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ANALYSIS AND APPROXIMATION OF THE GINZBURG–LANDAU MODEL OF SUPERCONDUCTIVITY*

QIANG DU†, MAX D. GUNZBURGER‡, AND JANET S. PETERSON§

Abstract. The authors consider the Ginzburg–Landau model for superconductivity. First some well-known features of superconducting materials are reviewed and then various results concerning the model, the resultant differential equations, and their solution on bounded domains are derived. Then, finite element approximations of the solutions of the Ginzburg–Landau equations are considered and error estimates of optimal order are derived.

Key words. superconductivity, Ginzburg–Landau equations, finite element approximations

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1. Introduction.

The superconductivity of certain metals (such as mercury, lead, and tin) at very low temperatures was discovered by H. Kamerlingh-Onnes in 1908. He observed that electrical resistance disappeared completely below some critical temperature. Indeed, closed currents in a ring of superconducting material have been observed to flow without decay for over two years, and the resistivity of some of these materials has been estimated to be no greater than 10^{-23} ohm-cm!

In addition to this *perfect conductivity* property, superconductors are also characterized by the property of *perfect diamagnetism*. This phenomenon was discovered in 1933 by W. Meissner and R. Ochsenfeld, and is also known as the *Meissner effect*. They observed that not only is a magnetic field excluded from a superconductor, i.e., if a magnetic field is applied to a superconducting material at a temperature below the critical temperature, it does not penetrate into the material, but also that a magnetic field is expelled from a superconductor, i.e., if a superconductor subject to a magnetic field is cooled through the critical temperature, the magnetic field is expelled from the material. Of course, sufficiently large magnetic fields cannot be excluded from the material, so that the Meissner effect also predicts the existence of a critical magnetic field above which the material ceases to be superconducting, even at temperatures below the critical temperature. Furthermore, passage through the critical temperature is reversible. Then, simple thermodynamic arguments can be used (see, e.g., [33]) to show that the transition from the normal to the superconducting state at zero applied magnetic field is not accompanied by any release of latent heat; thus, we have what is known as a *second-order transition*.

A good theoretical understanding of low-temperature superconductivity was not arrived at until the 1950s. Indeed, a completely acceptable *microscopic* theory did not exist until Bardeen, Cooper, and Schrieffer [4] published their landmark paper in 1957. However, even earlier, various *macroscopic* theories were proposed, most notably the homogeneous theory of London and London [29] in 1935, the nonlocal theory of Pippard [32] in 1950, and the theory of Ginzburg and Landau [19], also in 1950, a full 7 years before the Bardeen, Cooper, Schrieffer (BCS) theory! The Ginzburg–Landau (GL) theory was

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†Department of Mathematics, University of Chicago, Illinois 60637. Present address, Department of Mathematics, Michigan State University, East Lansing, Michigan 48224.

‡Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061. This research was supported by the Department of Energy at the Los Alamos National Laboratory, and by the U.S. Air Force Office of Scientific Research grant AFOSR-90-0179.

§Department of Mathematics, Virgina Polytechnic Institute and State University, Blacksburg, Virginia 24061. This research was supported by the Department of Energy at the Los Alamos National Laboratory.

itself based on a general theory, introduced by Landau in 1937, for second-order phase transitions in fluids that was based on minimizing the Helmholtz free energy. Ginzburg and Landau thought of the conducting electrons as being a "fluid" that could appear in two phases, namely superconducting and normal (nonsuperconducting.) Through a stroke of intuitive genius, Ginzburg and Landau added to the theory of phase transitions certain effects, motivated by quantum-mechanical considerations, to account for the fact that the electron "fluid" motion is affected by the presence of magnetic fields.

The GL theory was not widely accepted immediately, mainly due to its phenomenological character. However, in 1959, Gor'kov [21] showed that, in the appropriate limit, the macroscopic GL theory can be derived from the microscopic BCS theory. After this work of Gor'kov, the GL theory became accepted as a valid (macroscopic) model for low-temperature superconducting effects.

At the time that Ginzburg and Landau proposed their theory, it was thought that the transition between the superconducting and normal phases is always accompanied by positive surface energy, so that the minimum energy principle would lead to relatively few such transitions in a sample of material. Indeed, this agreed with experimental observations in what is now known as *type* I *superconductors*. Then, in 1957 (the same year as BCS), Abrikosov [1] investigated what would happen if the surface energy accompanying phase transitions was negative. The GL theory then predicts that, in order to minimize the energy, there would be relatively many phase transitions in a material sample, and that indeed the normal and superconducting state could coexist in what is known as the *mixed state*. About ten years later, such *type* II *superconductors* were observed experimentally. It is another remarkable feature of the GL theory that it allowed for such materials, even before their existence was known! From a technological standpoint, type II superconductors are the ones of greatest interest, mainly because they can retain superconductivity properties in the presence of large applied magnetic fields.

Due to the extremely low temperature necessary for known materials, e.g., metals, to become superconducting, their practical usefulness was very limited and therefore general interest in superconductivity waned. However, after the recent advances in cryogenics and, even more so, after the recent discovery of high-temperature superconductors, there has naturally been a resurgence in interest. One question that arises is the applicability of the GL theory, or some variant of it, to high-temperature superconductors. In this regard, no general consensus has been reached.

Our short introduction by no means does justice to the history of superconductivity, nor do we intend to give a full description of even the GL theory. There are, however, many excellent references that may be consulted for detailed descriptions of both the microscopic and macroscopic theories of low-temperature superconductivity. Among these are [13], [26], [33], and [38]. There has also been substantial interest in the GL model within the mathematical physics community, e.g., see [5], [8], [10], [16], [24], [31], [34], [35], [40], and [41]. For another recent survey, see [9].

Approximations of solutions of the GL model for superconductivity have been obtained by many authors. These approximations are usually obtained by using series solutions of one type or another; see, e.g., [1], [6], [16], [23], [25], [27], [28], and [39]. Monte Carlo–simulated annealing simulations have also been obtained by [15].

A main goal of our ongoing work is to develop robust and efficient codes that can be used to help determine, through comparisons with experimental observations, the extent to which GL models can be applied to high-temperature superconductors. Although the results and algorithms of this paper are presented in the context of GL models for low-temperature superconductivity, many of these apply equally well to GL models for

high-temperature superconductivity.

In §2, we briefly describe the GL model for superconductivity. Then, in §3, we obtain results concerning the model, some of which are new, and often verify that the model agrees with experimental observations; some of the results are generalizations of known mathematical results. In §4, we develop and analyze finite element algorithms for approximating solutions to the model. Finally, in §5, we briefly describe a periodic Ginzburg–Landau model that is actually used, instead of the boundary value model of §2, in most computational studies of superconductivity.

2. The Ginzburg–Landau model for superconductivity. We trust that many of our readers are not familiar with the Ginzburg–Landau model of superconductivity. Therefore, in this section we discuss the derivation of the GL model, and of some well-known features of superconductors that are well described by the model. Necessarily, our presentation will be rather sketchy. For details concerning the material of this section, any of the many books on superconductivity, e.g., [13], [26], [33], and [38] may be consulted.

2.1. The Ginzburg–Landau free energy. Ginzburg and Landau postulated that the Helmholtz free energy per unit volume of a superconducting material is given by

$$f = f_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{|\mathbf{h}|^2}{8\pi} + \frac{1}{2m_s}\left|\left(-i\hbar\nabla - \frac{e_s\mathbf{A}}{c}\right)\psi\right|^2. \tag{2.1}$$

Here, the constant f_n is the free energy of the normal (nonsuperconducting) state in the absence of magnetic fields, ψ is the (complex-valued) order parameter, $\mathbf{A}$ is the magnetic potential, $\mathbf{h} = \operatorname{curl}\mathbf{A}$ is the magnetic field, α and β are constants (with respect to the space variable $\mathbf{x}$) whose values depend on the temperature, c is the speed of light, e_s and m_s are the charge and mass, respectively, of the superconducting charge-carriers, and $2\pi\hbar$ is Planck's constant.

The microscopic electron pairing theory of superconductivity implies that $e_s = 2e$, where e is the electron charge. The value of m_s is somewhat more arbitrary since any change in its definition can be compensated by an attendant change in the magnitude of ψ. However, it is customary to either set $m_s = m$ or $m_s = 2m$, where m is the electron mass. We will adopt the latter value. In this case, $|\psi|^2 = n_s$, where n_s is the density of superconducting charge carriers (equaling half the density of electrons) in the sample; thus the magnitude of the order parameter ψ gives the density of superconducting electron pairs.

The appearance of an order parameter is a part of the Landau theory of second-order phase transitions. The need for a *complex*-valued order parameter is somewhat difficult to explain. Ginzburg and Landau thought of their order parameter as an "averaged wave-function of the superconducting electrons." The connection, made by Gor'kov, between the microscopic BCS theory and the macroscopic GL theory justified the need for ψ to be complex-valued. However, recall that the GL theory preceded the BCS theory by seven years, so that the arguments of Ginzburg and Landau were based on physical intuition. Adopting the wave-function view of ψ, one may, as was done above, normalize the order parameter so that the square of its magnitude is proportional to the density of superconducting charge-carriers. The phase of the order parameter ψ can then be shown to be related to the current in the superconductor. (More on this later.)

Let us briefly discuss the various terms appearing in (2.1). First, note that $f_n + |\mathbf{h}|^2/8\pi$ is the free energy density of the normal state in the presence of the magnetic field $\mathbf{h}$. Next, recall that the Landau theory of second-order phase transitions was based on the following three assumptions: (a) there exists an order parameter that goes to

zero at the transition; (b) the free energy may be expanded as a power series in the order parameter; and (c) the coefficients in the expansion are regular functions of the temperature. Various physical arguments can be invoked to deduce that the expansion postulated in (b) is in even powers of $|\psi|$. Thus, recalling that in the present context the transition is one between the normal and superconducting states, the first four terms on the right-hand side of (2.1) represent a truncation of this power series for small values of $|\psi|$, i.e., near the transition.

In the GL theory, the density of superconducting charge-carriers, and thus the order parameter, is allowed to be spatially varying. Then, another consequence of the interpretation of ψ as a wave-function is the existence of a kinetic energy density associated with spatial variations of ψ that must be accounted for in the free energy density. Variations in the order parameter should penalize the energy, so that it is natural to add to the free energy density a term proportional to $|\nabla\psi|^2$. On the other hand, the free energy should be gauge-invariant. Here, Ginzburg and Landau, through another stroke of intuitive genius, postulated that the last term in (2.1) is the added energy density, in gauge-invariant form, due to the spatial variations in ψ. Following [38], we can elucidate this point by noting that the last term in (2.1) may be rewritten in the form

$$\frac{1}{2m_s}\left[\hbar^2\left|(\nabla|\psi|)\right|^2+\left|\hbar\nabla\phi-\frac{e_s\mathbf{A}}{c}\right|^2|\psi|^2\right], \tag{2.2}$$

where ϕ is the phase of ψ, i.e.,

$$\psi=|\psi|e^{i\phi}. \tag{2.3}$$

The first term in (2.2) clearly penalizes the energy when $|\psi|$ varies, and the second term may be interpreted as a gauge-invariant kinetic energy density associated with currents in the superconductor.

In the presence of an applied magnetic field $\mathbf{H}$, the Gibbs free energy density g differs from f due to the work done by the electromagnetic force induced by the applied field. This work (per unit volume) is given by $-\mathbf{h}\cdot\mathbf{H}/4\pi$ so that the Gibbs free energy density is given by

$$g=f-\frac{\mathbf{h}\cdot\mathbf{H}}{4\pi}=f_n+\alpha|\psi|^2+\frac{\beta}{2}|\psi|^4+\frac{1}{2m_s}\left|\left(-i\hbar\nabla-\frac{e_s\mathbf{A}}{c}\right)\psi\right|^2+\frac{|\mathbf{h}|^2}{8\pi}-\frac{\mathbf{h}\cdot\mathbf{H}}{4\pi}.$$

If Ω denotes the region occupied by the superconducting sample, the Gibbs free energy $\mathcal{G}$ of the sample is then given by

$$\begin{aligned}\mathcal{G}(\psi,\mathbf{A})=\int_\Omega g\,d\Omega=&\int_\Omega\left(f_n+\alpha|\psi|^2+\frac{\beta}{2}|\psi|^4\right)\\&+\int_\Omega\left[\frac{1}{2m_s}\left|\left(-i\hbar\nabla-\frac{e_s\mathbf{A}}{c}\right)\psi\right|^2+\frac{|\mathbf{h}|^2}{8\pi}-\frac{\mathbf{h}\cdot\mathbf{H}}{4\pi}\right]d\Omega.\end{aligned} \tag{2.4}$$

The basic thermodynamic postulate of the GL theory is that the superconducting sample is in a state such that its Gibbs free energy is a minimum.

The dependence of the value of the constants α and β on the temperature can be obtained from the microscopic theory. The constant β is positive; otherwise, $\mathcal{G}$ would not have a minimum value. If α is positive as well, then one easily sees that $\psi=0$ and $\mathbf{h}=\mathbf{H}$ minimizes $\mathcal{G}$. This is the normal, or nonsuperconducting state. On the other hand, if α is negative, and in the absence of surface effects, it can be shown that $\mathcal{G}$ is minimized by

$\mathbf{A} = \mathbf{0}$ (so that $\mathbf{h} = \mathbf{0}$) and $\psi = (-\alpha/\beta)^{1/2}$. This is the ideal superconductor with a constant density of superconducting charge-carriers and an excluded magnetic field, i.e., a superconductor with a perfect Meissner effect. With the presence of surface effects, minimizers of $\mathcal{G}$ are not so trivial.

Remark. It will, at times, be convenient to consider the functional $\mathcal{E}$, defined by

$$\mathcal{E}(\psi, \mathbf{A}) = \mathcal{G} + \int_\Omega \left(\frac{\alpha^2}{2\beta} + \frac{\mathbf{H}\cdot\mathbf{H}}{8\pi} - f_n \right) d\Omega = \int_\Omega \frac{1}{2} \left(\sqrt{\beta}|\psi|^2 + \frac{\alpha}{\sqrt{\beta}} \right)^2 d\Omega + \int_\Omega \left[\frac{1}{2m_s} \left| \left(-i\hbar\nabla - \frac{e_s\mathbf{A}}{c} \right) \psi \right|^2 + \frac{|\mathbf{h}-\mathbf{H}|^2}{8\pi} \right] d\Omega, \tag{2.5}$$

instead of $\mathcal{G}$. Clearly, $(\psi, \mathbf{A})$ is a minimizer of $\mathcal{G}$ if and only if it is also a minimizer of $\mathcal{E}$. The main virtue of the functional $\mathcal{E}$ is that it is nonnegative.

Remark. High-temperature superconductors are generally *anisotropic* and *inhomogeneous*, as opposed to low-temperature superconductors. The free energy densities and functionals given above correspond to low-temperature superconductors. In principle, there is no difficulty in extending the GL model to high-temperature superconductors. For example, the constants α and β, whose value depends on the temperature, are replaced by spatially varying scalar-valued functions, and the constant m_s is replaced by a matrix, with possibly spatially varying entries. (Of course, we would replace $1/m_s$ with m_s^{-1}.) However, in practice, there are difficulties. In the first place, the functional form and values of α, β, and m_s are not known. In addition, the justification of the GL theory as an appropriate limit of a microscopic theory is not available in the high-temperature case. For these, and other reasons, the jury is still out as to whether or not high-temperature superconductors can be modeled by anisotropic and inhomogeneous generalizations of the GL theory.

2.2. The Ginzburg–Landau equations and boundary conditions. If we use standard techniques from the calculus of variations, the minimization of $\mathcal{G}$ with respect to variations in ψ and $\mathbf{A}$ yields the celebrated Ginzburg–Landau equations

$$\frac{1}{2m_s} \left(-i\hbar\nabla - \frac{e_s\mathbf{A}}{c} \right)^2 \psi + \alpha\psi + \beta|\psi|^2\psi = 0 \quad \text{in } \Omega \tag{2.6}$$

and

$$\operatorname{curl}\operatorname{curl}\mathbf{A} + \frac{2\pi i e_s\hbar}{m_s c}(\psi^*\nabla\psi - \psi\nabla\psi^*) + \frac{4\pi e_s^2}{m_s c^2}|\psi|^2\mathbf{A} = \operatorname{curl}\mathbf{H} \quad \text{in } \Omega, \tag{2.7}$$

where $(\cdot)^*$ denotes the complex conjugate. Note that the current is given by

$$\mathbf{j} = (c/4\pi)\operatorname{curl}\mathbf{h} = (c/4\pi)\operatorname{curl}\operatorname{curl}\mathbf{A},$$

so that

$$\begin{aligned} \mathbf{j} &= -\frac{ie_s\hbar}{2m_s}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{e_s^2}{m_s c}|\psi|^2\mathbf{A} + \frac{c}{4\pi}\operatorname{curl}\mathbf{H} \\ &= \left(\frac{e_s\hbar}{m_s}\nabla\phi - \frac{e_s^2}{m_s c}\mathbf{A} \right)|\psi|^2 + \frac{c}{4\pi}\operatorname{curl}\mathbf{H}. \end{aligned} \tag{2.8}$$

Note the dependence of the current $\mathbf{j}$ on the phase ϕ of the order parameter.

Candidate minimizers of $\mathcal{G}$ are not a priori constrained to satisfy any boundary conditions. Thus, the minimization process also yields the natural boundary conditions

$$\left(i\hbar\nabla\psi + \frac{e_s}{c}\mathbf{A}\psi\right)\cdot\mathbf{n} = 0 \quad \text{on } \Gamma \tag{2.9}$$

and

$$\operatorname{curl}\mathbf{A}\times\mathbf{n} = \mathbf{H}\times\mathbf{n} \quad \text{on } \Gamma\,, \tag{2.10}$$

where Γ denotes the boundary of Ω and $\mathbf{n}$ the unit outer normal vector to Γ.

More general boundary conditions have also been suggested. First, note that the GL free energy functionals are not valid near the boundary of the superconducting sample. (On the other hand, the GL differential equations are usually retained, even in the vicinity of the boundary.) A boundary condition is then determined by requiring that the normal component of the current be continuous at the boundary, i.e., $\mathbf{j}\cdot\mathbf{n} = (c/4\pi)\operatorname{curl}\mathbf{H}\cdot\mathbf{n}$ on Γ, or even more commonly, that the normal component of the current vanishes, i.e., $\mathbf{j}\cdot\mathbf{n} = (c/4\pi)\operatorname{curl}\mathbf{H}\cdot\mathbf{n} = 0$ on Γ. In either case, (2.8) then yields that

$$\psi^*\left(-i\hbar\nabla\psi - \frac{e_s}{c}\mathbf{A}\psi\right)\cdot\mathbf{n} + \psi\left(i\hbar\nabla\psi^* - \frac{e_s}{c}\mathbf{A}\psi^*\right)\cdot\mathbf{n} = 0 \quad \text{on } \Gamma\,.$$

It is easily seen that this relation implies that

$$\left(-i\hbar\nabla\psi - \frac{e_s}{c}\mathbf{A}\psi\right)\cdot\mathbf{n} = i\gamma\psi \quad \text{on } \Gamma \tag{2.11}$$

for some real valued function γ. The microscopic theory of superconductivity yields that $\gamma \neq 0$ for a superconductor-normal metal interface, and that $\gamma = 0$ for a superconductor-insulator (or vacuum) interface. Note that in the latter case, (2.9) and (2.11) coincide. We will, for convenience, adopt the boundary condition (2.9), although no real difficulties would be engendered by instead considering the more general boundary condition (2.11).

Remark. We have derived the GL equations (2.6) and (2.7) by minimizing the GL form of the Gibbs free energy. As has been already noted, these equations may also be derived as an appropriate limit of the BCS microscopic theory of superconductivity. Likewise, the boundary condition (2.9) may be derived from the microscopic theory.

Remark. The boundary condition (2.11) can also result directly from setting the first variation of $\mathcal{G}$ to zero if we add to the definition (2.4) the term $-\int_\Gamma i\gamma|\psi|^2\,d\Gamma$. We do not know if there is any physical justification for the addition of this term to the free energy.

Remark. The Ginzburg–Landau equations can be easily generalized to the case of high-temperature superconductors For example, (2.6) is replaced by

$$\frac{1}{2}\left(-i\hbar\nabla - \frac{e_s\mathbf{A}}{c}\right)\cdot m_s^{-1}\cdot\left(-i\hbar\nabla - \frac{e_s\mathbf{A}}{c}\right)\psi + \alpha\psi + \beta|\psi|^2\psi = 0 \quad \text{in } \Omega\,,$$

where e_s and α and β are scalar-valued functions and m_s is a matrix-valued function.

2.3. Fluxoid quantization; some important scales and parameters. Let $\partial\Sigma$ denote a closed curve lying in the material sample such that $|\psi| \neq 0$ everywhere on the curve, i.e., the curve $\partial\Sigma$ nowhere intersects a normal region. Let Σ denote a surface bounded by this closed curve. Consider the expression

$$\Phi' = \int_\Sigma \mathbf{h}\cdot d\Sigma + \frac{m_s c}{e_s^2}\int_{\partial\Sigma}\frac{\mathbf{j}}{|\psi|^2}\cdot d(\partial\Sigma)\,.$$

The first term on the right is the magnetic flux through the surface Σ, and Φ' is known as the *fluxoid*. Using Stokes' theorem and $\mathbf{h} = \operatorname{curl} \mathbf{A}$, we then have that

$$\Phi' = \int_{\partial\Sigma} \left(\mathbf{A} + \frac{m_s c}{e_s^2} \frac{\mathbf{j}}{|\psi|^2} \right) \cdot d(\partial\Sigma),$$

so that, using (2.3), (2.8), and the fact that $|\psi|$ is single valued,

$$\Phi' = \frac{\hbar c}{e_s} \int_{\partial\Sigma} \nabla\phi \cdot d(\partial\Sigma) = \Phi_0 n, \tag{2.12}$$

where $\Phi_0 = 2\pi\hbar c/e_s$ and n is an integer. Thus, the fluxoid Φ' is quantized.

For type I superconductors without holes or nonsuperconducting material inclusions, i.e., Ω is simply connected, fluxoid quantization is of no importance since in this case Σ always lies in the superconductor and we may choose $n = 0$. On the other hand, if there are holes or such inclusions in the sample, i.e., Ω is multiply connected, and if $\partial\Sigma$ encloses such a hole, then (2.12) implies that Φ' must be an integer multiple of Φ_0.

Fluxoid quantization can be used to explain the persistence of currents in a superconducting ring. (Our explanation is drawn from [38].) The current cannot decrease by arbitrarily small amounts, but only in finite jumps such that the fluxoid decreases by one or more integer multiples of Φ_0. If only a single, or a few electrons were involved, this could be easily accomplished. However, for a superconductor, we are requiring a quantum jump in (the phase of) ψ. Such a *macroscopic* change requires the *simultaneous* quantum jump by a very large ($> 10^{20}$) number of particles. Such an event is, of course, extremely improbable, so that the current in the superconductor persists.

For type II superconductors, (2.12), i.e., the quantization of the fluxoid, has some other important consequences, even in simply connected domains. In the mixed superconducting-normal state, we may place $\partial\Sigma$ in a superconducting region. Any region where the field penetrates that is enclosed by $\partial\Sigma$ must have a fluxoid value that is an integer multiple of Φ_0. This implies that such regions cannot be arbitrarily small, so that, even though the surface energy associated with the transition from one state to the other is negative, we do not have arbitrarily small scale transitions; they are limited by the requirement that the fluxoid is an integer multiple of Φ_0.

In the introduction, we mentioned that above a critical value of the field, superconductivity is destroyed. For a superconductor with perfect Meissner effect this critical field can be computed. First, observe that with a perfect Meissner effect, $|\mathbf{h}| = \mathbf{0}$ and $\psi = \psi_0 = (-\alpha/\beta)^{1/2}$ so that from (2.4), $g = f_n - \alpha^2/(2/\beta)$. On the other hand, above the critical field, superconductivity is lost, i.e., $\psi = 0$ and $\mathbf{h} = \mathbf{H}$, so that $g_n = f_n - |\mathbf{H}|^2/(8\pi)$. At the critical field $H_c = |\mathbf{H}|$, at which the transition from the superconducting to the normal state occurs, the value of the Gibbs free energy in each of the states must be the same, i.e., at $|\mathbf{H}| = H_c$ we have that $g = g_n$, so that

$$H_c = \sqrt{\frac{4\pi\alpha^2}{\beta}}. \tag{2.13}$$

We will use H_c as a fundamental scale for magnetic fields. It should be noted that for type I superconductors, H_c is the field at which superconductivity is lost. However, due to the existence of the mixed superconducting-normal state, type II superconductors are not in a perfect Meissner state and retain some superconductivity properties at fields above, and in some cases much above, H_c. This fact makes type II superconductors interesting from technological and design points of view. We will discuss this in more detail in §2.4.

Two length scales play important roles in the theory and understanding of superconductors. First, assume that the applied field $\mathbf{H}$ is constant. Assuming that the sample is perfectly conducting, i.e., $\psi = \psi_0 = (-\alpha/\beta)^{1/2}$, let us see how far the field penetrates into the sample. With the use of the relation $\mathbf{h} = \operatorname{curl} \mathbf{A}$ we deduce that

$$\operatorname{curl}\operatorname{curl}\mathbf{h} + \frac{1}{\lambda^2}\mathbf{h} = \mathbf{0}, \tag{2.14}$$

where

$$\lambda = \left(\frac{m_s c^2}{4\pi e_s^2 |\psi_0|^2}\right)^{1/2} = \left(-\frac{\beta m_s c^2}{4\pi\alpha e_s^2}\right)^{1/2}.$$

Equation (2.14) is known as the *London equation*, and the parameter λ, whose value depends on the temperature, is known as the (*London*) *penetration depth*. The latter terminology can be easily justified by examining, for example, a one-dimensional version of (2.14). Here we let h denote the single nontrivial component of the field and z the distance from the boundary of the sample. Then, (2.14) and (2.10) reduce to $d^2h/dz^2 - (1/\lambda^2)h = 0$ for $z > 0$ and $h(0) = H$, where H denotes the applied field. Then, $h = He^{-z/\lambda}$ so that λ clearly gives a measure of how far the field h penetrates into the sample.

Now, let us determine what is the length scale of variations of the order parameter. We now assume that there is a perfect Meissner effect, i.e., $\mathbf{A} = \mathbf{0}$. Then, the coefficients in (2.6) become real, and we may take ψ to be real as well, so that (2.6) reduces to

$$\Delta\psi - \frac{1}{\xi^2}\left(\psi + \frac{\beta}{\alpha}\psi^3\right) = 0, \tag{2.15}$$

where

$$\xi = \left(-\frac{\hbar^2}{2m_s\alpha}\right)^{1/2}.$$

The parameter ξ is known as the (*Ginzburg–Landau*) *coherence length*. This terminology can be justified by examining, for example, a one-dimensional, linearized (about the perfect conducting state) version of (2.15). Thus, letting x denote the single coordinate, setting $\psi = \psi_0(1 + \mu)$ with $\psi_0 = (-\alpha/\beta)^{1/2}$, and neglecting nonlinear terms in μ, we have that $d^2\mu/dx^2 - (2/\xi^2)\mu = 0$. Then, μ is proportional to $e^{\pm\sqrt{2}x/\xi}$ so that a small disturbance of ψ from ψ_0 will decay in a characteristic length of order ξ.

The ratio of the penetration length to the coherence length, i.e.,

$$\kappa = \frac{\lambda}{\xi} = \sqrt{\frac{\beta}{2\pi}}\left(\frac{m_s c}{e_c\hbar}\right),$$

is known as the *Ginzburg–Landau parameter*. Its importance lies in the fact that this single parameter, along with the given data, i.e., the applied field, completely determines the solution of the GL equations. In particular, as we shall see in §3, the value of κ determines whether the superconductor is of type I or II.

The central role that κ plays can be made evident by nondimensionalizing lengths by λ, fields by $\sqrt{2}H_c$ (the factor $\sqrt{2}$ is customary and convenient), currents by $(cH_c)/(2\sqrt{2}\pi\lambda)$, the magnetic potential by $\sqrt{2}\lambda H_c$, the order parameter by $\psi_0 = (-\alpha/\beta)^{1/2}$,

free energy densities by α^2/β, and the Gibbs free energy by $(\alpha^2\lambda^3)/\beta$. In this case, the Gibbs free energy, in nondimensionalized form, is given by

$$\mathcal{G}(\psi, \mathbf{A}) = \int_\Omega \left(f_n - |\psi|^2 + \frac{1}{2}|\psi|^4 + \left| \left(-\frac{i}{\kappa}\nabla - \mathbf{A} \right) \psi \right|^2 + |\mathbf{h}|^2 - 2\mathbf{h}\cdot\mathbf{H} \right), \tag{2.16}$$

where we have used the same notation to denote nondimensionalized variables used to denote variables with dimension. Likewise, the GL equations and boundary conditions (2.6)–(2.10) are, in nondimensional form, given by

$$\left(-\frac{i}{\kappa}\nabla - \mathbf{A} \right)^2 \psi - \psi + |\psi|^2\psi = 0 \quad \text{in } \Omega, \tag{2.17}$$

$$\mathbf{j} = \operatorname{curl} \mathbf{h} = \operatorname{curl}\operatorname{curl} \mathbf{A} = -\frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) - |\psi|^2\mathbf{A} + \operatorname{curl}\mathbf{H} \quad \text{in } \Omega, \tag{2.18}$$

$$\left(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi \right) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \tag{2.19}$$

and

$$\operatorname{curl}\mathbf{A} \times \mathbf{n} = \mathbf{H} \times \mathbf{n} \quad \text{on } \Gamma. \tag{2.20}$$

Now it is clear that κ, along with the data $\mathbf{H}$, completely determines minimizers of the Gibbs free energy, or solutions of the GL equations. It is important to note that high-temperature superconductors, and in general, inhomogeneous or anisotropic superconductors, cannot be characterized by a single, constant parameter such as κ, as is the case for low-temperature, homogeneous, isotropic superconductors.

2.4. Some properties of superconductors. We close this section by discussing some further properties of superconductors. Space limitations preclude us from giving any of the formal arguments that are used in the literature to explain these properties. Any of the references cited at the beginning of §2 may be consulted for all details. We do mention that all of these properties, as well as the ones discussed previously, have been verified experimentally.

We have already mentioned that if the applied field is sufficiently strong, then superconductivity is destroyed. (Here, we assume that the applied field is constant with magnitude H.) However, in general this critical field is not given by (2.13), which was derived assuming an ideal Meissner effect. It can be shown that if

$$H > H_{c2} = \sqrt{2}\kappa H_c, \tag{2.21}$$

then superconductivity is lost.

Remark. It should be noted that the derivation of this result, as given in various books on superconductivity, is somewhat formal. First, the derivation presupposes the existence of such a critical field in order to derive its value. In addition, the derivation also assumes that the order parameter is continuous at the transition from the normal to the superconducting states; this certainly is not true for type I superconductors. In spite of these apparent shortcomings, we proceed, as is customary, to apply this result.

For ideal type I superconductors (real ones behave in a somewhat more complex manner), $\kappa < 1/\sqrt{2}$ and (2.21) imply that as the field is reduced, the sample will remain

normal until a field $H_{c2} < H_c$ is reached, at which superconductivity will be total. This, in turn, implies a nonreversible loop, since as we increase the field, superconductivity is retained until $H = H_c$. In any case, the transition from the normal to the superconducting state at H_c or H_{c2} is a first-order one, i.e., at the critical field the order parameter jumps from zero to $\psi_0 = (-\alpha/\beta)^{1/2}$. This phenomenon is sketched in Fig. 2.1. (In real type I superconductors, there is field penetration at the surface of the sample; this complicates the idealized hysteresis loop just described.)

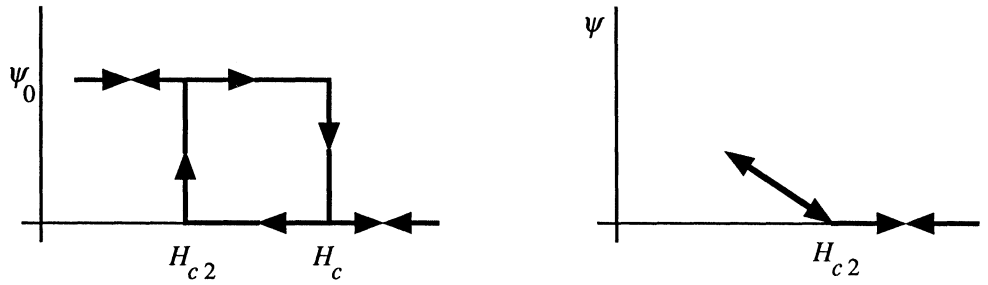

FIG. 2.1. *The transition from the normal to the superconducting state. (Arrows point in the direction of change in the applied field H.)* (a) *Hysteresis loop for type* I *superconductors* ($\kappa < 1/\sqrt{2}$). (b) *Reversible, second-order transition for type* II *superconductors* ($\kappa > 1/\sqrt{2}$).

For type II superconductors, $\kappa > 1/\sqrt{2}$ and we see that $H_{c2} > H_c$, i.e., superconductivity is not completely lost even at fields higher than the thermodynamic critical field H_c. Unlike type I superconductors, the transition at H_{c2} is a second-order one, and the order parameter is continuous there; the situation is sketched in Fig. 2.1. (Due to surface effects, which are neglected in the derivation of (2.21), real type II superconductors retain some superconductivity properties at fields even higher than H_{c2}.)

In the type II setting, there also exists a second critical field H_{c1}. For $H < H_{c1}$, the superconductor behaves very much like a type I superconductor, i.e., with a perfect Meissner effect, except near the surfaces. For an ideal superconductor, superconductivity is lost for $H > H_{c2}$. For $H_{c1} < H < H_{c2}$, we have the appearance of *vortices*, or *filaments*. The superconducting and normal states coexist. At the center of the vortices, the order parameter vanishes, i.e., we have the normal state. Near H_{c1}, these vortex filaments are well isolated, but as the field is increased, the vortices become more numerous and come closer and closer. At the same time, as the field is increased, the maximum value of the magnitude of the order parameter decreases, until at H_{c2} it vanishes. It can be shown that each vortex has associated with it one fluxoid, i.e., for each vortex $\Phi' = \Phi_0$.

This stark contrast betweeen type I and type II superconductors can be further demonstrated by examining the magnetization curves for each type. Here, we consider the magnetization M of the sample, given by $4\pi M = B - H$. (These are all bulk, or averaged quantities. In particular, B is the average value of the field and H the magnitude of the (constant) applied field.) For type I superconductors (neglecting any hysteresis effects), and for $H < H_c$, the sample is in the superconducting state, $B = 0$, and therefore $-4\pi M = H$; for $H > H_c$, the sample is normal, $B = H$, and therefore $4\pi M = 0$. The transition from $-4\pi M = H$ to $-4\pi M = 0$ at $H = H_c$ is discontinuous. Figure 2.2 gives a sketch of the magnetization curve for an ideal type I superconductor, i.e., $\kappa < 1/\sqrt{2}$.

For type II superconductors and $H < H_{c1} < H_c$, we again have the superconducting state $B = 0$ and $-4\pi M = H$, while for $H > H_{c2} > H_c$, we have the normal state $B = H$ and $-4\pi M = 0$. However, for $H_{c1} < H_c < H_{c2}$, we have a mixed state where $0 < B < H$ and therefore $0 < -4\pi M < H$. The transition between $-4\pi M = H$ at $H = H_{c1}$ to $-4\pi M = 0$ at $H = H_{c2}$ is continuous. Figure 2.2 also gives a sketch of

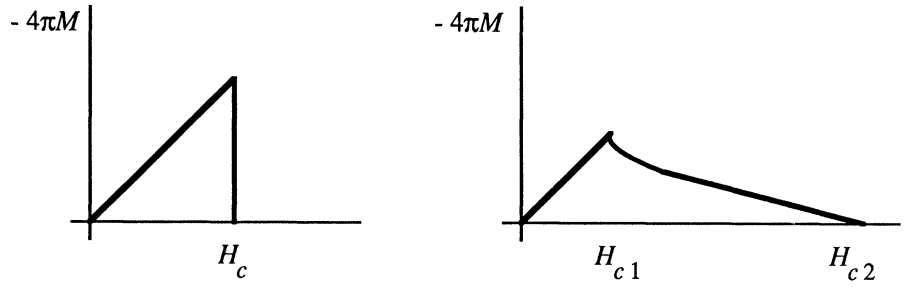

FIG. 2.2. *Magnetization M vs. applied field H.* (a) *Type* I *superconductors* ($\kappa < 1/\sqrt{2}$). (b) *Type* II *superconductors* ($\kappa > 1/\sqrt{2}$).

the magnetization curve for an ideal type II superconductor. Note that regardless of the value of κ, the area under the magnetization curve is the same, i.e.,

$$\int_0^\infty (-4\pi M)\, dH = \frac{H_c^2}{8\pi}.$$

We close this review section with some comments on the factors which limit the validity of the GL model. For example, we have assumed that higher powers can be neglected in the expansion of f in terms of powers of $|\psi|$. Also, the GL theory assumes a *local* relation between the current and the vector potential. In general, this relation is not local. It can be shown, both from the microscopic theory and from nonlocal macroscopic theories, that in order for the GL model to be valid, we must have that the temperature is close to the critical temperature at which the transition from the normal to the superconducting states occurs. It can also be shown that the temperature range over which the GL model is valid is larger for type II than it is for type I superconductors. Nevertheless, it should be emphasized that for temperatures sufficiently close to the critical transition temperature, the GL model is valid for both type I and II superconductors. Furthermore, experimental evidence suggests that, in fact, the GL model remains useful in describing correct physics even for temperatures so different from the critical temperature that a mathematical justification cannot be provided.

3. Minimizers of the Ginzburg–Landau functional. In this section we examine some questions about minimizers of the GL functional (2.4), or equivalently, of (2.5). The latter, in terms of nondimensionalized variables, is given by

$$\mathcal{E}(\psi, \mathbf{A}) = \int_\Omega \left[\frac{1}{2}\left(|\psi|^2 - 1\right)^2 + \left| \left(-\frac{i}{\kappa}\nabla - \mathbf{A} \right) \psi \right|^2 + |\operatorname{curl} \mathbf{A} - \mathbf{H}|^2 \right],$$

where Ω is a bounded, open subset of $\mathbb{R}^d$, d =2 or 3. Unless otherwise noted, we will also assume that Ω is a domain with "smooth" boundary or is a convex polyhedral domain. The boundary of Ω will be denoted by Γ. We will also make use of the functional

$$\mathcal{F}(\psi, \mathbf{A}) = \mathcal{E}(\psi, \mathbf{A}) + \int_\Omega |\operatorname{div} \mathbf{A}|^2\, d\Omega.$$

Throughout, for any nonnegative integer s, $H^s(\Omega)$ will denote the Sobolev space of real-valued functions having square integrable derivatives of order up to s. The corresponding spaces of complex-valued functions will be denoted by $\mathcal{H}^s(\Omega)$. Corresponding

spaces of vector-valued functions, each of whose d components belong to $H^s(\Omega)$, will be denoted by $\mathbf{H}^s(\Omega)$, i.e., $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^d$. Norms of functions belonging to $H^s(\Omega)$, $\mathbf{H}^s(\Omega)$, and $\mathcal{H}^s(\Omega)$ will all be denoted, without any possible ambiguity, by $\|\cdot\|_s$. For details concerning these spaces, consult [2]. A similar notational convention will hold for the Lebesgue spaces $L^p(\Omega)$ and their complex and vector-valued counterparts $\mathcal{L}^p(\Omega)$ and $\mathbf{L}^p(\Omega)$, respectively.

We will make use of the following subspaces of $\mathbf{H}^1(\Omega)$:

$$\mathbf{H}^1_n(\Omega) = \{\, \mathbf{Q} \in \mathbf{H}^1(\Omega) \;:\; \mathbf{Q}\cdot\mathbf{n} = 0 \text{ on } \Gamma \,\}$$

and

$$\mathbf{H}^1_n(\mathrm{div}\,;\Omega) = \{\, \mathbf{Q} \in \mathbf{H}^1(\Omega) \;:\; \mathrm{div}\,\mathbf{Q} = 0 \text{ in } \Omega \text{ and } \mathbf{Q}\cdot\mathbf{n} = 0 \text{ on } \Gamma \,\}.$$

We note that $(\|\mathrm{div}\,\mathbf{Q}\|_0^2 + \|\mathrm{curl}\,\mathbf{Q}\|_0^2)^{1/2}$ and $\|\mathrm{curl}\,\mathbf{Q}\|_0$ define norms on $\mathbf{H}^1_n(\Omega)$ and $\mathbf{H}^1_n(\mathrm{div}\,;\Omega)$, respectively, that are equivalent to the standard $\mathbf{H}^1(\Omega)$-norm $\|\mathbf{Q}\|_1$; see, e.g., [20].

3.1. Gauge invariance. We begin by giving a precise definition of gauge invariance. For any $\phi \in H^2(\Omega)$, let the linear transformation G_ϕ from $\mathcal{H}^1(\Omega)\times\mathbf{H}^1(\Omega)$ into itself be defined by

$$G_\phi(\psi,\mathbf{A}) = (\zeta,\mathbf{Q}) \in \mathcal{H}^1(\Omega)\times\mathbf{H}^1(\Omega) \quad \forall\,(\psi,\mathbf{A}) \in \mathcal{H}^1(\Omega)\times\mathbf{H}^1(\Omega)$$

if

$$\zeta = \psi e^{i\kappa\phi} \quad \text{and} \quad \mathbf{Q} = \mathbf{A} + \nabla\phi\,.$$

Note that if $(\zeta,\mathbf{Q}) = G_\phi(\psi,\mathbf{A})$, then $(\psi,\mathbf{A}) = G_{-\phi}(\zeta,\mathbf{Q})$.

DEFINITION. $(\psi,\mathbf{A})$ and $(\zeta,\mathbf{Q})$ are said to be *gauge equivalent* if and only if there exists a $\phi \in H^2(\Omega)$ such that $(\psi,\mathbf{A}) = G_\phi(\zeta,\mathbf{Q})$.

We have the following two results about gauge invariance with respect to $\mathcal{H}^1(\Omega)\times\mathbf{H}^1_n(\mathrm{div}\,;\Omega)$.

LEMMA 3.1. *Any* $(\zeta,\mathbf{Q}) \in \mathcal{H}^1(\Omega)\times\mathbf{H}^1(\Omega)$ *is gauge equivalent to an element of* $\mathcal{H}^1(\Omega)\times\mathbf{H}^1_n(\mathrm{div}\,;\Omega)$.

Proof. Let ϕ be defined by $\Delta\phi = \mathrm{div}\,\mathbf{Q}$ in Ω and $\partial\phi/\partial n = \mathbf{Q}\cdot\mathbf{n}$. Then, $\phi \in H^2(\Omega)$ and clearly $G_\phi(\zeta,\mathbf{Q}) \in \mathcal{H}^1(\Omega)\times\mathbf{H}^1_n(\mathrm{div}\,;\Omega)$. □

LEMMA 3.2. $(\psi,\mathbf{A})$ *and* $(\zeta,\mathbf{Q}) \in \mathcal{H}^1(\Omega)\times\mathbf{H}^1_n(\mathrm{div}\,;\Omega)$ *are gauge equivalent if and only if* $\mathbf{A} = \mathbf{Q}$ *and* $\zeta = \psi e^{ic}$ *for some real constant* c.

Proof. The proof is obvious. □

Of course, Ginzburg and Landau constructed their free energy functional so that it is gauge invariant. This is stated in the following proposition whose proof is merely a simple calculation.

PROPOSITION 3.3. *For all* $\phi \in H^2(\Omega)$ *and* $(\psi,\mathbf{A}) \in \mathcal{H}^1(\Omega)\times\mathbf{H}^1(\Omega)$, $\mathcal{E}(\psi,\mathbf{A}) = \mathcal{E}(G_\phi(\psi,\mathbf{A}))$, *i.e.,* $\mathcal{E}$ *is invariant under the gauge transformation* G_ϕ.

Thus, for example, if $(\psi,\mathbf{A})$ is a minimizer of $\mathcal{E}$, then so is $G_\phi(\psi,\mathbf{A})$, i.e., so is any $(\zeta,\mathbf{Q})$ that can be obtained from $(\psi,\mathbf{A})$ through a gauge transformation. In this way, it is possible to define equivalence classes of minimizers, i.e., minimizers that may be obtained from each other through gauge transformations. By choosing a particular gauge, one extracts a particular member of its equivalence class.

3.2. Existence of minimizers. We now give some results concerning the existence of minimizers of $\mathcal{F}$ and $\mathcal{E}$ in various spaces, and the relation between the various minimizers.

PROPOSITION 3.4. *$\mathcal{E}$ has at least one minimizer belonging to $\mathcal{H}^1(\Omega) \times \mathbf{H}^1_n(\mathrm{div}\,;\Omega)$.*

Proof. It is easy to check that $\mathcal{E}$ is nonnegative, is continuous in the strong topology, and lower semicontinuous in the weak topology. Then the proof proceeds using standard arguments; see, e.g., [24]. First, we have the existence of a minimizing sequence, i.e., $(\psi^n, \mathbf{A}^n) \in \mathcal{H}^1(\Omega) \times \mathbf{H}^1_n(\mathrm{div}\,;\Omega)$ such that $\mathcal{E}(\psi^n, \mathbf{A}^n) \searrow \inf \mathcal{E} \geq 0$. Then, due to the equivalence of norms, we have the boundedness of $(\psi^n, \mathbf{A}^n) \in \mathcal{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$. After extracting a subsequence, we see that the weak limit will be a minimizer. □

THEOREM 3.5. *$\mathcal{E}$ has at least one minimizer belonging to $\mathcal{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$. Moreover,*

$$\min_{\mathcal{H}^1(\Omega)\times\mathbf{H}^1(\Omega)} \mathcal{E} = \min_{\mathcal{H}^1(\Omega)\times\mathbf{H}^1_n(\mathrm{div}\,;\Omega)} \mathcal{E}\,.$$

Proof. The results follow from Propositions 3.3 and 3.4. □

COROLLARY 3.6. *Any minimizer of $\mathcal{E}$ in $\mathcal{H}^1(\Omega) \times \mathbf{H}^1_n(\mathrm{div}\,;\Omega)$ is also a minimizer of $\mathcal{E}$ in $\mathcal{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$. Any minimizer of $\mathcal{E}$ in $\mathcal{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$ is gauge equivalent to a minimizer of $\mathcal{E}$ in $\mathcal{H}^1(\Omega) \times \mathbf{H}^1_n(\mathrm{div}\,;\Omega)$.*

Proof. The results follow from the fact that $\mathcal{H}^1(\Omega) \times \mathbf{H}^1_n(div;\Omega) \subset \mathcal{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$. □

LEMMA 3.7. *For any $(\psi, \mathbf{A}) \in \mathcal{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$, let $\phi \in H^2(\Omega) \setminus \mathbb{R}$ satisfy*

$$\Delta\phi = \mathrm{div}\,\mathbf{A} \quad \textit{in } \Omega \quad \textit{and} \quad \partial\phi/\partial n = \mathbf{A}\cdot\mathbf{n} \quad \textit{on } \Gamma\,.$$

Then,

$$\mathcal{F}(G_\phi(\psi, \mathbf{A})) = \mathcal{F}(\psi, \mathbf{A}) - \int_\Omega |\mathrm{div}\,\mathbf{A}|^2 = \mathcal{E}(\psi, \mathbf{A})\,.$$

Proof. The results follow from Proposition 3.3 and the definitions of $\mathcal{E}$ and $\mathcal{F}$. □

THEOREM 3.8. *$\mathcal{F}$ has at least one minimizer in $\mathcal{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$. Moreover, all minimizers $(\psi, \mathbf{A})$ of $\mathcal{F}$ satisfy* $\mathrm{div}\,\mathbf{A} = 0$.

Proof. The proof is similar to that of Proposition 3.4. If $(\psi^n, \mathbf{A}^n)$ is a minimizing sequence, so is, by Lemma 3.7, $G_{\phi^n}(\psi^n, \mathbf{A}^n)$. The boundedness of the sequence $G_{\phi^n}(\psi^n, \mathbf{A}^n)$ is easily seen. Thus, we can proceed to obtain the existence of a minimizer. The fact that $\mathrm{div}\,\mathbf{A} = 0$ for minimizers also follows from Lemma 3.7. □

COROLLARY 3.9.

$$\min_{\mathcal{H}^1(\Omega)\times\mathbf{H}^1(\Omega)} \mathcal{F} = \min_{\mathcal{H}^1(\Omega)\times\mathbf{H}^1_n(\mathrm{div}\,;\Omega)} \mathcal{F}\,,$$

$$\min_{\mathcal{H}^1(\Omega)\times\mathbf{H}^1(\Omega)} \mathcal{F} = \min_{\mathcal{H}^1(\Omega)\times\mathbf{H}^1_n(\Omega)} \mathcal{F}\,,$$

and

$$\min_{\mathcal{H}^1(\Omega)\times\mathbf{H}^1(\Omega)} \mathcal{F} = \min_{\mathcal{H}^1(\Omega)\times\mathbf{H}^1(\Omega)} \mathcal{E}\,.$$

Proof. The first equality follows from Theorem 3.8 and the fact that if $\mathrm{div}\,\mathbf{A} = 0$, then $\mathcal{F}(G_\phi(\psi, \mathbf{A})) = \mathcal{F}(\psi, \mathbf{A})$, where ϕ is defined as in Lemma 3.7. Then, the second equality easily follows from the first. Finally, the third equality follows from the fact that $\mathcal{F}(\psi, \mathbf{A}) = \mathcal{E}(\psi, \mathbf{A})$ for $(\psi, \mathbf{A}) \in \mathcal{H}^1(\Omega) \times \mathbf{H}^1_n(\mathrm{div}\,;\Omega)$. □

Thus, we have shown that we can locate minimizers of $\mathcal{E}$ in $\mathcal{H}^1(\Omega)\times\mathbf{H}^1(\Omega)$ by looking for minimizers of $\mathcal{F}$ in $\mathcal{H}^1(\Omega) \times \mathbf{H}^1_n(\Omega)$. We shall see that the latter problem has some definite advantages, especially from a computational point of view.

3.3. The Ginzburg–Landau equations. We now derive the GL equations which we have formally introduced in §2.2. At the same time, by looking for minimizers of $\mathcal{F}$ in $\mathcal{H}^1(\Omega) \times \mathbf{H}^1_n(\Omega)$ we will be automatically choosing a gauge for $(\psi, \mathbf{A})$.

Taking the Frechét derivative of $\mathcal{F}(\psi, \mathbf{A})$ with respect to $\psi \in \mathcal{H}^1(\Omega)$ we obtain

$$\int_\Omega \left[\left(-\frac{i}{\kappa}\nabla\psi - \mathbf{A}\psi \right) \cdot \left(\frac{i}{\kappa}\nabla\tilde{\psi}^* - \mathbf{A}\tilde{\psi}^* \right) + \left(-\frac{i}{\kappa}\nabla\tilde{\psi} - \mathbf{A}\tilde{\psi} \right) \cdot \left(\frac{i}{\kappa}\nabla\psi^* - \mathbf{A}\psi^* \right) + (|\psi|^2 - 1)(\psi\tilde{\psi}^* + \psi^*\tilde{\psi}) \right] d\Omega = 0 \quad \forall\, \tilde{\psi} \in \mathcal{H}^1(\Omega)\,; \tag{3.1}$$

and taking the derivative with respect to $\mathbf{A} \in \mathbf{H}^1_n(\Omega)$ we get

$$\int_\Omega \left[\operatorname{div}\mathbf{A} \operatorname{div}\tilde{\mathbf{A}} + \operatorname{curl}\mathbf{A} \cdot \operatorname{curl}\tilde{\mathbf{A}} + |\psi|^2\mathbf{A}\cdot\tilde{\mathbf{A}} + \frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*)\cdot\tilde{\mathbf{A}} \right] d\Omega = \int_\Omega \mathbf{H}\cdot\operatorname{curl}\tilde{\mathbf{A}}\, d\Omega \quad \forall\, \tilde{\mathbf{A}} \in \mathbf{H}^1_n(\Omega)\,. \tag{3.2}$$

Integration by parts in (3.1) yields, should $(\psi, \mathbf{A})$ be sufficiently smooth, the first GL equation (2.17), and as a natural boundary condition

$$\nabla\psi \cdot \mathbf{n} = 0 \quad \text{on } \Gamma\,. \tag{3.3}$$

Of course, since $\mathbf{A}\cdot\mathbf{n} = 0$ on Γ, (3.3) is the same as (2.19).

Let $\tilde{\mathbf{A}} = \nabla q$, where $q \in H^2(\Omega)$ satisfies $\Delta q = \operatorname{div}\mathbf{A}$ in Ω and $\partial q/\partial n = 0$ on Γ. Then, $\tilde{\mathbf{A}} \in \mathbf{H}^1_n(\Omega)$. We substitute this $\tilde{\mathbf{A}}$ in (3.2) and integrate by parts to obtain

$$\int_\Omega (\operatorname{div}\mathbf{A})^2 = \int_\Omega q \operatorname{div}\left(|\psi|^2\mathbf{A} + \frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) \right) = 0\,,$$

where we have used (3.3) in the last equality. Thus, we have that $\operatorname{div}\mathbf{A} = 0$ almost everywhere in Ω, so that we are employing the (*London or Coulomb*) *gauge*, along with, of course, $\mathbf{A}\cdot\mathbf{n} = 0$ on Γ. Furthermore, (3.2) also yields the second GL equation (2.18) and, as a natural boundary condition, (2.20). Note that since $\operatorname{div}\mathbf{A} = 0$, (2.18) may be expressed in the form

$$-\Delta\mathbf{A} + \frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) + |\psi|^2\mathbf{A} = \operatorname{curl}\mathbf{H} \quad \text{in } \Omega\,.$$

This second-order system of elliptic partial differential equations is supplemented by the boundary conditions (2.20) and $\mathbf{A}\cdot\mathbf{n} = 0$.

3.4. Some properties of minimizers of the GL free energy and solutions of the GL equations. We now derive a series of properties possessed by minimizers of the GL free energy, or of solutions of the GL equations.

3.4.1. Nonexistence of local maxima. We first show that solutions of the GL equations cannot correspond to local maxima of the GL free energy.

PROPOSITION 3.10. *A solution of the Ginzburg–Landau equations* (2.17), (2.18) *cannot be a local maximum of $\mathcal{E}$, or of $\mathcal{F}$.*

Proof. It is obvious that for fixed $\psi \in \mathcal{H}^1(\Omega)$, $\mathcal{E}$ is a convex functional in $\mathbf{H}^1(\Omega)$. Moreover, if $(\psi, \mathbf{A})$ is a solution of the GL equations, and if $\mu(\epsilon) = \mathcal{F}(\epsilon\psi, \mathbf{A})$, we have

that $\mu(\epsilon)$ is locally convex at $\epsilon = 1$. Indeed, we need only check $d^2\mu/d\epsilon^2$. A simple calculation yields that

$$\left.\frac{d^2\mu}{d\epsilon^2}\right|_{\epsilon=1} = \int_\Omega \left(6|\psi|^4 - 2|\psi|^2 + 2\left|\left(-\frac{i}{\kappa}\nabla - \mathbf{A}\right)\psi\right|^2\right).$$

Since $(\psi, \mathbf{A})$ is presumed to be a solution of the GL equations, we may then show that

$$\left.\frac{d^2\mu}{d\epsilon^2}\right|_{\epsilon=1} = 4\int_\Omega \left(|\psi|^4 + \left|\left(-\frac{i}{\kappa}\nabla - \mathbf{A}\right)\psi\right|^2\right) \geq 0,$$

with equality holding if and only if $\psi = 0$. □

3.4.2. Boundedness of the order parameter. We next show that the magnitude of the order parameter is bounded, and in fact, is bounded by the ideal superconducting value, i.e., in nondimensionalized form, $|\psi| \leq 1$.

PROPOSITION 3.11. *If $(\psi, \mathbf{A})$ is a solution of the Ginzburg–Landau equations, then $|\psi| \leq 1$ almost everywhere.*

Proof. We follow, in the present context of bounded domains, the proof in [40] for this same result in the case of $\Omega = \mathbb{R}^3$.

Set $\tilde{\psi} = (|\psi| - 1)_+ f$, where $f = \psi/|\psi|$ and where $q_+ = q$ if $q \geq 0$ and $q_+ = 0$ if $q < 0$. Let $\Omega^+ = \{\mathbf{x} \in \Omega : |\psi| > 1\}$. Then, on Ω^+,

$$\frac{i}{\kappa}\nabla\tilde{\psi}^* - \mathbf{A}\tilde{\psi}^* = \frac{i}{\kappa}f^*\nabla|\psi| + (|\psi| - 1)\left(\frac{i}{\kappa}\nabla f^* - \mathbf{A}f^*\right),$$

and

$$-\frac{i}{\kappa}\nabla\psi - \mathbf{A}\psi = -\frac{i}{\kappa}f\nabla|\psi| + |\psi|\left(-\frac{i}{\kappa}\nabla f - \mathbf{A}f\right)$$

so that

$$\Re\left\{\left(-\frac{i}{\kappa}\nabla\psi - \mathbf{A}\psi\right)\cdot\left(\frac{i}{\kappa}\nabla\tilde{\psi}^* - \mathbf{A}\tilde{\psi}^*\right)\right\} = (\nabla|\psi|)^2 + |\psi|(|\psi| - 1)\left|-\frac{i}{\kappa}\nabla f - \mathbf{A}f\right|^2.$$

Then, since (3.1) implies that

$$\Re\left\{\int_\Omega \left[\left(-\frac{i}{\kappa}\nabla\psi - \mathbf{A}\psi\right)\cdot\left(\frac{i}{\kappa}\nabla\tilde{\psi}^* - \mathbf{A}\tilde{\psi}^*\right) + (|\psi|^2 - 1)\psi\tilde{\psi}^*\right] d\Omega\right\} = 0,$$

we have that

$$\int_{\Omega^+}\left[(\nabla|\psi|)^2 + |\psi|(|\psi| - 1)\left|-\frac{i}{\kappa}\nabla f - \mathbf{A}f\right|^2 + |\psi|(|\psi| + 1)(|\psi| - 1)^2\right] d\Omega = 0.$$

Then, since the integrand is positive, $\text{meas}(\Omega^+) = 0$ and therefore, $|\psi| \leq 1$ almost everywhere. □

3.4.3. Constant solutions and the existence of mixed-state superconducting solutions for $H < H_c$. The normal solution is characterized by $\psi = 0$. In this case, the

potential $\mathbf{A} = \hat{\mathbf{A}}$, where

$$\operatorname{curl}\operatorname{curl}\hat{\mathbf{A}} = \operatorname{curl}\mathbf{H} \quad \text{in } \Omega \quad \text{and} \quad \operatorname{curl}\hat{\mathbf{A}} \times \mathbf{n} = \mathbf{H} \times \mathbf{n} \quad \text{on } \Gamma \tag{3.4}$$

and, in the gauge we are employing,

$$\operatorname{div}\hat{\mathbf{A}} = 0 \quad \text{in } \Omega \quad \text{and} \quad \hat{\mathbf{A}} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \tag{3.5}$$

where $\mathbf{H}$ is the applied field. Note that $(0, \hat{\mathbf{A}})$ is a solution of the GL equations for any value of κ.

If the applied field *vanishes*, i.e., $\mathbf{H} = \mathbf{0}$, then the ideal superconducting solution (with perfect Meissner effect) is the unique global minimizer of the GL free energy.

LEMMA 3.12. *If* $\mathbf{H} = \mathbf{0}$, *then* $\psi = 1$ *and* $\mathbf{A} = \mathbf{0}$ *is the global minimizer of* $\mathcal{E}$.

Proof. If $\mathbf{H} = \mathbf{0}$, then $\mathcal{E}(1, \mathbf{0}) = 0$. However, for any $(\psi, \mathbf{A}) \in \mathcal{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$, $\mathcal{E}(\psi, \mathbf{A}) \geq 0$, so that for any such $(\psi, \mathbf{A})$, $\mathcal{E}(\psi, \mathbf{A}) \geq \mathcal{E}(1, \mathbf{0})$, with equality holding if and only if $\psi = 1$ and $\mathbf{A} = \mathbf{0}$. □

It should be noted that when $\mathbf{H} = \mathbf{0}$, this result does not rule out the existence of other critical points; see [24].

Next, we show that if the applied field *does not vanish*, then, in most cases, the only solution of the GL *equations* with $\psi =$ constant is the normal state $\psi = 0$ and $\mathbf{A} = \hat{\mathbf{A}}$.

LEMMA 3.13. *Let* $\mathbf{H} \neq \mathbf{0}$ *and, if* $\Omega \subset \mathbb{R}^3$ *with a continuous normal vector, let its boundary* Γ *have nonzero Euler number. Then, the only solution of the Ginzburg–Landau equations with* $\psi =$ *constant and* $\mathbf{A}$ *continuous is the normal solution* $\psi = 0$ *and* $\mathbf{A} = \hat{\mathbf{A}}$.

Proof. If $\psi =$ constant, say $\psi = k$, then, from (2.18), (2.20), and our choice of gauge, we have that

$$\operatorname{curl}\operatorname{curl}\mathbf{A} + k^2\mathbf{A} = \operatorname{curl}\mathbf{H} \quad \text{and} \quad \operatorname{div}\mathbf{A} = 0 \quad \text{in } \Omega \tag{3.6}$$

and

$$\operatorname{curl}\mathbf{A} \times \mathbf{n} = \mathbf{H} \times \mathbf{n} \quad \text{and} \quad \mathbf{A} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma. \tag{3.7}$$

Also, from (2.17), we have that $(|\mathbf{A}|^2 - 1 + |k|^2)k = 0$ so that either $k = 0$ or $|k|^2 = 1 - |\mathbf{A}|^2$. In the first case, we easily deduce that $\mathbf{A} = \hat{\mathbf{A}}$. Now, in the second case, if $|k|^2 = 1$, then $\mathbf{A} = \mathbf{0}$, which leads to $\mathbf{H} = \mathbf{0}$, a contradiction. We are left with the case $|k|^2 < 1$. In this case $\mathbf{A}$ defines a continuous vector field on Ω that has constant magnitude $|\mathbf{A}| = \sqrt{1 - k^2} < 1$. Meanwhile, since $\mathbf{A} \cdot \mathbf{n} = 0$ on Γ, $\mathbf{A}$ also defines a continuous tangential vector field on Γ, again with constant magnitude $|\mathbf{A}| = \sqrt{1 - k^2}$. If the normal vector to Γ is discontinuous, the fact that $\mathbf{A}$, restricted to Γ, is a continuous tangential vector field on Γ implies that $\mathbf{A} = \mathbf{0}$, which again leads to the contradictory result $\mathbf{H} = \mathbf{0}$.

At this point it is convenient to separate the cases of $\Omega \subset \mathbb{R}^2$ and $\Omega \subset \mathbb{R}^3$, since a straightforward calculus proof is available in the former case. For $\Omega \subset \mathbb{R}^2$, let $\tilde{\mathbf{H}} = \mathbf{H} - \operatorname{curl}\mathbf{A}$ so that (3.6) and (3.7) yield that

$$\operatorname{curl}\tilde{\mathbf{H}} = k^2\mathbf{A} \quad \text{in } \Omega \tag{3.8}$$

and

$$\tilde{\mathbf{H}} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma. \tag{3.9}$$

Of course, we also have that

$$\operatorname{div}\tilde{\mathbf{H}} = 0 \quad \text{in } \Omega. \tag{3.10}$$

It is easy to show that $\tilde{\mathbf{H}} = \tilde{H}\hat{\mathbf{k}}$, where $\hat{\mathbf{k}}$ denotes a unit vector perpendicular to the plane. Then, (3.8)–(3.10) imply that $|\text{grad}\,\tilde{H}| = \text{constant} > 0$, so that $\tilde{H}$ has no interior maxima or minima. But since $\tilde{H} = 0$ on Γ, this implies that $\tilde{H} = 0$ in Ω as well, a contradiction.

We now turn to the only remaining case, namely $k^2 < 1$ and $\Omega \subset \mathbb{R}^3$ such that the normal to Γ is continuous. In this case, if the Euler number of Γ is not zero, it is well known that one cannot have a nonvanishing continuous tangential vector field on Γ with constant magnitude. Thus, again, we are led to $\mathbf{A} = \mathbf{0}$ and the contradictory result $\mathbf{H} = \mathbf{0}$. □

Remark. The necessity of the continuity of $\mathbf{A}$ is demonstrated by the following counterexample, for which we thank A. J. Meir. Let Ω in $\mathbb{R}^2$ be a square, and let $\tilde{\mathbf{H}} = \tilde{H}\hat{\mathbf{k}}$, where $\tilde{H}$ is an Egyptian pyramid defined on Ω, i.e., a continuous piecewise linear function vanishing on Γ and having an appropriate maximum value at the centroid of the square. Then, from (3.8) we deduce that the components of $\mathbf{A}$ are piecewise constants over Ω, and that $\mathbf{A}$ satisfies all the requisite conditions.

Remark. Surfaces in $\mathbb{R}^3$ with Euler number zero are torus-like; see, e.g., [30]. The necessity of having a nonvanishing Euler number is demonstrated by the following counterexample, for which we thank W. Floyd and Y. Rong. Let Ω be a solid torus, let the origin be located at the center of the torus, and let the z-axis be the axis of rotation for the torus. Then, let

$$\mathbf{A} = \sqrt{1-k^2}\left(\frac{-y}{\sqrt{x^2+y^2}}\,,\ \frac{x}{\sqrt{x^2+y^2}}\,,\ 0\right).$$

It is easily verified that $\mathbf{A}$ satisfies all the requisite conditions.

Remark. Whenever Lemma 3.13 fails to hold, it is due to the fact that for $0 < |\psi| = |k| < 1$ it is possible to define fields $\mathbf{H}$ such that there exists a potential $\mathbf{A}$ such that (3.6) and (3.7) hold, and such that $|\mathbf{A}| = \sqrt{1-k^2}$ on $\overline{\Omega}$. However, given an arbitrary field $\mathbf{H}$, it is quite likely that the solution of $\mathbf{A}$ of (3.6) and (3.7) will not have constant magnitude, even when Ω is a solid torus.

We next show that for $H < H_c$ and for any value of κ, the normal solution cannot be a global minimizer. (Note that in terms of nondimensionalized variables, $H_c = 1/\sqrt{2}$.)

PROPOSITION 3.14. *Suppose the external field is smooth, and that* $H = \max_\Omega |\mathbf{H}|$. *Then, if* $H < 1/\sqrt{2}$, *the normal solution* $\psi = 0$ *and* $\mathbf{A} = \hat{\mathbf{A}}$, *where* $\hat{\mathbf{A}}$ *is defined by* (3.4), (3.5), *is not a global minimizer of* $\mathcal{E}$.

Proof. Note that $\mathcal{E}(0, \hat{\mathbf{A}}) = V/2$, where $V = \text{measure}(\Omega)$. On the other hand, $\mathcal{E}(1, 0) = \int_\Omega |\mathbf{H}|^2 \le H^2 V$. Thus, if $H < 1/\sqrt{2}$, $\mathcal{E}(1,0) < \mathcal{E}(0, \hat{\mathbf{A}})$, and thus the normal solution cannot be a global minimizer. □

Lemma 3.13 and Proposition 3.14 have the following immediate consequence. Note, through an examination of the proof of Lemma 3.13, that here we need not require that Γ have a nonvanishing Euler number.

COROLLARY 3.15. *For any value of* κ, *if the applied field* $\mathbf{H} \neq \mathbf{0}$ *is such that* $0 < \max_\Omega |\mathbf{H}| < 1/\sqrt{2}$, *then the Ginzburg–Landau equations have a solution such that* $\psi \neq$ *constant.*

Thus, if (in dimensional form) $0 < H < H_c$, then the GL equations have a solution such that ψ is not constant. This is a *mixed-state* solution, since there are places where ψ differs from both zero and one. Of course, the mixed-states of type I and II superconductors can be quite different. In type I superconductors, for $H < H_c$, the mixed-state is present near boundaries; away from boundaries we have basically the perfect Meissner state. For type II superconductors, there is the possibility of vortex-like solutions where a mixed-state can exist anywhere in Ω.

Remark. We close this section with a remark concerning the *regularity* of solutions of the GL equations. It is easily shown that this regularity is completely determined by the regularity of the *linear* problem

$$\Delta\psi = 0 \quad \text{and} \quad \Delta\mathbf{A} = -\operatorname{curl}\mathbf{H} \quad \text{in } \Omega \tag{3.11}$$

and

$$\nabla\psi\cdot\mathbf{n} = 0\,, \quad \mathbf{A}\cdot\mathbf{n} = 0\,, \quad \text{and} \quad \operatorname{curl}\mathbf{A}\times\mathbf{n} = \mathbf{H}\times\mathbf{n} \quad \text{on } \Gamma\,. \tag{3.12}$$

For sufficiently smooth Γ, it may be shown that solutions $(\psi, \mathbf{A})$ of (3.11), (3.12) belong to $\mathcal{H}^{m+1}(\Omega)\times\mathbf{H}^{m+1}(\Omega)$ whenever $\mathbf{H}\in\mathbf{H}^m(\Omega)$. This result may be obtained, e.g., for $m=1$ and Γ of class C^2, using the theory of [3]. However, for less smooth boundaries, e.g., convex polyhedral domains in $\mathbb{R}^2$ or $\mathbb{R}^3$, it is not known if solutions of (3.11), (3.12) belong to $\mathcal{H}^2(\Omega)\times\mathbf{H}^2(\Omega)$, even for smooth $\mathbf{H}$; it seems that at least additional compatibility conditions along edges and at vertices are needed. The culprit is the "nonstandard" boundary condition for $\mathbf{A}$ found in (3.12).

4. Finite element approximations. Finite element approximations to the solutions of the Ginzburg–Landau equations are defined in the usual manner. In order to keep the exposition simple, we assume that Ω is a convex polyhedral domain. We choose families of finite-dimensional subspaces $\mathcal{S}^h\subset\mathcal{H}^1(\Omega)$ and $\mathbf{V}^h\subset\mathbf{H}^1_n(\Omega)$, parametrized by a parameter h that tends to zero. These spaces are constructed, in a standard way, from partitions of Ω into finite elements; h is then some measure of the size of the finite elements in a partition. We assume that the subspaces satisfy the following approximation properties:

$$\inf_{\tilde\psi^h\in\mathcal{S}^h}\|\psi-\tilde\psi^h\|_1\to 0 \quad \text{as } h\to 0 \quad \forall\,\psi\in\mathcal{H}^1(\Omega)\,, \tag{4.1}$$

$$\inf_{\tilde{\mathbf{A}}^h\in\mathbf{V}^h}\|\mathbf{A}-\tilde{\mathbf{A}}^h\|_1\to 0 \quad \text{as } h\to 0 \quad \forall\,\mathbf{A}\in\mathbf{H}^1_n(\Omega)\,, \tag{4.2}$$

$$\inf_{\tilde\psi^h\in\mathcal{S}^h}\|\psi-\tilde\psi^h\|_1\le Ch^m\|\psi\|_{m+1} \quad \forall\,\psi\in\mathcal{H}^{m+1}(\Omega)\,, \tag{4.3}$$

and

$$\inf_{\tilde{\mathbf{A}}^h\in\mathbf{V}^h}\|\mathbf{A}-\tilde{\mathbf{A}}^h\|_1\le Ch^m\|\mathbf{A}\|_{m+1} \quad \forall\,\mathbf{A}\in\mathbf{H}^{m+1}(\Omega)\cap\mathbf{H}^1_n(\Omega)\,. \tag{4.4}$$

We may consult [11] for conditions on the finite element partitions such that (4.1)–(4.4) are satisfied.

Finite element approximations are then defined as follows: seek $\psi^h\in\mathcal{S}^h$ and $\mathbf{A}^h\in\mathbf{V}^h$ such that

$$\begin{aligned}\int_\Omega\Bigg[&\left(-\frac{i}{\kappa}\nabla\psi^h-\mathbf{A}^h\psi^h\right)\cdot\left(\frac{i}{\kappa}\nabla(\tilde\psi^h)^*-\mathbf{A}^h(\tilde\psi^h)^*\right)\\ &+\left(-\frac{i}{\kappa}\nabla\tilde\psi^h-\mathbf{A}^h\tilde\psi^h\right)\cdot\left(\frac{i}{\kappa}\nabla(\psi^h)^*-\mathbf{A}^h(\psi^h)^*\right)\\ &+(|\psi^h|^2-1)(\psi^h(\tilde\psi^h)^*+(\psi^h)^*\tilde\psi^h)\Bigg]=0 \quad \forall\,\tilde\psi^h\in\mathcal{S}^h,\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}\int_\Omega &[\operatorname{div}\mathbf{A}^h \operatorname{div}\tilde{\mathbf{A}}^h + \operatorname{curl}\mathbf{A}^h \cdot \operatorname{curl}\tilde{\mathbf{A}}^h + |\psi^h|^2\mathbf{A}^h\cdot\tilde{\mathbf{A}}^h \\ &+\frac{i}{2\kappa}((\psi^h)^*\nabla\psi^h - \psi^h\nabla(\psi^h)^*)\cdot\tilde{\mathbf{A}}^h]d\Omega \\ &= \int_\Omega \mathbf{H}\cdot\operatorname{curl}\tilde{\mathbf{A}}^h d\Omega \quad \forall\,\tilde{\mathbf{A}}^h \in \mathbf{V}^h.\end{aligned} \tag{4.6}$$

Note that (4.5), (4.6) is merely a discretization of (3.1), (3.2); as was seen in §3.3, the latter is a weak formulation of the GL equation with the gauge $\operatorname{div}\mathbf{A} = 0$ in Ω and $\mathbf{A}\cdot\mathbf{n} = 0$ on Γ.

4.1. Quotation of some results concerning the approximation of a class of nonlinear problems. The error estimates to be derived in §4.2 make use of results of [7] and [12] (see also [20]) concerning the approximation of a class of nonlinear problems. Similar results may be also found in [17] and [18]. Here, for the sake of completeness, we will state the relevant results, specialized to our needs. The nonlinear problems to be considered are of the type

$$F(\kappa, u) \equiv u + TG(\kappa, u) = 0, \tag{4.7}$$

where $T \in \mathcal{B}(Y; X)$, G is a C^2 mapping from $\Lambda \times X$ into Y, X and Y are Banach spaces, Λ is a compact interval of $\mathbb{R}$, and $\mathcal{B}(Y; X)$ denotes the space of bounded linear operators from Y into X. We say that $\{(\kappa, u(\kappa)) : \kappa \in \Lambda\}$ is a branch of solutions of (4.7) if $\kappa \to u(\kappa)$ is a continuous function from Λ into X such that $F(\kappa, u(\kappa)) = 0$. The branch is called a *regular branch* if we also have that $D_u F(\kappa, u(\kappa))$ is an isomorphism from X into X for all $\kappa \in \Lambda$. Here, D_u denotes the Frechét derivative with respect to u.

Approximations are defined by introducing a subspace $X^h \subset X$ and an approximating operator $T^h \in \mathcal{B}(Y; X^h)$. Then, we seek $u^h \in X^h$ such that

$$F^h(\kappa, u^h) \equiv u^h + T^h G(\kappa, u^h) = 0\,. \tag{4.8}$$

We will assume that there exists another Banach space Z, contained in Y, with continuous imbedding, such that

$$D_u G(\kappa, u) \in \mathcal{B}(X; Z) \quad \forall\,\kappa \in \Lambda \text{ and } u \in X\,. \tag{4.9}$$

Concerning the operator T^h, we assume the approximation properties

$$\lim_{h\to 0} \|(T^h - T)v\|_X = 0 \quad \forall\, v \in Y \tag{4.10}$$

and

$$\lim_{h\to 0} \|(T^h - T)\|_{\mathcal{B}(Z;X)} = 0\,. \tag{4.11}$$

Note that (4.11) follows from (4.10) whenever the imbedding $Z \subset Y$ is compact.

We now may state the results that will be used in the sequel.

THEOREM 4.1. *Let X and Y be Banach spaces and Λ a compact subset of $\mathbb{R}$. Assume that G is a C^2 mapping from $\Lambda \times X$ into Y and that all second Frechét derivatives of G are bounded on all bounded sets of $\Lambda \times X$. Assume that* (4.9)–(4.11) *hold and that $\{(\kappa, u(\kappa)); \kappa \in \Lambda\}$ is a branch of regular solutions of* (4.7). *Then there exists a neighborhood*

$\mathcal{O}$ of the origin in X and, for h sufficiently small, a unique C^2 function $\kappa \to u^h(\kappa) \in X^h$, such that $\{(\kappa, u^h(\kappa)); \kappa \in \Lambda\}$ is a branch of regular solutions of (4.8) *and $u^h(\kappa) - (\kappa) \in \mathcal{O}$ for all $\kappa \in \Lambda$. Moreover, there exists a constant $C > 0$, independent of h and κ, such that*

$$\|u(\kappa) - u^h(\kappa)\|_X \le C\|(T^h - T)G(\kappa, u(\kappa))\|_X \quad \forall\, \kappa \in \Lambda\,. \tag{4.12}$$

4.2. Error estimates. We begin by recasting the weak formulation (3.1), (3.2) and its discretization (4.5), (4.6) into a form that fits into the framework of §4.1. Let

$$X = \mathcal{H}^1(\Omega) \times \mathbf{H}^1_n(\Omega) \qquad Y = (\mathcal{H}^1(\Omega))' \times (\mathbf{H}^1_n(\Omega))',$$
$$Z = \mathcal{L}^{3/2}(\Omega) \times \mathbf{L}^{3/2}(\Omega) \qquad \text{and} \qquad X^h = \mathcal{S}^h \times \mathbf{V}^h,$$

where $(\cdot)'$ denotes the dual space. Note that $Z \subset Y$ with a compact imbedding.

Let the operator $T \in \mathcal{B}(Y; X)$ be defined in the following manner: $T(\xi, \mathbf{P}) = (\theta, \mathbf{Q})$ for $(\xi, \mathbf{P}) \in Y$ and $(\theta, \mathbf{Q}) \in X$, if and only if

$$\int_\Omega (\nabla\theta \cdot \nabla\tilde{\psi}^* + \nabla\tilde{\psi} \cdot \nabla\theta^* + \theta\tilde{\psi}^* + \tilde{\psi}\theta^*)d\Omega = \int_\Omega (\xi\tilde{\psi}^* + \xi^*\tilde{\psi})d\Omega \quad \forall\, \tilde{\psi} \in \mathcal{H}^1(\Omega) \tag{4.13}$$

and

$$\int_\Omega (\operatorname{div}\mathbf{Q}\operatorname{div}\tilde{\mathbf{A}} + \operatorname{curl}\mathbf{Q} \cdot \operatorname{curl}\tilde{\mathbf{A}})d\Omega = \int_\Omega \mathbf{P} \cdot \tilde{\mathbf{A}}d\Omega \quad \forall\, \tilde{\mathbf{A}} \in \mathbf{H}^1_n(\Omega)\,. \tag{4.14}$$

It is easily seen that (4.13) and (4.14) are weak formulations of two *uncoupled* Poisson-type equations for θ and $\mathbf{Q}$, and that T is the solution operator of these equations. Indeed, (4.13) is a weak formulation of

$$-\Delta\theta + \theta = \xi \quad \text{in } \Omega \quad \text{and} \quad \nabla\theta \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \tag{4.15}$$

and (4.14) is a weak formulation of

$$-\Delta\mathbf{Q} = \mathbf{P} \quad \text{in } \Omega \quad \text{and} \quad \mathbf{Q} \cdot \mathbf{n} = 0 \quad \text{and} \quad \operatorname{curl}\mathbf{Q} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma\,. \tag{4.16}$$

Let the operator $T^h \in \mathcal{B}(Y; X^h)$ be defined in the following manner: $T(\xi, \mathbf{P}) = (\theta^h, \mathbf{Q}^h)$ for $(\xi, \mathbf{P}) \in Y$ and $(\theta^h, \mathbf{Q}^h) \in X^h$, if and only if

$$\begin{aligned}\int_\Omega [\nabla\theta^h \cdot \nabla(\tilde{\psi}^h)^* + \nabla\tilde{\psi}^h \cdot \nabla(\theta^h)^* + \theta^h(\tilde{\psi}^h)^* + \tilde{\psi}^h(\theta^h)^*]d\Omega \\ = \int_\Omega [\xi(\tilde{\psi}^h)^* + \xi^*\tilde{\psi}^h]d\Omega \quad \forall\, \tilde{\psi}^h \in \mathcal{S}^h\end{aligned} \tag{4.17}$$

and

$$\int_\Omega (\operatorname{div}\mathbf{Q}^h\operatorname{div}\tilde{\mathbf{A}}^h + \operatorname{curl}\mathbf{Q}^h \cdot \operatorname{curl}\tilde{\mathbf{A}}^h)d\Omega = \int_\Omega \mathbf{P} \cdot \tilde{\mathbf{A}}^h d\Omega \quad \forall\, \tilde{\mathbf{A}}^h \in \mathbf{V}^h\,. \tag{4.18}$$

Now, (4.17) and (4.18) consist of two discrete Poisson-type problems that are discretizations of (4.13) and (4.14), respectively; also, T^h is the solution operator for these discrete problems.

Concerning the operators T and T^h, we have the following result.

LEMMA 4.2 *The operator T is well defined by* (4.13), (4.14). *Let the finite element subspaces satisfy the inclusions $\mathcal{S}^h \subset \mathcal{H}^1(\Omega)$ and $\mathbf{V}^h \subset \mathbf{H}^1_n(\Omega)$. Then, the operator T^h is*

also well defined by (4.17), (4.18). *Let the finite element spaces satisfy* (4.1), (4.2). *Then, as* $h \to 0$,

$$\|(T - T^h)(\xi, \mathbf{P})\|_X \to 0\,. \tag{4.19}$$

Also, if the finite element spaces satisfy (4.3), (4.4) *and if* $(\theta, \mathbf{Q}) = T(\xi, \mathbf{P})$ *satisfies* $(\theta, \mathbf{Q}) \in \mathcal{H}^{m+1}(\Omega) \times \mathbf{H}^{m+1}(\Omega) \cap \mathbf{H}^1_n(\Omega)$, *then*

$$\|(T - T^h)(\xi, \mathbf{P})\|_X \le Ch^m(\|\psi\|_{m+1} + \|\mathbf{A}\|m + 1)\,. \tag{4.20}$$

Proof. By well defined we mean, for example, that T does indeed belong to $\mathcal{B}(Y; X)$. The left-hand side of (4.13) defines a Hermitian, positive definite sesquilinear form on $\mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega)$ and the left-hand side of (4.14) defines a symmetric, positive definite bilinear form on $\mathbf{H}^1_n(\Omega) \times \mathbf{H}^1_n(\Omega)$. Moreover, whenever $(\xi, \mathbf{P}) \in Y$, the right-hand sides of (4.13) and (4.14) define bounded linear functionals on $\mathcal{H}^1(\Omega)$ and $\mathbf{H}^1_n(\Omega)$, respectively. Thus, by the Lax–Milgram theorem, both (4.13) and (4.14) have unique solutions and the solution operator is bounded, i.e., the operator T is well defined. Similarly, it can be shown that the operator T^h is also well defined.

Standard finite element arguments applied to the pairs (4.13) and (4.17) and (4.14) and (4.18) imply that

$$\|\theta - \theta^h\|_1 \le \inf_{\tilde{\psi}^h \in \mathcal{S}^h} \|\theta - \tilde{\psi}^h\|_1$$

and

$$\|\mathbf{Q} - \mathbf{Q}^h\|_1 \le \inf_{\tilde{\mathbf{A}}^h \in \mathbf{V}^h} \|\mathbf{Q} - \tilde{\mathbf{A}}^h\|_1\,,$$

respectively. Then, (4.19) follows from (4.1) and (4.2), and, if $\theta \in \mathcal{H}^{m+1}(\Omega)$ and $\mathbf{Q} \in \mathbf{H}^{m+1}(\Omega)$, (4.20) follows from (4.3) and (4.4). □

Let Λ be a compact subset of $\mathbb{R}_+$. Next, we defined the *nonlinear* mapping $G : \Lambda \times X \to Y$ as follows: $G(\kappa, (\theta, \mathbf{Q})) = (\xi, \mathbf{P})$ for $\kappa \in \Lambda$, $(\theta, \mathbf{Q}) \in X$, and $(\xi, \mathbf{P}) \in Y$, if and only if

$$\begin{aligned}\int_\Omega \xi\tilde{\psi}^* d\Omega = &\int_\Omega \left[\kappa^2(|\theta|^2 - 1) + |\mathbf{Q}|^2 - 1\right]\theta\tilde{\psi}^* d\Omega \\ &-i\int_\Omega \mathbf{Q}\cdot\left[\theta\nabla\tilde{\psi}^* - \tilde{\psi}^*\nabla\theta\right]d\Omega \quad \forall\,\tilde{\psi} \in \mathcal{H}^1(\Omega)\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}\int_\Omega \mathbf{P}\cdot\tilde{\mathbf{A}} d\Omega = &\int_\Omega \left(|\theta|^2\mathbf{Q}\cdot\tilde{\mathbf{A}} + \frac{i}{2}(\theta^*\nabla\theta - \theta\nabla\theta^*)\cdot\tilde{\mathbf{A}}\right)d\Omega \\ &-\int_\Omega (\kappa\mathbf{H}\cdot\operatorname{curl}\tilde{\mathbf{A}})d\Omega \quad \forall\,\tilde{\mathbf{A}} \in \mathbf{H}^1_n(\Omega)\,.\end{aligned} \tag{4.22}$$

It is easily seen, with the association $u = (\psi, \kappa\mathbf{A})$, that the system (3.1), (3.2) is equivalent to

$$(\psi, \kappa\mathbf{A}) + TG(\kappa, (\psi, \kappa\mathbf{A})) = 0 \tag{4.23}$$

and that the discrete system (4.5), (4.6) is equivalent to

$$(\psi^h, \kappa\mathbf{A}^h) + T^hG(\kappa, (\psi^h, \kappa\mathbf{A}^h)) = 0\,. \tag{4.24}$$

Thus, we have cast our problem into the form of §4.1.

A solution $(\psi(\kappa), \mathbf{A}(\kappa))$ of (3.1), (3.2), or equivalently, of (4.23), is called *regular* if the linear problem

$$T^{-1}(\hat{\psi}, \kappa\hat{\mathbf{A}}) + D_2G(\kappa, (\psi, \kappa\mathbf{A}))(\hat{\psi}, \kappa\hat{\mathbf{A}}) = (\hat{\xi}, \hat{\mathbf{P}}) \tag{4.25}$$

has a unique solution $(\hat{\psi}, \hat{\mathbf{A}}) \in X$ for every $(\hat{\xi}, \hat{\mathbf{P}}) \in Y$. An analogous definition holds for regular solutions of the discrete system (4.5), (4.6), or equivalently, of (4.24). In (4.25), $D_2G(\kappa, (\cdot))$ denotes the Frechét derivative of G with respect to the second argument. For given $(\theta, \mathbf{Q}) \in X$, a direct computation yields that $(\xi, \mathbf{P}) \in Y$ satisfies

$$(\xi, \mathbf{P}) = D_2G(\kappa, (\theta, \mathbf{Q}))(\hat{\theta}, \hat{\mathbf{Q}})$$

for $(\hat{\theta}, \hat{\mathbf{Q}}) \in X$, if and only if

(4.26)

$$\int_\Omega \xi\tilde{\psi}^* d\Omega = \int_\Omega \left([\kappa^2(\theta^2\hat{\theta}^* + 2\theta\theta^*\hat{\theta} - \hat{\theta}) + (2\theta\mathbf{Q}\cdot\hat{\mathbf{Q}} + |\mathbf{Q}|^2\hat{\theta} - \hat{\theta})]\tilde{\psi}^*\right) d\Omega$$
$$- \int_\Omega \left(i\mathbf{Q}\cdot[\hat{\theta}\nabla\tilde{\psi}^* - \tilde{\psi}^*\nabla\hat{\theta}] + i\hat{\mathbf{Q}}\cdot[\theta\nabla\tilde{\psi}^* - \tilde{\psi}^*\nabla\theta]\right) d\Omega \quad \forall\, \tilde{\psi} \in \mathcal{H}^1(\Omega)$$

and

(4.27)

$$\int_\Omega \mathbf{P}\cdot\tilde{\mathbf{A}} d\Omega = \int_\Omega \left([|\theta|^2\hat{\mathbf{Q}} + (\hat{\theta}\theta^* + \theta\hat{\theta}^*)\mathbf{Q}]\cdot\tilde{\mathbf{A}}\right)$$
$$+ \left(\frac{i}{2}(\hat{\theta}^*\nabla\theta - \hat{\theta}\nabla\theta^* + \theta^*\nabla\hat{\theta} - \theta\nabla\hat{\theta}^*)\cdot\tilde{\mathbf{A}}\right) d\Omega \quad \forall\, \tilde{\mathbf{A}} \in \mathbf{H}^1_n(\Omega).$$

Remark. We will assume throughout that the system (3.1), (3.2), or equivalently (4.23), has a branch of regular solutions for κ belonging to a compact interval of $\mathbb{R}_+$. It can be shown, using techniques similar to those employed for the Navier–Stokes equations (see [37] and the references cited therein) that for almost all values of κ and for almost all data $\mathbf{H}$, the system (3.1), (3.2), or equivalently, (4.23), is regular, i.e., is locally unique. See [31] for some other results in this direction.

Concerning the operator $G(\cdot, \cdot)$, we have the following result.

LEMMA 4.3. *Let Λ be a compact subset of $\mathbb{R}_+$, and let $G : \Lambda \times X \to Y$ be defined by* (4.21), (4.22). *Then, G is a C^2 mapping from $\Lambda \times X$ into Y. Let $D_2G(\cdot, \cdot)$, defined by* (4.26), (4.27), *denote the Frechét derivative of G with respect to the second argument. Then, for any $(\theta, \mathbf{Q}) \in X$,*

$$D_2G(\kappa, (\theta, \mathbf{Q})) \in \mathcal{B}(Z; X)\,. \tag{4.28}$$

Moreover, all second Frechét derivatives of G are bounded on bounded sets of $\Lambda \times X$.

Proof. Clearly, $G(\kappa, (\theta, \mathbf{Q}))$ is a polynomial map in κ, the components of $\mathbf{Q}$, and the real and imaginary parts of θ, and thus it can be shown that, in fact, G is a C^∞ mapping from $\Lambda \times X$ into Y.

Through an examination of (4.26), (4.27) we see that

$$D_2G(\kappa, (\theta, \mathbf{Q}))(\hat{\theta}, \hat{\mathbf{Q}}) = (\,C_1 + C_2\,,\, \mathbf{C}_3 + \mathbf{C}_4\,)\,,$$

where

$$C_1 = \kappa^2(\theta^2\hat{\theta}^* + 2\theta\theta^*\hat{\theta} - \hat{\theta}) + 2\theta\mathbf{Q}\cdot\hat{\mathbf{Q}} + |\mathbf{Q}|^2\hat{\theta} - \hat{\theta},$$
$$C_2 = 2i\mathbf{Q}\cdot\nabla\hat{\theta} + 2i\hat{\mathbf{Q}}\cdot\nabla\theta + i\hat{\theta}\mathrm{div}\,\mathbf{Q} + i\theta\mathrm{div}\,\hat{\mathbf{Q}},$$
$$\mathbf{C}_3 = |\theta|^2\hat{\mathbf{Q}} + (\hat{\theta}\theta^* + \theta\hat{\theta}^*)\mathbf{Q},$$
$$\mathbf{C}_4 = \frac{i}{2}(\hat{\theta}^*\nabla\theta - \hat{\theta}\nabla\theta^* + \theta^*\nabla\hat{\theta} - \theta\nabla\theta^*).$$

Since θ and $\hat{\theta}$ belong to $\mathcal{H}^1(\Omega)$, we have that θ and $\hat{\theta}$ belong to $\mathcal{L}^6(\Omega)$ and, for $i = 1, \cdots, d$, $\partial\theta/\partial x_i$ and $\partial\hat{\theta}/\partial x_i$ belong to $\mathcal{L}^2(\Omega)$. Likewise, since $\mathbf{Q}$ and $\hat{\mathbf{Q}}$ belong to $\mathbf{H}^1(\Omega)$, we have that $\mathbf{Q}$ and $\hat{\mathbf{Q}}$ belong to $\mathbf{L}^6(\Omega)$ and, for $i, j = 1, \cdots, d$, $\partial q_j/\partial x_i$ and $\partial\hat{q}_j/\partial x_i$ belong to $L^2(\Omega)$. From these we may conclude that $C_1 \in \mathcal{L}^2(\Omega)$, $C_2 \in \mathcal{L}^{3/2}(\Omega)$, $\mathbf{C}_3 \in \mathbf{L}^2(\Omega)$, and $\mathbf{C}_4 \in \mathbf{L}^{3/2}(\Omega)$. Then, $C_1 + C_2 \in \mathcal{L}^{3/2}(\Omega)$ and $\mathbf{C}_3 + \mathbf{C}_4 \in \mathbf{L}^{3/2}(\Omega)$ so that (4.28) holds.

Next, we recall the well-known result, see, e.g., [20], [22], or [36], that whenever u, v, w, and z belong to $H^1(\Omega)$, then,

$$\left|\int_\Omega uvwz\,d\Omega\right| \le C_5\|u\|_1\|v\|_1\|w\|_1\|z\|_1 \tag{4.29}$$

and

$$\left|\int_\Omega u\frac{\partial v}{\partial x_i}w\,d\Omega\right| \le C_6\|u\|_1\|v\|_1\|w\|_1 \tag{4.30}$$

for some constants C_5 and C_6. Then, a straightforward (but tedious) calculation shows that, as a result of (4.29) and (4.30), all second Frechét derivatives of G are bounded on bounded sets of $\Lambda \times X$. □

Using Theorem 4.1, we are led to the following result.

THEOREM 4.4. *Assume that Λ is a compact interval of $\mathbb{R}_+$ and that there exists a branch $\{\kappa, (\psi, \mathbf{A}) : \kappa \in \Lambda\}$ of regular solutions of the system* (3.1), (3.2). *Assume that the finite element spaces $\mathcal{S}^h$ and $\mathbf{V}^h$ satisfy the conditions* (4.1)–(4.4). *Then, there exists a neighborhood $\mathcal{O}$ of the origin in $X = \mathcal{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$ and, for h sufficiently small, a unique branch $\{\kappa, (\psi^h, \mathbf{A}^h)) : \kappa \in \Lambda\}$ of solutions of the discrete system* (4.5),(4.6) *such that $(\psi, \mathbf{A}) - (\psi^h, \mathbf{A}^h) \in \mathcal{O}$ for all $\kappa \in \Lambda$. Moreover,*

$$\|\psi(\kappa) - \psi^h(\kappa)\|_1 + \|\mathbf{A}(\kappa) - \mathbf{A}^h(\kappa)\|_1 \to 0 \tag{4.31}$$

as $h \to 0$, uniformily in κ.

If, in addition, the solution of the system (3.1), (3.2) *satisfies $(\psi, \mathbf{A}) \in \mathcal{H}^{m+1}(\Omega) \times \mathbf{H}^{m+1}(\Omega)$, then there exists a constant C, independent of h, such that*

$$\|\psi(\kappa) - \psi^h(\kappa)\|_1 + \|\mathbf{A}(\kappa) - \mathbf{A}^h(\kappa)\|_1 \le Ch^m(\|\psi(\kappa)\|_{m+1} + \|\mathbf{A}(\kappa)\|_{m+1}), \tag{4.32}$$

uniformly in κ.

Proof. First, as a consequence of Lemma 4.2, we see that (4.10) and (4.11) are satisfied. Then, (4.12) follows since the imbedding $Z \subset Y$ is compact. The remaining hypotheses of Theorem 4.1 are verified in Lemma 4.2. Then, the present results follow from Theorem 4.1. In particular, since $(T - T^h)G(\kappa, (\psi, \mathbf{A})) = -(\psi, \mathbf{A})$, (4.31) and (4.32) follow from (4.19) and (4.20), respectively. □

Remark. Using some additional results (that may be found in [7], [12], and [20]) concerning the approximation of problems of the type (4.7), we may derive error estimates

for $(\psi, \mathbf{A})$ in the $[\mathcal{L}^2(\Omega) \times \mathbf{L}^2(\Omega)]$-norm. In particular, if solutions of the (formal) adjoint of (4.25) can be shown to belong to $\mathcal{H}^2(\Omega) \times \mathbf{H}^2(\Omega)$ whenever $(\hat{\xi}, \hat{\mathbf{P}}) \in \mathcal{L}^2(\Omega) \times \mathbf{L}^2(\Omega)$, then it can be shown that

$$\|\psi(\kappa) - \psi^h(\kappa)\|_0 + \|\mathbf{A}(\kappa) - \mathbf{A}^h(\kappa)\|_0 \leq Ch(\|\psi(\kappa) - \psi^h(\kappa)\|_1 + \|\mathbf{A}(\kappa) - \mathbf{A}^h(\kappa)\|_1).$$

Thus, in this case, the error measured in the $[\mathcal{L}^2(\Omega) \times \mathbf{L}^2(\Omega)]$-norm converges at a higher rate (as $h \to 0$) than does the error measured in the $[\mathcal{H}^1(\Omega) \times \mathbf{H}^1(\Omega)]$-norm.

Remark. Any of the many standard iterative methods for the solution of nonlinear systems of equations may be employed to solve the nonlinear discrete system (4.5), (4.6). For example, using the notation of (4.25), Newton's method is defined as follows. Given a value for κ and an initial guess $(\psi^{(0)}, \mathbf{A}^{(0)})$ for $(\psi^h, \mathbf{A}^h)$, the sequence of Newton iterates $\{(\psi^{(s)}, \mathbf{A}^{(s)})\}_{s \geq 1}$ is defined by

$$\begin{aligned} T^{-1}(\psi^{(s+1)}, \mathbf{A}^{(s+1)}) &+ D_2 G(\kappa, (\psi^{(s)}, \mathbf{A}^{(s)}))(\psi^{(s+1)}, \mathbf{A}^{(s+1)}) \\ &= G(\kappa, (\psi^{(s)}, \mathbf{A}^{(s)})) - D_2 G(\kappa, (\psi^{(s)}, \mathbf{A}^{(s)}))(\psi^{(s)}, \mathbf{A}^{(s)}) \quad \text{for } s = 0, 1, \cdots, \end{aligned}$$

It can be shown (using techniques similar to those employed for the Navier–Stokes equations [20]) that if the initial guess $(\psi^{(0)}, \mathbf{A}^{(0)})$ is "sufficiently" close to a nonsingular solution $(\psi^h, \mathbf{A}^h)$ of the discrete system (4.5), (4.6), then the Newton iterates converge to $(\psi^h, \mathbf{A}^h)$ with a quadratic rate of convergence. Good initial guesses may be generated by using, for example, continuation methods wherein the solution at one value of κ is used to generate a good initial guess for the solution at another value of κ. Modern continuation methods also have the ability to compute the solution, as a function of κ, even in the presence of bifurcation points.

5. The periodic Ginzburg–Landau model. From a practical viewpoint, the model and associated finite element method presented and analyzed above are of use mostly for type I superconductors. In this case, solutions are relatively simple in structure, with most of the interesting phenomena occuring near the boundary. Thus, with perhaps some grid refinement near the boundary, the finite element discretization of §4 would be effective. On the other hand, solutions to the GL model for the case of type II superconductors exhibit much more complicated structures, e.g., vortex-like phenomena, with variations occuring over the order of 10^3 Ängstroms. Thus, in any computation of practical utility, the grid size necessary to resolve these structures would be prohibitively small. An alternate model, using quasi-periodic boundary conditions (see, e.g., [1], [15], [16], [28], and [31]), may be used for type II superconductors. Here, we briefly describe this periodic model. Further details and a discussion of finite element methods for this alternate model will be the subject of a future paper.

We assume that $\Omega \subset \mathbb{R}^2$. The periodic model is based on the assumption that away from bounding surfaces, certain physical variables exhibit periodic behavior. Specifically, it is assumed that the magnetic field ($\mathbf{h} = \operatorname{curl} \mathbf{A}$), the current ($\mathbf{j} = ((1/\kappa)\nabla\phi - \mathbf{A})|\psi|^2$), and the density of superconducting charge carriers ($|\psi|^2$) are periodic with respect to the lattice determined by the fixed vectors $\mathbf{t}_1$ and $\mathbf{t}_2$, i.e.,

$$\mathbf{h}(\mathbf{x}+\mathbf{t}_k) = \mathbf{h}(\mathbf{x}), \quad \mathbf{j}(\mathbf{x}+\mathbf{t}_k) = \mathbf{j}(\mathbf{x}), \quad \text{and} \quad |\psi(\mathbf{x}+\mathbf{t}_k)| = |\psi(\mathbf{x})|, \quad \mathbf{x} \in \mathbb{R}^2, \ k = 1, 2.$$

The direction and relative lengths of the lattice vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ are determined by the lattice symmetry. For example, for an equilateral triangular lattice, the angle between $\mathbf{t}_1$ and $\mathbf{t}_2$ is 60 degrees, and the two lattice vectors are of the same length. (It is well known,

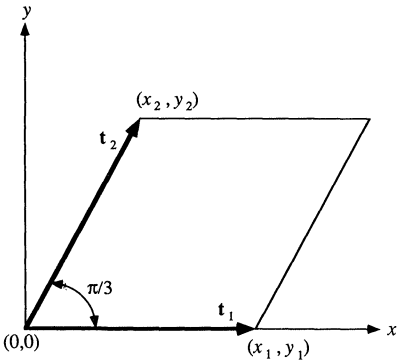

FIG. 5.1. *A cell of the lattice determined by the vectors* $\mathbf{t}_1$ *and* $\mathbf{t}_2$.

e.g., see any of the above papers, that the GL free energy is a minimum for a regular triangular lattice.) The size of a lattice cell, and thus the absolute lengths of the lattice vectors, is determined by the fluxoid quantization condition and is given by

$$|\Omega| = \frac{2\pi n}{\kappa \bar{B}},$$

where $|\Omega|$, $\bar{B}$, and n, respectively denote the area of the lattice cell, the average magnetic field, and the number of fluxoids carried by the each cell of the lattice.

The order parameter ψ and the magnetic potential $\mathbf{A}$ can then be shown to satisfy the "quasi-periodic" boundary conditions

$$\psi(\mathbf{x} + \mathbf{t}_k) = \psi(\mathbf{x}) e^{i\kappa g_k}, \qquad \mathbf{x} \in \mathbb{R}^2, \quad k = 1, 2, \tag{5.1}$$

and

$$\mathbf{A}(\mathbf{x} + \mathbf{t}_k) = \mathbf{A}(\mathbf{x}) + \nabla g_k, \qquad \mathbf{x} \in \mathbb{R}^2, \quad k = 1, 2, \tag{5.2}$$

where $g_k(\mathbf{x}; \mathbf{t}_k)$, $k = 1, 2$, is determined from the fluxoid quantization condition and, if there is one fluxoid associated with each cell, is given by

$$g_k = -\tfrac{1}{2}(\mathbf{t}_k \times \bar{B}\mathbf{k}_3) \cdot \mathbf{x}, \qquad k = 1, 2,$$

where $\mathbf{k}_3$ denotes a unit vector perpendicular to the (x, y)-plane.

Due to the "periodic" nature of ψ and $\mathbf{A}$, it is customary to focus on a single lattice cell, such as one with a corner at the origin. Figure 5.1 provides a sketch of such a cell for the equilateral triangular lattice, where without loss of generality, we can assume that one of the lattice vectors is aligned with a coordinate axis.

With respect to the cell depicted in Fig. 5.1, the "periodicity" conditions (5.1), (5.2) imply that

$$\mathbf{A}(x + x_2, y_2) = \mathbf{A}(x, 0) + \tfrac{1}{2}\bar{B}(x_2\mathbf{k}_2 - y_2\mathbf{k}_1) \quad \text{for } 0 < x < x_1, \tag{5.3}$$

$$\mathbf{A}\left(x_1 + \frac{x_2}{y_2}y, y\right) = \mathbf{A}\left(\frac{x_2}{y_2}y, y\right) + \frac{1}{2}\bar{B}x_1\mathbf{k}_2 \quad \text{for } 0 < y < y_2, \tag{5.4}$$

$$\psi(x + x_2, y_2) = \psi(x, 0) \exp\{-i\tfrac{1}{2}\kappa\bar{B}y_2 x\} \quad \text{for } 0 < x < x_1, \tag{5.5}$$

and

$$\psi\left(x_1 + \frac{x_2}{y_2}y, y\right) = \psi\left(\frac{x_2}{y_2}y, y\right)\exp\left\{i\frac{1}{2}\kappa\bar{B}x_1 y\right\} \quad \text{for } 0 < y < y_2\,, \tag{5.6}$$

where $\mathbf{k}_1$ and $\mathbf{k}_2$ denote the unit vectors in the direction of the x and y axes, respectively.

The "periodic" GL model is then to minimize the GL free energy (2.16) over all appropriate functions ψ and $\mathbf{A}$, i.e., functions having one square integrable derivative, that also satisfy (5.3)–(5.6), where in (2.16), Ω now denotes the cell depicted in Fig. 5.1. The application of standard techniques of the calculus of variations then yields that minimizers satisfy the GL equations (2.17), (2.18). Of course, (5.3)–(5.6) appear as essential boundary conditions. The minimization process also yields natural boundary conditions; it is easy to see that these imply that the current and the magnetic field are periodic with respect to the lattice. Since (5.5), (5.6) imply that $|\psi|$ is likewise periodic, we see that minimizing (2.16) over functions satisfying (5.3)–(5.6) yields all the required physical periodicity conditions.

There is one difficulty still to be overcome. The average magnetic field is given by

$$\bar{B} = \frac{1}{|\Omega|}\int_\Omega h\, d\Omega = \frac{1}{|\Omega|}\int_\Omega \operatorname{curl}\mathbf{A}\, d\Omega\,,$$

where, since we are dealing with a planar problem, $\mathbf{h} = (0, 0, h)^T$ and curl $\mathbf{A}$ may be viewed as a scalar-valued function. Obviously, $\bar{B}$ depends on the solution of the problem. On the other hand, the cell Ω and the lattice vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ that are used to define the problem, depend on $\bar{B}$. Thus it seems that not everything needed to pose the "periodic" problem is known. This difficulty is circumvented as follows. Instead of specifying κ and the constant applied field $\mathbf{H}$, we specify κ and the average field $\bar{B}$. Note that if $\mathbf{H}$ is a constant, it does not explicitly appear in the specification of the problem. Of course, it is also necessary to set the number n of fluxoids carried in each cell. For the triangular lattice determined from the cell depicted in Fig. 5.1, $n = 1$. With n, $\bar{B}$, and κ specified, all the data in the specification of the "periodic" problem are known. The applied field corresponding to the solution obtained is then easily deduced; see, e.g., [14].

Most of the results of §§3 and 4 can be extended to the periodic model. As was indicated above, details will be provided elsewhere.

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