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ON THE RELATIVISTIC VELOCITY COMPOSITION PARADOX AND THE THOMAS ROTATION

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The non–commutativity and the non–associativity of the composition law of the non–colinear velocities lead to an apparent paradox, which in turn is solved by the Thomas rotation. A 3×3 parametric, unimodular and orthogonal matrix elaborated by Ungar is able to determine the Thomas rotation. However, the algebra involved in the derivation of the Thomas rotation matrix is overwhelming. The aim of this paper is to present a direct derivation of the Thomas angle as the angle between the composite vectors of the non–colinear velocities, thus obtaining a simplicity with which the rotation can be expressed. This allows the formulation of an alternative to the statement related to the necessity of the Thomas rotation of the Cartesian axes by the statement implying the necessity of the rotation of the direct (inverse) relativistic composite velocity to coincide with the inverse (direct) relativistic composite velocity.

Key words: Special relativity, Thomas rotation.

1. INTRODUCTION

As Thomas pointed out,[1,3] two successive Lorentz transformations are not equivalent to a pure Lorentz transformation, but to a pure Lorentz transformation preceded, or followed, by a rotation (or precession) of the space coordinates. It is well known that a Lorentz transformation in one time dimension and one space dimension give rise to a group structure for the set of relativistically admissible velocities. In order to extend the 1+1 Lorentz transformation to one with 1+3 dimensions, the Thomas rotation is inevitable involved. This, in turn, provides the relativistic interpretation of the non–commutative and non–associative composition law of non–colinear velocities. The difficulties arising from the lack of commutativity and associativity of the composition law of the non–colinear velocities (denoted by Ungar as a paradox[4] but not listed by Sastry[5]) have been discussed in.[6] Denying the impact of the difficulties revealed in,[6] Ungar in a recently series of papers[4,7–14] not only resolved the relativistic velocity composition paradox, by means of Thomas rotation, but elaborated a very interesting formalism underlying a non–associative group structure for relativistically admissible velocities. A brief review of the relativistic composition paradox is the subject of Sec. 2 while the Sec. 3 is devoted to Ungar's solution of the Thomas rotation angle ε_U. The direct derivation of the angle ε_M between the composite velocities $\mathbf{u} \oplus \mathbf{v}$ and $\mathbf{v} \oplus \mathbf{u}$ of the non–colinear velocities $\mathbf{u}$ and $\mathbf{v}$ is the subject of Sec. 4. Comments of the results are discussed in the last section.

2. *THE MOCANU PARADOX*

Let Σ'', Σ', Σ be three inertial frames in standard form, i.e., their homologous axes are parallel and in direct succession. The velocity of Σ'' relative to Σ' is $\mathbf{v}$, while the velocity of Σ' relative to Σ is $\mathbf{u}$, as shown in Fig. 1a.

The velocity $\mathbf{v}_d = \mathbf{u} \oplus \mathbf{v}$ of Σ'' relative to Σ is given by[4]

$$\mathbf{u}_d = \mathbf{u} \oplus \mathbf{v} = \frac{\mathbf{u} + \mathbf{v}}{1 + \mathbf{u} \cdot \mathbf{v} / c^2} + \frac{1}{c^2} \frac{\gamma_u}{\gamma_u + 1} \frac{\mathbf{u} \times (\mathbf{u} \times \mathbf{v})}{1 + \mathbf{u} \cdot \mathbf{v} / c^2} \qquad \mathbf{u}, \mathbf{v} \in R_c^3,$$

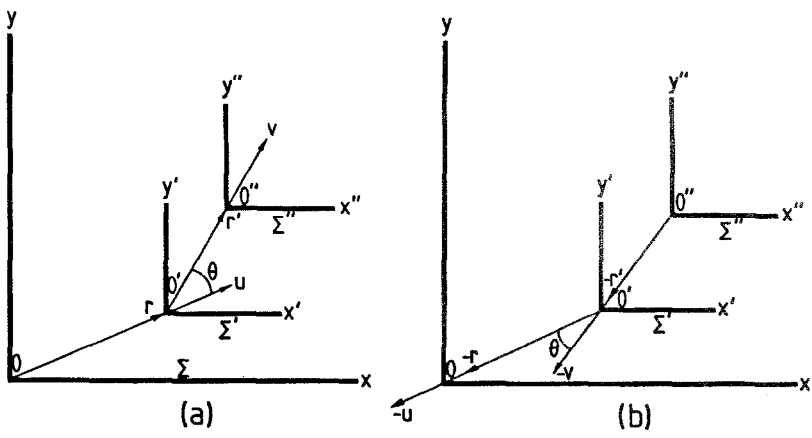

Fig. 1 (a) Frame Σ'' moves with velocity $\mathbf{v}$ relative to frame Σ', while Σ' moves with velocity $\mathbf{u}$ relative to frame Σ. As observed by an observer on Σ', the axes of frame Σ'' are parallel to those of frame Σ', and the axes of frame Σ' are parallel to those of frame Σ, (b) Frame Σ moves with velocity $-\mathbf{u}$ relative to frame Σ', while frame Σ' moves with velocity $-\mathbf{v}$ relative to frame Σ''. Time and one space dimension are suppressed for clarity.

and represents the direct relativistic composite of the non–colinear velocities **u** and **v**. In a slightly different form, $\mathbf{u}_d$ may be expressed as follows:[6]

$$\mathbf{u}_d = \mathbf{u} \oplus \mathbf{v} = \frac{\gamma_u \mathbf{u} + \mathbf{v} + \kappa_u \mathbf{u}\,(\mathbf{u}\cdot\mathbf{v})u^{-2}}{\gamma_u (1 + \mathbf{u}\cdot\mathbf{v}/c^2)}, \tag{1}$$

where

$$R_c^3 = \{\mathbf{u} \in R^3 \;;\; |\mathbf{u}| < c\}$$

is the set of 3 vectors in the Euclidean 3–space R^3 with magnitude smaller than the speed of light in free space, called the relativistically admissible velocity space; $\oplus$ is the relativistic velocity composition operator; $\times$ and $\cdot$ denote the usual vector and scalar product operator, and γ_u, defined by

$$\gamma_u = (1 - u^2/c^2)^{-1/2}, \quad \kappa_u = \gamma_u - 1, \tag{2}$$

is the Lorentz factor. Now let Σ, Σ', Σ'' be the set of three inertial frames in an inverse succession, such that the velocity of Σ relative to Σ' is $-\mathbf{u}$, while the velocity of Σ' relative to Σ'' is $-\mathbf{v}$, as shown in Fig. 1b. The velocity of Σ relative to Σ'' is given by[4]

$$-\mathbf{u}_i = \mathbf{v} \oplus \mathbf{u} = \frac{\mathbf{v} + \mathbf{u}}{1 + \mathbf{u}\cdot\mathbf{v}/c^2} + \frac{\gamma_v}{\gamma_v + 1}\,\frac{\mathbf{v} \times (\mathbf{v} \times \mathbf{u})}{1 + \mathbf{u}\cdot\mathbf{v}/c^2},$$

or, in a slightly different form,[6]

$$-\mathbf{u}_i = \mathbf{v} \oplus \mathbf{u} = \frac{\gamma_v \mathbf{v} + \mathbf{u} + \kappa_u \mathbf{v}\,(\mathbf{u}\cdot\mathbf{v})v^{-2}}{\gamma_v (1 + \mathbf{u}\cdot\mathbf{v}/c^2)} \tag{3}$$

is the inverse relativistic composite of the non–colinear velocities **v** and **u**, where

$$\gamma_v = (1 - v^2/c^2)^{-1/2}; \quad \kappa_v = \gamma_v - 1. \tag{4}$$

Although the composite velocities $\mathbf{u}_d$ and $\mathbf{u}_i$ have equal magnitudes[4,6]

$$(\mathbf{u} \oplus \mathbf{v})^2 = (\mathbf{v} \oplus \mathbf{u})^2 = \left|\frac{\mathbf{u} + \mathbf{v}}{1 + \mathbf{u}\cdot\mathbf{v}/c^2}\right|^2 - \frac{1}{c^2}\left|\frac{\mathbf{u} \times \mathbf{v}}{1 + \mathbf{u}\cdot\mathbf{v}/c^2}\right|^2, \tag{5}$$

they have different orientations,

$$\mathbf{u}_d \neq \mathbf{u}_i\,. \tag{6}$$

At first sight the inequality (6) does not comply with Einstein's principle of velocity reciprocity (EPVR), which he formulated as follows: "We postulate that the relation between the coordinates of the two systems is linear. Then the inverse transformation is also linear and the complete non–preference of the one or the other system demands that the transformation shall be identical with the original one, except for a change of **v** to –**v**."[15] By (1), the velocity of Σ'' relative to Σ is $\mathbf{u}_d$, while the velocity of Σ relative to Σ'' is $\mathbf{u}_i$. However, if **u** and **v** are non–colinear, then the inequality (6) holds. Thus we encounter the difficulty which Ungar[4] calls the Mocanu paradox: Which one is the "correct" velocity of Σ'' relative to Σ? It is $\mathbf{u} \oplus \mathbf{v}$ or $\mathbf{v} \oplus \mathbf{u}$? As a consequence, the Mocanu paradox rests on the impossibility of deriving, in STR, "a unique and coherent transformation law of non–colinear velocities."[6] Following Feynman,[16] "a paradox is a situation which gives one answer when analyzed one way, and a different answer when analyzed another way, so that we are left in somewhat of a quandary as to actually what would happen. Of course, in physics there are never any real paradoxes because there is one correct answer." Thus in physics "a paradox is only a confusion in our understanding." According to Ungar,[4,7–14] "the physical phenomenon which saves STR from the difficulties presented by the Mocanu paradox turns to be the Thomas rotation." Following a brief presentation of the Thomas rotation and some of its properties, the Ungar's solution of the Mocanu paradox will be presented in the next section.

3. THE UNGAR SOLUTION OF THE MOCANU PARADOX

Let (Σ_0, Σ) be a directly ordered pair of inertial frames, where Σ_0 is assumed to move uniformly with velocity **v**, with respect to Σ. A pure Lorentz transformation, also called "boost," is a Lorentz transformation without rotation of the space–time coordinates of an event measured in Σ_0 and Σ. The transformation equations, expressed in vector form, are given by[6,17–18]

$$\mathbf{r} = \mathbf{r}_0 + c^{-2}\gamma^2(\gamma+1)^{-1}\,\mathbf{v}\,(\mathbf{r}_0\cdot\mathbf{v}) + \gamma\,\mathbf{v}\,t_0,$$

$$t = \gamma\,(t_0 + \mathbf{r}_0\cdot\mathbf{v}\,/\,c_0^2),$$

or, in matrix form,

$$\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = B(\mathbf{v}) \begin{bmatrix} t_0 \\ x_0 \\ y_0 \\ z_0 \end{bmatrix}, \tag{7}$$

where $\mathbf{r}$ ($\mathbf{r}_0$) is the position vector at time t (t_0), x, y, z (x_0, y_0, z_0) are the corresponding Cartesian components of $\mathbf{r}$ ($\mathbf{r}_0$) relative to Σ (Σ_0), and B (**v**) is the boost of Eq. (7) parametrized by a velocity parameter $\mathbf{v}$ (v_x, v_y, v_z) with the matrix representation

$$B(\mathbf{v}) = \begin{bmatrix} \gamma \ , & c^{-2}\gamma v_x & , & c^{-2}\gamma v_y & , & c^{-2}\gamma v_z \\ \gamma v_x \ , 1+c^{-2}\gamma^2(\gamma+1)^{-1}v_x^2 & , & c^{-2}\gamma^2(\gamma+1)^{-1}v_x v_y & , & c^{-2}\gamma^2(\gamma+1)^{-1}v_x v_z \\ \gamma v_y \ , c^{-2}\gamma^2(\gamma+1)^{-1}v_y v_x & , & 1+c^{-2}\gamma^2(\gamma+1)^{-1}v_y^2 & , & c^{-2}\gamma^2(\gamma+1)^{-1}v_y v_z \\ \gamma v_z \ , c^{-2}\gamma^2(\gamma+1)^{-1}v_z v_x & , & c^{-2}\gamma^2(\gamma+1)^{-1}v_z v_y & , & 1+c^{-2}\gamma^2(\gamma+1)^{-1}v_z^2 \end{bmatrix}. \quad (8)$$

As Møller pointed out,[17] in general, two successive boosts are not equivalent to a boost; the combination of two successive boosts is equivalent to a net boost preceded or followed by a Thomas rotation[7]

$$B(\mathbf{u})\ B(\mathbf{v}) = B(\mathbf{u} \oplus \mathbf{v})\ \mathrm{Tom}[\mathbf{u};\mathbf{v}], \quad (9a)$$

and

$$B(\mathbf{u})\ B(\mathbf{v}) = \mathrm{Tom}[\mathbf{u};\mathbf{v}]\ B(\mathbf{v} \oplus \mathbf{u}), \quad (9b)$$

where Tom [**u**; **v**] is a 4×4 matrix representing space rotation of time–space coordinates by a Thomas rotation, given by the equation

$$\mathrm{Tom}[\mathbf{u};\mathbf{v}] = \begin{bmatrix} 1 \ , & 0 \\ 0 \ , & \mathrm{tom}[\mathbf{u};\mathbf{v}] \end{bmatrix}. \quad (10)$$

According to Ungar,[4] in the Fig. 1a, the axes of both frames Σ'' and Σ have been constructed parallel to those of Σ' as seen by observers accompanying the moving system Σ'. Nevertheless, an observer in Σ sees the axes of Σ'' rotated relative to his own axes by a Thomas rotation angle ε_U shown in Fig. 2.

This Thomas rotation is said to be generated by the velocities **u** and **v** and has been denoted in Eqs. (9) and (10) by tom [**u**; **v**]. If the rotation angle from **u** to **v** is denoted by θ (Fig. 2), then the Thomas rotation angle ε_U is given by the equations[4,7–9]

$$\cos\varepsilon_U = \frac{(k+\cos\theta)^2 - \sin^2\theta}{(k+\cos\theta)^2 + \sin^2\theta}, \quad (11a)$$

$$\sin\varepsilon_U = \frac{-2(k+\cos\theta)\sin\theta}{(k+\cos\theta)^2 + \sin^2\theta}, \quad (11b)$$

where the sign minus in the expressions of sin ε_U is the consequence of the opposite signs of ε_U and θ in the interval $[-\pi, \pi]$, and k denotes

$$k^2 = \frac{\gamma_u + 1}{\gamma_u - 1}\,\frac{\gamma_v + 1}{\gamma_v - 1}. \quad (11c)$$

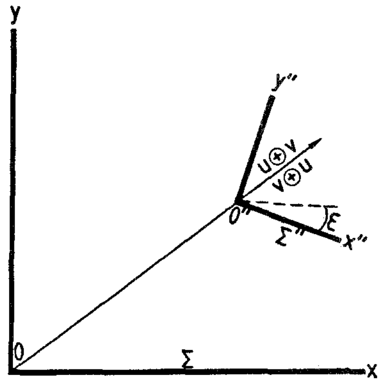

Fig.2 An observer in Σ sees the axes of Σ'' rotated relative to his own axes by a Thomas rotation angle ε_U.

In his words, the Ungar resolution of the Mocanu paradox is now clear: "Unlike the Galilean composite velocities $\mathbf{u} \oplus \mathbf{v}$ ($\mathbf{u}, \mathbf{v} \in R^3$), the relativistic composite velocities

$$\mathbf{u} \oplus \mathbf{v} \quad (\mathbf{u}, \mathbf{v} \in R_c^3)$$

embody space rotation tom [**u**; **v**]. When we say that the (composite) velocity of frame Σ'' relative to frame Σ is $\mathbf{u} \oplus \mathbf{v}$ (respectively $\mathbf{v} \oplus \mathbf{u}$) we mean that in the transition from the time–space coordinates of an event measured in Σ'' into the time–space coordinates of the event measured in Σ, we have to apply a boost with velocity parameter $\mathbf{u} \oplus \mathbf{v}$ (resp. $\mathbf{v} \oplus \mathbf{u}$) preceded (resp. followed) by the Thomas rotation tom [**u**; **v**]. Lorentz transformation in 1+3 dimensions cannot be parametrized by velocity alone. They are, rather, parametrized by velocities and orientations in such a way that composite Lorentz transformations correspond to composite velocities and composite orientations which are interrelated by Thomas rotation."[7]

There are various attempts in the literature to express the Thomas rotation in terms of its generating parameters, resulting in expressions having different forms.[19–29] The superiority of the expression in Eqs. (11) over other existing ones rests on the fact that it appears in a rotation matrix form to which standard results may be applied.

4. THE DERIVATION OF THE ANGLE BETWEEN COMPOSITE VELOCITIES

Ungar recognizes that the algebra involved in the attempt to express the Thomas rotation matrix tom [**u**; **v**], (10), is overwhelming, and in his own words "a Herculean task of the derivation of Eqs. (11) is necessary."[7] However, Fig. 2 (identical with the Fig. 2b in[4]) shows that the rotation of the axes of Σ'' related to Σ by the Thomas rotation angle ε_U implies the rotation of $\mathbf{u} \oplus \mathbf{v}$ (resp. $\mathbf{v} \oplus \mathbf{u}$) to coincide to $\mathbf{v} \oplus \mathbf{u}$ (resp. $\mathbf{u} \oplus \mathbf{v}$). This suggests the examination of the dependence of the Thomas rotation angle upon the angle ε_M between the composite velocities $\mathbf{u} \oplus \mathbf{v}$ and $\mathbf{v} \oplus \mathbf{u}$,

$$\varepsilon_M = \langle\, (\mathbf{u} \oplus \mathbf{v}, \mathbf{v} \oplus \mathbf{u}) \tag{12}$$

This problem has not drawn the attention of the investigators. Moreover, there is the temptation to admit the identity of ε_U and ε_M. In order to express the angle ε_M (12) as simple as possible, the scalar and vector products between $\mathbf{u} \oplus \mathbf{v}$ and $\mathbf{v} \oplus \mathbf{u}$ may be used:

$$\cos \varepsilon_M = \frac{(\mathbf{u} \oplus \mathbf{v}) \cdot (\mathbf{v} \oplus \mathbf{u})}{(\mathbf{u} \oplus \mathbf{v})^2}, \tag{13a}$$

$$\left| \sin \varepsilon_M \right| = \frac{\left| (\mathbf{u} \oplus \mathbf{v}) \times (\mathbf{v} \oplus \mathbf{u}) \right|}{(\mathbf{u} \oplus \mathbf{v})^2}. \tag{13b}$$

After simple operations, Eqs. (13) expressed in terms of γ_u, γ_v and θ, take the forms (A.8) and (A.13):

$$\cos \varepsilon_M = \frac{a_{M0} + a_{M1} \cos\theta + a_{M2} \cos^2\theta + a_{M3} \cos^3\theta}{b_{M0} + b_{M1} \cos\theta + b_{M2} \cos^2\theta}, \tag{14a}$$

$$\sin \varepsilon_M = \frac{-(s_{M0} + s_{M1} \cos\theta + s_{M2} \cos^2\theta) \sin\theta}{b_{M0} + b_{M1} \cos\theta + b_{M2} \cos^2\theta}, \tag{14b}$$

where the coefficients are given by (A.7) and (A.12) and the sign minus in the expressions of $\sin \varepsilon_M$ (14b) has the same meaning as in Eq. (11b).

Proceeding in the same way, Eqs. (11), expressed in terms of γ_u and γ_v, become (A.14)

$$\cos \varepsilon_U = \frac{a_{U0} + a_{U1} \cos\theta + a_{U2} \cos^2\theta}{b_{U0} + b_{U1} \cos\theta}, \tag{15a}$$

$$\sin \varepsilon_U = \frac{-(s_{U0} + s_{U1} \cos\theta) \sin\theta}{b_{U0} + b_{U1} \cos\theta}, \tag{15b}$$

where the coefficients are given by (A.14c).

Remark. The denominator and the nominator of $\sin \varepsilon_M$ (13b) being functions of $\cos\theta$ ($-\pi < \theta < \pi$) have positive as well as negative values, that is $-1 < \sin(\theta) < 1$. It is worth nothing that $\sin \varepsilon_U$ (11b) has been derived from the identity $\sin^2 \varepsilon_U + \cos^2 \varepsilon_U = 1$,[7] while sin ε_M and cos ε_M have been calculated using the scalar (13a) and vector products (13b). The identity $\sin^2 \varepsilon_M + \cos^2 \varepsilon_M = 1$ is easily verified.

At the first sight, the comparison of Eqs. (14) and (15) shows that $\cos \varepsilon_M \neq \cos \varepsilon_U$ and $\sin \varepsilon_M \neq \sin \varepsilon_U$. However two different analytical demonstrations given in the Appendix B, confirm the identities of Eqs. (14) and (15). According to the first demonstration, the multiplication of both the numerator and the denominator of Ungar's Eqs. (15) with the factor $\Psi(\gamma_u, \gamma_v, \theta)$ (B.2) leads to the Eqs. (14). The second demonstration assumes, ab initio, the equality of Eqs. (14) and (15) and the task of the calculation consists in the verification of that equality.

Summarizing, we can state that the most important result of this paper is the identification of the Thomas rotation angle ε_U (defined in Fig. 2 by the rotation angle of the axes of Σ'' with respect to the axes of Σ) with the angle ε_M between the composite velocities $\mathbf{u} \oplus \mathbf{v}$ and $\mathbf{v} \oplus \mathbf{u}$ of the non–colinear velocities $\mathbf{u}$ and $\mathbf{v}$.

Now let us compare the method of derivation of the Thomas angle, applied in this paper, with some of those existing in the literature. Since the Lorentz transformation in one time dimension and one space dimension can be represented by complex numbers, there are expressions of Thomas rotation in terms of hypercomplex numbers and elements of Clifford algebra.[26] Expressions of the Thomas rotation were presented by Ben–Menahem[19–20] and others.[21–25] Various approximations and evaluations of Thomas rotation were obtained by Salingaros in a series of papers[26] in which he used the Baker–Hausdorff formula, which corrects the product of non–commuting exponentials in the Clifford algebra. Thomas rotation is referred to as a Wigner rotation by several authors.[7] Objection to the use of the term Thomas rotation instead of Wigner rotation is expressed by Noz and Kim.[29] They reserve the term Wigner rotation for the space rotation defined in the Lorentz frame in which the boosted particle under consideration is at the rest. The physical aspects of the Thomas rotation have been given by Phipps.[28] In the literature, thus, the study of the Thomas rotation by means of vector and matrix algebra (several authors circumvent the difficulties by implying the Clifford formalism) is restricted by a severe complexity. In the present paper the author copes with the complexity of the above–mentioned methods of the derivation of the Thomas rotation, thus obtaining an astonishing simplicity with which the Thomas rotation angle can be expressed, and hence making the Thomas rotation problem accessible to a wider audience.

APPENDIX A

Denoting by N_c the numerator of cos ε_M (13a),

$$\begin{aligned} N_c &= [\gamma_u \mathbf{u} + \mathbf{v} + \kappa_u \mathbf{u}(\mathbf{u}\cdot\mathbf{v})u^{-2}]\cdot[\gamma_v \mathbf{v} + \mathbf{u} + \kappa_v \mathbf{v}(\mathbf{u}\cdot\mathbf{v})v^{-2}] = \\ &= \gamma_u u^2 + \gamma_v v^2 + (\gamma_u + \gamma_v + \gamma_u\gamma_v)uv\cos\theta + (\gamma_u\kappa_v u^2 + \\ &\quad \gamma_v\kappa_u v^2)\cos^2\theta + \kappa_u\kappa_v uv\cos^3\theta, \end{aligned} \tag{A.1}$$

and by n the denominator of cos ε_M,

$$\begin{aligned} n &= \gamma_u\gamma_v[(\mathbf{u}+\mathbf{v})^2 - c^{-2}(\mathbf{u}\times\mathbf{v})^2] = \gamma_u\gamma_v[u^2 + v^2 + 2uv\cos\theta - \\ &\quad c^{-2}u^2v^2\sin^2\theta] = \gamma_u\gamma_v[u^2 + v^2 - c^{-2}u^2v^2 + \\ &\quad 2uv\cos\theta + c^{-2}u^2v^2\cos^2\theta], \end{aligned} \tag{A.2}$$

$\cos \varepsilon_M$ (13a) is seen to take the form

$$\cos \varepsilon_M = \frac{N_c}{n}. \tag{A.3}$$

From Eqs. (2) and (4), the magnitudes of the velocities **u** and **v** as functions of γ_u and γ_v, are given by,

$$u = c\gamma_u^{-1}\sqrt{\gamma_u^2 - 1} \quad ; \quad v = c\gamma_v^{-1}\sqrt{\gamma_v^2 - 1}; \tag{A.4}$$

and taking into account the notations of κ_u (2) and κ_v (4), Eq. (A.1) may be expressed in terms of γ_u, γ_v and c, as follows:

$$\begin{aligned} N_c &= c^2\gamma_u^{-1}\gamma_v^{-1}\,[\gamma_u(\gamma_v^2 - 1) + \gamma_v(\gamma_u^2 - 1) + (\gamma_u + \gamma_v + \gamma_u\gamma_v - 1) \\ &\left(\sqrt{(\gamma_u^2 - 1)(\gamma_v^2 - 1)}\right)\cos\theta + (\gamma_u - 1)(\gamma_v - 1)(\gamma_u + \gamma_v + 2\gamma_u\gamma_v) \\ &\cos^2\theta + (\gamma_u - 1)(\gamma_v - 1)\left(\sqrt{(\gamma_u^2 - 1)(\gamma_v^2 - 1)}\right)\cos^3\theta]. \end{aligned} \tag{A.5}$$

Proceeding in a similar way with Eq. (A.2), we find

$$\begin{aligned} n = \lambda_u^{-1}\gamma_v^{-1}c^2 &\left[\gamma_u^2\gamma_v^2 - 1 + 2\gamma_u\gamma_v\left(\sqrt{(\gamma_u^2 - 1)(\gamma_v^2 - 1)}\right)\right. \\ &\left.\cos\theta + \left(\sqrt{(\gamma_u^2 - 1)(\gamma_v^2)}\right)\cos\theta\right]. \end{aligned} \tag{A.6}$$

On using the notations

$$\begin{aligned} &a_{M0} = (\gamma_u + \gamma_v)(\gamma_u\gamma_v - 1), \quad a_{M1} = (\gamma_u + \gamma_v + \gamma_u\gamma_v - 1)\sqrt{(\gamma_u^2 - 1)(\gamma_v^2 - 1)}, \\ &a_{M2} = (\gamma_u - 1)(\gamma_v - 1)(\gamma_u + \gamma_v + 2\gamma_u\gamma_v), \\ &a_{M3} = (\gamma_u - 1)(\gamma_v - 1)\sqrt{(\gamma_u^2 - 1)(\gamma_v^2 - 1)}, \\ &b_{M0} = \gamma_u^2\gamma_v^2 - 1, \quad b_{M1} = 2\gamma_u\gamma_v\sqrt{(\gamma_u^2 - 1)(\gamma_v^2 - 1)}, \\ &b_{M2} = (\gamma_u^2 - 1)(\gamma_v^2 - 1), \end{aligned} \tag{A.7}$$

Eq. (A.3) can be written in the form

$$\cos \varepsilon_M = \frac{a_{M0} + a_{M1}\cos\theta + a_{M2}\cos^2\theta + a_{M3}\cos^3\theta}{b_{M0} + b_{M1}\cos\theta + b_{M2}\cos^2\theta}. \tag{A.8}$$

Denoting by N_s the numerator of $\sin \varepsilon_M$ in (13b),

$$N_s = \left| (\gamma_u \mathbf{u} + \mathbf{v} + \kappa_u \mathbf{u}(\mathbf{u}\cdot\mathbf{v})u^{-2}) \times (\gamma_v \mathbf{v} + \mathbf{u} + \kappa_v \mathbf{v}(\mathbf{u}\cdot\mathbf{v})v^{-2}) \right| =$$

$$= \{(\gamma_u\gamma_v - 1)uv + [\gamma_u(\gamma_v - 1)u^2 + \gamma_v(\gamma_u - 1)v^2]\cos\theta +$$

$$(\gamma_u - 1)(\gamma_v - 1)uv\cos^2\theta\}\sin\theta, \tag{A.9}$$

$\sin \varepsilon_M$ may be expressed as

$$\sin\varepsilon_M = \frac{N_s}{n} \tag{A.10}$$

Replacing u and v by their values from (A.4), Eq. (A.9) expressed in terms of γ_u, γ_v and c becomes

$$N_s = \gamma_u^{-1}\gamma_v^{-1}c^2 \left\{ (\gamma_u\gamma_v - 1)\sqrt{(\gamma_u^2 - 1)(\gamma_v^2 - 1)} + (\gamma_u - 1)(\gamma_v - 1) \right.$$

$$\left. (\gamma_u + \gamma_v + 2\gamma_u\gamma_v)\cos\theta + (\gamma_u - 1)(\gamma_v - 1)\left(\sqrt{(\gamma_u^2 - 1)(\gamma_v^2 - 1)}\right)\cos^2\theta \right\}\sin\theta. \tag{A.11}$$

Using the notations,

$$s_{M0} = (\gamma_u\gamma_v - 1)\sqrt{(\gamma_u^2 - 1)(\gamma_v^2 - 1)},$$

$$s_{M1} = (\gamma_u - 1)(\gamma_v - 1)(\gamma_u + \gamma_v + 2\gamma_u\gamma_v),$$

$$s_{M2} = (\gamma_u - 1)(\gamma_v - 1)\sqrt{(\gamma_u^2 - 1)(\gamma_v^2 - 1)}, \tag{A.12}$$

and taking into account the expressions of n (A.6) and N_s (A.11) Eq. (A.10) takes the form,

$$\sin\varepsilon_M = \frac{-(s_{M0} + s_{M1}\cos\theta + s_{M2}\cos^2\theta)\sin\theta}{b_{M0} + b_{M1}\cos\theta + b_{M2}\cos^2\theta}, \tag{A.13}$$

where the sign minus has the same meaning as in the case of $\sin \varepsilon_U$ in (11b).

Replacement of the value of k (11c) into (11a) and (11b), leads to expressions for $\cos \varepsilon_U$ and $\sin \varepsilon_U$:

$$\cos\varepsilon_U = \frac{a_{U0} + a_{U1}\cos\theta + a_{U2}\cos^2\theta}{b_{U0} + b_{U1}\cos\theta}, \tag{A.14a}$$

$$\sin\varepsilon_U = \frac{-(s_{U0} + s_{U1}\cos\theta)\sin\theta}{b_{U0} + b_{U1}\cos\theta}, \tag{A.14b}$$

where

$$\begin{aligned} a_{U0} &= \gamma_u + \gamma_v, \quad a_{U1} = \sqrt{(\gamma_u^2 - 1)(\gamma_v^2 - 1)}, \\ a_{U2} &= (\gamma_u - 1)(\gamma_v - 1), \\ b_{U0} &= \gamma_u\gamma_v + 1, \quad b_{U1} = \sqrt{(\gamma_u^2 - 1)(\gamma_v^2 - 1)}, \\ s_{U0} &= \sqrt{(\gamma_u^2 - 1)(\gamma_v^2 - 1)}, \quad s_{U1} = (\gamma_u - 1)(\gamma_v - 1). \end{aligned} \tag{A.14c}$$

APPENDIX B

(a) First Demonstration of $\varepsilon_U = \varepsilon_M$.

The nominator of cos ε_M (A.8), denoted by N_{cM}, may be written as

$$N_{cM} = (a_{U0} + a_{U1}\cos\theta + a_{U2}\cos^2\theta)\,\Psi(\gamma_u, \gamma_v, \theta), \tag{B.1}$$

where a_{U0}, a_{U1}, and a_{U2} are given by (A.14c) and Ψ $(\gamma_u, \gamma_v, \theta)$ denotes the function

$$\Psi(\gamma_u, \gamma_v, \theta) = \gamma_u\gamma_v - 1 + \left(\sqrt{(\gamma_u^2 - 1)(\gamma_v^2 - 1)}\right)\cos\theta. \tag{B.2}$$

In a similar way the nominator of sin ε_M (A.13), denoted by N_{sM}, may be transformed as follows:

$$N_{sM} = -[(s_{U0} + s_{U1}\cos\theta)\sin\theta]\,\Psi(\gamma_u, \gamma_v, \theta), \tag{B.3}$$

where s_{U0} and s_{U1} are given by (A.14c).

As regards the denominator of cos ε_M (A.8), denoted by n_M, it may be written as

$$n_M = (b_{U0} + b_{U1}\cos\theta)\,\Psi(\gamma_u, \gamma_v, \theta), \tag{B.4}$$

where b_{U0} and b_{U1} are given by (A.14c).

Substitution of the expression of N_{cM} from (B.1) and that of n_M from (B.4) into the Eq. (A.8) yields

$$\cos \varepsilon_M = \frac{N_{cM}}{n_M} = \frac{(a_{U0} + a_{U1} \cos\theta + a_{U2} \cos^2 \theta)\Psi(\gamma_u, \gamma_v, \theta)}{(b_{U0} + b_{U1} \cos\theta)\Psi(\gamma_u, \gamma_v, \theta)}. \tag{B.5}$$

Comparing (B.5) with (A.14a) and simplifying the factor $\Psi(\gamma_u, \gamma_v, \theta)$, we have

$$\cos \varepsilon_M = \cos \varepsilon_U . \tag{B.6}$$

Similarly, the substitution of the expression of N_{sM} from (B.3) and that of n_M from (B.4) into Eq. (A.13) yields

$$\sin \varepsilon_M = \frac{N_{sM}}{n_M} = \frac{-(s_{M0} + s_{U1} \cos\theta)\Psi(\gamma_u, \gamma_v, \theta)}{(b_{U0} + b_{U1} \cos\theta)\Psi(\gamma_u, \gamma_v, \theta)}. \tag{B.7}$$

Comparison of (B.7) with (A.14b) and simplification of the factor $\Psi(\gamma_u, \gamma_v, \theta)$ leads to the equality

$$\sin \varepsilon_M = \sin \varepsilon_U . \tag{B.8}$$

From Eqs. (B.6) and (B.8), it follows that the Ungar's Eqs. (11) are identical with Eqs. (14) and, consequently, the Thomas rotation angle ε_U (solution of Eqs. (11)) is identical with the angle ε_M between the composite velocities $\mathbf{u} \oplus \mathbf{v}$ and $\mathbf{v} \oplus \mathbf{u}$:

$$\varepsilon_U = \varepsilon_M \tag{B.9}$$

(b) Second demonstration of $\varepsilon_U = \varepsilon_M$.

Let us assume the identities of Eqs. (11) and (16). In the case of Eqs. (11a) and (16a), we must have

$$\frac{a_{M0} + a_{M1} \cos\theta + a_{M2} \cos^2 \theta + a_{M3} \cos^3 \theta}{b_{M0} + b_{M1} \cos\theta + b_{M2} \cos^2 \theta} = \frac{a_{U0} + a_{U1} \cos\theta + a_{U2} \cos^2 \theta}{b_{U0} + b_{U1} \cos\theta},$$

which assumes

$$(a_{M0} b_{U0} - a_{U0} b_{M0}) + (a_{M0} b_{U1} - a_{M1} b_{U0} - a_{U1} b_{M0} - a_{U0} b_{M1}) \cos\theta + (a_{M1} b_{U1} + a_{M2} b_{U0} - a_{U2} b_{M0} - a_{U1} b_{M1} - a_{U0} b_{M2}) \cos^2 \theta + (a_{M2} b_{U1} + a_{M3} b_{U0} - a_{U2} b_{M1} - a_{U1} b_{M2}) \cos^3 \theta + (a_{M3} b_{U1} - a_{U2} b_{M2}) \cos^4 \theta = 0.$$

After a simple calculation of the coefficients which multiply $\cos^j \theta$, ($j = 0, 1, 2, 3, 4$), we have

$$K_i^{(c)} = 0 ,$$

that is, Eqs. (11a) and (16a) are identical. Similarly the identity of Eqs. (11b) and (16b) is derived.

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REFERENCES

1. L. H. Thomas, "The motion of the spinning electron," *Nature* **117**, 514 (1926).
2. L. H. Thomas, "The kinematics of an electron with an axis," *Phil. Mag.* **3**, 1–22 (1927).
3. L. H. Thomas, "Recollections of the discovery of the Thomas precessional frequency," *AIP Conference Proceedings*, No. 95 (High Energy Spin Physics), G. M. Bunce, ed. (Brookhaven National Laboratory, 1982), pp. 4–12.
4. A. A. Ungar, "The relativistic velocity composition paradox and the Thomas rotation," *Found. Phys.* **19**, 1385–1396 (1989).
5. G. P. Sastry, "Is Length contraction really paradoxical?," *Am. J. Phys.* **55**, 943–945 (1987).
6. C. I. Mocanu, "Some difficulties within the framework of relativistic electrodynamics," *Arch. Elektrotech.* **69**, 97–110 (1986).
7. A. A. Ungar, "Thomas rotation and the parametrization of the Lorentz transformation group," *Found. Phys. Lett.* **1**, 57–89 (1988).
8. A. A. Ungar, "The Thomas rotation formalism underlying a non–associative group structure for relativistic velocities," *Appl. Math. Lett.* **1**, 403–405 (1988).
9. A. A. Ungar, "The relativistic non–commutative non–associative group of velocities and the Thomas rotation," *Results in Mathematics* **16**, 168–179 (1989).
10. A. A. Ungar, "Axiomatic approach to the non–associative group of relativistic velocities," *Found Phys. Lett.* **2**, 199–203 (1989).
11. A. A. Ungar, "Weakly associative groups," *Results in Mathematics* **17**, 149–168 (1990).
12. A. A. Ungar, "Thomas precession and its associated group – like structure," *Am. J. Phys.* **54**, 824–834 (1991).
13. A. A. Ungar, "Quasidirect product groups and the Lorentz transformation group," in T. M. Rassias, ed. *Constantin Caratheodory: An International Tribute* (World Scientific, Singapore, 1991), pp. 1378–1392.
14. A. A. Ungar, "Successive Lorentz transformations of the electromagnetic field," *Found. Phys.* **21**, 569–589 (1991).
15. A. Einstein, *Sidelights on relativity* (Methuen, London, 1922).

16. R. P. Feynman, R. B. Leighton, M. Sands, *The Feynman Lectures* (Addison–Wesley, Reading, Massachusetts, 1964), Vol. II, Sec. 17–4.
17. M. C. Møller, *The Theory of Relativity* (Clarendon Press, Oxford, 1952).
18. W. Rindler, *Special Relativity* (Oliver & Boyd, London, 1986).
19. S. Ben–Menahem, "The Thomas precession and velocity space curvature," *J. Math. Phys.* **27**, 1234–1296 (1986).
20. S. Ben–Menahem, "Wigner rotation revisited," *Am. J. Phys.* **53**, 62–66 (1985).
21. S. Dencoff and D. R. Inglis, "On the Thomas precession of accelerated axes," *Phys. Rev.* **50**, 784 (1936).
22. G. P. Fisher, "The Thomas precession," *Am. J. Phys.* **40**, 1772–1781 (1972).
23. W. H. Furry, "Lorentz transformation and the Thomas precession," *Am. J. Phys.* **23**, 517–525 (1955).
24. A. C. Hirshfeld and Metzger, "A simple formula for combining rotations and Lorentz boosts," *Am. J. Phys.* **54**, 550–552 (1986).
25. E. G. P. Rowe, "The Thomas precession," *Eur. J. Phys.* **5**, 40–45 (1984).
26. N. Salingaros, "The Lorentz group and the Thomas precession: Exact results for the product of two boosts," *J. Math. Phys.* **27**, 157–162 (1986); also "Erratum: The Lorentz, *J. Math. Phys.* **29**, 1265 (1988).
27. David Shelupsky, "Derivation of the Thomas precession formula," *Am. J. Phys.* **35**, 650–651 (1967).
28. Thomas Phipps, *Heretical Verities: Mathematical Themes in Physical Description* (Classical Non–Fiction Library, Box 926 Urbana, Illinois 61801, 1986).
29. M. E. Noz and Y. S. *Kim, Special Relativity and Quantum Theory* (Kluwer Academic, Dordrecht, 1988).