### TORSION GRAVITY: A REAPPRAISAL

H. I. Arcos\* and J. G. Pereira Instituto de Física Teórica, Universidade Estadual Paulista Rua Pamplona 145, 01405-900 São Paulo, Brazil

The role played by torsion in gravitation is critically reviewed. After a description of the problems and controversies involving the physics of torsion, a comprehensive presentation of the teleparallel equivalent of general relativity is made. According to this theory, curvature and torsion are alternative ways of describing the gravitational field, and consequently related to the same degrees of freedom of gravity. However, more general gravity theories, like for example Einstein–Cartan and gauge theories for the Poincaré and the affine groups, consider curvature and torsion as representing independent degrees of freedom. By using an active version of the strong equivalence principle, a possible solution to this conceptual question is reviewed. This solution favors ultimately the teleparallel point of view, and consequently the completeness of general relativity. A discussion of the consequences for gravitation is presented.

#### I. INTRODUCTION

#### A. General Concepts: Riemann versus Weitzenböck

Gravitation presents a quite peculiar property: particles with different masses and different compositions feel it in such a way that all of them acquire the same acceleration and, given the same initial conditions, follow the same path. Such universality of response—usually referred to as *universality of free fall*—is the most fundamental characteristic of the gravitational interaction.<sup>1</sup> It is a unique property, peculiar to gravitation: no other basic interaction of Nature has it. Effects equally felt by all bodies were known since long. They are the so called *inertial* effects, which show up in non-inertial frames. Examples on Earth are the centrifugal and the Coriolis forces.

Universality of inertial effects was one of the hints used by Einstein towards general relativity, his theory for gravitation. Another ingredient was the notion of field. The concept allows the best approach to interactions coherent with special relativity. All known forces are mediated by fields on spacetime. If gravitation is to be represented by a field, it should, by the considerations above, be a universal field, equally felt by every particle. A natural solution is to assume that gravitation changes the spacetime itself. And, of all the fields present in a spacetime, the metric appears as the most fundamental. The simplest way to change spacetime, then, would be to change its metric. Furthermore, the metric does change when looked at from a non-inertial frame, where the (also universal) inertial effects are present. According to this approach, therefore, the gravitational field should be represented by the spacetime metric. In the absence of gravitation, the spacetime metric should reduce to the flat Minkowski metric.

A crucial point of this description of the gravitational interaction, which is a description fundamentally based on the universality of free fall, is that it makes no use of the concept of *force*. In fact, according to it, instead of acting through a force, the presence of gravitation is represented by a deformation of the spacetime structure. More precisely, the presence of a gravitational field is supposed to produce a *curvature* in spacetime, the gravitational interaction being achieved by letting (spinless) particles to follow the geodesics of the underlying spacetime. Notice that no other kind of spacetime deformation is supposed to exist. Torsion, for example, which would be another natural spacetime deformation, is assumed to vanish from the very beginning. This is the approach of general relativity, in which geometry replaces the concept of gravitational force, and the trajectories are determined, not by force equations, but by geodesics.<sup>2</sup> The spacetime underlying this theory is a pseudo Riemannian space. It is important to remark that only an interaction presenting the property of universality can be described by a geometrization of spacetime. It is also important to mention that, in the eventual absence of universality, the general relativity description of gravitation would break down.

On the other hand, like the other fundamental interactions of nature, gravitation can also be described in terms of a gauge theory. In fact, the teleparallel equivalent of general relativity, or teleparallel gravity for short, and be interpreted as a gauge theory for the translation group. In this theory, instead of torsion, curvature is assumed to vanish. The corresponding underlying spacetime is, in this case, a Weitzenböck spacetime. In spite of this fundamental difference, the two theories are found to yield equivalent descriptions of the gravitational interaction. This means that curvature and torsion are able to provide, each one, equivalent descriptions of the gravitational interaction.

<sup>\*</sup> Permanent address: Universidad Tecnológica de Pereira, A.A. 97, La Julita, Pereira, Colombia

Conceptual differences, however, show up. According to general relativity, curvature is used to geometrize spacetime. Teleparallelism, on the other hand, attributes gravitation to torsion, but in this case torsion accounts for gravitation not by geometrizing the interaction, but by acting as a force. As a consequence, there are no geodesics in the teleparallel equivalent of general relativity, but only force equations quite analogous to the Lorentz force equation of electrodynamics. We may then say that the gravitational interaction can be described in terms of curvature, as is usually done in general relativity, or alternatively in terms of torsion, in which case we have the so called teleparallel gravity. Whether gravitation requires a curved or a torsioned spacetime—or equivalently, a Riemann<sup>6</sup> or a Weitzenböck spacetime structure—turns out to be, at least classically, a matter of convention.

### B. Teleparallel Gravity and the Weak Equivalence Principle

Rephrasing the above arguments, we can say that universality of free fall is the reason for gravitation to present, in addition to the fundamental teleparallel gauge approach, the equivalent geometric description of general relativity. In fact, in order to attribute gravitation to curvature, it is essential that gravitation be universal, or equivalently, that the weak equivalence principle, which establishes the equality of inertial and gravitational masses, be true. Only under these circumstances is it possible to assure that all spinless particles of nature, independently of their internal constitution, feel gravitation the same and, for a given set of initial conditions, follow the same trajectory—a geodesic of the underlying Riemannian spacetime.

Now, as is widely known, the electromagnetic interaction is not universal: there exists no electromagnetic equivalence principle. Nevertheless, Maxwell's theory, a gauge theory for the Abelian group U(1), describes quite consistently the electromagnetic interaction. Relying then on the fact that Maxwell's theory and teleparallel gravity are both Abelian gauge theories, in which the equations of motion of test particles are not geodesics but force equations, the question then arises whether the gauge approach of teleparallel gravity would also be able to describe the gravitational interaction in the eventual lack of universality—that is, of the weak equivalence principle. The answer to this question is positive: teleparallel gravity does not require the validity of the equivalence principle to describe the gravitational interaction.<sup>7</sup> In other words, whereas the geometrical description of general relativity breaks down, the gauge description of teleparallel gravity remains as a consistent theory in the absence of universality. In spite of the equivalence with general relativity, therefore, teleparallel gravity seems to belong to a more general class of theory. This is a very important issue because, even though the equivalence principle has presently passed all experimental tests, there are many controversies related with its validity, mainly at the quantum level.

#### C. Teleparallel Coupling of the Fundamental Fields

The gravitational coupling of the fundamental fields in teleparallel gravity is a very controversial subject.<sup>10–15</sup> The basic difficulty lies in the definition of the spin connection, and consequently in the correct form of the gravitational coupling prescription. Since no experimental data are available to help decide, from the different possibilities, which is the correct one, the only one can do is to use consistency arguments grounded in physical principles. One such possibility is to rely on the alluded equivalence between general relativity and teleparallel gravity. According to this formulation, each one of the fundamental fields of nature—scalar, spinor, and electromagnetic—are required to couple to torsion in a such a way to preserve the equivalence between teleparallel gravity and general relativity. When this approach is applied to these fields, as we are going to see, a teleparallel spin connection can naturally be defined, which yields quite consistent results.

### D. The Physics of Torsion Beyond Teleparallel Gravity

As already discussed, in general relativity torsion is assumed to vanish from the very beginning, whereas in teleparallel gravity curvature is assumed to vanish. In spite of this fundamental difference, the two theories are found to yield equivalent descriptions of the gravitational interaction. An immediate implication of this equivalence is that curvature and torsion turns out to be simply alternative ways of describing the gravitational field, and consequently related to the same degrees of freedom of gravity. This property is corroborated by the fact that the symmetric matter energy-momentum tensor appears as source in both theories: as the source of curvature in general relativity, and as the source of torsion in teleparallel gravity.

On the other hand, theoretical speculations have since the early days of general relativity discussed the necessity of including torsion, in addition to curvature, in the description of the gravitational interaction.<sup>17</sup> For example, more general gravity theories, like Einstein-Cartan<sup>18</sup> and gauge theories for the Poincaré<sup>19</sup> and the affine groups,<sup>20</sup>

consider curvature and torsion as representing independent degrees of freedom. In these theories, differently from teleparallel gravity, torsion should become relevant only when spins are important. This could be the case either at the microscopic level or near a neutron star. According to these models, therefore, since torsion represents additional degrees of freedom in relation to curvature, new physical phenomena would be expected from its presence.

Now, the above described difference rises a conceptual question in relation to the actual role played by torsion in the description of the gravitational interaction. In fact, the two physical interpretations described above are clearly conflictive: if one is correct, the other is necessarily wrong. This is a typical problem to be solved by experiment. However, due to the weakness of the gravitational interaction, there are no available data on the gravitational coupling of the fundamental particles. In addition, no one has ever reported new gravitational phenomena near a neutron star, for example, where the effects of torsion would be relevant according to those more general gravity theories. Therefore, also in this general case, the only one can do in the search of a gravitational coupling prescription is to use consistency arguments grounded in known physical principles. A possible way to proceed is to remember that the general covariance principle—seen as an active version of the strong equivalence principle—naturally defines a gravitational coupling prescription. We can then use it to obtain the form of such prescription in the presence of curvature and torsion. It should be remarked that this procedure is general, and has already been consistently applied to obtain the coupling prescription in the specific case of teleparallel gravity.<sup>21</sup>

### E. Purposes and Strategy

The basic purpose of this paper is to critically review the role played by torsion in the description of the gravitational interaction. We will proceed according to the following scheme. In Section II, we introduce the basic definitions and set the notations we are going to use. In Section III, we present a comprehensive review of the basic properties of teleparallel gravity. In particular, the role played by torsion in this theory will be extensively discussed and clarified. In Section IV, by requiring compatibility with the strong equivalence principle, a gravitational coupling prescription in the presence of curvature and torsion is obtained. This prescription, as we are going to see, is found to be always equivalent with that of general relativity, a result that reinforces the completeness of this theory, as well as the teleparallel point of view, according to which torsion does not represent additional degrees of freedom for gravity, but simply an alternative way of representing the gravitational field.<sup>22</sup> An application to the case of a spinning particle will be presented.<sup>23</sup> Finally, a discussion of the main points of the review will be made, and the consequences for gravitation will be analyzed.

#### II. NOTATIONS AND DEFINITIONS

The geometrical setting of any gauge theory for gravitation is the tangent bundle, a natural construction always present in spacetime. In fact, at each point of spacetime—the base space of the bundle—there is always a tangent space attached to it—the fiber of the bundle—on which the gauge group acts. We use the Greek alphabet  $(\mu, \nu, \rho, \dots = 0, 1, 2, 3)$  to denote the holonomic indices related to spacetime, and the Latin alphabet  $(a, b, c, \dots = 0, 1, 2, 3)$  to denote the anholonomic indices related to the tangent space, assumed to be a Minkowski spacetime with the metric  $\eta_{ab} = \text{diag}(+1, -1, -1, -1)$ . The spacetime coordinates, therefore, will be denoted by  $x^{\mu}$ , whereas the tangent space coordinates will be denoted by  $x^a$ . Since these coordinates are functions of each other, the holonomic derivatives in these two spaces can be identified by

$$\partial_{\mu} = (\partial_{\mu} x^{a}) \ \partial_{a} \quad \text{and} \quad \partial_{a} = (\partial_{a} x^{\mu}) \ \partial_{\mu},$$
 (1)

where  $\partial_{\mu}x^{a}$  is a trivial—that is, holonomic—tetrad, with  $\partial_{a}x^{\mu}$  its inverse.

From the geometrical point of view, a connection specifies how a vector field is transported along a curve. In a local coordinate chart with basis vectors  $\{e_{\mu}\}=\{\partial_{\mu}\}$ , the connection coefficients  $\Gamma^{\lambda}{}_{\nu\mu}$  are defined by

$$\nabla_{e_{\nu}} e_{\mu} = e_{\lambda} \Gamma^{\lambda}{}_{\nu\mu}. \tag{2}$$

Once the action of  $\nabla$  on the basis vectors is defined, one can compute its action on any vector field  $V^{\rho}$ :

$$\nabla_{\mu}V^{\rho} = \partial_{\mu}V^{\rho} + \Gamma^{\rho}_{\nu\mu}V^{\nu}. \tag{3}$$

Now, given a nontrivial tetrad  $h^a_{\mu}$ , the spacetime and the tangent-space metrics are related by

$$g_{\mu\nu} = \eta_{ab} h^a{}_{\mu} h^b{}_{\nu}. \tag{4}$$

Of course, as far as  $e_{\mu}$  is a trivial tetrad, the metric  $g_{\mu\nu} = \eta_{ab} \, e^a{}_{\mu} \, e^b{}_{\nu}$  will be simply the Minkowski metric written in a different coordinate system. A connection  $\Gamma^{\rho}{}_{\lambda\mu}$  is said to be metric compatible if

$$\nabla_{\lambda} g_{\mu\nu} \equiv \partial_{\lambda} g_{\mu\nu} - \Gamma^{\rho}{}_{\lambda\mu} g_{\rho\nu} - \Gamma^{\rho}{}_{\lambda\nu} g_{\rho\mu} = 0. \tag{5}$$

On the other hand, a spin connection  $A_{\mu}$  is a connection assuming values in the Lie algebra of the Lorentz group,

$$A_{\mu} = \frac{1}{2} A^{ab}_{\ \mu} S_{ab},\tag{6}$$

with  $S_{ab}$  a representation of the Lorentz generators. Using the tetrad, a general connection  $\Gamma^{\rho}_{\nu\mu}$  can be related with the corresponding spin connection  $A^{a}_{b\mu}$  through

$$\Gamma^{\rho}{}_{\nu\mu} = h_a{}^{\rho}\partial_{\mu}h^a{}_{\nu} + h_a{}^{\rho}A^a{}_{b\mu}h^b{}_{\nu}. \tag{7}$$

The inverse relation is, consequently,

$$A^{a}{}_{b\mu} = h^{a}{}_{\nu} \partial_{\mu} h_{b}{}^{\nu} + h^{a}{}_{\nu} \Gamma^{\nu}{}_{\rho\mu} h_{b}{}^{\rho}. \tag{8}$$

Equations (7) and (8) are simply different ways of expressing the property that the total—that is, acting on both indices—derivative of the tetrad vanishes identically:

$$\partial_{\mu}h^{a}_{\ \nu} - \Gamma^{\rho}_{\ \nu\mu}h^{a}_{\ \rho} + A^{a}_{b\mu}h^{b}_{\ \nu} = 0. \tag{9}$$

In the present work, we will separate the notions of space and connections. From a formal point of view, curvature and torsion are in fact properties of a connection.<sup>24</sup> Strictly speaking, there is no such a thing as curvature or torsion of spacetime, but only curvature or torsion of connections. This becomes evident if we remember that many different connections are allowed to exist in the same spacetime.<sup>25</sup> Of course, when restricted to the specific case of general relativity, universality of gravitation allows the Levi–Civita connection to be interpreted as part of the spacetime definition as all particles and fields feel this connection the same. However, when considering several connections with different curvature and torsion, it seems far wiser and convenient to take spacetime simply as a manifold, and connections (with their curvatures and torsions) as additional structures.

The curvature and the torsion tensors of the connection  $A^a_{b\mu}$  are defined respectively by

$$R^{a}{}_{b\nu\mu} = \partial_{\nu}A^{a}{}_{b\mu} - \partial_{\mu}A^{a}{}_{b\nu} + A^{a}{}_{e\nu}A^{e}{}_{b\mu} - A^{a}{}_{e\mu}A^{e}{}_{b\nu}$$

$$\tag{10}$$

and

$$T^{a}_{\nu\mu} = \partial_{\nu}h^{a}_{\mu} - \partial_{\mu}h^{a}_{\nu} + A^{a}_{e\nu}h^{e}_{\mu} - A^{a}_{e\mu}h^{e}_{\nu}. \tag{11}$$

Using the relation (8), they can be expressed in a purely spacetime form, given by

$$R^{\rho}{}_{\lambda\nu\mu} \equiv h_a{}^{\rho} h^b{}_{\lambda} R^a{}_{b\nu\mu} = \partial_{\nu} \Gamma^{\rho}{}_{\lambda\mu} - \partial_{\mu} \Gamma^{\rho}{}_{\lambda\nu} + \Gamma^{\rho}{}_{\eta\nu} \Gamma^{\eta}{}_{\lambda\mu} - \Gamma^{\rho}{}_{\eta\mu} \Gamma^{\eta}{}_{\lambda\nu}$$

$$\tag{12}$$

and

$$T^{\rho}_{\nu\mu} \equiv h_a^{\ \rho} T^a_{\ \nu\mu} = \Gamma^{\rho}_{\ \mu\nu} - \Gamma^{\rho}_{\ \nu\mu}. \tag{13}$$

The connection coefficients can be conveniently decomposed according to

$$\Gamma^{\rho}{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu} + K^{\rho}{}_{\mu\nu},\tag{14}$$

where

$$\overset{\circ}{\Gamma}^{\sigma}{}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho} \left(\partial_{\mu}g_{\rho\nu} + \partial_{\nu}g_{\rho\mu} - \partial_{\rho}g_{\mu\nu}\right) \tag{15}$$

is the Levi-Civita connection of general relativity, and

$$K^{\rho}{}_{\mu\nu} = \frac{1}{2} \left( T_{\nu}{}^{\rho}{}_{\mu} + T_{\mu}{}^{\rho}{}_{\nu} + T^{\rho}{}_{\mu\nu} \right) \tag{16}$$

is the contortion tensor. Using the relation (7), the decomposition (14) can be rewritten in terms of the spin connections as

$$A^{c}{}_{a\nu} = \mathring{A}^{c}{}_{a\nu} + K^{c}{}_{a\nu}, \tag{17}$$

where  $\overset{\circ}{A}{}^{c}{}_{a\nu}$  is the Ricci coefficient of rotation, the spin connection of general relativity.

Teleparallel gravity, on the other hand, is characterized by the vanishing of the so called Weitzenböck spin connection:  $\mathring{A}^{a}{}_{b\mu}=0$ . In this case, the relation (17) assumes the form

$$\overset{\circ}{A}{}^{c}{}_{a\nu} = 0 - \overset{\bullet}{K}{}^{c}{}_{a\nu}. \tag{18}$$

Furthermore, from Eq. (7) we see that the corresponding Weitzenböck connection has the form

$$\Gamma^{\rho}_{\nu\mu} = h_a{}^{\rho}\partial_{\mu}h^a{}_{\nu}. \tag{19}$$

In the remaining of the paper, all magnitudes related with general relativity will be denoted with an over "o", whereas magnitudes related with teleparallel gravity will be denoted with an over "•".

Under a local Lorentz transformation  $\Lambda^a{}_b \equiv \Lambda^a{}_b(x)$ , the tetrad changes according to  $h'{}^a{}_\mu = \Lambda^a{}_b h^b{}_\mu$ , whereas the spin connection undergoes the transformation

$$A^{\prime a}{}_{b\mu} = \Lambda^{a}{}_{c} A^{c}{}_{d\mu} \Lambda_{b}{}^{d} + \Lambda^{a}{}_{c} \partial_{\mu} \Lambda_{b}{}^{c}. \tag{20}$$

In the same way, it is easy to verify that  $T^a_{\ \nu\mu}$  and  $R^a_{\ b\nu\mu}$  transform covariantly under local Lorentz transformations:

$$T'^{a}_{\nu\mu} = \Lambda^{a}_{b} T^{b}_{\nu\mu} \quad and \quad R'^{a}_{b\nu\mu} = \Lambda^{a}_{c} \Lambda_{b}^{d} R^{c}_{d\nu\mu}.$$
 (21)

This means that  $\Gamma^{\rho}_{\nu\mu}$ ,  $T^{\lambda}_{\mu\nu}$  and  $R^{\rho}_{\lambda\nu\mu}$  are all invariant under a local Lorentz transformation.

A nontrivial tetrad field defines naturally a non-coordinate basis for vector fields and their duals,

$$h_a = h_a{}^{\mu}\partial_{\mu} \quad \text{and} \quad h^a = h^a{}_{\mu}dx^{\mu}. \tag{22}$$

This basis is clearly non-holonomic,

$$[h_c, h_d] = f^a{}_{cd} h_a, \tag{23}$$

with

$$f^{a}{}_{cd} = h_{c}{}^{\mu} h_{d}{}^{\nu} (\partial_{\nu} h^{a}{}_{\mu} - \partial_{\mu} h^{a}{}_{\nu}) \tag{24}$$

the coefficient of anholonomy. In this non-coordinate basis, and using the fact that the last index of the spin connection is a tensor index,

$$A^{a}{}_{bc} = A^{a}{}_{b\mu} h_{c}{}^{\mu}, (25)$$

the curvature and torsion components are given respectively by<sup>26</sup>

$$R^{a}_{bcd} = h_{c}A^{a}_{bd} - h_{d}A^{a}_{bc} + A^{a}_{ec}A^{a}_{bd} - A^{a}_{ed}A^{e}_{bc} + f^{e}_{cd}A^{a}_{be}$$
(26)

and

$$T^{a}_{bc} = A^{a}_{cb} - A^{a}_{bc} - f^{a}_{bc}. (27)$$

## III. TELEPARALLEL DESCRIPTIONS OF GRAVITATION

# A. Fundamentals of Teleparallel Gravity

The notion of absolute parallelism (or teleparallelism) was introduced by Einstein in the late twenties, in his unsuccessful attempt to unify gravitation and electromagnetism.<sup>27</sup> About three decades later, after the pioneering works by Utiyama<sup>28</sup> and Kibble,<sup>19</sup> respectively on gauge theories for the Lorentz and the Poincaré groups, there was a gravitationally-related revival of those ideas,<sup>29–31</sup> which since then have received considerable attention, mainly in the context of gauge theories for gravitation,<sup>32–34</sup> of which teleparallel gravity,<sup>35–40</sup> a gauge theory for the translation group, is a particular case.

As a gauge theory for the translation group, the fundamental field of teleparallel gravity is the gauge potential  $B_{\mu}$ , a field assuming values in the Lie algebra of the translation group,

$$B_{\mu} = B^a{}_{\mu} P_a, \tag{28}$$

where  $P_a = \partial/\partial x^a$  are the translation generators, which satisfy

$$[P_a, P_b] = 0. (29)$$

A gauge transformation is defined as a local (point dependent) translation of the tangent-space coordinates,

$$x^{a'} = x^a + \alpha^a, (30)$$

with  $\alpha^a \equiv \alpha^a(x^\mu)$  the corresponding infinitesimal parameters. In terms of  $P_a$ , it can be written in the form

$$\delta x^a = \alpha^b P_b x^a. \tag{31}$$

Let us consider now a general source field  $\Psi \equiv \Psi(x^{\mu})$ . Its infinitesimal gauge transformation does not depend on the spin character, and is given by

$$\delta\Psi = \alpha^a P_a \Psi,\tag{32}$$

with  $\delta\Psi$  standing for the functional change at the same  $x^{\mu}$ , which is the relevant transformation for gauge theories. It is important to remark that the translation generators are able to act on the argument of any source field because of the identifications (1). Using the general definition of covariant derivative<sup>25</sup>

$$h_{\mu} = \partial_{\mu} + B^{a}{}_{\mu} \frac{\delta}{\delta \alpha^{a}},\tag{33}$$

the translational covariant derivative of  $\Psi$  is found to be

$$h_{\mu}\Psi = \partial_{\mu}\Psi + B^{a}_{\mu}P_{a}\Psi. \tag{34}$$

Equivalently, we can write<sup>5</sup>

$$h_{\mu}\Psi = h^{a}{}_{\mu} \partial_{a}\Psi, \tag{35}$$

where

$$h^a{}_{\mu} = \partial_{\mu}x^a + B^a{}_{\mu} \equiv h_{\mu}x^a \tag{36}$$

is a nontrivial—that is, anholonomic—tetrad field. As the generators  $P_a = \partial_a$  are derivatives which act on the fields through their arguments, every source field in nature will respond to their action, and consequently will couple to the translational gauge potentials. In other words, every source field in nature will feel gravitation the same. This is the origin of the concept of universality according to teleparallel gravity.

As usual in gauge theories, the field strength, denoted  $T^a_{\mu\nu}$ , is obtained from the commutation relation of covariant derivatives:

$$[h_{\mu}, h_{\nu}]\Psi = T^{a}_{\mu\nu} P_{a}\Psi. \tag{37}$$

We see from this expression that  $T^a_{\mu\nu}$  is also a field assuming values in the Lie algebra of the translation group. As an easy calculation shows,

$$T^{a}_{\mu\nu} = \partial_{\mu}B^{a}_{\nu} - \partial_{\nu}B^{a}_{\mu} \equiv \partial_{\mu}h^{a}_{\nu} - \partial_{\nu}h^{a}_{\mu}. \tag{38}$$

Now, from the covariance of  $h_{\mu}\Psi$ , we obtain the transformation of the gauge potentials:

$$B^{a'}{}_{\mu} = B^a{}_{\mu} - \partial_{\mu}\alpha^a. \tag{39}$$

By using the transformations (30) and (39), the tetrad is found to be gauge invariant:

$$h^{a'}_{\ \mu} = h^a_{\ \mu}.$$
 (40)

Consequently, as expected for an Abelian gauge theory,  $T^a_{\mu\nu}$  is also invariant under a gauge transformation. We remark finally that, whereas the tangent space indices are raised and lowered with the metric  $\eta_{ab}$ , the spacetime

indices are raised and lowered with the Riemannian metric  $g_{\mu\nu}$ , as given by Eq. (4). It should be stressed that, although representing the spacetime metric,  $g_{\mu\nu}$  plays no dynamic role in the teleparallel description of gravitation.

A nontrivial tetrad field induces on spacetime a teleparallel structure which is directly related to the presence of the gravitational field. In fact, given a nontrivial tetrad, it is possible to define the so called Weitzenböck connection

$$\Gamma^{\rho}_{\mu\nu} = h_a{}^{\rho}\partial_{\nu}h^a{}_{\mu},\tag{41}$$

which is a connection presenting torsion, but no curvature. As a natural consequence of this definition, the Weitzenböck covariant derivative of the tetrad field vanishes identically:

$$\overset{\bullet}{\nabla}_{\nu}h^{a}{}_{\mu} \equiv \partial_{\nu}h^{a}{}_{\mu} - \overset{\bullet}{\Gamma}^{\rho}{}_{\mu\nu}h^{a}{}_{\rho} = 0. \tag{42}$$

This is the absolute parallelism condition. The torsion of the Weitzenböck connection is

$$T^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu} - \Gamma^{\rho}_{\mu\nu}, \tag{43}$$

from which we see that the gravitational field strength is nothing but torsion written in the tetrad basis:

$$T^{a}_{\mu\nu} = h^{a}_{\phantom{a}\rho} T^{\rho}_{\phantom{\rho}\mu\nu}. \tag{44}$$

In terms of  $T^{\rho}_{\mu\nu}$ , the commutation relation (37) assumes the form

$$[h_{\mu}, h_{\nu}] = T^{\rho}_{\mu\nu} h_{\rho}, \tag{45}$$

from where we see that torsion plays also the role of the nonholonomy of the translational gauge covariant derivative. A nontrivial tetrad field can also be used to define a torsionless linear connection  $\overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu}$ , the Levi–Civita connection of the metric (4), given by Eq. (15). The Weitzenböck and the Levi–Civita connections are related by

$$\overset{\bullet}{\Gamma}{}^{\rho}{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu} + \overset{\bullet}{K}{}^{\rho}{}_{\mu\nu},\tag{46}$$

where

$$\overset{\bullet}{K}{}^{\rho}{}_{\mu\nu} = \frac{1}{2} \left( \overset{\bullet}{T}{}_{\mu}{}^{\rho}{}_{\nu} + \overset{\bullet}{T}{}_{\nu}{}^{\rho}{}_{\mu} - \overset{\bullet}{T}{}^{\rho}{}_{\mu\nu} \right) \tag{47}$$

is the contortion of the Weitzenböck torsion.

As already remarked, the curvature of the Weitzenböck connection vanishes identically:

$$\overset{\bullet}{R}{}^{\rho}{}_{\theta\mu\nu} = \partial_{\mu}\overset{\bullet}{\Gamma}{}^{\rho}{}_{\theta\nu} - \partial_{\nu}\overset{\bullet}{\Gamma}{}^{\rho}{}_{\theta\mu} + \overset{\bullet}{\Gamma}{}^{\rho}{}_{\sigma\mu}\overset{\bullet}{\Gamma}{}^{\sigma}{}_{\theta\nu} - \overset{\bullet}{\Gamma}{}^{\rho}{}_{\sigma\nu}\overset{\bullet}{\Gamma}{}^{\sigma}{}_{\theta\mu} \equiv 0. \tag{48}$$

Substituting  $\Gamma^{\rho}_{\mu\nu}$  as given by Eq. (46), we get

$$\overset{\bullet}{R}{}^{\rho}{}_{\theta\mu\nu} = \overset{\circ}{R}{}^{\rho}{}_{\theta\mu\nu} + \overset{\bullet}{Q}{}^{\rho}{}_{\theta\mu\nu} \equiv 0, \tag{49}$$

where  $\overset{\circ}{R}{}^{\theta}{}_{\rho\mu\nu}$  is the curvature of the Levi–Civita connection, and

$$\overset{\bullet}{Q}{}^{\rho}{}_{\theta\mu\nu} = \overset{\bullet}{D}_{\mu}\overset{\bullet}{K}{}^{\rho}{}_{\theta\nu} - \overset{\bullet}{D}_{\nu}\overset{\bullet}{K}{}^{\rho}{}_{\theta\mu} + \overset{\bullet}{K}{}^{\sigma}{}_{\theta\nu}\overset{\bullet}{K}{}^{\rho}{}_{\sigma\mu} - \overset{\bullet}{K}{}^{\sigma}{}_{\theta\mu}\overset{\bullet}{K}{}^{\rho}{}_{\sigma\nu}$$

$$(50)$$

is a tensor written in terms of the Weitzenböck connection only. Here,  $\overset{\bullet}{D}_{\mu}$  is the teleparallel covariant derivative, with a connection term for each index of  $\overset{\bullet}{K}{}^{\rho}{}_{\theta\nu}$ . Acting on a spacetime vector  $V^{\mu}$ , for example, its explicit form is

$$\overset{\bullet}{D}_{\rho}V^{\mu} \equiv \partial_{\rho}V^{\mu} + \left(\overset{\bullet}{\Gamma}^{\mu}{}_{\lambda\rho} - \overset{\bullet}{K}^{\mu}{}_{\lambda\rho}\right)V^{\lambda}. \tag{51}$$

Owing to the relation (46), we see that it is nothing but the Levi–Civita covariant derivative of general relativity rephrased in terms of the Weitzenböck connection.

Equation (49) has an interesting interpretation: the contribution  $\overset{\circ}{R}{}^{\rho}{}_{\theta\mu\nu}$  coming from the Levi–Civita connection compensates exactly the contribution  $Q^{\rho}{}_{\theta\mu\nu}$  coming from the Weitzenböck connection, yielding an identically zero curvature tensor  $\overset{\bullet}{R}{}^{\rho}{}_{\theta\mu\nu}$ . This is a constraint satisfied by the Levi–Civita and Weitzenböck connections, and is the fulcrum of the equivalence between the Riemannian and the teleparallel descriptions of gravitation.

### B. Spin Connection and Coupling Prescription

### 1. General Relativity Spin Connection

The interaction of a general matter field with gravitation can be obtained through the application of the so called minimal coupling prescription, according to which all ordinary derivatives must be replaced by covariant derivatives. Because they are used in the construction of these covariant derivatives, gauge connections (or potentials, in physical terminology) are the most important personages in the description of an interaction. The relevant spin connection of general relativity is the so called Ricci coefficient of rotation  $\mathring{A}_{\mu}$ , a torsionless connection assuming values in the Lie algebra of the Lorentz group:

$$\mathring{A}_{\mu} = \frac{1}{2} \mathring{A}^{ab}_{\mu} S_{ab}, \tag{52}$$

where  $S_{ab}$  is an element of the Lorentz Lie algebra written in some appropriate representation. The minimal coupling prescription in general relativity, therefore, amounts to replace

$$\partial_a \to \overset{\circ}{\mathcal{D}}_a = h_a{}^{\mu} \overset{\circ}{\mathcal{D}}_{\mu},\tag{53}$$

where

$$\mathring{\mathcal{D}}_{\mu} = \partial_{\mu} - \frac{i}{2} \mathring{A}^{ab}{}_{\mu} S_{ab} \tag{54}$$

is the Fock-Ivanenko covariant derivative.<sup>41</sup> It is important to remark that the spin connection  $\overset{\circ}{A}{}^{ab}{}_{\mu}$  is not an independent field. In fact, in terms of the Levi-Civita connection, it is written as<sup>42</sup>

$$\mathring{A}^{a}{}_{b\mu} = h^{a}{}_{\rho} \mathring{\Gamma}^{\rho}{}_{\mu\nu} h_{b}{}^{\mu} + h^{a}{}_{\rho} \partial_{\nu} h_{b}{}^{\rho} \equiv h^{a}{}_{\rho} \mathring{\nabla}_{\nu} h_{b}{}^{\rho}, \tag{55}$$

from where we see that it is totally determined by the tetrad—or equivalently, by the metric. This means that, in general relativity, the local Lorentz symmetry is not dynamical (gauged), but essentially a kinematic symmetry.

Now, as is well known, a tetrad field can be used to transform *Lorentz* into spacetime tensors, and vice-versa. For example, a Lorentz vector field  $V^a$  is related to the corresponding spacetime vector  $V^{\mu}$  through

$$V^a = h^a{}_\mu V^\mu. \tag{56}$$

As a consequence, the covariant derivative (54) of a general *Lorentz* tensor field reduces to the usual Levi-Civita covariant derivative of the corresponding *spacetime* tensor. For example, take the vector field  $V^a$  for which the appropriate Lorentz generator is<sup>43</sup>

$$(S_{ab})^c_d = i \left( \delta^c_a \, \eta_{bd} - \delta^c_b \, \eta_{ad} \right). \tag{57}$$

It is then an easy task to verify that

$$\overset{\circ}{\mathcal{D}}_{\mu}V^{a} = h^{a}{}_{\rho}\overset{\circ}{\nabla}_{\mu}V^{\rho} \ . \tag{58}$$

However, in the case of half-integer spin fields, the situation is completely different. In fact, as is well known, there exists no spacetime representation for spinor fields. This means that no Levi-Civita covariant derivative can be defined for these fields. Thus, the only possible form for the covariant derivative of a Dirac spinor  $\psi$ , for example, is that given in terms of the spin connection,

$$\mathring{\mathcal{D}}_{\mu}\psi = \partial_{\mu}\psi - \frac{i}{2} \mathring{A}^{ab}_{\mu} S_{ab} \psi, \tag{59}$$

where

$$S_{ab} = \frac{i}{4} [\gamma_a, \gamma_b] \tag{60}$$

is the Lorentz spin-1/2 generator, with  $\gamma_a$  the Dirac matrices. We may say, therefore, that the Fock-Ivanenko derivative  $\overset{\circ}{\mathcal{D}}_{\mu}$  is more fundamental than the Levi-Civita covariant derivative  $\overset{\circ}{\nabla}_{\mu}$  in the sense that it is able to describe, through the minimal coupling prescription, the gravitational interaction of both tensor and spinor fields.

### 2. Teleparallel Spin Connection

In order to obtain the teleparallel version of the minimal coupling prescription, it is necessary to find first the correct teleparallel spin connection. Inspired by the definition (55), which gives the general relativity spin connection, it is usual to start by making the following attempt,

$$\overset{\bullet}{A}{}^{a}{}_{b\mu} = h^{a}{}_{\rho} \overset{\bullet}{\Gamma}{}^{\rho}{}_{\nu\mu} h_{b}{}^{\mu} + h^{a}{}_{\rho} \partial_{\mu} h_{b}{}^{\rho} \equiv h^{a}{}_{\rho} \overset{\bullet}{\nabla}_{\mu} h_{b}{}^{\rho}. \tag{61}$$

However, as a consequence of the absolute parallelism condition (42), we see that  $\tilde{A}^a{}_{b\mu}=0$ . This does not mean that in teleparallel gravity the *dynamical* spin connection, that is, the spin connection defining the minimal coupling prescription, vanishes. In fact, notice that, due to the affine character of connection space, there exists infinitely more possibilities.

Let us then adopt a different procedure to look for the teleparallel spin connection. Our basic guideline will be to find a coupling prescription which results equivalent to the coupling prescription of general relativity. This can be achieved by taking Eq. (46) and rewriting it in the tetrad basis. By using the transformation properties (55) and (61), we get

$${}^{\circ}_{A}{}^{a}{}_{b\mu} = -K^{a}{}_{b\mu} + 0, \tag{62}$$

where

$$\overset{\bullet}{K}{}^{a}{}_{b\mu} = h^{a}{}_{\rho} \overset{\bullet}{K}{}^{\rho}{}_{\nu\mu} h_{b}{}^{\nu}, \tag{63}$$

and where we have already used that  $A^a{}_{b\mu}=0$ . Notice that the zero connection appearing in Eq. (62) is crucial in the sense that it is the responsible for making the right hand-side a true connection. Therefore, based on these considerations, we can say that the teleparallel spin connection is given by minus the contortion tensor plus a zero-connection:<sup>16</sup>

$$\stackrel{\bullet}{\Omega}{}^a{}_{b\mu} = -\stackrel{\bullet}{K}{}^a{}_{b\mu} + 0. \tag{64}$$

Like any connection assuming values in the Lie algebra of the Lorentz group,

$$\Omega_{\mu} = \frac{1}{2} \stackrel{\bullet}{\Omega}{}^{a}{}_{b\mu} S_{a}{}^{b},$$
(65)

 $\overset{\bullet}{\Omega}{}^{a}{}_{b\mu}$  is anti-symmetric in the first two indices. Furthermore, under an infinitesimal local Lorentz transformation with parameters  $\epsilon^{a}{}_{b} \equiv \epsilon^{a}{}_{b}(x^{\mu})$ , it changes according to

$$\delta \hat{\Omega}^a{}_{b\mu} = -\hat{\mathcal{D}}_{\mu} \epsilon^a{}_b, \tag{66}$$

where

$$\overset{\bullet}{\mathcal{D}}_{\mu}\epsilon^{a}{}_{b} = \partial_{\mu}\epsilon^{a}{}_{b} + \overset{\bullet}{\Omega}^{a}{}_{c\mu}\epsilon^{c}{}_{b} - \overset{\bullet}{\Omega}^{c}{}_{b\mu}\epsilon^{a}{}_{c} \tag{67}$$

is the covariant derivative with  $\Omega^a_{b\mu}$  as the connection. Equation (66) is the standard gauge potential transformation of non-Abelian gauge theories. We see in this way that, in fact,  $\Omega^a_{b\mu}$  plays the role of the spin connection in teleparallel gravity. Accordingly, the teleparallel Fock-Ivanenko derivative operator is to be written in the form

$$\mathcal{D}_{\mu} = \partial_{\mu} - \frac{i}{2} \stackrel{\bullet}{\Omega}{}^{a}{}_{b\mu} S_{a}{}^{b}. \tag{68}$$

Notice that (67) is a particular case of this covariant derivative, obtained by taking  $S_a{}^b$  as the spin-2 representation of the Lorentz generators.<sup>43</sup> The minimal coupling prescription of teleparallel gravity, therefore, can be written in the form

$$\partial_a \to \mathcal{D}_a = h_a{}^{\mu} \mathcal{D}_{\mu}, \tag{69}$$

with  $\mathcal{D}_{\mu}$  the teleparallel Fock-Ivanenko derivative (68).

The covariant derivative (68) presents all necessary properties to be considered as yielding the fundamental coupling prescription in teleparallel gravity. For example, it transforms covariantly under local Lorentz transformations:

$$\overset{\bullet}{\mathcal{D}}'_{\mu} = U\overset{\bullet}{\mathcal{D}}_{\mu}U^{-1}.\tag{70}$$

Another important property is that the teleparallel coupling prescription defined by the covariant derivative (68) turns out to be completely equivalent with the usual minimal coupling prescription of general relativity. Actually, it is just what is needed so that it turns out to be the minimal coupling prescription of general relativity rephrased in terms of magnitudes of the teleparallel structure. Analogously to general relativity, the teleparallel Fock-Ivanenko covariant derivative (68) is the only one available for spinor fields in teleparallel gravity. For tensor fields, on the other hand, there is also a spacetime covariant derivative which acts in the corresponding spacetime tensors. As an example, let us consider again a Lorentz vector field  $V^a$ . By using the vector generator (57), it is an easy task to show that

$$\overset{\bullet}{\mathcal{D}}_{\mu}V^{a} = h^{a}{}_{\rho}\overset{\bullet}{D}_{\mu}V^{\rho},\tag{71}$$

where  $D_{\mu}$  is the teleparallel covariant derivative (51).

#### 3. Further Remarks

Due to the fact that the Weitzenböck spin connection  $\overset{\bullet}{A}$  vanishes when written in the tetrad basis, it is usually asserted that, for spinor fields, when written in the form

$$\overset{\bullet}{\mathcal{D}}_{\mu} = \partial_{\mu} - \frac{i}{2} \overset{\bullet}{A}{}^{a}{}_{b\mu} S_{a}{}^{b}, \tag{72}$$

the teleparallel Fock-Ivanenko derivative coincides with the ordinary derivative:<sup>35</sup>

$$\overset{\bullet}{\mathcal{D}}_{\mu} = \partial_{\mu} \tag{73}$$

However, there are several problems associated to this coupling prescription. First, it is not compatible with the coupling prescription of general relativity, which is somehow at variance with the equivalence between the corresponding gravitational Lagrangians. Second, it results different to apply this coupling prescription in the Lagrangian or in the field equation, which is a rather strange result. Summing up, we could say that there is no compelling arguments supporting the choice of  $A^a_{b\mu}$  as the spin connection of teleparallel gravity.

On the other hand, several arguments favor the conclusion that the spin connection of teleparallel gravity is in fact given by

$$\overset{\bullet}{\Omega}{}^{a}{}_{b\mu} = -\overset{\bullet}{K}{}^{a}{}_{b\mu} + 0, \tag{74}$$

which leads to the coupling prescription (68), or equivalently, to

$$\overset{\bullet}{\mathcal{D}}_{\mu} = \partial_{\mu} + \frac{i}{2} \overset{\bullet}{K}{}^{a}{}_{b\mu} S_{a}{}^{b}, \tag{75}$$

where we have dropped the zero connection for simplicity of notation. First, it is covariant under local Lorentz transformations. Second, in contrast to the coupling prescription (72), it results completely equivalent to apply the minimal coupling prescription in the Lagrangian or in the field equation. And third, it is self-consistent, and agrees with general relativity. For example, it is well know that, in the context of general relativity, the total covariant derivative of the tetrad field vanishes,

$$\partial_{\nu}h_{b}^{\rho} + \overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu} h_{b}{}^{\mu} - \overset{\circ}{A}{}^{a}{}_{b\nu} h_{a}{}^{\rho} = 0, \tag{76}$$

which is actually the same as (55). The teleparallel version of this expression can be obtained by substituting  $\overset{\circ}{\Gamma}{}^{\rho}_{\mu\nu}$  and  $\overset{\circ}{A}{}^{a}{}_{b\nu}$  by their teleparallel counterparts. Using Eqs. (46) and (62), one gets

$$\partial_{\nu}h_{b}^{\rho} + \left(\mathring{\Gamma}^{\rho}_{\mu\nu} - \mathring{K}^{\rho}_{\mu\nu}\right)h_{b}^{\mu} + \mathring{K}^{a}_{b\nu}h_{a}^{\rho} = 0. \tag{77}$$

However, the contortion terms cancel out, yielding the absolute parallelism condition

$$\partial_{\nu}h_b{}^{\rho} + \Gamma^{\rho}{}_{\mu\nu}h_b{}^{\mu} = 0, \tag{78}$$

which is the fundamental equation of teleparallel gravity. This shows the consistency of identifying the teleparallel spin connection as *minus* the contortion tensor plus the zero connection. Finally, as the coupling prescription (75) is covariant under local Lorentz transformation and is equivalent with the minimal coupling prescription of general relativity, teleparallel gravity with this coupling prescription turns out to be completely equivalent to general relativity, even in the presence of spinor fields.

# C. Application to the Fundamental Fields

#### 1. Scalar Field

According to the Einstein–Cartan models, only a spin distribution could produce or feel torsion.<sup>45</sup> A scalar field, for example, should be able to feel curvature only.<sup>10</sup> However, since from the teleparallel point of view the interaction of gravitation with any field can be described alternatively in terms of curvature or torsion, and since a scalar field is known to couple to curvature, it might also couple to torsion.<sup>46</sup> To see that, let us consider the Lagrangian of a scalar field  $\phi$ , which in a Minkowski spacetime is given by

$$\mathcal{L}_{\phi} = \frac{1}{2} \left[ \eta^{ab} \, \partial_a \phi \, \partial_b \phi - \mu^2 \phi^2 \right], \tag{79}$$

where  $\mu = mc/\hbar$ . The corresponding field equation is the Klein–Gordon equation

$$\partial_a \partial^a \phi + \mu^2 \phi = 0. \tag{80}$$

The coupling with gravitation is obtained by applying the teleparallel coupling prescription

$$\partial_a \to \mathcal{D}_a \equiv h_a{}^{\mu} \mathcal{D}_{\mu} = h_a{}^{\mu} \left( \partial_{\mu} + \frac{i}{2} \overset{\bullet}{K}{}^{ab}{}_{\mu} S_{ab} \right)$$
 (81)

to the free Lagrangian (79). The result is

$$\mathcal{L}_{\phi} = \frac{h}{2} \left[ \eta^{ab} \stackrel{\bullet}{\mathcal{D}}_{a} \phi \stackrel{\bullet}{\mathcal{D}}_{b} \phi - \mu^{2} \phi^{2} \right], \tag{82}$$

or equivalently,

$$\mathcal{L}_{\phi} = \frac{h}{2} \left[ g^{\mu\nu} \, \overset{\bullet}{\mathcal{D}}_{\mu} \phi \, \overset{\bullet}{\mathcal{D}}_{\nu} \phi - \mu^2 \phi^2 \right], \tag{83}$$

where  $h = \det(h^a_{\mu})$ . In the specific case of a scalar field,  $S_{ab}\phi = 0$ , and consequently

$$\overset{\bullet}{\mathcal{D}}_{\mu}\phi = \partial_{\mu}\phi. \tag{84}$$

Using the identity

$$\partial_{\mu}h = h h_a{}^{\rho} \partial_{\mu}h^a{}_{\rho} \equiv h \stackrel{\bullet}{\Gamma}{}^{\rho}{}_{\rho\mu}, \tag{85}$$

it is easy to show that the corresponding field equation is

$$\Box \phi + \mu^2 \phi = 0, \tag{86}$$

where

$$\stackrel{\bullet}{\Box} \phi = h^{-1} \partial_{\rho} (h \partial^{\rho} \phi) \equiv \partial_{\mu} \partial^{\mu} \phi + \stackrel{\bullet}{\Gamma}{}^{\mu}{}_{\mu\rho} \partial^{\rho} \phi \tag{87}$$

is the teleparallel version of the Laplace–Beltrami operator. Because  $\Gamma^{\rho}_{\rho\mu}$  is not symmetric in the last two indices, the above expression is not the Weitzenböck covariant divergence of  $\partial^{\mu}\phi$ . From Eq. (43), however, we can write

$$\Gamma^{\mu}{}_{\mu\rho} = \Gamma^{\mu}{}_{\rho\mu} + T^{\mu}{}_{\rho\mu}, \tag{88}$$

and the expression for  $\Box \phi$  may be rewritten in the form

$$\overset{\bullet}{\Box}\phi = \left(\overset{\bullet}{\nabla}_{\rho} + \overset{\bullet}{T}^{\mu}{}_{\rho\mu}\right)\partial^{\rho}\phi,\tag{89}$$

from where we see that the scalar field is able to couple to torsion through the derivative  $\partial^{\rho}\phi$ . Making use of the identity

$$T^{\rho}_{\mu\rho} = -K^{\rho}_{\mu\rho}, \tag{90}$$

the teleparallel version of the Klein–Gordon equation turns out to be  $^{46}$ 

$$D_{\mu}\partial^{\mu}\phi + \mu^{2}\phi = 0, \tag{91}$$

with  $D_{\mu}$  the teleparallel covariant derivative (51).

#### 2. Dirac Spinor Field

In teleparallel gravity, the coupling of spinor fields with gravitation is a highly controversial subject.<sup>10–15</sup> The reason for this is that in teleparallel gravity the dynamical spin connection—that is, the connection that describes the interaction of a spinor field with gravitation—is assumed to vanish.<sup>35</sup> However, as we are going to see, if instead of zero the connection (62) is considered as the teleparallel spin connection, teleparallel gravity becomes consistent and fully equivalent with general relativity, even in the presence of spinor fields.

In Minkowski spacetime, the spinor field Lagrangian is

$$\mathcal{L}_{\psi} = \frac{ic\hbar}{2} \left( \bar{\psi} \, \gamma^a \partial_a \psi - \partial_a \bar{\psi} \gamma^a \, \psi \right) - mc^2 \, \bar{\psi} \psi. \tag{92}$$

The corresponding field equation is the free Dirac equation

$$i\hbar\gamma^a\,\partial_a\psi - mc\,\psi = 0. \tag{93}$$

The gravitationally coupled Dirac Lagrangian is obtained by applying the teleparallel coupling prescription (81), with  $S_{ab}$  the spinor representation (60). The result is

$$\mathcal{L}_{\psi} = h \left[ \frac{ic\hbar}{2} \left( \bar{\psi} \gamma^{\mu} \stackrel{\bullet}{\mathcal{D}}_{\mu} \psi - \stackrel{\bullet}{\mathcal{D}}_{\mu}^{*} \bar{\psi} \gamma^{\mu} \psi \right) - m c^{2} \bar{\psi} \psi \right], \tag{94}$$

where  $\gamma^{\mu} \equiv \gamma^{\mu}(x) = \gamma^a h_a^{\mu}$ . Using the identity  $\mathcal{D}_{\mu}(h\gamma^{\mu}) = 0$ , the teleparallel version of the coupled Dirac equation is found to be

$$i\hbar\gamma^{\mu} \overset{\bullet}{\mathcal{D}}_{\mu} \psi - mc \psi = 0. \tag{95}$$

Comparing the Fock–Ivanenko derivatives (54) and (75), we see that, whereas in general relativity the Dirac spinor couples to the Ricci coefficient of rotation  $\overset{\circ}{A}_{\mu}$ , in teleparallel gravity it couples to the contortion tensor  $\overset{\bullet}{K}_{\mu}$ .

#### 3. Electromagnetic Field

In the usual framework of torsion gravity, it is a commonplace to assert that the electromagnetic field cannot be coupled to torsion in order to preserve the local gauge invariance of Maxwell's theory. Equivalently, one can say that, in the presence of torsion, the requirement of gauge invariance precludes the existence of a gravitational minimal coupling prescription for the electromagnetic field.<sup>10</sup> To circumvent this problem, it is usually postulated that the electromagnetic field can neither produce nor feel torsion.<sup>47</sup> In other words, torsion is assumed to be irrelevant to the Maxwell's equations.<sup>48</sup> This "solution" to the problem of the interaction of torsion with the electromagnetic field is far from reasonable. A far more consistent solution is achieved by observing that, if the electromagnetic field couples to curvature, the equivalence between general relativity and teleparallel gravity implies necessarily that it must also couple to torsion. In addition, it should be remarked that the above postulate is not valid at a microscopic level

since, from a quantum point of view, one may always expect an interaction between photons and torsion.<sup>49</sup> The reason for this is that a photon, perturbatively speaking, can virtually disintegrate into an electron–positron pair, and as these particles are massive fermions which couple to torsion, the photon must necessarily feel the presence of torsion. Consequently, even not interacting directly with torsion, the photon field does feel torsion through the virtual pair produced by the vacuum polarization. Moreover, as all macroscopic phenomena must necessarily have an interpretation based on an average of microscopic phenomena, and taking into account the strictly attractive character of gravitation which eliminates the possibility of a vanishing average, the above hypothesis seems to lead to a contradiction as no interaction is postulated to exist at the macroscopic level. As we are going to see, in spite of the controversies, <sup>50</sup> provided the teleparallel spin connection be properly chosen, the electromagnetic field can consistently couple to torsion in teleparallel gravity.

In Minkowski spacetime, the electromagnetic field is described by the Lagrangian density

$$\mathcal{L}_{em} = -\frac{1}{4} F_{ab} F^{ab},\tag{96}$$

where

$$F_{ab} = \partial_a A_b - \partial_b A_a \tag{97}$$

is the electromagnetic field-strength. The corresponding field equation is

$$\partial_a F^{ab} = 0, (98)$$

which along with the Bianchi identity

$$\partial_a F_{bc} + \partial_c F_{ab} + \partial_b F_{ca} = 0, (99)$$

constitutes Maxwell's equations. In the Lorentz gauge  $\partial_a A^a = 0$ , the field equation (98) acquires the form

$$\partial_c \partial^c A^a = 0. ag{100}$$

Let us obtain now, by applying the coupling prescription (81), Maxwell's equation in teleparallel gravity.<sup>51</sup> In the specific case of the electromagnetic vector field, the Lorentz generators  $S_{ab}$  are written in the vector representation (57), and the Fock–Ivanenko derivative assumes the form

$$\overset{\bullet}{\mathcal{D}}_{\mu}A^{c} = \partial_{\mu}A^{c} - \overset{\bullet}{K^{c}}_{d\mu}A^{d}. \tag{101}$$

To obtain the corresponding covariant derivative of the spacetime vector field  $A^{\nu}$ , we substitute  $A^{d}=h^{d}_{\ \nu}A^{\nu}$  in the right-hand side. The result is

$$\overset{\bullet}{\mathcal{D}}_{\mu}A^{c} = h^{c}{}_{\rho}\overset{\bullet}{\mathcal{D}}_{\mu}A^{\rho},\tag{102}$$

with

$$\overset{\bullet}{D}_{\mu}A^{\rho} = \partial_{\mu}A^{\rho} + \left(\overset{\bullet}{\Gamma}{}^{\rho}{}_{\nu\mu} - \overset{\bullet}{K}{}^{\rho}{}_{\nu\mu}\right)A^{\nu} \tag{103}$$

the teleparallel covariant derivative. This means that, for the specific case of a vector field, the teleparallel version of the minimal coupling prescription (81) can alternatively be stated as

$$\partial_a A_b \to h_a{}^\mu h_b{}^\rho \overset{\bullet}{D}_\mu A_\rho. \tag{104}$$

As a consequence, the gravitationally coupled Maxwell Lagrangian in teleparallel gravity can be written as

$$\mathcal{L}_{em} = -\frac{h}{4} F_{\mu\nu} F^{\mu\nu},\tag{105}$$

where

$$F_{\mu\nu} = \overset{\bullet}{D}_{\mu} A_{\nu} - \overset{\bullet}{D}_{\nu} A_{\mu}. \tag{106}$$

Using the explicit form of  $\overset{\bullet}{D}_{\mu}$ , and the definitions of torsion and contortion tensors, it is easy to verify that

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}. \tag{107}$$

We notice in passing that this tensor is invariant under the U(1) electromagnetic gauge transformations. The corresponding field equation is

$$D_{\mu}F^{\mu\nu} = 0, \tag{108}$$

which yields the first pair of Maxwell's equation in teleparallel gravity. Assuming the teleparallel Lorentz gauge  $D_{\mu}A^{\mu} = 0$ , and using the commutation relation

$$\begin{bmatrix} \overset{\bullet}{D}_{\mu}, \overset{\bullet}{D}_{\nu} \end{bmatrix} A^{\mu} = -\overset{\bullet}{Q}_{\mu\nu} A^{\mu}, \tag{109}$$

where  $Q_{\mu\nu} = Q^{\rho}_{\mu\rho\nu}$ , with  $Q^{\rho}_{\mu\sigma\nu}$  given by Eq. (50), we obtain

$$\overset{\bullet}{D}_{\mu}\overset{\bullet}{D}^{\mu}A_{\nu} + \overset{\bullet}{Q}^{\mu}{}_{\nu}A_{\mu} = 0. \tag{110}$$

This is the teleparallel version of the first pair of Maxwell's equation. On the other hand, by using the same coupling prescription in the Bianchi identity (99), the teleparallel version of the second pair of Maxwell's equation is found to be

$$\partial_{\mu}F_{\nu\sigma} + \partial_{\sigma}F_{\mu\nu} + \partial_{\nu}F_{\sigma\mu} = 0. \tag{111}$$

Summing up: in the context of the teleparallel equivalent of general relativity, the electromagnetic field is able to couple to torsion, and that this coupling does not violate the U(1) gauge invariance of Maxwell's theory. Furthermore, using the relation (46), it is easy to verify that the teleparallel version of Maxwell's equations, which are equations written in terms of the Weitzenböck connection only, are completely equivalent with the usual Maxwell's equations in a Riemannian background, which are equations written in terms of the Levi–Civita connection only. We can then say that teleparallel gravity is able to provide a consistent description of the interaction of torsion with the electromagnetic field.<sup>51</sup>

#### D. Lagrangian and Field Equations

The Lagrangian of the teleparallel equivalent of general relativity is

$$\overset{\bullet}{\mathcal{L}} = \frac{h}{2k^2} \left[ \frac{1}{4} \overset{\bullet}{T}^{\rho}{}_{\mu\nu} \overset{\bullet}{T}{}_{\rho}{}^{\mu\nu} + \frac{1}{2} \overset{\bullet}{T}{}^{\rho}{}_{\mu\nu} \overset{\bullet}{T}{}^{\nu\mu}{}_{\rho} - \overset{\bullet}{T}{}_{\rho\mu}{}^{\rho} \overset{\bullet}{T}{}^{\nu\mu}{}_{\nu} \right], \tag{112}$$

where  $k^2 = 8\pi G/c^4$  and  $h = \det(h^a{}_{\mu})$ . The first term corresponds to the usual Lagrangian of gauge theories. In the gravitational case, however, owing to the presence of a tetrad field, algebra and spacetime indices can now be changed into each other, and in consequence new contractions turn out to be possible. It is exactly this possibility that gives rise to the other two terms of the above Lagrangian. If we define the tensor

$$\dot{S}^{\rho\mu\nu} = -\dot{S}^{\rho\nu\mu} = \left[ \dot{K}^{\mu\nu\rho} - g^{\rho\nu} \dot{T}^{\sigma\mu}{}_{\sigma} + g^{\rho\mu} \dot{T}^{\sigma\nu}{}_{\sigma} \right], \tag{113}$$

usually called superpotential,<sup>52</sup> the teleparallel Lagrangian (112) can be rewritten in the form<sup>53</sup>

$$\overset{\bullet}{\mathcal{L}} = \frac{h}{4k^2} \overset{\bullet}{T}_{\rho\mu\nu} \overset{\bullet}{S}^{\rho\mu\nu}.$$
(114)

Using the identity

$$T^{\mu}{}_{\mu\rho} = K^{\mu}{}_{\rho\mu},$$
 (115)

which follows from the contortion definition, it can still be written

$$\overset{\bullet}{\mathcal{L}} = \frac{h}{2k^2} \left( \overset{\bullet}{K}^{\mu\nu\rho} \overset{\bullet}{K}_{\rho\nu\mu} - \overset{\bullet}{K}^{\mu\rho}{}_{\mu} \overset{\bullet}{K}^{\nu}{}_{\rho\nu} \right).$$
(116)

On the other hand, from Eq. (49) it is possible to show that

$$-\stackrel{\circ}{R} = \stackrel{\bullet}{Q} \equiv \left(\stackrel{\bullet}{K}^{\mu\nu\rho}\stackrel{\bullet}{K}_{\rho\nu\mu} - \stackrel{\bullet}{K}^{\mu\rho}_{\mu}\stackrel{\bullet}{K}^{\nu}_{\rho\nu}\right) + \partial_{\mu}(2h\stackrel{\bullet}{T}^{\nu\mu}_{\nu}). \tag{117}$$

Therefore, we see that

$$\overset{\bullet}{\mathcal{L}} = \overset{\circ}{\mathcal{L}} - \partial_{\mu} \left( 2 h \, k^{-2} \, \overset{\bullet}{T}^{\nu \mu}_{\nu} \right), \tag{118}$$

where

$$\overset{\circ}{\mathcal{L}} = -\frac{\sqrt{-g}}{2k^2} \overset{\circ}{R},\tag{119}$$

represents the Einstein-Hilbert Lagrangian of general relativity, and where we have used the identification  $h = \sqrt{-g}$ , with  $g = \det(g_{\mu\nu})$ . Up to a divergence, therefore, the teleparallel Lagrangian is equivalent to the Einstein-Hilbert Lagrangian of general relativity. It is interesting to observe that the first-order Møller's Lagrangian of general relativity,<sup>29</sup>

$$\mathring{\mathcal{L}}_{\mathrm{M}} = \frac{h}{2k^2} \left( \mathring{\nabla}_{\mu} h^{a\nu} \mathring{\nabla}_{\nu} h_a^{\mu} - \mathring{\nabla}_{\mu} h_a^{\mu} \mathring{\nabla}_{\nu} h^{a\nu} \right), \tag{120}$$

which differs from the Einstein-Hilbert Lagrangian by a total divergence, when rewritten in terms of the Weitzeböck connection coincides exactly—that is, without any boundary term—with the teleparallel Lagrangian (114). Teleparallel gravity, therefore, can be considered as fully equivalent with the Møller's first–order formulation of general relativity.

Let us consider now the Lagrangian

$$\mathcal{L} = \overset{\bullet}{\mathcal{L}} + \mathcal{L}_m \tag{121}$$

where  $\mathcal{L}_m$  is the Lagrangian of a general matter field  $\Psi$ . By performing variations in relation to the gauge field  $B^a{}_{\rho}$ , we obtain the teleparallel version of the gravitational field equation

$$\partial_{\sigma}(h\overset{\bullet}{S}_{a}{}^{\rho\sigma}) - k^{2}(h\overset{\bullet}{J}_{a}{}^{\rho}) = k^{2}(h\Theta_{a}{}^{\rho}), \tag{122}$$

where  $\overset{\bullet}{S}_a{}^{\rho\sigma} = h_a{}^{\lambda} \overset{\bullet}{S}_{\lambda}{}^{\rho\sigma}$ , and

$$h\Theta_a{}^{\rho} \equiv -\frac{\delta \mathcal{L}_m}{\delta B^a{}_{\rho}} \equiv -\frac{\delta \mathcal{L}_m}{\delta h^a{}_{\rho}} = -\left(\frac{\partial \mathcal{L}_m}{\partial h^a{}_{\rho}} - \partial_{\lambda} \frac{\partial \mathcal{L}_m}{\partial_{\lambda} \partial h^a{}_{\rho}}\right)$$
(123)

is the matter energy-momentum tensor. Analogously to the Yang-Mills theories,

$$h \dot{\mathcal{J}}_{a}{}^{\rho} \equiv -\frac{\partial \dot{\mathcal{L}}}{\partial B^{a}{}_{o}} \equiv -\frac{\partial \dot{\mathcal{L}}}{\partial h^{a}{}_{o}} = \frac{h}{k^{2}} h_{a}{}^{\lambda} \dot{S}_{c}{}^{\nu\rho} \dot{T}^{c}{}_{\nu\lambda} - h_{a}{}^{\rho} \dot{\mathcal{L}}$$

$$(124)$$

stands for the gauge current, which in this case represents the energy and momentum of the gravitational field.<sup>54</sup> In a purely spacetime form, it becomes

$$\mathring{J}_{\mu}{}^{\rho} \equiv h^{a}{}_{\mu} \mathring{J}_{a}{}^{\rho} = \frac{1}{k^{2}} \left( \mathring{S}_{\sigma}{}^{\nu\rho} \mathring{T}^{\sigma}{}_{\nu\mu} - \frac{1}{4} \delta_{\mu}{}^{\rho} \mathring{S}_{\sigma}{}^{\nu\lambda} \mathring{T}^{\sigma}{}_{\nu\lambda} \right), \tag{125}$$

which has the same structure of the symmetrized<sup>55</sup> energy–momentum tensor of the electromagnetic field.<sup>56</sup> Now, by using Eq. (46), the left-hand side of the field equation (122), after a lengthy but straightforward calculation, can be shown to satisfy

$$\partial_{\sigma}(h_{S_a}^{\bullet}{}^{\rho\sigma}) - k^2 \left(h_{J_a}^{\bullet}{}^{\rho}\right) = h\left(\stackrel{\circ}{R}_a{}^{\rho} - \frac{1}{2}h_a{}^{\rho}\stackrel{\circ}{R}\right). \tag{126}$$

As expected, due to the equivalence between the corresponding Lagrangians, the teleparallel field equation (122) is equivalent to Einstein's equation

$${\stackrel{\circ}{R}}_{a}{}^{\rho} - \frac{1}{2} h_{a}{}^{\rho} {\stackrel{\circ}{R}} = k^{2} \Theta_{a}{}^{\rho}. \tag{127}$$

### E. Gravitational Energy-Momentum Current

The definition of an energy-momentum density for the gravitational field is one of the oldest and most controversial problems of gravitation. As a true field, it would be natural to expect that gravity should have its own local energy-momentum density. However, it is usually asserted that such a density cannot be locally defined because of the equivalence principle.<sup>2</sup> As a consequence, any attempt to identify an energy-momentum density for the gravitational field leads to complexes that are not true tensors. The first of such attempt was made by Einstein who proposed an expression for the energy-momentum density of the gravitational field which was nothing but the canonical expression obtained from Noether's theorem.<sup>57</sup> Indeed, this quantity is a pseudotensor, an object that depends on the coordinate system. Several other attempts have been made, leading to different expressions for the energy-momentum pseudotensor for the gravitational field.<sup>58</sup>

Despite the existence of some controversial points related to the formulation of the equivalence principle, it seems true that, in the context of general relativity, no tensorial expression for the gravitational energy-momentum density can exist. In spite of some skepticism, there has been a continuous interest in this problem. In particular, a quasilocal approach has been proposed which is highly clarifying. According to this approach, for each gravitational energy-momentum pseudotensor, there is an associated superpotential which is a Hamiltonian boundary term. The energy-momentum defined by such a pseudotensor does not really depend on the local value of the reference frame, but only on the value of the reference frame on the boundary of a region—then its quasilocal character. As the relevant boundary conditions are physically acceptable, this approach validates the pseudotensor approach to the gravitational energy-momentum problem.

However, in the gauge context of teleparallel gravity, the existence of a tensorial expression for the gravitational energy-momentum density seems to be possible. Accordingly, the absence of such expression should be attributed to the general relativity description of gravitation, which seems not to be the appropriate framework to deal with this problem.<sup>62</sup> In fact, as can be easily checked, the current  $\hat{\jmath}_a^\rho$  transforms covariantly under a general spacetime coordinate transformation, is invariant under local (gauge) translation of the tangent-space coordinates, and transforms covariantly under a tangent-space Lorentz transformation. This means that  $\hat{\jmath}_a^\rho$  is a true spacetime and gauge tensor. Since our interest is the gravitational energy-momentum current, let us consider the sourceless case, in which the gravitational field equation becomes

$$\partial_{\sigma}(hS_a^{\rho\sigma}) - k^2 (hJ_a^{\rho}) = 0. \tag{128}$$

Due to the anti-symmetry of  $\overset{\bullet}{S}_a{}^{\rho\sigma}$  in the last two indices,  $(h\overset{\bullet}{\jmath}_a{}^{\rho})$  is conserved as a consequence of the field equation:

$$\partial_{\rho}(h \mathcal{I}_{a}^{\bullet}{}^{\rho}) = 0. \tag{129}$$

Making use of the identity

$$\partial_{\rho}h \equiv h_{\Gamma}^{\bullet}{}^{\nu}{}_{\nu\rho} = h \left( \Gamma^{\nu}{}_{\rho\nu} - K^{\nu}{}_{\rho\nu} \right), \tag{130}$$

this conservation law can be rewritten in the manifestly covariant form

$$\overset{\bullet}{\mathcal{D}}_{\rho}\overset{\bullet}{J}_{a}^{\rho} \equiv \partial_{\rho}\overset{\bullet}{J}_{a}^{\rho} + \left(\overset{\bullet}{\Gamma}^{\rho}{}_{\lambda\rho} - \overset{\bullet}{K}^{\rho}{}_{\lambda\rho}\right)\overset{\bullet}{J}_{a}^{\lambda} = 0,$$
(131)

with  $\overset{\bullet}{\mathcal{D}}_{\rho}$  the teleparallel covariant derivative (51).

Let us find out now the relation between the current  $\mathcal{J}_a^{\rho}$  and the usual gravitational energy-momentum pseudotensor. By using Eq. (41) to express  $\partial_{\rho}h_a^{\lambda}$ , the field equation (128) can be rewritten in a purely spacetime form,

$$\partial_{\sigma}(h_{S_{\lambda}}^{\bullet})^{\rho\sigma} - k^{2} (h_{t_{\lambda}}^{\bullet})^{\rho} = 0, \tag{132}$$

where

$$h_{t_{\lambda}}^{\bullet}{}^{\rho} = k^{-2} h_{\Gamma}^{\mu}{}_{\nu\lambda} S_{\mu}{}^{\rho\nu} - \delta_{\lambda}{}^{\rho} \mathcal{L}$$

$$\tag{133}$$

stands for the canonical energy-momentum pseudotensor of the gravitational field.<sup>63</sup> It is important to notice that  $t_{\lambda}^{\rho}$  is not simply the gauge current  $\hat{J}_{a}^{\rho}$  with the algebraic index "a" changed to the spacetime index " $\lambda$ ". It incorporates also an extra term coming from the derivative term of Eq. (122):

$$\overset{\bullet}{t}_{\lambda}{}^{\rho} = h^{a}{}_{\lambda} \overset{\bullet}{J}_{a}{}^{\rho} + k^{-2} \overset{\bullet}{\Gamma}{}^{\mu}{}_{\lambda\nu} \overset{\bullet}{S}_{\mu}{}^{\rho\nu}.$$
(134)

We see thus clearly the origin of the connection-term which transforms the gauge current  $\hat{j}_a{}^{\rho}$  into the energy-momentum pseudotensor  $\hat{t}_{\lambda}{}^{\rho}$ . Through the same mechanism, it is possible to appropriately exchange further terms between the derivative and the current terms of the field equation (132), giving rise to different definitions for the energy-momentum pseudotensor, each one connected to a different superpotential  $(h\hat{S}_{\lambda}{}^{\rho\sigma})$ . It is important to remark finally that, like the gauge current  $(h\hat{J}_a{}^{\rho})$ , the pseudotensor  $(h\hat{t}_{\lambda}{}^{\rho})$  is conserved as a consequence of the field equation:

$$\partial_{\rho}(ht_{\lambda}^{\rho}) = 0. \tag{135}$$

However, in contrast to what occurs with  $\hat{J}_a^{\rho}$ , due to the pseudotensor character of  $t_{\lambda}^{\rho}$ , this conservation law cannot be rewritten in terms of the teleparallel covariant derivative.

Because of its simplicity and transparency, the teleparallel approach to gravitation seems to be much more appropriate than general relativity to deal with the energy problem of the gravitational field. In fact, Møller already noticed a long time ago that a satisfactory solution for the problem of the energy distribution in a gravitational field could be obtained in the framework of a tetrad theory. In our notation, his expression for the gravitational energy-momentum density is<sup>29</sup>

$$ht_{\lambda}{}^{\rho} = \frac{\partial \mathring{\mathcal{L}}_{\mathcal{M}}}{\partial \partial_{\rho} h^{a}{}_{\mu}} \partial_{\lambda} h^{a}{}_{\mu} - \delta_{\lambda}{}^{\rho} \mathring{\mathcal{L}}_{\mathcal{M}}, \tag{136}$$

which is nothing but the usual Noether's canonical energy-momentum density in the tetrad formulation of general relativity. Since Møller's Lagrangian, given by Eq. (120), is exactly (without any surface term) equivalent with the teleparallel Lagrangian (114), the Møller's expression (136) will correspond exactly with the teleparallel energy-momentum density (133).

### F. Noether's Theorem: Matter Conservation Law

Let us consider the action integral of a general matter field,

$$S_m = \frac{1}{c} \int \mathcal{L}_m \ d^4x. \tag{137}$$

We assume a first-order formalism, according to which the Lagrangian depends only on the fields and on their first derivatives. Under a general transformation of the spacetime coordinates,

$$x^{\prime \rho} = x^{\rho} + \xi^{\rho},\tag{138}$$

the tetrad transforms according to

$$\delta h_a^{\ \mu} = h_a^{\ \rho} \, \partial_o \xi^{\mu} - \xi^{\rho} \, \partial_o h_a^{\ \mu}. \tag{139}$$

The corresponding transformation of the action integral is written as

$$\delta \mathcal{S}_m = \frac{1}{c} \int \Theta_\mu{}^a \, \delta h_a{}^\mu \, h \, d^4 x, \tag{140}$$

where  $h\Theta_{\mu}{}^{a} = \delta \mathcal{L}_{m}/\delta h_{a}{}^{\mu}$  is the matter energy-momentum tensor. Substituting  $\delta h_{a}{}^{\mu}$  as given by Eq. (139), we obtain

$$\delta \mathcal{S}_m = \frac{1}{c} \int \Theta_\mu{}^a \left[ h_a{}^\rho \partial_\rho \xi^\mu - \xi^\rho \partial_\rho h_a{}^\mu \right] h \ d^4 x. \tag{141}$$

Now, from the absolute parallelism condition (78), we have that

$$\partial_{\rho}h_{a}{}^{\mu} = 0 - \overset{\bullet}{\Gamma}{}^{\mu}{}_{\lambda\rho} h_{a}{}^{\lambda}. \tag{142}$$

Substituting into (141), after some manipulations, we get

$$\delta S_m = \frac{1}{c} \int \left[ \Theta_c^{\ \rho} \left( \partial_\rho \xi^c + 0 \right) + \Theta_\mu^{\ \rho} \mathring{T}^\mu_{\ \lambda\rho} \xi^\lambda \right] h \ d^4 x. \tag{143}$$

Contrary to some claims,  $^{64}$  provided the action  $S_m$  is local Lorentz invariant, the energy–momentum tensor  $\Theta_{\mu}{}^{\rho}$  is necessarily symmetric.  $^{65}$  Consequently, the above variation assumes the form

$$\delta \mathcal{S}_m = \frac{1}{c} \int \Theta_c^{\rho} \left[ \partial_{\rho} \xi^c + 0 - \overset{\bullet}{K}{}^c{}_{b\rho} \xi^b \right] h d^4 x. \tag{144}$$

Integrating the first term by parts, neglecting the surface term, and considering the arbitrariness of  $\xi^c$ , it follows from the invariance of the action integral that

$$\partial_{\mu}(h\Theta_a{}^{\mu}) - (0 - K^{\bullet}{}_{a\mu}) h\Theta_b{}^{\mu} = 0.$$
 (145)

Making use of the identity (130), the above conservation law becomes

$$\partial_{\mu}\Theta_{a}{}^{\mu} + (\mathring{\Gamma}^{\mu}{}_{\rho\mu} - \mathring{K}^{\mu}{}_{\rho\mu}) \Theta_{a}{}^{\rho} - (0 - \mathring{K}^{b}{}_{a\mu}) \Theta_{b}{}^{\mu} = 0. \tag{146}$$

In a purely spacetime form, it becomes

$$\partial_{\mu}\Theta_{\lambda}^{\mu} + (\overset{\bullet}{\Gamma}{}^{\mu}{}_{\rho\mu} - \overset{\bullet}{K}{}^{\mu}{}_{\rho\mu})\Theta_{\lambda}^{\rho} - (\overset{\bullet}{\Gamma}{}^{\rho}{}_{\lambda\mu} - \overset{\bullet}{K}{}^{\rho}{}_{\lambda\mu})\Theta_{\rho}^{\mu} \equiv \overset{\bullet}{D}_{\mu}\Theta_{\lambda}^{\mu} = 0. \tag{147}$$

This is the conservation law of matter energy-momentum tensor. In teleparallel gravity, therefore, it is not the Weitzenböck covariant derivative  $\overset{\bullet}{\nabla}_{\mu}$ , but the teleparallel covariant derivative  $\overset{\bullet}{D}_{\mu}$  that yields the correct conservation law for the energy-momentum tensors of matter fields. Of course, because of the relation (46), it can be written in the form

$$\overset{\circ}{\nabla}_{\mu}\Theta_{\lambda}{}^{\mu} \equiv \partial_{\mu}\Theta_{\lambda}{}^{\mu} + \overset{\circ}{\Gamma}{}^{\mu}{}_{\rho\mu}\Theta_{\lambda}{}^{\rho} - \overset{\circ}{\Gamma}{}^{\rho}{}_{\lambda\mu}\Theta_{\rho}{}^{\mu} = 0, \tag{148}$$

which is the corresponding conservation law of general relativity. It is important to remark that these "covariant conservation laws" are not, strictly speaking, real conservation laws in the sense that they do not yield a conserved "charge". They are actually identities, called Noether identities, which govern the exchange of energy and momentum between the matter and the gravitational fields. <sup>66</sup>

### G. Bianchi Identities

Analogously to the Maxwell theory, the first Bianchi identity of the gauge theory for the translation group is 67

$$\partial_{\rho} T^{a}{}_{\mu\nu} + \partial_{\nu} T^{a}{}_{\rho\mu} + \partial_{\mu} T^{a}{}_{\nu\rho} = 0. \tag{149}$$

Through a tedious, but straightforward calculation, it can be rewritten in a purely spacetime form:

$$\overset{\bullet}{Q}{}^{\rho}{}_{\theta\mu\nu} + \overset{\bullet}{Q}{}^{\rho}{}_{\nu\theta\mu} + \overset{\bullet}{Q}{}^{\rho}{}_{\mu\nu\theta} = 0. \tag{150}$$

Then, by making use of relation (49), it is easy to verify that it coincides with the first Bianchi identity of general relativity:

$$\overset{\circ}{R}{}^{\rho}{}_{\theta\mu\nu} + \overset{\circ}{R}{}^{\rho}{}_{\nu\theta\mu} + \overset{\circ}{R}{}^{\rho}{}_{\mu\nu\theta} = 0. \tag{151}$$

On the other hand, similarly to general relativity, teleparallel gravity presents also a second Bianchi identity, which is given by

$$\overset{\bullet}{D}_{\sigma}\overset{\bullet}{Q}_{\rho\theta\mu\nu} + \overset{\bullet}{D}_{\nu}\overset{\bullet}{Q}_{\rho\theta\sigma\mu} + \overset{\bullet}{D}_{\mu}\overset{\bullet}{Q}_{\rho\theta\nu\sigma} = 0. \tag{152}$$

This identity is easily seen to be equivalent to the second Bianchi identity of general relativity

$$\overset{\circ}{\nabla}_{\sigma}\overset{\circ}{R}_{\rho\theta\mu\nu} + \overset{\circ}{\nabla}_{\nu}\overset{\circ}{R}_{\rho\theta\sigma\mu} + \overset{\circ}{\nabla}_{\mu}\overset{\circ}{R}_{\rho\theta\nu\sigma} = 0, \tag{153}$$

whose contracted form is

$$\overset{\circ}{\nabla}_{\rho} \left[ \overset{\circ}{R}_{\lambda}{}^{\rho} - \frac{1}{2} \delta_{\lambda}{}^{\rho} \overset{\circ}{R} \right] = 0. \tag{154}$$

Through a similar procedure, the contracted form of the teleparallel Bianchi identity (152) is found to  $be^{68}$ 

$$\overset{\bullet}{D}_{\rho} \left[ \partial_{\sigma} (h \overset{\bullet}{S}_{\lambda}{}^{\rho \sigma}) - k^{2} (h \overset{\bullet}{t}_{\lambda}{}^{\rho}) \right] = 0. \tag{155}$$

If we remember that, in the presence of a general source field, the teleparallel field equation is given by

$$\partial_{\sigma}(hS_{\lambda}^{\bullet\sigma}) - k^{2}(ht_{\lambda}^{\bullet\rho}) = k^{2}(h\Theta_{\lambda}^{\rho}), \tag{156}$$

with  $\Theta_{\lambda}^{\rho}$  the matter energy-momentum tensor, and taking into account that

$$\overset{\bullet}{D}_{\rho}h = 0, \tag{157}$$

the Bianchi identity (155) is seen to be consistent with the conservation law

$$D_{\rho}\Theta_{\lambda}{}^{\rho} = 0, \tag{158}$$

as obtained from Noether's theorem [see Eq. (147)].

### H. Role of Torsion in Teleparallel Gravity

1. Force Equation Versus Geodesics

To begin with, let us consider, in the context of teleparallel gravity, the motion of a spinless particle of mass m in a gravitational field  $B^a{}_{\mu}$ . Analogously to the electromagnetic case, <sup>56</sup> the action integral is written in the form

$$S = \int_{a}^{b} \left[ -m c \, d\sigma - m c \, B^{a}_{\mu} \, u_{a} \, dx^{\mu} \right], \tag{159}$$

where  $d\sigma = (\eta_{ab} dx^a dx^b)^{1/2}$  is the Minkowski tangent-space invariant interval,

$$u^a = h^a{}_{\mu} u^{\mu}, \tag{160}$$

is the anholonomic particle four-velocity, with

$$u^{\mu} = \frac{dx^{\mu}}{ds} \tag{161}$$

the holonomic four-velocity, which is written in terms of the spacetime invariant interval  $ds = (g_{\mu\nu}dx^{\mu}dx^{\nu})^{1/2}$ . It should be noticed that, in terms of the tangent-space line element  $d\sigma$ , the four-velocity  $u^a$  is holonomic:<sup>7</sup>

$$u^a = \frac{dx^a}{d\sigma}. (162)$$

The first term of the action (159) represents the action of a free particle, and the second the coupling of the particle's mass with the gravitational field. Notice that the separation of the action in these two terms is possible only in a gauge theory, like teleparallel gravity, being not possible in general relativity. It is, however, equivalent with the usual action of general relativity. In fact, if we introduce the identities

$$h_a{}^\mu u^a u_\mu = 1 \tag{163}$$

and

$$\frac{\partial x^{\mu}}{\partial x^{a}} u^{a} u_{\mu} = \frac{ds}{d\sigma},\tag{164}$$

the action (159) can easily be seen to reduce to its general relativity version

$$S = -\int_{a}^{b} m \, c \, ds.$$

In this case, the interaction of the particle with the gravitational field is described by the metric tensor  $g_{\mu\nu}$ , which is present in ds.

Variation of the action (159) yields the equation of motion

$$h^a{}_\mu \frac{du_a}{ds} = T^a{}_{\mu\rho} u_a u^\rho. \tag{165}$$

This is the force equation governing the motion of the particle, in which the teleparallel field strength  $T^a_{\mu\rho}$ —that is, torsion—plays the role of gravitational force. To write it in a purely spacetime form, we use the relation

$$h^{a}_{\mu}\frac{du_{a}}{ds} = \omega_{\mu} \equiv \frac{du_{\mu}}{ds} - \Gamma^{\theta}_{\mu\nu}u_{\theta}u^{\nu}, \tag{166}$$

where  $\omega_{\mu}$  is the spacetime particle four-acceleration. We then get

$$u^{\nu} \overset{\bullet}{\nabla}_{\nu} u_{\mu} \equiv \frac{du_{\mu}}{ds} - \overset{\bullet}{\Gamma}^{\theta}{}_{\mu\nu} u_{\theta} u^{\nu} = \overset{\bullet}{T}^{\theta}{}_{\mu\nu} u_{\theta} u^{\nu}. \tag{167}$$

The left-hand side of this equation is the Weitzenböck covariant derivative of  $u_{\mu}$  along the world-line of the particle. The presence of the torsion tensor on its right-hand side, as already stressed, shows that in teleparallel gravity torsion plays the role of gravitational force. By using the identity

$$T^{\theta}_{\mu\nu} u_{\theta} u^{\nu} = -K^{\theta}_{\mu\nu} u_{\theta} u^{\nu}, \tag{168}$$

this equation can be rewritten in the form

$$u^{\nu} \overset{\bullet}{D}_{\nu} u_{\mu} \equiv \frac{du_{\mu}}{ds} - \left( \overset{\bullet}{\Gamma}^{\theta}{}_{\mu\nu} - \overset{\bullet}{K}^{\theta}{}_{\mu\nu} \right) u_{\theta} u^{\nu} = 0. \tag{169}$$

The left-hand side of this equation is the teleparallel covariant derivative of  $u_{\mu}$  along the world-line of the particle. Using the relation (46), it is found to be

$$u^{\nu} \overset{\circ}{\nabla}_{\nu} u_{\mu} \equiv \frac{du_{\mu}}{ds} - \overset{\circ}{\Gamma}^{\theta}{}_{\mu\nu} u_{\theta} u^{\nu} = 0. \tag{170}$$

This is precisely the geodesic equation of general relativity, which means that the trajectories followed by spinless particles are geodesics of the underlying Riemann spacetime. In a locally inertial coordinate system, the first derivative of the metric tensor vanishes, the Levi–Civita connection vanishes as well, and the geodesic equation (170) becomes the equation of motion of a free particle. This is the usual version of the (strong) equivalence principle as formulated in general relativity.<sup>65</sup>

It is important to notice that, by using the torsion definition (43), the force equation (167) can be written in the form

$$\frac{du_{\mu}}{ds} - \overset{\bullet}{\Gamma}{}^{\theta}{}_{\mu\nu} u_{\theta} u^{\nu} = 0. \tag{171}$$

As  $\Gamma_{\theta\nu\mu}$  is not symmetric in the last two indices, this is not a geodesic equation. This means that the trajectories followed by spinless particles are not geodesics of the induced Weitzenböck spacetime. In a locally inertial coordinate system, the first derivative of the metric tensor vanishes, and the Weitzenböck connection  $\Gamma_{\theta\nu\mu}$  becomes skew-symmetric in the first two indices. In this coordinate system, therefore, owing to the symmetry of  $u^{\theta}$   $u^{\nu}$ , the force equation (171) becomes the equation of motion of a free particle. This is the teleparallel version of the (strong) equivalence principle.<sup>5</sup>

#### 2. Teleparallel Equivalent of the Kerr-Newman Solution

In spite of the equivalence of teleparallel gravity with general relativity, there are conceptual differences between these two theories. For example, whereas in general relativity gravitation is described in terms of the curvature tensor, in teleparallel gravity it is described in terms of torsion. In addition to the conceptual differences, there are also some formal differences. For example, the possibility of decomposing torsion into three irreducible parts under the group of global Lorentz transformations allows a better understanding of the physical meaning of torsion. To exemplify this fact, we are going to study in this section the teleparallel version of the Kerr-Newman solution. This solution has a great generality as it reduces to the Kerr and to Schwarzschild solutions for some specific values of its parameters.

We begin by computing the Kerr-Newman tetrad in the Boyer-Lindquist coordinates, in which the Kerr-Newman metric is written as  $^2$ 

$$ds^{2} = g_{00}dt^{2} + g_{11}dr^{2} + g_{22}d\theta^{2} + g_{33}d\phi^{2} + 2g_{03}d\phi dt,$$
(172)

where

$$g_{00} = 1 - \frac{Rr}{\rho^2}, \ g_{11} = -\frac{\rho^2}{\Delta}, \ g_{22} = -\rho^2,$$
 (173)

$$g_{33} = -\left(r^2 + a^2 + \frac{Rra^2}{\rho^2}\sin^2\theta\right)\sin^2\theta,\tag{174}$$

$$g_{03} = g_{30} = \frac{Rra}{\rho^2} \sin^2 \theta. \tag{175}$$

In these expressions,  $\Delta = r^2 - Rr + a^2$ ,  $R = 2m - q^2/r$ ,  $\rho^2 = r^2 + a^2\cos^2\theta$ , with a the angular momentum of a gravitational unit mass source, m the mass of the solution, and q the electric charge. For q = 0 the Kerr–Newman metric reduces to the Kerr metric, and for a = q = 0 it reduces to the standard form of the Schwarzschild solution.

Using the relation (4), it is possible to find the Kerr-Newman tetrad components. They are

$$h^{a}{}_{\mu} \equiv \begin{pmatrix} \gamma_{00} & 0 & 0 & \eta \\ 0 & \gamma_{11} \, s\theta \, c\phi & \gamma_{22} \, c\theta \, c\phi & -\beta \, s\phi \\ 0 & \gamma_{11} \, s\theta \, s\phi & \gamma_{22} \, c\theta \, s\phi & \beta \, c\phi \\ 0 & \gamma_{11} \, c\theta & -\gamma_{22} \, s\theta & 0 \end{pmatrix}, \tag{176}$$

with the inverse tetrad given by

$$h_{a}{}^{\mu} \equiv \begin{pmatrix} \gamma_{00}^{-1} & 0 & 0 & 0\\ -\beta g^{03} s\phi & \gamma_{11}^{-1} s\theta c\phi & \gamma_{22}^{-1} c\theta c\phi & -\beta^{-1} s\phi\\ \beta g^{03} c\phi & \gamma_{11}^{-1} s\theta s\phi & \gamma_{22}^{-1} c\theta s\phi & \beta^{-1} c\phi\\ 0 & \gamma_{11}^{-1} c\theta & -\gamma_{22}^{-1} s\theta & 0 \end{pmatrix},$$
(177)

where  $\beta^2 = \eta^2 - g_{33}$ ,  $\eta = g_{03}/\gamma_{00}$ , and the notations  $\gamma_{00} = \sqrt{g_{00}}$ ,  $\gamma_{ii} = \sqrt{-g_{ii}}$ ,  $s\theta = \sin\theta$  and  $c\theta = \cos\theta$  have been introduced. Using Eqs. (41) and (43), through a lengthy, but straightforward calculation, we obtain the non-zero components of the torsion tensor,

$$\dot{T}^{0}_{01} = -[\ln \sqrt{g_{00}}]_{,r} 
\dot{T}^{0}_{13} = \eta_{,r}/\gamma_{00} - kg^{03}(k_{,r} - \gamma_{11} s\theta) 
\dot{T}^{0}_{23} = \eta_{,\theta}/\gamma_{00} - kg^{03}(k_{,\theta} - \gamma_{22} c\theta) 
\dot{T}^{1}_{12} = -[\ln \sqrt{-g_{11}}]_{,\theta} 
\dot{T}^{2}_{12} = [\ln \sqrt{-g_{22}}]_{,r} - \gamma_{11}/\gamma_{22} 
\dot{T}^{3}_{13} = (k_{,r} - \gamma_{11} s\theta)/k 
\dot{T}^{3}_{23} = (k_{,\theta} - \gamma_{22} c\theta)/k,$$

where we have denoted ordinary derivatives by a "comma".

Now, as already mentioned, the torsion tensor can be decomposed in the form

$$T_{\lambda\mu\nu} = \frac{2}{3} \left( t_{\lambda\mu\nu} - t_{\lambda\nu\mu} \right) + \frac{1}{3} \left( g_{\lambda\mu} V_{\nu} - g_{\lambda\nu} V_{\mu} \right) + \epsilon_{\lambda\mu\nu\rho} A^{\rho}, \tag{178}$$

where  $t_{\lambda\mu\nu}$  is the purely tensor part, and  $V_{\mu}$  and  $A^{\rho}$  represent respectively the vector and axial parts of torsion. They are defined by

$$t_{\lambda\mu\nu} = \frac{1}{2} \left( T_{\lambda\mu\nu} + T_{\mu\lambda\nu} \right) + \frac{1}{6} \left( g_{\nu\lambda} V_{\mu} + g_{\nu\mu} V_{\lambda} \right) - \frac{1}{3} g_{\lambda\mu} V_{\nu}, \tag{179}$$

$$V_{\mu} = T^{\nu}_{\nu\mu},\tag{180}$$

$$A^{\mu} = \frac{1}{6} \epsilon^{\mu\nu\rho\sigma} T_{\nu\rho\sigma}. \tag{181}$$

For the specific case of the Kerr-Newman solution, the non-zero components of the vector torsion are

$$\overset{\bullet}{V}_{1} = -[\ln\sqrt{g_{00}}]_{,r} - [\ln\sqrt{-g_{22}}]_{,r} + \gamma_{11}/\gamma_{22} - [\ln\beta]_{,r} + \gamma_{11}\,\mathrm{s}\theta/\beta,$$

$$\overset{\bullet}{V}_2 = -[\ln \sqrt{-g_{11}}]_{\theta} - [\ln \beta]_{\theta} + \gamma_{22} c\theta/\beta,$$

whereas the non-zero components of the axial torsion are

$$\overset{\bullet}{A}^{(1)} \times (6h) = -2(g_{00}T^{0}_{23} + g_{03}T^{3}_{23})$$

$$\overset{\bullet}{A}^{(2)} \times (6h) = 2[g_{00}T^{0}_{13} + g_{03}(T^{3}_{13} + T^{0}_{01})].$$

To show the simplicity and transparency of teleparallel gravity, let us obtain the equation of motion of the particle's spin vector. By expanding the metric components up to first order in the angular momentum a, and by taking the weak-field limit, characterized by keeping terms up to first order in  $\alpha_m = 2m/r$  and  $\alpha_q = q^2/r^2$ , the axial-vector tensor reduces to

$$A^{(1)} \times (6h) = -2(g_{03})_{,\theta} \tag{182}$$

$$\mathring{A}^{(2)} \times (6h) = 2[\gamma_{00} (\eta)_{,r} - \eta (\gamma_{00})_{,r}], \tag{183}$$

where  $h = r^2 \sin \theta$ . Substituting the metric components, and keeping the weak-field approximation, the space components  $\mathbf{A} = \mathbf{A}^{(1)} \gamma_{11} \mathbf{e}_r + \mathbf{A}^{(2)} \gamma_{22} \mathbf{e}_\theta$  of the axial-vector torsion becomes

$$\mathbf{\hat{A}} \equiv \mathbf{\hat{A}}_{m} + \mathbf{\hat{A}}_{q} = \frac{\alpha_{m} a}{3r^{2}} [2\cos\theta \mathbf{e}_{r} + \sin\theta \mathbf{e}_{\theta}] - \frac{\alpha_{q} a}{3r^{2}} [2\cos\theta \mathbf{e}_{r} + 2\sin\theta \mathbf{e}_{\theta}], \tag{184}$$

where  $\stackrel{\bullet}{A}_m$  and  $\stackrel{\bullet}{A}_q$  represent respectively the mass and charge parts of the axial-vector tensor.

Let us consider now a Dirac particle in the presence of the axial–vector torsion  $\mathbf{\mathring{A}}$ . It has been shown by many authors<sup>35,69,70</sup> that the particle spin  $\mathbf{s}$  satisfies the equation of motion

$$\frac{d\mathbf{s}}{dt} = -\mathbf{b} \times \mathbf{s},\tag{185}$$

where  $\mathbf{b} = 3\mathbf{A}/2$ . Using Eq. (184), we get

$$\mathbf{b} = \frac{J}{r^3} [2\cos\theta \,\mathbf{e}_r + \sin\theta \,\mathbf{e}_\theta] - \frac{(q^2/m)J}{r^4} \left[2\cos\theta \,\mathbf{e}_r + 2\sin\theta \,\mathbf{e}_\theta\right],\tag{186}$$

with J=ma the angular momentum. In the particular case of the Kerr solution, q=0, and one can write

$$\mathbf{b} = \frac{G}{r^3} \left[ -\mathbf{J} + 3(\mathbf{J} \cdot \mathbf{e}_r) \,\mathbf{e}_r \right],\tag{187}$$

where  $J = J\mathbf{e}_z$ . This means that

$$\mathbf{b} = \omega_{LT},\tag{188}$$

where  $\omega_{LT}$  is the Lense-Thirring precession angular velocity, which in general relativity, as is well known, is produced by the gravitomagnetic component of the gravitational field.<sup>71</sup> We see in this way that, in teleparallel gravity, the axial-vector torsion  $\mathbf{A}$  represents the gravitomagnetic component of the gravitational field.<sup>72</sup> When  $q \neq 0$ , the electric charge contributes with an additional term to the Lense-Thirring precession angular velocity.

### 3. Dealing Without the Equivalence Principle

As is well known, the electromagnetic interaction is not universal: there is no an electromagnetic equivalence principle. In spite of this, Maxwell's theory, a gauge theory for the unitary group U(1), is able to consistently describe the electromagnetic interaction. Given the analogy between electromagnetism and teleparallel gravity, the question then arises whether the gauge approach of teleparallel gravity would also be able to describe the gravitational interaction in the lack of universality, that is, in the absence of the weak equivalence principle.

Let us then consider again the problem of the motion of a spinless particle in a gravitational field represented by the translational gauge potential  $B^a_{\mu}$ , supposing however that the gravitational mass  $m_g$  and the inertial mass  $m_i$  do not coincide. In this case, the action integral is written in the form

$$S = \int_{a}^{b} \left( -m_{i} c \, d\sigma - m_{g} \, c \, B^{a}_{\mu} \, u_{a} \, dx^{\mu} \right). \tag{189}$$

Variation of the action (189) yields the equation of motion

$$\left(\partial_{\mu}x^{a} + \frac{m_{g}}{m_{i}}B^{a}_{\mu}\right)\frac{du_{a}}{ds} = \frac{m_{g}}{m_{i}}\stackrel{\bullet}{T}^{a}_{\mu\rho}u_{a}u^{\rho}.$$
(190)

This is the force equation governing the motion of the particle, in which the teleparallel field strength  $T^a_{\mu\rho}$  plays the role of gravitational force. Similarly to the electromagnetic Lorentz force, which depends on the relation  $q/m_i$ , with q the electric charge of the particle, the gravitational force depends explicitly on the relation  $m_g/m_i$  of the particle.

The crucial point is to observe that, although the equation of motion depends explicitly on the relation  $m_i/m_g$  of the particle, neither  $B^a{}_{\mu}$  nor  $T^a{}_{\rho\mu}$  depends on this relation. This means essentially that the teleparallel field equation (122) can be consistently solved for the gravitational potential  $B^a{}_{\mu}$ , which can then be used to write down the equation of motion (190), independently of the validity or not of the weak equivalence principle. This means essentially that, even in the absence of the weak equivalence principle, teleparallel gravity is able to describe the motion of a particle with  $m_g \neq m_i$ .

Let us now see what happens in the context of general relativity. By using the identity (168), the force equation (190) can be rewritten in the form

$$\frac{du_{\mu}}{ds} - \overset{\circ}{\Gamma}{}^{\lambda}{}_{\mu\rho} u_{\lambda} u^{\rho} = \left(\frac{m_g - m_i}{m_g}\right) \partial_{\mu} x^a \frac{du_a}{ds},\tag{191}$$

where use has been made also of the relation (46). Notice that the violation of the weak equivalence principle produces a deviation from the geodesic motion, which is proportional to the difference between the gravitational and inertial masses. Notice furthermore that, due to the assumed non-universality of free fall, there is no a local coordinate system in which the gravitational effects are absent. Of course, when  $m_g = m_i$ , the equation of motion (191) reduces to the geodesic equation of general relativity. However, in the absence of the weak equivalence principle, it is not a geodesic equation, which means that it does not comply with the geometric description of general relativity, according to which all trajectories must be given by geodesic equations.

In order to reduce the force equation (190) to a geodesic equation, it is necessary to define a new tetrad field, which is given by

$$\bar{h}^{a}{}_{\mu} = \partial_{\mu}x^{a} + \frac{m_{g}}{m_{i}} B^{a}{}_{\mu}.$$
 (192)

However, in this case the solution of the gravitational field equations, which in general relativity is a tetrad or a metric tensor, would depend on the relation  $m_g/m_i$  of the test particle, and this renders the theory inconsistent. We can then conclude that, in the absence of the weak equivalence principle, the geometric description of gravitation provided by general relativity breaks down. Since the gauge potential  $B^a_{\mu}$  can always be obtained independently of any property of the test particle, teleparallel gravity remains as a consistent theory in the lack of universality. Accordingly,  $B^a_{\mu}$  can be considered as the most fundamental field representing gravitation.<sup>7</sup>

### I. Phase Factor Approach to Teleparallel Gravity

As we have just discussed, the fundamental field of teleparallel gravity is the gauge potential  $B^a_{\mu}$ . In this formulation, gravitation becomes quite analogous to electromagnetism. Based on this analogy, and relying on the phase-factor

approach to Maxwell's theory, a teleparallel nonintegrable phase-factor approach to gravitation can be developed,<sup>73</sup> which represents the quantum mechanical version of the classical gravitational Lorentz force.

As is well known, in addition to the usual differential formalism, electromagnetism presents also a global formulation in terms of a nonintegrable phase factor. According to this approach, electromagnetism can be considered as the gauge invariant action of a nonintegrable (path-dependent) phase factor. For a particle with electric charge q traveling from an initial point P to a final point Q, the phase factor is given by

$$\Phi_e(\mathsf{P}|\mathsf{Q}) = \exp\left[\frac{iq}{\hbar c} \int_{\mathsf{P}}^{\mathsf{Q}} A_\mu \, dx^\mu\right],\tag{193}$$

where  $A_{\mu}$  is the electromagnetic gauge potential. In the classical (non-quantum) limit, the nonintegrable phase factor approach yields the same results as those obtained from the Lorentz force equation

$$\frac{du^a}{ds} = \frac{q}{mc^2} \stackrel{\bullet}{T}{}^a{}_b u^b. \tag{194}$$

In this sense, the phase-factor approach can be considered as the *quantum* generalization of the *classical* Lorentz force equation. It is actually more general, as it can be used both on simply-connected and on multiply-connected domains. Its use is mandatory, for example, to describe the Bohm-Aharonov effect, a quantum phenomenon taking place in a multiply-connected space.

Now, in analogy with electromagnetism,  $B^a{}_{\mu}$  can be used to construct a global formulation for gravitation. To start with, let us notice that the electromagnetic phase factor  $\Phi_e(P|Q)$  is of the form

$$\Phi_e(\mathsf{P}|\mathsf{Q}) = \exp\left[\frac{i}{\hbar}\,\mathcal{S}_e\right],\tag{195}$$

where  $S_e$  is the action integral describing the interaction of the charged particle with the electromagnetic field. Now, in teleparallel gravity, the action integral describing the interaction of a particle of mass m with gravitation, according to Eq. (159), is given by

$$S_g = \int_{\mathsf{P}}^{\mathsf{Q}} m \, c \, B^a{}_{\mu} \, u_a \, dx^{\mu}. \tag{196}$$

Therefore, the corresponding gravitational nonintegrable phase factor turns out to be<sup>73</sup>

$$\Phi_g(\mathsf{P}|\mathsf{Q}) = \exp\left[\frac{imc}{\hbar} \int_{\mathsf{P}}^{\mathsf{Q}} B^a{}_{\mu} u_a \, dx^{\mu}\right]. \tag{197}$$

Similarly to the electromagnetic phase factor, it represents the *quantum* mechanical law that replaces the *classical* gravitational Lorentz force equation (165). In fact, in the classical limit it yields the same results as the gravitational Lorentz force.

The above global formulation for gravitation has already been applied to study the so called Colella-Overhauser-Werner (COW) experiment,<sup>75</sup> yielding the correct quantum phase-shift induced on the neutrons by their interaction with Earth's gravitational field. It has also been applied to study the quantum phase-shift produced by the coupling of the particle's kinetic energy with the gravitomagnetic component of the gravitational field, which corresponds to the gravitational analog of the Aharonov-Bohm effect.<sup>76</sup>

# IV. TORSION PHYSICS: INFINITELY MANY EQUIVALENT THEORIES

#### A. Physical Motivation

As stated in the Introduction, the classical equivalence between teleparallel gravity and general relativity implies that curvature and torsion might be simply alternative ways of describing the gravitational field, and consequently related to the same degrees of freedom of gravity. Whether this interpretation for torsion is universal or not is an open question. In other words, whether torsion has the same physical role in more general gravitation theories, in which curvature and torsion are simultaneously present, is a question yet to be solved. This is the problem we tackle next, where we review some attempts<sup>22,23</sup> to get a consistent answer to this puzzle.

Our approach will consist in studying the gravitational coupling prescription in the presence of curvature and torsion, independently of the theory governing the dynamics of the gravitational field. One has only to consider a gravitational field presenting curvature and torsion, and use it to obtain, from an independent variational principle, the particle or field equations in the presence of gravity. This, however, is not an easy task. The basic difficulty is that, differently from all other interactions of nature, where covariance does determine the gauge connection, in the presence of curvature and torsion, covariance alone is not able to determine the form of the gravitational coupling prescription. The reason for this indefiniteness is that the space of Lorentz connections is an affine space,<sup>24</sup> and consequently one can always add a tensor to a given connection without destroying the covariance of the theory. As a result of this indefiniteness, there will exist infinitely many possibilities for the gravitational coupling prescription. Notice that in the specific cases of general relativity and teleparallel gravity, characterized respectively by a vanishing torsion and a vanishing curvature, the above indefiniteness is absent since in these cases the connections are uniquely determined—and the corresponding coupling prescriptions completely specified—by the requirement of covariance. Notice furthermore that in the case of internal (Yang-Mills) gauge theories, where the concept of torsion is absent,<sup>77</sup> the above indefiniteness is not present either.

In order to consider the above problem, a strategy based on the equivalence principle will be used. Notice that, due to the intrinsic relation of gravitation with spacetime, there is a deep relationship between covariance (either under general coordinate or local Lorentz transformations) and the equivalence principle. In fact, an alternative version of this principle is the so called *principle of general covariance*. <sup>65</sup> It states that a special relativity equation will hold in the presence of gravitation if it is generally covariant, that is, it preserves its form under a general transformation of the spacetime coordinates. Now, in order to make an equation generally covariant, a connection is always necessary, which is in principle concerned only with the *inertial* properties of the coordinate system under consideration. Then, using the equivalence between inertial and gravitational effects, instead of representing inertial properties, this connection can equivalently be assumed to represent a true gravitational field. In this way, equations valid in the presence of gravitation are obtained from the corresponding special relativity equations. Of course, in a locally inertial coordinate system, they must go back to the corresponding equations of special relativity. The principle of general covariance, therefore, can be considered as an active version of the equivalence principle in the sense that, by making a special relativity equation covariant, and by using the strong equivalence principle, it is possible to obtain its form in the presence of gravitation. It should be emphasized that the general covariance alone is empty of any physical content as any equation can be made covariant. Only when use is made of the strong equivalence principle, and the inertial compensating term is assumed as representing a true gravitational field, principle of general covariance can be seen as a version of the equivalence principle.<sup>78</sup>

The above description of the general covariance principle refers to its usual *holonomic* version. An alternative, more general version of the principle can be obtained in the context of nonholonomic frames. The basic difference between these two versions is that, instead of requiring that an equation be covariant under a general transformation of the spacetime coordinates, in the nonholonomic-frame version the equation is required to transform covariantly under a *local* Lorentz rotation of the frame. Of course, in spite of the different nature of the involved transformations, the physical content of both approaches are the same.<sup>79</sup> The frame version, however, is more general in the sense that, contrary to the coordinate version, it holds for integer as well as for half-integer spin fields.

The crucial point now is to observe that, when the *purely inertial* connection is replaced by a connection representing a *true gravitational field*, the principle of general covariance naturally defines a covariant derivative, and consequently also a gravitational coupling prescription. For the cases of general relativity and teleparallel gravity, the nonholonomic-frame version of this principle has already been seen to yield the usual coupling prescriptions of these theories.<sup>21</sup> Our purpose here will then be to determine, in the general case characterized by the simultaneous presence of curvature and torsion, the form of the gravitational coupling prescription *implied by the general covariance principle*.

### B. Nonholonomic General Covariance Principle

Let us consider the Minkowski spacetime of special relativity, endowed with the Lorentzian metric  $\eta$ . In this spacetime one can take the frame  $\delta_a = \delta_a{}^{\mu}\partial_{\mu}$  as being a trivial (holonomous) tetrad, with components  $\delta_a{}^{\mu}$ . Consider now a *local*, that is, point-dependent Lorentz transformation  $\Lambda_a{}^b = \Lambda_a{}^b(x)$ . It yields the new frame

$$h_a = h_a{}^{\mu}\partial_{\mu},\tag{198}$$

with components  $h_a{}^{\mu} \equiv h_a{}^{\mu}(x)$  given by

$$h_a{}^\mu = \Lambda_a{}^b \delta_b{}^\mu. \tag{199}$$

Notice that, on account of the locality of the Lorentz transformation, the new frame  $h_a$  is nonholonomous, with  $f^a{}_{bc}$  as coefficient of nonholonomy [see Eq. (23)]. So, if one makes use of the orthogonality property of the tetrads, we see

from Eq. (199) that the Lorentz group element can be written in the form  $\Lambda_b{}^d = h_b{}^\rho \delta_\rho{}^d$ . From this expression, it follows that

$$(h_a \Lambda_b^d) \Lambda^c_d = \frac{1}{2} \left( f_b^c{}_a + f_a{}^c{}_b - f^c{}_{ba} \right). \tag{200}$$

On the other hand, the action describing a free particle in Minkowski spacetime is [see Eq. (159)]

$$S = -mc \int d\sigma. \tag{201}$$

Seen from the holonomous frame  $\delta_a$ , the corresponding equation of motion is

$$\frac{dv^a}{ds} = 0, (202)$$

where  $v^a = \delta^a{}_{\mu}u^{\mu}$ , with  $u^{\mu} = (dx^{\mu}/ds)$  the holonomous particle four-velocity. Seen from the nonholonomous frame  $h_a$ , a straightforward calculation shows that the equation of motion (202) is

$$\frac{du^c}{ds} + \frac{1}{2} \left( f_b{}^c{}_a + f_a{}^c{}_b - f^c{}_{ba} \right) u^a u^b = 0, \tag{203}$$

where  $u^c = \Lambda^c{}_d v^d = h^c{}_\mu u^\mu$ , and use has been made of Eq. (200). It is important to emphasize that, in the flat spacetime of special relativity one is free to choose any tetrad  $\{e_a\}$  as a moving frame. The fact that, for each  $x \in M$ , the frame  $h_a$  can be arbitrarily rotated introduces the compensating term  $\frac{1}{2}(f_b{}^c{}_a + f_a{}^c{}_b - f^c{}_{ba})$  in the free-particle equation of motion. This term, therefore, is concerned only with the inertial properties of the frame.

#### 1. Equivalence Between Inertial and Gravitational Effects

According to the general covariance principle, the equation of motion valid in the presence of gravitation can be obtained from the corresponding special relativistic equation by replacing the inertial compensating term by a connection  $A^c{}_{ab}$  representing a true gravitational field. Considering a general Lorentz-valued connection presenting both curvature and torsion, one can write<sup>26</sup>

$$A^{c}{}_{ba} - A^{c}{}_{ab} = T^{c}{}_{ab} + f^{c}{}_{ab}, (204)$$

with  $T^{c}_{ba}$  the torsion of the connection  $A^{c}_{ab}$ . Use of this equation for three different combination of indices gives

$$A^{c}{}_{ab} = \frac{1}{2} \left( f_{b}{}^{c}{}_{a} + f_{a}{}^{c}{}_{b} - f^{c}{}_{ba} \right) + \frac{1}{2} \left( T_{b}{}^{c}{}_{a} + T_{a}{}^{c}{}_{b} - T^{c}{}_{ba} \right). \tag{205}$$

Accordingly, the compensating term of Eq. (203) can be written in the form

$$\frac{1}{2} \left( f_b{}^c{}_a + f_a{}^c{}_b - f^c{}_{ba} \right) = A^c{}_{ab} - K^c{}_{ab}. \tag{206}$$

This equation is actually an expression of the equivalence principle. In fact, whereas its left-hand side involves only *inertial* properties of the frames, its right-hand side contains purely *gravitational* quantities. Using this expression in Eq. (203), one gets

$$\frac{du^c}{ds} + A^c{}_{ab} u^a u^b = K^c{}_{ab} u^a u^b. {(207)}$$

This is the particle equation of motion in the presence of curvature and torsion that follows from the principle of general covariance. It entails a very peculiar interpretation for contortion, which appears playing the role of a gravitational force.<sup>5</sup> Because of the identity (17), it is easy to see that the equation of motion (207) is equivalent with the geodesic equation of general relativity:

$$\frac{du^c}{ds} + \mathring{A}^c{}_{ab} u^a u^b = 0. {(208)}$$

### C. Gravitational Coupling Prescription

The equation of motion (207) can be written in the form

$$u^{\mu} \mathcal{D}_{\mu} u^{c} = 0, \tag{209}$$

with  $\mathcal{D}_{\mu}$  a generalized Fock–Ivanenko derivative, which when acting on a general vector field  $X^c$  reads

$$\mathcal{D}_{\mu}X^{c} = \partial_{\mu}X^{c} + (A^{c}_{a\mu} - K^{c}_{a\mu})X^{a}.$$
 (210)

We notice in passing that this covariant derivative satisfies the relation

$$\mathcal{D}_{\mu}X^{c} = h^{c}_{\rho} D_{\mu}X^{\rho}, \tag{211}$$

where

$$D_{\mu}X^{\rho} = \partial_{\mu}X^{\rho} + (\Gamma^{\rho}{}_{\lambda\mu} - K^{\rho}{}_{\lambda\mu})X^{\lambda}$$
(212)

is the corresponding spacetime derivative.

Now, using the vector representation (57) of the Lorentz generators, the generalized Fock–Ivanenko derivative (210) can be written in the form

$$\mathcal{D}_{\mu}X^{c} = \partial_{\mu}X^{c} - \frac{i}{2}(A^{ab}_{\mu} - K^{ab}_{\mu}) (S_{ab})^{c}_{d} X^{d}.$$
(213)

Furthermore, although obtained in the case of a Lorentz vector field (four-velocity), the compensating term (200) can be easily verified to be the same for any field. In fact, denoting by  $U \equiv U(\Lambda)$  the element of the Lorentz group in an arbitrary representation, it can be shown that

$$(h_a U)U^{-1} = -\frac{i}{4} \left( f_{bca} + f_{acb} - f_{cba} \right) J^{bc}, \tag{214}$$

with  $J^{bc}$  denoting the corresponding Lorentz generator. In this case, the covariant derivative (213) will have the form

$$\mathcal{D}_{\mu} = \partial_{\mu} - \frac{i}{2} \left( A^{ab}{}_{\mu} - K^{ab}{}_{\mu} \right) J_{ab}. \tag{215}$$

This means that, in the presence of curvature and torsion, the coupling prescription of fields carrying an arbitrary representation of the Lorentz group will be

$$\partial_a \equiv \delta^{\mu}{}_a \partial_{\mu} \to \mathcal{D}_a \equiv e^{\mu}{}_a \mathcal{D}_{\mu}. \tag{216}$$

Of course, due to the relation (17), it is clearly equivalent with the coupling prescription of general relativity.

# D. The Connection Space

# 1. Defining Translations

The mathematical validity of the coupling prescription (215) is rooted on the fact that a general connection space is an infinite, homotopically trivial affine space.<sup>80</sup> In the specific case of Lorentz connections, a point in this space will be a connection

$$A = A^{bc}_{\mu} J_{bc} dx^{\mu} \tag{217}$$

presenting simultaneously curvature and torsion. In the language of differential forms, they are defined respectively by

$$R = dA + AA \equiv \mathcal{D}_A A \tag{218}$$

and

$$T = dh + Ah \equiv \mathcal{D}_A h,\tag{219}$$

where  $\mathcal{D}_A$  denotes the covariant differential in the connection A. Now, given two connections A and  $\bar{A}$ , the difference

$$k = \bar{A} - A \tag{220}$$

is also a 1-form assuming values in the Lorentz Lie algebra, but transforming covariantly:

$$k = UkU^{-1}. (221)$$

Its covariant derivative is consequently given by

$$\mathcal{D}_A k = dk + \{A, k\}. \tag{222}$$

It is then easy to verify that, given two connections such that  $\bar{A} = A + k$ , their curvature and torsion will be related by

$$\bar{R} = R + \mathcal{D}_A k + k k \tag{223}$$

and

$$\bar{T} = T + k h. \tag{224}$$

The effect of adding a covector k to a given connection A, therefore, is to change its curvature and torsion 2-forms. We rewrite now Eq. (220) in components:

$$A^{a}{}_{bc} = \bar{A}^{a}{}_{bc} - k^{a}{}_{bc}. \tag{225}$$

Since  $k^a{}_{bc}$  is a Lorentz-valued covector, it is necessarily anti-symmetric in the first two indices. Separating  $k^a{}_{bc}$  in the symmetric and anti-symmetric parts in the last two indices, one gets

$$k^{a}_{bc} = \frac{1}{2}(k^{a}_{bc} + k^{a}_{cb}) + \frac{1}{2}(k^{a}_{bc} - k^{a}_{cb}). \tag{226}$$

Defining

$$k^{a}_{bc} - k^{a}_{cb} \equiv t^{a}_{cb} = -t^{a}_{bc}, \tag{227}$$

and using this equation for three combination of indices, it is easy to verify that

$$k^{a}_{bc} = \frac{1}{2}(t_{a}{}^{c}_{b} + t_{b}{}^{c}_{a} - t^{a}_{bc}). \tag{228}$$

This means essentially that the difference between any two Lorentz-valued connections has the form of a contortion tensor.

#### 2. Equivalence under Translations in the Connection Space

As already discussed, due to the affine character of the connection space, one can always add a tensor to a given connection without spoiling the covariance of the derivative (215). Since adding a tensor to a connection corresponds just to redefining the origin of the connection space, this means that covariance does not determine a preferred origin for this space. Let us then analyze the physical meaning of translations in the connection space. To begin with, we take again the connection appearing in the covariant derivative (215):

$$\Omega^a{}_{bc} \equiv A^a{}_{bc} - K^a{}_{bc}. \tag{229}$$

A translation in the connection space with parameter  $k^a{}_{bc}$  corresponds to

$$\bar{\Omega}^{a}{}_{bc} = \Omega^{a}{}_{bc} + k^{a}{}_{bc} \equiv A^{a}{}_{bc} - K^{a}{}_{bc} + k^{a}{}_{bc}. \tag{230}$$

Now, since  $k^a{}_{bc}$  has always the form of a contortion tensor, as given by Eq. (228), the above connection is equivalent to

$$\bar{\Omega}^{a}{}_{bc} = A^{a}{}_{bc} - \bar{K}^{a}{}_{bc}, \tag{231}$$

with  $K^a{}_{bc} = K^a{}_{bc} - k^a{}_{bc}$  another contortion tensor.

Let us then consider a few particular cases. First, we choose  $t^a_{bc}$  as the torsion of the connection  $A^a_{bc}$ , that is,  $t^a_{bc} = T^a_{bc}$ . In this case,  $k^a_{bc} = K^a_{bc}$ , and the last two terms of Eq. (230) cancel each other, yielding  $\bar{K}^a_{bc} = 0$ . This means that the torsion of  $A^a_{bc}$  vanishes, and we are left with the torsionless spin connection of general relativity:

$$\bar{\Omega}^a{}_{bc} = \mathring{A}^a{}_{bc}. \tag{232}$$

On the other hand, if we choose  $t^a{}_{bc}$  such that

$$t^{a}_{bc} = T^{a}_{bc} - f^{a}_{bc}, (233)$$

the connection  $A^a{}_{bc}$  vanishes, which characterizes teleparallel gravity. In this case, the resulting connection has the form  $^{16}$ 

$$\bar{A}^a{}_{bc} = -\bar{K}^a{}_{bc},\tag{234}$$

where  $K^a{}_{bc}$  is the contortion tensor written in terms of the Weitzenböck torsion  $T^a{}_{bc} = -f^a{}_{bc}$ . There are actually infinitely many choices for  $t^a{}_{bc}$ , each one defining a different translation in the connection space, and consequently yielding a connection  $A^a{}_{bc}$  with different curvature and torsion. All cases, however, are ultimately equivalent with general relativity as for all cases the *dynamical* spin connection is

$$\bar{\Omega}^{a}{}_{bc} = A^{a}{}_{bc} - \bar{K}^{a}{}_{bc} \equiv \mathring{A}^{a}{}_{bc}. \tag{235}$$

It is important to emphasize that, despite yielding physically equivalent coupling prescriptions, the physical equations are not covariant under a translation in the connection space. For example, under a particular translation, the geodesic equation of general relativity is lead to the force equation of teleparallel gravity, which are completely different equations. These two equations, however, as well as any other obtained through a general translation in the connection space, are equivalent in the sense that they describe the same physical trajectory.

#### E. The Spinning Particle

As an application of the gravitational minimal coupling prescription entailed by the covariant derivative (215), let us study the motion of a classical particle of mass m and spin s in a gravitational field presenting curvature and torsion.<sup>23</sup> According to the gauge approach, the action integral describing such a particle *minimally* coupled to gravitation is

$$S = \int_{a}^{b} \left[ -m \, c \, d\sigma - \frac{1}{c^{2}} B^{a}{}_{\mu} \, \mathcal{P}_{a} \, dx^{\mu} + \frac{1}{2} \, \Omega^{ab}{}_{\mu} \, \mathcal{S}_{ab} \, dx^{\mu} \right], \tag{236}$$

where  $\mathcal{P}_a = mcu_a$  is the Noether charge associated with the invariance of  $\mathcal{S}$  under translations,<sup>5</sup> and  $\mathcal{S}_{ab}$  is the Noether charge associated with the invariance of  $\mathcal{S}$  under Lorentz transformations.<sup>81</sup> In other words,  $\mathcal{P}_a$  is the momentum, and  $\mathcal{S}_{ab}$  is the spin angular momentum density, which satisfies the Poisson relation

$$\{\mathcal{S}_{ab}, \mathcal{S}_{cd}\} = \eta_{ac}\,\mathcal{S}_{bd} + \eta_{bd}\,\mathcal{S}_{ac} - \eta_{ad}\,\mathcal{S}_{bc} - \eta_{bc}\,\mathcal{S}_{ad}.\tag{237}$$

Notice that, according to this prescription, the particle's momentum couples minimally to the translational gauge potential  $B^a{}_{\mu}$ , whereas the spin of the particle couples minimally to the dynamical spin connection  $\Omega^{ab}{}_{\mu}$ , which is nothing but the Ricci coefficient of rotation:

$$\Omega^{ab}_{\ \mu} = (A^{ab}_{\ \mu} - K^{ab}_{\ \mu}) \equiv \mathring{A}^{ab}_{\ \mu}.$$

The Routhian arising from the action (236) is

$$\mathcal{R}_{0} = -m c \sqrt{u^{2}} \frac{d\sigma}{ds} - \frac{1}{c^{2}} B^{a}{}_{\mu} \mathcal{P}_{a} u^{\mu} + \frac{1}{2} \Omega^{ab}{}_{\mu} \mathcal{S}_{ab} u^{\mu}, \tag{238}$$

where the weak constraint  $\sqrt{u^2} \equiv \sqrt{u_a u^a} = \sqrt{u_\mu u^\mu} = 1$  has been introduced in the first term. The equation of motion for the particle trajectory is obtained from

$$\frac{\delta}{\delta x^{\mu}} \int \mathcal{R}_0 \, ds = 0,\tag{239}$$

whereas the equation of motion for the spin tensor follows from

$$\frac{dS_{ab}}{ds} = \{\mathcal{R}_0, S_{ab}\}. \tag{240}$$

Now, the four-velocity and the spin angular momentum density must satisfy the constraints

$$S_{ab}S^{ab} = 2s^2 \tag{241}$$

$$S_{ab}u^a = 0. (242)$$

However, since the equations of motions obtained from the Routhian  $\mathcal{R}_0$  do not satisfy the above constraints, it is necessary to include them into the Routhian. The simplest way to achieve this amounts to the following.<sup>82</sup> First, a new expression for the spin is introduced:

$$\tilde{\mathcal{S}}_{ab} = \mathcal{S}_{ab} - \frac{\mathcal{S}_{ac} u^c u_b}{u^2} - \frac{\mathcal{S}_{cb} u^c u_a}{u^2}.$$
(243)

This new tensor satisfies the Poisson relation (237) with the metric  $\eta_{ab} - u_a u_b/u^2$ . A new Routhian that incorporates the above constraints is obtained by replacing all  $S_{ab}$  in  $\mathcal{R}_0$  by  $\tilde{S}_{ab}$ , and by adding to it the term

$$\frac{du^a}{ds} \frac{S_{ab}u^b}{u^2}.$$

The new Routhian is then

$$\mathcal{R} = -m c \sqrt{u^2} \frac{d\sigma}{ds} - \frac{1}{c^2} B^a{}_{\mu} \mathcal{P}_a u^{\mu} + \frac{1}{2} \Omega^{ab}{}_{\mu} \mathcal{S}_{ab} u^{\mu} - \frac{\mathcal{D}u^a}{\mathcal{D}s} \frac{\mathcal{S}_{ab}u^b}{u^2}, \tag{244}$$

where

$$\frac{\mathcal{D}u^a}{\mathcal{D}s} = u^\mu \, \mathcal{D}_\mu u^a,$$

with  $\mathcal{D}_{\mu}$  the covariant derivative (215).

Using the Routhian (244), the equation of motion for the spin is found to be

$$\frac{\mathcal{D}\mathcal{S}_{ab}}{\mathcal{D}s} = \left(u_a \,\mathcal{S}_{bc} - u_b \,\mathcal{S}_{ac}\right) \frac{\mathcal{D}u^c}{\mathcal{D}s},\tag{245}$$

which coincides with the corresponding result of general relativity. Making use of the Lagrangian formalism, the next step is to obtain the equation of motion defining the trajectory of the particle. Through a tedious but straightforward calculation, it is found to be

$$\frac{\mathcal{D}}{\mathcal{D}s} \left( m \, c \, u_c \right) + \frac{\mathcal{D}}{\mathcal{D}s} \left( \frac{\mathcal{D}u^a}{\mathcal{D}s} \frac{\mathcal{S}_{ac}}{u^2} \right) = \frac{1}{2} \left( R^{ab}_{\ \mu\nu} - Q^{ab}_{\ \mu\nu} \right) \mathcal{S}_{ab} \, u^{\nu} \, h_c^{\ \mu}, \tag{246}$$

where

$$Q^{a}{}_{b\mu\nu} = \mathcal{D}_{\mu}K^{a}{}_{b\nu} - \mathcal{D}_{\nu}K^{a}{}_{b\mu} + K^{a}{}_{d\mu}K^{d}{}_{b\nu} - K^{a}{}_{d\nu}K^{d}{}_{b\mu}. \tag{247}$$

Using the constraints (241-242), it is easy to verify that

$$\frac{\mathcal{D}u^a}{\mathcal{D}s}\frac{\mathcal{S}_{ac}}{u^2} = u^a \frac{\mathcal{D}\mathcal{S}_{ca}}{\mathcal{D}s}.$$

As a consequence, Eq. (246) acquires the form

$$\frac{\mathcal{D}}{\mathcal{D}s} \left( mcu_c + u^a \frac{\mathcal{D}S_{ca}}{\mathcal{D}s} \right) = \frac{1}{2} (R^{ab}_{\ \mu\nu} - Q^{ab}_{\ \mu\nu}) \, \mathcal{S}_{ab} \, u^{\nu} \, h_c^{\ \mu}. \tag{248}$$

Defining the generalized four-momentum

$$\mathbb{P}_c = h_c^{\ \mu} \, \mathbb{P}_{\mu} \equiv m \, c \, u_c + u^a \, \frac{\mathcal{D}\mathcal{S}_{ca}}{\mathcal{D}s},\tag{249}$$

we get

$$\frac{\mathcal{DP}_{\mu}}{\mathcal{D}s} = \frac{1}{2} \left( R^{ab}_{\ \mu\nu} - Q^{ab}_{\ \mu\nu} \right) \mathcal{S}_{ab} u^{\nu}. \tag{250}$$

This is the equation governing the motion of the particle in the presence of both curvature and torsion. It is written in terms of a general spin connection, as well as in terms of its curvature and torsion. It can be rewritten in terms of the Ricci coefficient of rotation only, in which case it reduces to the ordinary Papapetrou equation<sup>83</sup>

$$\frac{\mathring{\mathcal{D}}\mathbb{P}_{\mu}}{\mathcal{D}_{S}} = \frac{1}{2} \mathring{R}^{ab}_{\mu\nu} \mathcal{S}_{ab} u^{\nu}. \tag{251}$$

It can also be rewritten in terms of the teleparallel spin connection (62), in which case it reduces to the teleparallel equivalent of the Papapetrou equation,

$$\frac{\stackrel{\bullet}{\mathcal{D}}\mathbb{P}_{\mu}}{\mathcal{D}s} = -\frac{1}{2} \stackrel{\bullet}{Q}{}^{ab}{}_{\mu\nu} \mathcal{S}_{ab} u^{\nu}, \tag{252}$$

with  $Q^{ab}_{cd}$  given by Eq. (50). Notice that the particle's spin, similarly to the electromagnetic field [see the teleparallel Maxwell equation (110)], couples to a curvature-like tensor, which is a tensor written in terms of torsion only. It is important to recall that, although physically equivalent, there are conceptual differences in the way the above equations of motion describe the gravitational interaction.

#### V. OUTLOOK AND PERSPECTIVES

This review is made up of two main parts. In the first, represented by Section III, a comprehensive description of the teleparallel equivalent of general relativity was presented. In spite of the mentioned equivalence, there are conceptual differences between general relativity and teleparallel gravity, the most significant being the different character of the fundamental field of the theories: whereas in general relativity it is a tetrad field  $h^a_{\mu}$  (or equivalently, a metric tensor  $g_{\mu\nu}$ ), in teleparallel gravity it is a translational gauge potential  $B^a_{\mu}$ , the nontrivial part of the tetrad field:

$$h^a_{\ \mu} = \partial_\mu x^a + B^a_{\ \mu}. \tag{253}$$

This apparently small difference has deep consequences. In fact, any gravitational theory whose fundamental field is a tetrad (or a metric), is necessarily a *geometrical theory*. On the other hand, a theory whose fundamental field is a gauge potential, like for example teleparallel gravity, is non–geometrical in essence. It is actually a gauge theory, and as such it is able to describe the gravitational interaction in the absence of the weak equivalence principle.

To understand this point, let us consider a particle whose gravitational mass  $m_g$  does not coincide with its inertial mass  $m_i$ . In this case, a geometrical theory for gravitation would require the introduction of a new tetrad field, given by<sup>7</sup>

$$\bar{h}^{a}{}_{\mu} = \partial_{\mu} x^{a} + \frac{m_{g}}{m_{i}} B^{a}{}_{\mu}. \tag{254}$$

Since the relation  $m_g/m_i$  of the test particle appears "inside" the tetrad definition, any theory in which  $\bar{h}^a{}_{\mu}$  is the fundamental field will be inconsistent in the sense that particles with different relations  $m_g/m_i$  will require different solutions of the field equations to keep a geometric description of gravitation, in which the trajectories are necessarily given by geodesics. On the other hand, we see from the tetrad (254) that the relation  $m_g/m_i$  appears "outside" the gauge potential  $B^a{}_{\mu}$ . This means essentially that, in this case, the gravitational field equations can be consistently solved for  $B^a{}_{\mu}$  independently of any test–particle property. This is the fundamental reason for teleparallel gravity to remain as a viable theory for gravitation, even in the absence of the weak equivalence principle. This result may have important consequences for a fundamental problem of quantum gravity, namely, the conceptual difficulty of reconciling the local character of the equivalence principle with the non–local character of the uncertainty principle.<sup>84</sup> Since teleparallel gravity can be formulated independently of the equivalence principle, the quantization of the gravitational field may possibly appear more consistent if considered in the teleparallel picture.

A further consequence that emerges from the conceptual differences between general relativity and teleparallel gravity is that, whereas in the former curvature is used to geometrize the gravitational interaction—spinless particles follow geodesics—in the latter torsion describes the gravitational interaction by acting as a force—trajectories are not given by geodesics, but by force equations. According to the teleparallel approach, therefore, the role played by

torsion is quite well defined: it appears as an alternative to curvature in the description of the gravitational field, and is consequently related with the same degrees of freedom of gravity. Now, this interpretation is completely different from that appearing in more general theories, like Einstein–Cartan and gauge theories for the Poincaré and the affine groups. In these theories, curvature and torsion are considered as independent fields, related with different degrees of freedom of gravity, and consequently with different physical phenomena. This is a conflicting situation as these two interpretations cannot be both correct: if one is correct, the other is necessarily wrong.

We come then to the second part of the review, presented in Section IV. In this part, as an attempt to solve the above described paradox, we have critically reviewed the physics of torsion in gravitation. More specifically, we have used the general covariance principle—seen as an alternative version of the strong equivalence principle—to study the gravitational coupling prescription in the presence of curvature and torsion. We recall that, according to this principle, in order to make an equation generally covariant, a connection is always necessary, which is in principle concerned only with the *inertial* properties of the frame under consideration. Then, by using the equivalence between inertial and gravitational effects, instead of representing inertial properties, this connection can equivalently be assumed to represent a true gravitational field. In this way, equations valid in the presence of gravitation can be obtained from the corresponding special relativity equations. Now, as we have seen, the inertial compensating connection  $\frac{1}{2}(f_b{}^c{}_a + f_a{}^c{}_b - f^c{}_{ba})$ , also known as object of anholonomy, is related to a general Lorentz spin connection  $A^c{}_{ab}$  through

$$\frac{1}{2}(f_b{}^c{}_a + f_a{}^c{}_b - f^c{}_{ba}) = A^c{}_{ab} - K^c{}_{ab}. \tag{255}$$

This equation is nothing but an expression of the equivalence principle: whereas its left-hand side involves only inertial properties of a given frame, its right-hand side contains purely gravitational quantities. Therefore, to replace inertial effects by a true gravitational field means to replace the left-hand side by the right-hand side of Eq. (255). This means essentially that the dynamical spin connection, that is, the spin connection defining the covariant derivative, and consequently the gravitational coupling prescription, is given by the right-hand side of Eq. (255). Even in the presence of curvature and torsion, therefore, torsion appears as playing the role of gravitational force. This result gives support to the point of view of teleparallel gravity, according to which torsion does not represent additional degrees of freedom of gravity, but simply an alternative way of representing the gravitational field. Furthermore, since  $A^c{}_{ab} - K^c{}_{ab} = \mathring{A}^c{}_{ab}$ , the ensuing coupling prescription will always be equivalent with the coupling prescription of general relativity, a result that reinforces the completeness of this theory.

It should be remarked that the object of anholonomy is sometimes replaced by the spin connection  $A^c{}_{ab}$  only. The resulting coupling prescription is the one usually assumed to hold in Einstein-Cartan and other gauge theories for gravitation. Although it can be made to satisfy the usual (passive) strong equivalence principle, this coupling prescription clearly violates its active version. Furthermore, it presents some drawbacks, of which the most important is, perhaps, the fact that, when used to describe the gravitational coupling of the electromagnetic field, it violates the U(1) gauge invariance of Maxwell's theory. On the other hand, the coupling prescription implied by the general covariance principle presents several formal advantages: it preserves the role played by torsion in teleparallel gravity, it is consistent with both the active and passive versions of the strong equivalence principle, it can be applied in the Lagrangian or in the field equations with the same result, and when used to describe the interaction of the electromagnetic field with gravitation, it does not violate the U(1) gauge invariance of Maxwell's theory.

Summing up, we can say that the main output of the developments presented in this review is a new interpretation for torsion in connection with the gravitational interaction. According to this view, curvature and torsion are simply alternative ways of describing the gravitational field. As a consequence, any gravitational phenomenon that can be interpreted in terms of curvature, can also be interpreted in terms of torsion. Of course, we are aware that the physical soundness of our arguments does not necessarily mean that they are correct, and that a definitive answer can only be provided by experiments. However, considering that, at least up to now, there are no compelling experimental evidences for *new physics associated with torsion*, we could say that the teleparallel point of view is favored by the available experimental data. For example, no new gravitational physics has ever been reported near a neutron star. On the other hand, it is true that, due to the weakness of the gravitational interaction, no experimental data exist on the coupling of the spin of the fundamental particles to gravitation. Anyway, precision experiments, <sup>85</sup> in laboratory or as astrophysical and cosmological tests, are expected to be available in the foreseeable future, when then a final answer will hopefully be achieved.

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