CHAPTER 17

THE TWISTOR APPROACH TO SPACE-TIME STRUCTURES

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An outline of twistor theory is presented. Initial motivations (from 1963) are given for this type of non-local geometry, as an intended scheme for unifying quantum theory and space-time structure. Basic twistor geometry and algebra is exhibited, and it is shown that this provides a complex-manifold description of classical (spinning) massless particles. Simple quantum commutation rules lead to a concise representation of massless particle wavefunctions, in terms of contour integrals or (more profoundly) holomorphic 1st cohomology. Non-linear versions give elegant representations of anti-self-dual Einstein (or Yang-Mills) fields, describing left-handed non-linear gravitons (or Yang-Mills particles). A brief outline of the current status of the 'googly problem' is provided, whereby the right-handed particles would also be incorporated.

1. Early Motivations and Fundamental Basis of Twistor Theory

Twistor theory's original motivations, prior to December 1963 (which marks its initiation, as a physical theory[a]), came from several different directions; but in general terms, the intention was for a theory that would represent some kind of scheme for unifying basic principles coming from both quantum mechanics and relativity. Yet, the theory did not arise out of an attempt, on my part, to "quantize" space-time structure in any conventional sense. A good measure of my own reasons for not adopting this more conventional "quantum-gravity" stance came from a suspicion that the very rules of quantum mechanics might well have to be changed in such a unification, in order that its disturbing paradoxes (basically, the various forms of the *measurement* paradox) might perhaps be satisfactorily resolved as part of the proposed unification. For, in my own view, a resolution of these paradoxes would *necessarily* involve an actual (though presumably subtle) modification of the underlying quantum-mechanical rules. Nevertheless, it is clear that the rules of quantum mechanics must apply very precisely to physical systems that are, in an appropriate sense, "small". Most particularly, I had always been profoundly impressed by the physical role that quantum theory had found for the *complex number field*. This is manifested most particularly in the quantum linear superposition rule. But since quantum linearity can be regarded as the main "culprit" with regard to the measurement paradox one may expect that some kind of (complex?) *non*-linearity might begin to show itself when systems get large. For *small* systems the complex-*linear* superposition rule is extraordinarily precise.

We must, however, keep in mind that the notion of "small" that is of relevance here does not refer (simply) to small *distances*. We now know from experimental findings with EPR (Einstein–Rosen–Podolsky[b]) systems, that quantum entanglements can stretch unattenuated to at least some 30 kilometers.[c] Moreover, as an "objectivist" who had been aware of the profound puzzles that EPR phenomena would present for a consistent space-time picture of "quantum reality", I had considered these phenomena to be indicative of a need for some kind of objective "non-local geometry". Even at the more elementary level of a wavefunction for a single particle, there is an essential non-locality, as the measurement of the particle at one place forbids its detection at some distant place, even thought the spread of the wavefunction may have to encompass both locations in order for possible quantum interference to be accommodated. We should bear in mind that all such non-localities refer not just to small distances. When a notion of "smallness" is relevant, it is more likely that its appropriate measure might refer to *mass* displacements between two components of a superposition[d] (e.g. when a physical detector becomes entangled with the system), and general relativity tells us that mass displacements refer to *space-time curvature* differences.

Some years previously,[e] I had initiated *spin-network* theory. This theory had close ties with EPR-Bohm situations, these being entanglements which are manifested in *spin* correlations between widely separated events. The original form of spin-network theory had provided a kind of discrete *quantum geometry* for 3-dimensional Euclidean space, where spatial notions are taken to be *derived* rather than built initially into the theory, and it could be said to provide a non-local geometry of this nature. But I had recognized the limitations inherent in the essentially non-relativistic nature of that theory, and in its inability to describe spatial displacements. Spin networks arose from a study of the representation theory of the rotation group SO(3), and a logical route to follow might seem to be to replace this group by the Poincaré group. However, that approach did not particularly appeal to me, partly owing to the Poincaré group's non-semi-simple nature,[f] and generalizing further to the *conformal group* of Minkowski space — essentially to SO(2,4) — had, for various reasons seemed to me to be a possibility more in line with what I had in mind.

The representations of SO(3) are described in terms of 2-spinors, these providing the fundamental representation space of the *spin group* SU(2), of SO(3), and I had been very struck by the precise geometrical association between (projective SU(2)) 2-spinors and actual directions in 3-dimensional *physical space*.[g] In the case of the twistor group SO(2,4), there turned out to be an even more remarkable geometrical space-time association. Being led to consider representations of SO(2,4), instead of SO(3), we study its spin group SU(2,2), in place of SU(2). The fundamental representation space of SU(2,2) — the *reduced* (or "half") spin space for SO(2,4) — is the complex 4-dimensional vector space which I refer to as *twistor space* $\mathbb{T}$. The full (unreduced) spin space for SO(2,4) is the direct sum of twistor space $\mathbb{T}$ with its dual space $\mathbb{T}^*$. By virtue of $\mathbb{T}$'s $(++--)$ Hermitian structure, $\mathbb{T}^*$ can also be identified as the *complex conjugate* of $\mathbb{T}$. This Hermitian structure tells us that twistor space $\mathbb{T}$ has a 7-real-dimensional subspace $\mathbb{N}$, consisting of twistors whose (squared) "norm" $\|\mathbf{Z}\|$, given by this structure, *vanishes*. The elements of $\mathbb{N}$ are called *null twistors*, and we find that the *projective* null twistors (elements of the projective subspace $\mathbb{PN}$ of the projective twistor space $\mathbb{PT}$) are in precise geometrical correspondence with *light rays* in Minkowski space $\mathbb{M}$, i.e. null geodesics. Here we must include the limiting light rays that are the generators of the light cone at *null infinity* $\mathscr{I}$ for the conformally compactified Minkowski space $\mathbb{M}^\sharp$, these being described by the elements of a complex line $\mathbb{P}\mathbf{I}$ lying in $\mathbb{PT}$ (where $\mathbf{I}$ is a certain complex 2-dimensional subspace of $\mathbb{T}$ representing space-time "infinity").

This geometrical fact is remarkable enough, but the relation between twistors and important physical quantities goes much farther than this. In the first place, the *real* scaling of a null twistor $\mathbf{Z}$ has a direct physical interpretation, assigning an actual (future-pointing) *4-momentum* (equivalently a frequency) to a massless particle with world-line defined by $\mathbb{P}\mathbf{Z}$ (taking

$\mathbf{Z} \in \mathbb{N} - \mathbf{I}$). There remains a phase freedom

$$\mathbf{Z} \mapsto e^{i\theta}\mathbf{Z} \quad (\theta \text{ real}),$$

not affecting this 4-momentum. More strikingly, it turns out that *every* element $\mathbf{Z}$ of $\mathbb{T} - \mathbf{I}$ (not just those in $\mathbb{N}$), up to this same phase freedom, has an interpretation describing the *kinematics* of a massless particle which can have a *non-zero spin*. The helicity s of this particle (whose modulus is the spin) turns out to be simply

$$s = \frac{1}{2}\,\|\mathbf{Z}\|$$

(taking units with $\hbar = 1$). When $s \neq 0$ there is no actual "world-line" defined (in a Poincaré-invariant way), and the particle is, to some extent, *non-localized*.

What this demonstrates, since $\mathbb{T}$ is a *complex* vector space, is that twistor theory reveals a hidden Poincaré-invariant *holomorphic* (i.e. complex-analytic) structure to the kinematics of a massless particles (this kinematics being extended by the above phase freedom). The more primitive fact that the *celestial sphere* of an observer, according to relativity theory, can naturally regarded as a *Riemann sphere*[h] (a complex 1-manifold) is a particular aspect of this holomorphic structure. It had long struck me as particularly pertinent fact that only in the $1+3$ dimensions of our observed universe can the space of light rays through a point — i.e. the celestial sphere — be regarded as a complex manifold, and that the (non-reflective) Lorentz group can then be regarded as the group of holomorphic self-transformations of this sphere. On this view, it might be possible to view the celestial sphere as a kind of "quantum spread" of two directions, somewhat similarly to the way in which The Riemann sphere of possible "directions of spin" for a massive spin-$\frac{1}{2}$ particle can be thought of as a "quantum spread" of two independent directions (say, "up" and "down").

Nevertheless, all this twistor structure is, as yet, fully classical, so twistor theory is actually revealing a hidden role for complex-number structure which is present already at the classical level of (special) relativistic physics. This is very much in line with the driving force behind twistor theory. Rather than trying to "quantize" geometry, in some conventional sense, one seeks out strands of connection between the underlying mathematics of the quantum formalism and that of space-time geometry and kinematics. In relation to what has just been said, it may be pointed out that although this complex structure is an immediate feature only of *massless* particles, it is now a conventional standpoint to regard massless particles as being in some sense primary, with mass entering at a secondary stage, via some specific (e.g. Higgs) mechanism. Although twistor theory is perfectly capable of handling massive particles (and is neutral with respect to the specific Higgs mechanism), there indeed is a special "primitive" role for massless particles in that theory.

Something analogous could said to apply also to the "primitive" nature of Minkowski space $\mathbb{M}$, since it is here that these holomorphic twistor-related structures are most evident. Things get considerably more complicated when the effects of mass (or energy) begin to play their roles, and we are driven to consider *general relativity*. Here, the correct twistorial procedures are still only partially understood, but there is, nevertheless, a tantalizing relation between holomorphic structure and Einstein's vacuum equations, as revealed particularly in the "non-linear graviton" construction[i] of 1975/6, and its extensions. Moreover, many years previously, in the years before 1963, there had been another significant driving force behind the origination of twistor theory, which had relevance to this. I had been impressed by hints of a hidden complex structure revealed in the roles that complex (holomorphic) functions play in numerous exact solutions of the Einstein (vacuum) equations (e.g. plane-fronted waves, Robinson–Trautman solutions, stationary axi-symmetric solutions, including, as I learned later, the Kerr solution). The image of an ice-berg had come to mind, where all that we normally perceive of this hidden complex structure represents but a tiny part of it. This suggested that some novel way of looking at (possibly curved) space-time geometry might reveal some kind of hidden complex structure, even at the classical level. I had felt that an understanding of this could be a pointer to understanding how curved space-time structure might somehow become intertwined with quantum-mechanical principles (and particularly quantum theory's complex structure) in some fundamental way.

There is one further (but interrelated) motivation that may be mentioned here, namely the idea that the sought-for complex geometry should in some way automatically incorporate the notion — fundamental to quantum field theory — of the splitting field amplitudes into their positive and negative frequency parts. This procedure is neatly expressed, for a function of a single (real) variable, by the holomorphic extendibility of that function into the "top half" S^+ or "bottom half" S^- of the Riemann sphere S^2, where S^2 is divided into these two hemispheres by the equator, this representing the real line $\mathbb{R}$ (compactified to a circle by adjoining to it a "point at infinity"). In view of the importance of this frequency splitting to quantum field theory, I had imagined that our sought-for complex geometry should somehow reflect this frequency splitting in a natural but more global way, applying now all at once to entire fields, as might be globally defined on $\mathbb{M}$. As we shall be seeing in §5, this motivation is indeed satisfied in twistor theory but, as it turned out, in a much more subtle way than I had ever imagined.

2. Basic Twistor Geometry and Algebra

Choose standard Minkowski coordinates r^0, r^1, r^2, r^3 for $\mathbb{M}$ (with r^0 as the time coordinate, with $c = 1$). These are to be related to the standard complex coordinates Z^0, Z^1, Z^2, Z^3 for the vector space $\mathbb{T}$, via the *incidence relation*

$$\begin{pmatrix} \mathsf{Z}^0 \\ \mathsf{Z}^1 \end{pmatrix} = \frac{\mathrm{i}}{\sqrt{2}} \begin{pmatrix} r^0 + r^3 & r^1 + \mathrm{i}r^2 \\ r^1 - \mathrm{i}r^2 & r^0 - r^3 \end{pmatrix} \begin{pmatrix} \mathsf{Z}^2 \\ \mathsf{Z}^3 \end{pmatrix}.$$

To find what locus $\mathbf{R}$ in $\mathbb{T}$ corresponds to a fixed point R in $\mathbb{M}$, we hold the coordinates r^a of R fixed and vary Z^α while maintaining the incidence relation. We have two homogeneous linear equations in the Z^α, giving us a linear subspace $\mathbf{R}$ of $\mathbb{T}$ of dimension 2 to represent R. In terms of the *projective* space $\mathbb{PT}$, we find a projective straight line $\mathbb{P}\mathbf{R}$ (which is a Riemann sphere) to represent R. Conversely, to find the locus in $\mathbb{M}$ corresponding to a particular Z^α, we hold Z^α fixed and ask for the family of r^a which satisfy the incidence relation. Now if we require the r^a to be *real* — as indeed we should, if we are properly concerned with Minkowski space $\mathbb{M}$, rather than its complexification $\mathbb{CM}$ — then we find that the condition

$$\bar{\mathsf{Z}}_\alpha \mathsf{Z}^\alpha = 0$$

for a *null* twistor must hold, where (noting the order of $\mathbf{Z}$'s components, in what follows)

$$\bar{\mathsf{Z}}_\alpha = (\bar{\mathsf{Z}}_0, \bar{\mathsf{Z}}_1, \bar{\mathsf{Z}}_2, \bar{\mathsf{Z}}_3) = \textit{complex conjugate} \text{ of } (\mathsf{Z}^2, \mathsf{Z}^3, \mathsf{Z}^0, \mathsf{Z}^1).$$

In fact, the quantity

$$\begin{aligned} \|\mathbf{Z}\| &= \bar{\mathsf{Z}}_\alpha \mathsf{Z}^\alpha \\ &= \bar{\mathsf{Z}}_0 \mathsf{Z}^0 + \bar{\mathsf{Z}}_1 \mathsf{Z}^1 + \bar{\mathsf{Z}}_2 \mathsf{Z}^2 + \bar{\mathsf{Z}}_3 \mathsf{Z}^3 \\ &= \frac{1}{2}\left(|\mathsf{Z}^0 + \mathsf{Z}^2|^2 + |\mathsf{Z}^1 + \mathsf{Z}^3|^2 - |\mathsf{Z}^0 - \mathsf{Z}^2|^2 - |\mathsf{Z}^1 - \mathsf{Z}^3|^2\right) \end{aligned}$$

is the twistor *norm* referred to in §1, this being a Hermitian form of signature $(++--)$ (because of the swapping of the first two components with the second in the definition of twistor complex conjugation). Assuming that $\|\mathbf{Z}\| = 0$, so $\mathbf{Z}$ is a null twistor, we find that the locus z of points in $\mathbb{M}$ which are incident with $\mathbf{Z}$ is indeed a *light ray* (null geodesic), in accordance with what was asserted in §1. At least this is strictly the case provided that Z^2 and Z^3 do not both vanish. If they do both vanish, then we can interpret the light ray z in conformally compactified Minkowski space $\mathbb{M}^\sharp$, as a generator of the light cone $\mathscr{I}$ at infinity.[j]

Note that the index positioning on $\bar{\mathsf{Z}}_\alpha$ is consistent with $\bar{\mathbf{Z}}$ being a *dual* twistor (element of $\mathbb{T}^*$). Generally, the operation of twistor complex conjugation interchanges the spaces $\mathbb{T}$ and $\mathbb{T}^*$, so the up/down nature of twistor indices are reversed under complex conjugation. Sometimes I shall use the script letter $\mathcal{C}$ to denote twistor complex conjugation, as applied to (abstract-)indexed twistor (or, later, 2-spinor) quantities. Thus, we have, in particular, $\mathcal{C}\mathsf{Z}^\alpha = \bar{\mathsf{Z}}_\alpha$ and $\mathcal{C}\bar{\mathsf{Z}}_\alpha = \mathsf{Z}^\alpha$.

The geometrical correspondence between $\mathbb{M}^\sharp$ and $\mathbb{PN}$, and also between the complexification $\mathbb{CM}^\sharp$ and $\mathbb{PT}$, has many intriguing features. I mention only a very few of these here. We have seen that points of $\mathbb{M}^\sharp$ correspond to projective lines lying in $\mathbb{PN}$, and that points of $\mathbb{PN}$ correspond to light rays in $\mathbb{M}^\sharp$. In $\mathbb{M}^\sharp$, incidence is represented by a point lying on a light ray; in $\mathbb{PN}$, incidence is represented, correspondingly, by a projective line passing through a point. A *general* projective line in $\mathbb{PT}$ (*not* necessarily restricted to lie in $\mathbb{PN}$) corresponds to a *complex* space-time point, i.e. a point of $\mathbb{CM}^\sharp$. (This is a classical correspondence of 19th century geometry, often referred to as the *Klein* correspondence, $\mathbb{CM}^\sharp$ being understood as a complex 4-quadric.[k]) A general point $\mathbb{P}\mathbf{Z}$ of $\mathbb{PT}$ corresponds to a 2-complex-dimensional locus Z referred to as an *α-plane* (a standard classical terminology), this being a "self-dual" 2-plane on which the complex metric (induced from $\mathbb{CM}^\sharp$) vanishes identically. There is another type of complex 2-surface on $\mathbb{CM}^\sharp$ whose metric vanishes identically, which is "anti-self-dual", called a *β-plane.* The twistor correspondence represents β-planes in $\mathbb{CM}^\sharp$ by *complex projective 2-planes* in $\mathbb{PT}$.

The Hermitian relationship $\mathbf{Z} \leftrightarrow \bar{\mathbf{Z}}$ provides a *duality* transformation of $\mathbb{PT}$ in which points go to complex 2-planes, and *vice versa*, so this complex conjugation interchanges α-planes with β-planes on $\mathbb{CM}^\sharp$. In general, the point $\mathbb{P}\mathbf{Z}$ of $\mathbb{PT}$ will not lie on its corresponding plane $\mathbb{P}\bar{\mathbf{Z}}$ in $\mathbb{PT}$, the condition for it to do so being $\|\mathbf{Z}\| = 0$. In terms of $\mathbb{CM}^\sharp$: an α-plane Z will not generally meet its complex conjugate β-plane $\bar{Z}$, but the condition for it to do so is $\|\mathsf{Z}\| = 0$. When they do meet, their intersection is the complexification $\mathbb{C}z$ of the light ray z that we obtained earlier (the part of $\mathbb{C}z$ lying in $\mathbb{M}^\sharp$ being the light ray z itself).

Finally, we note that the projective-space structure of $\mathbb{PT}$ completely fixes the complex-*conformal* structure of $\mathbb{CM}^\sharp$ in a very direct way. By the term "complex-conformal" structure, I simply mean the structure defined by the complex null cones, this being equivalent to a locally defined complex metric, up to general conformal rescalings. Two points R, S of $\mathbb{CM}^\sharp$ are *null separated* if and only if the corresponding lines $\mathbb{P}\mathbf{R}$ and $\mathbb{P}\mathbf{S}$ of $\mathbb{PT}$ *intersect*, whence, the *light cone* of a point R in $\mathbb{CM}^\sharp$ is represented in $\mathbb{PT}$ by the family of lines which meet $\mathbb{P}\mathbf{R}$. More generally, we can obtain the Minkowskian squared interval between R and S as $-4\mathbf{R{:}S}(\mathbf{R{:}I})^{-1}(\mathbf{S{:}I})^{-1}$, where $\mathbf{R{:}S}$ stands for $\frac{1}{2}\varepsilon_{\alpha\beta\rho\sigma}\mathsf{R}^{\alpha\beta}\mathsf{S}^{\rho\sigma}$, etc., and where each of $\mathsf{R}^{\alpha\beta}$, etc. is antisymmetrical and simple; i.e. has the form $\mathsf{R}^{\alpha\beta} = \mathsf{X}^{[\alpha}\mathsf{Y}^{\beta]}$, etc. Here, I have made use of the particular *infinity twistors* $\mathsf{I}^{\alpha\beta}$ and $\mathsf{I}_{\alpha\beta}$, representing the 2-dimensional "infinity" subspace $\mathbf{I}$ of $\mathbb{T}$ (referred to in §1) or line $\mathbb{P}\mathbf{I}$ of $\mathbb{P}\mathbf{T}$, subject to

$$\mathsf{I}_{\alpha\beta} = \tfrac{1}{2}\varepsilon_{\alpha\beta\rho\sigma}\mathsf{I}^{\rho\sigma} = \mathcal{C}\mathsf{I}^{\alpha\beta}\,, \quad \mathsf{I}^{\alpha\beta} = \tfrac{1}{2}\varepsilon^{\alpha\beta\rho\sigma}\mathsf{I}_{\rho\sigma} = \mathcal{C}\mathsf{I}_{\alpha\beta}\,,$$

$$\mathsf{I}_{[\alpha\beta}\mathsf{I}_{\rho]\sigma} = 0\,, \quad \mathsf{I}^{[\alpha\beta}\mathsf{I}^{\rho]\sigma} = 0\,, \quad \mathsf{I}^{\alpha\beta}\mathsf{I}_{\beta\gamma} = 0\,.$$

(Here $\boldsymbol{\varepsilon}_{\alpha\beta\rho\sigma}$ and $\boldsymbol{\varepsilon}^{\alpha\beta\rho\sigma}$ are skew-symmetrical Levi-Civita twistors, satisfying $\boldsymbol{\varepsilon}_{\alpha\beta\rho\sigma}\boldsymbol{\varepsilon}^{\alpha\beta\rho\sigma} = 24$ and $\mathcal{C}\boldsymbol{\varepsilon}^{\alpha\beta\rho\sigma} = \boldsymbol{\varepsilon}_{\alpha\beta\rho\sigma}$ and $\mathcal{C}\boldsymbol{\varepsilon}_{\alpha\beta\rho\sigma} = \boldsymbol{\varepsilon}^{\alpha\beta\rho\sigma}$). Conformal-invariance breaking can be achieved by the incorporation of $\mathsf{I}^{\alpha\beta}$ or $\mathsf{I}_{\alpha\beta}$ into expressions, as desired.

3. Momentum and Angular Momentum for Massless Particles

It is convenient to use a 2-spinor notation[1] for much of twistor theory. The components of a twistor naturally fall into two pairs, where the first two are the components of an upper unprimed 2-spinor $\boldsymbol{\omega}$ and the second two, the components of a lower primed spinor $\boldsymbol{\pi}$:

$$\omega^0 = \mathsf{Z}^0\,, \quad \omega^1 = \mathsf{Z}^1\,, \quad \pi_{0'} = \mathsf{Z}^2\,, \quad \pi_{1'} = \mathsf{Z}^3\,.$$

So we can write

$$\mathsf{Z}^\alpha = (\omega^A, \pi_{A'})$$

and

$$\bar{\mathsf{Z}}_\alpha = (\bar{\pi}_A, \bar{\omega}^{A'})\,,$$

so that

$$\bar{\mathsf{Z}}_\alpha \mathsf{Z}^\alpha = \bar{\pi}_A \omega^A + \bar{\omega}^{A'} \pi_{A'}$$

(bearing in mind that the primed spin-space is the complex conjugate of the unprimed spin-space). Then the incidence relation becomes

$$\omega^A = \mathrm{i}\, r^{AA'} \pi_{A'}$$

where a standard 2-spinor representation of 4-vectors is being used:

$$r^{AA'} = \frac{1}{\sqrt{2}} \begin{pmatrix} r^{00'} & r^{01'} \\ r^{10'} & r^{11'} \end{pmatrix}.$$

On change of origin, from the original origin O to a new origin Q whose position vector relative to O is q^a (i.e. $q^{AA'}$), we find

$$\omega^A \rightsquigarrow \omega^A - \mathrm{i}\, q^{AA'} \pi_{A'} \quad \text{and} \quad \pi_{A'} \rightsquigarrow \pi_{A'}$$

This turns out to be consistent with the physical interpretation of a twistor $\mathsf{Z}^\alpha = (\omega^A, \pi_{A'})$, as providing the 4-momentum/6-angular momentum kinematics for a massless particle. From the twistor Z^α we can construct quantities (with $\bar{\pi}_A = \mathcal{C}\pi_{A'}$ and $\bar{\omega}^{A'} = \mathcal{C}\omega^A$)

$$p_a = \bar{\pi}_A \pi_{A'} \quad \text{and} \quad M^{ab} = \mathrm{i}\, \omega^{(A} \bar{\pi}^{B)} \boldsymbol{\varepsilon}^{A'B'} - \mathrm{i} \boldsymbol{\varepsilon}^{AB} \bar{\omega}^{(A'} \pi^{B')},$$

(where in expressions such as these, I am adopting an abstract-index viewpoint,[m] which allows me to *equate* the vector/tensor (abstract) index "a" with pair of spinor (abstract) indices "AA'", etc.) and we find that they have the correct behaviour ($p_a \rightsquigarrow p_a$, $M^{ab} \rightsquigarrow M^{ab} - q^a p^b + q^b p^a$) under change of origin, for p_a to be the 4-momentum and M^{ab} the 6-angular momentum for a relativistic system. Moreover, the required conditions, for a *massless* particle, that p_a be null and future pointing, and that the Pauli-Lubanski spin vector S_a constructed from p_a and M^{ab} be *proportional* to p_a

$$S_a = \frac{1}{2}\, e_{abcd} p^b M^{cd} = s\, p_a$$

with s being the *helicity* (which is the standard requirement), are now automatically satisfied by the above twistorial definitions, where the only restriction on Z^α is that $\pi_{A'} \neq 0$, in order that the 4-momentum be non-zero. Conversely, given a 4-momentum p_a and a 6-angular momentum M^{ab}, subject to these "massless-particle" conditions, we find that such a Z^α always exists, uniquely up to the phase freedom

$$\mathsf{Z}^\alpha \mapsto \mathrm{e}^{\mathrm{i}\theta} \mathsf{Z}^\alpha.$$

Furthermore, it turns out that the helicity s is simply given by

$$s = \frac{1}{2}\bar{\mathsf{Z}}_\alpha \mathsf{Z}^\alpha = \frac{1}{2}\|\mathbf{Z}\|$$

as was asserted in §1.

All this has been entirely classical, and we have seen that twistor space, being a complex space, indeed provides us with a natural (Poincaré-invariant) complex structure associated with the classical kinematics of a massless particle. (More explicitly, owing to the above phase freedom, the complex space that we have exhibited — the twistor space $\mathbb{T}$ with the subspace I removed — is a circle bundle over the space of classical kinematics for a massless particle.) But what about the *quantum* kinematics? It turns out that all we need is to impose standard canonical commutation rules between Z^α and $\bar{\mathsf{Z}}_\alpha$:

$$[\mathsf{Z}^\alpha, \mathsf{Z}^\beta] = 0\,, \quad [\bar{\mathsf{Z}}_\alpha, \bar{\mathsf{Z}}_\beta] = 0\,, \quad [\mathsf{Z}^\alpha, \mathsf{Z}_\beta] = \delta^\alpha_\beta\,,$$

and then the standard commutators for p_a and M^{ab} (as Poincaré group generators, namely $[p_a, p_b] = 0$, $[p_a, M^{bc}] = 2\mathrm{i}g_a{}^{[b}p^{c]}$, $[M^{ab}, M_{cd}] = 4\mathrm{i}g^{[b}{}_{[c}M^{a]}{}_{d]}$) follow unambiguously. There are no factor ordering problems here (because of the symmetry brackets in the twistor expression for M^{ab}), but we must be slightly careful in the case of the helicity operator s, which nevertheless comes out unambiguously as

$$s = \frac{1}{4}(\mathsf{Z}^\alpha \bar{\mathsf{Z}}_\alpha + \bar{\mathsf{Z}}_\alpha \mathsf{Z}^\alpha)\,.$$

We can now consider the notion of a *twistor wavefunction* which, since Z^α and $\bar{\mathsf{Z}}_\alpha$ are conjugate variables, should be a function *either* of Z^α *or* of $\bar{\mathsf{Z}}_\alpha$, but not of both (which is analogous to the rules for ordinary position x^a and momentum p_a, where a wavefunction can be a function of x^a or of momentum p_a, but not both). But what does it mean for a function $f(\mathsf{Z}^\alpha)$ to be independent of $\bar{\mathsf{Z}}_\alpha$? The condition is $\partial f/\partial \bar{\mathsf{Z}}_\alpha = 0$, which provides us with the Cauchy–Riemann equations for f, asserting that f is *holomorphic* in Z^α. We could, alternatively, choose the conjugate (or dual) representation, where our wavefunction g is taken to be holomorphic in $\bar{\mathsf{Z}}_\alpha$, which means *anti*-holomorphic in Z^α. It is more convenient, with this representation, to use a dual twistor variable $\mathsf{W}_\alpha (= \bar{\mathsf{Z}}_\alpha)$ and to consider $g(\mathbf{W})$ simply as holomorphic in W_α.

To be definite, I shall tend phrase my arguments in terms of the Z^α-representation (and then the results using the W_α-representation follow essentially by symmetry). In the Z^α-representation, we can interpret the quantum operator $\bar{\mathsf{Z}}_\alpha$, according to

$$\bar{\mathsf{Z}}_\alpha = -\frac{\partial}{\partial \mathsf{Z}^\alpha}\,.$$

Note that this enables us to "re-instate holomorphicity" in expressions which may, classically, have to be described *non*-holomorphically. For example, the classical expression for *helicity* is given in terms of the distinctly non-holomorphic quantity $\bar{\mathsf{Z}}_\alpha \mathsf{Z}^\alpha$, whereas quantum-mechanically, we have the entirely holomorphic operator

$$\begin{aligned} s &= \frac{1}{4}(\mathsf{Z}^\alpha \bar{\mathsf{Z}}_\alpha + \bar{\mathsf{Z}}_\alpha \mathsf{Z}^\alpha) \\ &= \frac{1}{4}(2\mathsf{Z}^\alpha \bar{\mathsf{Z}}_\alpha - \delta^\alpha_\alpha) \\ &= \frac{1}{2}\left(-\mathsf{Z}^\alpha \frac{\partial}{\partial \mathsf{Z}}^\alpha - 2\right). \end{aligned}$$

We note that the operator

$$\Upsilon = \mathsf{Z}^\alpha \frac{\partial}{\partial \mathsf{Z}^\alpha}$$

is Euler's *homogeneity* operator, whose eigenfunctions are homogeneous functions in Z^α with eigenvalue the degree of homogeneity. Thus, if we wish to describe a massless particle whose helicity takes the specific value $n/2$, we can use a twistor function $f(\mathsf{Z}^\alpha)$ which is homogeneous of degree $-n-2$. For a photon, for example, we would use a function of degree 0 for the left-handed part and of degree -4 for the right-handed part.

4. Massless Fields and their Twistor Contour Integrals

What is the relation between such a twistor wavefunction for a particle of a specific helicity and the ordinary space-time description of such a particle? I shall use 2-spinor notation (where $\Box = \nabla_a \nabla^a = \nabla_{AA'}\nabla^{AA'}$, etc.). For helicity 0 we have

$$\Box \varphi = 0\,;$$

for negative helicity $n/2$ (< 0)

$$\nabla^{AA'}\phi_{AB\ldots L} = 0\,;$$

and for positive helicity $n/2$ (> 0)

$$\nabla^{AA'}\chi_{A'B'\ldots L'} = 0\,.$$

Here $\phi_{AB\ldots L}$ has $-n/2$ indices and $\chi_{A'B'\ldots L'}$ has $+n/2$ indices (a positive number in each case), and each is totally symmetric

$$\phi_{(AB\ldots L)} = \phi_{AB\ldots L}\,, \quad \chi_{(A'B'\ldots L')} = \chi_{A'B'\ldots L'}\,.$$

In the case $s = \pm 1$ (spin 1), we can relate these equations to the more familiar source-free Maxwell equations

$$\nabla^a F_{ab} = 0\,, \quad \nabla_{[a}F_{bc]} = 0\,,$$

given by $F_{ab}(= -F_{ba})$ defined by

$$F_{ab} = F_{AA'BB'} = \phi_{AB}\varepsilon_{A'B'} + \varepsilon_{AB}\chi_{A'B'}\,.$$

(Recall the abstract-index conventions noted in §3; also incorporated are the basic anti-symmetrical 2-spinor Levi-Civita quantities, used for raising or lowering 2-spinor indices, ε^{AB}, $\varepsilon^{A'B'}$, ε_{AB}, and $\varepsilon_{A'B'}$, where $g_{ab} = \varepsilon_{AB}\varepsilon_{A'B'}$.) In the case $s = \pm 2$, we obtain the linearized Einstein vacuum equations,[n] expressed in terms of the linearized curvature tensor K_{abcd}, with symmetry relations

$$K_{abcd} = K_{[cd][ab]}\,, \quad K_{[abc]d} = 0$$

and vacuum condition

$$K_{abc}{}^{a} = 0\,,$$

all these being automatic consequences of

$$K_{abcd} = \phi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \varepsilon_{AB}\varepsilon_{CD}\chi_{A'B'C'D'}$$

(abstract indices!), the Bianchi identity equation

$$\nabla_{[a}K_{bc]de} = 0\,,$$

now re-expressing the spin-2 massless free-field equation above.

I have given the above expressions for Maxwell and linear Einstein fields in the general case of *complex* fields, which is appropriate for the description of wavefunctions, but if we wish to describe a real field, we restrict to the case where the χ and ϕ fields are complex conjugates of one another:

$$\chi_{A'B'} = \bar{\phi}_{A'B'} = \mathcal{C}\phi_{AB}\,, \quad \chi_{A'B'C'D'} = \bar{\phi}_{A'B'C'D'} = \mathcal{C}\phi_{ABCD}\,.$$

In the case of complex fields (wavefunctions), we can specialize to

$$\phi_{AB\ldots L} = 0$$

which gives a *self-dual* field (for integer spin), describing a *positive*-helicity (right-handed) massless particle, or to

$$\chi_{A'B'\ldots L'} = 0$$

which gives an *anti-self-dual* field, describing a *negative*-helicity (left-handed) massless particle.

We ask for the relation between these massless field equations and a twistor wavefunction of homogeneity degree $-n-2$. The answer is largely expressed in the contour-integral expressions[o]

$$\varphi(\boldsymbol{x}) = c_0 \oint f(\boldsymbol{\omega}, \boldsymbol{\pi}) \delta \mathsf{Z}$$

for the case $n = 0$,

$$\chi_{A'B'\ldots L'}(\boldsymbol{x}) = c_n \oint \pi_{A'}\pi_{B'} \ldots \pi_{L'} f(\boldsymbol{\omega}, \boldsymbol{\pi}) \delta \mathsf{Z}$$

for $n > 0$, and

$$\phi_{AB\ldots L}(\boldsymbol{x}) = c_n \oint \frac{\partial}{\partial \omega^A} \frac{\partial}{\partial \omega^B} \cdots \frac{\partial}{\partial \omega^L} f(\boldsymbol{\omega}, \boldsymbol{\pi}) \delta \mathsf{Z}$$

for $n < 0$. (The constants c_n are here left undetermined, their most appropriate values to be perhaps fixed at some later date.) In each case, $\boldsymbol{\omega}$ is first to be eliminated by means of the *incidence relation*

$$\boldsymbol{\omega} = \mathrm{i} \boldsymbol{x} \boldsymbol{\pi}$$

before the integration is performed, the quantity $\delta \mathsf{Z}$ being defined by either of the following definitions

homogeneous case ($\oint$ with 1-dimensional real contour): $\delta \mathsf{Z} = \pi_{A'} \mathrm{d}\pi^{A'}$

inhomogeneous case ($\oint$ with 2-dimensional real contour): $\delta \mathsf{Z} = \frac{1}{2} \mathrm{d}\pi_{A'} \wedge \mathrm{d}\pi^{A'}$.

In either case, the integration removes the $\boldsymbol{\pi}$-dependence, and we are left with a function solely of $\boldsymbol{x}$. Moreover, it is a direct matter to verify that the appropriate massless field equation is indeed satisfied in each case.

In the homogeneous case we get a genuine contour integral — in the sense that the answer does not change under continuous deformations of the (closed) contour, within regions where f remains holomorphic — provided that the entire integrand (including the 1-form $\delta \mathsf{Z}$) has homogeneity degree zero, that being the condition that its exterior derivative vanishes. This condition is ensured by the nature of the 1-form $\delta \mathsf{Z}$ and the balancing of the homogeneities of the various terms, using the homogeneity prescription for f given above. Here the contour is just a one-dimensional real curve, so the geometry is normally very simple. The inhomogeneous case, on the other hand, involves a contour which is a two-dimensional real surface. This sometimes gives additional freedom, but the geometry is often not so transparent as in the homogeneous case. However, there is a more compelling reason to expect that the inhomogeneous case provides a broader-ranging viewpoint. This arises from the fact that we get a genuine contour integral even when the homogeneities do not balance, the exterior derivative of the integrand vanishing merely by virtue of its holomorphicity. The role the homogeneity balancing to zero is now simply that the integral will now *vanish* without this balancing.

There are many reasons why the inhomogeneous prescription gives a more powerful viewpoint, but perhaps the most transparent of these is that we can now describe the wavefunction of, say, a plane-polarized photon, which is not in an eigenstate of helicity — whereas this cannot be done directly in the homogeneous case — helicities $+1$ and -1 being now involved simultaneously. Using the inhomogeneous form, we can simply add together a twistor function of homogeneity degree -4 (to describe the right-handed part of the photon) and a twistor function of homogeneity degree 0 (to describe the left-handed part). This would be acted upon by the appropriate combination $\pi_{A'}\pi_{B'} + \partial^2/\partial\omega^{A'}\partial\omega^{B'}$. The cross-terms, for which the homogeneity does not balance to zero, simply disappear upon integration, and we are left with the appropriate space-time sum of a self-dual and an anti-self-dual complex Maxwell field.

In order to get a feeling for the nature of the contour-integral expressions for massless fields generally, let us consider the homogeneous case, so we just have a 1-dimensional contour, and take $n = 0$ with a very simple twistor function (homogeneity degree -2):

$$f(\mathsf{Z}) = \frac{1}{(A_\alpha \mathsf{Z}^\alpha)(B_\beta \mathsf{Z}^\beta)}\,.$$

The singularities of this function are simple poles, represented in $\mathbb{PT}$ as lying along two planes $\mathbb{P}\mathbf{A}$ and $\mathbb{P}\mathbf{B}$, corresponding to the vanishing of the respective terms $A_\alpha \mathsf{Z}^\alpha$ and $B_\beta \mathsf{Z}^\beta$. The point X in $\mathbb{CM}$ whose position vector is $\boldsymbol{x}$ is represented as a line $\mathbb{P}\mathbf{X}$ in $\mathbb{PT}$ (and this lies in $\mathbb{PN}$ whenever $\boldsymbol{x}$ is real). Recall that the line $\mathbb{P}\mathbf{X}$ is a Riemann sphere, and the function f has, as its singularities, a simple pole at each of the two points where $\mathbb{P}\mathbf{X}$ meets the planes $\mathbb{P}\mathbf{A}$ and $\mathbb{P}\mathbf{B}$. We choose a contour on this sphere which loops once between these poles, separating them, so that it cannot be shrunk away continuously without passing across one pole or the other. In this case the integration is easily performed, and we obtain a field $\varphi(\boldsymbol{x})$ that is a constant multiple of $\{(x_a - q_a)(x^a - q^a)\}^{-1}$, where q^a is the position vector of the point Q whose representation in $\mathbb{PT}$ is the line of intersection $\mathbb{P}\mathbf{Q}$ of the two planes $\mathbb{P}\mathbf{A}$ and $\mathbb{P}\mathbf{B}$. Notice that the field is singular only on the light cone of Q, which is when X and Q are null-separated, i.e. when the lines $\mathbb{P}\mathbf{X}$ and $\mathbb{P}\mathbf{Q}$ meet. This singularity arises when the poles on the Riemann sphere $\mathbb{P}\mathbf{X}$ "pinch" together, and the contour cannot pass between them.

For the wavefunction of a free particle, we require a condition of *positive frequency*. This is achieved by demanding that the space-time field extend holomorphically into the region of $\mathbb{CM}$ referred to as the *forward tube*, which consists of points whose (complex) position vectors have imaginary parts which are *past-timelike*. We find that this corresponds precisely to the family of lines lying in the upper region $\mathbb{PT}^+$ of projective twistor space. We see that this positive-frequency condition is easily satisfied for the particular scalar field φ just considered, if we arrange for $\mathbb{P}\mathbf{Q}$ to lie entirely in $\mathbb{PT}^-$, for then no line in $\mathbb{PT}^+$ can meet it.[p] It is clear, also, that something very similar will hold for any twistor function f whose singularities, in $\mathbb{PT}^+$, lie within two disjoint closed sets $\mathcal{A}$ and $\mathcal{B}$. For then, supposing that we get a non-zero field at all, with a contour on the Riemann sphere $\mathbb{P}\mathbf{X}$ separating $\mathbb{P}\mathbf{X} \cap \mathcal{A}$ from $\mathbb{P}\mathbf{X} \cap \mathcal{B}$, it follows from the fact that $\mathcal{A}$ and $\mathcal{B}$ are disjoint closed sets in $\mathbb{PT}^+$ that the contours can never get pinched. This situation clearly has considerable generality. It applies, for example, equally well to fields of arbitrary helicity and not simply to the case $n = 0$, and many types of positive-frequency wavefunctions are thereby obtained.

5. Twistor Sheaf Cohomology

We see, by means of this contour-integral representation, that twistor theory provides a powerful method of generating wavefunctions for massless fields, where the massless field equations seem to "dissolve" into pure complex analyticity. There is, however, a curious difficulty with this twistor representation, the resolution of which will lead us to a more sophisticated and fruitful point of view. One of the properties of wavefunctions that is evident in the conventional space-time representation is that such fields are Poincaré covariant (and in fact conformally covariant), and that they form a complex *linear space*. This would be clear also for the twistor representation (perhaps even more so, especially for conformal invariance), were it not for the awkward fact that we seem to need to assign some fixed region in which the singularities are to reside, and that any such assignment would destroy manifest Poincaré covariance. The reason for seeming to need to fix the singularity region is that if we add two twistor functions, the singularities of the sum are likely to occupy the union of the singularity regions of the two twistor functions individually. If these singularity regions are allowed too much freedom, then with extensive (and perhaps continuous) linear combinations, the resulting regions may not merely be complicated; in certain cases they could even preclude the finding of any appropriate contour whatsoever.

Compensating this difficulty, there is evidently some mobility in the singularity regions themselves. In the case of the particular twistor function f, considered in the previous section, we could replace B_α by adding a multiple of A_α

$$B_\alpha \rightsquigarrow B_\alpha + \lambda A_\alpha \,,$$

where λ is any fixed complex number, thereby *moving* the singularity region $\mathbf{B}$. It is not hard to see that this gives us a completely equivalent twistor function to the one that we had before, in the sense that if we subtract one from the other (while retaining a contour that works for both), then this "difference" twistor function finds its singularities all on one side of the contour, necessarily giving *zero* for the contour integral because the contour "slips off" on the other side. The same would apply if we choose to move $\mathbf{A}$, correspondingly, rather than $\mathbf{B}$. This indicates that a deeper perspective on twistor functions is needed.

As a pointer to this deeper perspective, let us consider the more general situation of the two disjoint closed sets of singularities $\mathcal{A}$ and $\mathcal{B}$ in $\mathbb{PT}^+$, as introduced above (and we should also bear in mind that we may want to generalize $\mathbb{PT}^+$ itself to some other appropriate region of interest lying within $\mathbb{PT}$, or perhaps even to some other complex manifold). We may re-express the conditions on f, $\mathcal{A}$, and $\mathcal{B}$ in the following curious-looking way: the region on which f is assigned to be holomorphic is the *intersection* $\mathcal{U}_1 \cap \mathcal{U}_2$ of two open sets

$$\mathcal{U}_1 = \mathbb{PT}^+ - \mathcal{A} \quad \text{and} \quad \mathcal{U}_2 = \mathbb{PT}^+ - \mathcal{B} \,,$$

where the *union* $\mathcal{U}_1 \cup \mathcal{U}_2$ is the entire region $\mathbb{PT}^+$ that we are interested in. What is the purpose of this? We shall see in a moment. First, this situation extends to more complicated open coverings $\{\mathcal{U}_i\}$ of $\mathbb{PT}^+$, which we shall require to be *locally finite* (i.e. only a finite number of the $\mathcal{U}_i$ contain any given point of $\mathbb{PT}^+$), though the sets $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots$ need not actually be finite in number. Since there are now liable to be many different intersections of pairs of these sets, our "twistor function" is now defined in terms of a *collection* $\{f_{ij}\}$ of various holomorphic functions f_{ij}, with

$$f_{ij} \text{ holomorphic on } \mathcal{U}_i \cap \mathcal{U}_j$$

subject to

$$f_{ij} = -f_{ji}$$

on intersecting pairs $\mathcal{U}_i \cap \mathcal{U}_j$, and

$$f_{ij} + f_{jk} = f_{ik}$$

on intersecting triples $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$. Such a collection $\{f_{ij}\}$ is called a *1-cochain* (of holomorphic sheaf cohomology), with respect to the covering $\{\mathcal{U}_i\}$ of $\mathbb{PT}^+$. If all the members of this cochain can simultaneously be expressed in the form

$$f_{ij} = h_i - h_j\,, \quad \text{with each } h_k \text{ holomorphic on } \mathcal{U}_k$$

then we say that the collection $\{f_{ij}\}$ is a *1-coboundary.* (We may check that the particular freedom that we encountered, with the replacement $B_\alpha \rightsquigarrow B_\alpha + \lambda A_\alpha$ in our example above, is an example of adding a coboundary to f.) The elements of *1st cohomology* (here *holomorphic* 1st cohomology)[q], with respect to the covering $\{\mathcal{U}_i\}$, are the 1-cochains *factored out* by the 1-coboundaries (so two 1-cochains are deemed equivalent, as elements of 1st cohomology, if their difference is a 1-coboundary).

Now, we would like to get rid of this reference to a particular covering of $\mathbb{PT}^+$ and to refer just to $\mathbb{PT}^+$ itself. The general procedure for doing this is to take what is called a "direct limit" of finer and finer coverings, but this is complicated and non-intuitive. Fortunately, one of the miracles of complex analysis now comes to our rescue, which tells us that if the sets $\mathcal{U}_i$ are what are called *Stein* manifolds, then we are already finished, and we do not need to take the direct limit! The cohomology relative to any Stein covering $\{\mathcal{U}_i\}$ of $\mathbb{PT}^+$ is *independent* of the choice of Stein covering, and therefore refers simply to $\mathbb{PT}^+$ itself, as a whole!

But what is a Stein manifold? The definition refers simply to its intrinsic complex-manifold structure, and it does not depend on any particular imbedding of it in a larger complex manifold (such as $\mathbb{PT}^+$). It is easiest not to rely on a full definition a Stein manifold here, but merely give some examples of certain widespread classes of Stein manifolds, so that we can see how easy it is to ensure that we do have a Stein covering. In the first place, any region of $\mathbb{C}^n$ that is delineated by the vanishing of a family of (holomorphic) polynomial equations is Stein (an "affine variety"). Secondly, any region of $\mathbb{CP}^n$ obtainable by *excluding* the zero locus of a single homogeneous polynomial will be Stein. Thirdly, any region of $\mathbb{C}^n$ which has a smooth boundary satisfying a certain "convexity" condition — referred to as *holomorphic pseudo-convexity* (specified by a positive-definiteness criterion on a certain Hermitian form defined by the equation of the boundary, called the Levy form[r]). This last property tells us that we can always find small Stein-manifold open neighbourhoods of any point of a complex manifold (e.g. a small spherical ball).

An important additional property is that the *intersection* of Stein manifolds is again always Stein. We can use this fact to compare 1st cohomology elements defined with respect to two different Stein coverings $\{\mathcal{U}_i\}$ and $\{\mathcal{V}_I\}$. All we need to do is find the *common refinement* of the two coverings, which is a covering $\{\mathcal{W}_{iJ}\}$ each of whose members is the intersection of one set from each of the two coverings:

$$\mathcal{W}_{iJ} = \mathcal{U}_i \cap \mathcal{V}_J$$

(where we may ignore the empty intersections) and this will also be a Stein covering. Thus, if we have cochains $\{f_{ij}\}$ with respect to $\{\mathcal{U}_i\}$ and $\{g_{IJ}\}$ with respect to $\{\mathcal{V}_I\}$, then we can form the sum "$f+g$", with respect to the common refinement $\{\mathcal{W}_{iJ}\}$ as the cochain $\{f_{ij}|_{IJ} + g_{IJ}|_{ij}\}$, where $f_{ij}|_{IJ}$ is f_{ij} on $\mathcal{U}_i \cap \mathcal{U}_j$, restricted to its the intersection with $\mathcal{V}_I \cap \mathcal{V}_J$, and $g_{IJ}|_{ij}$ is g_{IJ} on $\mathcal{V}_I \cap \mathcal{V}_J$, restricted to its the intersection with $\mathcal{U}_i \cap \mathcal{U}_j$. It is a direct matter to show that this leads to a sum of the corresponding cohomology elements. Hence, the notion of a 1st cohomology element of $\mathbb{PT}^+$ is *intrinsic* to $\mathbb{PT}^+$, and does not really care about how we have chosen to cover it with a collection of open (Stein) sets.[s] This now gets us out of our difficulty with the twistor representation of massless fields. But it leads us to a greater degree of sophistication in the mathematics needed for the description of physical fields than we might have expected.

One aspect of this sophistication is that our twistor description is already a *non-local one*, even for the description of certain things which, in ordinary space-time, are perfectly local, like a physical field. But we should bear in mind that this twistor description is primarily for *wavefunctions*, and we recall from §1 that the wavefunction of a single particle is *already* something with puzzling non-local features (since the detection of the particle in one place immediately forbids its detection at some distant place). Accordingly, twistor theory's essentially non-local description of wavefunctions is actually something rather closer to Nature than the conventional picture in terms of a space-time "field". Indeed, it is frequently pointed out that this *holistic* character of a wavefunction distinguishes it fundamentally from the kind of *local* behaviour that is exhibited by ordinary physical fields or wavelike disturbances, this distinction contributing to the common viewpoint that a wavefunction is not to be attributed any actual "physical reality". However, we see that the cohomological character of the *twistor* formulation of a wavefunction gives it precisely the kind of holistic (non-local) nature that wavefunctions actually posses, and I would contend that the twistor formulation of a wavefunction assigns just the right kind of mathematical "reality" to a physical wavefunction.

To emphasize the essential non-locality of the concept of cohomology, we may take note of the fact that the 1st (and higher) holomorphic cohomology of any Stein manifold always *vanishes*. (I have not defined cohomology higher than the 1st here; the basic difference is that for n-cochains, we need functions defined on $(n+1)$-ple intersections, with a consistency condition on $(n+2)$-ple intersections, the coboundaries being defined in terms of "h"s on n-ple intersections.[t]) It is important, therefore, that $\mathbb{PT}^+$ is *not* Stein (it is, indeed, not pseudo-convex at its boundary $\mathbb{PN}$). It is this that allows non-trivial holomorphic 1st cohomology elements to exist. However, from what has just been said, we see that 1st cohomology of an open region always vanishes *locally*, in the sense that it vanishes if we restrict it down to a small Stein set containing any chosen point. First (and higher) cohomology, for an open complex manifold, is indeed an essentially *non-local* notion.

To end this section, we take note of the remarkable fact that the *positive-frequency* condition for a wavefunction is now neatly taken care of by the fact that we are referring simply to the holomorphic (1st) cohomology of $\mathbb{PT}^+$. Correspondingly, *negative*-frequency complex massless fields would be those which are described by the holomorphic cohomology of $\mathbb{PT}^-$. This provides a very close analogy to the way in which the positive/negative frequency splitting of a (complex) function defined on the real line — compactified into a circle S^1 — can be described in terms of holomorphic extendibility into the northern or southern hemispheres S^+, S^- of a Riemann sphere (a $\mathbb{CP}^1$) whose equator represents this S^1 (and where I am arranging things so that the "north pole" is the point $-i$, with the "south pole" at $+$i). A complex function defined on S^1 can be split into a component which extends holomorphically into S^+, namely the positive-frequency part, and a component which extends holomorphically into S^-, the negative frequency part. This splitting is very closely analogous to the splitting of a complex 1st cohomology element defined on the "equator" $\mathbb{PN}$ of $\mathbb{PT}$ (a $\mathbb{CP}^3$) into its "positive frequency part", which extends holomorphically into $\mathbb{PT}^+$, and its "negative frequency part", extending holomorphically into $\mathbb{PT}^-$. A complex function can be thought of as an element of "0th cohomology", and the whole "splitting" procedure applies also to nth cohomology, defined on an analogue of $\mathbb{PN}$, dividing $\mathbb{CP}^{2n+1}$ into two halves analogous to $\mathbb{PT}^+$ and $\mathbb{PT}^-$.

In each case, we can consider that these functions may be "twisted" to a certain degree, which refers to fixing a particular homogeneity for the functions (cohomology elements), as defined on the non-projective space $\mathbb{C}^{2n+2}$. There is, however, a more serious subtlety if we wish these cohomology elements to form a *Hilbert space*, so that there is a (positive definite) norm, or Hermitian scalar product defined. This requires an appropriate notion of "fall-off" as the cohomology element approaches the boundary. We can ensure that this scalar product exists, however, if we demand *analyticity* at S^1, or at $\mathbb{PN}$ (or at the higher-dimensional analogue of $\mathbb{PN}$, for higher cohomology), which is adequate for most purposes in twistor theory.[u]

Some readers might be disturbed by the dual role that $\mathbb{PT}^+$ seems to be playing in this discussion. On the one hand, we have seen in §3 that it is associated with *positive helicity* but, on the other, we now see that it is associated with positive *frequency*, which means positive *energy*. However, this association between the signs of helicity and energy may be regarded as a consequence of the (arbitrary) choice that we have made to express things in terms of $\mathbb{T}$ (the Z^α-description) rather than $\mathbb{T}^*$ (the W_α-description). Had we used $\mathbb{T}^*$, we would find that the forward tube is represented in terms of projective lines in $\mathbb{T}^*$ (the space of dual twistors W_α for which the norm $\|\mathbf{W}\| = \bar{\mathsf{W}}^\alpha \mathsf{W}_\alpha$ is negative), whereas it would now be the massless kinematics for *negative* helicity which is represented by this space, and the association would be between the signs of *minus* the helicity and of energy.

6. The Non-Linear Graviton

There is a particular quality possessed by 1st cohomology that seems to provide a strong pointer to the future development of twistor theory. This is the existence of certain non-linear generalizations of 1st cohomology that have important (but as yet incomplete) relevance to the twistor descriptions of the known interactions of Nature: the Einstein gravitational interaction and those forces (electro-weak and strong) described by Yang–Mills theory. Recall that in 1st cohomology we have functions defined on overlaps of *pairs* of open sets, with a consistency relation on *triple* overlaps of open sets. This is closely analogous to the procedure for building a (perhaps "curved") manifold out of overlapping coordinate neighbourhoods. Here there are *transition functions* defined on overlaps of pairs, with a consistency condition on triple overlaps, and there is a condition of non-triviality for the resulting manifold that is analogous to the coboundary condition of cohomology.

The analogy can be made much more precise in the case of *small deformations* of some given complex manifold $\mathcal{X}$. Here we take a (locally finite) covering $\{\mathcal{U}_i\}$ of $\mathcal{X}$. We consider a cochain $\{F_{ij}\}$, where each F_{ij} is a holomorphic *vector field* on $\mathcal{U}_i \cap \mathcal{U}_j$. We are to think of the sets $\mathcal{U}_i$ as infinitesimally "sliding" over one another, as directed by this vector field. In fact, any non-trivial continuous deformation of $\mathcal{X}$ to a new complex manifold can be generated (infinitesimally) by such means, for some non-trivial 1st cohomology element defined by such a cochain. The converse is not quite true, as unless a certain 2nd cohomology element vanishes — which usually seems to be the case — we cannot guarantee that the given 1st cohomology element actually "exponentiates" consistently to give a finite deformation.

We shall first consider projective twistor space and, to be specific, let us take $\mathcal{X}$ to be either $\mathbb{PT}^+$ or an appropriate neighbourhood of some line $\mathbb{P}\mathbf{R}$ in $\mathbb{PT}$. The latter situation refers to the *local* space-time neighbouring some space-time point R. We imagine some point Q, near R, moving around, so as to sweep out some open neighbourhood $\mathcal{V}$ of R in $\mathbb{CM}$; then the corresponding line $\mathbb{P}\mathbf{Q}$ in $\mathbb{PT}$ will sweep out some small open region $\mathbb{P}\mathcal{R}$ in $\mathbb{PT}$, called a *tubular neighbourhood* of $\mathbb{P}\mathbf{R}$.

We are going to try to *deform* twistor space, so that it might, in some way, encode the structure of a *curved* space-time. The reason for considering $\mathbb{PT}^+$ or $\mathbb{P}\mathcal{R}$ is that it turns out that we cannot deform the *whole* of $\mathbb{PT}$, continuously, so as to obtain a distinct complex manifold. For there are rigorous theorems which tell us that there is no complex manifold with the same topology as $\mathbb{CP}^3$ whose complex structure actually differs from that of $\mathbb{CP}^3$. But if we restrict attention to $\mathbb{PT}^+$, or to our tubular neighbourhood $\mathbb{P}\mathcal{R}$ of the line $\mathbb{P}\mathbf{R}$, then many such deformations are possible. For definiteness in what follows, let us work with the tubular neighbourhood case $\mathbb{P}\mathcal{R}$.

In fact, we can use a twistor function f, homogeneous of degree $+2$ (this homogeneity corresponding to helicity $s = -2$, as is appropriate for a left-handed graviton), to generate the required deformation — where I am now assuming, for simplicity, that there are just two open sets $\mathcal{U}_1$ and $\mathcal{U}_2$ covering $\mathbb{P}\mathcal{R}$ (where we can take these sets to be the intersections of two Stein manifolds with $\mathbb{P}\mathcal{R}$, if we like, but $\mathcal{U}_1$ and $\mathcal{U}_2$ will not themselves be Stein), f being defined on $\mathcal{U}_1 \cap \mathcal{U}_2$. We may take it that the Riemann sphere $\mathbb{P}\mathbf{R}$ is divided, by its intersections with $\mathcal{U}_1$ and $\mathcal{U}_2$, into two slightly extended hemispherical open sets which overlap in an annular region, these two being "thickened out" to give us the two 3-complex-dimensional open regions $\mathcal{U}_1$ and $\mathcal{U}_2$. We shall generate our deformation by means of the vector field $\mathbf{F}$ on $\mathcal{U}_1 \cap \mathcal{U}_2$, defined (see §2) by

$$\mathbf{F} = \mathsf{I}^{\alpha\beta} \frac{\partial f}{\partial \mathsf{Z}^\alpha} \frac{\partial}{\partial \mathsf{Z}^\beta}$$

which we can rewrite as

$$\mathbf{F} = \varepsilon^{AB} \frac{\partial f}{\partial \omega^A} \frac{\partial}{\partial \omega^B} \,.$$

This vector field has homogeneity 0. In fact, in what follows, we shall first allow $\mathbf{F}$ to be a quite *general* holomorphic vector field on $\mathcal{U}_1 \cap \mathcal{U}_2$, homogeneous of degree 0. The special significance of the particular form given above will emerge a little later.

If we imagine sliding $\mathcal{U}_1$ over $\mathcal{U}_2$ by an infinitesimal amount, according to $\mathbf{F}$, then (by virtue of this homogeneity) we get an infinitesimal deformation of the complex structure of the projective space $\mathbb{P}\mathcal{R}$. In fact, we can now envisage *exponentiating* $\mathbf{F}$, so as to get a *finite* deformation $\mathbb{P}\mathcal{R}^{\sim}$ of $\mathbb{P}\mathcal{R}$. (Had we chosen a covering of $\mathbb{P}\mathcal{R}$ with more than two sets, then this might have been problematic with regard to getting the triple-overlap condition to behave consistently, and this is the reason for restricting attention the case of a two-set covering.) Roughly, this amounts to "breaking" $\mathbb{P}\mathcal{R}$ into two (overlapping) pieces and then re-gluing the pieces in a slightly displaced way, so as to obtain $\mathbb{P}\mathcal{R}^{\sim}$.

How are we to make use of $\mathbb{P}\mathcal{R}^{\sim}$ as a new kind of (projective) twistor space? We would like to have some "lines" in $\mathbb{P}\mathcal{R}^{\sim}$ that can be interpreted as the points of some sort of "space-time". We can't use the same lines as the $\mathbb{P}\mathbf{Q}$s that we had before, because these will have now become "broken" by this procedure. But fortunately, some theorems of Kodaira (1962, 1963) and Kodaira and Spencer (1958) now come to our rescue, telling us that (assuming that this deformation — though finite — is in an appropriate sense "not too big") there will always be a 4-complex-parameter family of holomorphic curves (Riemann spheres) lying in $\mathbb{P}\tilde{\mathcal{R}}$, characterized by the fact that they can be obtained *continuously* from the line $\mathbb{P}\mathbf{R}$ that we started with, lying in $\mathbb{PT}$ (as the deformation continuously proceeds). Each curve $\mathbb{P}\tilde{\mathbf{Q}}$ of this family — which I shall refer to as a *line* in $\mathbb{P}\tilde{\mathcal{R}}$ — is to be represented by a point $\tilde{Q}$ of a *new* complex manifold $\tilde{\mathcal{V}}$. The complex 4-manifold $\tilde{\mathcal{V}}$ is, indeed, *defined* as the space parameterizing these lines in $\mathbb{P}\tilde{\mathcal{R}}$.

In fact, the manifold $\tilde{\mathcal{V}}$ automatically acquires a complex *conformal* structure, purely from the incidence properties of $\mathbb{P}\tilde{\mathcal{R}}$. Recall from §2 that this was the case with the relationship between $\mathbb{CM}$ and $\mathbb{PT}$, since null separation between two points Q and S, in $\mathbb{CM}$, is represented, in $\mathbb{PT}$, simply by the meeting of the corresponding lines $\mathbb{P}\mathbf{Q}$ and $\mathbb{P}\mathbf{S}$. We adopt precisely the same procedure here, so the *light cone* of a point $\tilde{Q}$ in $\tilde{\mathcal{V}}$ is defined simply as the locus of points $\tilde{S}$ whose corresponding line $\mathbb{P}\tilde{\S}$ meets $\mathbb{P}\tilde{\mathbf{Q}}$. Again, it follows from general theorems that this light cone's vertex is an ordinary *quadratic* (i.e. not Finsler) one arising from a complex quadratic metric (which is thereby defined, locally, up to proportionality). We can now proceed to piece together several small regions $\tilde{\mathcal{V}}$, obtained from these "local" tubular neighbourhoods $\mathbb{P}\tilde{\mathcal{R}}$, so as to obtain a more extended complex conformal manifold $\mathcal{M}$, obtained from a more extended (projective) twistor space $\mathbb{P}\mathcal{T}$, generalizing complex Minkowski space $\mathbb{CM}$ and its relation to $\mathbb{PT}$.

Does $\mathcal{M}$ have any particular properties, by virtue of this construction? Indeed it does. Most significantly, the very existence of *points* in $\mathbb{P}\mathcal{T}$, leads to $\mathcal{M}$ being *anti-self dual* (where the "anti" part of this terminology is purely conventional, chosen here to fit in with standard twistor conventions and use of the Z^α-representation). Why is this? The lines in $\mathbb{P}\mathcal{T}$ constitute a 4-parameter family, so those passing through some given point $\mathbb{P}\mathbf{Z}$, of $\mathbb{P}\mathcal{T}$, (2 analytic conditions) will constitute a 2-parameter family. The points of $\mathcal{M}$ representing this 2-parameter family of lines will represent a complex 2-surface, which will be called the α-*surface* Z in $\mathcal{M}$. Now since each pair of lines through $\mathbb{P}\mathbf{Z}$ must intersect (at $\mathbb{P}\mathbf{Z}$), it follows that each pair of points in the α-surface Z must be *null separated*, so that, as in $\mathbb{M}$, any α-surface must be *totally null* (vanishing induced conformal metric), and therefore is either self dual or anti-self dual. Conventionally, we call the α-surfaces *self dual*.

Now we ask: what is the condition on the Weyl curvature tensor $C_{abc}{}^d$ (which is well defined, for a conformal manifold) for the existence of a 3-parameter family of (self-dual) α-surfaces? It is a straight-forward calculation to show that the self-dual part of the Weyl curvature must consequently vanish and, conversely, that the vanishing of the self-dual Weyl tensor of a conformal (complex) 4-manifold $\mathcal{M}$ is (locally) the condition for the existence (locally) of a 3-parameter family of α-surfaces. The existence of such a family enables us to construct $\mathcal{M}$'s (projective) *twistor space* $\mathbb{P}\mathcal{T}$, each point of $\mathbb{P}\mathcal{T}$ representing an α-surface in $\mathcal{M}$. The above procedure thus provides us with a direct way of constructing (any) conformally anti-self-dual (complex) 4-manifold from a suitable (but "generic") complex 3-manifold $\mathbb{P}\mathcal{T}$, the only technical difficulty, in this construction being actually *finding*[v] such a family of lines in $\mathbb{P}\mathcal{T}$. Their existence can be ensured by the procedure to follow.

The particular form of the vector field $\mathbf{F}$, as given at the beginning of this section in terms of a twistor function f, provides us only a restricted class of such projective twistor spaces $\mathbb{P}\mathcal{T}$, whose particular significance we shall be seeing in a moment. For the *general* projective twistor space $\mathbb{P}\mathcal{T}$, from which an anti-self-dual conformal manifold can be constructed, we simply generate our deformation using a *general* holomorphic vector field[w] $\mathbf{F}$ on $\mathcal{U}_1 \cap \mathcal{U}_2$, homogeneous of degree 0, in place of one of the above restricted form. By use of this kind of deformation — for deformations that are "not too big" — we are assured that $\mathbb{P}\mathcal{T}$ has the right form for it to have an appropriate 4-parameter family of lines enabling a generic anti-self-dual $\mathcal{M}$ to be constructed.

We take note of the fact that this construction provides us with a *complex* anti-self-dual conformal manifold $\mathcal{M}$. Such an $\mathcal{M}$ can be the complexification of a real Lorentzian conformal manifold only in the (relatively) uninteresting case of conformal flatness. For in the complex case, the Weyl conformal curvature tensor C_{abcd} can be written in a 2-spinor (abstract-index) form in just the same way as for the tensor K_{abcd} of §4

$$C_{abcd} = \Phi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \varepsilon_{AB}\varepsilon_{CD}\mathrm{X}_{A'B'C'D'}\,,$$

where Φ_{ABCD} and $\mathrm{X}_{A'B'C'D'}$ are each totally symmetrical. In the conformally self dual case we have

$$\Phi_{ABCD} = 0$$

and in the conformally anti-self-dual case we have

$$\mathrm{X}_{A'B'C'D'} = 0\,,$$

but in the Lorentzian case we must have

$$\mathrm{X}_{A'B'C'D'} = \mathcal{C}\Phi_{ABCD}\,,$$

so if one vanishes so must the other, whence C_{abcd} as a whole must vanish — the condition for (local) conformal flatness. It may be remarked, however, that in the positive-definite case (signature$++++$)[x] and the split-signature case (signature $++--$)[y] there is a large family of conformally (anti-)self-dual 4-manifolds. These spaces, and their twistor spaces, have a considerable pure-mathematical interest,[z] there being (different) "reality conditions" on each of the independent quantities Φ_{ABCD} and $\mathrm{X}_{A'B'C'D'}$ in these two cases.

Yet, *complex* (anti-)self-spaces are by no means devoid of *physical* interest, especially those which can arise from deformations of (parts of) $\mathbb{T}$ for which $\mathbf{F}$ has the special form $\mathbf{F} = \varepsilon^{AB}\partial f/\partial\omega^A \partial/\partial\omega^B$, as given at the beginning of this section. We note that, in these cases, the operator $\partial/\partial\pi_{A'}$ does not appear, and it follows that its infinitesimal action on $\mathsf{Z}^\alpha = (\omega^A, \pi_{A'})$ leaves $\pi_{A'}$ unaffected. Thus, $\mathbf{F}$ generates a deformation of (part of) $\mathbb{T}$ that preserves the projection

$$\mathbf{F} : \mathbb{T} \to \bar{\mathbb{S}}^*\,,$$

(where $\bar{\mathbb{S}}^*$ is the complex conjugate of the dual $\mathbb{S}^*$ of the spin space $\mathbb{S}$; note that $\mathbb{S}$ is the space of 2-spinors like ω^A, and $\bar{\mathbb{S}}^*$ is the space of those like $\pi_{A'}$). In each coordinate patch $\mathcal{U}_i$, with standard twistor coordinates, this projection takes the form

$$(\omega^A, \pi_{A'}) \mapsto \pi_{A'}\,,$$

and this now extends to a projection $\mathbf{F}$ that applies to the whole (non-projective) curved twistor space

$$\mathbf{F} : \mathcal{T} \to \bar{\mathbb{S}}^*$$

The inverse $\mathbf{F}^{-1}$ of this projection is a *fibration* of $\mathbb{P}\mathcal{T}$, each fibre being the entire complex 2-surface in $\mathcal{T}$ which projects down to a particular $\boldsymbol{\pi}$ in $\bar{\mathbb{S}}^*$. In this case, the *lines* in $\mathbb{P}\mathcal{T}$ can be neatly characterized as the projective versions of the holomorphic *cross-sections* of this fibration. These are the results of maps $\mathbf{R} : \bar{\mathbb{S}}^* \to \mathcal{T}$ (whose composition $\mathbf{F} \circ \mathbf{R}$ with $\mathbf{F}$ is the identity on $\bar{\mathbb{S}}^*$) which lift $\bar{\mathbb{S}}^*$ back into $\mathbb{P}\mathcal{T}$, and they generalize what in the canonical flat case would be expressed as $\pi_{A'} \mapsto (\mathrm{i}r^{AA'}\pi_{A'}, \pi_{A'})$, where r^a is the position vector of the point in $\mathbb{CM}$ that this cross-section defines. (Note the appearance of the basic incidence relation of §2.)

The 2-surfaces of this fibration have tangent directions annihilated by a simple closed 2-form $\boldsymbol{\tau}$ which is $(\frac{1}{2}\times)$ the exterior derivative of a 1-form $\boldsymbol{\iota}$ (of homogeneity degree 2):

$$2\boldsymbol{\tau} = \mathrm{d}\boldsymbol{\iota}\,, \quad \boldsymbol{\iota} \wedge \boldsymbol{\tau} = 0$$

(the latter condition ensuring simplicity of $\boldsymbol{\iota}$, since $\boldsymbol{\tau} \wedge \boldsymbol{\tau} = 0$). The forms $\boldsymbol{\iota}$ and $\boldsymbol{\tau}$ are part of the structure of $\mathcal{T}$ that is unchanged by the deformation generated by our special kind of $\mathbf{F}$ defined from f as given above; and in the flat case $\mathbb{T}$, these forms are the two alternative versions of the (Poincaré-invariant) "$\delta\mathsf{Z}$", used in the contour integrals of §4:

$$\boldsymbol{\iota} = \mathsf{I}_{\alpha\beta}\mathsf{Z}^{\alpha}\mathrm{d}\mathsf{Z}^{\beta} = \varepsilon^{A'B'}\pi_{A'}\mathrm{d}\pi_{B'}\,, \quad \boldsymbol{\tau} = \frac{1}{2}\mathsf{I}_{\alpha\beta}\mathrm{d}\mathsf{Z}^{\alpha} \wedge \mathrm{d}\mathsf{Z}^{\beta} = \frac{1}{2}\varepsilon^{A'B'}\mathrm{d}\pi_{A'} \wedge \mathrm{d}\pi_{B'}\,.$$

In addition, $\mathcal{T}$ has a ("volume") 4-form $\boldsymbol{\phi}$ which is ($\frac{1}{4}\times$) the exterior derivative of a ("projective-volume") 3-form $\boldsymbol{\theta}$ (each of homogeneity degree 4), also unchanged by the deformation, where in the flat case $\mathbb{T}$,

$$\boldsymbol{\theta} = \frac{1}{6}\varepsilon_{\alpha\beta\rho\sigma}\mathsf{Z}^{\alpha}\mathrm{d}\mathsf{Z}^{\beta} \wedge \mathrm{d}\mathsf{Z}^{\rho} \wedge \mathrm{d}\mathsf{Z}^{\sigma}\,, \quad \boldsymbol{\phi} = \frac{1}{24}\boldsymbol{\varepsilon}_{\alpha\beta\rho\sigma}\mathrm{d}\mathsf{Z}^{\alpha} \wedge \mathrm{d}\mathsf{Z}^{\beta} \wedge \mathrm{d}\mathsf{Z}^{\rho} \wedge \mathrm{d}\mathsf{Z}^{\sigma}\,,$$

and we have

$$4\boldsymbol{\phi} = \mathrm{d}\boldsymbol{\theta}\,, \quad \boldsymbol{\iota} \wedge \boldsymbol{\theta} = 0$$

This local structure possessed by $\mathcal{T}$ enables $\mathcal{M}$ to be assigned a metric $g_{ab} (= \varepsilon_{AB}\varepsilon_{A'B'})$, where the "$\varepsilon^{A'B'}$" in $g^{ab} = \varepsilon^{AB}\varepsilon^{A'B'}$ comes from $\boldsymbol{\tau}$ (the area form in the structure of $\bar{\mathbb{S}}^*$) and "ε_{AB}" comes from "$\boldsymbol{\phi} \div \boldsymbol{\tau}$" (the area form in the fibres of $\mathbb{F}^{-1}$). This metric determines a connection, and because the projection $\mathbb{F} : \mathcal{T} \to \mathbb{S}^*$ is preserved in our special deformation, it follows that there is a global parallelism for elements of $\bar{\mathbb{S}}^*$ (the space of $\pi_{A'}$-spinors). The self-dual part of the Weyl curvature being zero ($\mathrm{X}_{A'B'C'D'} = 0$), this turns out to imply the *vanishing of the Ricci tensor* ($R_{ab} = 0$). In fact, this argument reverses, and we find that any complex-Riemannian 4-manifold $\mathcal{M}$, which is both Ricci flat and conformally anti-self dual, has (locally) a twistor space $\mathcal{T}$ with the structure just provided above, from which $\mathcal{M}$ can be reconstructed by the foregoing procedure (in which the points of $\mathcal{M}$ are identified as holomorphic cross-sections of the fibration $\mathbf{F}^{-1}$).

How are we to interpret such a complex "space-time" $\mathcal{M}$ physically? The first place where such spaces were encountered, in a physical context, was with the $\mathcal{H}$-space construction of Ezra T. Newman (1976, 1979). Here, one considers the complexified future conformal infinity $\mathbb{C}\mathscr{I}^+$, of an asymptotically flat space-time (assumed analytic), and examines cross-sections ("cuts") of $\mathbb{C}\mathscr{I}^+$, which satisfy a condition of "vanishing complex asymptotic shear", this being a generalization to a curved space-time of a procedure which would locate the intersections, with $\mathbb{C}\mathscr{I}^+$, of the future light cones of points in $\mathbb{CM}$. In the case where $\mathcal{M} = \mathbb{CM}$, this procedure indeed enables $\mathbb{CM}$ to be reconstructed from the geometrical structure of its $\mathbb{C}\mathscr{I}^+$. However, in the general case of a gravitationally radiating (analytic) asymptotically flat space-time, one finds that this procedure does not reproduce the original space-time (complexified) but, instead, produces a Ricci-flat, conformally anti-self-dual complex 4-space called $\mathcal{H}$-space, which may be thought of as the "space-time" reconstructed from the anti-self-dual part of its outgoing gravitational field.[aa]

Subsequently it was proposed[bb] that such anti-self-dual Ricci-flat complex spaces, if subjected to an appropriate condition of "positive frequency" could be viewed as representing a *non-linear* description of the *wavefunction* of a left-handed graviton. Indeed this kind of interpretation is very much in line with the aims of twistor theory, as put forward in §1. In the standard perturbative viewpoint, a single "graviton" would be described by a solution of *linearized* general relativity, and it is an entirely flat-space quantity, where curvature and non-trivial causality structure does not arise. These geometrical properties, characteristic of general relativity proper, only occur when contributions involving an indefinitely large number of such "linear" gravitons are involved. But a non-linear graviton, as described by the above twistorial construction, is already a curved-space entity, and we may take the view that, in that description, each graviton carries its own measure of actual curvature. Moreover, being a non-linear entity, the concept of such a graviton moves us away from the standard linear structure of quantum mechanics, leading to a hope that eventually some non-linear quantum mechanics might arise, according to which the measurement paradox might eventually be resolved.

7. The Googly Problem; Further Developments

As yet, however, there is no clear relation between this kind of non-linearity and that which might be relevant to state reduction. Any such development would seem to require, as an initial step, a more complete description of the gravitational field than that which was outlined in the previous section. The "non-linear graviton" of §6 is, after all, only "half a graviton" in the sense that it restricts our consideration to only one of the two helicities that should be available to a graviton. Of course one could repeat the entire argument in terms of the *dual* twistor description — or W_α-representation — and then we should have a description of a right-handed non-linear graviton. But this is of no use to us if we wish for a comprehensive formalism in which, for example, plane-polarized gravitons might be described. In such a formalism, it would have to be possible also to describe a right-handed (i.e. *self*-dual) graviton while still using the Z^α-representation (or, equivalently, to describe a left-handed graviton using the W_{α^-}-representation). This would mean finding the appropriate non-linear version of a twistor function homogeneous of degree -6 (which, in the $\mathbf{Z}^\alpha$-representation, means helicity $s = +2$). This problem is referred to as the *googly problem* (a reference to the subtle bowling of a cricket ball with a right-handed spin about the direction of motion, using a bowling action which would appear to be imparting a left-handed spin).

I have referred to the googly problem only in the gravitational case, but there are analogues for electromagnetism and other Yang–Mills fields also. For left-handed photons (or the left-handed high-energy massless limit of W- or Z-bosons, or "gluons" of strong interactions) one would anticipate some sort of twistor deformation that is obtainable from the "exponentiation" of vector field obtainable from a twistor function homogeneous of degree 0. Indeed, there is such a construction, due to Richard Ward (1977) which provides the general anti-self-dual solution of the Yang–Mills equations, for a given specific group, in terms of such a (generally non-linear) twistor construction. Basically, what is required is to produce a (locally unconstrained) holomorphic vector bundle (for this group), over some suitable region $\mathcal{X}$ of $\mathbb{PT}$ where, as in §6, we may take $\mathcal{X}$ to be either $\mathbb{PT}^+$ or an appropriate tubular neighbourhood of some line $\mathbb{P}\mathbf{R}$ in $\mathbb{PT}$. Then, noting that holomorphicity basically fixes the bundle to be *constant over any line* in $\mathcal{X}$, we obtain a fibre over that point in $\mathbb{CM}$ which corresponds to this line. Thus, we can transfer the whole bundle over $\mathcal{X}$ to a bundle over the corresponding region $\mathcal{V}$, in $\mathbb{CM}$. From the fact that there is a single fibre over any specific point $\mathbb{P}\mathbf{Z}$ in $\mathcal{X}$, we find a natural connection defined on the bundle over $\mathcal{X}$ that is necessarily constant over the corresponding α-plane Z in $\mathcal{V}$. This makes the connection an *anti-self-dual* one, as required, and we find that this construction is reversible, showing the essential equivalence between anti-self-dual Yang–Mills fields on (appropriate) regions of $\mathbb{CM}$ and holomorphic bundles on the corresponding regions in $\mathbb{PT}$.

This is the geometrical essence of the Ward construction. It has found many applications, especially in the theory of integrable systems.[cc] It is clear, however, that for a full application of these ideas in basic physical theory, a satisfactory solution to the Yang–Mills googly problem is needed also. In view of its importance, it is perhaps remarkable how little interest this topic has aroused so far in the physics community, although the matter has recently received some attention in the twistor-string literature, which

I shall refer to briefly at the end of this section.

My own approach has been to concentrate attention more thoroughly on the gravitational googly problem, in the hope that the seemingly more severe restrictions on what types of construction are likely to be appropriate, in the gravitational case, may act as a guide to the correct approach. On the whole, however, there has been much frustration (for over 25 years!), and it is still not altogether clear whether the correct approach has been found. Accordingly, I shall only rather briefly outline what seem to be the three most promising modes of attack on this problem, concentrating mainly on the first. These may be classified crudely as follows:

- Geometric
- Functorial
- Twistor-string related.

It should be mentioned that a satisfactory solution of the googly problem should also inform us how the left and right helicities of the graviton are to *combine*, so we anticipate a twistorial ("Z^α"-)representation of solutions of the full (vacuum) Einstein equations.

The geometric approach[dd] is the most fully developed, and it has turned out to be possible to encode the information of a vacuum (i.e. Ricci-flat), analytic and appropriately asymptotically flat, complexified, space-time $\mathcal{M}$ in the structure of a deformed twistor space $\mathcal{T}$. The construction of $\mathcal{T}$ from $\mathcal{M}$ is completely explicit, and the data determining the structure of $\mathcal{M}$ seems to be given freely (i.e. without differential equations or awkward boundary conditions having to be solved). However the re-construction of $\mathcal{M}$ from $\mathcal{T}$ remains conjectural and somewhat problematic.

The local structure of this proposed complex 4-dimensional twistor space $\mathcal{T}$ can be defined in terms of the (non-vanishing) holomorphic forms $\boldsymbol{\iota}$ and $\boldsymbol{\theta}$ (a 1-form and a 3-form, respectively) that we had in §6, where

$$\boldsymbol{\iota} \wedge \mathrm{d}\boldsymbol{\iota} = 0\,, \quad \boldsymbol{\iota} \wedge \boldsymbol{\theta} = 0\,,$$

as before, from which we can infer that $\mathcal{T}$ is foliated, locally, by a family of holomorphic curves — which I shall refer to as *Euler curves* defined by $\boldsymbol{\theta}$, and of holomorphic 3-surfaces, defined by $\boldsymbol{\iota}$, each of these 3-surfaces being foliated by Euler curves, but now we are to specify each of $\boldsymbol{\iota}$ and $\boldsymbol{\theta}$ only up to proportionality. There are, however, some restrictions on how these forms can be jointly rescaled, which can be stated as the requirement that the following two quantities $\mathbf{\Pi}$ and $\mathbf{\Sigma}$ are to be invariant:

$$\mathbf{\Pi} = \mathrm{d}\boldsymbol{\theta} \otimes \mathrm{d}\boldsymbol{\theta} \otimes \boldsymbol{\theta}\,, \quad \mathbf{\Sigma} = \mathrm{d}\boldsymbol{\theta} \otimes \boldsymbol{\iota} = -2\boldsymbol{\theta} \oslash \mathrm{d}\boldsymbol{\iota}$$

where we demand that the two given expressions for $\mathbf{\Sigma}$ are to be equal, the bilinear operator "$\varnothing$" being defined by

$$\boldsymbol{\eta}\varnothing(\boldsymbol{\rho}\wedge\boldsymbol{\sigma}) = (\boldsymbol{\eta}\wedge\boldsymbol{\rho})\otimes\boldsymbol{\sigma} - (\boldsymbol{\eta}\wedge\boldsymbol{\sigma})\otimes\boldsymbol{\rho}$$

as applied to any r-form $\boldsymbol{\eta}$ and 1-forms $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$. We can express this $\varnothing$ in "index form" as twice the anti-symmetrization of the final index of $\boldsymbol{\eta}$ with the two indices of the 2-form which follows the $\varnothing$ symbol. (This generalizes to a $\varnothing$-operation between an r-form and a t-form, where we take $t\times$ the antisymmetrization of the final index of the r-form with all indices of the t-form.) The preservation of the two quantities $\mathbf{\Pi}$ and $\mathbf{\Sigma}$ is really just asserting that on the overlap of two open sets $\mathcal{U}'$ and $\mathcal{U}$, the quantities $\boldsymbol{\iota}$ and $\boldsymbol{\theta}$ must scale according to the (somewhat strange) rules

$$\boldsymbol{\iota}' = \kappa\boldsymbol{\iota}\,, \quad \boldsymbol{\theta}' = \kappa^2\boldsymbol{\theta}\,, \quad \text{and} \quad \mathrm{d}\boldsymbol{\theta}' = \kappa^{-1}\mathrm{d}\boldsymbol{\theta}\,,$$

for some scalar function κ. The *Euler homogeneity operator* $\boldsymbol{\Upsilon}$ (a vector field — see beginning of §3), which points along the Euler curves, can be defined, formally, by

$$\boldsymbol{\Upsilon} = \boldsymbol{\theta} \div \boldsymbol{\phi}$$

where we recall that the 4-form $\boldsymbol{\phi}$ of §6 is defined from $\boldsymbol{\theta}$ by $4\boldsymbol{\phi} = \mathrm{d}\boldsymbol{\theta}$. More precisely, we can define $\boldsymbol{\Upsilon}$ by

$$\mathrm{d}\xi\wedge\boldsymbol{\theta} = \boldsymbol{\Upsilon}(\xi)\boldsymbol{\phi}$$

for any scalar field ξ. We find, on the overlap between open regions $\mathcal{U}'$ and $\mathcal{U}$ (primed quantities κ' and $\boldsymbol{\Upsilon}'$ referring to $\mathcal{U}'$), that

$$\kappa' = \kappa^{-1}$$
$$\boldsymbol{\Upsilon}' = \kappa^3\boldsymbol{\Upsilon}$$

and, consequently,

$$\boldsymbol{\Upsilon}(\kappa) = 2\kappa^{-2} - 2\kappa$$

and, equivalently $\boldsymbol{\Upsilon}(\kappa^{-1}) = 2\kappa^2 - 2\kappa^{-1}$. We can deduce from all this that, on overlaps, κ^3 takes the form

$$\kappa^3 = 1 - f_{-6}(\mathsf{Z}^\alpha)$$

in standard flat-space terms, so we obtain the encoding of a twistor function f_{-6}, homogeneous of degree -6 (in a fully cohomological way).

This geometrical means of encoding a twistor function of the required "googly" homogeneity -6 may seem somewhat strange, where the information is stored in a curious non-linear deformation of the *scaling* of the Euler curves (the curves which collapse down to points in the passage from $\mathcal{T}$ to $\mathbb{P}\mathcal{T}$). This deformation destroys the clear notion of the homogeneity degree of a function defined on $\mathcal{T}$. On the other hand, no other procedure has yet emerged for the encoding of this self-dual curvature information in a deformation of twistor space. Moreover, this curious non-linear scaling of the Euler curves can actually be seen to arise in an well-defined construction of $\mathcal{T}$ in terms of the space-time geometry of $\mathcal{M}$, where the points of $\mathcal{T}$ are defined as solutions of an explicit differential equation defined at $\mathbb{C}\mathscr{I}^+$, in which the *self-dual* (as opposed to anti-self-dual) Weyl curvature appears as a coefficient in the equation determining this scaling.[ee] There also appears to be a clear algebraic role for these particular scalings in certain relevant expressions.

Nonetheless, to be fully confident that such procedures are really following "correct" lines, one would like to have a clear-cut way of seeing that the resulting gravitational theory is really left/right symmetric, despite the extreme lop-sidedness that seems to be involved in this geometry. One might imagine that this could be understood at a formal algebraic level; for the algebra generated by commuting quantities Z^0, Z^1, Z^2, Z^3 is formally identical with that generated by $\partial/\partial\mathsf{Z}^0$, $\partial/\partial\mathsf{Z}^1$, $\partial/\partial\mathsf{Z}^2$, $\partial/\partial Z^3$. But, can we see, for example, that if we translate (in some formal sense) a pure *left*-helicity ($s = -2$) "non-linear graviton" — as given by the prescriptions of §6 — from a construction in terms of ordinary "Z^α-coordinates" to one in terms of "$\partial/\partial\mathsf{Z}^\alpha$-coordinates", then this now behaves (being now pure right-handed, $s = +2$) as though it were a complex manifold, as constructed in the present section, with a *flat* $\mathbb{P}\mathcal{T}$ (i.e. for which $\mathbb{P}\mathcal{T}$, is a portion of flat projective twistor space $\mathbb{PT}$) so that a pure *right*-helicity graviton is now being described?

The required notions can be at least partially formulated in terms of *category theory*,[ff] where one finds a certain "functorial" relation between the multiplicative action of a quantity X and the derivative action of $\partial/\partial\mathsf{X}$, but where the latter is *dual* to the former, in the sense that the functorial arrows are reversed.[gg] One would hope to find that the above deformations generated by twistor functions of homogeneity degree $+2$ and -6 to be related in a similar way. As yet, this is not very clear, the issues being complicated by a basic obscurity about how one is to "dualize" the procedures that apply to building a manifold out of open coordinate patches. These matters are tied up with the issue of the "radius of convergence" of an analytic function defined by a power series in Z^α (so that the function has an appropriate open set on which it is defined) and whatever the corresponding notion should be for a "power series" in $\partial/\partial \mathsf{Z}^\alpha$.

It would appear to be probable that some insights into the appropriate "quantum twistor geometry" are to be obtained from the procedures of *non-commutative geometry*[hh] applied to the original twistor space $\mathbb{T}$, since here we have basic "coordinates" Z^α and $\bar{\mathsf{Z}}_\alpha$ which do not commute, these behaving formally like Z^α and $-\partial/\partial\mathsf{Z}^\alpha$ I am not aware of any detailed work in this direction, however. In the absence of this, I wish to make some pertinent comments that seem to address this kind of issue from a somewhat different angle.

There is at least one way in which the replacements

$$\bar{\mathsf{Z}}_\alpha \rightsquigarrow -\frac{\partial}{\partial \mathsf{Z}^\alpha} \quad \text{and} \quad \frac{\partial}{\partial \bar{\mathsf{Z}}_\alpha} \rightsquigarrow \mathsf{Z}^\alpha$$

do find a clear mathematical representation in important twistor expressions. This is in the (positive definite) Hermitian *scalar product* $\langle f|g\rangle$ between positive-frequency twistor functions (1st cohomology elements) f and g, each of a given homogeneity degree r. Let us choose another such twistor function h, but now of homogeneity $r-1$. Then we find the relations

$$\langle \frac{\partial f}{\partial \mathsf{Z}^\alpha}|h\rangle = -\langle f|\mathsf{Z}^\alpha h\rangle\,,$$

$$\langle \mathsf{Z}^\alpha h|f\rangle = -\langle h|\frac{\partial f}{\partial \mathsf{Z}^\alpha}\rangle\,,$$

which is consistent with the above replacements, where we must bear in mind that the quantities in the "$\langle\ldots|$" actually appear in complex-conjugate form and that in the first of these relations a minus sign comes about when the action of "$\partial/\partial\bar{\mathsf{Z}}_\alpha$" is transferred from leftward to rightward.

To see how to ensure that these relations are satisfied, we need the general form of the scalar product, but where (for the moment) I restrict attention to cases for which $r > -4$. We find that this scalar product takes the form

$$\langle f|g\rangle = c \oint \bar{f}(\mathsf{W}_\alpha)[\mathsf{W}_\alpha \mathsf{Z}^\alpha]_{-r-4}\, g(\mathsf{Z}^\alpha)\mathrm{d}^4\mathbf{W}\wedge \mathrm{d}^4\mathbf{Z}$$

where c is some constant, independent of r, where $\mathrm{d}^4\mathbf{W} = \frac{1}{24}\varepsilon^{\alpha\beta\rho\sigma}\mathrm{d}\mathsf{W}_\alpha \wedge \mathrm{d}\mathsf{W}_\beta \wedge \mathrm{d}\mathsf{W}_\beta \wedge \mathrm{d}W_\beta$ and correspondingly for $\mathrm{d}^4\mathbf{Z}$ (which is the 4-form $\boldsymbol{\phi}$ above), and where (with $n > 0$)

$$[x]_{-n} = -(-x)^{-n}(n-1)!$$

so that

$$[x]_{-1} = x^{-1} \quad \text{and} \quad \frac{\mathrm{d}[x]_{-n}}{\mathrm{d}x} = [x]_{-n-1}\,.$$

The scalar product $\langle \ldots | \ldots \rangle$ then satisfies the required relations, above (for $r > -4$).

Now, bearing in mind what was said earlier, towards the end of §4, about the need to consider twistor functions that are not necessarily homogeneous (this being reinforced by the discussion of the googly problem, earlier in this section, in which inhomogeneous expressions arise), we find that we shall need to replace the "$[\mathsf{W}_\alpha \mathsf{Z}^\alpha]_{-r-4}$" term in the scalar product by a *sum* of such terms in which different values of r are involved (taking note of the fact that the "cross-terms", where the integrand has homogeneity other than zero, must vanish). If we are concerned with only a finite number of these terms, then we have no problem, but for an infinite number of terms (again restricting to $r > -4$, for the moment), then we appear to be presented with the seriously divergent series

$$F(x) = x^{-1} - 1!x^{-2} + 2!x^{-3} - 3!x^{-4} + 4!x^{-5} - 5!x^{-6} + \ldots .$$

Yet, in the 18th century, Euler had already shown that — in a formal sense at least — this series can be equated to the *convergent* expression[ii]

$$\begin{aligned} E(x) = (-\gamma - \log x) + x(1 - \gamma - \log x) + \frac{x^2}{2!}(1 + \frac{1}{2} - \gamma - \log x) \\ + \frac{x^3}{3!}(1 + \frac{1}{2} + \frac{1}{3} - \gamma - \log x) \\ + \frac{x^4}{4!}(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \gamma - \log x) + \ldots , \end{aligned}$$

where γ is *Euler's constant*. The function $E(x)$ can be defined in various equivalent ways, for example by the (equivalent) integral formulae

$$\begin{aligned} E(x) &= \mathrm{e}^x \int_x^\infty \frac{\mathrm{e}^{-u}}{u}\,\mathrm{d}u \\ &= \int_0^\infty \frac{\mathrm{e}^{-u}}{(u+x)}\mathrm{d}u \end{aligned}$$

where we *formally* have

$$F(x) - E(x) = 0 .$$

This suggests that we might use $E(\mathsf{W}_\alpha \mathsf{Z}^\alpha)$ in place of our divergent $F(\mathsf{W}_\alpha \mathsf{Z}^\alpha)$, in the contour integral expression for $\langle f | g \rangle$ (and that this might also serve to extend the definition of $\langle \ldots | \ldots \rangle$ to values of r with $r \leq -4$).

However, this is not such a simple matter because the singularity structure of the terms in the series for $E(x)$ involve logarithms, and the standard contours surrounding the *poles* that occur in the formal series for $F(x)$ run into branch cuts, and cannot be directly used. The issues arising here appear to be somewhat subtle and complicated, and they are not fully resolved at present. Various approaches for handling such logarithmic terms have been used in the past, including regarding $\log x$, in a contour integral, as arising from a limit[jj] of $x^\nu\Gamma(\nu)$ as $\nu \to 0$. But the most fruitful and satisfactory procedure appears to be to regard such terms as being treated as though the contour has a *boundary* in place of the branch cut in $\log x$, the "$\log x$" term itself being replaced, more or less, by $2\pi\mathrm{i}$. In fact, according to a detailed study of twistor diagrams (the twistor analogue of Feynman diagrams) by Andrew Hodges (1985, 1998), this boundary should be taken at $x = k$, where k is some non-zero constant. Hodges also finds that the natural value of k appears to be given by $k = e^{-\gamma}$ (or possibly at $k = -e^{-\gamma}$), where γ is Euler's constant. This choice has to do with the requirement that infra-red divergences be regularized (as indeed they are with this prescription), and also that a certain idempotency requirement of twistor diagrams be satisfied.

It is interesting that this choice of boundary at $x = \mathrm{e}^{-\gamma}$ corresponds to a requirement that the leading term "$(-\gamma - \log x)$", in the expansion of Euler's function $E(x)$ above, should vanish. Indeed, we appear to resolve a difficulty with the contour topology, in the definition of $\langle \ldots | \ldots \rangle$, if we allow a "blow down" in the twistor space $\mathbb{T}$ where our (8-dimensional) contour, in $\mathbb{T} \times \mathbb{T}^*$, encounters $\mathsf{W}_\alpha \mathsf{Z}^\alpha = \mathrm{e}^{-\gamma}$, a place where $-\gamma - \log(\mathsf{W}_\alpha \mathsf{Z}^\alpha)$ *vanishes* on one branch of the logarithm, which seems to be a requirement for the consistency of this procedure. However the full significance of all this is not yet clear, and requires further understanding. This is work presently in progress, but there is significant hope that the procedures that Hodges has successfully developed in the theory of twistor diagrams may serve to illuminate, and to be illuminated by, this study of quantum twistor geometry.

Finally, some remarks concerning the recent *twistor string theory* are appropriate here. In December 2003, Edward Witten introduced the basis of this new body of ideas.[kk] Here, many of the procedures of string theory are united with those of twistor theory to provide some great simplifications and new insights in the theory of Yang–Mills scattering processes. These involve multiple "gluon" processes (in the massless limit), where in -and out- gluon states are taken to be pure helicity states. To some considerable extent, twistor theory is well set up to handle such situations (helicity states for massless particles being the natural building blocks of the physical interpretation of twistor theory). So it is perhaps not surprising that twistor theory can offer considerable simplifications in the description of such processes.

But it is likely that the string-theory perspective can also offer some new insights into the basis of twistor theory also. As has been emphasized at several places above, the information in a twistor wave-function is stored non-locally (in the form of 1st cohomology or in the non-linear construction of a deformed twistor space). There is no local information in these constructions. The situation is reminiscent of what happens in a topological quantum field theory, where again there is no local information, but a Lagrangian formalism can nevertheless be introduced. (An oft-cited example of this is "$(2 + 1)$-dimensional general relativity", where the "vacuum" is treated as a Ricci-flat region, as in standard 4-dimensional general relativity, but where in this 3-dimensional case the Weyl curvature also vanishes — automatically. There is now no local field information and no dynamics for this "gravitational field", yet a Lagrangian formalism can still be used.) Witten (1988) has shown that such a topological quantum field theory can be treated using string-theoretic procedures and non-trivial results thereby obtained (such as in the theory of knots and links in 3-dimensional space). The fact that there is no local dynamics both in the case of twistor theory and in the situations of a topological field theory suggests that there could be a link between the two. Indeed, Witten proposes such a link (this being more strictly a "holomorphic" than a topological theory[ll]), and there is considerable hope that this may open up new prospects for twistor theory, where up to now there has been little in the way of a Lagrangian basis for developing a comprehensive "twistor dynamics" leading to a genuine approach to a twistorial theory of physics that can stand on its own.

As for how this might relate to a full solution to the googly problem, no serious attempt seems to have been made, so far, to tackle the issues that arise in gravitational theory. But some developments in the case of Yang–Mills theory have been suggested. These may be regarded as taking the Ward construction as encompassing the anti-self-dual part of the Yang–Mills field, but then perturbing away from this so as to provide a full description in which both self-dual and anti-self-duel parts of the Yang–Mills field are described. Although this procedure does not yet provide a full resolution of the googly problem in the Yang–Mills case (let alone the

Einstein case) it seems to indicate some new directions of procedure which could open up promising lines of new development.

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Notes

a. See Penrose 1987, pp. 350, 359.
b. See Einstein, Podolsky, and Rosen (1935); Bohm (1951) Ch. 22, §§15–19; Bell (1987); Baggott (2004).
c. For results of this kind, see Tittel *et al.* (1998).
d. See Károlyházy, F. (1966); Diósi (1989); Penrose (1996, 2000, 2004).
e. This was around 1955, but only published later; see Penrose (1971, 1975, 2004). It should be mentioned that a version of spin-network theory is also used in the loop-variable approach to quantum gravity; see Ashtekar and Lewandowski (2004).
f. However, John Moussouris (1983) has had some success in pursuing this approach, in his (unpublished) Oxford D.Phil. thesis.
g. This is a well-known correspondence; see, for example, Penrose (2004), §22.9, Fig. 22.10.
h. See Terrell (1959); Penrose (1959, 2004 §18.5).
i. Penrose (1976).
j. See Penrose and Rindler (1986), Chapter 9; Penrose (2004) §§33.3, 5.
k. Penrose and Rindler (1986) §§9.2,3; Penrose (2004).
l. Penrose and Rindler (1984).
m. Penrose and Rindler (1984), Chapter 2.
n. Fierz and Pauli (1939); Fierz (1940); Penrose and Rindler (1984).
o. See Penrose (1968, 1969); Hughston (1979); Penrose and Rindler (1986); versions of these expressions can be traced back to Whittaker (1903) and Bateman (1904, 1944).
p. This type of non-singular field, termed an "elementary state", being of finite norm and positive frequency, plays an important role in twistor scattering theory (see Hodges 1985, 1998). These fields appear to have been first studied by C. Lanczos.
q. The more conventional term to use here, rather than "holomorphic cohomology" is "sheaf cohomology", with a "coherent analytic sheaf"; see Gunning and Rossi (1965), Wells (1991).

r. If the (smooth) boundary is defined by $L = 0$, where L is a smooth real-valued function of the holomorphic coordinates z_i and their complex conjugates $\bar{z}_i$, and whose gradient is non-vanishing at $L = 0$, then the Levy form is defined by the matrix of mixed partial derivatives $\partial^2 L/\partial z_i \partial \bar{z}_j$ restricted to the holomorphic tangent directions of the boundary $L = 0$. See Gunning and Rossi (1965); Wells (1991).

s. It may be remarked that a full Stein covering of $\mathbb{PT}^+$ must always involve an infinite number of open sets, because $\mathbb{PT}^+$ is not holomorphically pseudo-convex at its boundary $\mathbb{PN}$. In practice, however, one normally gets away with just a 2-set covering, encompassing a more extended region $\mathbb{PT}$ of than just $\mathbb{PT}^+$.

t. The descriptions of sheaf cohomology that I am providing here are being given only in the form of what is called Čech cohomology. This turns out to be by far the simplest for explicit representations. But there are other equivalent forms which are useful in various different contexts, most notably the Dolbeault (or $\bar{\partial}$) cohomology and that defined by "extensions" of exact sequences; see Wells (1991), Ward and Wells (1989).

u. For further details on these matters, see Eastwood, Penrose, and Wells (1981), Bailey, Ehrenpreis, and Wells (1982).

v. And checking the normal-bundle condition of the next note 23.

w. If we wish to exhibit $\mathbb{P}\mathcal{T}$ directly, rather than generating it in this way, we need to demand the existence of "lines" whose *normal bundle* is of the right holomorphic class. See Ward and Wells (1989).

x. Atiyah, Hitchin, and Singer (1978).

y. Dunajski (2002).

z. See, for example, Hitchin (1979, 1982); LeBrun (1990, 1998).

aa. See Penrose (1992).

bb. Penrose (1976).

cc. See, particularly, Mason and Woodhouse (1996) for an overview of these matters.

dd. Penrose (2001a); Frauendiener and Penrose (2001).

ee. Penrose (2001a).

ff. Eilenberg and Mac Lane (1945); Mac Lane (1988).

gg. See Penrose (2001b).

hh. Connes and Berberian (1995).

ii. See Hardy (1949).

jj. See Penrose (1968), although the needed Pochhammar-type contours were not understood at that time.

kk. Witten (2003); this was based partly on earlier results due to Parke and Taylor (1986) and by Berends and Giele (1988) on Gluon scatterings, and on twistor-related ideas of Nair (1988).

ll. See also Penrose (1988).

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