## On a General Sextic Equation Solved by the Rogers-Ramanujan Continued fraction

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**Keywords**: Sextic equation; *j*-invariant; Ramanujan; Continued fraction; Algebraic Equations; Algebraic Numbers; Elliptic Functions; Modular equations;

#### Abstract

In this article we solve a general class of sextic equations. The solution follows if we consider the j-invariant and relate it with the polynomial equation's coefficients. The form of the solution is a relation of Rogers-Ramanujan continued fraction. The inverse technique can also be used for the evaluation of the Rogers-Ramanujan continued fraction, in which the equation is not now the depressed equation but another quite more simplified equation.

# 1 Introductory Definitions

We will solve the following equation

$$\frac{b^2}{20a} + bX^3 + aX^6 = C_1 X^5 : (eq)$$

or equivalent

$$\frac{b^2}{20a} + bX + aX^2 = C_1 X^{5/3} \tag{1}$$

using the j-invariant and the Rogers-Ramanujan continued fraction.

For |q| < 1, the Rogers Ramanujan continued fraction (RRCF) (see [2],[3],[4]) is defined as

$$R(q) := \frac{q^{1/5}}{1+} \frac{q^1}{1+} \frac{q^2}{1+} \frac{q^3}{1+} \cdots$$
 (2)

From the Theory of Elliptic functions the j-invariant (see [5],[8]) is

$$j_r := \left[ \left( \frac{\eta(\frac{1}{2}\sqrt{-r})}{\eta(\sqrt{-r})} \right)^{16} + 16 \left( \frac{\eta(\sqrt{-r})}{\eta(\frac{1}{2}\sqrt{-r})} \right)^8 \right]^3, \tag{3}$$

where

$$\eta(\tau) := e^{\pi i \tau / 12} \prod_{n=1}^{\infty} \left( 1 - e^{2\pi i n \tau} \right)$$
(4)

is the Dedekind's eta function and

$$\tau = \frac{1+\sqrt{-r}}{2}$$
 ,  $\tau = \sqrt{-r}$  ,  $r$  positive real.

We have also in the q-notation

$$f(-q) := \prod_{n=1}^{\infty} (1 - q^n).$$
 (5)

In what follows we use the following known result (see Wolfram pages for 'Rogers-Ramanujan Continued Fraction' and [17]):

$$R = R(e^{-2\pi\sqrt{r}}),$$

then:

$$j_r = -\frac{\left(R^{20} - 228R^{15} + 494R^{10} + 228R^5 + 1\right)^3}{R^5 \left(R^{10} + 11R^5 - 1\right)^5} \tag{6}$$

From ([3],[4]) we have

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{qf^6(-q^5)} \tag{7}$$

The general hypergeometric function is defined as

$$_{p}F_{q}\left[\left\{a_{1}, a_{2}, \dots, a_{p}\right\}, \left\{b_{1}, b_{2}, \dots, b_{q}\right\}, x\right] = \sum_{n=0}^{\infty} \frac{(a_{1})_{n}(a_{2})_{n}, \dots, (a_{p})_{n}}{(b_{1})_{n}(b_{2})_{n}, \dots, (b_{q})_{n}} \frac{x^{n}}{n!}$$

where  $(c)_n = c(c+1) \dots (c+n-1)$ , hence  $(1)_n = n!$ .

The standard definition of the elliptic integral of the first kind (see [7],[8],[15]) is:

$$K(x) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - x^2 \sin^2(t)}}$$
 (8)

$$K(x) = \frac{\pi}{2} {}_{2}F_{1}\left(\{1/2, 1/2\}; \{1\}; x^{2}\right) = \frac{\pi}{2} {}_{2}F_{1}\left(1/2, 1/2; 1; x^{2}\right)$$
(9)

In the notation of Mathematica we have

$$K(x) = \text{EllipticK}[x^2] \tag{10}$$

The elliptic singular modulus  $k = k_r$  is defined to be the solution of the equation:

$$\frac{K\left(\sqrt{1-k^2}\right)}{K(k)} = \sqrt{r}.\tag{11}$$

In Mathematica's notation

$$k = k_r = k[r] = \text{InverseEllipticNomeQ}[e^{-\pi\sqrt{r}}]^{1/2}.$$
 (12)

The complementary modulus is given by  $k'_r = \sqrt{1 - k_r^2}$ . (For evaluations of  $k_r$  see [5],[15],[16]).

Also we call  $w_r := \sqrt{k_r k_{25r}}$  noting that if one knows  $w = w_r$  then (see [2]), knows  $k_r$  and  $k_{25r}$ .

## 2 Theorems

## Proposition 1. (see [2])

If  $q = e^{-\pi\sqrt{r}}$  and r real positive then we define

$$A = A_r := \frac{f^6(-q^2)}{q^2 f^6(-q^{10})} = R(q^2)^{-5} - 11 - R(q^2)^5$$
(13)

then

$$A_r = a_{4r} = \frac{(k_r k_r')^2}{(w_r w_r')^2} \left( \frac{w_r}{k_r} + \frac{w_r'}{k_r'} - \frac{w_r w_r'}{k_r k_r'} \right)^3$$
 (14)

#### Theorem 1.

Let  $a, b, C_1$  be constants. One can solve the equation

$$\frac{b^2}{20a} + bX + aX^2 = C_1 X^{5/3},\tag{15}$$

finding r > 0 such that

$$j_r = 250C_1^3 a^{-2} b^{-1}. (16)$$

Then (15) have solution

$$X = \frac{b}{250a} A_r = \frac{b}{250a} \frac{f(-e^{-2\pi\sqrt{r}})^6}{e^{-2\pi\sqrt{r}} f(-e^{-10\pi\sqrt{r}})^6}.$$
 (17)

## Proof.

For to solve the equation (15) find r such that

$$j_r^{1/3} = \frac{5 \cdot 2^{1/3} C_1}{a^{2/3} b^{1/3}} \tag{18}$$

Consider also the transformation of the constants

$$3125m = \frac{b^2}{20a} , 250ml^{-1} = \frac{b^2}{250a} \left( b + \frac{b^2}{20a} \right)^{-1}$$

and

$$ml^{-2} = \frac{b^2}{62500} \left( b + \frac{b^2}{20a} \right)^{-2},$$

with inverse

$$l = \frac{b(20a+b)}{20a}$$
,  $m = \frac{b^2}{62500a}$ .

Then

$$X = \frac{250m}{l(l-3125m)}x_1 = \frac{250m}{l-3125m}x = \frac{b}{250a}x,$$

where  $x_1$  satisfies

$$3125m + 250x_1ml^{-1} + x_1^2ml^{-2} = ml^{-5/3}j^{1/3}x_1^{5/3}$$

If we set  $x_1 = lx$ , then it is

and the proof is complete.

$$3125 + 250x + x^2 = j^{1/3}x^{5/3}$$

or equivalently

$$3125 + 250A_r + A_r^2 = j_r^{1/3} A_r^{5/3}$$
(19)

Relation (19) is equivalent to equation (6), in view of (7). Hence from Proposition 1

$$X = A_r = a_{4r} = \frac{b}{250a} \frac{f(-e^{-2\pi\sqrt{r}})^6}{e^{-2\pi\sqrt{r}}f(-e^{-10\pi\sqrt{r}})^6} =$$

$$= \frac{b}{250a} \frac{(k_r k_r')^2}{(w_r w_r')^2} \left(\frac{w_r}{k_r} + \frac{w_r'}{k_r'} - \frac{w_r w_r'}{k_r k_r'}\right)^3 = \frac{b}{250a} \left(R^{-5}(q^2) - 11 - R^5(q^2)\right) (20)$$

The j-invariant is connected with the singular modulus from the equation

$$j_r = \frac{256(k_r^2 + k_r^{\prime 4})^3}{(k_r k_r^{\prime})^4}.$$
 (21)

We can solve (21) and express  $k_r$  in radicals to an algebraic function of  $j_r$ . The 5th degree modular equation which connects  $k_{25r}$  and  $k_r$  is (see [3]):

$$k_r k_{25r} + k_r' k_{25r}' + 2 \cdot 4^{1/3} (k_r k_{25r} k_r' k_{25r}')^{1/3} = 1$$
 (22)

We will evaluate the root of (1) first with parametrization and second with Rogers-Ramanujan continued fraction and the Elliptic-K function. For this it have been showed (see [19]) that if

$$k_{25r}k_r = w_r^2 = w^2, (23)$$

setting the following parametrization of w:

$$w = \sqrt{\frac{L(18+L)}{6(64+3L)}},\tag{24}$$

we get

$$\frac{(k_{25r})^{1/2}}{w^{1/2}} = \frac{w^{1/2}}{(k_r)^{1/2}} = \frac{1}{2}\sqrt{4 + \frac{2}{3}\left(\frac{L^{1/6}}{M^{1/6}} - 4\frac{M^{1/6}}{L^{1/6}}\right)^2} + \frac{1}{2}\sqrt{\frac{2}{3}}\left(\frac{L^{1/6}}{M^{1/6}} - 4\frac{M^{1/6}}{L^{1/6}}\right)$$
(25)

where

$$M = \frac{18 + L}{64 + 3L}$$

From the above relations we get also

$$-\frac{k_r - w}{\sqrt{k_r w}} = \frac{k_{25r} - w}{\sqrt{k_{25r} w}} = \sqrt{\frac{2}{3}} \left( \frac{L^{1/6}}{M^{1/6}} - 4 \frac{M^{1/6}}{L^{1/6}} \right)$$
(26)

Hence we can consider the above equations as follows: Taking an arbitrary number L we construct an w. Now for this w we evaluate the two numbers  $k_{25r}$  and  $k_r$ . Thus when we know the w, the  $k_r$  and  $k_{25r}$  are given from (24),(25),(26). The result is: We can set a number L and from this calculate the two inverse elliptic nome's. But we don't know the r. One can see (from the definition of  $k_r$ ) that the r can evaluated from equation

$$r = \frac{K^2(\sqrt{1 - k_r^2})}{K^2(k_r)} \tag{27}$$

Hence we define

$$r = k^{(-1)}(x) := \frac{K^2(\sqrt{1-x^2})}{K^2(x)}$$
 (28)

However is very difficult to evaluate the r in a closed form, such as roots of polynomials or else when a number x is given. Some numerical evaluations indicate us that even if x are algebraic numbers, (not trivial as with  $k^{(-1)} (2^{-1/2}) = 1$  or the cases  $x = k_r$ ,  $r = 1, 2, 3, \ldots$ ) the r are not rational and may even not algebraics.

## The algebraic representation of X

We know that (see [2]):

$$X = X(L) = \frac{b}{250a} \frac{x_L^2 (1 - x_L^2)}{(w_L w_L')^2} \left( \frac{w_L}{x_L} + \frac{w_L'}{\sqrt{1 - x_L^2}} - \frac{w_L w_L'}{x_L \sqrt{1 - x_L^2}} \right)^3$$
(29)

where  $x_L = k_r$  is the singular modulus which corresponds to some L.

$$C_1 = \frac{a^{2/3}b^{1/3}}{5 \cdot 2^{1/3}} j_{r_L}^{1/3}. \tag{30}$$

The procedure is to select a number L and from (24),(25) evaluate  $w_L$ ,  $x_L$  and

$$w_L' = \sqrt{\sqrt{1 - \frac{w_L^4}{x_L^2}} \sqrt{1 - x_L^2}}. (31)$$

The solution X = X(L) of (1) is (29) and for this L holds

$$r_L = k^{(-1)}(x_L) (32)$$

$$X = \frac{b}{250a} \frac{f\left(-e^{-2\pi\sqrt{k^{(-1)}(x_L)}}\right)^6}{e^{-2\pi\sqrt{k^{(-1)}(x_L)}}f\left(-e^{-10\pi\sqrt{k^{(-1)}(x_L)}}\right)^6}$$
(33)

$$j_{r_L} = 250C_1^3 a^{-2} b^{-1} (34)$$

$$j_{r_L} = \frac{256(x_L^2 + (1 - x_L^2)^2)^3}{x_L^4 (1 - x_L^2)^2} = 250 \frac{C_1^3}{a^2 b}$$
 (35)

Hence we get the next:

## Theorem 2.

One can find parametric solutions of (1) if for a given L construct the  $w_L$ ,  $x_L$  and the complementary  $w_L'$  (these values are given from (23),(24),(25),(31)). Also  $x_L' = \sqrt{1 - x_L^2}$ . The  $C_1$  must be

$$C_1 = \sqrt[3]{\frac{a^2 b j_{r_L}}{250}} \tag{36}$$

The solution  $X = X_L$  is given from

$$X = X(L) = \frac{b}{250a} \frac{x_L^2 (1 - x_L^2)}{(w_L w_L')^2} \left( \frac{w_L}{x_L} + \frac{w_L'}{\sqrt{1 - x_L^2}} - \frac{w_L w_L'}{x_L \sqrt{1 - x_L^2}} \right)^3$$
(37)

#### Note.

- i) The above solution (37) works for parametric solutions (setting a L), as also for solutions which we know r,  $k_r$  and  $k_{25r}$ . (For a related method on solving the quintic see Wolfram pages 'Quintic Equation')
- ii) In [16] it have been shown that when one knows for some  $r_0$  the  $k_{r_0}$  and  $k_{r_0/25}$  then can evaluate any  $k_{25^n r_0}$  in radicals closed form for all n positive integers. But in general the values  $k_r$  and  $k_{25r}$  can given from tables or with a simple PC (see [4],[5],[13],[15],[17]).

## The inverse functions method

From the analysis in [2], the solution X of (1) can reduced with inverse functions as follows:

Consider the function

$$U(x) = \frac{256(x^2 + (1 - x^2)^2)^3}{x^4(1 - x^2)^2},$$
(38)

The equation U(x) = t have known solution with respect to x, which we will call  $x = U^{(-1)}(t)$ . Hence

$$\frac{256(k_r^2 + (1 - k_r^2)^2)^3}{k_r^4 (1 - k_r^2)^2} = 250 \frac{C_1^3}{a^2 b}$$
 (39)

or

$$U(k_r) = 250 \frac{C_1^3}{a^2 b}$$
$$k_r = U^{(-1)} \left( 250 \frac{C_1^3}{a^2 b} \right)$$

or

$$r = k^{(-1)} \left( U^{(-1)} \left( 250 \frac{C_1^3}{a^2 b} \right) \right)$$

The function  $k^{(-1)}(x)$  is that of (28).

## Theorem 3.

The equation (1) have solution

$$X = \frac{b}{250a} \left( R \left( e^{-2\pi\sqrt{k^{(-1)}(\alpha)}} \right)^{-5} - 11 - R \left( e^{-2\pi\sqrt{k^{(-1)}(\alpha)}} \right)^{5} \right) \tag{40}$$

where

$$\alpha = U^{(-1)} \left( 250 \frac{C_1^3}{a^2 b} \right) \tag{41}$$

#### Notes.

- 1) Observe here that we don't need the value of w and the class invariant j.
- 2) From [10] we have

$$R(e^{-x}) = e^{-x/5} \frac{\vartheta_4(3ix/4, e^{-5x/2})}{\vartheta_4(ix/4, e^{-5x/2})}, \forall x > 0$$

Hence the solution can expressed also in theta functions. That is if  $\alpha=k_r$ ,  $r=1,2,3,\ldots$  then the solution of (1) reduced to that of evaluation of Rogers-Ramanujan continued fraction R(q) with  $q=e^{-\pi\sqrt{r}}$ . In view of [10] we have

$$X = \frac{b}{250a} \left[ e^{2\pi\sqrt{r}} \left( \frac{\vartheta_4 \left( 3i\pi\sqrt{r}/2, e^{-5\pi\sqrt{r}} \right)}{\vartheta_4 \left( i\pi\sqrt{r}/2, e^{-5\pi\sqrt{r}} \right)} \right)^{-5} - 11 - e^{-2\pi\sqrt{r}} \left( \frac{\vartheta_4 \left( 3i\pi\sqrt{r}/2, e^{-5\pi\sqrt{r}} \right)}{\vartheta_4 \left( i\pi\sqrt{r}/2, e^{-5\pi\sqrt{r}} \right)} \right)^{5} \right]$$

## Example 1.

The equation

$$X^2 + 3X + \frac{9}{20} = \frac{26}{5\sqrt[3]{3}}X^{5/3}$$

have

$$\alpha = U^{(-1)} \left( \frac{35152}{9} \right) = \frac{\sqrt{3}}{2}$$

hence a solution is

$$X = \frac{3}{250} \left( R \left( e^{-2\pi\sqrt{r}} \right)^{-5} - 11 - R \left( e^{-2\pi\sqrt{r}} \right)^{5} \right)$$

where

$$r = \frac{K\left(\frac{1}{2}\right)^2}{K\left(\frac{\sqrt{3}}{2}\right)^2}$$

For this r the X is a solution.

Continuing one can set to

$$X^{5} = \frac{b^{2}}{20aC_{1}} + \frac{b}{C_{1}}X^{3} + \frac{a}{C_{1}}X^{6}$$

$$\tag{42}$$

any value  $X = X_0$  and  $C_1 = 1$  then evaluate

$$a = \frac{-5bX_0^3 + 5X_0^5 + \sqrt{5}\sqrt{4b^2X_0^6 - 10bX_0^8 + 5X_0^{10}}}{10X_0^6}$$
(43)

equation (42) holds always and we get that

$$R\left(e^{-2\pi\sqrt{r}}\right)^{-5} - 11 - R\left(e^{-2\pi\sqrt{r}}\right)^{5} =$$

$$= \frac{25\left(-5b + 5X_0^2 + \sqrt{5}\sqrt{4b^2 - 10bX_0^2 + 5X_0^4}\right)}{bX_0^2}$$

where  $j_r = 250a^{-2}b^{-1}$ .

$$j_r = \frac{25000X_0^6}{b\left(-5b + 5X_0^2 + \sqrt{5}\sqrt{4b^2 - 10bX_0^2 + 5X_0^4}\right)^2}$$
(44)

The result is the following parametrized evaluation of the Rogers-Ramanujan continued fraction

## Theorem 4.

$$A_r = R \left( e^{-2\pi\sqrt{r}} \right)^{-5} - 11 - R \left( e^{-2\pi\sqrt{r}} \right)^5 =$$

$$=\frac{25\left(-5b+5t^5+\sqrt{5}\sqrt{4b^2-10bt^2+5t^4}\right)}{b}\tag{45}$$

and

$$j_r = \frac{25000t^6}{b\left(-5b + 5t^2 + \sqrt{5}\sqrt{4b^2 - 10bt^2 + 5t^4}\right)^2}$$
(46)

## Corollary.

If

$$\sqrt[3]{A_r^2 j_r} = \text{rational}$$

then  $A_r$  is of the form

$$A_r = \frac{A + B\sqrt{D}}{C}$$

where A, B, C, D rationals

## Theorem 5.

If for a certain r > 0 we know the value of  $R(e^{-\pi\sqrt{r}})$  in radicals, then we can evaluate both  $k_r$  and  $k_{25r}$  and the opposite.

#### Proof.

Suppose we know for a certain r>0 the value of  $R(e^{-\pi\sqrt{r}})$ , (the correspondence between  $R(e^{-\pi\sqrt{r}})$  and  $R(e^{-2\pi\sqrt{r}})$  is given by (96) bellow or see [13]). Then from (6) we know the value of  $j_r$  and from (21) we know  $k_r$ . Let also  $q=e^{-\pi\sqrt{r}}$ , r>0 and  $v_r=R(q)$ , then it have been proved by Ramanujan that

$$v_{r/25}^5 = v_r \frac{1 - 2v_r + 4v_r^2 - 3v_r^3 + v_r^4}{1 + 3v_r + 4v_r^2 + 2v_r^3 + v_r^4},$$

Hence we can get the value of  $R(e^{-\pi\sqrt{r}/5})$ . Hence again form (6) we find  $j_{r/25}$  and from (21) the value of  $k_{r/25}$ . But from relation (53) bellow knowing  $k_r$  and  $k_{r/25}$  we can evaluate all  $k_{25^n r}$ ,  $n = 1, 2, \ldots$  and consequently  $k_{25r}$  as a special case. The oposite follow from Proposition 1.

#### Theorem 6.

The solution  $U_0$  of the equation

$$U_0 = j_r^{1/3} \left( 125 - \sqrt{12500 + U_0} \right)^{5/3} \tag{47}$$

is

$$U_0 = U(j_r) = \sum_{n=1}^{\infty} \frac{j_r^{n/3}}{n!} \left[ \frac{d^{n-1}}{da^{n-1}} \left( 125 - \sqrt{12500 + a} \right)^{5n/3} \right]_{a=0}$$

If  $X = x_0$  is root of

$$U_0 = X^2 + 250X + 3125$$

then

$$3125 + 250x_0 + x_0^2 = j_r^{1/3} (-1)^{1/3} x_0^{5/3}$$
(47a)

and

$$x_0 = X = X_r = A_r = R(e^{-2\pi\sqrt{r}})^{-5} - 11 - R(e^{-2\pi\sqrt{r}})^5$$
 (47b)

#### Proof.

Consider (1), then make the change of variable  $U_0 = X^2 + 250X + 3125$ , we arrive to (47). The Legendre inversion theorem states that the solution of y = af(y) (see [7] pg.132-133) is

$$y = \sum_{n=1}^{\infty} \frac{a^n}{n!} \left[ \frac{d^{n-1}}{dx^{n-1}} f(x)^n \right]_{x=0}$$

This theorem works for  $j_r$  small, for example with  $j_1 = 1728$  it converges very slowly but for r such that  $j_r = 800$ , (r-complex) we get numerical evaluations and hence also theoretical.

## Theorem 7.

Tf

$$c_n := \left[ \frac{d^{n-1}}{da^{n-1}} \left( 125 - \sqrt{12500 + a} \right)^{5n/3} \right]_{a=0}$$

then

$$c_n = \frac{5^6}{3} (-1)^{n+1} n \cdot 10^{-5n/3} {}_2F_1 \left[ \frac{5n}{6}, \frac{5n+3}{6}; \frac{2(n+3)}{3}; \frac{1}{5} \right] \frac{\Gamma(5n/3)}{\Gamma(2+2n/3)}$$

#### Proof.

Recall a theorem of Euler (see [18] pg.306-307). If the root of

$$aqx^p + x^q = 1$$

is x, then

$$x^{n} = \frac{n}{q} \sum_{k=0}^{\infty} \frac{\Gamma(\{n+pk\}/q)(-qa)^{k}}{\Gamma(\{n+pk\}/q - k + 1)k!}$$

Hence from the fact that

$$w = \frac{250(125 - \sqrt{12500 + x})}{3125 - x}$$

is solution of

$$acb^{-2}w^2 + w = 1$$
,  $a = 1$ ,  $b = -250$ ,  $c = -3125 + x$ 

we get

$$\left(-125 + \sqrt{12500 + x}\right)^n = \frac{n}{250^n} \sum_{k=0}^{\infty} \frac{\Gamma(n+2k)(-1)^k}{\Gamma(n+k+1)62500^k k!} (-3125 + x)^{k+n}$$
 (48)

Using the formula

$$f^{(\nu)}(x_0) = \sum_{n=0}^{\infty} \frac{f^{(\nu+n)}(x_0)}{n!} (-x_0)^n$$

the result follows.

Theorem 7 is for numerical evaluations since the hypergeometric series can more easily computed than the (n-1)th derivative of the 5n/3 power of  $125 - \sqrt{12500 + x}$ .

## Corollary.

For every 'suitable' value of  $x_0$  such that  $X_r = x_0 : (a)$ ,  $X_r$  is of (47b) exists a r solution of (a) such that

$$X_r^2 + 250X_r + 3125 =$$

$$=3^{-1}\cdot 5^{6}\sum_{n=1}^{\infty}(-1)^{n+1}n\frac{\Gamma(5n/3)}{\Gamma(2+2n/3)}{}_{2}F_{1}\left[\frac{5n}{6},\frac{5n+3}{6};\frac{2(n+3)}{3};\frac{1}{5}\right]\frac{(10^{-5}j_{r})^{n/3}}{n!}$$

## Example 2.

For  $X_r = x_0 = -12$ , we have r = -0.186710441... - i0.251574161... and

$$X_r^2 + 250X_r + 3125 = 269 = \sum_{n=1}^{\infty} \frac{(-j_r)^{n/3}}{n!} \left[ \frac{d^{n-1}}{dz^{n-1}} \left( 125 - \sqrt{12500 + z} \right)^{5n/3} \right]_{z=0}$$

## Example 3.

Consider the equation

$$X^{2} + 250X + 3125 = 2(-1)^{1/3}10^{2/3}X^{5/3}$$

Then clearly  $j = j_r = 800$  and a solution is

$$X = x_0 = -125 + \sqrt{12500 + \sum_{n=1}^{\infty} \frac{(2\sqrt[3]{100})^n}{n!} \left[ \frac{d^{n-1}}{dz^{n-1}} \left( 125 - \sqrt{12500 + z} \right)^{5n/3} \right]_{z=0}}$$

# 3 Applications

#### Example 4.

Set L = 1/3 then

$$w_N = w_1(L) = w_1\left(\frac{1}{3}\right) = \frac{1}{3}\sqrt{\frac{11}{78}}$$

and

$$k_N = x_L = x_1(L) = x_1\left(\frac{1}{3}\right) =$$

$$= \frac{\frac{1}{3}\sqrt{\frac{11}{78}}}{\left(\frac{-4(\frac{11}{13})^{1/6} + (\frac{13}{11})^{1/6}}{\sqrt{6}} + \frac{1}{2}\sqrt{4 + \frac{2}{3}\left(-4\left(\frac{11}{13}\right)^{1/6} + \left(\frac{13}{11}\right)^{1/6}\right)^2}\right)^2}$$

and

$$k_{25N} = x_2(L) = x_2\left(\frac{1}{3}\right) =$$

$$= \frac{1}{3}\sqrt{\frac{11}{78}} \left(\frac{-4(\frac{11}{13})^{1/6} + (\frac{13}{11})^{1/6}}{\sqrt{6}} + \frac{1}{2}\sqrt{4 + \frac{2}{3}\left(-4\left(\frac{11}{13}\right)^{1/6} + \left(\frac{13}{11}\right)^{1/6}\right)^2}\right)^2$$

where the N is given by

$$N = r_L = r_{1/3} = \frac{K^2 \left( \sqrt{1 - x_1 \left(\frac{1}{3}\right)^2} \right)}{K^2 \left( x_1 \left(\frac{1}{3}\right) \right)}$$

From the value of  $x_L$  we obtain  $j_{r_L}$  and hence the corresponding  $C_1$  in radicals-closed form and hence  $X = X_L$  from (37) and (31). The numbers a, b take arbitrary values.

We note here that in future application of this method one must tabulate values of  $(r, w_r)$  and not  $j_r$  or  $k_r$  which follow from these of  $w_r$ . This can be done in some cases using the Main Theorem in [16] and the solution (37) of Theorem 2 of the present paper.

Form [16] we have if

$$Q(x) = \frac{\left(-1 - e^{\frac{1}{5}y} + e^{\frac{2}{5}y}\right)^5}{\left(e^{\frac{1}{5}y} - e^{\frac{2}{5}y} + 2e^{\frac{3}{5}y} - 3e^{\frac{4}{5}y} + 5e^y + 3e^{\frac{6}{5}y} + 2e^{\frac{7}{5}y} + e^{\frac{8}{5}y} + e^{\frac{9}{5}y}\right)}$$
(49)

$$y = \operatorname{arcsinh}\left(\frac{11+x}{2}\right)$$

$$Y = U_0(X) = \sqrt{-\frac{5}{3X^2} + \frac{25}{3X^2h(X)} + \frac{X^4}{h(X)} + \frac{h(X)}{3X^2}}$$
 (50)

$$h(x) = \left(-125 - 9x^6 + 3\sqrt{3}\sqrt{-125x^6 - 22x^{12} - x^{18}}\right)^{1/3}$$

$$U_1(Y) = X = \sqrt{-\frac{1}{2Y^2} + \frac{Y^4}{2} + \frac{\sqrt{1 + 18Y^6 + Y^{12}}}{2Y^2}}.$$
 (51)

and

$$P(x) = P[x] = U_0[Q^{1/6}[U_1[x]]] \text{ and } P^{(n)}(x) = (P \underbrace{\circ \dots \circ}_n P)(x)$$
 (52)

then

$$k_{25^{n}r_{0}} = \sqrt{1/2 - 1/2\sqrt{1 - 4\left(k_{r_{0}}k'_{r_{0}}\right)^{2} \prod_{j=1}^{n} P^{(j)} \left[\sqrt[12]{\frac{k_{r_{0}}k'_{r_{0}}}{k_{r_{0}/25}k'_{r_{0}/25}}}\right]^{24}}$$
(53)

Example 5.

$$k_{1/5} = \sqrt{\frac{9 + 4\sqrt{5} + 2\sqrt{38 + 17\sqrt{5}}}{18 + 8\sqrt{5}}}$$

$$k_5 = \sqrt{\frac{9 + 4\sqrt{5} - 2\sqrt{38 + 17\sqrt{5}}}{18 + 8\sqrt{5}}}$$

$$k_{125} = \sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1 - (9 - 4\sqrt{5})P[1]^2}}$$
(54)

Example 6.

It is

$$k_1 = \frac{1}{\sqrt{2}}$$

$$k_{25} = \frac{1}{\sqrt{2\left(51841 + 23184\sqrt{5} + 12\sqrt{37325880 + 16692641\sqrt{5}}\right)}}$$

Hence

$$k_{625}k'_{625} = \frac{1}{2(161 + 72\sqrt{5})}P\left[161 - 72\sqrt{5}\right]$$

and hence

$$k_{625} = \sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1 - \left(\frac{P\left[161 - 72\sqrt{5}\right]}{161 + 72\sqrt{5}}\right)^2}}$$
 (55)

By this way we can evaluate every  $k_r$  which is  $r=4^l9^m25^nr_0$  when  $k_{r_0}$  and  $k_{r_0/25}$  are known,  $l,m,n\in\mathbf{N}$ .

**Note.** In the case that

$$\left(\frac{L}{M}\right)^{1/6} = A = \frac{f}{g} = 3^a \frac{2p+1}{2h+1},\tag{56}$$

where f, g positive integers, with f < g and  $p, h \neq 0 \pmod{4}$ ,  $(a \in \mathbf{Z} - \{-1, 0, 1\})$ , we can find w from a given  $x = k_r$  which is of the form

$$x = k_r = \frac{t_1 \sqrt{t_2}}{\left(t_3 + \sqrt{t_4}\right)^2}$$

where  $t_i$ , i = 1, 2, 3, 4 rationals.

In view of (25) and the action of the command recognize, (which is needed to put number x into his form) the output will be an octic equation with step 2 containing nested square roots:

<< Number Theory' Recognize'

$$Solve[Reduce[N[x,1000],16,v]==0,v]$$

The smallest root it will be

$$\sqrt{D} = \sqrt{4096 + 88\left(3^a \frac{2p+1}{2h+1}\right)^6 + \left(3^a \frac{2p+1}{2h+1}\right)^{12}}$$

One can see that for these x's the f and g are given from the Diofantie equation

$$9D = g^{12} \left( 4096 + 88 \frac{f^6}{g^6} + \frac{f^{12}}{g^{12}} \right) \tag{57}$$

hence the number A will be known and

$$w^{2} = \frac{4096 - 20A^{6} + A^{12} + (-64 + A^{6})\sqrt{4096 + 88A^{6} + A^{12}}}{108A^{6}}$$
 (58)

Hence we have the value of X in radicals.

If for example

$$x = k_r = \frac{-37754085\sqrt{3} + 3\sqrt{476791023769787}}{77\sqrt{2}\left(-435 + \sqrt{224799}\right)^2}$$

then for all  $C_1$ , a, b such that

$$j_r = \frac{256(x^2 + (1 - x^2)^2)^3}{x^4(1 - x^2)^2} = 250\frac{C_1^3}{a^2b}$$

with Mathematica and the package 'Recognize' we evaluate

$$Solve[Recognize[N[x, 1000], 16, v] == 0, v]$$

which gives the value of x in the desired form. The solution that corresponds to x have smallest square root

$$\sqrt{D} = \sqrt{1430373071309361}$$

the command 'Reduce' give us the f and g

$$Reduce[9D == (4096 + 88(f/g)^6 + (f/g)^{12})g^{12}, \{f, g\}, Integers]$$

Hence we get the values f = 7, g = 11 and w. The solution (29) is

$$X = \frac{A}{35153041^3} [5579801448 - 11724990\sqrt{224799} +$$

$$+\sqrt{6362897839 \left(9487950991-20011160 \sqrt{224799}\right)}]^3$$

where

$$A = A_1 + B_1 - \frac{1}{2}\sqrt{A_2 + B_2\sqrt{224799}}$$

 $A_1 = \frac{93573266991461291403517623659291588}{29148873138738228269392700625}$ 

9155201445598248892391809778079434882871423635278599557

= \frac{100475821799491332014433982488923918097780794348828714230332783993373}{849656805258255011387371218620407164266412150030875390625}

 $B_{2} = \frac{12578872085638673940246389099496155002610886190948544424950380549}{12578872085638673940246389099496155002610886190948544424950380549}$ 

#### Theorem 8.

When  $(L/M)^{1/6}$  is rational, we can always find  $k_{25r}$  from  $k_r$ .

## Example 7.

Set  $a = (k_r k_r')^2$ , b = (ww'), then equation (1) have solution

$$X = \frac{m_5^3}{250},$$

where  $m_5$  is the multiplier (see [3]).

Hence

$$3125\frac{(ww')^4}{(k_rk_r')^2} + 250(ww')^2m_5^3 + (k_rk_r')^2m_5^6 = (k_rk_r')^{4/3}(ww')^{2/3}j_r^{1/3}m_5^5$$

## Example 8.

We will find a solution of the equation

$$\frac{3125}{16} + 125X + 4X^2 = 132X^{5/3} \tag{59}$$

in radicals using Theorem 3.

## Solution.

It is  $a=4,\,b=125,$  and we have to solve  $j_r=287496$  or equivalently r=4. Hence a solution of (59) is:

$$X = \frac{125e^{4\pi}}{250 \cdot 4} \frac{f(-e^{-4\pi})^6}{f(-e^{-20\pi})^6} = \frac{1}{8} \left( R(e^{-4\pi})^{-5} - 11 - R(e^{-4\pi})^5 \right)$$
 (60)

The exact root in radicals can be found but is very large and complicated with our method. We give a way how one can obtain it:

It is known that

$$R(e^{-2\pi}) = -\frac{1+\sqrt{5}}{2} + \sqrt{\frac{5+\sqrt{5}}{2}}$$
 (61)

But from the duplication formula (see [4],[13]): If u = R(q) and  $\nu = R(q^2)$ , then

$$\frac{\nu - u^2}{\nu + u^2} = u\nu^2. \tag{62}$$

Hence we find the value of  $R(e^{-4\pi})$  in radicals and hence the solution of (59) using (60),(61),(62).

The root using the program Mathematica is

$$X = \frac{143375}{16} + \frac{64125\sqrt{5}}{16} + \frac{1}{2}\sqrt{\frac{20553203125}{32} + \frac{9191671875\sqrt{5}}{32}}$$

In this case it is more convenient to use Mathematica's command Solve. But in other cases these solutions can not found.

From the above result we have shown that

$$\frac{1}{8} \left( R(e^{-4\pi})^{-5} - 11 - R(e^{-4\pi})^5 \right) =$$

$$= \frac{143375}{16} + \frac{64125\sqrt{5}}{16} + \frac{1}{2} \sqrt{\frac{20553203125}{32} + \frac{9191671875\sqrt{5}}{32}}.$$

One can see that if we set

$$Y_{\tau} := \frac{b}{250a} \left( R \left( e^{2\pi i \tau} \right)^{-5} - 11 - R \left( e^{2\pi i \tau} \right)^{5} \right)$$
 : (a)

Then if

$$\tau = \frac{1 + \sqrt{-r}}{2}$$
 or  $\tau = \sqrt{-r}$ ,

where r positive integer in some cases we can evaluate  $Y_{\tau}$  solving directly the equation (1), with parameters  $a=4,\,b=125$  and  $C_1$  depended on  $j_{\tau}$ . Some examples are

$$\frac{1}{16} \left[ R \left( e^{\pi(i - \sqrt{51})} \right)^{-5} - 11 - R \left( e^{\pi(i - \sqrt{51})} \right)^{5} \right] =$$

$$= -\frac{125}{16} \left[ 5541103 + 1343914\sqrt{17} + \sqrt{61407604829690 + 14893531819350\sqrt{17}} + 4\sqrt{\left\{ \frac{1}{6140760482969 + 1489353181935\sqrt{17}} (94272348104055803848937570 + 22864402871059934148609270\sqrt{17} + 22663164169063100077\sqrt{170} \left( 6140760482969 + 1489353181935\sqrt{17} \right) + 8506643792036854023\sqrt{61407604829690 + 14893531819350\sqrt{17}} \right].$$
(63)

$$Y_{\sqrt{-1/5}} = \frac{5\sqrt{5}}{8}. (64)$$

$$Y_{\sqrt{-2/5}} = \frac{5}{8} \left( 5 + 2\sqrt{5} \right). \tag{65}$$

$$Y_{\sqrt{-3/5}} = \frac{5}{16} \left( 25 + 11\sqrt{5} \right). \tag{66}$$

$$Y_{\sqrt{-4/5}} = \frac{5}{16} \left( 25 + 13\sqrt{5} + 5\sqrt{58 + 26\sqrt{5}} \right). \tag{67}$$

$$Y_{\sqrt{-5/5}} = \frac{125}{8} \left( 2 + \sqrt{5} \right). \tag{68}$$

$$Y_{\sqrt{-6/5}} = \frac{5}{8} \left( 50 + 35\sqrt{2} + 3\sqrt{5(99 + 70\sqrt{2})} \right). \tag{69}$$

$$Y_{\sqrt{-9/5}} = \frac{5}{8} \left( 225 + 104\sqrt{5} + 10\sqrt{1047 + 468\sqrt{5}} \right). \tag{70}$$

$$Y_{\sqrt{-12/5}} = \frac{5}{16} \left( 1690 + 975\sqrt{3} + 29\sqrt{6755 + 3900\sqrt{3}} \right). \tag{71}$$

$$Y_{\sqrt{-14/5}} = \frac{5}{8} \left( 1850 + 585\sqrt{10} + 7\sqrt{5\left(27379 + 8658\sqrt{10}\right)} \right). \tag{72}$$

$$Y_{\sqrt{-17/5}} = \frac{5}{8} \left( 5360 + 585\sqrt{85} + 4\sqrt{3613670 + 391950\sqrt{85}} \right). \tag{73}$$

We describe the method bellow.

For some r positive rational we find the value of  $j_{r/5}$ ; this can be done with the command 'Recognize' of the program Mathematica (if  $j_{r/5}$  is root of a small degree algebraic polynomial equation). Then we find  $C_1$  (from (16)) and for the values a=4, b=125 there will be

$$Y_{\tau} = \text{root of equation (1)}.$$

In many cases of such r, equation (1) can solved in radicals with Mathematica (we have not find the reason yet), but still in others not. Hence we get relations like (63)-(73).

## 4 More Theorems and Results

Theorem 9. (Conjecture)

For every positive real r, we have

$$Y_{\sqrt{-r/5}}Y_{\sqrt{-r^{-1/5}}} = \frac{125}{64}. (74)$$

If l, m, t and d are integers and

$$Y_{\sqrt{-r/5}} = \frac{l + m\sqrt{d}}{t} \tag{75}$$

then

$$l^2 - m^2 d = t^2 \frac{125}{64} \tag{76}$$

In general we conjecture that

## **Theorem 10.** (Conjecture)

If  $r = a_1/b_1$  with  $a_1, b_1 \in \mathbf{N}$  and  $GCD(a_1, 5) = 1$ ,  $GCD(b_1, 5) = 1$  then

$$deg\left(Y_{\sqrt{-r/5}}\right) = deg\left(j_{\sqrt{-r/5}}\right) \tag{77}$$

For example if  $deg\left(Y_{\sqrt{-r/5}}\right) = 4$ , then

$$Y_{\sqrt{-r/5}} = A + B\sqrt{D} \tag{78}$$

where deg(A) = deg(D) = 2 and

$$A^2 - B^2 D = \frac{125}{64} U \tag{79}$$

where deg(U)=2 or U=1. If  $U\neq 1$  then  $U=l+m\sqrt{d}$  and also if  $j_{\sqrt{-r/5}}$  have smallest nested square root  $\sqrt{d}$ , then  $UU^*=l^2-m^2d=1$ . The symbol \* denotes the algebraic conjugate.

Hence for example if r = 6 then d = 2 and

$$j_{\sqrt{-6/5}} = 8640[25551735275 - 18067805280\sqrt{2} -$$

$$-196\sqrt{10\left(3399058140008707-2403497060447490\sqrt{2}\right)}]$$

then  $U = l + m\sqrt{2}$  with

$$l^2 - 2m^2 = 1.$$

We solve the above Pell's equation. The solution we looking for, taking the smallest to higher order solutions, for this example with r=6 is  $l_1=99$  and  $m_1=70$ . Hence  $A^2-B^2D=\frac{125}{64}(99+70\sqrt{2})$ .

Now we assume that  $A = k_1 + l_1 \sqrt{d}$ , again with d = 2 and  $D = k_2 + l_2 \sqrt{d}$ , etc... We proceed solving Pell's equations.

## Theorem 11.

For a given  $r \in \mathbb{N}$  and  $deg\left(Y_{\sqrt{-r/5}}\right) = 2, 4, \text{ or } 8, \text{ if the smallest nested root of }$ 

 $j_{\sqrt{-r/5}}$  is  $\sqrt{d}$  then we can evaluate the Rogers-Ramanujan continued fraction with integer parameters.

i) In the case  $deg\left(Y_{\sqrt{-r/5}}\right) = 2$  then

$$Y_{\sqrt{-r/5}} = \frac{l + m\sqrt{d}}{t} \tag{80}$$

where

$$l^2 - m^2 d = 1 \text{ and } l, m, d \in \mathbf{N}$$
 (81)

- ii) In the case  $deg\left(Y_{\sqrt{-r/5}}\right) = 4$  we have
- a) If  $U \neq \frac{125}{64}$ , then

$$Y_{\sqrt{-r/5}} = \frac{5}{8} \sqrt{\left(a_0 + \sqrt{-1 + a_0^2}\right)} \left(\sqrt{5 + p} - \sqrt{p}\right)$$
 (82)

where

$$Y_{\sqrt{-r/5}}Y_{\sqrt{-r/5}}^* = \frac{125}{64} \left( a_0 + \sqrt{a_0^2 - 1} \right), \tag{83}$$

with,  $a_0$  positive integer, is solution of  $l^2 - m^2 d = 1$ . Hence  $l = a_0$  and  $m = d^{-1/2} \sqrt{a_0^2 - 1}$  is positive integer. The parameter p is positive rational can be found from the numerical value of  $Y_{\sqrt{-r/5}}$ .

**b)** If  $U = \frac{125}{64}$ , then

$$Y_{\sqrt{-r/5}} = A + \frac{1}{8}\sqrt{-125 + 64A^2},\tag{84}$$

where we set  $A = k + l\sqrt{d}$ . Then a starting point for the evaluation of the integers k, l will be the relation

$$l^2 = \frac{(A-k)^2}{d} = \text{ square of integer}$$
 (85)

iii) If  $deg\left(Y_{\sqrt{-r4^{-1}5^{-1}}}\right) = 4$ , then we can evaluate  $Y_{\sqrt{-r5^{-1}}}$ .

It holds  $deg\left(Y_{\sqrt{-r^{5^{-1}}}}\right)=8$ , the minimal polynomial of  $Y_{\sqrt{-r^{5^{-1}}}}/Y_{\sqrt{-r^{4^{-1}5^{-1}}}}$  is of degree 4 or 8 and symmetric. Hence it can be reduced in at most 4th degree polynomial, hence it is solvable. Thus it remains the evaluation of  $Y_{\sqrt{-r^{4^{-1}5^{-1}}}}$ , which can be done with the help of step (ii).

$$Y_{\sqrt{-r5^{-1}}} = \frac{5}{8} \sqrt{a_0 + \sqrt{-1 + a_0^2}} \left( \sqrt{p+5} - \sqrt{p} \right) 2^{-1} \left( \sqrt{x+4} - \sqrt{x} \right)$$
 (86)

where  $x = a_1 + b_1\sqrt{d} + c\sqrt{a_2 + b_2\sqrt{d}}$ ,  $a_1$ ,  $b_1$ ,  $a_2$ ,  $b_2$ , c integers and

$$Y_{\sqrt{-r5^{-1}4^{-1}}} = \frac{5}{8}\sqrt{a_0 + \sqrt{-1 + a_0^2}} \left(\sqrt{p+5} - \sqrt{p}\right)$$

## Example 9.

For  $r = 68 = 4 \cdot 17$  and from (73) we have d = 85

$$x = a_1 + b_1 \sqrt{85} + c\sqrt{a_2 + b_2 \sqrt{85}}$$

$$Y_{\sqrt{-68/5}}/Y_{\sqrt{-17/5}} = 2^{-1} \left(\sqrt{x+4} - \sqrt{x}\right)$$

$$a_1 = 2891581250, b_1 = 313636050, c = 12960$$

$$a_2 = 99557521554, b_2 = 10798529365$$

hence

$$Y_{\sqrt{-68/5}} = Y_{\sqrt{-17/5}} 2^{-1} \left( \sqrt{x+4} - \sqrt{x} \right) =$$

$$= \frac{5}{16} \left( 5360 + 585\sqrt{85} + 4\sqrt{3613670 + 391950\sqrt{85}} \right) \left( \sqrt{x+4} - \sqrt{x} \right)$$

#### Theorem 12.

If  $r = a_1/b_1$  with  $deg(j_{r/5}) = \nu \le 4$ , then  $deg(A_{r/5}) = \nu$  and equation (1) (with a, b rationals) can solved in radicals.

## Application.

If r = 3/4 then  $deg(j_{3/20}) = 4$  and  $A_{r/5}$  is solution of

$$15625 - 2112500v + 443375v^2 - 16900v^3 + v^4 = 0$$

hence

$$A_{3/20} = R \left( e^{-\pi\sqrt{3/5}} \right)^{-5} - 11 - R \left( e^{-\pi\sqrt{3/5}} \right)^{5} =$$

$$= \frac{5}{2} \left( 1690 - 975\sqrt{3} + 29\sqrt{6755 - 3900\sqrt{3}} \right)$$

## Theorem 13.

If  $Q(x) := x^5$  then

$$\frac{1}{j_{\tau}^{1/3}} \left[ R \left( e^{2\pi i \tau} \right)^{-5} - 11 - R \left( e^{2\pi i \tau} \right)^{5} \right]^{1/3} = \sqrt[3]{\frac{-125}{j_{\tau}}} + \sqrt{\frac{12500}{j_{\tau}^{2}}} + Q \left( \sqrt[3]{\frac{-125}{j_{\tau}}} + \dots \right)$$
(87)

#### Proof.

Equation (1) for  $a=1,\,b=250j_{\tau}^{-1},\,C_1=1$  can be written in the form

$$(X^3 - a_1)^2 - b_1 = X^5 + c_1, (88)$$

where  $a_1 = -125j_{\tau}^{-1}$ ,  $b_1 = 12500j_{\tau}^{-2}$ ,  $c_1 = 0$ 

Hence  $Y_{\tau}$  we can be expressed in nested periodical functions. This completes the proof.

Example 10.

If

$$C_1^3 = 32a^2b$$

then

$$X = \frac{b}{250a} \left( R(e^{-2\pi\sqrt{2}})^{-5} - 11 - R(e^{-2\pi\sqrt{2}})^5 \right)$$

## Equation (1) and the Derivative of Rogers-Ramanujan Continued fraction

From [11] it is known that if

$$N(q) = q^{5/6} f(-q)^{-4} \frac{R'(q)}{R(q)}$$
(89)

and  $N(q^2) = u(q) = u$ ,  $N(q^3) = h(q) = h$  and N(q) = v(q) = v, then

$$5u^6 - u^2v^2 - 125u^4v^4 + 5v^6 \stackrel{?}{=} 0 (90)$$

and

$$125h^{12} + h^3v^3 + 1125h^9v^3 + 1125h^3v^9 + 1953125h^9v^9 - 125v^{12} \stackrel{?}{=} 0 \tag{91}$$

which are solvable. But from [12] we have

$$\frac{5R'(q)}{R(q)(R(q)^{-5} - 11 - R(q)^5)^{1/6}} = f^4(-q)q^{-5/6}$$
(92)

or

$$N(q) = \frac{1}{5} \left( R(q)^{-5} - 11 - R(q)^5 \right)^{1/6}$$
 (93)

Hence the solution of (1) can also given in the form

$$X = X_r = \frac{125b}{2a}N(q^2)^6 \tag{94}$$

and from (85) we have

$$2^{2/3}a^{1/3}b^{1/3}(X_rX_{4r})^{1/3} + \frac{10 \cdot 2^{1/3}a}{b^{1/3}}(X_rX_{4r})^{2/3} - 2a^{2/3}(X_r + X_{4r}) = 0 \quad (95)$$

## Note.

One can prove relation (90) using (89),(93) and the duplication formula (see [13]):

$$\frac{R(q^2) - R^2(q)}{R(q^2) + R^2(q)} = R(q)R^2(q^2)$$
(96)

The same method can work and with other higher modular equations of the derivative but the evaluations are very difficult even for a program.

Another interesting note that can simplify the problem is the singular moduli of the fifth base (see [14],[15]):

$$u(x) = {}_{2}F_{1}\left(\frac{1}{6}, \frac{5}{6}; 1; x\right). \tag{97}$$

In this case we have

$$j_r = \frac{432}{\beta_r (1 - \beta_r)} = \frac{250C_1^3}{a^2 b},\tag{98}$$

where  $\beta_r$  is the solution of

$$\frac{{}_{2}F_{1}\left(\frac{1}{6},\frac{5}{6};1;1-\beta_{r}\right)}{{}_{2}F_{1}\left(\frac{1}{6},\frac{5}{6};1;\beta_{r}\right)} = \sqrt{r}$$

$$(99)$$

The moduli  $\beta_r$  can evaluated from  $k_r$  and the opposite from the relation

$$\frac{256(k_r^2 + (1 - k_r^2)^2)^3}{k_r^4 (1 - k_r^2)^2} = \frac{432}{\beta_r (1 - \beta_r)}$$
(100)

## Proposition 2.

The equation (1) have solution

$$X = \frac{b}{250a} \left( R \left( e^{-2\pi\sqrt{\beta^{(-1)}(\alpha)}} \right)^{-5} - 11 - R \left( e^{-2\pi\sqrt{\beta^{(-1)}(\alpha)}} \right)^{5} \right), \tag{101}$$

where

$$\alpha = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{216a^2b}{125C_1^3}}, \ \beta^{(-1)}(x) = \left(\frac{u(1-x)}{u(x)}\right)^2.$$

#### Corollary.

The equation

$$aX^{2} + bX + \frac{b^{2}}{20a} = \frac{6a^{2/3}b^{1/3}}{5\beta_{r}^{1/3}(1 - \beta_{r})^{1/3}}X^{5/3}$$

admits solution  $X = A_r$ .

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