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ON MULTIPOLE EXPANSIONS IN THE THEORY OF ELECTROMAGNETIC RADIATION

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Synopsis

A new method is developed for expanding the electromagnetic field of radiating charges and currents in multipole components. Outside a sphere enclosing all sources, the field is represented in terms of Debye potentials which are shown to be closely related to the radial components of the electric and magnetic vectors. Attention is drawn to remarkably simple source representations of these radial components. A discussion of vector-potential versus Debye-potential representation of multipole fields is included. Only familiarity with ordinary vector calculus is required.

1. *Introduction*. In this paper we shall investigate the problem of multipole expansion of the electromagnetic field produced by a given volume distribution of electric currents varying harmonically with time and located in some bounded region of free space.

Let R denote any sphere of finite radius enclosing all currents. In addition to the right-handed system of rectangular coordinates x, y, z with origin at the centre of R, we introduce spherical polar coordinates r, ϑ, φ in the usual fashion. Vectors are printed in heavy type. A function of position may be denoted in either of three ways: $f(x, y, z)$, $f(r, \vartheta, \varphi)$, and $f(\mathbf{r})$. To distinguish between field points and integration points we shall attach a prime to the coordinates of the latter; the same applies to the familiar differential operators as curl, div and grad. Without specification to the contrary, integrals that follow are volume integrals extended over the interior of R.

Throughout the paper a time dependence exp $(-ikct)$ is assumed but always suppressed. Physical quantities can be obtained from the associated complex-valued time-independent quantities of this paper by multiplying the latter quantities by exp $(-ikct)$ and then taking real parts. We shall use the Gaussian system of units.

Let $\mathbf{i}$ denote the given current density. Until further notice it will be assumed that $\mathbf{i}$ has continuous second-order partial derivatives everywhere and is zero on and outside the sphere R.

The electromagnetic field $\mathbf{E}$, $\mathbf{H}$ produced by the given currents $\mathbf{i}$ obeys Maxwell's equations

$$\left.\begin{aligned} &\operatorname{curl}\mathbf{H} + ik\,\mathbf{E} = (4\pi/c)\mathbf{i}, \quad \operatorname{div}\mathbf{H} = 0, \\ &\operatorname{curl}\mathbf{E} - ik\,\mathbf{H} = \mathbf{0}, \quad \operatorname{div}\mathbf{E} = 4\pi\varrho, \end{aligned}\right\} \tag{1}$$

in which ϱ is the charge density, connected with the current density by the equation of continuity

$$c^{-1} \operatorname{div} \mathbf{i} = ik\varrho. \tag{2}$$

As is well known, eqs (1) admit one and only one solution that is compatible with Sommerfeld's radiation condition at infinity. This unique solution is conveniently expressed in terms of an auxiliary vector:

$$\mathbf{H} = \operatorname{curl} \mathbf{A}, \;\; \mathbf{E} = (4\pi/ikc)\, \mathbf{i} - (1/ik) \operatorname{curlcurl} \mathbf{A}, \tag{3}$$

in which the so-called vector potential $\mathbf{A}$ is defined by

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c} \int \mathbf{i}(\mathbf{r}') \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}V'. \tag{4}$$

It may be recalled that the vector $\mathbf{A}$ is not uniquely defined by eqs (3) alone. These equations remain valid under a transformation of gauge, that is, if $\mathbf{A}$ is replaced by $\mathbf{A} + \operatorname{grad} \chi$, where χ is an arbitrary scalar function of position.

Equations (3) express $\mathbf{E}$ and $\mathbf{H}$ in terms of three scalar functions, namely, the three rectangular components of $\mathbf{A}$. The freedom we have in choosing a particular vector potential by a change of gauge is perhaps an indication that the same field $\mathbf{E}$, $\mathbf{H}$ may be derived from only *two* scalar functions, if not generally then at least so in special cases. For example, Green and Wolf [1]) recently succeeded in proving this for any region which is free of currents and charges. As was pointed out by Bremmer [2]), the theory of Green and Wolf can be extended so as to include regions with sources.

Another example is provided by the two scalar functions introduced a long time ago by Debye [3]) in his solution for the problem of diffraction of a plane electromagnetic wave by a sphere. Generalizing Debye's result we shall show that, if we confine ourselves to the space exterior to R, the field given by eqs (3) and (4) may also be derived from

$$\left.\begin{aligned} \mathbf{E} &= \operatorname{curlcurl} (\mathbf{r}\Pi_1) + ik \operatorname{curl} (\mathbf{r}\Pi_2), \\ \mathbf{H} &= -ik \operatorname{curl} (\mathbf{r}\Pi_1) + \operatorname{curlcurl} (\mathbf{r}\Pi_2), \end{aligned}\right\} \tag{5}$$

in which Π_1 and Π_2 are two scalar functions that are readily expressible in terms of the currents flowing in the interior of R.

For all points exterior to R, the field produced by the currents $\mathbf{i}$ can be conceived of as a superposition of electric and magnetic multipole fields whose sources are located at the centre of R. Once we have fixed the origin of coordinates and defined an appropriate basic set of multipole fields, we may ask how the coefficients of the multipole expansion can be calculated from the given currents. Several authors have succeeded in answering this question [4])–[9]). We strongly believe, however, that a rigorous and yet simple method as presented in this paper has several advantages over other methods.

It will be shown that the Debye potentials [8]) automatically lead to a simple solution. Our method may appeal to most physicists and engineers because of its simplicity, not requiring the usual techniques of quantum mechanics (group theory) and tensor calculus but only the familiar vector calculus. Our theory is based on some simple properties of the dot products $\mathbf{r}.\mathbf{E}$ and $\mathbf{r}.\mathbf{H}$. These two scalar functions (i) determine the whole field exterior to R in a unique way, (ii) have simple source representations, (iii) are solutions of the homogeneous wave equation in free space, and (iv) are closely related to and immediately found from the Debye potentials.

2. *A uniqueness theorem.* Let us consider a space domain D between two concentric spheres

$$0 < r_1 \leqslant r \leqslant r_2 \leqslant \infty, \; 0 \leqslant \varphi \leqslant 2\pi, \; 0 \leqslant \vartheta \leqslant \pi.$$

It will be assumed that D is free of currents and charges. Let there be in D an analytic electromagnetic field $\mathbf{E}$, $\mathbf{H}$ with vanishing radial components E_r and H_r. It then follows that $\mathbf{E}$ and $\mathbf{H}$ are identically zero within D.

To prove this, we write Maxwell's equations in free space (eqs (1) with $\mathbf{i} = \mathbf{0}$, $\varrho = 0$) in spherical components. If we then substitute $E_r = H_r = 0$ we are left with the following equations:

$$\begin{aligned} (\partial/\partial\vartheta)\,(\sin\vartheta\, H_\varphi) &= \partial H_\vartheta/\partial\varphi, & (\partial/\partial\vartheta)\,(\sin\vartheta\, E_\varphi) &= \partial E_\vartheta/\partial\varphi, \\ ikr\, E_\vartheta &= (\partial/\partial r)\,(rH_\varphi), & -\, ikr\, H_\vartheta &= (\partial/\partial r)\,(rE_\varphi), \\ -\, ikr\, E_\varphi &= (\partial/\partial r)\,(rH_\vartheta), & ikr\, H_\varphi &= (\partial/\partial r)\,(rE_\vartheta). \end{aligned}$$

The integration with respect to r can be carried out immediately. From the last four equations we get

$$\begin{aligned} rE_\vartheta &= A_1(\vartheta, \varphi)\, e^{ikr} + B_1(\vartheta, \varphi)\, e^{-ikr}, \\ rH_\varphi &= A_1(\vartheta, \varphi)\, e^{ikr} - B_1(\vartheta, \varphi)\, e^{-ikr}, \\ rH_\vartheta &= A_2(\vartheta, \varphi)\, e^{ikr} - B_2(\vartheta, \varphi)\, e^{-ikr}, \\ rE_\varphi &= -\, A_2(\vartheta, \varphi)\, e^{ikr} - B_2(\vartheta, \varphi)\, e^{-ikr}. \end{aligned}$$

Substitution of these expressions in the remaining two differential equations yields

$$(\partial/\partial\vartheta)\,(C \sin\vartheta) = \mp\, i\, \partial C/\partial\varphi, \;\; C = A_1 \pm iA_2 \text{ or } C = B_1 \pm iB_2. \qquad (6)$$

The general solution of eq. (6) is easy to obtain. Introducing the new variable $\psi = \mp \operatorname{logtan}(\frac{1}{2}\vartheta)$ and the function $f = C \sin\vartheta$, we readily get $\partial f/\partial\psi = i\, \partial f/\partial\varphi$ whose general solution is $f = g(\varphi + i\psi)$, where g is any function of the complex variable $\varphi + i\psi$. Consequently

$$C(\vartheta, \varphi) = (\sin\vartheta)^{-1}\, g(\varphi \mp i \operatorname{logtan} \tfrac{1}{2}\vartheta).$$

In our physical problem, the function g must be single-valued and analytic in the whole complex plane of $\varphi + i\psi$. At $\vartheta = 0$ and $\vartheta = \pi$, the function C must be finite, that is, $g(\varphi \pm i\infty) = 0$. Moreover it must be periodic in φ of period 2π. Hence g is uniformly bounded in the whole complex plane. From Liouville's theorem of the theory of complex functions it follows that g must be constant. Since it vanishes at infinity in the direction of the imaginary axis, it must be identically zero. Therefore all functions A and B are zero. This completes our proof that $\mathbf{E}$ and $\mathbf{H}$ are identically zero in D.

Equivalently we have proved that any electromagnetic field satisfying Maxwell's equations in the empty space between two concentric spheres is completely determined by its radial components E_r and H_r. In particular, the field $\mathbf{E}$, $\mathbf{H}$ of the currents $\mathbf{i}$ is outside R fully characterized by the dot products $\mathbf{r}.\mathbf{E}$ and $\mathbf{r}.\mathbf{H}$.

3. *Source representations of the radial components.* By straightforward differentiation in rectangular coordinates it is easy to verify that

$$\begin{aligned}(\Delta + k^2)\,(\mathbf{r}.\mathbf{v}) &= 2\,\mathrm{div}\,\mathbf{v} + \mathbf{r}.(\Delta\mathbf{v} + k^2\mathbf{v}) \\ &= 2\,\mathrm{div}\,\mathbf{v} + \mathbf{r}.(\mathrm{graddiv}\,\mathbf{v} - \mathrm{curlcurl}\,\mathbf{v} + k^2\,\mathbf{v}),\end{aligned} \tag{7}$$

in which $\mathbf{v}$ is any vector and Δ the three-dimensional Laplacian. If we apply this identity to $\mathbf{v} = \mathbf{H}$ and use Maxwell's equations (1) to eliminate all field vectors, we are left with the simple relation

$$(\Delta + k^2)\,(\mathbf{r}.\mathbf{H}) = -\,(4\pi/c)\,(\mathbf{r}.\mathrm{curl}\,\mathbf{i}). \tag{8}$$

Equation (8) shows that $\mathbf{r}.\mathbf{H}$ is a solution of the homogeneous wave equation at any point of free space.

The technique of solving eq. (8) is well known, as may be recalled from the usual derivation of eq. (4) for the vector potential $\mathbf{A}$. The only solution of eq. (8) that is compatible with Sommerfeld's radiation condition is given by

$$\mathbf{r}.\mathbf{H}(\mathbf{r}) = \frac{1}{c}\int (\mathbf{r}'.\mathrm{curl}'\,\mathbf{i})\,\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}V', \tag{9}$$

which is a remarkably simple representation of $\mathbf{r}.\mathbf{H}$ in terms of its sources $\mathbf{r}.\mathrm{curl}\,\mathbf{i}$.

An analogous expression for $\mathbf{r}.\mathbf{E}$ is found if eq. (7) is applied to $\mathbf{v} = \mathbf{E}$. The analogue of eq. (8) turns out to be

$$(\Delta + k^2)\,(\mathbf{r}.\mathbf{E}) = -\,4\pi\,\{(ik/c)\,\mathbf{r}.\mathbf{i} - \mathbf{r}.\mathrm{grad}\,\varrho - 2\varrho\}, \tag{10}$$

which on integration gives

$$\mathbf{r}.\mathbf{E}(\mathbf{r}) = \int \left(\frac{ik}{c}\,\mathbf{r}'.\mathbf{i} - \mathbf{r}'.\,\mathrm{grad}'\,\varrho - 2\varrho\right)\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}V'. \tag{11}$$

A simpler source representation for $\mathbf{r}.\mathbf{E}$ is obtained if eq. (7) is applied to $\mathbf{v} = \mathbf{E} - (4\pi/ikc)\,\mathbf{i}$. The differential equation becomes

$$(\Delta + k^2)\left\{\mathbf{r}.\left(\mathbf{E} - \frac{4\pi}{ikc}\,\mathbf{i}\right)\right\} = \frac{4\pi}{ikc}\,(\mathbf{r}.\mathrm{curlcurl}\,\mathbf{i}), \tag{12}$$

so that

$$\mathbf{r}.\mathbf{E}(\mathbf{r}) = \frac{4\pi}{ikc}\,\mathbf{r}.\mathbf{i} - \frac{1}{ikc}\int (\mathbf{r}'.\mathrm{curl}'\mathrm{curl}'\,\mathbf{i})\,\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}V'. \tag{13}$$

The source representations of $\mathbf{r}.\mathbf{E}$ and $\mathbf{r}.\mathbf{H}$ given in this section are valid for any point of space. In free space these dot products satisfy the homogeneous wave equation. For points exterior to R they can be expanded in a series of eigenfunctions of the wave equation by simply expanding the free-space Green's function $e^{ik|\mathbf{r}-\mathbf{r}'|}/|\mathbf{r}-\mathbf{r}'|$ in terms of these eigenfunctions. This will be done in the next section.

4. *Series expansions for the radial components.* In the construction of elementary solutions of the scalar wave equation by the method of separation of variables we encounter cylinder functions and spherical harmonics. In the notation of Watson [10]) for Bessel and Hankel functions, let

$$j_n(x) = (\pi/2x)^{\frac{1}{2}}\,J_{n+\frac{1}{2}}(x), \quad h_n(x) = (\pi/2x)^{\frac{1}{2}}\,H^{(1)}_{n+\frac{1}{2}}(x), \tag{14}$$

where $x \geqslant 0$ and $n = 0, 1, 2, \ldots$. Further, let $P_n^m(x)$, $-1 \leqslant x \leqslant 1$, denote Legendre's function in the notation of Barnes and Hobson [11]); n and m are integers not necessarily positive. We then define, for $-n \leqslant m \leqslant n$, the surface harmonic of degree n and order m by

$$Y_n^m(\vartheta, \varphi) = \left[(2n+1)\frac{(n-m)\,!}{(n+m)\,!}\right]^{\frac{1}{2}} P_n^m(\cos\vartheta)\,e^{im\varphi}, \tag{15}$$

in which the factor of normalization is chosen so as to make the mean value of $|\,Y_n^m\,|^2$ over all directions ϑ, φ equal to unity independently of n and m. Note that

$$Y_n^{-m} = (-1)^m\,\overline{Y_n^m}, \tag{16}$$

where the bar indicates the conjugate complex value.

A complete set of elementary solutions of the wave equation, orthogonal over the unit sphere, satisfying Sommerfeld's radiation condition at infinity and being everywhere regular except at the origin, is

$$\Pi_n^m(\mathbf{r}) = h_n(kr)\,Y_n^m(\vartheta, \varphi) \qquad (n \geqslant 0;\ m = 0, \pm 1, \pm 2, \ldots, \pm n). \tag{17}$$

A similar set of functions not obeying Sommerfeld's condition at infinity but remaining finite at the orgin is obtained from eq. (17) by replacing h_n by j_n. The latter set will only be needed in the coordinates of integration, so that there will not arise any confusion if we denote them by the same symbol:

$$\Pi_n^m(\mathbf{r}') = j_n(kr')\,Y_n^m(\vartheta', \varphi') \quad (n \geqslant 0;\ m = 0, \pm 1, \pm 2, \ldots, \pm n). \tag{18}$$

It is well known that the free-space Green's function can be expanded in terms of the functions of eqs (17) and (18). By the addition theorem of cylinder functions [10]) we first have

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = ik\,\Sigma_{n=0}^{\infty}\,(2n+1)\,j_n(kr')\,h_n(kr)\,P_n\,(\cos\Theta),$$

in which $r' < r$ and $\cos\Theta = \cos\vartheta\,\cos\vartheta' + \sin\vartheta\,\sin\vartheta'\,\cos\,(\varphi - \varphi')$. Secondly, by the addition theorem of Legendre functions [11]),

$$P_n\,(\cos\Theta) = \Sigma_{m=-n}^{n}\,(-1)^m\,P_n^m\,(\cos\vartheta)\,P_n^{-m}\,(\cos\vartheta')\,e^{im(\varphi-\varphi')},$$

so that we ultimately arrive at

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = ik\,\Sigma_{n,m}\,(-1)^m\,\Pi_n^{-m}\,(\mathbf{r}')\,\Pi_n^m(\mathbf{r}), \tag{19}$$

valid if $r' < r$.

Let us now substitute the expansion (19) in eq. (9). We then find that

$$\mathbf{r}.\mathbf{H}(\mathbf{r}) = (ik/c)\,\Sigma_{n,m}\,(-1)^m\,\Pi_n^m(\mathbf{r}) \int (\mathbf{r}'.\mathrm{curl}'\,\mathbf{i})\,\Pi_n^{-m}(\mathbf{r}')\,\mathrm{d}V',$$

this series being absolutely and uniformly convergent in the space exterior to the sphere R enclosing all currents. In virtue of

$$(\mathbf{r}'.\mathrm{curl}'\,\mathbf{i})\,\Pi_n^{-m} = \mathbf{i}.\mathrm{curl}'\,(\mathbf{r}'\,\Pi_n^{-m}) + \mathrm{div}'\,(\mathbf{i}\times\mathbf{r}'\,\Pi_n^{-m}),$$

Gauss's theorem and the fact that $\mathbf{i}$ vanishes on R, we have

$$\int (\mathbf{r}'.\mathrm{curl}'\,\mathbf{i})\,\Pi_n^{-m}(\mathbf{r}')\,\mathrm{d}V' = \int \mathbf{i}.\mathrm{curl}'\,(\mathbf{r}'\Pi_n^{-m})\,\mathrm{d}V',$$

so that the expansion of $\mathbf{r}.\mathbf{H}$ becomes

$$\mathbf{r}.\mathbf{H}(\mathbf{r}) = (ik/c)\,\Sigma_{n,m}\,(-1)^m\,\Pi_n^m(\mathbf{r}) \int \mathbf{i}.\,\mathrm{curl}'\,(\mathbf{r}'\Pi_n^{-m})\,\mathrm{d}V'. \tag{20}$$

The analogous expansion of $\mathbf{r}.\mathbf{E}$ follows from eq. (13) in exactly the same way. Noting that the first term of the right-hand side of eq. (13) vanishes outside R, we get (outside R)

$$\mathbf{r}.\mathbf{E}(\mathbf{r}) = -c^{-1}\,\Sigma_{n,m}\,(-1)^m\,\Pi_n^m(\mathbf{r}) \int \mathbf{i}.\mathrm{curl}'\,\mathrm{curl}'\,(\mathbf{r}'\Pi_n^{-m})\,\mathrm{d}V'. \tag{21}$$

In eqs (20) and (21) the summation extends over all positive integers n while m goes through all integers between $-n$ and $+n$, inclusive. There is no term with $n = 0$ because $\mathrm{curl}'\,(\mathbf{r}'\Pi_0^0)$ vanishes identically. The expansions (20) and (21) are divergent in the interior of the sphere just surrounding all currents (that is, the R of smallest radius possible).

5. *Construction of the Debye potentials.* We now define two scalar functions by dividing successive terms in the series (20) and (21) by $n(n+1)$ and show that the functions so obtained are the Debye potentials pertaining to the electromagnetic field produced by the given currents $\mathbf{i}$:

$$\Pi_1(\mathbf{r}) = -\frac{1}{c}\,\Sigma_{n,m}\,\frac{(-1)^m}{n(n+1)}\,\Pi_n^m(\mathbf{r}) \int \mathbf{i}.\mathrm{curl}'\,\mathrm{curl}'\,(\mathbf{r}'\Pi_n^{-m})\,\mathrm{d}V', \tag{22}$$

$$\Pi_2(\mathbf{r}) = \frac{ik}{c} \Sigma_{n,m} \frac{(-1)^m}{n(n+1)} \Pi_n^m(\mathbf{r}) \int \mathbf{i}.\text{curl}' (\mathbf{r}' \Pi_n^{-m}) \, \mathrm{d}V'. \tag{23}$$

First of all, it is obvious that in the domain of convergence (that is, the space exterior to R) *both functions are solutions of the homogeneous scalar wave equation*. It is not difficult to verify that this is a *sufficient* condition to ensure that the field derived from eqs (22) and (23) according to eqs (5) satisfies the homogeneous free-space Maxwell's equations. To show that the latter field is identical with that following from eqs (3) and (4), it is sufficient to show that the corresponding radial fields are the same, in virtue of the uniqueness theorem of section 2. This, however, is evidently true from the way we derived eqs (22) and (23) from eqs (21) and (20), respectively, in combination with the identities

$$\mathbf{r}.\text{curl}\,(\mathbf{r}\Pi_n^m) = 0, \quad \mathbf{r}.\text{curlcurl}\,(\mathbf{r}\Pi_n^m) = n(n+1)\,\Pi_n^m, \tag{24}$$

of which the first is trivial and the second may be easily proved using spherical coordinates and a few familiar relations for $h_n(kr)$ and Y_n^m. This completes our proof that Π_1 ans Π_2 defined by eqs (22) and (23) are the Debye potentials of the field produced by the currents **i**.

Inspection of eqs (22) and (23) shows that we need no longer assume that **i** has continuous derivatives of the second order. It is sufficient that **i** be absolutely integrable.

If Π is either of the two Debye potentials we may add to $\mathbf{r}\Pi$ the gradient of an arbitrary scalar χ without altering the corresponding field **E**, **H**. If we restrict grad χ to be of the form $\mathbf{r}\psi$, then ψ is necessarily a function of r alone. If we further require ψ to be a solution of the wave equation satisfying the radiation condition at infinity, then ψ must be a constant times Π_0^0. Therefore, except for the addition of terms proportional to Π_0^0 the Debye potentials are uniquely defined by the field **E**, **H**. Consequently there is much less freedom in choosing the Debye potentials Π_1 and Π_2 than in choosing a vector potential; see eqs (3).

6. *Electric and magnetic multipole fields.* In the preceding section we have seen that the electromagnetic field produced by the given currents can, outside R, be calculated from eqs (5) with

$$\Pi_1 = \Sigma_{n,m}\, a_n^m\, \Pi_n^m(\mathbf{r}), \quad \Pi_2 = \Sigma_{n,m}\, b_n^m\, \Pi_n^m(\mathbf{r}), \tag{25}$$

in which the coefficients are defined by

$$a_n^m = -\frac{1}{c}\frac{1}{n(n+1)} \int \mathbf{i}(\mathbf{r}').\text{curl}'\text{curl}' \{\mathbf{r}' j_n(kr')\, \overline{Y_n^m}\, (\vartheta', \varphi')\}\, \mathrm{d}V', \tag{26}$$

$$b_n^m = \frac{ik}{c}\frac{1}{n(n+1)} \int \mathbf{i}(\mathbf{r}').\text{curl}' \{\mathbf{r}' j_n(kr')\, \overline{Y_n^m}(\vartheta', \varphi')\}\, \mathrm{d}V'. \tag{27}$$

We now *define* a unit electric multipole field of degree n and order m to be that combination of **E** and **H** vectors which is obtained from eqs (5) by letting $\Pi_1 = \Pi_n^m$ and $\Pi_2 = 0$. Similarly, a unit magnetic multipole field of degree n and order m is that combination of **E** and **H** vectors which is obtained by letting $\Pi_1 = 0$ and $\Pi_2 = \Pi_n^m$. In other words, unit (n, m)-poles produce the fields

$$\left.\begin{array}{l}\text{electric multipole: } \mathbf{E} = \operatorname{curlcurl}(\mathbf{r}\Pi_n^m),\ \mathbf{H} = -ik\operatorname{curl}(\mathbf{r}\Pi_n^m),\\ \text{magnetic multipole: } \mathbf{H} = \operatorname{curlcurl}(\mathbf{r}\Pi_n^m),\ \mathbf{E} = ik\operatorname{curl}(\mathbf{r}\Pi_n^m).\end{array}\right\} \quad (28)$$

These multipole fields satisfy the homogeneous Maxwell's equations except at the origin of coordinates and they also obey Sommerfeld's radiation condition at infinity.

It should be observed that, with our definition of basic multipole fields, the field of an electric Hertzian dipole located at the origin and directed along the x-axis (that is, an infinitesimal current element of length δx carrying a uniform current i_x) is a superposition of electric multipole fields of degree $n = 1$ and orders $m = \pm 1$, corresponding to the Debye-potential representation

$$\Pi_1 = -(k/c\sqrt{6})\, i_x\delta x\, \{\Pi_1^1 - \Pi_1^{-1}\},\ \Pi_2 = 0.$$

Similarly, for an electric Hertzian dipole in the y direction:

$$\Pi_1 = (ik/c\sqrt{6})\, i_y\delta y\, \{\Pi_1^1 + \Pi_1^{-1}\},\ \Pi_2 = 0,$$

and for an electric Hertzian dipole in the z direction:

$$\Pi_1 = -(k/c\sqrt{3})\, i_z\delta z\, \Pi_1^0,\ \Pi_2 = 0.$$

Analogously, any electric quadrupole field is a certain combination of the basic electric multipole fields of degree $n = 2$ and orders $m = 0, \pm 1, \pm 2$. For those readers who visualize multipoles in terms of derivatives of a monopole, we may recall the known relation [12] [13])

$$\Pi_n^m(\mathbf{r}) = i^{-n}\left\{(2n+1)\frac{(n-m)!}{(n+m)!}\right\}^{\frac{1}{2}}\left[\left\{\frac{1}{ik}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right\}^m P_n^{(m)}\left(\frac{1}{ik}\frac{\partial}{\partial z}\right)\right]\Pi_0^0(\mathbf{r}),$$

valid when $m \geqslant 0$, where $P_n^{(m)}$ is the mth derivative of Legendre's polynomial $P_n(u)$ with respect to u.

Equations (25) to (27) show that the field due to an arbitrary current distribution **i** can be expanded in a series of multipole fields if we restrict ourselves to that part of free space which lies outside a spherical surface enclosing all currents *). The individual multipole terms all have their sources at the centre of this sphere, the multipole strengths being given by the coefficients a_n^m and b_n^m of eqs (26) and (27). It should be borne in mind that these coefficients are not identical with the *static* multipole moments. In

*) It may be noted that for field points in free space *within the convex enclosure of the system of currents* an expansion in multipole fields is impossible.

the limiting case $ka \to 0$ (where a denotes the radius of R), however, the leading terms in the expansions according to powers of ka of a_n^m and b_n^m are proportional to the static electric and magnetic multipole moments respectively.

7. *Intrinsic properties of multipole fields.* To get an insight into the intrinsic properties of the rather complicated multipole fields (28) we may resolve their field vectors in rectangular components, for example. These components are solutions of the wave equation and can therefore be written as linear combinations of the functions Π_n^m. The actual calculations, although tedious, are straightforward, requiring differential and recursive relations of Bessel and Legendre functions. We obtain for the unit electric multipole field:

$$E_x \pm iE_y = k\Big[\mp (n+1)\left\{\frac{(n \mp m)(n \mp m - 1)}{(2n-1)(2n+1)}\right\}^{\frac{1}{2}} \Pi_{n-1}^{m\pm 1} \pm \pm n\left\{\frac{(n \pm m + 1)(n \pm m + 2)}{(2n+1)(2n+3)}\right\}^{\frac{1}{2}} \Pi_{n+1}^{m\pm 1}\Big],$$

$$E_z = k\Big[(n+1)\left\{\frac{(n-m)(n+m)}{(2n-1)(2n+1)}\right\}^{\frac{1}{2}} \Pi_{n-1}^{m} + n\left\{\frac{(n-m+1)(n+m+1)}{(2n+1)(2n+3)}\right\}^{\frac{1}{2}} \Pi_{n+1}^{m}\Big],$$

$$H_x \pm iH_y = k\,[n \mp m)(n \pm m + 1)]^{\frac{1}{2}}\, \Pi_n^{m\pm 1}, \quad H_z = -km\,\Pi_n^m.$$

The analogous expressions for the unit magnetic multipole field are obtained by replacing **E** and **H** by **H** and —**E** respectively.

An alternative description of the multipole fields (28) is in terms of their spherical polar components:

electric multipole	magnetic multipole
$E_r = n(n+1)\frac{1}{r} h_n Y_n^m$	$E_r = 0$
$E_\vartheta = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rh_n)\frac{\partial}{\partial\vartheta} Y_n^m$	$E_\vartheta = -km\, h_n Y_n^m/\sin\vartheta$
$E_\varphi = im\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rh_n)\, Y_n^m/\sin\vartheta$	$E_\varphi = -ik\, h_n \frac{\partial}{\partial\vartheta} Y_n^m$
$H_r = 0$	$H_r = n(n+1)\frac{1}{r} h_n Y_n^m$
$H_\vartheta = km\, h_n Y_n^m/\sin\vartheta$	$H_\vartheta = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rh_n)\frac{\partial}{\partial\vartheta} Y_n^m$
$H_\varphi = ik\, h_n \frac{\partial}{\partial\vartheta} Y_n^m$	$H_\varphi = im\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rh_n)\, Y_n^m/\sin\vartheta$

These equations show several characteristic features of the multipole fields. In the first place, we see again that all components are identically zero if $n = m = 0$; that is, there are no radiating monopoles. Secondly, the radial magnetic field H_r vanishes for any electric multipole. Therefore, electric multipole fields are spherical transverse magnetic waves (*TM* waves) or spherical electric waves (*E* waves), in the terminology of the radio engineer. Similarly, magnetic multipole fields are spherical transverse electric waves (*TE* waves) or spherical magnetic waves (*H* waves) because E_r vanishes. Stating that radiating monopoles do not exist is equivalent to stating that there are no spherical transverse electromagnetic waves (*TEM* waves) in free space (see also section 2).

A further feature of our multipole fields is that rE_r and rH_r, if not identically zero, are equal to $n(n+1)\, h_n Y_n^m = n(n+1)\, \Pi_n^m$. This feature, mathematically expressed in eqs (24), was in fact the key to finding the connection between the Debye potentials (or equivalently the multipole expansion) and the simple source representations of $\mathbf{r}.\mathbf{E}$ and $\mathbf{r}.\mathbf{H}$.

Fairly simple approximations for the multipole fields can be given at large distances from the origin. Ignoring terms of order $1/r^2$ and recalling that $h_n(x)$ is asymptotically equal to $i^{-n-1}e^{ix}/x$ as $x \to \infty$, we easily find for the electric multipoles

$$\left.\begin{aligned} \mathbf{E} &\sim i^{-n}\frac{e^{ikr}}{r}\left\{\frac{\partial Y_n^m}{\partial\vartheta}\,\mathbf{a}_\vartheta + \frac{im}{\sin\vartheta}\,Y_n^m\,\mathbf{a}_\varphi\right\} = i^{-n}\,e^{ikr}\,\mathrm{grad}\,Y_n^m, \\ \mathbf{H} &\sim i^{-n}\frac{e^{ikr}}{r}\left\{\frac{\partial Y_n^m}{\partial\vartheta}\,\mathbf{a}_\varphi - \frac{im}{\sin\vartheta}\,Y_n^m\,\mathbf{a}_\vartheta\right\} = i^{-n}\,e^{ikr}\,(\mathbf{a}_r \times \mathrm{grad}\,Y_n^m), \end{aligned}\right\} \qquad (29)$$

where $\mathbf{a}_r$, $\mathbf{a}_\vartheta$ and $\mathbf{a}_\varphi$ are unit vectors in the r, ϑ and φ directions. Similar approximations for magnetic multipole fields are obtained on replacing $\mathbf{E}$ and $\mathbf{H}$ by $\mathbf{H}$ and $-\mathbf{E}$ respectively. In the wave zone the three vectors $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{r}$ form a right-handed orthogonal triad.

To calculate the energy radiated by a unit multipole in the unit of time, we multiply the vectors (29) by $\exp(-ikct)$, take the real parts, calculate Poynting's vector, and integrate over all directions. The time average of Poynting's vector at large distances from the origin, for either type of multipole, is

$$\frac{c}{8\pi}\,\mathrm{Re}\,(\mathbf{E}\times\mathbf{H}) \sim \frac{c}{8\pi}\,|\mathbf{E}|^2\,\mathbf{a}_r \sim \frac{c}{8\pi}\,|\mathbf{H}|^2\,\mathbf{a}_r \sim \frac{c}{8\pi}\,|\mathrm{grad}\,Y_n^m|^2\,\mathbf{a}_r.$$

Using

$$\begin{aligned} &\lim_{r\to\infty}\,(c/8\pi)\,r^2 \textstyle\int_0^{2\pi} d\varphi \int_0^{\pi} |\mathrm{grad}\,Y_n^m|^2 \sin\vartheta\, d\vartheta \\ &\qquad = (c/8\pi)\,n(n+1) \textstyle\int_0^{2\pi} d\varphi \int_0^{\pi} |Y_n^m|^2 \sin\vartheta\, d\vartheta = \tfrac{1}{2}n(n+1)\,c, \end{aligned}$$

we find the radiated power of the unit multipole to be

$$P = \tfrac{1}{2}n(n+1)\,c. \qquad (30)$$

Our basic set of multipole fields (28) is not only complete but also hermitian orthogonal in the sense that

$$\int \text{curlcurl}\,(\mathbf{r}\Pi_n^m)\,.\,\overline{\text{curl}\,(\mathbf{r}\Pi_{n'}^{m'})}\,\mathrm{d}\Omega = 0,$$

$$\int \text{curl}\,(\mathbf{r}\Pi_n^m)\,.\,\overline{\text{curl}\,(\mathbf{r}\Pi_{n'}^{m'})}\,\mathrm{d}\Omega = 4\pi\, n(n+1)\,|h_n(kr)|^2\,\delta_n^{n'}\,\delta_m^{m'},$$

$$\int \text{curlcurl}\,(\mathbf{r}\Pi_n^m)\,.\,\overline{\text{curlcurl}\,(\mathbf{r}\Pi_{n'}^{m'})}\,\mathrm{d}\Omega$$

$$= \frac{4\pi\, n(n+1)\,k^2}{2n+1}\{(n+1)\,|h_{n-1}(kr)|^2 + n\,|h_{n+1}(kr)|^2\}\,\delta_n^{n'}\,\delta_m^{m'},$$

in which the integrations extend over all directions in space with $\mathrm{d}\Omega = \sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi$. These relations may be readily proved by evaluating the scalar products in rectangular or in spherical coordinates.

In virtue of the orthogonality of the multipole fields, in particular as $r \to \infty$, it follows that the total energy radiated by the currents in the unit of time is given by

$$P_{total} = \tfrac{1}{2}c\,\Sigma_n\, n(n+1)\,\Sigma_m\,(|a_n^m|^2 + |b_n^m|^2). \tag{31}$$

8. *Vector potential versus Debye potentials.* So far as we know, Hansen [9]) first succeeded in expanding the electromagnetic field of an arbitrary system of currents in multipole fields. Hansen's method is simple, not using group theory or vector dyadics. The main difference with our method as outlined above is that Hansen based his theory on the vector potential **A** of eq. (4). In this connection it is interesting to investigate how the vector potential is related to the Debye potentials.

Referring for details to Hansen's paper cited, let us quote his expansion for the vector potential. In our notation we have

$$\mathbf{A}(\mathbf{r}) = \frac{i}{kc}\,\Sigma\,(-1)^m\,\text{grad}\,\Pi_n^m \int \mathbf{i}\,.\,\text{grad}'\,\Pi_n^{-m}\,\mathrm{d}V'$$

$$+\frac{ik}{c}\,\Sigma\,\frac{(-1)^m}{n(n+1)}\,\text{curl}\,(\mathbf{r}\Pi_n^m)\int \mathbf{i}\,.\,\text{curl}'\,(\mathbf{r}'\Pi_n^{-m})\,\mathrm{d}V'$$

$$+\frac{i}{kc}\,\Sigma\,\frac{(-1)^m}{n(n+1)}\,\text{curlcurl}\,(\mathbf{r}\Pi_n^m)\int \mathbf{i}\,.\,\text{curl}'\text{curl}'\,(\mathbf{r}'\Pi_n^{-m})\,\mathrm{d}V'.$$

This result of Hansen can be simply expressed in the two Debye potentials and a third scalar wave function:

$$ik\mathbf{A} = \text{curlcurl}\,(\mathbf{r}\Pi_1) + ik\,\text{curl}\,(\mathbf{r}\Pi_2) + \text{grad}\,\Phi, \tag{32}$$

valid for all points exterior to R, in which Φ is defined by

$$\Phi(\mathbf{r}) = -\frac{1}{c}\,\Sigma\,(-1)^m\,\Pi_n^m(\mathbf{r})\int \mathbf{i}\,.\,\text{grad}'\,\Pi_n^{-m}\,\mathrm{d}V'.$$

It is not difficult to see that Φ is nothing but the ordinary scalar potential.

On integrating by parts, applying the equation of continuity and using eq. (19) we find

$$\begin{aligned}\Phi(\mathbf{r}) &= c^{-1}\,\Sigma\,(-1)^m\,\Pi_n^m(\mathbf{r}) \int (\operatorname{div}'\,\mathbf{i})\,\Pi_n^{-m}\,\mathrm{d}V' \\ &= ik\,\Sigma\,(-1)^m\,\Pi_n^m(\mathbf{r}) \int \varrho\,\Pi_n^{-m}\,\mathrm{d}V' \\ &= \int \varrho(\mathbf{r}')\,\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}V',\end{aligned}$$

which proofs our assertion.

Conversely, we can derive Hansen's expansion of the vector potential from eq. (32), which is a consequence of the first eq. (5) and the known equation $\mathbf{E} = ik\mathbf{A} - \operatorname{grad}\Phi$, the latter being equivalent to the second eq. (3) in virtue of the relation $\operatorname{div}\mathbf{A} = ik\Phi$.

The curlcurl in the right-hand side of eq. (32) can be eliminated and we are then left with

$$ik\mathbf{A} = k^2\mathbf{r}\Pi_1 + ik\,\operatorname{curl}\,(\mathbf{r}\Pi_2) + \operatorname{grad}\,(\Phi + \Pi_1 + \mathbf{r}.\operatorname{grad}\Pi_1), \qquad (33)$$

and by taking the curl of both sides we see that the first eq. (3) and the second eq. (5) are consistent with each other.

It is therefore a matter of taste whether the theory of multipole expansion is most conveniently based on the vector potential $\mathbf{A}$ or the Debye potentials Π_1 and Π_2, the results being completely identical. However, we believe that the Debye potentials are more advantageous in that they preserve right from the start the symmetry between the electric and the magnetic multipoles, represented by Π_1 and Π_2 respectively. The only disadvantage of the Debye potentials is that they cannot represent the field in the interior of R, whereas eqs (3) and (4) hold everywhere in space. This distinction is, however, irrelevant for the problem of multipole expansion because such an expansion is only convergent outside R.

We have deliberately refrained from introducing special symbols for the vector fields defined by eqs (28) because an abundance of symbols and indices is likely causing hard reading. In addition it is easier to visualize multipole expansion in terms of the scalar wave functions Π_1 and Π_2 than in terms of the complicated multipole fields themselves.

By substituting eq. (19) in eq. (4), we see that each of the three rectangular components of the vector potential can be expanded in a series of functions (17), convergent at all points exterior to R.. It is therefore interesting to investigate what combination of multipole fields is represented by the general term of this series. As an example, consider the field whose vector potential is given by

$$\mathbf{A} = (A_x, A_y, A_z) = (\Pi_n^m, 0, 0). \qquad (34)$$

The corresponding scalar potential, $\Phi = (1/ik)$ div $\mathbf{A}$, is found to be

$$\Phi = -\frac{i}{2}\left[\left\{\frac{(n+m)(n+m-1)}{(2n-1)(2n+1)}\right\}^{\frac{1}{2}} \Pi_{n-1}^{m-1} + \left\{\frac{(n-m+1)(n-m+2)}{(2n+1)(2n+3)}\right\}^{\frac{1}{2}} \Pi_{n+1}^{m-1} - \left\{\frac{(n-m)(n-m-1)}{(2n-1)(2n+1)}\right\}^{\frac{1}{2}} \Pi_{n-1}^{m+1} - \left\{\frac{(n+m+1)(n+m+2)}{(2n+1)(2n+3)}\right\}^{\frac{1}{2}} \Pi_{n+1}^{m+1}\right].$$

The radial electric and magnetic field vectors can be calculated in a straightforward manner, and the Debye potentials follow from them by dividing coefficients by $n(n+1)$:

$$\Pi_1 = \frac{i}{2n(n+1)}\left[(n+1)\left\{\left(\frac{(n+m)(n+m-1)}{(2n-1)(2n+1)}\right)^{\frac{1}{2}} \Pi_{n-1}^{m-1} - \left(\frac{(n-m)(n-m-1)}{(2n-1)(2n+1)}\right)^{\frac{1}{2}} \Pi_{n-1}^{m+1}\right\} - n\left\{\left(\frac{(n-m+1)(n-m+2)}{(2n+1)(2n+3)}\right)^{\frac{1}{2}} \Pi_{n+1}^{m-1} - \left(\frac{(n+m+1)(n+m+2)}{(2n+1)(2n+3)}\right)^{\frac{1}{2}} \Pi_{n+1}^{m+1}\right\}\right],$$

$$\Pi_2 = -\frac{i}{2n(n+1)}\left[\{(n+m)(n-m+1)\}^{\frac{1}{2}} \Pi_n^{m-1} + \{(n-m)(n+m+1)\}^{\frac{1}{2}} \Pi_n^{m+1}\right].$$

(If $n = 0$, $m = 0$, these formulae reduce to

$$\Phi = \Pi_1 = \frac{i}{\sqrt{6}}\{\Pi_1^1 - \Pi_1^{-1}\},\ \Pi_2 = 0.)$$

Therefore, the field derived from the simple vector potential of eq. (34) is a *linear combination of two magnetic and four electric multipole fields* (but for exceptions like $n = 0$). This clearly demonstrates that the vector potential is not very suitable for multipole representation.

The case of eq. (34) and its analogue, $\mathbf{A} = (0, \Pi_n^m, 0)$, can be combined into the linear combinations

$$A_x \pm iA_y = 0, \quad A_x \mp iA_y = \Pi_n^m, \quad A_z = 0 \tag{35}$$

whose corresponding scalar potentials are

$$\Phi = \pm\frac{i}{2}\left[\left\{\frac{(n \mp m)(n \mp m - 1)}{(2n-1)(2n+1)}\right\}^{\frac{1}{2}} \Pi_{n-1}^{m\pm 1} + \left\{\frac{(n \pm m+1)(n \pm m+2)}{(2n+1)(2n+3)}\right\}^{\frac{1}{2}} \Pi_{n+1}^{m\pm 1}\right],$$

while the equivalent Debye potentials are given by

$$\Pi_1 = \mp\frac{i}{2n(n+1)}\left[(n+1)\left\{\frac{(n \mp m)(n \mp m - 1)}{(2n-1)(2n+1)}\right\}^{\frac{1}{2}} \Pi_{n-1}^{m\pm 1} - n\left\{\frac{(n \pm m + 1)(n \pm m + 2)}{(2n+1)(2n+3)}\right\}^{\frac{1}{2}} \Pi_{n+1}^{m\pm 1}\right],$$

$$\Pi_2 = -\frac{i}{2n(n+1)}[(n \mp m)(n \pm m + 1)]^{\frac{1}{2}} \Pi_n^{m\pm 1}.$$

For the remaining case,

$$\mathbf{A} = (0, 0, \Pi_n^m), \tag{36}$$

with the scalar potential

$$\Phi = i\left[-\left\{\frac{(n-m)(n+m)}{(2n-1)(2n+1)}\right\}^{\frac{1}{2}} \Pi_{n-1}^m + n\left\{\frac{(n-m+1)(n+m+1)}{(2n+1)(2n+3)}\right\}^{\frac{1}{2}} \Pi_{n+1}^m\right],$$

the Debye potentials are

$$\Pi_1 = \frac{i}{n(n+1)}\left[(n+1)\left\{\frac{(n-m)(n+m)}{(2n-1)(2n+1)}\right\}^{\frac{1}{2}} \Pi_{n-1}^m + \right.$$
$$\left. + n\left\{\frac{(n-m+1)(n+m+1)}{(2n+1)(2n+3)}\right\}^{\frac{1}{2}} \Pi_{n+1}^m\right],$$

$$\Pi_2 = \frac{im}{n(n+1)} \Pi_n^m.$$

From the preceding formulae the representation in terms of Debye potentials follows at once if the representation in terms of the vector potential is known. As has been remarked before, the Debye potentials are unique except for the addition of Π_0^0. We now turn to the converse problem: Given the Debye potentials, what is the vector potential? There is, of course, no unique answer to this question, because the gradient of an arbitrary function may be added to the vector potential without altering the field. Let it suffice to mention three simple cases:

(i) The electromagnetic field following from $\Pi_1 = 0$, $\Pi_2 = \Pi_n^m$ is identical with that following from

$$\Phi = 0,\ A_z = -im\,\Pi_n^m,\ A_x \pm iA_y = i\,[(n \mp m)(n \pm m + 1)]^{\frac{1}{2}}\,\Pi_n^{m\pm 1}.$$

(ii) The field of $\Pi_1 = \Pi_n^m$, $\Pi_2 = 0$ is identical with that of

$$\Phi = n\Pi_n^m,\ A_z = -i\left[\frac{2n+1}{2n-1}(n-m)(n+m)\right]^{\frac{1}{2}} \Pi_{n-1}^m,$$

$$A_x \pm iA_y = \pm i\left[\frac{2n+1}{2n-1}(n \mp m)(n \mp m - 1)\right]^{\frac{1}{2}} \Pi_{n-1}^{m\pm 1}.$$

(iii) The field of $\Pi_1 = \Pi_n^m$, $\Pi_2 = 0$ is also identical with that of

$$\Phi = -(n+1)\,\Pi_n^m,\ A_z = -i\left[\frac{2n+1}{2n+3}(n-m+1)(n+m+1)\right]^{\frac{1}{2}} \Pi_{n+1}^m,$$

$$A_x \pm iA_y = \mp i\left[\frac{2n+1}{2n+3}(n \pm m + 1)(n \pm m + 2)\right]^{\frac{1}{2}} \Pi_{n+1}^{m\pm 1}.$$

These statements may be confirmed by taking suitable linear combinations of the cases treated in eqs (34) to (36) and by increasing or decreasing n's and m's by one unit. Subtracting the fields of the cases (ii) and (iii) should yield zero field; in fact, the difference of the two vector potentials turns out to be the gradient of a scalar. Further, if we multiply all quantities of case (ii) by $n + 1$ and of case (iii) by n, add the results, and divide by $2n + 1$, we get a representation of the field in which the scalar potential vanishes, so that $\mathbf{A}$ becomes essentially identical with the electric vector: $\mathbf{E} = ik\mathbf{A}$, whose explicit form has been given in section 6.

9. *Concluding remarks.* In connection with the uniqueness theorem covered by section 2, it is interesting to mention the following theorem:

Let K denote the class of complex-valued functions $f(\vartheta, \varphi)$ defined on the unit sphere such that f has the mean value zero and is further expressible in a uniformly convergent series of surface harmonics (cf. ref. 11, Chap. VII). Let f_s $(s = 1, 2)$ be any two elements of K. There is then one and only one solution of the free-space harmonic-time-dependent Maxwell equations outside the sphere of radius a, satisfying the radiation condition at infinity, such that the radial components of the field vectors assume the boundary values $E_r = f_1$, $H_r = f_2$ at the sphere $r = a$.

Instead of proving this theorem, let it suffice to indicate the relevant Debye potentials:

$$\Pi_s(\mathbf{r}) = a\, \Sigma_{n,m} \frac{\beta^m_{n,s}}{n(n+1)} \frac{h_n(kr)}{h_n(ka)} Y^m_n(\vartheta, \varphi), \qquad (r \geqslant a)$$

in which the coefficients are given by

$$\beta^m_{n,s} = \frac{1}{4\pi} \int f_s\,(\vartheta', \varphi')\, \overline{Y^m_n}(\vartheta', \varphi')\, \mathrm{d}\Omega'. \qquad (s = 1, 2)$$

The restriction to zero-mean-value functions $f(\vartheta, \varphi)$ is significant. As we have seen in section 2, the radial components E_r and H_r, if not identically zero, necessarily depend on the direction ϑ, φ. In other words, a system of currents isotropically radiating in all directions of free space is physically impossible. For an alternative proof, see M a t h i s [14]).

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