THE APPARENT SHAPE OF A RELATIVISTICALLY MOVING SPHERE

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It would be natural to assume that, according to the special theory of relativity, an object moving with a speed comparable with that of light should *appear* to be flattened in the direction of motion on account of its FitzGerald–Lorentz contraction. It will be shown here, however, that this is by no means generally the case. It turns out, in particular, that the appearance of a sphere, no matter how it is moving, is always such as to present a *circular* outline to any observer. Thus an instantaneous photograph* of a rapidly moving sphere has the same outline as that of a stationary sphere.

This result may seem paradoxical at first. For example, it might be thought that for a distant sphere moving perpendicularly to the line joining its centre to the observer, the flattening in its direction of motion would certainly be apparent. As the tangents from the observer to the flattened sphere are all nearly the same length, it might seem that the finite velocity of light is irrelevant here. However, the light which appears to the observer to be coming from the leading part of the sphere leaves the sphere at a later ime, in the observer's frame, than that which appears to come from the trailing part. In fact the light from the trailing part reaches the observer from *behind* the sphere, which it can do since the sphere is continually moving out of its way.) The length of the image of the sphere in the direction of motion is thus greater than might otherwise be expected, so that if it were not for the flattening the sphere would appear to be elongated.

In order to prove the exact result that the sphere always presents a circular outline, it is more convenient to consider the sphere as being at rest and the observer moving, this being allowable according to the special principle of relativity. If the sphere is accelerating, in order that it may reasonably be said to remain a sphere throughout its motion, it must be always instantaneously at rest and spherical in *some* Lorentz frame. The sphere will then satisfy the Born conditions for a rigid body (see Synge (3), p. 36). Now the light received by the observer, O, at an instant of his time, from what appears to him to be the outline of the sphere, comes from the sphere at one instant of *its* time. This shows that the acceleration of the sphere is irrelevant. Furthermore, a stationary observer at O clearly sees something with a circular outline. All this is evident from the symmetry. It is therefore only necessary to consider what transformation of the field of vision must be employed when passing from a stationary to a moving observer at the same point, and to show that this transformation is one which sends apparent circles into apparent circles.

* This is not a 'snapshot' in the sense of Synge ((3), p. 120) in which the sphere would indeed appear flattened. I am concerned, here, with a world-picture rather than a world-map (see Milne (1), p. 107), so that the finite velocity of the light coming from the sphere must be taken into account.

One way of doing this is to use the relativistic aberration formula*

$$\tan \tfrac{1}{2}\theta' = \tan \tfrac{1}{2}\theta \sqrt{\frac{c-v}{c+v}}.$$

A stereographic projection of the unit sphere with centre O, from the point for which $\theta = \pi$, sends circles into circles, and the above formula leads simply to an expansion of the plane of projection by an amount $\surd\{(c-v)/(c+v)\}$.

Alternatively, the following purely geometrical (space-time) argument, suggested to me by J. L. Synge, may be used. Consider the intersection of the past null cone of the event O with any hyperplane through O containing a time-like vector. The hyperplane represents the history of a plane moving with a constant velocity and the past null cone of O represents the history of a sphere with fixed centre and uniformly decreasing radius. They intersect in a circle which converges on O along a right circular cone. Hence, a stationary observer at the event O sees a circle only. Now this holds equally well in any Lorentz frame and therefore shows that a cone of light which appears as a circle to a stationary observer at O also appears as a circle to any moving observer at O.

There is yet another way of obtaining this result, namely by using properties of two-component spinors. Any past null vector (t, x, y, z) with $t < 0, t^2-x^2-y^2-z^2 = 0$ can be represented as a hermitian matrix which is minus the product of a two-component spinor (a, b) with its conjugate (see Veblen (4)). Thus,

$$\begin{pmatrix} t-z & x+iy \\ x-iy & t+z \end{pmatrix} = -\begin{pmatrix} \bar{a}a & \bar{a}b \\ \bar{b}a & \bar{b}b \end{pmatrix}.$$

A direction along the past light cone is then uniquely associated with the corresponding spinor ray, i.e. with the ratio $\lambda = b/a$. Now the points of the light cone at time $t = -1$ constitute a sphere $\mathscr{S}$, whose equations are

$$x^2+y^2+z^2 = 1; \quad t = -1.$$

The field of vision of a stationary observer at the origin may be conveniently represented by this sphere. But $\mathscr{S}$ may be projected stereographically from the point $(-1, 0, 0, -1)$ into the plane $z = 0, t = -1$, with the point $(-1, x, y, z)$ projecting into $(-1, x/(1+z), y/(1+z), 0)$. This plane may be taken as the Argand plane of the complex number

$$\frac{x}{1+z}+i\,\frac{y}{1+z} = \frac{x+iy}{z-t} = -\frac{\bar{a}b}{\bar{a}a} = -\lambda.$$

The sphere $\mathscr{S}$ can then be regarded as the Argand sphere of $-\lambda$. But a proper Lorentz transformation of the directions through the origin corresponds to a linear transformation of the spinors (a, b), i.e. to a bilinear transformation of λ. This sends circles into circles on $\mathscr{S}$ as required, so that the result follows.

* This is not the usual formula, which is $\cos\theta = \dfrac{\cos\theta' - v/c}{1-(v/c)\cos\theta'}$ (see Synge (3), p. 147). The two forms are easily shown to be equivalent.

It is perhaps worth remarking here that the fact that any proper homogeneous Lorentz transformation is determined by its effect on any three null directions (see Synge (3), p. 99) is here seen to be equivalent to the corresponding property for complex numbers under bilinear transformations (or points of a projective line under a projectivity).

These considerations can also be applied to non-spherical objects moving with uniform velocity. The appearance of such an object is always a circular transform (i.e. product of inversions) of what it would appear in some orientation when stationary. Thus, straight lines appear circular (or straight). Since a stationary circle may appear elliptical, a moving circle can appear boomerang shaped (inverse of an ellipse). In view of this, it is doubtful whether it would be correct to say that a sphere always appears *spherical* rather than just saying it has a circular outline, since the intersection of two spheres does not necessarily appear circular. Nevertheless, there is no other consistent shape which presents a circular outline when stationary, to any observer.

Finally, it may be remarked that all the above considerations apply equally well to a de Sitter space (Minkowski 4-sphere, see Schrödinger (2)) owing to its symmetry. The foregoing arguments apply almost without change.

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A NEW EXPRESSION FOR EINSTEIN'S LAW OF GRAVITATION

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1. *Introduction.* The result of the present paper on Riemannian geometry arose (following a suggestion of G. J. Whitrow) in the course of a more general investigation by one of the authors (4). It seems not to have been noticed before, and is therefore worth publishing separately.

Consider the usual formulation of Riemannian geometry in terms of vectors (e.g. (1), pp. 366 *et seq*). We shall use the notation $\mathscr{E}_\mu$ because the present formulation is slightly