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Fractional time-scales Noether theorem with Caputo Δ derivatives for Hamiltonian systems

Xue Tian[a], Yi Zhang[b,*]

[a] School of Science, Nanjing University of Science and Technology, Nanjing 210094, China
[b] College of Civil Engineering, Suzhou University of Science and Technology, Suzhou 215011, China

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ABSTRACT

This paper examines a new Noether theorem for Hamiltonian systems with Caputo Δ derivatives based on fractional time-scales calculus, which overcomes the difficulties unified to study the Noether theorems of fractional continuous systems and fractional discrete systems. To begin with, the fractional time-scales definitions and properties with Caputo Δ derivatives are introduced. Next, the fractional time-scales Hamilton canonical equations with Caputo Δ derivatives are formulated. For the fractional time-scales Hamiltonian system, the definitions and criteria of Noether symmetries without transforming time and with transforming time are given, respectively. Furthermore, the corresponding Noether theorem without transforming time and its Noether theorem with transforming time are obtained. The latter one can reduce to the time-scales Noether theorem with Δ derivatives or the fractional Noether theorem with Caputo derivatives for Hamiltonian systems. Finally, the fractional time-scales damped oscillator and Kepler problem are taken as examples to verify the correctness of the results.

1. Introduction

In 1918, Noether [1], a German female mathematician, published an influential paper in German called “invariant variational problems”. Later, this article was translated into English [2] so that more people could read and study it. The paper mainly gave two theorems: the first theorem concerns symmetries and conserved quantities in classical mechanics, and the second theorem involves in general relativity. The first theorem of Noether (Noether theorem) has become the basis of symmetries and conserved quantities not only for the study of classical mechanics and classical field theory but also for the study of quantum mechanics and quantum field theory. Not only can Noether theorem help to find the classical conservation law of Newtonian mechanics, the generalized momentum conservation law and generalized energy conservation law of Lagrangian mechanics, but also find more conservation laws. The conservation laws of mechanical systems are of great significance to the study of dynamic behavior, stability and calculation of mechanical systems [3]. Therefore, Noether theorem plays a crucial role in analytical mechanics. With the hot research on Noether theorem in recent decades, the research on Noether theorem of classical mechanics [4,5], Lagrangian mechanics [6–8], Hamiltonian mechanics [6,7,9], nonholonomic mechanics [10,11] and Birkhoffian mechanics [6,7] has been becoming more and more perfect. Not just for these continuous

* Corresponding author.
E-mail addresses: crystaltianxue@njust.edu.cn (X. Tian), zhy@mail.usts.edu.cn (Y. Zhang).

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systems, but Noether theorems for discrete systems have obtained some results [12–14]. In recent years, some scholars have used the time-scales theory to unify the study of Noether theorem for continuous systems and discrete systems.

The concept of time-scales is a unified theory of continuous and discrete calculus proposed by Hilger in 1988 [15]. He found a deep connection between continuous calculus and difference equations. Instead of being two separate theories, they were really two strands of the same thing. Accordingly, Hilger devised the time-scales theory to summarize them. It is paramount to note that not only can the time-scales theory unify the study of continuous and discrete systems, but also it can study quantum systems simultaneously. There is no doubt that this advantageous mathematical tool has brought a lot of conveniences to build models and has been applied in economics, biology and other disciplines [16–18]. As for the time-scales Noether theorem, it dates back to 2008 when Bartosiewicz and Torres studied the time-scales Noether theorem with Δ derivatives for Lagrangian systems by using the technique of time-re-parameterization [19]. Then, in 2011, Bartosiewicz et al. established the time-scales second Euler-Lagrange equation with Δ derivatives [20]. Subsequently, time-scales Noether theorems were extended to Hamiltonian mechanics [21–24], nonconservative mechanics [25–27], nonholonomic mechanics [28–30], Birkhoffian mechanics [31–33] and so on. However, in 2016, Anerot et al. pointed out the error of Bartosiewicz [19,20] and reproved the time-scales Noether theorem by using the generalized Jost's method [34,35]. Consequently, the time-scales Noether theorems obtained by referring to the technique of time-re-parameterization in [19] and [20] are open to question.

Due to the rapid development of time-scales calculus and the need for more practical mathematical models, the emergence of fractional time-scales calculus was promoted. As is known to all, in mathematical concept, fractional calculus is a natural extension of integral calculus, and in practice, it is found that fractional damping does exist in nature and practical problems. Compared with integral calculus, fractional model can provide more practical models for engineering, materials and other fields. In 1999, Podlubny [36] gave the definitions of fractional derivatives for continuous systems. If the time-scale satisfies $\mathbb{T}=\mathbb{R}$, then the fractional time-scale calculus is consistent with the classical fractional calculus in the monograph [36]. In 2007, Atici and Eloe [37] proposed the definition of discrete fractional derivatives and finite fractional differences. If the time-scale is taken to be $\mathbb{T}=\mathbb{Z}$, the fractional time-scale calculus becomes the calculus of the discrete system with step size $\mu=1$ in [37]. If the time-scale is $\mathbb{T}=q^{\mathbb{N}_0}$, the fractional time-scale calculus becomes the discrete fractional q-calculus in [38]. Therefore, continuous, discrete and quantum fractional systems can be processed systematically with different values of time-scales $\mathbb{T}$. Based on the fractional calculus on time-scales, the similarities and differences between continuous and discrete fractional systems can be analyzed, which is beneficial to avoid the repeated proof of some problems. In addition, the step size can not only be a constant but also become a function with variables. Based on the different choices of time-scales, more and more general results can be obtained, which can describe the physical essences of continuous and discrete fractional systems as well as other complex fractional dynamic systems more clearly and accurately. Because of these advantages of fractional time-scales calculus, in recent years some results have been obtained, including fractional time-scales principles and inequalities [39], existence and uniqueness of solutions for fractional time-scales dynamic equations [40,41], the theory of fractional time-scales calculus [42–44], fractional time-scales operators with application to dynamic equations [45,46], fractional time-scales optimal control [47], fractional time-scales chaotic systems [48–50], fractional time-scales recurrent neural networks [51]. Although fractional Noether theorems for Lagrangian systems [52], Birkhoffian systems [53–55] and nonconservative systems [56–58] have been studied, there are no reports on fractional time-scales Noether theorem so far.

In this paper, the fractional time-scales Noether theorem with Caputo Δ derivatives for Hamiltonian systems is investigated. Compared with Lagrangian systems, Hamiltonian systems are symmetrical and symplectic in structure, which led to the development of geometric mechanics. In addition, in solving many complex mechanical problems, such as celestial mechanics and vibration theory, it is more convenient to have a general discussion in Hamiltonian systems. Moreover, under certain conditions, the fractional time-scales Noether theorem for Hamiltonian systems can reduce to the one for Lagrangian systems. This paper proceeds as follows. In Section 2, some basics and fundamental properties of fractional time-scales calculus are recalled. In Section 3, the fractional time-scales Hamilton canonical equations with Caputo Δ derivatives are formulated. The time-scales Noether theorems with Caputo Δ derivatives for Hamiltonian systems without transforming time and with transforming time are obtained in Section 4 and Section 5, respectively. In the next section, two examples are presented. Section 7 summarizes the results of this investigation.

2. Preliminaries

The definitions of time-scales are given in detail in the books [16,18,59], including forward jump operator σ, backward jump operator ρ, graininess function μ, Δ derivative $f^{\Delta}(t)$, $\mathbb{T}^k$ and the set of rd-continuous functions C^1_{rd}. In this section, some main definitions and properties of fractional time-scales derivatives are recalled.

Definition 1 [59]**.** The rd-continuous functions $h_k:\mathbb{T}\times\mathbb{T}\to\mathbb{R},\ k\in\mathbb{N}_+$ are defined by

$$h_k(t,s)=\int_s^t h_{k-1}(\tau,s)\Delta\tau,\ \ k\in\mathbb{N}_+,\ \ t,s\in\mathbb{T},$$

where $h_0(t,s)=1$. So, it is easy to get that

$$h_1(t,s)=\int_s^t \Delta\tau=t-s,$$

$$h_k^{\Delta_t}(t,s) = h_{k-1}(t,s), \ \ k \in \mathbb{N}_+, \ \ t,s \in \mathbb{T}.$$

Definition 2 [59]**.** Let $i \in \{1, 2, \cdots, n\}$ be fixed. Assuming that $f : \Lambda^n \to \mathbb{R}$ is regulated with respect to t_i and $\sup \mathbb{T}_i = \infty$, the Laplace transform of f with respect to t_i is denoted by

$$\mathscr{L}_i(f)(t_1, \cdots, t_{i-1}, z_i, t_{i+1}, \cdots, t_n) = \int_0^\infty e_{\ominus z_i}^{\sigma_i}(t_i, 0) f(t) \Delta_i t_i, \tag{1}$$

where $\ominus z_i = -\frac{z_i}{1+\mu_i(t_i)z_i}$, $1 + \mu_i(t_i) z_i \neq 0$ for all $t_i \in \mathbb{T}_i^\kappa$, $z_i \in \mathbb{C}$ and the improper integral (1) exists.

Lemma 1 [59]**.** *Let $z \in \mathbb{C} \backslash \{0\}$ and $1 + z\mu(x) \neq 0$ for $x \in \mathbb{T}_0$ and $n \in \mathbb{N}_0$, then*

$$\mathscr{L}(h_n(x,0))(z) = \frac{1}{z^{n+1}},$$

and

$$\lim_{x \to \infty} (h_n(x,0) e_{\ominus z}(x,0)) = 0.$$

Definition 3 [59]**.** The generalized Δ power function $h_\alpha(t, t_0)$ on $\mathbb{T}$ is defined by

$$h_\alpha(t, t_0) = \mathscr{L}^{-1}\left(\frac{1}{z^{\alpha+1}}\right)(t), \ \ t \geq t_0, \ \ \alpha \in \mathbb{R}$$

for all $z \in \mathbb{C} \backslash \{0\}$ such that $\mathscr{L}^{-1}$ exists, $t \geq t_0$. The shift of $h_\alpha(t, t_0)$ is denoted by

$$h_\alpha(t,s) = \widehat{h_\alpha(\cdot, t_0)}(t,s), \quad t,s \in \mathbb{T}, \quad t \geq s \geq t_0.$$

Definition 4 [39]**.** Let $\mathbb{T}$ be a time scale, $\alpha > 0$, $a, b \in \mathbb{R}$, $a < b$. For a function $f : [a,b] \in \mathbb{R}$, the left Riemann-Liouville fractional Δ integral is

$${}_aI_{\Delta,t}^\alpha f(t) = \int_a^t h_{\alpha-1}(t, \sigma(\tau)) f(\tau) \Delta\tau, \ \ t > a,$$

and the right Riemann-Liouville fractional Δ integral is

$${}_tI_{\Delta,b}^\alpha f(t) = \int_t^b h_{\alpha-1}(\sigma(\tau), t) f(\tau) \Delta\tau, \ \ t < b.$$

Remark 1. If $\alpha = 0$, then ${}_aI_{\Delta,t}^\alpha f(t) = f(t)$ and ${}_tI_{\Delta,b}^\alpha f(t) = f(t)$.

Remark 2 [36]**.** If $\mathbb{T} = \mathbb{R}$, i.e., $h_{\alpha-1}(t,a) = \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}$ and $h_{\alpha-1}(b,t) = \frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)}$, then

$${}_aI_t^\alpha f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) d\tau, \ \ t > a,$$

and

$${}_tI_b^\alpha f(t) = \int_t^b \frac{(\tau-t)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) d\tau, \ \ t < b.$$

Remark 3 [37]**.** If $\mathbb{T} = \mathbb{Z}$, i.e.,

$$h_{\alpha-1}(t,s) = \frac{(t-s)^{(\alpha-1)}}{\Gamma(\alpha)}, \ \ t \geq s \geq a+1,$$

where define

$$t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}, \ \ t \in \mathbb{T}, \ \ \alpha \in \mathbb{R},$$

then

$$\begin{aligned} {}_aI_{\Delta,t}^\alpha f(t) &= \int_a^t h_{\alpha-1}(t, \sigma(s)) f(s) \Delta s = \int_a^t \frac{(t-\sigma(s))^{(\alpha-1)}}{\Gamma(\alpha)} f(s) \Delta s \\ &= \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-1} (t-s-1)^{(\alpha-1)} f(s), \ \ t \geq a+1, \end{aligned}$$

and

$${}_tI_{\Delta,b}^\alpha f(t) = \int_t^b h_{\alpha-1}(\sigma(s), t) f(s) \Delta s = \int_t^b \frac{(\sigma(s)-t)^{(\alpha-1)}}{\Gamma(\alpha)} f(s) \Delta s$$

$$= \frac{1}{\Gamma(\alpha)}\sum_{s=t}^{b-1}(s+1-t)^{(\alpha-1)}f(s),\ \ t\le b-1.$$

Remark 4 [59]**.** If $\mathbb{T}=u^{\mathbb{N}_0},\ u>1,$ let $\alpha\in[0,\infty)\backslash\mathbb{N}_0$. The u-factorial function is defined by

$$(t-a)_u^{\alpha}=t^{\alpha}\prod_{n=0}^{\infty}\frac{1-\frac{a}{t}u^n}{1-\frac{a}{t}u^{\alpha+n}}=t^{\alpha}\prod_{n=0}^{\infty}\frac{t-au^n}{t-au^{\alpha+n}},\ \ a,t\in\mathbb{T},\ \ t\ge a.$$

And the u-gamma function $\Gamma_u:\mathbb{R}\backslash\mathbb{Z}\to\mathbb{R}$ is

$$\Gamma_u\left(\frac{1}{2}\right)=1,\ \Gamma_u(\alpha)\frac{u^{\alpha}-1}{u-1}=\Gamma_u(\alpha-1),\ \alpha\in\mathbb{R}\backslash\mathbb{Z}.$$

Then

$$h_{\alpha}(t,a)=\Gamma_u(\alpha)(t-a)_u^{\alpha},$$

and

$$ {}_aI^{\alpha}_{\Delta,t}f(t)=\int_a^t h_{\alpha-1}(t,\sigma(s))f(s)\Delta s=\Gamma_u(\alpha-1)\int_a^t(t-us)_u^{\alpha-1}f(s)\Delta s,$$
$$t,s\in\mathbb{T},\ \ t\ge s\ge a.$$

Similarly,

$$h_{\alpha}(b,t)=\Gamma_u(\alpha)(b-t)_u^{\alpha},$$

and

$$ {}_tI^{\alpha}_{\Delta,b}f(t)=\int_t^b h_{\alpha-1}(\sigma(s),t)f(s)\Delta s=\Gamma_u(\alpha-1)\int_t^b(us-t)_u^{\alpha-1}f(s)\Delta s,$$
$$t,s\in\mathbb{T},\ \ t\le s\le b.$$

Definition 5. For a function $f:\mathbb{T}\to\mathbb{R},\ f^{\Delta^n}$ is denoted as $D^n_{\Delta}f,\ n\in\mathbb{N}_0$. Let $\mathbb{T}$ be a time scale, $\alpha\ge 0,\ m=\overline{[\alpha]}+1$. For $a,b\in\mathbb{T}^{\kappa^m},\ a<b,$ the left Riemann-Liouville fractional Δ derivative is

$$ {}_aD^{\alpha}_{\Delta,t}f(t)=D^m_{\Delta}\left({}_aI^{m-\alpha}_{\Delta,t}f(t)\right)=D^m_{\Delta}\left(\int_a^t h_{m-\alpha-1}(t,\sigma(\tau))f(\tau)\Delta\tau\right),$$
$$t\in\mathbb{T},\ \ t>a.$$

And the right Riemann-Liouville fractional Δ derivative is

$$ {}_tD^{\alpha}_{\Delta,b}f(t)=-D^m_{\Delta}\left({}_tI^{m-\alpha}_{\Delta,b}f(t)\right)=-D^m_{\Delta}\left(\int_t^b h_{m-\alpha-1}(\sigma(\tau),t)f(\tau)\Delta\tau\right),$$
$$t\in\mathbb{T},\ \ t<b.$$

Definition 6. Let $\mathbb{T}$ be a time scale, $t\in\mathbb{T},\ \alpha\ge 0,$ the left Caputo fractional Δ derivative is

$$ {}^C_aD^{\alpha}_{\Delta,t}f(t)={}_aI^{1-\alpha}_{\Delta,t}f^{\Delta}(t)=\int_a^t h_{-\alpha}(t,\sigma(\tau))f^{\Delta}(\tau)\Delta\tau$$
$$={}_aD^{\alpha}_{\Delta,t}\left(f(t)-\sum_{k=0}^{m-1}h_k(t,a)f^{\Delta^k}(a)\right),$$

and the right Caputo fractional Δ derivative is

$$ {}^C_tD^{\alpha}_{\Delta,b}f(t)=-{}_tI^{1-\alpha}_{\Delta,b}f^{\Delta}(t)=-\int_t^b h_{-\alpha}(\sigma(\tau),t)f^{\Delta}(\tau)\Delta\tau$$
$$={}_tD^{\alpha}_{\Delta,b}\left(f(t)-\sum_{k=0}^{m-1}h_k(b,t)f^{\Delta^k}(b)\right),$$

where $m=\overline{[\alpha]}+1$ if $\alpha\notin\mathbb{N},\ m=\overline{[\alpha]}$ if $\alpha\in\mathbb{N}$.

Lemma 2 [16]**.** *Assume that $f,g:\mathbb{T}\to\mathbb{R}$ are Δ differentiable at $t\in\mathbb{T}^k$. For $a,b\in\mathbb{T}$, there is the following property of Δ derivatives:*

$$(fg)^{\Delta}(t)=f^{\Delta}(t)g(t)+f^{\sigma}(t)g^{\Delta}(t)=f(t)g^{\Delta}(t)+f^{\Delta}(t)g^{\sigma}(t). \tag{2}$$

Lemma 3 [60]**.** *Let $(X,\mathscr{U},u_\Delta)$ and $(Y,\mathscr{V},\nu_\Delta)$ be two finite-dimensional time-scales measure spaces. If $f: X\times Y\to\mathbb{R}$ is a Δ integrable function and define the functions*

$$\varphi(y)=\int_X f(x,y)du_\Delta(x)$$

for all each $y\in Y$ and

$$\psi(y)=\int_Y f(x,y)d\nu_\Delta(y)$$

for all each $x\in X$, then φ is Δ integrable on Y and ψ is Δ integrable on X and

$$\int_X du_\Delta(x)\int_Y f(x,y)d\nu_\Delta(y)=\int_Y d\nu_\Delta(y)\int_X f(x,y)du_\Delta(x).$$

Lemma 4 [16]**.** *Let $f\in C_{rd},\ f:[a,b]\to\mathbb{R}^n$, then*

$$\int_a^b f^T(t)g^\Delta(t)\Delta t=0$$

for all $g\in C_{rd}^1$ with $g(a)=g(b)=0$ holds if and only if

$$f(t)=c,$$

where $c\in\mathbb{R}^n$.

Lemma 5 [16]**.** *Let $\nu:\mathbb{T}\to\mathbb{R}$ be a strictly increasing function, then $\bar{\mathbb{T}}=\nu(\mathbb{T})$ is a new time scale. The jump function on $\bar{\mathbb{T}}$ is denoted by $\bar\sigma$ and the derivative on $\bar{\mathbb{T}}$ is denoted by $\bar\Delta$. Then $\nu\circ\sigma=\bar\sigma\circ\nu$. Let $\omega:\bar{\mathbb{T}}\to\mathbb{R}$. If $\nu^\Delta(t)$ and $\omega^{\bar\Delta}(\nu(t))$ exist for $t\in\mathbb{T}^\kappa$, then*

$$(\omega\circ\nu)^\Delta=\left(\omega^{\bar\Delta}\circ\nu\right)\nu^\Delta. \tag{3}$$

If $\omega=\nu^{-1}:\bar{\mathbb{T}}\to\mathbb{T}$, then

$$\frac{1}{\nu^\Delta}=\left(\nu^{-1}\right)^{\bar\Delta}\circ\nu \tag{4}$$

at points where $\nu^\Delta\neq0$. If $f:\mathbb{T}\to\mathbb{R}$ is rd-continuous and ν is differentiable with rd-continuous derivative, then

$$\int_a^b f(t)\nu^\Delta(t)\Delta t=\int_{\nu(a)}^{\nu(b)}\left(f\circ\nu^{-1}\right)(s)\bar\Delta s \tag{5}$$

for $a,b\in\mathbb{T}$.

Lemma 6 [47]**.** *The formulas of fractional time-scales integration by parts are as follows*

$$\int_a^b g(t)\,{}_a^C D_{\Delta,t}^\alpha f(t)\Delta t=\int_a^b f^\sigma(t)\,{}_t D_{\Delta,b}^\alpha g(t)\Delta t+\left[f(t)\,{}_t I_{\Delta,b}^{1-\alpha}g(t)\right]\Big|_a^b, \tag{6}$$

$$\int_a^b g(t)\,{}_t^C D_{\Delta,b}^\alpha f(t)\Delta t=\int_a^b f^\sigma(t)\,{}_a D_{\Delta,t}^\alpha g(t)\Delta t-\left[f(t)\,{}_a I_{\Delta,t}^{1-\alpha}g(t)\right]\Big|_a^b. \tag{7}$$

3. Fractional time-scales Hamiltonian canonical equations with Caputo Δ derivatives

Assume the configuration of a time-scales mechanical system is determined by n generalized coordinates $q_k(k=1,2,\ldots,n)$, and the time-scales Lagrangian of the system with Caputo derivatives is $L\left(t,q_k^\sigma(t),{}_a^C D_{\Delta,t}^\alpha q_k(t)\right)$, $\alpha\in[0,1)$. Then, the generalized momenta are

$$p_k=\frac{\partial L}{\partial\,{}_a^C D_{\Delta,t}^\alpha q_k}, \tag{8}$$

and its Hamiltonian is

$$H=H\left(t,q_k^\sigma(t),p_k(t)\right)=p_k\cdot{}_a^C D_{\Delta,t}^\alpha q_k-L\left(t,q_k^\sigma(t),{}_a^C D_{\Delta,t}^\alpha q_k(t)\right). \tag{9}$$

So the Hamiltonian action with Caputo Δ derivatives is

$$S[q(\cdot),p(\cdot)]=\int_a^b\left[p_k(t)\,{}_a^C D_{\Delta,t}^\alpha q_k(t)-H\left(t,q_k^\sigma(t),p_k(t)\right)\right]\Delta t. \tag{10}$$

The fractional time-scales Hamilton principle is

$$\delta S=0, \tag{11}$$

which satisfies the relation

$$\delta{}_a^C D_{\Delta,t}^{\alpha} q_k = {}_a^C D_{\Delta,t}^{\alpha} \delta q_k,\quad (\delta q_k)^{\sigma} = \delta q_k^{\sigma} \tag{12}$$

and the boundary conditions

$$\delta q_k(t)|_{t=a} = 0,\quad \delta q_k(t)|_{t=b} = 0. \tag{13}$$

Taking the calculation of the variation to Eq. (10), it follows that

$$\delta S = \int_a^b \left[\delta p_k \cdot {}_a^C D_{\Delta,t}^{\alpha} q_k + p_k \cdot \delta{}_a^C D_{\Delta,t}^{\alpha} q_k - \frac{\partial H}{\partial q_k^{\sigma}} \delta q_k^{\sigma} - \frac{\partial H}{\partial p_k} \delta p_k \right] \Delta t. \tag{14}$$

From formulas (6) and (12), then

$$\begin{aligned}\int_a^b p_k \cdot \delta{}_a^C D_{\Delta,t}^{\alpha} q_k \Delta t &= \int_a^b p_k \cdot {}_a^C D_{\Delta,t}^{\alpha} \delta q_k \Delta t \\ &= \int_a^b \delta q_k^{\sigma} \cdot {}_t D_{\Delta,b}^{\alpha} p_k \Delta t + \left[\delta q_k \cdot {}_t I_{\Delta,b}^{1-\alpha} p_k \right] \Big|_a^b \\ &= \int_a^b \delta q_k^{\sigma} \cdot {}_t D_{\Delta,b}^{\alpha} p_k \Delta t. \end{aligned} \tag{15}$$

According to the above formula and Eq. (2), it concludes that

$$\begin{aligned}&\int_a^b \left[p_k \cdot \delta{}_a^C D_{\Delta,t}^{\alpha} q_k - \frac{\partial H}{\partial q_k^{\sigma}} \delta q_k^{\sigma} \right] \Delta t = \int_a^b \left[\left({}_t D_{\Delta,b}^{\alpha} p_k - \frac{\partial H}{\partial q_k^{\sigma}} \right) \delta q_k^{\sigma} \right] \Delta t \\ &= \int_a^b \left\{ \left[\left(\int_a^t \left({}_\theta D_{\Delta,b}^{\alpha} p_k - \frac{\partial H}{\partial q_k^{\sigma}} \right) \Delta\theta \right) \delta q_k \right]^{\Delta} - \left(\int_a^t \left({}_\theta D_{\Delta,b}^{\alpha} p_k - \frac{\partial H}{\partial q_k^{\sigma}} \right) \Delta\theta \right) (\delta q_k)^{\Delta} \right\} \Delta t \\ &= \left[\left(\int_a^t \left({}_\theta D_{\Delta,b}^{\alpha} p_k - \frac{\partial H}{\partial q_k^{\sigma}} \right) \Delta\theta \right) \delta q_k \right] \Bigg|_a^b - \int_a^b \left(\int_a^t \left({}_\theta D_{\Delta,b}^{\alpha} p_k - \frac{\partial H}{\partial q_k^{\sigma}} \right) \Delta\theta \right) (\delta q_k)^{\Delta} \Delta t \\ &= \int_a^b \left[\int_a^t \left(\frac{\partial H}{\partial q_k^{\sigma}} - {}_\theta D_{\Delta,b}^{\alpha} p_k \right) \Delta\theta \right] (\delta q_k)^{\Delta} \Delta t. \end{aligned} \tag{16}$$

Substituting (14) into (11) and considering formula (16), then

$$\delta S = \int_a^b \left\{ \left[\int_a^t \left(\frac{\partial H}{\partial q_k^{\sigma}} - {}_\theta D_{\Delta,b}^{\alpha} p_k \right) \Delta\theta \right] (\delta q_k)^{\Delta} + \left({}_a^C D_{\Delta,t}^{\alpha} q_k - \frac{\partial H}{\partial p_k} \right) \delta p_k \right\} \Delta t = 0. \tag{17}$$

Differentiating both sides of formula (9) with respect to p_k, it follows that

$$\frac{\partial H}{\partial p_k} = {}_a^C D_{\Delta,t}^{\alpha} q_k. \tag{18}$$

Substituting (18) into (17), then

$$\int_a^b \left[\int_a^t \left(\frac{\partial H}{\partial q_k^{\sigma}} - {}_\theta D_{\Delta,b}^{\alpha} p_k \right) \Delta\theta \right] (\delta q_k)^{\Delta} \Delta t = 0. \tag{19}$$

By Lemma 4, formula (19) can be expressed as

$$\int_a^t \left(\frac{\partial H}{\partial q_k^{\sigma}} - {}_\theta D_{\Delta,b}^{\alpha} p_k \right) \Delta\theta = const. \tag{20}$$

Differentiating both sides of formula (20) with respect to t, it follows that

$$\frac{\partial H}{\partial q_k^{\sigma}} = {}_t D_{\Delta,b}^{\alpha} p_k. \tag{21}$$

From Eqs. (18) and (21), the fractional time-scales Hamilton canonical equations are obtained

$${}_a^C D_{\Delta,t}^{\alpha} q_k = \frac{\partial H}{\partial p_k},\quad {}_t D_{\Delta,b}^{\alpha} p_k = \frac{\partial H}{\partial q_k^{\sigma}}. \tag{22}$$

Remark 5. If $\alpha = 1$, Eqs. (22) reduce to the time-scales Hamilton canonical equations with Δ derivatives [22]

$$q_k^{\Delta} = \frac{\partial H}{\partial p_k},\quad p_k^{\Delta} = -\frac{\partial H}{\partial q_k^{\sigma}}. \tag{23}$$

If $\mathbb{T}=\mathbb{R}$, Eqs. (22) reduce to the fractional Hamilton canonical equations with Caputo derivatives

$$ {}_a^C D_t^\alpha q_k=\frac{\partial H}{\partial p_k},\ \ {}_t D_b^\alpha p_k=\frac{\partial H}{\partial q_k}. \tag{24}$$

If $\alpha=1$ and $\mathbb{T}=\mathbb{R}$, Eqs. (22) reduce to the classical Hamilton canonical equations

$$\dot q_k=\frac{\partial H}{\partial p_k},\ \ \dot p_k=-\frac{\partial H}{\partial q_k}. \tag{25}$$

4. Fractional time-scales Noether theorem for Hamiltonian systems without transforming time

Introduce the infinitesimal transformations of a one-parameter group without transforming time

$$\begin{aligned}\bar q_k(t)&=q_k(t)+\varepsilon\xi_k(t,q_i,p_i)+o(\varepsilon),\\ \bar p_k(t)&=p_k(t)+\varepsilon\eta_k(t,q_i,p_i)+o(\varepsilon),\end{aligned} \tag{26}$$

where $\xi_k,\ \eta_k(k=1,2,\dots,n)$ are the infinitesimal generators and ε is an infinitesimal parameter.

Definition 7. The Hamiltonian action (10) is invariant under the infinitesimal transformations (26) if

$$\begin{aligned}&\int_{t_a}^{t_b}\left[p_k(t){}_a^C D_{\Delta,t}^\alpha q_k(t)-H\big(t,q_k^\sigma(t),p_k(t)\big)\right]\Delta t\\ &=\int_{t_a}^{t_b}\left[\bar p_k(t){}_a^C D_{\Delta,t}^\alpha \bar q_k(t)-H\big(t,\bar q_k^\sigma(t),\bar p_k(t)\big)\right]\Delta t\end{aligned} \tag{27}$$

for any subinterval $[t_a,t_b]\subseteq[a,b]$ with $t_a,t_b\in\mathbb{T}$.

Criterion 1. If the Hamiltonian action (10) is invariant under the infinitesimal transformations (26), then

$$p_k\cdot{}_a^C D_{\Delta,t}^\alpha\xi_k-\frac{\partial H}{\partial q_k^\sigma}\xi_k^\sigma=0. \tag{28}$$

Proof. By Definition 7, since the condition (27) holds for any subinterval $[t_a,t_b]\subseteq[a,b]$, it follows that

$$\begin{aligned}&p_k(t){}_a^C D_{\Delta,t}^\alpha q_k(t)-H\big(t,q_k^\sigma(t),p_k(t)\big)\\ &=(p_k+\varepsilon\eta_k){}_a^C D_{\Delta,t}^\alpha(q_k+\varepsilon\xi_k)-H\big(t,q_k^\sigma+\varepsilon\xi_k^\sigma,p_k+\varepsilon\eta_k\big).\end{aligned} \tag{29}$$

Differentiating (29) with respect to ε, then putting $\varepsilon=0$, and using fractional time-scales definitions and properties with Caputo Δ derivatives, then

$$0=\eta_k\cdot{}_a^C D_{\Delta,t}^\alpha q_k+p_k\cdot{}_a^C D_{\Delta,t}^\alpha\xi_k-\frac{\partial H}{\partial q_k^\sigma}\xi_k^\sigma-\frac{\partial H}{\partial p_k}\eta_k. \tag{30}$$

From the first equation of (22), Criterion 1 is proved. □

Theorem 1. *If the Hamiltonian action (10) is invariant in the sense of Definition 7, then*

$$p_k\cdot{}_a^C D_{\Delta,t}^\alpha\xi_k-\xi_k^\sigma\cdot{}_t D_{\Delta,b}^\alpha p_k=0. \tag{31}$$

The proof follows from the second equation of (22) and formula (28).
The symbol $\mathscr{D}_\Delta^\alpha$ is defined by

$$\mathscr{D}_\Delta^\alpha[f,g]=g\cdot{}_a^C D_{\Delta,t}^\alpha f-f^\sigma\cdot{}_t D_{\Delta,b}^\alpha g. \tag{32}$$

Formula (31) can be written as

$$\mathscr{D}_\Delta^\alpha[\xi_k,p_k]=0. \tag{33}$$

5. Fractional time-scales Noether theorem for Hamiltonian systems with transforming time

Next, introduce the infinitesimal transformations of the $(\Delta,\mathbb{T})$-admissible projectable group [35] with an infinitesimal parameter ε in the case of transforming time

$$\begin{aligned}\bar t&=T_\varepsilon(t)=t+\varepsilon\zeta(t,q_i,p_i)+o(\varepsilon),\\ \bar q_k\big(\bar t\big)&=Q_\varepsilon^k(t,q_i,p_i)=q_k(t)+\varepsilon\xi_k(t,q_i,p_i)+o(\varepsilon),\\ \bar p_k\big(\bar t\big)&=P_\varepsilon^k(t,q_i,p_i)=p_k(t)+\varepsilon\eta_k(t,q_i,p_i)+o(\varepsilon),\end{aligned} \tag{34}$$

where $\zeta,\ \xi_k,\ \eta_k(k=1,2,\dots,n)$ are the infinitesimal generators.

Definition 8. The Hamiltonian action (10) is invariant under a one-parameter $(\Delta, \mathbb{T})$-admissible projectable group of transformations (34) if and only if

$$\begin{aligned}&\int_{t_a}^{t_b}\left[p_k(t){}_a^C D_{\Delta,t}^{\alpha} q_k(t) - H\big(t, q_k^{\sigma}(t), p_k(t)\big)\right]\Delta t\\ &= \int_{\nu(t_a)}^{\nu(t_b)}\left[\bar{p}_k\big(\bar{t}\big){}_{\nu(a)}^{C} D_{\bar{\Delta},\bar{t}}^{\alpha}\bar{q}_k\big(\bar{t}\big) - H\Big(\bar{t}, \bar{q}_k^{\bar{\sigma}}\big(\bar{t}\big), \bar{p}_k\big(\bar{t}\big)\Big)\right]\bar{\Delta}\bar{t} \end{aligned} \tag{35}$$

for any subinterval $[t_a, t_b] \subseteq [a, b]$ with $t_a, t_b \in \mathbb{T}$.

Criterion 2. If the Hamiltonian action (10) is invariant under the infinitesimal transformations (34), then

$$\begin{aligned}&\eta_k \cdot {}_a^C D_{\Delta,t}^{\alpha} q_k + p_k \cdot {}_a^C D_{\Delta,t}^{\alpha}\xi_k - (\alpha - 1)p_k \cdot \zeta^{\Delta} \cdot {}_a^C D_{\Delta,t}^{\alpha} q_k - H\zeta^{\Delta}\\ &-\left(\frac{\partial H}{\partial t}\zeta + \frac{\partial H}{\partial q_k^{\sigma}}\xi_k^{\sigma} + \frac{\partial H}{\partial p_k}\eta_k\right) = 0. \end{aligned} \tag{36}$$

Proof. Note that Definition 6 and Lemma 5 imply

$$\begin{aligned}&{}_{\nu(a)}^{C} D_{\bar{\Delta},\bar{t}}^{\alpha}\bar{q}_k\big(\bar{t}\big)\\ &= \int_{\nu(a)}^{\nu(t)} h_{-\alpha}\big(\bar{t}, \bar{\sigma}\big(\bar{\theta}\big)\big)\bar{q}_k^{\bar{\Delta}}\big(\bar{\theta}\big)\bar{\Delta}\big(\bar{\theta}\big)\\ &= \int_a^t h_{-\alpha}(\nu(t), (\bar{\sigma}\circ\nu)(\theta))\bar{q}_k^{\bar{\Delta}}(\nu(\theta))\nu^{\Delta}(\theta)\Delta(\theta)\\ &= \int_a^t h_{-\alpha}\big(t\cdot\nu^{\Delta}(t), \sigma(\theta)\nu^{\Delta}(t)\big)\cdot(\bar{q}_k\circ\nu(\theta))^{\Delta}\Delta(\theta)\\ &= \frac{{}_a^C D_{\Delta,t}^{\alpha}(\bar{q}_k\circ\nu)(t)}{A_{\alpha}\big(\nu^{\Delta}(t)\big)}, \end{aligned} \tag{37}$$

where A_α is defined by

$$A_{\alpha}(x(t)) = x^{\alpha} \tag{38}$$

and it satisfies $A_{\alpha}^{\Delta} = \alpha A_{\alpha-1}$, $A_\alpha \cdot A_\beta = A_{\alpha+\beta}$. By Definition 8, it leads to

$$\begin{aligned}&\int_{t_a}^{t_b}\left[p_k(t){}_a^C D_{\Delta,t}^{\alpha} q_k(t) - H\big(t, q_k^{\sigma}(t), p_k(t)\big)\right]\Delta t\\ &= \int_{\nu(t_a)}^{\nu(t_b)}\left[\bar{p}_k\big(\bar{t}\big){}_{\nu(a)}^{C} D_{\bar{\Delta},\bar{t}}^{\alpha}\bar{q}_k\big(\bar{t}\big) - H\Big(\bar{t}, \bar{q}_k^{\bar{\sigma}}\big(\bar{t}\big), \bar{p}_k\big(\bar{t}\big)\Big)\right]\bar{\Delta}\bar{t}\\ &= \int_{t_a}^{t_b}\left[(\bar{p}_k\circ\nu)(t)\frac{{}_a^C D_{\Delta,t}^{\alpha}(\bar{q}_k\circ\nu)(t)}{A_{\alpha}\big(\nu^{\Delta}(t)\big)} - H(\nu(t), (\bar{q}_k\circ\bar{\sigma}\circ\nu)(t), (\bar{p}_k\circ\nu)(t))\right]\nu^{\Delta}(t)\Delta t\\ &= \int_{t_a}^{t_b}\left[P_{\varepsilon}^k\frac{{}_a^C D_{\Delta,t}^{\alpha} Q_{\varepsilon}^k}{A_{\alpha}\big(T_{\varepsilon}^{\Delta}\big)} - H\Big(T_{\varepsilon}, \big(Q_{\varepsilon}^k\big)^{\sigma}, P_{\varepsilon}^k\Big)\right]T_{\varepsilon}^{\Delta}(t)\Delta t. \end{aligned} \tag{39}$$

Since $[t_a, t_b]$ is any subinterval of $[a, b]$, Eq. (39) is equivalent to

$$p_k(t){}_a^C D_{\Delta,t}^{\alpha} q_k(t) - H\big(t, q_k^{\sigma}(t), p_k(t)\big) = \left[P_{\varepsilon}^k\frac{{}_a^C D_{\Delta,t}^{\alpha} Q_{\varepsilon}^k}{A_{\alpha}\big(T_{\varepsilon}^{\Delta}\big)} - H\Big(T_{\varepsilon}, \big(Q_{\varepsilon}^k\big)^{\sigma}, P_{\varepsilon}^k\Big)\right]T_{\varepsilon}^{\Delta}. \tag{40}$$

Differentiating (40) with respect to ε and put $\varepsilon = 0$, it follows that

$$\begin{aligned}0 &= \eta_k \cdot {}_a^C D_{\Delta,t}^{\alpha} q_k + p_k \cdot {}_a^C D_{\Delta,t}^{\alpha}\xi_k - (\alpha - 1)p_k \cdot \zeta^{\Delta} \cdot {}_a^C D_{\Delta,t}^{\alpha} q_k\\ &\quad - H\zeta^{\Delta} - \left(\frac{\partial H}{\partial t}\zeta + \frac{\partial H}{\partial q_k^{\sigma}}\xi_k^{\sigma} + \frac{\partial H}{\partial p_k}\eta_k\right). \end{aligned} \tag{41}$$

□

Remark 6. Taking into account that

$$T_{\varepsilon} = T_{\varepsilon}^{\sigma} - \mu(t)T_{\varepsilon}^{\Delta}, \tag{42}$$

the invariance condition (35) can be written as

$$\begin{aligned}&\int_{t_a}^{t_b}\left[p_k(t){}_a^C D_{\Delta,t}^{\alpha} q_k(t)-H\left(t,q_k^{\sigma}(t),p_k(t)\right)\right]\Delta t\\&=\int_{t_a}^{t_b}\left[P_{\varepsilon}^k\frac{{}_a^C D_{\Delta,t}^{\alpha} Q_{\varepsilon}^k}{A_{\alpha}\left(T_{\varepsilon}^{\Delta}\right)}-H\left(T_{\varepsilon},\left(Q_{\varepsilon}^k\right)^{\sigma},P_{\varepsilon}^k\right)\right]T_{\varepsilon}^{\Delta}(t)\Delta t.\end{aligned}\tag{43}$$

Next, introduce the extended Lagrangian denoted by $\tilde{\mathbb{L}}:\mathbb{R}\times[a,b]\times\mathbb{R}^n\times\mathbb{R}^*\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$

$$\tilde{\mathbb{L}}(\tau;t,q,w,v,p)=\left[p\frac{v}{A_{\alpha}(w)}-H(t-\mu(\tau)w,q,p)\right]w.\tag{44}$$

The action is defined by $S_{\tilde{\mathbb{L}}}$ and denoted by

$$S_{\tilde{\mathbb{L}}}(t,q,p)=\int_{t_a}^{t_b}\tilde{\mathbb{L}}\left(\tau;t^{\sigma}(\tau),\left(q_k^{\sigma}\circ t\right)(\tau),t^{\Delta}(\tau),{}_a^C D_{\Delta,t}^{\alpha} q_k(\tau),p_k(\tau)\right)\Delta\tau.\tag{45}$$

The time-scales bundle path class $\mathscr{F}$ is denoted by

$$\begin{aligned}\mathscr{F}=\Big\{&(t,q,p)\in C_{rd}^{1,\Delta}(\mathbb{T})\times C_{rd}^{1,\Delta}(\mathbb{T})\times C_{rd}^{1,\Delta}(\mathbb{T});\\&\tau\mapsto(t(\tau),(q\circ t)(t),(p\circ t)(t))=(\tau,q(\tau),p(\tau))\}.\end{aligned}\tag{46}$$

When $t(\tau)=\tau$, i.e., $t^{\Delta}=1$, then

$$\begin{aligned}&\tilde{\mathbb{L}}\left(\tau;t^{\sigma}(\tau),q_k^{\sigma}(\tau),\tau^{\Delta},{}_a^C D_{\Delta,t}^{\alpha} q_k(\tau),p(\tau)\right)\\&=p(\tau){}_a^C D_{\Delta,\tau}^{\alpha} q_k(\tau)-H\left(\tau,q_k^{\sigma}(\tau),p_k(\tau)\right).\end{aligned}\tag{47}$$

The invariance condition (43) over $\mathscr{F}$ can be written as

$$S_{\tilde{\mathbb{L}}}(t,q,p)=\int_{t_a}^{t_b}\tilde{\mathbb{L}}\left(\tau;T_{\varepsilon}^{\sigma},\left(Q_{\varepsilon}^k\right)^{\sigma},T_{\varepsilon}^{\bar{\Delta}},{}_a^C D_{\bar{\Delta},\tau}^{\alpha} Q_{\varepsilon}^k,P_{\varepsilon}^k\right)\bar{\Delta}\tau.\tag{48}$$

Differentiating (48) with respect to ε and putting $\varepsilon=0$, it follows that

$$\partial_t\tilde{\mathbb{L}}(*)\zeta^{\sigma}+\partial_q\tilde{\mathbb{L}}(*)\xi_k^{\sigma}+\partial_w\tilde{\mathbb{L}}(*)\zeta^{\Delta}+\partial_v\tilde{\mathbb{L}}(*){}_a^C D_{\bar{\Delta},\tau}^{\alpha}\xi_k+\partial_p\tilde{\mathbb{L}}(*)\eta_k=0,\tag{49}$$

where $(*)\triangleq\left(\tau;\tau^{\sigma},q_k^{\sigma}(\tau),\tau^{\bar{\Delta}},{}_a^C D_{\bar{\Delta},\tau}^{\alpha} q_k(\tau),p(\tau)\right)$. The relation (44) leads to

$$\begin{cases}\partial_t\tilde{\mathbb{L}}(\tau;t^{\sigma},q,w,v,p)=-\partial_t H(t^{\sigma}-\mu(\tau)w,q,p)w\\\partial_q\tilde{\mathbb{L}}(\tau;t^{\sigma},q,w,v,p)=-\partial_q H(t^{\sigma}-\mu(\tau)w,q,p)w\\\partial_w\tilde{\mathbb{L}}(\tau;t^{\sigma},q,w,v,p)=\partial_t H(t^{\sigma}-\mu(\tau)w,q,p)\mu(\tau)w-\frac{(\alpha-1)vp}{A_{\alpha}(w)}\\\qquad\qquad-H(t^{\sigma}-\mu(\tau)w,q,p)\\\partial_v\tilde{\mathbb{L}}(\tau;t^{\sigma},q,w,v,p)=\frac{pw}{A_{\alpha}(w)}\\\partial_p\tilde{\mathbb{L}}(\tau;t^{\sigma},q,w,v,p)=\left[\frac{v}{A_{\alpha}(w)}-\partial_p H(t^{\sigma}-\mu(\tau)w,q,p)\right]w.\end{cases}\tag{50}$$

Reducing Eq. (50) over $\mathscr{F}$, they can be written as

$$\begin{cases}\frac{\partial}{\partial\tau}\tilde{\mathbb{L}}\left(\tau;\tau^{\sigma},q_k^{\sigma}(\tau),1,{}_a^C D_{\Delta,\tau}^{\alpha} q_k(\tau),p_k(\tau)\right)=-\frac{\partial}{\partial\tau}H\left(\tau,q_k^{\sigma}(\tau),p_k(\tau)\right)\\\frac{\partial}{\partial q_k^{\sigma}}\tilde{\mathbb{L}}\left(\tau;\tau^{\sigma},q_k^{\sigma}(\tau),1,{}_a^C D_{\Delta,\tau}^{\alpha} q_k(\tau),p_k(\tau)\right)=-\frac{\partial}{\partial q_k^{\sigma}}H\left(\tau,q_k^{\sigma}(\tau),p_k(\tau)\right)\\\frac{\partial}{\partial\tau^{\Delta}}\tilde{\mathbb{L}}\left(\tau;\tau^{\sigma},q_k^{\sigma}(\tau),1,{}_a^C D_{\Delta,\tau}^{\alpha} q_k(\tau),p_k(\tau)\right)=\frac{\partial}{\partial\tau}H\left(\tau,q_k^{\sigma}(\tau),p_k(\tau)\right)\mu(\tau)\\\quad-(\alpha-1)p_k(\tau){}_a^C D_{\Delta,\tau}^{\alpha} q_k(\tau)-H\left(\tau,q_k^{\sigma}(\tau),p_k(\tau)\right)\\\frac{\partial}{\partial{}_a^C D_{\Delta,t}^{\alpha} q_k}\tilde{\mathbb{L}}\left(\tau;\tau^{\sigma},q_k^{\sigma}(\tau),1,{}_a^C D_{\Delta,\tau}^{\alpha} q_k(\tau),p_k(\tau)\right)=p_k(\tau)\\\frac{\partial}{\partial p_k}\tilde{\mathbb{L}}\left(\tau;\tau^{\sigma},q_k^{\sigma}(\tau),1,{}_a^C D_{\Delta,\tau}^{\alpha} q_k(\tau),p_k(\tau)\right)={}_a^C D_{\Delta,\tau}^{\alpha} q_k(\tau)-\frac{\partial}{\partial p_k}H\left(\tau,q_k^{\sigma}(\tau),p_k(\tau)\right).\end{cases}\tag{51}$$

Substituting (51) into (49), it concludes that

$$\begin{aligned}&-\frac{\partial H(\bullet)}{\partial\tau}\zeta^{\sigma}-\frac{\partial H(\bullet)}{\partial q_k^{\sigma}}\xi_k^{\sigma}-\left[H(\bullet)+(\alpha-1)p_k(\tau){}_a^C D_{\Delta,\tau}^{\alpha} q_k(\tau)-\frac{\partial H(\bullet)}{\partial\tau}\mu(\tau)\right]\zeta^{\Delta}\\&\qquad+p_k(\tau){}_a^C D_{\Delta,\tau}^{\alpha}\xi_k+\eta_k\cdot{}_a^C D_{\Delta,\tau}^{\alpha} q_k(\tau)-\frac{\partial H(\bullet)}{\partial p_k}\eta_k=0,\end{aligned}\tag{52}$$

where $(\bullet) \triangleq \left(\tau, q_k^\sigma(\tau), p_k(\tau)\right)$. By use of the fractional time-scales Hamilton canonical Eqs. (22), (52) can be expressed as

$$-\frac{\partial H(\bullet)}{\partial \tau}\zeta^\sigma - \left[H(\bullet) + (\alpha-1)p_k(\tau)\frac{\partial H(\bullet)}{\partial p_k} - \frac{\partial H(\bullet)}{\partial \tau}\mu(\tau)\right]\zeta^\Delta + \mathscr{D}_\Delta^\alpha[\xi_k, p_k] = 0. \tag{53}$$

Let

$$\mathscr{H}(\bullet) = H(\bullet) + (\alpha-1)p_k(\tau)\frac{\partial H(\bullet)}{\partial p_k} - \frac{\partial H(\bullet)}{\partial \tau}\mu(\tau). \tag{54}$$

Since $[\mathscr{H}(\bullet)\zeta]^\Delta = \mathscr{H}(\bullet)\zeta^\Delta + \mathscr{H}^\Delta(\bullet)\zeta^\sigma$, then the above formula can be written as

$$\left[\mathscr{H}^\Delta(\bullet) - \frac{\partial H(\bullet)}{\partial \tau}\right]\zeta^\sigma - [\mathscr{H}(\bullet)\zeta]^\Delta + \mathscr{D}_\Delta^\alpha[\xi_k, p_k] = 0. \tag{55}$$

Integrating (55), the fractional time-scales Noether theorem for Hamiltonian systems is obtained.

Theorem 2. *If the Hamiltonian action (10) is invariant in the sense of Definition 8, then there exists a conserved quantity*

$$I = \int_a^t \left[\mathscr{H}^\Delta(\bullet) - \frac{\partial H(\bullet)}{\partial \tau}\right]\zeta^\sigma \Delta\tau - \mathscr{H}\left(t, q_k^\sigma(t), p_k(t)\right)\zeta + \xi_k \cdot {}_tI_{\Delta,b}^{1-\alpha} p_k = const. \tag{56}$$

Remark 7. If $\alpha = 1$, formula (55) becomes

$$\left[\mathscr{H}^\Delta(\star) - \frac{\partial H(\star)}{\partial \tau}\right]\zeta^\sigma - [\mathscr{H}(\star)\zeta - p_k \cdot \xi_k]^\Delta = 0, \tag{57}$$

where $(\star) = \left(\tau, q_k^\sigma(\tau), p_k(\tau)\right)$, $p_k = \frac{\partial L\left(\tau, q_k^\sigma, q_k^\Delta\right)}{\partial q_k^\Delta}$, $H(\star) = p_k \cdot q_k^\Delta - L\left(\tau, q_k^\sigma, q_k^\Delta\right)$ and $\mathscr{H}(\star) = H(\star) - \frac{\partial H(\star)}{\partial \tau}\mu(\tau)$. Integrating it between a and t, then the time-scales Noether theorem for Hamiltonian systems is obtained as follows

$$I = \int_a^t \left[\mathscr{H}^\Delta(\star) - \frac{\partial H(\star)}{\partial \tau}\right]\zeta^\sigma \Delta\tau - \mathscr{H}\left(t, q_k^\sigma(t), p_k(t)\right)\zeta + p_k \cdot \xi_k = const. \tag{58}$$

Remark 8. If $\mathbb{T} = \mathbb{R}$, i.e., $\mu = 0$, formula (55) becomes

$$\left[\dot{\mathscr{H}}(*) - \frac{\partial H(*)}{\partial \tau}\right]\zeta - \frac{d}{dt}[\mathscr{H}(*) \cdot \zeta] + \mathscr{D}^\alpha[\xi_k, p_k] = 0, \tag{59}$$

where $(*) = (\tau, q_k(\tau), p_k(\tau))$, $p_k = \frac{\partial L(\tau, q_k, {}_a^C D_t^\alpha q_k)}{\partial {}_a^C D_t^\alpha q_k}$ and $\mathscr{H}(*) = H(*) + (\alpha-1)p_k(\tau)\frac{\partial H(*)}{\partial p_k}$, $\mathscr{D}^\alpha[\xi_k, p_k] = p_k \cdot {}_a^C D_t^\alpha \xi_k - \xi_k \cdot {}_tD_b^\alpha p_k$. Considering the property of fractional derivatives

$$\int_a^b g(t) \cdot {}_a^C D_t^\alpha f(t)\Delta t = \int_a^b f(t) \cdot {}_tD_b^\alpha g(t)\Delta t + \left[f(t) \cdot {}_tI_b^{1-\alpha} g(t)\right]\Big|_a^b, \tag{60}$$

then Theorem 2 reduces to the fractional Noether theorem for Hamiltonian systems

$$I = \int_a^t \left[\dot{\mathscr{H}}(*) - \tfrac{\partial H(*)}{\partial \tau}\right]\zeta d\tau - \mathscr{H}(t, q_k(t), p_k(t))\zeta + \xi_k \cdot {}_tI_b^{1-\alpha} p_k = const. \tag{61}$$

6. Examples

Example 1. Because fractional models have a unique ability in describing anomalous behavior and memory effects, fractional dynamic equations lead to better results than integer equations in many practical systems. The fractional damped oscillator is a generalization of the classical damped oscillator equation, which can be considered to describe the dynamics of certain gases dissolved in a fluid and the dynamics of a sphere immersed in an incompressible viscous fluid [61]. The time-scales theory is introduced to provide a new model for studying the fractional damping oscillator of continuous variables, discrete variables or piecewise continuous variables. The study of the fractional time-scales damped oscillator Noether theorem is not only helpful to understand the property and physical meaning of the system but also to find the solution of the system. Here, a nonlinear equation with fractional damping is studied on the time-scale $\mathbb{T} = \{t_k = a + kh, k \in \mathbb{N}\}$, i.e., the graininess function $\mu = h$, which the Lagrangian is

$$L = \frac{1}{2}\exp(\gamma t)\left[\left({}_a^C D_{\Delta,t}^\alpha q\right)^2 - (q^\sigma)^2\right]. \tag{62}$$

Then the generalized momentum is

$$p = \frac{\partial L}{\partial {}_a^C D_{\Delta,t}^{\alpha} q} = \exp(\gamma t) {}_a^C D_{\Delta,t}^{\alpha} q, \tag{63}$$

and the fractional time-scales Hamiltonian is

$$H = p\left({}_a^C D_{\Delta,t}^{\alpha} q\right) - L = \frac{1}{2}\exp(-\gamma t) p^2 + \frac{1}{2}\exp(\gamma t) \cdot (q^{\sigma})^2. \tag{64}$$

The fractional time-scales Hamilton canonical Eqs. (22) are given by

$${}_a^C D_{\Delta,t}^{\alpha} q = \frac{\partial H}{\partial p} = \exp(-\gamma t) p, \ \ {}_t D_{\Delta,b}^{\alpha} p = \frac{\partial H}{\partial q^{\sigma}} = \exp(\gamma t) q^{\sigma}. \tag{65}$$

If $\mathbb{T} = \mathbb{R}$, $\alpha = 1$, from formulas (63) and (65), the classical damped oscillator is obtained

$$\ddot{q} + \gamma \dot{q} + q = 0. \tag{66}$$

So Eqs. (65) can be called the fractional time-scales damped oscillator for Hamiltonian systems. Criterion 2 gives

$$\begin{aligned} &\eta\left({}_a^C D_{\Delta,t}^{\alpha} q - \frac{\partial H}{\partial p}\right) + p \cdot {}_a^C D_{\Delta,t}^{\alpha} \xi - \left[\left(\alpha - \frac{1}{2}\right) p^2 \exp(-\gamma t) + \frac{1}{2}\exp(\gamma t) \cdot (q^{\sigma})^2\right] \zeta^{\Delta} \\ &\quad + \left[\frac{\gamma}{2} p^2 \exp(-\gamma t) - \frac{\gamma}{2}\exp(\gamma t) \cdot (q^{\sigma})^2\right] \zeta - \exp(\gamma t) q^{\sigma} \xi^{\sigma} = 0. \end{aligned} \tag{67}$$

Eq. (67) has a solution

$$\zeta = 1, \ \ \xi = -\frac{\gamma}{2} q. \tag{68}$$

There is no limit to η. From Theorem 2, the conserved quantity is

$$I = \int_a^t \left[\frac{\gamma}{2}\exp(-\gamma\tau) \cdot (p(\tau))^2 - \frac{\gamma}{2}\exp(\gamma\tau) \cdot (q^{\sigma}(\tau))^2\right] \Delta\tau - \frac{\gamma q}{2} \cdot {}_t I_{\Delta,b}^{1-\alpha} p. \tag{69}$$

If $a = 0$, $b = 10$, $\gamma = 1$ and $\alpha = 0.5$, then the time-scale $\mathbb{T} = \{t_k = kh, k \in \mathbb{N}\}$ is the interval $[1; 9]$ with $h = 1$. And the initial conditions satisfy $q_0 = 1$, $q_1 = 0.5$. According to formulas (65) and (69), the simulation is obtained as follows.

In order to simplify the calculation, the value of I^{Δ} is just considered in the simulation. From Fig. 1(c), it's obvious that $I^{\Delta} \equiv 0$ which can verify that I is a conserved quantity. Moreover, when α takes any decimal of $(0, 1)$, the Δ derivative of its conserved quantity I^{Δ} is always zero. In addition, if $\alpha = 1$, the time-scales conserved quantity with Caputo fractional derivatives (69) reduces to the one with integer order. The simulations on the interval $[0; 10]$ with $h = 1$ and on the interval $[0; 3.5]$ with $h = 0.1$ are as follows.

When $\alpha = 1$ and the initial conditions remain unchanged, it can be seen from Figs. 2 and 3 that when the step sizes $h = 1$ and $h = 0.1$, their conserved quantities are the same constant and $I = 0.25$.

Example 2. As is known to all, in a Hamiltonian system with two degrees of freedom, the orbits of all finite motions are closed in the Kepler problem. This is due to a hidden symmetry caused by the existence of third integral of motion in the Kepler problem. Eleonskiĭ et al. [62] presented that these properties also apply to the fractional Kepler problem. Zhai and Zhang [63] worked the Lie symmetries and conserved quantities of the time-scales Kepler problem. The Lagrangian with Caputo Δ derivatives of two interacting particles of mass one in the Kepler problem can be defined by

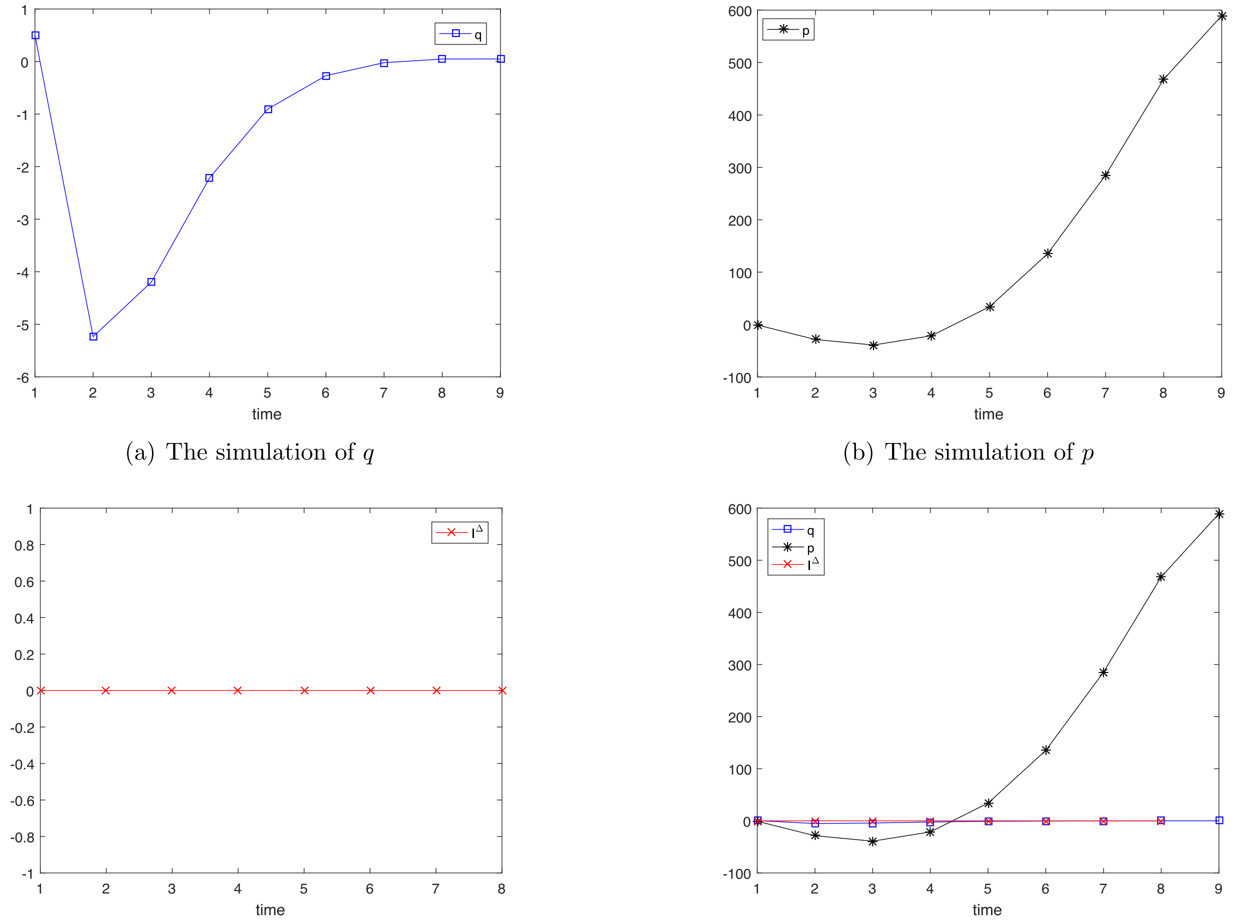
$$\begin{aligned} &L\left(q_1^{\sigma}, q_2^{\sigma}, {}_a^C D_{\Delta,t}^{\alpha} q_1, {}_a^C D_{\Delta,t}^{\alpha} q_2\right) \\ &\quad = \frac{1}{2}\left[\left({}_a^C D_{\Delta,t}^{\alpha} q_1\right)^2 + \left({}_a^C D_{\Delta,t}^{\alpha} q_2\right)^2\right] - \left[(q_1^{\sigma})^2 + (q_2^{\sigma})^2\right]^{-1/2} \end{aligned} \tag{70}$$

on $\left(\mathbb{R}^2 \backslash \{0\}\right) \times \mathbb{R}^2$. So its Hamiltonian is

$$H(q_1^{\sigma}, q_2^{\sigma}, p_1, p_2) = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \left[(q_1^{\sigma})^2 + (q_2^{\sigma})^2\right]^{-1/2}. \tag{71}$$

According to the fractional time-scales Hamilton canonical equations (22), it concludes that

$$\begin{aligned} {}_a^C D_{\Delta,t}^{\alpha} q_1 &= \frac{\partial H}{\partial p_1} = p_1, \\ {}_a^C D_{\Delta,t}^{\alpha} q_2 &= \frac{\partial H}{\partial p_2} = p_2, \\ {}_t D_{\Delta,b}^{\alpha} p_1 &= \frac{\partial H}{\partial q_1^{\sigma}} = -q_1^{\sigma}\left[(q_1^{\sigma})^2 + (q_2^{\sigma})^2\right]^{-3/2}, \\ {}_t D_{\Delta,b}^{\alpha} p_2 &= \frac{\partial H}{\partial q_2^{\sigma}} = -q_2^{\sigma}\left[(q_1^{\sigma})^2 + (q_2^{\sigma})^2\right]^{-3/2}. \end{aligned} \tag{72}$$



(a) The simulation of q (b) The simulation of p

(c) The simulation of I^{Δ} (d) The simulation of q, p and I^{Δ}

Fig. 1. The simulation of q, p and I^{Δ} on $[1;9]$ with $\alpha = 0.5$ and $h = 1$.

Criterion 2 leads to

$$\begin{aligned}&\eta_1 \cdot {}_a^C D_{\Delta,t}^{\alpha} q_1 + \eta_2 \cdot {}_a^C D_{\Delta,t}^{\alpha} q_2 + p_1 \cdot {}_a^C D_{\Delta,t}^{\alpha} \xi_1 + p_2 \cdot {}_a^C D_{\Delta,t}^{\alpha} \xi_2 \\ &- (\alpha - 1) p_1 \cdot \zeta^{\Delta} \cdot {}_a^C D_{\Delta,t}^{\alpha} q_1 - (\alpha - 1) p_2 \cdot \zeta^{\Delta} \cdot {}_a^C D_{\Delta,t}^{\alpha} q_2 - H\zeta^{\Delta} \\ &- \left(\frac{\partial H}{\partial t}\zeta + \frac{\partial H}{\partial q_1^{\sigma}}\xi_1^{\sigma} + \frac{\partial H}{\partial q_2^{\sigma}}\xi_2^{\sigma} + \frac{\partial H}{\partial p_1}\eta_1 + \frac{\partial H}{\partial p_2}\eta_2 \right) = 0. \end{aligned} \tag{73}$$

Two sets of infinitesimals can be solved

$$\zeta = -1, \ \ \xi_1 = \xi_2 = 0, \tag{74}$$

and

$$\zeta = 0, \ \ \xi_1 = -q_2, \ \ \xi_2 = q_1. \tag{75}$$

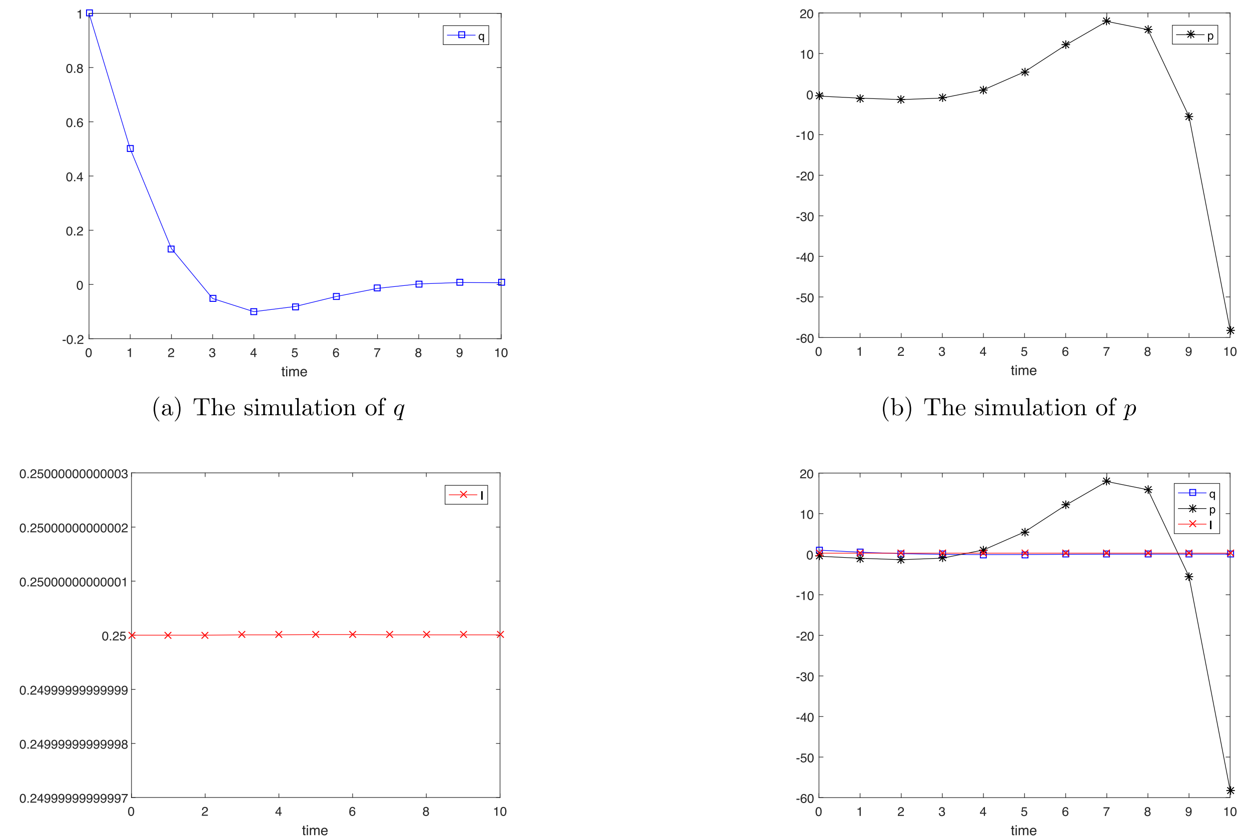
Both sets of solutions have no limit to η_1 and η_2. According to Theorem 2, two conserved quantities are found

$$I_1 = \int_a^t \mathscr{H}^{\Delta} \Delta\tau - \mathscr{H} = const. \tag{76}$$

and

$$I_2 = -q_{2t} I_{\Delta,b}^{1-\alpha} p_1 + q_{1t} I_{\Delta,b}^{1-\alpha} p_2 = const. \tag{77}$$

where $\mathscr{H} = H + (\alpha - 1)\left(p_1^2 + p_2^2\right)$. If $\alpha = 1$, then $I_1 = \int_a^t H^{\Delta} \Delta\tau - H$ and $I_2 = -q_2 p_1 + q_1 p_2$, which are consistent with the results in [35].



(a) The simulation of q (b) The simulation of p

(c) The simulation of I (d) The simulation of q, p and I

Fig. 2. The simulation of q, p and I on $[0; 10]$ with $\alpha = 1$ and $h = 1$.

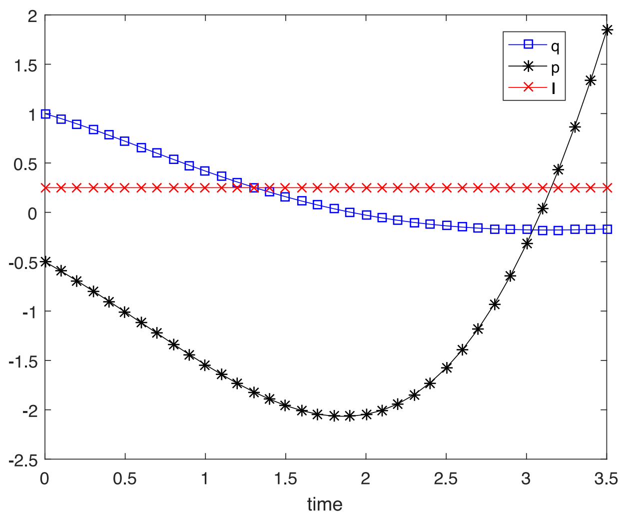
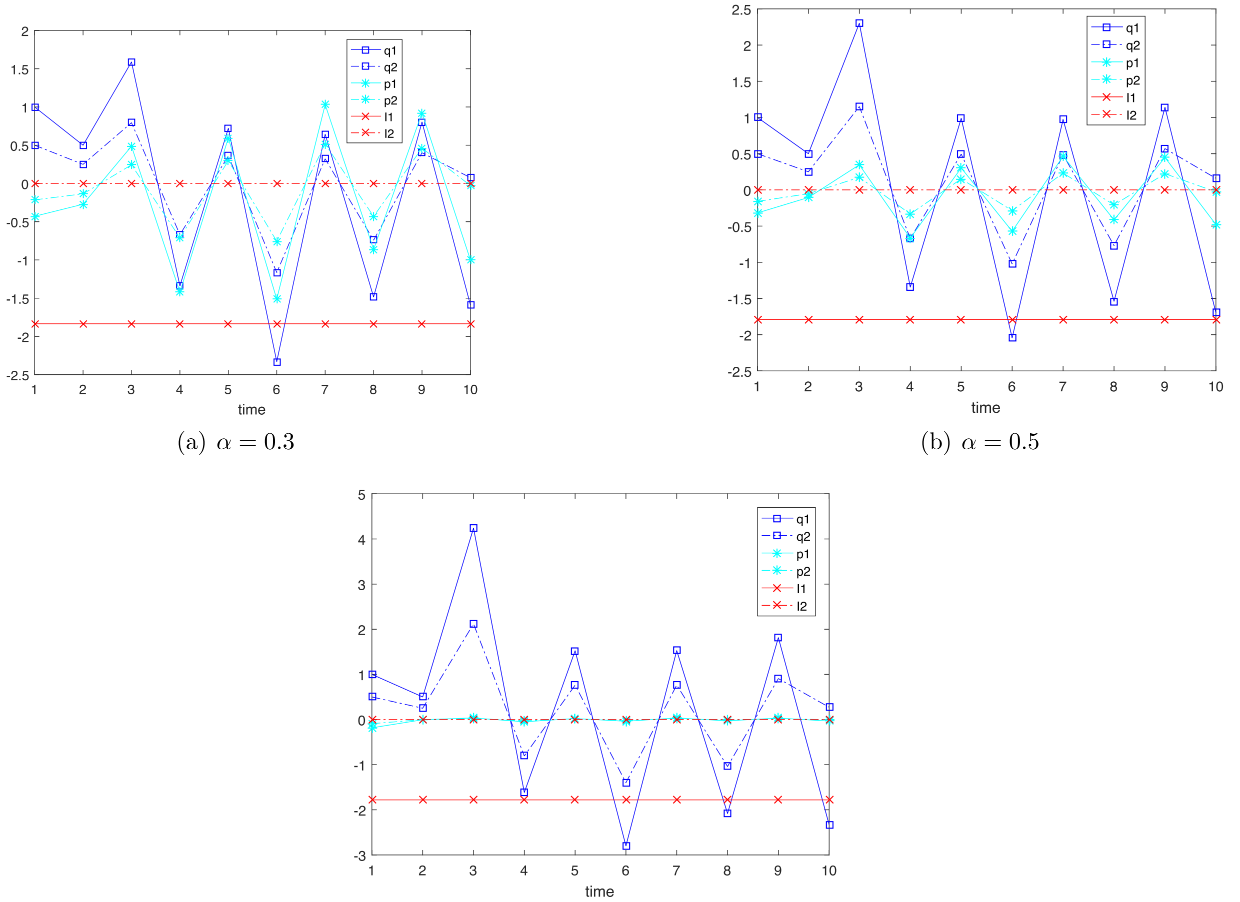


Fig. 3. The simulation of q, p and I on $[0; 3.5]$ with $\alpha = 1$ and $h = 0.1$.

If the initial conditions satisfy $q_1(1) = 1$, $q_1(2) = 0.5$, $q_2(1) = 0.5$, $q_2(2) = 0.25$, the time-scale $\mathbb{T} = \mathbb{N}_+$ is the interval with $h = 1$. Then, the simulations when $\alpha = 0.3$, $\alpha = 0.5$ and $\alpha = 0.7$ are obtained as follows.

From Fig. 4, it shows that the conserved quantities obtained from formulas (76) and (77) are constants, which verify the correctness of Theorem 2. It is worth noting that when α is different, the values of I_1 are different, but the values of I_2 are both zero. Therefore, I_2 of this example is a trivial conserved quantity under the initial conditions.



(a) $\alpha = 0.3$

(b) $\alpha = 0.5$

(c) $\alpha = 0.7$

Fig. 4. The simulation of q_1, q_2, p_1, p_2, I_1 and I_2 on [1; 10] with $h = 1$.

7. Conclusions

The fractional time-scales Noether theorems without transforming time (Theorem 1) and with transforming time (Theorem 2) for Hamiltonian systems are studied. In fact, Theorem 1 can become the results of literature [19,22,31] under certain conditions. Since the time-scales Jost's method [35] can deal with time-scales Noether theorem and the fractional Jost's method [52] can work out fractional Noether theorem, the fractional time-scales Noether theorem can be solved by combining these two methods. Then, Theorem 2 is obtained, which can not reduce to the time-scales Noether theorem [19,22,31] obtained by using the technique of time-re-parameterization when $\alpha = 1$. But, it can reduce to the time-scales Noether theorem [35] in case of $\alpha = 1$ and the fractional Noether theorem [52] in the case of $\mathbb{T} = \mathbb{R}$.

Therefore, the two examples are simulated by using MATLAB, which illustrate the correctness of our Theorem 2. In addition, Theorem 2 includes the time-scales Noether theorem with integer Δ derivatives and the fractional Noether theorem with Caputo derivatives for Hamiltonian systems. It is worth noting that due to the difficulty of fractional time-scales calculus, the case with step size $\mu = 1$ is only simulated. Discussions of different step sizes and their structure-preserving algorithm are likely to be a follow-up work.

On account of the characteristics of time-scales and fractional calculus, the study of the fractional time-scales theory is beneficial to develop program to deal with some engineering problems. Since the fractional time-scales theory is still in its infancy, the theory for Birkhoffian systems and nonholonomic systems as well as their symmetries and conserved quantities can be further studied.

Declaration of Competing Interest

The authors declare that they have no conflict of interest.

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