Mathisson-Papapetrou equations in metric and gauge theories of gravity in a Lagrangian formulation

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Abstract

We present a simple method to derive the semiclassical equations of motion for a spinning particle in a gravitational field. We investigate the cases of classical, rotating particles, i.e. the so-called pole-dipole particles, as well as particles with an additional intrinsic spin. We show that, starting with a simple Lagrangian, one can derive equations for the spin evolution and momentum propagation in the framework of metric theories of gravity (general relativity) and in theories based on a Riemann-Cartan geometry (Poincaré gauge theory), without explicitly referring to matter current densities (spin and stress-energy). Our results agree with those derived from the multipole expansion of the current densities by the conventional Papapetrou method and from the WKB analysis for elementary particles.

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1 Introduction

It is of interest to have an alternative method of deriving the Mathisson-Papapetrou equations (see [1] and [2]) not only because the conventional procedure is rather lengthy, but more importantly, because one might get a better insight into the interpretation of the various quantities involved (see also [3] for a detailed analysis of the classical method). For instance, in Riemann-Cartan space, there exists a large variation of possibilities of how to define the momentum vector which eventually lead to differences in the final equations (see [4] and [5], especially the remarks at the end of [5]). Furthermore, if we consider a specific solution for the metric and the torsion fields, it would be convenient to have a Lagrangian into which we can plug the results and derive the equations of motion

by varying the Lagrangian, instead of plugging the solution directly into the equations of motion. For instance, in general relativity, one usually substitutes the Schwarzschild metric into the Lagrangian $L = mg_{ik}u^iu^k$ and not into the geodesic equation, which would require to evaluate the complete Christoffel symbols. In this way, one can take full advantage of the symmetries of the problem. The aim of this article is therefore to find suitable generalizations of $L = mg_{ik}u^iu^k$ that allow for the description of particles with spin (classical and intrinsic) moving in a Riemann-Cartan geometry.

To avoid confusion, let us make a remark on the language we will use throughout this article. We will use the expression classical spin of a body to denote its orbital angular momentum in its (center of mass) rest frame. The word classical is used to distinguish it from the intrinsic spin which is of quantum mechanical origin, and the word spin is used to emphasize that we are always talking about the angular momentum in the rest frame. When talking about intrinsic spin, we either refer to the intrinsic spin of an elementary particle, or to the total intrinsic spin of a macroscopic body as a result of the polarization of its constituent particles, for instance in a ferromagnet or in a neutron star. The word spin may refer to both classical and intrinsic spin.

Several attempts have been made for a Lagrangian description in the past and results were achieved not only in general relativity for the classical multipole particle [6], but also for point particles with intrinsic spin in a Riemannian space [7], [8]. We will review these cases, discuss differences in our derivation method and generalize to particles with intrinsic spin and dipole moment (classical spin) in a general Riemann-Cartan space.

Let us use the example of a test particle in an electromagnetic field to illustrate how we will manage to achieve equations for the position as well as for the spin evolution, from a single Lagrangian, without referring to the field equations or the Bianchi identities.

The Lagrangian of a charged particle reads

$$L = \frac{m}{2}g_{ik}u^iu^k - eA_iu^i.$$

The charge e is of course the integrated current j^m and its conservation law $\dot{e} = 0$ (the dot denotes the time derivative with respect to proper time τ) can be derived from the relation $j^m_{;m} = 0$ which is a consequence of the field equations. However, we do not want to refer to the field equations at all and we will ignore the relation $\dot{e} = 0$ for the moment. The equations of motions are then derived in the form

$$m\hat{\mathbf{D}}u^i = eF^i_{\ k}u^k + A^i\frac{\mathrm{d}e}{\mathrm{d}\tau}.$$

The symbol D denotes the covariant derivative with respect to the proper time τ . Without knowing the Maxwell equations, we certainly know that they are gauge invariant with

respect to $A_m \to A_m + f_{,m}$ and consequently this should be the case for the equations of motion too. We therefore have to require $\dot{e} = 0$. In this way, we are let to the charge conservation equation as well as to the correct equation of motion. How do we know that the field equations will be gauge invariant? Well, the action for the Maxwell field is constructed under exactly this requirement. The gauge invariance is more fundamental then the field equations themselves. It incidentally leads to the unique Lagrangian F^2 . (This will not be the case for the local SO(3,1) gauge invariance of Poincaré gauge theory, which allows for a whole family of Lagrangians.)

In the same way, we will derive the equations for the evolution of the intrinsic spin and/or classical spin by investigating the covariance properties of the momentum equation with respect to coordinate transformations and/or Lorentz gauge transformations.

In the next section, we derive in this way the Mathisson-Papapetrou equations for the pole-dipole particle in general relativity. In section 3, we generalize the result to Riemann-Cartan geometry and finally in sections 4 and 5 we investigate the case of particles with intrinsic spin and of macroscopic spin polarized bodies. To get more confidence in our procedure, we will re-derive in section 6 the equations of motion for the case of the Dirac particle using an alternative method. In section 7 we consider as an illustrative example the precession equation of a spinning particle in the torsion field arising in Brans-Dicke theory generalized to Riemann-Cartan space. Finally, in section 8, we analyze the symmetry properties of the Lagrangian itself, as opposed to the equations of motion.

2 The classical Mathisson-Papapetrou equations

In this section, we consider a classical (i.e. without intrinsic spin) pole-dipole particle moving in a curved, Riemannian spacetime.

Instead of postulating a Lagrangian, we will try to derive it from simple considerations. The following manipulations, although not mathematically rigorous, will provide us with a strong motivation for the Lagrangian (8) which will be our starting point.

In general relativity, the geodesic equations for a body of mass m are derived from the following Lagrangian

$$L = \frac{m}{2} g_{ik} u^i u^k.$$

First, consider the body to be constituted from freely moving test particles, moving all in a given exterior field. Then, the Lagrangian can be written as

$$L = \sum_{l} \frac{m_{(l)}}{2} g_{ik}(x_{(l)}) u_{(l)}^{i} u_{(l)}^{k}. \tag{1}$$

However, in order to describe an extended, rotating body, the coordinates of the different constituents will have to fulfill certain constraints, due to the rigid structure of the body. We now follow the method that is usually used on post-Newtonian Lagrangians (see [9] for instance) and also in classical mechanics, but we will apply it directly to the covariant form (1). Let X^i denote the center of mass of the body. (We avoid the difficulties and ambiguities concerning the definition of the center of mass in general relativity. We just suppose that there exists a vector X^i whose spatial components X^{α} , $\alpha = 1, 2, 3$ agree in some limit with the Newtonian center of mass.)

Then, for the coordinate of a mass element of the body we can write (we omit the index denoting the mass element)

$$x^i = X^i + \rho^i, \tag{2}$$

and for the velocity of the same mass element

$$u^i = V^i + \omega^i_{\ k} \rho^k, \tag{3}$$

with antisymmetric ω^{ik} and $V^i = \mathrm{d}X^i/\mathrm{d}\tau$. Just as in the original work of Papapetrou [2], we suppose $\rho^0 = 0$ in a certain reference frame. If we set in the same frame $\omega^{i0} = 0$, the quantity ω^{ik} is clearly the angular velocity in four dimensional form, i.e. we have for the spatial part $\varepsilon^{\nu\mu\lambda}\omega_{\lambda} = \omega^{\mu\nu}$ and the three velocity reads $\vec{u} = \vec{V} + \vec{\omega} \times \vec{\rho}$. Consequently, ω^{ik} is related to the angular momentum (the classical spin) through

$$S^{ik} = I\omega^{ik},\tag{4}$$

where I is the moment of inertia of the body, defined as in classical mechanics, i.e. we have in the spherical case $I = I^{xx} = I^{yy} = I^{zz}$, with $I^{\mu\nu} = \int (-g^{\mu\nu}\rho^2 - \rho^{\mu}\rho^{\nu}) dm$.

We will use the three dimensional relation

$$\frac{1}{2}Ig^{\mu\nu} = -\int \rho^{\mu}\rho^{\nu} \, \mathrm{d}m. \tag{5}$$

This relation can be found in [10] (§106, problem 4) and also in [9]. It is easily proved by taking its trace and using the fact that $I^{xx} = I^{yy} = I^{zz}$. (In the expression for the moment of inertia, the metric can be considered to be independent or ρ , since the integrant is already of order ρ^2 . See expansion below.) Note that (5) is essentially a Newtonian order relation. There is no univoque generalization of the concept of moment of inertia in the framework of general relativity.

Since by definition, X^i is the particle's center of mass, we have

$$\int \rho^i \mathrm{d}m = 0. \tag{6}$$

This equation, again, can only be correctly understood at the Newtonian level. We do not specify the integration meter dm in general, but just assume that it has the correct Newtonian limit. We now write (1) in the form $L = \int \frac{1}{2}g_{ik}(x)u^iu^k dm$ and expand the metric $g_{ik}(x) = g_{ik}(X) + g_{ik,l}(X)\rho^l$. By using (3)-(6), and retaining only the first order metric perturbations and terms at most linear in ω^{ik} we find

$$L = \frac{m}{2} g_{ik} V^i V^k - \frac{1}{2} g_{ik,m} S^{im} V^k,$$
 (7)

where $V^i = \mathrm{d}X^i/\mathrm{d}\tau$. Note that the terms linear in ρ do not contribute because of (6). Since ρ^i has been eliminated, we can now leave the frame where $\rho^0 = \omega^{i0} = 0$ and consider S^{im} as a four dimensional antisymmetric tensor.

Using the symmetry properties of the Christoffel symbols $\hat{\Gamma}^l_{km}$ and of S^{im} , we can finally write (7) in the simple form (we change the notation from V^i to u^i)

$$L = \frac{m}{2} g_{ik} u^i u^k - \frac{1}{2} \hat{\Gamma}^l_{km} S_l^{\ k} u^m.$$
 (8)

This Lagrangian (generalized to higher order multipole terms) has been used in [6], starting from Newtonian quantities and looking for four dimensional generalizations. A Lagrangian approach for spinning media can also be found in [11].

Let us emphasize once again that the whole derivation that led to (8) is essentially based on Newtonian physics and can therefore not be used to conclude, or even to prove, that (8) is the correct Lagrangian for the description of the dipole particle. The arguments make however clear that (8) is probably a good candidate to start with, because by construction, it will certainly lead to equations that possess the correct Newtonian limit. Beyond that, we can only compare the equations of motion derived from (8) with those derived by other means (as the Papapetrou method) or directly with experiment. In this sense, we simply postulate the Lagrangian (8) and forget about its origin.

Since S^{ik} is an integrated quantity, it is a function of proper time only, without explicit x^i -dependence. Therefore, the Euler-Lagrange equations are found to be

$$m\hat{D}u_m = -\frac{1}{2}(\hat{\Gamma}_{ki,m}^l - \hat{\Gamma}_{km,i}^l)S_l^{\ k}u^i + \frac{1}{2}\hat{\Gamma}_{km}^l\dot{S}_l^{\ k}.$$
 (9)

Next, the authors of [6] proceed as follows. In a coordinate system where $\hat{\Gamma}_{ik}^l = 0$, equation (9) can be written as

$$m\hat{\mathbf{D}}u_m = -\frac{1}{2}\hat{R}^l_{\ kmi}S_l^{\ k}u^i,\tag{10}$$

and since this is a tensor equation, it has to be valid in every coordinate system. In this way, starting with a more general Lagrangian, they were able to derive equations of motion very effectively, including up to octupole moment contributions. Apart from (10), the spin evolution equation was written down as covariant generalization of the Newtonian precession equation. In brief, this method consists in looking for covariant equations that have the correct Newtonian limit. In a Riemannian theory, this is a very powerful tool and is infallible in the sense that there is only one covariant generalization to a given Newtonian equation, as long as we suppose that the minimal coupling prescription is valid (that is, matter fields should not couple directly to curvature).

Here, we will follow a different procedure. Since we will later on generalize our geometry to Riemann-Cartan space, there will in general be more than one way to write a certain Newtonian equation in a covariant form. For instance, both the geodesic and the autoparallel equation, although they differ in a general Riemann-Cartan geometry and although they are both covariant, reduce to the correct, free particle motion in the absence of gravitational fields (curvature and torsion). Therefore, the mere knowledge of the Newtonian limit is not enough to fix the general form of an equation.

We proceed as in the electromagnetic analogy in the introduction. We claim that (9) is already covariant as it stands and look at the consequences of this claim. If we complete the first two terms of the r.h.s. of (9) to form the curvature tensor, we find

$$m\hat{\mathbf{D}}u_{m} = -\frac{1}{2}(\hat{\Gamma}_{ki,m}^{l} - \hat{\Gamma}_{km,i}^{l} + \hat{\Gamma}_{nm}^{l}\hat{\Gamma}_{ki}^{n} - \hat{\Gamma}_{ni}^{l}\hat{\Gamma}_{km}^{n})S_{l}^{k}u^{i} + \frac{1}{2}(\hat{\Gamma}_{nm}^{l}\hat{\Gamma}_{ki}^{n} - \hat{\Gamma}_{ki}^{l}\hat{\Gamma}_{km}^{n})S_{l}^{k}u^{i} + \frac{1}{2}\hat{\Gamma}_{km}^{l}\dot{S}_{l}^{k}.$$

It is now easy to recognize the covariant derivative $\hat{\mathbf{D}}S_l^{\ k}$ in the second row. Thus, we have

$$m\hat{D}u_{m} = -\frac{1}{2}\hat{R}^{l}{}_{kmi}S_{l}{}^{k}u^{i} + \frac{1}{2}\hat{\Gamma}^{l}{}_{km}\hat{D}S_{l}{}^{k}.$$
(11)

For this equation to be covariant, we have to require in addition

$$\hat{\mathbf{D}}S^{ik} = 0. \tag{12}$$

In this way, we finally get equations (10) and (12) from the Lagrangian (8). They are in agreement with the Mathisson-Papapetrou equations in the required order of precision, i.e. if one identifies momentum with mu^i (see below).

To summarize our procedure, we suppose that equations (11) are correct (apart from higher multipole terms of course) and since ultimately, they were derived from (1), they have to be covariant. This claim leads to a constraint, the spin evolution equation. There

is a certain similarity between this method and the original derivation of Papapetrou [2]. Papapetrou begins with a symmetric stress-energy tensor and finds, after having expanded up to the dipole order, an expression for the integrated stress-energy tensor that is not apparently symmetric. The claim that it should be symmetric then leads to a constraint which is just the spin evolution equation.

Last, we remark that there is a tiny difference between (10) and (12) and the equations originally derived by Papapetrou in the sense that the momentum P^i of the latter is identified with mu^i in our equations, whereas the correct relation reads $P_i = mu_i + \hat{D}S_{ik}u^k$. We will derive the correct equations in the next section; we note however that the difference can be shown to be of second order in the spin and of first order in the curvature (see [3] or [12]) and is thus negligibly small compared to the other terms. On the other hand, it has been shown in [13] that (10) and (12) can actually be derived from the original Papapetrou equations by a slight redefinition of the center of mass of the body and can thus equally well be considered as exact (since we have not specified our definition of center of mass). (See also [9] on the issue of the center of mass redefinition.)

3 The pole-dipole particle in Riemann-Cartan space

We now turn to a classical pole-dipole particle moving in Riemann-Cartan geometry. We know that, since torsion couples only to the intrinsic spin, the classical pole-dipole particle should behave exactly in the same way as in a purely Riemannian space. It is however necessary to have at our disposal a consistent method of re-deriving the same equations in the full Riemann-Cartan framework for the later generalization to particles with intrinsic spin.

Let us first give a short review of the basic concepts of Riemann-Cartan geometry and fix our notations and conventions. For a complete introduction into the subject, the reader may consult [14]-[17] and the extensive reference list in [17].

Latin letters from the beginning of the alphabet (a, b, c...) run from 0 to 3 and are (flat) tangent space indices. Especially, η_{ab} is the Minkowski metric diag(1, -1, -1, -1) in tangent space. Latin letters from the middle of the alphabet (i, j, k...) are indices in a curved spacetime with metric g_{ik} as before. We introduce the independent gauge fields, the tetrad e_m^a and the connection Γ^{ab}_m (antisymmetric in ab), as well as the corresponding field strengths, the curvature and torsion tensors

$$\begin{array}{lcl} R^{ab}_{\ lm} & = & \Gamma^{ab}_{\ m,l} - \Gamma^{ab}_{\ l,m} + \Gamma^{a}_{\ cl} \Gamma^{cb}_{\ m} - \Gamma^{a}_{\ cm} \Gamma^{cb}_{\ l} \\ T^{a}_{\ lm} & = & e^{a}_{m,l} - e^{a}_{l,m} + e^{b}_{m} \Gamma^{a}_{\ bl} - e^{b}_{l} \Gamma^{a}_{\ bm}. \end{array}$$

The spacetime connection Γ^i_{ml} and the spacetime metric g_{ik} can now be defined through

$$e^a_{m,l} + \Gamma^a_{bl} e^b_m = e^a_i \Gamma^i_{ml}$$
$$e^a_i e^b_k \eta_{ab} = g_{ik}.$$

It is understood that there exists an inverse to the tetrad, such that $e_i^a e_b^i = \delta_b^a$. It can easily be shown that the connection splits into two parts,

$$\Gamma^{ab}_{\ m} = \hat{\Gamma}^{ab}_{\ m} + K^{ab}_{\ m},$$

such that $\hat{\Gamma}^{ab}_{m}$ is torsion-free and is essentially a function of e_m^a . K^{ab}_{m} is the contortion tensor (see below). Especially, the spacetime connection $\hat{\Gamma}^{i}_{ml}$ constructed from

$$e_{m,l}^a + \hat{\Gamma}^a_{bl} e_m^b = e_i^a \hat{\Gamma}^i_{ml}$$

is just the (symmetric) Christoffel connection of general relativity, a function of the metric only. This notation is consistent with that of the previous section.

The gauge fields e_m^a and Γ^{ab}_m are covector fields with respect to the spacetime index m. Under a local Lorentz transformation in tangent space, $\Lambda^a_b(x^m)$, they transform as

$$e_m^a \to \Lambda^a_{\ b} e_m^b, \quad \Gamma^a_{\ bm} \to \Lambda^a_{\ c} \Lambda_b^{\ d} \Gamma^c_{\ dm} - \Lambda^a_{\ c} m \Lambda_b^{\ c}.$$
 (13)

The torsion and curvature are Lorentz tensors with respect to their tangent space indices as is easily shown. The contortion $K^{ab}_{\ m}$ is also a Lorentz tensor and is related to the torsion through $K^i_{\ lm} = \frac{1}{2}(T_{l\ m}^{\ i} + T_{m\ l}^{\ i} - T^i_{\ lm})$, with $K^i_{\ lm} = e^i_a e_{lb} K^{ab}_{\ m}$ and analogously for $T^i_{\ lm}$. The inverse relation is $T^i_{\ lm} = -2K^i_{\ [lm]}$.

Although we introduce the torsion tensor, which can be interpreted as the field strength corresponding to the translational group, we do not consider (local) translations here. We require our theory to be invariant with respect to local (tangent space) Lorentz transformations and general coordinate (spacetime) transformations only. The identification of the tetrad (or its nontrivial part) as translational gauge potential needs a special treatment and is not needed here. (See however section 8.)

All quantities constructed from the torsion-free connection $\hat{\Gamma}^{ab}_{\ m}$ or $\hat{\Gamma}^i_{lm}$ will be denoted with a hat, for instance $\hat{R}^{il}_{\ km} = e^i_a e^l_b \hat{R}^{ab}_{\ km}$ is the usual Riemann curvature tensor. Furthermore, we will use covariant derivatives with respect to proper time along a trajectory in the form

$$DP_m = \frac{\mathrm{d}P_m}{\mathrm{d}\tau} - \Gamma^i_{mk} P_i u^k$$

$$DP_a = \frac{\mathrm{d}P_a}{\mathrm{d}\tau} - \Gamma^b_{ak} P_b u^k.$$

It is easily shown that $DP_m = e_m^a DP_a$, which justifies the use of the same symbol D. Of course, the same derivative when referring to the torsion-less connection will be denoted by \hat{D} .

For the sake of completeness, and to avoid confusion concerning sign conventions, let us also derive the conservation laws for the stress-energy tensor and the intrinsic spin density (for details, see [14]). The stress-energy and spin density are defined by variation of the matter Lagrangian density \mathcal{L}_m as follows

$$T_i^a = \frac{1}{2e} \frac{\delta \mathcal{L}_m}{\delta e_a^i} \tag{14}$$

$$\sigma_{ab}^{i} = \frac{1}{e} \frac{\delta \mathcal{L}_{m}}{\delta \Gamma^{ab}_{i}}.$$
 (15)

Under an infinitesimal Lorentz transformation (13), with $\Lambda^a_{\ b} = \delta^a_b + \varepsilon^a_{\ b}$, where $\varepsilon^{ab} = -\varepsilon^{ba}$, the fields transform as

$$\delta\Gamma^{ab}_{\ m} = -\varepsilon^{ab}_{\ ,m} - \Gamma^{a}_{\ cm}\varepsilon^{cb} - \Gamma^{b}_{\ cm}\varepsilon^{ac}, \quad \delta e^{m}_{a} = \varepsilon_{a}^{\ c}e^{m}_{c}. \tag{16}$$

Therefore, the variation of the matter Lagrangian reads, omitting a total divergence

$$\delta \mathcal{L}_{m} = \frac{\delta \mathcal{L}_{m}}{\delta e_{a}^{m}} \delta e_{a}^{m} + \frac{\delta \mathcal{L}_{m}}{\delta \Gamma^{ab}_{m}} \delta \Gamma^{ab}_{m} = e(2T^{[ac]} + D_{m}\sigma^{acm})\varepsilon_{ac}, \tag{17}$$

where D_m is defined to act with Γ^{ab}_m on tangent space indices, and with $\hat{\Gamma}^l_{ki}$ (torsion free) on spacetime indices. The requirement of Lorentz invariance, i.e. $\delta \mathcal{L}_m = 0$, then leads to

$$2T^{[ac]} + \mathcal{D}_m \sigma^{acm} = 0. \tag{18}$$

On the other hand, the connection $\Gamma^{ab}_{\ m}$ and the tetrad field e^a_m are covariant spacetime vectors with respect to m (thus, e^m_a is a contravariant vector), and under an infinitesimal coordinate transformation

$$\tilde{x}^i = x^i + \xi^i, \tag{19}$$

the fields transform as

$$\tilde{e}_a^m(\tilde{x}) = e_a^m(x) + \xi_k^m e_a^k, \quad \tilde{\Gamma}_m^{ab}(\tilde{x}) = \Gamma_m^{ab}(x) - \xi_{,m}^k \Gamma_k^{ab}. \tag{20}$$

We are interested in the change of the Lagrangian under active transformations. In order to evaluate the change of the fields at the same point x, we have to express the transformed fields with the old coordinates, i.e.

$$\tilde{e}_a^m(x) = \tilde{e}_a^m(\tilde{x}) - \tilde{e}_{a,k}^m(\tilde{x})\xi^k, \quad \tilde{\Gamma}_m^{ab}(x) = \tilde{\Gamma}_m^{ab}(\tilde{x}) - \xi^k \Gamma_{m,k}^{ab}(\tilde{x}). \tag{21}$$

We then find for the variation, to first order in ξ

$$\delta e_a^m = \tilde{e}_a^m(x) - e_a^m(x) = \xi_{,k}^m e_a^k - \xi^k e_{a,k}^m, \tag{22}$$

$$\delta\Gamma^{ab}_{\ m} = \tilde{\Gamma}^{ab}_{\ m}(x) - \Gamma^{ab}_{\ m}(x) = -\xi^k_{\ m}\Gamma^{ab}_{\ k} - \xi^k\Gamma^{ab}_{\ m,k}. \tag{23}$$

The change in the Lagrangian therefore reads

$$\delta \mathcal{L}_{m} = \frac{\delta L_{m}}{\delta e_{a}^{m}} \delta e_{a}^{m} + \frac{\delta \mathcal{L}_{m}}{\delta \Gamma^{ab}_{m}} \delta \Gamma^{ab}_{m}$$

$$= -2(eT_{m}^{a} e_{a}^{k})_{,k} \xi^{m} - 2eT_{m}^{a} e_{a,k}^{m} \xi^{k} + (e\sigma_{ab}^{m} \Gamma^{ab}_{k})_{,m} \xi^{k} - e\sigma_{ab}^{m} \Gamma^{ab}_{m,k} \xi^{k}. \quad (24)$$

Requiring $\delta L_m = 0$ and regrouping carefully the terms, we finally get

$$(D_m T^m_{\ b}) e^b_k + T^m_{\ b} T^b_{\ mk} = \frac{1}{2} R^{ab}_{\ mk} \sigma_{ab}^{\ m} + \frac{1}{2} \Gamma^{ab}_{\ k} (D_m \sigma_{ab}^{\ m} + 2T_{[ab]}). \tag{25}$$

The last term vanishes with (18). The first term, written in terms of the usual (torsionless) covariant derivatives, (denoted with a semicolon;) reads

$$D_{m}T^{m}_{b}e^{b}_{k} = T^{m}_{b;m}e^{b}_{k} - K^{a}_{bm}T^{m}_{a}e^{b}_{k}$$
$$= T^{m}_{k:m} - K^{l}_{km}T^{m}_{l},$$

and the second term, using the relation $T^{i}_{lm} = -2K^{i}_{[lm]}$,

$$T^{m}_{b}T^{b}_{mk} = -T^{m}_{l}(K^{l}_{mk} - K^{l}_{km}).$$

Therefore, the first two terms of (25) simplify to $T_{k;m}^m - T_l^m K_{mk}^l$ and the equation of motion finally takes the form

$$T_{k;m}^{m} - K_{k}^{im} T_{mi} = \frac{1}{2} R_{mk}^{ab} \sigma_{ab}^{m}.$$
 (26)

Recall that $K^{im}_{\ k}$ is antisymmetric in im, so that classical matter (with $T_{[mi]}=0$ and $\sigma_{ab}^{\ m}=0$) will follow the general relativity relation $T^{km}_{\ ;m}=0$.

Let us now return to the classical pole dipole particle without intrinsic spin. We will re-derive the equations of the previous section using the concepts of Riemann-Cartan geometry.

Based on the form of equation (8), we will try the following Lagrangian

$$L = e_i^a P_a u^i - \frac{1}{2} \hat{\Gamma}^{ab}_{\ m} S_{ab} u^m. \tag{27}$$

Of course, we have coupled the (classical) spin to the torsion-less connection only. In (27), the momentum vector P_a and the spin tensor S_{ab} are considered as parameters and are neither functions of the coordinates nor of the velocities. They can be considered as the charges of the corresponding gauge fields e_m^a and $\hat{\Gamma}_m^{ab}$.

The Euler-Lagrange equations are then derived in a straightforward manner:

$$e_{i,m}^{a}u^{i}P_{a} - e_{m,i}^{a}u^{i}P_{a} - \frac{1}{2}(\hat{\Gamma}_{i,m}^{ab} - \hat{\Gamma}_{m,i}^{ab})S_{ab}u^{i} = e_{m}^{a}\dot{P}_{a} - \frac{1}{2}\Gamma_{m}^{ab}\dot{S}_{ab}.$$
 (28)

We use the same trick as before and express \dot{P}_a with the help of $\mathrm{D}P_a$ and \dot{S}_{ab} with $\hat{\mathrm{D}}S_{ab}$. The result is

$$DP_m - T^a_{\ mi} u^i P_a + K^a_{\ im} u^i P_a = -\frac{1}{2} \hat{R}^{ab}_{\ mi} S_{ab} u^i + \frac{1}{2} \hat{\Gamma}^{ab}_{\ m} [\hat{D} S_{ab} - u_b P_a + u_a P_b].$$
 (29)

It is easily shown, using $DP_m = \hat{D}P_m - K^l_{mi}P_lu^i$ and $T^l_{mi} = -2K^l_{[mi]}$, that the left hand side is just $\hat{D}P_m$.

In order for the equation to be Lorentz gauge invariant, we have to require

$$\hat{D}S_{ik} = u_k P_i - u_i P_k,\tag{30}$$

and then the momentum equation reduces to

$$\hat{\mathbf{D}}P_m = -\frac{1}{2}R^{lk}{}_{im}S_{lk}u^m. \tag{31}$$

If we define the mass through $P_i u^i = m$ and require $u_i u^i = 1$, we can derive from (30) the relation

$$P_i = mu_i + \hat{D}S_{ik}u^k. (32)$$

Equations (30) and (31), together with (32) are exactly the equations derived by Papapetrou in [2]. The equations derived in the previous paragraph differ by the second term in (32), which, as we have said, is of order S^2 .

The derivation used in this paragraph shows clearly that the same equations are valid in every type of Riemann-Cartan spacetime. In the final equations, there is no torsion involved. The same equations hold, for instance, in a teleparallel theory. Torsion effects will only arise if we consider particles with intrinsic spin.

4 Particles with intrinsic spin

Particles with intrinsic spin require a more careful treatment, because one can easily run into problems, as has been clarified in [4]. In this section, we have in mind elementary

particles (extended bodies with intrinsic spin will be treated in the next section). Therefore, it seems, at first sight, obvious, to treat them as monopole particles, without dipole and higher order moments. Let us briefly review the Papapetrou method, as it has been applied in a Riemann-Cartan framework in [4] and [5], neglecting higher order poles.

We start from equations (18) and (26). It is convenient to write them with spacetime indices only, and to use the following form

$$(e\sigma^{ikm})_{,m} = -2eT^{[ik]} - \Gamma^{i}_{lm}e\sigma^{lkm} - \Gamma^{k}_{lm}e\sigma^{ilm}$$
(33)

$$(eT_{k}^{m})_{,m} - \hat{\Gamma}_{km}^{l} eT_{l}^{m} - K_{k}^{im} eT_{mi} = \frac{1}{2} eR_{l}^{il} {}_{mk} \sigma_{il}^{m},$$
(34)

where $e = \det e_m^a = \sqrt{-g}$. Then, one considers a worldline X^m and develops all the fields around this worldline, $x^m = X^m + \delta x^m$. Apart from (33) and (34), consider the following equations

$$(e\sigma^{ikm}x^n)_{,m} = e\sigma^{ikn} + x^n(e\sigma^{ikm})_{,m} \tag{35}$$

$$(eT_k^m x^i)_{,m} = eT_k^i + x^i (eT_k^m)_{,m}. (36)$$

Integrating over three dimensional space and neglecting integrals containing δx^m (monopole approximation), one derives the following equations from (35) and (36) (for details, see [4]):

$$u^{i}P_{k} = u^{0} \int eT_{k}^{i} d^{3}x, \quad \text{with} \quad P_{k} = \int eT_{k}^{0} d^{3}x, \tag{37}$$

$$\sigma^{lm}u^k = u^0 \int e\sigma^{lmk} d^3x, \text{ with } \sigma^{ik} = \int e\sigma^{ik0} d^3x,$$
 (38)

where $u^i = dX^i/d\tau$. It is easy to show that σ^{ik} and P_k as defined above are tensors in the monopole approximation. Using those relations in (33) and (34), we arrive at the following equations

$$D\sigma^{ik} = (P^i u^k - P^k u^i) (39)$$

$$\hat{D}P_k - K^{im}_{\ k} u_m P_i = \frac{1}{2} R^{il}_{\ mk} \sigma_{il} u^m. \tag{40}$$

This completes the Papapetrou analysis of the monopole particle with intrinsic spin. However, as has been clarified in [4], there are problems with the above approach. Indeed, the relation (38) cannot be considered to be generally valid. This is easily seen by considering the Dirac particle, where σ^{ikl} is totally antisymmetric. For such a spin density,

the relation (38) leads to $\sigma^{ik} = 0$, as can be seen by setting one index to zero and using the antisymmetry properties of σ^{ikl} .

Applying WKB methods on Lagrangians for particles with integer and half-integer spins, the following relation that replaces (38) was derived in [18]

$$u^{0} \int e\sigma^{lmk} d^{3}x = \sigma^{lm}u^{k} + \frac{1}{2s} [\sigma^{kl}u^{m} + \sigma^{mk}u^{l}], \tag{41}$$

for a particle with spin s. This reduces to (38) in the limit of large spin, which coincides with the usual result for the Weyssenhoff spin fluid. Thus, (38) can be considered to hold for macroscopic spin polarized bodies, whereas elementary particles obey (41). For spin one-half, we get from (41) a totally antisymmetric spin density.

In order to avoid the relation (38), that arises in the monopole approximation of the Papapetrou approach, the authors of [4] concluded that one has to include dipole moments of the particle, i.e. to introduce a classical spin besides the intrinsic spin. The same has been done in [5], where the Mathisson-Papapetrou equations in Riemann-Cartan space were derived for the first time. This seems a rather strange concept for an elementary particle and is problematic from a practical point of view. How can one discern experimentally the spin and the rotational momentum of an elementary particle? How can we define the intrinsic spin of an electron if it is not the total angular momentum in the particle's rest frame? Therefore, throughout this article, we take the point of view that the intrinsic spin of an elementary particle corresponds to its total angular momentum in the rest frame of the particle. There are possible objections to this, especially when one considers compound particles, like the proton for instance. In such cases, one could say that the spin is partly due to the spin of the constituent quarks and partly of orbital nature (some kind of rotation of the quarks around the center of mass of the proton, in a semiclassical picture). Indeed, experiments are carried out that aim at determining these different parts. However, the fact that the total spin appears always to be exactly 1/2 leads us to believe that the proton as a whole can be treated quantum mechanically as a spin 1/2 particle. If the spin is partly of non-intrinsic nature, one has to answer the question why the orbital part cannot be transferred to other particles and why it always sums up with the intrinsic part to exactly 1/2. The final answer to this question, however, should belong to the experiment. To this aim, the full equations, containing intrinsic spin as well as dipole correction terms, as derived in [4] and [5], could eventually be used.

In order to avoid the problem with equation (38) without taking into account dipole terms, we will choose another way of dealing with (semiclassical) elementary particles. Having in mind that the connection $\Gamma^{ab}_{\ m} = \hat{\Gamma}^{ab}_{\ m} + K^{ab}_{\ m}$ couples linearly to the spin density in elementary particle Lagrangians, we conclude from the fact that the spin density is of

the form (41), that the spin-torsion coupling will be of the form

$$K^{ik}_{l}[\sigma_{ik}u^{l} + \frac{1}{2s}\sigma^{l}_{i}u_{k} + \frac{1}{2s}\sigma^{l}_{k}u_{i}] = K^{*ik}_{l}\sigma_{ik}u^{l}.$$
(42)

The left hand side determines K^{*ik}_{l} to which we will refer as the effective torsion

$$K^{*ik}_{l} = K^{ik}_{l} + \frac{1}{2s}K_{l}^{ik} + \frac{1}{2s}K_{l}^{k}^{i}. \tag{43}$$

Only this part of the torsion will couple to the spin. For a macroscopic spin-polarized body, $K^{*ik}{}_l$ reduces to the full $K^{ik}{}_l$ and for the Dirac particle (s=1/2), it reduces to the totally antisymmetric part of the torsion. All quantities formed from the effective torsion will be denoted with a star, especially $\Gamma^{*ab}{}_m = \hat{\Gamma}^{ab}{}_m + K^{*ab}{}_m$.

Let us now derive the equations of motion for spin and position for a point particle with intrinsic spin. If we suppose that in a purely Riemannian space, the particle with intrinsic spin will behave just like a particle with classical spin, we have to introduce the term $-\frac{1}{2}\hat{\Gamma}^{ab}_{m}\sigma_{ab}u^{m}$ into our Lagrangian (see (27)). If we add to this the spin-torsion coupling (42), we are led to the following Lagrangian

$$L = e_i^a P_a u^i - \frac{1}{2} \Gamma^{*ab}_{\ m} \sigma_{ab} u^m. \tag{44}$$

The ultimate physical justification for the use of a specific Lagrangian can only be the correctness of the equations of motion derived from it.

Just as in the previous sections, we readily derive the Euler-Lagrange equations in the form

$$D^* P_m - T^{*a}_{mi} u^i P_a = -\frac{1}{2} R^{*ab}_{mi} \sigma_{ab} u^i + \frac{1}{2} \Gamma^{*ab}_{m} [D^* \sigma_{ab} - P_a u_b + P_b u_a]. \tag{45}$$

Requiring the gauge invariance under a local Lorentz transformation, we finally get

$$D^* P_m - T^{*a}_{mi} u^i P_a = -\frac{1}{2} R^{*ab}_{mi} \sigma_{ab} u^i, \tag{46}$$

$$D^* \sigma_{ik} = P_i u_k - P_k u_i. \tag{47}$$

In order to compare with the classical geodesics, we can rewrite the first equation as

$$\hat{D}P_m - \frac{1}{2}K_{lim}^*(P^lu^i - P^iu^l) = -\frac{1}{2}R_{mi}^{*ab}\sigma_{ab}u^i.$$
(48)

For a macroscopic spin-polarized body, these equations agree completely with the result from the monopole approximation in [4], i.e. with equations (39) and (40). However,

such a macroscopic body will also possess dipole moments and the equations have to be adopted to that case (see next section). For the spin half particle (totally antisymmetric torsion), the results agree, if we ignore again the quantity $P^{[i}u^{k]}$, which is of order σ^2 , with the results from the WKB analysis of the Dirac equation, as carried out in [19]. Also, in the same approximation, there is an agreement with the results of [20] for the spin 3/2 particle and of [21] and [22] for the spin 1 or Proca particle.

Thus, there is no need to consider dipole moments while dealing with elementary particles. The Lagrangian (44) provides a very comfortable method of deriving the correct equations of motion without reference neither to the gravitational field equations, nor to the specific matter Lagrangian for the particle under consideration. The only additional information we needed was relation (41).

Equations (46) and (47) were derived in [7] for the case of a purely Riemannian space $(T_{lm}^a = 0)$, following a slightly different procedure. The term $\Gamma^{ab}_{\ m}\sigma_{ab}u^m$ was interpreted as the Hamiltonian part of a Routhian R and the spin evolution equations were then derived using the Heisenberg equations $\frac{\mathrm{d}\sigma_{ab}}{\mathrm{d}\tau} = i[R, \sigma_{ab}]$ and the commutation relations for the spin tensor. However, in order to get the correct equations, they had to redefine the spin tensor and to modify slightly the Routhian. This was due to the fact that their Lagrangian part was taken to be of the form $m\sqrt{u_iu^i}$. On the other hand, if we start with the form $e_i^a P_a u^i$ and suppose that P_a and σ_{ab} obey the commutation relations of the Poincaré algebra, one gets the correct precession equation following the same procedure as in [7]. We can see this as an independent confirmation of our method, since the mere requirement of Lorentz gauge invariance leads to the same result.

In [8], a different method was used, involving additional variables to describe the spin degrees of freedom. However, although initially intended to describe elementary particles in Riemann-Cartan spacetimes, the equations actually correspond to the special case $s \to \infty$. The same holds true for the equations given in [23] and [24]. Another, more recent attempt to generalize the multipole formalism to Riemann-Cartan geometry can be found in [25].

5 Macroscopic bodies with intrinsic spin

Finally, we are ready to consider macroscopic spin-polarized bodies. This is probably the experimentally most important case, since gravitational effects on elementary particles are usually very small. (See however [26]-[28] for experiments involving gravity on a quantum mechanical scale.) In this section, we have in mind an extended body, like a neutron star, for which the spins of the constituent particles are (partially or fully) aligned. In addition, the body may rotate. From the last two sections, we may expect a coupling of

the classical spin S_{ab} to the Riemannian connection $\hat{\Gamma}^{ab}_{m}$, and a coupling of the intrinsic spin σ_{ab} to the effective connection, which coincides in the macroscopic case with the full connection $\Gamma^{*ab}_{m} = \Gamma^{ab}_{m}$. Hence we start with

$$L = e_i^a P_a u^i - \frac{1}{2} \hat{\Gamma}^{ab}{}_m S_{ab} u^m - \frac{1}{2} \Gamma^{ab}{}_m \sigma_{ab} u^m.$$
 (49)

We derive the following equations

$$\hat{D}P_{m} - \frac{1}{2}K_{lim}(P^{l}u^{i} - P^{i}u^{l}) = -\frac{1}{2}\hat{R}^{lk}{}_{mi}S_{lk}u^{i} - \frac{1}{2}R^{ab}{}_{mi}\sigma_{ab}u^{i}
+ \frac{1}{2}\Gamma^{ab}{}_{m}[D\sigma_{ab} - P_{a}u_{b} + P_{b}u_{a}] + \frac{1}{2}\hat{\Gamma}^{ab}{}_{m}\hat{D}S_{ab}. \quad (50)$$

We can now regroup all the terms proportional to $\Gamma^{ab}_{\ m}$, or equivalently to $\hat{\Gamma}^{ab}_{\ m}$, and then require Lorentz covariance. The result is

$$\hat{D}P_m - \frac{1}{2}K_{lim}(P^lu^i - P^iu^l) = -\frac{1}{2}\hat{R}^{lk}{}_{mi}S_{lk}u^i - \frac{1}{2}R^{ab}{}_{mi}\sigma_{ab}u^i - \frac{1}{2}K^{ik}{}_{m}\hat{D}S_{ik}$$
 (51)

and for the spin evolution

$$D\sigma_{ik} + \hat{D}S_{ik} = P_i u_k - P_k u_i. \tag{52}$$

We can use (52) in (51) to eliminate the apparent coupling of the torsion to the rotational moment in the last term. The equations are more or less what could have been expected right from the start, except maybe for the last term in (51). Our equations agree with those derived in [4] (for the macroscopic limit $s \to \infty$) using the Papapetrou method in the dipole approximation.

These equations alone are of course not sufficient to determine the behavior of the body. An additional relation between the internal and the classical spin has to be assumed. For a simple model of a neutron star, let us suppose a strong coupling of the form $S_{ik} = g\sigma_{ik}$ with constant g. In this case, we can write (52) in the form

$$\frac{1+g}{g} \left[\hat{D}S_{ik} + \frac{1}{1+g} K^l_{im} S_{lk} u^m + \frac{1}{1+g} K^l_{km} S_{il} u^m \right] = P_i u_k - P_k u_i.$$

Now, if we introduce the connection

$$\tilde{\Gamma}_{lm}^{i} = \hat{\Gamma}_{lm}^{i} + \frac{1}{1+g} K^{i}_{lm}, \tag{53}$$

we can write

$$\frac{1+g}{g}\,\tilde{\mathbf{D}}S_{ik} = P_i u_k - P_k u_i.$$

Multiplying with u^i , defining the mass as $m = P_i u^i$ and using $1 = u_i u^i$, we find

$$P_k = mu_k + \frac{1+g}{g} \tilde{D}S_{ki} u^i, (54)$$

and the precession equation finally reads

$$\tilde{\mathbf{D}}S_{ik} = (u_k \tilde{\mathbf{D}}S_{il} - u_i \tilde{\mathbf{D}}S_{kl})u^l. \tag{55}$$

Thus, the intrinsic spin (and also the classical as well as the total spin) is Fermi-Walker transported with respect to the connection (53).

Introducing the constraint $S_{ik} = g\sigma_{ik}$ directly into (49), we see that the connection (32) governs the whole propagation, and not only the spin evolution. The equation for momentum propagation is easily derived then.

We used the constraint $S_{ik} = g\sigma_{ik}$ mainly as an illustrative example. For realistic neutron star models, you can consult [29] and [30] for instance. You should however have in mind that these models rely strongly on general relativity as underlying gravitational theory. Since the use of our equations of motion naturally suppose that the underlying theory is a Poincaré gauge theory with dynamical torsion fields, these models have to be revisited in view of the modified gravitational interaction. This might lead to severe changes, since the interior of neutron stars is governed by very strong fields. To derive such models is beyond the scope of this article.

6 The Dirac particle

In order to gain more confidence in the procedure we followed in section 4, mainly the in introduction of the effective torsion (43), we will have a closer look at the most important example, that of a spin 1/2 particle and present an alternative derivation of the equations of motion.

The Dirac particle is described by the following Lagrangian density

$$\mathcal{L} = \frac{i}{2} \left[\bar{\psi} \gamma^m D_m \psi - \bar{D}_m \bar{\psi} \gamma^m \psi \right] + m \bar{\psi} \psi, \tag{56}$$

where $D_m \psi = (\partial_m - \frac{i}{2} \Gamma^{ab}_{\ m} \sigma_{ab}) \psi$, $\bar{D}_m \bar{\psi} = \partial \bar{\psi} + \frac{i}{2} \Gamma^{ab}_{\ m} \bar{\psi} \sigma_{ab}$ and $\gamma^m = e^m_a \gamma^a$, with γ^a the usual Dirac matrices and $\sigma_{ab} = \frac{i}{4} [\gamma_a, \gamma_b]$. The Dirac equation reads $(T_m = T^i_{\ mi})$

$$i(\gamma^m D_m - \gamma^m T_m)\psi = m\psi. (57)$$

The stress-energy tensor $(2e)^{-1} \delta(e\mathcal{L})/\delta e_a^i$ is found to be

$$T_i^a = \frac{i}{4} [\bar{\psi}\gamma^a D_i \psi - \bar{D}_i \bar{\psi}\gamma^a \psi]. \tag{58}$$

Several remarks are in order at this point. It is a well known fact that only the totally antisymmetric part of the torsion couples to the Dirac particle. This follows immediately from (57), which can be written as

$$i\gamma^m D_m^* \psi = m\psi, \tag{59}$$

where D_m^* is constructed from the connection (43) with s = 1/2. Furthermore, the Lagrangian (56) is numerically equal to

$$\mathcal{L}^* = \frac{i}{2} \left[\bar{\psi} \gamma^m D_m^* \psi - \bar{D}_m^* \bar{\psi} \gamma^m \psi \right] + m \bar{\psi} \psi. \tag{60}$$

This Lagrangian is preferable from the point of view that it incorporates a minimal coupling principle that can be applied equally well in the Lagrangian or in the field equations. However, (56) and (60), although they both lead to the same Dirac equation, are not completely equivalent, since for the latter the stress-energy tensor becomes

$$T^{*a}_{\ i} = \frac{i}{4} [\bar{\psi}\gamma^a D_i^* \psi - \bar{D}_i^* \bar{\psi}\gamma^a \psi]. \tag{61}$$

which differs from (58) in the non-axial torsion parts contained in the latter. Once again (61) seems preferable, since the Dirac particle does not couple to these torsion parts. However, it makes of course a difference which of these tensors actually represents the source term of the gravitational equations. If we remain in the logic of a Lorentz gauge theory and have in mind that there might be additional sources with different spin (coupling to other torsion parts), the only consistent way is to write down only Lagrangians that contain the full connection Γ^{ab}_{m} , as is the case for (56), even though they might contain parts which do not couple to the particle in question. Otherwise we can not carry out the variation with respect to Γ^{ab}_{m} .

However, as long as we are interested only in the equations of motion for the Dirac particle (or their semiclassical limit), and not in the gravitational field equations, the use of either $\Gamma^{ab}_{\ m}$ or $\Gamma^{*ab}_{\ m}$ is equivalent. Equations (57) and (59) are exactly the same. Under this aspect, the use of $\Gamma^{*ab}_{\ m}$ is certainly preferable. It can be seen as the physically relevant connection since it is the connection that really couples to the Dirac particle.

What does this have to do with our equations of motion? Well, our point is that, if we start, as in the usual Papapetrou procedure, from the divergence relations for the stress-energy tensor and the spin density, and then define the momentum as $P_i = \int eT_i^0 d^3x$,

this is already very unfortunate right from the start, since this momentum vector contains implicitly non-axial torsion parts from (58) which should not couple to the Dirac particle at all. This explains at least partly why the monopole approximation does not lead to correct results for elementary particles [4]. A first step should be to look for a divergence relation involving the tensor T^{*a}_{i} and carry out the Papapetrou method on this tensor.

The above considerations are also useful in another aspect. The use of the covariant derivative D_m^* allows us to generalize the proper time formalism, i.e. the covariant generalization of the Heisenberg equations, to the case of a Riemann-Cartan space. We will briefly discuss the concepts and refer to [31]-[34] and the references therein for further details. Notably, in [34], the resulting equations were used to extract for the first time the precession frequency of the spin of a Dirac particle in an axial torsion field.

One starts with the operator H whose eigenvalue is (the negative of) the mass, $H\psi = -m\psi$. (The mass is the only available covariant generalization of the classical energy, apart from the stress-energy tensor.) Then identify (for any operator A)

$$i[H, A] = \frac{\mathrm{d}A}{\mathrm{d}\tau},\tag{62}$$

where τ is interpreted as proper time along the semiclassical trajectory. In flat spacetime, the Hamiltonian is just $H = -\gamma^i p_i$, with $p_i = i\partial_i$ and EM fields can easily be taken into account by $p_i \to \pi_i = i\partial_i + eA_i$. With this in mind, and having a look at (59), we find that the Hamiltonian for our case has the form

$$H = -\gamma^m P_m, (63)$$

with

$$P_m = iD_m^* = i(\partial_m - \frac{i}{2}\Gamma_m^{ab*}\sigma_{ab}), \tag{64}$$

where Γ^{ab*}_{m} is again the connection which contains only the totally antisymmetric part of the torsion. For the velocity operator, we find

$$u^{i} = \frac{\mathrm{d}x^{i}}{\mathrm{d}\tau} = i[H, x^{i}] = \gamma^{i}. \tag{65}$$

We thus have the expected relation $H = -u^m P_m = -m$. Let us have a look at the Heisenberg algebra for momentum and position operators. We have:

$$[x^i, x^k] = 0, \quad [P_i, x^k] = i\delta_i^k,$$

$$[P_i, P_k] = \frac{i}{2} R^{*cd}_{ik} \sigma_{cd}.$$

This is in complete analogy with the EM case (just recall the relation $[\pi_i, \pi_k] = ieF_{ik}$). A useful relation is also $[P_i, \gamma^k] = -i\gamma^l \Gamma^{*k}_{li} = -i\gamma^l (T^{*k}_{il} + \Gamma^{*k}_{il})$.

Writing down the commutation relations for P_m , γ^i and σ_{ab} with the Hamiltonian, it is straightforward to derive the following equations of motion

$$D^* \sigma_{lk} = P_l u_k - P_k u_l \tag{66}$$

$$D^* P_m = -\frac{1}{2} R^{*cd}_{mk} u^k \sigma_{cd} + T^{*k}_{ml} u^l P_k$$
 (67)

$$D^*u^i = -\sigma^{mi}P_m. (68)$$

Of course, D* appearing here is again the proper time derivative acting on tensors (not on spinors) with Γ^{*i}_{lm} . Relation (68) has been related to the zitterbewegung and has no classical counterpart (see [31]). The other two are exactly our equations (46)-(47).

Both derivations, the one of section 4 and the present one, although conceptionally quite different, lead to the correct results of the WKB analysis [19]. In both cases, the crucial step was to use the correct connection containing only the part of the torsion that effectively couples to the spin density.

7 Spin-polarized test body in Brans-Dicke theory

Finally, we will take a look at a specific example involving the equations derived in section 5. Since in most practical cases, intrinsic spin effects will be very small (see however [26]-[28]), we turn to a theory that gives rise to a torsion field even in the absence of a spinning source, namely Brans-Dicke theory generalized to Riemann-Cartan geometry. The theory is based on the action

$$S = \int e \left(\varphi R - \frac{\omega + \frac{3}{2}}{\varphi} \varphi_{,m} \varphi^{,m} - \mathcal{L}_m \right) d^4 x.$$
 (69)

It is easily shown by solving the equation that arises from variation with respect to the connection, that in the case of a spinless matter Lagrangian (the source), we get a torsion field of the form (cf. [35], [36]),

$$K^{lm}_{\ i} = \frac{1}{2\varphi} (\delta^l_i \varphi^{,m} - \delta^m_i \varphi^{,l}). \tag{70}$$

The field φ and the metric g_{ik} are subject to the classical Brans-Dicke field equations, as can be seen by plugging (49) into the equations for φ and e_m^a . The torsion (70) is entirely determined by its trace alone. It is often claimed (probably having the Dirac particle in

mind), that such a torsion field is pure gauge, and does not lead to physical consequences. However, if the spin of the test particle moving in this field is not totally antisymmetric, there might well be a measurable effect.

We are interested in a macroscopic test body with intrinsic spin moving in the field of a classical, spherically symmetric source. (For equations of motion of a classical point mass in Brans-Dicke theory, see for instance [37] and references therein.) For simplicity, although not quite realistic, let us suppose that the test body does not rotate. The spin evolution is then described by equation (52) with $S_{ik}=0$. We proceed as in [4] and introduce the spin vector $\sigma^i=\frac{1}{2}\eta^{iklm}u_k\sigma_{lm}$ where $\eta^{iklm}=\frac{1}{e}\varepsilon^{iklm}$. If we impose the condition $\sigma_{kl}u^k=0$ we can invert to $\sigma^{ik}=-\eta^{iklm}u_l\sigma_m$. Omitting the higher order terms, we are left with

$$D\sigma^{i} = \hat{D}\sigma^{i} + K^{i}{}_{lm}\sigma^{l}u^{m} = 0.$$
(71)

We are not interested in the momentum equation, because the deviations from the geodesics are expected to be very small.

Since the metric and scalar field are determined from the same equations as in classical Brans-Dicke theory, we can directly use the results from the post-Newtonian expansion as given in [38] for instance, and write

$$\varphi = 1 + c\frac{m}{r} \tag{72}$$

$$g_{00} = 1 - 2a \frac{m}{r} \tag{73}$$

$$g_{\alpha\beta} = \delta_{\alpha\beta}(-1 - 2b\frac{m}{r}). \tag{74}$$

The coefficients read

$$a = 1 - \frac{s}{2 + \omega} \tag{75}$$

$$b = \frac{1+\omega}{2+\omega} \left(1 + \frac{s}{1+\omega}\right) \tag{76}$$

$$c = \frac{1-2s}{2+\omega}. (77)$$

Here, m is the mass of the source body and s its sensitivity (see [38] and references therein). The sensitivity is basically the binding energy per unit mass. Recall that the static spherically symmetric solution of Brans-Dicke theory is not a one parameter solution as in general relativity, but depends on more properties of the central source. In the parameterized post-Newtonian formalism, it turned out that an astrophysical body can conveniently be described by its mass and its sensitivity. The latter takes values from

 10^{-6} for a usual star (like the sun) to one half for black holes. Neutron stars have values around 0.1 - 0.3 (see [38]).

We now introduce (72)-(74) into (70) and (71). From the construction of σ^i , we have $\sigma^i u_i = 0$. Using this relation and parameterizing the equation with the time coordinate, we get the result

$$\frac{d\vec{\sigma}}{dt} = \frac{m}{r^3} \left[\frac{a + 2b + c}{2} (\vec{L} \times \vec{\sigma}) - \frac{a}{2} (\vec{x}(\vec{\sigma} \cdot \vec{v}) + \vec{v}(\vec{\sigma} \cdot \vec{x})) + b\vec{\sigma}(\vec{x} \cdot \vec{v}) + \frac{c}{4} (\vec{v}(\vec{x} \cdot \vec{\sigma}) + \vec{x}(\vec{v} \cdot \vec{\sigma})) \right],$$
(78)

with $\vec{L} = \vec{v} \times \vec{x}$ the orbital angular momentum per unit mass of the test body. We have separated terms antisymmetric and symmetric in (\vec{x}, \vec{v}) . Just as in general relativity, the latter can easily be shown to have mean values over one orbit that vanish for closed orbits and that are negligible for the quasi-closed orbits we consider here (i.e. orbits closed, up to a small spin precession correction).

Thus, the relevant precession equation reads:

$$\frac{\mathrm{d}\vec{\sigma}}{\mathrm{d}t} = \frac{m}{r^3} \frac{a + 2b + c}{2} \left(\vec{L} \times \vec{\sigma} \right). \tag{79}$$

The torsion corrections come from the term in c.

In the limit $\omega \to \infty$, the parameters take the values a=1,b=1,c=0 and we get the well known factor $\frac{3}{2}$ from general relativity in the spin precession equation, but this time for intrinsic spin. This confirms once again the result that in general relativity, a particle with intrinsic spin behaves just like a particle with classical spin. More precisely, in the $\omega \to \infty$ limit, the theory goes over not to general relativity, but to Einstein-Cartan theory (see [39] for a recent review). Since we consider only spinless sources however, there is no difference in the resulting solutions. (Even for a spinning source, the differences vanish outside the source.)

In Brans-Dicke theory, the only case where the torsion vanishes is when $s = \frac{1}{2}$, i.e. when the source is a black hole. In this case, c = 0 and the scalar field is constant.

The interesting cases are of course when $c \neq 0$. The largest influence of the torsion is found for ordinary stars, when $s \approx 0$. We can evaluate for this case the ratio of the torsional part and the other parts for different values of ω , for instance

$$\omega \approx 500 \quad \rightarrow \quad \frac{c}{a+2b} \approx 1, 4 \cdot 10^{-3}$$

$$\omega \approx 1000 \quad \rightarrow \quad \frac{c}{a+2b} \approx 6 \cdot 10^{-4}.$$

These results can be interpreted in two ways: Within the framework of Brans-Dicke theory with torsion, they tell us the difference in the precession equation of a classical rotating test body (which does not couple to torsion) and a spin polarized test body. Thus, we can in principle determine, whether a body is just rotating or possesses intrinsic spin.

On the other hand, for a spin-polarized body, they tell us the difference between the precession in classical Brans-Dicke theory without torsion for which the spin precession is the same for rotational and intrinsic spin and the precession of the same body in Brans-Dicke theory in a Riemann-Cartan geometry.

The results are easily generalized to the more realistic case of a rotating test body using (55). More work is involved to get results for the two body problem (for instance a binary consisting of two rotating, spin polarized neutron stars). However, as far as the test body is concerned, we have shown that already the pure spin terms are very small. If there is an additional rotation, the relevant connection will be of the form (53), where for a realistic neutron star model $(S_{ik} >> \sigma_{ik})$, we will have a large g and therefore, the torsion effects will become even smaller.

As far as the source is concerned, in a realistic neutron star model, the rotational effects will by far dominate the spin effects, so that the field will essentially be a Kerr-like analogue of Brans-Dicke theory. In any case, the torsion outside the source will again be produced by the scalar field only, because the spin of the source gives rise only to non-dynamical torsion fields (vanishing outside of the spin density), just as in Einstein-Cartan theory. A discussion of geodesics and autoparallels (for classical point particles) in such a Kerr-Brans-Dicke geometry can be found in [40]. These results may easily be generalized to take into account the spin effects.

Nevertheless, the analysis shows that there are physical consequences of the vector torsion, although very small, and we cannot consider it as a pure gauge field.

8 Symmetries of the Lagrangian

Throughout this paper, we have obtained the spin evolution equation by requiring the momentum equation that results from the Euler-Lagrange equations of the Lagrangian under investigation to be either covariant under spacetime diffeomorphisms in the Riemannian case or under local Lorentz transformations in the Riemann-Cartan case. One might wonder whether the same argument can be applied directly to the Lagrangian, i.e. if the requirement of the invariance of the Lagrangian under those transformations leads to the equation for the spin evolution. We will investigate this subject case by case.

Let us begin with the Lagrangian (8). The only symmetry that occurs in Riemannian

geometry is the diffeomorphism invariance, which can be written in the infinitesimal form

$$x^i \to x^i + \varepsilon^i(x).$$
 (80)

This contains Lorentz transformations ($\varepsilon^i(x) = \varepsilon_{ik}x^k$ with ε_{ik} antisymmetric and constant) as well as translations ($\varepsilon^i = a^i$ with constant a^i) as subcases. The requirement of the Lagrangian to be invariant (up to a proper time derivative) under (80) leads to the Euler-Lagrange equations if the Lagrangian is a function of x^i and u^i alone. However, in the case of (8), there are additional quantities S_{ik} that transform as a tensor under coordinate transformations. Therefore, the requirement $\delta L = 0$ leads to the equation $\dot{S}_{ik} = 0$, as is easily shown. This cannot be used as a constraint; it is neither covariant, nor physically acceptable as equation of motion. Therefore, we can conclude that the Lagrangian (8) is not a scalar. Covariant under (80) are only the final equations of motions.

However, in the case of the Lagrangians (27), (44) and (49), no such problems occur, since the quantities P_a and S_{ab} transform as scalars under coordinate transformations. The requirement for L to be invariant under (80) simply leads to the Euler-Lagrange equations. In those cases, the scalar character of the Lagrangians is obvious anyway, since the connection as well as the tetrad field transform as covariant spacetime vectors.

Let us turn to the gauge transformations. We consider the Lagrangian

$$L = e_i^a P_a u^i - \frac{1}{2} \Gamma^{ab}_{\ m} S_{ab} u^m, \tag{81}$$

which stands exemplary for the Lagrangians (27), (44) or (49). (Remember that the connections, $\Gamma^{ab}_{\ m}$, $\hat{\Gamma}^{ab}_{\ m}$ and $\Gamma^{*ab}_{\ m}$ all transform in the same way.) Before turning to the Lorentz transformations, let us make a remark on gauge translations.

In Poincaré gauge theory, the ten gauge fields Γ^{ab}_{m} (Lorentz) and Γ^{a}_{m} (translations) alone are not sufficient to write down an invariant Lagrangian. This is due to the fact that the translational field Γ^{a}_{m} does not transform homogeneously as Lorentz vector. (It might even vanish for instance and is certainly not invertible, and thus not suitable to be used as a tetrad field.) One therefore introduces an additional field ξ^{a} (the coset parameters) transforming as $\xi^{a} \to \xi^{a} + \varepsilon^{a}_{\ b} \xi^{b} + \varepsilon^{a}$ under Poincaré transformations and defines the tetrad field as $e^{a}_{m} = \Gamma^{a}_{\ m} + D_{m} \varepsilon^{a}$. Then, this tetrad field transforms homogeneously as Lorentz vector (i.e. it is invariant under translations). Therefore, every Lagrangian expressed in terms of the fields $\Gamma^{ab}_{\ m}$ and e^{a}_{m} is trivially invariant under translations. The translational gauge symmetry is hidden. From this point, we started our discussion of Riemann-Cartan geometry in section 3. The details and the foundation of this interpretation of Poincaré gauge theory can be found in [41].

This leaves us with the local Lorentz transformations. The field transformations as given in section 3 can be written for infinitesimal Lorentz transformations with parameters

 $\varepsilon^{ab}(x) = -\varepsilon^{ba}$ in the form

$$\delta\Gamma^{ab}_{\ m} = -D_m \varepsilon^{ab} \quad \text{and} \quad \delta e^a_m = \varepsilon^a_{\ b} e^b_m.$$
 (82)

Let us for one moment assume that the quantities P_a and S_{ab} in (81) are invariant (i.e. scalars). Then, the change in the Lagrangian reads

$$\delta L = \varepsilon^a_{\ b} e^b_m P_a u^m + \frac{1}{2} \mathcal{D}_m \varepsilon^{ab} S_{ab}. \tag{83}$$

Dropping a total proper time derivative and using the antisymmetry of ε^{ab} , this can be written in the form

$$\delta L = -\frac{1}{2}\varepsilon^{ab}(DS_{ab} - (P_a u_b - P_b u_a)). \tag{84}$$

Thus, the requirement $\delta L = 0$ leads to the correct spin evolution equation $DS_{ab} = P_a u_b - P_b u_a$.

However, this result was derived assuming that S_{ab} and P_a are invariant under Lorentz transformations. Since they are ultimately related to spacetime tensors via $S_{ik} = e^a_i e^b_k S_{ab}$ and $P_i = e^a_i P_a$, we know that they have to transform as Lorentz tensors, as has been assumed throughout the article. Including this additional variation in (83) leads to $\delta L = -\frac{1}{2} \varepsilon^{ab} \dot{S}_{ab}$. If we require this to vanish, we are let to equation $\dot{S}_{ab} = 0$, which not only is not covariant, but has also to be rejected on physical grounds.

Therefore, we conclude that the Lagrangians (27), (44) and (49) are not invariant under local Lorentz gauge transformations. Covariant are only the final equations of motion.

9 Conclusion

Our main result is summarized in the Lagrangians (44) for elementary particles and (49) for macroscopic bodies. Varying with respect to the coordinates and requiring gauge or diffeomorphism invariance of the resulting equation, the equations of motion for spin and position are obtained very easily and without any ambiguities.

The method can be justified by the fact that the results are in agreement with the WKB analysis of elementary particles and with those obtained from the pole-dipole expansion following Papapetrou's method.

We did not discuss in general the final equations in this paper, because they are not new, except for the result that a rotating, spin-polarized body with a strong spin-rotation coupling $S = g\sigma$, couples to the connection given by (53). Nevertheless, we took the opportunity to show briefly that the vector-torsion arising in generalized Brans-Dicke

theory affects the spin precession of a macroscopic body and should not be referred to as pure gauge.

Finally, we showed that the Lagrangians under investigation are not scalars under the transformations under consideration, nor does the requirement for them to be invariant lead to the correct spin evolution equations. This means that the processes of varying with respect to the coordinates and requiring gauge or diffeomorphism invariance do not commute.

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