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SUPERTWISTORS AND CONFORMAL SUPERSYMMETRY *

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Supertwistors that extend the twistor concept to supersymmetry are introduced and studied. Using them the superconformal algebra $SU(2,2|N)$ and its supermultiplets are derived.

1. Introduction

The objective of twistor theory is a description of space-time in terms emphasizing its conformal structure. The aim of supersymmetry theory is the establishment of symmetry constraints in models of interactions among Bose and Fermi particles. Major proponents of both twistors [1] and supersymmetry [2] have suggested that these two theories are related. However, to the knowledge of this author, no clear-cut relation has yet been established [3].

In this paper we give an extension of classical twistor theory which encompasses many elements of supersymmetry theory. In particular, we show how the Lie superalgebra [4] $SU(2,2|N)$ and its non-linear realization as the superconformal algebra [5] arise in a natural manner.

2. Model of a spinning particle

Consider a classical zero-mass spinning particle. To describe the spin and possible internal degrees of freedom of the particle, we take for the relativistic configuration space not just Minkowski space, but a superspace. Its Bose coordinates are the components of a real Minkowski vector $X^{AA'}$ †. The Fermi coordinates are $2N$ complex

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† The Weyl spinor formalism used here is the generalization of the standard formalism [6] to the graded case. The only real differences have to do with factor ordering. In particular, the complex conjugation operation must be modified so that the complex conjugate of a product is the product of the complex conjugates of the factors written in reversed order [7,8].

quantities forming the components of N Weyl spinors $\theta^{\alpha A}$ ($\alpha = 1, 2, \dots N$). The $\theta^{\alpha A}$ and their complex conjugates $\bar\theta^{\alpha A'}$ are anticommuting variables and so generate a Grassmann algebra spanned by 2^{4N} elements. This description of a spinning particle differs from that of other authors [7,9] who instead use anticommuting vector quantities. We take for the first-order form of the action [10]

$$A = \int \mathrm{d}\tau\, p_{AA'}[\dot X^{AA'} - \tfrac{1}{2}p^{AA'} - \tfrac{1}{2}i(\bar\theta^{\alpha A'}\dot\theta^{\alpha A} + \theta^{\alpha A}\dot{\bar\theta}^{\alpha A'})] \,. \tag{1}$$

Here the dot means differentiation with respect to the real parameter τ. The equations of motion are

$$\dot p_{AA'} = 0 \,, \tag{2a}$$

$$\dot X^{AA'} = p^{AA'} + \tfrac{1}{2}i(\bar\theta^{\alpha A'}\dot\theta^{\alpha A} + \theta^{\alpha A}\dot{\bar\theta}^{\alpha A'}) \,, \tag{2b}$$

$$p_{AA'}\dot\theta^{\alpha A} = 0 \,. \tag{2c}$$

These are subject to the zero-mass constraint $p_{AA'}p^{AA'} = 0$. By solving (2b) for $p_{AA'}$, one obtains the second-order form of the action

$$A = \tfrac{1}{2}\int \mathrm{d}\tau \left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^2 = \tfrac{1}{2}\int \mathrm{d}\tau \frac{\mathrm{d}Y^{AA'}}{\mathrm{d}\tau}\frac{\mathrm{d}Y_{AA'}}{\mathrm{d}\tau} \,, \tag{3}$$

where

$$\mathrm{d}Y^{AA'} = \mathrm{d}X^{AA'} - \tfrac{1}{2}i(\bar\theta^{\alpha A'}\mathrm{d}\theta^{\alpha A} + \theta^{\alpha A}\mathrm{d}\bar\theta^{\alpha A'}) \,.$$

We note that $\mathrm{d}s^2$ is the non-Riemannian rank-four line element on the configuration superspace [8,11,12].

The action (1) is invariant under a number of infinitesimal transformations, in particular the extended super-Poincaré algebra [13] of X-translations $T^{AA'}$, homogeneous Lorentz transformations L^{AB}, and anticommuting supertranslations $\phi^{\alpha A}$. These transformations generate a subalgebra of a particular contraction of OSp(N|4) [14]. This contraction also yields a set of internal O(N) charges. The action (1) is invariant under this O(N), and even under a larger U(N) group. However, for our purposes only the extended super-Poincaré subalgebra is required. Under this the transformation laws of the dynamical variables are

$$\delta X^{AA'} = T^{AA'} + L^A_B X^{BA'} + \bar L^{A'}_{B'}X^{AB'} + \tfrac{1}{2}i(\bar\phi^{\alpha A'}\theta^{\alpha A} + \phi^{\alpha A}\bar\theta^{\alpha A'}) \,, \tag{4a}$$

$$\delta p_{AA'} = -L^B_A p_{BA'} - \bar L^{B'}_{A'}p_{AB'} \,, \tag{4b}$$

$$\delta\theta^{\alpha A} = L^A_B\theta^{\alpha B} + \phi^{\alpha A} \,. \tag{4c}$$

The conserved generators are then

$$P_{AA'} = p_{AA'}\,, \tag{5a}$$

$$M_{AA'BB'} = \tfrac{1}{2}(p_{AA'}X_{BB'} - p_{BB'}X_{AA'}) + \tfrac{1}{4}i(p_{AB'}\bar{\theta}^{\alpha}_{A'}\theta^{\alpha}_{B} - p_{BA'}\bar{\theta}^{\alpha}_{B'}\theta^{\alpha}_{A})\,, \tag{5b}$$

$$J^{\alpha}_{A'} = -p_{AA'}\theta^{\alpha A}\,. \tag{5c}$$

We now express the generators in terms of a minimal set of variables by first observing that since $P_{AA'}$ is Hermitian, null, and assumed future-directed, it can be written as

$$P_{AA'} = \lambda_A \bar{\lambda}_{A'}\,. \tag{6}$$

The components of the spinor λ_A are ordinary complex numbers and are determined by $P_{AA'}$ up to an arbitrary phase factor $\alpha(\tau)$. We now check that the angular momentum $M_{AA'BB'}$ satisfies the Pauli-Lubanski condition, which in spinor form is

$$p^{BB'}M_{AB'BA'} = -\tfrac{1}{2}iSP_{AA'}\,. \tag{7}$$

The spin (helicity) S is found to be a real pseudoscalar even element of the Grassmann algebra

$$S = -\tfrac{1}{2}p^{BB'}\bar{\theta}^{\beta}_{B'}\theta^{\beta}_{B}\,. \tag{8}$$

The satisfaction of (6) and (7) means that $M_{AA'BB'}$ can be written [15] in terms of λ_A and another spinor $\mu_{A'}$ as

$$M_{AA'BB'} = \tfrac{1}{4}i\,[(\bar{\mu}_A\lambda_B + \bar{\mu}_B\lambda_A)\,\epsilon_{A'B'} - (\mu_{A'}\bar{\lambda}_{B'} + \mu_{B'}\bar{\lambda}_{A'})\,\epsilon_{AB}]\,. \tag{9}$$

The $\mu_{A'}$ are *not* ordinary numbers but are in general even elements of the Grassmann algebra generated by the θ's. They are fixed up to the factor $\alpha(\tau)$. Finally, we see from (5c) that $J^{\alpha}_{A'}\lambda^{A'} = 0$, so that J^{α}_{A} can be expressed as

$$J^{\alpha}_{A'} = \xi^{\alpha}\bar{\lambda}_{A'}\,. \tag{10}$$

The N anticommuting variables ξ^{α} are again determined up to the factor $\alpha(\tau)$. In terms of the variables $(\lambda^A, \mu_{A'}, \xi^{\alpha})$ the spin S can be written as

$$S = -\tfrac{1}{2}(\bar{\mu}_A\lambda^A + \bar{\lambda}^{A'}\mu_{A'}) = -\tfrac{1}{2}\bar{\xi}^{\alpha}\xi^{\alpha}\,. \tag{11}$$

Taken together, eqs. (6), (9) and (10) express the supersymmetry generators in terms of the 4 + N – soon to be identified as supertwistor – variables $(\lambda^A, \mu_{A'}, \xi^{\alpha})$.

Of these λ^A and $\mu_{A'}$ are commuting Lorentz spinors and ξ^α are anticommuting Lorentz scalars. At first sight this appears to violate the spin-statistics connection. In fact, this is not so. As will be detailed in sect. 4, in the space $(\lambda^A, \mu_{A'}, \xi^\alpha)$ there acts a representation of $SU(2,2|N)$. For this representation it does not matter whether the λ's and μ's are Bose and the ξ's Fermi or *vice versa*. In either case the transformations that take λ's and μ's into ξ's and ξ's into λ's and μ's are Fermi and the remaining ones Bose. We have made the statistics assignment in such a way that for $N = 0$ supertwistors should reduce to the standard ungraded twistors of Penrose.

3. Supertwistors

We now change our point of view and think of the space of all "triples" $(\lambda^A, \mu_{A'}, \xi^\alpha)$ as being more basic than the original configuration superspace. This is done by considering a point (X, θ) not as the fundamental object, but merely as an *incidence* of certain particle trajectories [16]. The trajectories are given *via* eqs. (6), (9) and (10) by particular values of $(\lambda^A, \mu_{A'}, \xi^\alpha)$. Thus, we ask the following question: What are the allowed values of $(\lambda^A, \mu_{A'}, \xi^\alpha)$ for all trajectories *through* the point (X, θ)? This is answered by first observing that $P_{AA'}$ and thus λ^A can be arbitrary. Then, the simultaneous satisfaction of eqs. (5b) and (9) and eqs. (5c) and (10) requires

$$\mu_{A'} = -i(X_{A'A} + \tfrac{1}{2} i \bar{\theta}^\alpha_{A'} \theta^\alpha_A)\lambda^A , \tag{12a}$$

$$\xi^\alpha = \theta^\alpha_A \lambda^A . \tag{12b}$$

There is, however, one further consideration. Trajectories corresponding to triples $(c\lambda^A, c\mu_{A'}, c\xi^\alpha)$ for $(\lambda^A, \mu_{A'}, \xi^\alpha)$ fixed and c ranging over all non-zero complex numbers differ only by rescaling of the parameter τ, and thus are identical. We call the class of all such identified triples a *supertwistor* and denote it by just $(\lambda^A, \mu_{A'}, \xi^\alpha)$. The space of all supertwistors is thus a *complex projective superspace*. All supertwistors satisfying (12) for a given (X, θ) form a "surface" in supertwistor space which we denote by $\Sigma(X, \theta)$.

4. The superconformal algebra

We started from the action (1) in order to provide a simple dynamical derivation of the expressions of supersymmetry generators in terms of the supertwistor variables. As a matter of fact these relations have a more general algebraic basis. Here we simply abstract them from the particular dynamical context of the previous sections and immediately enlarge the superalgebra (4) to the more important superconformal algebra. To that effect we now consider the bilinear form on supertwistor space given by

$$F(1,2) = \bar{F}(2,1) = \bar{\mu}_{1A}\lambda_2^A + \bar{\lambda}_1^{A'}\mu_{2A'} - \bar{\xi}_1^\alpha \xi_2^\alpha \,. \tag{13}$$

This form characterizes the $\Sigma(X, \theta)$ "surfaces" in the sense that for any two points on such a surface $F(1,2) = 0$. We are thus led to investigate the algebra of linear transformations preserving the form $F(1,2)$. The action of such a linear transformation can be expressed as

$$\begin{pmatrix} \delta\lambda^A \\ \delta\mu_{A'} \\ \delta\xi^\alpha \end{pmatrix} = \begin{pmatrix} L_B^A + \frac{1}{2}(-D + iG)\,\delta_B^A & -iK^{AB'} & \psi^{\beta A} \\ -iT_{A'B} & -\bar{L}_{A'}^{B'} + \frac{1}{2}(D + iG)\,\delta_{A'}^{B'} & \bar{\phi}_{A'}^\beta \\ \phi_B^\alpha & \bar{\psi}^{\alpha B'} & (2i/N)\,G\delta^{\alpha\beta} + S^{\alpha\beta} \end{pmatrix} \begin{pmatrix} \lambda^B \\ \mu_{B'} \\ \xi^\beta \end{pmatrix} . \tag{14}$$

Here D and G are real, $T_{A'B}$ and $K^{AB'}$ are Hermitean, L_B^A and $S^{\alpha\beta}$ are elements of the algebras SL(2,C) and SU(N) respectively, and ϕ_B^α and $\psi^{\beta A}$ are anticommuting spinors. These transformations form the Lie superalgebra SU(2,2|N). The algebra is *special* unitary, i.e., the supertrace of the matrix of (14) is zero, since any overall multiplicative factor is meaningless for a projective space. Now, using eqs. (12) we find that while the form F(1,2) is invariant under (14), the surface * $\Sigma(X, \theta)$ is transformed into the surface $\Sigma(X + \delta X, \theta + \delta\theta)$ where

$$\begin{aligned} \delta X_{A'A} = {} & T_{A'A} - L_A^B X_{A'B} - \bar{L}_{A'}^{B'} X_{B'A} + D X_{A'A} + K^{BB'} X_{A'B} X_{B'A} \\ & - \tfrac{1}{4} K^{BB'} \bar{\theta}_{A'}^\alpha \theta_B^\alpha \bar{\theta}_{B'}^\beta \theta_A^\beta + \tfrac{1}{2} i (\bar{\phi}_{A'}^\alpha \theta_A^\alpha + \phi_A^\alpha \bar{\theta}_{A'}^\alpha) \\ & + \tfrac{1}{2} (\bar{\psi}^{\alpha B'} X_{B'A} \bar{\theta}_{A'}^\alpha - \psi^{\alpha B} X_{A'B} \theta_A^\alpha) \\ & - \tfrac{1}{4} i (\bar{\psi}^{\beta B} \theta_A^\beta \bar{\theta}_{A'}^\alpha \theta_B^\alpha + \bar{\psi}^{\beta B'} \bar{\theta}_{A'}^\beta \bar{\theta}_{B'}^\alpha \theta_A^\alpha) \,, \end{aligned} \tag{15a}$$

$$\begin{aligned} \delta\theta_A^\alpha = {} & -L_A^B \theta_B^\alpha + \tfrac{1}{2} D \theta_A^\alpha + K^{BA'} (X_{A'A} + \tfrac{1}{2} i \bar{\theta}_{A'}^\beta \theta_A^\beta)\, \theta_B^\alpha + \phi_A^\alpha \\ & - \psi^{\beta B} \theta_A^\beta \theta_B^\alpha - i \bar{\psi}^{\alpha A'} (X_{A'A} + \tfrac{1}{2} i \bar{\theta}_{A'}^\beta \theta_A^\beta) \\ & + \left(\frac{2}{N} - \frac{1}{2}\right) iG\theta_A^\alpha + S^{\alpha\beta} \theta_A^\beta \,. \end{aligned} \tag{15b}$$

* The surface $\Sigma(X, \theta)$ can be given a somewhat more concrete meaning similar to the coset space interpretation of configuration superspace [2]. This is done by exponentiating the matrix of eq. (14) for finite $T_{AA'} = X_{AA'}$ and $\phi_A^\alpha = \theta_A^\alpha$ and acting with this "supergroup" element on the supertwistor $(\lambda^A, 0, 0)$. The resulting supertwistor $(\lambda^A, \mu_{A'}, \xi^\alpha)$ is then just that given by eqs. (12). Thus as λ^A ranges over all complex values, the surface $\Sigma(X, \theta)$ is generated.

The meaning of these transformations now becomes clear. They form the Wess-Zumino superconformal algebra [5] generalized from SU(2,2|1) to SU(2,2|N). The non-linear action of this algebra as point transformations on superspace was derived in ref. [12] where the θ's were treated as Majorana spinors. The eqs. (15) are just the Weyl spinor form of eq. (8a) of this reference. The Bose parameters are X-translations $T_{A'A}$, homogeneous Lorentz transformations $L^{A}_{B'}$, dilatations D, conformal boosts $K^{AA'}$, phase or γ_5-transformations G, and internal SU(N) transformations $S^{\alpha\beta}$. The Fermi parameters are supertranslations ϕ^{α}_{A} and superconformal boosts $\psi^{\alpha A}$. It will be convenient in the next section to use modified configuration superspace coordinates (Z,θ) where $Z_{A'A} = X_{A'A} + \frac{1}{2}i\bar{\theta}^{\alpha}_{A'}\theta^{\alpha}_{A}$. In terms of these, eqs. (15) take a somewhat simplified form

$$\delta Z_{A'A} = T_{A'A} - L^{B}_{A}Z_{A'B} - \bar{L}^{B'}_{A'}Z_{B'A} + DZ_{A'A} + K^{B'B}Z_{A'B}Z_{B'A}$$
$$+ i\bar{\phi}^{\alpha}_{A'}\theta^{\alpha}_{A} - \psi^{\alpha B}Z_{A'B}\theta^{\alpha}_{A}\,, \tag{16a}$$

$$\delta\theta^{\alpha}_{A} = -L^{B}_{A}\theta^{\alpha}_{B} + \tfrac{1}{2}D\theta^{\alpha}_{A} + K^{BA'}Z_{A'A}\theta^{\alpha}_{B} + \phi^{\alpha}_{A} - \bar{\psi}^{\beta B}\theta^{\beta}_{A}\theta^{\alpha}_{B}$$
$$- i\bar{\psi}^{\alpha A'}Z_{A'A} + \left(\frac{2}{N} - \frac{1}{2}\right) iG\theta^{\alpha}_{A} + S^{\alpha\beta}\theta^{\beta}_{A}\,. \tag{16b}$$

5. Zero-mass supermultiplets

In this next section we will show how to extend the twistor contour integral method for generating zero-mass fields of arbitrary spin [17] to supertwistors. Consider the integral

$$\Omega^{A_1 \cdots A_{2S}}(Z,\theta) = \oint \mathrm{d}\lambda^{B}\lambda_{B}\lambda^{A_1} \ldots \lambda^{A_{2S}}\omega[\lambda^{C}, -iZ_{C'C}\lambda^{C}, \theta^{\gamma}_{C}\lambda^{C}]\,, \tag{17}$$

where ω is some generalized complex analytic function in supertwistor space and must be Grassmann even/odd with respect to $\theta^{\alpha A}$ for the spin S integral/half odd-integral if the usual spin-statistics connection is to hold. For (17) to be meaningful as a projective space integral, ω must be homogeneous of degree $-2S-2$. The λ-contour induces for each point (Z,θ) a contour in the surface $\Sigma(Z,\theta)$ enclosing singularities of ω. From eq. (6) the λ^{A} can be chosen to be ordinary complex numbers. With this choice the right-hand side of eq. (17) is an ordinary contour integral. When superconformal transformations are performed on the supertwistors (see below) then the λ^{A} will in general not be ordinary complex numbers but even elements of the Grassmann algebra generated by the θ's. Although the integral (17) is then not an ordinary contour integral, it can be used without further mathematical specification as a formal device for deriving the superconformal transformation properties of superfields (see eq. (23) below).

We now expand ω in a power series in the N anticommuting variables ξ^α to give

$$\omega[\lambda^C, -iZ_{C'C}\lambda^C, \theta^\gamma_C\lambda^C]$$
$$= \sum_{K=0}^{N} \omega_{\alpha_1 \dots \alpha_K}[\lambda^C, -iZ_{C'C}\lambda^C]\, \theta^{\alpha_1}_{A_1}\lambda^{A_1} \dots \theta^{\alpha_K}_{A_K}\lambda^{A_K} . \tag{18}$$

Inserting this into (17) yields

$$\Omega^{A_1 \dots A_{2S}}(Z,\theta) = \sum_{K=0}^{N} U^{A_1 \dots A_{2S+K}}_{\alpha_1 \dots \alpha_K}(Z)\, \theta^{\alpha_1}_{A_{2S+1}} \dots \theta^{\alpha_K}_{A_{2S+K}} , \tag{19}$$

where

$$U^{A_1 \dots A_{2S+K}}_{\alpha_1 \dots \alpha_K}(Z) = \oint \mathrm{d}\lambda^B\lambda_B\lambda^{A_1} \dots \lambda^{A_{2S+K}}\omega_{\alpha_1 \dots \alpha_K}[\lambda^C, -iZ_{C'C}\lambda^C] . \tag{20}$$

Recalling that $Z_{A'A} = X_{A'A} + \frac{1}{2}i\bar{\theta}^\alpha_{A'}\theta^\alpha_A$ itself is an even Grassmann variable, the U fields can be further expanded as

$$U^{A_1 \dots A_{2S+K}}_{\alpha_1 \dots \alpha_K}(Z) = \exp\left(\tfrac{1}{2}i\bar{\theta}^\beta_{B'}\theta^\beta_B \frac{\partial}{\partial X_{B'B}}\right) U^{A_1 \dots A_{2S+K}}_{\alpha_1 \dots \alpha_K}(X) . \tag{21}$$

In this form the relation of the material given here to that of ref. [8] dealing with chiral superfields is most direct.

Each field $A^{A_1 \dots A_{2S+K}}_{\alpha_1 \dots \alpha_K}(X)$ is totally symmetric in its spinor indices and by the method of construction identically satisfies (since $\lambda_A\lambda^A = 0$) the zero-mass field equations for spin $S + \frac{1}{2}K$

$$\frac{\partial U^{A_1 \dots A_{2S+K}}_{\alpha_1 \dots \alpha_K}}{\partial X^{A_1 A_1'}} = 0 , \qquad \text{for } S, K \neq 0 , \tag{22a}$$

$$\frac{\partial^2 U^{A_1 \dots A_{2S+K}}_{\alpha_1 \dots \alpha_K}}{\partial X^{BB'}\partial X_{BB'}} = 0 , \qquad \text{for all } S, K . \tag{22b}$$

Each field $U^{A_1 \dots A_{2S+K}}_{\alpha_1 \dots \alpha_K}(X)$ is totally antisymmetric in its internal indices. This uniquely determines the SU(N) representation for each multiplet member $K = 0, 1, 2, \dots N$.

The supermultiplets given by this construction thus have the particle content

$$[S, 1] \oplus [S + \tfrac{1}{2}, N] \oplus [S + 1, \tfrac{1}{2}N(N-1)] \oplus \dots \oplus [S + \tfrac{1}{2}(N-1), N]$$
$$\oplus [S + \tfrac{1}{2}N, 1] ,$$

where the symbol $[J, M]$ corresponds to spin J and SU(N) multiplicity M. These supermultiplets are closely related to those obtained by the more standard Clifford algebra technique [18], and they probably do not exhaust all possible superconformal supermultiplets. It is an interesting problem to extend our construction to the case of a general superconformal supermultiplet. Unlike the ordinary Clifford technique, our construction is rooted in superspace and has the advantage of directly yielding the properly constrained superfield.

We now investigate the action of the superconformal algebra on these supermultiplets. We first consider a transformation as merely a change of coordinates. On configuration superspace it is a relabeling of coordinates from (Z, θ) to $(\tilde{Z}, \tilde{\theta})$. We define the transform of (17) to be

$$\tilde{\Omega}^{A_1 \cdots A_{2S}}(\tilde{Z}, \tilde{\theta}) = \oint \mathrm{d}\tilde{\lambda}^B \tilde{\lambda}_B \tilde{\lambda}^{A_1} \ldots \tilde{\lambda}^{A_{2S}} \tilde{\omega}[\tilde{\lambda}^C, -i\tilde{Z}_{C'C}\tilde{\lambda}^C, \tilde{\theta}^\gamma_C \tilde{\lambda}^C] \,.$$

Here the supertwistor $(\tilde{\lambda}^A, \tilde{\mu}_{A'}, \tilde{\xi}^\alpha)$ on $\Sigma(\tilde{Z}, \tilde{\theta})$ is the image of $(\lambda^A, \mu_{A'}, \xi^\alpha)$ on $\Sigma(Z, \theta)$ under the transformation, and the kernel transforms as a scalar, i.e., $\tilde{\omega}[\tilde{\lambda}^C, -i\tilde{Z}_{C'C}\tilde{\lambda}^C, \tilde{\theta}^\gamma_C\tilde{\lambda}^C] = \omega[\lambda^C, -iZ_{C'C}\lambda^C, \theta^\gamma_C\lambda^C]$. From (12) and (14) we then have

$$\tilde{\lambda}^A = M^A_B \lambda^B \,,$$

where

$$M^A_B = \delta^A_B + L^A_B + \tfrac{1}{2}(\;D + iG)\,\delta^A_B - K^{AA'}Z_{A'B} + \psi^{\alpha A}\theta^\alpha_B \,.$$

A change of integration variable back to λ^A then gives

$$\tilde{\Omega}^{A_1 \cdots A_{2S}}(\tilde{Z}, \tilde{\theta}) = M M^{A_1}_{B_1} \ldots M^{A_{2S}}_{B_{2S}} \Omega^{B_1 \cdots B_{2S}}(Z, \theta) \,,$$

$$= M^{S+1} \hat{M}^{A_1}_{B_1} \ldots \hat{M}^{A_{2S}}_{B_{2S}} \Omega^{B_1 \cdots B_{2S}}(Z, \theta) \,.$$

Here M and $\hat{M}^A_B$ are the Jacobian determinant and unimodular parts of M^A_B respectively. Since M^A_B is actually an infinitesimal transformation, this gives the passive variation

$$\begin{aligned}
&\delta^* \Omega^{A_1 \cdots A_{2S}}(Z, \theta) \\
&\quad = (S+1)(-D + iG - K^{BB'}Z_{BB'} + \psi^{\beta B}\theta^\beta_B)\,\Omega^{A_1 \cdots A_{2S}}(Z, \theta) \\
&\quad + (L^C_D - K^{CC'}Z_{C'D} + \tfrac{1}{2}K^{EE'}Z_{EE'}\delta^{C'}_D \\
&\quad + \psi^{\gamma C}\theta^\gamma_D - \tfrac{1}{2}\psi^{\gamma E}\theta^\gamma_E\delta^C_D)\,\sigma^{DA_1 \cdots A_{2S}}_{CB_1 \cdots B_{2S}}\,\Omega^{B_1 \cdots B_{2S}}(Z, \theta) \,,
\end{aligned} \tag{23a}$$

where σ_C^D are the spin S-matrix generators of SL(2,C). The active or functional variation is related to this by

$$\delta\Omega^{A_1 \cdots A_{2S}} = \delta^*\Omega^{A_1 \cdots A_{2S}} - \frac{\partial}{\partial Z_{BB'}}\Omega^{A_1 \cdots A_{2S}}\delta Z_{BB'}$$

$$-\frac{\partial}{\partial\theta_B^\beta}\Omega^{A_1 \cdots A_{2S}}\delta\theta_B^\beta , \tag{23b}$$

with $\delta Z_{BB'}$ and $\delta\theta_B^\beta$ given by (16). The θ-expansions given above can now be inserted to give the variations of the U fields. Since this result is rather complicated in general, we give it here explicitly only for the simplest case where $N = 1$ and $S = 0$. We write the expansion (19) in this case as

$$\Omega(Z, \theta) = U(Z) + V^A(Z)\,\theta_A .$$

The further expansion by (21) then gives

$$\Omega(Z, \theta) = U(X) + V^A(X)\,\theta_A + \tfrac{1}{2}i\frac{\partial}{\partial X_{A'A}}\,U(X)\,\bar\theta_{A'}\theta_A .$$

Higher-order terms in the θ's vanish because of the field equations (22). $U(X)$ and $V^A(X)$ are ordinary fields on Minkowski space. $U(X)$ is a scalar Bose field while $V^A(X)$ is a Weyl spin-$\frac{1}{2}$ Fermi field. This content is as expected for the spin-zero supermultiplet. From eqs. (23) we find

$$\delta U(X) = (-D + iG - K^{BB'}X_{BB'})\,U(X)$$

$$+ (\phi_B - i\bar\psi^{B'}X_{B'B})\,V^B(X) - \Delta X_{B'B}\frac{\partial}{\partial X_{B'B}}\,U(X) ,$$

$$\delta V^A(X) = L_B^A\,V^B(X) + (-\tfrac{3}{2}D + \tfrac{1}{2}iG - K^{BB'}X_{BB'})\,V^A(X)$$

$$- K^{AA'}X_{A'B}V^B(X) + \psi^A\,U(X) - i(\bar\phi_{A'} + i\psi^B X_{A'B})\frac{\partial}{\partial X_{A'A}}U(X)$$

$$- \Delta X_{B'B}\frac{\partial}{\partial X_{B'B}}\,V^A(X) ,$$

where

$$\Delta X_{B'B} = T_{B'B} - L_B^A X_{B'A} - \bar L_{B'}^{A'} X_{BA'} + DX_{B'B} + K^{AA'}X_{B'A}X_{A'B} .$$

$\Delta X_{A'A}$ is just the variation of the Minkowski coordinates $X_{A'A}$ under the action of the ordinary conformal algebra. These variations of the member fields of a scalar supermultiplet are precisely those derived in ref. [5] provided that the arbitrary conformal weight parameter n, present there is set to $\frac{1}{2}$. For this value of n, and for the spinor field satisfying the zero-mass Dirac equation, the auxiliary fields of this reference are not needed in the transformation law and can be set to zero.

6. Discussion

We have generalized Penrose twistors from the ordinary case to the graded case of superconformal supersymmetry. The action of the ordinary conformal algebra on the 4 Minkowski coordinates is non-linear. By contrast, twistors have simple conformal SU(2,2) properties. In the same way, the 4 + N coordinates of superspace support a highly non-linear realization of the superconformal algebra SU(2,2|N), and again the supertwistors transform in a simple way. This is the real reason why the particle content of a supermultiplet is so easily obtained by the supertwistor formalism.

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