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Solutions of the Zero-Rest-Mass Equations

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By means of contour integrals involving arbitrary analytic functions, general solutions of the zero-rest-mass field equations in flat space–time can be generated for each spin. If the contour surrounds only a simple (respectively, low-order) pole of the function, the resulting field is null (respectively, algebraically special).

1. THE CONTOUR INTEGRAL

It is possible to generate a very wide class of solutions to the zero-rest-mass field equations for each spin, $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots$, in flat space–time by means of a certain contour-integral expression. By choosing the integrand and contour suitably, the resulting field can be made to have certain prescribed properties, e.g., it may be made null or algebraically special. The expression arises naturally in the theory of twistors,[1] but the result can be given quite readily without using twistors. The present note gives the main results without going into the general theory.

Let x^0, x^1, x^2, x^3 be standard Minkowskian coordinates and set

$$u = 2^{-\frac{1}{2}}(x^0 + x^1), \quad v = 2^{-\frac{1}{2}}(x^0 - x^1),$$
$$\zeta = 2^{-\frac{1}{2}}(x^2 + ix^3), \tag{1.1}$$

so that the metric becomes $ds^2 = 2du\, dv - 2d\zeta\, d\bar{\zeta}$. Let f be an analytic function of three complex variables. Choose a nonnegative integer $2s$ and, for $r = 0, 1, \cdots, 2s$, put

$$\phi_r = \frac{1}{2\pi i}\oint \lambda^r f(\lambda, u + \lambda\bar{\zeta}, \zeta + \lambda v)\, d\lambda, \tag{1.2}$$

where the contour surrounds (but does not encounter) singularities of f and varies continuously with u, v, and ζ. Then we have

$$\frac{\partial \phi_r}{\partial \bar{\zeta}} = \frac{\partial \phi_{r+1}}{\partial u}, \quad \frac{\partial \phi_r}{\partial v} = \frac{\partial \phi_{r+1}}{\partial \zeta}, \quad r = 0, \cdots, 2s - 1, \tag{1.3}$$

(if $s > 0$) and

$$\left\{\frac{\partial^2}{\partial u \partial v} - \frac{\partial^2}{\partial \zeta \partial \bar{\zeta}}\right\}\phi_r = 0, \quad r = 0, \cdots, 2s. \tag{1.4}$$

Equations (1.4) is simply the wave equation in the coordinates (1.1) while Eqs. (1.3) are the spin-s zero-rest-mass equations in a suitable notation. For if we put

$$\phi_0 = \phi_{000\cdots 0}, \quad \phi_1 = \phi_{100\cdots 0}, \quad \phi_2 = \phi_{110\cdots 0}, \cdots,$$
$$\phi_{2s} = \phi_{111\cdots 1}, \tag{1.5}$$

where $\phi_{ABC\cdots K}$ has $2s$ indices and is symmetric: $\phi_{AB\cdots K} = \phi_{(AB\cdots K)}$; then Eqs. (1.3) can be written

$$\frac{\partial}{\partial x_{AP'}}\phi_{ABC\cdots K} = 0 \tag{1.6}$$

(summation convention assumed), where the 2-spinor notation $x_{00'} = v$, $x_{01'} = -\bar{\zeta}$, $x_{10'} = -\zeta$, and $x_{11'} = u$ is being used. Equation (1.6) is simply the Dirac–Fierz spinor equation[2,3] for mass zero and spin s. If $s = 1$, we can put

$$\phi_0 = \tfrac{1}{2}(F^{12} - F^{02} - iF^{13} + iF^{03}),$$
$$\phi_1 = \tfrac{1}{2}(F^{01} - iF^{23}),$$
$$\phi_2 = \tfrac{1}{2}(F^{12} + F^{02} + iF^{13} + iF^{03}),$$

and Eqs. (1.3) become Maxwell's equations

$$\frac{\partial}{\partial x^a}F^{ab} = 0, \quad \frac{\partial}{\partial x^a}F_{bc} + \frac{\partial}{\partial x^b}F_{ca} + \frac{\partial}{\partial x^c}F_{ab} = 0.$$

Similarly, for $s = 2$, we can get the linearized Einstein equations in gauge-invariant ("curvature-tensor") form.[3,4,5]

2. NULL AND ALGEBRAICALLY SPECIAL FIELDS

Suppose the contour in (1.2) surrounds only a k-order pole of the function f. Let $\lambda = \eta$ be the pole for given u, v, ζ (so η is a function of u, v, ζ, $\bar{\zeta}$). Then

$$\oint (\lambda - \eta)^k \lambda^r f(\lambda, u + \lambda\bar{\zeta}, \zeta + \lambda v)\, d\lambda = 0.$$

Thus, if $2s \geq k$,

$$\phi_{r+k} - k\phi_{r+k-1}\eta + \tfrac{1}{2}k(k-1)\phi_{r+k-2}\eta^2 - \cdots$$
$$+ \phi_r(-\eta)^k = 0, \quad r = 0, \cdots, 2s - k \tag{2.1}$$

[see (1.2)]. We can rewrite Eqs. (2.1), using the notation (1.5), as

$$\phi_{AB\cdots DE\cdots K}\xi^A\xi^B\cdots\xi^D = 0, \tag{2.2}$$

where $A, B, \cdots, D$ are k in number and where

$$\xi^0 = -\eta, \quad \xi^1 = 1. \tag{2.3}$$

[1] R. Penrose, J. Math. Phys. 8, 345 (1967).

[2] P. A. M. Dirac, Proc. Roy. Soc. (London) A155, 447 (1936).

[3] M. Fierz, Helv. Phys. Acta 13, 45 (1940); M. Fierz and W. Pauli, Proc. Roy. Soc. (London) A173, 211 (1939).

[4] R. K. Sachs and P. G. Bergmann, Phys. Rev. 112, 674 (1958).

[5] R. Penrose, Proc. Roy. Soc. (London) A284, 159 (1965); Ann. Phys. (N.Y.) 10, 171 (1960).

Equation (2.2) is, in fact, precisely the condition for a spinor ξ^A to represent a $(2s - k + 1)$-fold *principal null direction*[5] for the field $\phi_{AB\cdots K}$. Thus, when $k = 1$, so that the contour surrounds only a simple pole of f, all $2s$ principal null directions coincide in the direction of ξ^A so that we get a *null field*.[5] [In the Maxwell case this means $F_{ab}F^{ab} = 0 = F_{ab}F_{cd}\epsilon^{abcd}$ i.e., $\phi_0\phi_2 = (\phi_1)^2$.] More generally, whenever $k < 2s$ *at least two* principal null directions coincide (in the direction of ξ^A), that is to say, the field is algebraically special. (Such fields, when $s = 2$, are of interest in gravitation theory.[6,7])

The direction of ξ^A is given by the vector translation $\xi^A\bar{\xi}^{B'}$, i.e., by

$$du:d\zeta:d\bar{\zeta}:dv = \xi^0\bar{\xi}^{0'}:\xi^0\bar{\xi}^{1'}:\xi^1\bar{\xi}^{0'}:\ \xi^1\bar{\xi}^1$$
$$= \eta\bar{\eta}:-\eta:-\bar{\eta}:1.$$

Thus η defines the null direction[8] given by

$$du + \eta\, d\bar{\zeta} = 0 = d\zeta + \eta\, dv. \tag{2.4}$$

It is known[5–7,9] that the multiple principal null direction of an algebraically special field is tangential to a shear-free congruence of null geodesics (here, straight lines). In the above case, this follows also by a theorem of Kerr[1] which states that such a congruence is defined by (2.4) if we *specify an analytic relation connecting* η, $u + \eta\bar{\zeta}$, and $\zeta + \eta v$. In the present situation, the analytic equation $\{f(\lambda, u + \lambda\bar{\zeta}, \zeta + \lambda v)\}^{-1} = 0$ defines the poles $\lambda = \eta$ of f, verifying that the directions (2.4) are indeed geodetic and shear free.

In addition, Kerr's theorem states that, conversely, *any* shear-free geodetic null congruence in flat space–time (except for certain rather special limiting cases) can be obtained from such a analytic relation. This indicates the generality of expression (1.2) for the construction of null fields. Robinson[9] showed how, *starting* from any shear-free geodetic null congruence, it is possible to construct all the corresponding null solutions of Maxwell's equations by the arbitrary specification of an analytic function of two complex variables. The integral (1.2) achieves effectively the same thing (in flat space–time, but now for fields of arbitrary spin $s \geq 1$). For, by Kerr's result, the given shear-free congruence can (normally) be defined by (2.4) subject to $h(\eta, u + \eta\xi, \zeta + \eta v) = 0$, where h is some analytic function with simple zeros. Into the integrand of (1.2) we can substitute $f = gh^{-1}$, where g is an analytic function regular at the (relevant) zeros of h. The freedom of choice for the residues in (1.2) at the poles of f is, h being given, simply the freedom in the choice of g at $h = 0$. This is essentially one complex function of two complex variables ($h = 0$ being a two-complex-dimensional set) in agreement with Robinson's result.

Indeed, we can go somewhat further since algebraically special fields can also be treated by the method given here. For example, if $s = 2$, an expression gh^{-3} yields, when substituted into (1.2), a general type of algebraically special linearized gravitational field. (Here it is the values of g and its first and second derivatives, at $h = 0$, which are relevant.) We can also consider the slightly more general fields given when $k = 2s$ in (1.2). The directions (2.4) are evidently (by Kerr's theorem) still geodetic and shear free but they are now just *simple* principal null directions and the field is not algebraically special. Since, for a general field, the principal null directions are neither shear free nor geodetic, it follows that the fields given by $k = 2s$ in (1.2) are still of a rather special type. (The case $k = 2s = 1$ defines what we might tentatively call a "null neutrino field.") However, more general fields are generated if the contour surrounds a pole of higher order than $2s$ [for then (2.4) will not even be a principal null direction of the field], or more than one pole of f (in which case the resulting field will be a finite linear combination of fields of the type we have just been considering), or singularities or singular regions of more complicated types.

It is not hard to construct a function f for most of the simple types of fields normally encountered (e.g., plane waves, monopole, or multipole solutions, etc.). Also, provided the contour can be chosen consistently, we can obtain linear combinations of such fields in the form (1.2) simply by taking the corresponding linear combinations of f's. This process may fail if too extensive (continuous) linear combinations of f's are taken, since the resulting singularities may leave no room for the contour. Nevertheless, it is evident that there is considerable generality in the expression (1.2).

The full discussion of (1.2) and of its transformation properties is best carried out in terms of twistors.[1] The twistor description will be given elsewhere.[10]

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[6] I. Robinson and A. Trautman, Proc. Roy. Soc. (London) **A265**, 463 (1962).

[7] J. N. Goldberg and R. K. Sachs, Acta Phys. Polon. **22**, 13 (1962).

[8] The value $\eta = \infty$ also gives a well-defined null direction, although this would not arise from the integral (1.2) as given. To obtain null and algebraically special fields in a way similar to the above, but in which this exceptional null direction could also be represented, we would have to transform (1.2) suitably. It is in the transformation properties of (1.2) that the different spin values play a role. [Equation (1.2) is curiously oblivious to the value of s here!] The complete manifestly (conformally) covariant expression, of which (1.2) is a particular realization, requires the use of twistors.

[9] I. Robinson, J. Math. Phys. **2**, 290 (1960).

[10] *Note Added in Proof.* Due to the delay in the publishing of this paper, this description has already appeared; see R. Penrose, Intern. J. Theoret. Phys. **1**, 61 (1968).