms of the *bivector*

$$\pi^{ik} \leftrightarrow \beta^J\beta^K\epsilon^{J'K'} + \epsilon^{JK}\bar{\beta}^{J'}\bar{\beta}^{K'}. \tag{2.7}$$

This is real (and null), and it determines a *half-plane element* (the "flag plane") tangent to the light cone along the vector α^i. However, the sign of β^J cannot be realized *locally*[15] geometrically, since a continuous rotation through 2π changes β^J into $-\beta^J$. If we are interested in β^J only up to *proportionality,* then the complete geometrical realization is simply as a null *direction.*

Let us now represent, in spinor terms, the quantities n^i, m^{ii}, which define the null line L. We have

$$n^i \leftrightarrow n^{JJ'} = \lambda^J\bar{\lambda}^{J'} \tag{2.8}$$

and, from (2.1),

$$m^{ik} \leftrightarrow l^{JJ'}\lambda^K\bar{\lambda}^{K'} - \lambda^J\bar{\lambda}^{J'}l^{KK'} = i\epsilon^{JK}\mu^{(J'}\bar{\lambda}^{K')} - i\mu^{(J}\lambda^{K)}\epsilon^{J'K'}, \tag{2.9}$$

where

$$\mu_{A'} = -i\lambda^A l_{AA'}. \tag{2.10}$$

(Round brackets denote symmetrization. The factor $-i$ is for later convenience.) Thus λ^A and $\mu_{A'}$ together determine l^i and m^{ik}. We may think of λ^A as defining the *direction* of L and $\mu_{A'}$ as effectively giving us the *moment* of λ^A ("acting" at P) about O. It is clear from (2.10) that, if λ^A is multiplied by any complex factor, then L is unchanged if $\mu_{A'}$ is multiplied by the same factor. Furthermore, (2.10) implies that $\mu_{A'}$ is independent of the choice of P on L (i.e., if $l_{AA'} \to l_{AA'} + a\lambda_A\bar{\lambda}_{A'}$, then $\mu_{A'}$ is unchanged since $\lambda^A\lambda_A = 0$).

Note a particular choice of P which is of interest, namely the intersection of L with the null cone of O. (This intersection exists uniquely provided L does not lie in any null hyperplane through O.) Then l^i or $-l^i$ has the form (2.6) and it follows from (2.10) that

$$l_{AA'} = i(\lambda^B\bar{\mu}_B)^{-1}\bar{\mu}_A\mu_{A'}. \tag{2.11}$$

[14] E. T. Whittaker, Proc. Roy. Soc. (London) **A158**, 38 (1937); W. T. Payne, Am. J. Phys. **20**, 253 (1952); R. Penrose, "Null Hypersurface Initial Data", in P. G. Bergmann's Aeronautical Research Lab. Tech. Documentary Rept. 63-56 (Office of Aerospace Research, U. S. Air Force, 1963).

[15] To give a rigorous definition of a spinor which takes into account its sign, it is usual to appeal to the theory of fibre bundles. This is not essential, however, and an elementary (nonlocal) geometrical description will be given in an appendix to a forthcoming book by R. Penrose and W. Rindler on the applications of spinors in relativity.

Thus, the null direction defined by $\mu_{A'}$[cf., (2.6)] is that of the null line through O which meets L (Fig. 1). The exceptional case $\lambda^B\bar{\mu}_B = 0$ corresponds to L lying in a null hyperplane through O. [This follows from (2.10), since n^i would be necessarily orthogonal to any choice of l_i.] In this case, λ^A and $\bar{\mu}^A$ are proportional, so the null direction of $\mu_{A'}$ is that of L. More exceptionally still, L might pass through O. This is the case $\mu_{A'} = 0$. On the other hand, for the present, we do not allow $\lambda^A = 0$ (although this will be reconsidered shortly).

The null line L can now be assigned, for coordinates, the three complex ratios of the four complex quantities

$$L^0 = \lambda^0, \qquad L^1 = \lambda^1, \qquad L^2 = \mu_{0'}, \qquad L^3 = \mu_{1'}, \tag{2.12}$$

which we write

$$(L^\alpha) = (\lambda^A, \mu_{A'}). \tag{2.13}$$

These three complex ratios are equivalent to *six* real parameters, so we must expect to find one real relation connecting λ^A and $\mu_{A'}$. This is obtained from (2.10), since the *reality* of the vector l^i implies $l_{AA'}$ is Hermitian, whence

$$\operatorname{Re}(\lambda^A\bar{\mu}_A) = 0. \tag{2.14}$$

Provided $\lambda^A \neq 0$, condition (2.14) is also *sufficient* to ensure the existence of a null line L associated with λ^A and $\mu_{A'}$. For, if $\lambda^A\bar{\mu}_A$ is pure imaginary and nonvanishing, then (2.11) gives us a point P, through which we choose L with direction given by λ^A. On the other hand, if $\lambda^A\bar{\mu}_A = 0$, we can easily solve (2.10) to obtain P, with OP spacelike ($l_{AA'}$ of the form $\lambda_A\bar{\nu}_{A'} + \nu_A\bar{\lambda}_{A'}$).

A quantity with components (L^α) given as in (2.13) will be called a *null twistor* (of valence $\left[{1 \atop 0}\right]$) if (2.14) holds.[16] We shall also be interested in such quantities when (2.14) does *not* hold (and also when λ^A is allowed to vanish). We may regard such a quantity as describing, in some sense, a "complexified" null line [when (2.14) fails], since if we allow l^i to be a *complex* vector, then (2.14) would, in general, be violated. [This is not a straightforward complexification of the null lines in M in the ordinary sense, however, since the real dimensionality of the system of lines is only increased from five to six by dropping (2.14); but cf., Sec. VI.] If $\operatorname{Re}(\lambda^A\bar{\mu}_A) > 0$, we refer to L^α as a *right-handed twistor* (valence $\left[{1 \atop 0}\right]$) and if $\operatorname{Re}(\lambda^A\bar{\mu}_A) < 0$, a *left-handed twistor.* The reason for this terminology will become evident later. When $\lambda^A = 0$, we shall still refer to L^α as a *null* twistor even though no finite null line L is represented. The role of such twistors will emerge shortly.

[16] Any temptation to identify the twistor (2.13) with a Dirac spinor should be rejected here, since their transformation properties are quite different [cf., for example, (7.13), (7.17)].

III. INCIDENCE OF NULL LINES IN TWISTOR TERMS

The algebraic rules for the manipulation of twistors will have, as their basis, the idea of *incidence* between null lines. Let us consider, in addition to L, a second null line X, described by the null twistor X^α:

$$(X^\alpha) = (\xi^A, \eta_{A'}), \tag{3.1}$$

where ξ^A and $\eta_{A'}$ are the analogs of λ^A, $\mu_{A'}$ above. That is, ξ^A defines the direction of X and we have

$$\eta_{A'} = -i\xi^A x_{AA'}, \tag{3.2}$$

where x^i is the position vector of a point on X (see Fig. 2). Suppose now that X and L do intersect. Then we may choose $l^i = x^i$ for the coordinate vector of this intersection point, whence by (2.10) and (3.2),

$$\xi^A\bar{\mu}_A = i\xi^A l_{AA'}\bar{\lambda}^{A'} = i\xi^A x_{AA'}\bar{\lambda}^{A'} = -\eta_{A'}\bar{\lambda}^{A'}. \tag{3.3}$$

Let us define the *complex conjugate* of a twistor L^α to be $\bar{L}_\alpha$ (valence $\left[{0 \atop 1}\right]$), where

$$(\bar{L}_\alpha) = (\bar{\mu}_A, \bar{\lambda}^{A'}) \tag{3.4}$$

when L^α is given by (2.13) [irrespective of the condition (2.14), or whether or not λ^A vanishes]. In component form[17]:

$$\bar{L}_0 = \overline{L^2}, \qquad \bar{L}_1 = \overline{L^3}, \qquad \bar{L}_2 = \overline{L^0}, \qquad \bar{L}_3 = \overline{L^1}. \tag{3.5}$$

Then, (3.3) tells us that a necessary condition for

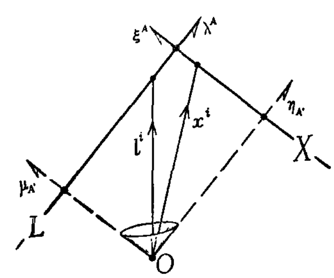

Fig. 2. Incidence of two null lines.

[17] The operation of twistor (or spinor) complex conjugation is denoted by a bar extending only over the base symbol involved and not over the indices. If the bar extends also over the indices, this denotes simply the complex conjugate of the complex number that the symbol represents.

the null lines L and X to meet, in terms of their corresponding twistors, is

$$X^\alpha \bar{L}_\alpha = 0, \tag{3.6}$$

since

$$X^\alpha \bar{L}_\alpha = \xi^A \bar{\mu}_A + \eta_{A'} \bar{\lambda}^{A'}. \tag{3.7}$$

Note, further, that the condition (2.14) for L^α to represent a real null line is simply

$$L^\alpha \bar{L}_\alpha = 0. \tag{3.8}$$

We can think of (3.8) as a special case of (3.6) since L intersects itself! We also have, here,

$$X^\alpha \bar{X}_\alpha = 0. \tag{3.9}$$

Condition (3.6) is, conversely, also *sufficient* for the null lines L and X to intersect, where we assume, for the moment, that λ^A and ξ^A are not proportional, so that L and X are not parallel. For, we then have $\xi^A\lambda_A \neq 0$ and we can construct the (complex) vector

$$p_i \leftrightarrow p_{JJ'} = (i/\xi^A\lambda_A)(\lambda_J \eta_{J'} - \xi_J \mu_{J'}). \tag{3.10}$$

We observe that

$$\mu_{A'} = -i\lambda^A p_{AA'}, \qquad \eta_{A'} = -i\xi^A p_{AA'}.$$

Thus, when p_i is *real*, we can satisfy (2.10) and (3.2) by putting $l_i = p_i = x_i$, whence L and X must intersect. (In fact, it is valid to regard the "complex point" with position vector p^i as the "intersection" of L and X even in the cases when p_i is not real, as we shall see later.) Now p_i is real if and only if $p_{JJ'}$ is Hermitian. Since λ^A and ξ^A are not proportional, we can test the Hermiticity of $p_{JJ'}$ by taking components with respect to λ^A, ξ^A:

$$\lambda^A \bar{\lambda}^{A'}(p_{AA'} - \bar{p}_{A'A}) = i\mu_{A'}\bar{\lambda}^{A'} + i\lambda^A\bar{\mu}_A = iL^\alpha \bar{L}_\alpha,$$

$$\xi^A \bar{\xi}^{A'}(p_{AA'} - \bar{p}_{A'A}) = i\eta_{A'}\bar{\xi}^{A'} + i\xi^A\bar{\eta}_A = iX^\alpha \bar{X}_\alpha,$$

$$\xi^A \bar{\lambda}^{A'}(p_{AA'} - \bar{p}_{A'A}) = i\eta_{A'}\bar{\lambda}^{A'} + i\xi^A\bar{\mu}_A = iX^\alpha \bar{L}_\alpha.$$

Thus, the reality of p_i is a consequence of (3.8), (3.9), and (3.6), so that in the case of twistors representing nonparallel null lines, we can state the condition that the lines meet simply as the orthogonality condition (3.6) between the twistors.

We may further ask what is the geometrical meaning of this condition $X^\alpha \bar{L}_\alpha = 0$ when X and L are real *parallel* null lines. Perhaps the simplest way to see the result is to use a limiting argument. Keep L_α fixed but vary X^α, so that $X^\alpha \bar{L}_\alpha = 0$ throughout the motion. Suppose L and X are initially not parallel, but become parallel in the limit. The intersection point recedes to infinity along L. The possible positions of X, for each *finite* position of the intersection point, are the generators of the null cone of this point. As the point recedes to infinity, the null cone becomes, in the limit, the null hyperplane through X. Thus, when L and X are parallel, the condition $X^\alpha \bar{L}_\alpha = 0$ becomes the condition that *L and X lie in the same null hyperplane.*

In fact, it is convenient to regard all the null lines of a null hyperplane as intersecting in a single *point at infinity*. Thus, we adjoin to our space M a set of ∞^3 such points at infinity, one for each of the null hyperplanes in M. The geometrical role played by the twistors L_α with $\lambda^A = 0$ (but $\mu_{A'} \neq 0$) now emerges. For, if $\lambda^A = 0$, the condition $X^\alpha \bar{L}_\alpha = 0$ becomes $\xi^A \bar{\mu}_A = 0$ [cf., (3.7)]. That is to say, *the direction of X coincides with the null direction represented by $\mu_{A'}$*, so that for *fixed* L^α the corresponding lines X are all parallel. Each null hyperplane of parallel lines X gives rise to *one* point at infinity; the aggregate of all these points at infinity, corresponding to all these parallel hyperplanes, gives an ∞^1 system of points at infinity, namely the points of the *null line at infinity* L. In fact, to complete the picture, we must add one further point at infinity, not lying on any finite line, which we call I. The point I is common to all the lines L at infinity. The fact that any two null lines at infinity must, for consistency here, be considered to intersect, follows from (3.7), since if $\lambda^A = 0$ and also $\xi^A = 0$, then $X^\alpha \bar{L}_\alpha = 0$ automatically follows. The point I then plays the part of the *vertex* of a *null cone at infinity* the generators of which are the lines at infinity considered above (see Fig. 3).

The structure of the completed ("compactified") Minkowski space arrived at in this way, by adding a (closed) null cone at infinity, is one which has been considered by a number of authors.[18,7] We henceforth use the symbol M, here, to refer to the *entire* completed space and not just to the set of finite points. The set of finite points is instead denoted by $M\{I\}$ to indicate that the null cone of the

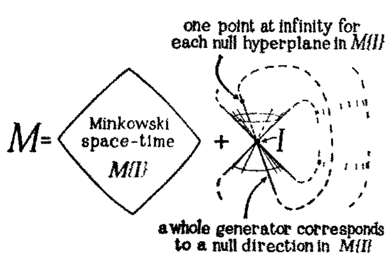

FIG. 3. The construction of the compact manifold M.

[18] N. H. Kuiper, Ann. Math. **50**, 916 (1949); H. Rudberg, dissertation, University of Uppsala, Uppsala, Sweden (1958); R. Penrose, in *Proceedings of the 1962 Conference on Relativistic Theories of Gravitation, Warsaw* (Polish Academy of Science, Warsaw, 1965) A. Uhlmann, Acta Phys. Polon. **24**, 293 (1963). Rudberg also mentions the four-valuedness of spinors.

particular point I has been removed from M. The geometry of M is briefly as follows: M is a compact nonsingular manifold without boundary which has a well-defined conformal structure everywhere (i.e., a metric defined up to proportionality). It has a transitive ∞^{15} group of motions preserving this conformal structure, every point being on an equal footing with every other point. M contains ∞^5 null "straight" lines (geodesics) each of which is topologically a circle. Through each point P of M pass ∞^2 of these lines generating the (closed up) null cone of P. The removal of any one of these null cones leaves a space with Euclidean topology which can be assigned a Minkowskian metric (unique up to dilatations) consistent with the given conformal structure. The entire space M has the topology $S^3 \times S^1$ and can be realized as a real projective quadric fourfold with signature $(++----)$.

The null lines in M are in one-to-one correspondence with the proportionality classes of twistors L^α (i.e., κL^α represents the same line as L^α, $\kappa \neq 0$) which satisfy $L^\alpha \bar{L}_\alpha = 0$ and $L^\alpha \neq 0$. Incidence between null lines in M is expressed as orthogonality ($X^\alpha \bar{L}_\alpha = 0$) between the corresponding twistors.

IV. ROBINSON CONGRUENCES

Thus far, the geometrical description of a twistor given here has been restricted to the case of null twistors (of valence $[^1_0]$). As mentioned earlier, if on the other hand $L^\alpha \bar{L}_\alpha \neq 0$, then we may think of L^α as representing, in some sense, a kind of "complexified" null line L. This is, in fact, the way we tend to view "geometrically" a general twistor of valence $[^1_0]$. However, it is significant that a much more precise realization can be given in terms of congruences[19] of null lines. In order to fix our ideas, we regard such a congruence as representing, rather, a twistor of valence $[^0_1]$ (e.g., $\bar{L}_\alpha$), although this makes little difference in practice.

Now, any (real) null line L in M can be completely characterized by *the system of all null lines which meet it*. In twistor terms, that is, if we know the set of all X^α satisfying $X^\alpha \bar{L}_\alpha = 0$, $X^\alpha \bar{X}_\alpha = 0$ (where $L^\alpha \bar{L}_\alpha = 0$, $L^\alpha \neq 0$), then we know L^α up to proportionality. Thus, while we think of L^α as describing the null line L in M, we can think of $\bar{L}_\alpha$ as describing the *congruence* L of null lines (i.e., a three-dimensional system[19] of lines in M) which meet L. In exactly the same way we can represent a general twistor R_α of valence $[^0_1]$, in terms of the congruence R of lines X whose twistors X^α (with $X^\alpha \bar{X}_\alpha = 0$) satisfy $X^\alpha R_\alpha = 0$. If $R_\alpha \bar{R}^\alpha \neq 0$, such a congruence is referred to here as a *Robinson congruence*.[20] [If $R^\alpha \bar{R}^\alpha = 0$, then we denote by $\bar{R}$ the line which is met by all the lines of the congruence R. On the other hand, if L denotes the line, then $\bar{L}$ denotes the corresponding congruence. In general, we adopt the convention that the operation of applying a bar to any aymbol is its own inverse. From this it follows that the definition of $\bar{R}^\alpha$, given R_α, exactly mirrors the definition (3.5) of $\bar{L}_\alpha$, given L^α, since we could put $L^\alpha = \bar{R}^\alpha$. Thus,[17] $(\bar{R}^\alpha) = (\bar{R}^0, \bar{R}^1, \bar{R}^2, \bar{R}^3) = (\overline{R_2}, \overline{R_3}, \overline{R_0}, \overline{R_1})$.] If $R_\alpha \bar{R}^\alpha > 0$, we refer to the Robinson congruence R as right-handed. If $R_\alpha \bar{R}^\alpha < 0$, it is left-handed.

Since Robinson congruences occupy such a basic position in the geometry of twistors, it seems worthwhile to examine a particular such congruence in some detail here. (In fact, any particular case is completely representative of the general case, since it turns out that any two Robinson congruences can be transformed one into the other by a Poincaré transformation, perhaps involving a reflection.)

Choose a real number ϵ and put

$$(R_\alpha) = (\epsilon/\sqrt{2}, 0, 1, 0) \tag{4.1}$$

so that

$$R_\alpha \bar{R}^\alpha = \epsilon\sqrt{2}. \tag{4.2}$$

Let X^α be as in (3.1): $(X^\alpha) = (\xi^0, \xi^1, \eta_{0'}, \eta_{1'})$, so that the condition for the line X to belong to the conrgruence R is

$$\xi^0 \epsilon/\sqrt{2} + \eta_{0'} = 0. \tag{4.3}$$

Write the position vector x^i of a point on X as

$$x^0 = t, \qquad x^1 = z, \qquad x^2 = x, \qquad x^3 = y.$$

Equation (3.2), with (2.4), gives

$$(\eta_{0'}, \eta_{1'}) = -(i/\sqrt{2})(\xi^0, \xi^1)\begin{bmatrix} t - z & -x + iy \\ -x - iy & t + z \end{bmatrix}.$$

Using (4.3), we then get $-\epsilon\xi^0 = i\xi^0(-t + z) + i\xi^1(x + iy)$, i.e.,

$$\xi^0 : \xi^1 = x + iy : t - z + i\epsilon. \tag{4.4}$$

Now $\xi^J \bar{\xi}^{J'}$ corresponds to a tangent vector to X, so that

$$\begin{bmatrix} \xi^0\bar{\xi}^{0'} & \xi^0\bar{\xi}^{1'} \\ \xi^1\bar{\xi}^{0'} & \xi^1\bar{\xi}^{1'} \end{bmatrix} \propto \begin{bmatrix} dt + dz & dx + i\,dy \\ dx - i\,dy & dt - dz \end{bmatrix}.$$

[19] The term "congruence" is used to denote a system of curves (or surfaces, etc.) for which there is just one member of the system (or at most a discrete number) through a general point in the space.

[20] First investigated by I. Robinson (private communication).

Combining this with (4.4) we get the differential equations for the line X

$$dt + dz : dx - i\,dy = dx + i\,dy : dt - dz$$
$$= x + iy : t - z + i\epsilon. \quad (4.5)$$

The solution of this is

$$\left.\begin{aligned} t - z + i\epsilon &= (x + iy)\alpha, \\ (x - iy) - (t + z)\alpha &= \beta, \end{aligned}\right\} \quad (4.6)$$

where α and β are complex constants defining the particular line X of R, for consistency satisfying

$$\operatorname{Im}(\bar{\beta}\alpha) = \epsilon. \quad (4.7)$$

In order to get a more visualizable picture of the situation, let us consider the intersection with a spacelike hyperplane $t = \tau$ (τ const). Each null line X meets this hyperplane in one point (τ, x, y, z). The direction of X there can be represented by considering the projection into the hyperplane of the tangent vector to X at this point. These projected tangent vectors are then tangents to a series of curves in the hyperplane which should give some kind of a picture of the structure of the Robinson congruence. To find the differential equations of these curves, we simply replace t by τ in (4.5) and dt by $ds = (dx^2 + dy^2 + dz^2)^{\frac{1}{2}}$. Then we get

$$(x + iy)(ds - dz) = (\tau - z + i\epsilon)(dx + i\,dy).$$

The solutions of this equation are given (apart from the spurious $x + iy = \text{const}$) by

$$\left.\begin{aligned} x^2 + y^2 + (z - \tau)^2 - 2\epsilon(x \sin\varphi & \\ + y\cos\varphi)\tan\theta &= \epsilon^2, \\ z - \tau = (x\cos\varphi - y\sin\varphi)\tan\theta, & \end{aligned}\right\} \quad (4.8)$$

where θ and φ are constants defining the different curves. These curves are evidently circles, being intersections of spheres with planes. They twist around one another in such a way that every pair of circles is linked. The twisting has a *positive* screw sense if $\epsilon > 0$, i.e., if R_α is right-handed.[21] They lie on the set of coaxial tori[22] obtained by eliminating φ between the two equations. [These tori are the rotations about the z axis of a system of coaxial circles in the (x, z) plane.]

From the point of view of the completed space-time M, we should regard the hyperplane $t = \tau$ as being completed (conformally) by a point (namely I) at infinity. It then becomes topologically a three-dimensional sphere S^3 (of which the hyperplane $t = \tau$ may be regarded as the stereographic projection). The vector field on S^3 is everywhere nonsingular and nowhere vanishing. The circles constitute an example of what is known as the Hopf fibring of S^3 (one example being "Clifford parallels" on S^3).

Note that all the circles in the hyperplane thread through the particular (smallest) circle of radius $|\epsilon|$ and centre $z = \tau$, $x = y = 0$ given when $\theta = 0$. If ϵ is small, this circle describes, as τ increases, a path approximating that of a null line. If ϵ is zero, the path is exactly the null line $z = t$, $x = y = 0$. For small ϵ, we may think of the lines of the Robinson congruence as defining an approximate null line, but the lines twist around one another and never quite meet. The twisting has a positive or negative screw sence according as R_α is right- or left-handed. [In the limit $\epsilon \to 0$, the circles (4.8) all touch the z axis at $z = \tau$. The tangents to these circles are orthogonal to the spheres touching the (x, y) plane at $z = \tau$, these spheres being the intersections of $z = \tau$ with the null cones with vertices on $z = t$, $x = y = 0$. The lines of the congruence are then just the generators of these null cones in this case, as expected.]

V. THE ASSOCIATED SPINOR FIELD OF A TWISTOR

We saw above that the tangent vectors to the lines of the Robinson congruence constituted a field of null vectors which was regular and nonvanishing everywhere on M. The general Robinson congruence R can be written compactly in spinor terms in a way which exhibits this (and other) facts very simply. This leads to an interpretation of twistors in terms of certain spinor fields.

Put $R_\alpha = \bar{L}_\alpha$, so we can use the notation of Secs. II and III. The condition $X^\alpha \bar{L}_\alpha = 0$, for X to belong to the congruence $\bar{L}$ is then, by (3.7), (3.2),

$$0 = \xi^A \bar{\mu}_A + \eta_{A'} \bar{\lambda}^{A'} = \xi^A(\bar{\mu}_A - i x_{AA'} \bar{\lambda}^{A'}),$$

from which it follows that ξ_A is proportional to the expression in the bracket. Since X defines X^α only up to proportionality, we are at liberty to choose the scale factor for ξ^A so that

$$\xi_A = \bar{\mu}_A - i x_{AA'} \bar{\lambda}^{A'}. \quad (5.1)$$

We can think of (5.1) as defining a *spinor field*, since ξ_A is a function of x_i, the associated null vectors $v^i \leftrightarrow \xi^J \bar{\xi}^{J'}$ being tangents to the lines of

[21] The screw sense arising here depends, of course, on the "handedness" of the choice of matrices in (2.4).

[22] I am grateful to J. Terryl and J. E. Reeve for this observation concerning the Hopf fibring.

the congruence $\bar{L}$. Since $\lambda^A x_{AA'}\bar{\lambda}^{A'}$ is real, it follows that

$$\mathrm{Re}\,(\lambda^A \xi_A) = \mathrm{Re}\,(\lambda^A \bar{\mu}_A) = \tfrac{1}{2}L^\alpha \bar{L}_\alpha, \qquad (5.2)$$

so that ξ_A cannot vanish unless L^α is null. The field of null directions defined by (5.1), i.e., by the Robinson congruence $\bar{L}$, is thus *well defined and regular* throughout $M\{I\}$. The regularity at infinity also follows from the conformal transformation properties of (5.1) as we are to see shortly.

Now write

$$\nabla_{BB'} \equiv \partial/\partial x^{BB'}, \qquad (5.3)$$

so that $\nabla_{BB'}x_{AA'} = \epsilon_{AB}\epsilon_{A'B'}$. Then we have, from (5.1),

$$\nabla_{BB'}\xi_A = i\epsilon_{BA}\bar{\lambda}_{B'}. \qquad (5.4)$$

Thus, the field ξ_A satisfies the differential equation

$$\nabla^{B'}_{(B}\xi_{A)} = 0. \qquad (5.5)$$

Conversely, from this equation we can reconstruct the form of the expression (5.1). For (5.5) implies that for *some* spinor $i\bar{\lambda}_{B'}$, the derivative $\nabla_{BB'}\xi_A$ has the form given in (5.4). Furthermore, this $\bar{\lambda}_{B'}$ must be constant because $\nabla^{C'}_C\nabla^{B'}_B\xi_A = 0$ [as follows from (5.5), since skew-symmetry in B, A *and* in C, A implies that the expression vanishes]. Integration of (5.4) leads straight to (5.1), where $\bar{\mu}_A$ is another constant.

We now have the important result that *there is a one-to-one relation between twistors of valence* $\left[{0 \atop 1}\right]$ *and spinor fields satisfying* (5.5). This gives us a more complete representation of a twistor than just as a Robinson congruence, since the *factor of proportionality* of L^α is now also represented. Note that $L^\alpha\bar{L}_\alpha$ is also expressible simply in terms of ξ_A as

$$L^\alpha\bar{L}_\alpha = \mathrm{Im}\,(\bar{\xi}^{B'}\nabla^A_{B'}\xi_A). \qquad (5.6)$$

But since we are interested in twistors in relation to the *whole* of M, rather than just $M\{I\}$, we should also ask how the field ξ_A is to be defined at infinity. Essentially all that is required here is to verify that (5.5) is invariant under conformal transformation [e.g., under inversions, since then I becomes a finite point,[7] cf., (7.18)]. This invariance is achieved[23] by specifying that ξ_A transform as a *conformal density of weight* $\frac{1}{2}$. Then the field equation (5.5) is also satisfied at infinity, in the sense that it holds with respect to some metric which is regular on the null cone of I. This, in particular, defines the Robinson congruence $\bar{L}$ regularly at infinity. (In fact, there is one line of $\bar{L}$ totally at infinity, namely that corresponding to the direction of λ^A.) The quantity (5.6) is also invariant[23] under conformal transformation (ξ_A of weight $\frac{1}{2}$), so we can refer to $L^\alpha\bar{L}_\alpha$ as the *conformal invariant* of the spinor field ξ_A or of the twistor L^α.

Some subtleties remain, however, concerning the extension of the field ξ_A across infinity. The possibility of defining two-valued (spinor) fields in $M\{I\}$ is well known. But here we require a spinor field defined *globally* on M. The multiple connectedness of M is such as to cause a slight difficulty in this respect. But in any case, we can think, instead, in terms of $\xi_A\xi_B$ or the associated bivector [cf., (2.7)], since these are not two-valued. Then we might expect to find a representation of a twistor up to sign. However, it turns out [cf., (7.25)] that twistors of odd total valence are *four*-valued under the conformal group. Thus, any such conformally invariant representation of a twistor (of odd valence), cannot distinguish the twistor from i times the twistor. (This is analogous to the fact that spinors of odd valence cannot be completely represented in terms of tensors, in a Lorentz covariant way, whereby the spinors are distinguished from their negatives.) In the present case, when we extend the field ξ^A across infinity, we find that the field on the two sides of the null cone of I cannot be exactly matched, but ξ^A on one side must be matched with $i\xi^A$ (or $-i\xi^A$) on the other side. [We can see this explicitly in terms of the inversion of (7.18), for which the conformal factor is $\Omega = x^i x_i/2a^2$. The null cone of I is transformed from the null cone, $\Omega = 0$, of the origin, across which Ω changes sign. The transformed ξ^A field picks up the factor $\Omega^{-\frac{1}{2}}$.] Thus, the spinor field (5.1), when defined over the *whole* of M, must be thought of as *four-valued*; if ξ_A is one value, then $i\xi_A$, $-\xi_A$, $-i\xi_A$ are the other three.

Since spinors can be represented geometrically (up to sign), this implies a geometrical realization of any twistor of valence $\left[{0 \atop 1}\right]$ up to a multiple of 1, i, -1, or $-i$. The phase of ξ_A defines, as we have seen [cf., (2.7)], a half-plane element tangent to the light cone; $-\xi_A$ defines the same half-plane element; but $i\xi_A$ and $-i\xi_A$ define the opposite half-plane element. Thus, a complete unoriented null plane element is defined at each point by the phase of $\bar{L}_\alpha$. As we follow one of the lines X belonging to the Robinson congruence $\bar{L}$, we find that this plane element rotates about X in the opposite direction from the neighboring lines of the congruence and twice as fast. Owing to the way connections are

[23] See Ref. 7, Eqs. (10.1), (10.7), and (10.8) for the relevant formulas.

made at infinity, this results in the aggregate of plane elements at points of X, effectively constituting a Möbius strip, which explains why the plane elements cannot be consistently oriented along X.

To see how these plane elements rotate about X, consider $(\xi^B\bar{\xi}^{B'}\nabla_{BB'})\xi_A = i\bar{\xi}^{B'}\bar{\lambda}_{B'}\xi_A$ [see (5.4)]. The real part of $2i\bar{\xi}^{B'}\bar{\lambda}_{B'}$ measures the rate of extension of the vector represented by ξ_A and the imaginary part measures the rate of rotation of the half-plane element about the null direction of ξ_A. By (5.2),

$$\mathrm{Im}\,(2i\bar{\xi}^{B'}\bar{\lambda}_{B'}) = -2\,\mathrm{Re}\,(\xi_B\lambda^B) = -L^\alpha\bar{L}_\alpha,$$

whence the half-plane element rotates in a negative or positive sense according as the twistor L^α is right or left handed. (It might appear that the rate of rotation is constant here but this is misleading since the scaling is different at different points of X.) Note that if L^α is null, there is no rotation of the half-plane element. We still get a Möbius strip, however, since a half-twist always occurs, in effect, at infinity.

To see how the neighboring lines of the congruence rotate about X, consider

$$\xi^A\bar{\xi}^{B'}\nabla_{BB'}\bar{\xi}_A = -i\bar{\xi}^{B'}\bar{\lambda}_{B'}\xi_B. \qquad (5.7)$$

Here, the real part of $-i\bar{\xi}^{B'}\bar{\lambda}_{B'}$ measures the convergence of the null lines and the imaginary part measures their rotation about one another.[24] The rotation is in the opposite sense from that of the half-plane elements and half as fast.

The expansion of the null lines and the change in the magnitude of ξ_A are not conformally covariant concepts, whereas the rotations are. Another conformally covariant concept for a congruence of null lines is the *shear* of these lines.[11,24] In the present case we have

$$\xi^B\xi^A\nabla_{BB'}\xi_A = 0, \qquad (5.8)$$

which states that the shear *vanishes*.

VI. THE COMPLEX PROJECTIVE SPACE C

We have seen that the null lines in M form an ∞^5 system which can be extended to an ∞^6 system by including "complexified" null lines (the latter being representable in terms of certain related structures called Robinson congruences). The members of this ∞^6 system can be given complex projective coordinates $(L^\alpha) = (L^0, L^1, L^2, L^3)$. That is, it is just the three complex ratios L^0: L^1: L^2: L^3 which are significant, the only restriction on $L^0, \cdots, L^3$ being that they must not *all* vanish. This ∞^6 system we may think of as constituting a *three-dimensional complex projective space* which we denote by the letter C. The "points" of C are just the "complexified" null lines (and the null lines) of M.

[24] P. Jordan, J. Ehlers, and R. Sachs, Akad. Wiss. Lit. Mainz, no. 1 (Mainz 2) (1961); E. T. Newman and R. Penrose, J. Math. Phys. 3, 566 (1962).

In fact, we have two alternative pictures of any given situation, namely the one in terms of M and the one in terms of C. For example, we may think of an object L, with projective coordinates (L^α) *either* as, say, a "complexified" null line of the real space–time M (M picture) *or* simply as a point in a certain projective three space (C picture). These are just two different ways of *visualizing* what is the same physical situation in each case. In order that the two pictures be completely equivalent, however, we need to be able to interpret, in C, the condition of "*reality*" of a null line in M, of *incidence* between null lines in M, and finally of *points* in M. In effect, this requires that the conjugation relation $L^\alpha \leftrightarrow \bar{L}_\alpha$ should have a meaning with regard to the C picture. Now, we have seen that a twistor L^α (valence $[^1_0]$) refers to a point L of C; a twistor R_α (valence $[^0_1]$) therefore refers to the dual concept of a *plane* R in C, namely the plane of all points X for which $X^\alpha R_\alpha = 0$. (This is, of course, a plane in the *complex* sense. As a *real* manifold it is a four-dimensional subset of the six-real-dimensional manifold C.) The conjugation relation $L^\alpha \leftrightarrow \bar{L}_\alpha$ therefore describes a point $\leftrightarrow$ plane correspondence in C, which we may refer to as a *Hermitian correlation* of signature $(++--)$. The signature here refers to the Hermitian form

$$X^\alpha\bar{X}_\alpha = X^0\overline{X^2} + X^1\overline{X^3} + X^2\overline{X^0} + X^3\overline{X^1}. \qquad (6.1)$$

We regard this Hermitian correlation as being an *intrinsic* part of the geometric structure of C.

The (real) null lines in M are the points of the five-real-dimensional subset N (with topology $S^3 \times S^2$) of C defined by the equation $X^\alpha\bar{X}_\alpha = 0$. Thus, N is a hypersurface if we regard C as a real six-dimensional manifold, but it is not a hypersurface in the sense of the *complex* structure of C. We refer to the subset of C for which $X^\alpha\bar{X}_\alpha > 0$ holds as C^+ and the part for which $X^\alpha\bar{X}_\alpha < 0$ as C^-. The two sets C^+ and C^- are then disconnected from one another and have N as their common boundary (Fig. 4). If L is any point of C, we may regard the plane $\bar{L}$ as the *polar plane* of L with respect to N, since "polarizing" $X^\alpha\bar{X}_\alpha$ with L^α yields $X^\alpha\bar{L}_\alpha = 0$ (or $L^\alpha\bar{X}_\alpha = 0$), the equation of the plane $\bar{L}$. The Robinson congruence associated with L is now the intersection (topology S^3) of the plane $\bar{L}$ with N. (A slight inconsistency of notation arises here in that it is the Robinson congruence $\bar{L} \cap N$ that had been previously labeled $\bar{L}$, rather than the entire plane. The Robinson congruence includes only "real" null lines of M, by definition. In cases where there is possibility of confusion, we refer to "the Robinson congruence $\bar{L}$" or "the plane $\bar{L}$" as the case may be, but in general $\bar{L}$ refers to an entire plane in C.) When L lies on N, the plane $\bar{L}$ can be thought of as the *complex tangent plane to N at L*. This is also just the case when *L lies on its polar plane.*

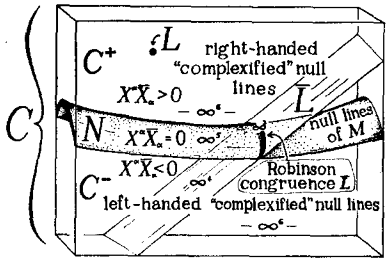

FIG. 4. The C picture.

Now, if we wish to represent a *point of M* in terms of the C picture, we may do this using incidence properties of null lines in M. Any point P in M can be uniquely represented by an ∞^2 system of null lines in M, namely by the generators of the null cone of P. Let K and L be two null lines in M through P. The generators of the null cone of P are then the null lines common to both $\bar{K}$ and $\bar{L}$ (i.e., the generators must meet both K and L). In the C picture this is an ∞^2 system[25] of lines on N which lie on the intersection of the two planes $\bar{K}$ and $\bar{L}$. This intersection is simply a complex projective (straight) line in C. Thus, we have the result that any point of M is represented, in the C picture, by a complex projective line which lies entirely on N. (A complex projective line is topologically a sphere S^2, in terms of its real structure. This agrees with the topology of the set of null lines through P, i.e., of the null directions at P.) We do not distinguish, notationally, here, between the point P in M and the system of null lines in M through P. Then we are at liberty to denote also by P the *line* in the C picture which represents this *point* P in the M picture.

Conversely, *any* line P in the C picture, which lies entirely on N, represents some point P in the M picture. (In the terminology of projective geometry, in the C picture, "line" always implies "complex projective straight line.") To see this, consider the C picture and let the line P lie entirely on N. Let K and L be two points on P. Then we have $K^\alpha\bar{K}_\alpha = 0$, $L^\alpha\bar{L}_\alpha = 0$, and, more generally $(K^\alpha + \beta L^\alpha)(\bar{K}_\alpha + \bar{\beta}\bar{L}_\alpha) = 0$ for all complex β (the general point on the line P having a twistor of the form $K^\alpha + \beta L^\alpha$). Hence $L^\alpha\bar{K}_\alpha = 0$. Thus, in the M picture, the null lines L and K must intersect. This holds for *any* pair of null lines belonging to the ∞^2 system represented by P. These null lines must therefore all meet in a *point* and, in fact, must be the generators of the null cone of this point. We thus label this point P, and the situation is as before.

Note that, in the C picture, the condition that a point L lies on a line P, both L and P lying on N, is interpreted in the M picture as the condition that the null line L passes through the point P. We have, in fact, a kind of duality correspondence between M and N. To sum up, we have the following correspondence between the M picture and the subset N in the C picture:

There is a one-to-one relation between the null lines in M and the points in N. (6.2)

There is a one-to-one relation between the points in M and the complex lines in N. (6.3)

The condition for a point to lie on a null line in M is interpreted, in N, as the condition for the corresponding line to pass through the corresponding point. (6.4)

From (6.4), we see that the condition that two points in M have a null separation is simply the condition that their corresponding lines in N should intersect; the condition that two null lines in M should meet is the condition that the join of the corresponding points in N should lie entirely in N.

We may ask how we should interpret the lines of C which do *not* lie entirely on N. Lines in a projective three space form a four-dimensional system, so that in terms of real dimensions these lines describe an ∞^8 system of objects in M. This suggests that a general line in C represents a *complexified* point in M, where now the straightforward doubling of dimensions suggests that these are complexified points in the usual sense, i.e., their position vectors are allowed to have imaginary parts. To see that this is indeed a consistent interpretation, consider a general line in the C picture as a join of two general points L, X of C. Using the notation of Secs. II and III, we can form the expression (3.10). We may regard this *complex vector p_i* as the position vector, the M picture, of the complex point of intersection of the two null lines ("complexified" or otherwise) in M, defined by L and X. (Note that in this sense *any*

[25] A symbol ∞^r indicates r dimensionality in the *real* sense.

two distinct null lines in M have a complex point in common.)

Any line P of C falls into one of six classes. If q^i is the *imaginary part* of the position vector p^i for the corresponding complex point P in M, then [from (3.10)]

$$P \subset C^+ \Leftrightarrow q^i \text{ timelike, future-pointing;} \tag{6.5}$$

$$P \subset C^+ \cup N,\ P \not\subset N,\ P \not\subset C^+ \Leftrightarrow q^i \text{ null, future-pointing;} \tag{6.6}$$

$$P \text{ intersects all of } C^-,\ N,\ C^+ \Leftrightarrow q^i \text{ spacelike, nonzero;} \tag{6.7}$$

$$P \subset C^- \cup N,\ P \not\subset N,\ P \not\subset C^- \Leftrightarrow q^i \text{ null, past-pointing;} \tag{6.8}$$

$$P \subset C^- \Leftrightarrow q^i \text{ timelike, past-pointing;} \tag{6.9}$$

$$P \subset N \Leftrightarrow q^i = 0. \tag{6.10}$$

In cases (6.6) and (6.8), the line P in C *touches* M. The set (6.5) (in the M picture) is sometimes referred to as a "future-tube."[10]

There is a representation of lines in projective three-space, well known to geometers, called the Klein (or Grassmann) representation.[26] This is a four- (complex-) dimensional quadric, the points of which correspond to the lines of complex projective three-space. Thus, in our case, we may regard the fully complexified version M^* of M as the Klein representation of the lines in C. We noted in Sec. III that M itself was essentially a real $(++----)$ quadric fourfold. M^* is the complexified version of this quadric. There are two systems of planes on M^*, called α planes and β planes. The α planes correspond to points in C and the β planes to planes in C. We may regard M as a submanifold of M^*. The α planes which meet M intersect M in the null lines of M. Similarly with the β planes. Thus, when properly complexified, the null lines in M become, in effect, pairs of complex projective planes, namely one α plane and one β plane for each fully complexified null line. The "complexification" of the null lines achieved here by the use of twistors amounts to selecting only the α planes to represent the null lines.

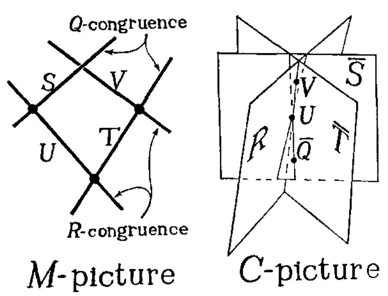

FIG. 5. The condition for $\bar{Q}^\alpha R_\alpha = 0$ is that, in the M picture, V and S should meet, and that in the C picture, $V \in \bar{S}$, whence $\bar{Q} \in R$.

Much of the earlier discussion could have been carried out in terms of the manifold M^*. The emphasis here, however, has been to try and represent twistors as far as possible in terms of "real" structures in the space–time M. One final condition remains to be represented, however, in order that the geometry of C can be completely realized in terms of M. This is that we must be able to interpret the vanishing of a twistor scalar product in geometrical terms[27] in M.

Consider the condition

$$\bar{Q}^\alpha R_\alpha = 0. \tag{6.11}$$

If $\bar{Q}$ and $\bar{R}$ are both null lines in M, we have seen that this is the condition for the lines to intersect. If $\bar{Q}$ is a null line and R a Robinson congruence, then (6.11) is simply the condition that $\bar{Q}$ should belong to the congruence R. It remains to consider the case when *both* Q and R are Robinson congruences in M. Let S and T be two null lines belonging to the Q congruence and let U and V be null lines belonging to the R congruence. Suppose the three pairs of lines (S, U), (U, T), (T, V) intersect. *Then a necessary and sufficient condition for* (6.11) *to hold is that the pair* (V, S) *also intersect* (Fig. 5). To see this, consider the points V, U, $\bar{Q}$ and the planes $\bar{S}$, $\bar{T}$, R in the C picture. We have $S \in Q$, $T \in Q$, whence $\bar{Q} \in \bar{S}$, $\bar{Q} \in \bar{T}$. Also $U \in R$, $V \in R$ and $U \in \bar{S}$, $U \in \bar{T}$, $V \in \bar{T}$. Thus, the line UV must be the intersection of the planes R and $\bar{T}$ (Fig. 5). If (6.11) holds, then Q also lies on this line, whence $V \in \bar{S}$. Conversely if $V \in \bar{S}$ then $\bar{Q} \in R$. Thus, in the M picture, the condition for (6.11) to hold is that the null lines V and S should meet.

This establishes the geometrical equivalence of the M picture with the C picture, whereby the conformal geometry of Minkowski space is completely represented in twistor terms. The *metric* geometry of Minkowski space, on the other hand, requires the introduction of a fixed metric twistor. This is done in Sec. X.

[26] See any standard work on classical algebraic geometry, for example, J. A. Todd, *Projective and Analytical Geometry* (I. Pitman, London, 1947); J. G. Semple and L. Roth, *Algebraic Geometry* (Clarendon Press, Oxford, England, 1949).

[27] Strictly, we should also show that the concept of a Robinson congruence is "geometrical" in M. An explicit construction in terms of incidence of null lines is given in Sec.IX, but in any case, the geometric (and conformally invariant) nature of a Robinson congruence is already implied by Sec.IV and V.

VII. TWISTOR TRANSFORMATIONS

Although the transformation properties of twistors have not been given explicitly as yet, they are implicit in the preceding discussion. Any continuous transformation of M into itself, which preserves its conformal structure (and, hence, the null cone and null line structure of M) must, by (6.4), correspond to a continuous transformation of N into itself which preserves its linearity structure. Furthermore, since the concept of a Robinson congruence is conformally invariant (cf., Sec. V), this transformation of N extends uniquely to a transformation of the whole space C. We have seen that the orthogonality relation (6.11) between points and planes in C can be stated in terms of (conformally invariant) incidence properties in M. Hence, collinearity of points in C has a conformally invariant interpretation in M. It follows that the conformal transformations of M are represented, in the C picture, by continuous point transformations of C, which preserve its linearity structure, and for which the submanifold N is invariant.

If we restrict ourselves to conformal transformations of M which are continuous with the identity, then the corresponding transformations of C is also continuous with the identity. Such transformations, preserving the linearity structure of C, must therefore be *projective point transformations*[26] of C (in the complex sense). If, on the other hand, we allow transformations of M which involve space or time reflections, then we must also consider *anti-projective* transformations of C, that is, transformations which combine a complex conjugation operation with a projective transformation (so that cross ratios become complex conjugated). We may also consider transformations of C which are not point transformations but *correlations*[26] (duality correspondences) in which points of C are mapped to planes, and planes to points. These would not strictly correspond to transformations of M, because of the way C has been constructed here, but it is convenient to think of them as representing transformations of the *complexified* version M^* of M. The *natural* (analytic) extension to M^* of a transformation defined on M sometimes turns out to correspond to a correlation in C.

The basic operation $\mathfrak{C}$ of *complex conjugation* in M^*, which leaves the real space M invariant, corresponds, in the C picture, simply to the *Hermitian correlation* given by $X^\alpha \leftrightarrow \bar{X}_\alpha$. A transformation of M equivalent to a *space reflection* corresponds, in the C picture, to a point transformation of C, which is antiprojective and interchanges C^- and C^+. (The twist orientation of Robinson congruences is reversed under space reflection.) However, if we wish to extend this to a space reflection $\mathfrak{P}$ of the whole of M^* we must, since $\mathfrak{P}$ is analytic, remove the antiprojective nature of the transformation of C and consider, instead, a *correlation* which sends points X of $C^+[C^-]$ into planes $\bar{Y}$ for which Y is in $C^-[C^+]$. The antiprojective point transformation just considered, which interchanges C^+ and C^- would then correspond to $\mathfrak{C}\mathfrak{P}$. Similarly, a point transformation of C which represents a *time reflection* of M is antiprojective, here transforming each of C^-, C^+ into itself. It therefore really represents $\mathfrak{C}\mathfrak{T}$, where $\mathfrak{T}$ describes the (analytic) extension to M^* of this time reflection of M. The transformation of C representing $\mathfrak{T}$ is a *correlation* sending each point X of $C^+[C^-]$ into a plane $\bar{Y}$ with $Y \in C^+[C^-]$. The transformation $\mathfrak{P}\mathfrak{T}$, being the product of two projective correlations, is a projective point transformation of C. It interchanges C^- with C^+, and is not continuous with the identity unless we widen the group of transformations of M^* to include *complex* conformal transformations. (These would not be point transformations of M, but we may view them as transformations on the line systems in M which transform Robinson congruences into one another. They are represented, in the C picture, as projective transformations of C which do not preserve N.) The transformation $\mathfrak{P}\mathfrak{C}\mathfrak{T}$ is an antiprojective correlation in C.

Let us examine more explicitly the effect of an allowable transformation on a general twistor $A^{\alpha\beta\cdots\delta}_{\rho\cdots\tau}$ of valence $[{}^r_s]$. (The indices $\alpha, \beta, \cdots, \delta$ are r in number and $\rho, \cdots, \tau$ are s in number, each ranging over four values 0, 1, 2, 3.) We may define a general twistor, in terms of twistors of valence $[{}^1_0]$ and $[{}^0_1]$, in any of the standard ways that "tensors" may be built up from "vectors," e.g., as linear combinations of outer products, or in terms of multilinear mappings of one-index twistors into the scalars. Alternatively, we may simply use the transformation properties to define a twistor $A^{\alpha\cdots\delta}_{\rho\cdots\tau}$ of valence $[{}^r_s]$ (considering for the moment only transformations continuous with the identity):

$$\bar{A}^{\alpha\beta\cdots\delta}_{\rho\cdots\tau} = A^{\kappa\lambda\cdots\nu}_{\varphi\cdots\psi} t^\alpha_\kappa t^\beta_\lambda \cdots t^\delta_\nu T^\varphi_\rho \cdots T^\psi_\tau. \tag{7.1}$$

The matrices (t^α_β), (T^α_β) are inverses[28] of each other

$$t^\alpha_\beta T^\beta_\gamma = \delta^\alpha_\gamma = T^\alpha_\beta t^\beta_\gamma, \tag{7.2}$$

[28] In fact, the transformations of C would be the same if we specified only that $(T_\beta{}^\alpha)$ be *proportional* to the inverse of $(t_\beta{}^\alpha)$ since the X^α are *projective* coordinates for C. However, the stronger requirement (7.2) is adopted here since the factor of proportionality of a twistor is required when the more complete representation in accordance with Sec. V is used.

where twistors X^α, Y_α, of respective valence $[{}^1_0]$, $[{}^0_1]$, transform as

$$\tilde{X}^\alpha = t^\alpha_\beta X^\beta, \qquad \tilde{Y}_\alpha = T^\beta_\alpha Y_\beta \tag{7.3}$$

under a conformal transformation of M continuous with the identity (i.e., a projective transformation of C preserving C^+). But since $\bar{X}_\alpha$ is a twistor of valence $[{}^0_1]$, it follows that

$$T^\alpha_\beta = \bar{t}^\alpha_\beta, \quad \text{i.e.,} \quad t^\alpha_\beta \bar{t}^\beta_\gamma = \delta^\alpha_\gamma, \tag{7.4}$$

where the convention used earlier for twistor indices under complex conjugation [cf., (3.5)] is being employed here, namely that, under complex conjugation, upper and lower index positions are interchanged and also, the pairs 0, 1 and 2, 3 are interchanged, i.e.,[17]

$$\bar{t}^0_0 = \overline{t^2_2}, \quad \bar{t}^0_1 = \overline{t^3_2}, \quad \bar{t}^0_2 = \overline{t^0_2}, \quad \bar{t}^0_3 = \overline{t^1_2},$$
$$\bar{t}^1_0 = \overline{t^2_3}, \cdots, \bar{t}^3_2 = \overline{t^0_1}, \quad \bar{t}^3_3 = \overline{t^1_1}.$$

Condition (7.4) states that the matrix (t^α_β) is pseudo-unitary in the sense that the form (6.1) is left invariant, that is, in matrix notation,

$$(t^\alpha_\beta)^* \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} (t^\gamma_\delta) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \tag{7.5}$$

where the asterisk denotes Hermitian conjugate in the usual sense.[29]

Equation (7.4) ensures that the operation of complex conjugation, according to the rules just stated, is a twistor operation. That is, if $A^{\alpha\beta\cdots\delta}_{\rho\cdots\tau}$ is a twistor of valence $[{}^q_r]$, then $\bar{A}^{\rho\cdots\tau}_{\alpha\beta\cdots\delta}$ is a twistor of valence $[{}^r_q]$ (where,[17] for example, $\bar{A}^{3\cdots0}_{21\cdots3} = \overline{A^{03\cdots1}_{1\cdots2}}$, etc.). Clearly, the operations of addition, outer multiplication, contraction (between upper and lower indices), and index permutation (not mixing upper with lower indices) are also twistor operations. [That is to say, they commute with (7.1).] We might also consider complexified twistor transformations (7.1) for which (7.4) does not hold. These correspond to *complex* conformal transformations of M^*, not preserving M, and the relation between $A^{\alpha\cdots\delta}_{\rho\cdots\tau}$ and $\bar{A}^{\rho\cdots\tau}_{\alpha\cdots\delta}$ would not be preserved. Such transformations will not be discussed here.

It will be appropriate to impose one final condition on the matrix (t^α_β), namely that it should be *unimodular*:

$$|t^\alpha_\beta| = 1. \tag{7.6}$$

This amounts to requiring that the Levi-Civita symbols $\epsilon_{\alpha\beta\gamma\delta}$, $\epsilon^{\alpha\beta\gamma\delta}$ [with fixed components +1, −1, 0 according as $(\alpha, \beta, \gamma, \delta)$ is an even permutation, an odd permutation, or no permutation of (0, 1, 2, 3)] shall be *twistors* rather than just "twistor densities." In fact, this condition is *necessary* if a strict interpretation of twistors according to the scheme of Sec. V is to be adhered to. The effect of (7.6) on the twistor algebra is only to enrich it slightly in that the operation of forming *duals* of skew-symmetric twistors is now allowable. Note also that

$$\epsilon_{\alpha\beta\gamma\delta} = \bar{\epsilon}_{\alpha\beta\gamma\delta}, \qquad \epsilon^{\alpha\beta\gamma\delta} = \bar{\epsilon}^{\alpha\beta\gamma\delta}, \tag{7.7}$$

since (2, 3, 0, 1) is an even permutation of (0, 1, 2, 3)

The transformations of the form (7.1) are not the only allowable twistor transformations since, as we have seen, the operations $\mathfrak{C}$, $\mathfrak{P}$, $\mathfrak{T}$, $\mathfrak{CP}$, $\mathfrak{CT}$, and $\mathfrak{PCT}$, give conformal transformations of M but they do not correspond to projective point transformations of C. On the other hand, $\mathfrak{PT}$ *does* have the form (7.1), with (7.2) and (7.6) holding, but where (7.4) is replaced by

$$T^\alpha_\beta = -\bar{t}^\alpha_\beta; \qquad t^\alpha_\beta \bar{t}^\beta_\gamma = -\delta^\alpha_\gamma. \tag{7.8}$$

Thus, for $\mathfrak{PT}$, we have $\bar{\bar{A}}\,{}^{\cdots}_{\cdots} = -\bar{\bar{A}}\,{}^{\cdots}_{\cdots}$ if the total valence of $A^{\cdots}_{\cdots}$ is odd. (This, again, is assuming that we impose (7.2). Had we chosen $t^\alpha_\beta T^\beta_\gamma = -\delta^\alpha_\gamma$ instead, then $\bar{\bar{A}}\,{}^{\cdots}_{\cdots} = \bar{\bar{A}}\,{}^{\cdots}_{\cdots}$, *but* $\bar{L}^\alpha \bar{R}_\alpha = -L^\alpha R_\alpha$, etc.) The operation $\mathfrak{C}$ is given by

$$A^{\alpha\beta\cdots\delta}_{\rho\cdots\tau} \rightarrow \bar{A}^{\rho\cdots\tau}_{\alpha\beta\cdots\delta}, \tag{7.9}$$

while $\mathfrak{PCT}$ is a combination of (7.9) with a transformation of the form (7.1) satisfying (7.8). The operations $\mathfrak{P}$ and $\mathfrak{T}$ belong to a class of twistor transformations given by

$$\tilde{A}^{\rho\cdots\tau}_{\alpha\beta\cdots\delta} = (\mp 1)^s A^{\kappa\lambda\cdots\tau}_{\varphi\cdots\psi} u_{\alpha\kappa} u_{\beta\lambda} \cdots u_{\delta\nu} \bar{u}^{\rho\varphi} \cdots \bar{u}^{\gamma\psi}, \tag{7.10}$$

where $(u_{\alpha\beta})$ and the transpose of $\mp(\bar{u}^{\alpha\beta})$ are unimodular inverse matrices:

$$u_{\beta\alpha}\bar{u}^{\beta\gamma} = \mp\delta^\gamma_\alpha; \qquad |u_{\alpha\beta}| = 1. \tag{7.11}$$

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[29] The representation of twistors in terms of components given in this paper is perhaps the most convenient, but is by no means the only one possible. It amounts to insisting that the coordinate basis twistors $E^\alpha{}_{(0)}$, $E^\alpha{}_{(1)}$, $E^\alpha{}_{(2)}$, $E^\alpha{}_{(3)}$ are null and satisfy $E^\alpha{}_{(0)}\bar{E}_{(2)\alpha} = 1 = E^\alpha{}_{(1)}\bar{E}_{(3)\alpha}$, with all the other scalar products vanishing. Another possible choice would be to require $E^\alpha{}_{(a)}\bar{E}_{(b)\alpha} = (-1)^b\delta_{ab}$ (b not summed!) in which case twistor complex conjugation would take the form $\bar{L}_0 = \overline{L^0}$, $\bar{L}_1 = -\overline{L^1}$, $\bar{L}_2 = \overline{L^2}$, $\bar{L}_3 = -\overline{L^3}$, instead of (3.5), and the fixed matrix in (7.5) would become diag $(1, -1, 1, -1)$. (The simple connection with spinors (2.13) would be lost, however.) The only *essential* restriction on the way in which the twistors are represented is that the signature of the form $X^\alpha\bar{X}_\alpha$ must be $(++--)$ [cf., (6.1)].

The minus sign applies when a reversal of space orientation is involved and the plus sign when it is the time orientation that is reversed. Equation (7.11) implies that the matrix $(u_{\alpha\beta})$ is also pseudo-unitary in the sense of (7.4) and (7.5), or (7.8). These transformations give the projective correlations of C. The antiprojective point transformations of C such as $\mathcal{C}\mathcal{P}$ and $\mathcal{C}\mathcal{T}$, are given by the combination of (7.10) with (7.9).

Let us now examine some twistor transformations in a little more detail. The *translation* given by

$$\tilde{x}^i = x^i + a^i, \qquad \tilde{\xi}^A = \xi^A \tag{7.12}$$

[using the notation of (3.1), (3.2)] results in

$$(\tilde{X}^\alpha) = (\xi^A,\ \eta_{A'} - i\xi^B a_{BA'}),$$

whence[30]

$$(t^\alpha_\beta) = \begin{bmatrix} -\epsilon^A{}_B & 0 \\ -ia_{A'B} & \epsilon_{A'}{}^{B'} \end{bmatrix}. \tag{7.13}$$

The *Lorentz rotation* given, with $(b^A{}_B)$ unimodular, by

$$\tilde{x}^{JJ'} = x^{KK'}b^J{}_K\bar{b}^{J'}{}_{K'}, \qquad \tilde{\xi}^A = \xi^B b^A{}_B, \tag{7.14}$$

results in

$$(\tilde{X}^\alpha) = (b^A{}_B\xi^B,\ -\bar{b}_{A'}{}^{B'}\eta_{B'}),$$

whence

$$(t^\alpha_\beta) = \begin{bmatrix} b^A{}_B & 0 \\ 0 & -\bar{b}_{A'}{}^{B'} \end{bmatrix}. \tag{7.15}$$

Combining (7.14) with (7.12) and (7.15) with (7.13), we get the general restricted Poincaré transformation

$$\tilde{x}^{JJ'} = x^{KK'}b^J{}_K\bar{b}^{J'}{}_{K'} + a^{JJ'}, \qquad \tilde{\xi}^A = \xi^B b^A{}_B \tag{7.16}$$

[actually, "inhomogeneous $SL(2, C)$ transformation"[10]] represented by

$$(t^\alpha_\beta) = \begin{bmatrix} b^A{}_B & 0 \\ -ia_{CA'}b^C{}_B & -\bar{b}_{A'}{}^{B'} \end{bmatrix} \tag{7.17}$$

so that matrices of the form (7.17) give a (two-valued) representation of the Poincaré group.

The *inversion*

$$\tilde{x}^i = 2a^2x^i(x^kx_k)^{-1} \tag{7.18}$$

$(a^4 > 0)$ is given if we put $\tilde{\xi}^{A'} = a^{-1}\eta^{A'}$, $\tilde{\eta}_A = -a\xi_A$, i.e.,

$$\tilde{X}_\alpha = (-a\xi_A,\ a^{-1}\eta^{A'})$$

since this agrees with (3.2) [and hence with (3.10)]. Thus, in this case

$$(u_{\alpha\beta}) = \begin{bmatrix} a\epsilon_{AB} & 0 \\ 0 & a^{-1}\epsilon^{A'B'} \end{bmatrix}. \tag{7.19}$$

Here a reversal in time orientation $(a^2 > 0)$ *or* space orientation $(a^2 < 0)$ is involved, so we get (7.10) rather than (7.1). The space–time reflection

$$\tilde{x}^i = -x^i, \qquad \tilde{\xi}^A = -\xi^A \tag{7.20}$$

is given if

$$\tilde{X}^\alpha = (-\xi^A,\ \eta_{A'})$$

so that[30]

$$(t^\alpha_\beta) = \begin{bmatrix} \epsilon^A{}_B & 0 \\ 0 & \epsilon_{A'}{}^{B'} \end{bmatrix}. \tag{7.21}$$

For a space reflection or a time reflection, we must introduce a timelike vector v^i at the origin. For convenience we normalize v^i to have length $\sqrt{2}$, so that

$$\tilde{x}^i = \mp x^i \pm v^iv_kx^k; \qquad v_kv^k = 2. \tag{7.22}$$

The upper sign refers to space reflection and the lower sign to time reflection. In spinor terms we have

$$\tilde{x}^{JJ'} = \pm x^{KK'}v_K^{J'}v_{K'}^{J}; \quad \tilde{\xi}^{A'} = \xi^B v_B^{A'}, \quad \tilde{\eta}_A = \pm\eta_{B'}v_A^{B'}$$

with $v_A^{B'}v_{B'}^C = -\delta_A^C$. Thus,

$$\tilde{X}_\alpha = (\pm v_A^{B'}\eta_{B'},\ v_B^{A'}\xi^B),$$

so that

$$(u_{\alpha\beta}) = \begin{bmatrix} 0 & \pm v_A{}^{B'} \\ v^{A'}{}_B & 0 \end{bmatrix}. \tag{7.23}$$

Note that $u_{\alpha\beta}$ is *symmetric* for a space reflection and *skew-symmetric* for a time reflection. Recall, also, that $u_{\alpha\beta}$ was skew-symmetric for the inversion (7.18) [cf., (7.19)]. The significance of this lies in the fact that there are two distinct kinds of projective correlation in C which are involutory (i.e., whose squares are the identity), namely polarity with respect to a quadric (corresponding to $u_{\alpha\beta}$ symmetric) and null-polarity with respect to a linear complex[26] ($u_{\alpha\beta}$ skew-symmetric). The null-polarity is distinguished by the fact that any point in C lies *on* the plane into which it is transformed, whence any null line in M meets the null line into which it is transformed. Thus, in addition to the cases of time reflection or inversion just considered, $u_{\alpha\beta}$ is skew-

[30] The staggering of the spinor indices is to indicate that, in each case, the left-hand index labels rows and right-hand index labels columns. Also, for notational consistency, $-\epsilon^A{}_B$, is used here instead of $\delta^A{}_B$, and $\epsilon_{A'}{}^{B'}$ instead of $\delta_{A'}{}^{B'}$, although they are all numerically equal!

symmetric in the case of a space reflection in a plane (that is, in a timelike hyperplane). The linear complex involved is, in each case, the system of lines in C which are left invariant by the correlation. Some of these lines lie in N corresponding to the points of M which are invariant under the transformation. These points constitute a *hypersphere* in M which is spacelike [as in (7.18) with $a^2 > 0$, or (7.22) with the lower sign] if a time reversal is involved, or timelike [e.g., (7.18) with $a^2 < 0$] if it is a change of spatial orientation that is involved. (A hyperplane is to be regarded as a case of a hypersphere in M.) Transformations for which $u_{\alpha\beta}$ is *symmetric* include, in addition to the space reflection in the origin (or, more correctly, in a timelike line), a reflection in a spacelike line which is accompanied by a time reflection. In these cases, the points of M left invariant by the transformation are represented by the generators which lie in N of the quadric (equation: $X^\alpha X^\beta u_{\alpha\beta} = 0$) defining the polarity in C. These invariant points in M constitute either a timelike circle (a timelike straight line being one case) or a pair of spacelike circles (a spacelike straight line together with a "circle" at infinity being one case).

The involutory projective *point* transformations of C are called harmonic perspectives.[26] They fall into two main classes, depending on whether the invariant points of C constitute two skew lines (when $t_\alpha^\alpha = 0$) or a point and a plane ($t_\alpha^\alpha = \pm 2, \pm 2i$). The *first* class is further subdivided according as the two skew lines both lie in N, both cross N, or lie one in C^+ and one in C^-. (No other cases are possible.) If the lines both lie on N, they correspond to two special invariant points of M. The spacelike 2-sphere in M of intersection of the null cones of these two special points consists also of invarient points. Examples of this type of transformation are the space–time reflection (7.20) in the origin O (whence O and I are the special invariant points and the invariant 2-sphere is at infinity) or a reflection in a plane together with a time reversal (in which case the plane is the invariant "2-sphere" and the special invariant points are both at infinity). If the two skew lines both cross N, we get a timelike 2-sphere of invariant points in M (e.g., a timelike 2-plane, for the case of an ordinary reflection in a line). If neither skew line meets N, there are *no* invariant points in M [e.g., the inversion (7.18) with $a^2 > 0$, followed by the time reflection of (7.22)]. The transformations of the *second* class are also interesting in that no points of M are left invariant. The plane R of invariant points of C meets N in a set of points representing a Robinson congruence. The lines of the Robinson congruence are all invariant under the transformation but not pointwise invariant. Noninvolutory transformations with this property also exist. These are intimately related to twistors and their *four-valuedness* and so will be described briefly next.

Choose a right-handed twistor R_α normalized so that

$$R_\alpha \bar{R}^\alpha = 1.$$

Define

$$t(\theta)_\beta^\alpha = e^{i\theta}(\delta_\beta^\alpha - \bar{R}^\alpha R_\beta) + e^{-3i\theta}\bar{R}^\alpha R_\beta \qquad (7.24)$$

for each real θ. Then $t(\theta)_\beta^\alpha$ satisfies (7.4) and (7.6), so it represents an allowable twistor transformation. Also

$$t(\theta)_\beta^\alpha t(\varphi)_\gamma^\beta = t(\theta + \varphi)_\gamma^\alpha$$

so that these transformations form a one-parameter subgroup of twistor transformations. We can represent any line L of the *Robinson congruence* R, in M, by a twistor L^α satisfying $L^\alpha R_\alpha = 0$ (and $L^\alpha \bar{L}_\alpha = 0$). Thus,

$$t(\theta)_\beta^\alpha L^\beta = e^{i\theta} L^\alpha$$

whence each line of R must be left invariant under the transformation. Furthermore, except for those values

$$\theta = \tfrac{1}{2}n\pi \qquad (n = \cdots, -2, -1, 0, 1, 2, \cdots)$$

for which $t(\theta)_\beta^\alpha$ is proportional to δ_β^α, there are *no points* of M invariant under the transformation. (This follows because, in the C picture, the lines which are left invariant are those which either pass through the point $\bar{R}$ or lie in the plane R. In neither case can these lines lie on N, since $\bar{R}$ is not on N and R does not touch N.)

Finally, observe that for $\theta = \frac{1}{2}\pi$ we have

$$t\left(\frac{\pi}{2}\right)_\beta^\alpha = i\,\delta_\beta^\alpha. \qquad (7.25)$$

This gives the identity transformation on C and therefore also the identity transformation on M. But it *multiplies every twistor of valence* $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ *by* i. Also (7.25) is continuous with the identity twistor transformation via (7.24) with $0 \le \theta \le \frac{1}{2}\pi$. Hence *twistors are essentially four-valued under conformal transformations of* M.

VIII. THE KERR THEOREM

Consider the problem of finding all null solutions of Maxwell's equations (i.e., with $F_{ij}F^{ij} = 0 =$

$F_{ij}F_{kl}\epsilon^{ijkl}$) in free space. Robinson[11] showed that the problem could be reduced to that of finding all the shear-free null geodetic congruences in the space-time [cf., (5.8)], since associated with any null field was a congruence of this type, while conversely, given such a congruence, all the corresponding solutions of Maxwell's equations could then be defined in terms of arbitrary complex functions. The result applies to curved space-times as well as flat. We are concerned here only with space-times which are (conformally) flat. With this restriction, Robinson's result applies also to null zero rest-mass fields of arbitrary spin ($>\frac{1}{2}$). (The result for spin 2 has a bearing on the construction of null solutions of Einstein's nonlinear field equations.[31])

The solution of the remainder of the problem for flat space-time, namely the construction of all the shear-free null congruences, is given by a remarkable theorem due to Kerr.[32] The theorem takes the following very natural form when stated in twistor terms.

A congruence of null lines in M is shear-free if and only if it is representable in C as the intersection of N with a complex analytic surface S in C (or as a limiting case of such an intersection). (8.1)

"Complex analytic" means here analytic in the sense of the complex structure of C, where S is a "surface" in the complex sense (i.e., it has four real dimensions). Thus, (8.1) tells us that the shear-free condition [cf., (5.8)] is, effectively, a Cauchy-Riemann type of relation in the C picture. [The reason for the final parenthetic remark in (8.1) is to enable certain exceptional cases to be incorporated, for which the congruence does not form an analytic system even in the real sense. These exceptional cases appear to occur only when the rotation of the congruence also vanishes, cf., (5.7).]

Suppose, first, that a congruence of null lines in M is defined by an equation

$$\varphi(X^\alpha) \equiv \varphi(\xi^A, \eta_{A'}) = 0, \tag{8.2}$$

where φ is analytic and homogeneous in the four complex variables X^α so that (8.2) defines a complex analytic surface S in C. We see that the congruence must be shear free. Putting $\eta_{A'} = -i\xi^A x_{AA'}$, as in (3.2), Eq. (8.2) can be used to solve for ξ^A, up to proportionality, as a function of $x_{AA'}$, i.e., to give the direction of the line X as a function of a point on X with position vector x^i. Euler's condition on φ, for it to be homogeneous (degree n), gives

$$X^\alpha \frac{\partial\varphi}{\partial X^\alpha} \equiv \xi^A \frac{\partial\varphi}{\partial\xi^A} + \eta_{A'} \frac{\partial\varphi}{\partial\eta_{A'}} = n\varphi.$$

Hence, by (3.2) and (8.2),

$$\xi^A\left(\frac{\partial\varphi}{\partial\xi^A} - ix_{AA'}\frac{\partial\varphi}{\partial\eta_{A'}}\right) = 0,$$

whence

$$\frac{\partial\varphi}{\partial\xi^A} - ix_{AA'}\frac{\partial\varphi}{\partial\eta_{A'}} = \kappa\xi_A \tag{8.3}$$

for some κ. The derivative of (8.2) with respect to $x_{AA'}$ [cf., (5.3)] must also vanish

$$\begin{aligned} 0 = \nabla^{BB'}\varphi &= \frac{\partial\varphi}{\partial\xi^A}\nabla^{BB'}\xi^A + \frac{\partial\varphi}{\partial\eta_{A'}}\nabla^{BB'}\eta_{A'} \\ &= \frac{\partial\varphi}{\partial\xi^A}\nabla^{BB'}\xi^A \\ &\quad + \frac{\partial\varphi}{\partial\eta_{A'}}(-ix_{AA'}\nabla^{BB'}\xi^A - i\xi^A\nabla^{BB'}x_{AA'}) \\ &= \kappa\xi_A\nabla^{BB'}\xi^A - i\frac{\partial\varphi}{\partial\eta_{B'}}\xi^B, \end{aligned}$$

by (3.2) and (8.3). Assuming $\kappa \neq 0$, contraction with ξ_B gives $\xi_B\xi_A\nabla^{BB'}\xi^A = 0$, i.e., the condition (5.8) for the null directions defined by ξ^A to be tangent to *shear-free* null straight lines, as required. Finally, κ *cannot* vanish if (8.2) represents a genuine condition ξ^A, since the left-hand side of (8.3) is simply the derivative, with respect to ξ^A, of φ considered as a function of ξ^A and $x_{AA'}$.

The converse result that (8.2) represents essentially the most *general* shear-free congruence in M is somewhat less straightforward because of the existence of exceptional cases. However, if we assume that the congruence is *analytic in the real sense*—and any nonanalytic congruence (presumably necessarily rotation-free) can be approximated arbitrarily closely by analytic ones—then we can see from the form of (5.8) that the congruence is *determined* once the directions of the lines (i.e., of ξ^A) are known on any *spacelike 2-sphere F* in M. [If $\xi^0/\xi^1 = \zeta$, then (5.8) becomes $\partial\zeta/\partial x^{00'} = -\zeta\partial\zeta/\partial x^{10'}$, $\partial\zeta/\partial x^{11'} = -\zeta^{-1}\partial\zeta/\partial x^{01'}$. This defines the propagation of ζ

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[31] I. Robinson and A. Trautman, Proc. Roy. Soc. (London) **A265**, 463 (1962).

[32] R. P. Kerr, private communication; cf. also R. P. Kerr, Phys. Rev. Letters **11**, 238 (1963); R. P. Kerr and A. Schild, in Proceedings of the American Mathematical Society Symposium, April 1964. According to Kerr's original construction, a general shear-free null geodetic congruence is defined by an analytic relation $\phi(\eta, x^{00'} + \eta x^{10'}, x^{01'} + \eta x^{11'}) = 0$, where $dx^{00'} + \eta dx^{10'} = 0 = dx^{01'} + \eta dx^{11'}$.

off the 2-plane $x^{00'} = x^{11'} = 0$. On this 2-plane—which from the point of view of M is a 2-sphere—ζ can be given arbitrarily as a function of the complex variable $x^{01'}$ and its complex conjugate $x^{10'}$.] The 2-sphere F is the intersection of two null cones (vertices A, B) in M and is therefore represented, in the C picture by the lines lying in N which meet two skew lines A, B in N. The *complexification* F^* of F is, in the C picture, the system of *all* lines in C meeting both A and B. Now (in the M picture), F is a *real environment*[10] for F^*, so that any complex analytic function on F extends uniquely to a complex analytic function on F^*. The null directions of the congruence, defined at points of F, are represented in the C picture as an ∞^2 system of points lying on the lines of the F system (describing a *real* analytic surface). These extend uniquely to an ∞^4 system of points S on the lines of the F^* system (describing a *complex* analytic surface). The intersection of S with N then defines the given shear-free congruence as required (since the congruence defined by S agrees with the given one on F).

The Kerr theorem provides a very convenient means of studying the structure of shear-free null congruences in M, in general. Only a few results are briefly indicated here. For example, the lines of a shear-free null congruence along which the *rotation vanishes* are represented in C by the points where lines of N touch S. We can also generally form the *reciprocal*[26] $\bar{S}$ of S with respect to the Hermitian correlation defined by N (i.e., the envelope of polar planes with respect to N of points of S). Then $\bar{S}$ defines a shear-free null congruence in M which is, in a sense, "reciprocal" to the original one. The congruences which are *everywhere rotation-free* are the ones which are *self-reciprocal* (although the individual lines of the congruence do not generally reciprocate to themselves). In this case, S is a *ruled surface*[26] in C, with ∞^1 of its generators lying in N. This ∞^1 system corresponds, in M, to a *curve* and the lines of the congruence are just the null lines meeting this curve.[33] This curve in M is *null* if the ruled surface in C is *developable*.

Of special interest are the *algebraic* shear-free null congruences.[34] In this case S is algebraic variety[26] and φ can be given as a homogeneous polynomial

$$\varphi(X^\alpha) \equiv S_{\alpha\beta\cdots\delta}X^\alpha X^\beta \cdots X^\delta = 0, \tag{8.4}$$

where $S_{\alpha\beta\cdots\delta}$ is symmetric, of valence $\left[{0 \atop n}\right]$. In terms of ξ^A and $\eta_{A'} = -i\xi^A x_{AA'}$, (8.4) becomes

$$\xi^A\xi^B \cdots \xi^D\Phi_{AB\cdots D} = 0, \tag{8.5}$$

where the spinor field $\Phi_{AB\cdots D}$ is defined by

$$\Phi_{AB\cdots D} = \overset{0}{\sigma}_{(AB\cdots D)} + \overset{1}{\sigma}_{A'(B\cdots D}x^{A'}_{A)} + \cdots + \overset{n}{\sigma}_{A'B'\cdots D'}x^{A'}_{(A}x^{B'}_{B} \cdots x^{D'}_{D)}, \tag{8.6}$$

the $\overset{r}{\sigma}\ldots$'s being constants which are essentially the coefficients $S_{\alpha\beta\cdots\delta}$. (The round brackets denote symmetrization.) The $\Phi_{AB\cdots D}$ is symmetric, satisfies the "field equation"

$$\nabla^{E'}_{(E}\Phi_{AB\cdots D)} = 0 \tag{8.7}$$

by virtue of (8.6) and has a "canonical decomposition"[7]

$$\Phi_{AB\cdots D} = \overset{1}{\xi}_{(A}\overset{2}{\xi}_{B} \cdots \overset{n}{\xi}_{D)}, \tag{8.8}$$

where each ξ of (8.8) satisfies (8.5). Conversely, *every* solution of (8.7) has the form (8.6) and each resulting ξ of (8.8) [or (8.5)] defines (one branch of) the corresponding algebraic shear-free null congruence. This generalizes the results of Sec. V (for which $n = 1$) and gives us a representation of an arbitrary symmetric twistor of valence $\left[{0 \atop n}\right]$, in terms of a symmetric spinor field satisfying (8.7).

Robinson's construction of the general, null, zero rest-mass field from its associated shear-free congruence can also conveniently be represented in twistor terms: the field can be defined in terms of a complex function on S. However, such matters are not entered into here. The twister description of physical fields, generally, is left to a later paper.

IX. GEOMETRICAL APPLICATIONS OF TWISTORS

Many interesting geometrical properties arise from the interplay between the geometric structure of M^* and that of C. Some of these result simply from the fact that the former is the Klein representation of the latter and are, therefore, essentially classical results of algebraic geometry.[26] Others, however, take into account the reality structure of M and so have a more direct relevance to the structure of the physical world. The natural algebra of the C picture—namely twistor algebra—can be used to derive certain geometrical properties of M. A small selection is given here.

A linear space of dimension r in C can be represented by a (*simple*) *skew-symmetric twistor* of valence $\left[{r+1 \atop 0}\right]$ or by its *dual* of valence $\left[{0 \atop 3-r}\right]$, ($r =$

[33] The rotation-free, shear-free null congruences which are *not* analytic in the real sense emerge here simply as the system of null lines meeting a *nonanalytic* curve in M.

[34] Explicit shear-free null congruences of this type have been used to generate explicit solutions of Einstein's equations. For example, in Kerr's construction of the field of a rotating body, $\varphi(x^\alpha)$ is quadratic. For details, see Ref. 32.

0, 1, 2). The relation between general skew-symmetric twistors and their duals is

$$A_{\alpha\beta\gamma} = A^{\delta}\epsilon_{\alpha\beta\gamma\delta}, \quad B_{\alpha\beta} = \tfrac{1}{2}B^{\gamma\delta}\epsilon_{\alpha\beta\gamma\delta}, \quad C_{\alpha} = -\tfrac{1}{6}C^{\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta}, \tag{9.1}$$

where conversely

$$A^{\alpha} = -\tfrac{1}{6}A_{\beta\gamma\delta}\epsilon^{\alpha\beta\gamma\delta}, \quad B^{\alpha\beta} = \tfrac{1}{2}B_{\gamma\delta}\epsilon^{\alpha\beta\gamma\delta}, \quad C^{\alpha\beta\gamma} = C_{\delta}\epsilon^{\alpha\beta\gamma\delta}. \tag{9.2}$$

Because of (7.7) and the symmetry between (9.1) and (9.2), the operations of complex conjugation and of forming the dual commute. The meaning of, for example, $\bar{B}^{\alpha\beta}$, given $B^{\alpha\beta}$, is therefore unambiguous. We call $B^{\alpha\beta}$ *real* if $B^{\alpha\beta} = \bar{B}^{\alpha\beta}$ (whence also $B_{\alpha\beta} = \bar{B}_{\alpha\beta}$).

A skew-symmetric twistor $P^{\alpha\beta}$ is *simple* if it is of the form

$$P^{\alpha\beta} = X^{\alpha}Y^{\beta} - Y^{\alpha}X^{\beta}, \tag{9.3}$$

whence $P_{\alpha\beta}$, $\bar{P}^{\alpha\beta}$, $\bar{P}_{\alpha\beta}$ are all also simple. ($C^{\alpha\beta\gamma}$ and $A_{\alpha\beta\gamma}$ are necessarily simple.) Equivalent alternative conditions for $P^{\alpha\beta}$ to be simple are

$$P^{\alpha\beta}P_{\alpha\gamma} = 0 \quad \text{or} \quad P^{\alpha\beta}P_{\alpha\beta} = 0. \tag{9.4}$$

As given by (9.3), $P^{\alpha\beta}$ represents, in the C picture, the *line* joining two points X and Y. In the M picture, $P^{\alpha\beta}$ represents a complex *point* P (i.e., point of M^*). Thus, we can use *simple skew-symmetric twistors* $P^{\alpha\beta}$ *for projective coordinates in* M^*.

Now, the Hermitian correlation defined by N effects the correspondence $P^{\alpha\beta} \leftrightarrow \bar{P}_{\alpha\beta}$. This is the complex conjugation operation in M^* [cf., (7.9)] so, in terms of our coordinates $P^{\alpha\beta}$, complex conjugation in M^* is defined by $P^{\alpha\beta} \leftrightarrow \bar{P}^{\alpha\beta}$, the points of M being given when $P^{\alpha\beta}$ is *real* (in the above sense). Corresponding to (6.5)–(6.10) we can also express, in terms of $P^{\alpha\beta}$, the space–time nature of the imaginary part q^i of the position vector of a complex point P of M^*:

$$P^{\alpha\beta}\bar{P}_{\alpha\beta} \gtreqless 0 \text{ according as } q^i q_i \gtreqless 0, \tag{9.5}$$

and if $q^i q_i \geq 0$, then for all Z_{α}

$$Z_{\alpha}\bar{Z}^{\beta}P^{\alpha\gamma}\bar{P}_{\beta\gamma}\begin{cases} \geq 0 & \text{if } q^i \text{ future-pointing,} \\ \leq 0 & \text{if } q^i \text{ past-pointing.}\end{cases} \tag{9.6}$$

[The point of intersection of the plane Z with the line P in C is represented by $Z_{\alpha}P^{\alpha\gamma}$ and, cf., (6.5), (6.6), (6.8), and (6.9).]

A complex point of M^* can be realized in terms of a *real* structure in M in various ways; for example, in terms of the involutory transformation (see Sec. VII) which is represented in C by the harmonic perspective whose axes are the lines P and $\bar{P}$ in C (assuming P and $\bar{P}$ are skew, i.e., $P_{\alpha\beta}\bar{P}^{\alpha\beta} \neq 0$); or in terms of linear systems of Robinson congruences. However, if q^i is *spacelike*, we have a more easily visualizable representation.[35] In this case, the lines P, $\bar{P}$ in C each meets N in an ∞^1 system of points. In the M picture, these become two ∞^1 systems of null lines. No two null lines belonging to the same system can intersect, but every null line of the P system meets every null line of the $\bar{P}$ system (since in the C picture, all the points of the line P are conjugate under the Hermitian correlation to all the points of the line $\bar{P}$.) We can refer to the P system of lines in M as a *null regulus* and the $\bar{P}$ system as its *complementary* null regulus.[26] Any two nonintersecting null lines in M belong to a unique null regulus; the null transversals of these two null lines generate the complementary null regulus. The null reguli in M thus geometrically represent the points in M^* whose position vectors have spacelike imaginary parts.

A *real* skew-symmetric twistor $B^{\alpha\beta}(=\bar{B}^{\alpha\beta})$, which is *not* simple, also has a direct interpretation in M. We can, in fact, normalize $B^{\alpha\beta}$ so that

$$B^{\alpha\beta}B_{\alpha\gamma} = \pm\delta_{\gamma}^{\beta}. \tag{9.7}$$

Then, if we put $B_{\alpha\beta} = u_{\alpha\beta}$ in (7.10) we get an involutory projective correlation, in C, of the type for which the invariant points in M constitute a hypersphere (cf., Sec. VII). Thus, $B^{\alpha\beta}$ represents a *hypersphere* in M which is spacelike or timelike according as the upper or lower sign occurs in (9.7). (In the limiting case when $B^{\alpha\beta}$ becomes simple, the hypersphere becomes a null cone with vertex B.)

We have seen that A^{α} represents a point A in C and that $\bar{A}_{\alpha}$ represents its "polar plane" with respect to N (cf., Sec. VI). A lies on N if and only if $A^{\alpha}\bar{A}_{\alpha} = 0$, which is also the condition for $\bar{A}$ to touch N. In terms of the dual twistor $A_{\alpha\beta\gamma}$, this condition is

$$\bar{A}^{\alpha\beta\gamma}A_{\alpha\beta\gamma} = 0. \tag{9.8}$$

Let X^{α}, Y^{α}, Z^{α} be three (nonzero) null twistors so

[35] When q^i is timelike we may represent P as a parallelism on M (with torsion; left-handed if $P \subset C^+$) which is closely related to Clifford parallelism on S^3. The sets of *null* directions which are to be regarded as parallel are those of the Robinson congruences represented by the points of the line P in C (i.e., by planes through $\bar{P}$). A transitive four-parameter group of (conformal) motions of M preserves this parallelism, namely that given by twistor transformations (7.1) for which the line P is left pointwise invariant. This group is readily seen to be the group of unitary (2×2) matices and leads to Uhlmann's representation (see Ref. 18) of the points of M in terms of such matrices.

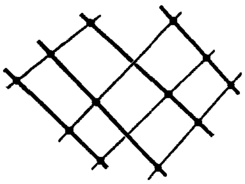

FIG. 6. A geometrical theorem concerning null lines in M.

that, in the M picture, X, Y, and Z are three null lines. Put

$$\bar{A}^{\alpha\beta\gamma} = X^{[\alpha}Y^{\beta}Z^{\gamma]}. \tag{9.9}$$

(Square brackets denote skew-symmetrization.) In the M picture, A is generally a "complexified null line". But if (9.8) is satisfied, then A is a real null line in M which meets each of X, Y, and Z. ($\bar{A}_\alpha X^\alpha = 0$ follows from $\bar{A}^{[\alpha\beta\gamma}X^{\delta]} = 0$, etc.) Thus, *the condition for the three null lines X, Y, Z to have a common null transversal* is simply (9.9) substituted into (9.8), i.e.,

$$\{XY\}\{YZ\}\{ZX\} = 1, \tag{9.10}$$

where

$$\{XY\} = -(X^\alpha \bar{Y}_\alpha)(Y^\beta \bar{X}_\beta)^{-1} = \{YX\}^{-1}$$

etc. (since X^α, Y^α, and Z^α are null). We assume no two of X, Y, Z meet so that (9.10) is well defined. If X, Y, and Z have *two* common transversals, then (9.9) vanishes and X, Y, and Z belong to a null regulus.

It is possible to derive a host of geometrical theorems concerning incidence of null lines in M, from the condition (9.10). For example, if four null lines in M, of which no two meet, are such that there is a null transversal to each of three different selections of triplets of the lines, then there is also a null transversal to the remaining triplet. (The configuration is that depicted in Fig. 6—except for the case when there is a single common transversal to all four lines.) Some theorems of this general type can be generated by means of a diagrammatic notation: a graph made up of triangles whose *vertices* represent null lines in M, can be used. The triangles represent triples of lines with a common null transversal. Now, if any one of these triangles represents a circuit in the graph linearly dependent on the circuits given by other triangles, then, because of the form of (9.10), we have a geometrical theorem. For the case just considered (cf., Fig. 6), the graph would be that of a tetrahedron.[36] Any other polyhedron of triangles would also give rise to a theorem on null lines.

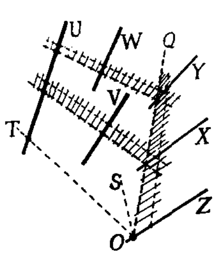

FIG. 7. Construction of the Robinson congruence containing three given null lines U, V, and W.

Finally, let us consider the *construction of a Robinson congruence* R containing three given null lines U, V, W in general position in M. (In the C picture, R is the plane of the three points U, V, W.) We wish to construct the line of R which passes through a general point O in M. Now, a Robinson congruence is characterized by the fact that, if any two null lines X, Y belong to the congruence, then so does every line of the null regulus containing X and Y. (In the C picture, the join of any two points in a plane also lies on the plane.) We have seen that the null regulus containing X and Y is simply the set of null transversals of any pair of null transversals of X and Y. Thus, if we can find X and Y belonging, respectively, to the null regulus containing U, V and to that containing U, W such that the null regulus containing X and Y also contains a line Z through O (see Fig. 7), the construction is complete. What is required, in fact, is to show that there is a null line Q through O which meets a line (namely X) of the U, V regulus and a distinct line (namely Y) of the U, W regulus; i.e., the triplets U, V, Q and U, W, Q must each have a null transversal. In fact, there is, in general, a unique such line Q through O. For, putting $Q^\alpha = S^\alpha + \zeta T^\alpha$, where T and S are null lines through O, with T meeting U, and substituting in (9.10), we get two simultaneous linear equations in ζ and $\bar{\zeta}$. The condition for these equations to have a unique solution reduces to $\{UV\}\{VT\}\{TW\}\{WU\} \neq 1$ which, for general positions of O, is indeed satisfied.[37]

X. THE METRIC TWISTOR

If we wish to use twistors to describe the Minkowski metric structure of M (i.e., $M\{I\}$), rather than just its conformal structure, we may do this by introducing a *metric twistor* $I^{\alpha\beta}$ which represents the point I of M, according to the scheme of Sec. IX. Thus, $I^{\alpha\beta}$ is simple, skew-symmetric, and real (in the sense of Sec. IX):

$$I^{\alpha\beta}I_{\alpha\gamma} = 0, \tag{10.1}$$

$$I^{\alpha\beta} = \bar{I}^{\alpha\beta} \tag{10.2}$$

[36] Since a real null line in M defines both a point in C and a plane in C through this point, the configuration of Fig. 6 is represented in the C picture as a pair of mutually inscribed and circumscribed tetrahedra—a configuration familiar to geometers. We may note that the *full* complexification of a null line in M leads, in the C picture, strictly to a point in C *together* with a plane through it. This gives a five-complex-dimensional system as we would expect.

[37] It only fails if there is a circle through O meeting U, V, and W which lies on a null cone through U.

(with $I^{\alpha\beta} = -I^{\beta\alpha}$). The twistor transformations which leave $I^{\alpha\beta}$ invariant give the *Poincaré group.* (The transformations which leave $I^{\alpha\beta}$ invariant up to *proportionality* give the Poincaré group plus the *dilatations.* These would be just the conformal transformations of M which leave the point I—and its null cone—invariant.)

We may also treat *de Sitter space–time* in a very similar way. In this case there is a hypersphere at infinity rather than a null cone at infinity. As we have seen in Sec. IX, this can be described by a real, skew-symmetric twistor which is *not simple,* i.e., we just drop condition (10.1). Other conformally flat space–times can also be treated. For example, if we retain (10.1) but drop (10.2) [and replace (10.3) by its modulus], we can describe the Einstein static universe. The matter is not pursued further here, however.

We can generate Poincaré covariant operations simply by employing twistor algebra and admitting, as basic elements of the algebra (in addition to $\epsilon_{\alpha\beta\rho\sigma}$, $\epsilon^{\alpha\beta\rho\sigma}$, δ_β^α), the twistors $I^{\alpha\beta}$, $I_{\alpha\beta}$, satisfying (10.1), (10.2). We can represent points P, of $M^*\{I\}$ by simple skew-symmetric twistors $P^{\alpha\beta}$ normalized, for convenience, so that

$$P^{\alpha\beta}I_{\alpha\beta} = 2. \tag{10.3}$$

($P \in M\{I\}$ if $P^{\alpha\beta} = \bar{P}^{\alpha\beta}$.) Then the correspondence between P and $P^{\alpha\beta}$ is unique.[38] If $Q^{\alpha\beta}$ and $R^{\alpha\beta}$ similarly represent points of $M^*\{I\}$, then an example of a Poincaré covariant operation is

$$aP^{\alpha\beta} + bQ^{\alpha\beta} + cR^{\alpha\beta}$$
$$- \tfrac{1}{2}\{bcQ^{\gamma\delta}R_{\gamma\delta} + caR^{\gamma\delta}P_{\gamma\delta} + abP^{\gamma\delta}Q_{\gamma\delta}\}I^{\alpha\beta}. \tag{10.4}$$

The significance of this particular expression is that if

$$a + b + c = 1 \tag{10.5}$$

then (10.4) represents the point in $M^*\{I\}$ whose position vector is $ap^i + bq^i + cr^i$, where p^i, q^i, r^i are the respective position vectors of P, Q, R. [This vector operation is clearly Poincaré covariant if (10.5) holds and represents a weighted mean.] Expression (10.4) generalizes to any number of points in an obvious way. Other less involved expressions than (10.4) can, of course, also be given and define Poincaré covariant operations in $M\{I\}$. For example, $P^{\alpha\beta} + aI^{\alpha\beta}$ and $P^{\alpha\beta} - Q^{\alpha\beta}$ represent, respectively, a hypersphere center P and the hyperplane bisecting PQ orthogonally. (These twistors are not simple, so they represent "hyperspheres" rather than points.) We can check the coefficients in (10.4) using

$$P^{\alpha\beta}Q_{\alpha\beta} = -(p_i - q_i)(p^i - q^i), \tag{10.6}$$

which is a familiar Poincaré invariant expression.

The derivation of expressions such as (10.6) is facilitated if we specialize our twistor structure still further by the introduction of an *origin twistor* $O^{\alpha\beta}$ which represents a particular point O (the "origin") in M. Here, $O^{\alpha\beta}$ is to be simple, skew-symmetric, real and normalized as in (10.3)

$$O^{\alpha\beta}O_{\alpha\gamma} = 0, \qquad \bar{O}^{\alpha\beta} = O^{\alpha\beta}, \qquad O^{\alpha\beta}I_{\alpha\beta} = 2. \tag{10.7}$$

By putting $Q^{\alpha\beta} = O^{\alpha\beta}$ in (10.4) and (10.6), we obtain the twistor expressions for the Lorentz covariant vector operations of linear combination and squared magnitude.

Since the transformation group leaving both $I^{\alpha\beta}$ and $O^{\alpha\beta}$ invariant is the Lorentz group, we can expect to be able to express *spinors,* referred to the origin O, in *terms* of twistors. In effect, we carry out the construction given in Secs. II and III for twistors in terms of spinors, but in reverse. This gives us a correspondence between spinor indices and "reduced" twistor indices for which

$$\begin{aligned} O^{\alpha\beta} &\leftrightarrow \epsilon^{AB}, & I_{\alpha\beta} &\leftrightarrow \epsilon_{AB}, \\ O_{\alpha\beta} &\leftrightarrow \epsilon^{A'B'}, & I^{\alpha\beta} &\leftrightarrow \epsilon_{A'B'}. \end{aligned} \tag{10.8}$$

The *orthogonal idempotents* which reduce the twistor space—to a direct sum of two spinor spaces (one unprimed and one primed)—are

$$\begin{aligned} J_\beta^\alpha &= O^{\alpha\gamma}I_{\beta\gamma} \leftrightarrow \delta_B^A = \epsilon^{AG}\epsilon_{BG}, \\ \bar{J}_\beta^\alpha &= O_{\beta\gamma}I^{\alpha\gamma} \leftrightarrow \delta_{A'}^{B'} = \epsilon^{B'G'}\epsilon_{A'G'}. \end{aligned} \tag{10.9}$$

We have [from (10.1), (10.2), (10.7), etc.]

$$\begin{aligned} J_\beta^\alpha J_\gamma^\beta = J_\gamma^\alpha, \qquad J_\beta^\alpha \bar{J}_\gamma^\beta = 0 = \bar{J}_\beta^\alpha J_\gamma^\beta, \\ \bar{J}_\beta^\alpha \bar{J}_\gamma^\beta = \bar{J}_\gamma^\alpha, \qquad J_\beta^\alpha + \bar{J}_\beta^\alpha = \delta_\gamma^\alpha. \end{aligned} \tag{10.10}$$

A general twistor $P_{\rho\ldots\tau}^{\alpha\beta\ldots\delta}$, of valence $[{}^r_s]$ corresponds to a set of 2^{r+s} spinors. To obtain such a spinor, each index of $P_{\rho\ldots\tau}^{\alpha\beta\ldots\delta}$ is transvected with either a J_β^α or a $\bar{J}_\beta^\alpha$. The resulting twistor then represents a spinor with an unprimed index corresponding to each free J_β^α index and a primed index, in the *reverse position,* corresponding to each free $\bar{J}_\beta^\alpha$ index. For example,

$$P^{\kappa\lambda\mu}{}_{\varphi\chi}J_\kappa^\alpha \bar{J}_\lambda^\beta J_\mu^\gamma \bar{J}_\rho^\varphi J_\sigma^\chi \leftrightarrow \Pi^A{}_{B'}{}^{GR'}{}_S.$$

[38] In practice, the X^α and Y^α of the decomposition (9.3) (which may be specilized if desired) often turn out to be more convenient coordinates than $P^{\alpha\beta}$. This is of value in connection with physical fields and will be discussed elsewhere.

With the different possibilities for J^{α}_{β} and $\bar{J}^{\alpha}_{\beta}$, these give 2^{r+s} different spinors, which [by (10.10)] together determine $P^{\alpha\beta\cdots\delta}_{\rho\cdots\tau}$. Any spinor operation can thus be mirrored in twistor terms, using J^{α}_{β} and $\bar{J}^{\alpha}_{\beta}$. The correspondences of (10.8) and (10.9) are consistent with this.

The basic relations (2.13) and (3.1) are expressed as

$$X^{\beta}J^{\alpha}_{\beta} \leftrightarrow \xi^{A}, \qquad X^{\beta}\bar{J}^{\alpha}_{\beta} \leftrightarrow \eta_{A'},$$

$$L^{\beta}J^{\alpha}_{\beta} \leftrightarrow \lambda^{A}, \qquad L^{\beta}\bar{J}^{\alpha}_{\beta} \leftrightarrow \mu_{A'}.$$

If we put

$$\nu P^{\alpha\beta} = X^{\alpha}L^{\beta} - L^{\alpha}X^{\beta}$$

[cf., (9.3)] with ν chosen so that (10.3) holds, we get $\nu = \xi_A\lambda^A$. Then,

$$\left.\begin{aligned} &P^{\rho\sigma}J^{\alpha}_{\rho}J^{\beta}_{\sigma} \leftrightarrow \epsilon^{AB}, \\ &P^{\rho\sigma}J^{\alpha}_{\rho}\bar{J}^{\beta}_{\sigma} \leftrightarrow -ip^{A}_{B'}, \\ &P^{\rho\sigma}\bar{J}^{\alpha}_{\rho}\bar{J}^{\beta}_{\sigma} \leftrightarrow -\tfrac{1}{2}p_ip^i\epsilon_{A'B'}, \end{aligned}\right\} \qquad (10.11)$$

where $p_{AB'}$ is given as in (3.10). In the C picture P is the line joining two points X and L; in M^*, P is the (complex) point of intersection of two ("complexified") null lines. Thus, according to (3.10), p^i represents the *position vector*, with respect to O, of the point P in M^*. We can express (10.11) in matrix form as

$$(P^{\alpha\beta}) = \begin{pmatrix} \epsilon^{AB} & -ip^{A}{}_{B'} \\ ip_{A'}{}^{B} & -\tfrac{1}{2}p_ip^i\epsilon_{A'B'} \end{pmatrix}.$$

Expressions such as (10.6) and (10.4) then follow at once.

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