Algebraic Electromagnetism

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***Abstract*—This paper introduces the concept of Algebraic Electromagnetism to solve the problem of finding stable spatial discretizations of the electromagnetic field for large scale, ultra-wide-band electromagnetic systems, composed of possibly nonlinear subsystems with memory and/or hysteresis effects. It is a thorough approach to exact discrete electromagnetism, given by an algebraic construction of general material operators that have the property that solving Maxwell's equations with these is exactly equivalent to solving a corresponding system of ordinary differential equations.**

I. Introduction

This paper contains the improved, condensed and streamlined mathematical framework developed to derive spatial discretizations for the simulation of ultra-wide-band high power electromagnetic fields coupling into complex infrastructure, like for example an high altitude electromagnetic pulse hitting a warship. Some of its prior development steps are documented in [1]. As required for that application this formulation has no implicit limits on the spectral width and scale of the problems to model. It further allows the systematic full bidirectional coupling of any number of subsystems, as well as nonlinear materials with memory and/or hysteresis effects. In addition it allows to model techniques like local mesh refinement as coupling between subsystems, by treating each level of refinement as subsystem. However, this generality comes at the cost of not leading to a concrete implementation of a single algorithm, but to a method to systematically construct a huge class of possibly viable algorithms. It is a tool to build tools, a meta-tool. However, well known algorithms, like the finite integration technique (FIT) [3] and some edge element based finite element methods (FEM) [6] can be derived straightforwardly as special cases of this approach.

Algebraic electromagnetism focuses on the creation of approximate material operators for Maxwell's equations by simple algebraic means combined with human insight into the problem to model, so that solving Maxwell's equations with these approximate materials is exactly equivalent to solving a system of ordinary differential equations (ode). Human insight is required since the mathematical tools used are too simple to allow a mathematical investigation of the quality of the chosen approximations.

However, because of its generality it is not validated by testing example implementations, but by mathematically proofing its central concepts.

II. Notations

The superscript † annotates a reflexive generalized inverse, which always exists [2], and is given by the properties:

$$\boldsymbol{Q}\boldsymbol{Q}^{\dagger}\boldsymbol{Q} = \boldsymbol{Q} \qquad\qquad \boldsymbol{Q}^{\dagger}\boldsymbol{Q}\boldsymbol{Q}^{\dagger} = \boldsymbol{Q}^{\dagger} \tag{1}$$

The Moore Penrose pseudoinverse would be a possible choice, but, with regard to computational efficiency or simplicity, probably not the best one.

For function composition and application we omit parentheses analog to the notation often used for the linear case. For example $\mathfrak{G}\boldsymbol{A}\eta x$ is short for $\mathfrak{G}(\boldsymbol{A}(\eta(x)))$.

III. Maxwell's Equations

For brevity we restrict this investigation to the curl equations of Maxwell's equations, since these are the essential part for describing the dynamics of electromagnetic fields. That is we investigate

$$\begin{array}{rcccccc} \mathbf{curl}\vec{\boldsymbol{H}} & = & \frac{\partial}{\partial t}\epsilon\vec{\boldsymbol{E}} & + & \kappa\vec{\boldsymbol{E}} & + & \vec{\boldsymbol{J}}_e \\ -\,\mathbf{curl}\vec{\boldsymbol{E}} & = & \frac{\partial}{\partial t}\mu\vec{\boldsymbol{H}} & + & \eta\vec{\boldsymbol{H}} & + & \vec{\boldsymbol{J}}_m \end{array}$$

where ϵ is the permittivity operator, κ is the conductivity operator, μ is the permeability operator, and η is a magnetic conductivity operator. These operators may be arbitrarily general, and for example include nonlinear and or memory/hysteresis effects. The extension of the proposed ideas to the other equations in Maxwell's equations is straight forward.

IV. Projections By Sampling And Interpolation

The central concept to construct the required approximate materials are algebraic projections that are constructed by careful selection of sampling and interpolation operators. For that, first a function space $\mathcal{H}$, for example a Banach or Hilbert space (neither is required), that contains the spatial aspects of fields, is selected. That is, for a physical field $\vec{\boldsymbol{F}}(\vec{x},t)$ we have for all time points t that $\vec{\boldsymbol{F}}(\cdot,t) \in \mathcal{H}$. Next, the following definitions are required:

A **spatial sampling operator** is a time-invariant linear operator $\mathfrak{S} : \mathcal{H} \to \mathbb{R}^m$ operating on spatial coordinates. Applied to a time-dependent field $\vec{\boldsymbol{F}}(\vec{x},t)$ it yields a signal vector $\boldsymbol{f}(t) := \mathfrak{S}\vec{\boldsymbol{F}}(\cdot,t)$.

A **spatial interpolation operator** is a time-invariant linear operator $\mathfrak{I} : \mathbb{R}^n \to \mathcal{H}^n$, where $\mathcal{H}^n \subset \mathcal{H}$ is a finite dimensional subspace. Applied to a time-dependent signal vector $\boldsymbol{g}(t)$ it yields a time dependent field $\vec{\boldsymbol{G}}(\vec{x},t) := (\mathfrak{I}\boldsymbol{g}(t))(\vec{x})$.

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A **Q-matrix** is the matrix $\boldsymbol{Q} := \mathcal{S}\mathcal{J}$.

For example, to construct a finite-element-like method, let $\vec{\psi}_j$ annotate the j-th of n FEM ansatz functions, $\mathcal{E}_i$ be the j-th of the m edges of a mesh and the sampling and interpolation operators be

$$\mathcal{S} := \vec{\boldsymbol{F}} \mapsto \sum_{i=1}^{m} \left(\int_{\mathcal{E}_i} \vec{\boldsymbol{F}} \cdot \vec{\psi}_i \mathrm{d}s \right) \boldsymbol{e}_i \quad \mathcal{J} := \boldsymbol{x} \mapsto \sum_{j=1}^{n} \boldsymbol{x}_j \vec{\psi}_j \quad (2)$$

with the corresponding Q-matrix $\boldsymbol{Q}_{ij} = \int_{\mathcal{E}_i} \vec{\psi}_i \cdot \vec{\psi}_j \mathrm{d}s$ where $\boldsymbol{e}_i$ is the i-th canonical basis vector of $\mathbb{R}^m$. It should be noted, that it is not required that $n = m$.

With this definitions given we can show that it is possible to construct algebraic projections, that is a mapping that is linear and idempotent, for every possible choice of sampling and interpolation operators:

Lemma 1 Projection of $(\mathcal{H}, \mathcal{S}, \mathcal{J}, \boldsymbol{Q}^\dagger)$:
Let $\mathcal{H}$ be a vector space and $(\mathcal{S},\mathcal{J})$ be spatial sampling and interpolation operators, and $\boldsymbol{Q}^\dagger$ a reflexive generalized inverse of $\boldsymbol{Q} = \mathcal{S}\mathcal{J}$ then

$$\mathcal{P} := \mathcal{J}\boldsymbol{Q}^\dagger\mathcal{S} \quad (3)$$

is a projection onto img$(\mathcal{J}\boldsymbol{Q}^\dagger)$.

Proof: First we proof that $\mathcal{P}$ is idempotent. We have $\mathcal{P}\mathcal{P} = \mathcal{J}\boldsymbol{Q}^\dagger\mathcal{S}\mathcal{J}\boldsymbol{Q}^\dagger\mathcal{S} = \mathcal{J}\boldsymbol{Q}^\dagger\boldsymbol{Q}\boldsymbol{Q}^\dagger\mathcal{S} = \mathcal{J}\boldsymbol{Q}^\dagger\mathcal{S} = \mathcal{P}$. Next we can proof that img$(\mathcal{J}\boldsymbol{Q}^\dagger)$ is an invariant subspace since $\mathcal{P}\mathcal{J}\boldsymbol{Q}^\dagger = \mathcal{J}\boldsymbol{Q}^\dagger\mathcal{S}\mathcal{J}\boldsymbol{Q}^\dagger = \mathcal{J}\boldsymbol{Q}^\dagger\boldsymbol{Q}\boldsymbol{Q}^\dagger = \mathcal{J}\boldsymbol{Q}^\dagger$. Finally we proof that img$(\mathcal{J}\boldsymbol{Q}^\dagger)$ is the complete invariant subspace. Assume there were a $\vec{\boldsymbol{F}}$ with $\mathcal{P}\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}$, but $\vec{\boldsymbol{F}} \neq$ img$(\mathcal{J}\boldsymbol{Q}^\dagger)$. Then we have $\vec{\boldsymbol{F}} = \mathcal{P}\vec{\boldsymbol{F}} = \mathcal{J}\boldsymbol{Q}^\dagger\mathcal{S}\vec{\boldsymbol{F}}$, which is a contradiction. ■

In contrast to the discussion in [1] all formal constraints regarding the selection of sampling and interpolation operators to construct a projection have been dropped here. Since reflexive generalized inverses always exist [2], a projection as constructed exists for every choice of spatial sampling and interpolation operator. This generality has the disadvantage that the space projected onto can be arbitrarily simple, in the worst case the vector space $\{0\}$. However, in a crude algebraic way in conjunction with human insight, that is with the problem-aware selection of spatial sampling and interpolation operators, we can state that $\vec{\boldsymbol{F}} \approx \mathcal{P}\vec{\boldsymbol{F}}$ for a field $\vec{\boldsymbol{F}}$ of a problem to model. To mathematically investigate the quality of this approximation some adequate form of metric must be selected for $\mathcal{H}$. This falls into the realm of functional analysis and is beyond the required mathematical tools for this approach, which requires linear algebra and a little vector analysis only.

V. Algebraic Material Approximations

From a given set of general material operators $\epsilon, \kappa, \mu, \eta$ we can construct algebraic approximates using projections in the sense discussed in the previous section. To do this we select a function space, spatial sampling and interpolation operators as well as a reflexive generalized inverse of the implied Q-matrix, for each field type, that is $(\mathcal{H}_e, \mathcal{S}_e, \mathcal{J}_e, \boldsymbol{Q}_e^\dagger)$ for the $\vec{\boldsymbol{E}}$ field , $(\mathcal{H}_h, \mathcal{S}_h, \mathcal{J}_h, \boldsymbol{Q}_h^\dagger)$ for the $\vec{\boldsymbol{H}}$ field, $(\mathcal{H}_d, \mathcal{S}_d, \mathcal{J}_d, \boldsymbol{Q}_d^\dagger)$ for the $\vec{\boldsymbol{D}}$ field and $(\mathcal{H}_b, \mathcal{S}_b, \mathcal{J}_b, \boldsymbol{Q}_b^\dagger)$ for the $\vec{\boldsymbol{B}}$ field, and simply apply the associated projections to the input and output of each material operator. That is:

$$\tilde{\epsilon} := \mathcal{P}_d \epsilon \mathcal{P}_e \qquad \tilde{\kappa} := \mathcal{P}_d \kappa \mathcal{P}_e \quad (4)$$

$$\tilde{\mu} := \mathcal{P}_b \mu \mathcal{P}_h \qquad \tilde{\eta} := \mathcal{P}_b \eta \mathcal{P}_h \ . \quad (5)$$

The spatially discrete core of these approximate operators in the sense that, for example, $\tilde{\epsilon} = \mathcal{J}_d \boldsymbol{M}_\epsilon \mathcal{S}_e$, is given by the operators

$$\boldsymbol{M}_\epsilon := \boldsymbol{Q}_d^\dagger \mathcal{S}_d \epsilon \mathcal{J}_e \boldsymbol{Q}_e^\dagger \qquad \boldsymbol{M}_\kappa := \boldsymbol{Q}_d^\dagger \mathcal{S}_d \kappa \mathcal{J}_e \boldsymbol{Q}_e^\dagger \quad (6)$$

$$\boldsymbol{M}_\mu := \boldsymbol{Q}_b^\dagger \mathcal{S}_b \mu \mathcal{J}_h \boldsymbol{Q}_h^\dagger \qquad \boldsymbol{M}_\eta := \boldsymbol{Q}_b^\dagger \mathcal{S}_b \eta \mathcal{J}_h \boldsymbol{Q}_h^\dagger \quad (7)$$

In the same spirit we define discretized currents for later use, as $\vec{\tilde{\boldsymbol{J}}}_{\boldsymbol{e}} := \mathcal{P}_d \vec{\boldsymbol{J}}_e$, $\vec{\tilde{\boldsymbol{J}}}_{\boldsymbol{m}} := \mathcal{P}_b \vec{\boldsymbol{J}}_m$, $\boldsymbol{j}_e := \boldsymbol{Q}_d^\dagger \mathcal{S}_d \vec{\boldsymbol{J}}_e$, and $\boldsymbol{j}_m := \boldsymbol{Q}_b^\dagger \mathcal{S}_b \vec{\boldsymbol{J}}_m$.

VI. Fundamental Results

With these preparations we can proof the fundamental theorem of algebraic electromagnetism, provided that the selection of the sampling and interpolation operators satisfies one crucial constraint: The existence of discrete curl operators.

Theorem 1 Fundamental Theorem :
With the abbreviations above let $\boldsymbol{C}_b^{\mathcal{J}}$ and $\boldsymbol{C}_d^{\mathcal{J}}$ be matrices, so-called discrete curl operators, with $\mathbf{curl}\mathcal{J}_h \boldsymbol{Q}_h^\dagger = \mathcal{J}_d \boldsymbol{Q}_d^\dagger \boldsymbol{C}_d^{\mathcal{J}}$ and $\mathbf{curl}\mathcal{J}_e \boldsymbol{Q}_e^\dagger = \mathcal{J}_b \boldsymbol{Q}_b^\dagger \boldsymbol{C}_b^{\mathcal{J}}$. Then the following holds:

If $(\boldsymbol{e}, \boldsymbol{h})$ is a solution of

$$\boldsymbol{Q}_d^\dagger \boldsymbol{C}_d^{\mathcal{J}} \boldsymbol{h} = \frac{\partial}{\partial t} \boldsymbol{M}_\epsilon \boldsymbol{e} + \boldsymbol{M}_\mu \boldsymbol{e} + \boldsymbol{j}_e \quad (8)$$

$$-\boldsymbol{Q}_b^\dagger \boldsymbol{C}_b^{\mathcal{J}} \boldsymbol{e} = \frac{\partial}{\partial t} \boldsymbol{M}_\mu \boldsymbol{h} + \boldsymbol{M}_\eta \boldsymbol{h} + \boldsymbol{j}_m \quad (9)$$

then $(\vec{\boldsymbol{E}}, \vec{\boldsymbol{H}})$ is a solution of

$$\mathbf{curl}\vec{\boldsymbol{H}} = \frac{\partial}{\partial t} \tilde{\epsilon} \vec{\boldsymbol{E}} + \tilde{\kappa} \vec{\boldsymbol{E}} + \vec{\tilde{\boldsymbol{J}}}_{\boldsymbol{e}} \quad (10)$$

$$-\mathbf{curl}\vec{\boldsymbol{E}} = \frac{\partial}{\partial t} \tilde{\mu} \vec{\boldsymbol{H}} + \tilde{\eta} \vec{\boldsymbol{H}} + \vec{\tilde{\boldsymbol{J}}}_{\boldsymbol{m}} \quad (11)$$

where $\vec{\boldsymbol{E}} := \mathcal{J}_e \boldsymbol{Q}_e^\dagger \boldsymbol{e}$ and $\vec{\boldsymbol{H}} := \mathcal{J}_h \boldsymbol{Q}_h^\dagger \boldsymbol{h}$ and $\mathcal{P}_e \vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}$ as well as $\mathcal{P}_h \vec{\boldsymbol{H}} = \vec{\boldsymbol{H}}$.

Proof: Let $\vec{\boldsymbol{E}} := \mathcal{J}_e \boldsymbol{Q}_e^\dagger \boldsymbol{e}$ and $\vec{\boldsymbol{H}} := \mathcal{J}_h \boldsymbol{Q}_h^\dagger \boldsymbol{h}$. Then we have $\mathcal{P}_e \vec{\boldsymbol{E}} = \mathcal{P}_e \mathcal{J}_e \boldsymbol{Q}_e^\dagger \boldsymbol{e} = \mathcal{J}_e \boldsymbol{Q}_e^\dagger \boldsymbol{e} = \vec{\boldsymbol{E}}$ and analog for the magnetic field. Further the existence of $\boldsymbol{C}_d^{\mathcal{J}}$ implies that img$(\mathbf{curl}\mathcal{J}_h \boldsymbol{Q}_h^\dagger) \subset$ img$(\mathcal{J}_d \boldsymbol{Q}_d^\dagger)$ which implies $\mathcal{P}_d \, \mathbf{curl}\mathcal{J}_h \boldsymbol{Q}_h^\dagger = \mathbf{curl}\mathcal{J}_h \boldsymbol{Q}_h^\dagger$. With that we have $\mathcal{P}_d \, \mathbf{curl}\vec{\boldsymbol{H}} = \mathcal{P}_d \, \mathbf{curl}\mathcal{J}_h \boldsymbol{Q}_h^\dagger \boldsymbol{h} = \mathbf{curl}\mathcal{J}_h \boldsymbol{Q}_h^\dagger \boldsymbol{h} = \mathbf{curl}\vec{\boldsymbol{H}}$ and analog for the electric field. With this the ode can be rewritten:

$$\begin{aligned}
& \boldsymbol{Q}_d^\dagger \boldsymbol{C}_d^{\mathcal{J}} \boldsymbol{h} = \frac{\partial}{\partial t} \boldsymbol{Q}_d^\dagger \mathcal{S}_d \epsilon \mathcal{J}_e \boldsymbol{Q}_e^\dagger \boldsymbol{e} + \boldsymbol{Q}_d^\dagger \mathcal{S}_d \kappa \mathcal{J}_e \boldsymbol{Q}_e^\dagger \boldsymbol{e} + \boldsymbol{Q}_d^\dagger \mathcal{S}_d \vec{\boldsymbol{J}}_e \\
\Rightarrow \quad & \mathcal{J}_d \boldsymbol{Q}_d^\dagger \boldsymbol{C}_d^{\mathcal{J}} \boldsymbol{h} = \frac{\partial}{\partial t} \mathcal{J}_d \boldsymbol{Q}_d^\dagger \mathcal{S}_d \mathcal{P}_d \epsilon \vec{\boldsymbol{E}} + \mathcal{J}_d \boldsymbol{Q}_d^\dagger \mathcal{S}_d \mathcal{P}_d \kappa \vec{\boldsymbol{E}} + \mathcal{J}_d \boldsymbol{Q}_d^\dagger \mathcal{S}_d \vec{\boldsymbol{J}}_e \\
\Rightarrow \quad & \mathbf{curl}\mathcal{J}_h \boldsymbol{Q}_h^\dagger \boldsymbol{h} = \frac{\partial}{\partial t} \mathcal{P}_d \mathcal{P}_d \epsilon \mathcal{P}_e \vec{\boldsymbol{E}} + \mathcal{P}_d \mathcal{P}_d \kappa \mathcal{P}_e \vec{\boldsymbol{E}} + \mathcal{P}_d \vec{\boldsymbol{J}}_e \\
\Rightarrow \quad & \mathbf{curl}\vec{\boldsymbol{H}} = \frac{\partial}{\partial t} \mathcal{P}_d \epsilon \mathcal{P}_e \vec{\boldsymbol{E}} + \mathcal{P}_d \kappa \mathcal{P}_e \vec{\boldsymbol{E}} + \mathcal{P}_d \vec{\boldsymbol{J}}_e
\end{aligned}$$

and analog for the other equation[1]. ∎

It should be noted that special care must be taken, to ensure that the discrete operators exist and are well-defined. This is, for example, not the case for common combinations of first order edge elements for $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{H}}$ on the same mesh.

The above theorem does not make any assumption about the uniqueness of the solution. Provided that using the approximated material operators still yields a well-posed pde problem, the number of possible solutions is equal to the number of possible boundary configurations that are expressible in the function spaces projected onto. The following discretization theorem is thus useful to construct reasonable initial and boundary conditions:

Theorem 2 Discretization Theorem :
With the abbreviations above and discrete curl operators as in the fundamental theorem, the following holds:

If $(\vec{\boldsymbol{E}}, \vec{\boldsymbol{H}})$ is a solution of

$$\mathbf{curl}\vec{\boldsymbol{H}} = \frac{\partial}{\partial t}\tilde{\epsilon}\vec{\boldsymbol{E}} + \tilde{\kappa}\vec{\boldsymbol{E}} + \vec{\tilde{\boldsymbol{J}}}_{\boldsymbol{e}} \tag{12}$$

$$-\mathbf{curl}\vec{\boldsymbol{E}} = \frac{\partial}{\partial t}\tilde{\mu}\vec{\boldsymbol{H}} + \tilde{\eta}\vec{\boldsymbol{H}} + \vec{\tilde{\boldsymbol{J}}}_{\boldsymbol{m}} \tag{13}$$

under the constraint $\mathcal{P}_e\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}$ and $\mathcal{P}_h\vec{\boldsymbol{H}} = \vec{\boldsymbol{H}}$ then $(\boldsymbol{e}, \boldsymbol{h})$ is a solution of

$$\boldsymbol{Q}_d^\dagger \boldsymbol{C}_d^{\mathcal{I}} \boldsymbol{h} = \frac{\partial}{\partial t}\boldsymbol{M}_\epsilon \boldsymbol{e} + \boldsymbol{M}_\mu \boldsymbol{e} + \boldsymbol{j}_e \tag{14}$$

$$-\boldsymbol{Q}_b^\dagger \boldsymbol{C}_b^{\mathcal{I}} \boldsymbol{e} = \frac{\partial}{\partial t}\boldsymbol{M}_\mu \boldsymbol{h} + \boldsymbol{M}_\eta \boldsymbol{h} + \boldsymbol{j}_m \tag{15}$$

where $\boldsymbol{e} := \mathcal{S}_e\vec{\boldsymbol{E}}$ and $\boldsymbol{h} := \mathcal{S}_h\vec{\boldsymbol{H}}$ and $\boldsymbol{e} = \boldsymbol{Q}_e\boldsymbol{Q}_e^\dagger\boldsymbol{e}$ as well as $\boldsymbol{h} = \boldsymbol{Q}_h\boldsymbol{Q}_h^\dagger\boldsymbol{h}$.

Proof: Let $\boldsymbol{e} := \mathcal{S}_e\vec{\boldsymbol{E}}$ and $\boldsymbol{h} := \mathcal{S}_h\vec{\boldsymbol{H}}$ then $\mathbf{curl}\vec{\boldsymbol{H}} = \mathbf{curl}\mathcal{P}_h\vec{\boldsymbol{H}} = \mathbf{curl}\mathcal{I}_h\boldsymbol{Q}_h^\dagger\mathcal{S}_h\vec{\boldsymbol{H}} = \mathcal{I}_d\boldsymbol{Q}_d^\dagger\boldsymbol{C}_d^{\mathcal{I}}\mathcal{S}_h\vec{\boldsymbol{H}} = \mathcal{I}_d\boldsymbol{Q}_d^\dagger\boldsymbol{C}_d^{\mathcal{I}}\boldsymbol{h}$ and thus, with the abbreviations inserted:

$$\mathbf{curl}\vec{\boldsymbol{H}} = \frac{\partial}{\partial t}\mathcal{P}_d\epsilon\mathcal{P}_e\vec{\boldsymbol{E}} + \mathcal{P}_d\kappa\mathcal{P}_e\vec{\boldsymbol{E}} + \mathcal{P}_d\boldsymbol{J}_e$$

$$\Rightarrow \quad \mathbf{curl}\vec{\boldsymbol{H}} = \frac{\partial}{\partial t}\mathcal{I}_d\boldsymbol{Q}_d^\dagger\boldsymbol{S}_d\epsilon\mathcal{I}_e\boldsymbol{Q}_e^\dagger\mathcal{S}_e\vec{\boldsymbol{E}} + \mathcal{P}_d\kappa\mathcal{I}_e\boldsymbol{Q}_e^\dagger\mathcal{S}_e\vec{\boldsymbol{E}} + \mathcal{P}_d\boldsymbol{J}_e$$

$$\Rightarrow \quad \mathcal{S}_d\,\mathbf{curl}\vec{\boldsymbol{H}} = \mathcal{S}_d\frac{\partial}{\partial t}\mathcal{I}_d\boldsymbol{Q}_d^\dagger\boldsymbol{S}_d\epsilon\mathcal{I}_e\boldsymbol{Q}_e^\dagger\boldsymbol{e} + \mathcal{S}_d\mathcal{P}_d\kappa\mathcal{I}_e\boldsymbol{Q}_e^\dagger\boldsymbol{e} + \boldsymbol{S}_d\mathcal{P}_d\boldsymbol{J}_e$$

$$\Rightarrow \quad \mathcal{S}_d\mathcal{I}_d\boldsymbol{Q}_d^\dagger\boldsymbol{C}_d^{\mathcal{I}}\boldsymbol{h} = \mathcal{S}_d\mathcal{I}_d\boldsymbol{Q}_d^\dagger\frac{\partial}{\partial t}\mathcal{S}_d\epsilon\mathcal{I}_e\boldsymbol{Q}_e^\dagger\boldsymbol{e} + \mathcal{S}_d\mathcal{I}_d\boldsymbol{Q}_d^\dagger\mathcal{S}_d\kappa\mathcal{I}_e\boldsymbol{Q}_e^\dagger\boldsymbol{e} + \mathcal{S}_d\mathcal{I}_d\boldsymbol{Q}_d^\dagger\mathcal{S}_d\boldsymbol{J}_e$$

$$\Rightarrow \quad \boldsymbol{Q}_d\boldsymbol{Q}_d^\dagger\boldsymbol{C}_d^{\mathcal{I}}\boldsymbol{h} = \boldsymbol{Q}_d\boldsymbol{Q}_d^\dagger\frac{\partial}{\partial t}\mathcal{S}_d\epsilon\mathcal{I}_e\boldsymbol{Q}_e^\dagger\boldsymbol{e} + \boldsymbol{Q}_d\boldsymbol{Q}_d^\dagger\mathcal{S}_d\kappa\mathcal{I}_e\boldsymbol{Q}_e^\dagger\boldsymbol{e} + \boldsymbol{Q}_d\boldsymbol{Q}_d^\dagger\boldsymbol{S}_d\boldsymbol{J}_e$$

$$\Rightarrow \quad \boldsymbol{Q}_d^\dagger\boldsymbol{C}_d^{\mathcal{I}}\boldsymbol{h} = \frac{\partial}{\partial t}\boldsymbol{Q}_d^\dagger\mathcal{S}_d\epsilon\mathcal{I}_e\boldsymbol{Q}_e^\dagger\boldsymbol{e} + \boldsymbol{Q}_d^\dagger\mathcal{S}_d\kappa\mathcal{I}_e\boldsymbol{Q}_e^\dagger\boldsymbol{e} + \boldsymbol{Q}_d^\dagger\boldsymbol{S}_d\boldsymbol{J}_e$$

and analog for the other equation. Further we have $\boldsymbol{e} = \mathcal{S}_e\vec{\boldsymbol{E}} = \mathcal{S}_e\mathcal{P}_e\vec{\boldsymbol{E}} = \boldsymbol{Q}_e\boldsymbol{Q}_e^\dagger\mathcal{S}_e\vec{\boldsymbol{E}} = \boldsymbol{Q}_e\boldsymbol{Q}_e^\dagger\boldsymbol{e}$ and analog for the other equation. ∎

This theorem implies, that the initial and boundary conditions can be constructed by first applying the projections and then sample the data. For example, if the unapproximated initial electric field is given by $\vec{\boldsymbol{E}}_0$ then the initial data for the approximated discrete set of equations $\boldsymbol{e}_0 := \vec{\boldsymbol{S}}_e\mathcal{P}_e\vec{\boldsymbol{E}}_0$. Constructing the required constraint for the boundary conditions is a bit more involved. It can be expressed by the restriction $(\mathcal{I}_e\boldsymbol{Q}_e^\dagger\boldsymbol{e} - \mathcal{P}_e\vec{\boldsymbol{E}}^\partial)\Big|_{\partial\Omega} = 0$ where $\vec{\boldsymbol{E}}^\partial$ is any field, not necessarily the solution, that satisfies the unapproximated boundary conditions, for the boundary of some domain Ω. Making that condition explicit requires further calculations that are beyond the scope of this paper.

[1]It should be noted, that $\frac{\partial}{\partial t}\mathcal{I} = \mathcal{I}\frac{\partial}{\partial t}$ for all possible choices of interpolation operators, since $\mathcal{I}$ is a linear mapping from a finite dimensional space to a finite dimensional space, and thus continuous. However $\frac{\partial}{\partial t}\mathcal{S} \neq \mathcal{S}\frac{\partial}{\partial t}$ for many possible sampling operator choices.

VII. Constructing Discrete Curl Operators

With the above results the central challenge to finding a valid discretization is to find sets of sampling and interpolation operators that have discrete curl operators as required. However, for integral based sampling operators, it is often quite easy to find discrete curl operators of the form $\mathcal{S}_y\,\mathbf{curl} = \mathfrak{C}_x^{\mathcal{S}}\mathcal{S}_x$ by careful application of Stokes' theorem.

Provided that the interpolation spaces are chosen carefully enough, so that an interpolation operator as required is guaranteed to exist, the following lemma allows usage of the sampling-based discrete curl operator instead:

Lemma 2 :
Assume there exist a matrix $\boldsymbol{C}_x^{\mathcal{S}}$ so that $\mathcal{S}_y\,\mathbf{curl} = \boldsymbol{C}_x^{\mathcal{S}}\mathcal{S}_x$ and a matrix $\boldsymbol{C}_y^{\mathcal{I}}$ so that $\mathbf{curl}\mathcal{I}_x\boldsymbol{Q}_x^\dagger = \mathcal{I}_y\boldsymbol{Q}_y^\dagger\boldsymbol{C}_y^{\mathcal{I}}$ then

$$\boldsymbol{C}_x^{\mathcal{S}}\boldsymbol{Q}_x\boldsymbol{Q}_x^\dagger = \boldsymbol{Q}_y\boldsymbol{Q}_y^\dagger\boldsymbol{C}_y^{\mathcal{I}} \tag{16}$$

Proof: By definition we have $\mathcal{S}_y\,\mathbf{curl}\mathcal{I}_x\boldsymbol{Q}_x^\dagger = \mathcal{S}_y\mathcal{I}_y\boldsymbol{Q}_y^\dagger\boldsymbol{C}_y^{\mathcal{I}} = \boldsymbol{Q}_y\boldsymbol{Q}_y^\dagger\boldsymbol{C}_y^{\mathcal{I}}$ and $\mathcal{S}_y\,\mathbf{curl}\mathcal{I}_x\boldsymbol{Q}_x^\dagger = \boldsymbol{C}_x^{\mathcal{S}}\mathcal{S}_x\mathcal{I}_x\boldsymbol{Q}_x^\dagger = \boldsymbol{C}_x^{\mathcal{S}}\boldsymbol{Q}_x\boldsymbol{Q}_x^\dagger$. ∎

In the case that $\boldsymbol{Q}^\dagger$ is a right-inverse of $\boldsymbol{Q}$, this yields the convenient result that $\boldsymbol{C}_x^{\mathcal{S}} = \boldsymbol{C}_y^{\mathcal{I}}$.

VIII. System Theoretic Perspective

The above methodology can be used to transform Maxwell's equations into a system state form, if there exist reflexive generalized inverses for the material operators. For example, the first curl equation can be written as

$$\frac{\partial}{\partial t}\vec{\boldsymbol{D}} = \mathbf{curl}(\mu^-\vec{\boldsymbol{B}}) - \kappa\epsilon^-\vec{\boldsymbol{D}} - \boldsymbol{J}_e\ .$$

The generalized reflexive operators are given be the properties $\mu\mu^-\mu = \mu$ and $\mu^-\mu\mu^- = \mu^-$ and analog for ϵ. Switching to approximate materials without abbreviations yields

$$\frac{\partial}{\partial t}\vec{\boldsymbol{D}} = \mathbf{curl}\mathcal{P}_h\mu^-\mathcal{P}_b\vec{\boldsymbol{B}} - \mathcal{P}_d\kappa\epsilon^-\mathcal{P}_d\vec{\boldsymbol{D}} - \mathcal{P}_d\boldsymbol{J}_e$$

which corresponds (by analog theorems and proofs) to a discrete form

$$\frac{\partial}{\partial t}\boldsymbol{d} = \boldsymbol{Q}_d\boldsymbol{Q}_d^\dagger\boldsymbol{C}_b^{\mathcal{I}}\mathcal{S}_h\mu^-\mathcal{I}_b\boldsymbol{Q}_b^\dagger\boldsymbol{b} - \boldsymbol{Q}_d\boldsymbol{Q}_d^\dagger\mathcal{S}_d\kappa\epsilon^-\mathcal{I}_d\boldsymbol{Q}_d^\dagger\boldsymbol{d} - \boldsymbol{j}_e\ . \tag{17}$$

This however is a variant of the Trotter-Kato theorem [5] using the projections introduced above. The details are thus omitted. The advantage of this formulation is, that a time discretization does not involve the inversion of a general discrete, possibly nonlinear, material operator, only the computation of a generalized reflexive inverse, which is constant.

IX. Coupled Systems and Local Refinements

So far, this approach entails, well known formulations like FIT [3] or edge-element-based finite element approaches as in [4], [6] or [7]. Its primary purpose is to model the spatial discretization of huge complex systems and the basic method to do this is simple. Imagine the model that requires discretization, for example a warship. Then for each part and/or subsystem of the model, select a known algebraic electromagnetism formulation of Maxwell's equations (for example a FIT discretization with an adequate resolution or a FEM specialized for cables) and cover the whole model with these. For each of these discretizations corresponding spatial sampling and interpolation operators along with discrete curls are defined. Let k be the number of these sub-discretizations and their operators be indexed by an upper script integer. Then the composed discretization for the complete model is defined by composed sampling and interpolation operators, which are given by

$$\mathfrak{S}_e := \begin{pmatrix} \mathfrak{S}_e^0 \\ \vdots \\ \mathfrak{S}_e^k \end{pmatrix} \qquad \mathfrak{I}_e := \begin{pmatrix} \mathfrak{I}_e^0 & \dots & \mathfrak{I}_e^k \end{pmatrix} \tag{18}$$

for the electric field and analog the other fields. The $\boldsymbol{Q}_e$ matrix is then given by

$$\boldsymbol{Q}_e = \begin{pmatrix} \boldsymbol{Q}_e^0 & \mathfrak{S}_e^0\mathfrak{I}_e^1 & \dots & \mathfrak{S}_e^0\mathfrak{I}_e^k \\ \mathfrak{S}_e^1\mathfrak{I}_e^0 & \boldsymbol{Q}_e^1 & \dots & \mathfrak{S}_e^1\mathfrak{I}_e^k \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{S}_e^k\mathfrak{I}_e^0 & \mathfrak{S}_e^k\mathfrak{I}_e^1 & \dots & \boldsymbol{Q}_e^k \end{pmatrix} \tag{19}$$

and the other $\boldsymbol{Q}$-matices and material operators have an analog layout. To see, that using the fundamental theorem with these operators implies the description of a fully coupled discretization, imagine, that, for example, all sub-discretizations are done with finite elements on nowhere overlapping domains. Then all off diagonal elements in all operators are zero and the resulting discrete equations describes k independent systems.

In the general case, the selected sub-discretizations could be anything, like mesh refinements and/or maybe completely overlapping different types of discretizations. However, the choice of each sub-discretization must be done with care and insight. In particular, the so created approximate material operators do need to actually approximate the initial problem in an analytical sense, not only in the crude algebraic sense required for the fundamental theorem. Furthermore discrete curl operators must exist for the composed operators.

Describing algorithms to select good approximations requires elaborate (functional) analysis and is beyond the scope of this discussion. A general construction of discrete curl operators is not trivial either. However, if all $\boldsymbol{Q}$-matrices are invertible, i.e. $\boldsymbol{Q}^\dagger = \boldsymbol{Q}^{-1}$, the construction of the discrete curl of the coupled system is easy, as for example

$$\begin{aligned} \mathbf{curl}\mathfrak{I}_h\boldsymbol{Q}_h^\dagger &= \mathbf{curl}\mathfrak{I}_h\overline{\boldsymbol{Q}}_h^\dagger\overline{\boldsymbol{Q}}_h\boldsymbol{Q}_h^\dagger = \mathfrak{I}_d\overline{\boldsymbol{Q}}_d^\dagger\overline{\boldsymbol{C}}_d^{\mathfrak{I}}\overline{\boldsymbol{Q}}_h\boldsymbol{Q}_h^\dagger \\ &= \mathfrak{I}_d\boldsymbol{Q}_d^\dagger\underbrace{\boldsymbol{Q}_d\overline{\boldsymbol{Q}}_d^\dagger\overline{\boldsymbol{C}}_d^{\mathfrak{I}}\overline{\boldsymbol{Q}}_h\boldsymbol{Q}_h^\dagger}_{=:\boldsymbol{C}_d^{\mathfrak{I}}} \end{aligned}$$

where $\overline{\boldsymbol{Q}}_h$ is a block diagonal matrix with the $\boldsymbol{Q}_h$-matrices of the sub-discretizations on the diagonal,$\overline{\boldsymbol{Q}}_d$ is a block diagonal matrix with the $\boldsymbol{Q}_d$-matrices of the sub-discretizations on the diagonal and $\overline{\boldsymbol{C}}_d^{\mathfrak{I}}$ is block diagonal matrix with the corresponding discrete curls of the sub-discretizations on the diagonal.

X. Stability

The solution of the discrete system of the fundamental theorem is equivalent to a solution of Maxwell's equations and stability can thus be investigated using the field energy and Poynting's theorem. Usually, a system is unstable, if and only if, its energy will go to infinity with time. For non-pathological cases this usually means that the discrete system is stable as long as the discretizations at the boundary do not cause an on average non-vanishing approximation-error-based energy influx into the system. With that in mind, doing stable numerical time integration "just" requires the selection of time integrators that have bounded errors in the energy over the time evolution. Symplectic methods for example [8] and [9] are good candidates as well as exponential integrators [10].

XI. Summary

We have proposed a construction for an encompassing class of algorithms for exact discrete electromagnetism, that uses simple mathematical means in conjunction with human insight, to model a virtually unrestricted class of large scale electromagnetic problems. However, it remains to be investigated what kind of not already established practically useful algorithms can be constructed with this methodology. In upcoming work we will use this methodology to investigate the coupling of algorithms like FIT and a FEM cable representation with this methodology.

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