Twistors and Aspects of Integrability of self-dual SYM Theory

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Abstract

With the help of the Penrose-Ward transform, which relates certain holomorphic vector bundles over the supertwistor space to the equations of motion of self-dual SYM theory in four dimensions, we construct hidden infinite-dimensional symmetries of the theory. We also present a new and shorter proof (cf. hep-th/0412163) of the relation between certain deformation algebras and hidden symmetry algebras. This article is based on a talk given by the author at the Workshop on Supersymmetries and Quantum Symmetries 2005 at the BLTP in Dubna, Russia.

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1. Introduction and conclusions

By analyzing the linearized [1] and full [2] field equations and by virtue of the Penrose-Ward transform [3], it was shown that there is a one-to-one correspondence between the moduli space of holomorphic Chern-Simons theory on supertwistor space and of self-dual $\mathcal{N}=4$ SYM theory in four dimensions.² This correspondence has then been used for a twistorial construction of hidden infinite-dimensional symmetry algebras in the self-dual truncation of SYM theory [7]. Therein, the results known for the purely bosonic self-dual YM equations (see, e.g., Refs. [8]–[13]) have been generalized to the supersymmetric setting. Here, we shall briefly report on those results thereby also presenting a new and shorter proof of the relation between certain deformation algebras on the twistor side and symmetry algebras on the gauge theory side. For the sake of clarity, the discussion presented below is given in the complex setting only but, of course, it is also possible to implement real structures (see, e.g., [2, 7] for details).

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2. Preliminaries

2.1. Supertwistor space

The starting point of our discussion is the complex projective supertwistor space $\mathbb{C}P^{3|\mathcal{N}}$ with homogeneous coordinates $[z^{\alpha}, \lambda_{\dot{\alpha}}, \eta_i]$ obeying the equivalence relation

$$(z^{\alpha}, \lambda_{\dot{\alpha}}, \eta_i) \sim (tz^{\alpha}, t\lambda_{\dot{\alpha}}, t\eta_i)$$
 (2.1)

for any $t \in \mathbb{C}^*$. Here, the spinorial indices $\alpha, \beta, \ldots, \dot{\alpha}, \dot{\beta}, \ldots$ run from 1 to 2 and the R-symmetry indices i, j, \ldots from 1 to \mathcal{N} . In the following, we are interested in the open subset $\mathcal{P}^{3|\mathcal{N}} := \mathbb{C}P^{3|\mathcal{N}} \setminus \mathbb{C}P^{1|\mathcal{N}}$ defined by $\lambda_{\dot{\alpha}} \neq 0$. This space can be covered by two patches, say \mathcal{U}_+ and \mathcal{U}_- , for which $\lambda_{\dot{1}} \neq 0$ and $\lambda_{\dot{2}} \neq 0$, respectively. On those patches we have the coordinates

$$z_{+}^{\alpha} := \frac{z^{\alpha}}{\lambda_{1}}, \qquad z_{+}^{3} := \frac{\lambda_{2}}{\lambda_{1}} =: \lambda_{+} \quad \text{and} \quad \eta_{i}^{+} := \frac{\eta_{i}}{\lambda_{1}} \quad \text{on} \quad \mathcal{U}_{+},$$

$$z_{-}^{\alpha} := \frac{z^{\alpha}}{\lambda_{2}}, \quad z_{-}^{3} := \frac{\lambda_{1}}{\lambda_{2}} =: \lambda_{-} \quad \text{and} \quad \eta_{i}^{-} := \frac{\eta_{i}}{\lambda_{2}} \quad \text{on} \quad \mathcal{U}_{-},$$

$$(2.2)$$

which are related by

$$z_{+}^{\alpha} = \frac{1}{z^{3}} z_{-}^{\alpha}, \qquad z_{+}^{3} = \frac{1}{z^{3}} \quad \text{and} \quad \eta_{i}^{+} = \frac{1}{z^{3}} \eta_{i}^{-}$$
 (2.3)

on $\mathcal{U}_+ \cap \mathcal{U}_-$. This in particular shows that $\mathcal{P}^{3|\mathcal{N}}$, which we simply call supertwistor space, is a holomorphic fibration over the Riemann sphere $\mathbb{C}P^1$,

$$\mathcal{P}^{3|\mathcal{N}} = \mathcal{O}(1) \otimes \mathbb{C}^2 \oplus \Pi \mathcal{O}(1) \otimes \mathbb{C}^{\mathcal{N}} \to \mathbb{C}P^1. \tag{2.4}$$

²For reviews of twistor theory, we refer to [4, 5, 6, 12].

From this definition it is clear that global holomorphic sections of the fibration (2.4) are degree one polynomials. In a given trivialization, they are locally of the form

$$z_{\pm}^{\alpha} = x^{\alpha\dot{\alpha}}\lambda_{\dot{\alpha}}^{\pm} \quad \text{and} \quad \eta_{i}^{\pm} = \eta_{i}^{\dot{\alpha}}\lambda_{\dot{\alpha}}^{\pm}$$
 (2.5)

and parametrized by the moduli $(x^{\alpha\dot{\alpha}}, \eta_i^{\dot{\alpha}}) \in \mathbb{C}^{4|2\mathcal{N}}$. Here, we also introduced the common abbreviations

$$(\lambda_{\dot{\alpha}}^+) := \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix} \quad \text{and} \quad (\lambda_{\dot{\alpha}}^-) := \begin{pmatrix} \lambda_- \\ 1 \end{pmatrix}.$$
 (2.6)

Therefore, $\mathcal{P}^{3|\mathcal{N}}$ naturally fits into the following double fibration

$$\begin{array}{c|c}
\mathcal{F}^{5|2\mathcal{N}} \\
\pi_2 & \pi_1 \\
\mathcal{P}^{3|\mathcal{N}} & \mathbb{C}^{4|2\mathcal{N}}
\end{array} (2.7)$$

where $\mathcal{F}^{5|2\mathcal{N}} \cong \mathbb{C}^{4|2\mathcal{N}} \times \mathbb{C}P^1$ is called the correspondence space. The (holomorphic) projections are given according to

$$\pi_1 : (x^{\alpha\dot{\alpha}}, \lambda_{\pm}, \eta_i^{\dot{\alpha}}) \mapsto (x^{\alpha\dot{\alpha}}, \eta_i^{\dot{\alpha}}),$$

$$\pi_2 : (x^{\alpha\dot{\alpha}}, \lambda_{\pm}, \eta_i^{\dot{\alpha}}) \mapsto (z_{\pm}^{\alpha} = x^{\alpha\dot{\alpha}}\lambda_{\dot{\alpha}}^{\pm}, z_{\pm}^3 = \lambda_{\pm}, \eta_i^{\pm} = \eta_i^{\dot{\alpha}}\lambda_{\dot{\alpha}}^{\pm}).$$

$$(2.8)$$

Next let us take a closer look at the relations (2.5). Fixing a point $(z_{\pm}^{\alpha}, \lambda_{\pm}, \eta_{i}^{\pm})$ in supertwistor space and solving (2.5) for $(x^{\alpha\dot{\alpha}}, \eta_i^{\dot{\alpha}})$, one determines an isotropic (null) plane $\mathbb{C}^{2|\mathcal{N}}$ in $\mathbb{C}^{4|2\mathcal{N}}$. On the other hand, a fixed point $(x^{\alpha\dot{\alpha}}, \eta_i^{\dot{\alpha}}) \in \mathbb{C}^{4|2\mathcal{N}}$ gives a holomorphic embedding of the Riemann sphere into supertwistor space. Thus, we have

(i) a point
$$p \in \mathcal{P}^{3|\mathcal{N}} \longleftrightarrow$$
 an isotropic plane $\mathbb{C}_p^{2|\mathcal{N}} \hookrightarrow \mathbb{C}^{4|2\mathcal{N}}$,
(ii) $\mathbb{C}P_{x,p}^1 \hookrightarrow \mathcal{P}^{3|\mathcal{N}} \longleftrightarrow$ a point $(x,\eta) \in \mathbb{C}^{4|2\mathcal{N}}$.

(ii)
$$\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}} \longleftrightarrow \text{a point } (x,\eta) \in \mathbb{C}^{4|2\mathcal{N}}.$$

2.2. Holomorphy and self-dual SYM theory in the twistor approach

In order to study super gauge theory, some additional data on the manifolds appearing in the double fibration (2.7) is required. Let us consider a rank n holomorphic vector bundle $\mathcal{E} \to \mathcal{P}^{3|\mathcal{N}}$ which is characterized by the transition function $f = \{f_{+-}\}$ and its pull-back $\pi_2^*\mathcal{E}$ to the supermanifold $\mathcal{F}^{5|2\mathcal{N}}$. For notational reasons, we denote the pulledback transition function by the same letter f. By definition of a pull-back, the transition function f is constant along the fibers of $\pi_2: \mathcal{F}^{5|2\mathcal{N}} \to \mathcal{P}^{3|\mathcal{N}}$. Therefore, it is annihilated by the vector fields

$$D_{\alpha}^{\pm} := \lambda_{\pm}^{\dot{\alpha}} \partial_{\alpha \dot{\alpha}} \quad \text{and} \quad D_{\pm}^{i} := \lambda_{\pm}^{\dot{\alpha}} \partial_{\dot{\alpha}}^{i},$$
 (2.9)

where $\partial_{\alpha\dot{\alpha}} = \partial/\partial x^{\alpha\dot{\alpha}}$ and $\partial_{\dot{\alpha}}^i = \partial/\partial \eta_i^{\dot{\alpha}}$. Spinorial indices are raised and lowered via the ϵ -tensors, $\epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = -\epsilon_{12} = -\epsilon_{\dot{1}\dot{2}} = 1$, together with the normalizations $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_{\alpha}^{\gamma}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}}=\delta_{\dot{\alpha}}^{\dot{\gamma}}$. Let $\bar{\partial}_{\mathcal{P}}$ and $\bar{\partial}_{\mathcal{F}}$ be the anti-holomorphic parts of the exterior derivatives on the supertwistor space and the correspondence space, respectively. Then we have $\pi_2^* \bar{\partial}_{\mathcal{P}} = \bar{\partial}_{\mathcal{F}} \circ \pi_2^*$, and hence, the transition function of $\pi_2^* \mathcal{E}$ is also annihilated by $\bar{\partial}_{\mathcal{F}}$. Next we want to assume that the bundle $\mathcal{E} \to \mathcal{P}^{3|\mathcal{N}}$ is holomorphically trivial when

restricted to any projective line $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$. This condition implies that there exist

some smooth $GL(n, \mathbb{C})$ -valued functions $\psi = \{\psi_{\pm}\}$, which define a trivialization of $\pi_2^*\mathcal{E}$, such that $f = \{f_{+-}\}$ can be decomposed as

$$f_{+-} = \psi_{+}^{-1}\psi_{-} \tag{2.10}$$

and

$$\bar{\partial}_{\mathcal{F}}\psi_{+} = 0. \tag{2.11}$$

In particular, this formula implies that the ψ_{\pm} depend holomorphically on λ_{\pm} . Applying the vector fields (2.9) to (2.10), we realize by virtue of an extension of Liouville's theorem that the expressions

$$\psi_{+}D_{\alpha}^{+}\psi_{+}^{-1} = \psi_{-}D_{\alpha}^{+}\psi_{-}^{-1}$$
 and $\psi_{+}D_{+}^{i}\psi_{+}^{-1} = \psi_{-}D_{+}^{i}\psi_{-}^{-1}$ (2.12)

must be at most linear in λ_{\pm} . Therefore, we may introduce a Lie-algebra valued one-form \mathcal{A} such that

$$D_{\alpha}^{\pm} \, \mathcal{A} := \mathcal{A}_{\alpha}^{\pm} := \lambda_{\pm}^{\dot{\alpha}} \mathcal{A}_{\alpha \dot{\alpha}} = \psi_{\pm} D_{\alpha}^{\pm} \psi_{\pm}^{-1}, D_{\pm}^{i} \, \mathcal{A} := \mathcal{A}_{\pm}^{i} := \lambda_{\pm}^{\dot{\alpha}} \mathcal{A}_{\dot{\alpha}}^{i} = \psi_{\pm} D_{\pm}^{i} \psi_{\pm}^{-1},$$
(2.13)

and hence

$$\lambda_{\pm}^{\dot{\alpha}}(\partial_{\alpha\dot{\alpha}} + \mathcal{A}_{\alpha\dot{\alpha}})\psi_{\pm} = 0,$$

$$\lambda_{\pm}^{\dot{\alpha}}(\partial_{\dot{\alpha}}^{i} + \mathcal{A}_{\dot{\alpha}}^{i})\psi_{\pm} = 0$$
(2.14)

and $\bar{\partial}_{\mathcal{F}}\psi_{\pm}=0$. The compatibility conditions for the linear system (2.14) read as

$$\left[\nabla_{\alpha(\dot{\alpha}}, \nabla_{\beta\dot{\beta}}\right] = 0, \quad \left[\nabla^{i}_{(\dot{\alpha}}, \nabla_{\alpha\dot{\beta}}\right] = 0 \quad \text{and} \quad \left\{\nabla^{i}_{(\dot{\alpha}}, \nabla^{j}_{\dot{\beta}}\right\} = 0, \tag{2.15}$$

where we have introduced

$$\nabla_{\alpha\dot{\alpha}} := \partial_{\alpha\dot{\alpha}} + \mathcal{A}_{\alpha\dot{\alpha}} \quad \text{and} \quad \nabla^{i}_{\dot{\alpha}} := \partial^{i}_{\dot{\alpha}} + \mathcal{A}^{i}_{\dot{\alpha}}. \tag{2.16}$$

Eqs. (2.15) have been known for quite some time and it has been shown that they are equivalent to the equations of motion of \mathcal{N} -extended self-dual SYM theory [14, 15] on four-dimensional space-time. Note that Eqs. (2.14) imply that the gauge potentials $\mathcal{A}_{\alpha\dot{\alpha}}$ and $\mathcal{A}^i_{\dot{\alpha}}$ do not change when we perform transformations of the form $\psi_{\pm} \mapsto \psi_{\pm} h_{\pm}$, where the $h = \{h_{\pm}\}$ are annihilated by the vector fields (2.9) and $\bar{\partial}_{\mathcal{F}}$. Under such transformations the transition function $f = \{f_{+-}\}$ of $\pi_2^*\mathcal{E}$ transform into a transition function $h_+^{-1}f_{+-}h_-$ of a bundle which is said to be equivalent to $\pi_2^*\mathcal{E}$. On the other hand, gauge transformations of the gauge potentials are induced by transformations of the form $\psi_{\pm} \mapsto g^{-1}\psi_{\pm}$ for some smooth λ -independent $GL(n,\mathbb{C})$ -valued g. Under such transformations the transition function f is unchanged. In fact, we have

Theorem 1 There is a one-to-one correspondence between equivalence classes of holomorphic vector bundles over the supertwistor space which are holomorphically trivial when restricted to any $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$ and gauge equivalence classes of solutions to the equations of motion of \mathcal{N} -extended self-dual SYM theory in four dimensions. In fact, Eqs. (2.13) give the Penrose-Ward transform, i.e., the relation between fields on supertwistor space and fields on space-time.

3. Hidden symmetries

3.1. Infinitesimal deformations

In order to study solutions to the linearized equations of motion (i.e., symmetries), one considers small perturbations of the transition functions $f = \{f_{+-}\}$ of a holomorphic vector bundle $\mathcal{E} \to \mathcal{P}^{3|\mathcal{N}}$ and its pull-back $\pi_2^*\mathcal{E} \to \mathcal{F}^{5|2\mathcal{N}}$, respectively. Note that any infinitesimal perturbation of f is allowed, as small enough perturbations will, by Kodaira's theorem on deformation theory, preserve its trivializability properties on the curves $\mathbb{C}P^1_{x,n} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$. This follows directly from $H^1(\mathbb{C}P^1,\mathcal{O}) = 0$. Thus, we find

$$f_{+-} + \delta f_{+-} = (\psi_{+} + \delta \psi_{+})^{-1} (\psi_{-} + \delta \psi_{-})$$
(3.1)

for the deformed transition function of $\pi_2^*\mathcal{E}$. Upon introducing the Lie-algebra valued function

$$\phi_{+-} := \psi_{+}(\delta f_{+-})\psi_{-}^{-1}, \tag{3.2}$$

and linearizing Eq. (3.1), we have to find the splitting

$$\phi_{+-} = \phi_{+} - \phi_{-}. \tag{3.3}$$

Here, the Lie-algebra valued functions ϕ_{\pm} can be extended to holomorphic functions in λ_{\pm} on the respective patches, and which eventually yield

$$\delta\psi_{\pm} = -\phi_{\pm}\psi_{\pm}.\tag{3.4}$$

Moreover, we point out that finding such ϕ_{\pm} from ϕ_{+-} means to solve the infinitesimal variant of the Riemann-Hilbert problem. Obviously, the splitting (3.3) and hence solutions to the Riemann-Hilbert problem are not unique, as we certainly have the freedom to consider new $\tilde{\phi}_{\pm}$ shifted by some function function ω which is globally defined, i.e., $\tilde{\phi}_{\pm} = \phi_{\pm} + \omega$. In fact, such shifts eventually correspond to infinitesimal gauge transformations.

Infinitesimal variations of the linear system (2.14) yield

$$\delta \mathcal{A}_{\alpha}^{\pm} = \lambda_{\pm}^{\dot{\alpha}} \delta \mathcal{A}_{\alpha \dot{\alpha}} = \lambda_{\pm}^{\dot{\alpha}} \nabla_{\alpha \dot{\alpha}} \phi_{\pm} \quad \text{and} \quad \delta \mathcal{A}_{\pm}^{i} = \lambda_{\pm}^{\dot{\alpha}} \delta \mathcal{A}_{\dot{\alpha}}^{i} = \lambda_{\pm}^{\dot{\alpha}} \nabla_{\dot{\alpha}}^{i} \phi_{\pm}, \quad (3.5)$$

where the covariant derivatives have been introduced in (2.16). Note that they act adjointly in these equations. The λ -expansion of Eqs. (3.5) eventually gives the infinitesimal transformation $\delta \mathcal{A}_{\alpha\dot{\alpha}}$ and $\delta \mathcal{A}_{\dot{\alpha}}^i$, which satisfy by construction the linearized equations of motion. Note that the equivalence relations as defined at the end of Sec. 2.2 have an infinitesimal counterpart. Therefore, we altogether have

Corollary 1 There is a one-to-one correspondence between equivalence classes of deformations of the transition functions of holomorphic vector bundles over the supertwistor space which are holomorphically trivial when restricted to any $\mathbb{C}P_{x,\eta}^1 \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$ and equivalence classes of symmetries of \mathcal{N} -extended self-dual SYM theory in four dimensions.

3.2. Hidden symmetry algebras

Suppose we are given some indexed set $\{\delta_a\}$ of infinitesimal deformations $\delta_a f_{+-}$ of the transition function of our holomorphic vector bundle $\pi_2^* \mathcal{E}$. Suppose further that the δ_a s satisfy a deformation algebra of the form

$$[\delta_a, \delta_b\} f_{+-} = f_{ab}{}^c \delta_c f_{+-}, \tag{3.6}$$

where the $f_{ab}{}^c$ s are generically structure functions and $[\cdot, \cdot]$ denotes the graded commutator. Let us assume that the $f_{ab}{}^c$ s are constant. Above we have seen that any such deformation $\delta_a f_{+-}$ yields a symmetry of \mathcal{N} -extended self-dual SYM theory. So, given such an algebra, what is the corresponding symmetry algebra on the gauge theory side? To answer this question, we consider

$$[\delta_1, \delta_2] = (-)^{p_a p_b} \varepsilon^a \varrho^b [\delta_a, \delta_b], \tag{3.7}$$

where ε^a and ϱ^b are the infinitesimal parameters of the transformations δ_1 and δ_2 , respectively, and p_a denotes the Graßmann parity of the transformation δ_a . Explicitly, we may write

$$[\delta_1, \delta_2] \mathcal{A}_{\alpha}^{\pm} = \delta_1 (\mathcal{A}_{\alpha}^{\pm} + \delta_2 \mathcal{A}_{\alpha}^{\pm}) - \delta_1 \mathcal{A}_{\alpha}^{\pm} - \delta_2 (\mathcal{A}_{\alpha}^{\pm} + \delta_1 \mathcal{A}_{\alpha}^{\pm}) + \delta_2 \mathcal{A}_{\alpha}^{\pm}$$
(3.8)

and similarly for \mathcal{A}_{\pm}^{i} ; cf. also (3.5). Then one easily checks that

$$[\delta_1, \delta_2] \mathcal{A}_{\alpha}^{\pm} = \lambda_{\pm}^{\dot{\alpha}} \nabla_{\alpha \dot{\alpha}} \Sigma_{12}^{\pm}, \quad \text{with} \quad \Sigma_{12}^{\pm} := \delta_1 \phi_{\pm}^2 - \delta_2 \phi_{\pm}^1 + [\phi_{\pm}^1, \phi_{\pm}^2]. \quad (3.9)$$

Note that we use the notation $\phi_{\pm}^1 = \varepsilon^a \phi_{\pm a}$ and similarly for ϕ_{\pm}^2 . Next one considers the commutator

$$[\delta_1, \delta_2] f_{+-} = \delta_1(f_{+-} + \delta_2 f_{+-}) - \delta_1 f_{+-} - \delta_2(f_{+-} + \delta_1 f_{+-}) + \delta_2 f_{+-}. \tag{3.10}$$

Using the definition (3.2) and the resulting splittings (3.3) for the deformations $\delta_{1,2}f_{+-}$, one can show after some tedious but straightforward algebraic manipulations that the commutator (3.10) is given by

$$[\delta_1, \delta_2] f_{+-} = \psi_+^{-1} (\Sigma_{12}^+ - \Sigma_{12}^-) \psi_-, \tag{3.11}$$

where Σ_{12}^{\pm} has been introduced (3.9). By hypothesis (3.6), it must also be equal to

$$[\delta_1, \delta_2] f_{+-} = \delta_3 f_{+-}, \quad \text{with} \quad \delta_3 = (-)^{p_a p_b} \varepsilon^a \rho^b f_{ab}{}^c \delta_c, \quad (3.12)$$

i.e.,

$$[\delta_1, \delta_2] f_{+-} = \psi_+^{-1} (\phi_+^3 - \phi_-^3) \psi_-, \tag{3.13}$$

where $\phi_{\pm}^3 = (-)^{p_a p_b} \varepsilon^a \varrho^b f_{ab}{}^c \phi_{\pm c}$. By comparing this equation with the result (3.11), we therefore conclude

$$\Sigma_{12}^{\pm} = \phi_{+}^{3} + \omega^{3} = (-)^{p_{a}p_{b}} \varepsilon^{a} \rho^{b} (f_{ab}{}^{c} \phi_{+c} + \omega_{ab}), \tag{3.14}$$

since the $f_{ab}{}^c$ s are assumed to be constant. Here, ω^3 (respectively, ω_{ab}) is some function independent of λ_{\pm} and therefore representing an infinitesimal gauge transformation. Combining this result with Eq. (3.9), we get the following

Theorem 2 Suppose we are given a deformation algebra of the form (3.6) with constant $f_{ab}{}^{c}$. Then the corresponding symmetry algebra on the gauge theory side has exactly the same form modulo possible gauge transformations.

It should be stressed that this theorem does, however, not give the explicit form of the gauge parameter ω_{ab} . In order to compute it, one has to perform the splitting procedure explicitly; see [7] for details.

3.3. Examples: Affine extensions of gauge type and superconformal symmetries

So far, we have been quite general. Let us now exemplify our discussion. Let X_a be some generator of the gauge algebra $\mathfrak{gl}(n,\mathbb{C})$ and consider

$$\delta_a^m f_{+-} := \lambda_+^m [X_a, f_{+-}], \quad \text{for} \quad m \in \mathbb{Z}.$$
 (3.15)

For m=0, the transformations of the components of the gauge potential are given by

$$\delta_a^0 \mathcal{A}_{\alpha\dot{\alpha}} = [X_a, \mathcal{A}_{\alpha\dot{\alpha}}] \quad \text{and} \quad \delta_a^0 \mathcal{A}_{\dot{\alpha}}^i = [X_a, \mathcal{A}_{\dot{\alpha}}^i].$$
 (3.16)

Thus, they represent a gauge type transformation with constant gauge parameter (a global symmetry transformation). The corresponding deformation algebra is easily computed to be

$$[\delta_a^m, \delta_b^n] f_{+-} = f_{ab}{}^c \delta_c^{m+n} f_{+-}, \tag{3.17}$$

where the $f_{ab}{}^c$ s are the structure constants of $\mathfrak{gl}(n,\mathbb{C})$, i.e., we get a centerless Kac-Moody algebra. By virtue of our above theorem, we will get the same algebra (modulo gauge transformations) on the gauge theory side (for explicit calculations see also [7]). Note that such deformations can be used for the explicit construction of solutions to the field equations [16].

Another example is concerned with affine extensions of superconformal symmetries. In [7], it was shown that the generators of the superconformal algebra when viewed as vector fields have to be pulled back to the correspondence space (and hence to the supertwistor space) in a very particular fashion. Their pull-backs are explicitly given by

$$\widetilde{P}_{\alpha\dot{\alpha}} = P_{\alpha\dot{\alpha}}, \qquad \widetilde{Q}_{i\alpha} = Q_{i\alpha}, \qquad \widetilde{Q}_{\dot{\alpha}}^{i} = Q_{\dot{\alpha}}^{i},
\widetilde{D} = D,
\widetilde{K}^{\alpha\dot{\alpha}} = K^{\alpha\dot{\alpha}} + x^{\alpha\dot{\beta}}Z_{\dot{\beta}}{}^{\dot{\alpha}}, \qquad \widetilde{K}^{i\alpha} = K^{i\alpha}, \qquad \widetilde{K}_{i}^{\dot{\alpha}} = K_{i}^{\dot{\alpha}} + \eta_{i}^{\dot{\beta}}Z_{\dot{\beta}}{}^{\dot{\alpha}},
\widetilde{J}_{\alpha\beta} = J_{\alpha\beta}, \qquad \widetilde{J}_{\dot{\alpha}\dot{\beta}} = J_{\dot{\alpha}\dot{\beta}} - \frac{1}{2}Z_{\dot{\alpha}\dot{\beta}},
\widetilde{T}_{i}^{j} = T_{i}^{j}, \qquad \widetilde{A} = A,$$
(3.18)

where the untilded quantities, commonly denoted by N_a in the sequel, are the usual vector field expressions for the superconformal generators on four-dimensional superspace-time and

$$Z_{\dot{\alpha}\dot{\beta}} := \lambda_{\dot{\alpha}}^{\pm} \lambda_{\dot{\beta}}^{\pm} \partial_{\lambda_{\pm}} + \hat{\lambda}_{\dot{\alpha}}^{\pm} \hat{\lambda}_{\dot{\beta}}^{\pm} \partial_{\bar{\lambda}_{\pm}}. \tag{3.19}$$

Here, $(\hat{\lambda}_{\dot{\alpha}}^+) := {}^t(-\bar{\lambda}_+, 1)$ and $\hat{\lambda}_{\dot{\alpha}}^+ = \bar{\lambda}_-^{-1}\hat{\lambda}_{\dot{\alpha}}^-$. Let us define the following (holomorphic) action on the transition function:

$$\delta_a^m f_{+-} := \lambda_+^m \widetilde{N}_a f_{+-}, \quad \text{for} \quad m \in \mathbb{Z},$$
 (3.20)

where \widetilde{N}_a represents any of the generators given above. Note that the $\bar{\lambda}$ -derivative drops out as f_{+-} is holomorphic in λ_{\pm} . For m=0, we find

$$\delta_a^0 \mathcal{A}_{\alpha\dot{\alpha}} = \mathcal{L}_{N_a} \mathcal{A}_{\alpha\dot{\alpha}} \quad \text{and} \quad \delta_a^0 \mathcal{A}_{\dot{\alpha}}^i = \mathcal{L}_{N_a} \mathcal{A}_{\dot{\alpha}}^i, \quad (3.21)$$

where \mathcal{L}_{N_a} is the Lie derivative along N_a . Furthermore, one straightforwardly deduces

$$[\delta_a^m, \delta_b^n] f_{+-} = (f_{ab}^c + ng_a \delta_a^c - (-)^{p_a} mg_b \delta_a^c) \delta_c^{m+n} f_{+-},$$
(3.22)

what represents a centerless Kac-Moody-Virasoro type algebra. Here, the $f_{ab}{}^c$ s are the structure constants of the superconformal algebra. The g_a s are abbreviations for $\lambda_+^{-1} \tilde{N}_a^{\lambda_+}$, where $\tilde{N}_a^{\lambda_+}$ represents the ∂_{λ_+} -component of \tilde{N}_a . Hence, this time we obtain structure functions rather than structure constants and therefore we have to restrict our discussion to a certain subalgebra of the superconformal algebra in order to apply the above theorem. The most naive way of doing this is simply by dropping the special conformal generators $\tilde{K}^{\alpha\dot{\alpha}}$ and $\tilde{K}_i^{\dot{\alpha}}$ and the rotation generators $J_{\dot{\alpha}\dot{\beta}}$. Then one eventually obtains honest structure constants and can therefore use the theorem. However, in [7] we have seen that one need not to exclude the rotation generators $J_{\dot{\alpha}\dot{\beta}}$, since the structure functions for the maximal subalgebra of the superconformal algebra which does not contain $\tilde{K}^{\alpha\dot{\alpha}}$ and $\tilde{K}_i^{\dot{\alpha}}$ are only dependent on λ_{\pm} . By inspecting the formulas (3.5), we see that such λ -dependent functions do not spoil the generic form of the transformations on the gauge theory side, i.e., the corresponding symmetry algebra still closes. This is not the case when $\tilde{K}^{\alpha\dot{\alpha}}$ and $\tilde{K}_i^{\dot{\alpha}}$ are included as the structure functions also depend on $x^{\alpha\dot{\alpha}}$ and $\eta_i^{\dot{\alpha}}$, respectively. For more details see [7].

Finally, let us stress that the existence of such such algebras originates from the fact that the full group of continuous transformations acting on the space of holomorphic vector bundles over the supertwistor space is a semi-direct product of the group of local holomorphic automorphisms (i.e., complex structure preserving maps) of the supertwistor space and of the group of one-cochains for a certain covering of the supertwistor space with values in the sheaf of holomorphic maps of the supertwistor space into the gauge group. This can be shown by following and generalizing the lines presented in the case of the purely bosonic self-dual YM equations [12, 13].

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