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# Behavior of a Hollow Superconducting Cylinder in a Magnetic Field

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*The behavior of a hollow superconducting cylinder of arbitrary dimensions in an external magnetic field is investigated in detail within the Ginzburg–Landau macroscopic theory. The thermodynamic potential of the system is presented in a compact form, which enables us to give a simpler description of the transitions of the system between the quantized levels in a magnetic field than in previous work on this subject. The general theory is illustrated by a number of cases, which show the dependence of the order parameter, the total flux in a cavity, and the magnetic moment of the cylinder on the magnitude of the external field and on temperature. The phase transition curves and the hysteresis boundaries are found. The tricritical points, where the difference between first- and second-order phase transitions vanishes, are established. The oscillations in a magnetic field of the specific heat of the cylinder are investigated. The formulas are presented for the case of a thin-walled cylinder and are valid for arbitrary values of the order parameter $\psi$, inner and outer radii of the cylinder, external field, and temperature. The results are discussed in connection with experiment and other papers on this subject.*

## 1. INTRODUCTION

A hollow superconducting cylinder in a magnetic field comprises a system with a number of interesting properties. Attention has been given to theoretical and experimental investigation of the flux quantization in such a cylinder,[1–12] to the oscillations of the critical temperature of a cylinder in an external magnetic field $H_0$,[13–19] and to the transitions between the quantized levels.[20–25] The surface currents in massive hollow cylinders in a strong external field have also been investigated.[26–29] Other work has been devoted to effects arising when a transport current passes the hollow cylinder (the structure of the intermediate state, paramagnetic effects, and so on).



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In this paper we discuss in detail within the framework of Ginzburg–Landau theory some aspects of the behavior of a hollow superconducting cylinder in an external magnetic field. We present general exact expressions describing the system as well as the results of numerical computations which illustrate the dependences of different physical quantities. Our main concern is with questions that have not been sufficiently treated in the literature. Thus, we investigate in detail the dependence of the Ginzburg–Landau functional and of the superconducting order parameter on the external field and temperature, consider the transitions between the quantized levels, and establish the boundaries of hysteresis. We find the magnetic moment and the total flux in the cylinder as functions of the external field and temperature, consider oscillations of the specific heat of the cylinder in transitions between quantum levels near $T_c$, investigate the role of impurities, and so on. We also compare our results with those of previous papers and with experiment.

## 2. THE FUNCTIONAL OF THE SYSTEM

The properties of a hollow superconducting cylinder in an external magnetic field $H_0$ are described through the functional (compare, for example, Refs. 20 and 21)

$$\mathcal{G}_s(H_0) = F_s(H_0) - \int_V \frac{\mathbf{H}_0\mathbf{B}}{4\pi}\, dv + \frac{H_1^2}{8\pi} V_1 \tag{1}$$

where $\mathbf{B} = \operatorname{rot} \mathbf{A}$ is the field inside the superconductor, $H_1$ is the field in the cavity, $V_1$ is the volume of the cavity, $V_s$ is the volume occupied by the superconductor, $V = V_1 + V_s$ is the total volume occupied by the cylinder, and $F_s$ is the free energy of the superconductor, which has the usual form[30]

$$F_s = F_{n0} + \int \left\{ \frac{B^2}{8\pi} - \alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4 + \frac{1}{2m^*}\left|\hbar\,\nabla\Psi - \frac{ie^*}{c}\mathbf{A}\Psi\right|^2 \right\} dv \tag{2}$$

Here we use the description in terms of the macroscopic wave function of the superconductor $\Psi = |\Psi|\, e^{i\varphi}$, where $\varphi$ is the phase; $m^* = 2m$ is the mass and $e^* = 2e$ is the charge of the Cooper pairs. The functional (1) is minimal if $T = \text{const}$, $H_0 = \text{const}$, i.e., for arbitrary variations in the system we have $\Delta\mathcal{G}_s \leqslant 0$.[20,21] The variation of (1) with respect to $\Psi^*$ gives the Ginzburg–Landau equation

$$-\alpha\Psi + \beta|\Psi|^2\Psi - \frac{\hbar^2}{2m^*}\left(\nabla + \frac{ie^*}{\hbar c}\mathbf{A}\right)^2 \Psi = 0 \tag{3}$$

and the variation of (1) with respect to $\mathbf{A}$ gives

$$\operatorname{rot}\operatorname{rot}\mathbf{A}=(4\pi/c)\,\mathbf{j} \tag{4}$$

$$\mathbf{j}=\mathbf{j}_s=\frac{(e^*)^2}{m^*c}|\Psi|^2\left(-\mathbf{A}+\frac{\hbar c}{e^*}\nabla\varphi\right) \tag{5}$$

Below we restrict ourselves to the case when $|\Psi|$ does not depend on coordinates, $|\Psi|=\text{const}$. This is true in the case of a sufficiently thin cylinder, with the thickness of the walls subject to conditions

$$\xi_0<d<\delta_L(T),\,\xi(T) \tag{6}$$

Here $\xi_0$ is the correlation length at $T=0$, and $\delta_L(T)$ and $\xi(T)$ are the temperature-dependent London length and coherence length. The condition $\xi_0<\delta_L(T)$ defines the region of applicability of the Ginzburg–Landau theory. For type I superconductors ($\kappa<1/\sqrt{2}$) this region is rather small: $T_c-T\leqslant\kappa^2T_c$. For thick-walled cylinders, where conditions (6) do not hold, we can nonetheless approximately put $|\Psi|=\text{const}$, understanding by $|\Psi|$ some value averaged over coordinates. The results obtained in this case will be correct, strictly speaking, only qualitatively.

Introducing the reduced variables by[30]

$$\begin{gathered}H_{\text{cm}}^2=\frac{4\pi\alpha^2}{\beta}=\frac{\Phi_0^2}{8\pi^2\xi^2(T)\delta_L^2(T)},\qquad \Psi_0^2=\frac{\alpha}{\beta},\qquad \psi=\frac{|\Psi|}{|\Psi_0|}\\ \delta_L^2=\frac{m^*c^2\beta}{4\pi(e^*)^2\alpha},\qquad \xi^2(T)=\frac{\hbar^2}{2m^*\alpha},\qquad \kappa^2=\frac{\delta_L^2}{\xi^2}=\frac{(m^*)^2c^2\beta}{2\pi(e^*)^2\hbar^2}\end{gathered} \tag{7}$$

($\Phi_0=hc/e^*$ is the flux quantum), we can rewrite (1) in the form[20,21]

$$\mathscr{F}=\frac{\mathscr{G}_s-\mathscr{G}_n}{V_sH_{\text{cm}}^2/8\pi}=\psi^4-2\psi^2-\frac{4\pi}{V_sH_{\text{cm}}^2}\mathbf{M}\mathbf{H}_0+n\frac{\Phi_0}{8\pi}\frac{H_1-H_0}{V_sH_{\text{cm}}^2/8\pi} \tag{8}$$

Here $\mathbf{M}$ is the magnetic moment of the hollow cylinder, which can be evaluated from[10]

$$\mathbf{M}=\frac{1}{2c}\int_{V_s}[\mathbf{j}_s(\mathbf{r})\times\mathbf{r}]\,dv=\int_V\frac{\mathbf{B}-\mathbf{H}_0}{4\pi}\,dv \tag{9}$$

In (8), $\mathscr{G}_n=\mathscr{G}_s(\psi=0)$ and we take into account that in the case of a hollow cylinder the phase factor can be written in the form $e^{i\varphi}=e^{in\theta}$, where $n$ is an integer indicating how many flux quanta are contained in the cavity, and $\theta$ is the azimuthal angle in the cylindrical coordinate system. The expression (8) differs from the formula obtained by Ginzburg[31] in the case of a solid cylinder by the last term, which takes into account the presence of a cavity. The thermodynamic potential in the form (8) is equivalent to the potential used in Ref. 22, where it is written in different notation. The form (8) is more useful because the contribution due to the magnetic moment of the system is shown explicitly.

To find the distribution of the current and field over the cylinder cross section it is necessary to solve the electrodynamic equations (4) and (5) combined with (3). In the case $|\Psi|=\text{const}$ the problem is simplified and was solved in a number of papers.[6–11] In cylindrical coordinates Eqs. (4) and (5) can be written in the form

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rA)\right]=\frac{1}{\lambda^2}\left(A-\frac{\hbar cn}{e^*r}\right) \tag{10}$$

where $A$ is the azimuthal component of $\mathbf{A}$, and $\lambda=\delta_L(T)/\psi$. The general solution of (10) is

$$A=\frac{\hbar cn}{e^*r}+C_1I_1\left(\frac{r}{\lambda}\right)+C_2K_1\left(\frac{r}{\lambda}\right) \tag{11}$$

$$B=\frac{1}{r}\frac{d}{dr}(rA)=\frac{1}{\delta}\left[C_1I_0\left(\frac{r}{\lambda}\right)-C_2K_0\left(\frac{r}{\lambda}\right)\right] \tag{12}$$

where $I_n$ and $K_n$ are modified Bessel functions. The arbitrary constants $C_1$ and $C_2$ are to be determined from the boundary conditions (here $r_1$ and $r_2$ are the inner and outer radii of the cylinder). As a result, the solution (11) is expressed in terms of the external field $H_0$ and the field inside the cavity $H_1$, which is still unknown. The condition

$$\oint_{r_1}\mathbf{A}\,d\mathbf{l}=\pi r_1^2H_1 \tag{13}$$

(the contour integral is taken over the inner surface of the cylinder, and $\pi r_1^2H_1$ is the magnetic flux inside the cavity) provides an additional equation from which to find $H_1$. Introducing the dimensionless notation

$$\phi_1=\frac{\Phi_1}{\Phi_0},\qquad \phi_2=\frac{\Phi_2}{\Phi_0},\qquad \phi_{a1}=\frac{\pi r_1^2H_0}{\Phi_0},\qquad \phi_{a2}=\frac{\pi r_2^2H_0}{\Phi_0} \tag{14}$$

[here $\Phi_1=\pi r_1^2H_1$, and $\Phi_2$ is the total flux in the cylinder, including the flux contained within the walls; $\Phi_2$ is equal to the integral (13) taken over the external surface $r_2$], we find the final expressions

$$\phi_1=n\frac{D}{D_1}+\frac{2}{\tilde{D}_1}\phi_{a1} \tag{15}$$

$$\phi_2=n\frac{\tilde{D}_1-2}{\tilde{D}_1}+\left(1+\frac{\tilde{D}_2-4/\tilde{D}_1}{\rho_2^2D}\right)\phi_{a2} \tag{16}$$

$$\frac{4\pi M}{\Phi_0}=\phi_1\frac{\tilde{D}_1-2}{\rho_1^2 D}+\phi_{a2}\frac{\tilde{D}_2-2}{\rho_2^2 D}\equiv\phi_2-\phi_{a2} \tag{17}$$

Here $\rho_1=r_1\psi/\delta_L$, $\rho_2=r_2\psi/\delta_L$,

$$\begin{aligned} D&=K_0(\rho_1)I_0(\rho_2)-I_0(\rho_1)K_0(\rho_2)\\ D_1&=K_2(\rho_1)I_0(\rho_2)-I_2(\rho_1)K_0(\rho_2), \qquad \tilde{D}_1=\rho_1^2 D_1\\ D_2&=K_2(\rho_2)I_0(\rho_1)-I_2(\rho_2)K_0(\rho_1), \qquad \tilde{D}_2=\rho_2^2 D_2 \end{aligned} \tag{18}$$

Using (14)–(18), we can rewrite the functional (8) in the form

$$\mathcal{F}=\psi^4-2\psi^2+\frac{8\xi^2(T)\psi^2}{r_2^2-r_1^2}\left[\frac{D}{\tilde{D}_1}(n-p\phi_{a1})^2+q\phi_{a1}^2\right] \tag{19}$$

$$p=\frac{\tilde{D}_1-2}{\rho_1^2 D}, \qquad q=\frac{4-\tilde{D}_1-\tilde{D}_2}{\rho_1^4 D} \tag{20}$$

The compact form (19) of the thermodynamic potential enables one to see immediately the important peculiarity of the behavior of the hollow superconducting cylinder in an external field. Indeed, as will be shown, the factor $p$ in (19) is of order unity, and $q \ll 1$. So for the time being we drop the term $q\phi_{a1}^2$ and consider the remaining term in brackets in (19). It is clear that when the applied field $\phi_{a1}$ is increased, it is favorable for the system to undergo transitions and change its quantum number $n \to n+1$, so that the quantity $|n-p\phi_{a1}|$ remains $\leqslant 1/2$. Consequently, the potential (19) will be minimal. In the absence of the term $q\phi_{a1}^2$ the system would continuously undergo transitions to higher and higher levels $n$ if $\phi_{a1}$ were increased, and there would be strict periodicity relative to the field, with period $\delta\phi_{a1}=1/p$. In dimensioned variables this period is equal to $\delta\Phi_{a1}=\Phi_0/p$, i.e., it does not strictly coincide with one flux quantum. This circumstance was mentioned in Ref. 6.

Note that if instead of $\phi_{a1}$ we introduce in (19) a new variable $\phi_{a*}=p\phi_{a1}$, then the period of $\mathcal{F}$ relative to the variable $\phi_{a*}$ [as determined mainly by the term $(n-\phi_{a*})^2$] would be strictly equal to unity, $\delta\phi_{a*}=1$ (i.e., to one flux quantum). The same can be seen from Eq. (15), which can be rewritten in the form $\phi_1=\phi_{a1}+(n-\phi_{a*})D/D_1$, i.e., the field in a cavity is strictly equal to the external field at the values of $\phi_{a*}=n$ ($n=0, 1, 2, \ldots$). The difference in the variables $\phi_{a1}$ and $\phi_{a*}$ is connected with the different definitions of the area to which the external field is applied. In the case of the variable $\phi_{a1}=\pi r_1^2 H_0/\Phi_0$ the area of the inner cavity is involved, while in the case of the variable $\phi_{a*}=p\phi_{a1}$ the effective area $\pi r_*^2$ enters, where $r_*^2=pr_1^2$. Below it will be shown that for $d=r_2-r_1 \ll r_1$

we have $p \approx r_2/r_1$, and $r_* = (r_1 r_2)^{1/2}$, i.e., the effective radius $r_*$ is approximately equal to the geometrical average of $r_1$ and $r_2$. In a number of experiments[12,18,19] periodic dependences were measured for a cylinder in an external field (in particular, the dependence of the total flux in a cavity). In some experiments a period equal to one flux quantum was found,[32,33] while in others this period was less than one quantum. These contradictions relate to the question of which quantity, $\phi_{a1}$ or $\phi_{a*}$, had its periodicity studied. If the applied flux was defined as $\Phi_{a*} = \pi r_*^2 H_0$, where $r_* \approx (r_1 r_2)^{1/2}$ is the effective radius, then a period equal strictly to one flux quantum should be observed. (On the figures presented below the periodicity is best revealed relative to the variable $\phi_{a*}$.)

We stress that exact periodicity is violated by the presence of the parabolic term $q\phi_{a1}^2$ in (19). This term, which grows quadratically with the field, ensures that sooner or later the superconductivity is suppressed by the field and transition to the normal state occurs. (In the first theoretical papers,[16,17] where the effect of oscillations of $T_c$ of a thin-walled cylinder in a magnetic field was studied, the parabolic term was neglected. The importance of the parabolic term to the picture of $T_c$ oscillations was noted in Refs. 20 and 21.)

From (15) and (19) it can be seen that at the points $n = p\phi_{a1}$ the first term in the brackets in (19) vanishes; here we have $\phi_1 = \phi_{a1}$, i.e., the field inside and outside the cylinder is the same. The cylindrical specimen in this case resembles a flat superconducting plate in a field parallel to its surface. It is easy to see that when $r_1 \to \infty$, $d = \text{const}$, the term $q\phi_{a1}^2$ in (19) ensures the correct limiting transition to the case of a superconducting plate in a parallel field[34]:

$$\mathscr{F}_{\text{plate}} = \psi^4 - 2\psi^2 + \frac{H_0^2}{H_{\text{cm}}^2(T)}\left[1 - \frac{2\delta_L}{\psi d}\,\text{th}\,\frac{\psi d}{2\delta_L}\right] \tag{21}$$

$$\mathscr{F}_{\text{film}}\big|_{d/\delta_L \ll 1} = \psi^4 - 2\psi^2 + \frac{H_0^2}{H_{\text{cm}}^2(T)}\left[\frac{1}{12}\left(\frac{d}{\delta_L}\right)^2\psi^2 - \frac{1}{120}\left(\frac{d}{\delta_L}\right)^4\psi^4 + \cdots\right] \tag{21a}$$

where $H_{\text{cm}}^2(T)$ is defined in (7).

In concluding this section we give the formulas for the dependence $\xi(T)$ for clean and dirty superconductors[35,36]:

$$\xi(T) = \xi(0)(1 - T/T_c)^{-1/2}, \qquad \delta_L(T) = \kappa\xi(T) \tag{22}$$

In the case of clean superconductors

$$\xi(0) = 0.74\xi_0 \tag{23}$$

and in the case of dirty superconductors

$$\xi(0) = 0.85(\xi_0 l)^{1/2} \tag{24}$$

where $\xi_0$ is the correlation length and $l$ is the free scattering length. For $l \ll \xi_0$ we have $\kappa_{\text{dirty}} = \kappa_{\text{clean}} 0.755\xi_0/l \gg \kappa_{\text{clean}}$.

## 3. THE FIRST- AND SECOND-ORDER PHASE TRANSITIONS

The functional $\mathcal{F}$ of the system depends on the reduced order parameter $\psi$, as well as on the external field, the temperature, and the integer $n$. In accordance with the Ginzburg–Landau theory,[30,35,36] the parameter $\psi$ is found from the minimum condition on the functional: $\partial\mathcal{F}/\partial\psi = 0$, $\partial^2\mathcal{F}/\partial\psi^2 > 0$. Below we shall write down the corresponding equation, but first it is useful to describe qualitatively the characteristic behavior of $\mathcal{F}(\psi)$.

Two types of behavior of the function $\mathcal{F}(\psi)$ are possible, as presented schematically in Fig. 1. The curves of Fig. 1a correspond to the case of first-order phase transitions. The position of the minimum of $\mathcal{F}$, corresponding to $\psi_0 > 0$, is marked on the curves by circles. When the field (or the temperature) increases, the curves of Fig. 1a move upward. The inflection point $\psi_{00}$ lies on curve 4; here the superconducting state is metastable and maximally "superheated." On curve 5 the superconducting state with $\psi_0 \neq 0$ is absent and here the system must jump from the state with $\psi_{00} \neq 0$ into the normal state, $\psi = 0$ (or into a superconducting state with a different value of $n$). The condition when the inflection point vanishes ($\mathcal{F}' = \mathcal{F}'' = 0$ at $\psi \neq 0$) gives the "superheating" (sh) boundary.

If the cylinder was originally in the normal state with $\psi = 0$ (curve 5), then by reducing the field (or the temperature; curves 4 and 3), one can keep the specimen in the normal state $\psi = 0$ in spite of the appearance of

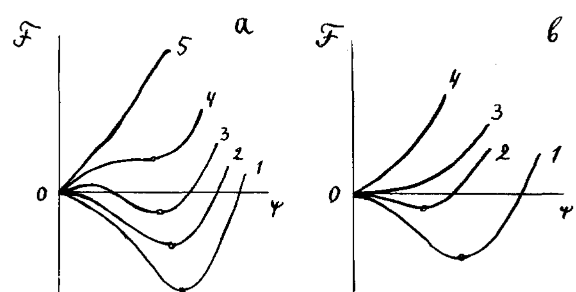


Fig. 1. Schematic depiction of the dependence of the thermodynamic potential; (a) first-order transitions, (b) second-order transitions.

the superconducting minimum on curve 3. This case corresponds to a "supercooled" normal state. When the maximum on curve 3 vanishes, the system must jump into the superconducting state (see curve 2). The condition for the maximum on curve 3 to vanish at $\psi \ll 1$ gives the "supercooling" (sc) boundary. This condition has the form $\mathcal{F}' = \mathcal{F}'' = 0$ at $\psi = 0$.

The dependence of $\mathcal{F}(\psi)$ shown in Fig. 1b corresponds to a second-order phase transition: the order parameter $\psi_0$ vanishes gradually with increasing field or temperature. The second-order phase transition point (so) is also defined by the condition $\mathcal{F}' = \mathcal{F}'' = 0$ at $\psi = 0$.

Now we study the behavior of the system in detail. We rewrite Eq. (19) for $\mathcal{F}$ in the equivalent form

$$\mathcal{F} = \psi^4 - 2\psi^2 + 4Q_0(an^2 - 2bn\phi_{a1} + c\phi_{a1}^2)$$

$$Q_0 = \frac{2\xi^2(T)}{r_2^2 - r_1^2}\frac{\delta_L^2(T)}{r_1^2} \tag{25}$$

$$a = \frac{D}{D_1}, \qquad b = 1 - \frac{2}{\rho_1^2 D_1}, \qquad c = \frac{4}{\rho_1^4 D D_1} - \frac{r_2^2}{r_1^2}\frac{D_2}{D}$$

Introducing the notation $\dot{f} = \psi\, \partial f/\partial\psi$, we write the condition $\partial\mathcal{F}/\partial\psi = 0$ in the form

$$\phi_{a1}^2 - 2\phi_{a1} n \frac{\dot{b}}{\dot{c}} + n^2 \frac{\dot{a}}{\dot{c}} + \frac{\psi^2(1-\psi^2)}{\dot{c} Q_0} = 0 \tag{26}$$

The condition (26) establishes a complicated transcendental dependence $\psi(\phi_{a1})$ between the value $\psi$ at the extremal point of the functional $\mathcal{F}$ and the field $\phi_{a1}$ [recall that $\psi$ enters the argument of the Bessel functions (11)]. It is more convenient to find the inverse dependence $\phi_{a1}(\psi)$. Solving the quadratic Eq. (26) relative to $\phi_{a1}$, we get

$$\phi_{a1} = n\frac{\dot{b}}{\dot{c}} \pm \left(\frac{\psi^2(1-\psi^2)}{Q_0 \dot{c}} - n^2 \frac{\dot{a}\dot{c} - \dot{b}^2}{\dot{c}^2}\right)^{1/2} \tag{27}$$

where

$$\dot{a} = \frac{D_1 \dot{D} - D\dot{D}_1}{D_1^2}, \qquad \dot{b} = \frac{2}{\rho_1^2 D_1^2}(2D_1 + \dot{D}_1)$$

$$\dot{c} = \frac{r_2^2}{r_1^2}\left(\frac{D_2\dot{D} - \dot{D}_2 D}{D^2}\right) - \frac{4}{\rho_1^4 D^2 D_1}(\dot{D}D_1 + D\dot{D}_1) - \frac{16}{\rho_1^4 D D_1}$$

Here $\dot{D}$, $\dot{D}_1$, and $\dot{D}_2$ are given in terms of differentials of the functions (18): $\dot{D} = \rho_1\, \partial D/\partial\rho_1 + \rho_2\, \partial D/\partial\rho_2$, and analogously for $\dot{D}_1$ and $\dot{D}_2$.

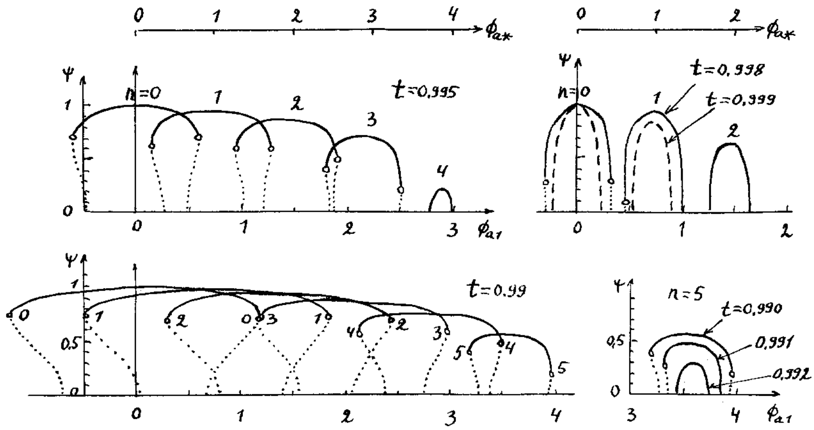

Fig. 2. The dependence of the order parameter $\psi$ on the field for different temperatures $t = T/T_c$ for a cylinder with parameter set $\mathscr{P}_1$ (see footnote below). The integer $n$ on the curves shows the number of the quantum state.

Equation (27) enables one to find the superconducting order parameter (which correspond to the extremum of $\mathscr{F}$) in a quantum state $n$ depending on the external field $\phi_{a1}$ and temperature $T$ for arbitrary dimensions $r_1$ and $r_2$. The corresponding dependences are illustrated by Fig. 2.* The solid lines show the values of $\psi$ that correspond to the minimum of $\mathscr{F}$, and the broken lines show the values of $\psi$ that correspond to the maximum of $\mathscr{F}$ (these values correspond to unstable states, which cannot be realized experimentally). At the points $\psi_{00}$ marked by a circle the minimum and the maximum of $\mathscr{F}$ coincide. These are the inflection points of $\mathscr{F}$. The inflection points are simultaneously points of maximal "superheating," or points where the superconducting state ends. Upon attaining the point $\psi_{00}$ the system undergoes a first-order transition into the normal state $\psi = 0$ or into a state with a different value of $n$.

The value $\phi_{a1} = \phi_{sc}$ corresponding to $\psi \to 0$, i.e., the "supercooling" boundary, can also be found. These values are found from (27) by a limiting transition $\psi \to 0$:

$$\phi_{sc}\,(=\phi_{so}) = \frac{2n}{r^2+1} \pm \left[\frac{2}{r^2+1}\,\frac{r_1^2}{\xi^2(T)} - 4n^2\left(\frac{\mathscr{L}}{r^4-1} - \frac{1}{(r^2+1)^2}\right)\right]^{1/2} \qquad (28)$$

*In the numerical calculations we have used two sets of parameters: $\mathscr{P}_1\{\xi_0 = 1\times10^{-5}$ cm, $\kappa = 0.2$, $r_1 = 6\times10^{-5}$ cm, $r_2 = 8\times10^{-5}$ cm$\}$; this set corresponds to the microcylinders examined in Ref. 12; and a set corresponding to cylinders of somewhat larger radius: $\mathscr{P}_2\{\xi_0 = 1\times10^{-5}$ cm, $\kappa = 0.2$, $r_1 = 20\times10^{-5}$ cm, $r_2 = 22\times10^{-5}$ cm$\}$. Section 4 compares the results for clean and dirty superconductors with free scattering length $l = 1\times10^{-6}$ cm [see Eqs. (23), (24)].

or ($t = T/T_c$)

$$t_{sc} = (t_{so} =) = 1 - \mathscr{P}$$

$$\mathscr{P} = \frac{\xi^2(0)}{r_1^2}\left(\phi_{a1}^2 \frac{r^2+1}{2} - 2n\phi_{a1} + \frac{2n^2\mathscr{L}}{r^2-1}\right) \tag{28a}$$

$$\mathscr{L} = \ln \frac{r_2}{r_1}$$

Here the notation $r = r_2/r_1$ has been introduced and expression (22) has been used.

The maximal $n$ at which the superconducting state can exist can be found from the condition of the vanishing square root in (28):

$$n_{\max} = \frac{r_1}{\xi(T)}\left[2\left(\frac{\mathscr{L}}{r^2-1} - \frac{1}{r^2+1}\right)\right]^{-1/2} \tag{29}$$

[the integer part of (29) is understood]. The number $n_{\max}$ depends on the cylinder dimensions as well as on temperature.

The phase diagram of the superconducting cylinder is shown in Fig. 3 in the $(\phi_{a1}, T)$ plane. The solid lines represent the values of $\phi_{sc}$, Eq. (28), while the broken lines represent the values of $\phi_{sh}$, corresponding to

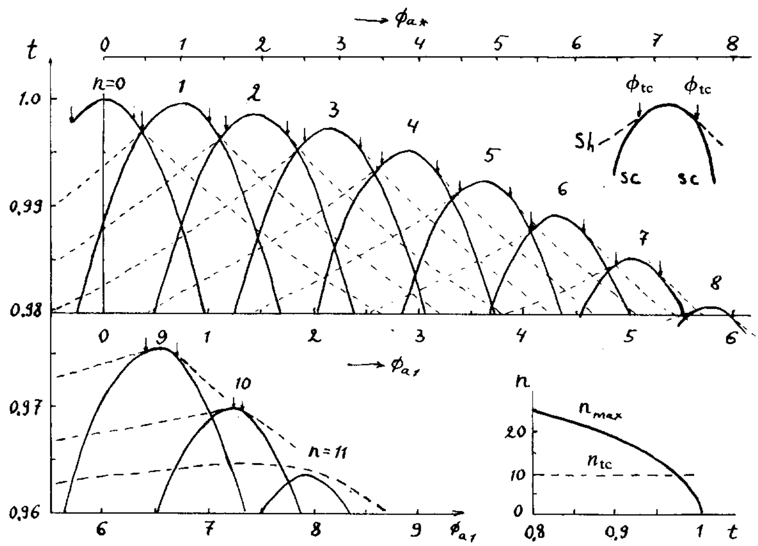


Fig. 3. The phase diagram dividing the superconducting and normal states. The solid lines are the curves $t_{sc}(\phi_{a1})$ and $t_{so}(\phi_{a1})$ (see text), the broken lines are the "superheating" curves $t_{sh}(\phi_{a1})$. The tricritical points $\phi_{tc}$ are marked by arrows. The cylinder parameters set are $\mathscr{P}_1$.

the points $\psi_{00}$, i.e., to the points of maximal "superheating" of the superconducting state (see light circles on Fig. 2). The region between $\phi_{sc}$ and $\phi_{sh}$ corresponds to the metastable "superheated" state. At tricritical points $(\phi_{tc}, T_{tc})$ (marked by arrows in Fig. 3) the curves $\phi_{sc}$ and $\phi_{sh}$ merge; at these points the difference between first- and second-order phase transitions vanishes because the first-order jump takes place at $\psi \to 0$.

The tricritical point $(\phi_{tc}, T_{tc})$ can be found by using the expansion of the functional $\mathscr{F}$, Eq. (27) [or (19)], at small $\psi \ll 1$:

$$\mathscr{F} = a_2\psi^2 + a_4\psi^4 + a_6\psi^6 + \cdots \tag{30}$$

$$a_2 = -2 + \frac{2\mathscr{P}}{1-t}, \qquad a_4 = 1 - \frac{1}{\kappa^2}(\alpha\phi_{a1}^2 - \beta n\phi_{a1} + \gamma n^2), \qquad a_6 > 0$$

$$\alpha = \frac{r^4 + r^2 - 2}{6}, \qquad \beta = \frac{3r^2 - 1}{4} - \frac{\mathscr{L}}{r^2 - 1}, \qquad \gamma = 1 - \frac{2\mathscr{L}}{r^2 - 1}$$

$\mathscr{P}$ and $\mathscr{L}$ are defined in (28a). (These expansions coincide with those obtained in Ref. 22.) The condition for the second-order phase transition ($\mathscr{F}' = 0$ at $\psi = 0$) corresponds to $a_2 = 0$. We find the curve $\phi_{so}(T)$, which is given by the same formula (28) as $\phi_{sc}(T)$. The tricritical point corresponds to the case when the coefficients $a_2$ and $a_4$ in (30) simultaneously vanish. The condition $a_4 = 0$ does not depend on $T$; from it we find $\phi_{tc}$:

$$\phi_{tc} = \frac{n\beta \pm [n^2\beta^2 + 4\alpha(\kappa^2 - \gamma n^2)]^{1/2}}{2\alpha} \tag{31}$$

For a given $n$ there are two tricritical points (see Fig. 3). From the condition $a_2 = 0$, using the dependence (22), we find $T_{tc}$:

$$\frac{T_{tc}}{T_c} = 1 - \frac{\xi^2(0)}{r_1^2}\left(\phi_{tc}\frac{r^2+1}{2} + n^2\frac{2\mathscr{L}}{r^2-1} - 2n\phi_{tc}\right) \tag{32}$$

The tricritical points (or Landau points[22]) are not present on the phase curves for all values of $n$ (see Fig. 3). As can be seen from (31), the square root vanishes for large $n$. This condition gives the maximal $n_{tc}$ for which the tricritical point exists:

$$n_{tc} = \kappa\left(\frac{4\alpha}{4\alpha\gamma - \beta^2}\right)^{1/2} \tag{33}$$

(here the integer part of $n_{tc}$ is understood). The number $n_{tc}$ does not depend on temperature and is defined only by the cylinder dimensions and by the value of $\kappa$.

The insert to Fig. 3 shows $n_{\max}$ [Eq. (29)] and $n_{\rm tc}$ [Eq. (33)] for a cylinder with parameters $\mathscr{P}_1$ (see footnote to p. 417).

As can be seen from Fig. 3, there are regions on the phase diagram where the superconducting state with number $n$ is destroyed by a first-order phase transition (in the region between the curves $\phi_{\rm sc}$ and $\phi_{\rm sh}$) or by a second-order transition (in the region near the curve $\phi_{\rm so}$, between two tricritical points). From Fig. 2 and 3 it can also be seen that when the external field and the number $n$ increase, the superconducting state is divided into intervals between which only a normal state with $\psi = 0$ is possible. The appearance of such isolated superconducting regions on the phase diagram of a thin-walled cylinder was pointed out in Refs. 20 and 37 (reenterant superconductivity.

The oscillations of the critical temperature $T_c$ of a thin-walled cylinder in a magnetic field (as observed in a number of experiments—the so-called Little–Parks effect; see Refs. 16, 17, 35, and 36 for details) are connected with transitions of the system from one quantum level $n$ to another. As we have pointed out, neglect of the role of the parabolic term $q\phi_{a1}^2$ in (19) led to the theoretical prediction of strict periodicity in $T_c(\phi_{a1})$ for arbitrary large fields $\phi_{a1}$. The anomalies observed in the experiment (in particular, the presence of a parabolic background) were attributed to the nonexact orientation of the cylinder axis along the magnetic field.[16,19] However, by taking into account the parabolic term $q\phi_{a1}^2$, some of these anomalies can be explained in a different way.[20] Moreover, taking account of the term $q\phi_{a1}^2$ automatically provides for the suppression of the superconductivity in high fields and the transition to the normal state. (A more detailed discussion of the role played by the parabolic term in the case of thin-walled cylinders is given elsewhere.[20,24])

The disappearance of tricritical points on the phase diagram (see Fig. 3) is also due to the presence in (19) of a parabolic term. If the temperature is reduced in a sufficiently strong applied field $\phi_{a1}$ (starting from the normal

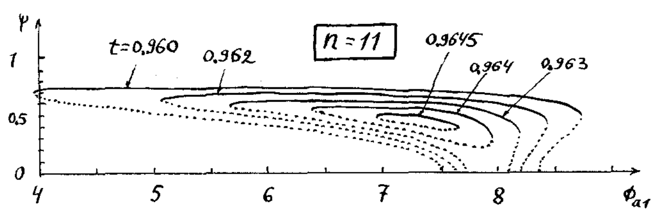


Fig. 4. The dependence of the order parameter $\psi$ in the state $n = 11$ (see Fig. 3) on the field and temperature. The superconducting state is destroyed by a first-order phase transition (having a finite $\psi$) if the temperature is increased.

state), then the superconducting state can appear only by a jump or by a first-order phase transition. (Figure 4 illustrates the temperature dependence of the order parameter in the state $n = 11$, when the tricritical point is already absent. One can see that $\psi$ vanishes by a jump if the temperature is increased.) As a result, the sudden appearance of the superconducting state with finite $\psi$ might be more difficult compared with the case when the superconductivity appears by going through those regions of the second-order phase transition curve where the order parameter is small, $\psi \to 0$ (these regions are between two neighboring tricritical points).

This may explain the fact that for not too thin cylinders and for sufficiently low temperatures the hysteretic phenomena in a magnetic field are more pronounced.

Figures 5 and 6 show the flux inside the cavity of the cylinder $\phi_1$ and the magnetic moment $M$, as found from (15)–(17) taking account of the dependence $\psi(\phi_{a1})$ according to (27).*

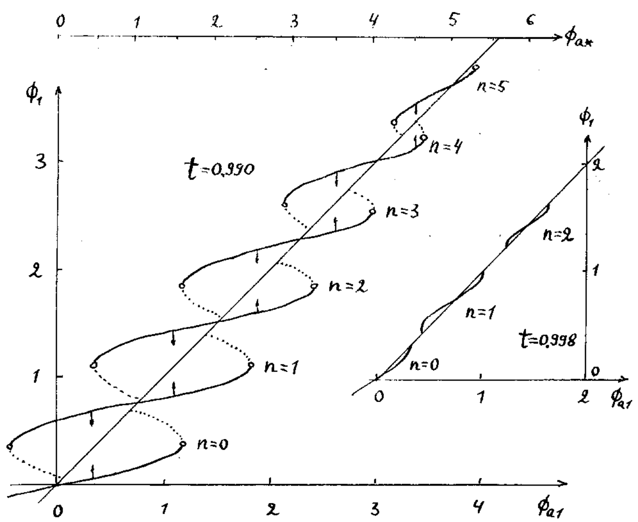


Fig. 5. The dependence of the flux in the cylinder cavity $\phi_1$ on the applied field $\phi_{a1}$ for different temperatures. The cylinder parameters are set $\mathscr{P}_1$.

*Note that the calculated dependences of the magnetic moment of the hollow cylinder on the field and temperature found in the present paper disagree with the curves drawn in Ref. 22. Unfortunately, no straightforward formula for the magnetic moment is given in Ref. 22, so comparison of the results is difficult. This disagreement may be due to a computational error in Ref. 22 or be connected with the temperature dependence of $\kappa(T)$ introduced there (in our calculations $\kappa = \text{const}$).

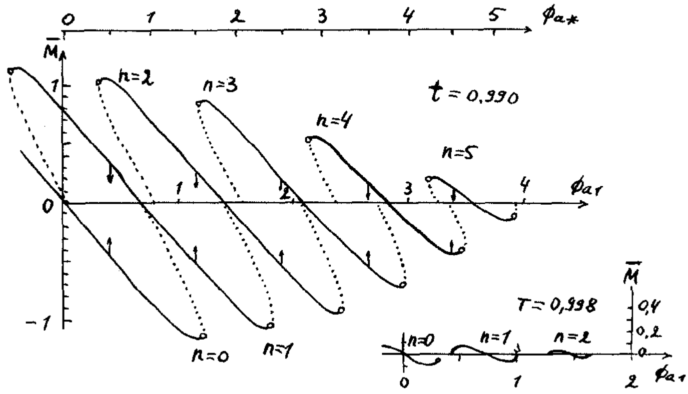


Fig. 6. The dependence of the magnetic moment $\bar{M} = 4\pi M/\Phi_0$ [see Eq. (17)] on the field and temperature. The cylinder parameters are set $\mathscr{P}_1$.

The dependence of the thermodynamic potential $\mathscr{F}$ in (19) [or (25)] on the field at the point $\psi = \psi_0$, which corresponds to the minimum of $\mathscr{F}$, is given in Fig. 7. The temperature dependences of various quantities are shown in Fig. 8, including that of the total captured flux inside a cavity $\phi_2(0)$ in the state $n = 1$ for $\phi_{a1} = 0$.

## 4. THE LIMITING CASE OF A THIN-WALLED CYLINDER

The curves of Figs. 2–10 were obtained by using the exact formulas of Sections 2 and 3, containing Bessel functions. In some limiting cases

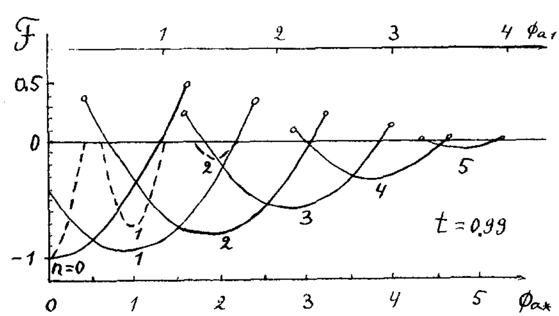


Fig. 7. The value of the thermodynamic potential $\mathscr{F}$ at its minimum [see Eqs. (19), (3)] for a cylinder with parameter set $\mathscr{P}_1$. The solid lines are for $t = 0.990$, the broken lines for $t = 0.998$. The equilibrium transitions from one level to another ($n \to n+1$) take place at the points $\phi_{a*} = n + 1/2$.

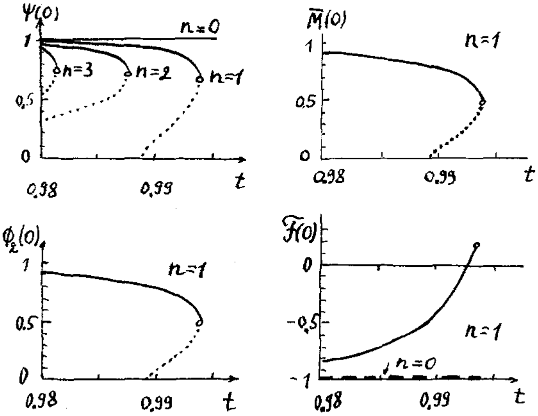


Fig. 8. The dependence of $\psi$, $\bar{M}$, $\phi_2$, and $\mathscr{F}$ on the temperature in the state $n=1$ having trapped flux in the absence of an external field ($\phi_{a1}=0$).

simpler expressions can be found. Thus, in the case of a thin-walled cylinder, $d/\delta_L(T)\ll 1, d/r_1\ll 1$, expanding the Bessel functions in the small parameter $\Delta=\psi d/\delta_L(T)$, we find*

$$\mathscr{F}=-2\psi^2+\psi^4+\frac{2A(n-p\phi_{a1})^2\psi^2}{1+\mu\psi^2/2}+Q_2\phi_{a1}^2\psi^2-Q_4\phi_{a1}^2\psi^4 \tag{34}$$

$$\phi_1=n\frac{\mu_1\psi^2/2}{1+\mu\psi^2/2}+\frac{\phi_{a1}}{1+\mu\psi^2/2}$$
$$\phi_2=n\frac{\mu\psi^2/2}{1+\mu\psi^2/2}+\phi_{a2}\left(1-\frac{\mu_1\psi^2/2}{1+\mu\psi^2/2}\right) \tag{35}$$

$$\frac{4\pi M}{\Phi_0}=\phi_2-\phi_{a2},\qquad \mu=\frac{r_1d}{\delta_L^2}\left(1+\frac{d_1}{2}\right),\qquad \mu_1=\frac{r_1d}{\delta_L^2}\left(1-\frac{d_1}{2}\right) \tag{36}$$

$$A=\frac{\xi^2(T)}{r_1^2}\left(1-\frac{d_1}{2}\right),\qquad Q_2=\frac{2}{3}\frac{\xi^2(T)}{r_1^2}d_1^2,\qquad Q_4=\frac{1}{15}\frac{d_1^4}{\kappa^2}$$
$$p=1+d_1,\qquad d_1=d/r_1 \tag{37}$$

*Note that by expanding the $D_1$ function in (19) in the small parameter $\Delta\ll 1$, terms of the type $\mu=\rho_1\Delta=\psi^2r_1d/\delta_L^2(T)$ appear in the denominator, which can be large if $r_1\gg d$. We keep these terms in the denominator and do not expand them in powers of $d$. As a result, expression (34) for $\mathscr{F}$ in the case of a thin-walled cylinder is more general than, for instance, the expansion of $\mathscr{F}$ in powers of $\psi$ obtained by Douglass,[11] which is, strictly speaking, valid only for $\psi\ll 1$. Expression (34) for $\mathscr{F}$ is valid for all values of $\psi$. The screening factor $\mu$ in (34) plays the important role of determining the character of the phase transition, i.e., whether it is of first- or second-order (for further detail see Refs. 10 and 20).

At the points $n = p\phi_{a1}$ we have $\phi_1 = \phi_{a1}$ and $\phi_2 = \phi_{a2}$ and $\mathcal{F}$ coincides with the potential of a thin film in a parallel field (21a).

The condition of the extremum of the functional $\mathcal{F}$ ($\partial\mathcal{F}/\partial\psi = 0$) now takes the form*

$$-1+\psi^2+\frac{1}{2}Q_2\phi^2-Q_4\phi^2\psi^2+\frac{A(\phi-n)^2}{(1+\mu\psi^2/2)^2}=0 \tag{38}$$

where $\phi = p\phi_{a1} \equiv \phi_{a*}$. Equation (38) determines the value of $\psi$ that corresponds to the extremum of $\mathcal{F}$ as a function of external field and temperature. This equation reduces to a cubic equation and can be solved by the Cardan formulas. It is simpler to find from (38) not the dependence $\psi(\phi)$, but $\phi(\psi)$:

$$\phi=\frac{n\pm\{P(1-\psi^2)[1+P(\frac{1}{2}Q_2-Q_4\psi^2)]-n^2P(\frac{1}{2}Q_2-Q_4\psi^2)\}^{1/2}}{1+P(\frac{1}{2}Q_2-Q_4\psi^2)} \tag{39}$$

Here $P=(1+\mu\psi^2/2)^2/A$.

The point $\psi_{00}$ where the superconducting solutions end (the inflection point in Figs. 1a, 2, and 3) can be found from the condition $\partial^2\mathcal{F}/\partial\psi^2=0$:

$$-1+\frac{1}{2}Q_2\phi^2+3\psi^2(1-Q_4\phi^2)+\frac{A(\phi-n)^2}{(1+\mu\psi^2/2)^2}-\frac{2\mu A(\phi-n)^2\psi^2}{(1+\mu\psi^2/2)^3}=0 \tag{40}$$

Simultaneous solution of the system (38), (40) gives

$$\psi_{00}^2=\left[\frac{2}{3}-\frac{2}{3\mu}(1-Q_4\phi_{\mathrm{sh}}^2)-\frac{1}{3}Q_2\phi_{\mathrm{sh}}^2\right]\Big/(1-Q_4\phi_{\mathrm{sh}}^2) \tag{41}$$

where $\phi_{\mathrm{sh}}(T)$ is a root of the equation

$$\mu A(\phi-n)^2(1-Q_4\phi^2)^2=[\tfrac{2}{3}+\tfrac{1}{3}\mu-\phi^2(\tfrac{1}{6}\mu Q_2+\tfrac{2}{3}Q_4)]^3 \tag{42}$$

Equation (42) defines, in the $(\phi, T)$ plane, the curve of the maximal "superheating" of the superconducting state. Noting that the quantities $\mu A$, $\mu Q_2$, and $Q_4$ do not depend on temperature, which only enters the term $\mu/3$, we can easily solve Eq. (42) versus $T$ and find the dependence $T=T_{\mathrm{sh}}(\phi)$:

$$\frac{T_{\mathrm{sh}}}{T_c}=1-\frac{\xi^2(0)}{r_1^2}\Big\{\tfrac{1}{3}d_1^2(1+\tfrac{2}{5}d_1)\phi^2+3\Big(1-\frac{d_1}{2}\Big)\Big[\Big(\frac{\kappa^2}{d_1}\Big)^2(\phi-n)^2$$
$$\times\Big(1-\frac{d_1^4}{15\kappa^2}\phi^2\Big)^2\Big]^{1/3}-2\Big(1+\frac{d_1}{2'}\Big)\frac{\kappa^2}{d_1}\Big\} \tag{43}$$

*The formulas for the thin-walled cylinder can be obtained by starting from the general expressions of Section 3. We prefer to start from the simpler expression (34) and derive the corresponding formulas anew.

The condition of the second-order phase transition ($\mathscr{F}'=\mathscr{F}''=0$ at $\psi=0$) now takes the form [according to (38) or (40)]

$$-1+\tfrac{1}{2}Q_2\phi^2+A(\phi-n)^2=0 \tag{44}$$

This relation determines the curve, in the ($\phi$, $T$) plane, of the second-order phase transitions (this curve is between the neighboring tricritical points in Fig. 3).

The boundary of the "supercooled" state is also found from the conditions $\mathscr{F}'=\mathscr{F}''=0$ at $\psi=0$, i.e., we again obtain the relation (44). From (44) it is easy to find the temperature that corresponds to maximal "supercooling":

$$\frac{T_{sc}}{T_c}=1-\frac{\xi^2(0)}{r_1^2}\left[\tfrac{1}{3}d_1^2\phi^2+\left(1-\frac{d_1}{2}\right)(\phi-n)^2\right] \tag{45}$$

The tricritical point for the case $d_1 \ll 1$ is simpler to find from (31) and (32):

$$\phi_{tc}=n(1-d_1)\pm\frac{\kappa}{d_1^{1/2}}\left(1-n^2\frac{d_1^4}{15\kappa^2}\right)^{1/2} \tag{46}$$

$$\frac{T_{tc}}{T_c}=1-\frac{\xi^2(0)}{r_1^2}[(\phi_{tc}-n)^2+d_1(\phi_{tc}^2-n^2)] \tag{47}$$

$$n_{tc}=\sqrt{15}\,\frac{\kappa}{d_1^2}(1+d_1), \qquad n_{max}=\frac{\sqrt{3}}{d_1}\frac{r_1}{\xi(T)} \tag{48}$$

[$n_{max}$ at $d_1 \ll 1$ is found from (29)]. From (48) we find $n_{tc}/n_{max}=\sqrt{5}\,\delta_L(T)/d$, i.e., for $d<\sqrt{5}\,\delta_L(T)$ we have $n_{tc}>n_{max}$. In other words, all the phase curves of a very thin cylinder possess tricritical points. When $d$ is diminished (or the number of impurities is increased, i.e., $\kappa$ is increased; see Fig. 9) the tricritical point shifts into the region of lower temperatures

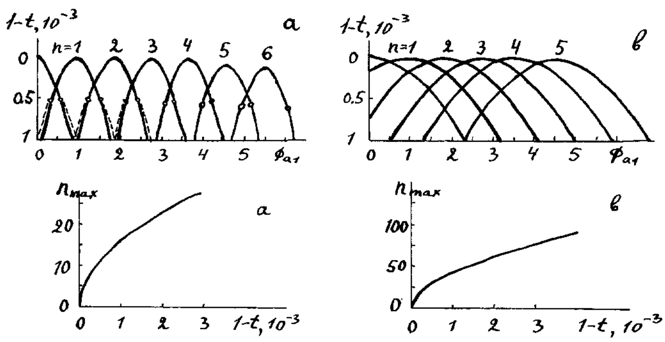


Fig. 9. The phase diagrams for a cylinder with parameter set $\mathscr{P}_2$. (a) The clean case [Eq. (23)]; (b) the dirty case [Eq. (24)] ($l=10^{-6}$ cm). The dependence $n_{max}(t)$ is also shown. $n_{tc}$ is 87 for (a) and 658 for (b).

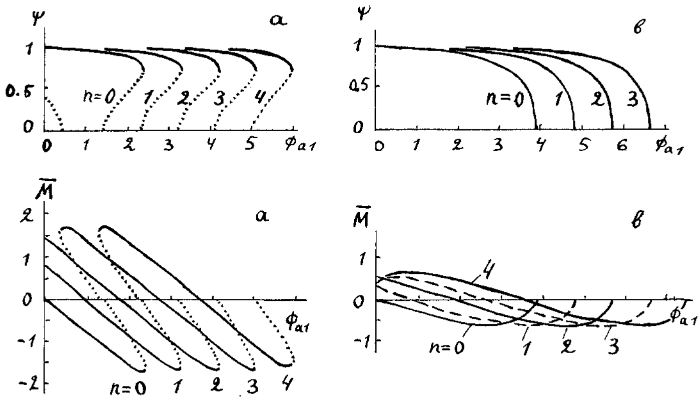


Fig. 10. The field dependence of $\psi$ and $\bar{M} = 4\pi M/\Phi_0$ at $t = 0.997$ for a cylinder with parameter set $\mathcal{P}_2$. (a) The clean case, (b) the dirty case ($l = 10^{-6}$ cm).

and larger fields (e.g., in the dirty case, Fig. 9b, the tricritical point for $n = 0$ is at $\phi_{\text{tc}} \approx 5$).

From (46) we find the value $n = n_{1/2}$ for which the tricritical point $\phi_{\text{tc}}$ takes the value $\phi_{\text{tc}} \approx n + 1/2$, namely, $n_{1/2} \approx \kappa/d_1$. If $n < n_{1/2}$ the phase diagram (of the type of Fig. 9) ends in hysteresis-free curves of second-order phase transitions, $t_{\text{so}}(\phi)$. Thus, the first-order hysteretic transition from the superconducting into the normal state and the accompanying jumps in physical quantities in the case of thin-walled cylinders can be observed only for sufficiently large fields and low temperatures (at $n > n_{1/2}$).

Figures 10 and 11 illustrate the influence of increasing number of impurities on the behavior of the order parameter, the magnetic moment,

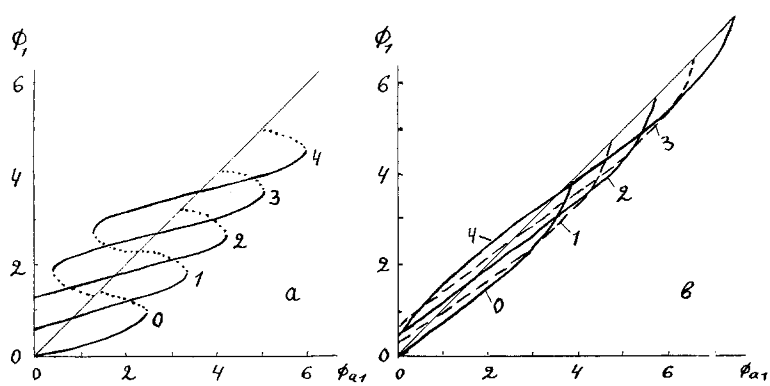


Fig. 11. The dependence of the flux $\phi_1$ on the external field at $t = 0.997$ for a cylinder with parameter set $\mathcal{P}_2$. (a) The clean case, (b) the dirty case ($l = 10^{-6}$ cm).

and the field inside the cavity for the cylinder with parameter set $\mathscr{P}_2$ (see footnote on p. 417). Note that the approximate formulas of Section 4 give a good description of the behavior of a thin-walled cylinder over the entire range of variation of the order parameter $\psi$ and they are also valid for large values of the screening parameter $\mu \approx r_1 d/\delta_L^2(T)$ [see the denominator in (34)]. The analogous formulas in, e.g., Refs. 11 and 12, may be used only for $\psi \ll 1$, i.e., near the second-order phase transition curve.

The case of cylinders with large radii $\rho_{1,2} \gg 1$ but an arbitrary value of $\Delta = \rho_2 - \rho_1$ can be considered on the basis of the general formulas of Section 2. We quote here only the formula for the magnetic flux $\phi_1$ trapped in the cavity of a cylinder at $\phi_{a1} = 0$. From (15) we find, using the expansions of the Bessel functions for large arguments,

$$\phi_1 = n\frac{D}{D_1} \approx n\frac{e^{\Delta}(1+1/8\rho_2-1/8\rho_1)-e^{-\Delta}(1-1/8\rho_2+1/8\rho_1)}{e^{\Delta}(1+15/8\rho_1+1/8\rho_2)-e^{-\Delta}(1-1/8\rho_2-1/8\rho_1)} \tag{49}$$

Note that the formula (4) for the trapped flux given in Ref. 12 is wrong. At $d \to 0$ it leads to a divergence of the type $1/d$, and, moreover, does not contain the parameter $\psi$. The correct limiting expression for $\phi_1$ at $\Delta = \psi d/\delta_L \ll 1$ can be obtained from (49):

$$\phi_1 = n\frac{\frac{1}{2}\rho_1\Delta}{1+\frac{1}{2}\rho_1\Delta} \tag{50}$$

This coincides with the expression found in Refs. 10 and 20. At $d \to 0$, (50) also agrees with (36).

## 5. OSCILLATIONS OF THE SPECIFIC HEAT IN A MAGNETIC FIELD

In addition to the oscillations of the critical temperature of a hollow cylinder in a magnetic field (the Little–Parks effect), the system's specific heat should also oscillate, which is again a direct consequence of the transitions between quantized levels (compare Ref. 25). Indeed, the difference in the specific heat of a cylinder in the superconducting and normal states is

$$\Delta C = C_s - C_n = T\frac{d}{dT}(S_s - S_n) \tag{51}$$

where $S$ is the entropy,

$$S_s - S_n = -\frac{1}{V_s}\frac{d}{dT}(\mathscr{G}_s - \mathscr{G}_n) \tag{52}$$

and the difference in the total thermodynamic potential $\Delta\mathscr{G} = \mathscr{G}_s - \mathscr{G}_n$ is related to the reduced potential difference $\mathscr{F}$ in (8) by the formula

$$\Delta\mathscr{G} = V_s \frac{H_{\rm cm}^2(T)}{8\pi}\mathscr{F} \tag{53}$$

Combining expressions (51)–(53), we find

$$\Delta C = T\frac{d}{dT}\left[-\frac{H_{\rm cm}(T)}{4\pi}\frac{dH_{\rm cm}(T)}{dT}\mathscr{F} - \frac{H_{\rm cm}^2}{8\pi}\frac{d\mathscr{F}}{dT}\right] \tag{54}$$

where the full derivative $d\mathscr{F}/dT = \partial\mathscr{F}/\partial T + (\partial\mathscr{F}/\partial\psi)(\partial\psi/\partial T)$. Thus, because $\mathscr{F}$ is an oscillating function of $H_0$, the difference $\Delta C$ should also oscillate in a magnetic field.

We also consider the case of second-order phase transitions in a magnetic field, i.e., the region in the vicinity of the phase curve (so) on the state diagram Fig. 3. Because the second-order phase transitions take place at $\psi \to 0$, we can use the expansion $\mathscr{F} = a_2\psi^2 + a_4\psi^4 + \cdots$ [see (30)]. Putting this expansion into (54) and calculating the temperature derivatives at $\psi \to 0$, we obtain

$$\Delta C = T\left[-\frac{H_{\rm cm}^2(T)}{8\pi}\frac{\partial a_2}{\partial T}\frac{\partial\psi^2}{\partial T}\right] \tag{55}$$

The condition $\partial\mathscr{F}/\partial\psi = 0$ gives $a_2 + 2a_4\psi^2 = 0$, from which

$$\frac{\partial\psi^2}{\partial T} = -\frac{1}{2a_4}\frac{\partial a_2}{\partial T}$$

and consequently

$$\Delta C = T\frac{H_{\rm cm}^2(T)}{8\pi}\frac{(\partial a_2/\partial T)^2}{2a_4} \tag{56}$$

Introducing $H_{\rm cm}^2(T) = H_{\rm cm}^2(0)(1-t)$, $t = T/T_c$, and $a_2 = -2 + 2\mathscr{P}/(1-t)$, we find finally

$$\Delta C = C_0\frac{t_{\rm so}(\phi_{a1})}{a_4(\phi_{a1})}, \qquad C_0 = \frac{H_{\rm cm}^2(0)}{4\pi T_c} \tag{57}$$

As transition temperature $t$ we use the temperature of the second-order phase transition $t_{\rm so} = 1 - \mathscr{P}$ [see (28a)]. The dependence of $\Delta C$ on the field is shown in Fig. 12. At tricritical points [where the coefficient $a_4$ in (30) vanishes] we have $\Delta C \to \infty$.

In the case of a first-order phase transition to the normal state it can be seen that $\Delta C$ is infinity on the whole phase transition curve, since in the case of a first-order transition a finite amount of heat is discharged at

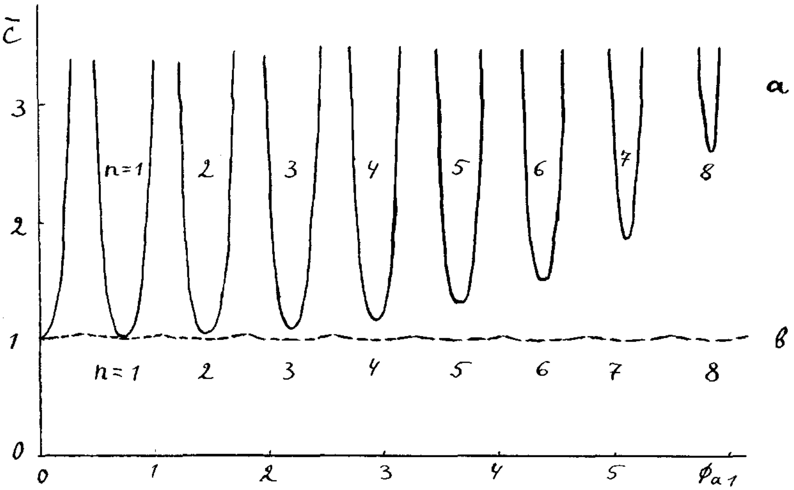

Fig. 12. The oscillations of the specific heat $\bar{C} = \Delta C/C_0$ in (57) by second-order transitions in an external field for a cylinder with parameter set $\mathscr{P}_1$. At the tricritical points $\bar{C} \to \infty$. The solid lines are for the clean case, the broken lines for the dirty case.

$\Delta T \to 0$. We will not write down the corresponding formulas and note only that by shifting the point of the first-order phase transition to the tricritical point, we obtain the expression (57), as expected.

## 6. CONCLUSION

In this paper we have reduced the general expression for the thermodynamic potential of the system (8) (which is equivalent to the one obtained in a number of other papers) to the compact form (19). Equation (19) reveals the periodic dependence on magnetic field, due to consecutive transitions of the system between quantized levels, and also the destructive influence of a strong external field (the parabolic term $q\phi_{a1}^2$). This formula generalizes the results of Tinkham,[16,17,35,36] who has described the Little–Parks effect[13,14] in the case of a thin-walled cylinder and second-order phase transitions. By using (19), we can describe the dependence of a hollow cylinder of arbitrary dimensions and predict the possibility of first-order hysteretic phase transitions. On the basis of the general expressions (19), (25), and (27) a detailed study has been made of the dependences of the order parameter, the total flux, and the magnetic moment of a cylinder on the temperature and applied field; the boundaries of the hysteresis and the "superheating" and "supercooling" fields have been obtained, the oscillations of the specific heat of a cylinder in a magnetic field examined, and the influence of nonmagnetic impurities estimated. An expression [Eq. (34)] for the thermodynamic potential of the system in the case of a thin-walled cylinder has been obtained, which is valid for arbitrary values of $\psi$ (not for $\psi \ll 1$ only, as in the case of second-order phase transitions). Graphs have been presented to illustrate the general theory.

Aspects of the general picture have been reported elsewhere.[20–25] The main task of the present paper was to give a consistent and systematic presentation of the predictions of the theory. We hope that this work will be useful for interpreting experimental results and for setting up new experiments with hollow superconducting cylinders.

In conclusion, we note that in Ref. 38 we obtained a general expression for the thermodynamic potential of a hollow cylinder in the presence of a thermoelectric current of normal excitations $j_n = b\, \nabla T$ (see also Refs. 39 and 40). This expression is analogous to (19) (with $\phi_{a1}$ to be replaced by $j_n$) and it indicates that transitions to higher quantized levels $n$ can be stimulated not only by an external field, but also by a temperature gradient $\nabla T$ or a normal current $j_n$ (even in the absence of an external field).[38] The deep analogy between the behavior of a hollow cylinder in an external field and a thermoelectric ring under the influence of a temperature gradient should be kept in mind in setting up and interpreting the corresponding experiments.

Note also that hysteretic transitions in a hollow cylinder are closely connected with fluctuations in superconductors, and consequently the study of such transitions should be of additional interest (compare Ref. 38).

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