The Quaternion Calculus

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COROLLARY 4. *Let* $a + b = 0$ *and* $-1 < a\tau < 1$. *Then the solution* x *of* (1) *and* (2) *approaches a limit as* $t \to \infty$:

$$x(t) \to \frac{1}{1 + b\tau} \left[\phi(0) + b \int_{-\tau}^{0} \phi(s)\, ds \right].$$

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THE QUATERNION CALCULUS

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1. Introduction. Most students, upon completing a first course in complex analysis, have glimpsed the immense power and elegance of the subject, particularly in treating two dimensional physical problems. The question then arises as to whether an analogous calculus exists for three dimensions. Lack of an appropriate hypercomplex number system seems to prevent any attempt along this line from going

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very far. Nevertheless, there exists an extensively developed four dimensional calculus, little known in this country, which was developed by R. Fueter [1] in the decade following 1935. A good bibliography to papers on the subject is found in [2]. Rose's work on quaternion velocity potentials for axisymmetric fluid flow appears to be the only paper on the subject to appear in English.

Fueter defines both right and left regular functions of a quaternion variable and develops the associated theory by producing analogues of both Cauchy Theorems, Liouville's Theorem, and Laurent series developments. In quaternion [4] Abelian functions having four periods are constructed and their properties studied.

Some of the essential aspects of Fueter's calculus will be discussed in this paper, using a somewhat different approach. The author has found that selected topics from this subject provide excellent optional topics for courses in complex variables, especially for the more enquiring students. Once acquainted with quaternions students often guess and prove theorems analogous to those which they have recently learned in the course. Science and engineering students gain greatly from such exposure as the use of quaternions provides them with a "concrete" example of an algebra more complicated than that of ordinary complex numbers. (Students never seem to view matrices in this manner.)

The compact quaternion form of Maxwell's equations which has been discovered repeatedly by undergraduates over the years is included along with several other topics of classroom interest.

2. Quaternions. Quaternions were invented in 1843 by the Irish mathematician William Rowen Hamilton after a lengthy struggle to extend the theory of complex numbers to three dimensions. An account of Hamilton's ultimate rejection of the commutative law of multiplication and the ensuing quaternion wars which raged afterwards is to be found in [5] and [6].

The algebra of quaternions has the distinction of being one of the three associative division algebras possible. Linear combinations are formed of the four units 1, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ using coefficients taken from the real number field. The quaternion thus formed, $w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ will be denoted $\mathbf{q}$ or $w + \mathbf{r}$, where $\mathbf{r}$ is the usual radius vector of three dimensions. The w component of $\mathbf{q}$ is called the **scalar part** of the quaternion and $\mathbf{r}$ is termed its **vector part.** Quaternion addition and scalar multiplication are defined in the usual manner as to constitute a linear algebra. The symbol 1 behaves as the ordinary number one in multiplication while the other units satisfy: $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$, $\mathbf{jk} = \mathbf{i} = -\mathbf{kj}$, $\mathbf{ki} = \mathbf{j} = -\mathbf{ik}$. Products of quaternions are formed using the above rules and the distributive law. Thus

$$(a + \mathbf{A})(b + \mathbf{B}) = ab - \mathbf{A} \cdot \mathbf{B} + a\mathbf{B} + b\mathbf{A} + \mathbf{A} \times \mathbf{B},$$

where the dot and cross indicate the usual three dimensional scalar and vector cross products respectively. For any quaternion $\mathbf{q} = w + \mathbf{r}$ there exists a **conjugate quaternion,** $\bar{\mathbf{q}} = w - \mathbf{r}$, satisfying $\mathbf{q}\bar{\mathbf{q}} = \bar{\mathbf{q}}\mathbf{q} = w^2 + x^2 + y^2 + z^2 = |\mathbf{q}|^2$. The non-negative quantity $|\mathbf{q}|$ is termed the **norm** of $\mathbf{q}$. The conjugation operation satisfies the equation $\overline{\mathbf{AB}} = \overline{\mathbf{B}}\,\overline{\mathbf{A}}$. Quaternion multiplication is not commutative but all other algebraic properties of the real and complex numbers hold.

The skew field of quaternions is isomorphic to a subset of 4 by 4 matrices under the mapping:

$$\mathbf{q} \rightarrow \begin{bmatrix} w & x & y & z \\ -x & w & -z & y \\ -y & z & w & -x \\ -z & -y & x & w \end{bmatrix}$$

or to a set of 2 by 2 complex matrices related to the Pauli spin matrices [8]. The topological properties of the quaternion group are discussed in [7]. We shall consider functions of a quaternion variable $\mathbf{q}$ which will be written $\mathbf{F}(\mathbf{q})$; such functions can be decomposed into a scalar and vector part which we shall write as $\mathbf{F}(\mathbf{q}) = \phi + \boldsymbol{\psi}$. The vector part of $\mathbf{F}$ will be expressed in component form as $\boldsymbol{\psi} = \psi_1\mathbf{i} + \psi_2\mathbf{j} + \psi_3\mathbf{k}$. Generally, the four components of $\mathbf{F}$ will be required to possess continuous partial derivatives up to a certain order, usually first or second, for our proofs to hold but we shall not belabor this point.

In the sequel, D is a simply connected domain of E^4 with subdomain σ having as its boundary the closed hypersurface $\partial\sigma$. Volume elements of σ are denoted dV while the (quaternion) oriented, outwardly directed surface elements of $\partial\sigma$ are denoted $d\mathbf{Q}$. Introducing the **quaternion gradient operator**

$$\Box = \frac{\partial}{\partial w} + \boldsymbol{\nabla} = \frac{\partial}{\partial w} + \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z},$$

we have the following useful result.

THEOREM 2.1. *Let* $\mathbf{F} = \phi + \boldsymbol{\psi}$ *be a function of the quaternion variable* $\mathbf{q} = w + \mathbf{r}$, *then*

$$\int_{\partial\sigma} (d\mathbf{Q})\mathbf{F} = \int_{\sigma} \Box\mathbf{F}\,dV. \tag{1}$$

Proof. Equation (1) is a quaternion form of the Gauss divergence theorem for four dimensions. Let $d\mathbf{Q} = dQ_0 + dQ_1\mathbf{i} + dQ_2\mathbf{j} + dQ_3\mathbf{k}$. If M is the matrix

$$\begin{bmatrix} \phi & \psi_1 & \psi_2 & \psi_3 \\ -\psi_1 & \phi & -\psi_3 & \psi_2 \\ -\psi_2 & \psi_3 & \phi & -\psi_1 \\ -\psi_3 & -\psi_2 & \psi_1 & \phi \end{bmatrix}$$

and $[d\mathbf{q}] = (dQ_0, dQ_1, dQ_2, dQ_3)$ is a row vector having the same components as the quaternion $d\mathbf{Q}$, then, the matrix product $[dq]M$ is a row vector with the same components as the quaternion product $d\mathbf{QF}$. By the Gauss Theorem

$$\int_{\partial\sigma} [d\mathbf{q}]M = \int_{\sigma} \operatorname{div}(M)\, dV,$$

where the matrix divergence of M is to be taken.

It is readily verified that $\operatorname{div}(M)$ is a row vector whose four components are the same as those of the quaternion

$$\begin{aligned} \Box\mathbf{F} &= \left(\frac{\partial}{\partial w} + \nabla\right)(\phi + \psi) \\ &= \frac{\partial \phi}{\partial w} + \nabla\phi + \frac{\partial \psi}{\partial w} - \nabla \cdot \psi + \nabla \times \psi, \end{aligned} \tag{2}$$

which establishes the result.

Similarly, we may demonstrate the alternate form of this result:

$$\int_{\partial\sigma} \mathbf{F}d\mathbf{Q} = \int_{\sigma} \mathbf{F}\,\Box\, dV,$$

where the gradient operator is understood to act on the function F to its left.

3. Regular quaternion functions. In seeking to construct a differential and integral calculus of quaternion functions the first step would seem to be definition of a derivative. A (right) quaternion derivative of the function $\mathbf{F}$ might be formed by requiring the limit

$$d\mathbf{F}/d\mathbf{q} = \lim(\mathbf{F}(\mathbf{q} + \Delta\mathbf{q}) - \mathbf{F}(\mathbf{q}))/\Delta\mathbf{q}$$

to exist as $\Delta\mathbf{q} \to 0$ and be independent of path for all increments $\Delta\mathbf{q}$. By considering four linearly independent increments Δw, $\Delta x\mathbf{i}$, $\Delta y\mathbf{j}$, $\Delta z\mathbf{k}$ one can derive a set of over-determined partial differential equations to be satisfied relating the components of $\mathbf{F}$ under such conditions. This approach leads to nothing productive since, even for the simple function $\mathbf{q}^2$, the ratio of $\Delta\mathbf{F}$ to $\Delta\mathbf{q}$ is not independent of $\Delta\mathbf{q}$, as was first observed by Hamilton [9]. The best one can do is to define scalar directional derivatives under the definition

$$d_{\mathbf{n}}\mathbf{F} = \lim(\mathbf{F}(\mathbf{q} + \varepsilon\mathbf{n}) - \mathbf{F}(\mathbf{q}))/\varepsilon$$

with ε real, $\varepsilon \to 0$, and $\mathbf{n}$ a unit quaternion in the desired direction. The vector Taylor series expansion theorem in any direction can be then obtained but no real calculus results since only directionally dependent quantities are encountered. These ideas were first put forward by Hamilton himself in his *Elements of Quaternions*, [9].

To avoid the above difficulties, a weaker condition than path independence of the differential ratio must be adopted. For a continuous function of the complex variable $z = x + iy$, the assertion

$$\int_C f(z)dz = 0$$

for every closed contour, C, in a domain of the z-plane is equivalent to the regularity of f in that domain (Morera's theorem). An alternate approach which suggests itself is the following.

A function $\mathbf{F}$ of the quaternion variable $\mathbf{q}$ is said to be **left regular** in D if

$$\int_{\partial\sigma} d\mathbf{Q}\mathbf{F} = 0 \tag{3}$$

for every closed hypersurface, $\partial\sigma$, in D.

A **right regular** function is defined in similar manner by requiring the vanishing of the integral $\int_{\partial\sigma} \mathbf{F}(\mathbf{q})d\mathbf{Q}$ under the same circumstances. The following properties of regular functions are easily established.

LEMMA 3.1. *If $\mathbf{F}(\mathbf{q})$ is right (left) regular in D and $\mathbf{q}_0$ is a constant quaternion, then $\mathbf{F}(\mathbf{q} - \mathbf{q}_0)$ is also right (left) regular in D.*

LEMMA 3.2. *If $\mathbf{F}$ is right regular in D and $\mathbf{G}$ is left regular in D then $\int_{\partial\sigma} \mathbf{F} d\mathbf{Q}\mathbf{G} = 0$ for any closed hypersurface, $\partial\sigma$, in D.*

THEOREM 3.1. *The function $\mathbf{F} = \phi + \boldsymbol{\psi}$ is left regular in D if and only if*

$$\frac{\partial\phi}{\partial w} = \nabla \cdot \boldsymbol{\psi} \tag{4}$$

$$\nabla\phi = -\frac{\partial\boldsymbol{\psi}}{\partial w} - \nabla \times \boldsymbol{\psi}. \tag{5}$$

Proof. This result follows directly from (1) and (2) since (4) and (5) are equivalent to the single quaternion equation $\Box\mathbf{F} = 0$.

The equations satisfied by the components of a right regular function are identical to (4) and (5) with the sign preceding the cross product in (5) changed to plus and the identical to the quaternion equation $\mathbf{F}\Box = 0$. If a function is simultaneously left and right regular or, briefly, regular, then $\nabla \times \boldsymbol{\psi} = 0$ and $\boldsymbol{\psi}$ is the gradiant of a scalar potential function, $\boldsymbol{\psi} = \nabla\Phi$. In this case, (4) and (5) are replaced by

$$\frac{\partial\phi}{\partial w} = \Delta\Phi, \nabla\left(\phi + \frac{\partial\Phi}{\partial w}\right) = 0,$$

where Δ denotes the three dimensional Laplacian operator in x, y and z. These last two equations have some application in the study of fluid flow [**3**].

COROLLARY 3.1.1. *Each component of a left or right regular function satisfies Laplace's equation in the four variables w, x, y and z.*

Proof. Taking the divergence of both sides of (5) we obtain

$$\Delta\phi = -\nabla\cdot\frac{\partial\psi}{\partial w} = -\frac{\partial}{\partial w}(\nabla\cdot\psi).$$

From (4)

$$\Delta\phi = -\frac{\partial}{\partial w}\left(\frac{\partial\phi}{\partial w}\right) = -\frac{\partial^2\phi}{\partial w^2}.$$

Thus,

$$\Delta\phi + \frac{\partial^2\phi}{\partial w^2} = \Delta_4\phi = 0$$

as required for the scalar part of $\mathbf{F}$. From (5) we derive

$$\begin{aligned}-\frac{\partial^2\psi}{\partial w^2} &= \frac{\partial}{\partial w}(\nabla\times\psi) + \frac{\partial}{\partial w}\nabla\phi \\ &= -\nabla\times(\nabla\phi + \nabla\times\psi) + \nabla(\nabla\cdot\psi) = \Delta\psi,\end{aligned}$$

so that

$$\Delta\psi + \frac{\partial^2\psi}{\partial w^2} = 0.$$

As might be expected, given a scalar function ϕ sufficient differentiability, a vector function ψ can be found so that $\phi + \psi$ constitutes a regular function of $\mathbf{q}$, [**1**]. Due to the well-known maximum principle for solutions of Laplace's equation we have the following analogue of Liouville's theorem.

COROLLARY 3.1.2. *The only quaternion function regular with bounded norm in all E^4 is a constant.*

The concept of regularity may be extended to include functions regular in $\bar{\mathbf{q}}$.

DEFINITION. A function $\mathbf{F} = \phi + \psi$ is said to be **left regular** in $\bar{\mathbf{q}}$ for a domain D provided

$$\int_{\partial\sigma} d\bar{\mathbf{Q}}\mathbf{F} = 0$$

for every closed hypersurface, $\partial\sigma$, in D.

Right regularity *in* $\bar{\mathbf{q}}$ is defined in the obvious manner through the vanishing of $\int_{\partial\sigma}\mathbf{F}d\bar{\mathbf{Q}}$ in D. Necessary and sufficient conditions for $\mathbf{F}$ to be left (right) regular in $\mathbf{q}$ are $\bar{\Box}\mathbf{F} = 0(\mathbf{F}\bar{\Box} = 0)$ in D where

$$\bar{\Box} = \frac{\partial}{\partial w} - \nabla.$$

Regular functions of $\dot{\mathbf{q}}$ also satisfy Corollaries 3.1.1 and 3.1.2. A function, $\mathbf{F}$, is left (right) regular in $\bar{\mathbf{q}}$ only if its conjugate, $\bar{\mathbf{F}}$, is right (left) regular in $\mathbf{q}$. Further, the only functions simultaneously regular in both $\mathbf{q}$ and $\bar{\mathbf{q}}$ are constants.

4. Generation of regular functions. Under the foregoing definitions, one hopes that a norm convergent quaternion power series of the form

$$\sum_{n=0}^{\infty} \mathbf{a}_n(\mathbf{q}-\mathbf{q}_0)^n,$$

where the $\mathbf{a}_n$ are constant quaternions, would be a regular function of $\mathbf{q}$. Thus, for every regular function of the complex variable z one could generate an analogous regular function of $\mathbf{q}$ by formally replacing z by $\mathbf{q}$ in the power series expansion. It is the perversity of the quaternion calculus that even simple powers of q are not regular functions. For example, the scalar part of $\mathbf{q}^2$ is $w^2 - \mathbf{r}\cdot\mathbf{r}$ which does not satisfy Laplace's equation and hence cannot be regular in $\mathbf{q}$. Nevertheless, there is a close connection between convergent quaternion power series and regular functions. We shall term quaternion functions defined by norm convergent power series to be **analytic functions** and shall restrict ourselves to power series with real coefficients.

The formal device of replacing z by $\mathbf{q}$ in a series expansion can be carried out in a more systematic manner. Let $f(z) = u(x,y) + iv(x,y)$ be a regular function of the complex variable $x+iy$ in some domain. We generate a quaternion function $\mathbf{F}$ from f by replacing x with w, y with $r = (x^2+y^2+z^2)^{\frac{1}{2}}$ and i with $\mathbf{e}_r = \mathbf{r}/r$ so that

$$\mathbf{F}(\mathbf{q}) = u(w,r) + \mathbf{e}_r v(w,r).$$

Since z^n is replaced by $\mathbf{q}^n$ this method yields the same result as the power series substitution. We inquire as to whether or not the function $\mathbf{F}$ thus generated is regular in q. Instead of attempting to verify (4) and (5), we shall check the necessary conditions $\Delta_4(u+\mathbf{e}_r v) = 0$. We find that

$$\frac{\partial^2}{\partial w^2}(u+\mathbf{e}_r v) = \frac{\partial^2 u}{\partial w^2} + \mathbf{e}_r \frac{\partial^2 v}{\partial w^2},$$

$$\Delta(u+\mathbf{e}_r v) = \frac{2}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + 2\left(\frac{1}{r}\frac{\partial v}{\partial r} - \frac{1}{r^2}v\right)\mathbf{e}_r + \frac{\partial^2 v}{\partial r^2}\mathbf{e}_r.$$

Since

$$\frac{\partial^2 u}{\partial w^2} + \frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 v}{\partial u^2} + \frac{\partial^2 v}{\partial r^2} = 0,$$

then

$$\Delta_4(u+\mathbf{e}_r v) = \frac{2}{r}\frac{\partial u}{\partial r} + 2\left(\frac{1}{r}\frac{\partial v}{\partial r} - \frac{1}{r^2}v\right)\mathbf{e}_r. \tag{6}$$

The only functions generated in this manner whose components satisfy Laplace's equation are constants or linear functions of q. Using $\partial u/\partial r = -\partial v/\partial w$, we may rewrite (6) as

$$\Delta_4(u + \mathbf{e}_r v) = 2\left[-\frac{\partial}{\partial w}\left(\frac{v}{r}\right) + \mathbf{e}_r \frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right].$$

Since the special variables x, y, z only occur in the combination r, this result appears to be a special case of the more general equation

$$(7) \qquad \Delta_4(u + \mathbf{e}_r v) = 2\left(-\frac{\partial}{\partial w} + \boldsymbol{\nabla}\right)\left(\frac{v}{r}\right).$$

Since

$$\Delta_4(\mathbf{u} + \mathbf{e}_r v) = \Box\,\bar{\Box}(u + \mathbf{e}_r v) = -2\,\bar{\Box}\left(\frac{v}{r}\right),$$

we deduce that

$$(8) \qquad \Box\,\mathbf{F} = -2\frac{v}{r},$$

as may be readily verified. The equation (8) holds only for functions $\mathbf{F}$, constructed in the above manner. If $\mathbf{F}$ is generated from a function regular in the complex variable $\bar{z} = x - iy$, the corresponding result obtained is

$$(9) \qquad \bar{\Box}\,\mathbf{F} = 2\frac{v}{r}.$$

Equations (8) and (9) yield the relation

$$(10) \qquad \bar{\Box}\,\mathbf{F} = 2\left(\frac{\partial \mathbf{F}}{\partial w} + \frac{v}{r}\right),$$

which may be applied if $\mathbf{F}$ is generated from a regular function of z.

The symmetry of the generating process shows that the generated function must be regular (both left and right) if it is either left or right regular and, therefore, must satisfy $\Box\,\mathbf{F} = 0$. Equation (8) shows that functions generated from regular functions are not regular; however,

$$\Delta_4\left(\frac{v}{r}\right) = \frac{1}{r}\left(\frac{\partial^2 v}{\partial w^2} + \frac{\partial^2 v}{\partial r^2}\right) = 0.$$

We have proved the following:

THEOREM 4.1. *If $\mathbf{F}$ is generated from a regular function f of z then the function $\Delta_4\mathbf{F}$ is a regular function of $\mathbf{q}$.*

COROLLARY 4.1.1. *The norm convergent series $\Sigma \mathbf{a}_n \Delta_4 \mathbf{q}^n$ is regular in $\mathbf{q}$.*

COROLLARY 4.1.2. *Each component of a function $\mathbf{F}$ generated as above satisfies the biharmonic equation $\Delta_4\Delta_4\mathbf{F} = 0$.*

COROLLARY 4.1.3. *Let v be harmonic in w and r. Then the quaternion function $\bar{\Box}(v/r)$ is regular in* $\mathbf{q}$.

THEOREM 4.2. *Let* $\mathbf{F}$ *be generated from the function f regular in z then*

$$\Delta_4\mathbf{F} = \frac{2}{r}\mathbf{e}_r\left(\frac{\partial\mathbf{F}}{\partial w} - \frac{v(w,r)}{r}\right). \tag{11}$$

Proof. Applying the operator $\bar{\Box}$ to both sides of (8), we find

$$\bar{\Box}\,\Box\mathbf{F} = \Delta_4\mathbf{F} = -2\bar{\Box}\left(\frac{v}{r}\right) = -\frac{2}{r}\left(\bar{\Box}v + \frac{v}{r}\mathbf{e}_r\right). \tag{12}$$

Since v may be written as $v(w,r) = \frac{1}{2}(\mathbf{e}_r\bar{\mathbf{F}} - \mathbf{e}_r\mathbf{F})$, we have

$$\bar{\Box}v = \tfrac{1}{2}\bar{\Box}(\mathbf{e}_r\bar{\mathbf{F}}) - \tfrac{1}{2}\bar{\Box}(\mathbf{e}_r\mathbf{F}). \tag{13}$$

The function $\mathbf{e}_r\bar{\mathbf{F}}$ is generated from the function $i\bar{f}$ which is regular in $\bar{z}$ while $\mathbf{e}_r\mathbf{F}$ is generated from if which is regular in z, so by (9) and (10), equation (13) becomes

$$\bar{\Box}v = \frac{1}{2}\left(2\frac{u}{r}\right) - \frac{1}{2}\left(2\mathbf{e}_r\frac{\partial\mathbf{F}}{\partial w} + \frac{u}{r}\right) = -\mathbf{e}_r\frac{\partial\mathbf{F}}{\partial w}.$$

Equation (11) now follows from (12). Fueter's two formulas for $\Delta_4\mathbf{q}^n$ and $\Delta_4\mathbf{q}^{-n}$ [**1**, p. 316] are special cases of (11).

5. The Cauchy-Fueter integral formula. Cauchy's integral formula expresses the value of a regular function at a point interior to a closed contour in terms of the integral of its values on the contour. An analogous but more complicated theorem holds for regular functions of $\mathbf{q}$. We shall need the following fundamental theorem.

THEOREM 5.1. *Let $\partial\sigma$ be a closed hypersurface in E^4 containing the point* $\mathbf{q}_0$, *then*

$$\int_{\partial\sigma}\Delta_4(\mathbf{q}-\mathbf{q}_0)^n d\mathbf{Q} = \begin{cases} 0 & n = 0, 1, \cdots \\ 8\pi^2 & n = -1 \\ 0 & n = -2, -3, \cdots \end{cases} \tag{14}$$

Proof. For n a non-negative integer $\Delta_4(\mathbf{q}-\mathbf{q}_0)^n$ is regular in E^4 and the result follows directly from the definition of regularity. If n is a negative integer the desired results can all be obtained from the case $n = -1$ by differentiation under the integral sign with respect to the scalar part of $\mathbf{q}_0$. In fact, if

$$\int_{\partial\sigma}\Delta_4(\mathbf{q}-\mathbf{q}_0)^{-1}d\mathbf{Q} = 8\pi^2, \tag{15}$$

then

$$\frac{\partial^k}{\partial w_0^k}\int_{\partial\sigma}\Delta_4(\mathbf{q}-\mathbf{q}_0)^{-1}d\mathbf{Q}=\int_{\partial\sigma}\Delta_4\frac{\partial^k}{\partial w_0^k}(\mathbf{q}-\mathbf{q}_0)^{-1}d\mathbf{Q}$$

$$=(k!)\int_{\partial\sigma}\Delta_4(\mathbf{q}-\mathbf{q}_0)^{-(1+k)}d\mathbf{Q}=0,$$

with $\mathbf{q}_0 = w_0 + \mathbf{r}_0$ and $k = 1,2,\cdots$.

All that remains is to prove (15). In view of Lemma 3.1 we need only to establish the case where $\mathbf{q}_0 = 0$ and $\partial\sigma$ is a hypersurface enclosing the point $\mathbf{q}_0 = 0$. Since $\Delta_4\mathbf{q}^{-1}$ is regular except at $\mathbf{q} = 0$,

$$\int_{\partial\sigma}\Delta_4\mathbf{q}^{-1}d\mathbf{Q}=-\int_{|q|=1}\Delta_4\mathbf{q}^{-1}d\mathbf{Q}.$$

Equation (7) can be used to find $\Delta_4\mathbf{q}^{-1}$. Because $v(w,r) = -r/\rho^2$ where $\rho^2 = w^2+r^2$, then $\Delta_4\mathbf{q}^{-1} = -(4/\rho^2)\mathbf{q}^{-1}$. The scalar surface element of a sphere having radius $|\mathbf{q}|$ in E^4 is $|\mathbf{q}|^3 dS$, where dS is the surface element of the corresponding unit sphere in E^4, [**10**, p. 677]. The oriented surface element for a sphere of radius $|\mathbf{q}|$ is therefore

$$d\mathbf{Q} = |\mathbf{q}|^2\mathbf{q}dS. \tag{16}$$

The integral in question becomes

$$-\int_{|q|=1}\Delta_4\mathbf{q}^{-1}d\mathbf{Q}=4\int_{|q|=1}\mathbf{q}^{-1}\mathbf{q}dS=8\pi^2,$$

since the surface area of the unit sphere in E^4 is $2\pi^2$, [**10**, p. 677].

The previous theorem leads one to expect that the functions $\Delta_4\mathbf{q}^n$ will play roughly the same role in the quaternion calculus that the functions z^n play in ordinary complex analysis. Given a function $\mathbf{F}$ defined by a Laurent type series

$$\mathbf{F}(\mathbf{q})=\sum_{n=-\infty}^{\infty}\mathbf{a}_n(\mathbf{q}-\mathbf{q}_0)^n$$

we deduce formally

$$\Delta_4\mathbf{F}(\mathbf{q})(\mathbf{q}-\mathbf{q}_0)^{-k}=\sum_{n=-\infty}^{\infty}\mathbf{a}_n\Delta_4(\mathbf{q}-\mathbf{q}_0)^{n-k},$$

from which we derive

$$\mathbf{a}_{k-1}=\mathbf{F}(\mathbf{q}_0)=\frac{1}{8\pi^2}\int_{\partial\sigma}\Delta_4\mathbf{F}(\mathbf{q})(\mathbf{q}-\mathbf{q}_0)^{-k}d\mathbf{Q}.$$

Of more interest is the following analogue of the Cauchy integral formula.

THEOREM 5.2. *Let* $\mathbf{F}$ *be a regular function of* $\mathbf{q}$ *in* D. *If* $\partial\sigma$ *is a hypersurface in in* D *containing the point* $\mathbf{q}_0$, *then*

$$\mathbf{F}(\mathbf{q}_0) = \frac{1}{8\pi^2}\int_{\partial\sigma} \mathbf{F}(\mathbf{q})d\mathbf{Q}\Delta_4(\mathbf{q}-\mathbf{q}_0)^{-1}.$$

Proof. For ε small enough, the hypersurface centered at $\mathbf{q}_0$ defined by $|\mathbf{q}-\mathbf{q}_0| = \varepsilon$ lies inside $\partial\sigma$. In the region between the surface of the ε-sphere and $\partial\sigma$ both $\mathbf{F}$ and $\Delta_4(\mathbf{q}-\mathbf{q}_0)^{-1}$ are regular so that, using Lemma 3.2 we can show that

$$\frac{1}{8\pi^2}\int_{\partial\sigma} \mathbf{F}(\mathbf{q})d\mathbf{Q}\Delta_4(\mathbf{q}-\mathbf{q}_0)^{-1} = \frac{1}{8\pi^2}\int_{|q-q_0|=\varepsilon} \mathbf{F}(\mathbf{q})d\mathbf{Q}\Delta_4(\mathbf{q}-\mathbf{q}_0)^{-1}.$$

The surface element for the last integral is found by replacing q with $q - q_0$ in (16); thus,

$$d\mathbf{Q} = |\mathbf{q}-\mathbf{q}_0|^2(\mathbf{q}-\mathbf{q}_0)dS.$$

The function $F(q)$ is to have sufficient differentiability so that

$$\mathbf{F}(\mathbf{q}) = \mathbf{F}(\mathbf{q}_0) + O(|\mathbf{q}-\mathbf{q}_0|),\ |\mathbf{q}-\mathbf{q}_0| \to 0.$$

The limit of the last integral as $\varepsilon \to 0$ is therefore found to be

$$\lim \frac{4}{8\pi^2}\int_{|q-q_0|=1} \mathbf{F}(\mathbf{q})\varepsilon^2(\mathbf{q}-\mathbf{q}_0)\varepsilon^{-2}(\mathbf{q}-\mathbf{q}_0)^{-1}dS$$

$$= \lim \frac{1}{2\pi^2}\int_{|q-q_0|=1} (\mathbf{F}(\mathbf{q}_0) + O(\varepsilon))dS = \mathbf{F}(\mathbf{q}_0)$$

as required.

It is essential in Theorem 5.2 that the terms in the integral be separated by the differential $d\mathbf{Q}$ since $\mathbf{F}\cdot\Delta_4(\mathbf{q}-\mathbf{q}_0)^{-1}$ is not generally regular even if $\mathbf{F}$ is. Many properties of regular functions such as the existence of series expansions, mean value theorems, etc., can be proved from (17) in much the same manner as is done in complex analysis. We give only one such example, the familiar Poisson integral formula for $n = 4$, [**10**, p. 265].

COROLLARY 5.2.1. *Let* $\mathbf{F} = \phi + \psi$ *be regular in* $\mathbf{q}$ *for* $|\mathbf{q}| \leqq \rho$ *and let* $\mathbf{q}_0 = w_0 + \boldsymbol{r}_0$ *be a point such that* $|\boldsymbol{q}_0| = R$, *where* $\rho > R$, *then*

$$\phi(\boldsymbol{q}_0) = \frac{\rho^2(\rho^2-R^2)}{2\pi^2}\int_{|q-q_0|=1} \frac{\phi(\boldsymbol{q})dS}{(\rho^2+R^2-2\rho R\cos(\theta))^2} \tag{18}$$

where $\cos(\theta)$ *is defined by*

$$\cos(\theta) = (ww_0 + \mathbf{r}\cdot\mathbf{r}_0)/|\mathbf{q}||\mathbf{q}_0|. \tag{19}$$

Proof. From (19)

$$\begin{aligned}|\mathbf{q}||\mathbf{q}_0|\cos(\theta) &= ww_0 + \boldsymbol{r}\cdot\boldsymbol{r}_0\\ &= |\mathbf{q}|^2 + |\mathbf{q}_0|^2 - |\mathbf{q}-\mathbf{q}_0|^2\end{aligned}$$

so that (18) becomes

$$\phi(\mathbf{q}_0) = \frac{\rho^2(\rho^2 - R^2)}{2\pi^2} \int_{|q-q_0|=1} \frac{\phi(\mathbf{q})dS}{|\mathbf{q}-\mathbf{q}_0|^4} \tag{20}$$

From Theorem 5.2,

$$\begin{aligned} \mathbf{F}(\mathbf{q}_0) &= \frac{1}{8\pi^2}\int_{|q-q_0|=\rho} \mathbf{F}(q)d\mathbf{Q}\Delta_4(\mathbf{q}-\mathbf{q}_0)^{-1} \\ &= \frac{1}{2\pi^2}\int_{|q-q_0|=1} \frac{\mathbf{F}(\mathbf{q})\rho^2\mathbf{q}dS(\mathbf{q}-\mathbf{q}_0)^{-1}}{|\mathbf{q}-\mathbf{q}_0|^2} \end{aligned} \tag{21}$$

If $\mathbf{q}_0^* = \rho^2\bar{\mathbf{q}}_0^{-1}$ then $|\mathbf{q}_0^*| > \rho^2/R > \rho$ and

$$\begin{aligned} O &= \frac{1}{8\pi^2}\int_{|q-q_0|=\rho} \mathbf{F}(\mathbf{q})d\mathbf{Q}(\mathbf{q}-\mathbf{q}_0)^{-1} \\ &= \frac{1}{2\pi^2}\int_{|q-q_0|=1}\left(\frac{R}{\rho}\right)^2 \frac{\rho^2\mathbf{q}dS\mathbf{q}^{-1}(\bar{\mathbf{q}}_0-\bar{\mathbf{q}})^{-1}\bar{\mathbf{q}}_0}{|\mathbf{q}-\mathbf{q}_0|^2}. \end{aligned} \tag{22}$$

Dividing out the constant $(R/\rho)^2$ from (22) and subtracting (21) from (22) we find

$$\begin{aligned} \mathbf{F}(\mathbf{q}_0) &= \frac{1}{2\pi^2}\int_{|q-q_0|=1} \frac{\mathbf{F}(\mathbf{q})\rho^2}{|\mathbf{q}-\mathbf{q}_0|^2}[\mathbf{q}(\mathbf{q}-\mathbf{q}_0)^{-1} + (\bar{\mathbf{q}}-\bar{\mathbf{q}}_0)^{-1}\bar{\mathbf{q}}_0]dS \\ &= \frac{\rho^2-R^2}{2\pi^2}\int_{|q-q_0|=1}\frac{\mathbf{F}(\mathbf{q})dS}{|\mathbf{q}-\mathbf{q}_0|^4} \end{aligned}$$

which proves the equation (20) and hence the theorem when the scalar parts of the last equation are equated.

6. Applications. Aside from older mechanics texts which sometimes treat rotations in the quaternion form, they are seldom encountered except with their cousins octernions and Clifford numbers in the factorization of relativistic energy equations, [**11**]. Instead of studying Laplace's equation in 4 variables, one generally wants to consider the wave operator,

$$\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}.$$

Formal replacement of w to ict changes one equation into the other. The equations for right regular quaternion functions then become

$$-\frac{i}{C}\frac{\partial\phi}{\partial t} = \nabla\cdot\psi, \tag{23}$$

$$\nabla\phi = \frac{i}{C}\frac{\partial\psi}{\partial t} + \nabla\times\psi. \tag{24}$$

The resemblance of (23) to a conservation equation suggests the further substitution $\phi = -i\lambda$ to obtain from (23) and (24)

$$\frac{1}{c}\frac{\partial\lambda}{\partial t} + \nabla \cdot \psi = 0, \tag{25}$$

$$\nabla\lambda = -\frac{1}{c}\frac{\partial\psi}{\partial t} + i\nabla \times \psi. \tag{26}$$

We expect λ to be real and ψ to have real components. Equations (25) and (26) describe a variety of physical systems. If we identify $\lambda = ce$ and $\psi = c^2\pi$ where c is the speed of light *in vacuo*, e and π are the relativistic energy and momentum densities, respectively, of the system under consideration, then, (25) and (26) break into the three equations

$$\frac{\partial e}{\partial t} + \nabla \cdot (c^2\pi) = 0$$

$$\frac{1}{c}\frac{\partial\pi}{\partial t} + \nabla\left(\frac{e}{c}\right) = 0$$

$$\nabla \times \pi = 0.$$

These three equations, the first of which is the conservation of mass-energy, constitute the basis of relativistic mechanics in the absence of electromagnetic forces, [**11**, p. 272]. Defining the relativistic momentum quaternion $\mathbf{P} = -ie + c\pi$ and the operator

$$\square^* = \frac{-i}{c}\frac{\partial}{\partial t} + \nabla$$

we have the following quaternion expression for these equations:

$$\square^*\mathbf{P} = 0.$$

Thus, we have proved the following result.

THEOREM 6.1. *In the absence of electromagnetic forces, the momentum quaternion,* $\mathbf{P}$, *is a (formal) regular function of the quaternion variable* $ict + r$.

Maxwell's equations

$$\nabla \cdot \mathbf{H} = 0, \quad \nabla \cdot \mathbf{E} = \rho$$

$$\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} = 0$$

$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} = \nabla \times \mathbf{H}$$

are likewise expressed in the simple quaternion form

$$\Box^*(\mathbf{E} + i\mathbf{H}) = -\rho + \frac{i}{c}\mathbf{J}. \tag{27}$$

If ϕ and $\mathbf{A}$ are the usual Hertzian scalar and vector potentials for $\mathbf{E}$ and $\mathbf{H}$, [**12**, p. 212], the electromagnetic field is derivable from the quaternion potential function $i\phi + \mathbf{A}$ through the equation

$$\bar{\Box}^*(i\phi + \mathbf{A}) = i(\mathbf{E} + i\mathbf{H}). \tag{28}$$

From (27) and (28) we obtain

$$\Box^*\bar{\Box}^*(i\phi + \mathbf{A}) = \left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \Delta\right)(i\phi + \mathbf{A}). \tag{29}$$

In component form (29) yield the equations relating the electromagnetic potential with the charge and current densities

$$\Box^2\phi = \rho, \quad \Box^2\mathbf{A} = \frac{1}{c}\mathbf{J},$$

where

$$\Box^2 = \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}.$$

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