Gravitation, Gauge Theories And Differential Geometry

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1. Introduction

Advances in mathematics and physics have often occurred together. The development of Newton's theory of mechanics and the simultaneous development of the techniques of calculus constitute a classic example of this phenomenon. However, as mathematics and physics have become increasingly specialized over the last several decades, a formidable language barrier has grown up between the two. It is thus remarkable that several recent developments in theoretical physics have made use of the ideas and results of modern mathematics and, in fact, have elicited the direct participation of a number of mathematicians. The time therefore seems ripe to attempt to break down the language barriers between physics and certain branches of mathematics and to re-establish interdisciplinary communication (see, for example, Robinson [1977]; Mayer [1977]).

The purpose of this article is to outline various mathematical ideas, methods, and results, primarily from differential geometry and topology, and to show where they can be applied to Yang–Mills gauge theories and Einstein's theory of gravitation.

We have several goals in mind. The first is to convey to physicists the bases for many mathematical concepts by using intuitive arguments while avoiding the detailed formality of most textbooks. Although a variety of mathematical theorems will be stated, we will generally give simple examples motivating the results instead of presenting abstract proofs.

Another goal is to list a wide variety of mathematical terminology and results in a format which allows easy reference. The reader then has the option of supplementing the descriptions given here by consulting standard mathematical references and articles such as those listed in the bibliography.

Finally, we intend this article to serve the dual purpose of acquainting mathematicians with some basic physical concepts which have mathematical ramifications; physical problems have often stimulated new directions in mathematical thought.

1.1. Gauge theories

By way of introduction to the main text, let us give a brief survey of how mathematicians and physicists noticed and began to work on certain problems of mutual interest. One crucial step was taken by Yang and Mills [1954] when they introduced the concept of a non-abelian gauge theory as a generalization of Maxwell's theory of electromagnetism. The Yang–Mills theory involves a self-interaction among gauge fields, which gives it a certain similarity to Einstein's theory of gravity (Utiyama [1956]). At about the same time, the mathematical theory of fiber bundles had reached the advanced stage described in Steenrod's book (Steenrod [1953]) but was generally unknown to the physics community. The fact that Yang–Mills theories and the affine geometry of principal fiber bundles are one and the same thing was eventually recognized by various authors as early as 1963 (Lubkin [1963]; Hermann [1970]; Trautman [1970]), but few of the implications were explored. The potential utility of the differential geometric methods of fiber bundles in gauge theories was pointed out to the bulk of the physics community by the paper of Wu and Yang [1975]. For example, Wu and Yang showed how the long-standing problem of the Dirac string for magnetic monopoles (Dirac [1931]) could be resolved by using overlapping coordinate patches with gauge potentials differing by a gauge transformation; for mathematicians, the necessity of using coordinate patches is a trivial consequence of the fact that non-trivial fiber bundles cannot be described by a single gauge potential defined over the whole coordinate space.

Almost simultaneously with the Wu–Yang paper, Belavin, Polyakov, Schwarz and Tyupkin [1975] discovered a remarkable finite-action solution of the Euclidean SU(2) Yang–Mills gauge theory, now generally known as the "instanton" or, sometimes, the "pseudoparticle". The instanton has self-dual or anti-dual field strength and carries a non-vanishing topological quantum number; from the mathematical point of view, this number is the integral of the second Chern class, which is an integer characterizing the topology of an SU(2) principal fiber bundle. 't Hooft [1976a, 1977] recognized that the instanton provided a mechanism for breaking the chiral $U(1)$ symmetry and solving the long-standing problem of the ninth axial current, together with a possible mechanism for the violation of CP symmetry and fermion number.

Another important consequence of the instanton is that it revealed the existence of a periodic structure of the Yang–Mills vacua (Jackiw and Rebbi [1976b]; Callan, Dashen and Gross [1976]). The instanton action gives the lowest order approximation to the quantum mechanical tunneling amplitude between these states. The true ground state of the theory becomes the coherent mixture of all such vacuum states.

Following the BPST instanton, which had topological index ± 1 for self-dual or anti-dual field strength, Witten [1977], Corrigan and Fairlie [1977], Wilczek [1977], 't Hooft [1976b] and Jackiw, Nohl and Rebbi [1977] found ways of constructing "multiple instanton" solutions characterized by (anti)-self-dual field strength and arbitrary integer topological index $\pm k$. At this point, the question was whether or not the parameter space of the k-instanton solution was exhausted by the $(5k+4)$ parameters of the Jackiw–Nohl–Rebbi solution (for $k=1$ and $k=2$, the number of parameters reduces to 5 and 13, respectively). The answer was provided both by mathematicians and physicists. Schwarz [1977] and Atiyah, Hitchin and Singer [1977] used the Atiyah–Singer index theorem [1968] to show that the parameter space was $(8k-3)$-dimensional. The same result was found by Jackiw and Rebbi [1977] and Brown, Carlitz and Lee [1977] using physicists' methods. It was also noted that the Dirac equation in the presence of the k-(anti)-instanton field would have k zero frequency modes of chirality ± 1. Physicists' arguments leading to this result were found by Coleman [1976], who integrated the local equation for the Adler–Bell–Jackiw anomaly (Adler [1969]; Bell and Jackiw [1969]). The number of parameters for self-dual Yang–Mills solutions for general Lie groups was worked out by Bernard, Christ, Guth and Weinberg [1977] and by Atiyah, Hitchin and Singer [1978]. It became apparent that the same class of problems was being attacked simultaneously by mathematicians and physicists and that a new basis existed for mutual discourse.

The attention of the mathematicians was now drawn to the problem of constructing Yang–Mills solutions with index k which exhausted the available free parameters for a given gauge group. The first concrete steps in this direction were taken by Ward [1977] and by Atiyah and Ward [1977] who adapted Penrose's twistor formalism to Yang–Mills theory to show how the problem could be solved. Atiyah, Hitchin, Drinfeld and Manin [1978] then used a somewhat different approach to give a construction of the most general solutions with self-dual field strength. The remarkable fact about this construction is that powerful tools of algebraic geometry made it possible to reduce the non-linear Yang–Mills differential equations to *linear algebraic equations*. The final link in the chain was provided by Bourguignon, Lawson and Simons [1979], who showed that, for compactified Euclidean space-time, *all* stable finite action solutions of the Euclidean Yang–Mills equations have self-dual field strength. Thus all stable finite action solutions of the Euclidean Yang–Mills equations are, in principle, known.

Finally, we note an interesting parallel development concerning the choice of gauge in a Yang–Mills theory. Gribov [1977, 1978] and Mandelstam [1977] noticed that the traditional Coulomb gauge choice does not determine a unique gauge potential; there exist an infinite number of gauge-equivalent fields all obeying the Coulomb gauge condition. The gauge-choice ambiguity can be avoided if the underlying space-time is a flat space (see, e.g., Coleman [1977]). However, Singer [1978a] showed that the Gribov ambiguity was incurable if he assumed a compactified Euclidean space-time manifold. Singer's calculation introduced powerful methods for examining the functional space of the path-integral using the differential geometry of infinite-dimensional fiber bundles; the exploitation of such techniques may eventually lead to a more satisfactory understanding of the path integral approach to the quantization of gauge theories.

1.2. Gravitation

The methods of differential geometry have always been essential in Einstein's theory of gravity (see, e.g., Trautman [1964]; Misner, Thorne and Wheeler [1973]). However, the discovery of the Yang–Mills instanton and its relevance to the path integral quantization procedure led to the hope that similar new approaches might be used in quantum gravity. The groundwork for the path integral approach to quantum gravity was laid by De Witt [1967a,b,c]. Prescriptions were subsequently developed for giving an appropriate boundary correction to the action (Gibbons and Hawking [1977]) and for avoiding the problem of negative gravitational action (Gibbons, Hawking and Perry [1978]).

The problem was then to determine which classical Euclidean Einstein solutions might be important in the gravitational path integral and which, if any, might play a physical role similar to that of the Yang–Mills instanton. The Euler–Poincaré characteristic χ and the signature τ were identified by Belavin and Burlankov [1976] and by Eguchi and Freund [1976] as gravitational analogs of the Yang–Mills topological index k. Eguchi and Freund went on to suggest the Fubini–Study metric on two-dimensional complex projective space as a possible gravitational instanton, but the absence of well-defined spinors on this manifold lessens its appeal. Hawking [1977] then proposed a Euclidean Taub–NUT metric with self-dual curvature as a gravitational instanton, and furthermore presented a new multiple-center solution reminiscent of the $k>1$ Yang–Mills solutions. However, Hawking's metrics had a distorted asymptotic behavior at infinity and, in fact, resembled magnetic monopoles more than instantons. It was also noted by Eguchi, Gilkey and Hanson [1978], by Römer and Schroer [1977] and by Pope [1978] that special care was required to compute the topological invariants for manifolds with boundary, such as those Hawking considered; here, the Atiyah–Patodi–Singer index theorem [1973, 1975a,b, 1976] with boundary corrections was applied to the study of physical questions arising in quantum gravity.

Starting from the idea that since the Yang–Mills instanton potential is asymptotically a pure gauge, a gravitational instanton should have an asymptotically flat metric, Eguchi and Hanson [1978] derived a new Euclidean Einstein metric with self-dual curvature which seems to be the closest gravitational analog of the Yang–Mills instanton. Although this metric is asymptotically flat, the manifold's boundary at infinity is not the three-sphere of ordinary Euclidean space, but is a three-sphere with opposite points identified (Belinskii et al. [1978]). Essentially this same metric was found independently by Calabi [1979] as the solution to an abstract mathematical problem. Gibbons and Hawking [1978] subsequently realized that this metric was the first of a class of metrics found by making a simple modification to Hawking's original multicenter metric (Hawking [1977]). The metrics in this new class are all *asymptotically locally Euclidean*: they are asymptotically flat, but the boundaries are three-spheres with points identified under the action of some discrete group. The manifolds described by these metrics are distinguished by the signature τ, which takes on all integer values and plays the role of the Yang–Mills index k. An explicit construction by Hawking and Pope [1978b] and an index theory calculation by Hanson and Römer [1978] show that the metrics with signature τ give a spin 3/2 anomaly 2τ, but do not contribute at all to the spin 1/2 axial anomaly as did the Yang–Mills index k. This distinction appears to have its origins in the existence of supersymmetry. Hitchin [1979] has now discussed further generalizations of these metrics and pointed out the existence of complex algebraic manifolds whose asymptotic boundaries are three-spheres identified under the action of all possible groups. He has also suggested that these manifolds may admit metrics with self-dual curvatures. These manifolds appear to exhaust the class of asymptotically locally Euclidean Einstein solutions with self-dual curvature, and thus provide a complete classification of this type of gravitational instanton. In principle, the Penrose construction can be used to find the self-dual metrics on each of these manifolds, so that the gravitational problem is nearing the same degree of completeness that exists for the Yang–Mills theory.

1.3. Outline

In the main body of this article, we will attempt to provide a physicist with the mathematical ideas underlying the sequence of discoveries just described. In addition, we wish to provide a mathematician with a feeling for some of the physical problems to which mathematical methods might apply. In section 2, we introduce the basic concepts of manifolds and differential forms, and then discuss the elements of de Rham cohomology. In section 3, we consider Riemannian geometry and explain the relationship between classical tensor analysis and modern differential geometric notation. Section 4 is devoted to an exposition of the geometry of fiber bundles. We introduce the concepts of connections and curvatures on fiber bundles in section 5 and give some physical examples. In section 6, we develop the theory of characteristic classes, which are the topological invariants used to classify fiber bundles. The Atiyah–Singer index theorem for manifolds without boundary is discussed in section 7. The generalization of the index theorem to manifolds with boundary is presented in section 8. Section 9 contains a brief discussion of Yang–Mills instantons and a list of mathematical results relevant to Yang–Mills theories, while section 10 treats gravitational instantons and gives a list of mathematical results associated with gravitation.

A number of basic mathematical formulas are collected in the appendices, while the bibliography contains suggestions for further reading.

Due to limitations of time and space, we have not been able to provide detailed treatments of a number of interesting mathematical and physical topics; brief discussions of some such topics are given in sections 9 and 10. We also note that many of the "mathematical" results we present have also been discovered by physicists using different methods of calculation; we have made no attempt to treat in detail these alternative derivations, but refer the reader instead to the bibliography for appropriate review articles elaborating on the conventional physical approaches.

2. Manifolds and differential forms

Manifolds are generalizations of the familiar ideas of lines, planes and their higher dimensional analogs. In this section, we introduce the basic concepts of manifolds, differential forms and de Rham cohomology (see, for instance, Flanders [1963]). Various examples are given to show how these tools can be used in physical problems.

2.1. *Definition of a manifold*

A real (complex) n-dimensional *manifold* M is a space which looks like a Euclidean space $\mathbb{R}^n(\mathbb{C}^n)$ around *each point.* More precisely, a manifold is defined by introducing a set of neighborhoods U_i covering M, where each U_i is a subspace of $\mathbb{R}^n(\mathbb{C}^n)$. Thus, a manifold is constructed by pasting together many pieces of $\mathbb{R}^n(\mathbb{C}^n)$.

In fig. 2.1, we show some examples of manifolds in one dimension: fig. 2.1a is a line segment of $\mathbb{R}^1$, the simplest possible manifold. Figure 2.1b shows the circle S^1; this is a non-trivial manifold which requires at least two neighborhoods for its construction. Figure 2.2 shows some spaces which are *not* manifolds: no neighborhood of a multiple junction looks like $\mathbb{R}^n$.

Examples 2.1

Let us discuss some of the typical n-dimensional manifolds which we will encounter.

1. $\mathbb{R}^n$ itself and $\mathbb{C}^n$ itself are the most trivial examples. These are noncompact manifolds.
2. The n-sphere S^n defined by the equation

$$\sum_{i=1}^{n+1} x_i^2 = c^2, \qquad c = \text{constant}. \tag{2.1}$$

The "zero-sphere" S^0 is just the two points $x = \pm c$. S^1 is a circle or ring and S^2 is a sphere like a balloon.

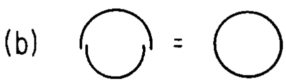

Fig. 2.1. One-dimensional manifolds: (a) is a line segment of $\mathbb{R}^1$. (b) shows the construction of S^1 using two neighborhoods.

Fig. 2.2. One-dimensional spaces which are not manifolds. The condition that the space looks locally like $\mathbb{R}^1$ is violated at the junctions.

3. *Projective spaces.* Complex projective space, $P_n(\mathbb{C})$, is the set of lines in $\mathbb{C}^{n+1}$ passing through the origin. If $z = (z_0, \ldots, z_n) \neq 0$, then z determines a complex line through the origin. Two points z, z' determine the same line if $z = cz'$ for some $c \neq 0$. We introduce the equivalence relation $z \simeq z'$ if there is a non-zero constant such that $z = cz'$; $P_n(\mathbb{C})$ is $\mathbb{C}^{n+1} - \{0\}$ modulo this identification.

We define neighborhoods U_k in $P_n(\mathbb{C})$ as the set of lines for which $z_k \neq 0$ (this condition is unchanged by replacing z by a scalar multiple). The ratio $z_i/z_k = cz_i/cz_k$ is well-defined on U_k. Let

$$\zeta_i^{(k)} = z_i/z_k \qquad \text{on } U_k$$

and $\zeta^{(k)} = (\zeta_0^{(k)}, \ldots, \zeta_n^{(k)})$ where we omit $\zeta_k^{(k)} = 1$. This gives a map from U_k to $\mathbb{C}^n$ and defines complex coordinates on U_k. We see that

$$\zeta_i^{(j)} = \frac{z_i}{z_k}\frac{z_k}{z_j} = \zeta_i^{(k)}(\zeta_j^{(k)})^{-1}$$

is well-defined on $U_j \cap U_k$. The $(n+1)$ z_i's are "homogeneous coordinates" on $P_n(\mathbb{C})$. Later we will show that the z_i's can be regarded as sections to a line bundle over $P_n(\mathbb{C})$. The n $\zeta_i^{(k)}$'s defined in each U_k are local "inhomogeneous coordinates".

Real projective space, $P_n(\mathbb{R})$, is the set of lines in $\mathbb{R}^{n+1}$ passing through the origin. It may also be regarded as the sphere S^n in $\mathbb{R}^{n+1}$ where we identify antipodal points. (Two unit vectors x, x' determine the same line in $\mathbb{R}^{n+1}$ if $x = \pm x'$.)

Remark: $P_1(\mathbb{C}) = S^2$ and $P_3(\mathbb{R}) = \mathrm{SO}(3)$.

4. *Group manifolds* are defined by the space of free parameters in the defining representation of a group. Several group manifolds are easily identifiable with simple topological manifolds:

(a) $\mathbb{Z}_2$ is the group of addition modulo 2, with elements (0, 1); $\mathbb{Z}_2$ may also be thought of as the group generated by multiplication by (-1), and thus has elements ± 1. This latter representation shows its equivalence to the zero-sphere,

$$\mathbb{Z}_2 = S^0.$$

(b) $U(1)$ is the group of multiplication by unimodular complex numbers, with elements $e^{i\theta}$. Since θ, $0 \leq \theta < 2\pi$ parametrizes a circle, we see that

$$U(1) = S^1.$$

(c) SU(2). A general SU(2) matrix can be written as

$$u = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix},$$

where $a = x_1 + ix_2$, $b = x_3 + ix_4$, bar denotes complex conjugation and

$$\det u = |a|^2 + |b|^2 = \sum_{i=1}^{4} x_i^2 = 1.$$

Hence we can identify the parameter space of SU(2) with the manifold of the three-sphere S^3

$$SU(2) = S^3.$$

(d) SO(3). It is well-known that SU(2) is the double-covering of SO(3), so that SO(3) can be written as the manifold

$$SO(3) = SU(2)/\mathbb{Z}_2 = P_3(\mathbb{R})$$

where $P_3(\mathbb{R})$ is three-dimensional real projective space.

Boundary of a manifold. The boundary of a line segment is the two end points; the boundary of a disc is a circle. Thus we may, in general, determine another manifold of dimension $(n-1)$ by taking the boundary of an n-manifold. We denote the boundary of a manifold M as ∂M.

Note: The boundary of a boundary is always empty, $\partial\partial M = \emptyset$.

Coordinate systems. One of the important themes in manifold theory is the idea of coordinate transformations relating adjacent neighborhoods. Suppose we have a covering $\{U_i\}$ of a manifold M and some coordinate system ϕ_i in each neighborhood U_i. ϕ_i is a mapping from U_i to $\mathbb{R}^n$. Then we need to know how to relate two coordinate systems ϕ_i and ϕ_j in the overlapping region $U_i \cap U_j$, the shaded area in fig. 2.3. The answer is the following: we take ϕ_i^{-1} to be the mapping *back* from $\mathbb{R}^n$, so the transformation from the coordinate system ϕ_i to the coordinate system ϕ_j is given by the transition function

$$\phi_{ji} = \phi_j \cdot \phi_i^{-1}.$$

This map is required to be C^∞ (have continuous partial derivatives of all orders). If the ϕ_{ji} are real analytic, then M is said to be a real analytic manifold. If the ϕ_{ji} are holomorphic (i.e., complex valued functions with complex power series), then M is said to be a complex manifold.

Examples 2.1 (Continued)

5. *Two sphere.* On S^2 we may choose just two neighborhoods, U_1 and U_2, which cover the northern and southern hemisphere, respectively, and one transition function ϕ_{12}, where

$$\phi_{12}(x, y) = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$$

in the intersection $U_1 \cap U_2$ of the neighborhoods. In terms of complex coordinates, $z = x + iy$,

$$\phi_{12}(z) = 1/z.$$

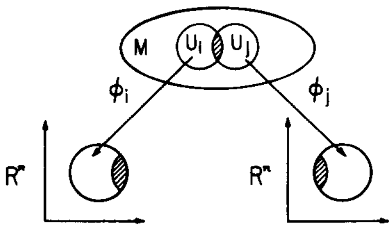

Fig. 2.3. Overlapping neighborhoods of a manifold M and their coordinate systems. ϕ_i is a map from U_i to an open subspace of $\mathbb{R}^n$.

Since this transition function is not only smooth but holomorphic, S^2 has the structure of a complex manifold (namely $P_1(\mathbb{C})$).

6. *Projective space.* $P_n(\mathbb{C})$ is also a complex manifold because its transition functions are holomorphic

$$\phi_{ji}(z_0, \ldots, z_n) = \left(\frac{z_i}{z_j} z_0, \ldots, \frac{z_i}{z_j} z_n\right)$$

on $U_i \cap U_j$ (where we recall $z_i \neq 0$, $z_j \neq 0$).

7. *Lie groups in general.* If A is a matrix, then $\exp(A) = I + A + \cdots + A^n/n! + \cdots$ converges to an invertible matrix. Let G be one of the groups: $\mathrm{GL}(k, \mathbb{C})$, $\mathrm{GL}(k, \mathbb{R})$, $U(k)$, $\mathrm{SU}(k)$, $O(k)$, $\mathrm{SO}(k)$ and let $\mathfrak{g}$ be the Lie-algebra of G. $\mathfrak{g}$ is a linear set of matrices and $\exp: \mathfrak{g} \to G$ is a diffeomorphism from a neighborhood of the origin in $\mathfrak{g}$ to the identity I in G. This defines a coordinate system near $I \in G$; we can define a coordinate system near *any* $g_0 \in G$ by mapping $\mathfrak{g}$ into $g_0 \exp \mathfrak{g}$. The transition functions are thus given by left multiplication in the group. G is a real analytic manifold.

2.2. *Tangent space and cotangent space*

One of the most important concepts used to study the properties of a manifold M is the tangent space $T_p(M)$ at a point $p \in M$. To develop the idea of the tangent space, let us first consider a curve $y = f(x)$ in a plane as shown in fig. 2.4. Consider a point $x = p + v$ very close to p; then we may expand $f(x)$ in a Taylor series, yielding

$$f(x = p + v) = f(p) + v\, \mathrm{d}f(x)/\mathrm{d}x\,|_{x=p} + \cdots. \tag{2.2}$$

The slope of the curve, $\mathrm{d}f/\mathrm{d}x$ at $x = p$, is represented in fig. 2.4. If we had an n-dimensional surface with coordinates x^i, there would be n different directions, so the second term in (2.2) would become

$$v^i\, \partial f(\boldsymbol{x})/\partial x^i\,|_{\boldsymbol{x}=\boldsymbol{p}}.$$

(Here we introduce the convention of implied summation on repeated indices.) We can thus begin to see that, regardless of the particular details of the manifold considered, the directional derivative

$$v^i\, \partial/\partial x^i\,|_{\boldsymbol{x}=\boldsymbol{p}} \tag{2.3}$$

has an intrinsic meaning. $\{\partial/\partial x^i\}$ at $x = p$ defines a basis for the tangent space of M at p. A collection of these directional derivatives at each point in M with smoothly varying coefficients $v^i(x)$ is called a *vector field.*

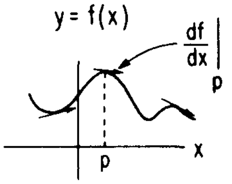

Fig. 2.4. Tangent to a curve $y = f(x)$.

The tangent space $T_p(M)$ is thus defined as the vector space spanned by the tangents at p to all curves passing through p in the manifold (see fig. 2.5). No matter how curved the manifold may be, $T_p(M)$ is always an n-dimensional vector space at each point p.

The tangent space occurs naturally in classical mechanics. We consider a Lagrangian $L(q^i(t), \dot{q}^i(t))$ and recall that t-derivatives can be defined using the implicit function rule

$$\mathrm{d}/\mathrm{d}t = \partial/\partial t + \dot{q}^i\,\partial/\partial q^i. \tag{2.4}$$

Comparison with eq. (2.3) shows that the second term in the above equation has the structure of a vector field. *Velocity space* in Lagrangian classical mechanics corresponds exactly to the tangent space of the configuration space: if M has coordinates $\{q^i\}$, then $T_q(M)$ has coordinates $\{\dot{q}^i\}$. Equation (2.4) shows that the operators $\{\partial/\partial q^i\}$ form a basis for $T_q(M)$.

The cotangent space $T_p^*(M)$ of a manifold at $p \in M$ is defined as the dual vector space to the tangent space $T_p(M)$. A *dual vector space* is defined as follows: given an n-dimensional vector space V with basis E_i, $i = 1, \ldots, n$, the basis e^i of the dual space V^* is determined by the inner product

$$\langle E_i, e^j\rangle = \delta^j_i.$$

When we take the basis vectors $E_i = \partial/\partial x^i$ for $T_p(M)$, we write the basis vectors for $T_p^*(M)$ as the differential line elements

$$e^i = \mathrm{d}x^i.$$

Thus the inner product is given by

$$\langle \partial/\partial x^i, \mathrm{d}x^j\rangle = \delta^j_i.$$

Now consider the vector field

$$V = v^i\,\partial/\partial x^i$$

and the covector field

$$U = u_i\,\mathrm{d}x^i.$$

Under general coordinate transformations $x \to x'(x)$, V and U are invariant, but since

$$\mathrm{d}x'^i = \frac{\partial x'^i}{\partial x^j}\mathrm{d}x^j \qquad \frac{\partial}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x^j},$$

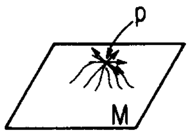

Fig. 2.5. Curves through a point p of M. The tangents to these curves span the tangent space $T_p(M)$.

the components v^i and u_i change according to

$$v'^i = v^j \, \partial x'^i / \partial x^j$$
$$u'_i = u_j \, \partial x^j / \partial x'^i .$$

(The invariance of V and U in fact is the origin of the transformation law for contravariant and covariant vectors, respectively.) Thus the inner product

$$\langle V, U \rangle = v^i u_i = v'^i u'_i$$

is invariant under general coordinate transformations.

The idea of the cotangent space also occurs in classical mechanics. Whereas tangent space corresponds to velocity space, cotangent space corresponds to *momentum space*. Here the basis vectors are given by the differential line elements $\mathrm{d}q^i$, so the cotangent vector fields are expressed as

$$p_i \, \mathrm{d}q^i$$

where we identify

$$p_i = \partial L(q^i, \dot{q}^i) / \partial \dot{q}^i .$$

Using the basis elements of $T_p(M)$ and $T_p^*(M)$, we may now extend the concept of a field to include tensor fields over M with l covariant and k contravariant indices, which we write

$$w^{(k)}_{(l)} = w^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_l} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_l} .$$

The tensor product symbol $\otimes$ implies no symmetrization or antisymmetrization of indices – each basis element is taken to act independently of the others.

2.3. *Differential forms*

A special class of tensor fields, the totally *antisymmetric covariant tensor fields* are useful for many practical calculations.

We begin by defining Cartan's *wedge product*, also known as the exterior product, as the antisymmetric tensor product of cotangent space basis elements

$$\begin{aligned} \mathrm{d}x \wedge \mathrm{d}y &= \tfrac{1}{2}(\mathrm{d}x \otimes \mathrm{d}y - \mathrm{d}y \otimes \mathrm{d}x) \\ &= -\mathrm{d}y \wedge \mathrm{d}x . \end{aligned}$$

Note that, by definition,

$$\mathrm{d}x \wedge \mathrm{d}x = 0 .$$

The differential line elements $\mathrm{d}x$ and $\mathrm{d}y$ are called *differential 1-forms* or 1-forms; thus the wedge product is a rule for constructing *2-forms* out of pairs of 1-forms. It is easy to show that the 2-form made in this way has the properties we expect of a differential *area* element. Suppose we change variables to $x'(x, y)$, $y'(x, y)$; then we find

$$\begin{aligned} \mathrm{d}x' \wedge \mathrm{d}y' &= \left(\frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y} - \frac{\partial x'}{\partial y}\frac{\partial y'}{\partial x}\right) \mathrm{d}x \wedge \mathrm{d}y \\ &= \text{Jacobian } (x', y'; x, y)\, \mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

Cartan's wedge product thus is designed to produce the required *signed* Jacobian every time we change variables. Let $\Lambda^p(x)$ be the set of anti-symmetric p-tensors at a point x. This is a vector space of dimension $n!/p!(n-p)!$. The $\Lambda^p(x)$ patch together to define a bundle over M as we shall discuss later. $C^\infty(\Lambda^p)$ is the space of smooth p-forms, represented by anti-symmetric tensors $f_{ij}\ldots(x)$ having p indices contracted with the wedge products of p differentials. The elements of $C^\infty(\Lambda^p)$ may then be written explicitly as follows:

$$\begin{array}{rll} C^\infty(\Lambda^0) = \{f(x)\} & \text{dimension} = 1 & \\ C^\infty(\Lambda^1) = \{f_i(x)\,\mathrm{d}x^i\} & \dim = n & \\ C^\infty(\Lambda^2) = \{f_{ij}(x)\,\mathrm{d}x^i \wedge \mathrm{d}x^j\} & \dim = n(n-1)/2! & \\ C^\infty(\Lambda^3) = \{f_{ijk}(x)\,\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k\} & \dim = n(n-1)(n-2)/3! & \\ \vdots & \vdots & \\ C^\infty(\Lambda^{n-1}) = \{f_{i_1\ldots i_{n-1}}\,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_{n-1}}\} & \dim = n & \\ C^\infty(\Lambda^n) = \{f_{i_1\ldots i_n}\,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_n}\} & \dim = 1. & \qquad (2.5) \end{array}$$

Several important properties emerge: First, we see that Λ^p and Λ^{n-p} have the same dimension as vector spaces. In particular, $C^\infty(\Lambda^n)$ is representable by a single function times the n-volume element. Furthermore, we deduce that $\Lambda^p = 0$ for $p > n$, since some differential would appear twice and be annihilated.

Now it is clear that the wedge product may be used to make $(p+q)$-forms out of a given p-form and a given q-form. But since one gets zero for $p + q > n$, the resulting forms always belong to the original set of spaces, which we write

$$\Lambda^* = \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2 \oplus \cdots \oplus \Lambda^n.$$

The space Λ^* of all possible antisymmetric covariant tensors therefore reproduces itself under the wedge product operation: Λ^* is a graded algebra called Cartan's *exterior algebra* of differential forms.
Remark: Let α_p be an element of Λ^p, β_q an element of Λ^q. Then

$$\alpha_p \wedge \beta_q = (-1)^{pq}\beta_q \wedge \alpha_p.$$

Hence odd forms anticommute and the wedge product of identical 1-forms will always vanish.

Exterior derivative: Another useful tool for manipulating differential forms is the exterior derivative operation, which takes p-forms into $(p+1)$-forms according to the rule

$$C^\infty(\Lambda^0) \xrightarrow{\mathrm{d}} C^\infty(\Lambda^1); \qquad \mathrm{d}(f(x)) = \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i$$

$$C^\infty(\Lambda^1) \xrightarrow{\mathrm{d}} C^\infty(\Lambda^2); \qquad \mathrm{d}(f_j(x)\,\mathrm{d}x^j) = \frac{\partial f_j}{\partial x^i}\,\mathrm{d}x^i \wedge \mathrm{d}x^j$$

$$C^\infty(\Lambda^2) \xrightarrow{\mathrm{d}} C^\infty(\Lambda^3); \qquad \mathrm{d}(f_{jk}(x)\,\mathrm{d}x^j \wedge \mathrm{d}x^k) = \frac{\partial f_{jk}}{\partial x^i}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$$

etc.

Here we have taken the convention that the new differential line element is always inserted *before* any previously existing wedge products. Note also that, to be precise, only the totally antisymmetric parts of the partial derivatives contribute.

An important property of the exterior derivative is that it gives *zero* when applied twice:

$$\mathrm{dd}\omega_p = 0.$$

This identity follows from the equality of mixed partial derivatives, as we can see from the following simple example:

$$C^\infty(\Lambda^0) \xrightarrow{\mathrm{d}} C^\infty(\Lambda^1) \xrightarrow{\mathrm{d}} C^\infty(\Lambda^2)$$

$$\mathrm{d}f = \partial_j f\,\mathrm{d}x^j$$

$$\mathrm{dd}f = \partial_i\,\partial_j f\,\mathrm{d}x^i \wedge \mathrm{d}x^j = \tfrac{1}{2}(\partial_i\,\partial_j f - \partial_j\,\partial_i f)\,\mathrm{d}x^i \wedge \mathrm{d}x^j = 0.$$

In vector notation, $\mathrm{dd}\omega_p = 0$ is equivalent to the familiar statements that

$$\mathrm{curl}\cdot\mathrm{grad}\, f = 0$$

$$\mathrm{div}\cdot\mathrm{curl}\, f = 0, \qquad \text{etc.}$$

We note also the rule for differentiating the wedge product of a p-form α_p and a q-form β_q:

$$\mathrm{d}(\alpha_p \wedge \beta_q) = \mathrm{d}\alpha_p \wedge \beta_q + (-1)^p\,\alpha_p \wedge \mathrm{d}\beta_q.$$

Note: The exterior derivative anticommutes with 1-forms.

Examples 2.3

1. Possible p-forms α_p in two-dimensional space are

$$\alpha_0 = f(x, y)$$

$$\alpha_1 = u(x, y)\,\mathrm{d}x + v(x, y)\,\mathrm{d}y$$

$$\alpha_2 = \phi(x, y)\,\mathrm{d}x \wedge \mathrm{d}y.$$

The exterior derivative of a line element gives the two-dimensional curl times the area:

$$\mathrm{d}(u(x,y)\,\mathrm{d}x + v(x,y)\,\mathrm{d}y) = (\partial_x v - \partial_y u)\,\mathrm{d}x \wedge \mathrm{d}y.$$

2. The three-space p-forms α_p are

$$\begin{aligned}
\alpha_0 &= f(x)\\
\alpha_1 &= v_1\,\mathrm{d}x^1 + v_2\,\mathrm{d}x^2 + v_3\,\mathrm{d}x^3\\
\alpha_2 &= w_1\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 + w_2\,\mathrm{d}x^3 \wedge \mathrm{d}x^1 + w_3\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\\
\alpha_3 &= \phi(x)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3.
\end{aligned}$$

We see that

$$\begin{aligned}
\alpha_1 \wedge \alpha_2 &= (v_1 w_1 + v_2 w_2 + v_3 w_3)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\\
\mathrm{d}\alpha_1 &= (\varepsilon_{ijk}\,\partial_j v_k)\tfrac{1}{2}\varepsilon_{ilm}\,\mathrm{d}x^l \wedge \mathrm{d}x^m\\
\mathrm{d}\alpha_2 &= (\partial_1 w_1 + \partial_2 w_2 + \partial_3 w_3)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3.
\end{aligned}$$

We thus recognize the usual operations of three-dimensional vector calculus.

2.4. Hodge star and the Laplacian

As we have seen from eq. (2.5) and the examples, the number of independent functions in $C^\infty(\Lambda^p)$ is the same as that in $C^\infty(\Lambda^{n-p})$: there exists a *duality* between the two spaces. We are thus motivated to introduce an operator, the *Hodge* $*$ or *duality transformation*, which transforms p-forms into $(n-p)$-forms; in a flat Euclidean space the operator is defined by

$$*(\mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_p}) = \frac{1}{(n-p)!}\varepsilon_{i_1 i_2 \ldots i_p i_{p+1} \ldots i_n}\,\mathrm{d}x^{i_{p+1}} \wedge \mathrm{d}x^{i_{p+2}} \wedge \cdots \wedge \mathrm{d}x^{i_n}.$$

Here $\varepsilon_{ijk\ldots}$ is the totally antisymmetric tensor in n-dimensions.

Note: Later, when we introduce a metric, we will have to be careful about raising and lowering indices and multiplying by $g^{1/2}$. For now, this point is inessential and will be postponed.

Repeating the $*$ operator on a p-form ω_p gives

$$**\,\omega_p = (-1)^{p(n-p)}\omega_p.$$

We note that for $p = n$,

$$\mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_n} = \varepsilon_{i_1 i_2 \ldots i_n}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n. \tag{2.6}$$

Inner product: Letting α_p and β_p be p-forms, we define the inner product as the integral

$$(\alpha_p, \beta_p) = \int_M \alpha_p \wedge * \beta_p.$$

For general p-forms α_p, β_p with coefficient functions $f_{ijk\dots}$ and $g_{ijk\dots}$, it is easy to show that

$$(\alpha_p, \beta_p) = p! \int_M f_{ijk\dots} g_{ijk\dots} \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n.$$

The inner product has the further property that

$$(\alpha_p, \beta_p) = (\beta_p, \alpha_p)$$

because of the identity

$$\alpha_p \wedge * \beta_p = \beta_p \wedge * \alpha_p$$

which follows from (2.6).

Adjoint of exterior derivative: Examining the inner product $(\alpha_p, \mathrm{d}\beta_{p-1})$ and integrating by parts, we find

$$(\alpha_p, \mathrm{d}\beta_{p-1}) = (\delta\alpha_p, \beta_{p-1}),$$

where the adjoint of d is

$$\delta = (-1)^{np+n+1} * \mathrm{d} *.$$

Note that for n *even* and all p,

$$\delta = -* \mathrm{d} *,$$

while for n odd,

$$\delta = (-1)^p * \mathrm{d} *.$$

(*Remark*: Additional factors of (-1) occur for spaces with negative signature.) δ *reduces* the degree of a differential form by one unit, whereas d increases the degree:

$$\mathrm{d}\colon C^\infty(\Lambda^p) \to C^\infty(\Lambda^{p+1})$$
$$\delta\colon C^\infty(\Lambda^p) \to C^\infty(\Lambda^{p-1}).$$

Like d, δ acting on forms produces conventional tensor calculus operations – for example, with $n = 3$ and $p = 1$, we find

$$\delta(\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x}) = -*(\boldsymbol{\nabla} \cdot \boldsymbol{v}) \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 = -\boldsymbol{\nabla} \cdot \boldsymbol{v}.$$

We note that, like d, δ gives *zero* when repeated:

$$\delta\delta\omega_p = 0.$$

Laplacian: The Laplacian on a manifold can be constructed once d and δ are known (this would, in general, require knowledge of a metric, but we will continue to use a flat metric for the time being). The Laplacian is

$$\Delta = (\mathrm{d} + \delta)^2 = \mathrm{d}\delta + \delta\mathrm{d}. \tag{2.7}$$

We sometimes add a subscript to d and δ to remind ourselves what kind of form we are acting on. Thus we may write the Laplacian on p-forms as

$$\Delta\omega_p = \mathrm{d}_{p-1}\,\delta_p\omega_p + \delta_{p+1}\,\mathrm{d}_p\,\omega_p.$$

The Laplacian clearly takes p-forms back into p-forms,

$$\Delta\colon C^\infty(\Lambda^p) \to C^\infty(\Lambda^p).$$

For example, on 1-forms, we find

$$\Delta(\boldsymbol{v}\cdot \mathrm{d}\boldsymbol{x}) = -\frac{\partial^2 \boldsymbol{v}}{\partial x^k\,\partial x^k}\cdot \mathrm{d}\boldsymbol{x}.$$

Thus Δ is called a *positive* operator because its Fourier transform introduces a factor of i^2 which cancels the minus sign. An elegant way of proving the positivity of the Laplacian follows from taking the inner product of the two p-forms ω_p and $\Delta\omega_p$. Using (2.7) we find that, provided there are no boundary terms,

$$\begin{aligned}(\omega_p, \Delta\omega_p) &= (\omega_p, \mathrm{d}\delta\omega_p) + (\omega_p, \delta\,\mathrm{d}\omega_p)\\ &= (\delta\omega_p, \delta\omega_p) + (\mathrm{d}\omega_p, \mathrm{d}\omega_p),\end{aligned}$$

which is necessarily ≥ 0. As a corollary, we see that for sufficiently well-behaved forms, ω_p is *harmonic*, that is

$$\Delta\omega_p = 0,$$

if and only if ω_p *is closed*,

$$\mathrm{d}\omega_p = 0$$

and *co-closed*,

$$\delta\omega_p = 0.$$

A p-form ω_p which can be written globally as the exterior derivative of some $(p-1)$-form α_{p-1},

$$\omega_p = \mathrm{d}\alpha_{p-1},$$

is called an *exact* p-form. Similarly, a p-form ω_p which can be expressed globally as

$$\omega_p = \delta\alpha_{p+1}$$

is called a *co-exact* p-form.

Hodge's theorem: Hodge [1952] has shown that if M is a compact manifold without boundary, any p-form ω_p can be uniquely decomposed as a sum of exact, co-exact and harmonic forms,

$$\omega_p = \mathrm{d}\alpha_{p-1} + \delta\beta_{p+1} + \gamma_p$$

where γ_p is a *harmonic* p-form. For many applications, the essential properties of ω_p lie entirely in the harmonic piece γ_p.

Stokes' theorem: If M is a p-dimensional manifold with a non-empty boundary ∂M, then Stokes' theorem says that for any $(p-1)$-form ω_{p-1},

$$\int_M \mathrm{d}\omega_{p-1} = \int_{\partial M} \omega_{p-1}.$$

If ∂M has several parts, the right-hand side is an *oriented* sum. For $p = 1$, where M is a line segment from a to b, we find the fundamental theorem of calculus,

$$\int_a^b \mathrm{d}f(x) = f(b) - f(a).$$

For $p = 2$, we find

$$\int_{\text{surface}} \mathrm{d}(\boldsymbol{A} \cdot \mathrm{d}\boldsymbol{x}) = \oint_{\text{line}} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{x}.$$

In 3 dimensions, where we may make the identification

$$\mathrm{d}(\boldsymbol{A} \cdot \mathrm{d}\boldsymbol{x}) = \tfrac{1}{2}(\partial_i A_j - \partial_j A_i)\, \mathrm{d}x^i \wedge \mathrm{d}x^j \equiv \tfrac{1}{2}\varepsilon_{ijk} B_k\, \mathrm{d}x^i \wedge \mathrm{d}x^j,$$

we recognize the formula for the magnetic flux going through a surface,

$$\int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \oint \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{x}.$$

For $p = 3$, we examine the 2-form

$$\omega = \tfrac{1}{2}\varepsilon_{ijk}\, E_k\, \mathrm{d}x^i \wedge \mathrm{d}x^j$$

obeying

$$\mathrm{d}\omega = \nabla \cdot \boldsymbol{E}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 .$$

Then Stokes' theorem becomes

$$\int \nabla \cdot \boldsymbol{E}\, \mathrm{d}^3x = \int_{\text{volume}} \mathrm{d}\omega = \int_{\text{surface}} \omega = \int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S}$$

and we recognize Gauss' law.

Examples 2.4

1. *Two-dimensions* ($n = 2$):
   Basis of Λ^*: $(1, \mathrm{d}x, \mathrm{d}y, \mathrm{d}x \wedge \mathrm{d}y)$
   Hodge $*$: $*(1, \mathrm{d}x, \mathrm{d}y, \mathrm{d}x \wedge \mathrm{d}y) = (\mathrm{d}x \wedge \mathrm{d}y, \mathrm{d}y, -\mathrm{d}x, 1)$
   δ operation:

   $$\delta f(x, y) = 0$$

   $$\delta(u\, \mathrm{d}x + v\, \mathrm{d}y) = -(\partial_x u + \partial_y v)$$

   $$\delta\phi\, \mathrm{d}x \wedge \mathrm{d}y = -\partial_x\phi\, \mathrm{d}y + \partial_y\phi\, \mathrm{d}x$$

   Laplacian: acting on, for instance, 0-forms,

   $$\Delta f = -(\partial_x^2 f + \partial_y^2 f).$$

2. *Euclidean Maxwell's equation* ($\mu = 1, 2, 3, 4;\ i = 1, 2, 3$)

   Gauge potential: $A = A_\mu(x)\, \mathrm{d}x^\mu$
   Gauge transform: $A' = A + \mathrm{d}\Lambda(x)$
   Field strength: $F = \mathrm{d}A = \mathrm{d}A'$
   (gauge invariant $= \tfrac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$
   due to $\mathrm{dd}\Lambda = 0$) $= \tfrac{1}{2} F_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$

   E and B: $F = E_i\, \mathrm{d}x^i \wedge \mathrm{d}x^4 + \tfrac{1}{2} B_i \varepsilon_{ijk}\, \mathrm{d}x^j \wedge \mathrm{d}x^k$

   $$*F = \tfrac{1}{2} E_i \varepsilon_{ijk}\, \mathrm{d}x^j \wedge \mathrm{d}x^k + B_i\, \mathrm{d}x^i \wedge \mathrm{d}x^4$$

   duality: $F \leftrightarrow *F,\ E \leftrightarrow B$

Euler eqn. = inhomogeneous eqns: $\delta F = j$

$$\delta F = -\nabla \cdot \boldsymbol{E}\, \mathrm{d}x^4 + (\partial_4 \boldsymbol{E} + \nabla \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{x}$$

$$j = j_\mu\, \mathrm{d}x^\mu = \boldsymbol{j} \cdot \mathrm{d}\boldsymbol{x} + j_4\, \mathrm{d}x^4$$

Bianchi identity = homogeneous eqns: $\mathrm{d}F = \mathrm{d}\mathrm{d}A = 0$

$$\mathrm{d}F = \nabla \cdot \boldsymbol{B}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \tfrac{1}{2}(\partial_4 \boldsymbol{B} + \nabla \times \boldsymbol{E})_i \varepsilon_{ijk}\, \mathrm{d}x^j \wedge \mathrm{d}x^k \wedge \mathrm{d}x^4 = 0.$$

Note: If $j = 0$, then $\mathrm{d}F = \delta F = 0$, so F is *harmonic*, $\Delta F = 0$.

3. *Dirac magnetic monopole* (Dirac [1931]). In order to describe a magnetic charge, we introduce two coordinate patches $U_\pm$ covering the $z > -\varepsilon$ and the $z < +\varepsilon$ regions of $\mathbf{R}^3 - \{0\}$, with overlap region $U_+ \cap U_-$ effectively equal to the x–y plane at $z = 0$ minus the origin. The gauge potentials which are well-defined in these respective regions are taken as

$$A_\pm = \frac{1}{2r} \frac{1}{z \pm r} (x\, \mathrm{d}y - y\, \mathrm{d}x) = \tfrac{1}{2}(\pm 1 - \cos\theta)\, \mathrm{d}\phi$$

where $r^2 = x^2 + y^2 + z^2$. A_+ and A_- have the Dirac string singularity at $\theta = \pi$ and $\theta = 0$, respectively. Note that A_+ and A_- are related by a gauge transformation:

$$A_+ = A_- + \mathrm{d} \tan^{-1}(y/x) = A_- + \mathrm{d}\phi.$$

In the overlap region $\theta = \pi/2$, $r > 0$, both potentials are regular. The field is given by $F = \mathrm{d}A_\pm$ in $U_\pm$, so

$$F = \frac{1}{2r^3}(x\, \mathrm{d}y \wedge \mathrm{d}z + y\, \mathrm{d}z \wedge \mathrm{d}x + z\, \mathrm{d}x \wedge \mathrm{d}y)$$

or

$$\boldsymbol{B} = \boldsymbol{x}/2r^3.$$

Remark: *Dirac strings*. In the modern approach to the magnetic monopole, $A_\pm$ are defined only in their respective coordinate patches $U_\pm$. In Dirac's formulation of the monopole, coordinate patches were not used and $A_\pm$ were used over all of $\mathbf{R}^3$. This led to the appearance of fictitious "string singularities" on the $\pm z$ axis.

2.5. *Introduction to homology and cohomology*

We conclude this section with a brief treatment of the concepts of homology and de Rham cohomology, which form a crucial link between the topological aspects of manifolds and their differentiable structure.

Homology: Homology is used to distinguish topologically inequivalent manifolds. For a treatment more mathematically precise than the one given here, see Greenberg [1967] or Spanier [1966].

Let M be a smooth connected manifold. A *p-chain* a_p is a formal sum of the form $a_p = \Sigma_i c_i N_i$ where the N_i are smooth p-dimensional oriented submanifolds of M. If the coefficients c_i are real (complex), then a_p is a real (complex) chain; if the coefficients c_i are integers, a_p is an integral chain; if the coefficients $c_i \in \mathbb{Z}_2 = \{0,1\}$, then a_p is a $\mathbb{Z}_2$ chain. There are other coefficients which could be considered, but these are the only ones we shall be interested in.

Let ∂ denote the operation of taking the oriented boundary. We define $\partial a_p = \Sigma_i c_i \, \partial N_i$ to be a $(p-1)$-chain. Let $Z_p = \{a_p\colon \partial a_p = \emptyset\}$ be the set of *cycles* (i.e., p-chains with no boundaries) and let $B_p = \{\partial a_{p+1}\}$ be the set of *boundaries* (i.e., those chains which can be written as $a_p = \partial a_{p+1}$ for some a_{p+1}). Since the boundary of a boundary is always empty, $\partial\partial a_p = \emptyset$, B_p is a subset of Z_p.

We define the *simplicial homology* of M by

$$H_p = Z_p/B_p.$$

H_p is the set of equivalence classes of cycles $z_p \in Z_p$ which differ only by boundaries; that is $z'_p \simeq z_p$ provided that $z'_p = z_p + \partial a_{p+1}$. We can think of representative cycles in H_p as manifolds patched together to "surround" a hole; we ignore cycles which can be "filled in".

We may choose different coefficient groups to define $H_p(M;\mathbb{R})$, $H_p(M;\mathbb{C})$, $H_p(M;\mathbb{Z})$, or $H_p(M;\mathbb{Z}_2)$. There are simple relations $H_p(M;\mathbb{R}) = H_p(M;\mathbb{Z}) \otimes \mathbb{R}$ and $H_p(M;\mathbb{C}) = H_p(M;\mathbb{R}) \otimes \mathbb{C} = H_p(M;\mathbb{Z}) \otimes \mathbb{C}$. In other words, modulo finite groups (i.e., *torsion*), $H_p(M;\mathbb{R})$, $H_p(M;\mathbb{Z})$, and $H_p(M;\mathbb{C})$ are essentially the same.

The integral homology is fundamental. We can regard any integral cycle as real by embedding $\mathbb{Z}$ in $\mathbb{R}$. We can reduce any integral cycle mod 2 to get a $\mathbb{Z}_2$ cycle. The *universal coefficient theorem* gives a formula for the homology with $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Z}_2$ coefficients in terms of the integral homology. In particular, real homology is obtained from integral homology by replacing all the "$\mathbb{Z}$" factors by "$\mathbb{R}$" and by throwing away any torsion subgroups.

It is clear that $H_p(M;G) = 0$ for $p > \dim(M)$. If M is connected, $H_0(M;G) = G$. If M is orientable, then $H_n(M;G) = G$. If G is a field, then we have Poincaré duality, $H_p(M;G) = H_{n-p}(M;G)$, for orientable M ($G = \mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_2$ but not $\mathbb{Z}$).

Examples

1. *Torus*. We illustrate the computation of homology for the torus T^2. In fig. 2.6, the curves a and b belong to the same homology class because they bound a two-dimensional strip σ (shown as a shaded area).

$$\partial\sigma = a - b.$$

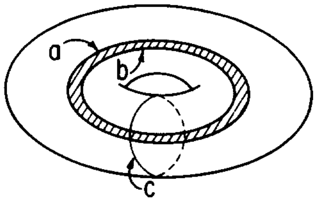

Fig. 2.6. Homology classes of the torus. a and b, which bound the shaded area, are homologous. a and c are not.

Curves a and c however do *not* belong to the same homology class. The homology groups of the torus, $M = T^2$, are

$$H_0(M;\mathbb{R}) = \mathbb{R}$$
$$H_1(M;\mathbb{R}) = \mathbb{R} \oplus \mathbb{R}$$
$$H_2(M;\mathbb{R}) = \mathbb{R}.$$

The generators of H_1 are given by the two curves a and c.

2. *Torsion and homology of* $P_3(\mathbb{R}) = \mathrm{SO}(3)$.

The concept of torsion and the effect of different coefficient groups can be illustrated by examining $M = P_3(\mathbb{R}) = \mathrm{SO}(3)$. Let ρ map S^3 to $P_3(\mathbb{R})$ by antipodal identification of the points of S^3.

Let S^2 be the equator of S^3, let S^1 be the equator of S^2, and let D^k be the upper hemisphere of S^k. Then $\rho(D^k)$ is a k-chain on $P_3(\mathbb{R})$ and

$\partial\rho(D^3) = 0$	(this is a cycle and generates H_3 with any coefficients)
$\partial\rho(D^2) = 2\rho(D^1)$	(this is a cycle in $\mathbb{Z}_2$ but not in $\mathbb{R}$ or $\mathbb{Z}$)
$\partial\rho(D^1) = 0$	(this is a cycle. Over $\mathbb{R}$ we have $\rho(D^1) = \partial\frac{1}{2}\rho(D^2)$ so this is a boundary. It is not a boundary over $\mathbb{Z}$ or $\mathbb{Z}_2$ and generates H_1 for these groups).

In $\mathbb{Z}_2$ homology $\rho(D^k)$ gives the generators of $H_k(P_3(\mathbb{R}); \mathbb{Z}_2)$. The homology groups of $P_3(\mathbb{R})$ can be shown to be the following:

$$\begin{array}{lll}
H_0(M;\mathbb{Z}) = \mathbb{Z} & H_0(M;\mathbb{R}) = \mathbb{R} & H_0(M;\mathbb{Z}_2) = \mathbb{Z}_2 \\
H_1(M;\mathbb{Z}) = \mathbb{Z}_2 & H_1(M;\mathbb{R}) = 0 & H_1(M;\mathbb{Z}_2) = \mathbb{Z}_2 \\
H_2(M;\mathbb{Z}) = 0 & H_2(M;\mathbb{R}) = 0 & H_2(M;\mathbb{Z}_2) = \mathbb{Z}_2 \\
H_3(M;\mathbb{Z}) = \mathbb{Z} & H_3(M;\mathbb{R}) = \mathbb{R} & H_3(M;\mathbb{Z}_2) = \mathbb{Z}_2.
\end{array}$$

These groups are different because of the existence of torsion.

de Rham cohomology

If G is a field ($G = \mathbb{R}, \mathbb{C}, \mathbb{Z}_2$), the homology group $H_p(M; G)$ is a vector space over G. We define the cohomology group $H^p(M; G)$ to be the dual vector space to $H_p(M, G)$. (The definition of $H^p(M;\mathbb{Z})$ is slightly more complicated and we shall omit it.) The remarkable fact is that $H^p(M;\mathbb{R})$ or $H^p(M;\mathbb{C})$ may be understood using differential forms. We define the *de Rham cohomology groups* $H^p_{\mathrm{DR}}(M;\mathbb{R})$ as follows: recall that a p-form ω_p is closed if $\mathrm{d}\omega_p = 0$ and exact if $\omega_p = \mathrm{d}\alpha_{p-1}$. Let

$Z^p_{\mathrm{DR}} = \{\omega_p : \mathrm{d}\omega_p = 0\}$	(the *closed* forms)
$B^p_{\mathrm{DR}} = \{\omega_p : \omega_p = \mathrm{d}\alpha_{p-1}\}$	(the *exact* forms)
$H^p_{\mathrm{DR}}(M;\mathbb{R}) = Z^p/B^p$	(closed modulo exact forms).

The de Rham cohomology is the set of equivalence classes of closed forms which differ only by exact forms; that is

$$\omega_p \simeq \omega'_p$$

if $\omega_p = \omega'_p + \mathrm{d}\alpha_{p-1}$ for some α_{p-1}.

Remark: The space H^0 is special because there are no (-1)-forms, and thus no 0-forms can be expressed as exterior derivatives. Since the exterior derivative of a constant is zero,

$$H^0 = \{\text{space of constant functions}\}$$

and

$$\dim(H^0) = \text{number of connected pieces of the manifold.}$$

Poincaré lemma: The de Rham cohomology of Euclidean space $\mathbb{R}^n$ is trivial,

$$\dim H^p(\mathbb{R}^n) = 0 \qquad p > 0$$

$$(\dim H^0(\mathbb{R}^n) = 1),$$

since any closed form can be expressed as the exterior derivative of a lower form in $\mathbb{R}^n$. For example, in $\mathbf{R}^3$, any closed 1-form can be expressed as the gradient of a scalar function,

$$\nabla \times \boldsymbol{A} = 0 \rightarrow \boldsymbol{A} = \nabla\varphi .$$

Therefore any closed form can be expressed as an exact form in any local $\mathbf{R}^n$ coordinate patch of the manifold. *Non-trivial de Rham cohomology therefore occurs only when the local coordinate neighborhoods are patched together in a globally non-trivial way.*

de Rham's theorem: The inner product of a cycle $c_p \in Z_p$ and a closed form $\omega_p \in Z^p_{\mathrm{DR}}$ is defined as

$$\pi(c_p, \omega_p) = \int_{c_p} \omega_p$$

where $\pi(c, \omega) \in \mathbb{R}$ is called a *period*. We note that by Stokes' theorem, when $c_p \in Z_p$ and $\omega_p \in Z^p_{\mathrm{DR}}$, then

$$\int_{c_p} \omega_p + \mathrm{d}\alpha_{p-1} = \int_{c_p} \omega_p + \int_{\partial c_p} \alpha_{p-1} = \int_{c_p} \omega_p ,$$

and

$$\int_{c_p + \partial a_{p+1}} \omega_p = \int_{c_p} \omega_p + \int_{a_{p+1}} \mathrm{d}\omega_p = \int_{c_p} \omega_p .$$

This pairing is thus independent of the choice of the representatives of the equivalence classes and defines a map

$$\pi: H_p(M;\mathbf{R})\otimes H^p_{\mathrm{DR}}(M;\mathbf{R})\to\mathbf{R}.$$

de Rham has proven the following fundamental theorems when M is a compact manifold without boundary:

Let $\{c_i\}$, $i=1,\ldots,\dim H_p(M;\mathbf{R})$, be a set of independent p-cycles forming a basis for $H_p(M;\mathbf{R})$.

First theorem: Given any set of periods ν_i, $i=1,\ldots,\dim H_p$, there exists a closed p-form ω for which

$$\nu_i=\pi(c_i,\omega)=\int_{c_i}\omega,\qquad i=1,\ldots,\dim H_p.$$

Second theorem: If all the periods for a p-form α vanish,

$$0=\pi(c_i,\alpha)=\int_{c_i}\alpha,\qquad i=1,\ldots,\dim H_p$$

then α is *exact*.

In other words, if $\{\omega_j\}$ is a basis for $H^p_{\mathrm{DR}}(M;\mathbf{R})$, then the period matrix

$$\pi_{ij}=\pi(c_i,\omega_j)$$

is invertible. Thus $H^p_{\mathrm{DR}}(M;\mathbf{R})$ is dual to $H_p(M;\mathbf{R})$ with respect to the inner product π. Therefore de Rham cohomology H^p_{DR} and simplicial cohomology H^p are *naturally isomorphic*,

$$H^p_{\mathrm{DR}}(M;\mathbf{R})\simeq H^p(M;\mathbf{R}),$$

and henceforth will be identified.

We define

$$b_p=\dim H_p(M;\mathbf{R})=\dim H^p(M;\mathbf{R})$$

as the pth *Betti number* of M. The alternating sum of the Betti numbers is the Euler characteristic

$$\chi(M)=\sum_{p=0}^{n}(-1)^p b_p.$$

The de Rham theorem relates the topological Euler characteristic calculated from H_p to the analytic Euler characteristic calculated from de Rham cohomology. The Gauss–Bonnet theorem gives a formula for $\chi(M)$ in terms of curvature as we shall see later.

We say that a cohomology class is *integral* if $\pi(c,\omega)\in\mathbf{Z}$ for any integral cycle c. There is always a natural embedding of $H^p(M;\mathbb{Z})$ in $H^p\ (M;\mathbb{Z})\otimes\mathbb{R}\simeq H^p(M;\mathbb{R})$. However, $H^p(M;\mathbb{Z})$ is not isomorphic to the set of integral de Rham classes since torsion elements are lost during the embedding; $H^p(M;\mathbb{Z})$ in general has torsion elements while $H^p(M;\mathbb{R})$ (and $H^p(M;\mathbb{C})$) do not.

Pullback mappings. If $f: M\to N$ and if ω_p is a p-form on N, then we can pull back ω_p to define $f^*\omega_p$ as a p-form on M. For example, if $x^\mu\in M$, $y^i\in N$, $f^i(x^\mu)=y^i$ and $\omega=g_i(y)\,\mathrm{d}y^i$, then we find $f^*\omega = g_i(f(x))\ \partial_\mu f^i(x)\,\mathrm{d}x^\mu$. Since $\mathrm{d}(f^*\,\omega_p)=f^*\,\mathrm{d}\omega_p$, f^* pulls back closed forms to closed forms and exact forms to exact forms. This defines a map f^*: $H^p(N;\mathbb{R})\to H^p(M;\mathbb{R})$. The dual map f_*: $H_p(M;\mathbb{R})\to H_p(N;\mathbb{R})$ goes the other way. f_* is defined on the chain level by using the map f to "push forward" chains on M to chains on N. It is easy to check that f_* maps cycles to cycles and boundaries to boundaries. f^* is a zero map if $p>\dim M$ or $\dim N$. We also note that

$$\pi(c,f^*\omega)=\pi(f_*c,\omega).$$

Ring structure: The wedge product of two closed forms is again closed; the wedge product of an exact and a closed form is exact. Wedge product preserves the cohomology equivalence relation and induces a map from $H^p(M;\mathbb{R})\otimes H^q(M;\mathbb{R})\to H^{p+q}(M;\mathbb{R})$. This defines a ring structure on $H^*(M;\mathbb{R})=\bigoplus_p H^p(M;\mathbb{R})$. Since

$$f^*(\theta\wedge\omega)=f^*\theta\wedge f^*\omega,$$

pulling back preserves the ring structure. $H^*(M;\mathbb{Z})$ and $H^*(M;\mathbb{Z}_2)$ have ring structures similar to $H^*(M;\mathbb{R})$.

Poincaré duality: If M is a compact orientable manifold without boundary, then $H^n(M;\mathbb{R})=\mathbb{R}$ because any $\omega_n\in H^n(M;\mathbb{R})$ may be written up to a total differential as

$$\omega_n=\text{const}\times(\text{volume element in } M).$$

Poincaré duality states that $H^p(M;\mathbb{R})$ is dual to $H^{n-p}(M;\mathbb{R})$ with respect to the inner product

$$(\omega_p,\omega_{n-p})=\int_M \omega_p\wedge\omega_{n-p}.$$

Consequently H^p and H^{n-p} are isomorphic as vector spaces and

$$\dim H^p(M;\mathbb{R})=\dim H^{n-p}(M;\mathbb{R}).$$

Hence the Betti numbers are related by

$$b_p=b_{n-p}.$$

Poincaré duality is valid with $\mathbb{Z}_2$ coefficients regardless of whether or not M is orientable.

Product formulas: If $M = M_1 \times M_2$, then

$$H^k(M;\mathbf{R}) \simeq \bigoplus_{p+q=k} H^p(M_1;\mathbf{R}) \otimes H^q(M_2;\mathbf{R}),$$

so $H^*(M;\mathbf{R}) \simeq H^*(M_1;\mathbf{R}) \otimes H^*(M_2;\mathbf{R})$. Furthermore, this is a ring isomorphism. This is the *Kunneth formula*. This formula is *not* valid with $\mathbf{Z}$ or $\mathbf{Z}_2$ coefficients. Since the Betti numbers are related by

$$b_k(M) = \sum_{p+q=k} b_p(M_1)\, b_q(M_2),$$

we find that the Euler characteristics obey the relation

$$\chi(M = M_1 \times M_2) = \chi(M_1)\,\chi(M_2).$$

Harmonic forms and de Rham cohomology

If M is a compact manifold without boundary, we can express each de Rham cohomology class as a harmonic form using the Hodge decomposition theorem,

$$\omega = \mathrm{d}\alpha + \delta\beta + \gamma,$$

where γ is harmonic. If $\mathrm{d}\omega = 0$ then $\mathrm{d}\delta\beta = 0$ so $\delta\beta = 0$ and $\omega = \mathrm{d}\alpha + \gamma$. This shows that every cohomology class contains a harmonic representative. If ω is harmonic, then $\delta\mathrm{d}\alpha = 0$, so $\mathrm{d}\alpha = 0$ and $\omega = \gamma$. This establishes an isomorphism between $H^p(M;\mathbf{R})$ and the set of harmonic p-forms $\mathrm{Harm}^p(M;\mathbf{R})$. This is always finite-dimensional, so $H^p(M;\mathbf{R})$ is finite. (If M has a boundary, we must use suitable boundary conditions to obtain this isomorphism.)

If $M = M_1 \times M_2$, θ_1 is harmonic on M_1 and θ_2 is harmonic on M_2, then $\theta_1 \wedge \theta_2$ is harmonic on $M_1 \times M_2$. This defines the isomorphism

$$\mathrm{Harm}^k(M = M_1 \times M_2;\mathbf{R}) \simeq \bigoplus_{p+q=k} \mathrm{Harm}^p(M_1;\mathbf{R}) \otimes \mathrm{Harm}^q(M_2;\mathbf{R}),$$

which is equivalent to the Kunneth formula defined above.

Note: In general the wedge product of two harmonic forms will *not* be harmonic so the ring structure is not given in terms of harmonic forms.

Note: If M is oriented, the Hodge operator maps $\Lambda^p \to \Lambda^{n-p}$ with $** = (-1)^{p(n-p)}$. The $*$ operator commutes with the Laplacian and induces an isomorphism

$$*: \mathrm{Harm}^p(M;\mathbf{R}) \simeq \mathrm{Harm}^{n-p}(M;\mathbf{R}).$$

Therefore

$$\dim H^p(M;\mathbf{R}) = \dim H^{n-p}(M;\mathbf{R}).$$

This is another way of looking at Poincaré duality.

Equivariant cohomology: An *isometry* of M is a map of M to itself which preserves a given Riemannian metric on M. Let M be a manifold on which a finite group G acts by isometries without fixed points and let $N = M/G$. If ω_p is harmonic on M and $g \in G$, then the pullback $g * \omega_p$ on M is harmonic. If

$$g * \omega_p = \omega_p, \qquad \text{for all } g \in G,$$

then ω_p is called *G-invariant*. The harmonic p-forms on $N = M/G$ can be identified with the G-invariant harmonic p-forms on M.

Examples 2.5

1. *de Rham cohomology of* $\mathbb{R}^n$. All closed forms are exact on $\mathbb{R}^n$ except for the scalar functions which belong to H^0. If f is a function and $df = 0$, then all the partial derivatives of f vanish so f is constant. $\dim H^0(\mathbb{R}^n; \mathbb{R}) = 1$, $\dim H^k(\mathbb{R}^n; \mathbb{R}) = 0$ for $k \neq 0$.

2. *de Rham cohomology of* S^n. Only H^0 and H^n are non zero for S^n and both have dimension 1. H^0 consists of the constant functions and H^n consists of the constant multiples of the volume element. These are the harmonic forms.

3. *de Rham cohomology of the torus*, $T^2 = S^1 \times S^1$. Let θ_1 and θ_2, $0 \leq \theta_i < 2\pi$, be coordinates on each of the two circles making up the torus. The differential forms $d\theta_i$ are then *closed* but not *exact*, since the θ_i are defined only modulo 2π and are therefore not global coordinates. Thus $d\theta_1$ and $d\theta_2$ form a basis for $H^1(T^2; \mathbb{R})$ and $\dim H^1(T^2; \mathbb{R}) = 2$. By the Kunneth formula, $H^2(T^2 = S^1 \times S^1; \mathbb{R}) = H^1(S^1; \mathbb{R}) \otimes H^1(S^1; \mathbb{R})$ and so $H^2(T^2; \mathbb{R})$ is generated by $d\theta_1 \wedge d\theta_2$ where $\dim H^2(T^2; \mathbb{R}) = 1$. Obviously $\dim H^0(T^2; \mathbb{R}) = 1$ also.

It is instructive to work out the Hodge decomposition theorem explicitly for T^2 by expanding $C^\infty(\Lambda^p)$ in Fourier series using the coordinates θ_i. We find

$$\omega_0 = \sum a_{nm}\, e^{in\theta_1}\, e^{im\theta_2}$$

$$\omega_1 = \sum b^{(1)}_{nm}\, e^{in\theta_1}\, e^{im\theta_2}\, d\theta_1 + \sum b^{(2)}_{nm}\, e^{in\theta_1}\, e^{im\theta_2}\, d\theta_2$$

$$\omega_2 = \sum c_{nm}\, e^{in\theta_1}\, e^{im\theta_2}\, d\theta_1 \wedge d\theta_2.$$

Now we compute the Laplacians

$$\Delta\omega_0 = \delta d\omega_0 = \sum (n^2 + m^2) a_{nm}\, e^{in\theta_1}\, e^{im\theta_2}$$

$$\Delta\omega_1 = (d\delta + \delta d)\omega_1 = \sum (n^2 + m^2)(b^{(1)}_{nm}\, d\theta_1 + b^{(2)}_{nm}\, d\theta_2)\, e^{in\theta_1}\, e^{im\theta_2}$$

$$\Delta\omega_2 = d\delta\omega_2 = \sum (n^2 + m^2) c_{nm}\, e^{in\theta_1}\, e^{im\theta_2}\, d\theta_1 \wedge d\theta_2$$

and introduce the Green's functions G_p of the form

$$G_0 \cdot \omega_0 = \sum_{(n,m) \neq (0,0)} a_{nm}\, e^{in\theta_1}\, e^{im\theta_2}/(n^2 + m^2), \qquad \text{etc.}$$

Then we may write each element of $C^\infty(\Lambda^p)$ as the sum of a closed, a co-closed, and a harmonic form as follows:

$$
\begin{aligned}
\omega_0 &= \Delta G_0\omega_0 + a_{00} = 0 + \delta(\mathrm{d}G_0\omega_0) + a_{00}\\
\omega_1 &= \Delta G_1\omega_1 + b_{00}^{(1)}\,\mathrm{d}\theta_1 + b_{00}^{(2)}\,\mathrm{d}\theta_2\\
&= \mathrm{d}(\delta G_1\omega_1) + \delta(\mathrm{d}G_1\omega_1) + b_{00}^{(1)}\,\mathrm{d}\theta_1 + b_{00}^{(2)}\,\mathrm{d}\theta_2\\
\omega_2 &= \Delta G_2\omega_2 + c_{00}\,\mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2\\
&= \mathrm{d}(\delta G_2\omega_2) + 0 + c_{00}\,\mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2.
\end{aligned}
$$

We verify explicitly the dimensions of each cohomology class from the harmonic representatives in the decomposition of ω_p.

4. *de Rham cohomology of* $P_n(\mathbb{C})$. There is an element $x \in H^2(P_n(\mathbb{C}); \mathbb{R})$ such that x^k generates $H^{2k}(P_n(\mathbb{C}); \mathbb{R}) \simeq \mathbb{R}$ for $k = 0, \ldots, n$. $H^j(P_n(\mathbb{C}); \mathbb{R}) = 0$ if j is odd or if $j > 2n$. x will be the first Chern class of a line bundle as discussed later. It has integral periods as does x^k for $k = 0, \ldots, n$. $x^{n+1} = 0$ since this would be a $2n+2$ form. There is a natural inclusion of $\mathbb{C}^n$ into $\mathbb{C}^{n+1}$ which induces an inclusion of $P_{n-1}(\mathbb{C})$ into $P_n(\mathbb{C})$ which we denote by i. Then $i^*: H^k(P_n(\mathbb{C}); \mathbb{R}) \to H^k(P_{n-1}(\mathbb{C}); \mathbb{R})$ is an isomorphism for $k < 2n$. Consequently, x is universal; we can view x as belonging to $H^2(P_n(\mathbb{C}); \mathbb{R})$ for any n. (x is the normalized Kähler form of $P_n(\mathbb{C})$; see example 3.4.3.)

5. *de Rham cohomology of* $U(n)$. Let g be an $n \times n$ unitary matrix $g \in U(n)$. $g^{-1}\,\mathrm{d}g$ is a complex matrix of 1-forms. Let $\omega_k = \mathrm{Tr}(g^{-1}\,\mathrm{d}g)^k$ for $k = 1, 2, \ldots, 2n-1$. Then ω_k is a complex k-form which is closed; $\omega_k = 0$ if k is even. The $\{\omega_1, \omega_3, \ldots, \omega_{2n-1}\}$ generate $H^*(U(n); \mathbb{C})$. By adding appropriate factors of $\sqrt{-1}$ to make everything real, we could get corresponding generators for $H^*(U(n); \mathbb{R})$. (If we add appropriate scaling factors, these become integral classes which generate $H^*(U(n); \mathbb{Z})$.) If we then take the mod 2 reduction, we get classes which generate $H^*(U(n); \mathbb{Z}_2)$. $g^{-1}\,\mathrm{d}g$ is the Cartan form which will be discussed later. For example, if $n = 2$, then:

$$
\begin{aligned}
&H^0(U(2); \mathbb{C}) \simeq \mathbb{C}, \qquad H^1(U(2); \mathbb{C}) \simeq \mathbb{C} \quad (\text{generator } \omega_1)\\
&H^2(U(2); \mathbb{C}) \simeq 0, \qquad H^3(U(2); \mathbb{C}) \simeq \mathbb{C} \quad (\text{generator } \omega_3)\\
&H^4(U(2); \mathbb{C}) \simeq \mathbb{C}\ (\text{generator } \omega_1 \wedge \omega_3), \qquad H^k(U(2); \mathbb{C}) = 0 \text{ for } k > 4.
\end{aligned}
$$

Of course, $U(2) = U(1) \times \mathrm{SU}(2) = S^1 \times S^3$ topologically (although not as a group). Up to a scaling factor ω_1 is $\mathrm{d}\theta$ on S^1 and ω_3 is the volume element on S^3. $H^*(S^1 \times S^3; \mathbb{C}) = H^*(S^1; \mathbb{C}) \otimes H^*(S^3; \mathbb{C})$ is just an illustration of the Kunneth formula.

6. *de Rham cohomology of* $\mathrm{SU}(n)$. $\mathrm{SU}(n)$ is a subgroup of $U(n)$; let i: $\mathrm{SU}(n) \to U(n)$ be the inclusion map. The $i^*\omega_k \in H^k(\mathrm{SU}(n); \mathbb{R})$ are generators for $k = 3, \ldots, 2n-1$. ($H^1 = 0$ since $\mathrm{Tr}(g^{-1}\,\mathrm{d}g) = 0$ for $\mathrm{SU}(n)$.) Topologically, $U(n) = S^1 \times \mathrm{SU}(n)$ and $H^*(U(n)) = H^*(S^1) \otimes H^*(\mathrm{SU}(n))$.

7. *The de Rham cohomology of* $P_n(\mathbb{R})$ is a good example involving torsion.

(a) With real coefficients, we argue that

$$
H^k(P_n(\mathbb{R}); \mathbb{R}) = H_k(P_n(\mathbb{R}); \mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } k = 0 \text{ or } k = n,\ n \text{ odd}\\ 0 & \text{otherwise.} \end{cases}
$$

If $k \neq 0$, n, then there are no harmonic forms on the universal cover S^n and hence $H^k(P_n(\mathbb{R}); \mathbb{R}) = 0$, for

$k \neq 0, n$. Since $P_n(\mathbb{R})$ is connected, $H^0(P_n(\mathbb{R}); \mathbb{R}) = \mathbb{R}$. Finally, if n is odd, the antipodal map $f(x) = -x$ on S^n preserves the volume element and hence $P_n(\mathbb{R})$ is orientable and $H^n(P_n(\mathbb{R}); \mathbb{R}) = \mathbb{R}$. If n is even, the antipodal map reverses the sign of the volume form so there is no equivariant harmonic n-form and $H^n(P_n(\mathbb{R}); \mathbb{R}) = 0$. $P_n(\mathbb{R})$ is not orientable if n is even.

(b) With $\mathbb{Z}_2$ coefficients there is an element $x \in H^1(P_n(\mathbb{R}); \mathbb{Z}_2)$ so that x^k generates $H^k(P_n(\mathbb{R}); \mathbb{Z}_2) \simeq \mathbb{Z}_2$ for $k = 0, \ldots, n$. If i: $P_{n-1}(\mathbb{R}) \to P_n(\mathbb{R})$ is the natural inclusion, then $i^*x = x$ so i^*: $H^k(P_n(\mathbb{R}); \mathbb{Z}_2) \to H^k(P_{n-1}(\mathbb{R}); \mathbb{Z}_2)$ is an isomorphism for $k = 0, \ldots, n-1$. (x is a Stiefel–Whitney class.)

(c) With integer coefficients,

$$H^k(P_n(\mathbb{R}); \mathbb{Z}) = \mathbb{Z}, 0, \mathbb{Z}_2, 0, \mathbb{Z}_2, \ldots;$$

$$H^n(P_n(\mathbb{R}); \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = \text{odd} \\ \mathbb{Z}_2 & \text{if } n = \text{even} \end{cases}$$

$$H_k(P_n(\mathbb{R}); \mathbb{Z}) = \mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0, \ldots;$$

$$H_n(P_n(\mathbb{R}); \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = \text{odd} \\ 0 & \text{if } n = \text{even}. \end{cases}$$

The shift in the relative positions of the $\mathbb{Z}_2$ terms in H^k and H_k is a consequence of the universal coefficient theorem (see, e.g., Spanier [1966]).

3. Riemannian manifolds

We now consider manifolds endowed with a metric. We apply the tools of the previous section and present classical Riemannian geometry in a modern notation which is convenient for practical calculations. A still more abstract approach to Riemannian manifolds will be given when we treat connections on fiber bundles.

3.1. Cartan structure equations

Suppose we are given a 4-manifold M and a metric $g_{\mu\nu}(x)$ on M in local coordinates x^μ. Then the distance $\mathrm{d}s$ between two infinitesimally nearby points x^μ and $x^\mu + \mathrm{d}x^\mu$ is given by

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu$$

where the $g_{\mu\nu}$ are the components of a symmetric covariant second-rank tensor.

We now decompose the metric into vierbeins (solder forms) or tetrads $e^a{}_\mu(x)$ as follows:

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$$
$$\eta^{ab} = g^{\mu\nu} e^a{}_\mu e^b{}_\nu.$$

Here η_{ab} is a flat, usually Cartesian, metric such as the following:

Euclidean space:

$$\eta_{ab} = \delta_{ab}, \qquad a, b = 1, 2, 3, 4;$$

Minkowski space:

$$\eta_{ab} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \qquad a, b = 0, 1, 2, 3.$$

$e^a{}_\mu$ is, in some sense, the *square root* of the metric.

Throughout this section, Greek indices $\mu, \nu, \ldots$ will be raised and lowered with $g_{\mu\nu}$ or its inverse $g^{\mu\nu}$ and Latin indices $a, b, \ldots$ will be raised and lowered by η_{ab} and η^{ab}. We define the inverse of $e^a{}_\mu$ by

$$E_a{}^\mu = \eta_{ab} g^{\mu\nu} e^b{}_\nu$$

which obeys

$$E_a{}^\mu e^b{}_\mu = \delta^b_a$$
$$\eta^{ab} E_a{}^\mu E_b{}^\nu = g^{\mu\nu} \qquad \text{etc.}$$

Thus $e^a{}_\mu$ and $E_a{}^\mu$ are used to interconvert Latin and Greek indices when necessary.

We therefore see that $e^a{}_\mu$ is the matrix which transforms the coordinate basis $\mathrm{d}x^\mu$ of $T^*_x(M)$ to an orthonormal basis of $T^*_x(M)$,

$$e^a = e^a{}_\mu \, \mathrm{d}x^\mu .$$

(Note that while the coordinate basis $\mathrm{d}x^\mu$ is always an exact differential, e^a is not necessarily an exact 1-form.) Similarly, $E_a{}^\mu$ is a transformation from the basis $\partial/\partial x^\mu$ of $T_x(M)$ to the orthonormal basis of $T_x(M)$,

$$E_a = E_a{}^\mu \, \partial/\partial x^\mu .$$

(Note that E_a and E_b do not necessarily commute, while $\partial/\partial x^\mu$ and $\partial/\partial x^\nu$ do commute.)

We now introduce the ***affine spin connection one-form*** $\omega^a{}_b$ **and define**

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b \equiv T^a \equiv \tfrac{1}{2} T^a{}_{bc} e^b \wedge e^c . \tag{3.1}$$

This is called the *torsion* 2-form of the manifold. The *curvature* 2-form is defined as

$$R^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \tfrac{1}{2} R^a{}_{bcd} e^c \wedge e^d . \tag{3.2}$$

Equations (3.1) and (3.2) are called *Cartan's structure equations*.

Consistency conditions: Taking the exterior derivative of (3.1) we find

$$\mathrm{d}T^a + \omega^a{}_b \wedge T^b = R^a{}_b \wedge e^b . \tag{3.3}$$

Differentiating (3.2), we find the *Bianchi identities*:

$$\mathrm{d}R^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0. \tag{3.4}$$

We define the *covariant derivative* of a differential form $V^a{}_b$ of degree p as

$$\mathrm{D}V^a{}_b = \mathrm{d}V^a{}_b + \omega^a{}_c \wedge V^c{}_b - (-1)^p V^a{}_c \wedge \omega^c{}_b. \tag{3.5}$$

The consistency condition (3.4) then reads

$$\mathrm{D}R^a{}_b = 0.$$

Gauge transformations: Consider an orthogonal rotation of the orthonormal frame

$$e^a \to e'^a = \Phi^a{}_b e^b,$$

where

$$\eta_{ab}\Phi^a{}_c\Phi^b{}_d = \eta_{cd}.$$

Note that

$$(\mathrm{d}\Phi)^a{}_b(\Phi^{-1})^b{}_c = -\Phi^a{}_b(\mathrm{d}\Phi^{-1})^b{}_c.$$

Then we find

$$T'^a = \mathrm{d}e'^a + \omega'^a{}_b \wedge e'^b$$

where

$$T'^a = \Phi^a{}_b T^b$$

and the new connection is

$$\omega'^a{}_b = \Phi^a{}_c\omega^c{}_d(\Phi^{-1})^d{}_b + \Phi^a{}_c(\mathrm{d}\Phi^{-1})^c{}_b.$$

The transformation law for the curvature 2-form is given by

$$R'^a{}_b = \mathrm{d}\omega'^a{}_b + \omega'^a{}_c \wedge \omega'^c{}_b = \Phi^a{}_c R^c{}_d(\Phi^{-1})^d{}_b.$$

A similar exercise shows that under a change of frame, the "covariant derivative" (3.5) in fact transforms covariantly,

$$(\mathrm{D}V)'^a{}_b = \Phi^a{}_c(\mathrm{D}V)^c{}_d(\Phi^{-1})^d{}_b.$$

3.2. Relation to classical tensor calculus

The Cartan differential form approach is, of course, equivalent to the conventional tensor formulation of Riemannian geometry. Here we summarize the relationships among various quantities appearing in the two approaches. Figure 3.1 is a caricature of classical tensor calculus.

Volume and inner product: The invariant oriented volume element in n dimensions is

$$\mathrm{d}V = e^1 \wedge e^2 \wedge \cdots \wedge e^n = |g|^{1/2}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n \tag{3.6}$$

where g is the determinant of the metric tensor.

In curved space, the Hodge $*$ operation will involve the metric. If

$$\epsilon_{\mu_1\mu_2,\ldots,\mu_n} = \begin{cases} 0 & \text{any two indices repeated} \\ +1 & \text{even permutation} \\ -1 & \text{odd permutation,} \end{cases}$$

then

$$\epsilon_{\mu_1\mu_2,\ldots,\mu_n} = g\epsilon^{\mu_1\mu_2,\ldots,\mu_n}$$

and we define the standard tensor densities

$$E_{\mu_1,\ldots,\mu_n} = |g|^{1/2}\epsilon_{\mu_1,\ldots,\mu_n}$$

$$E^{\mu_1,\ldots,\mu_n} = |g|^{-1/2}\epsilon^{\mu_1,\ldots,\mu_n}.$$

The Hodge $*$ is then defined as the operation which correctly produces the curved space inner product. The inner product for 1-forms is defined using the Hodge $*$ as

$$\alpha \wedge *\beta = g^{\mu\nu}\alpha_\mu\beta_\nu|g|^{1/2}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n. \tag{3.7}$$

Hodge $*$ is therefore defined as

$$*(\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}) = \frac{|g|^{1/2}}{(n-p)!}\epsilon^{\mu_1,\ldots,\mu_p}{}_{\mu_{p+1},\ldots,\mu_n}\,\mathrm{d}x^{\mu_{p+1}} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}.$$

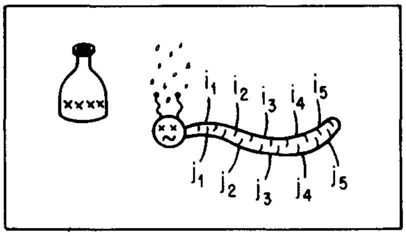

Fig. 3.1. Classical tensor calculus intoxicated by the plethora of indices.

Because of (3.6) we can rewrite this in the form

$$*(e^{a_1} \wedge \cdots \wedge e^{a_p}) = \frac{1}{(n-p)!} \epsilon^{a_1,\ldots,a_p}{}_{a_{p+1},\ldots,a_n} e^{a_{p+1}} \wedge \cdots \wedge e^{a_n}$$

where $\epsilon_{ab\ldots}$ has its indices raised and lowered by the flat metric η_{ab}. If we convert Greek to Latin indices using the vierbeins, e.g.,

$$\alpha = \alpha_\mu \, \mathrm{d}x^\mu = \alpha_a e^a$$

we recover the inner product (3.7):

$$\alpha \wedge *\beta = \eta^{ab}\alpha_a\beta_b(e^1 \wedge e^2 \wedge \cdots \wedge e^n) = g^{\mu\nu}\alpha_\mu\beta_\nu(|g|^{1/2}\,\mathrm{d}^n x).$$

The various tensors that we have defined with flat indices $a, b, \ldots$ are, of course, related to the tensor objects with curved indices by multiplication with $e^a{}_\mu$, $E^a{}_\mu$. The curvature two-form is first decomposed as

$$R^a{}_b = \tfrac{1}{2}R^a{}_{bcd}e^c \wedge e^d = \tfrac{1}{2}R^a{}_{b\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu,$$

and then the Riemann tensor is written

$$\text{Riemann tensor} = R^\alpha{}_{\beta\mu\nu} = E_a{}^\alpha e^b{}_\beta R^a{}_{b\mu\nu}.$$

Similarly, the torsion is

$$T^a = \tfrac{1}{2}T^a{}_{bc}e^b \wedge e^c = \tfrac{1}{2}T^a{}_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$$

$$T^\alpha{}_{\mu\nu} = E_a{}^\alpha T^a{}_{\mu\nu}.$$

Levi–Civita connection: The covariant derivative in the tensor formalism is defined using the Levi–Civita connection $\Gamma^\mu_{\alpha\beta}$, which physicists generally refer to as the Christoffel symbol. The Levi–Civita connection is determined by two conditions, the covariant constancy of the metric and the absence of torsion. In the tensor notation, these conditions are

$$\textit{metricity}: \qquad g_{\mu\nu;\alpha} = \partial_\alpha g_{\mu\nu} - \Gamma^\lambda_{\alpha\mu}g_{\lambda\nu} - \Gamma^\lambda_{\alpha\nu}g_{\mu\lambda} = 0 \tag{3.8}$$

$$\textit{no torsion}: \qquad T^\mu{}_{\alpha\beta} = \tfrac{1}{2}(\Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha}) = 0. \tag{3.9}$$

The Christoffel symbol is then uniquely determined in terms of the metric to be

$$\Gamma^\mu_{\alpha\beta} = \tfrac{1}{2}g^{\mu\nu}(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}).$$

In Cartan's method, the Levi–Civita *spin connection* is obtained by restricting the affine spin connection

ω_{ab} in an analogous way. The conditions (3.8) and (3.9) are replaced by

$$\textit{metricity:}\qquad \omega_{ab} = -\omega_{ba} \tag{3.10}$$

$$\textit{no torsion:}\qquad T^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0. \tag{3.11}$$

$\omega^a{}_{b\mu}$ is then determined in terms of the vierbeins and inverse vierbeins and is related to $\Gamma^{\mu}_{\alpha\beta}$ by

$$\begin{aligned}\omega^a{}_{b\mu} &= e^a{}_\nu E_b{}^\nu{}_{;\mu} = e^a{}_\nu(\partial_\mu E_b{}^\nu + \Gamma^\nu_{\mu\lambda} E_b{}^\lambda)\\ &= -E_b{}^\nu e^a{}_{\nu;\mu} = -E_b{}^\nu(\partial_\mu e^a{}_\nu - \Gamma^\lambda_{\mu\nu} e^a{}_\lambda).\end{aligned}$$

From

$$0 = \delta^a{}_{b;\mu} = e^a{}_{\nu;\mu} g^{\nu\lambda} e_{b\lambda} + e^a{}_\nu g^{\nu\lambda}{}_{;\mu} e_{b\lambda} + e^a{}_\nu g^{\nu\lambda} e_{b\lambda;\mu}$$

we see that (3.10) is indeed a consequence of covariant constancy of the metric, (3.8). Similarly, if we write eq. (3.11) as

$$\begin{aligned}0 &= \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + E^{a\alpha} e_{b\alpha;\mu} e^b{}_\nu - E^{a\alpha} e_{b\alpha;\nu} e^b{}_\mu\\ &= \delta^a_b(\Gamma^\lambda_{\nu\mu} e^b{}_\lambda - \Gamma^\lambda_{\mu\nu} e^b{}_\lambda)\end{aligned}$$

we recognize the torsion-free condition (3.9).

The curvature can be extracted from Cartan's equations by computing

$$\partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + \omega^a{}_{c\mu}\omega^c{}_{b\nu} - \omega^a{}_{c\nu}\omega^c{}_{b\mu} = e^a{}_\alpha E_b{}^\beta R^\alpha{}_{\beta\mu\nu}$$

where

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\nu\beta} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\nu\beta} - \Gamma^\alpha_{\nu\gamma}\Gamma^\gamma_{\mu\beta}. \tag{3.12}$$

Weyl tensor: A useful object in n-dimensional geometry is the *Weyl tensor*, defined as

$$W_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} + \frac{\mathcal{R}}{(n-1)(n-2)}(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}) - \frac{1}{(n-2)}(g_{\alpha\mu}\mathcal{R}_{\beta\nu} - g_{\alpha\nu}\mathcal{R}_{\beta\mu} - g_{\beta\mu}\mathcal{R}_{\alpha\nu} + g_{\beta\nu}\mathcal{R}_{\alpha\mu}),$$

where $\mathcal{R}_{\mu\nu} = R_{\mu\alpha\nu\beta}g^{\alpha\beta}$ and $\mathcal{R} = \mathcal{R}_{\mu\nu}g^{\mu\nu}$ are the Ricci tensor and the scalar curvature. The Weyl tensor is traceless in all pairs of indices.

Examples 3.2

We will for simplicity look only at Levi-Civita connections ($T^a = 0$, $\omega_{ab} = -\omega_{ba}$), so the vierbeins determine ω_{ab} uniquely.

1. *Coordinate transform of flat Cartesian coordinates to polar coordinates.* The Riemannian curvature remains zero, although the connection may be nontrivial.

a. *Two dimensions*; $\mathbf{R}^2$. $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$; $e^1 = \mathrm{d}x$, $e^2 = \mathrm{d}y$. If $x = r\cos\theta$, $y = r\sin\theta$, then

$$\begin{pmatrix} e^r = \mathrm{d}r \\ e^\theta = r\,\mathrm{d}\theta \end{pmatrix} = \frac{1}{r}\begin{pmatrix} x & y \\ -y & x \end{pmatrix}\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix}.$$

Action of Hodge $*$: $\quad *(\mathrm{d}x, \mathrm{d}y) = (\mathrm{d}y, -\mathrm{d}x)$

$$*(\mathrm{d}r, r\,\mathrm{d}\theta) = (r\,\mathrm{d}\theta, -\mathrm{d}r).$$

Structure equations:

$$\mathrm{d}e^r - \omega \wedge e^\theta = 0 - \omega \wedge r\,\mathrm{d}\theta = 0$$

$$\mathrm{d}e^\theta + \omega \wedge e^r = \mathrm{d}r \wedge \mathrm{d}\theta + \omega \wedge \mathrm{d}r = 0.$$

Connection and curvature:

$$\omega = \mathrm{d}\theta$$

$$R = \mathrm{d}\omega = 0.$$

b. *Four dimensions*; $\mathbf{R}^4$. $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}t^2$. We define polar coordinates by

$$x + \mathrm{i}y = r\cos\frac{\theta}{2}\exp\frac{\mathrm{i}}{2}(\psi + \varphi)$$

$$z + \mathrm{i}t = r\sin\frac{\theta}{2}\exp\frac{\mathrm{i}}{2}(\psi - \varphi)$$

$$0 \le \theta < \pi, \qquad 0 \le \varphi < 2\pi, \qquad 0 \le \psi < 4\pi$$

$$\begin{pmatrix} e^0 = \mathrm{d}r \\ e^1 = r\sigma_x \\ e^2 = r\sigma_y \\ e^3 = r\sigma_z \end{pmatrix} = \frac{1}{r}\begin{pmatrix} x & y & z & t \\ -t & -z & y & x \\ z & -t & -x & y \\ -y & x & -t & z \end{pmatrix}\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \\ \mathrm{d}t \end{pmatrix}$$

σ_x, σ_y and σ_z obey the relation $\mathrm{d}\sigma_x = 2\sigma_y \wedge \sigma_z$, cyclic. The connections and curvatures are given by

$$\omega^1{}_0 = \omega^2{}_3 = \sigma_x, \quad \omega^2{}_0 = \omega^3{}_1 = \sigma_y, \quad \omega^3{}_0 = \omega^1{}_2 = \sigma_z$$

$$\begin{aligned} R^0{}_1 &= \mathrm{d}\omega^0{}_1 + \omega^0{}_2 \wedge \omega^2{}_1 + \omega^0{}_3 \wedge \omega^3{}_1 \\ &= -2\sigma_y \wedge \sigma_z + (-\sigma_y) \wedge (-\sigma_z) + (-\sigma_z) \wedge \sigma_y = 0, \qquad \text{etc.} \end{aligned}$$

Remark: σ_x, σ_y and σ_z are the left-invariant 1-forms on the manifold of the group $\mathrm{SU}(2) = S^3$ and will appear also in our treatment of the geometry of Lie groups.

2. *Two-sphere*. The metric on S^2 is easily found by setting $r = \text{constant}$ in the flat $\mathbf{R}^3$ metric:

$$ds^2 = r^2\,d\theta^2 + r^2 \sin^2\theta\,d\varphi^2 = (e^1)^2 + (e^2)^2.$$

We choose

$$e^1 = r\,d\theta, \qquad e^2 = r\sin\theta\,d\varphi$$

so the structure equations

$$de^1 = 0 = -\omega^1{}_2 \wedge e^2$$
$$de^2 = r\cos\theta\,d\theta \wedge d\varphi = -\omega^2{}_1 \wedge e^1$$

give the connection

$$\omega^1{}_2 = -\cos\theta\,d\varphi$$

and the curvature

$$R^1{}_2 = R^1{}_{212} e^1 \wedge e^2$$
$$= d\omega^1{}_2 = \frac{1}{r^2} e^1 \wedge e^2.$$

The Gaussian curvature is thus $K = R_{abab} = 2/r^2$, showing that S^2 has constant positive curvature.

3. *4-Sphere with polar coordinates*. The de Sitter metric on S^4 with radius R is

$$ds^2 = (dr^2 + r^2(\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2))/(1 + (r/2R)^2)^2.$$

e^a with $a = 0, 1, 2, 3$ is defined by

$$(1 + (r/2R)^2)e^a = \{dr, r\sigma_x, r\sigma_y, r\sigma_z\}.$$

From the structure equations, we find

$$\omega_{i0} = (1 - (r/2R)^2)e^i/r = \sigma_i \frac{1 - (r/2R)^2}{1 + (r/2R)^2}$$

$$\tfrac{1}{2}\epsilon_{ijk}\omega_{jk} = (1 + (r/2R)^2)e^i/r = \sigma_i$$

$$R^{ab} = \frac{1}{R^2} e^a \wedge e^b.$$

The Weyl tensor vanishes identically.

3.3. Einstein's equations and self-dual manifolds

Defining the Ricci tensor and scalar curvature in 4 dimensions as

$$\mathcal{R}^{\mu}{}_{\nu} = g^{\alpha\beta} R^{\mu}{}_{\alpha\nu\beta}, \qquad \mathcal{R} = g_{\mu\nu}\mathcal{R}^{\mu\nu}, \tag{3.13}$$

we write Einstein's equations with cosmological term Λ as

$$\mathcal{R}^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu}\mathcal{R} = T^{\mu\nu} - \Lambda g^{\mu\nu}. \tag{3.14}$$

If the matter energy-momentum tensor $T^{\mu\nu}$ and Λ vanish, Einstein's equations imply the vanishing of the Ricci tensor, which we write in the flat vierbein basis as

$$0 = \mathcal{R}^{a}{}_{b} = e^{a}{}_{\mu} E_{b}{}^{\nu} \mathcal{R}^{\mu}{}_{\nu} = R^{a}{}_{cbd}\eta^{cd}. \tag{3.15}$$

We note that in Einstein's theory we always work with the torsion-free Levi–Civita connection, so the consistency condition (3.3) becomes the *cyclic identity*:

$$R^{a}{}_{b} \wedge e^{b} = 0 \rightarrow \epsilon_{abcd} R^{e}{}_{bcd} = 0. \tag{3.16}$$

Now let us define the dual of the Riemann tensor as

$$\tilde{R}_{abcd} = \tfrac{1}{2}\epsilon_{abmn} R_{mncd}. \tag{3.17}$$

Suppose the Riemann tensor is (anti)-self-dual,

$$R_{abcd} = \pm \tilde{R}_{abcd}.$$

Then the cyclic identity implies Einstein's empty space equations,

$$\begin{aligned} 0 = \epsilon_{abcd} R_{ebcd} &= \pm \tfrac{1}{2}\epsilon_{abcd}\epsilon_{ebmn} R_{mncd} \\ &= \pm(\mathcal{R}\delta_{ae} - 2\mathcal{R}_{ae}). \end{aligned}$$

Remark: A similar argument can be used to show that Einstein's empty space equations may be written as

$$\tilde{R}^{a}{}_{b} \wedge e^{b} = 0, \qquad (\tilde{R}_{ab} = \tfrac{1}{2}\epsilon_{abcd} R^{cd} = \tfrac{1}{2}\tilde{R}_{abcd} e^{c} \wedge e^{d}).$$

The equivalence of the cyclic identity and Einstein's equations for self-dual $R^{a}{}_{b}$ is then obvious.

From the relation between R_{ab} and ω_{ab},

$$\begin{aligned} R_{01} &= d\omega_{01} + \omega_{02} \wedge \omega_{21} + \omega_{03} \wedge \omega_{31} \\ R_{23} &= d\omega_{23} + \omega_{20} \wedge \omega_{03} + \omega_{21} \wedge \omega_{13} \qquad \text{etc.,} \end{aligned}$$

we notice that R_{ab} is self-dual, $R_{ab} = \pm\tilde{R}_{ab}$, if ω_{ab} is self-dual,

$$\omega_{ab} = \pm\tilde{\omega}_{ab}.$$

Therefore one way to generate a solution of Einstein's equations is to find a metric with self-dual connection.

Remark: Suppose $R_{ab} = \pm\tilde{R}_{ab}$ but $\omega_{ab} \neq \pm\tilde{\omega}_{ab}$. Then we decompose ω_{ab} into self-dual and anti-self-dual parts. Using an $O(4)$ gauge transformation one can always remove the piece of ω_{ab} with the wrong duality. The only change in R_{ab} under the gauge transformation is a rotation by an orthogonal matrix which preserves its duality properties. Thus any self-dual R_{ab} can be considered to come from a self-dual connection ω_{ab} if we work in an appropriate "self-dual gauge".

Self-dual and conformally self-dual structures in 4 dimensions

In the case of four dimensions some simplification occurs since the dual of the curvature 2-form is also a 2-form. Let us define self-dual and anti-self-dual bases for Λ^2 using the vierbein one-forms e^a:

$$\text{basis of } \Lambda^2_\pm = \begin{cases} \lambda^1_\pm = e^0 \wedge e^1 \pm e^2 \wedge e^3 \\ \lambda^2_\pm = e^0 \wedge e^2 \pm e^3 \wedge e^1, \\ \lambda^3_\pm = e^0 \wedge e^3 \pm e^1 \wedge e^2 \end{cases} \qquad *\lambda^i_\pm = \pm\lambda^i_\pm.$$

The curvature tensor can then be viewed as a 6×6 matrix R mapping $\Lambda^2_\pm$ into $\Lambda^2_\pm$ (see, e.g., Atiyah, Hitchin and Singer [1978]),

$$R\Lambda^2 = \left(\begin{array}{c|c} A & C^+ \\ \hline C^- & B \end{array}\right)\begin{pmatrix}\lambda^i_+ \\ \lambda^i_-\end{pmatrix},$$

where A is the 3×3 matrix whose first column is

$$A_{11} = R_{0101} + R_{0123} + R_{2301} + R_{2323}$$

$$A_{21} = R_{0201} + R_{0223} + R_{3101} + R_{3123}$$

$$A_{31} = R_{0301} + R_{0323} + R_{1201} + R_{1223}.$$

That is,

$$A_{ij} = +(R_{0i0j} + \tfrac{1}{2}\epsilon_{jkl}R_{0ikl}) + \tfrac{1}{2}\epsilon_{imn}(R_{mn0j} + \epsilon_{jkl}R_{mnkl})$$

and B and $C^\pm$ are defined by changing the four signs in the definition of A as follows:

$$A \sim (+,+,+,+)$$

$$B \sim (+,-,-,-)$$

$$C^+ \sim (+,-,+,-)$$

$$C^- \sim (+,+,-,+).$$

The Hodge $*$ duality transformation acts on R from the left as the matrix

$$* = \begin{pmatrix} I_3 & 0 \\ 0 & -I_3 \end{pmatrix}.$$

Now if we let

$$S = \operatorname{Tr} A = \operatorname{Tr} B$$

and subtract the trace, we find

$$W = R - \tfrac{5}{6} I_6 = \begin{pmatrix} W_+ & C^+ \\ C^- & W_- \end{pmatrix}$$

where

C^+ = tracefree Ricci tensor
$W_+ + W_-$ = tracefree Weyl tensor.

The interesting spaces can then be categorized as

Einstein: $C^\pm = 0$ $\qquad (\mathcal{R}_{\mu\nu} = \Lambda g_{\mu\nu})$
Ricci flat: $C^\pm = 0, \quad S = 0$ $\qquad (\mathcal{R}_{\mu\nu} = 0)$
Conformally flat: $W_\pm = 0$
Self-dual: $W_- = 0, \quad C^\pm = 0$
Anti-self-dual: $W_+ = 0, \quad C^\pm = 0$
Conformally self-dual: $W_- = 0$
Conformally anti-self-dual: $W_+ = 0$.

Beware: What physicists refer to as *self-dual metrics* are those which have self-dual Riemann tensor and which mathematicians may call "half-flat". The spaces which a physicist describes as having a *self-dual Weyl tensor* or as *conformally self-dual* may be called simply "self-dual" by mathematicians.

Examples 3.3

1. *Schwarzschild metric*. The best-known solution to the empty space Einstein equations is the Schwarzschild "black hole" metric:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{R}\right)\mathrm{d}t^2 + \frac{1}{1 - 2M/R}\,\mathrm{d}R^2 + R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)$$

$$0 \le \theta < \pi, \quad 0 \le \varphi < 2\pi.$$

Choosing the vierbeins

$$e^0 = \left(1 - \frac{2M}{R}\right)^{1/2}\mathrm{d}t, \quad e^1 = \left(1 - \frac{2M}{R}\right)^{-1/2}\mathrm{d}R, \quad e^2 = R\,\mathrm{d}\theta, \quad e^3 = R\sin\theta\,\mathrm{d}\varphi$$

and raising and lowering Latin indices with $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$, we find the connections

$$\begin{aligned} \omega^0{}_1 &= \frac{M}{R^2}\,\mathrm{d}t & \omega^2{}_3 &= -\cos\theta\,\mathrm{d}\varphi \\ \omega^0{}_2 &= 0 & \omega^3{}_1 &= (1-2M/R)^{1/2}\sin\theta\,\mathrm{d}\varphi \\ \omega^0{}_3 &= 0 & \omega^1{}_2 &= -(1-2M/R)^{1/2}\,\mathrm{d}\theta. \end{aligned}$$

Then the curvature 2-forms are

$$\begin{aligned} R^0{}_1 &= \frac{2M}{R^3}e^0\wedge e^1 & R^2{}_3 &= \frac{2M}{R^3}e^2\wedge e^3 \\ R^0{}_2 &= \frac{-M}{R^3}e^0\wedge e^2 & R^3{}_1 &= \frac{-M}{R^3}e^3\wedge e^1 \\ R^0{}_3 &= \frac{-M}{R^3}e^0\wedge e^3 & R^1{}_2 &= -\frac{M}{R^3}e^1\wedge e^2, \end{aligned}$$

and we easily verify that the Schwarzschild metric satisfies the Einstein equations outside the singularity at $R = 0$.

2. *Self-dual Taub–NUT metric*. One example of a metric which satisfies the Euclidean Einstein equations with self-dual Riemann tensor is the self-dual Taub–NUT metric (Hawking [1977]):

$$\mathrm{d}s^2 = \frac{1}{4}\frac{r+m}{r-m}\,\mathrm{d}r^2 + (r^2-m^2)(\sigma_x{}^2+\sigma_y{}^2) + 4m^2\frac{r-m}{r+m}\sigma_z{}^2$$

where σ_x, σ_y and σ_z are defined in example 3.2.1 and m is an arbitrary constant. We choose

$$e^a = \left\{\frac{1}{2}\left(\frac{r+m}{r-m}\right)^{1/2}\mathrm{d}r,\quad (r^2-m^2)^{1/2}\sigma_x,\quad (r^2-m^2)^{1/2}\sigma_y,\quad 2m\left(\frac{r-m}{r+m}\right)^{1/2}\sigma_z\right\}$$

and find the connections

$$\begin{aligned} \omega^0{}_1 &= \frac{2r}{r+m}\sigma_x & \omega^2{}_3 &= \frac{2m}{r+m}\sigma_x \\ \omega^0{}_2 &= \frac{2r}{r+m}\sigma_y & \omega^3{}_1 &= \frac{2m}{r+m}\sigma_y \\ \omega^0{}_3 &= \frac{4m^2}{(r+m)^2}\sigma_z & \omega^1{}_2 &= \left(2-\frac{4m^2}{(r+m)^2}\right)\sigma_z, \end{aligned}$$

and curvatures

$$R^0{}_1 = -R^2{}_3 = \frac{-m}{(r+m)^3}(e^0 \wedge e^1 - e^2 \wedge e^3)$$

$$R^0{}_2 = -R^3{}_1 = \frac{-m}{(r+m)^3}(e^0 \wedge e^2 - e^3 \wedge e^1)$$

$$R^0{}_3 = -R^1{}_2 = \frac{2m}{(r+m)^3}(e^0 \wedge e^3 - e^1 \wedge e^2).$$

3. *Metric of Eguchi and Hanson* [1978]. Another solution of the Euclidean Einstein equations with self-dual curvature is given by

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1-(a/r)^4} + r^2(\sigma_x{}^2 + \sigma_y{}^2 + (1-(a/r)^4)\sigma_z{}^2)$$

where a is an arbitrary constant. Choosing the vierbeins

$$e^a = \{(1-(a/r)^4)^{-1/2}\,\mathrm{d}r,\, r\sigma_x,\, r\sigma_y,\, r(1-(a/r)^4)^{1/2}\sigma_z\}$$

we find self-dual connections

$$\omega^0{}_1 = -\omega^2{}_3 = -(1-(a/r)^4)^{1/2}\sigma_x$$

$$\omega^0{}_2 = -\omega^3{}_1 = -(1-(a/r)^4)^{1/2}\sigma_y$$

$$\omega^0{}_3 = -\omega^1{}_2 = -(1+(a/r)^4)\sigma_z,$$

and curvatures

$$R^0{}_1 = -R^2{}_3 = \frac{2a^4}{r^6}(-e^0 \wedge e^1 + e^2 \wedge e^3)$$

$$R^0{}_2 = -R^3{}_1 = \frac{2a^4}{r^6}(-e^0 \wedge e^2 + e^3 \wedge e^1)$$

$$R^0{}_3 = -R^1{}_2 = -\frac{4a^4}{r^6}(-e^0 \wedge e^3 + e^1 \wedge e^2).$$

The apparent singularities in the metric at $r = a$ can be removed by choosing the angular coordinate ranges

$$0 \le \theta < \pi, \quad 0 \le \varphi < 2\pi, \quad 0 \le \psi < 2\pi.$$

Thus the boundary at ∞ becomes $P_3(\mathbf{R})$. (If $0 \le \psi < 4\pi$, it would have been S^3.) See section 10 for further discussion.

3.4. Complex manifolds

M is a complex manifold of dimension n if we can find complex coordinates with holomorphic transition functions in a real manifold with real dimension $2n$. Let $z_k = x_k + \mathrm{i}y_k$ be local complex coordinates; the conjugate coordinates are $\bar{z}_k = x_k - \mathrm{i}y_k$. We define:

$$\partial/\partial z_k = \tfrac{1}{2}(\partial/\partial x_k - \mathrm{i}\,\partial/\partial y_k) \qquad \partial/\partial \bar{z}_k = \tfrac{1}{2}(\partial/\partial x_k + \mathrm{i}\,\partial/\partial y_k)$$

$$\mathrm{d}z_k = \mathrm{d}x_k + \mathrm{i}\,\mathrm{d}y_k \qquad \mathrm{d}\bar{z}_k = \mathrm{d}x_k - \mathrm{i}\,\mathrm{d}y_k.$$

Then it is easily checked that

$$\mathrm{d}f = \sum_k (\partial f/\partial z_k)\,\mathrm{d}z_k + \sum_k (\partial f/\partial \bar{z}_k)\,\mathrm{d}\bar{z}_k = \partial f + \bar{\partial} f \tag{3.18}$$

where

$$\partial f = \sum_k (\partial f/\partial z_k)\,\mathrm{d}z_k$$

$$\bar{\partial} f = \sum_k (\partial f/\partial \bar{z}_k)\,\mathrm{d}\bar{z}_k.$$

If $f(z)$ is a holomorphic function of a single variable,

$$\bar{\partial} f = (\partial f/\partial \bar{z})\,\mathrm{d}\bar{z} = 0.$$

In general, a function f on $\mathbb{C}^n$ is holomorphic if $\partial f/\partial \bar{z}_k = 0$ for $k = 1, \ldots, n$ or equivalently if $\bar{\partial} f = 0$.

If w_k is another set of local complex coordinates, then

$$\mathrm{d}w_k = \partial w_k + \bar{\partial} w_k = \partial w_k = \sum_j \frac{\partial w_k}{\partial z_j}\,\mathrm{d}z_j$$

$$\mathrm{d}\bar{w}_k = \sum_j \frac{\partial \bar{w}_k}{\partial \bar{z}_j}\,\mathrm{d}\bar{z}_j.$$

We define the complex tangent and cotangent spaces in terms of their local bases as follows:

$$T_{\mathrm{c}}(M) = \{\partial/\partial z_j\} \qquad \bar{T}_{\mathrm{c}}(M) = \{\partial/\partial \bar{z}_j\}$$

$$T^*_{\mathrm{c}}(M) = \{\mathrm{d}z_j\} \qquad \bar{T}^*_{\mathrm{c}}(M) = \{\mathrm{d}\bar{z}_j\}.$$

In fact, these spaces are invariantly defined independent of the particular local complex coordinates which are chosen. We note that $T(M) \otimes \mathbf{C} = T_{\mathrm{c}}(M) \oplus \bar{T}_{\mathrm{c}}(M)$ and $T^*(M) \otimes \mathbf{C} = T^*_{\mathrm{c}}(M) \oplus \bar{T}^*_{\mathrm{c}}(M)$.

We can define *complex* exterior forms $\Lambda^{p,q}$ which have bases containing p factors of $\mathrm{d}z_k$ and q

factors of $d\bar{z}_k$. The operators ∂ and $\bar{\partial}$ act as

$$\partial\colon C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p+1,q}), \qquad \bar{\partial}\colon C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p,q+1}).$$

Clearly we can define $d\omega = \partial\omega + \bar{\partial}\omega$ for any form $\omega \in \Lambda^{p,q}$. These operators satisfy the relations:

$$\partial\partial\omega = 0, \qquad \bar{\partial}\bar{\partial}\omega = 0, \qquad \partial\bar{\partial}\omega = -\bar{\partial}\partial\omega. \tag{3.19}$$

We define the conjugate operators with respect to the inner product by

$$\delta = (-1)^{np+n+1} * d * \equiv d^* = \partial^* + \bar{\partial}^*. \tag{3.20}$$

There are then three kinds of Laplacians:

$$\Delta = (d + \delta)^2$$

$$\Delta' = 2(\partial + \partial^*)^2$$

$$\Delta'' = 2(\bar{\partial} + \bar{\partial}^*)^2.$$

Almost complex structure: A manifold M has an almost complex structure if there exists a linear map J from $T(M)$ to $T(M)$ such that $J^2 = -1$. For example, take a Cartesian coordinate system (x, y) on $\mathbf{R}^2$ and define J by the 2×2 matrix

$$J\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

$$J^2\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix}.$$

Clearly J is equivalent to multiplication by $i = \sqrt{-1}$,

$$i(x + iy) = ix - y$$

$$i^2(x + iy) = -(x + iy).$$

As an operator, J has eigenvalues $\pm i$. We note that, obviously, no J can be found on odd-dimensional manifolds.

Kähler manifolds: Let us consider a Hermitian metric on M given by

$$ds^2 = g_{a\bar{b}}\, dz^a\, d\bar{z}^b, \tag{3.21}$$

where $g_{a\bar{b}}$ is a Hermitian matrix. We define the Kähler form

$$K = \frac{i}{2} g_{a\bar{b}}\, dz^a \wedge d\bar{z}^b .$$

Then

$$\bar{K} = -\frac{i}{2} \bar{g}_{a\bar{b}}\, d\bar{z}^a \wedge dz^b = \frac{i}{2} g_{b\bar{a}}\, dz^b \wedge d\bar{z}^a = K$$

is a real 2-form.

A metric is said to be a Kähler metric if $dK = 0$, i.e., the Kähler form is closed. M is a Kähler manifold if it admits a Kähler metric. Any Riemann surface (real dimension 2) is automatically Kähler since $dK = 0$ for any 2-form. There are, however, complex manifolds of real dimension 4 which admit no Kähler metric.

If $dK = 0$, then, in fact, K is *harmonic* and

$$dK = \delta K = 0 .$$

For a Kähler metric, all the Laplacians are equal; $\Delta = \Delta' = \Delta''$. A Kähler manifold is *Hodge* if there exists a holomorphic line bundle whose first Chern form is the Kähler form of the manifold. Hodge manifolds are given by algebraic equations in $P_n(\mathbb{C})$ for some large n.

If a metric is a Kähler metric, then the set of the forms

$$K, K \wedge K, \ldots, \underset{(n \text{ times})}{K \wedge K \wedge \cdots \wedge K}$$

are all non-zero and harmonic. They define cohomology classes in $H^p(M; \mathbb{R})$ for $p = 2, \ldots, 2n$. (If the metric is Hodge, then these are all integral classes.) $P_n(\mathbb{C})$ is a Kähler manifold and all of its cohomology classes are generated by scalar multiples of the set of forms given above.

If M is any complex manifold, it has a natural orientation defined by requiring that

$$\int_M K \wedge \cdots \wedge K > 0 .$$

Examples 3.4

1. *Flat two space.* Taking $z = x + iy$, we choose the flat metric

$$ds^2 = dx^2 + dy^2 = (dx + i\,dy)(dx - i\,dy) = dz\, d\bar{z} .$$

Hence the Kähler form is

$$K = \frac{i}{2}(dx + i\,dy) \wedge (dx - i\,dy) = dx \wedge dy$$

which is obviously closed and coclosed.

2. *Two sphere*, $S^2 = P_1(\mathbb{C})$. We convert the standard metric on S^2 with radius $\frac{1}{2}$ into complex

coordinates:

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{(1+x^2+y^2)^2} = \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{(1+z\bar{z})^2}.$$

The Kähler form is then

$$K = \frac{\mathrm{i}}{2}\frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1+z\bar{z})^2} = \frac{\mathrm{d}x \wedge \mathrm{d}y}{(1+x^2+y^2)^2}.$$

Choosing vierbeins $e^1 = \mathrm{d}x/(1+x^2+y^2)$, $e^2 = \mathrm{d}y/(1+x^2+y^2)$ we find

$$K = e^1 \wedge e^2$$

$$*K = 1$$

so K is harmonic. We note that

$$K = \frac{\mathrm{i}}{2}\partial\bar{\partial}\ln(1+z\bar{z}).$$

3. *Fubini–Study metric on $P_n(\mathbb{C})$*. The Fubini–Study metric on $P_n(\mathbb{C})$ is given by the Kähler form

$$K = \frac{\mathrm{i}}{2}\partial\bar{\partial}\ln\Big(1 + \sum_{\alpha=1}^{n} z^\alpha \bar{z}^\alpha\Big)$$

$$= \frac{\mathrm{i}}{2}\frac{\mathrm{d}z^\alpha \wedge \mathrm{d}\bar{z}^\beta}{(1+\Sigma\, z^\gamma \bar{z}^\gamma)^2}[\delta_{\alpha\beta}(1+\Sigma\, z^\gamma\bar{z}^\gamma) - \bar{z}^\alpha z^\beta].$$

For $P_2(\mathbb{C})$, we find

$$\mathrm{d}s^2 = \frac{\Sigma\,\mathrm{d}z^\alpha\,\mathrm{d}\bar{z}^\alpha}{1+\Sigma\, z^\gamma\bar{z}^\gamma} - \frac{\Sigma\,\bar{z}^\alpha\,\mathrm{d}z^\alpha\,\bar{z}^\beta\,\mathrm{d}z^\beta}{(1+\Sigma\, z^\gamma\bar{z}^\gamma)^2}$$

$$= \frac{\mathrm{d}r^2 + r^2(\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2)}{1+r^2} - \frac{r^2(\mathrm{d}r^2 + r^2\sigma_z{}^2)}{(1+r^2)^2}$$

$$= \frac{\mathrm{d}r^2 + r^2\sigma_z{}^2}{(1+r^2)^2} + \frac{r^2(\sigma_x{}^2 + \sigma_y{}^2)}{1+r^2}.$$

Choosing the vierbein one-forms

$$e^0 = \mathrm{d}r/(1+r^2), \qquad e^1 = r\sigma_x/(1+r^2)^{1/2}$$
$$e^2 = r\sigma_y/(1+r^2)^{1/2}, \qquad e^3 = r\sigma_z/(1+r^2)$$

we find the connection one-forms

$$\omega^0{}_1 = -\frac{1}{r}e^1 \qquad \omega^2{}_3 = \frac{1}{r}e^1$$

$$\omega^0{}_2 = -\frac{1}{r}e^2 \qquad \omega^3{}_1 = \frac{1}{r}e^2$$

$$\omega^0{}_3 = \frac{r^2-1}{r}e^3 \qquad \omega^1{}_2 = \frac{1+2r^2}{r}e^3.$$

The curvatures are constant:

$$R_{01} = e^0 \wedge e^1 - e^2 \wedge e^3 \qquad R_{23} = -e^0 \wedge e^1 + e^2 \wedge e^3$$

$$R_{02} = e^0 \wedge e^2 - e^3 \wedge e^1 \qquad R_{31} = -e^0 \wedge e^2 + e^3 \wedge e^1$$

$$R_{03} = 4e^0 \wedge e^3 + 2e^1 \wedge e^2 \qquad R_{12} = 2e^0 \wedge e^3 + 4e^1 \wedge e^2.$$

We find that the Ricci tensor is

$$\mathcal{R}_{ab} = 6\delta_{ab}$$

so Einstein's equation

$$\mathcal{R}_{ab} - \tfrac{1}{2}\delta_{ab}\mathcal{R} = -\Lambda\delta_{ab}$$

is solved with the cosmological constant,

$$\Lambda = +6.$$

The Weyl tensor for the Fubini–Study metric is

$$W_{abcd} = R_{abcd} - 2(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}),$$

so the two-forms $W_{ab} = \frac{1}{2}W_{abcd}\, e^c \wedge e^d$ are *self-dual*:

$$W_{01} = W_{23} = -e^0 \wedge e^1 - e^2 \wedge e^3$$

$$W_{02} = W_{31} = -e^0 \wedge e^2 - e^3 \wedge e^1$$

$$W_{03} = W_{12} = 2e^0 \wedge e^3 + 2e^1 \wedge e^2.$$

More geometrical properties of $P_2(\mathbb{C})$ will be explored later.

4. $S^1 \times S^{2n-1}$. Let c be a complex number with $|c| > 1$. On $\mathbb{C}^n - \{0\}$ we introduce the equivalence relation $z \simeq z'$ if $z = c^k z'$ for some integer k. The resulting quotient manifold will be a complex manifold and will be topologically equivalent to $S^1 \times S^{2n-1}$. We suppose $n \geq 2$, then $H^*(S^1; \mathbb{R}) \otimes H^*(S^{2n-1}; \mathbb{R}) = H^*(S^1 \times S^{2n-1}; \mathbb{R})$ implies that $H^2(S^1 \times S^{2n-1}; \mathbb{R}) = 0$, so this complex manifold does not admit any Kähler metric. It is worth noting that different values of the constant c yield inequivalent complex manifolds (although the underlying topological type is unchanged).

5. *Metrics on the group manifolds of* $U(n)$, $\mathrm{SU}(n)$, $O(n)$, $\mathrm{SO}(n)$. Let $g(t)$, $h(t)$: $[0, \varepsilon) \to G$ be two curves with $g(0) = h(0) = g_0$. We define a metric on G by defining the inner product of the two tangent vectors $(g'(0), h'(0)) = -\mathrm{Tr}(g_0^{-1} g'(0) g_0^{-1} h'(0))$. It is easily verified that this is a positive definite metric which is both left and right invariant on these groups; i.e., multiplication on either the right or the left is an isometry which preserves this metric. Up to a scaling factor, this is the *Killing metric*.

4. Geometry of fiber bundles

Many important concepts in physics can be interpreted in terms of the geometry of fiber bundles. Maxwell's theory of electromagnetism and Yang–Mills theories are essentially theories of connections on principal bundles with a given gauge group G as the fiber. Einstein's theory of gravitation deals with the Levi–Civita connection on the frame bundle of the space-time manifold.

In this section, we shall define the notion of a fiber bundle and study the geometrical properties of a variety of interesting bundles. We begin for simplicity with vector bundles and then go on to treat principal bundles.

4.1. Fiber bundles

We begin our treatment of fiber bundles with an informal discussion of the basic concepts. We shall then outline a more formal mathematical approach. Suppose we are given some manifold M which we shall call the *base manifold* as well as another manifold F which we shall call the *fiber*. A *fiber bundle E* over M with fiber F is a manifold which is locally a direct product of M and F. That is, if M is covered by a set of local coordinate neighborhoods $\{U_i\}$, then the bundle E is topologically described in each neighborhood U_i by the product manifold

$$U_i \times F$$

as shown in fig. 4.1.

A little thought shows that the local direct-product structure still leaves a great deal of information about the *global* topology of E undetermined. To completely specify the bundle E, we must provide a set of *transition functions* $\{\Phi_{ij}\}$ which tell how the fiber manifolds match up in the overlap between two neighborhoods, $U_i \cap U_j$. We write Φ_{ij} as a mapping

$$\Phi_{ij}: F|_{U_i} \to F|_{U_j} \text{ in } U_i \cap U_j, \tag{4.1}$$

as illustrated in fig. 4.2. Thus, although the local topology of the bundle is trivial, the global topology

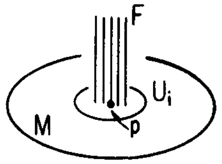

Fig. 4.1. Local direct-product structure of a fiber bundle. A vertical line represents a fiber associated to each point, such as p, in U_i.

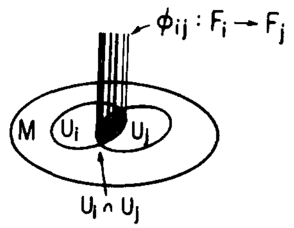

Fig. 4.2. The transition function Φ_{ij} defines the mapping of the coordinates of the fibers over U_i to those over U_j in the overlap region $U_i \cap U_j$.

determined by the transition functions may be quite complicated due to the relative twisting of neighboring fibers. For this reason, fiber bundles are sometimes called *twisted products* in the mathematical literature.

Example: *The Möbius strip.* A simple non-trivial fiber bundle is the Möbius strip, which we may construct as follows: Let the base manifold M be the circle S^1 parametrized by the angle θ. We cover S^1 by two semicircular neighborhoods $U_\pm$ as shown in fig. 4.3a,

$$U_+ = \{\theta: -\epsilon < \theta < \pi + \epsilon\}, \qquad U_- = \{\theta: \pi - \epsilon < \theta < 2\pi + \epsilon = 0 + \epsilon\}.$$

We take the fiber F to be an interval in the real line with coordinates $t \in [-1, 1]$. The bundle then consists of the two local pieces shown in fig. 4.3b,

$$U_+ \times F \text{ with coordinates } (\theta, t_+), \qquad U_- \times F \text{ with coordinates } (\theta, t_-),$$

and the transition functions relating t_+ to t_- in $U_+ \cap U_-$. This overlap consists of two regions I and II illustrated in fig. 4.3c. We choose the transition functions to be:

$$t_+ = t_- \text{ in region I} = \{\theta: -\epsilon < \theta < \epsilon\}$$

$$t_+ = -t_- \text{ in region II} = \{\theta: \pi - \epsilon < \theta < \pi + \epsilon\}.$$

Identifying t with $-t$ in region II twists the bundle and gives it the non-trivial global topology of the Möbius strip, as shown in fig. 4.3d.

Trivial bundles: If all the transition functions can be taken to be the identity, the global topology of the bundle is that of the direct product

$$E = M \times F.$$

Such bundles are called *trivial fiber bundles* or sometimes simply trivial bundles. For example, if we had set $t_+ = t_-$ in both regions I and II in the example above, we would have found a trivial bundle equal to the cylinder $S^1 \times [-1, 1]$.

It is a theorem that *any fiber bundle over a contractible base space is trivial.* Thus, for example, all

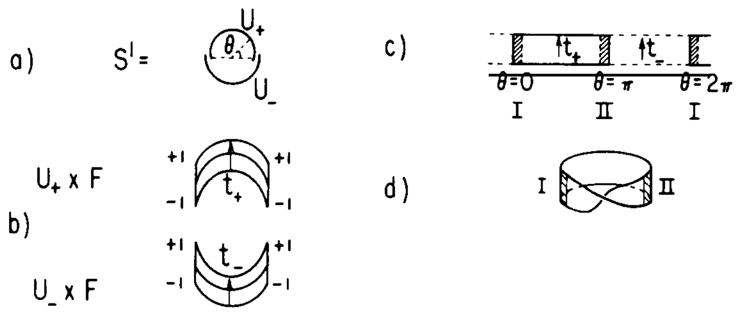

Fig. 4.3. Möbius strip. (a) The base space S^1 is covered by two neighborhoods $U_\pm$ which overlap at $\theta \sim 0$ and $\theta \sim \pi$. (b) Pieces of the bundle formed by taking the direct product of $U_\pm$ with the fiber $[-1, +1]$ having coordinates $t_\pm$. (c) The overlapping regions I and II of $U_+ \times F$ and $U_- \times F$. (d) A non-trivial bundle, the Möbius strip, is obtained by setting $t_+ = t_-$ in region I and $t_+ = -t_-$ in region II.

fiber bundles over a coordinate ball in $\mathbb{R}^n$ or over the sphere S^n minus a single point are necessarily trivial. Non-trivial fiber bundles can only be constructed when the global topology of the base space is non-trivial.

Sections: A *cross section* or simply a *section* s of a fiber bundle E is a rule which assigns a preferred point $s(x)$ on each fiber to each point x of the base manifold M, as illustrated in fig. 4.4. A *local section* is a section which is only defined over a subset of M. We can always define local sections in the local patches $U_i \times F$ from which the bundle is constructed. These sections are simply functions from U_i into F. The existence of *global* sections depends on the global geometry of the bundle E. There exist fiber bundles which have no global sections.

Formal approach to fiber bundles

A more sophisticated description of fiber bundles requires us to define a *projection* π which maps the fiber bundle E onto the base space M by shrinking each fiber to a point. If $x \in M$, $\pi^{-1}(x)$ is the *fiber over* x; $\pi^{-1}(x)$ acts like a flashlight shining through a hole at x to produce a "light ray" equal to the fiber. We sometimes denote the fiber F over x as F_x.

We let $\pi^{-1}(U_i)$ denote the subset of E which projects down to the neighborhood U_i in M. By assumption, there exists an isomorphism which maps $U_i \times F$ to $\pi^{-1}(U_i)$. This amounts to an assignment of local coordinates in the bundle often referred to as a *trivialization*. It is important to observe that this isomorphism is not *canonical*; we cannot simply identify $U_i \times F$ with $\pi^{-1}(U_i)$. We are now ready to give our formal description:

Formal definition of a fiber bundle: A fiber bundle E with fiber F over the base manifold M consists of a

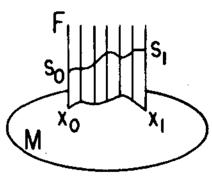

Fig. 4.4. A local cross-section or section of a bundle is a mapping which assigns a point s of the fiber to each point x of the base.

topological space E together with a projection $\pi: E \to M$ which satisfies the local triviality condition: For each point $x \in M$, there exists a neighborhood U_i of x and an isomorphism Φ_i which maps $U_i \times F$ to the subset $\pi^{-1}(U_i)$ of the bundle E. Letting (x, f) denote a point of $U_i \times F$, we require that $\pi(\Phi_i(x, f)) = x$ as a consistency condition. When we ignore the action of $\Phi_i(x, f)$ on the argument x, we may regard it as an x-dependent map $\Phi_{ix}(f)$ taking F into F_x.

The *transition functions* are defined as

$$\Phi_{ij} = \Phi_i^{-1}\Phi_j \tag{4.2}$$

in the overlap of the neighborhoods U_i and U_j. For each fixed $x \in U_i \cap U_j$, this is a map from F onto F; Φ_{ij} relates the local product structure over U_i to that over U_j. We require that these transition functions belong to a group G of transformations of the fiber space F. G is called the *structure group* of the fiber bundle.

The transition functions satisfy the cocycle conditions:

$$\Phi_{ii} = \text{identity}$$

$$\Phi_{ij}\Phi_{jk} = \Phi_{ik} \text{ for } x \in U_i \cap U_j \cap U_k.$$

A set of transition functions can be used to define a consistent procedure for gluing together local pieces of a bundle if and only if the cocycle conditions are satisfied. A bundle is completely determined by its transition functions.

Pullback bundles: Let E be a fiber bundle over the base manifold M with fiber F and suppose that $h: M' \to M$ is a map from some other manifold M' to M. The *pullback bundle* E' denoted by h^*E, is defined by copying the fiber of E over each point $x = h(x')$ in M over the point x' in M'. If we denote a point of $M' \times E$ by the pair (x', e), then

$$E' = h^*E = \{(x', e) \in M' \times E \text{ such that } \pi(e) = h(x')\}. \tag{4.3}$$

Thus E' is a *subset* of $M' \times E$ obtained by restricting oneself to the curve $\pi(e) = h(x')$. [Example: let h be the identity map and let $E = M = M' = \mathbb{R}$; then $x = x'$ is a line in $\mathbb{R}^2 = M' \times M$ and $E' = \mathbb{R}$.] If $\{U_i\}$ is a covering of M such that E is locally trivial over U_i and if $\Phi_{ij}(x)$ are the transition functions of E, then $\{h^{-1}(U_i)\}$ is a covering of M' such that $E' = h^*E$ is locally trivial. The corresponding transition functions of the pullback bundle are:

$$\Phi'_{ij}(x') = (h^*\Phi_{ij})(x') = \Phi_{ij}(h(x')). \tag{4.4}$$

It is clear that if $M = M'$ and if $h(x) = x$ is the identity map, then $h^*(E)$ can be naturally identified with the original bundle E.

Homotopy axiom: If h and g are two maps from M' to M, we say that they are *homotopic* if there exists a map $H: M' \times [0, 1] \to M$ such that $H(x', 0) = h(x')$ and $H(x', 1) = g(x')$. If we let $h_t(x') = h(x', t)$, then we are simply smoothly pushing the map $h = h_0$ to the map $g = h_1$. It is a theorem that if h and g are homotopic then h^*E is isomorphic to g^*E. For example, if M is contractible, we can let $h(x) = x$ be the identity map and let $g(x) = x_0$ be the map which collapses all of M to a point. These maps are homotopic so $E = h^*E$ is isomorphic to $g^*E = M \times F$; this proves that E is trivial if M is contractible.

4.2. Vector bundles

Let us consider a bundle E with a k-dimensional real fiber $F = \mathbb{R}^k$ over an n-dimensional base space M; k is commonly called the *bundle dimension* and we shall write $\dim(E) = k$ even though this is in reality the dimension of the fiber alone. (The total dimension of E is of course $(n + k)$.) E is called a *vector bundle* if is transition functions belong to $\mathrm{GL}(k, \mathbb{R})$ rather than to the full group of diffeomorphisms (differentiable transformations which are 1–1 and onto) of $\mathbb{R}^k$. Since $\mathrm{GL}(k, \mathbb{R})$ preserves the usual operations of addition and scalar multiplication on a vector space, the fibers of E inherit the structure of a vector space. We can think of a vector bundle as being a family of vector spaces (the fibers) which are parametrized by the base space M. Clearly there is a similar notion of a complex vector bundle if we replace $\mathbb{R}^k$ by $\mathbb{C}^k$ and $\mathrm{GL}(k, \mathbb{R})$ by $\mathrm{GL}(k, \mathbb{C})$.

Vector space structure on the set of sections: We can use the vector space structure on the fibers of a vector bundle to define the pointwise addition or scalar multiplication of sections. We write sections of a vector bundle in the form $s(x)$ to emphasize their vector-valued nature. Thus if $s(x)$ and $s'(x)$ are two local sections to E, we can define the local section $(s + s')(x) = s(x) + s'(x)$ by adding the values in the fibers. If $f(x)$ is a smooth continuous function on M, we can define the new section $[fs](x) = s(x)f(x)$ by pointwise scalar multiplication in the fibers.

Zero section: The origin $\{0\}$ of $\mathbb{C}^k$ or $\mathbb{R}^k$ is preserved by the general linear group and represents a distinguished element of the fiber of a vector bundle. Let $s(x) = 0$; this defines a global section called the *zero-section* of the vector bundle. We can always choose a non-zero section in any single neighborhood U_i. If we assume that this section is zero near the boundary of U_i, we can extend this section continuously to be zero outside of U_i. Therefore, any vector bundle has many global sections, although there may be no global sections which are everywhere non-zero.

Moving frames: At each point x of some neighborhood U of M, we can choose a basis $\{e_1(x), \ldots, e_k(x)\}$ for the k-dimensional fiber over x. We assume that the basis varies continuously with x if it varies at all; such a collection of bases defined for all x in U is called a *frame*. If we have chosen a local trivialization of $U \times \mathbb{C}^k \to \pi^{-1}(U)$, then we can regard the $e_i(x)$ as vector-valued functions from U into $\mathbb{C}^k$ and the entire frame as a matrix-valued function from U into $\mathrm{GL}(k, \mathbb{C})$. The coordinate frame is then the set of constant sections:

$$
\begin{aligned}
e_1(x) &= (1, 0, \ldots, 0) \\
e_2(x) &= (0, 1, \ldots, 0) \\
&\;\vdots
\end{aligned}
$$

We remark that one may still discuss the notion of a frame without necessarily having chosen a local trivialization.

A choice of frame $\{e_j(x)\}$ may in fact be used to specify the isomorphism Φ mapping $U \times \mathbb{C}^k \to \pi^{-1}(U)$. If $x \in U$ in M and if $z = (z_1, \ldots, z_k) \in \mathbb{C}^k$, we define

$$\Phi(x, z) = \sum_{j=1}^{k} e_j(x)\, z^j(x). \tag{4.5}$$

This introduces a local trivialization. Clearly $\Phi(x; 0, \ldots, 1, 0, \ldots, 0) = e_j(x)$ is just the vector in the fiber $\pi^{-1}(x)$ associated with the section $e_j(x)$.

Change of frames: Let U and U' be two neighborhoods in M and suppose that we have frames $\{e_i\}$ and $\{e'_i\}$ over U and U'. Let $\{z^i\}$ and $\{z''\}$ be the respective fiber coordinates, and let $\Phi = \Phi_{UU'}$ be the $\mathrm{GL}(k, \mathbb{C})$-valued transition function on $U \cap U'$. Then the frames, coordinates, and transition functions are related as follows:

$$\begin{aligned} e'_j(x) &= e_i(x)\, \Phi^{-1}_{ij}(x) \\ z'^i(x) &= \Phi_{ij}(x)\, z^j(x) \\ \Phi_{ij} &= (i, j) \text{ element of the matrix } \Phi_{U'U}. \end{aligned} \tag{4.6}$$

Hence

$$e_i z^i = e'_i z'^i$$

as required. *Note*: reversing the order in which the transition matrix acts would interchange the roles of right and left multiplication and would change the sign convention in the curvature from $R = \mathrm{d}\omega + \omega \wedge \omega$ to $R = \mathrm{d}\omega - \omega \wedge \omega$.

Line bundles: A line bundle is a vector bundle with a one-dimensional vector space as fiber. It is a family of lines parametrized by the base space M. If we replace the interval $[-1, 1]$ by the real line $\mathbb{R}$ in the Möbius strip example, we find a non-trivial real line bundle over the circle. If we replace $[-1, 1]$ by the complex numbers $\mathbb{C}$, the resulting line bundle is isomorphic to $S^1 \times \mathbb{C}$ and is therefore trivial. Note that $\mathrm{GL}(1, \mathbb{C})$ and $\mathrm{GL}(1, \mathbb{R})$ are *Abelian* groups so right and left multiplication commute; consequently, for line bundles, it does not matter whether we write the transition function on the left or on the right.

Tangent and cotangent bundles: We let the *tangent bundle* $T(M)$ and the *cotangent bundle* $T^*(M)$ be the real vector bundles whose fibers at a point $x \in M$ are given by the tangent space $T_x(M)$ or the cotangent space $T^*_x(M)$. These spaces were discussed earlier; we observe that if $x = (x_1, \ldots, x_n)$ is a local coordinate system defined on some neighborhood U in M, then we can choose the following standard bases for the local frames:

$$\begin{aligned} &\{\partial/\partial x_1, \ldots, \partial/\partial x_n\} \quad \text{for the tangent bundle } T(M) \\ &\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\} \qquad\quad\;\; \text{for the cotangent bundle } T^*(M). \end{aligned} \tag{4.7}$$

If U' is another neighborhood in M with local coordinates x', the transition functions in $U \cap U'$ are given by:

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x'_j} \cdot \frac{\partial x'_j}{\partial x_i} \qquad \text{on } T(M)$$
$$\mathrm{d}x_i = \mathrm{d}x'_j \cdot \frac{\partial x_i}{\partial x'_j} \qquad \text{on } T^*(M). \tag{4.8}$$

The *complexified* tangent and cotangent bundles $T(M) \otimes \mathbb{C}$ and $T^*(M) \otimes \mathbb{C}$ of a real manifold M are defined by permitting the coefficients of the frames $\{\partial/\partial x_i\}$ and $\{\mathrm{d}x_i\}$ to be complex.

If M is a complex manifold with local complex coordinates z_j, we define the *complex tangent bundle* $T_c(M)$ to be the sub-bundle of $T(M) \otimes \mathbb{C}$ which is spanned by the holomorphic tangent vectors $\partial/\partial z_j$. The (complex) dimension of $T_c(M)$ is half the real dimension of $T(M)$. If we forget the complex structure on $T_c(M)$ and consider $T_c(M)$ as a real bundle, then $T_c(M)$ is isomorphic to $T(M)$.

Constructions on vector bundles

If V is a vector space, we define the dual space V^* to be the set of linear functionals. If V and W are a pair of vector spaces, we can define the Whitney sum $V \oplus W$ and the tensor product $V \otimes W$. These and other constructions can be carried over to the vector bundle case as we describe in what follows.

Digression on dual vector spaces: We first recall some facts concerning the dual space V^* of linear functionals. An element $v^* \in V^*$ is just a linear map $v^*: V \to \mathbb{R}$. The sum and scalar multiple of linear maps are again linear maps so V^* is a vector space. If $\{e_1, \ldots, e_k\}$ is a basis for V and $v^* \in V^*$, then $v^*(e_j z^j) = z^j v^*(e_j)$, so the action of v^* on a section is determined by the value of the linear map on the basis. We define the *dual basis* $\{e^{*1}, \ldots, e^{*k}\}$ of the dual space V^* of linear functionals by

$$e^{*i}(e_j) = \delta^i_j \qquad \text{i.e.} \qquad e^{*i}(e_j z^j) = z^i.$$

These equations show that we can regard the e^{*i} themselves as defining coordinates on V. Similarly, the e_j define coordinates on V^*. We see that

$$\dim(V) = \dim(V^*) = k.$$

If we change bases and set $e_i = e'_j \Phi_{ji}$, then the new dual basis is given by

$$e^{*i} = \Phi^{-1}_{ij} e^{*\prime j} = e^{*\prime j} (\Phi^{\mathrm{t}})^{-1}_{ji}. \tag{4.9}$$

The dual basis transforms just as a set of coordinates on V transforms.

Dual vector spaces arise naturally whenever we have two vector spaces V and W together with a non-singular inner product $(v, w) \in \mathbb{R}$ or $\mathbb{C}$ where $v \in V$, $w \in W$. Since (v, w) is a linear functional on v, we can regard w as an element of the dual space V^* whose action is defined by

$$w(v) = (v, w).$$

Since the inner product is non-singular, we may identify W with V^*. Conversely, V and V^* possess a natural inner product defined by the action of v^* on v:

$$(v, v^*) = v^*(v).$$

We may regard V itself as a space of linear functionals dual to V^* if we define the action of $v \in V$ by

$$v(v^*) = (v, v^*) = v^*(v).$$

If V is finite dimensional, we find that $V^{**} = V$; this conclusion is false if V is infinite dimensional.

A simple example: Let V be the vector space of all polynomials of degree 1 or 0. Let $V = W$ and define an inner product by $(v, w) = \int_0^1 v(x)\, w(x)\, \mathrm{d}x$. If $\{1, x\}$ is a basis for V, the corresponding dual basis for $W \simeq V^*$ relative to this pairing is $\{4 - 6x, -6 + 12x\}$.

Dual bundles: Let E be a vector bundle with fiber F_x; let E^* be the *dual vector bundle* with pointwise fiber F_x^*. If $\{\boldsymbol{e}_i\}$ is a local frame for E, we have the dual frame $\{\boldsymbol{e}^{*j}\}$ for E^* defined by $\boldsymbol{e}^{*i}(\boldsymbol{e}_j) \equiv (\boldsymbol{e}_j, \boldsymbol{e}^{*i}) = \delta_{ij}$. If the transition functions of E are given by $k \times k$ matrices Φ, then the transition functions of E^* are given by the $k \times k$ matrices $(\Phi^{\mathrm{t}})^{-1}$.

If $E = T(M)$ is the tangent bundle, then $E^* = T^*(M)$ is the cotangent bundle. The $\{\partial/\partial x_i\}$ and the $\{\mathrm{d}x_i\}$ are dual bases in the usual sense and the transition matrices given earlier satisfy all the required properties.

Whitney sum bundle: The Whitney sum $V \oplus W$ of two vector spaces V and W is defined to be the set of all pairs (v, w). The vector space structure of (v, w) is

$$(v, w) + (v', w') = (v + v', w + w') \quad \text{and} \quad \lambda(v, w) = (\lambda v, \lambda w).$$

If we identify v with $(v, 0)$ and w with $(0, w)$, then V and W are subspaces of $V \oplus W$. If $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{f}_j\}$ form bases for V and W, respectively, then $\{\boldsymbol{e}_i, \boldsymbol{f}_j\}$ is a basis for $V \oplus W$ so $\dim(V \oplus W) = \dim(V) + \dim(W)$.

If E and F are vector bundles over M, the fiber of the *Whitney sum bundle* $E \oplus F$ is obtained by taking the Whitney direct sum of the fibers of E and F at each point $x \in M$. If $\dim(E) = j$ and $\dim(F) = k$ and if the transition functions of E and F are the $j \times j$ matrices Φ and the $k \times k$ matrices Ψ, respectively, then the transition matrices of $E \oplus F$ are just the $(j + k) \times (j + k)$ matrices $\Phi \oplus \Psi$ given by:

$$\begin{pmatrix} \Phi & 0 \\ 0 & \Psi \end{pmatrix} = \Phi \oplus \Psi. \tag{4.10}$$

If $\{\boldsymbol{e}_i\}$, $\{\boldsymbol{f}_l\}$ are local frames for E and F, then $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_j, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_k\}$ is a local frame for $E \oplus F$. Clearly, $\dim(E \oplus F) = \dim(E) + \dim(F) = j + k$.

Tensor product bundle: The tensor product bundle $E \otimes F$ of E and F is obtained by taking the tensor product of the fibers of E and of F at each point $x \in M$. The transition matrices for $E \otimes F$ are obtained by taking the tensor product of the transition functions of E and the transition functions of F. A local frame for $E \otimes F$ is given by $\{e_i \otimes f_l\}$ so $\dim(E \otimes F) = \dim(E)\dim(F)$.

Bundles of linear maps: If V and W are vector spaces, we define $\mathrm{Hom}(V, W)$ to be the *space of all linear maps* from V into W. For example, $\mathrm{Hom}(V, \mathbb{R}) = V^*$ since V^* is by definition the space of all linear maps from V to $\mathbb{R}$. If $\dim(V) = j$ and $\dim(W) = k$, then $\mathrm{Hom}(V, W)$ can be identified with the set of all $k \times j$ matrices and is a vector space in its own right. If E and F are vector bundles, we define $\mathrm{Hom}(E, F)$ to be the vector bundle whose fiber is Hom of the fibers of E and F. There is a natural isomorphism $\mathrm{Hom}(V, W) = V^* \otimes W$ and similarly $\mathrm{Hom}(E, F) = E^* \otimes F$. Since $E^{**} = E$, the isomorphism $\mathrm{Hom}(E^*, F) = E \otimes F$ can be used to give an alternative definition of the tensor product.

Other constructions: Let $\otimes^p(E) = E \otimes \cdots \otimes E$ be the bundle of p-tensors. $\Lambda^p(E)$ is the bundle of antisymmetric p-tensors and $S^p(E)$ is the bundle of symmetric p-tensors; these are both sub-bundles of $\otimes^p(E)$. If $\dim(E) = k$, then

$$\dim(\otimes^p(E)) = k^p, \qquad \dim(\Lambda^p(E)) = \binom{k}{p}, \qquad \dim(S^p(E)) = \binom{k+p-1}{p}.$$

The transition functions of $\Lambda^p(E)$ and $S^p(E)$ are p-fold tensor products of the transition functions of E with the appropriate symmetry properties. Note that $C^\infty(\Lambda^p(T^*(M)))$ is just the space of p-forms on M.

Complementary bundles (*normal bundles*): If E is a real or complex vector bundle over M with fiber V of dimension k, we can always construct a (nonunique) complementary bundle $\bar{E}$ such that the Whitney sum $E \oplus \bar{E} \simeq M \times \mathbb{C}^l$ is a trivial bundle with fiber $\mathbb{C}^l$ for some $l > k$. A frequent application of this fact occurs in the construction of the *tangent* and *normal bundles* of a manifold. If M is an n-dimensional complex manifold embedded in $\mathbb{C}^m$, the bundle of tangent vectors $T_c(M)$ (dimension $= n$) and the bundle of normal vectors $N_c(M)$ (dimension $= m - n$) are both non-trivial in general. However, the Whitney sum is the trivial $n + (m - n) = m$-dimensional bundle I_m:

$$T_c(M) \oplus N_c(M) = I_m = M \times \mathbb{C}^m. \tag{4.11}$$

Fiber metrics (*inner products*): *A fiber metric* is a pointwise inner product between two sections of a vector bundle which allows us to define the *length* of a section at a point x of the base. In local coordinates, a fiber metric is a positive definite symmetric matrix $h_{ij}(x)$. The inner product of two sections is then

$$(s, s') = h_{ij}(x)\, z^i(x)\, \bar{z}'^j(x), \tag{4.12}$$

where $\bar{z}$ denotes complex conjugation if the fiber is complex. Under a change of frame, we obviously find

$$h \to (\Phi^{\mathrm{t}})^{-1} h \bar{\Phi}^{-1}.$$

A fiber metric defines a (conjugate) linear isomorphism between E and E^*.

If E is a real vector bundle with a fiber metric, the fiber metric defines a pairing of E with itself and gives an isomorphism between E and E^*. If $E = T(M)$, the fiber metric is simply a Riemannian metric on M; thus *$T(M)$ is always isomorphic to $T^*(M)$.*

If E is a complex vector bundle, the fiber metric is conjugate linear in the second factor. This defines a conjugate linear pairing of E with itself and gives a conjugate linear isomorphism between E and E^*. Thus in the complex case, E need not be isomorphic to E^*; this fact can sometimes be detected by the characteristic classes, as we shall see later.

Examples 4.2

1. *Tangent and cotangent bundles of S^2*: Let $U = S^2 - \{(0, 0, -\frac{1}{2})\}$ and let $U' = S^2 - \{(0, 0, \frac{1}{2})\}$ be spheres of unit diameter minus the south/north poles. We stereographically project these two neighborhoods to the plane to define coordinates $\boldsymbol{x} = (x, y)$ and $\boldsymbol{x}' = (x', y')$. Let $r^2 = x^2 + y^2$. In these coordinates, the standard metric is given by:

$$ds^2 = (1 + r^2)^{-2}(dx^2 + dy^2).$$

The U' coordinates are related to the U coordinates by the inversion

$$\boldsymbol{x}' = r^{-2}\boldsymbol{x},$$

so

$$d\boldsymbol{x}' = r^{-4}(r^2\, d\boldsymbol{x} - 2\boldsymbol{x}\,(\boldsymbol{x} \cdot d\boldsymbol{x})).$$

The transition functions $|\partial x'/\partial x|$ for $T^*(S^2)$ are therefore given by:

$$\Phi_{U'U}(\boldsymbol{x}) = r^{-4}(\delta_{ij}r^2 - 2x_ix_j) \quad \text{on} \quad U' \cap U.$$

We introduce polar coordinates on $\mathbb{R}^2 - (0, 0)$ and restrict to $r = 1$, so that we are effectively working on the equator S^1 of the sphere. Then we find

$$\Phi_{U'U}(\cos\theta, \sin\theta) = \begin{pmatrix} -\cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}. \tag{4.13}$$

(The transposed inverse matrix is of course the transition matrix for $T(S^2)$.) $\Phi_{U'U}$ represents a non-trivial map of $S^1 \to \mathrm{GL}(2, \mathbb{R})$. This map is just twice the generator of $\pi_1(\mathrm{GL}(2, \mathbb{R})) = \mathbb{Z}$.

The bundles $T(S^2)$ and $T^*(S^2)$ are non-trivial and isomorphic. Let I denote the trivial bundle over S^2. We can identify I with the normal bundle of S^2 in $\mathbb{R}^3$ so $T(S^2) \oplus I = T(\mathbb{R}^3) = I^3$ is the trivial bundle of dimension 3 over S^2. Similarly $T^*(S^2) \oplus I = T^*(\mathbb{R}^3) = I^3$. If we regard the transition map $\Phi_{U'U} \oplus I$ given above as a map from S^1 to $\mathrm{GL}(3, \mathbb{R})$, then it is still twice the generator. Since $\pi_1(\mathrm{GL}(3, \mathbb{R})) = Z_2$, the map is null homotopic and $T^*(S^2) \oplus I$ is trivial.

2. *The natural line bundle over projective space.* We defined $P_n(\mathbb{C})$ to be the set of lines through the origin in $\mathbb{C}^{n+1}$. Let $I^{n+1} = P_n(\mathbb{C}) \times \mathbb{C}^{n+1}$ be the trivial bundle of dimension $n + 1$ over $P_n(\mathbb{C})$. We denote a point of I^{n+1} by the pair (p, z); scalar multiplication and addition are performed on the second factor while leaving the first factor unchanged in this expression. Let L be the sub-bundle of I^{n+1} defined by:

$$L = \{(p, z) \in I^{n+1} = P_n(\mathbb{C}) \times \mathbb{C}^{n+1} \text{ such that } z \in p\}. \tag{4.14}$$

In other words, the fiber of L over a point p of $P_n(\mathbb{C})$ is just the set of points in $\mathbb{C}^{n+1}$ which belong to the line p.

In example 2.1.3 we defined coordinates $\zeta_i^{(j)} = z_i/z_j$ on neighborhoods $U_j = \{p: z_j(p) \neq 0\}$. On U_j, we define the section s_j to L by:

$$s_j(p) = (\zeta_0^{(j)}(p), \ldots, 1, \ldots, \zeta_n^{(j)}(p)).$$

The transition functions are 1×1 complex matrices – i.e. scalars:

$$s_k(p) = (\zeta_k^{(j)})^{-1} s_j(p).$$

Since the transition functions are holomorphic, L is a holomorphic line bundle.

The dual bundle L^* has sections s_j^* so that $s_j^*(s_j) = 1$. (*Note*: since we have a line bundle, a frame is given by a single section. The subscripts here refer to different coordinate systems and not to elements of a frame.) The transition functions act as

$$s_k^* = s_j^* \zeta_k^{(j)}.$$

We now interpret the $\{s_j^*\}$ as homogeneous coordinates on $P_n(\mathbb{C})$, since it is clear that

$$s_j^*(p) = z_j.$$

Note that $s_j^* = 0$ whenever $z_j = 0$, i.e. whenever p is *not* in the neighborhood U_j. The ratio of these global sections may be used to define the inhomogeneous coordinates $\zeta_i^{(j)}$.

Note: L^* has global holomorphic sections s_j^* whose zeroes lie in the complement of U_j, which is just a projective space of dimension $(n-1)$. The bundle L does *not* have any global holomorphic sections; since $s_j s_j^* = 1$ and $s_j^* = 0$ on the complement of U_j, s_j must blow up like z_j^{-1} on the complement of U_j. The s_j are therefore *meromorphic* sections of L.

We define the line bundle L^k by:

$$\begin{aligned} &L^* \otimes \cdots \otimes L^* \quad \text{if } k < 0 \\ &L^0 = I \quad \text{(the trivial line bundle)} \\ &L \otimes \cdots \otimes L \quad \text{if } k > 0. \end{aligned} \tag{4.15}$$

Because $L \otimes L^* = I$, $L^j \otimes L^k = L^{j+k}$ for all integers j, k. Any line bundle over $P_n(\mathbb{C})$ is isomorphic to L^k for some uniquely defined integer k. The integer k is related to the first Chern class of L^k as we shall see later.

Let $T_c(P_n(\mathbb{C}))$ and $T_c^*(P_n(\mathbb{C})) = \Lambda^{1,0}(P_n(\mathbb{C}))$ be the complex tangent and cotangent spaces. Then:

$$I \oplus T_c(P_n(\mathbb{C})) = L^* \oplus \cdots \oplus L^* \quad \text{(a total of } n+1 \text{ times)}$$

$$I \oplus T_c^*(P_n(\mathbb{C})) = L \oplus \cdots \oplus L \quad \text{(a total of } n+1 \text{ times).}$$

(This identity does not preserve the holomorphic structures but is an isomorphism between complex vector bundles.)

3. *Relationship between $T(S^2)$ and L^k*. Using the relations $S^2 = P_1(\mathbb{C})$ and $T(S^2) = T_c(P_1(\mathbb{C}))$, we may combine the two previous examples for $n = 1$ to show

$$T^*(S^2) = L \otimes L, \qquad T(S^2) = L^* \otimes L^*. \tag{4.16}$$

We prove these relationships by recalling that we may choose complex coordinates on S^2 of the form $\zeta_0 = z_1/z_0$ on U_0 and $\zeta_1 = z_0/z_1 = \zeta_0^{-1}$ on U_1. We choose the basis of $T^*(S^2)$ to be $\mathrm{d}\zeta_0$ on U_0 and $-\mathrm{d}\zeta_1$ on U_1. The transition functions are given by

$$(-\mathrm{d}\zeta_1) = \zeta_0^{-2}(\mathrm{d}\zeta_0).$$

The local sections

$$s_0 = (1, \zeta_0), \qquad s_1 = (\zeta_0^{-1}, 1)$$

of L give the transition function $s_1 = \zeta_0^{-1} s_0$. The $L \otimes L$ transition functions are thus

$$s_1 \otimes s_1 = \zeta_0^{-2} s_0 \otimes s_0,$$

so $T^*(S^2)$ and $L \otimes L$ are isomorphic bundles. The isomorphism between $T(S^2)$ and $L^* \otimes L^*$ is obtained by dualizing the preceding argument.

4.3. *Principal bundles*

A vector bundle is a fiber bundle whose fiber F is a linear vector space and whose transition functions belong to the general linear group of F. A *principal bundle* P is a fiber bundle whose fiber is a Lie group G (which is a manifold); the transition functions of P belong to G and act on G by *left* multiplication. We can define a *right* action of G on P because left and right multiplication commute. This action is a map from $P \times G \to P$ which commutes with the projection π, i.e.

$$\pi(p \cdot g) = \pi(p) \quad \text{for any } g \in G \text{ and } p \in P.$$

We remind the reader that the roles of left and right multiplication may be reversed if desired.

We can construct a principal bundle P known either as the *frame bundle* or as the *associated principal bundle* from a given vector bundle E. The fiber G_x of P at x is the set of all frames of the vector space F_x which is the fiber of E over the point x. In order to be specific, let us consider the case of the complex vector space of k dimensions, $F = \mathbb{C}^k$. Then the fiber G of the frame bundle P is the collection of the $k \times k$ non-singular matrices which form the group $\mathrm{GL}(k, \mathbb{C})$; i.e., G is the structure group of the vector bundle E.

The associated principal bundle P has the same transition functions as the vector bundle E. These transition functions are $\mathrm{GL}(k, \mathbb{C})$ group elements and they act on the fiber G by *left* multiplication. On the other hand the *right* action of the group $G = \mathrm{GL}(k, \mathbb{C})$ on the principal G bundle P takes a frame $e = \{e_1, \ldots, e_k\}$ to a new frame in the same fiber

$$e \cdot g = \{e_i g_{i1}, \ldots, e_i g_{ik}\} \qquad \text{(sum over } i \text{ is implied)} \tag{4.17}$$

for $|g_{ij}| \in \mathrm{GL}(k, \mathbb{C})$.

If P is a principal G bundle and if ρ is a representation of G on a finite-dimensional vector space V, we can define the *associated vector bundle* $P \times_\rho V$ by introducing the equivalence relation on $P \times V$:

$$(p, \rho(g) \cdot v) \simeq (p \cdot g, v) \quad \text{for all } p \in P, v \in V, g \in G. \tag{4.18}$$

The transition functions on $P \times_\rho V$ are given by the representation $\rho(\Phi)$ applied to the transition functions Φ of P. If P is the frame bundle of E and if ρ is the identity representation of G on the fiber F, then $P \times_\rho F = E$. In this way we may pass from a vector bundle E to its associated principal bundle P and back again by changing the space on which the transition functions act from the vector space to the general linear group and back.

Unitary frame bundles: If E is a vector bundle with an inner product, we can apply the Gram–Schmidt process to construct unitary frames. The bundle of unitary frames is a $U(k)$ principal bundle if E is complex and an $O(k)$ principal bundle if E is real. If E is an oriented real bundle, we may consider the set of oriented frames to define an $\mathrm{SO}(k)$ principal bundle.

If E is a complex vector bundle with an inner product and if the transition functions are unitary with determinant 1, we can define an $\mathrm{SU}(k)$ principal bundle associated with E. However, not every vector bundle admits $\mathrm{SU}(k)$ transition functions; the first Chern class must vanish.

Local sections: If $\gamma(x)$ is a local section to P over a neighborhood U in M, we can use right multiplication to define a map

$$\Phi\colon U \times G \to \pi^{-1}(U),$$

where $\Phi(x, g) = \gamma(x) \cdot g$. This gives a local trivialization of P. A principal bundle P is trivial if and only if it has a global section; non-trivial principal bundles do not admit global sections. (The identity element of G is *not* invariant so there is no analog of the zero section to a vector bundle.)

Lie algebras: The Lie algebra $\mathcal{G}$ of G is the tangent space $T_e(G)$ at the identity element e of G. By using *left* translation in the group, we may identify $\mathcal{G}$ with the set of left-invariant vector fields on G. Let $\mathcal{G}^*$ be the dual space. We can identify $\mathcal{G}^*$ with the left-invariant 1-forms on G. Let $\{L_a\}$ be a basis for $\mathcal{G}$ and let $\{\phi_a\}$ be the dual basis for $\mathcal{G}^*$. The $\{L_a\}$ obey the Lie bracket algebra

$$[L_a, L_b] = f_{abc} L_c, \tag{4.19}$$

where the f_{abc} are the structure constants for $\mathcal{G}$. The Maurer–Cartan equation

$$\mathrm{d}\phi_a = \tfrac{1}{2} f_{abc} \phi_b \wedge \phi_c \tag{4.20}$$

is the corresponding equation for $\mathcal{G}^*$.

Examples 4.3

1. *Principal Z_2 bundle.* One of the simplest examples of a principal fiber bundle is obtained from the Möbius strip example with $M = S^1$ by replacing the line-interval fiber $F = [-1, 1]$ by its end points ± 1. These end points form a group under multiplication

$$\boldsymbol{Z}_2 = S^0 = \{+1, -1\},$$

and we have a fiber which is a group manifold. The transition functions Φ are $\mathbb{Z}_2$ group elements and act on the fiber $F = \mathbb{Z}_2$ by the group multiplication. We let $M = S^1$ be covered by two neighborhoods, so there are two overlapping regions I and II. Then we can construct two different types of bundles in the following way;

$$\text{trivial bundle: } \Phi_{\mathrm{I}} = \Phi_{\mathrm{II}} \qquad E = S^1 \times \mathbb{Z}_2 = \text{two circles;}$$

$$\text{non-trivial bundle: } \Phi_{\mathrm{I}} = -\Phi_{\mathrm{II}}, \qquad E = \text{double covering of a circle.}$$

These bundles correspond to the boundaries of a cylinder and a Möbius strip.

2. *Magnetic monopole bundle.* We shall see later that Dirac's magnetic monopole corresponds to a principal $U(1)$ bundle over S^2. We construct this bundle by taking

$$\text{Base } M = S^2; \qquad \text{coordinates } (\theta, \phi),\ 0 \le \theta < \pi,\ 0 \le \phi < 2\pi$$

$$\text{Fiber } F = U(1) = S^1; \qquad U(1) \text{ coordinate } e^{i\psi}.$$

We break S^2 into two hemispherical neighborhoods $H_\pm$ with $H_+ \cap H_-$ a thin strip parametrized by the equatorial angle ϕ, as shown in fig. 4.5. Locally, the bundle looks like

$$H_+ \times U(1), \quad \text{coordinates } (\theta, \phi; e^{i\psi_+})$$

$$H_- \times U(1), \quad \text{coordinates } (\theta, \phi; e^{i\psi_-}).$$

The transition functions must be functions of ϕ along $H_+ \cap H_-$ and must be elements of $U(1)$ to give a principal bundle. We therefore choose to relate the H_+ and H_- fiber coordinates as follows:

$$e^{i\psi_-} = e^{in\phi}\, e^{i\psi_+}. \tag{4.21}$$

n must be an integer for the resulting structure to be a manifold; the fibers must fit together exactly when we complete a full revolution around the equator in ϕ. This is in essence a topological version of the Dirac monopole quantization condition.

For $n = 0$, we have a trivial bundle

$$P(n = 0) = S^2 \times S^1.$$

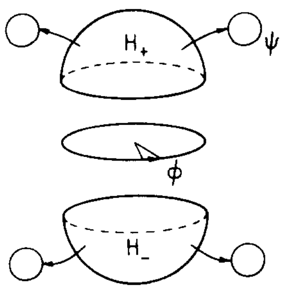

Fig. 4.5. The magnetic monopole bundle, showing the two hemispherical neighborhoods $H_\pm$ covering the base manifold $M = S^2$. A fiber $U(1) = S^1$ parametrized by ψ is attached to each point of $H_\pm$. The intersection of $H_\pm$ at $\theta \sim \pi/2$ is a strip parametrized by ϕ.

The case $n = 1$ is the famous Hopf fibering (Steenrod [1951]; Trautman [1977]) of the three-sphere

$$P(n = 1) = S^3$$

and describes a singly-charged Dirac monopole. For general n, we have a more complicated bundle corresponding to a monopole of charge n.

Remark: n corresponds to the *first Chern class* and characterizes inequivalent monopole bundles.

3. *Instanton bundle*. Another interesting principal bundle corresponds to the Yang–Mills instanton. We take the base space to be compactified Euclidean space-time, namely the four-sphere, and the fiber to be the group SU(2):

$$\begin{aligned} &\text{Base } M = S^4; && \text{coordinates } (\theta, \phi, \psi, r) \\ &\text{Fiber } F = \mathrm{SU}(2) = S^3; && \text{coordinates } (\alpha, \beta, \gamma). \end{aligned}$$

We split S^4 into two "hemispheres" $H_\pm$ whose boundaries are S^3's. Thus we may parametrize the thin intersection of H_+ with H_- along the "equator" of S^4 by the Euler angles (θ, ϕ, ψ) of S^3. Using the standard construction, we have a representation $h(\theta, \phi, \psi)$ of SU(2),

$$h = \frac{t - \mathrm{i}\boldsymbol{\lambda} \cdot \boldsymbol{x}}{r}, \qquad \begin{cases} x + \mathrm{i}y = r \cos\dfrac{\theta}{2} \exp\dfrac{\mathrm{i}}{2}(\psi + \phi) \\ z + \mathrm{i}t = r \sin\dfrac{\theta}{2} \exp\dfrac{\mathrm{i}}{2}(\psi - \varphi), \end{cases}$$

where the $\boldsymbol{\lambda}$ are the Pauli matrices. The fiber coordinates are similarly given by SU(2) matrices $g(\alpha, \beta, \gamma)$ depending on the group Euler angles (α, β, γ).

Thus we have the local bundle patches

$$\begin{aligned} &H_+ \times \mathrm{SU}(2), && \text{coordinates } (\theta, \phi, \psi, r; \alpha_+, \beta_+, \gamma_+) \\ &H_- \times \mathrm{SU}(2), && \text{coordinates } (\theta, \phi, \psi, r; \alpha_-, \beta_-, \gamma_-). \end{aligned}$$

In $H_+ \cap H_-$, we construct the transition from the SU(2) fibers $g(\alpha_+, \beta_+, \gamma_+)$ to $g(\alpha_-, \beta_-, \gamma_-)$ using multiplication by the SU(2) matrix $h(\theta, \phi, \psi)$;

$$g(\alpha_-, \beta_-, \gamma_-) = h^k(\theta, \phi, \psi)\, g(\alpha_+, \beta_+, \gamma). \tag{4.22}$$

The power k of the matrix $h(\theta, \phi, \psi)$ must be an integer to give a well-defined manifold.

For $k = 1$, we get the Hopf fibering of S^7 (Steenrod [1951]; Trautman [1977]),

$$P(k = 1) = S^7.$$

This is the bundle described by the single-instanton solution (Belavin et al. [1975]). More general instanton solutions describe bundles with other values of k.

Remark: k corresponds to the *second Chern class* and characterizes the equivalence classes of instanton bundles.

4.4. Spin bundles and Clifford bundles

We have concentrated in most of this section on vector bundles and principal bundles whose fibers had structure groups such as $O(k)$ and $U(k)$. Another important type of vector space which may appear as a fiber is a space of *spinors*. The structure group of a spinor space is the spin group, Spin(k). For example, the spin group corresponding to SO(3) is just its double covering, Spin(3) = SU(2). The principal spin bundles associated with a bundle of spinors have fibers lying in Spin(k). We note that not all base manifolds admit well-defined spinor structures; spinors arising from the tangent space can only be defined for manifolds where the second Stiefel–Whitney class (described in section 6) vanishes.

Spinors must in general belong to an algebra of anticommuting variables. Such variables are a special case of the more general notion of a *Clifford algebra*, which may also be used to define a type of fiber bundle. For example, if we start with a real vector bundle E of dimension k, we can construct the corresponding *Clifford bundle*, Cliff(E), as follows. The sections of Cliff(E) are constructed from sections $s(x)$ and $s'(x)$ of E by introducing the Clifford multiplication

$$s \cdot s' + s' \cdot s = 2(s, s'), \tag{4.23}$$

where (s, s') is the vector bundle inner product. Cliff(E) is then a 2^k-dimensional bundle containing E as a sub-bundle. The Clifford algebra acts on itself by Clifford multiplication; relative to a matrix basis, this action admits a $2^k \times 2^k$ dimensional representation of the algebra. For $k = 1$, we find a 2×2 Pauli matrix representation, while for $k = 2$, we have the 4×4 Dirac matrices.

We note that there is a natural isomorphism between the exterior algebra bundle $\Lambda^*(E)$ and the Clifford bundle, Cliff(E). For example, the 16 independent Dirac matrix components 1, γ_5, γ_μ, $\gamma_\mu\gamma_5$ and $[\gamma_\mu, \gamma_\nu]$ can be matched with the elements 1, $dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$, dx^μ, $\epsilon_{\mu\nu\lambda\sigma}\, dx^\nu \wedge dx^\lambda \wedge dx^\sigma$ and $dx^\mu \wedge dx^\nu$ of Λ^*.

For further details, see Chevalley [1954] and Atiyah, Bott and Shapiro [1964].

5. Connections on fiber bundles

So far, we have only considered fiber bundles as global geometric constructions. The notion of a connection plays an essential role in the local differential geometry of fiber bundles. A connection defines a covariant derivative which contains a gauge field and specifies the way in which a vector in the bundle E is to be parallel-transported along a curve lying in the base M. We shall first describe connections on vector bundles and then proceed to treat connections on principal bundles. We shall give several examples, including the Dirac monopole and the Yang–Mills instanton.

5.1. Vector bundle connections

The Levi-Civita connection on a surface in $\mathbb{R}^3$

The modern concept of a connection arose from the attempt to find an intrinsic definition of differentiation on a curved two dimensional surface embedded in the three dimensional space $\mathbb{R}^3$ of physical experience. We take the unit sphere S^2 in $\mathbb{R}^3$ as a specific example. Let the coordinates

$$x(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \quad 0 \le \theta \le \pi, 0 \le \phi \le 2\pi$$

parametrize the sphere. We observe that $\boldsymbol{x}(\theta, \phi)$ is also the unit normal. The Riemannian metric induced by the chosen embedding is given by:

$$g_{ij} = \begin{pmatrix} \partial_\theta \boldsymbol{x} \cdot \partial_\theta \boldsymbol{x} & \partial_\theta \boldsymbol{x} \cdot \partial_\phi \boldsymbol{x} \\ \partial_\theta \boldsymbol{x} \cdot \partial_\phi \boldsymbol{x} & \partial_\phi \boldsymbol{x} \cdot \partial_\phi \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

so that

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2 .$$

The two vector fields

$$\boldsymbol{u}_1 = \partial_\theta \boldsymbol{x} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$

$$\boldsymbol{u}_2 = \partial_\phi \boldsymbol{x} = (-\sin\theta\sin\phi, \sin\theta\cos\phi, 0)$$

are tangent to the surface and span the tangent space provided that $0 < \theta < \pi$ (i.e., away from the north and south poles, where this parametrization is singular). Clearly, any derivative can be decomposed as shown in fig. 5.1 into tangential components proportional to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, and a normal component $\hat{n}$ proportional to $\boldsymbol{x}$. We identify $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ with the bases $\partial/\partial\theta$ and $\partial/\partial\phi$ for the tangent space because

$$\partial f(\boldsymbol{x})/\partial\theta = \boldsymbol{u}_1 \cdot \partial f/\partial \boldsymbol{x}, \qquad \partial f(\boldsymbol{x})/\partial\phi = \boldsymbol{u}_2 \cdot \partial f/\partial \boldsymbol{x}$$

where $f(\boldsymbol{x})$ is a function on $\mathbb{R}^3$.

Our goal is now to differentiate tangential vector fields in a way which is intrinsic to the surface and not to the particular *embedding* involved.

First we compute the ordinary partial derivatives

$$\partial_\theta(u_1) = (-\sin\theta\cos\phi, -\sin\theta\sin\phi, -\cos\theta) = -\boldsymbol{x}$$

$$\partial_\phi(u_1) = \partial_\theta(u_2) = (-\cos\theta\sin\phi, \cos\theta\cos\phi, 0) = \frac{\cos\theta}{\sin\theta}\boldsymbol{u}_2$$

$$\partial_\phi(u_2) = (-\sin\theta, \cos\phi, -\sin\theta\sin\phi, 0) = -\sin^2\theta\,\boldsymbol{x} - \cos\theta\sin\theta\,\boldsymbol{u}_1 .$$

We define intrinsic covariant differentiation ∇_X with respect to a given tangent vector X by taking the

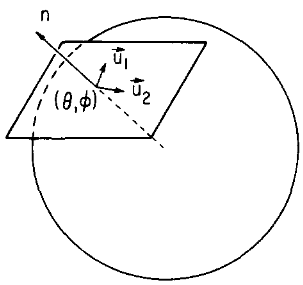

Fig. 5.1. Normal direction $\hat{n}$ and tangential directions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ at a point (θ, ϕ) of S^2 embedded in R^3.

ordinary derivative and projecting back to the surface. ∇_X is then the directional derivative obtained by throwing away the normal component of the ordinary partial derivatives:

$$\nabla_{u_1}(u_1) = 0$$

$$\nabla_{u_1}(u_2) = \nabla_{u_2}(u_1) = \cot\theta\, u_2$$

$$\nabla_{u_2}(u_2) = -\cos\theta\sin\theta\, u_1.$$

∇ is the *Levi–Civita connection* on S^2. Using the identification of (u_1, u_2) with $(\partial/\partial\theta, \partial/\partial\phi)$, we write

$$\nabla_{\partial/\partial\theta} \equiv \nabla_{u_1}, \qquad \nabla_{\partial/\partial\phi} \equiv \nabla_{u_2}.$$

Now the Christoffel symbol is defined by

$$\nabla_{u_i}(u_j) = u_k \Gamma^k{}_{ij} \quad \text{or} \quad \nabla_{\partial_i}(\partial_j) = \Gamma^k{}_{ij}\,\partial_k$$

where $\partial_1 = \partial/\partial\theta$, $\partial_2 = \partial/\partial\phi$. Then, in our example, we find

$$\Gamma^2{}_{12} = \Gamma^2{}_{21} = \cot\theta, \qquad \Gamma^1{}_{22} = -\cos\theta\sin\theta, \qquad \Gamma^k{}_{ij} = 0 \text{ otherwise}.$$

Geodesic equation: Suppose $x(t)$ is a curve lying on S^2. This curve is a *geodesic* if there is no shear, i.e., the acceleration $\ddot{x}$ has only components normal to the surface. This condition may be written

$$\nabla_{\dot{x}}(\dot{x}) = 0. \tag{5.1}$$

For example, if we consider a parallel to latitude $x(t) = x(\theta = \theta_0, \phi = t)$ then $\dot{x} = u_2$ and $\nabla_{\dot{x}}(\dot{x}) = -\cos\theta_0 \sin\theta_0\, u_1$. This curve is a geodesic on the equator, $\theta_0 = \pi/2$. The curves $x(t) = x(\theta = t, \phi = \phi_0)$ always satisfy the geodesic equations because $\dot{x} = u_1$ and $\nabla_{\dot{x}}(\dot{x}) = 0$; these are great circles through the north and south poles.

Parallel transport: The Levi–Civita connection provides a rule for the parallel transport of vectors on a surface. Let $x(t)$ be a curve in S^2 and let $s(t)$ be a vector field defined along the curve. We say that s is parallel transported along the curve if it satisfies the equation

$$\nabla_{\dot{x}}(s) = 0,$$

i.e., $\dot{s}$ is normal to the surface. Given an initial vector $s(t_0)$ and the connection, $s(t)$ is uniquely determined by the parallel transport equation.

Parallel translation around a closed curve need not be the identity. For example, let x be the geodesic triangle in S^2 connecting the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. x consists of 3 great circles:

$$x(t) = \begin{cases} (\cos(t), \sin(t), 0) & t \in [0, \pi/2] \\ (0, \sin(t), -\cos(t)) & t \in [\pi/2, \pi] \\ (-\sin(t), 0, -\cos(t)) & t \in [\pi, 3\pi/2]. \end{cases}$$

Let $s(0)$ be the initial tangent vector

$$s(0) = (0, \alpha, \beta)$$

at $(1, 0, 0)$. When we parallel transport $s(0)$ along $x(t)$ using the Levi–Civita connection we find

$$s(t) = \begin{cases} (-\alpha \sin(t), \alpha \cos(t), \beta) & t \in [0, \pi/2] \\ (-\alpha, \beta \cos(t), \beta \sin(t)) & t \in [\pi/2, \pi] \\ (\alpha \cos(t), -\beta, -\alpha \sin(t)) & t \in [\pi, 3\pi/2]. \end{cases}$$

One may verify that $s(t)$ is continuous at the corners $\pi/2$, π and satisfies $\nabla_{\dot{x}}(s) = 0$, since $\partial s/\partial t$ is normal to the surface. Parallel translation around the geodesic triangle changes s from $s(0) = (0, \alpha, \beta)$ to $s(3\pi/2) = (0, -\beta, \alpha)$, which represents a rotation through $\pi/2$ (see fig. 5.2). Note that $\pi/2$ is the area of the spherical triangle.

Holonomy: Holonomy is the process of assigning to each closed curve the linear transformation measuring the rotation which results when a vector is parallel transported around the given curve. In our example, the holonomy matrix changing $s(0)$ to $s(3\pi/2)$ is

$$H_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} .$$

The set of holonomy matrices forms a group called the *holonomy group*. The non-triviality of holonomy is related to the existence of curvature on the sphere: parallel transport around a closed curve in a plane gives no rotation.

General definitions of the connection

In the general case, there is no natural embedding of a manifold M in Euclidean space. Thus, even for the tangent bundle, it is meaningless to talk about normals to M. The problem is even more difficult for a general vector bundle. Therefore, we now proceed to abstract the intrinsic features of the Levi–Civita connection which allowed us to discuss parallel translation.

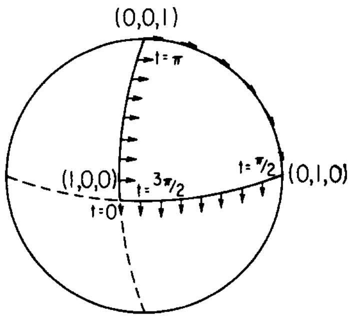

Fig. 5.2. Parallel transport of a vector around a spherical triangle.

Background: Let E be a general vector bundle. On each neighborhood U we choose a local frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_k\}$ and express vectors in $\pi^{-1}(U)$ in the form

$$Z = \sum_{i=1}^{k} \boldsymbol{e}_i z^i .$$

This gives a local trivialization of $\pi^{-1}(U) \approx U \times F$ and defines local coordinates (x, z). The vectors $\boldsymbol{e}_i$ themselves have the form

$$\boldsymbol{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$$

in each *local* frame. This, however, does not mean that $\boldsymbol{e}_i$ is a constant vector on M since the local frames may be different in each neighborhood. The dependence of $\boldsymbol{e}_i$ on x due to the change of the local frame is dictated by the rule of covariant differentiation described below. A *local section* to the bundle is a smooth map from U to the fiber and can be regarded as a vector-valued function,

$$s(x) = \sum_{i=1}^{k} \boldsymbol{e}_i(x)\, z^i(x).$$

The *tangent space* $T(E)$ and the *cotangent space* $T^*(E)$ of the bundle may be assigned the local bases

$$T(E):\quad (\partial/\partial x^\mu, \partial/\partial z^i)$$

$$T^*(E):\ (\mathrm{d}x^\mu, \mathrm{d}z^i).$$

We now give a series of equivalent definitions of a connection on a vector bundle.

(1) *Parallel transport approach.* The Levi–Civita connection lets us take the directional derivative of a tangent vector field and get another tangent vector field. We generalize this concept for vector bundles as follows: Let X be a tangent vector and let s be a section to E. A connection ∇ is a rule $\nabla_X(s)$ for taking the directional derivative of s in the direction X and getting another section to E. The assignment of a connection ∇ in a general vector bundle E provides a rule for the parallel transport of sections.

Let $\boldsymbol{x}(t)$ be a curve in M; we say that $s(t)$ is parallel-transported along $\boldsymbol{x}$ if s satisfies the differential equation

$$\nabla_{\dot{x}}(s) = 0. \tag{5.2}$$

There always exists a unique solution to this equation for given initial conditions. The generalized Christoffel symbols $\Gamma^j{}_{\mu i}$ giving the action of a connection ∇ on a frame of the bundle E are defined by

$$\nabla_{\partial/\partial x^\mu}(\boldsymbol{e}_i) = \boldsymbol{e}_j \Gamma^j{}_{\mu i}.$$

We recall that we may associate the operator $\mathrm{d}/\mathrm{d}t$ with $\dot{x}^\mu$ because

$$\mathrm{d}f(x)/\mathrm{d}t = \dot{x}^\mu\, \partial f/\partial x^\mu .$$

In terms of the Christoffel symbols, the parallel transport equation takes the form

$$\begin{aligned}\nabla_{\dot{x}}(s) &= \nabla_{\mathrm{d/d}t}(e_i z^i) = \nabla_{\mathrm{d/d}t}(e_i) z^i + e_j \dot{z}^j \\ &= \dot{x}^\mu (\nabla_{\partial/\partial x^\mu}(e_i) z^i + e_j \, \partial_\mu z^j) \\ &= \dot{x}^\mu e_j (\Gamma^j{}_{\mu i} z^i + \partial_\mu z^j) = 0.\end{aligned}$$

Note: we have implicitly made use of various properties of $\nabla_X(s)$ which we will formalize later.

(2) *Tangent space approach*. Parallel transport along a curve $x(t)$ lets us compare the fibers of the bundle E at different points of the curve. Thus it becomes natural to think of *lifting* a curve $x(t)$ in M to a curve

$$c(t) = (x^\mu(t), z^i(t))$$

in the bundle. Differentiation along $c(t)$ is defined by

$$\frac{\mathrm{d}}{\mathrm{d}t} = \dot{x}^\mu \frac{\partial}{\partial x^\mu} + \dot{z}^i \frac{\partial}{\partial z^i} \quad ,$$

where $\dot{z}^i$ is given by solving the parallel transport equation:

$$\dot{z}^i + \Gamma^i{}_{\mu j} \dot{x}^\mu z^j = 0. \tag{5.3}$$

Thus we may write

$$\frac{\mathrm{d}}{\mathrm{d}t} = \dot{x}^\mu \left(\frac{\partial}{\partial x^\mu} - \Gamma^i{}_{\mu j} z^j \frac{\partial}{\partial z^i}\right) = \dot{x}^\mu \mathrm{D}_\mu \quad ,$$

where

$$\mathrm{D}_\mu = \frac{\partial}{\partial x^\mu} - \Gamma^i{}_{\mu j} z^j \frac{\partial}{\partial z^i} \tag{5.4}$$

is the operator in $T(E)$ known as the *covariant derivative*.

We are thus led to define a splitting of $T(E)$ at $x \in U$ into *vertical* component $V(E)$ with basis $\{\partial/\partial z^i\}$ lying strictly in the fiber and a *horizontal* component $H(E)$ with basis $\{\mathrm{D}_\mu\}$:

$$T_x(E) = V_x(E) \oplus H_x(E)$$

$$\text{basis} = \left(\frac{\partial}{\partial z^i}, \mathrm{D}_\mu\right).$$

This splitting is illustrated schematically in fig. 5.3.

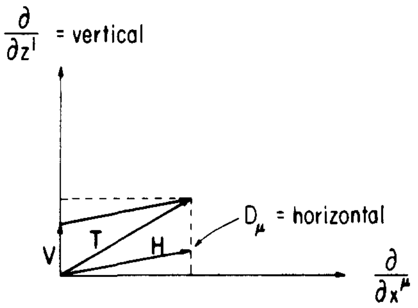

Fig. 5.3. Splitting the tangent space $T(E)$ of the bundle into vertical and horizontal components.

(3) *Cotangent space approach.* In the cotangent space approach, one considers a vector-valued one-form

$$\omega^i = \mathrm{d}z^i + \Gamma^i{}_{\mu j}\, \mathrm{d}x^\mu\, z^j \tag{5.5}$$

in $T^*(E)$ which is essentially the deviation from the parallel transport law given above. We observe that ω^i is the unique non-trivial solution to the conditions

$$\begin{aligned} &\langle \omega^i, \mathrm{D}_\mu \rangle = 0. \\ &\langle \omega^i, \partial/\partial z^j \rangle = \delta_{ij}. \end{aligned} \tag{5.6}$$

Conversely, these conditions determine D_μ if ω^i is given. The connection one-form ω^i annihilates the horizontal subspace of $T(E)$, and is, in some sense, dual to it.

We now introduce the matrix-valued connection one-form Γ, where

$$\Gamma^i{}_j = \Gamma^i{}_{\mu j}\, \mathrm{d}x^\mu.$$

The *total covariant derivative* $\nabla(s)$ is defined by

$$\nabla(s) = \boldsymbol{e}_i \otimes \mathrm{d}z^i(x) + \boldsymbol{e}_i \otimes \Gamma^i{}_j z^j(x) \tag{5.7}$$

which maps $C^\infty(E)$ to $C^\infty(E \otimes T^*(M))$. Note that this is the pullback to M (using the section $z^i(x)$) of a covariant derivative in the *bundle* given by

$$\nabla(\boldsymbol{Z}) = \boldsymbol{e}_i \otimes \omega^i, \qquad (\boldsymbol{Z} = \boldsymbol{e}_i z^i \in \pi^{-1}(U)),$$

where ω^i belongs to $T^*(E)$ rather than $T^*(M)$. The total covariant derivative contains all the directional derivatives at the same time in the same way that $\mathrm{d}f = (\partial f/\partial x^\mu)\,\mathrm{d}x^\mu$ contains all the partial derivatives of f.

(4) *Axiomatic approach.* We began this section by discussing covariant differentiation as a directional derivative. We now formalize the properties of covariant differentiation that we have been using implicitly in the previous approaches. The axiomatic properties of the connection $\nabla_X(s)$ are

1. Linearity in s:

$$\nabla_X(s+s') = \nabla_X(s) + \nabla_X(s')$$

2. Linearity in X:

$$\nabla_{X+X'}(s) = \nabla_X(s) + \nabla_{X'}(s)$$

3. Behaves like a first-order differential operator:

$$\nabla_X(sf) = s \cdot X(f) + (\nabla_X(s))f$$

4. Tensoriality in X:

$$\nabla_{fX}(s) = f\nabla_X(s)$$

where $s(x)$ is a section to E, X is vector field on M and $f(x)$ is a scalar function. These are clearly desirable properties which are straightforward generalizations of the features of the Levi–Civita connection.

The axiomatic properties of the *total* covariant derivative ∇ are:

1. Linearity in s:

$$\nabla(s+s') = \nabla(s) + \nabla(s')$$

2. Behaves like a first-order differential operator:

$$\nabla(sf) = s \otimes \mathrm{d}f + \nabla(s)f.$$

The relationship between these two differential operators is given by

$$\begin{aligned} &1.\ \nabla(s) = \nabla_{\partial/\partial x^\mu}(s) \otimes \mathrm{d}x^\mu \\ &2.\ \nabla_X(s) = \langle \nabla(s), X \rangle, \end{aligned} \tag{5.8}$$

where $X \in C^\infty(T(M))$ and $\nabla(s) \in C^\infty(E \otimes T^*(M))$.

One can extend total covariant differentiation to p-form-valued sections of E by the rule

$$\nabla(s \otimes \theta) = \nabla(s) \wedge \theta + s \otimes \mathrm{d}\theta \tag{5.9}$$

where $s \in C^\infty(E)$ and $\theta \in C^\infty(\Lambda^p(M))$. ∇ thus extends to a differential operator with the following domain and range:

$$\nabla\colon C^\infty(E \otimes \Lambda^p(M)) \to C^\infty(E \otimes \Lambda^{p+1}(M)).$$

(5) *Change of frame approach.* Under a change of frame,

$$e'_j = e_i \Phi^{-1}_{ij}(x), \qquad z'^i = \Phi_{ij}(x) z^j,$$

and sections are invariant:

$$s(x) = e_i z^i = e'_i z'^i = s'(x).$$

We see that

$$\nabla(e'_j) = \nabla(e_i) \otimes \Phi^{-1}_{ij} + e_i \otimes \mathrm{d}\Phi^{-1}_{ij} = e'_i \Gamma^i{}_j{}'$$

where

$$\Gamma'^i{}_j = \Phi_{ik} \Gamma^k{}_l \Phi^{-1}_{lj} + \Phi_{ik}\, \mathrm{d}\Phi^{-1}_{kj}, \tag{5.10}$$

so the connection 1-form $\Gamma^i{}_j$ transforms as a gauge field rather than as a tensor. We may in fact *define* a connection as a collection of one-forms $\Gamma^i{}_j$ obeying the transformation law (5.10).

Using eq. (5.10), we can check that ∇ is independent of the choice of frame and is thus well-defined in the overlap region $U \cap U'$. We find

$$\nabla(s) = \nabla(e_i z^i) = e_j \otimes \Gamma^j{}_i z^i + e_j \otimes \mathrm{d}z^j = e'_j \otimes \Gamma'^j{}_i z'^i + e'_j \otimes \mathrm{d}z'^j.$$

5.2. *Curvature*

The curvature of a fiber bundle characterizes its geometry. It can be calculated in several different equivalent ways corresponding to the different approaches to the connection.

(1) *Parallel transport.* Curvature measures the extent to which parallel transport is path-dependent. If the curvature is zero and $x(t)$ is a path lying in a coordinate ball of M, then the result of parallel transport is always the identity transformation (this need not be true if the path encloses a hole, as we shall see later when we discuss locally flat bundles). For curved manifolds, we get non-trivial results: parallel transport around a geodesic triangle on S^2 gives a rotation equal to the area of the spherical triangle.

A quantitative measure of the curvature can be calculated using parallel transport as follows: Let $(x^1, x^2, \ldots)$ be a local coordinate chart and take a square path $x(t)$ with vertices, say, in the 1–2 plane. Let $H_{ij}(\tau)$ be the holonomy matrix obtained by traversing the path with vertices $(0, 0, 0, \ldots)$, $(0, \tau^{1/2}, 0, \ldots)$, $(\tau^{1/2}, \tau^{1/2}, 0, \ldots)$, $(\tau^{1/2}, 0, 0, \ldots)$. Then the curvature matrix in the 1–2 plane is

$$R_{ij}(1,2) = \frac{\mathrm{d}}{\mathrm{d}\tau} H_{ij}(\tau)\Big|_{\tau=0}. \tag{5.11}$$

The correspondence between this curvature and those to be introduced below may be found by expanding the connection in Taylor series.

(2) *Tangent space.* The curvature is defined as the commutator of the components D_μ of the basis for

the horizontal subspace of $T(E)$,

$$[D_\mu, D_\nu] = -R^i{}_{j\mu\nu} z^j \, \partial/\partial z^i, \tag{5.12}$$

where $R^i{}_{j\mu\nu}$ can be expressed in terms of Christoffel symbols as

$$R^i{}_{j\mu\nu} = \partial_\mu \Gamma^i{}_{\nu j} - \partial_\nu \Gamma^i{}_{\mu j} + \Gamma^i_{\mu k}\Gamma^k_{\nu j} - \Gamma^i_{\nu k}\Gamma^k_{\mu j}.$$

Note that the right-hand side of eq. (5.12) has only *vertical* components. $R^i{}_{j\mu\nu}$ is interpretable as the obstruction to integrability of the horizontal subspace.

(3) *Cotangent space.* In this approach, the curvature appears as a matrix-valued 2-form

$$R^i{}_j = d\Gamma^i{}_j + \Gamma^i{}_k \wedge \Gamma^k{}_j = \tfrac{1}{2} R^i{}_{j\mu\nu}\, dx^\mu \wedge dx^\nu. \tag{5.13}$$

We observe that $R^i{}_j z^j$ is the covariant differential of the one-form $\omega^i \in T^*(E)$:

$$R^i{}_j z^j = d\omega^i + \Gamma^i{}_j \wedge \omega^j.$$

Note that although ω^i has dz^k components, they cancel out in $R^i{}_j$.

(4) *Axiomatic formulation.* Curvature measures the extent to which covariant differentiation fails to commute. We define the *curvature operator* as

$$R(X, Y)(s) = \nabla_X \nabla_Y(s) - \nabla_Y \nabla_X(s) - \nabla_{[X,Y]}(s), \tag{5.14}$$

where

$$R\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right)(e_i) = e_j R^j{}_{i\mu\nu}.$$

The axiomatic properties of the curvature operator are

1. Multilinearity:

$$R(X + X', Y)(s) = R(X, Y)(s) + R(X', Y)(s)$$

2. Anti-symmetry:

$$R(X, Y)(s) = -R(Y, X)(s)$$

3. Tensoriality:

$$\begin{aligned} R(fX, Y)(s) &= R(X, fY)(s) = R(X, Y)(fs) \\ &= fR(X, Y)(s) \end{aligned}$$

where X and Y are vector fields, $s(x)$ is a section and $f(x)$ is a scalar function.

The *total curvature* R is a matrix-valued 2-form given by

$$\begin{aligned} R(s) &= \nabla^2(s) = \nabla(e_j \otimes \Gamma^j{}_i z^i + e_j \otimes \mathrm{d}z^j) \\ &= e_k \otimes \Gamma^k{}_j \wedge \Gamma^j{}_i z^i + e_k \otimes (\mathrm{d}\Gamma^k{}_i z^i - \Gamma^k{}_j \wedge \mathrm{d}z^j) + e_k \otimes \Gamma^k{}_j \wedge \mathrm{d}z^j + 0 \\ &= e_k \otimes R^k{}_i z^i . \end{aligned} \tag{5.15}$$

The matrix $R = \|R^i{}_j\|$ is also given by

$$R = \tfrac{1}{2} R\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$$

acting on a section s. The axiomatic property of R is just the statement that it is a 2-form valued linear map from $E \to E$.

(5) *Change of frame*. By using (5.13), we find that $R^i{}_j$ transforms as

$$R'^i{}_j = \Phi^i{}_k R^k{}_l (\Phi^{-1})^l{}_j$$

under the change of frame (4.6) and (5.10). Hence by (5.15) $R(s)$ is in fact invariant under a change of frame.

The curvature can be regarded as an obstruction to finding locally flat (i.e., covariant constant) frames. Given e_i, let us attempt to find a new frame $e'_i = e_j \Phi^{-1}_{ji}$ which is locally flat. If we set $\nabla(e'_i) = 0$, we find the matrix differential equation

$$\Phi \Gamma \Phi^{-1} + \Phi \, \mathrm{d}\Phi^{-1} = 0 .$$

This equation is solved if Γ is a pure gauge,

$$\Gamma^i{}_j = -(\mathrm{d}\Phi^{-1})^i{}_k \, \Phi^k{}_j = (\Phi^{-1})^i{}_k \, \mathrm{d}\Phi^k{}_j .$$

If Γ obeys this equation, the curvature vanishes. Conversely, by the Frobenius theorem, if the curvature vanishes, Γ can be written as a pure gauge.

5.3. Torsion and connections on the tangent bundle

One advantage of the cotangent space formulation (5.7) of the vector bundle connection ∇ is that it is independent of the coordinate system $\{x^\mu\}$ on M. Furthermore, multiple covariant differentiation of an invariant one-form such as $p_\mu \, \mathrm{d}x^\mu$ is independent of the connection chosen on the cotangent bundle $T^*(M)$. However, if we choose to differentiate the individual tensor components $z^i{}_{;\mu}$ of the covariant derivative of a section $s(x) = e_i z^i(x)$ of a vector bundle, we must specify in addition a connection on $T^*(M)$ to treat the "μ" index. (We will show in the next section that connections on $T(M)$ give natural connections on $T^*(M)$, and vice-versa.) *Torsion* is a property of the connection on the tangent bundle which must be introduced when we examine the double covariant derivative. We have already encountered torsion in section 3 when we studied metric geometry on Riemannian manifolds. Here we extend the notion to general vector bundles.

Let $\{\Gamma^i{}_{\mu j}\}$ be the Christoffel symbols on the vector bundle E, and let $\{\gamma^\nu{}_{\mu\lambda}\}$ be the Christoffel symbols on $T(M)$. We define the components of the double covariant derivative of a section $s(x) = e_i z^i(x)$ as

$$z^i{}_{;\mu;\nu} = \partial_\nu(\partial_\mu z^i + \Gamma^i{}_{\mu j} z^j) + \Gamma^i{}_{\nu j}(\partial_\mu z^j + \Gamma^j{}_{\mu k} z^k) - \gamma^\lambda{}_{\mu\nu}(\partial_\lambda z^i + \Gamma^i{}_{\lambda j} z^j).$$

(The sign in front of $\gamma^\lambda{}_{\mu\nu}$ follows from the requirement for lowering indices to get the connection on $T^*(M)$.) The commutator of double covariant differentiation on a section yields the formula

$$z^i{}_{;\mu;\nu} - z^i{}_{;\nu;\mu} = -R^i{}_{j\mu\nu} z^j - T^\lambda{}_{\mu\nu} z^i{}_{;\lambda}, \tag{5.16}$$

where we have introduced a new tensor, the *torsion*,

$$T^\lambda{}_{\mu\nu} = \gamma^\lambda{}_{\mu\nu} - \gamma^\lambda{}_{\nu\mu}.$$

Multiple covariant differentiation can be written schematically in the form

$$C^\infty(E) \xrightarrow{\nabla} C^\infty(E \otimes T^*(M)) \xrightarrow{\nabla} C^\infty(E \otimes T^*(M) \otimes T^*(M)),$$

which again emphasizes the requirement for a connection on $T^*(M)$, or equivalently on $T(M)$.
Note: We remark that the multiple covariant derivative treated here is *not* the operator ∇^2 used to define the curvature 2-form, since ∇^2 is independent of the connection on $T^*(M)$ and has values in $C^\infty(E \otimes \Lambda^2(T^*(M)))$.

Axiomatic approach to torsion: We define the torsion operator on $T(M)$ by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

This is a vector field with components

$$T\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = (\gamma^\lambda{}_{\mu\nu} - \gamma^\lambda{}_{\nu\mu})\frac{\partial}{\partial x^\lambda}.$$

Levi–Civita connection: Once a metric $(X, Y) = g_{\mu\nu} x^\mu y^\nu$ has been chosen, the Levi–Civita connection on $T(M)$ is uniquely defined by the properties

$$\begin{aligned} &1.\ \text{Torsion-free:} \quad T(X, Y) = 0 \\ &2.\ \text{Covariant constancy of metric:} \quad \mathrm{d}(X, Y) = (\nabla X, Y) + (X, \nabla Y). \end{aligned} \tag{5.17}$$

These conditions were discussed in detail in section 3.

5.4. Connections on related bundles

Dual bundles: If E and E^* are dual vector bundles with dual frame bases $\{e_i\}$ and $\{e^{*i}\}$, the connection

∇^* on E^* is defined by the requirement that the natural inner product between sections s and s^* be differentiated according to the following rule:

$$\mathrm{d}\langle s, s^*\rangle = \langle \nabla(s), s^*\rangle + \langle s, \nabla^*(s^*)\rangle.$$

In other words,

$$\nabla(e_i) = e_j \Gamma^j{}_{\mu i}\, \mathrm{d}x^\mu$$

$$\nabla^*(e^{*i}) = -e^{*j}\Gamma^i{}_{\mu j}\, \mathrm{d}x^\mu.$$

If E has a fiber metric, we may identify E with E^* using a conjugate linear isomorphism. The connection ∇ is said to be *Riemannian* if $\nabla = \nabla^*$, i.e.,

$$\Gamma^i{}_{\mu j} = -\Gamma^j{}_{\mu i} \tag{5.18}$$

relative to an orthonormal frame basis. The curvature of a Riemannian connection relative to an orthonormal frame basis is anti-symmetric:

$$R^i{}_j = -R^j{}_i. \tag{5.19}$$

The Levi–Civita connection on $T(M)$ is the unique torsion-free Riemannian connection.

Whitney sum bundle: If E and F are vector bundles with connections ∇ and ∇', there is a natural connection $\nabla \oplus \nabla'$ defined on $E \oplus F$ by the following rule:

$$(\nabla \oplus \nabla')(s \oplus s') = \nabla(s) \oplus \nabla'(s').$$

In other words,

$$(\nabla \oplus \nabla')(e_i \oplus f_j) = e_k \otimes \Gamma^k{}_{\mu i}\, \mathrm{d}x^\mu \oplus f_l \otimes \Gamma'^l{}_{\mu j}\, \mathrm{d}x^\mu. \tag{5.20}$$

The curvature is given by the direct sum of the curvatures of E and F.

Tensor product bundle: There is a natural connection ∇'' defined on $E \otimes F$ by the following rule:

$$\nabla''(s \otimes s') = (\nabla \otimes 1 + 1 \otimes \nabla')(s \otimes s') = \nabla(s) \otimes s' + s \otimes \nabla'(s').$$

The curvature of ∇'' is given by

$$R'' = R \otimes 1 + 1 \otimes R'. \tag{5.21}$$

Pullback bundle: Let f: $M \to M'$ and let ∇' be a connection on the vector bundle E' over M'. There is a natural pullback connection $\nabla = f^*\nabla'$ with Christoffel symbols which are the pullback of the Christoffel symbols of ∇', that is:

$$\Gamma^i{}_{\mu j} = \Gamma'^i{}_{\alpha j} \frac{\partial x'^\alpha}{\partial x^\mu}.$$

The curvature of ∇ is the pullback of the curvature of ∇':

$$R^i{}_{j\mu\nu} = \tfrac{1}{2} R'^i{}_{j\alpha\beta} \left(\frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} - \frac{\partial x'^\alpha}{\partial x^\nu} \frac{\partial x'^\beta}{\partial x^\mu} \right).$$

Projected connections: Let E be a sub-bundle of F and let π: $F \to E$ be a projection. If ∇ is a connection on F, we can define the projected connection ∇^π on E by

$$\nabla^\pi(s) = \pi(\nabla(s))$$

where s is a section of F belonging to the *sub-bundle* E. Note that the curvature of ∇^π may be non-trivial even if the curvature of ∇ is zero. (Our introductory example deriving the Levi–Civita connection on S^2 embedded in $\mathbb{R}^3$ was in fact of this type.)

If π is an orthogonal projection relative to some fiber metric and ∇ is Riemannian, then ∇^π is Riemannian.

Examples 5.4

1. *Complex line bundle of* $P_1(\mathbb{C})$. Let L be the line bundle over $P_1(\mathbb{C})$ defined in example 4.2.2. This is a natural sub-bundle of $P_1(\mathbb{C}) \times \mathbb{C}^2$. We denote a point of the bundle L by $(x; z_0, z_1)$, where (z_0, z_1) lie on the line in $\mathbb{C}^2$ corresponding to the point x in $P_1(\mathbb{C})$. The natural fiber metric on L is given by

$$((x; z_0, z_1), (x; w_0, w_1)) = z_0 \bar{w}_0 + z_1 \bar{w}_1.$$

(This is induced by the canonical metric on $\mathbb{C}^2$.)

Now let

$$h(x; z_0, z_1) = |z_0|^2 + |z_1|^2$$

be the length of a point in L and form a connection ω lying in $T^*(L)$ given by

$$\omega = h^{-1}\, \partial h = \frac{\bar{z}_0\, \mathrm{d}z_0 + \bar{z}_1\, \mathrm{d}z_1}{|z_0|^2 + |z_1|^2}.$$

The curvature then is

$$\Omega = \mathrm{d}\omega + \omega \wedge \omega = (\partial + \bar{\partial})(h^{-1}\, \partial h) + 0 = -\partial\bar{\partial} \ln h.$$

In order to carry out practical computations, we choose a gauge (that is a local section of L) with coordinates $(x; \zeta_0^{(1)}, 1)$.

Here

$$\zeta_0^{(1)} = z_0/z_1 = u + \mathrm{i}v$$

for $u, v \in \mathbb{R}$. Then we compute

$$h = 1 + u^2 + v^2$$

$$\Omega = -\partial\bar{\partial} \ln(1 + u^2 + v^2) = \frac{2\mathrm{i}\, \mathrm{d}u \wedge \mathrm{d}v}{(1 + u^2 + v^2)^2} \ .$$

We recognize this from section 3.4 on Kähler manifolds as (2i) times the Kähler form for $S^2 = P_1(\mathbb{C})$. We thus can read off the metric directly from Ω.

Remark 1: In some sense $\omega = h^{-1}\,\partial h$ is a "pure gauge" with respect to a curvature involving only ∂. We find non-trivial full curvature because Ω involves $\mathrm{d} = (\partial + \bar{\partial})$.

Remark 2: The Fubini–Study metric on $P_n(\mathbb{C})$ can be defined in this same manner by taking

$$h(x; z_0, z_1, \ldots, z_n) = \sum_{i=0}^{n} |z_i|^2.$$

Remark 3: The same construction works for an arbitrary holomorphic line bundle over an arbitrary complex manifold once a fiber metric is chosen.

2. *Vector bundles over S^n*. If we let $n = 2l$, the trivial bundle $S^n \times \mathbb{C}^{2^l}$ can be split into a sum of non-trivial bundles $E_\pm$ by constructing a projection operator $\Pi_\pm$: $S^n \times \mathbb{C}^{2^l} \to E_\pm$. To accomplish this, we embed S^n in $\mathbb{R}^{n+1}$ using coordinates $(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$ and consider the set of $2^l \times 2^l$ self-adjoint complex matrices $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ obeying

$$\lambda_i\lambda_j + \lambda_j\lambda_i = 2\delta_{ij}$$

$$\lambda_0\lambda_1 \ldots \lambda_n = \mathrm{i}^l I$$

where I is the identity matrix. The $\{\lambda_i\}$ are Pauli matrices ($\lambda_0 = \tau_3, \lambda_1 = \tau_1, \lambda_2 = \tau_2$) for $l = 1$ and Dirac matrices ($\lambda_0 = \gamma_5, \lambda_1 = \gamma_1, \lambda_2 = \gamma_2, \lambda_3 = \gamma_3, \lambda_4 = \gamma_4$) for $l = 2$. We now define the complex matrix

$$\lambda(x) = \sum_{j=0}^{n} x_j\lambda_j$$

with $\{x_j\}$ lying on S^n, so that

$$\lambda^2(x) = I.$$

$\lambda(x)$ is a map from $\mathbb{C}^{2^l}$ to $\mathbb{C}^{2^l}$ which depends on the point x of the base space S^n. Since $\lambda^2(x) = I$, we may decompose its action on vectors $z \in \mathbb{C}^{2^l}$ into the two eigenspaces with eigenvalues ± 1,

$$\lambda(x) \cdot z = \pm z.$$

We then choose as our projection the matrix

$$\Pi_{\pm}(x)=\tfrac{1}{2}(1\pm\lambda(x))$$

which selects the 2^{l-1} dimensional vector space in $\mathbb{C}^{2^l}$ with $\lambda\cdot z=\pm z$.

We denote by $E_\pm$ the complex vector bundles over S^n whose fibers at each point $x\in S^n$ are defined by the action of $\pi_\pm(x)$. If $l=1$, we obtain complex line bundles over S^2. Clearly

$$E_+\oplus E_-=S^n\times\mathbb{C}^{2^l}.$$

We choose as our connections on $E_\pm$ the projection $\nabla_\pm$ of the flat connection ∇ acting on a section of $E_\pm$. To carry out this procedure, we choose a constant frame e_+^0 of E_+ at a point x_0 and generalize it to arbitrary x using the projection;

$$e_+(x)=\Pi_+(x)e_+^0$$

is a frame of E_+ everywhere. Since $\Pi_+(x_0)\,e_+(x_0)=\Pi_+(x_0)\,e_+^0\equiv e_+(x_0)$, we may take $e_+(x_0)=e_+^0$. The flat connection just acts by exterior differentiation, $\nabla(e)=\mathrm{d}e$; while the projected connection is difficult to calculate in general, it can be evaluated at x_0 as the projection of the flat connection since $e_\pm(x_0)=e_\pm^0$,

$$\nabla_\pm(e_\pm)|_{x_0}=\Pi_\pm\,\mathrm{d}e_\pm|_{x_0}=(\Pi_\pm\,\mathrm{d}\Pi_\pm)|_{x_0}e^0{}_\pm.$$

The curvature is obtained in a similar way;

$$(\nabla_\pm)^2(e_\pm)|_{x_0}=\Pi_\pm\,\mathrm{d}(\Pi_\pm\,\mathrm{d}\Pi_\pm\,\mathrm{e}^0{}_\pm)=\Pi_\pm\,\mathrm{d}\Pi_\pm\wedge\mathrm{d}\Pi_\pm\,e^0{}_\pm.$$

Hence the curvature 2-form at x_0 is

$$\Omega_\pm(x_0)=\Pi_\pm(x_0)\,\mathrm{d}\Pi_\pm(x_0)\wedge\mathrm{d}\Pi_\pm(x_0).$$

Remark 1: Note that although the connection and curvature matrices used here are double the correct dimension, all traces of products of these matrices involve only the meaningful portion of the matrices. The rank of the matrices equals the fiber dimension.
Remark 2: To evaluate an invariant polynomial of $\Omega_\pm$, it in fact suffices to perform the calculation at x_0 alone. One may thus show that

$$\mathrm{Tr}(\Omega_\pm{}^l)=\frac{n!(2\mathrm{i})^l}{2^{n+1}}\,\mathrm{d(vol)},$$

where d(vol) is the n-form volume element of S^n. This formula will be used later to examine the characteristic classes of this bundle.
Remark 3: If $l=1$, the associated principal bundles to $E_\pm$ describe the Dirac magnetic monopole.

5.5. Connections on principal bundles

We recall that a principal bundle P is a fiber bundle whose fiber and transition functions both belong to the same matrix group. The gauge potentials of Maxwell's theory of electromagnetism and Yang–Mills gauge theories are identifiable with connections on principal bundles. Here we give a brief treatment of the special aspects of connections on principal bundles.

Maurer–Cartan forms and the Lie algebra: We let G be a matrix group and $\mathcal{G}$ be its Lie algebra. The Maurer–Cartan form $g^{-1}\,dg$ is a matrix of one-forms belonging to the Lie algebra $\mathcal{G}$. This form is invariant under the left action by a constant group element g_0,

$$(g_0 g)^{-1}\,\mathrm{d}(g_0 g) = g^{-1}\,\mathrm{d}g.$$

Let $\{\Phi_a\}$ be a basis for the left-invariant one-forms. We then express the Cartan–Maurer form as

$$g^{-1}\,\mathrm{d}g = \Phi_a\,\frac{\lambda_a}{2\mathrm{i}}, \tag{5.22}$$

where $\lambda_a/2\mathrm{i}$ is a constant matrix in $\mathcal{G}$. Since $\mathrm{d}(g^{-1}\,\mathrm{d}g) + g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g = 0$, we find that Φ_a obeys the *Maurer–Cartan equations*

$$\mathrm{d}\Phi_a + \tfrac{1}{2} f_{abc}\Phi_b \wedge \Phi_c = 0, \tag{5.23}$$

where the f_{abc} are the structure constants of $\mathcal{G}$.

The *dual* of Φ_a is the differential operator

$$L_a = \mathrm{Tr}\left(g\,\frac{\lambda_a}{2\mathrm{i}}\,\frac{\partial}{\partial g^{\mathrm{T}}}\right) = \frac{1}{2\mathrm{i}}\,g_{jk}[\lambda_a]_{kl}\,\frac{\partial}{\partial g_{jl}}$$

obeying

$$\langle \Phi_a, L_b\rangle = \delta_{ab}, \qquad [L_a, L_b] = f_{abc}L_c. \tag{5.24}$$

$\{L_a\}$ is a left-invariant basis for the tangent space of G.

The corresponding *right* invariant objects are defined by

$$\mathrm{d}g\,g^{-1} = \frac{\lambda_a}{2\mathrm{i}}\,\bar{\Phi}_a, \qquad \bar{L}_a = \mathrm{Tr}\left(\frac{\lambda_a}{2\mathrm{i}}\,g\,\frac{\partial}{\partial g^{\mathrm{T}}}\right) \tag{5.25}$$

where

$$\begin{aligned} &\mathrm{d}\bar{\Phi}_a - \tfrac{1}{2} f_{abc}\bar{\Phi}_b \wedge \bar{\Phi}_c = 0 \\ &\langle \bar{\Phi}_a, \bar{L}_b\rangle = \delta_{ab}, \qquad [\bar{L}_a, \bar{L}_b] = -f_{abc}\bar{L}_c. \end{aligned} \tag{5.26}$$

That is, all structure equations have a reversed sign. Note that L_a and $\bar{L}_b$ *commute*:

$$[L_a, \bar{L}_b] = 0. \tag{5.27}$$

L_a and $\bar{L}_b$ generalize the familiar physical distinction between the space-fixed and body-fixed rotation generators of a quantum-mechanical top.

Parallel transport: Let P be a principal bundle. If we choose a local trivialization, then we have coordinates (x, g) for P, where $g \in G$. A local section of P is a smooth map from a neighborhood U to G. The assignment of a connection on a principal bundle provides a rule for the parallel transport of sections. A connection A of a principal fiber bundle is a Lie-algebra valued matrix of 1-forms in $T^*(M)$,

$$A(x) = A^a{}_\mu(x) \frac{\lambda_a}{2\mathrm{i}} \, \mathrm{d}x^\mu. \tag{5.28}$$

If $x(t)$ is a curve in M, the section $g_{ij}(t)$ is defined to be parallel-transported along x if the following differential equation is satisfied:

$$\dot{g}_{ik} + A_{\mu ij}(x) \, \dot{x}^\mu \, g_{jk} = 0, \tag{5.29}$$

where A_μ is the connection on P. We may rewrite this as:

$$g^{-1} \frac{\mathrm{d}g}{\mathrm{d}t} + g^{-1} \left(A^a{}_\mu(x) \frac{\lambda_a}{2\mathrm{i}} \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \right) g = 0.$$

Tangent space approach: Parallel transport along a curve $x(t)$ lets us compare the fibers of P at different points of the curve. In analogy to the methods used for vector bundle connections, we may lift curves $x(t)$ in M to curves in P. We define differentiation along the lifted curve by

$$\frac{\mathrm{d}}{\mathrm{d}t} = \dot{x}^\mu \frac{\partial}{\partial x^\mu} + \dot{g}_{ij} \frac{\partial}{\partial g_{ij}} = \dot{x}^\mu \left(\frac{\partial}{\partial x^\mu} - A^a{}_\mu(x) \frac{(\lambda^a)_{ik}}{2\mathrm{i}} g_{kj} \frac{\partial}{\partial g_{ij}} \right) = \dot{x}^\mu \left(\frac{\partial}{\partial x^\mu} - A^a_\mu(x) \bar{L}_a \right)$$

where we have used the parallel transport equation for $\dot{g}_{ij}$. Now the covariant derivative is defined as

$$\mathrm{D}_\mu = \frac{\partial}{\partial x^\mu} - A^a_\mu(x) \bar{L}_a. \tag{5.30}$$

We are thus led to define a splitting of $T(P)$ into horizontal component $H(P)$ with basis D_μ, and a vertical component $V(P)$ lying in $T(G)$:

$$T(P) = H(P) \oplus V(P).$$

This splitting is invariant under right multiplication by the group.

The curvature is defined by

$$[\mathrm{D}_\mu, \mathrm{D}_\nu] = -F^a_{\mu\nu} \bar{L}_a,$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{abc} A^b_\mu A^c_\nu. \tag{5.31}$$

As expected, the commutator of covariant derivatives has only vertical components.

Cotangent space approach: We may regard the connection on P as a $\mathcal{G}$-valued one-form ω in $T^*(P)$ whose vertical component is the Maurer–Cartan form $g^{-1}\,\mathrm{d}g$. In local coordinates, we may write

$$\omega = g^{-1}Ag + g^{-1}\,\mathrm{d}g,$$

where $A(x) = A_\mu^a(x)\,(\lambda_a/2\mathrm{i})\,\mathrm{d}x^\mu$. We observe that, as in the vector bundle case, ω annihilates the horizontal basis of $T(P)$ and is constant on the vertical basis:

$$\langle \omega, \mathrm{D}_\mu \rangle = 0, \qquad \langle \omega, L_a \rangle = \lambda_a/2\mathrm{i}. \tag{5.32}$$

Under the right action of the group, $g \to gg_0$, A remains invariant and ω transforms tensorially,

$$\omega \to g_0^{-1}\omega g_0.$$

The curvature in this approach is a Lie-algebra valued matrix 2-form defined by

$$\Omega = \mathrm{d}\omega + \omega \wedge \omega = g^{-1}Fg \tag{5.33}$$

where

$$F = \mathrm{d}A + A \wedge A = \tfrac{1}{2}F_{\mu\nu}{}^a \frac{\lambda_a}{2\mathrm{i}}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu.$$

Ω obeys the Bianchi identity,

$$\mathrm{d}\Omega + \omega \wedge \Omega - \Omega \wedge \omega = 0. \tag{5.34}$$

Note that Ω has no vertical components. It transforms tensorially under right action,

$$\Omega \to g_0^{-1}\Omega g_0.$$

Gauge transformation: The transition functions of a principal bundle act on fibers by left multiplication. Let us consider two overlapping neighborhoods U and U' and a transition function $\Phi_{U'U} = \Phi$. The local fiber coordinates g and g' in U and U' are related by

$$g' = \Phi g.$$

Then, in order for the connection 1-form ω to be well-defined in the overlapping region $U \cap U'$, A must transform as

$$A' = \Phi A \Phi^{-1} + \Phi\,\mathrm{d}\Phi^{-1}. \tag{5.35}$$

We verify that

$$\omega = g^{-1}Ag + g^{-1}\,\mathrm{d}g = g'^{-1}A'g' + g^{-1}\,\mathrm{d}g',$$

so ω is indeed well-defined in $T^*(P)$. The transformation (5.35) is the *gauge transformation* of A. Using

(5.31), we find the gauge transformation of F to be

$$F' = \Phi F \Phi^{-1}.$$

It is easy to check that the curvature 2-form Ω is also consistently defined over the manifold,

$$\Omega = g^{-1} F g = g'^{-1} F' g'.$$

Pullback to base space: By choosing a section $g = g(x)$, one can pull back ω and Ω to the base space. A and F are equivalent to the pullbacks $g^*\omega$ and $g^*\Omega$, which are sometimes denoted simply as ω and Ω. Gauge transformations of A and F correspond to changes of the section.

In the theory of gauge fields, the structure group G is called the *gauge group*: the choice $G = U(1)$, for instance, gives the theory of electricity and magnetism and $G = \mathrm{SU}(3)$ gives the color theory of strong interactions. The (pulled-back) connection A of a principal bundle is the gauge potential and the (pulled-back) curvature F gives the strength of the gauge field. When matter fields are present in the gauge theory, they are described by the associated vector bundles.

Examples 5.5

1. *Dirac magnetic monopole*. We now put a connection on the $U(1)$ principal fiber bundle over the base space S^2 described in example 4.3.2. If we choose a particular connection which satisfies Maxwell's equations, the physical system described corresponds to Dirac's magnetic monopole. As before, we split S^2 into hemispheres $H_\pm$ and assign $U(1)$ connection 1-forms to each half of the bundle,

$$\omega = \begin{cases} A_+ + \mathrm{d}\psi_+ & \text{on } H_+ \\ A_- + \mathrm{d}\psi_- & \text{on } H_- \end{cases} .$$

(For $U(1)$, we conventionally factor out the (i) arising from our convention that Lie algebras are represented by antihermitian matrices: $g^{-1}\,\mathrm{d}g = \mathrm{e}^{-\mathrm{i}\psi}\,\mathrm{d}\mathrm{e}^{\mathrm{i}\psi} = \mathrm{i}\,\mathrm{d}\psi \to \mathrm{d}\psi$.) Then the choice of the transition function (4.21)

$$\mathrm{e}^{\mathrm{i}\psi_-} = \mathrm{e}^{\mathrm{i}n\phi}\,\mathrm{e}^{\mathrm{i}\psi_+}$$

implies the gauge transformation,

$$A_+ = A_- + n\phi.$$

Gauge potentials which satisfy Maxwell's equations (in $\mathbb{R}^3 - \{0\}$) and are regular in H_+ and H_- are given by (see example 2.4.3),

$$A_\pm = \frac{n}{2}(\pm 1 - \cos\theta)\,\mathrm{d}\phi = \frac{n}{2r}\frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{z \pm r}.$$

The curvature is given by

$$F = \mathrm{d}A_\pm = \frac{n}{2}\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi = \frac{n}{2r^3}(x\,\mathrm{d}y \wedge \mathrm{d}z + y\,\mathrm{d}z \wedge \mathrm{d}x + z\,\mathrm{d}x \wedge \mathrm{d}y).$$

It is easy to see that although the $A_\pm$ are regular in $H_\pm$, they have a string singularity in $H_\mp$. We will allow $A_\pm$ to be used only in its regular neighborhood. It is clear that F is *closed* but not exact, since $\mathrm{d}A_\pm$ is only defined locally in $H_\pm$.

Remark 1: We shall see in the next section that the monopole charge is minus the first Chern number C_1 characterizing the bundle:

$$-C_1 = -\int_{S^2} c_1 = +\frac{1}{2\pi}\int_{S^2} F = +\frac{1}{2\pi}\left[\int_{H_+} F_+ + \int_{H_-} F_-\right] = n.$$

Remark 2: It is instructive to carry out the above calculations using the S^2 metric $(\mathrm{d}x^2+\mathrm{d}y^2)/(1+x^2+y^2)^2 = (\mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2)/(1+r^2)^2$ obtained by projection from the north or south pole onto $\mathbb{R}^2$. In this case the "string singularity" occurs at $r=0$ or $r=\infty$. This treatment closely resembles the instanton case described below.

2. *BPST Instanton in* SU(2) *Yang–Mills* (Belavin et al. [1975]). The instanton solution of Euclidean SU(2) Yang–Mills theory is a connection on a principal bundle with

$$\text{Base } M = S^4, \qquad \text{Fiber } G = \mathrm{SU}(2) = S^3.$$

We take the S^4 metric (see example 3.2.3)

$$\mathrm{d}s^2 = \frac{\mathrm{d}x_\mu\,\mathrm{d}x_\mu}{(1+r^2/a^2)^2} = \frac{\mathrm{d}r^2 + r^2(\sigma_x^2+\sigma_y^2+\sigma_z^2)}{(1+r^2/a^2)^2} = \sum_{a=0}^{3}(e^a)^2$$

obtained by projection from the north or south pole onto $\mathbb{R}^4$.

As in example 4.3.3, we split S^4 into "hemispheres" $H_\pm$. In the overlap region

$$H_+ \cap H_- \simeq S^3,$$

we relate the SU(2) fibers by the transition functions

$$g_- = [h(x)]^k \cdot g_+,$$

where k is an integer, $h = (t - \mathrm{i}\boldsymbol{\lambda}\cdot\boldsymbol{x})/r$ and $\boldsymbol{\lambda}$ are the SU(2) Pauli matrices. We note that

$$h^{-1}\,\mathrm{d}h = \mathrm{i}\lambda_k\sigma_k = \mathrm{i}\lambda_k\eta^k{}_{\mu\nu}x_\mu\,\mathrm{d}x_\nu/r^2$$

$$\mathrm{d}h\,h^{-1} = -\mathrm{i}\lambda_k\bar{\sigma}_k = -\mathrm{i}\lambda_k\bar{\eta}^k{}_{\mu\nu}x_\mu\,\mathrm{d}x_\nu/r^2$$

where η, $\bar{\eta}$ are 't Hooft's eta tensors ('t Hooft [1977]; see appendix C).

The connection 1-forms in the two neighborhoods of the bundle then may be written as

$$\omega = \begin{cases} g_+^{-1}Ag_+ + g_+^{-1}\,\mathrm{d}g_+ & \text{on } H_+ \\ g_-^{-1}A'g_- + g_-^{-1}\,\mathrm{d}g_- & \text{on } H_- \end{cases}$$

where

$$A'(x) = h^k(x)\,A(x)\,h^{-k}(x) + h^k(x)\,\mathrm{d}h^{-k}(x).$$

In the case $k = 1$, we have the single instanton solution,

$$H_+\colon A = \frac{r^2}{r^2 + a^2}\cdot h^{-1}\,\mathrm{d}h = \frac{r^2}{r^2 + a^2}\,\mathrm{i}\,\lambda_k\sigma_k$$

which is singular at the "south-pole" at $r = \infty$, and the gauge-transformed solution,

$$H_-\colon A' = h\left[\frac{r^2}{r^2 + a^2} h^{-1}\,\mathrm{d}h\right] h^{-1} + h\,\mathrm{d}h^{-1} = -\frac{\mathrm{d}h\,h^{-1}}{1 + r^2/a^2} = \frac{\mathrm{i}\,\lambda_k\bar{\sigma}_k}{1 + r^2/a^2}$$

which is singular at the "north-pole" at $r = 0$. (*Note*: A and A' are the Yang–Mills analogs of the two gauge-equivalent Dirac monopole solutions with Dirac strings in the upper and lower hemispheres of S^2.)

The field strengths in $H_\pm$ are easily computed to be

$$H_+\colon\quad F_+ = \mathrm{d}A + A \wedge A = \mathrm{i}\,\lambda_k \frac{2}{a^2}\left(e^0 \wedge e^k + \frac{1}{2}\epsilon_{kij} e^i \wedge e^j\right)$$

$$H_-\colon\quad F_- = \mathrm{d}A' + A' \wedge A' = hF_+h^{-1}.$$

Since F is self-dual,

$$*F = F,$$

the Bianchi identities imply that the Yang–Mills equations

$$\mathrm{D}_A *F = \mathrm{d}*F + A \wedge *F - *F \wedge A = 0$$

are satisfied. Replacing $h(x)$ by $h^{-1}(x)$ and interchanging σ_k and $\bar{\sigma}_k$ throughout would give us an anti-self-dual solution.

Remark 1: In the next section, we will see that the "instanton number" k is minus the second Chern number C_2 characterizing the bundle:

$$k = -C_2 = -\int_{S^4} c_2 = -\frac{1}{8\pi^2}\int_{S^4} \operatorname{Tr} F \wedge F$$

$$= -\frac{1}{8\pi^2}\left[\int_{H_+} \operatorname{Tr} F_+ \wedge F_+ + \int_{H_-} \operatorname{Tr} F_- \wedge F_-\right] = -\frac{1}{8\pi^2}\left(-\frac{48}{a^4}\right)\int_{S^4} e^0 \wedge e^1 \wedge e^2 \wedge e^3 = +1.$$

(Recall that the volume of S^4 with radius $a/2$ is $\pi^2 a^4/6$. See appendix A.)

Remark 2: Note that $A_{(\pm)} = A^a_{(\pm)}(\lambda^a/2i)$ for $k = \pm 1$ are derivable from the self-dual or anti-self-dual combinations of the $O(4)$ connections ω_{ab} of S^4 given in example 3.2.3,

$$A^1_{(+)} = -\omega_{01} - \omega_{23} = -2\sigma_x \frac{(r/a)^2}{1+(r/a)^2}, \qquad \text{cyclic},$$

$$A^1_{(-)} = +\omega_{01} - \omega_{23} = -\frac{2\sigma_x}{1+(r/a)^2}, \qquad \text{cyclic}.$$

Here the diameter $2R$ of S^4 is identified with the instanton size a. This is related to the fact that the $k = 1$ bundle is the Hopf fibration of S^7.

Remark 3: Under an $O(4)$ transformation, the $k = 1$ instanton transforms into itself up to a gauge transformation. Under an $O(5)$ transformation of S^4, it also transforms into itself up to a gauge transformation; the BPST instanton solution is unique in possessing the $O(5)$ symmetry (see, e.g., Jackiw and Rebbi [1976a]).

6. Characteristic classes

We have now seen explicitly how the construction of nontrivial fiber bundles involves certain integers characterizing the transition functions. Furthermore, we observed in passing that when we put connections on the bundles, these same integers corresponded to integrals involving a bundle's curvature. In this section, we will develop more thoroughly the concept of the *characteristic classes* distinguishing inequivalent fiber bundles. The manipulation of characteristic classes plays an essential role in index theory, which is the subject of the next section.

In the preceding sections we have been careful to distinguish among connection 1-forms and curvature 2-forms used for different purposes: $\omega^a{}_b$ and $R^a{}_b$ were used for Riemannian geometry in an orthonormal frame basis, $\Gamma^i{}_j$ and $R^i{}_j$ were used for vector bundles, and A and F were used for principal bundles. The notation ω was also used for connections lying in T^* of the bundle rather than in T^* of the base, while Ω was used for the corresponding curvature. In this section, we loosen these distinctions for notational convenience and employ the symbols ω and Ω to denote the values of the connection and curvature forms pulled back using sections of a bundle.

We shall deal with the following four categories of characteristic classes.

1. *Chern classes* $c_1, \ldots, c_k$ are defined for a complex vector bundle of dimension k (or equivalently for GL(k, $\mathbb{C}$) principal bundles). $c_i \in H^{2i}(M)$.

2. *Pontrjagin classes* $p_1, \ldots, p_j$ are defined for a real vector bundle of dimension k (or equivalently for GL(k, $\mathbb{R}$) principal bundles). $p_i \in H^{4i}(M)$. ($j = [k/2]$ is the greatest integer in $k/2$.)

3. *The Euler class* e is defined for an oriented bundle of even dimension k with a fiber metric (or equivalently for SO(k) principal bundles). $e \in H^k(M)$.

4. *Stiefel–Whitney classes* $w_1, \ldots, w_k$ are defined for a real vector bundle of dimension k (or equivalently for GL(k, $\mathbb{R}$) principal bundles). They are $\mathbb{Z}_2$ characteristic classes and are not given by curvature. $w_i \in H^i(M; \mathbb{Z}_2)$.

6.1. General properties of Chern classes

We begin our study of characteristic classes by examining the Chern classes associated with bundles having GL(k, $\mathbb{C}$) transition functions. Many of the methods we discuss will then be applicable to other groups and characteristic classes.

Invariant polynomials: Let α be a complex $k \times k$ matrix and $P(\alpha)$ be a polynomial in the components of α. $P(\alpha)$ is called an *invariant polynomial* or a *characteristic polynomial* if

$$P(\alpha) = P(g^{-1}\alpha g) \tag{6.1}$$

for all $g \in \mathrm{GL}(k, \mathbb{C})$. If α has eigenvalues $\{\lambda_1, \ldots, \lambda_k\}$, $P(\alpha)$ is a symmetric function of the eigenvalues. If $S_j(\lambda)$ is the jth symmetric polynomial,

$$S_j(\lambda) = \sum_{i_1 < i_2 < \cdots < i_j} \lambda_{i_1}\lambda_{i_2}, \ldots, \lambda_{i_j},$$

then $P(\alpha)$ is a polynomial in the $S_j(\lambda)$:

$$P(\alpha) = a + bS_1(\lambda) + cS_2(\lambda) + d[S_1(\lambda)]^2 + \cdots .$$

Examples of invariant polynomials are

$$\mathrm{Det}(I + \alpha) = 1 + S_1(\lambda) + S_2(\lambda) + \cdots + S_k(\lambda) \tag{6.2}$$

and $\mathrm{Tr}(\exp \alpha)$, which are used to define the Chern class and the Chern character.

If a matrix-valued curvature 2-form Ω is substituted for the matrix α in an invariant polynomial, we find the following properties:

(1) *$P(\Omega)$ is closed*

(2) *$P(\Omega)$ has topologically invariant integrals.*

We will prove these assertions following Chern [1967]. Suppose $P(\alpha_1, \ldots, \alpha_r)$ is a homogeneous invariant polynomial of degree r. Using the invariance of the polynomial under an infinitesimal transformation $g = I + g'$, we can deduce

$$\sum_{1 \le i \le r} P(\alpha_1, \ldots, g'\alpha_i - \alpha_i g', \ldots, \alpha_r) = 0.$$

Then if θ is a $k \times k$ Lie-algebra valued matrix of 1-forms and the $\{\alpha_i\}$ are $k \times k$ Lie-algebra valued forms of degree d_i, we find

$$\sum_{1 \le i \le r} (-1)^{d_1 + \cdots + d_{i-1}} P(\alpha_1, \ldots, \theta \wedge \alpha_i, \ldots, \alpha_r) - \sum_{1 \le i \le r} (-1)^{d_1 + \cdots + d_i} P(\alpha_1, \ldots, \alpha_i \wedge \theta, \ldots, \alpha_r) = 0. \tag{6.3}$$

Therefore if we choose θ to be the connection 1-form ω, we may write

$$\mathrm{d}P(\alpha_1, \ldots, \alpha_r) = \sum_{1 \le i \le r} (-1)^{d_1 + \cdots + d_{i-1}} P(\alpha_1, \ldots, \mathrm{D}\alpha_i, \ldots, \alpha_r)$$

where

$$\mathrm{D}\alpha_i = \mathrm{d}\alpha_i + \omega \wedge \alpha_i - (-1)^{d_i}\alpha_i \wedge \omega$$

is the covariant derivative of the form α_i.

If $\alpha_i = \Omega$, the curvature 2-form, we conclude that

$$\mathrm{d}P(\Omega) = 0$$

because of the Bianchi identity (5.34).

Now let ω and ω' be two connections on the bundle and Ω and Ω' their curvatures. We consider the interpolation between ω and ω',

$$\omega_t = \omega + t\eta \qquad 0 \le t \le 1,$$

where $\eta = \omega' - \omega$.

Then

$$\Omega_t = \mathrm{d}\omega_t + \omega_t \wedge \omega_t = \Omega + t\mathrm{D}\eta + t^2 \eta \wedge \eta,$$

where $\mathrm{D}\eta = \mathrm{d}\eta + \omega \wedge \eta + \eta \wedge \omega$.

Let $P(\alpha_1, \ldots, \alpha_r)$ be a symmetric polynomial and let

$$q(\beta, \alpha) = rP(\beta, \underbrace{\alpha, \ldots, \alpha}_{r-1}).$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t} P(\Omega_t) = q(\mathrm{D}\eta, \Omega_t) + 2tq(\eta \wedge \eta, \Omega_t).$$

On the other hand

$$\begin{aligned}\mathrm{D}\Omega_t &= t\mathrm{D}^2\eta + t^2(\mathrm{D}\eta \wedge \eta - \eta \wedge \mathrm{D}\eta) = t(\Omega \wedge \eta - \eta \wedge \Omega) + t^2(\mathrm{D}\eta \wedge \eta - \eta \wedge \mathrm{D}\eta) \\ &= t(\Omega_t \wedge \eta - \eta \wedge \Omega_t),\end{aligned}$$

so that

$$\begin{aligned}\mathrm{d}q(\eta, \Omega_t) &= q(\mathrm{D}\eta, \Omega_t) - r(r-1)P(\eta, \mathrm{D}\Omega_t, \Omega_t, \ldots, \Omega_t) \\ &= q(\mathrm{D}\eta, \Omega_t) - r(r-1)tP(\eta, (\Omega_t \wedge \eta - \eta \wedge \Omega_t), \Omega_t, \ldots, \Omega_t).\end{aligned}$$

Eq. (6.3) with $\theta = \alpha_1 = \eta$, $\alpha_2 = \cdots = \alpha_r = \Omega_t$ gives

$$2q(\eta \wedge \eta, \Omega_t) + r(r-1)P(\eta, (\Omega_t \wedge \eta - \eta \wedge \Omega_t), \Omega_t, \ldots, \Omega_t) = 0.$$

Combining the last two equations, we get

$$\mathrm{d}q(\eta, \Omega_t) = q(\mathrm{D}\eta, \Omega_t) + 2tq(\eta \wedge \eta, \Omega_t) = \frac{\mathrm{d}}{\mathrm{d}t} P(\Omega_t).$$

Hence

$$P(\Omega') - P(\Omega) = \mathrm{d}\int_0^1 q(\omega' - \omega, \Omega_t)\,\mathrm{d}t \equiv \mathrm{d}Q(\omega', \omega). \tag{6.4}$$

Since $P(\Omega')$ and $P(\Omega)$ differ by an exact form $\mathrm{d}Q$, their integrals over manifolds without boundary give the same results. Thus we have proven both properties (1) and (2).

Chern form: The Chern form of a complex vector bundle E over M with $\mathrm{GL}(k, \mathbb{C})$ transition functions and a connection ω is obtained by substituting the curvature 2-form $\Omega \in \mathrm{gl}(k, \mathbb{C})$ into the invariant polynomial $\mathrm{Det}(1 + \alpha)$. We define the *total Chern form* as

$$c(\Omega) = \mathrm{Det}\left(I + \frac{\mathrm{i}}{2\pi}\Omega\right) = 1 + c_1(\Omega) + c_2(\Omega) + \cdots, \tag{6.5}$$

where the individual Chern forms $c_j(\Omega)$ are polynomials of degree j in Ω:

$$\begin{aligned}
c_0 &= 1 \\
c_1 &= \frac{\mathrm{i}}{2\pi}\mathrm{Tr}\,\Omega \\
c_2 &= \frac{1}{8\pi^2}\{\mathrm{Tr}\,\Omega \wedge \Omega - \mathrm{Tr}\,\Omega \wedge \mathrm{Tr}\,\Omega\} \\
c_3 &= \frac{\mathrm{i}}{48\pi^2}\{-2\,\mathrm{Tr}\,\Omega \wedge \Omega \wedge \Omega + 3(\mathrm{Tr}\,\Omega \wedge \Omega) \wedge \mathrm{Tr}\,\Omega - \mathrm{Tr}\,\Omega \wedge \mathrm{Tr}\,\Omega \wedge \mathrm{Tr}\,\Omega\} \\
&\ \vdots
\end{aligned}$$

The explicit expressions for c_j are obtained from the eigenvalue expansion of $\alpha = \mathrm{diag}(\lambda_1, \ldots, \lambda_k)$:

$$\begin{aligned}
\mathrm{Det}\left(I + \frac{\mathrm{i}}{2\pi}\alpha\right) &= \left(1 + \frac{\mathrm{i}}{2\pi}\lambda_1\right)\left(1 + \frac{\mathrm{i}}{2\pi}\lambda_2\right)\cdots\left(1 + \frac{\mathrm{i}}{2\pi}\lambda_k\right) \\
&= 1 + \frac{\mathrm{i}}{2\pi}S_1(\lambda) + \left(\frac{\mathrm{i}}{2\pi}\right)^2 S_2(\lambda) + \cdots
\end{aligned}$$

where the $S_j(\lambda)$ are the elementary symmetric functions defined earlier. For example,

$$\left(\frac{\mathrm{i}}{2\pi}\right)^2 \sum_{j<l}^{k} \lambda_j\lambda_l = \left(\frac{\mathrm{i}}{2\pi}\right)^2 \left(\frac{1}{2}\right)((\mathrm{Tr}\,\alpha)^2 - \mathrm{Tr}(\alpha^2))$$

gives c_2 if the matrix α is replaced by Ω. Since $c_j(\Omega) \in \Lambda^{2j}(M)$, we see that

$$c_j = 0 \qquad \text{for } 2j > n = \dim M.$$

Thus $c(\Omega)$ is always a finite sum.

Since any invariant polynomial $P(\alpha)$ can be expressed in terms of the elementary symmetric functions, $P(\alpha)$ can be expressed as a polynomial in the Chern forms. Thus the Chern forms generate the characteristic ring.

Chern classes and cohomology: Since $P(\Omega)$ is closed, any homogeneous polynomial in the expansion of an invariant polynomial $P(\Omega)$ is closed:

$$\mathrm{d}c_j(\Omega) = 0. \tag{6.6}$$

We may verify this explicitly using the Bianchi identities; for example,

$$\mathrm{d}c_1(\Omega) = \frac{\mathrm{i}}{2\pi}\operatorname{Tr}\mathrm{d}(\mathrm{d}\omega + \omega\wedge\omega) = \frac{\mathrm{i}}{2\pi}\operatorname{Tr}(\Omega\wedge\omega - \omega\wedge\Omega) \equiv 0.$$

We conclude that the Chern forms $c_j(\Omega)$ define $2j$th cohomology classes,

$$c_j(\Omega) \in H^{2j}(M). \tag{6.7}$$

This cohomology class, which we will often denote by $c_j(E)$, is independent of the connection because $P(\Omega) - P(\Omega')$ is *exact* for any characteristic polynomial.

Chern numbers and topological invariance: It is a remarkable fact that the cohomology classes to which the Chern forms $c_j(\Omega)$ belong are actually *integer* classes. If we integrate $c_j(\Omega)$ over any $2j$-cycle in M with integer coefficients, we obtain an integer which is independent of the connection. The *Chern numbers* of a bundle are the numbers which result from integrating characteristic polynomials over the entire manifold; for example, if $n = 4$, the only two Chern numbers are

$$C_2(E) = \int_M c_2(\Omega)$$

$$C_1^2(E) = \int_M c_1(\Omega)\wedge c_1(\Omega).$$

Characteristic classes of unitary bundles: One can show that the $U(k)$ and $\mathrm{GL}(k, \mathbb{C})$ characteristic polynomials can be identified. Therefore their characteristic classes can be identified. This is *not* true for $\mathrm{GL}(k, \mathbb{R})$ and $O(k)$ or $\mathrm{SO}(k)$. The $\mathrm{SU}(k)$ characteristic classes are generated by $(c_2, \ldots, c_k)$ because $c_1 = 0$. Note that if $c_1 \neq 0$ for a complex vector bundle E, there is no associated $\mathrm{SU}(k)$ principal bundle. (Warning: there exist bundles with $c_1 = 0$ which also do not admit an $\mathrm{SU}(k)$ structure.)

Chern classes of composite bundles: Let $c(E) = c_0(E) + \cdots + c_k(E)$, with $c_j(E) \in H^{2j}(M)$, denote the total Chern class for a k-dimensional complex vector bundle E over M. Then we find

(1) Whitney sum: $c(E \oplus F) = c(E)\wedge c(F)$.
(2) $c_1(L \otimes L') = c_1(L) + c_1(L')$ for L, L' = line bundles.
(3) Pullback class: $c(f^*E) = f^*c(E)$, where $f\colon M' \to M$ and $E' = f^*E$ is the pullback of E over M'.

These properties plus the requirement that $C_1(L) = -1$ for the line bundle L over $P_1(\mathbb{C})$ are sometimes used as an axiomatic definition of the Chern class.

The Weil homomorphism: It is well-known that the "Casimir invariants" or polynomials in the center of a Lie algebra $\mathcal{G}$ with matrix basis $\{X_i\}$ are generated by the determinant

$$\mathrm{Det}(t \cdot I + a_i X_i) = \sum_k t^k P_k(a_i).$$

The Chern classes are thus obtained by substituting the Lie-algebra valued curvature 2-forms into each of the resulting characteristic polynomials.

6.2. Classifying spaces

We motivate the concept of classifying spaces for fiber bundles by showing how the standard complex line bundle L over $P_{n-1}(\mathbb{C})$ may be used to classify other line bundles. Let E be a complex line bundle over M and assume that we can find a complementary bundle $\bar{E}$ such that

$$E \oplus \bar{E} = M \times \mathbb{C}^n$$

for some $n > 1$ (this is always possible). The fibers of E are then lines in $\mathbb{C}^n$. We define a map $f(x)$ from the points $x \in M$ to $P_{n-1}(\mathbb{C})$ which associates to each point x the line in $\mathbb{C}^n$ given by the fiber F_x. *Then the line bundle E is isomorphic to the pullback of the natural line bundle L over* $P_{n-1}(\mathbb{C})$:

$$E \approx f^* L.$$

We can generalize this construction by considering the *Grassmann manifold* $\mathrm{Gr}(m, k, \mathbb{C})$ of k-planes in $\mathbb{C}^m$; just as the points of $\mathrm{Gr}(m, 1, \mathbb{C}) \equiv P_{m-1}(\mathbb{C})$ correspond to lines through the origin in $\mathbb{C}^m$, each point of $\mathrm{Gr}(m, k, \mathbb{C})$ corresponds to a k-plane through the origin. The *natural k-plane bundle* $L(m, k, \mathbb{C})$ over $\mathrm{Gr}(m, k, \mathbb{C})$ has as its fiber the k-plane in $\mathbb{C}^m$ over the corresponding point in $\mathrm{Gr}(m, k, \mathbb{C})$; $L(m, 1, \mathbb{C})$ is just the natural line bundle L over $P_{m-1}(\mathbb{C})$. We now quote without proof a basic theorem (see, e.g., Chern [1972]):

Theorem: Let M be a manifold of dimension n and E any k-dimensional complex vector bundle over M. Then there exists an integer m_0 (depending on n) such that for $m \geq m_0$,

(a) there exists a map f: $M \to \mathrm{Gr}(m, k, \mathbb{C})$ such that $E = f^* L(m, k, \mathbb{C})$;

(b) given any two maps f and g mapping $M \to \mathrm{Gr}(m, k, \mathbb{C})$, then $f^* L(m, k, \mathbb{C}) \approx g^* L(m, k, \mathbb{C})$ if and only if f and g are homotopic.

As a consequence of this theorem, the set of isomorphism classes of k-dimensional vector bundles is itself isomorphic to the set of homotopy classes of maps from M to $\mathrm{Gr}(m, k, \mathbb{C})$; in this manner, questions about the classification of vector bundles are reduced to questions about homotopy theory in algebraic topology.

Classifying spaces of principal bundles: $P(m, k, \mathbb{C})$, the bundle of frames of $L(m, k, \mathbb{C})$, is a principal $\mathrm{GL}(k, \mathbb{C})$ bundle over $\mathrm{Gr}(m, k, \mathbb{C})$. For $m \geq m_0$, very large, $P(m, k, \mathbb{C})$ and $L(m, k, \mathbb{C})$ are described by the same set of homotopy classes of maps from $M \to \mathrm{Gr}(m, k, \mathbb{C})$. In fact, we can make the identification

$$\mathrm{Gr}(m, k, \mathbb{C}) = \mathrm{GL}(m, \mathbb{C})/\mathrm{GL}(k, \mathbb{C}) \times \mathrm{GL}(m-k, \mathbb{C})$$
$$P(m, k, \mathbb{C}) = \mathrm{GL}(m, \mathbb{C})/\mathrm{GL}(m-k, \mathbb{C}) \tag{6.8}$$

where the projection π: $P(m, k, \mathbb{C}) \to \mathrm{Gr}(m, k, \mathbb{C})$ projects out the fiber $\mathrm{GL}(k, \mathbb{C})$. Clearly similar constructions can be carried out for $\mathrm{GL}(k, \mathbf{R})$ principal bundles, $\mathrm{SO}(k)$ principal bundles, $\mathrm{SU}(k)$ principal bundles, etc.

Universal classifying spaces: We define the universal Grassmannian $\mathrm{Gr}(\infty, k, \mathbb{C})$ by taking the union of the natural inclusion maps of $\mathrm{Gr}(m, k, \mathbb{C})$ into $\mathrm{Gr}(m+1, k, \mathbb{C})$. We denote the universal classifying bundles corresponding to $\mathrm{Gr}(\infty, k, \mathbb{C})$ by $L(\infty, k, \mathbb{C})$ and $P(\infty, k, \mathbb{C})$. The cohomology of $\mathrm{Gr}(\infty, k, \mathbb{C})$ is simpler than that of $\mathrm{Gr}(m, k, \mathbb{C})$ and is a polynomial algebra with generators $c_i = c_i(L(\infty, k, \mathbb{C})) = c_i(P(\infty, k, \mathbb{C}))$. Given a k-dimensional bundle E and a map

$$f: M \to \mathrm{Gr}(\infty, k, \mathbb{C})$$
$$f^* L(\infty, k, \mathbb{C}) \approx E,$$

we see that

$$c_i(E) = f^* c_i.$$

f is defined uniquely up to homotopy so the cohomology classes are all well-defined and depend only on the bundle E.
Note: from this approach, it is obvious that $U(k)$ bundles and $\mathrm{GL}(k, \mathbb{C})$ bundles both have the same classifying space $\mathrm{Gr}(\infty, k, \mathbb{C})$, and thus the same characteristic classes.

6.3. The splitting principle

Algebraic identities involving characteristic classes are a central part of index theory. Such manipulations are made vastly simpler by the use of a tool called the *splitting principle* (see e.g. Hirzebruch [1966]).

We gave above a brief description of the characteristic classes $c_i(E)$ using our knowledge of the cohomology of the classifying spaces $\mathrm{Gr}(m, k, \mathbb{C})$, the Grassmann manifolds. This is an approach based on algebraic topology; from this viewpoint the splitting principle is the idea that even though a given bundle is not, in general, a direct sum of one-dimensional line bundles, characteristic class manipulations can be performed as though this were the case. We also discussed the characteristic classes using invariant polynomials and curvature. From this differential geometric point of view, the splitting principle is simply the assertion that the diagonalizable matrices are dense.

We illustrate the concepts of the splitting principle with the familiar identity

$$\mathrm{Det}[\alpha] = \exp(\mathrm{Tr}\ln[\alpha]).$$

If α is a diagonalizable matrix with eigenvalues $\{\lambda_i\}$, then it is clear that

$$\mathrm{Det}[\alpha] = \prod_i \lambda_i = \exp\Big(\sum_i \ln \lambda_i\Big) = \exp(\mathrm{Tr}\ln[\alpha]).$$

Since both sides of this equation are continuous and since we can approximate any matrix by a diagonalizable matrix, this identity holds true for any matrix. Thus to prove an invariant identity of this sort, we may in fact assume that the matrix α is diagonal.

Now let Ω be an $n \times n$ matrix of curvature 2-forms. If we imagine that Ω is diagonalizable into n 2-forms Ω_j, then the Chern class becomes

$$\begin{aligned} c(E) &= \mathrm{Det}\Big(1 + \frac{\mathrm{i}}{2\pi}\Omega\Big) = \mathrm{Det}\begin{pmatrix} 1+\frac{\mathrm{i}}{2\pi}\Omega_1 & & 0 \\ & \ddots & \\ 0 & & 1+\frac{\mathrm{i}}{2\pi}\Omega_n \end{pmatrix} \\ &= \prod_{j=1}^{k}\Big(1 + \frac{\mathrm{i}}{2\pi}\Omega_j\Big) = \prod_{j=1}^{k}(1+x_j) \end{aligned} \tag{6.9}$$

where we will henceforth use the formal notation

$$x_j = \frac{\mathrm{i}}{2\pi}\Omega_j.$$

Each of the terms $(1+x_j)$ can be interpreted as the Chern class of a one-dimensional line bundle L_j,

$$c(L_j) = 1 + c_1(L_j) = 1 + \frac{\mathrm{i}}{2\pi}\Omega_j.$$

If we imagine that a k-dimensional vector bundle E has a decomposition

$$E = L_1 \oplus \cdots \oplus L_k,$$

then

$$c(E) = \prod_j c(L_j) = \prod_j (1+x_j).$$

Thus $c_l(E)$ can be thought of as the lth elementary symmetric function of the variables $\{x_j\}$:

$$c_1 = \sum_j x_j, \qquad c_2 = \sum_{i<j} x_i x_j, \qquad \ldots, \qquad c_k = x_1 x_2 \ldots x_k. \tag{6.10}$$

Sums of bundles: If A and B are matrices, then

$$\mathrm{Det}(A \oplus B) = \mathrm{Det}\,A \cdot \mathrm{Det}\,B.$$

Consequently, if E and F are vector bundles, then

$$c(E \oplus F) = c(E) \wedge c(F),$$

since this is true on the form level when we use the Whitney sum connection. From the point of view of algebraic topology, this identity is first proved for bundles E and F which actually split into a sum of line bundles. The splitting principle is then invoked to deduce the identity for the general case.

Chern character: Many essential manipulations in index theory involve not only Whitney sums of bundles but also tensor products of bundles. The total Chern class behaves well for Whitney sums, but not for product bundles. We are thus motivated to put aside $c(E) = \Pi_j(1 + x_j)$ and to find some other polynomial in the $\{x_j\}$ which has simple properties for product bundles as well as Whitney sums. One such polynomial is the Chern character $\mathrm{ch}(E)$. In terms of matrices, the Chern character is defined by the following invariant polynomial:

$$\mathrm{ch}(\alpha) = \mathrm{Tr}\exp\left(\frac{\mathrm{i}}{2\pi}\alpha\right) = \sum_j \frac{1}{j!}\mathrm{Tr}\left(\frac{\mathrm{i}}{2\pi}\alpha\right)^j. \tag{6.11}$$

Since

$$\mathrm{ch}(\alpha \oplus \beta) = \mathrm{ch}(\alpha) + \mathrm{ch}(\beta)$$

$$\mathrm{ch}(\alpha \otimes \beta) = \mathrm{ch}(\alpha)\,\mathrm{ch}(\beta),$$

these identities still hold when we substitute the curvature 2-form Ω to define $\mathrm{ch}(E)$. *Note*: since $\mathrm{Tr}(\Omega)^j = 0$ for $j > n/2$, we in fact have a finite sum.

The Chern character of E has the splitting principle expansion

$$\mathrm{ch}(E) = \sum_{j=1}^{k} \mathrm{e}^{x_j} = k + c_1(E) + \tfrac{1}{2}(c_1{}^2 - 2c_2)(E) + \cdots. \tag{6.12}$$

Other characteristic polynomials: Using the splitting principle, we may define characteristic classes by their generating functions. For example, the total Chern class has the generating function $\Pi(1 + x_j)$, while the Chern character has the generating function $\Sigma\, \mathrm{e}^{x_j}$. Another class which appears in the index theorem is the *Todd class* which has the generating function

$$\mathrm{td}(E) = \prod_{j=1}^{k} \frac{x_j}{1 - \mathrm{e}^{-x_j}} = 1 + \tfrac{1}{2}c_1(E) + \tfrac{1}{12}(c_1^2 + c_2)(E) + \cdots. \tag{6.13}$$

The Todd class is multiplicative for Whitney sums,

$$\mathrm{td}(E \oplus F) = \mathrm{td}(E)\,\mathrm{td}(F).$$

We can define other multiplicative characteristic classes by using other generating functions. Two other such functions are the *Hirzebruch L-polynomial*

$$L(E)=\prod_j \frac{x_j}{\tanh x_j} \tag{6.14}$$

which appears in the signature index formula, and the $\hat{A}$ *polynomial*

$$\hat{A}(E)=\prod_j \frac{x_j/2}{\sinh(x_j/2)} \tag{6.15}$$

which appears in the spin index formula.

Examples 6.3

1. *Chern class of $P_1(\mathbb{C})$ line bundle.* Let L be the natural line bundle over base manifold $M=S^2=P_1(\mathbb{C})$ (see example 5.4.1) with the natural curvature

$$\Omega=-\partial\bar{\partial}\ln(1+|z|^2)=\frac{-\mathrm{d}z\wedge\mathrm{d}\bar{z}}{(1+z\bar{z})^2}.$$

Then

$$c_1(L)=\frac{\mathrm{i}}{2\pi}\Omega=-\frac{1}{\pi}\frac{\mathrm{d}x\wedge\mathrm{d}y}{(1+x^2+y^2)^2}=-\frac{1}{\pi}\frac{r\,\mathrm{d}r\wedge\mathrm{d}\theta}{(1+r^2)^2},$$

so the Chern number characterizing the bundle is

$$C_1(L)=\int_{S^2}c_1(L)=-\frac{1}{\pi}\int_0^\infty\frac{r\,\mathrm{d}r}{(1+r^2)^2}\int_0^{2\pi}\mathrm{d}\theta=-1.$$

Dual line bundle: The natural curvature on the dual line bundle L^* is the complex conjugate of that for L,

$$\Omega(L^*)=\bar{\Omega}(L)=-\Omega(L),$$

so that the Chern class reverses in sign:

$$c_1(L^*)=-c_1(L)$$

$$C_1(L^*)=\int c_1(L^*)=+1.$$

Alternatively, we may derive this result from the fact that the tensor product bundle

$L^*\otimes L=I$ [trivial complex line bundle over $P_1(\mathbb{C})$]

is *trivial* because the transition functions $(z_i/z_j)(z_j/z_i) = 1$ are trivial. Thus we know that the total Chern class is

$$c(L^* \otimes L) = c\ (\text{trivial bundle}) = 1 + 0.$$

But then the Chern character formula for product bundles gives us our result:

$$0 = c_1(L^* \otimes L) = c_1(L^*) + c_1(L).$$

Tangent and cotangent bundles of $S^2 = P_1(\mathbb{C})$: We showed in example 4.2.3 that

$$T(S^2) = T_c(P_1(\mathbb{C})) \approx L^* \otimes L^*$$
$$T^*(S^2) = T_c^*(P_1(\mathbb{C})) \approx L \otimes L.$$

From the product bundle formula we immediately find

$$C_1(T(S^2)) = C_1(L^*) + C_1(L^*) = +2$$

$$C_1(T^*(S^2)) = C_1(L) + C_1(L) = -2.$$

2. *Chern classes of* $P_n(\mathbb{C})$. We next consider the natural line bundle L over $P_n(\mathbb{C})$ and its dual L^*. Choosing the Fubini–Study metric (example 3.4.3) on $P_n(\mathbb{C})$, we compute x from the Kähler form,

$$x = c_1(L^*) = \frac{\mathrm{i}}{2\pi}\,\Omega(L^*) = \frac{1}{\pi}\,K\ (\text{Fubini–Study}).$$

The factor $\mathrm{i}/2\pi$ is chosen so that the integral of x over $P_1(\mathbb{C})$ is equal to 1. It can be shown that the integral of x^n over $P_n(\mathbb{C})$ for any n is also 1. The 2-form x generates the cohomology ring of $P_n(\mathbb{C})$ with integer coefficients. The Betti numbers are

$$b_0 = b_2 = b_4 = \cdots = b_{2n} = 1$$

$$b_1 = b_3 = \cdots = b_{2n-1} = 0$$

and the Euler characteristic is

$$\chi(P_n(\mathbb{C})) = \sum (-1)^k b_k = n + 1.$$

To compute the Chern classes of $P_n(\mathbb{C})$, we first consider the bundle E_{n+1} consisting of the Whitney sum of $(n+1)$ copies of L^*. The total Chern class is then

$$c(E_{n+1}) = c(L^* \oplus L^* \oplus \cdots \oplus L^*) = (1 + x)^{n+1}.$$

There is a natural embedding of $T_c(P_n(\mathbb{C}))$ in E_{n+1}. The quotient or complementary bundle is trivial.

Thus

$$E_{n+1} = L^* \oplus \cdots \oplus L^* = T_c(P_n(\mathbb{C})) \oplus I.$$

Therefore we find that

$$c(E_{n+1}) = (1+x)^{n+1} = c(T_c(P_n(\mathbb{C}))) \cdot c(I) = c(T_c(P_n(\mathbb{C}))).$$

It is customary to define the Chern class of a complex manifold to be the Chern class of its complex tangent space,

$$c(M) = c(T_c(M)).$$

Thus in particular,

$$c(P_n(\mathbb{C})) = (1+x)^{n+1}.$$

We note that $c_n(P_n(\mathbb{C})) = (n+1)x^n$, so

$$\int_{P_n(\mathbb{C})} c_n(P_n(\mathbb{C})) = n+1.$$

It is no accident that this is the Euler characteristic of $P_n(\mathbb{C})$. The expression

$$\int c_n(M) = \chi(M)$$

is, in fact, the Gauss–Bonnet theorem for a complex manifold of complex dimension n.

3. *Vector bundles over S^n*. Let $n = 2l$ and let $E_\pm$ be the complex vector bundles over S^n introduced in example 5.4.2. We recall that $E_\pm$ was defined using the projection operator $\Pi_\pm$, that $E_+ \oplus E_- = S^n \times \mathbb{C}^{2^l}$ and that the fiber of $E_\pm$ has dimension 2^{l-1}. Choosing the curvature $\Omega_\pm(x_0) = \Pi_\pm(x_0)\, \mathrm{d}\Pi_\pm(x_0) \wedge \mathrm{d}\Pi_\pm(x_0)$, we recall that

$$\mathrm{Tr}(\Omega_\pm)^l = \pm \frac{n!(2\mathrm{i})^l}{2^{n+1}} \mathrm{d(vol)}.$$

Consequently

$$\int_{S^n} \mathrm{ch}(\Omega_\pm) = \pm \mathrm{i}^n,$$

where we take only the lth component (the n-form portion) of the ch polynomial. This shows that the bundles $E_\pm$ are non-trivial.

If $n = 2$ ($l = 1$), the fiber dimension is one and we have complex line bundles with (the 2-form part of) ch equal to $c_1(E_\pm)$, so

$$\int_{S^2} c_1(E_\pm) = \mp 1.$$

The associated principal bundles for these line bundles are the magnetic monopole bundles discussed earlier with charge ± 1.

Remark: While the matrix $\lambda(x)$ defined in example 5.4.2 maps $S^n \to \mathrm{GL}(2^l, \mathbb{C})$, the projection $\Pi_\pm(x)$ acts as

$$\Pi_\pm(x)\colon S^n \to \mathrm{Gr}(2^l, 2^{l-1}, \mathbb{C}).$$

The vector bundles $E_\pm$ are simply the pullbacks under $\Pi_\pm$ of the classifying bundles, $\Pi_\pm{}^* L(2^l, 2^{l-1}, \mathbb{C})$. This example illustrates the relationship between the homology of the embedding of S^n in $\mathrm{Gr}(2^l, 2^{l-1}, \mathbb{C})$ and the cohomology of the bundles characterized by the Chern classes of the classifying space.

4. *Chern class of $U(1)$ bundle.* We now turn from vector bundles to the Chern classes of principal bundles. We recall that for a $U(1)$ principal bundle P the curvature is purely imaginary. Thus we may write

$$\Omega = \mathrm{i}F$$

and so find the total Chern class

$$c(P) = \mathrm{Det}\left(1 + \frac{\mathrm{i}}{2\pi}\Omega\right) = 1 + \frac{\mathrm{i}}{2\pi}\mathrm{i}F = 1 - \frac{F}{2\pi}.$$

Hence

$$c_1(P) = -F/2\pi.$$

We noted in example 5.5.1 that the integral of c_1 for the Dirac monopole $U(1)$ bundle over S^2 was the integer giving the monopole charge,

$$C_1 = \int_{S^2} c_1 = -n.$$

Proof of topological invariance: The first Chern class of the monopole bundle ($M = S^2, F = S^1 = U(1)$) depends only on the bundle transition functions and is independent of whether the connection $A(x)$ satisfies Maxwell's equations.

As before, the gauge transformation on the equator is given by

$$A_+(x) = A_-(x) + n\,\mathrm{d}\phi.$$

Applying Stokes' theorem, we obtain,

$$-\int_{S^2} c_1 = \frac{1}{2\pi}\left[\int_{H_+} \mathrm{d}A_+ + \int_{H_-} \mathrm{d}A_-\right] = \frac{1}{2\pi}\int_{S^1} (A_+ - A_-),$$

where the sign change occurs because $\partial H_- = S^1$ has the opposite orientation from $\partial H_+ = S^1$. Using the relation between A_+ and A_-, we find

$$-C_1 = \frac{1}{2\pi}\int_{S^1} n\, \mathrm{d}\phi = n.$$

Only the *gauge transformation* enters into the computation.

5. *Chern class of G-bundle.* Let $\lambda_a/2\mathrm{i}$ be a matrix basis for the adjoint representation of the Lie algebra $\mathscr{G}$ of the group G with $\mathrm{Tr}\, \lambda_a\lambda_b = 2\delta_{ab}$. Then the curvature is written as

$$\Omega = g^{-1}F^a(x)\frac{\lambda_a}{2\mathrm{i}}\, g.$$

Since the factors of g^{-1} and g annihilate one another in the determinant, the Chern class of a principal G-bundle P over M is

$$c(P) = \mathrm{Det}\left(1 + \frac{1}{4\pi}\lambda_a F^a\right).$$

For $G = \mathrm{SU}(2)$, we can take the λ_a to be Pauli matrices. We find

$$c(P) = 1 - \frac{1}{(4\pi)^2}F^a \wedge F^a = 1 + \frac{1}{8\pi^2}\mathrm{Tr}(F \wedge F)$$

so that

$$c_1(P) = 0$$

$$c_2(P) = \frac{1}{8\pi^2}\mathrm{Tr}(F \wedge F).$$

We noted in example 5.5.2 that the integral of $c_2(P)$ for the self-dual Yang–Mills instanton connection on an SU(2) bundle over S^4 was

$$-C_2 = -\int_{S^4} c_2 = +k,$$

where k is the "instanton number".

Proof of topological invariance: Let us demonstrate the topological invariance of c_2 for the instanton G-bundle. We take $M = S^4$ to be covered by $H_\pm$, with $H_+ \cap H_- \simeq S^3$, and consider the gauge transformation

$$A_- = \Phi A_+ \Phi^{-1} + \Phi\, \mathrm{d}\Phi^{-1}$$
$$F_- = \Phi F_+ \Phi^{-1}.$$

Using the Bianchi identities and $\mathrm{Tr}(A \wedge A \wedge A \wedge A) = 0$, one can show

$$\mathrm{Tr}(F \wedge F) = \mathrm{d}\,\mathrm{Tr}(F \wedge A - \tfrac{1}{3} A \wedge A \wedge A).$$

Then, by using Stokes' theorem, we see that

$$C_2 = \int_{S^4} c_2 = \frac{1}{8\pi^2}\Big[\int_{H_+} \mathrm{Tr}(F_+ \wedge F_+) + \int_{H_-} \mathrm{Tr}(F_- \wedge F_-)\Big]$$
$$= \frac{1}{8\pi^2}\int_{S^3}\Big[\mathrm{Tr}\Big(F_+ \wedge A_+ - \frac{1}{3}(A_+)^3\Big) - \mathrm{Tr}\Big(F \wedge A_- - \frac{1}{3}(A_-)^3\Big)\Big].$$

When we substitute the expressions for A_- and F_- using the gauge transformation, we find

$$C_2 = \int_{S^4} c_2 = \frac{1}{8\pi^2}\int_{S^3} \mathrm{Tr}\Big[\frac{1}{3}\Phi\,\mathrm{d}\,\Phi^{-1} \wedge \Phi\,\mathrm{d}\Phi^{-1} \wedge \Phi\,\mathrm{d}\Phi^{-1} - \mathrm{d}(A_+ \wedge \mathrm{d}\Phi^{-1}\,\Phi)\Big]$$
$$= \frac{1}{24\pi^2}\int_{S^3} \mathrm{Tr}(\Phi\,\mathrm{d}\Phi^{-1})^3.$$

The entire value of C_2 is given by the winding number of the *gauge transformation* $\Phi\,\mathrm{d}\Phi^{-1}$ at the equator $H_+ \cap H_- \simeq S^3$.

Remark: Clearly the transition functions $\Phi(x)$ of the topological bundle fall into equivalence classes characterized by the value of the integer C_2. If C_2 is unchanged by taking

$$\Phi(x) \to h(x)\,\Phi(x),$$

$h(x)$ is referred to in the physics literature as a *small gauge transformation*; such functions are homotopic to the identity map. If C_2 is altered, $h(x)$ is called a *large gauge transformation*; choosing such a transition function modifies the topology of the bundle. A typical large gauge transformation in an SU(2) bundle is

$$h(x) = \frac{t - \mathrm{i}\boldsymbol{\lambda} \cdot \boldsymbol{x}}{r}, \qquad \{\boldsymbol{\lambda}\} = \text{Pauli matrices}.$$

If $\Phi = h^k$, we find that the bundle has $C_2 = -k$.

6.4. Other characteristic classes

6.4.1. Pontrjagin classes

We now discuss the characteristic classes of real vector bundles and their associated principal bundles. The bundle transition functions and the fibers of the principal bundles then belong to $GL(k, \mathbb{R})$. If one puts a fiber metric on a real vector bundle, the bundle transition functions can be reduced to $O(k)$. The associated bundle of orthonormal frames is an $O(k)$ principal bundle. There are some subtleties present in the real case which are absent in the complex case. While the characteristic forms of real vector bundles whose structure groups are $O(k)$ and $GL(k, \mathbb{R})$ are different, their characteristic classes are in fact the same.

Since we can always reduce the structure group to $O(k)$ and choose a Riemannian connection on the bundle, we first consider this case.

The *total Pontrjagin class* of a real $O(k)$ bundle E with curvature Ω lying in the Lie algebra of $O(k)$ is defined by the invariant polynomial

$$p(E) = \mathrm{Det}\left(I - \frac{1}{2\pi}\Omega\right) = 1 + p_1 + p_2 + \cdots. \tag{6.16}$$

Since $\Omega = -\Omega^t$, the only non-zero polynomials are of even degree in Ω. Thus $p_j(\Omega) \in \Lambda^{4j}(M)$ and the series expansion of $p(E)$ terminates either when $4j > n = \dim M$ or when $2j > k = \dim E$. $p_j(\Omega)$ is always closed and the cohomology class it represents is independent of the metric and the connection chosen; we let $p_j(E)$ denote this cohomology class. It is clear that the total Pontrjagin class obeys the Whitney sum formula

$$p(E \oplus F) = p(E)\, p(F).$$

Any invariant polynomial for a real bundle can be expanded in the Pontrjagin forms p_j, in the following sense: if $Q(\Omega)$ is a $GL(k, \mathbb{R})$-invariant polynomial and Ω is a $gl(k, \mathbb{R})$-valued curvature 2-form, then

$$Q = R(p_1, p_2, \ldots, p_{\max}) + S(\Omega)$$

where R is a polynomial and $S = 0$ when Ω lies in $O(k)$. Furthermore, the cohomology class represented by $S(\Omega)$ (for example: $S = \mathrm{Tr}\,\Omega$) will always be zero, even though $S(\Omega) \neq 0$ on $gl(k, \mathbb{R})$. Thus the $GL(k, \mathbb{R})$ and $O(k)$ characteristic classes are the *same*, while their characteristic forms may differ.

Pontrjagin classes in terms of Chern classes: In many applications, it turns out to be convenient to express the Pontrjagin classes of a real bundle in terms of the Chern classes of a complex bundle. If E is a real bundle, we can define $E_c = E \oplus \mathbb{C}$ as the complexification of E. (This is defined by the natural inclusion of $GL(k, \mathbb{R})$ into $GL(k, \mathbb{C})$.) If A is a skew-adjoint real matrix, we have the identity:

$$\det\left(I + \frac{i}{2\pi}A\right) = 1 - p_1(A) + p_2(A) \ldots$$

where the factors of -1 arise from the i^2 terms. This yields the identity:

$$p_k(E) = (-1)^k c_{2k}(E_c). \tag{6.17}$$

Conversely, given a complex bundle E of dimension k, we can form the corresponding real bundle E_r of dimension $2k$ by forgetting the complex structure on E. (This is called the "forgetful functor".) If we then form $(E_r)_c$, this is a complex vector bundle of complex dimension $2k$. Let $\bar{E}$ denote the complex conjugate bundle, which is, in fact, isomorphic to the dual bundle E^*. Then

$$(E_r)_c = E \oplus \bar{E} = E \oplus E^*.$$

Since

$$c(\bar{E}) = 1 - c_1(E) + c_2(E) - c_3(E) \ldots,$$

we find

$$c((E_r)_c) = 1 - p_1(E_r) + p_2(E_r) - \cdots = c(E)\, c(\bar{E})$$
$$= [1 + c_1(E) + c_2(E) + \cdots]\,[1 - c_1(E) + c_2(E) \ldots].$$

Half the terms cancel out. Identifying the remaining terms yields:

$$p_1(E_r) = (c_1^2 - 2c_2)(E)$$

$$p_2(E_r) = (c_2^2 - 2c_1c_3 + 2c_4)(E), \text{etc.}$$

Using the splitting principle, we find the equivalent polynomial expressions:

$$p_1(E_r) = \sum_i x_i^2$$
$$p_2(E_r) = \sum_{i<j} x_i^2 x_j^2 \tag{6.18}$$

and so forth. The form of these polynomials is related to the fact that the eigenvalues of a skew-symmetric matrix occur in complex conjugate pairs with purely imaginary eigenvalues.
Example: $P_n(\mathbb{C})$. The total Pontrjagin class of a complex manifold such as $P_n(\mathbb{C})$ is computed by using the forgetful functor to obtain the real tangent space $T(P_n(\mathbb{C}))$ from the complex tangent space $T_c(P_n(\mathbb{C}))$ and computing the Pontrjagin class of $T(P_n(\mathbb{C}))$. From example 6.3.2, we know that

$$c(T_c(P_n(\mathbb{C}))) = (1+x)^{n+1}$$
$$c(\bar{T}_c(P_n(\mathbb{C}))) = (1-x)^{n+1}$$

where x is the generator of the integral cohomology of $P_n(\mathbb{C})$. Then we find

$$c(T(P_n(\mathbb{C}))\otimes\mathbb{C}) = c(T_c(P_n(\mathbb{C})))c(\bar{T}_c(P_n(\mathbb{C}))) = (1-x^2)^{n+1}$$
$$= 1 - p_1(T(P_n(\mathbb{C}))) + p_2 \dots$$

so that the total Pontrjagin class is

$$p(T(P_n(\mathbb{C}))) = 1 + p_1 + p_2 + \cdots = (1+x^2)^{n+1}.$$

6.4.2. The Euler class

The transition functions of an oriented real k-dimensional vector bundle E can always be reduced to $\mathrm{SO}(k)$ transition functions. If $k = 2r$ is even, we can define an additional $\mathrm{SO}(k)$-invariant polynomial $e(\alpha)$ called the *Pfaffian*. This polynomial is not invariant under the orientation-preserving group $\mathrm{GL}_+(k, \mathbb{R})$. Thus the corresponding characteristic class can only be computed using a Riemannian connection, not a general linear connection. There exist bundles E with $e(E) \neq 0$ which nevertheless admit *flat* non-Riemannian connections. We recall that, in contrast, the Pontrjagin forms could be computed using a general linear connection.

Let $|\alpha_{ij}|$ be a real anti-symmetric $k\times k$ matrix in the Lie algebra $\mathrm{SO}(k)$. Taking $\{z^i\}$ to be local fiber coordinates in E. We construct the 2-form

$$\alpha = \tfrac{1}{2}\alpha_{ij}\,\mathrm{d}z^i \wedge \mathrm{d}z^j.$$

$e(\alpha)$ is then defined by the r-fold wedge product

$$\frac{1}{r!}\left(\frac{\alpha}{2\pi}\right)^r = e(\alpha)\,\mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^k. \tag{6.19}$$

The Pfaffian $e(\alpha)$ is $\mathrm{SO}(k)$-invariant. The Euler form of the bundle E is found by substituting the bundle's $\mathrm{SO}(k)$-valued curvature 2-form Ω for α:

$$\text{Euler form} = e(\Omega).$$

The Euler form is always closed and the characteristic class $e(E)$ is independent of the particular Riemannian metric and connection chosen.

Properties of the Euler class: While a real anti-symmetric matrix like $|\alpha_{ij}|$ cannot be diagonalized, it can be put in the form

$$\begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & x_r \\ & & & -x_r & 0 \end{pmatrix}$$

The splitting-principle formula for $e(E)$ is thus

$$e(E) = x_1 x_2 \dots x_r.$$

Since

$$p_r(E) = x_1^2 x_2^2 \dots x_r^2 = e^2(E),$$

we conclude that $e(E)$ is a *square root* of the highest Pontrjagin class. If we change the orientation of E, we replace $e(E)$ by $-e(E)$, and change the sign of the square root.

It is clear that e is multiplicative for Whitney sums:

$$e(E \oplus F) = e(E)\, e(F),$$

where we define $e(E) = 0$ for odd-dimensional bundles.

Complex bundles: If E is a complex vector bundle of dimension r, then its real $2r$-dimensional counterpart E_r inherits a natural orientation. Then we know that

$$e(E_r)^2 = p_r(E_r) = c_r(E)^2.$$

In fact, the signs work out so that $e(E_r)$ is just the *top Chern class* of E,

$$e(E_r) = [p_r(E_r)]^{1/2} = c_r(E).$$

Gauss–Bonnet theorem: The Gauss–Bonnet theorem for an even-dimensional manifold M relates the Euler characteristic to the Euler class by

$$\chi(M) = \int_M e(T(M)). \tag{6.20}$$

(If M is odd-dimensional, both $e(T(M)) = 0$ and $\chi(M) = 0$.) The example of $P_n(\mathbb{C})$ was worked out in 6.3.2.

Stable and unstable characteristic classes: In some circumstances, the Euler class may be non-zero even for bundles with *vanishing* Pontrjagin classes. For example, consider the tangent bundle of the sphere $T(S^m)$ for even m. Since $\chi = 2$, the Gauss–Bonnet theorem gives

$$e(T(S^m)) = 2 \cdot V(S^m),$$

where $V(S^m) \in H^m(S^m)$ is the normalized S^m volume element. Since

$$T(S^m) \oplus I = I^{m+1}$$

is a trivial bundle over S^m, we find

$$p(T(S^m) \oplus I) = p(T(S^m)) \cdot p(I) = p(T(S^m)) = p(I^{m+1}) = 1.$$

Thus the Pontrjagin classes of $T(S^m)$ are trivial. Pontrjagin classes are stable characteristic classes, while the Euler class is an *unstable* characteristic class; *stabilization* is the process of adding a trivial bundle to eliminate low fiber-dimensional pathologies of which the Euler characteristic is an example (see the discussion of K-theory given below).

Examples: The Euler classes for two or four-dimensional Riemannian manifolds M are given by

$$n = 2: \quad e(T(M)) = \frac{1}{2\pi} R_{12} = \frac{1}{4\pi} \epsilon_{ab} R^{ab} = \frac{1}{2\pi} R_{1212} e^1 \wedge e^2$$

$$n = 4: \quad e(T(M)) = \frac{1}{32\pi^2} \epsilon_{abcd} R^{ab} \wedge R^{cd},$$

where R^{ab} is the curvature 2-form in the orthonormal cotangent space basis. Since R^{ab} as a matrix belongs to so(n), we can see from the Weil homorphism construction how $e(T(M))$ emerges as a "square root" of a Pontrjagin class which would itself be zero when curvatures were substituted. For $n = 2$, we have

$$\mathrm{Det}\left[I - \frac{1}{2\pi}\begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix}\right] = 1 + \frac{\lambda^2}{(2\pi)^2} = 1 + p_1,$$

so we take $\lambda = R_{12}$ to find

$$e = (p_1)^{1/2} = \frac{R_{12}}{2\pi}.$$

For $n = 4$, with $R_{i4} = E_i$, $R_{jk} = \frac{1}{2}\epsilon_{ijk} B_i$, we have

$$\mathrm{Det}\left[I - \frac{1}{2\pi}\begin{pmatrix} 0 & B_3 & -B_2 & E_1 \\ -B_3 & 0 & B_1 & E_2 \\ B_2 & -B_1 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{pmatrix}\right] = 1 + p_1 + p_2$$

$$= 1 + \frac{1}{(2\pi)^2}(\boldsymbol{E}^2 + \boldsymbol{B}^2) + \frac{1}{(2\pi)^4}(\boldsymbol{E} \cdot \boldsymbol{B})^2$$

$$= 1 - \frac{1}{8\pi^2} R_{ab} R_{ba} + \frac{1}{(2\pi)^4}\frac{1}{64}(\epsilon_{abcd} R^{ab} R^{cd})^2.$$

Hence we find the first Pontrjagin class

$$p_1 = -\frac{1}{8\pi^2} \mathrm{Tr}\, R \wedge R$$

and the Euler class

$$e(T(M)) = (p_2)^{1/2} = \frac{1}{32\pi^2}\epsilon_{abcd}R^{ab}\wedge R^{cd}.$$

Similar formulas hold for all even dimensional cases.

Remark: Clearly the existence of the Euler class as a "square root" follows from the fact that the determinant of an anti-symmetric even-dimensional matrix is a *perfect square*. For odd dimensions, this determinant vanishes, and, in fact, the Euler class for n odd is always zero.

6.4.3. Stiefel–Whitney classes

The Stiefel–Whitney classes of a real bundle E over M with k dimensional fiber are the $\mathbb{Z}_2$ cohomology classes. In contrast to the other characteristic classes we have given earlier, they are not integral cohomology classes and are not given in terms of curvature. We identify the Stiefel–Whitney classes as

$$w_i \in H^i(M;\mathbb{Z}_2) \qquad i = 1, \ldots, n-1.$$

For $i = n$ (n even), w_n has values in $\mathbb{Z}$ rather than $\mathbb{Z}_2$ and is identifiable with the Euler class discussed above. The total Stiefel–Whitney class is, as usual, defined by

$$w(E) = 1 + w_1 + w_2 + \cdots + w_n.$$

The *first Stiefel–Whitney class* $w_1(T(M))$ is zero if and only if M is orientable.

The *second Stiefel–Whitney class* $w_2(T(M))$ is of great importance in physics because it determines whether or not parallel transport of Dirac spinors can be globally defined on $E = T(M)$. If

$$w_1(T(M)) = w_2(T(M)) = 0,$$

then spinors are well-defined and M is a *spin-manifold*. If

$$w_2(T(M)) \neq 0,$$

then there is a sign ambiguity when spinors are parallel-transported around some path in M: *such manifolds do not admit a spin structure.*

Example 1. Stiefel–Whitney classes of $P_n(\mathbb{C})$: The Stiefel–Whitney classes can be computed in closed form from the expression for the cohomology of $T(P_n(\mathbb{C}))$. The total class is just (Milnor and Stasheff [1974])

$$w(T(P_n(\mathbb{C}))) = (1+x)^{n+1} = 1 + w_2 + w_4 + \cdots + w_{2n},$$

where x is the 2-form c_1 of the natural line bundle and all coefficients of x^k are taken *mod 2* except for w_{2n}. Hence we find for $P_n(\mathbb{C})$

$$w_2 = (n+1)\big|_{\mathrm{mod}\,2}\cdot x = \begin{cases} 0 & n \text{ odd} \\ 1\cdot x & n \text{ even.} \end{cases}$$

In particular, $P_2(\mathbb{C})$, $P_4(\mathbb{C})$, ... do *not* admit a spin structure, while $P_1(\mathbb{C})$, $P_3(\mathbb{C})$, ..., do. Since $w_{2n} = (n+1) \cdot x$, we recover our previous result that the Euler characteristic is $(n+1)$. In addition, all the manifolds $P_n(\mathbb{C})$ are orientable since $w_1 = 0$.

Example 2: The total Stiefel–Whitney class of S^n is

$$w(S^n) = 1 + (1 + (-1)^n)\, V(S^n)$$

where $V(S^n)$ is the normalized n-form volume element. Hence $w_2 = 0$ and all n-spheres are spin manifolds.

Remark: For $S^2 = P_1(\mathbb{C})$, $w_2 = 2x$ plays a double role: the Euler characteristic $= 2$, and $2 \pmod 2) = 0$ implies that a spin structure exists.

6.5. K-theory

K-theory is concerned with the study of formal differences of vector bundles and plays an essential role in index theory. From the standpoint of algebraic topology, K-theory is an exotic cohomology theory, although we shall not adopt this viewpoint here (see, Atiyah [1967]).

Problems with formal differences of vector bundles: In the preceding sections we have studied the properties and characteristic classes of Whitney sum bundles such as $E \oplus F$. If $E \oplus F \approx E' \oplus F$, then it is tempting to introduce a formal difference operation which would allow us to cancel the vector bundle F from both sides of this equation and to conclude that $E \approx E'$. Unfortunately this cancellation does not work in general, as we may see from the following example:

Consider the manifold $M = S^2$ to be embedded in $\mathbb{R}^3$, and let $T(S^2)$ and $N(S^2)$ be the tangent and normal bundles, respectively. Letting I^k denote the trivial real vector bundle of dimension k, we note that $N(S^2) \simeq I$, the trivial line bundle. Then we find that

$$T(S^2) \oplus N(S^2) = T(\mathbb{R}^3) \simeq I^3$$

$$I^2 \oplus N(S^2) \simeq I^2 \oplus I = I^3.$$

If we could perform the formal cancellation of $N(S^2)$, then we would conclude that $T(S^2) \simeq I^2$, which is false. There are similar examples also for complex bundles.

Stable equivalence of vector bundles: The problems with formal differences of vector bundles can be resolved by replacing the notion of vector bundle isomorphism by the broader relationship of stable equivalence. If E and E' are two vector bundles, not necessarily of the same dimension, we say that E and E' are ***stably equivalent*** and write $E \underset{s}{\approx} E'$ provided that

$$E \oplus I^l \simeq E' \oplus I^j$$

for some integers l and j.

Taking the Whitney sum with trivial bundles serves to eliminate pathologies arising from low fiber dimension; this process is called *stabilization*. Two vector bundles of the same fiber dimension $k > \dim(M)$ are stably equivalent if and only if they are isomorphic; these two notions correspond if the fiber dimension is large enough. Since $E \underset{s}{\simeq} E'$ and $E' \underset{s}{\simeq} E''$ implies $E \underset{s}{\simeq} E''$, then $\underset{s}{\simeq}$ is an *equivalence relation*.

Definition of $K_0(M)$: If $E \oplus F \simeq E' \oplus F$, then E need not be isomorphic to E', but it is stably equivalent to E',

$$E \underset{s}{\simeq} E'.$$

If we define $K_0(M)$ to be the set of stable equivalence classes, then formal differences are well-defined on $K_0(M)$. Thus, for example, $T(S^2) \underset{s}{\simeq} I^2$ and $T(S^2)$ is stably trivial. Let $\mathrm{Vect}_k(M)$ be the set of isomorphism classes of vector bundles of fiber dimension k. We say that k is in the *stable range* provided that:

$k > \dim(M)$ (if we are working with real vector bundles)

$k > \frac{1}{2}\dim(M)$ (if we are working with complex vector bundles),

where $\dim(M)$ denotes the real dimension of M. We can identify $\mathrm{Vect}_k(M)$ with $K_0(M)$ in the stable range. In other words, once k is large enough, given any bundle E there is a bundle E' with fiber dimension k such that $E \underset{s}{\simeq} E'$. Furthermore, if $E \underset{s}{\simeq} E''$ is another such bundle, then E' and E'' are actually isomorphic.

If E is a vector bundle, we can always find a complementary bundle F such that $E \oplus F \simeq I^l$ is trivial for some integer l. The isomorphism class of F is not uniquely defined, but the stable equivalence class of F is unique and defines an element of $K_0(M)$. Since I^l represents the trivial or "zero" element of $K_0(M)$, F is the formal inverse of E. We thus have a group structure on $K_0(M)$. *Formal subtraction* of the bundle E is defined by taking the Whitney sum with the complementary bundle $E^{-1} = F$. Since $K_0(M) = \mathrm{Vect}_k(M)$ for k in the stable range, this also defines a group structure on $\mathrm{Vect}_k(M)$.

Unreduced K-theory: $K_0(M)$ does not distinguish between trivial bundles of different dimension since $I^l \underset{s}{\simeq} I^k$ for any k and l. We define a new group $K(M)$ using the following construction of Grothendieck (see Atiyah [1967]). If E and F are vector bundles, we define the *virtual bundle* $E \ominus F$ representing their formal difference. $K(M)$ is the Abelian group whose elements are virtual bundles. Thus $T(S^2)$ and I^2 represent the same element of $K(S^2)$.

The *virtual dimension* of $E \ominus F$ is $\dim(E) - \dim(F)$. $K_0(M)$ can be identified as the subgroup of $K(M)$ with vanishing virtual dimension.

Note that the tensor product is distributive with respect to the Whitney sum and thus defines a multiplication or ring structure on both $K(M)$ and $K_0(M)$.

Rational K-theory. We define $K(M)$ by allowing objects of the form jE where j could be 0, positive or negative. If j is positive, this is just $E \oplus \cdots \oplus E$, while if j is negative, this is a formal object involving formal differences. It is convenient to consider other coefficient groups in this context just as we did for homology and for cohomology. $K(M; \mathbb{Q})$ and $K_0(M; \mathbb{Q})$ are the groups which arise when we consider objects of the form qE where q is rational:

$$K(M; \mathbb{Q}) = K(M) \times \mathbb{Q} \qquad K_0(M; \mathbb{Q}) = K_0(M) \times \mathbb{Q}.$$

So far, we have not really distinguished between the complex and real case except to note that the stable range is greater in the complex case. We shall reserve the notation $K(M)$ and $K_0(M)$ for the group of complex bundles and shall use the notation $K^r(M)$ and $K_0^r(M)$ for the group of real vector bundles.

The Chern isomorphism: The Chern character provides the bridge between rational K-theory and rational cohomology. We recall that the Chern character satisfies the identities

$$\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F), \qquad \mathrm{ch}(E \otimes F) = \mathrm{ch}(E)\,\mathrm{ch}(F).$$

We can, in fact, extend the Chern character to K theory so that

$$\mathrm{ch}(E \ominus F) = \mathrm{ch}(E) - \mathrm{ch}(F).$$

This relationship is one of the important consequences of the Grothendieck construction.

The Chern character is a ring isomorphism from $K(M; \mathbb{Q})$ to the even-dimensional cohomology of M; it is a map

$$\mathrm{ch:}\quad K(M; \mathbb{Q}) \to \bigoplus_j H^{2j}(M; \mathbb{Q}).$$

If we restrict the Chern character to the subgroup $K_0(M; \mathbb{Q})$, then ch provides an isomorphism

$$\mathrm{ch:}\quad K_0(M; \mathbb{Q}) \simeq \bigoplus_{j>0} H^{2j}(M; \mathbb{Q}).$$

In other words, if M has non-trivial even cohomology, then M will have non-trivial vector bundles. In the real case, $c_j(E) = 0$ if j is odd so

$$\mathrm{ch:}\quad K^r(M; \mathbb{Q}) \simeq \bigoplus_j H^{4j}(M; \mathbb{Q}).$$

Thus, for example, any *real* vector bundle over S^2 is stably trivial since there is no real cohomology in dimensions divisible by 4 above H^0. On S^4, by contrast, there are many non-trivial bundles which are parametrized by the first Pontrjagin class p_1 because $H^4(S^4; \mathrm{Q}) = \mathrm{Q}$.

Torsion in K-theory: Suppose $k > \frac{1}{2}\dim(M)$ is in the stable range and consider the set of all cohomology classes of the form $\mathrm{ch}(E)$ as E ranges over all possible bundles with fiber dimension k. This set spans the even rational cohomology of M. Furthermore, if $\mathrm{ch}(E) = \dim(E)$ (i.e., $c_l(E) = 0$ for $l > 0$), then some multiple of E is stably trivial: there exists an integer j such that

$$E \oplus \cdots \oplus E \simeq I^{j \cdot \dim(E)}.$$

In other words, $jE = 0$ in K-theory so E is a *torsion element* of $K(M)$. The Chern character permits us to compute $K(M)$ *modulo torsion.*

The existence of torsion elements in K-theory can be illustrated by the following example: consider $P_2(\mathbb{R})$, which is S^2 modulo the identification of antipodal points, $x \sim -x$. We define L as the bundle over

$P_2(\mathbb{R})$ obtained by identifying $(x, z) \sim (-x, -z)$ in $S^2 \times \mathbb{C}$ (this is a generalization of the Möbius bundle). A section s of L over $P_2(\mathbb{R})$ is simply a function s on S^2 satisfying the identity $-s(x) = s(-x)$. Since any such function must have a zero, L is non-trivial (and in fact is not stably trivial so L represents a non-zero element of $K(P_2(\mathbb{R}))$). A frame for $L \oplus L$ is just a map g: $S^2 \to \mathrm{GL}(2, \mathbb{C})$ such that $g(x) = -g(-x)$. If we define:

$$g(x) = \begin{pmatrix} x_1 & x_2 + \mathrm{i}x_3 \\ x_2 - \mathrm{i}x_3 & -x_1 \end{pmatrix}$$

then $g(x)^2 = I$ for $x \in S^2$. Thus $L \oplus L \simeq I^2$ on $P_2(\mathbb{R})$ and L represents a torsion element of $K(P_2(\mathbb{R}))$.

If M has only even dimensional free cohomology, then there are *no* torsion elements in $K(M)$ so we can identify $K(M)$ with $\oplus_j H^{2j}(M; \mathbb{Z})$ additively (the ring structures are different). Since both S^n and $P_n(\mathbb{C})$ satisfy these hypotheses, we conclude that:

$$K(S^n) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} \text{ if } n \text{ is even} \\ \mathbb{Z} \text{ if } n \text{ is odd} \end{cases} \qquad K(P_n(\mathbb{C})) = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \quad (n+1 \text{ times})$$

$$K_0(S^n) = \begin{cases} \mathbb{Z} \text{ if } n \text{ is even} \\ 0 \text{ if } n \text{ is odd} \end{cases} \qquad K_0(P_n(\mathbb{C})) = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \quad (n \text{ times}).$$

For example: if n is odd and if $\dim(E) > \frac{1}{2}n$, then E is trivial on S^n since $K_0(S^n) = 0$. If n is even and if $k > \frac{1}{2}n$, we may identify

$$\mathrm{Vect}_k(S^n) = K_0(S^n) = \mathbb{Z}.$$

In other words, the stable equivalence class of any bundle E over S^n can be determined from the integer

$$\int_{S^n} c_l(E) \qquad \text{for } l = n/2.$$

The bundles constructed in example 5.4.2 give the generators for $K_0(S^n)$ if n is even.

Bott periodicity is the statement that the stable homotopy groups of $U(k)$ are periodic. This means that:

$$\pi_j(U(k)) = \begin{cases} \mathbb{Z} \text{ for } j \text{ odd and } j < 2k \\ 0 \text{ for } j \text{ even and } j < 2k. \end{cases}$$

This is related to the calculation of $K_0(S^{n+1}) = \mathrm{Vect}_k(S^{n+1})$ as follows: let E be a k-dimensional bundle over S^{n+1} and let $D_\pm$ be the upper and lower hemispheres of S^{n+1}. These are contractible so E is trivial over these sets. Let $e_\pm$ be unitary frames for E over $D_\pm$ and let $e_- = g(x)e_+$ on $S^n = D_+ \cap D_-$. $g(x)$ is the transition function defining E and gives a map g: $S^n \to U(k)$ which represents an element of $U(k)$. This map is in fact an isomorphism in the stable range. Therefore:

$$\pi_n(U(k)) = K_0(S^{n+1}) = \begin{cases} \mathbb{Z} & \text{if } n+1 \text{ is even} \quad (\text{i.e. } n \text{ is odd}) \\ 0 & \text{if } n+1 \text{ is odd} \quad (\text{i.e. } n \text{ is even}). \end{cases}$$

For example, we find that $\pi_1(U(k))=\mathbb{Z}$, so $\mathrm{Vect}_k(S^2)=\mathbb{Z}$ for all k. Since $\pi_2(U(k))=0$, we conclude that $\mathrm{Vect}_k(S^3)\simeq 0$ for all k.

Another way of stating Bott periodicity is to take $k=\infty$ and write

$$\pi_n(U(\infty))=\pi_{n+2}(U(\infty)).$$

A similar but somewhat more involved argument for the real groups $O(k)$ leads to the formula

$$\pi_n(O(\infty))=\pi_{n+8}(O(\infty)).$$

Remark: Difference bundles of the type treated by K-theory play an essential role in the mathematical definition of high-spin fields, such as the spin $\frac{3}{2}$ Rarita–Schwinger field. K-theory is implicitly used in the applications of index theorems to high-spin fields described in section 10.

7. Index theorems: Manifolds without boundary

The index theorem states the existence of a relationship between the analytic properties of differential operators on fiber bundles and the topological properties of the fiber bundles themselves. The simplest example is the Gauss–Bonnet theorem, which relates the number of harmonic forms on the manifold (Betti numbers) to the topological Euler characteristic given by integrating the Euler form over the manifold. In this case, the relevant differential operator is the exterior derivative mapping $C^\infty(\Lambda^p)\to C^\infty(\Lambda^{p+1})$, and the analytic property in question is the number of zero-frequency solutions to Laplace's equation. In general, the index theorem gives analogs of the Gauss–Bonnet theorem for other differential operators. The index of an operator, determined by the number of zero-frequency solutions to a generalized Laplace's equation, is expressed in terms of the characteristic classes of the fiber bundles involved. Thus the index theorem gives us useful information concerning various types of differential equations provided we understand the topology of the fiber bundles upon which the differential operators are defined.

We will first discuss the general formulation of the index theorem and then apply it to the classical elliptic complexes. We work out the index theorem explicitly in dimensions two and four for the de Rham, signature, Dolbeault and spin complexes. The index theorems for these complexes correspond to the Gauss–Bonnet theorem, the Hirzebruch signature theorem, the Riemann–Roch theorem, and the index theorem for the $\hat{A}$-genus. We conclude with a discussion of the Lefschetz fixed point theorem and the G-index theorem.

7.1. *The index theorem*

We begin for the sake of completeness with a fairly abstract description of the index theorem of Atiyah and Singer [1968a, b; 1971a, b]. The reader who is interested in specific applications may proceed directly to the appropriate subsequent sections. For an alternative treatment using heat equation methods, see, for example, Gilkey [1974], and references quoted therein.

Let M be a compact smooth manifold without boundary of dimension n. We will consider the case of manifolds with boundary in section 8. Let E and F be vector bundles over M and let D: $C^\infty(E)\to C^\infty(F)$ be a first-order differential operator. We choose local bundle coordinates for E and for F, with

$\{x_j\}$ being local coordinates on M. Then we can decompose D in the form

$$D = a_j(x)\, \partial/\partial x_j + b,$$

where the a_j and b are matrix-valued.

Symbol of an operator: The symbol of an operator is its *Fourier transform*. Let (x, k) be local coordinates on $T^*(M)$; we regard k as the Fourier-transform variable. Let $\tilde{f}(k)$ be the Fourier transform of $f(x)$ and recall that

$$\begin{aligned} Df(x) &= a_j(x)\frac{\partial f(x)}{\partial x_j} + bf \\ &= \int [\mathrm{i}a_j(x)\, k_j + b]\, \tilde{f}(k)\, \mathrm{e}^{\mathrm{i}x\cdot k}\, \mathrm{d}k. \end{aligned}$$

The *leading symbol* $\tilde{D}$ of D is the highest-order part of its Fourier transform,

$$\tilde{D}(x, k) = \sigma_{\mathrm{L}}(D)\,(x, k) = \mathrm{i}a_j(x)k_j.$$

This is a linear map from E to F.

Elliptic complexes: If $E = F$ and if $\tilde{D}(x, k)$ is invertible for $k \neq 0$, then D is said to be an *elliptic operator*. A similar definition holds for higher order operators.

Let $\{E_p\}$ be a finite sequence of vector bundles over M and let D_p: $C^\infty(E_p) \to C^\infty(E_{p+1})$ be a sequence of differential operators. We assume that this sequence is a *complex*, i.e., $D_{p+1}D_p = 0$. Figure 7.1 gives the standard graphical depiction of such a complex. Now let D_p^*: $C^\infty(E_{p+1}) \to C^\infty(E_p)$ be the dual map and let

$$\Delta_p = D_p^* D_p + D_{p-1} D_{p-1}^*$$

be the associated Laplacian. The complex is *elliptic* if Δ_p is an elliptic operator on $C^\infty(E_p)$. Equivalently, the complex is elliptic (or exact on the symbol level) if

$$\operatorname{Ker} \tilde{D}_p(x, k) = \operatorname{image} \tilde{D}_{p-1}(x, k), \qquad k \neq 0.$$

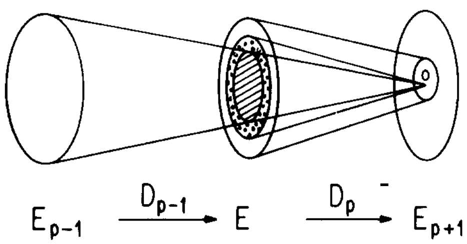

Fig. 7.1. A piece of a complex with $D_pD_{p-1} = 0$. The hatched area is Im D_{p-1}. The dotted area is Ker D_p/Im D_{p-1}.

Here the exactness on the symbol level plays a role analogous to that played by the Poincaré lemma in de Rham cohomology. These properties define an *elliptic complex*, denoted by $(E, D) = (\{E_p\}, \{D_p\})$.

Cohomology of elliptic complexes: There is a generalization of the Hodge decomposition theorem for an elliptic complex (E, D). If $f_p \in C^\infty(E_p)$, then f_p can be uniquely decomposed as a sum

$$f_p = D_{p-1}f_{p-1} + D_p^* f_{p+1} + h_p$$

where h_p is harmonic in the sense that $\Delta_p h_p = 0$.

We observe that

$$\operatorname{Ker} D_p \supset \operatorname{Image} D_{p-1}$$

because $D_p D_{p-1} = 0$. We may thus define cohomology groups for the elliptic complex (E, D) by (see fig. 7.1)

$$H^p(E, D) = \operatorname{Ker} D_p/\operatorname{Image} D_{p-1}. \tag{7.1}$$

As in de Rham cohomology, each cohomology class contains a unique harmonic representative, so we have an isomorphism

$$H^p(E, D) \approx \operatorname{Ker} \Delta_p. \tag{7.2}$$

These cohomology groups are finite-dimensional.

The index of an elliptic complex (E, D) is

$$\begin{aligned} \operatorname{index}(E, D) &= \sum_p (-1)^p \dim H^p(E, D) \\ &= \sum_p (-1)^p \dim \operatorname{Ker} \Delta_p. \end{aligned} \tag{7.3}$$

Example: Let $E_p = \Lambda^p(M)$ and let $D_p = \mathrm{d}$ be exterior differentiation on p-forms. Then

$$H^p(E, D) = H^p_{\mathrm{DR}}(M) = H^p(M; \mathbf{R})$$

by the de Rham theorem. The index of this complex is therefore the Euler characteristic,

$$\operatorname{index}(\Lambda^*, \mathrm{d}) = \sum_p (-1)^p \dim H^p(M; \mathbf{R}) = \sum_p (-1)^p b_p = \chi(M). \tag{7.4}$$

Note that the leading symbol of the Laplacian is $\tilde{D}(x, k) = +|k|^2$, so the complex is indeed elliptic.

Rolling up the complex: It is possible to construct a convenient two-term elliptic complex with the same index as a given complex (E, D). Let

$$F_0 = \bigoplus_p E_{2p} \qquad F_1 = \bigoplus_p E_{2p+1}$$

be the *even* and *odd* bundles, respectively. Then we consider the operators

$$A = \bigoplus_p (D_{2p} + D^*_{2p-1})$$

$$A^* = \bigoplus_p (D^*_{2p} + D_{2p-1})$$

where A: $C^\infty(F_0) \to C^\infty(F_1)$ and A^*: $C^\infty(F_1) \to C^\infty(F_0)$. The associated Laplacians are

$$\Box_0 = A^*A = \bigoplus_p \Delta_{2p}$$

$$\Box_1 = AA^* = \bigoplus_p \Delta_{2p+1}.$$

Therefore

$$\begin{aligned} \operatorname{index}(F, A) &= \dim \operatorname{Ker} \Box_0 - \dim \operatorname{Ker} \Box_1 \\ &= \sum_p (-1)^p \dim \operatorname{Ker} \Delta_p = \operatorname{index}(E, D). \end{aligned} \tag{7.5}$$

We note that if $k \neq 0$, the leading symbol $\tilde{A}(x, k)$ of A is an invertible matrix mapping F_0 to F_1. In particular, these two bundles have the same dimension.
Example: Let $(E, D) = (\Lambda^*, \mathrm{d})$ be the de Rham complex. Then F_0 is the bundle of even forms, F_1 is the bundle of odd forms, and $A = d + \delta$. The Euler characteristic is the sum of the *even* Betti numbers minus the sum of the *odd* Betti numbers.

The index theorem: The general index theorem may be described as follows: Let (x, k) be local coordinates for $T^*(M)$ and choose the "symplectic orientation" $\mathrm{d}x_1 \wedge \mathrm{d}k_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}k_n$. Let $D(M)$ be the unit disk bundle in $T^*(M)$ defined

$$D(M) = \{(x, k): |k|^2 \leq 1\}$$

and let the unit sphere bundle $S(M)$,

$$S(M) = \{(x, k): |k|^2 = 1\}$$

be its boundary. Now take *two* copies $D_\pm(M)$ of the unit disk bundles and glue them together along their common boundary $S(M)$ to define a new fiber bundle $\Psi(M)$ over M with *fiber* S^n. $\Psi(M)$ is the *compactified tangent bundle* of M. The orientation on $\Psi(M)$ is chosen to be that of $D_+(M)$. Finally, let ρ be the projection,

$$\rho: \Psi(M) \to M \tag{7.6}$$

and let $\rho_\pm$ be the restrictions of ρ to the "hemisphere bundles" $D_\pm(M)$,

$$\rho_\pm: D_\pm(M) \to M. \tag{7.7}$$

Given this structure, we wish to compute the index of an elliptic complex (E, D), which we roll up to form a two-term elliptic complex (F, A). Let $\tilde{A}(x, k)$ be the leading symbol of the operator A. Now consider the pullback bundles

$$\begin{aligned} F_+ &= \rho_+^*(F_0) \quad \text{over } D_+(M) \\ F_- &= \rho_-^*(F_1) \quad \text{over } D_-(M). \end{aligned} \tag{7.8}$$

Intuitively, we are placing the two bundles of the complex over the two hemispheres of $\Psi(M)$. We would now like to glue these bundles together to form a smooth bundle over $\Psi(M)$.

We can regard $\tilde{A}(x, k) = \sigma_L(A)(x, k)$ as a map from F_+ to F_- over $S(M) = D_+(M) \cap D_-(M)$. Because the complex is elliptic, $\tilde{A}(x, k)$ is an isomorphic map from F_+ to F_- over $S(M)$. We use this isomorphism to define the vector bundle $\Sigma(A)$ obtained by gluing F_+ to F_- using the transition function $\tilde{A}(x, k)$ over $S(M)$. $\Sigma(A)$ is sometimes called the *symbol bundle.*

Let $\mathrm{td}(M)$ be the Todd class of $T(M)$ and $\mathrm{ch}(\Sigma(A))$ be the Chern character of the symbol bundle. Then the *Atiyah–Singer index theorem* states that

$$\mathrm{index}(E, D) = \mathrm{index}(F, A) = \int_{\Psi(M)} \mathrm{ch}(\Sigma(A)) \wedge \rho^* \, \mathrm{td}(M). \tag{7.9}$$

We include in the integrand only those terms of dimension $2n = \dim \Psi(M)$. For the four classical elliptic complexes, this formula reduces to the form

$$\mathrm{index}(E, D) = (-1)^{n(n+1)/2} \int_M \mathrm{ch}\Big(\bigoplus_p (-1)^p E_p\Big) \frac{\mathrm{td}(M)}{e(M)} \cdot \tag{7.10}$$

where $e(M)$ is the Euler form and the division is heuristic.

Note: The index of any elliptic complex over an odd-dimensional manifold is zero; this would not be true if we considered *pseudo*-differential operators. For example, let

$$\begin{aligned} M &= S^1 \\ F_0 &= F_1 = S^1 \times \mathbb{C} \\ A &= \mathrm{e}^{-\mathrm{i}\theta}(-\mathrm{i}\,\partial_\theta + (-\partial_\theta{}^2)^{1/2}) - (\mathrm{i}\,\partial_\theta + (-\partial_\theta{}^2)^{1/2}) \\ \tilde{A}(\theta, k) &= \mathrm{e}^{-\mathrm{i}\theta}(k + |k|) + (k - |k|). \end{aligned}$$

This is a pseudo-differential elliptic complex with index = 1.

7.2. *The de Rham complex*

The exterior algebra $\Lambda^*(M)$ can be split into two distinct elliptic complexes. In this subsection we discuss the first, the de Rham complex, which is related to the Euler characteristic. We will discuss the second, the signature complex, in the following subsection.

The de Rham complex arises from the decomposition of the exterior algebra into even and odd forms:

$$\Lambda^{\text{even}} = \Lambda^0 \oplus \Lambda^2 \oplus \cdots$$
$$\Lambda^{\text{odd}} = \Lambda^1 \oplus \Lambda^3 \oplus \cdots$$

The operator for this elliptic complex is $\mathrm{d}+\delta$ where

$$(\mathrm{d}+\delta)\colon C^\infty(\Lambda^{\text{even}}) \to C^\infty(\Lambda^{\text{odd}}).$$

The index of the de Rham complex is the Euler characteristic $\chi(M)$,

$$\operatorname{index}(\Lambda^{\text{even,odd}}, \mathrm{d}+\delta) = \chi(M). \tag{7.11}$$

When we apply the index theorem to the de Rham complex, we recover the Gauss–Bonnet theorem,

$$\chi(M) = \int_M e(M), \tag{7.12}$$

where $e(M)$ is the Euler form. Using the results of the previous section, we may express $e(M)$ explicitly to show

$n = 2$:

$$\chi(M) = \frac{1}{4\pi} \int_M R_{ijij} \, \mathrm{d\,vol}$$

$$= \frac{1}{4\pi} \int_M \epsilon_{ab} R_{ab} = \frac{1}{2\pi} \int_M R_{12},$$

$n = 4$:

$$\chi(M) = \frac{1}{16\pi^2} \int_M \left(\tfrac{1}{2} R_{ijij} R_{klkl} - 2R_{ijik} R_{ljlk} + \tfrac{1}{2} R_{ijkl} R_{ijkl}\right) \mathrm{d\,vol}$$

$$= \frac{1}{32\pi^2} \int_M \epsilon_{abcd} R_{ab} \wedge R_{cd},$$

where R_{ab} is the curvature 2-form of M.

It is worth noting that we can use these integrals to evaluate $\chi(M)$ even if M is *not* orientable by regarding $(\mathrm{d\,vol})$ as a measure rather than as an n-form. The remaining index theorems will only apply to oriented manifolds.

Examples: (1) If $M = S^n$, then $\chi = 0$ for $n = $ odd, $\chi = 2$ for $n = $ even. (2) If $M = P_n(\mathbb{C})$, then $\chi = n + 1$.

7.3. *The signature complex*

The second splitting of the exterior algebra leads to the signature complex. We restrict ourselves henceforth to oriented manifolds of even dimension, $n = 2l$. We recall that the Euler characteristic $\chi(M)$ can be regarded either as a topological invariant or as the index of the de Rham complex. Similarly, the signature can be regarded either topologically, or as the index of an elliptic complex.

Topological signature. Let θ and ϕ belong to the middle cohomology group $H^l(M;\mathbb{R})$ and define the inner product

$$\sigma(\theta, \phi) = \int_M \theta \wedge \phi.$$

This inner product is *symmetric* if l = even (so n is divisible by 4) and *anti-symmetric* if l = odd. By Poincaré duality, this inner product is non-degenerate: for any $\theta \neq 0$, there is a ϕ such that $\sigma(\theta, \phi) \neq 0$. The *topological signature* $\tau(M)$ is defined as the signature of this quadratic form, i.e., the number of positive eigenvalues minus the number of negative eigenvalues. Note that if l = odd (i.e., n was not divisible by 4), then $\tau(M)$ vanishes automatically.

If $n = 4k$, we may relate the signature to the space of harmonic forms $H^{2k}(M;\mathbb{R})$. Since $*^2 = 1$ on $H^{2k}(M;\mathbb{R})$, we may decompose the harmonic forms into subspaces $H^{2k}_{\pm}(M;\mathbb{R})$ with eigenvalues ± 1 under the action of Hodge $*$. Since $\sigma(\theta, \phi)$ is related to the standard inner product by

$$\sigma(\theta, \phi) = (\theta, *\phi) = \int_M \theta \wedge \phi,$$

the decomposition of H^{2k} into $H^{2k}_{\pm}$ diagonalizes the quadratic form. Therefore, we may express the signature of M as

$$\begin{aligned} \tau(M) &= \dim H^{2k}_{+}(M;\mathbb{R}) - \dim H^{2k}_{-}(M;\mathbb{R}) \\ &= b^{+}_{2k} - b^{-}_{2k}, \end{aligned} \tag{7.13}$$

where we have split the middle dimension Betti number into $b_{2k} = b^{+}_{2k} + b^{-}_{2k}$.

Examples: (1) If $M = S^{2l}$, then $n = 2l$ and $b_l = 0$, so $\tau = 0$. (2) If $M = P_{2l}(\mathbb{C})$, then $n = 4l$ and $b_{2l} = b^{+}_{2l} = 1$, so $\tau = 1$.

Signature complex: We may use the above relationship to compute $\tau(M)$ as the index of an elliptic complex. We define an operator ω acting on p-forms by

$$\omega = \mathrm{i}^{p(p-1)+n/2} *,$$

where $\omega = *$ on Λ^{2k} if $n = 4k$. It is easy to show that

$$\omega(\mathrm{d} + \delta) = -(\mathrm{d} + \delta)\omega$$

$$\omega^2 = +1.$$

(Note that $(-\mathrm{i})^{n/2}\omega$ is just Clifford multiplication by the volume form.) Now let $\Lambda^{\pm}$ be the ± 1 eigenspaces of ω. Since ω anticommutes with $\mathrm{D} = \mathrm{d} + \delta$, we may define the elliptic complex

$$(\mathrm{d}+\delta):\quad C^{\infty}(\Lambda^{+}) \to C^{\infty}(\Lambda^{-}).$$

This is the *signature complex*. The contributions of the harmonic forms with eigenvalues ± 1 under ω cancel except in the middle dimension. *The index of the signature complex is the signature* $\tau(M)$,

$$\operatorname{index}(\Lambda^{\pm}, \mathrm{d}+\delta) = \dim H^{2k}_{+}(M;\mathbb{R}) - \dim H^{2k}_{-}(M;\mathbb{R}) = \tau(M). \tag{7.14}$$

When we apply the index theorem to the signature complex, we recover the Hirzebruch signature theorem,

$$\tau(M) = \int_M L(M), \tag{7.15}$$

where $L(M)$ is the Hirzebruch L-polynomial

$$L(M) = \prod_j \frac{x_j}{\tanh x_j} = 1 + \frac{1}{3}p_1 + \frac{1}{45}(7p_2 - p_1^2) + \cdots .$$

We only evaluate the integral for the part of $L(M)$ which is an n-form, and so $\tau(M) = 0$ if n is not a multiple of 4. Since the formula depends on the orientation of M, $\tau(M)$ changes sign when we reverse the orientation. Using the results of the previous section, we may express $L(M)$ explicitly to show

$n = 2$:

$$\tau(M) = 0$$

$n = 4$

$$\tau(M) = \frac{1}{3}\int_M p_1(T(M)) = -\frac{1}{24\pi^2}\int_M \operatorname{Tr}(R \wedge R).$$

Twisted signature complex (Atiyah, Bott and Patodi [1973, 1975]). Although $\tau(M) = 0$ for $n = 2, 6, 10, \ldots$, we can obtain a non-trivial index problem by taking coefficients in another vector bundle V. We can extend $(\mathrm{d}+\delta)$ to an operator $(\mathrm{d}+\delta)_V$, where

$$(\mathrm{d}+\delta)_V:\quad C^{\infty}(\Lambda^{+} \otimes V) \to C^{\infty}(\Lambda^{-} \otimes V).$$

The index theorem then becomes

$$\operatorname{index}(\Lambda^{\pm} \otimes V, (\mathrm{d}+\delta)_V) = \int_M L(M) \wedge \widetilde{\operatorname{ch}}(V), \tag{7.16}$$

where $\widetilde{\mathrm{ch}}$ is the Chern character with Ω replaced by 2Ω, i.e.,

$$\widetilde{\mathrm{ch}}(V) = \sum_k \left(\frac{\mathrm{i}}{2\pi}\right)^k \frac{2^k}{k!} \mathrm{Tr}(\Omega^k). \tag{7.17}$$

Thus, in particular, we find

$n = 2$:

$$\text{index} = \int_M 2c_1(V) = \frac{\mathrm{i}}{\pi} \int_M \mathrm{Tr}\,\Omega$$

$n = 4$:

$$\text{index} = \dim(V)\frac{1}{3}\int_M p_1 + \int_M (2c_1^2(V) - 4c_2(V))$$

$$= -\frac{\dim(V)}{24\pi^2}\int_M \mathrm{Tr}\,R \wedge R - \frac{1}{2\pi^2}\int_M \mathrm{Tr}\,\Omega \wedge \Omega$$

where Ω is the curvature of the bundle V. (Recall that if F is a 2-form corresponding to physical gauge field strengths, then $\Omega = \mathrm{i}F$ for $U(1)$ bundles, $\Omega = (\lambda^a/2\mathrm{i})F_a$ for $\mathrm{SU}(n)$ bundles, etc.)

If we perform the corresponding construction for the de Rham complex to define $(\mathrm{d}+\delta)_V$: $C^\infty(\Lambda^{\text{even}} \otimes V) \to C^\infty(\Lambda^{\text{odd}} \otimes V)$, then the index of this elliptic complex is just $\dim(V)\chi(M)$; the twisting is not detected by the de Rham complex. However, the signature complex is quite sensitive to the twisting, which can be used to produce an elliptic complex with non-zero index even in dimensions not divisible by 4.

7.4. *The Dolbeault complex*

If M is a complex manifold of real dimension n (complex dimension $n/2$), we may split the exterior algebra in yet another way. In section 3.4, we examined complex manifolds and defined the operator

$$\bar{\partial}: C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p,q+1}).$$

The Dolbeault complex is obtained by taking $p = 0$. We write the index of this complex as

$$\text{index}(\bar{\partial}) = \sum_{q=0}^{n/2} (-1)^q \dim H^{0,q}(M),$$

where $H^{p,q}$ is the cohomology group of $\bar{\partial}$ on $C^\infty(\Lambda^{p,q})$. *The index of the Dolbeault complex is the arithmetic genus* of the manifold and is the complex analog of the Euler characteristic. If the metric is Kähler, there is a natural identification

$$H^k(M;\mathbb{R}) = \bigoplus_{p+q=k} H^{p,q}(M),$$

so that the $H^{p,q}$ can be regarded as a refinement of de Rham cohomology.

When we apply the index theorem to the Dolbeault complex, we recover the Riemann–Roch theorem:

$$\text{index}(\bar{\partial}) = \int_M \text{td}(T_c(M)), \tag{7.18}$$

where $T_c(M)$ is the complex tangent space introduced in section 3.4 and td is the Todd class:

$$\text{td}(T_c(M)) = \prod_j \frac{x_j}{1-e^{-x_j}} = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_2 + c_1^2) + \cdots .$$

In the special cases $n = 2$ and $n = 4$, we can relate the arithmetic genus to the signature and the Euler characteristic as follows:

$n = 2$:

$$\text{index}(\bar{\partial}) = \tfrac{1}{2}\chi(M)$$

$n = 4$:

$$\text{index}(\bar{\partial}) = \tfrac{1}{4}(\chi(M) + \tau(M)).$$

Examples: (1) If $M = P_n(\mathbb{C})$, index $(\bar{\partial}) = 1$. (2) If $M = S^1 \times S^1$, index $(\bar{\partial}) = 0$.
Remark: We can use these formulas to show that certain manifolds do not admit complex structures. For example, if $M = S^4$, then $\chi = 2$, $\tau = 0$ and index $(\bar{\partial}) = \frac{1}{2}$, which shows S^4 is not complex. $P_2(\mathbb{C})$ with the proper orientation has index $(\bar{\partial}) = \frac{1}{4}(3+1) = 1$ and is complex; $P_2(\mathbb{C})$ with the opposite orientation is *not* complex since index $(\bar{\partial}) = \frac{1}{4}(3-1) = \frac{1}{2}$.

Twisted Dolbeault complex: Just as in the case of the signature complex, we can consider the tensor product bundle $\Lambda^{0,q} \otimes V$ and obtain a corresponding elliptic complex. The index theorem then becomes

$$\text{index}(\bar{\partial}_V) = \int_M \text{td}(T_c(M)) \wedge \text{ch}(V) \tag{7.19}$$

where ch(V) is the *ordinary* Chern character of V without any additional powers of 2. Thus, in particular, we find

$n = 2$:

$$\begin{aligned} \text{index}(\bar{\partial}_V) &= \frac{1}{2}\dim(V) \int_M c_1(T_c(M)) + \int_M c_1(V) \\ &= \frac{1}{2}\dim(V)\,\chi(M) + \frac{i}{2\pi}\int_M \text{Tr}\,\Omega \end{aligned}$$

$n = 4$:

$$\mathrm{index}(\bar\partial_V) = \frac{1}{12}\dim(V)\int_M [c_2(T_c(M)) + c_1^2(T_c(M))] + \frac{1}{2}\int_M [c_1(T_c(M)) \wedge c_1(V) + c_1^2(V) - 2c_2(V)].$$

In particular, if we take $V = \Lambda^{p,0}$, then we can compute $\Sigma_q\,(-1)^q \dim H^{p,q}(M)$ for any value of p, not just for $p = 0$.

7.5. The spin complex

The spin complex is perhaps the most subtle and interesting of the classical elliptic complexes. The deepest insight into its mathematical structure can be achieved using Clifford algebra bundles (Atiyah, Bott and Shapiro [1964]). Clifford algebras also provide a unified context for treating *all four* of the classical elliptic complexes. In fact, one may use the Clifford algebra approach to show that the spin complex is interpretable as the *square-root of plus or minus the de Rham complex*. Here we shall give a more mundane treatment of the spin complex.

We begin by restricting ourselves to a four-dimensional Euclidean-signature Riemannian spin manifold M. We choose Dirac matrices obeying

$$\{\gamma^a, \gamma^b\} \equiv \gamma^a\gamma^b + \gamma^b\gamma^a = -2\delta_{ab}$$

and take the representation

$$\gamma^a = \begin{pmatrix} 0 & \mathrm{i}\alpha_a \\ -\mathrm{i}\bar\alpha_a & 0 \end{pmatrix}; \qquad \alpha_a = (I, -\mathrm{i}\boldsymbol{\lambda}), \quad \bar\alpha_a = (I, \mathrm{i}\boldsymbol{\lambda})$$

where $\{\boldsymbol{\lambda}\}$ are the 2×2 Pauli matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the chiral operator γ_5 is diagonal,

$$\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

and we may split the space of Dirac spinors $\{\psi_\alpha\}$ into two eigenspaces of chirality ± 1:

$$\gamma_5\psi_\pm = \pm\psi_\pm.$$

The Dirac operator D is defined using the covariant derivative with respect to the basis of orthonormal frames of $T_x^*(M)$. Thus we take

$$\begin{aligned} D &= \gamma^a E_a{}^\mu(x)\, D_\mu(x) \\ &= \gamma^a E_a{}^\mu(x)\left(\frac{\partial}{\partial x^\mu} + \tfrac{1}{4}[\gamma_b, \gamma_c]\,\omega_\mu{}^{bc}(x)\right), \end{aligned}$$

where $E_a{}^{\mu}$ is an inverse vierbein of the metric on M and $\omega_\mu{}^{bc}\,\mathrm{d}x^\mu$ is the *spin connection* introduced in section 3. We observe that

$$\begin{aligned} D^\dagger D = DD^\dagger &= -g^{\mu\nu}D_\mu D_\nu + \tfrac{1}{4}[\gamma_a,\gamma_b]\tfrac{1}{4}[\gamma_c,\gamma_d]R^{abcd} \\ &= -g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} + \cdots, \end{aligned}$$

so the leading part of the operator is *elliptic* for metrics with Euclidean signature.

Clearly the spinors $\psi_+(x)$ upon which D acts are the analogs of C^∞ sections of the *fibers* of the bundles we treated in previous examples. We therefore must introduce a pair of corresponding *spin bundles* $\Delta_\pm$ over M with local coordinates

$$\Delta_\pm : (x^\mu, \psi_\pm).$$

Thus we finally arrive at the following definition of the *spin complex*

$$\begin{aligned} D&: \quad C^\infty(\Delta_+) \to C^\infty(\Delta_-) \\ D^\dagger&: \quad C^\infty(\Delta_-) \to C^\infty(\Delta_+). \end{aligned}$$

The index of the spin complex is

$$\begin{aligned} \mathrm{index}(\Delta_\pm, D) &= \dim \operatorname{Ker} D - \dim \operatorname{Ker} D^\dagger \\ &= n_+ - n_- \end{aligned} \tag{7.20}$$

where

$$n_\pm = (\text{number of chirality} = \pm 1 \text{ normalizable zero-frequency Dirac spinors}).$$

When we apply the index theorem to the spin complex, we find

$$n_+ - n_- = \int_M \hat{A}(M) \tag{7.21}$$

where the *A-roof genus* is given by

$$\hat{A}(M) = \prod_{i=1}^{n/2} \frac{x_i/2}{\sinh(x_i/2)} = 1 - \frac{1}{24}p_1 + \frac{1}{5760}(7p_1{}^2 - 4p_2) + \cdots$$

when $n = \dim M$ is a multiple of 4. For $n = 4$, we find

$$n_+ - n_- = -\frac{1}{24}P_1 \equiv -\frac{1}{24}\int_M p_1(T(M)) = +\frac{1}{24\cdot 8\pi^2}\int_M \operatorname{Tr}(R \wedge R).$$

Hence P_1 is a multiple of 24 for any compact 4-dimensional spin manifold without boundary.

Twisted spin complex: As for the other complexes, we can take the tensor product of the spin complex with a vector bundle V to produce a twisted spin complex,

$$\Delta_{\pm} \otimes V.$$

The Dirac spinors then have two sets of indices, one set of spinor indices for $\Delta_{\pm}$ and one set of "isospin" indices for V. In a typical physical application, the connection on V would be taken as

$$A^a_\mu(x)\frac{t^a}{2\mathrm{i}}$$

where A_μ is the Yang–Mills connection on the associated principal bundle and $\{t^a\}$ are $\dim(V)\times\dim(V)$ matrices giving a representation of the corresponding Lie algebra. When the Dirac operator D is extended to the operator D_V including the connection on V, the index theorem becomes

$$\mathrm{index}(\Delta_{\pm}\otimes V, D_V) = \int_M \hat{A}(M)\wedge \mathrm{ch}(V). \tag{7.22}$$

The index itself is the difference between the number of positive and negative chirality spinors in the kernel of the combined Dirac–Yang–Mills operator D_V,

$$\mathrm{index}(\Delta_{\pm}\otimes V) \equiv \nu_+ - \nu_-.$$

For $n = 2$, the index theorem for the twisted spin complex reduces to

$$\nu_+ - \nu_- = \int_M c_1(V) = \frac{\mathrm{i}}{2\pi}\int_M \mathrm{Tr}\,\Omega.$$

For $n = 4$, we find

$$\nu_+ - \nu_- = -\frac{\dim V}{24}\int_M p_1(T(M)) + \tfrac{1}{2}\int_M (c_1(V)^2 - 2c_2(V))$$

$$= +\frac{\dim V}{24\cdot 8\pi^2}\int_M \mathrm{Tr}(R\wedge R) - \frac{1}{8\pi^2}\int_M \mathrm{Tr}(\Omega\wedge\Omega).$$

Examples: 1. *$U(1)$ principal bundle in 2 dimensions*. Since $\Omega = \mathrm{i}F$ where $F = \frac{1}{2}F_{\mu\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$ is the Maxwell-field 2-form, we have

$$\nu_+ - \nu_- = -\frac{1}{2\pi}\int_M F.$$

2. *SU(2) principal bundle over S^4*. We choose spinors transforming according to the spin $\frac{1}{2}$ representation of SU(2), so dim $V = 2$. Since $\operatorname{Tr} R \wedge R = 0$ for S^4 and $c_1 = 0$, we find for the index

$$\nu_+ - \nu_- = -\int_M c_2(V) = -\frac{1}{8\pi^2}\int_M \operatorname{Tr}(\Omega \wedge \Omega) \equiv k$$

where $\Omega = \frac{1}{2}(\lambda_a/2\mathrm{i})F^a_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. Note that the "instanton number" k defined by $k = \nu_+ - \nu_-$ is minus the 2nd Chern number; k is positive if Ω is self-dual and negative for anti-self-dual Ω. In the actual instanton solutions, $\nu_\pm = 0$ for $k<0$, $k>0$, respectively.

For spinors ψ belonging to a $(2t+1)$-dimensional representation of SU(2) labeled by $t = 0, 1/2, 1, 3/2, \ldots$, the curvature Ω_t must be expressed as a matrix in the representation of ψ. If we define

$$k = -\frac{1}{8\pi^2}\int_M \operatorname{Tr}(\Omega \wedge \Omega)$$

where $\Omega = \Omega_{1/2}$ is a matrix in the spin 1/2 representation, then the index theorem for $(2t+1)$-dimensional SU(2) spinors can be shown to be

$$\nu_+(t) - \nu_-(t) = -\frac{1}{8\pi^2}\int_M \operatorname{Tr}(\Omega_t \wedge \Omega_t) = \tfrac{2}{3}t(t+1)(2t+1)k.$$

See Grossman [1977] for solutions of the Dirac equation with arbitrary k and an explicit verification of the index theorem for the twisted spin complex.

7.6. G-index theorems

The G-index theorem is a generalization of the ordinary index theorem. It is applicable when one is given in addition to the elliptic complex a suitable map f which takes the base manifold into itself, f: $M \to M$, and which therefore acts on the cohomology of the complex. For the de Rham complex, f may be any smooth map; for the signature complex, f must be an orientation-preserving isometry; f must be holomorphic for the Dolbeault complex, and, for the spin complex, f must be an orientation-preserving isometry which also preserves the spin structure.

The ordinary index theorem computes the alternating sum of dimensions of the cohomology groups of the elliptic complex in terms of characteristic classes; the G-index theorem computes the alternating sum of the trace of the action of f on the cohomology groups (the Lefschetz number) in terms of generalized characteristic classes.

We first examine the Lefschetz fixed-point theorem, which is a special case of the G-index theorem for the de Rham complex. Then we briefly outline the application of the G-index theorem to each of the classical elliptic complexes and present a number of examples.

7.6.1. Lefschetz fixed point theorem

Lefschetz numbers: Let M be a compact real manifold of dimension n without boundary and let $H^p(M;\mathbb{R})$ be the pth cohomology class of M. Let f: $M \to M$ be a smooth map and let f^*_p be the pull-back map on $H^p(M;\mathbb{R})$. Then if we choose a suitable basis, f_p^*: $H^p(M;\mathbb{R}) \to H^p(M;\mathbb{R})$ can be represented as a matrix with integer entries. The *Lefschetz number* $L(f)$ is the integer

$$L(f) = \sum_{p=0}^{n} (-1)^p \operatorname{Tr}(f_p^*).$$

$L(f)$ is a homotopy invariant of f. If $f(x) = x$ is the identity map, then $f_p^* = I_{\dim(H^p)}$ is the identity map on $H^p(M, \mathbb{R})$, so

$$L(\text{identity}) = \sum_{p=0}^{n} (-1)^p \dim(H^p) = \chi(M)$$

is the index of the de Rham complex. Thus the Lefschetz number can be thought of as a generalization of the Euler characteristic.

Lefschetz fixed-point theorem: We consider first the special case of an isometry f: $M \to M$. Then the fixed point set of f consists of totally geodesic submanifolds μ_j of M. Lefschetz proved that

$$L(f) = \sum_{\{\mu_j\}} \chi(\mu_j). \tag{7.23}$$

(If f is not an isometry, there are additional conditions which f must satisfy; in this situation, the terms in the sum are signed according to the direction of the normal derivative of f.) When f is homotopic to the identity and has only isolated fixed points, then the Euler characteristic of M equals the number of fixed points of f,

$$\chi(M) = (\text{number of fixed points of } f).$$

Vector fields: Let $V = v^\mu(x)\, \partial/\partial x^\mu$ be a vector field with isolated non-degenerate zeroes on a manifold M and let the map $f(t, x)$ be the infinitesimal *flow* of V:

$$f^\mu(0, x) = x^\mu$$

$$\frac{\partial f^\mu}{\partial t}(t, x) = v^\mu(f(t, x)).$$

$f(t, x_0)$ is the trajectory of the flow of V beginning at x_0. Since the flow is homotopic to the identity map, the Lefschetz number of the flow is the Euler characteristic of the manifold M. Furthermore, the *fixed points* of the flow correspond to the *zeroes* of the vector field. We conclude that the Euler characteristic of M is equal to the number of zeroes of V:

$$\chi(M) = (\text{number of zeroes of vector field}). \tag{7.24}$$

We note that if the flow is not an isometry (i.e., V is not a Killing vector field), then the zeroes of V have associated plus or minus signs; the Euler characteristic is then the *signed* sum of the zeroes of V.
Example: $S^2 = P_1(\mathbb{C})$. We know that $\chi(S^2) = 2$.

Case (*1*) The map $z \to e^{i\alpha} z$

is an isometry which is the flow of a vector field $r\alpha\, \partial/\partial\theta$ where $z = r\, e^{i\theta}$. It has two fixed isolated non-degenerate fixed points at $z = 0$ and $z = \infty$, each of which appears with a positive sign.

Case (*2*) The map $z \to z + 1$

is the flow of the vector field $\partial/\partial x$, where $z = x + iy$, and has a degenerate *double* fixed point at ∞.

7.6.2. G-index theorem

For the remainder of this section, we will only consider maps with non-degenerate isolated fixed points, although there are corresponding formulas for maps with higher dimensional invariant sets. With this restriction, we treat the G-index theorem for the four standard elliptic complexes.

We begin by choosing local coordinates $x^\mu \in U$ on M such that the map f can be written in the form

$$f^\mu(x) = f^\mu(x_0) + (x^\nu - x_0^\nu)\, \partial f^\mu(x_0)/\partial x^\nu + \cdots$$

where x_0 is a fixed point of the map. We denote the *Jacobian matrix* f' of the map by

$$f'(x_0) = |\partial f^\mu(x_0)/\partial x^\nu|.$$

We assume that f is non-degenerate, i.e., there are no tangent vectors left infinitesimally fixed by f' at x_0. This is equivalent to requiring that f' does not have the eigenvalue 1:

$$\mathrm{Det}(I - f') \neq 0.$$

Let (E_+, E_-) denote the rolled-up elliptic complex under consideration, and let f^* denote the pull-back operation mapping $E_\pm \to E_\pm$. Let $H^\pm$ denote the cohomology of the elliptic complex and let f^* act on the cohomology by the pullback. The Lefschetz number of the elliptic complex is then defined to be

$$L_{\mathrm{E}}(f) \equiv \mathrm{Tr}(f^* H^+) - \mathrm{Tr}(f^* H^-).$$

The G-index theorem expresses the global invariant $L_{\mathrm{E}}(f)$ in terms of local geometric information:

$$L_{\mathrm{E}}(f) = \sum_{\{\text{fixed points } x_0\}} \frac{\mathrm{Tr}\, f^*(x_0)E_+ - \mathrm{Tr}\, f^*(x_0)E_-}{|\mathrm{Det}(I - f'(x_0))|}.$$

We next apply this formula to the four classical elliptic complexes; for more details, see Atiyah and Singer [1968b].

de Rham complex. Let $E_+ = \Lambda^{\mathrm{even}}(M, \mathbb{R})$, $E_- = \Lambda^{\mathrm{odd}}(M, \mathbb{R})$. Then the G-index theorem becomes

$$L_{\text{de Rham}}(f) = \sum_{\left\{\substack{\text{fixed}\\ \text{points}}\right\}} \frac{\mathrm{Tr}(f^*\Lambda^{\mathrm{even}}) - \mathrm{Tr}(f^*\Lambda^{\mathrm{odd}})}{|\mathrm{Det}(I - f')|}.$$

After some algebra, an application of the splitting principle shows that

$$L_{\text{de Rham}}(f) = \sum_{\substack{\text{fixed}\\\text{points}}} \frac{\operatorname{Det}(I-f')}{|\operatorname{Det}(I-f')|} = \sum_{\substack{\text{fixed}\\\text{points}}} \operatorname{sign}\operatorname{Det}(I-f').$$

When f is an isometry, $\operatorname{Det}(I-f')=1$, so $L_{\text{de Rham}}(f)$ is just the number of fixed points of f.
Example: Analysis of local behavior of an isometry near a fixed point. Let $n=2$ and let f be an isometry which has the local form

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(*Note*: We need not specify M globally, because any orientation-preserving isometry has this local form.) f' is a rotation about the fixed point at the origin:

$$f' = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

As bases for $\Lambda^{\text{even,odd}}$, we choose

$$\Lambda^{\text{even}} = \begin{pmatrix} 1 \\ \mathrm{d}x \wedge \mathrm{d}y \end{pmatrix}, \quad \Lambda^{\text{odd}} = \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix}.$$

Then

$$f^*\Lambda^{\text{even}} = \begin{pmatrix} 1 \\ \mathrm{d}f_1 \wedge \mathrm{d}f_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \mathrm{d}x \wedge \mathrm{d}y \end{pmatrix}$$

$$f^*\Lambda^{\text{odd}} = \begin{pmatrix} \mathrm{d}f_1 \\ \mathrm{d}f_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix}$$

so $\operatorname{Tr}(f^*\Lambda^{\text{even}}) - \operatorname{Tr}(f^*\Lambda^{\text{odd}}) = 2 - 2\cos\theta$.
We verify that this agrees with $\operatorname{Det}(I-f') = 2-2\cos\theta$. There is one local fixed point at $x=y=0$, so the contribution to the Lefschetz formula is

$$\frac{2-2\cos\theta}{|2-2\cos\theta|} = 1.$$

Signature complex: Let M be an oriented manifold of even-dimension $n=2l$ and let f be an orientation-preserving isometry. Let $E_\pm = \Lambda^\pm(T^*(M))$ be the signature complex and let $H^\pm$ be the corresponding cohomology groups. We define

$$L_{\text{sign}}(f) = \operatorname{Tr} f^*H^+ - \operatorname{Tr} f^*H^- = \operatorname{Tr} f^*H^{l,+} - \operatorname{Tr} f^*H^{l,-},$$

since all terms cancel except those in the middle dimensional cohomology class. The G-index theorem is then

$$L_{\text{sign}}(f) = \sum_{\substack{\text{fixed}\\ \text{points}}} \frac{\operatorname{Tr} f^* \Lambda^{l,+} - \operatorname{Tr} f^* \Lambda^{l,-}}{\operatorname{Det}(I - f')},$$

where the determinant is positive because f is an isometry.

Example: Let $n = 2$ and let f be the same local map used in the de Rham complex example. As bases for $\Lambda^{\pm}$ we choose

$$\Lambda^{\pm} = \{\mathrm{d}x \pm \mathrm{i}\,\mathrm{d}y).$$

We verify that under the action of the signature operator $\omega = \mathrm{i}\,*$, the bases behave as they should:

$$\omega(\mathrm{d}x \pm \mathrm{i}\,\mathrm{d}y) = \pm(\mathrm{d}x \pm \mathrm{i}\,\mathrm{d}y).$$

Applying the pullback map, we find

$$f^*\Lambda^+ = \mathrm{d}f_1 + \mathrm{i}\,\mathrm{d}f_2 = \mathrm{e}^{+\mathrm{i}\theta}(\mathrm{d}x + \mathrm{i}\,\mathrm{d}y)$$

$$f^*\Lambda^- = \mathrm{d}f_1 - \mathrm{i}\,\mathrm{d}f_2 = \mathrm{e}^{-\mathrm{i}\theta}(\mathrm{d}x - \mathrm{i}\,\mathrm{d}y).$$

Again, there is one fixed point at the origin, so the contribution to the G-signature theorem reads

$$\frac{\mathrm{e}^{+\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}}{2(1-\cos\theta)} = +\mathrm{i}\,\frac{\sin\theta}{1-\cos\theta} = \mathrm{i}\cot(\theta/2).$$

We may extend this result to higher even dimensions $n = 2l$ as follows: Let f' be an orthogonal matrix which we may think of as a rotation about a fixed point at the origin. We decompose this rotation into a product of commuting 2×2 rotations through angles θ_j, $j = 1, \ldots, l$. Then we may show that the local contribution to the fixed point formula at the fixed point is

$$\prod_{j=1}^{l} \frac{\mathrm{i}\sin\theta_j}{1-\cos\theta_j} = \prod_{j=1}^{l} \mathrm{i}\cot(\theta_j/2).$$

Dolbeault complex. Let M be a holomorphic manifold and let f be a holomorphic map. Let $E_+ = \Lambda^{0,\text{even}}$ and $E_- = \Lambda^{0,\text{odd}}$ be the bundles of the Dolbeault complex. Then

$$L_{\text{Dol}}(f) = \operatorname{Tr} f^* H^{0,\text{even}} - \operatorname{Tr} f^* H^{0,\text{odd}}$$

The G-index theorem is

$$L_{\text{Dol}}(f) = \sum_{\substack{\text{fixed}\\ \text{points}}} \frac{\operatorname{Tr} f^*\Lambda^{0,\text{even}} - \operatorname{Tr} f^*\Lambda^{0,\text{odd}}}{|\operatorname{Det}(I - f')|}.$$

Example: Let $n=2$, take f to be the local rotation about the origin used above, and choose the bases

$$\Lambda^{0,\mathrm{even}}=\{1\}, \qquad \Lambda^{0,\mathrm{odd}}=\{\mathrm{d}\bar{z}=\mathrm{d}x-\mathrm{i}\,\mathrm{d}y\}.$$

Then the pullback acts as

$$f^*\Lambda^{0,\mathrm{even}}=1, \qquad f^*\Lambda^{0,\mathrm{odd}}=\mathrm{d}f_1-\mathrm{i}\,\mathrm{d}f_2=\mathrm{e}^{-\mathrm{i}\theta}\,\mathrm{d}\bar{z}.$$

The contribution to the G-index theorem is therefore

$$\frac{1-\mathrm{e}^{-\mathrm{i}\theta}}{2-2\cos\theta}.$$

In higher dimensions the contribution is given by the product of such terms.

Spin complex: Let M be a spin manifold and let f be an orientation-preserving spin isometry. Let $E_{\pm}=\Delta_{\pm}$ be the bundles of the spin complex and let $H^{\mathrm{spin},\pm}$ be the corresponding cohomology groups (or the harmonic spaces) of the Dirac operator. Then

$$L_{\mathrm{spin}}(f)=\mathrm{Tr}\,f^*H^{\mathrm{spin},+}-\mathrm{Tr}\,f^*H^{\mathrm{spin},-},$$

and the G-index theorem becomes

$$L_{\mathrm{spin}}(f)=\sum_{\substack{\text{fixed}\\ \text{points}}}\frac{\mathrm{Tr}\,f^*\Delta_+-\mathrm{Tr}\,f^*\Delta_-}{|\mathrm{Det}(I-f')|}.$$

Example: As before, let $n=2$ and take f' to be the local rotation around the origin. The spinor bases for $\Delta_{\pm}$,

$$\Delta_+=\begin{pmatrix}1\\0\end{pmatrix}, \qquad \Delta_-=\begin{pmatrix}0\\1\end{pmatrix},$$

transform under the rotation f' as

$$\begin{pmatrix}1\\0\end{pmatrix}\to\mathrm{e}^{+\mathrm{i}\theta/2}\begin{pmatrix}1\\0\end{pmatrix}, \qquad \begin{pmatrix}0\\1\end{pmatrix}\to\mathrm{e}^{-\mathrm{i}\theta/2}\begin{pmatrix}0\\1\end{pmatrix}.$$

Thus the contribution to the G-spin theorem becomes

$$\frac{\mathrm{e}^{+\mathrm{i}\theta/2}-\mathrm{e}^{-\mathrm{i}\theta/2}}{2-2\cos\theta}=+\frac{\mathrm{i}}{2\sin(\theta/2)}.$$

The contribution to the G-spin index for higher dimensions is a product of such terms.

Examples 7.6

1. Let G be a compact Lie group of dimension $n>0$. Let $g(t)$ for $t\in[0,1]$ be a curve in G with $g(0)=I$ and $g(t)\neq I$ for $t>0$. Let $f_t(X)=g(t)\cdot X$. Then $f_0(X)=X$ so f_0 is the identity map and $L(f_0)=\chi(G)$. For $t>0$, $f_t(X)=g(t)\cdot X\neq X$ since $g(t)\neq I$. Thus f has no fixed points, so $L(f_t)=0$. Since $L(f_0)=L(f_t)$, $\chi(G)=0$. This shows that the following Euler characteristics vanish: $\chi(U(k))=\chi(\mathrm{SU}(k))=\chi(O(k))=\chi(\mathrm{SO}(k))=0$ for $k>1$. If $k=1$, then we cannot use this argument; for example, $\chi(O(1))=2$ since $O(1)$ consists of two points ±1.

2. Let $M=P_n(\mathbb{C})$ for n even (so the dimension of M is divisible by 4). Let $x\in H^2(P_n(\mathbb{C});\mathbb{R})$ be the generator discussed in 6.3.2; $x^k\in H^{2k}(P_n(\mathbb{C});\mathbb{R})$ is a generator for $k=1,\ldots,n$. Let $f\colon M\to M$ and $f^*x=\lambda x$. Since f^* preserves the ring structure, $f^*(x^k)=\lambda^k x^k$. Therefore

$$L(f)=1+\lambda+\cdots+\lambda^n.$$

If n is even, this has no real roots so $L(f)\neq0$. Therefore f must have a fixed point.

3. Let $M=S^1\times S^1$ and let $f(\theta_1,\theta_2)=(\theta_2,\theta_1)$ be the interchange. Let $\{1,\mathrm{d}\theta_1,\mathrm{d}\theta_2,\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2\}$ be the basis for $H^*(M;\mathbb{R})$ discussed earlier. Then

$$f^*(1)=1\qquad f^*(\mathrm{d}\theta_1)=\mathrm{d}\theta_2\qquad f^*(\mathrm{d}\theta_2)=\mathrm{d}\theta_1\qquad f^*(\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2)=-\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2$$
$$\mathrm{Tr}\,f_0^*=1\qquad \mathrm{Tr}\,f_1^*=0\qquad \mathrm{Tr}\,f_2^*=-1$$

so $L(f)=1-0+(-1)=0$. The fixed point set of f is the diagonal S^1 so $L(f)=\chi(S^1)=0$. If $g(\theta_1,\theta_2)=(-\theta_2,\theta_1)$ then

$$g^*(1)=1\qquad g^*(\mathrm{d}\theta_1)=-\mathrm{d}\theta_2\qquad g^*(\mathrm{d}\theta_2)=\mathrm{d}\theta_1\qquad g^*(\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2)=\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2$$

so $L(g)=1-0+1=2$. g has two isolated fixed points $(0,0)$ and (π,π).

Let $M=S^2\times S^2$. The cohomology ring of M has generators $1\in H^0(M;\mathbb{R})\simeq\mathbb{R}$, ω_1, $\omega_2\in H^2(M;\mathbb{R})\simeq\mathbb{R}\oplus\mathbb{R}$, $\omega_1\wedge\omega_2\in H^4(M;\mathbb{R})\simeq\mathbb{R}$ where the $\omega_i\in H^2(S^2;\mathbb{R})$ for each factor. If $f(x,y)=(y,x)$ then

$$f^*(1)=1\qquad f^*(\omega_1)=\omega_2\qquad f^*(\omega_2)=\omega_1\qquad f^*(\omega_1\wedge\omega_2)=\omega_2\wedge\omega_1=\omega_1\wedge\omega_2$$

so $L(f)=1-0+1=2$. The fixed point set of f is the diagonal S^2 so $L(f)=\chi(S^2)=2$. If $g(x,y)=(-y,x)$, then

$$g^*(1)=1\qquad g^*(\omega_1)=-\omega_2\qquad g^*(\omega_2)=\omega_1\qquad g^*(\omega_1\wedge\omega_2)=-\omega_1\wedge\omega_2$$

so $L(g)=1-0+(-1)=0$. In this case g has no fixed points.

4. Let $M=S^1\times S^1$ be the 2-torus with generators $\mathrm{d}\theta_1$ and $\mathrm{d}\theta_2$. Then (with $\omega=\mathrm{i}\,*$)

$$\omega\cdot\mathrm{d}\theta_1=\mathrm{i}\,\mathrm{d}\theta_2,\qquad \omega\cdot\mathrm{d}\theta_2=-\mathrm{i}\,\mathrm{d}\theta_1,$$

so $(\mathrm{d}\theta_1\pm\mathrm{i}\,\mathrm{d}\theta_2)$ spans $H^1_\pm(M;\mathbb{R})$. If $f(\theta_1,\theta_2)=(\theta_1,\theta_2)$ is the identity map, then $\mathrm{Tr}\,f_+^*-\mathrm{Tr}\,f_-^*=1-1=\tau(M)=0$. Suppose that $g(\theta_1,\theta_2)=(-\theta_2,\theta_1)$, then

$$g^* \mathrm{d}\theta_1 = -\mathrm{d}\theta_2 \qquad g^* \mathrm{d}\theta_2 = \mathrm{d}\theta_1 \qquad g^*(\mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2) = \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2$$
$$g^*(\mathrm{d}\theta_1 + \mathrm{i}\,\mathrm{d}\theta_2) = \mathrm{i}(\mathrm{d}\theta_1 + \mathrm{i}\,\mathrm{d}\theta_2)$$
$$g^*(\mathrm{d}\theta_1 - \mathrm{i}\,\mathrm{d}\theta_2) = -\mathrm{i}(\mathrm{d}\theta_1 - \mathrm{i}\,\mathrm{d}\theta_2)$$
$$L_{\mathrm{sign}}(g) = \ \mathrm{i} - (-\mathrm{i}) = 2\mathrm{i}.$$

Since $L_{\mathrm{sign}}(f)$ is a homotopy invariant, we use an argument similar to that given for the ordinary Lefschetz number to show $\tau(M) = 0$ either if M is a compact Lie group or if M admits a Killing vector field without zeroes.

8. Index theorems: Manifolds with boundary

The applications of the index theorem described in the previous section hold only for bundles with base manifolds M which are closed and compact without boundary. Many interesting physical situations deal with base manifolds M which have nonempty boundaries or which, for M noncompact, can be treated as limiting cases of manifolds with boundary. This section is devoted to the extension of the index theorem to manifolds with boundary carried out by Atiyah, Patodi and Singer [1973, 1975a, 1975b, 1976].

Euler characteristic boundary corrections: In order to understand more clearly the necessity for boundary corrections to a topological index, let us review the familiar case of the Euler characteristic of a two-dimensional disc. The general formula can be written

$$\chi[M, \partial M] = \frac{1}{2\pi} \int_M R + \frac{1}{2\pi} \int_{\partial M} \frac{\mathrm{d}s}{\rho} + \frac{1}{2\pi} \sum_i (\pi - \theta_i).$$

Here R is the curvature 2-form (essentially the Gaussian curvature), $1/\rho$ is the geodesic curvature on the boundary and θ_i is the interior angle of each vertex, as shown in fig. 8.1. We illustrate the application of the formula to the three special cases depicted in fig. 8.2:

(a) *Flat, n-sided polygon*: We simply recover the fact that

$$\sum_{i=1}^{n} \theta_i = (n-2)\pi$$

implies

$$\chi = 0 + 0 + 1.$$

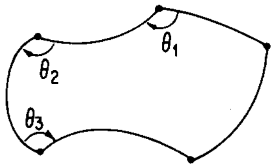

Fig. 8.1. An arbitrary two-dimensional surface with the topology of a disc.

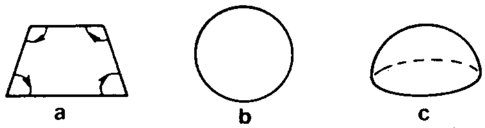

Fig. 8.2. Special cases: (a) polygon, (b) circle, (c) hemisphere.

(b) *Flat circle* of radius r; with $ds = r\,d\phi$ and $\rho = r$, only the geodesic term contributes:

$$\chi = 0 + 1 + 0.$$

(c) *Hemisphere*: the geodesics normal to the equator are parallel at the equator, so $\rho = \infty$, $R = (1/r^2)\, r^2\, d\phi\, d\cos\theta$ and only the Gaussian curvature term contributes,

$$\chi = 1 + 0 + 0.$$

We conclude that although the Euler characteristic of a disc is always $\chi = 1$, the Gaussian curvature and the boundary terms interact in complicated ways to maintain the topological invariance of the formula.

Remark: The area of a spherical polygon can be computed from the formula above using $\chi = 1$. Taking the sphere to have unit radius, we find

$$\text{Spherical area} = \int_{\text{polygon}} R = \sum \theta_i - (n-2)\pi.$$

For flat polygons (sphere of infinite radius), the "area" vanishes and we recover $\Sigma\, \theta_i = (n-2)\pi$. On a hyperboloid, the curvature is negative and the effective area is the *angular defect*,

$$\text{Hyperboloidal area} = (n-2)\pi - \sum \theta_i.$$

8.1. Index theorem with boundary

When we consider manifolds with boundary, we must first study the boundary conditions which determine the spectra of the operators. Ideally, one would like to find an index theorem using conventional local boundary conditions such as those appearing in ordinary physical problems. However, Atiyah and Bott [1964] have shown that in general there exist topological obstructions to finding good local boundary conditions. The spin, signature, and Dolbeault complexes in particular do not admit local boundary conditions, although the de Rham complex does. Therefore if one wants a general index theorem for a manifold with boundary, one must consider non-local boundary conditions. Atiyah, Patodi and Singer discovered that appropriate non-local boundary conditions could indeed be used to formulate an index theorem for elliptic complexes over manifolds with boundary.

We now outline the general nature of the Atiyah–Patodi–Singer index theorem. We begin by considering a classical elliptic complex (E, D) over a manifold M with nonempty boundary ∂M. For simplicity, we assume that $\{E\}$ is rolled up to a 2-term complex, D: $E_0 \to E_1$. In order to formulate the index theorem, we require analytic information on the boundary in addition to the purely topological information which sufficed in the case without boundary.

Boundary condition: We assume for the time being that M admits a product metric

$$ds^2 = f(\tau_0)\, d\tau^2 + g_{ij}(\tau_0, \theta_k)\, d\theta^i\, d\theta^j$$

on the boundary, where $\tau = \tau_0$ defines the boundary manifold ∂M. (We will deal later with the case when M does not admit a product metric.) Then we construct from D a Hermitian operator whose eigenfunctions ϕ are subject to the boundary condition

$$\phi \sim e^{-k\tau} \qquad k > 0 \tag{8.1}$$

near the boundary.

The index: We now define cohomology classes $H^p(E, D, \partial M)$ whose representatives obey the required boundary conditions. The corresponding index is then taken to be

$$\mathrm{index}(E, D, \partial M) = \sum_p (-1)^p H^p(E, D, \partial M).$$

Form of the index theorem: The extended index theorem of Atiyah–Patodi–Singer for manifolds with boundary takes the form

$$\mathrm{index}(E, D, \partial M) = V[M] + S[\partial M] + \xi[\partial M]. \tag{8.2}$$

Here

$V[M]$ = the integral over M of the same characteristic classes as in the $\partial M = \emptyset$ case. V is computable from the curvature alone.

$S[\partial M]$ = the integral over ∂M of the Chern–Simons form, described below. S is computable from the connection, the curvature, and the second fundamental form determined by a choice of the normal to the boundary.

$\xi[\partial M] = c\eta[M]$ = a constant c times the Atiyah–Patodi–Singer η-invariant of the boundary, described below. The η-invariant is determined by the eigenvalues of the tangential part of D restricted to the boundary ∂M. For several important cases, η can be computed algebraically.

The surface correction $S[\partial M]$ is present only if one uses a metric on M which does not become a product metric at the boundary. The $\xi[\partial M]$ correction is absent for the de Rham complex, but plays a crucial role in the Dolbeault, signature and spin complex index theorems.

General nature of the boundary corrections: One can develop an intuitive feeling for the nature of the boundary corrections to the index theorem by examining a pair of manifolds M and M' with the same boundary

$$L = \partial M = \partial M'.$$

We give M and M' each a metric and a connection and assume that they admit the same product metric near their boundaries. Thus we may sew M and M' together smoothly along their common boundary to form a new manifold $M \cup M'$ *without* boundary.

Now assume M and M' are 4-dimensional and consider, for example, the signature τ of $M \cup M'$. By the no-boundary index theorem,

$$\tau(M \cup M') = -\frac{1}{24\pi^2} \int_{M \cup M'} \mathrm{Tr}(\Omega \wedge \Omega)$$

where Ω is the curvature of the assumed metrics on M and M'.

Now we break the integral into two parts, one involving M, the other M' with the *opposite* orientation to its orientation in $M \cup M'$ (this gives M and M' the same relative orientation). If we call Ω' the curvature in M' with the new orientation, we find

$$\int_{M \cup M'} \mathrm{Tr}(\Omega \wedge \Omega) = \int_M \mathrm{Tr}(\Omega \wedge \Omega) - \int_{M'} \mathrm{Tr}(\Omega' \wedge \Omega').$$

Since with our chosen orientation the Novikov formula gives (see, e.g. Atiyah and Singer [1968b])

$$\tau(M \cup M') = \tau(M) - \tau(M'),$$

we find

$$\tau(M) + \frac{1}{24\pi^2} \int_M \mathrm{Tr}(\Omega \wedge \Omega) = \tau(M') + \frac{1}{24\pi^2} \int_{M'} \mathrm{Tr}(\Omega' \wedge \Omega').$$

Hence the quantity

$$-\eta_S[L] = \tau(M) + \frac{1}{24\pi^2} \int_M \mathrm{Tr}(\Omega \wedge \Omega)$$

depends only on the metric on $L = \partial M$. The index theorem gives an alternative expression for η_S in terms of the eigenvalues of the signature operator restricted to ∂M.

Next, suppose that we have a metric $\tilde{g}$ on M which is *not* a product metric on the boundary. Let $\tilde{\omega}$ be the connection obtained from $\tilde{g}$ and let $\tilde{\Omega}$ be its curvature. Then, as shown in section 6, the difference between $\mathrm{Tr}\, \tilde{\Omega} \wedge \tilde{\Omega}$ and $\mathrm{Tr}\, \Omega \wedge \Omega$ is a total derivative,

$$\mathrm{d}Q\,(\tilde{\omega}, \omega) = (\mathrm{Tr}\, \tilde{\Omega} \wedge \tilde{\Omega} - \mathrm{Tr}\, \Omega \wedge \Omega),$$

where Ω is the curvature of the metric g which is a product metric on ∂M. This expression gives an additional analytic correction to the index,

$$S[\partial M] = -\frac{1}{24\pi^2} \int_{\partial M} Q.$$

We now turn to a precise definition of the η-invariant.

8.2. The η-invariant

We consider our 2-term elliptic complex (E, D) with D: $E_0 \to E_1$ a linear operator obeying the boundary conditions (8.1). We choose $\partial/\partial\tau$ to represent the outward normal derivative on ∂M. We write D as

$$D = A \cdot \partial + B\, \partial/\partial\tau = B(B^{-1}A \cdot \partial + \partial/\partial\tau)$$

where A and B are matrices and $A \cdot \partial$ represents the tangential part of D. Whereas D itself might not have a true eigenvalue spectrum because $E_0 \neq E_1$ in general, the operator

$$\hat{D} = B^{-1}A \cdot \partial|_{\partial M}$$

maps $E_0 \to E_0$ on ∂M and does have a well-defined spectrum. We let $\{\lambda_i\}$ denote the eigenvalues of the tangential operator $\hat{D}$ acting on ∂M.

The η-invariant of Atiyah–Patodi–Singer is then defined by examining a natural generalization of the spectral Riemann zeta function for non-positive eigenvalues:

$$\eta_D[s, \partial M] = \sum_{\substack{\{\lambda_i\} \\ \lambda_i \neq 0}} \operatorname{sign}(\lambda_i)|\lambda_i|^{-s}, \qquad s > n/2 \quad (n = \dim M).$$

It has been shown that, despite the apparent singularities at $s = 0$, this expression possesses a regular analytic extension to $s = 0$; this continuation defines the η-invariant:

$$\eta_D[\partial M] \equiv \eta_D[s = 0, \partial M]. \tag{8.3}$$

Harmonic correction: If the elliptic operator D in question admits zero eigenvalues (as does the Dirac operator), then one must be careful to account for the missing zero eigenvalues in the definition of η_D. The correct prescription is to add $h_D(\partial M)$, which is the dimension of the space of functions harmonic under $\hat{D}$

$$\eta_D \to \eta_D + h_D.$$

Intuitively, it is clear that η_D counts the asymmetry between the number of positive and negative eigenvalues on the boundary. Furthermore, η_D is independent of the scale of the metric, and hence is independent of the numerical values of the $\{\lambda_i\}$. If the spectrum $\{\lambda_i\}$ varies with some parameter, typically a parameter specifying the location of the boundary surface, the smallest positive eigenvalue (say λ_k), may change sign at some point: one sees immediately that then there is one less positive eigenvalue and one more negative one, so η_D jumps by *two*:

$$\eta_D \to \eta_D - 2.$$

(Clearly many jumps with either sign can occur.) However, we note that *exactly* at the point where $\lambda_k = 0$, we must omit λ_k from the sum and add one, the dimension h_D of the new harmonic space; thus there is no change in η_D until $\lambda_k < 0$.

Computation of η_D: There are variety of special circumstances in which η_D can be calculated directly, e.g., when D = the signature or Dirac operator. The simplest situation is that in which the metric on ∂M possesses an orientation-reversing isometry; in this case

$$\eta_D[\partial M] = 0.$$

(If D is the Dirac operator, one must also assume that M is simply connected.)

Another case which has been calculated directly is that where the metric on ∂M is that of a distorted S^3,

$$ds^2 = \sigma_x{}^2 + \sigma_y{}^2 + \lambda^2\sigma_z{}^2.$$

Hitchin [1974] has shown by solving for the eigenvalues of the Dirac operator that

$$\eta_{\mathrm{Dirac}} = \tfrac{1}{6}(1-\lambda^2)^2.$$

When $\lambda^2 = 1$, the S^3 metric has an orientation-reversing isometry and η_{Dirac} vanishes as it must.

If one takes the symmetric ($\lambda = 1$) S^3 metric and identifies opposite points to get a metric on $P_3(\mathbb{R})$, η_S remains zero but η_{Dirac} may be non-zero because $P_3(\mathbb{R})$ is not simply connected and possesses two inequivalent spin-structures. In fact, the η-invariants for the standard operators can be calculated fairly straightforwardiy using G-index theory when the metric on ∂M is that of S^3 modulo a discrete group. We define the *Lens spaces* of S^3 by taking $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ and identifying the first $\mathbb{R}^2$ with itself when rotated by $e^{i\theta_1}$, then doing the same thing for the second $\mathbb{R}^2$ rotated by $e^{i\theta_2}$, where θ_1 and θ_2 have rational periods. The simplest case, $P_3(\mathbb{R})$, is obtained by setting $\theta_1 = \theta_2 = \pi$.

Let $m\theta_1 = m\theta_2 = 2\pi$. Then the general formulas for the η-invariant corrections to the indices for Lens space boundaries are (Atiyah, Patodi and Singer [1975b]; Atiyah [1978]; Hanson and Römer [1978]):

Signature: $$\xi_S = \frac{1}{m}\sum_{k=1}^{m-1}{}' \cot\tfrac{1}{2}k\theta_1 \cot\tfrac{1}{2}k\theta_2$$
$$(=0 \qquad \text{for } P_3(\mathbb{R}))$$

Dirac: $$\xi_{\mathrm{Dirac}} = -\frac{1}{4m}\sum_{k=1}^{m-1}{}' \frac{1}{\sin\tfrac{1}{2}k\theta_1 \sin\tfrac{1}{2}k\theta_2}$$
$$(=-\tfrac{1}{8} \qquad \text{for } P_3(\mathbb{R}))$$

Rarita–Schwinger: $$\xi_{\mathrm{RS}} = -\frac{1}{4m}\sum_{k=1}^{m-1}{}' \frac{2\cos k\theta_1 + 2\cos k\theta_2 - 1}{\sin\tfrac{1}{2}k\theta_1 \sin\tfrac{1}{2}k\theta_2}$$
$$(=+\tfrac{5}{8} \qquad \text{for } P_3(\mathbb{R})).$$

(See section 10 for additional cases with physical applications.)

8.3. Chern–Simons invariants and secondary characteristic classes

In our treatment of characteristic classes in section 6, we introduced the expression

$$Q(\omega', \omega) = r \int_0^1 P(\omega' - \omega, \Omega_t, \ldots, \Omega_t)\, dt$$

derived from an invariant polynomial $P(\Omega)$ of degree r with

$$\Omega_t = d\omega_t + \omega_t \wedge \omega_t.$$
$$\omega_t = t\omega' + (1-t)\omega.$$

The exterior derivative of Q was just the difference of the two invariant polynomials,

$$dQ = P(\Omega') - P(\Omega).$$

If M has no boundary, the integral of dQ vanishes. However, if $\partial M \neq \emptyset$, then by Stokes' theorem,

$$\int_M dQ = \int_{\partial M} Q$$

is not necessarily zero. In this case the forms $Q(\omega', \omega)$ are characteristic classes in their own right and are of independent interest (Chern [1972]; Chern and Simons [1974]).

Yang–Mills surface terms: The Chern–Simons formulas are equally valid for Riemannian connections and for Yang–Mills connections on a principal bundle. In the Yang–Mills case, if we set

$$P(F) = \mathrm{Tr}(F \wedge F)$$
$$F = dA + A \wedge A$$
$$A' = 0,$$

we find

$$Q(A, 0) = \mathrm{Tr}(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A).$$

Thus the familiar physicists' formula

$$\mathrm{Tr}\, F_{\mu\nu} \tilde{F}_{\mu\nu} = \partial_\mu J_\mu$$

where

$$J_\mu = 2\epsilon_{\mu\alpha\beta\gamma}\, \mathrm{Tr}(A_\alpha \partial_\beta A_\gamma + \tfrac{2}{3} A_\alpha A_\beta A_\gamma)$$

is simply a special case of the Chern–Simons formula.

Other cases of the formula appear in discussions of Yang–Mills "surface terms" (see, e.g., Gervais, Sakita and Wadia [1975]). Choosing $A' \neq 0$ in the Chern–Simons formula for $\mathrm{Tr}(F \wedge F)$ and setting

$\alpha = A - A'$, we find

$$Q(A, A') = \mathrm{Tr}(2\alpha \wedge F - \alpha \wedge \mathrm{d}\alpha - 2\alpha \wedge A \wedge \alpha + \tfrac{2}{3}\alpha \wedge \alpha \wedge \alpha).$$

Second fundamental form: Now let us consider the Levi–Civita connection one-form ω on M following from a metric which is not a product metric on ∂M. Then we choose a product metric on M which agrees with the original metric on ∂M; the connection one-form ω_0 of this metric will have only tangential components on ∂M. The *second fundamental form*

$$\theta = \omega - \omega_0$$

is a matrix of one-forms which is covariant under changes of frame and has only *normal* components on ∂M. As usual, we take

$$\omega_t = t\omega + (1-t)\omega_0, \qquad R_t = \mathrm{d}\omega_t + \omega_t \wedge \omega_t,$$

and observe that

$$\theta = \mathrm{d}\omega_t/\mathrm{d}t.$$

In four dimensions with $P = \mathrm{Tr}(R \wedge R)$, we find

$$\mathrm{Tr}(R \wedge R) = \mathrm{d}Q(\omega, \omega_0),$$

where

$$Q(\omega, \omega_0) = 2\int_0^1 \mathrm{Tr}(\theta \wedge R_t)\,\mathrm{d}t$$

$$= \mathrm{Tr}(2\theta \wedge R + \tfrac{2}{3}\theta \wedge \theta \wedge \theta - 2\theta \wedge \omega \wedge \theta - \theta \wedge \mathrm{d}\theta),$$

and we note that $\mathrm{Tr}(R_0 \wedge R_0) = 0$ for a product metric. The formula for Q simplifies considerably at the boundary, where the non-zero components of the matrix θ_{ab} are the normal components of the connection ω_{ab},

$$\theta_{0i} = \omega_{0i}, \qquad \theta_{23} = \theta_{31} = \theta_{12} = 0.$$

Using $R = \mathrm{d}\omega + \omega \wedge \omega$, we find after some algebra that

$$Q(\omega, \omega_0)|_{\text{boundary}} = 2\omega_{0i} \wedge R_{i0} = \mathrm{Tr}(\theta \wedge R).$$

Surface corrections to the index theorem: We now use the Chern–Simons formula to correct the Atiyah–Patodi–Singer index theorem for the case where the metric is not a product metric on ∂M (for a treatment of the signature complex, see Gilkey [1975]). Suppose the standard index theorem integral over curvature can be written in terms of an invariant polynomial $P(R)$ as

$$V[M] = c \int_M P(R)$$

for some constant c. Then the surface correction is

$$S[\partial M] = -c \int_{\partial M} Q(\omega, \omega_0).$$

The correction may be understood intuitively by noting that

$$V[M] + S[\partial M] = c \int_M (P(R) - \mathrm{d}Q(\omega, \omega_0)) \tag{8.4}$$

is *effectively* the integral over $cP(R_0)$. But since M may not admit a product metric with curvature R_0 away from ∂M, $P(R_0)$ cannot always be integrated over M. The surface correction circumvents this difficulty.

Locally flat bundles: The Chern–Simons invariants appear in place of ordinary characteristic classes in a variety of problems involving odd-dimensional manifolds. One interesting case is the study of the holonomy of locally flat bundles; this problem is closely related to the Bohm–Aharonov effect in a region free of electromagnetic fields.

As a simple example, let us take a connection

$$\omega = -\mathrm{i}q\, \mathrm{d}\theta$$

on a bundle $E = S^1 \times \mathbb{C}$, where $0 \leq \theta < 2\pi$ are coordinates on the base space S^1. Then we choose sections

$$s(\theta) = \mathrm{e}^{\mathrm{i}q\theta}$$

such that $s(\theta)$ is parallel-transported, $\nabla s = 0$. As θ ranges from 0 to 2π, we find a holonomy or phase shift $\mathrm{e}^{2\pi \mathrm{i}q}$ resulting from the traversal of a circuit around the base space S^1. The secondary characteristic class corresponding to the first Chern class $c_1 = (\mathrm{i}/2\pi)\,\mathrm{Tr}\, \Omega$ is

$$Q(\omega, 0) = \frac{\mathrm{i}}{2\pi} \int_0^1 \omega \, \mathrm{d}t = \frac{q}{2\pi} \mathrm{d}\theta.$$

The Chern–Simons invariant is interpretable as a charge:

$$\int_{S^1} Q(\omega, 0) = q.$$

Another example is provided by taking the flat connection on the line bundle $E = S^2 \times \mathbb{C}$ and using the induced connection on the $P_3(\mathbb{R})$ line bundle $\tilde{E}$ obtained by identifying the points (x, z) with $(-x, -z)$ in E. If γ is a path traversing half a great circle in S^3, it is a closed loop in $P_3(\mathbb{R})$ which represents the non-zero element of $\pi_1(P_3(\mathbb{R})) = \mathbb{Z}_2$. A phase factor of -1 is obtained by integrating the secondary characteristic class over γ.

8.4. *Index theorems for the classical elliptic complexes*

Here we briefly summarize the results of the Atiyah–Patodi–Singer index theorem for the classical elliptic complexes in four dimensions.

de Rham complex. Let $R^a{}_b$ be the curvature 2-form and $\theta^a{}_b = \omega^a{}_b - (\omega_0)^a{}_b$ the second fundamental form. Then the index theorem for the de Rham complex is (see Chern [1945]),

$$\chi(M) = \frac{1}{32\pi^2} \int_M \epsilon_{abcd} R^a{}_b \wedge R^c{}_d - \frac{1}{32\pi^2} \int_{\partial M} \epsilon_{abcd} (2\theta^a{}_b \wedge R^c{}_d - \tfrac{4}{3}\theta^a{}_b \wedge \theta^c{}_e \wedge \theta^e{}_d). \tag{8.5}$$

Signature complex. For the Hirzebruch signature complex, we find the index theorem

$$\tau(M) = -\frac{1}{24\pi^2} \int_M \mathrm{Tr}(R \wedge R) + \frac{1}{24\pi^2} \int_{\partial M} \mathrm{Tr}(\theta \wedge R) - \eta_S(\partial M). \tag{8.6}$$

Dolbeault complex. The index theorem for the Dolbeault complex with boundary involves additional subtleties which we will not discuss here. See Donnelly [1977] for further details.

Spin complex. The index theorem for the spin complex takes the form

$$\mathrm{index}(\Delta_\pm, D) = \frac{1}{24 \cdot 8\pi^2} \int_M \mathrm{Tr}(R \wedge R) - \frac{1}{24 \cdot 8\pi^2} \int_{\partial M} \mathrm{Tr}(\theta \wedge R) - \tfrac{1}{2}[\eta_{\mathrm{Dirac}}(\partial M) + h(\partial M)]. \tag{8.7}$$

Explicit examples are worked out at the end of this subsection.

Twisted spin complex. The treatment of twisted complexes over manifolds with boundary is straightforward in principle. We work out the index formulas for the twisted spin complex as an illustration. One first chooses a connection and a combined Dirac–Yang–Mills operator D_V on the twisted complex $\Delta_\pm \otimes V$. The index is the difference in the number of positive and negative chirality spinors in the kernel of D_V *obeying the Atiyah–Patodi–Singer boundary conditions.* (Recall that these are *nonlocal* boundary conditions and thus may not correspond to those which one might be tempted to use from physical considerations.) We write

$$\mathrm{index}(\Delta_\pm \otimes V, \partial M) = \nu_+(\partial M) - \nu_-(\partial M).$$

The twisted η-invariant $\eta(\Delta_\pm \otimes V, \partial M)$ must be computed from the appropriate spectrum $\{\lambda_i\}$ of D_V restricted to ∂M; computing η could in general be quite difficult. If the given metric is not a product metric on the boundary, we choose the desired second fundamental form and add the Chern–Simons correction to the tangent bundle curvature term; no analogous correction is required for the vector bundle piece. Hence for $n = 4$, we find the index theorem

$$\begin{aligned}\operatorname{index}(\Delta_\pm \otimes V, \partial M) &\equiv \nu_+(\partial M) - \nu_-(\partial M)\\ &= \frac{\dim V}{24 \cdot 8\pi^2}\left[\int_M \operatorname{Tr}(R \wedge R) - \int_{\partial M} \operatorname{Tr}(\theta \wedge R)\right] \\ &\quad - \frac{1}{8\pi^2}\int_M \operatorname{Tr}(F \wedge F) - \frac{1}{2}[\eta_{D_V}(\Delta_\pm \otimes V, \partial M) + h_{D_V}(\Delta_\pm \otimes V, \partial M)].\end{aligned} \tag{8.8}$$

Examples 8.4

1. *Self-dual Taub–NUT metric* (Eguchi, Gilkey and Hanson [1978]). Consider the metric

$$ds^2 = \frac{r+m}{r-m}\,dr^2 + (r^2 - m^2)\left[\sigma_x{}^2 + \sigma_y{}^2 + \left(\frac{2m}{r+m}\right)^2 \sigma_z{}^2\right]$$

and the product metric

$$ds_0{}^2 = \frac{r_0+m}{r_0-m}\,dr^2 + (r_0{}^2 - m^2)\left[\sigma_x{}^2 + \sigma_y{}^2 + \left(\frac{2m}{r_0+m}\right)^2 \sigma_z{}^2\right].$$

The connections are

$$\omega_{01} = -\frac{r}{r+m}\sigma_x, \qquad \omega_{02} = -\frac{r}{r+m}\sigma_y, \qquad \omega_{03} = -\frac{2m^2}{(r+m)^2}\sigma_z$$

$$\omega_{23} = -\frac{m}{r+m}\sigma_x, \qquad \omega_{31} = -\frac{m}{r+m}\sigma_y, \qquad \omega_{12} = \left(\frac{2m^2}{(r+m)^2} - 1\right)\sigma_z$$

and

$$(\omega_0)_{0i} = 0, \qquad (\omega_0)_{12} = \left(\frac{2m^2}{(r_0+m)^2} - 1\right)\sigma_z$$

$$(\omega_0)_{23} = -\frac{m}{r_0+m}\sigma_x, \qquad (\omega_0)_{31} = -\frac{m}{r_0+m}\sigma_y.$$

Hence the second fundamental form at the boundary $r = r_0$ is

$$\theta_{01} = -\frac{r_0}{r_0+m}\sigma_x, \qquad \theta_{02} = -\frac{r_0}{r_0+m}\sigma_y, \qquad \theta_{03} = -\frac{2m^2}{(r_0+m)^2}\sigma_z$$

$$\theta_{23} = \theta_{31} = \theta_{12} = 0.$$

Then the Dirac index is

$$\operatorname{index}(\text{Dirac}, r_0) = \frac{1}{24\cdot 8\pi^2}\left(\int_{M(r_0)} \operatorname{Tr} R\wedge R - \int_{S^3 \text{ at } r_0} \operatorname{Tr}\theta\wedge R\right) - \frac{1}{12}\left[1 - 2\left(\frac{2m}{r_0+m}\right)^2 + \left(\frac{2m}{r_0+m}\right)^4\right],$$

where we used Hitchin's formula [1974] for the η-invariant. Performing the integrals (the r-integration is from m to r_0), we find

$$\begin{aligned}\operatorname{index}(\text{Dirac}, r_0) &= \left[\frac{4m^3(m-2r_0)}{3(r_0+m)^4} - \left(-\frac{1}{12}\right)\right] - \frac{2m^2(r_0-m)^2}{3(r_0+m)^4} - \frac{1}{12}\left[1 - \frac{8m^2}{(r_0+m)^2} + \frac{16m^4}{(r_0+m)^4}\right]\\ &= 0.\end{aligned}$$

Thus the Atiyah–Patodi–Singer index theorem states that there is *no asymmetry* between positive and negative chirality Dirac spinors obeying the appropriate boundary conditions.

2. *Index theorems for the metric of Eguchi and Hanson* (Atiyah [1978]; Hanson and Römer [1978]). We take the metric treated in example 3.3.3,

$$ds^2 = \frac{dr^2}{(1-(a/r)^4)} + r^2(\sigma_x{}^2 + \sigma_y{}^2 + (1-(a/r)^4)\sigma_z{}^2),$$

where σ_x, σ_y, σ_z range over $P_3(\mathbb{R})$, and choose the product metric at $r = r_0$ to be

$$ds_0{}^2 = \frac{dr^2}{(1-(a/r_0)^4)} + r_0{}^2(\sigma_x{}^2 + \sigma_y{}^2 + (1-(a/r_0)^4)\sigma_z{}^2).$$

The second fundamental form $\theta = \omega - \omega_0$ at the boundary $r = r_0$ is then

$$\theta_{01} = -(1-(a/r_0)^4)^{1/2}\sigma_x, \qquad \theta_{02} = -(1-(a/r_0)^4)^{1/2}\sigma_y, \qquad \theta_{03} = -(1+(a/r_0)^4)\sigma_z$$

$$\theta_{12} = \theta_{23} = \theta_{31} = 0.$$

We choose the orientation $dr\wedge\sigma_x\wedge\sigma_y\wedge\sigma_z$ to be positive.

Integrating the appropriate forms for the Euler characteristic over the manifold M and its boundary $P_3(\mathbb{R})$ with $r_0\to\infty$, we find both a 4-volume term and a boundary correction,

$$\chi(M) = \tfrac{3}{2} - (-\tfrac{1}{2}) = 2.$$

The integral of the first Pontrjagin class for this metric is

$$P_1[M] = -\frac{1}{8\pi^2}\int_M \operatorname{Tr}(R\wedge R) = -3,$$

while the Chern–Simons boundary correction vanishes,

$$-Q_1[\partial M = P_3(\mathbf{R})] = \frac{1}{8\pi^2} \int_{P_3(\mathbf{R})} \mathrm{Tr}(\theta \wedge R) = 0.$$

The signature complex η-invariant correction for the $P_3(\mathbf{R})$ boundary is

$$\xi_S = \frac{1}{2} \cot^2 \frac{\pi}{2} = 0,$$

so the signature is

$$\tau(M) = \tfrac{1}{3} P_1 + \xi_S = -1.$$

The index of the spin $\frac{1}{2}$ Dirac operator is

$$I_{1/2} = \mathrm{index}(\mathrm{Dirac}, \partial M) = -\tfrac{1}{24} P_1 + \xi_{\mathrm{Dirac}}.$$

For $P_3(\mathbf{R})$, ξ_{Dirac} is $\frac{1}{2}$ the G-index,

$$\xi_{\mathrm{Dirac}} = \frac{1}{2}\left(\frac{\mathrm{i}}{2 \sin(\pi/2)} \times \frac{\mathrm{i}}{2 \sin(\pi/2)}\right) = -\frac{1}{8}.$$

Thus there is no asymmetry between positive and negative chirality Dirac spinors,

$$I_{1/2} = -\tfrac{1}{24}(-3) - \tfrac{1}{8} = 0.$$

The spin $\frac{3}{2}$ Rarita–Schwinger operator index theorem reads

$$I_{3/2} = \mathrm{index}(\text{Rarita–Schwinger}, \partial M) = \tfrac{21}{24} P_1 + \xi_{\mathrm{RS}}$$

where

$$\xi_{\mathrm{RS}} = -\frac{1}{2} \frac{(2\cos\theta_1 + 2\cos\theta_2 - 1)}{(2\sin\frac{1}{2}\theta_1)(2\sin\frac{1}{2}\theta_2)}.$$

For $P_3(\mathbf{R})$ boundaries ($\theta_1 = \theta_2 = \pi$), we have

$$I_{3/2} = \tfrac{21}{24}(-3) + \tfrac{5}{8} = -2.$$

Hence

$$I_{3/2} = 2\tau$$

and there does exist an asymmetry between positive and negative chirality Rarita–Schwinger spinors for this metric.

9. Differential geometry and Yang–Mills theory

In this section, we first give a brief introduction to the path-integral method for quantizing Yang–Mills theories and then describe some of the Yang–Mills instanton solutions. The last part of the section contains a list of mathematical results concerning Yang–Mills theories whose detailed treatment is beyond the scope of this article.

9.1. Path-integral approach to Yang–Mills theory

The most useful approach to the quantization of gauge theories appears to be Feynman's path integral method. From a geometric point of view, the path integral has the advantage of being able to take the global topology of the gauge potentials into account, while the canonical perturbation theory approach to quantization is sensitive only to the local topology.

At present, a mathematically precise theory of path integration can be formulated only for spacetimes with positive signatures $(+,+,+,+)$; we refer to such spacetimes as "Euclidean" or "imaginary time" manifolds. Physically meaningful answers are obtainable by continuing the results of the Euclidean path integration back to the Minkowski regime with signature $(-,+,+,+)$.

In the Euclidean path-integral approach to quantization, each field configuration $\varphi(x)$ is weighted by the "Boltzmann factor", i.e., the exponential of minus its Euclidean action $S[\varphi]$:

$$(\text{contribution of } \phi(x)) = \exp(-S[\phi]).$$

For Yang–Mills theories, the Euclidean action is

$$S[A] = +\frac{1}{4}\int_M F^a_{\mu\nu} F^a_{\mu\nu}\, g^{1/2}\, \mathrm{d}^4x = -\frac{1}{2}\int_M \operatorname{Tr} F \wedge *F, \tag{9.1}$$

which is positive definite. The contribution of each gauge potential or connection $A_\mu(x)$ to the path integral is therefore bounded and well-behaved.

The complete generating functional for the transition amplitudes of a theory is obtained by summing (or functionally integrating) over all inequivalent field configurations. Since the first-order functional variation of the action vanishes for solutions of the equations of motion, these configurations correspond to stationary points in the functional space. Therefore, in the path-integral approach, we first seek solutions to the Euclidean field equations with minimum action and then compute quantum-mechanical fluctuations around them.

The Yang–Mills field equations found by varying the action may be written as

$$\mathrm{d}*F + A \wedge *F - *F \wedge A = 0,$$

while the Bianchi identities are

$$\mathrm{d}F + A \wedge F - F \wedge A = 0.$$

These two equations together imply that the curvature F is *harmonic* in a suitable sense.

Minima of the action: In order to find the minimum action configurations of the Yang–Mills theory, let us consider the inequality

$$\int_M (F^a_{\mu\nu} \pm {}^*F^a_{\mu\nu})^2 g^{1/2}\, \mathrm{d}^4x \geq 0.$$

This bound is saturated by the self-dual field configurations

$$F = \pm * F. \tag{9.2}$$

In fact, these field configurations solve the Yang–Mills field equations since the Bianchi identities imply the field equations. The action now becomes

$$S = -\tfrac{1}{2}\int \mathrm{Tr}\, F \wedge *F = \mp\tfrac{1}{2}\int \mathrm{Tr}\, F \wedge F = 4\pi|k|,$$

where

$$-C_2 = k = -\frac{1}{8\pi}\int_M \mathrm{Tr}\, F \wedge F \tag{9.3}$$

is the integral of the 2nd Chern class. 't Hooft [1976a] called such special field configurations "instantons" since in the case $|k| = 1$ their field strength is centered around some point in space-time and thus attains its maximum value at some "instant of time".

Physical interpretation of instantons: The instanton can be interpreted as a quantum-mechanical tunneling phenomenon in Yang–Mills gauge theories. It induces a transition between homotopically inequivalent vacua. The true ground state of Yang–Mills theory then becomes a coherent mixture of all these vacuum states. For more details on this subject, see, for example, Jackiw [1977]. One-loop quantum-mechanical fluctuations about the instanton have been explicitly calculated by 't Hooft [1977], who showed that the instanton solved the long-standing $U(1)$ problem via its coupling to the anomaly of the ninth axial current.

9.2. *Yang–Mills instantons*

The dominant contribution to the Euclidean path integral comes from the instanton solutions obeying the self-duality condition

$$F = \pm * F.$$

All gauge-potentials or connections satisfying the Yang–Mills equations with self-dual curvature are now, in principle, known (see section 9.3).

1. *BPST solution* (Belavin et al. [1975]) [see examples 4.3.3. and 5.5.2]. The instanton of Belavin, Polyakov, Schwarz and Tyupkin solves the Yang–Mills equations with $k = \pm 1$. Although the spacetime of the solution appears to be $\mathbb{R}^4$, the boundary conditions at ∞ allow the space to be compactified to S^4. Hence the BPST instanton is a connection with self-dual curvature on an SU(2) principal bundle over S^4 with second Chern number $C_2 = -1$. Since the action of the BPST instanton is $S = 4\pi$, it has the least action possible for a nontrivial topology and thus is the most important solution in Yang–Mills theory. We note that the BPST instanton is, in fact, a connection on the Hopf fibering π: $S^7 \to S^4$ (Trautman [1977]) and for this reason can be obtained from self-dual combinations of the standard Riemannian connections on S^4 (see example 5.5.2).

2. *Multi-center* SU(2) *solutions*. A special class of self-dual solutions of the SU(2) Yang–Mills equations for arbitrary "instanton-number" k is obtained by the following simple ansatz ('t Hooft [1976b]; Wilczek [1976]; Corrigan and Fairlie [1977]),

$$A^a_\mu = -\bar{\eta}^a_{\mu\nu}\,\partial_\nu \ln \phi\,, \tag{9.4}$$

where the constants $\eta^a_{\mu\nu}$ and $\bar{\eta}^a_{\mu\nu}$ are given in appendix C. Imposing the self-duality condition, one obtains

$$\Box\phi/\phi = 0.$$

't Hooft gave the following solution to this equation,

$$\phi(x) = 1 + \sum_{i=1}^{k} \frac{\rho_i}{(x - x_i)^2}\,.$$

x_i and ρ_i are interpreted as the position and the size of the ith instanton and the solution describes the k-instanton configuration. The k-anti-instanton solution is obtained by replacing $\bar{\eta}$ by η.

This class of solutions was further generalized by Jackiw, Nohl and Rebbi [1977] who noticed that the 't Hooft solution is not invariant under conformal transformations and can, in fact, be generalized as

$$\phi(x) = \sum_{i=1}^{k+1} \frac{\lambda_i}{(x - y_i)^2}\,.$$

This solution again describes a k-instanton configuration and possesses $5k + 4$ parameters (overall scale is irrelevant). Here, however, the parameters λ_i and y_i are not directly related to the size and location of the ith instanton. In the special cases of $k = 1$ and 2, the solution possesses 5 and 13 parameters, respectively, when one excludes parameters associated with gauge transformations.

9.3. Mathematical results concerning Yang–Mills theories

There exist a variety of mathematical results concerning Yang–Mills theories and differential geometry whose detailed treatment is beyond the scope of this work. We present here a list of assorted mathematical facts which we feel might be of relevance to physics.

1. *Parameter space for instanton solutions.* Schwarz [1977] and Atiyah, Hitchin and Singer [1977] have applied the index theorem to an elliptic complex corresponding to the Yang–Mills equations. This complex allows one to analyze small self-dual fluctuations around the instanton solution. Determination of the index of the complex then allows one to compute the *number of possible free parameters* in an instanton solution. They found that for the k-instanton SU(2) solution,

$$\text{no. of free parameters} = 8k - 3,$$

in agreement with the results of Jackiw and Rebbi [1977] and Brown, Carlitz and Lee [1977] who used physicists' methods. Thus the Jackiw–Nohl–Rebbi solution exhausts the number of available parameters only for $k = 1$ and $k = 2$.

The analysis of small self-dual oscillations around instanton solutions was then extended to include all Lie groups (Atiyah, Hitchin and Singer [1978]; Bernard, Christ, Guth and Weinberg [1977]). The dimension of the space of parameters for irreducible self-dual connections on principal G-bundles over S^4 with $C_2 = -k$ is given in table 9.1 for each G. We also list restrictions on k which must hold if there are to exist irreducible connections which are not obtained by embedding the connection of a smaller group.

Table 9.1

Group	Dimension of parameter space	Irreducibility condition
SU(n)	$4nk - n^2 + 1$	$k \geq n/2$
Spin(n)	$4(n-2)k - n(n-1)/2$	$k \geq n/4$ ($n \geq 7$)
Sp(n)	$4(n+1)k - n(2n+1)$	$k \geq n$
G_2	$16k - 14$	$k \geq 2$
F_4	$36k - 52$	$k \geq 3$
E_6	$48k - 78$	$k \geq 3$
E_7	$72k - 133$	$k \geq 3$
E_8	$120k - 248$	$k \geq 3$

Thus, for example, SU(3) solutions have $12k - 8$ parameters and for $k \geq 2$ there exist irreducible SU(3) solutions which are not obtained from SU(2) solutions.

We remark that physicists often refer to the dimension of the parameter space as the number of *zero-frequency modes*, while mathematicians may refer to the same thing as the *dimension of the moduli space*.

2. *Explicit solutions for the most general self-dual connections.* The $(5k+4)$-parameter Jackiw–Nohl–Rebbi solutions for SU(2) instantons do not exhaust the $(8k-3)$-dimensional parameter space for $k \geq 3$. The problem of finding the most general solutions (e.g., with $8k - 3$ parameters for SU(2)) was attacked using twistor theory (Ward [1977]; Atiyah and Ward [1977]), and the method of universal connections and algebraic geometry (Atiyah, Hitchin, Drinfeld and Manin [1978]). It was shown that the problem of determining the most general self-dual connection for virtually any principal bundle over S^4 is reducible to a problem in algebraic geometry concerning holomorphic vector bundles over $P_3(\mathbb{C})$.

In fact, the whole procedure can be reduced to ordinary linear algebra. For example, to calculate the self-dual SU(2) connection for the bundle with Chern class $C_2 = -k$ one starts with a $(k+1)\times k$

dimensional quarternion-valued matrix

$$\Delta = a + bx.$$

(Physicists may prefer to think of a_{ij}, b_{ij} and x as having values in SU(2), so $x = x^0 - \mathrm{i}\, \boldsymbol{\lambda} \cdot \boldsymbol{x}$ etc., where $\{\boldsymbol{\lambda}\}$ are the Pauli matrices.)

Then one determines the universal connection $\omega = V^{+} \mathrm{d}V$ by solving the equations

$$\begin{aligned} &V^{+}\Delta = 0 \\ &V^{+}V = 1 \\ &1 = VV^{+} + \Delta \frac{1}{\Delta^{+}\Delta} \Delta^{+} \\ &\Delta^{+}\Delta = \text{a real number} \end{aligned} \tag{9.5}$$

for V. The number of free parameters in $V^{+} \mathrm{d}V$ which are not gauge degrees of freedom turns out to be exactly the required number. There are deep reasons, based on algebraic geometry, for the success of this construction (see e.g. Hartshorne [1978]). Propagators in these instanton fields were obtained by Christ, Weinberg and Stanton [1978] and Corrigan, Fairlie, Templeton and Goddard [1978] which generalized the result of Brown, Carlitz, Creamer and Lee [1977] for propagators in the 't Hooft, Jackiw–Nohl–Rebbi solution. We refer the reader to the original literature for further details.

3. *Universal connections* (Narasimhan and Ramanan [1961, 1963]; Dubois-Violette and Georgelin [1979]). In the derivation of the most general self-dual connections, the method of universal connections played an essential role. The theorem of Narasimhan and Ramanan shows that all fiber bundles with a given set of characteristic classes are viewable as particular projections of a more general bundle called a "universal classifying space". Typical classifying spaces are Grassmannian manifolds $\mathrm{Gr}(m, k)$, the space of all k-manifolds embedded in m-space, with m usually taken to approach infinity. Both the base manifold *and* the fiber of a given fiber bundle are included in the classifying space; complicated projections must be taken to describe bundles with complicated base manifolds.

One can write any connection on a fiber bundle in terms of a projection down from a universal connection on the classifying space. In particular, for sufficiently large m, the connection on a $U(k)$ principal bundle can always be written in terms of an $m \times k$ complex matrix V as

$$\omega = V^{+} \mathrm{d}V$$

where

$$V^{+}V = 1_k, \qquad VV^{+} = P(x) = (m \times m \text{ projection}).$$

Choosing a local cross-section $V(x)$ of the classifying space gives the Yang–Mills potential in a certain gauge,

$$A(x) = V^{+}(x)\, \mathrm{d}V(x).$$

$A(x)$ is *not* a pure gauge here because V is not a $k \times k$ matrix. The curvature

$$\begin{aligned} F &= \mathrm{d}A + A \wedge A \\ &= \mathrm{d}V^{+}(1 - P(x))\,\mathrm{d}V \end{aligned}$$

is, in general, non-trivial. Gauge transformations are obviously effected by multiplying V on the right by a $k \times k$ matrix $\Lambda(x)$,

$$V(x) \to V(x)\Lambda(x)$$

so that

$$\begin{aligned} A' &= (\Lambda^{+}V^{+})\,\mathrm{d}(V\Lambda) = \Lambda^{+}(V^{+}\mathrm{d}V)\Lambda + \Lambda^{+}(V^{+}V)\,\mathrm{d}\Lambda \\ &= \Lambda^{+}A\Lambda + \Lambda^{+}1_k\,\mathrm{d}\Lambda. \end{aligned}$$

The covariant derivative has a straightforward interpretation in terms of the action of the projection $P(x) = VV^{+}$ on the m-dimensional extension of the k-dimensional wave function Ψ,

$$\bar{\Psi} = V\Psi.$$

When one projects the exterior derivative of $\bar{\Psi}$, one finds the extension of the covariant derivative of the ordinary wave function Ψ:

$$P\,\mathrm{d}\bar{\Psi} = P\,\mathrm{d}V\Psi + PV\,\mathrm{d}\Psi = V(V^{+}\mathrm{d}V\Psi + \mathrm{d}\Psi) \equiv V\,\mathrm{D}\Psi.$$

4. *Compactifiability of finite-action Yang–Mills connections* (Uhlenbeck [1978]). Suppose $A(x)$ is a section of a connection one-form on a manifold M which is a compact manifold M lacking the origin, i.e.,

$$\hat{M} = M - \{0\}.$$

Suppose also that $F = \mathrm{d}A + A \wedge A$ is harmonic and that the Yang–Mills action is finite.

Then there exist gauge transformations near $\{0\}$ which extend A to all M. In fact, it has been shown that all Euclidean finite-action Yang–Mills solutions over $M - \{0\}$ are smoothly extended to the *compact* manifold M.

This theorem tells us that any self-dual finite-action solution to the Euclidean Yang–Mills equations must describe a bundle with a compactified spacetime base manifold.

5. *Stability of all self-dual solutions* (Bourguignon, Lawson and Simons [1979]). The stability of Yang–Mills solutions has also been studied. One can show that if the base manifold M is S^4, all stable Yang–Mills solutions are *self-dual*. Combined with Uhlenbeck's theorem given above, this theorem allows us to conclude that *all finite-action stable* Yang–Mills solutions (connections with harmonic curvatures) are self-dual.

6. *Index theorems in open spaces* (Callias [1978]; Bott and Seeley [1978]). An extension of the index theorem to Yang–Mills theories in open Euclidean spaces of odd-dimension d has been given by

Callias. This result has interesting applications to the Dirac equation in $(d+1)$-dimensional Minkowski spacetime.

7. *Meron solutions.* Besides the instantons, which are non-singular solutions to the Euclidean Yang–Mills field equations, there is a class of singular solutions called merons (Callan, Dashen and Gross [1977]) which were first discovered by De Alfaro, Fubini and Furlan [1976]. As compared with instantons whose topological charge density $\epsilon_{\mu\nu\rho\sigma}F_{\mu\nu}(x)F_{\rho\sigma}(x)$ is a smooth function of x, the topological charge density of merons vanishes everywhere except at the singular points.

For instance, the SU(2) 2-meron solution is given by

$$A = \tfrac{1}{2}g_1^{-1}\,\mathrm{d}g_1 + \tfrac{1}{2}g_2^{-1}\,\mathrm{d}g_2,$$

where

$$g_i = \frac{(t-t_i) - \mathrm{i}\boldsymbol{\lambda}\cdot(\boldsymbol{x}-\boldsymbol{x}_i)}{[(t-t_i)^2 + (\boldsymbol{x}-\boldsymbol{x}_i)^2]^{1/2}}\,.$$

The topological charge density of this solution is a sum of two δ-functions centered at x_1 and x_2, each of which gives $\frac{1}{2}$ unit of the quantized topological charge. Therefore, in some sense, the meron is a split instanton.

Glimm and Jaffee [1978] considered an axially-symmetric multimeron configuration and the existence of a solution for this configuration was proved by Jonsson, McBryan, Zirilli and Hubbard [1979].

8. *Absence of global gauge conditions in functional space of connections* (Singer [1978a]). The Feynman path-integral approach to the quantization of field theories is based on the use of the functional space of the field variables. In the case of Yang–Mills theories, the fields in question are the connections on the principal bundle, which are defined only up to gauge transformations. Hence the *functional space of connections* is a complicated infinite-dimensional fiber bundle whose projection carries all gauge-equivalent connections into the same point in the base space or *moduli space* of the bundle.

Physical quantities are calculated by integrating over the moduli space to avoid the meaningless infinities which would result from integrating over gauge-equivalent connections. Gribov [1977, 1978] discovered that there exist gauge-equivalent connections which obey the Coulomb gauge condition, so that defining functional integration over the moduli space could be potentially troublesome.

The mathematical nature of the problem of defining the moduli space of the functional space of connections was examined by Singer using techniques of global analysis. He has shown that for compact simply-connected spacetimes the infinite-dimensional bundle in question is nontrivial; hence a single global gauge condition could never be used to define a global section, and thus could not unambiguously define the moduli space. He showed that the manifold described by any given gauge condition eventually turned back on itself to intersect a given fiber of the functional bundle an infinite number of times. Thus the moduli space over which the path integration for gauge theories must be performed can be defined only in local patches.

9. *Natural metric on the functional space of connections and the Faddeev–Popov determinant* (Singer [1978b]; Babelon and Viallet [1979]). Before one can integrate over a functional space, one must know the *measure* of the integration element. To get the proper transformation properties of the functional measure, physicists multiply the integrand by a factor called the Faddeev–Popov determinant. It is now known that this measure follows from a *natural metric* on the moduli space of the functional space of

connections. The Faddeev–Popov determinant arises naturally as the standard $g^{1/2}$ Jacobian multiplying the naive measure.

10. *Ray–Singer torsion and the functional integral* (Singer [1978c]; Schwarz [1978, 1979a, b]). Functional determinants obtained by calculating the quadratic fluctuations around instantons are essential elements of the quantized Yang–Mills theory. Thus it is interesting to note that these functional determinants are intimately related to a mathematical construction by Ray and Singer [1971, 1973] introduced many years ago. Additional insights into the functional integral in Yang–Mills theory might be gained by the exploration of the Ray–Singer analytic torsion.

10. Differential geometry and Einstein's theory of gravitation

The intimate relationship between Einstein's theory of gravity and Riemannian geometry has been thoroughly explored over the years. Here we will attempt to outline some of the more recent ideas concerning the physics of gravitation and the relevance of modern differential geometry to gravitation. We begin with an introduction to current work on quantum gravity and gravitational instantons. We then present a list of mathematical results which are of specific interest to the study of gravity.

10.1. Path integral approach to quantum gravity

Quantization of the theory of gravitation is one of the most outstanding problems in theoretical physics. Due to the non-polynomial character of the theory the standard methods of quantization do not work for gravity. At present, Feynman's path integral approach appears to be the most viable procedure for quantizing gravity. Path integration has the advantage of being able to take into account the global topology of the space-time manifold as opposed to other quantization schemes. However, since the theory of gravity is not renormalizable in the usual sense, we always encounter the difficulties of non-renormalizable divergences in practical calculations.

As in the Yang–Mills case, we work with the Euclidean version of the theory and the Euclidean (imaginary time) path integral. Our field variables $g_{\mu\nu}$ are metrics having a Riemannian signature $(+,+,+,+)$, and the (imaginary time) gravitational action is given by

$$S[g] = -\frac{1}{16\pi G}\int_M \mathcal{R} g^{1/2}\,\mathrm{d}^4x - \frac{1}{8\pi G}\int_{\partial M} K\,\mathrm{d}^3\Sigma + C \tag{10.1}$$

where G is Newton's constant, $\mathcal{R}$ is the Ricci scalar curvature and K is the trace of the second fundamental form of the boundary in the metric g. The second term is a surface correction required when ∂M is nonempty (York [1972]; Gibbons and Hawking [1977]). C is a (possibly infinite) constant chosen so that $S[g]=0$ when the metric $g_{\mu\nu}$ is the flat space metric. Einstein's field equations in empty space are given by

$$\mathcal{R}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\mathcal{R} = 0. \tag{10.2}$$

As in the Yang–Mills theory, there exist finite action solutions to the Euclidean Einstein equations which possess interesting global topological properties. We describe these solutions in the next subsection.

Non-positive-definiteness of the Einstein action: Unlike the Yang–Mills case, the gravitational action is linear in the curvature and not necessarily positive. In particular, by introducing a rapidly varying conformal factor into a metric, one can make its action negative and arbitrarily large. This causes a divergence in the path integration over the conformal factor. To get around this difficulty, Gibbons, Hawking and Perry [1978] proposed the following procedure for the evaluation of the path integral:

- separate the functional space of metrics into conformal equivalence classes;
- in each class, choose the metric g for which the Ricci scalar $\mathcal{R} = 0$;
- rotate the contour of integration of the conformal factor λ to be parallel to the pure imaginary axis in order to achieve the convergence of the integration. Namely, we put $\lambda = 1 + i\xi$ and integrate over real ξ;
- integrate over all conformal equivalence classes.

Positive action conjecture: For the metric in a given conformal equivalence class with $\mathcal{R} = 0$, the gravitational action consists entirely of the surface term. Since the physically reasonable boundary condition for the metric is asymptotic flatness, one would hope that the action is positive in this case. This leads to *the positive action conjecture* (Gibbons, Hawking and Perry [1978]):

$S \geq 0$ for all asymptotically Euclidean positive definite metrics with $\mathcal{R} = 0$.

Asymptotically Euclidean metrics are those which approach the flat metric in all spacetime directions at ∞ and whose global topology is the same as $\mathbb{R}^4$ at ∞. It can be shown that $S = 0$ only for the flat metric on $\mathbb{R}^4$ (Gibbons and Pope [1979]). The positive action conjecture has recently been proven by Schoen and Yau [1979a].

A natural modification of the positive action conjecture was suggested by the discovery of a new type of metric (Eguchi and Hanson [1978]) which is locally flat at ∞, but has a global topology different from that of $\mathbb{R}^4$ at ∞ (Belinskii, Gibbons, Page and Pope [1978]). This class of metrics is called asymptotically locally Euclidean (ALE). *The generalized positive action conjecture* (Gibbons and Pope [1979]) states that

$S \geq 0$ for any complete non-singular positive definite asymptotically locally Euclidean metric with $\mathcal{R} = 0$; $S = 0$ if and only if the curvature is self-dual.

Spacetime foam (Hawking [1978]; Perry [1979]; Hawking, Page and Pope [1979]): Since the theory of gravity is not renormalizable, one expects strong quantum fluctuations at short distances, i.e., at the size of the Planck length. These fluctuations might be viewed as a "spacetime foam" which is the basic building block of the universe. Thus the spacetime in quantized gravity theory is expected to be highly curved at small distances, while at large distances the curvature is expected to cancel and give an almost flat spacetime. Spacetime foam is an important subject for future research in quantized gravity.

10.2. Gravitational instantons

As in the Yang–Mills theory, there also exist finite action solutions to the classical field equations in the theory of gravitation. Such solutions are called *gravitational instantons* because of the close analogy to the Yang–Mills instantons. A variety of solutions of Einstein's equations with instanton-like properties have been discovered. Those with self-dual curvature are especially appealing because they have interesting mathematical properties and bear the strongest similarity to the self-dual Yang–Mills instantons. For a review, see Eguchi and Hanson [1979].

1. ***The metric of Eguchi and Hanson*** [1978] [see example 3.3.3]. This is the metric which most closely resembles the Yang–Mills instanton of Belavin et al. [1975]. It has a self-dual Riemannian curvature which falls off rapidly in all spacetime directions and has $\chi = 2$, $\tau = -1$. The boundary at ∞ is $P_3(\mathbb{R}) = S^3/\mathbb{Z}_2$ (Belinskii et al. [1978]), and thus it is the simplest example of an asymptotically locally Euclidean metric. The global manifold is $T^*(P_1(\mathbb{C}))$.

2. *Multi-center self-dual metrics* (Hawking [1977]; Gibbons and Hawking [1978]). This class of metrics is given by

$$ds^2 = V^{-1}(\boldsymbol{x})(\mathrm{d}\tau + \boldsymbol{\omega}\cdot \mathrm{d}\boldsymbol{x})^2 + V(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\cdot \mathrm{d}\boldsymbol{x},$$

where

$$\nabla V = \pm \nabla \times \boldsymbol{\omega}$$

$$V = \epsilon + 2m \sum_{i=1}^{k} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}_i|}.$$

The connection and the curvature are both self-dual in this coordinate system. The case $\epsilon = 1$, $k = 1$ is the self-dual Taub–NUT metric discussed in example 3.3.2, but in a different coordinate frame. When $\epsilon = 1$ for general k, we find the multi-Taub–NUT metric. These metrics approach a flat metric in the spatial direction $|\boldsymbol{x}| \to \infty$, but are periodic in the variable τ.

When $\epsilon = 0$ the asymptotic behavior of the metric changes completely and the metric $g_{\mu\nu}$ approaches the flat metric at 4-dimensional ∞ modulo the identification of points of spacetime under the action of a discrete group. The case $\epsilon = 0$, $k = 1$ turns out to be just a coordinate transformation of the flat space metric. When $\epsilon = 0$, $k = 2$ the metric is a coordinate transformation of the Eguchi–Hanson metric discussed above (Prasad [1979]). For general k, the metric represents a $(k-1)$-instanton configuration whose boundary at ∞ is the lens space $L(k, 1)$ of S^3. ($L(k, m)$ is defined by identifying the points of S^3 = [boundary of $\mathbb{C}^2$] related by the map

$$(z_1, z_2) \to (\mathrm{e}^{2\pi i/k} z_1, \mathrm{e}^{2\pi i m/k} z_2).)$$

The $\epsilon = 0$ general-k metric has $\chi = k$, $|\tau| = k - 1$. The possibility of self-dual metrics on manifolds whose boundaries are given by S^3 modulo other discrete groups has been considered by Hitchin [1979] and Calabi [1979] and will be discussed below.

3. *Fubini–Study metric on* $P_2(\mathbb{C})$ (Eguchi and Freund [1976]; Gibbons and Pope [1978]) [see example 3.4.3]. The manifold $P_2(\mathbb{C})$ is closed and compact without boundary and has $\chi = 3$, $\tau = 1$. Except for the fact that $P_2(\mathbb{C})$ fails to admit well-defined Dirac spinors, the Fubini–Study metric on $P_2(\mathbb{C})$ would be an appealing gravitational instanton; this metric satisfies Einstein's equations with nonzero cosmological constant and has a self-dual Weyl tensor, rather than a self-dual curvature.

4. $K3$ *surface*. The K3 surface is the *only* compact regular simply-connected manifold without boundary which admits a nontrivial metric with self-dual curvature (Yau [1977]). While the explicit form of the metric is not known, it must exist; since its curvature is self-dual it will solve Einstein's equations

with zero cosmological constant. For the K3 surface, $\chi = 24$ and $\tau = -16$. (*Remark*: The natural structure on the K3 surface is, precisely speaking, *anti*-self-dual (see Atiyah, Hitchin and Singer [1978]).)

5. *Miscellaneous solutions.* Among other interesting solutions are the Euclidean de Sitter space metric (i.e., the standard metric on S^4), the non-self-dual Taub–NUT metric with horizon and the compact rotating metric on $P_2(\mathbb{C}) \oplus P_2(\mathbb{C})$ found by Page [1978a,b], and the rotating Taub–NUT-like metric of Gibbons and Perry [1979].

10.3. Nuts and bolts

The gravitational instantons listed above can be described in terms of interesting mathematical structures called "nuts" and "bolts" by Gibbons and Hawking [1979]. Let us examine a general Bianchi type IX metric of the following form

$$ds^2 = d\tau^2 + a^2(\tau)\,\sigma_x^2 + b^2(\tau)\,\sigma_y^2 + c^2(\tau)\,\sigma_z^2.$$

The manifold described by this metric is regular provided the functions a, b and c are finite and nonsingular at finite proper distance τ. However, the manifold can be regular even in the presence of apparent singularities.

Let us, for simplicity, consider singularities occurring at $\tau = 0$. A metric has a removable *nut singularity* provided that near $\tau = 0$,

$$a^2 = b^2 = c^2 = \tau^2.$$

Then this apparent singularity is nothing but a coordinate singularity of the polar coordinate system in $\mathbb{R}^4$ centered at $\tau = 0$. The singularity is removed by changing to a local Cartesian coordinate system near $\tau = 0$ and adding the point $\tau = 0$ to the manifold. Nut singularities may also be understood from the viewpoint of global topology as fixed points of the Killing vector field; by the Lefschetz fixed point theorem (see section 7), each such fixed point (or nut) adds *one unit* to the Euler characteristic of the manifold.

A metric has a removable *bolt singularity* if near $\tau = 0$,

$$a^2 = b^2 = \text{finite}$$

$$c^2 = n^2\tau^2, \qquad n = \text{integer}.$$

Here $a^2 = b^2$ implies the canonical S^2 metric $\frac{1}{4}(d\theta^2 + \sin^2\theta\, d\phi^2)$ for the $(a^2\sigma_x{}^2 + b^2\sigma_y{}^2)$ part of the metric, while at constant (θ, ϕ), the $(d\tau^2 + c^2\sigma_z{}^2)$ part of the metric looks like

$$d\tau^2 + n^2\tau^2\tfrac{1}{4}\, d\psi^2.$$

Provided the range of ψ is adjusted so $n\psi/2$ runs from 0 to 2π, the apparent singularity at $\tau = 0$ is just a coordinate singularity of the polar coordinate system in $\mathbb{R}^2$ at the origin. This singularity can again be removed using Cartesian coordinates. The topology of the manifold is locally $\mathbb{R}^2 \times S^2$ and the $\mathbb{R}^2$ shrinks to a point on S^2 as $\tau \to 0$. This S^2 is a fixed surface of the Killing vector field. According to the G-index theorem (see section 7), each such fixed submanifold contributes its own Euler characteristic to the Euler characteristic of the entire manifold; thus each bolt contributes *two units* to the Euler characteristic.

The self-dual Taub–NUT metric (example 3.3.2)

$$ds^2 = \frac{1}{4}\frac{r+m}{r-m}\,dr^2 + \tfrac{1}{4}(r^2-m^2)(d\theta^2 + \sin^2\theta\, d\phi^2) + m^2\left(\frac{r-m}{r+m}\right)(d\phi + \cos\theta\, d\psi)^2$$

behaves at $r = m + \epsilon$ as

$$ds^2 \approx d\tau^2 + \tau^2(\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2),$$

where $\tau = (2m\epsilon)^{1/2}$. Thus the apparent singularity at $r = m$ is a removable nut singularity. In contrast, the Eguchi–Hanson metric (example 3.3.3),

$$ds^2 = \frac{dr^2}{1-(a/r)^4} + r^2(\sigma_x{}^2 + \sigma_y{}^2 + (1-(a/r)^4)\sigma_z{}^2),$$

behaves near $r = a$, with fixed θ and ϕ, as

$$ds^2 \approx \tfrac{1}{4}(du^2 + u^2\, d\psi^2),$$

where $u^2 = r^2[1-(a/r)^4]$. Therefore, the apparent singularity at $r = a$ is a removable bolt singularity provided that the range of ψ is chosen to be that of the usual polar coordinates on $\mathbb{R}^2$,

$$0 \le \psi < 2\pi.$$

This explains why the boundary of the manifold of this metric is $P_3(\mathbb{R}) = S^3/\mathbb{Z}_2$, rather than S^3, which would have $0 \le \psi < 4\pi$. Next, we examine the $P_2(\mathbb{C})$ metric (example 3.4.3)

$$ds^2 = \frac{dr^2 + r^2\sigma_z{}^2}{(1+\Lambda r^2/6)^2} + \frac{r^2(\sigma_x{}^2 + \sigma_y{}^2)}{1+\Lambda r^2/6}.$$

Near $r = 0$, we obviously have a nut. On the other hand, at large r and fixed θ and ϕ, the metric behaves as

$$ds^2 \approx (\Lambda/6)^{-2}(du^2 + \tfrac{1}{4}u^2\, d\psi^2),$$

where $u = 1/r$. Thus the singularity at $u = 0$ $(r \to \infty)$ is a removable bolt singularity if

$$0 \le \psi < 4\pi.$$

Finally, we note that the Gibbons–Hawking k-center metrics can be shown to have k nut singularities.

10.4. Mathematical results pertinent to gravitation

Because of the close relationship between Einstein's theory of gravitation and differential geometry, any distinction between physical knowledge about gravitation and mathematical knowledge is necessarily somewhat arbitrary. In this section we collect a variety of useful facts pertinent to gravitation which seem to us primarily mathematical in flavor.

1. *Restrictions on four-dimensional Einstein manifolds.* A number of mathematical results are known which restrict the types of four-dimensional Euclidean-signature Einstein manifolds; these are precisely the manifolds which might be expected to be important in the Euclidean path integral for gravity.

We first restrict our attention to compact simply-connected four-dimensional spin manifolds M, and note that the Euler characteristic χ and the signature τ *nearly* characterize the manifold uniquely (recall that $|\tau|$ is a multiple of 8 for a spin manifold):

Case A: $|\tau| \neq \chi - 2 \;\Rightarrow\;$ M determined up to homotopy

Case B: $|\tau| = \chi - 2 \;\Rightarrow\;$ *unknown* whether M is determined up to homotopy.

It is not known if these conditions determine M up to a homeomorphism type.

It is instructive to study a manifold's properties in terms of its Betti numbers $(b_0, b_1, b_2, b_3, b_4)$; b_2 can be broken up into two parts,

$$b_2 = b_2^+ + b_2^-,$$

where b_2^+ is the number of self-dual harmonic 2-forms and b_2^- is the number of anti-self-dual harmonic 2-forms. We know the following results:

(1) Poincaré duality for compact orientable manifolds implies $b_0 = b_4$, $b_1 = b_3$
(2) $b_0 = b_4 =$ number of disjoint pieces of M
(3) $b_1 = b_3 = 0$ if M is simply connected
(4) $\chi = b_0 - b_1 + b_2 - b_3 + b_4 = 2b_0 - 2b_1 + b_2^+ + b_2^-$
(5) $\tau = b_2^+ - b_2^-$.

Thus for M compact and simply-connected,

$$\chi = 2 - 0 + b_2^+ + b_2^-$$
$$b_2^+ = \tfrac{1}{2}(\tau + \chi - 2)$$
$$b_2^- = \tfrac{1}{2}(-\tau + \chi - 2).$$

An *Einstein manifold* is defined as a manifold which admits a metric which obeys

$$\mathcal{R}_{\mu\nu} = \Lambda g_{\mu\nu}.$$

We state the following theorems:

I. (Berger [1965]). $\chi \geq 0$ for a 4-dimensional compact Einstein manifold M with $\chi = 0$ only if M is flat.

II. (Hitchin [1974b]).

$$\chi \geq \tfrac{3}{2}|\tau|$$

for a 4-dimensional compact Einstein manifold M, with

$$\chi = \tfrac{3}{2}|\tau|$$

only if M is flat or its universal covering is a K3 surface.

III. (Hitchin [1974b]). If M is a compact 4-dimensional Einstein manifold with non-negative (or non-positive) sectional curvature, then

$$\chi \geq (\tfrac{3}{2})^{3/2}|\tau|$$

with equality only if M is flat.

IV. (Gibbons and Pope [1979]). Suppose M is non-compact, and its non-compactness is completely characterized by removing N asymptotically Euclidean regions from a compact manifold $\bar{M}$. Then, if M is an Einstein space,

$$\chi(M) \geq N + \tfrac{3}{2}|\tau(M)|$$
$$\chi(\bar{M}) \geq 2N + \tfrac{3}{2}|\tau(\bar{M})|.$$

Examples:

$$\text{Einstein:}\quad S^4, S^2\times S^2, P_2(\mathbb{C}), 2P_2(\mathbb{C}), 3P_2(\mathbb{C})$$
$$\text{not Einstein:}\quad S^1\times S^3, 2T^4, nP_2(\mathbb{C})\ \text{for}\ n \geq 4.$$

2. *K3 surface*. The K3 surface and the four-torus T^4, are the *only* closed, compact manifolds admitting metrics with self-dual Riemann curvature. (Conversely, all Ricci flat manifolds are self-dual if they are closed and compact.) For T^4, the self-dual metric is the trivial flat metric. For the K3 surface, the self-dual metric is nontrivial but unknown, although Yau [1978] has, in principle, given a way to construct it numerically. Other approaches to finding the K3 metric have been described by Page [1978c] and by Gibbons and Pope [1979]. Only the K3 surface and the Enriques surface (whose universal covering is K3) or the quotient of an Enriques surface by a free antiholomorphic involution with $\pi_1 = \mathbb{Z}_2\times\mathbb{Z}_2$ saturate Hitchin's bound [1974b]

$$\chi = \tfrac{3}{2}|\tau|$$

with $\chi \neq 0$. We show below that $\chi = 24$, $|\tau| = 16$ and note that K3 is a complex manifold with first Betti number $b_1 = 0$, $b_2^+ = 19$, $b_2^- = 3$, and first Chern class $c_1 = 0$.

The K3 surface is definable as the solution to $f_4(z) = 0$ where f_4 is a homogeneous polynomial of degree 4 in the homogeneous coordinates z_0, z_1, z_2, z_3 of $P_3(\mathbb{C})$. It is thus instructive to examine it in the general context of polynomials $f_m(z) = 0$ of degree m in $P_3(\mathbb{C})$ (Back, Freund and Forger [1978]). We let V be the corresponding two-dimensional complex surface in $P_3(\mathbb{C})$ and split the tangent bundle of $P_3(\mathbb{C})$ in parts normal and tangential to V:

$$T(P_3(\mathbb{C})) = T(V) \oplus N(V).$$

The Chern classes for Whitney sums of bundles and for $P_n(\mathbb{C})$ itself are given by

$$c(T(P_3(\mathbb{C}))) = c(T(V))\, c(N(V))$$
$$c(T(P_n(\mathbb{C}))) = (1+x)^{n+1},$$

where x is $c_1(L^*)$, the normalized Kähler 2-form of the Fubini–Study metric on $P_n(\mathbb{C})$. Finally, we note that if V is given by $f_m(z) = 0$, the Chern class of $N(V)$ is given by

$$c(N(V)) = 1 + mx,$$

since m is the number of Riemann sheets of $f_m(z) = 0$. Letting

$$R = \mathrm{i} * x = \text{projection of the 2-form } x \text{ onto } V,$$

we combine the equations to give

$$(1+R)^4 = c(T(V))\,(1+mR)$$

and use the splitting principle to get (with $R \to r$)

$$\begin{aligned} c(T(V)) &= \frac{1 + 4r + 6r^2 + \cdots}{1 + mr} \\ &= 1 + (4-m)r + (m^2 - 4m + 6)r^2 = 1 + c_1 + c_2. \end{aligned}$$

Now, since

$$\int_V R \wedge R = m = \text{number of Riemann sheets}$$

and

$$p_1 = c_1{}^2 - 2c_2 = [(4-m)^2 - 2(m^2 - 4m + 6)]R \wedge R = (4 - m^2)R \wedge R,$$

we can calculate all the properties of K3 by setting $m = 4$:

(1) $c_1 = (4-m)R = 0, \qquad c_2 = (m^2 - 4m + 6)R \wedge R = 6R \wedge R$

(2) $\tau = \frac{1}{3}P_1 = \dfrac{1}{3}\displaystyle\int_V p_1 = \frac{1}{3}m(4 - m^2) = -16$

(3) $\chi = \int\limits_V c_2 = m(m^2 - 4m + 6) = 24$

$= \frac{3}{2}|\tau|$

(4) $\hat{A} = -\frac{1}{8}\tau = -\frac{1}{24}m(4 - m^2) = +2$

(5) $I_{\bar{\partial}} = \frac{1}{4}(\chi + \tau) = \frac{1}{4}(24 - 16) = +2.$

We thus see from (4) and (5) that K3 can be a spin manifold and a complex manifold.

3. *Harmonic spinors.* A very useful result concerning the Dirac equation on curved Euclidean (positive signature) manifolds is *Lichnerowicz's theorem* (Lichnerowicz [1963]):

If the scalar curvature $\mathcal{R}$ of a compact spin manifold is positive,

$$\mathcal{R} > 0,$$

then there are *no harmonic spinors* on the manifold.

However, there is no expression for the dimension of the space of harmonic spinors in terms of the topological invariants of the manifold: Hitchin [1974a] has shown that although the dimension of the space of harmonic spinors is conformally invariant, *it depends on the metric* used to define the Dirac operator.

4. *Spin structures.* As we observed in the section on characteristic classes, one can define spinors unambiguously on a manifold only if its second Stiefel–Whitney class *vanishes*: such a manifold is called a *spin-manifold*. However, the spinor phase ambiguity which occurs for non-spin manifolds can be cancelled by introducing an additional structure such as an electromagnetic field (a $U(1)$ principal bundle). This additional structure, the spin$_c$ structure, gives a new type of more general spin manifold. For instance, although the manifold $P_2(\mathbb{C})$ does not admit a spin structure, one can still define a spin$_c$ structure by introducing magnetic monopoles with half the Dirac charge (Trautman [1977]; Hawking and Pope [1978]). Back, Freund and Forger [1978] discuss interesting physical applications of the idea of the spin$_c$ structure.

5. *Deformations of conformally self-dual manifolds.* Singer [1978d] has examined the general case of the number of conformally self-dual deformations of a compact conformally self-dual manifold. This number is interesting to a physicist because it gives the number of free parameters, or the number of *zero-frequency modes*, of a given solution of Einstein's equations. By constructing an appropriate elliptic complex, Singer applies the index theorem and finds the number of conformally self-dual deformations to be the index of the complex:

$$I = \tfrac{1}{2}(29|\tau| - 15\chi) + \dim(\text{conformal group}) + (\text{correction for absence of vanishing theorem if scalar curvature} \leq 0).$$

Note that scale factors of the metric are not included here. This is the index of the gravitational deformations (see Gibbons and Perry [1978]) taking solutions to solutions, but the value of the action is not necessarily preserved.

Examples:

A. S^4. Here $\tau = 0$, $\chi = 2$, the conformal group is 15-dimensional and since $\mathcal{R} > 0$, there is no correction:

$$I = \tfrac{1}{2}(-30) + 15 + 0 = 0.$$

Thus a conformally self-dual metric on S^4 has no zero-frequency modes aside from a scale.

B. $P_2(\mathbb{C})$. Here $\tau = 1$, $\chi = 3$, the conformal group is 8-dimensional and $\mathcal{R} > 0$, so there is no correction:

$$I = \tfrac{1}{2}(29 - 45) + 8 + 0 = 0.$$

Thus the Fubini–Study metric, which has self-dual Weyl tensor, allows no conformally self-dual deformations apart from a scale.

C. *K3 surface*. For this manifold, $|\tau| = 16$, $\chi = 24$, the conformal group is empty, but there is no vanishing theorem because the manifold is *self-dual*; it has self-dual Riemann tensor in addition to self-dual Weyl tensor. Singer has shown that there are 5 covariant constant objects in W_-, which constitute the vanishing theorem correction. Thus

$$I = \tfrac{1}{2}(29 \times 16 - 15 \times 24) + 0 + 5 = 57.$$

Including a scale, we get 58 parameters for the K3 metric, in agreement with Hawking and Pope [1978]. This same result may also be found by observing that for the K3 surface, $b_2^+ = 19$, $b_2^- = 3$, so that one may explicitly construct the required deformations from the harmonic forms. One finds

$$I = 3 \times 19 = 57$$

as before.

The basic formula given above, of course, needs modification when the manifold in question has a boundary. The number of zero-frequency modes for self-dual (Riemann tensor) asymptotically locally Euclidean spaces with boundary $L(k+1, 1)$ has been determinated directly (Hawking and Pope [1978]). The result is

$$I = 3(k+1) - 6 = 3k - 3$$

plus a scale. Thus the Eguchi–Hanson metric [1978], which has $k = 1$, possesses no self-dual deformations apart from a scale.

6. *Asymptotically locally Euclidean self-dual manifolds*. The general concept of manifolds with self-dual Riemann tensor and asymptotic regions which are lens spaces $L(k+1, 1)$ of S^3 was introduced earlier (10.2.2). Hitchin [1979] and Calabi [1979] have examined the most general possible regular self-dual manifolds with asymptotically locally Euclidean (ALE) infinities. The complete classification of the spherical forms of S^3 is well-known (Wolf [1967]); the possible spaces which correspond to ALE infinities are:

Series A_k:	*cyclic group* of order k (=lens spaces $L(k+1, 1)$)
Series D_k:	*dihedral group* of order k
T:	*tetrahedral group*
O:	*octahedral group* $\approx$ *cubic group*
I:	*icosahedral group* $\approx$ *dodecahedral group*.

A_1 corresponds to the Eguchi–Hanson metric [1978] and A_k to the multicenter generalization of Gibbons and Hawking [1978]. We note that one must actually use the binary or double-covering groups D_k^*, T^*, O^*, I^* of D_k, T, O, I to avoid singularities in physical ALE spaces.

Complex algebraic manifolds whose boundaries correspond to each spherical form have been identified as follows, where x, y and z are all complex:

$$\begin{array}{ll} \text{Group} & \text{Algebraic 4-manifold} \\ \hline A_k & \begin{cases} z^{k+1} = xy \\ z^{k+1} + x^2 + y^2 = 0 \end{cases} \\ D_k & z^{k-1} + x^2 + y^2 z = 0 \\ T & x^2 + y^3 + z^4 = 0 \\ O & x^2 + y^3 + yz^3 = 0 \\ I & x^2 + y^3 + z^5 = 0. \end{array} \tag{10.3}$$

These equations are, in fact, prominent in algebraic geometry (Brieskorn [1968]); they are the unique set of algebraic equations of their type which possess resolvable singularities.

The Atiyah–Patodi–Singer η-invariant, the Euler characteristic, and the signature have been calculated for each of these cases by Gibbons, Pope and Römer [1979]. They find (our signs differ):

	χ	τ	$-\xi_{1/2} = \frac{1}{2}\eta_{\text{Dirac}}$
A_k	$k+1$	$-k$	$[(k+1)^2 - 1]/12(k+1)$
D_k^*	$k+1$	$-k$	$[4(k-2)^2 + 12(k-2) - 1]/48(k-2)$
T^*	7	-6	167/288
O^*	8	-7	383/576
I^*	9	-8	1079/1440

The values of the spin $\frac{1}{2}$ index all vanish, while the spin $\frac{3}{2}$ index for each case is 2τ.

7. *Proof of positivity of the energy and the action in general relativity* (Schoen and Yau [1978, 1979a, b, c]). The positivity of the gravitational mass or energy has long been conjectured on physical grounds, but until recently, mathematical proofs existed only for special cases. Recently Schoen and Yau produced a general proof of the positive energy conjecture using differential geometry and classical analysis.

By using the observation (Gibbons, Hawking and Perry [1978]) that the positivity of the energy in five dimensions is closely related to the positivity of the action in four dimensions, Schoen and Yau then succeeded in proving the (original) positive action conjecture stated in the previous section 10.1.

The Euclidean path integral approach to gravity, which depends in part on the positivity of the action, is on a much firmer mathematical footing as a consequence of these results.

8. *Applications of the index theorems to gravity*. We have already noted that the anomalous divergences of axial currents noted by physicists are, when integrated, closely related to mathematical index theorems. (The anomalous divergence of the axial vector current in an external gravitational field was first computed using physicists' methods before the relation of the anomaly to index theory was realized. See Delbourgo and Salam [1972] and Eguchi and Freund [1976].) A great deal of attention has consequently been paid to the application of index theory to operators in the presence of Euclidean

gravity, i.e., operators on Riemannian manifolds (Eguchi, Gilkey and Hanson [1978]; Römer and Schroer [1977]; Nielsen, Römer and Schroer [1977, 1978]; Pope [1978]; Christensen and Duff [1978]; Nielsen, Grisaru, Römer and Van Nieuwenhuizen [1978]; Perry [1978]; Critchley [1978]; Hawking and Pope [1978b]; Hanson and Römer [1978]; Christensen and Duff [1979]; Römer [1979]). One can, of course, also treat the case where connections on principal bundles are included. We present here a discussion of some of the major results. A tabulation of formulas and the index properties of various manifolds is given in the appendices.

Euler characteristic: The Euler characteristic χ is the index of the Euler complex, which deals with the exterior derivative mapping even-dimensional forms to odd-dimensional forms. The Euler characteristic gives the number of zeroes of vector fields on the manifold. If the manifold has a boundary, the index formula has differential geometric surface corrections (Chern [1945]), but no nonlocal or analytic corrections.

Hirzebruch signature: The Hirzebruch signature τ is the index I_S of the signature complex, which deals with the exterior derivative operator mapping self-dual forms to anti-self dual forms. The signature is nonzero in dimensions which are multiples of 4 and gives the difference between the number of harmonic self-dual forms and anti-self-dual forms of the middle dimension. The signature is one-third the Pontrjagin number P_1 in 4 dimensions,

$$I_S = \tau = \tfrac{1}{3}P_1.$$

If the manifold has a boundary, there exist both a local surface correction and a non-local Atiyah–Patodi–Singer (APS) η-invariant correction; the meaning of the signature is altered to include only (anti)-self-dual harmonic forms which obey the APS boundary conditions.

$\hat{A}$ genus (Dirac, spin 1/2 index): The $\hat{A}$ genus is the index $I_{1/2}$ of the Dirac complex, which deals with the spin $\frac{1}{2}$ Dirac operator mapping positive chirality spinors into negative chirality spinors. The $\hat{A}$ genus is an integer if the manifold is a spin manifold, and gives the difference between the number of positive chirality and negative chirality normalizable zero-frequency solutions to the Dirac equation. In 4 dimensions the Dirac index formula is related to the signature by

$$I_{1/2} = \hat{A} = -\tfrac{1}{8}\tau = -\tfrac{1}{24}P_1.$$

If the manifold has a boundary, there are both local boundary corrections and nonlocal η-invariant corrections; the corresponding zero-frequency solutions to the Dirac equation must obey the APS boundary conditions.

Rarita–Schwinger, spin 3/2 index: This index theorem deals with the spin $\frac{3}{2}$ Rarita–Schwinger operator mapping positive chirality spin $\frac{3}{2}$ wave functions into negative chirality spin $\frac{3}{2}$ wave functions. The spin $\frac{3}{2}$ wave functions are familiar to physicists, but the corresponding bundles are mathematically subtle; the accepted practice at present (Römer [1979]) is to define the Rarita–Schwinger $\pm$ chirality bundles as the *virtual bundles* (see section 6.5 on K theory)

$$\Delta^+_{3/2}(M) = \Delta_{1,1/2}(M) \ominus 2\Delta_{1/2,0}(M)$$

$$\Delta^-_{3/2}(M) = \Delta_{1/2,1}(M) \ominus 2\Delta_{0,1/2}(M),$$

where

$$\Delta_{m/2,n/2}(M) = S^m\Delta_+(M) \otimes S^n\Delta_-(M).$$

$\Delta_\pm(M)$ are the $\pm$ chirality bundles and S^r denotes the r-fold symmetric tensor product. The Rarita–Schwinger index is related to the signature by

$$I_{3/2} = \tfrac{21}{8}\tau = \tfrac{21}{24}P_1,$$

where $I_{3/2}$ is the difference between the number of positive chirality and the negative chirality zero frequency solutions of the Rarita–Schwinger equation. If the manifold has a boundary, there are both local boundary corrections and nonlocal η-invariant corrections, and the corresponding zero-frequency wave functions must obey the APS boundary conditions. The calculation of the η-invariant corrections is nontrivial; at present, they have been computed only for cases where the G-index theorem could be used to reduce the calculation to an algebraic form (Hanson and Römer [1978]; Römer [1979]; Gibbons, Pope and Römer [1979]). Direct construction of spin $\frac{3}{2}$ zero-frequency modes can be carried out using the method of Hawking and Pope [1978b], but it is difficult to show that there are no other solutions satisfying the Atiyah–Patodi–Singer boundary conditions without using the index theorem.
General spin index theorems: Christensen and Duff [1979] and Römer [1979] have examined the general-spin elliptic complexes

$$D_{m/2,n/2}\colon \Delta_{m/2,n/2}(M) \to \Delta_{n/2,m/2}(M)$$

where $\Delta_{m/2,n/2}$ was defined above and $D_{m/2,n/2}$ is an appropriate elliptic operator. They find that the index theorem takes the form

$$I_{m/2,n/2}[M] = -\frac{(m+1)(n+1)}{720}\{n(n+2)(3n^2+6n-14) - m(m+2)(3m^2+6m-14)\}\, P_1[M]. \quad (10.4)$$

In particular, one recovers the Dirac results

$$I_{1/2} \equiv I_{1/2,0} = -\tfrac{1}{24}P_1[M].$$

If the manifold has a boundary, surface corrections and η-invariant corrections must be applied. Römer [1979] has calculated the η-invariant corrections for a variety of interesting cases using G-index theory. For example, for the Eguchi–Hanson metric [1978], which has $P_3(\mathbb{R})$ as the boundary and no local surface corrections, the non-local boundary correction to the index is

$$\xi_{m/2,n/2}[P_3(\mathbb{R})] = \tfrac{1}{32}(m+1)(n+1)[(-1)^m - (-1)^n].$$

When one includes the effect of a principal G-bundle or vector bundle V_G with structure group G for a 4-dimensional manifold with no boundary corrections, Römer [1979] finds the full index

$$I^G_{m/2,n/2} = \dim V_G \cdot I_{m/2,n/2}[M] + \tfrac{1}{6}(m+1)(n+1)[m(m+2) - n(n+2)]\,\mathrm{ch}_2(V_G[M]), \quad (10.5)$$

where ch_2 denotes the Chern character on V_G integrated over its Λ^4 component.

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Appendix A: Miscellaneous formulas

1. *Manifolds.*

Tangent frame basis: $E_\mu = \dfrac{\partial}{\partial x^\mu}$; $E'_\nu = E_\mu \phi^\mu{}_\nu$

Cotangent frame basis: $e^\mu = \mathrm{d}x^\mu$; $e'^\mu = (\phi^{-1})^\mu{}_\nu e^\nu$

Transition function: $(\phi_{UU'})^\mu{}_\nu = \partial x^\mu / \partial x'^\nu$

Inner product: $\langle \partial/\partial x^\mu, \mathrm{d}x^\nu \rangle = \delta_\mu{}^\nu$

Vector field: $V = v^\mu\, \partial/\partial x^\mu$

Covector field: $P = p_\mu\, \mathrm{d}x^\mu$

Boundary: If $\dim(M) = n$, then $\dim(\partial M) = n - 1$.

$\partial\partial M = \emptyset$ (empty).

2. *Differential forms.* n = dimension of manifold. $\omega_p = p$-form.

Wedge product: $\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x$, $\mathrm{d}x \wedge \mathrm{d}x = 0$

p form: $\omega_p = f_{\mu_1 \ldots \mu_p}\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}$

Exterior derivative: $\mathrm{d}\omega_p = \mathrm{d}(f_{\mu\nu\ldots}(x)\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \ldots)$

$= \partial_\lambda f_{\mu\nu\ldots}(x)\, \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \cdots = (p+1)$-form

$\mathrm{d}\mathrm{d}\omega_p = 0$

Dual: $$*(\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}) = \frac{|g|^{1/2}}{(n-p)!} \varepsilon^{\mu_1, \ldots, \mu_p}{}_{\mu_{p+1}, \ldots, \mu_n}\, \mathrm{d}x^{\mu_{p+1}} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}$$

General forms: $\omega_p \wedge \omega_q = (-1)^{qp} \omega_q \wedge \omega_p$

$\mathrm{d}(\omega_p \wedge \omega_q) = \mathrm{d}\omega_p \wedge \omega_q + (-1)^p \omega_p \wedge \mathrm{d}\omega_q$

$** \omega_p = (-1)^{p(n-p)} \omega_p$

$\omega_p \wedge * \omega_q = \omega_q \wedge * \omega_p$

Coderivative: $\delta\omega_p = (-1)^{np+n+1} * \mathrm{d} * \omega_p = (p-1)$-form
(for positive signature metrics)

$\delta\delta\omega_p = 0$

Inner product: Let α_p and β_p be p-forms, M compact, $\partial M = \emptyset$.

$$(\alpha_p, \beta_p) = (\beta_p, \alpha_p) = \int_M \alpha_p \wedge * \beta_p$$

$$(\mathrm{d}\alpha_p, \beta_{p+1}) = (\alpha_p, \delta\beta_{p+1}); (\alpha_p, \mathrm{d}\beta_{p-1}) = (\delta\alpha_p, \beta_{p-1})$$

Laplacian: $\Delta\omega_p = (\mathrm{d} + \delta)^2\omega_p = (\mathrm{d}_{p-1}\delta_p + \delta_{p+1}\,\mathrm{d}_p)\omega_p = p\text{-form}$

Coordinate Laplacian: $\Delta\phi(x) = -|g|^{-1/2}\,\partial_\mu(g^{\mu\nu}|g|^{1/2}\,\partial_\nu)\phi(x)$

Stokes' Theorem: $\int_M \mathrm{d}\omega_{p-1} = \int_{\partial M} \omega_{p-1}$, where $\dim(M) = p$.

Hodge's theorem: $\omega_p = \mathrm{d}\alpha_{p-1} + \delta\beta_{p+1} + \gamma_p, \qquad \Delta\gamma_p = 0$

3. *Homology and cohomology.*

Homology: Z_p = cycles (p-chains a_p, with $\partial a_p = \emptyset$)

B_p = boundaries (p-chains b_p, with $b_p = \partial a_{p+1}$ for some a_{p+1})

$H_p = Z_p/B_p$ (homology = cycles modulo boundaries)

Cohomology: Z^p = closed forms (p-forms ω_p, with $\mathrm{d}\omega_p = 0$)

B^p = exact forms (p-forms ω_p, with $\omega_p = \mathrm{d}\alpha_{p-1}$ for some α_{p-1})

$H^p = Z^p/B^p$ (cohomology = closed modulo exact forms)

de Rham's theorem: H^p (de Rham) $\approx H^p$ (simplicial) $\approx H_p$ (simplicial)

Poincaré duality: $\dim H^p(M;\mathbb{R}) = \dim H^{n-p}(M;\mathbb{R})$, M orientable

Betti numbers: $b_p = \dim H^p = \dim H_p$ = number of harmonic p-forms γ_p, $\Delta\gamma_p = 0$

4. *Riemannian manifolds.* $g_{\mu\nu}$ = curved metric on M, η_{ab} = flat metric

Metric: $\mathrm{d}s^2 = \mathrm{d}x^\mu g_{\mu\nu}\,\mathrm{d}x^\nu = e^a\eta_{ab}e^b$

Vierbein basis of $T^*(M)$: $e^a = e^a{}_\mu\,\mathrm{d}x^\mu$

$$T(M):\quad E_a = E_a{}^\mu \frac{\partial}{\partial x^\mu} = \eta_{ab}g^{\mu\nu}e^b{}_\nu \frac{\partial}{\partial x^\mu}$$

Connection one-form: $\omega^a{}_b = \omega^a{}_{b\mu}\,\mathrm{d}x^\mu$

Cartan structure equations:

$$\text{torsion} = T^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b$$

$$\text{curvature} = R^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

Cartan identities: $\mathrm{d}T^a + \omega^a{}_b \wedge T^b = R^a{}_b \wedge e^b$

$$\mathrm{d}R^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0 \ (\textit{Bianchi identity})$$

Frame change: $\eta_{ab}\Phi^a{}_c\Phi^b{}_d = \eta_{cd}$

$$e'^a = \Phi^a{}_b e^b$$

$$\omega'^a{}_b = (\Phi\omega\Phi^{-1} + \Phi\,\mathrm{d}\Phi^{-1})^a{}_b$$

$$T'^a = \Phi^a{}_b T^b$$

$$R'^a{}_b = (\Phi R \Phi^{-1})^a{}_b$$

Levi–Civita
connection: 1. $T^a = 0$ (torsion free)

2. $\omega_{ab} = -\omega_{ba}$ (covariant constant metric)

These imply the *cyclic identity*, $R^a{}_b \wedge e^b = 0$.

5. *Complex manifolds.* $z_k = x_k + \mathrm{i}y_k$, $\bar{z}_k = x_k - \mathrm{i}y_k$

$$\partial f = \frac{\partial f}{\partial z_k}\,\mathrm{d}z^k = \frac{1}{2}\left(\frac{\partial f}{\partial x_k} - \mathrm{i}\frac{\partial f}{\partial y_k}\right)\mathrm{d}z^k$$

$$\bar{\partial} f = \frac{\partial f}{\partial \bar{z}_k}\,\mathrm{d}\bar{z}_k = \frac{1}{2}\left(\frac{\partial f}{\partial x_k} + \mathrm{i}\frac{\partial f}{\partial y_k}\right)\mathrm{d}\bar{z}_k$$

Exterior derivative: $\mathrm{d} = \partial + \bar{\partial}$

Hermitian metric: $\mathrm{d}s^2 = g_{jk}\,\mathrm{d}z^j\,\mathrm{d}\bar{z}^k$, g_{jk} = hermitian

Kähler form: $K = \bar{K} = \frac{\mathrm{i}}{2} g_{jk}\,\mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k$, g_{jk} = hermitian

6. *Some useful differential forms for practical calculations.*
Two dimensions: $x = r\cos\theta$, $y = r\sin\theta$ $0 \le \theta < 2\pi$

$$\begin{pmatrix} \mathrm{d}r \\ r\,\mathrm{d}\theta \end{pmatrix} = \begin{pmatrix} x/r & y/r \\ -y/r & x/r \end{pmatrix}\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix}, \qquad \mathrm{d}x \wedge \mathrm{d}y = r\,\mathrm{d}r \wedge \mathrm{d}\theta$$

Three dimensions: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$$\rho^2 = x^2 + y^2 = r^2\sin^2\theta \qquad 0 \le \theta < \pi, \quad 0 \le \phi < 2\pi$$

$$\begin{pmatrix} \mathrm{d}r \\ r\,\mathrm{d}\theta \\ r\sin\theta\,\mathrm{d}\phi \end{pmatrix} = \begin{pmatrix} x/r & y/r & z/r \\ xz/r\rho & yz/r\rho & -\rho/r \\ -y/\rho & x/\rho & 0 \end{pmatrix}\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix}$$

$$\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = r^2\sin\theta\,\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi$$

$$r^{-3}(x\,\mathrm{d}y \wedge \mathrm{d}z + y\,\mathrm{d}z \wedge \mathrm{d}x + z\,\mathrm{d}x \wedge \mathrm{d}y) = \sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi$$

Four dimensions:

(Instead of using the ordinary polar coordinates, we exploit the relationship between S^3 and SU(2))

$$z_1 = x + \mathrm{i}y = r\cos\frac{\theta}{2}\exp\frac{\mathrm{i}}{2}(\psi + \phi)$$

$$z_2 = z + \mathrm{i}t = r\sin\frac{\theta}{2}\exp\frac{\mathrm{i}}{2}(\psi - \phi)$$

$$0 \le \theta < \pi, \quad 0 \le \phi < 2\pi, \quad 0 \le \psi < 4\pi$$

$$\begin{pmatrix} \mathrm{d}r \\ r\sigma_x \\ r\sigma_y \\ r\sigma_z \end{pmatrix} = \frac{1}{r}\begin{pmatrix} x & y & z & t \\ -t & -z & y & x \\ z & -t & -x & y \\ -y & x & -t & z \end{pmatrix}\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \\ \mathrm{d}t \end{pmatrix} = \frac{1}{2r}\begin{pmatrix} \bar z_1 & \bar z_2 & z_1 & z_2 \\ \mathrm{i}z_2 & -\mathrm{i}z_1 & -\mathrm{i}\bar z_2 & \mathrm{i}\bar z_1 \\ z_2 & -z_1 & \bar z_2 & -\bar z_1 \\ -\mathrm{i}\bar z_1 & -\mathrm{i}\bar z_2 & \mathrm{i}z_1 & \mathrm{i}z_2 \end{pmatrix}\begin{pmatrix} \mathrm{d}z_1 \\ \mathrm{d}z_2 \\ \mathrm{d}\bar z_1 \\ \mathrm{d}\bar z_2 \end{pmatrix}$$

$\mathrm{d}\sigma_x = 2\sigma_y \wedge \sigma_z$, cyclic (Maurer–Cartan structure equation)

$$\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}t = r^3\,\mathrm{d}r \wedge \sigma_x \wedge \sigma_y \wedge \sigma_z = \tfrac{1}{4}\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}\bar z_1 \wedge \mathrm{d}\bar z_2$$

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}t^2 = \mathrm{d}r^2 + r^2({\sigma_x}^2 + {\sigma_y}^2 + {\sigma_z}^2) = \mathrm{d}z_1\,\mathrm{d}\bar z_1 + \mathrm{d}z_2\,\mathrm{d}\bar z_2$$

Minkowski space: $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1), \qquad \epsilon_{0123} = +1$

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}$$

Hodge $*$: $\quad *\mathrm{d}t = -\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$

$*(\mathrm{d}x^1 \wedge \mathrm{d}t) = +\mathrm{d}x^2 \wedge \mathrm{d}x^3$, cyclic

$*(\mathrm{d}x^2 \wedge \mathrm{d}x^3) = -\mathrm{d}x^1 \wedge \mathrm{d}t$, cyclic

Laplacian: $\quad \Delta = \mathrm{d}\delta + \delta\mathrm{d} = +\dfrac{\partial^2}{\partial t^2} - \dfrac{\partial^2}{\partial\boldsymbol{x}\cdot\partial\boldsymbol{x}}$

Maxwell's equations: $\quad A = -A^0\,\mathrm{d}t + \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{x}$

$\boldsymbol{B} = \nabla\times\boldsymbol{A}, \quad E^i = F^{0i} = -(\partial A^i/\partial t + \partial A^0/\partial x^i)$

$F = \mathrm{d}A = \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{x} \wedge \mathrm{d}t + \tfrac{1}{2}B_i\epsilon_{ijk}\,\mathrm{d}x^j \wedge \mathrm{d}x^k$

$**F = -F; \quad *F = \pm\mathrm{i}F \to \boldsymbol{E} = \pm\mathrm{i}\boldsymbol{B}.$

7. *Determining the Levi–Civita connection*. Let $\omega_{ab} = -\omega_{ba}$ and $\mathrm{d}e^a = c^a_{0i}e^0 \wedge e^i + c^a_{23}e^2 \wedge e^3 + c^a_{31}e^3 \wedge e^1 + c^a_{12}e^1 \wedge e^2 = -{\omega^a}_b \wedge e^b$. Then

$${\omega^0}_1 = e^0[-c^0_{01}] + e^1[-c^1_{01}] + e^2(\tfrac{1}{2})(c^0_{12} - c^1_{02} - c^2_{01}) + e^3(\tfrac{1}{2})(-c^0_{31} - c^3_{01} - c^1_{03})$$

$${\omega^0}_2 = e^0[-c^0_{02}] + e^1(\tfrac{1}{2})(-c^0_{12} - c^1_{02} - c^2_{01}) + e^2(-c^2_{02}) + e^3(\tfrac{1}{2})(c^0_{23} - c^2_{03} - c^3_{02})$$

$$\omega^0{}_3 = e^0[-c^0_{03}] + e^1(\tfrac{1}{2})(c^0_{31} - c^3_{01} - c^1_{03}) + e^2(\tfrac{1}{2})(-c^0_{23} - c^2_{03} - c^3_{02}) + e^3(-c^3_{03})$$
$$\omega^2{}_3 = e^0(\tfrac{1}{2})(c^3_{02} - c^0_{23} - c^2_{03}) + e^1(\tfrac{1}{2})(c^2_{31} + c^3_{12} - c^1_{23}) + e^2(-c^2_{23}) + e^3(-c^3_{23})$$
$$\omega^3{}_1 = e^0(\tfrac{1}{2})(c^1_{03} - c^0_{31} - c^3_{01}) + e^1(-c^1_{31}) + e^2(\tfrac{1}{2})(c^3_{12} + c^1_{23} - c^2_{31}) + e^3(-c^3_{31})$$
$$\omega^1{}_2 = e^0(\tfrac{1}{2})(c^2_{01} - c^0_{12} - c^1_{02}) + e^1(-c^1_{12}) + e^2(-c^2_{12}) + e^3(\tfrac{1}{2})(c^1_{23} + c^2_{31} - c^3_{12})$$

8. *n-sphere metrics.* $R^2 = \Sigma_{i=1}^{n+1} X_i^2$ = the (constant) radius of the sphere.

S^2: $\mathrm{d}s^2 = R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$

$$e^i = (R\,\mathrm{d}\theta, R\sin\theta\,\mathrm{d}\phi); \quad \omega^1{}_2 = -\cos\theta\,\mathrm{d}\phi; \quad R^1{}_2 = \frac{1}{R^2} e^1 \wedge e^2$$

S^3: $\mathrm{d}s^2 = R^2(\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2)$

$$e^i = (R\sigma_x, R\sigma_y, R\sigma_z); \quad \omega^2{}_3 = \sigma_x, \quad \omega^3{}_1 = \sigma_y, \quad \omega^1{}_2 = \sigma_z$$

$$R^2{}_3 = \frac{1}{R^2} e^2 \wedge e^3, \text{ cyclic}$$

S^4: $\mathrm{d}s^2 = (\mathrm{d}r^2 + r^2[\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2])/[1 + (r/2R)^2]^2$

$$[1 + (r/2R)^2]e^a = (\mathrm{d}r, r\sigma_x, r\sigma_y, r\sigma_z)$$
$$\omega_{i0} = \sigma_i(1 - (r/2R)^2)/(1 + (r/2R)^2)$$
$$\omega_{23} = \sigma_x, \quad \omega_{31} = \sigma_y, \quad \omega_{12} = \sigma_z$$
$$R^{ab} = \frac{1}{R^2} e^a \wedge e^b$$

S^n Cartesian metric: $r^2 = \sum_{i=1}^{n} (x^i)^2$

$$\mathrm{d}s^2 = \mathrm{d}x^i\,\mathrm{d}x^i/[1 + (r/2R)^2]^2$$
$$e^i = \mathrm{d}x^i/[1 + (r/2R)^2]$$

volume element $= e^1 \wedge e^2 \cdots \wedge e^n = \mathrm{d}^n x/[1 + (r/2R)^2]^n$

$V(S^n)$ = volume $= 2\pi^{(n+1)/2} R^n/\Gamma(\tfrac{1}{2}(n+1))$

$$V(S^0, S^1, \ldots) = \left(2, 2\pi R, 4\pi R^2, 2\pi^2 R^3, \frac{8\pi^2}{3} R^4, \ldots\right)$$

$$\omega^i{}_j = \frac{x^i\,\mathrm{d}x^j - x^j\,\mathrm{d}x^i}{2R^2[1 + (r/2R)^2]}$$

$$R^i{}_j = \frac{1}{R^2} e^i \wedge e^j; \quad R_{ijkl} = \frac{1}{R^2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

$$\mathcal{R}_{ij} = \frac{N-1}{R^2}\delta_{ij}, \qquad \mathcal{R} = \frac{N(N-1)}{R^2}, \qquad W_{ijkl} = 0.$$

9. *$P_n(\mathbb{C})$ metrics.*

Kähler form: $K = \frac{i}{2}\partial\bar{\partial}\ln\left(1 + \sum_{\alpha=1}^{n} z^\alpha \bar{z}^\alpha\right) = \pi c_1(L^*)$

Metric: $ds^2 = \frac{dz^\alpha\, d\bar{z}^\beta}{(1 + z^\gamma \bar{z}^\gamma)^2}[\delta_{\alpha\beta}(1 + z^\gamma \bar{z}^\gamma) - \bar{z}^\alpha z^\beta]$

Appendix B: Index theorem formulas

1. *Index theorems for Yang–Mills theory.*
Characteristic classes; $\dim(M) = 2, 4$; bundle V with curvature F.

$$C_1[V] = -\frac{1}{2\pi}\int_{M_2} \operatorname{Tr} F$$

$$C_2[V] = -k = +\frac{1}{8\pi^2}\int_{M_4} \operatorname{Tr} F \wedge F$$

Self-dual Yang–Mills index:

SU(2): $I_{\mathrm{YM}} = 8k - 3$

SU(3): $I_{\mathrm{YM}} = 12k - 8, \qquad k \geq 2\ (k = 1 \text{ is } \approx \mathrm{SU}(2))$

Spin $\frac{1}{2}$ index for $(2t + 1)$-dimensional representation of SU(2):

$I_{1/2}(t) = \frac{2}{3}t(t + 1)(2t + 1)k$

$I_{1/2}(1/2) = k$

2. *Index theorems for gravity.*
Characteristic classes, $\dim(M) = 4$;

$$P_1[M] = -\frac{1}{8\pi^2}\int_M \operatorname{Tr} R \wedge R$$

$$Q_1[\partial M] = -\frac{1}{8\pi^2}\int_{\partial M} \operatorname{Tr}(\theta \wedge R)$$

$\theta = \omega - \omega_0 =$ 2nd fundamental form, a connection with only normal components on ∂M

η-invariant:

$$\eta[\partial M, g] = \sum_{\{\lambda_i \neq 0\}} \operatorname{sign}(\lambda_i)|\lambda_i|^{-s}\Big|_{s=0}$$

Topological invariants:
Signature:

$$\tau = \tfrac{1}{3}(P_1 - Q_1) + \xi_S, \quad \xi_S = -\eta_S$$

Euler characteristic:

$$\chi = \frac{1}{32\pi^2}\left[\int_M \epsilon_{abcd} R^a{}_b \wedge R^c{}_d - \int_{\partial M} \epsilon_{abcd}(2\theta^a{}_b \wedge R^c{}_d - \tfrac{4}{3}\theta^a{}_b \wedge \theta^c{}_e \wedge \theta^e{}_d)\right]$$

Spin $\frac{1}{2}$ index:

$$I_{1/2}[M, g] = -\tfrac{1}{24}(P_1[M] - Q_1[\partial M]) + \xi_{1/2}$$
$$\xi_{1/2} = -\tfrac{1}{2}[\eta_{1/2} + h_{1/2}]$$
$$h_{1/2} = \text{dimension of harmonic space}$$

Spin $\frac{3}{2}$ index:

$$I_{3/2}[M, g] = +\tfrac{21}{24}(P_1[M] - Q_1[\partial M]) + \xi_{3/2}$$

Index of conformally self-dual gravitational perturbations; self-dual ALE metrics with infinity = $L(k+1, 1)$:

$$I_G = 3k - 3 + (\text{scale}) = 3k - 2.$$

3. *Combined Yang–Mills and gravity index.*
Let V be a bundle over a 4-manifold M, $\partial M = \emptyset$.
Spin $\frac{1}{2}$ index:

$$I_{1/2} = -\tfrac{1}{24}\dim(V)\, P_1[M] - C_2[V]$$

Spin $\frac{3}{2}$ index:

$$I_{3/2} = \tfrac{21}{24}\dim(V)\, P_1[M] + 3C_2[V]$$

If $\partial M \neq \emptyset$, replace P_1 by $P_1 - Q_1$ and add the appropriate η-invariant term.

Appendix C: Yang–Mills instantons

Yang–Mills potentials; $A = A^a_\mu \dfrac{\lambda_a}{2\mathrm{i}}\,\mathrm{d}x^\mu$

Yang–Mills field strengths: $F = \mathrm{d}A + A \wedge A = \dfrac{1}{2} F^a_{\mu\nu} \dfrac{\lambda_a}{2\mathrm{i}}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$

Yang–Mills equations: $d(*F)+A\wedge(*F)-(*F)\wedge A=J$

Bianchi identities: $dF+A\wedge F-F\wedge A\equiv 0$

1. *Belavin, Polyakov, Schwarz and Tyupkin* [1975] SU(2) *solution*. We take a, i, j to range from 1 to 3, μ, ν to range from 0 to 3, and define

Pauli matrices: $\lambda_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $\quad \lambda_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$, $\quad \lambda_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

't Hooft matrices: $\eta_{a\mu\nu}=\eta_{aij}=\epsilon_{aij} \qquad a,i,j=(1,2,3)$

$\eta_{ai0}=\delta_{ai} \qquad a,i=(1,2,3)$

$\eta_{a\mu\nu}=-\eta_{a\nu\mu}$

$\bar{\eta}_{a\mu\nu}=(-1)^{\delta_{\mu0}+\delta_{\nu0}}\eta_{a\mu\nu}$

$O(4)$ matrices: self-dual, $\sigma_{\mu\nu}=\lambda_a\eta_{a\mu\nu}$: $\quad \sigma_{ij}=\frac{1}{2}\epsilon_{ijk}\lambda_k$

$\sigma_{0i}=\frac{1}{2}\lambda_i$

and self-dual, $\bar{\sigma}_{\mu\nu}=\lambda_a\bar{\eta}_{a\mu\nu}$: $\quad \bar{\sigma}_{ij}=\sigma_{ij}$

$\bar{\sigma}_{0i}=-\frac{1}{2}\lambda_i$

If we set $g(x)=(t-i\boldsymbol{\lambda}\cdot\boldsymbol{x})/r$, $\quad r^2=t^2+\boldsymbol{x}^2$, then

$$g^{-1}\,dg=i\lambda_a\sigma_a=i\lambda_a\eta_{a\mu\nu}x^\mu\,dx^\nu/r^2$$
$$dg\,g^{-1}=-i\lambda_a\bar{\sigma}_a=-i\lambda_a\bar{\eta}_{a\mu\nu}x^\mu\,dx^\nu/r^2,$$

where

$$\sigma_x=\frac{1}{r^2}(y\,dz-z\,dy+x\,dt-t\,dx), \qquad d\sigma_x=+2\sigma_y\wedge\sigma_z, \qquad \text{cyclic in } (x,y,z)$$

$$\bar{\sigma}_x=\frac{1}{r^2}(y\,dz-z\,dy-x\,dt+t\,dx), \qquad d\bar{\sigma}_x=+2\bar{\sigma}_y\wedge\bar{\sigma}_z, \qquad \text{cyclic in } (x,y,z).$$

Then the BPST solutions are

Instanton $(k=1, F=*F)$.
first gauge:

$$A=\frac{r^2}{r^2+a^2}\,i\lambda_b\sigma_b=\frac{\lambda_b}{2i}\,dx^\mu\left(-2\frac{\eta_{b\nu\mu}x^\nu}{r^2+a^2}\right)$$
$$=g^{-1}\hat{A}g+g^{-1}\,dg$$
$$F=\frac{2ia^2\lambda_b}{(r^2+a^2)^2}(dr\wedge r\sigma_b+\tfrac{1}{2}r^2\epsilon_{bcd}\sigma_c\wedge\sigma_d)$$

second gauge:

$$\hat{A} = \frac{a^2}{r^2+a^2}\,\mathrm{i}\lambda_b\bar{\sigma}_b = \frac{-\lambda_b}{2\mathrm{i}}\,\mathrm{d}x^\mu\left(+2\frac{\bar{\eta}_{b\nu\mu}x^\nu}{r^2+a^2}\left(\frac{a^2}{r^2}\right)\right)$$

$$= \frac{-\lambda_b}{2\mathrm{i}}\,\bar{\eta}_{b\mu\nu}\,\mathrm{d}x^\mu\partial^\nu \ln\left(1+\frac{a^2}{r^2}\right)$$

Anti-instanton $(k=-1, F=-*F)$.
first gauge:

$$A = \frac{r^2}{r^2+a^2}\,\mathrm{i}\lambda_b\bar{\sigma}_b = \frac{-\lambda_b}{2\mathrm{i}}\,\mathrm{d}x^\mu\left(+2\frac{\bar{\eta}_{b\nu\mu}x^\nu}{r^2+a^2}\right)$$

second gauge:

$$\hat{A} = +\frac{a^2}{r^2+a^2}\,\mathrm{i}\lambda_b\sigma_b = \frac{\lambda_b}{2\mathrm{i}}\,\mathrm{d}x^\mu\left(-2\frac{\eta_{b\nu\mu}x^\nu}{r^2+a^2}\left(\frac{a^2}{r^2}\right)\right)$$

$$= g^{-1}Ag + g^{-1}\,\mathrm{d}g$$

$$\hat{F} = \frac{2\mathrm{i}a^2\lambda_b}{(r^2+a^2)^2}\left(-\mathrm{d}r \wedge r\sigma_b + \tfrac{1}{2}r^2\epsilon_{bcd}\sigma_c \wedge \sigma_d\right)$$

2. *'t Hooft* [1976b] *and Jackiw–Nohl–Rebbi* [1977] SU(2) *solutions*. Let

$$A^{(+)} = \frac{-\lambda^a}{2\mathrm{i}}\,\mathrm{d}x^\mu\,\bar{\eta}_{a\mu\nu}\,\partial^\nu \ln\phi(x)$$

and

$$A^{(-)} = \frac{-\lambda^a}{2\mathrm{i}}\,\mathrm{d}x^\mu\,\eta_{a\mu\nu}\,\partial^\nu \ln\phi(x).$$

Then if

$$\Box\phi/\phi = 0,$$

where

$$\Box = \sum_{\mu=0}^{3}(\partial^2/\partial x^\mu\,\partial x^\mu),$$

we find that

$A^{(+)}$ has $F=+*F$ (instantons)

$A^{(-)}$ has $F=-*F$ (anti-instantons).

The solutions for ϕ yielding instanton number $|k|$ are:

$$\phi = 1 + \sum_{j=1}^{k} \frac{\rho_j}{(x - x_j)^2}: \qquad \text{'t Hooft}$$

$$\phi = \sum_{j=1}^{k+1} \frac{\rho_j}{(x - x_j)^2}: \qquad \text{Jackiw–Nohl–Rebbi.}$$

Note that the $k = 1$ 't Hooft solution is obviously equal to the BPST instanton in the second gauge.

3. *Other explicit instanton solutions.* We refer the reader to Christ, Weinberg and Stanton [1978] and Corrigan, Fairlie, Templeton and Goddard [1978] for explicit applications of the results of Atiyah, Hitchin, Drinfeld and Manin [1978].

Appendix D: Gravitational instantons

Metric: $\mathrm{d}s^2 = \mathrm{d}x^\mu\, g_{\mu\nu}(x)\, \mathrm{d}x^\nu = e^a \eta_{ab} e^b$

Vierbein: $e^a = e^a{}_\mu\, \mathrm{d}x^\mu, \qquad \eta_{ab} = \text{flat}$

Levi–Civita connection: $\mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0$

$$\omega_{ab} = -\omega_{ba} = \omega_{ab\mu}\, \mathrm{d}x^\mu$$

Curvature: $R^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2} R^a{}_{bcd} e^c \wedge e^d$

Cyclic identity: $R^a{}_b \wedge e^b = 0$

Bianchi identities: $\mathrm{d}R^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0$

Empty-space Einstein equations: $(\mathcal{R}_{ab} = R_{ambn}\eta^{mn}, \quad \mathcal{R} = \mathcal{R}_{ab}\eta^{ab})$

$$\mathcal{R}_{ab} - \tfrac{1}{2}\eta_{ab}\mathcal{R} = 0 \qquad \text{(alternate form: } \tilde{R}^a{}_b \wedge e^b = 0, \text{ where } \tilde{R}^a{}_b = \tfrac{1}{4}\epsilon^a{}_{bef} R^{ef}{}_{cd} e^c \wedge e^d).$$

Einstein equations with matter and a cosmological constant

$$\mathcal{R}_{ab} - \tfrac{1}{2}\eta_{ab}\mathcal{R} = T_{ab} - \Lambda \eta_{ab}.$$

We list a variety of explicitly known metrics and give a table of the properties of the metrics and their corresponding manifolds.

1. *Metric of Eguchi and Hanson* [1978].

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{[1 - (a/r)^4]} + r^2(\sigma_x{}^2 + \sigma_y{}^2 + [1 - (a/r)^4]\sigma_z{}^2)$$

curvature is self-dual.

Table D.1
Properties of four-dimensional gravitational instantons
* Denotes entries which are unavailable or involve issues too complex to be abbreviated in the table. – Denotes undefined items. "No. param" gives the number of parameters of the metric. (The number of actual zero-frequency modes may be larger.)

Metric	M	∂M	Λ	Self-dual: Riemann Weyl Neither	Kähler: Yes No	χ	τ	$I_{1/2}$	$I_{3/2}$	No. param	Action
Flat space	$\mathbf{R}^4$	S^3	0	R = 0	Y	1	0	0	0	0	0
Torus	T^4	$\emptyset$	0	R = 0	Y	0	0	0	0	0	0
de Sitter	S^4	$\emptyset$	>0	W = 0	N	2	0	0	0	1	$3\pi/\Lambda$
Page	$P_2+\bar{P}_2$	$\emptyset$	>0	N	N	4	0	–	–	1	$1.8\pi/\Lambda$
$S^2\times S^2$	$S^2\times S^2$	$\emptyset$	>0	N	Y	4	0	0	0	1	$2\pi/\Lambda$
Schwarzschild	$\mathbf{R}^2\otimes S^2$	$S^1\times S^2$	0	N	N	2	0	0	*	1	$4\pi M^2$
Kerr	$\mathbf{R}^2\times S^2$	$S^1\times S^2$	0	N	N	2	0	0	*	2	$\pi M/\kappa$
Eguchi–Hanson	$T^*(P_1(\mathbf{C}))$	$P_3(\mathbf{R})$	0	R	Y	2	−1	0	−2	1	0
Taub–NUT	$\mathbf{R}^4$	distorted S^3	0	R	N	1	0	0	*	1	$4\pi M^2$
Fubini–Study	$P_2(\mathbf{C})$	$\emptyset$	>0	W	Y	3	1	–	–	1	$9\pi/4\Lambda$
$\epsilon = 1$ Gibbons–Hawking	*	*	0	R	N	k	$\pm k$	0	*	*	$4\pi kM^2$
A_k ($\epsilon = 0$ Gibbons–Hawking)	*	$S^3/\mathbf{Z}_{k+1} = L(k+1, 1)$	0	R	Y	$k+1$	$-k$	0	$-2k$	$3k-2$	0
D_k^*	*	S^3/D_k^*	0	R	Y	$k+1$	$-k$	0	$-2k$	*	0
T^*	*	S^3/T^*	0	R	Y	7	−6	0	−12	*	0
O^*	*	S^3/O^*	0	R	Y	8	−7	0	−14	*	0
I^*	*	S^3/I^*	0	R	Y	9	−8	0	−16	*	0
Taub-bolt	$P_2(\mathbf{C})-\{0\}$	distorted S^3	0	N	N	2	−1	–	–	1	$\frac{4}{5}\cdot 4\pi M^2$
Rotating Taub-bolt	*	*	0	N	N	2	−1	–	–	2	$4\pi\|N\|M$
K3 (unknown)	K3	$\emptyset$	0	R	Y	24	−16	+2	−42	58	0

2. *Euclidean Taub–NUT metric* (*Hawking* [1977]).

$$ds^2 = \frac{1}{4}\frac{r+m}{r-m}\,dr^2 + (r^2-m^2)(\sigma_x{}^2+\sigma_y{}^2) + 4m^2\frac{r-m}{r+m}\,\sigma_z{}^2$$

curvature is self-dual.

3. *Fubini–Study metric on $P_2(\mathbf{C})$.*

$$ds^2 = \frac{dr^2 + r^2\sigma_z{}^2}{(1+\Lambda r^2/6)^2} + \frac{r^2(\sigma_x{}^2+\sigma_y{}^2)}{1+\Lambda r^2/6}$$

self-dual Weyl tensor
cosmological term Λ.

4. *Taub–NUT–De Sitter metrics* (include 1, 2, 3 in appropriate limits).

$$ds^2 = \frac{L^2-\rho^2}{4\Delta}\,d\rho^2 + (\rho^2 - L^2)(\sigma_x{}^2 + \sigma_y{}^2) + \frac{4L^2\Delta}{\rho^2 - L^2}\cdot\sigma_z{}^2$$

$$\Delta = \rho^2 - 2m\rho + l^2 + \frac{\Lambda}{4}(l^4 + 2l^2\rho^2 - \tfrac{1}{3}\rho^4)$$

these metrics are not necessarily regular
cosmological term Λ.

5. *Gibbons–Hawking multi-center metrics* [1978].

$$ds^2 = V^{-1}(\boldsymbol{x})(d\tau + \boldsymbol{\omega}\cdot d\boldsymbol{x})^2 + V(\boldsymbol{x})\,d\boldsymbol{x}\cdot d\boldsymbol{x}$$

$$\nabla V = \pm\nabla\times\boldsymbol{\omega}$$

$$V = \epsilon + 2m\sum_{i=1}^{k}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_i|}$$

$\epsilon = 1$ multi-Taub–NUT ($k = 1 \to$ Taub–NUT)

$\epsilon = 0$ multi-asymptotically locally Euclidean $\begin{cases} k = 1 \to \text{ flat} \\ k = 2 \to \text{Eguchi–Hanson}\end{cases}$

self-dual or anti-self-dual curvature.

6. *Euclidean Schwarzschild metric.* (t has period $8\pi M$.)

$$ds^2 = (1 - 2M/R)\,dt^2 + \frac{1}{1-2M/R}\,dR^2 + R^2(d\theta^2 + \sin^2\theta\,d\phi^2)$$

7. *Euclidean Kerr metric.* (t has period $2\pi/\kappa$, ϕ has period $2\pi\alpha/\sqrt{M^2+\alpha^2}$.)

$$ds^2 = (r^2 - \alpha^2\cos^2\theta)\left(\frac{dr^2}{r^2 - 2Mr - \alpha^2} + d\theta^2\right) + \frac{\sin^2\theta}{r^2-\alpha^2\cos^2\theta}(\alpha\,dt - (r^2-\alpha^2)\,d\phi)^2$$

$$+\frac{r^2 - 2Mr - \alpha^2}{r^2 - \alpha^2\cos^2\theta}(dt - \alpha\sin^2\theta\,d\phi)^2$$

$\alpha = J/M$, Kerr parameter $\kappa = \sqrt{M^2+\alpha^2}/\{2M(M + \sqrt{M^2+\alpha^2})\}$

8. *de Sitter metric on* S^4.

$$ds^2 = [1 + (r/2R)^2]^{-2}(dr^2 + r^2\sigma_x{}^2 + r^2\sigma_y{}^2 + r^2\sigma_z)$$

curvature is not self-dual
Weyl tensor vanishes.

9. *$S^2 \times S^2$ metric.*

$$ds^2 = (1 - \Lambda r^2)\,d\tau^2 + \frac{dr^2}{(1 - \Lambda r^2)} + \frac{1}{\Lambda}(d\theta^2 + \sin^2\theta\, d\phi^2)$$

curvature is not self-dual
cosmological term Λ.

10. *Page metric* [1978b] on $P_2(\mathbb{C}) \oplus \overline{P_2(\mathbb{C})}$.

$$ds^2 = 3\Lambda^{-1}(1 + \nu^2)\left\{\frac{(1 - \nu^2 x^2)}{[3 - \nu^2 - \nu^2(1 + \nu^2)x^2]}\frac{dx^2}{1 - x^2}\right.$$
$$\left. + 4(\sigma_x{}^2 + \sigma_y{}^2)\frac{1 - \nu^2 x^2}{3 + 6\nu^2 - \nu^4} + 4\sigma_z{}^2(1 - x^2)\frac{[3 - \nu^2 - \nu^2(1 + \nu^2)x^2]}{(3 + \nu^2)^2(1 - \nu^2 x^2)}\right\}$$

curvature is not self-dual
cosmological term Λ.

11. *Taub-bolt metric* (Page [1978a]).

$$ds^2 = \frac{r^2 - N^2}{r^2 - 2.5Nr + N^2}\cdot dr^2 + 16N^2\frac{r^2 - 2.5Nr + N^2}{r^2 - N^2}\cdot \sigma_z{}^2 + 4(r^2 - N^2)(\sigma_x{}^2 + \sigma_y{}^2)$$

curvature is not self-dual.

12. *Rotating Taub-bolt metric* (Gibbons and Perry [1979]).

$$ds^2 = \Xi(r, \theta)\left(\frac{dr^2}{\Delta} + d\theta^2\right) + \frac{\sin^2\theta}{\Xi(r, \theta)}(\alpha\, dt + P_r\, d\phi)^2$$
$$+ \frac{\Delta}{\Xi(r, \theta)}(dt + P_\theta\, d\phi)^2$$

$$\Delta = r^2 - 2Mr + N^2 - \alpha^2$$

$$P_\theta = -\alpha\sin^2\theta + 2N\cos\theta - \frac{\alpha N^2}{N^2 - \alpha^2}$$

$$P_r = r^2 - \alpha^2 - \frac{N^4}{N^2 - \alpha^2}$$

$$\Xi(r, \theta) = P_r - \alpha P_\theta = r^2 - (\alpha\cos\theta + N)^2$$

curvature is not self-dual.

13. *K3 metric.* The K3 metric with self-dual curvature is not known. For a discussion of approximations to the K3 metric, see Page [1979c].

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