Operational Method of Circuit Analysis

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For that broad group of electrical engineers not familiar with Heaviside's operational method and its value in the analysis of electrical circuits, this article presents briefly some of the fundamental characteristics of the method and explains how it is applied; its application is illustrated by the formulation of operational equations for a few typical circuits. Although intended primarily to impart only the fundamentals of the subject to an engineer who has had ordinary mathematical training, it is hoped that the article also may serve as an introduction and guide to further study.

APPLICATION of operational methods to circuit studies has been growing rapidly within recent years, and more and more this form of analysis is achieving greater importance as a tool for regular use in handling engineering problems. The operational calculus is found to be of invaluable service in the solution of transients and in the treatment of dynamical systems, whether of an electrical or a mechanical nature. The steady advance made in the general knowledge and use of the Heaviside operational method is shown by the increasing number of technical papers that employ it.

This article has been written because of the spreading interest definitely indicated in operational circuit analysis, and the desire on the part of many who are unfamiliar with it to understand something of the principles of its application. The intention here is not to treat with the mathematics as such or with recent developments in the operational calculus. Merely, it is the aim to present what to the writer are some of the fundamental characteristics of the operational method, to show the formulation of the operational equations for a few circuits, and to discuss briefly some of the methods by which the solutions to these equations may be obtained. Stress is given the physical reasoning behind the steps.

Nature of Transients

Studies in engineering and physics give rise to problems on simple circuits, networks, and systems, these problems relating to any one of the major fields of electricity, mechanics, heat flow, sound, etc. Those of particular interest here are on transients in electrical circuits. The divisions of the field of electrical engineering in which transient analysis is of importance are many, as indicated by the applications Gardner has listed.[1]

In general, the study of electrical transients is the investigation of the response of such systems to disturbances produced in them. Specifically, the analysis of transients in electrical circuits involves the determination of the variation of charge, current, voltage, power, or energy with time while the system is undergoing a readjustment. It is inherent in all physical systems that even though the disturbing force may be applied instantaneously, time is required for the system to pass from one steady state condition to another. Time, therefore, is the independent variable, the quantity under investigation the dependent variable, to be expressed in the end as a function of time.

Heaviside's Objective

Briefly stated, the objective of Heaviside in analyzing circuits was to express the solutions of the problems in terms of functions of his operators, and then to assign such significance to these relations that their interpretations would be the *correct* solutions to his problems. The solutions were to be obtained as directly as possible, and were to give answers without further adjustment by way of any consideration of constants or functions of integration to include boundary conditions, as must be done in solving differential equations. The great majority of problems with which Heaviside treated involved systems initially at rest, the disturbing forces thus being zero for all values of time less than zero, and taking on their normal variations after zero time. Heaviside took advantage of this fact and sought to obtain solutions that automatically applied to such boundary conditions, his developments leading to this form of operational mathematics.

The operational equation, which is the initial expression for a problem in terms of operators and the circuit constants, must contain, therefore, all information necessary to solve the problem completely. The solution to the operational equation is to be the same as that for the differential equation after initial conditions have been included, and is to yield at once the total result, which for circuit transients is the transient plus the steady state variations of the quantity studied.

Heaviside's own method of interpreting his operational functions, and in converting them into explicit functions of time, appears to be one almost entirely experimental. It would seem that he must have applied his method to problems having known solutions, and in this way determined how well his general rules worked and what he could or could not do. No doubt such steps provided him with the body of rules which he used in his analyses but on which he gave little or no discussion in his technical

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1. For all numbered references see list at end of article.

papers. Because he studied circuits of the linear type on which a direct voltage was impressed at time $t = 0$, he had as a guide the solutions of the differential equations for those circuits. The solutions of simple problems were not difficult to obtain. His aim was to secure the same results by means of his operators without having to make recourse to differential equations or to problem boundary conditions, once the operational equations were written. By comparing operational forms for the same problems with the solutions obtained otherwise, the significance of these forms was indicated.

It may be well to state at this point that once an operational function is found to be related definitely to a given form of solution, the same operational expression met in any other problem will carry the same interpretation. Its translation in terms of a time function may be written immediately. The operational expressions with their time function equivalents, as a table, therefore become fundamental to the working use of the operational calculus as the integral table is to the ordinary calculus.

The correlation existing between the operational calculus and classical mathematics has been brought about in relatively recent years, and the mathematical material presented today in support of operational methods was not touched upon by Heaviside. Very much of it, of course, did not exist then, and mathematical rigor as a defense for his developments did not appeal to Heaviside, particularly in view of the fact that he was obtaining correct results for complicated problems. He was concerned primarily that his mathematics give him accurate answers in a quick and satisfactory way. Guided greatly by intuition and his wealth of knowledge on the physics behind his circuit studies, he developed the operational calculus now ascribed to his name.

Formulation of the Problem

The first step toward the solution of a problem is the derivation, from the fundamental *physical* relations of the system, of the *analytical* equations that adequately are to represent the system. The mathematical expressions thus obtained become the shorthand statements of the conditions surrounding the problem, and are to convey the same thoughts that otherwise would be stated in a longer way by words. That is, the mathematical formulation of engineering problems is nothing else than the symbolizing of physical ideas. The effort required to symbolize a problem depends upon the end desired, whether steady state conditions only are to be considered, or partial studies of the aspects of transient phenomena are to be carried out, or the complete solution including both transient and steady state relations is to be found.

Not all problems can be formulated in this manner because of the complexities encountered, and the difficulty in considering all the variables and the way in which they possibly should be introduced. The phenomena under treatment in this discussion are those that may be described by linear differential equations or the operational calculus.

Methods of Solution

The second step in the solution of a problem consists of carrying out the formal mathematical operations by which the initial circuit equations may be said to be solved. The resulting system equations usually will be expressed also in symbolic form, consequent calculations giving numerical results. Fundamentally, the methods of solving engineering problems fall into 2 major divisions. The first of these is qualitative, based upon reasoning from the physics of the problem together with reasoning from past experiences. The second division is quantitative, under which experimental schemes and mechanical devices may be employed as well as mathematics. Because of its power as a tool in the analysis of circuits, the operational calculus is becoming more prominent and is supplanting other forms of solution in many instances.

The Operator and the Unit Function

The operational calculus generally is typified by 2 symbols, the operator p, and the unit function **1**. The operator in its use probably is more mathematical than physical, the unit function more physical than mathematical. Operators are symbolic quantities indicating certain steps to be followed, or calling attention to given interpretations to be recognized. Operators are not new and engineers are familiar with many of them, for example, the trigonometric symbols "sin," "cos," "tan," etc.; the logarithmic and exponential notations; the operators j, and a (the latter met in studies on symmetrical components); and others such as D ($= d/dx, d/dt$, etc.). These quantities point to definite processes to be performed, although they are not always considered as operators.

The operator p in the Heaviside calculus [2,3] initially is to represent the time differentiator d/dt. Further, it is desired that this operator bear the reciprocal relation such that $1/p$ denote an integration. To make the operator as useful as possible it also should obey the ordinary rules of algebra. In addition to p indicating an operation, those who are acquainted with the solutions to linear differential equations will observe that it is the variable in the characteristic equation. The symbol p as used in the Heaviside operational calculus thus has several different interpretations.

If the purposes Heaviside had in mind are to be achieved, the operator p may not be defined presumptively. Its interpretation depends upon the physical nature of the problem at hand, and its meaning will change with transformations of the operational equations that contain it. The operational equation may be obtained from the corresponding differential equation by making the substitutions for the time differentiators, i. e., $p = d/dt$, ... $p^n = d^n/dt^n$, with the accompanying reciprocal relations $1/p = \int dt$, etc. In problems on transmission lines and in heat flow appear expressions containing p to half powers, i. e., $p^{1/2}$, $p^{3/2}$, etc. In no sense, though, does $p^{1/2}$ imply $(d/dt)^{1/2}$. Such a relation does not exist, and no significance may be attached to it. Fractional powers of p must be interpreted as they are met, and in view of the problem itself. The operator p, as it appears in its various manners, is not subject to any single definition and may not be characterized completely as yet.

The unit function,[3, 4] **1**, is a discontinuous function of time, and is defined as the function plotted in figure 1*a;* it is a function that is continuously zero until the time $t = 0$, and continuously unity thereafter. It thus displays the discontinuous form of the disturbing force, as illustrated further by the 3 other graphs of figure 1. The unit function is not

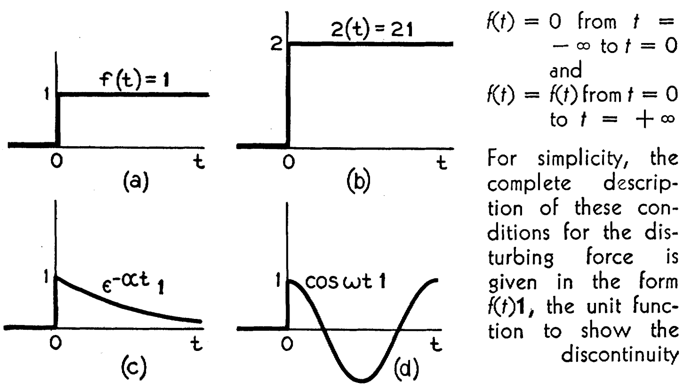


Fig. 1. Significance of the unit function

mentioned explicitly by all writers on operational calculus, but even when omitted in discussion or in their mathematics it usually is implied. Its principal use is to avoid a great many unnecessary words, for once it has been defined only $E(t)\cdot\mathbf{1}$ need be written to indicate an electromotive force that is applied at time $t = 0$ and thereafter has the value $E(t)$. Without the notation it would be necessary to explain that this is what is meant.

The following characteristics thus represent essentials in the operational method:

1. $p = d/dt,\ p^2 = d^2/dt^2, \ldots\ldots p^n = d^n/dt^n$; p is the time differentiator.

2. $\dfrac{1}{p} K\mathbf{1} = \displaystyle\int_0^t K\mathbf{1}dt = Kt$ (which may be written $Kt\mathbf{1}$, if desired); the definite integral from zero time to time t

   $\dfrac{1}{p^2} K\mathbf{1} = \displaystyle\int_0^t\int_0^t K\mathbf{1}dt^2 = \dfrac{Kt^2}{2}, \ldots\ldots$

   $\dfrac{1}{p^n} K\mathbf{1} = \dfrac{Kt^n}{n!}$, by repeated applications of $\dfrac{1}{p} K\mathbf{1}$; the inverse process of differentiation, but a definite integration.[5]

3. $p\dfrac{1}{p} K\mathbf{1} = K\mathbf{1}$; p obeys the ordinary rules of algebra.

Formulation of Operational Equations

With the foregoing discussions kept in mind, one turns next to the formulation of the operational equations for several simple circuits in order to investigate the manner in which they are expressed. Figure 2 shows a few typical circuit diagrams, these having been chosen purposely to illustrate certain fundamental features in deriving the operational equation. Space does not permit the inclusion of merited discussions on other circuit conditions of importance; these must be left to the reader to study.

Figure 2*a* represents a circuit of resistance and inductance to which is applied a direct voltage of E volts at $t = 0$, the current response being desired. From Kirchhoff's second law that, when taken with the proper algebraic signs, the sum of the voltages around a closed loop is zero, the differential equation for the circuit becomes

$$-Ri - L\frac{di}{dt} + E = 0$$

or

$$L\frac{di}{dt} + Ri = E \qquad (1)$$

The 2 terms involving the current carry negative signs with respect to the impressed electromotive force because these 2 voltages are in opposition to it.

Replacing the time derivative by the operator[6] p and introducing the unit function to call attention to the form of the applied voltage, the operational equation for the circuit becomes

$$Lpi + Ri = E\mathbf{1} \qquad (2)$$

Treating p algebraically, the operational solution for the current is

$$i = \frac{1}{Lp + R}E\mathbf{1} = \frac{E}{L}\frac{1}{p + \dfrac{R}{L}}\mathbf{1}$$

$$= \frac{E}{L}\frac{1}{p + \alpha}\mathbf{1} \qquad (3)$$

where $\alpha = R/L$. The operational impedance function for the circuit is designated by $Z(p) = (Lp + R)$. Solving the problem means finding a correct interpretation of $1/(p + \alpha)$ operating upon unit function.

For figure 2*b* there results (all capacitances in the examples are assumed initially uncharged)

$$Ri + \frac{\int idt}{C} = E \qquad (4)$$

which in operational symbols becomes

$$Ri + \frac{1}{C}\frac{1}{p}i = E\mathbf{1} \qquad (5)$$

and from which

$$i = \frac{1}{R + \dfrac{1}{pC}}E\mathbf{1}$$

$$= \frac{E}{R}\frac{p}{p + \alpha}\mathbf{1} \qquad (6)$$

where α now is $1/RC$. Just as a note of interest, those familiar with the elements of circuit theory will recognize in α, for both of these circuits, the reciprocal of the circuit time constants.

In like manner to the above developments, the operational solution for the total current in the circuit given by figure 2c is

$$i = \frac{1}{Z(p)}E1 = \frac{1}{r + \dfrac{(R+pL)\left(\dfrac{1}{pC}\right)}{R + pL + \dfrac{1}{pC}}}E1$$

$$= E\frac{p^2LC + pCR + 1}{p^2LCr + p(CRr + L) + (R + r)}1 \qquad (7)$$

It may be observed that the operational impedance function as first written in terms of the circuit elements is identical in form to that obtained for the equivalent impedance of a series-parallel circuit when expressed in terms of complex quantities. The operator p replaces $j\omega$. One initially may handle circuits in this manner, later substituting the proper operational equivalents for the corresponding circuit elements and reducing the resulting algebraic equation to its simplest form.

The circuit of figure 2d assumes a current of constant magnitude i suddenly forced through a parallel combination of capacitance and inductance. The resulting circuit voltage, which now is desired, is

$$e = Z(p)i1 = \frac{pL \cdot \dfrac{1}{pC}}{pL + \dfrac{1}{pC}}i1 = \frac{pL}{p^2LC + 1}i1$$

$$= \frac{i}{C}\frac{p}{p^2 + \alpha^2}1 \text{ (as usually expressed)} \qquad (8)$$

where for this circuit $\alpha = 1/\sqrt{LC}$.

The circuit of figure 2e is the same as that of figure 2d, but the current now is sinusoidal in form, the switch being closed at the instant the current is passing through zero and increasing positively. Since the impedance function for the circuit is the same as that just found, the voltage across the circuit elements becomes

$$e = Z(p)i1$$

$$= \frac{I}{C}\frac{p}{p^2 + \alpha^2}(\sin \omega t)1 \qquad (9)$$

The particular point to be emphasized here is that the operand is the entire part of the expression $(\sin \omega t)1$, and $p/(p^2 + \alpha^2)$ operates upon it as a whole. In other words, sin wt may not be treated simply as a multiplying factor, thus to retain its identity, nor may $p/(p^2 + \alpha^2)$ be evaluated by itself. The entire right-hand member, except for the constant term I/C, must be interpreted as a unit. This process is to be shown later.

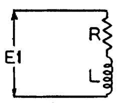
In order to handle properly the right hand member of equation 9 it is convenient to convert the time function sin ωt to its operational equivalent which later is shown to be

$$\sin \omega t = \frac{p\omega}{p^2 + \omega^2}1 \qquad (10)$$

The operational equation for the voltage therefore is found to be

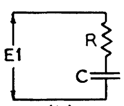
$$e = \frac{I}{C}\left(\frac{p}{p^2 + \alpha^2}\frac{p\omega}{p^2 + \omega^2}\right)1 \qquad (11)$$

from which, with correct interpretation of the



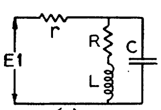
(a)

$$L\frac{di}{dt} + Ri = E \qquad Lpi + Ri = E1 \qquad i = \frac{E}{Lp + R}1 = \frac{E}{L}\frac{1}{p + \dfrac{R}{L}}1 = \frac{E}{R}\left(1 - \epsilon^{-\frac{R}{L}t}\right)$$



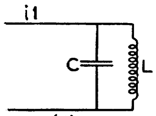
(b)

$$Ri + \frac{\int i\,dt}{C} = E \qquad Ri + pC\,i = E1 \qquad i = \frac{E}{R + \dfrac{1}{pC}}1 = \frac{E}{R}\frac{p}{p + \dfrac{1}{RC}}1 = \frac{E}{R}\epsilon^{-\frac{t}{RC}}$$



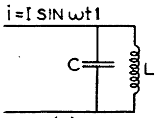
(c)

$$iZ(p) = E1 \qquad Z(p) = r + \frac{(R + pL)\left(\dfrac{1}{pC}\right)}{R + pL + \dfrac{1}{pC}} \qquad i = \frac{p^2LC + pCR + 1}{p^2LCr + p(CRr + L) + (R + r)}E1$$



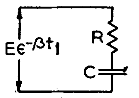
(d)

$$e = Z(p)i1 \qquad Z(p) = \frac{pL\dfrac{1}{pC}}{pL + \dfrac{1}{pC}} \qquad e = \frac{pL}{p^2LC + 1}i1 = \frac{i}{C}\frac{p}{p^2 + \dfrac{1}{LC}}1 = i\sqrt{\frac{L}{C}}\sin\frac{t}{\sqrt{LC}}$$



(e)

$$e = Z(p)i1 = \frac{pL}{p^2LC + 1}I\sin\omega t1 \qquad e = \frac{I}{C}\left(\frac{p}{p^2 + \dfrac{1}{LC}}\frac{p\omega}{p^2 + \omega^2}\right)1 = \frac{IL\omega}{\omega^2LC - 1}\left(\cos\frac{t}{\sqrt{LC}} - \cos\omega t\right)$$



(f)

$$Ri + \frac{i}{pC} = E\epsilon^{-\beta t}1 \qquad i = \frac{1}{R + \dfrac{1}{pC}}E\epsilon^{-\beta t}1 \qquad i = \frac{E}{R}\frac{p}{p + \dfrac{1}{CR}}\frac{p}{p + \beta}1 = \frac{E}{1 - RC\beta}\left(\frac{1}{R}\epsilon^{-\frac{t}{RC}} - C\beta\epsilon^{-\beta t}\right)$$

Fig. 2. Circuit diagrams and associated equations illustrating the application of the operational method

quantity within the parentheses operating upon unit function, the solution is reached.

The last illustration, given by figure 2*f*, involves the same point made in the previous example. The expression for the current here is sought, whence

$$i = \frac{1}{R + \frac{1}{pC}} E\epsilon^{-\beta t}\,\mathbf{1}$$

$$= \frac{E}{R}\left(\frac{p}{p+\alpha}\,\frac{p}{p+\beta}\right)\mathbf{1} \qquad (12)$$

where $\alpha = 1/RC$. The same statements hold as before for the operation of the entire quantity within the parentheses on unit function. Both problems indicate the requirement of converting a time function, other than a constant, in the operand into its equivalent operational form before proceeding to the final solution. For the last circuit, however, there is another method, Heaviside's shifting theorem, which will remove the exponential term from the operand. There are several ways of treating this problem.

To summarize briefly, the operational equation is an expression in terms of constants of the circuit and functions of the operator p. It is simply the formulated statement, expressed in symbols, of the physical problem under consideration. For a single circuit it stands exactly for the differential equation with its surrounding conditions which otherwise might be written, and its initial form may be obtained directly from the differential equation if desired. For a system or a network of n loops, there results a system of n simultaneous operational equations just as there also may be derived a system of n simultaneous differential equations. The operational equations, properly interpreted, must give solutions identical to those found for the differential equations with their boundary conditions included. Operational equations hence may be said to represent in abbreviation the actual equations, differential or integral, for the problem, together with the special form of boundary conditions which implies that the system initially is at rest.

Solutions to the Operational Equations

It is necessary next to determine the interpretations for the operational equations, and, for value to the reader, to solve those developed in the preceding section of this article. Setting up the operational equation for a circuit is relatively simple, but to obtain its solution is an entirely different matter. Leaving aside any discussion of the so-called classical methods of solving circuits, i. e., by differential or integral equations, attention is restricted to purely operational modes of attack of which there are in general several for most problems.

Solutions to operational equations may be found in several ways, prominent ones of which are:

1. Interpretations of operational expressions from known solutions. The operational equation for a specific circuit must yield the same result as that found by other means, e. g., by differential equations. Heaviside interpreted the operator $p^{1/2}\mathbf{1}$, for instance, as $(\pi t)^{-1/2}$ from the known solutions to heat flow problems.[7]
2. Long division of the operational equation to obtain a series, each term of which then is evaluated. The method includes binomial theorem expansions. Heaviside referred to this method as "algebrizing."[8]
3. Heaviside's expansion theorem.[9–17]
4. Partial fraction expansion of the operational equation into recognizable parts, a method identical to that employed in the integral calculus to assist in the evaluation of integrals, and a standard method in the theory of differential equations for many years.[18,19]
5. Heaviside's shifting theorem, to remove exponential time factors from the operand.[20–22]
6. Borel's theorem, relating to the product of 2 functions.[23–25]
7. Contour integration of the equivalent Fourier integral.[26–30]
8. Carson's integral equation.[31–35]

Some of these methods will be illustrated by application to the problems presented in figure 2.

Reasoning From Known Solutions

The solutions for the currents in the circuits of figures 2*a* and 2*b* are known readily—in fact, these 2 problems are perhaps the first ones discussed in any elementary study of transients. After the general solutions of the respective differential equations have been obtained, substitution of initial conditions gives the particular solutions. Determination of the constants of integration depends upon the physical relations that, in the first circuit $i = 0$ when $t = 0$, and in the second circuit, $i = E/R$ when $t = 0$. The currents are expressed explicitly as functions of time and are, respectively

$$i = \frac{E}{R}(1 - \epsilon^{-(R/L)t}) \qquad (13)$$

and

$$i = \frac{E}{R}\epsilon^{-t/RC} \qquad (14)$$

Since the solutions of the operational equations are to yield the same results, of necessity, then the interpretations of equations 3 and 6 follow immediately. Comparison of equations 3 and 13 gives

$$\frac{E}{L}\frac{1}{p+\alpha}\mathbf{1} = \frac{E}{R}(1 - \epsilon^{-(R/L)t})$$

whence

$$\frac{1}{p+\alpha}\mathbf{1} = \frac{1}{\alpha}(1 - \epsilon^{-\alpha t}) \qquad (15)$$

as a general relation. Similarly, comparison of equations 6 and 14 gives

$$\frac{E}{R}\frac{p}{p+\alpha}\mathbf{1} = \frac{E}{R}\epsilon^{-t/(RC)}$$

whence

$$\frac{p}{p+\alpha}\mathbf{1} = \epsilon^{-\alpha t} \qquad (16)$$

It may be observed that whatever α may be

$$\frac{p}{p+\alpha}\mathbf{1} = p\left(\frac{1}{p+\alpha}\mathbf{1}\right) = \frac{d}{dt}\left[\frac{1}{\alpha}(1 - \epsilon^{-\alpha t})\right] \qquad (17)$$

In like manner

$$\frac{1}{p+\alpha}\mathbf{1} = \frac{1}{p}\left(\frac{p}{p+\alpha}\mathbf{1}\right) = \int_0^t \epsilon^{-\alpha t}dt \quad (18)$$

The time derivative or integral obviously is taken of the equivalent time function for the operational expression. It should be understood that the juxtaposition of 2 operational functions does not mean the product of their respective time functions. An operation is implied, and the entire quantity must so be treated.

Although these 2 examples are given for special and the simplest of circuits, the time function equivalents of the corresponding operational expressions are perfectly general and may be used on all other circuits that yield the same operational forms. There is only one time function that is related to a given operational expression.

To illustrate the value of what has just been presented, consider the operational equivalent of sin ωt (refer to equation 10):

$$\sin \omega t = \frac{1}{2j}(\epsilon^{j\omega t} - \epsilon^{-j\omega t}) = \frac{1}{2j}\left(\frac{p}{p-j\omega} - \frac{p}{p+j\omega}\right)\mathbf{1}$$

$$= \frac{p\omega}{p^2+\omega^2}\mathbf{1} \quad (19)$$

Similarly

$$\cos \omega t = \frac{p^2}{p^2+\omega^2}\mathbf{1} \quad (20)$$

Both relations are of importance in operational circuit analysis.

Operational Division

The second manner of evaluating operational forms is that of long division, which for equations 3 and 6 gives the binomial expansions for those forms. To illustrate, when applied to equation 6 there results

$$\begin{aligned}\frac{p}{p+\alpha}\mathbf{1} &= p(p+\alpha)^{-1}\mathbf{1}\\ &= p(p^{-1} - p^{-2}\alpha + p^{-3}\alpha^2 - p^{-4}\alpha^3 + \ldots\ldots)\,\mathbf{1}\\ &= \left(1 - \frac{\alpha}{p} + \frac{\alpha^2}{p^2} - \frac{\alpha^3}{p^3} + \ldots\ldots\right)\mathbf{1}\\ &= \left(1 - \alpha t + \frac{\alpha^2 t^2}{2!} - \frac{\alpha^3 t^3}{3!} + \ldots\ldots\right)\\ &= \epsilon^{-\alpha t}\end{aligned} \quad (21)$$

The fourth step is obtained from the third by application of $\frac{1}{p^n}\mathbf{1} = \frac{t^n}{n!}$, and the resulting series is recognized as the expansion of $\epsilon^{-\alpha t}$. The same method of expansion holds for $\frac{1}{p+\alpha}\mathbf{1}$.

The solutions thus obtained are complete; they include the transient and the steady state variations of the currents. In the method of operational division, or where p appears in the denominator as it does in the foregoing expressions, it is the added restriction of making the reciprocal of the operator p the definite integral from zero time to time t that has the effect of including the constants of integration in the solution. Other methods of solution, one of which has been illustrated in the preceding section, give the correct result without recourse to integration.

More complicated operational expressions may be treated similarly through some convenient form of expansion of the operator into a power series, often by formal division of the numerator by the denominator of the operational fraction. The objection to operational expansions of this kind, however, is that they lead to infinite series which at times are extremely difficult, if not impossible, to recognize and sum. Furthermore, they may be of no, or very limited, value for numerical computations, though they are sometimes very useful in obtaining approximate results.

One point should be presented before passing on. The process of "algebrizing" the operational equation will give 2 series, depending upon the manner of expansion. The binomial theorem, for example, always can be written in 2 ways $(p+\alpha)^{-1}$ or $(\alpha+p)^{-1}$. In the first instance p appears in the denominators of the consequent terms, and in the second instance in the numerators. Where p occurs in descending powers, as it does in equation 21, the time function may be evident in its relation to the correct problem solution. With p appearing in whole powers in the numerators, the time function being $\mathbf{1}$, a series of zeros results and correspondence with the true solution is not observed. In the example just given it has been necessary to expand in inverse powers of p as shown.

Series expansions of operators may lead to a convergent series, or to a divergent or asymptotic solution. One is satisfactory for numerical computation for small values of time, the other for large values of time. Failure, however, may occur with either series, the algebrizing process breaking down. In brief, Heaviside's method of series expansion of the operator will not always give correct results automatically; caution, therefore, must be exercised in obtaining such forms of solution. One should investigate the correctness of the methods employed and the completeness of his results.

The Expansion Theorem

With but little discussion, Heaviside gave, in volume 2 of Electromagnetic Theory, and volume 2 of Electrical Papers, his expansion theorem. The theorem is a formal means by which the solutions of operational equations may be found directly. It is applicable to most problems, including a great many on transmission lines and heat flow where the constants are distributed rather than lumped. When the theorem does apply it is probably the most convenient method to use.

No derivation of the expansion theorem is given here, merely the statement of it:

$$i = E\left[\frac{1}{Z(0)} + \sum_{p_1}^{p_n}\frac{\epsilon^{p_k t}}{p_k Z'(p_k)}\right]\mathbf{1} \quad (22)$$

In brief, $Z(0)$ is the operational impedance function in which p and its powers have been replaced by zero; $p_1, p_2, \ldots\ldots p_n$ are the n roots of the equation $Z(p) = 0$; and $Z'(p_k)$ is $d/dp\ [Z(p)]$, after which the particular root p_k, for the kth term under consideration, is substituted for the general root p.

To show the steps in the use of the theorem, consider the R-L circuit of figure 2a. Here,

$$Z(p) = Lp + R = 0 \tag{23}$$

and only one root, $p = p_1 = -R/L$, exists; $Z(0) = R$ and $Z'(p) = L$, whence $p_1 Z'(p_1) = -R$. The solution for the current is

$$i = E\left[\frac{1}{R} - \frac{1}{R}\epsilon^{-(R/L)t}\right] \tag{24}$$

The circuit of figure 2c is solved more conveniently by the expansion theorem than by other means, although when written out completely in symbolic form the algebraic expressions are somewhat cumbersome to manipulate. With circuit data: $E = 100$ volts; $r = 100$ ohms; $R = 150$ ohms; $L = 0.1$ henry; and $C = 80$ microfarads, the solution for the current is

$$i = 0.40 + 0.6075\epsilon^{-223t} - 0.0075\epsilon^{-1402t} \tag{25}$$

which the reader may verify.

It may be seen that the term $Z(0)$ leads to the steady state current in the examples shown, the exponentials yielding the transient components. With an alternating voltage impressed on the circuits, the term $1/Z(0)$ disappears, and the steady state solutions come from the exponentials, those particular ones having imaginary indices $\pm j\omega t$ which lead to sin ωt or cos ωt, the form of the applied voltage.

The expansion theorem often is easily handled, and, when such is the case, leads to solutions that readily can be evaluated numerically. The greatest difficulty encountered in its use enters in determining the roots of $Z(p) = 0$, particularly when $Z(p)$ is of degree in p higher than the fourth. This same difficulty, however, is met in factoring all algebraic equations and is not peculiar in any way to the expansion theorem. One of the restrictions on the expansion theorem is that in its development zero and repeated roots are excluded. This restriction is not one of much consequence because physical problems tend only to approach such conditions (as expressed in this mathematical manner), and that but very seldom.

Partial Fraction Expansion

Partial fraction expansion, the basic principle underlying the expansion theorem, is the operation of decomposing a given fraction into a group of simpler fractions. These partial fractions have denominators that are factors of the denominator of the given fraction, and hence the latter is converted into a sum of expressions each one of which can be treated individually and more conveniently. The given fraction usually is reduced to a proper fraction before being decomposed, unless the degree of the numerator is already less than that of the denominator. It is assumed that the reader will recall the method of partial fraction expansion.

This method applied to equation 8, which was derived for the circuit of figure 2d, gives for the voltage,

$$\begin{aligned} e &= \frac{i}{C}\frac{p}{p^2+\alpha^2}\mathbf{1} \\ &= \frac{i}{C}\left[\frac{1}{2(p+j\alpha)} + \frac{1}{2(p-j\alpha)}\right]\mathbf{1} \\ &= \frac{i}{C}\left[\frac{1}{2j\alpha}(1-\epsilon^{-j\alpha t}) - \frac{1}{2j\alpha}(1-\epsilon^{j\alpha t})\right] = \frac{i}{C\alpha}\frac{\epsilon^{j\alpha t}-\epsilon^{-j\alpha t}}{2j} \\ &= i\sqrt{\frac{L}{C}}\sin\frac{t}{\sqrt{LC}} \end{aligned} \tag{26}$$

Observe also, that

$$e = i\sqrt{\frac{L}{C}}\frac{p\alpha}{p^2+\alpha^2}\mathbf{1} = i\sqrt{\frac{L}{C}}\sin\frac{t}{\sqrt{LC}}$$

an application of the relation given by equation 19. This problem can be attacked equally as well by use of the expansion theorem.

Solving equation 11, for the circuit of figure 2e, the method of partial fraction expansion gives for the voltage

$$\begin{aligned} e &= I\sqrt{\frac{L}{C}}\left(\frac{\alpha\omega}{\omega^2-\alpha^2}\frac{p^2}{p^2+\alpha^2} - \frac{\alpha\omega}{\omega^2-\alpha^2}\frac{p^2}{p^2+\omega^2}\right)\mathbf{1} \\ &= \frac{IL\omega}{\omega^2 LC - 1}\left(\cos\frac{t}{\sqrt{LC}} - \cos\omega t\right) \end{aligned} \tag{27}$$

Heaviside's Shifting Theorem

To show the manner in which an exponential time function is removed from the operand of an operational equation, Heaviside's shifting theorem is employed. The theorem, not proved here, allows an exponential, such as $\epsilon^{-\beta t}$, to be shifted from the operand by the substitution of $(p-\beta)$ for all p's in the rest of the operational equation. That is,

$$\frac{p}{p+\alpha}\epsilon^{-\beta t}\mathbf{1} = \epsilon^{-\beta t}\frac{p-\beta}{p+\alpha-\beta}\mathbf{1} \tag{28}$$

The exponential time function now becomes a multiplying factor only.

Equation 28 is a second form of the operational equation representing the circuit of figure 2f, whence, by partial fraction expansion of the operand, or by use of the expansion theorem

$$i = \frac{E}{1-RC\beta}\left[\frac{1}{R}\epsilon^{-t/(RC)} - C\beta\epsilon^{-\beta t}\right] \tag{29}$$

Space does not permit further discussion of methods of solving operational equations, but the methods presented, though in brief, should give some idea of the operational attack on physical problems.

Fractional Powers of p

Because questions frequently are asked about the significance of fractional differentiation and integration, a word should be given before closing on that phase of operational calculus. In his investigations on telegraph cables, Heaviside obtained series solutions for many problems in which the operational equations contained the operator to half powers. Both ascending and descending series were found, and it was necessary to interpret these half-power operators.

The appearance of terms such as $p^{3/2}1$, $p^{5/2}1$, $p^{-7/2}1$, etc., called particularly for interpretation of $p^{1/2}1$. By comparing his operational mathematics with classical heat flow problems having known solutions, Heaviside determined that

$$p^{1/2}1 = \frac{1}{\sqrt{\pi t}} \tag{30}$$

Other half powers of p then became simply the derivatives or integrals of $1/\sqrt{\pi t}$, for example

$$p^{-1/2}1 = \frac{1}{p}\, p^{1/2}1 = \frac{2}{\sqrt{\pi}}\, t^{1/2}$$

$$p^{5/2}1 = \frac{d^2}{dt^2}\, \frac{1}{\sqrt{\pi}}\, t^{-1/2} = \frac{1}{\frac{2}{1} \cdot \frac{2}{3} \cdot \sqrt{\pi}}\, t^{-5/2}$$

Again may be noted the experimental manner by which the time function equivalent to an operational form had been obtained.

Whether the terminologies "fractional differentiation" or "fractional integration" have meaning depends perhaps upon the reader's viewpoint. The entire interpretation of the fractional power operator is based upon equation 30, after which the operations are in integral powers of p. In this sense, these terminologies have lost the significance that might be carried over to the operational calculus from the ordinary calculus.

Tables of Operational Equivalents

From what few problems have been discussed thus far in this article, the time function equivalents of several operational forms have been determined. The solution of a larger group of problems obviously would increase the number of related expressions. Further, by use of the described methods of solution it is possible to devise operational forms and to seek their time function equivalents without the necessity of having specific physical problems in view.

The advantage of equivalent expressions is that once they are obtained they need not again be derived, but the time function may be written immediately whenever its corresponding operational form is met, regardless of the problem giving rise to it. Tables of operational expressions with their related time function equivalents are given in the textbooks mentioned. In addition to these, attention also is called to the extensive table of Fourier transforms presented by Campbell and Foster.[36] Since the operational calculus readily can be interpreted in terms of the Fourier integral, as in the work of Bromwich and March, this table properly may be regarded as belonging to the class of operational equivalents. As stated by Bush, "Most of them (Fourier transforms) become immediately available as operational formulas by noting that the operand is $p1$ in our notation." One approaching the operational calculus can obtain familiarity with many of its fundamentals by deriving for himself operational formulas he will find in these several tables.

Quite frequently one finds equivalent expressions in which the unit function accompanies both members, e. g., $\frac{p}{p+\alpha}1 = \epsilon^{-\alpha t}1$. Whether or not one cares to retain the unit function with both members is perhaps optional. Unit function has exactly the same physical significance for both terms, and in the sense of being a signpost serves only to call attention to the nature of the impressed force, and to the fact that the ultimate solution may be computed using values of time beginning with $t = 0$. For that reason its retention is fully justifiable. One may argue, though, that, once having operated upon unit function to obtain the corresponding time function, unit function perhaps should disappear. One also may question the mathematical correctness of the statement "operating upon unit function." As the writer views the ideas underlying the operational method, $\frac{p}{p+\alpha} = {}^{-}\alpha^{\alpha t}$ whether or not the unit function is exhibited explicitly in either term. We always are left with a far greater physical significance of the unit function than with one that is mathematical.

As stated at the outset, this article has been directed toward the reader who has had little acquaintance with the Heaviside operational calculus but who has desired to have some of its fundamentals presented to him. It has been the aim primarily to discuss the physical ideas underlying the material given. The treatment has been specific, with the view that in definitely applying the operational method to particular circuits its elements may be grasped more readily.

The scope of the operational method is fairly large. In the great majority of circuits cause and effect are directly proportional to each other, and the coefficients of the terms involving differentials or derivatives in the problem statements are constant. As a consequence, the differential equations representing such a system are said to be linear with constant coefficients. To be precise, these statements are not strictly true, but for all practical purposes they are assumed to hold. The coefficients are the system parameters R, L, C, etc., (either singly or grouped) which usually are called the circuit constants.

It is to the linear circuit with constant parameters that the operational method may be applied. It is not restricted, however, only to that type of problem described by the ordinary differential equation; it includes also certain circuits that may be represented by partial differential equations, e. g., the transmission line, and some problems in heat flow. As yet the operational calculus has not been applied satisfactorily from an engineering point of view to systems that are nonlinear or that have variable "circuit constants."

Many phases of operational circuit analysis and its related features, of course, have not been touched upon. Listing several of such subjects of consideration, there are: circuits the past history of which must be taken into account, changes of circuit, coupled circuits, net-works and systems of equations, the theorem of superposition, the infinite integral equation, transmission lines and cables, heat flow, series solutions, transfer operators, traveling waves.

These topics are beyond the scope of this article and must be left to the reader to follow. It has been the hope, however, that in the treatment given here he will be led to a better understanding of the operational method of approach to the solution of engineering problems.

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Multielement Operation of the Cathode Ray Oscillograph

With the aid of specially designed auxiliary devices, the cathode ray oscillograph may be operated as a multielement instrument, in circuits that permit of the repeated application of the wave to be observed. The number of waves that may be traced with apparent simulataneity on the oscillograph screen is limited only by the complexity of the control circuit and the crowding of the various curves on the relatively small screen.

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THE cathode ray oscillograph has undergone a very rapid development during the past few years, largely because of its promise as a receiver in television apparatus. Laboratory workers have benefited from this development, but its rapidity perhaps has resulted in an as yet incomplete realization of its potentialities.

For the observation and measurement of quick transients in electric circuits perhaps the only important disadvantage of the cathode ray oscillograph, in comparison with the bifilar type, is that it is designed to trace only a single curve. To overcome this disadvantage, there have been developed in the electrical engineering department at the Massachusetts Institute of Technology devices for making possible the operation of the instrument as a multielement oscillograph, in circuits that allow the repeated application of the waves to be measured. The waves may be either steady-state or transient.

Figure 1 shows the simple apparatus required, in addition to the standard oscillograph and sweep circuit, for handling 2 transients, and figure 2 a similar device developed to handle up to 4 transients simultaneously. In the apparatus of figure 1 a small synchronous motor drives 3 insulating disks bearing copper insets on portions of their peripheries. The function of the device in brief is to act as a commutator and to switch to the measuring plates of the oscillograph the various voltage waves to be traced. At the same time the device must trip the sweep circuit at the proper time so that the curves will remain stationary on the oscillograph screen.

Figure 3 indicates the layout of the 3 disks that

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