Resolving the London Equation Quandry

September 25, 2020

It's quite a mess we got into, right? Here's my resolution. There's nothing new here, but it's organized to my satisfaction. I have been sloppy with constants in various places (what electromagnetic units are we using, anyway?).

Start by deriving the London equation from the Ginzburg-Landau equations. Take the expression for the current density

$$\boldsymbol{j} = \frac{q}{m}\mathrm{Re}\left[\psi^*(\frac{\hbar}{i}\nabla - q\boldsymbol{A})\psi\right] \tag{1}$$

and assume the order parameter has constant magnitude, $\psi = |\psi|e^{i\theta}$. Inserting this form into the current density gives (since $\nabla\psi = i(\nabla\theta)\psi$)

$$\boldsymbol{j} = \frac{q}{m}\left[|\psi|e^{-i\theta}\left(\hbar(\nabla\theta) - q\boldsymbol{A}\right)|\psi|e^{i\theta}\right] = \frac{q^2|\psi|^2}{m}\left(\frac{\hbar}{q}\nabla\theta - \boldsymbol{A}\right) \tag{2}$$

and identify $q^2|\psi|^2/m$ as $1/\lambda^2$. Then substituting $\boldsymbol{j} = \nabla\times\nabla\times\boldsymbol{A}$, and fixing the gauge with $\nabla\cdot\boldsymbol{A} = 0$, we have

$$\lambda^2\nabla^2\boldsymbol{A} = \boldsymbol{A} - \frac{\hbar}{q}\nabla\theta \tag{3}$$

So, not quite the Helmholtz equation for the vector potential, but rather an inhomogeneous Helmholtz equation for the interior of the superconductor (and $\nabla^2\boldsymbol{A} = 0$ outside). Taking the curl of (2) gives the "usual" London equation $\nabla\times\boldsymbol{j} = \lambda^{-2}\boldsymbol{B}$ since $\nabla\times\nabla\theta = 0$. But equation (3) makes clear that there are not unique solutions to the London equation.

How is θ selected? We know that $\nabla\cdot\boldsymbol{j} = 0$, which combined with the Coulomb gauge condition yields $\nabla^2\theta = 0$ by taking the divergence of (2). Besides that, θ is not yet determined, and it need not even be single valued. When the fluxoid is nonzero it isn't single-valued, since it's the integral of $\frac{\hbar}{q}\nabla\theta$ along a closed curve which is the fluxoid.

What principle fixes θ? It is the minimization of the magnetic Gibbs free energy, when ψ has the form we assumed above. The G-L Helmholtz free energy density reduces to $\frac{m}{2q}|\boldsymbol{j}|^2$ (the electronic kinetic energy density) plus the magnetic energy density proportional to $|\nabla\times\boldsymbol{A}|^2$ (the former only inside the superconductor, the latter including the surrounding vacuum). Subtract $\boldsymbol{\mu}\cdot\boldsymbol{H}$

with $\boldsymbol{\mu}$ the sample magnetic moment and $\boldsymbol{H}$ the applied magnetic field from the Helmholtz free energy to get the magnetic Gibbs free energy.

This would seem to be pretty awful, since θ is a function of position everywhere in the superconductor, but since θ satisfies the Laplace equation it is sufficient to specify its value on the surface of the sample as well as along both sides of one or more "cut surfaces" to allow for non-single-valuedness. (To be more precise, θ should be specified just on one side of a cut surface, plus an integer which specifies the discontinuity $2\pi n$ across the surface.) The various cut surfaces and associated discontinuities determine families of solutions. "Vortices" require introducing a hole (radius equal to the coherence length) around what would otherwise be an internal edge of a cut surface.

The expression from London for the energy of a superconductor (just the kinetic energy?) may be helpful. But there seems to be no way to avoid a minimimization of the free energy, in which one must solve a Laplace equation followed by an inhomogeneous Helmholtz equation at each step. The advantage of minimizing the London free energy compared to the full G-L free energy is that the functional to be minimized involves θ on a surface rather than ψ throughout the volume. It's not clear to me if there is any way to cast this as a root-finding exercise to which Newton's method can be applied, since the functional dervatives of the free energy with respect to the values of θ on the surface don't have any explicit form. Formally there is the Green's function for the Laplace equation which allows one to express θ everywhere in the superconductor in terms of θ on the surface, and another Green's function for the equation that determines the vector potential, but I don't see at this time how those formal expressions allow one to do anything in practice.