Foundations of Physics Letters, Vol. 1, No. 1, 1988

THOMAS ROTATION AND THE PARAMETRIZATION OF THE LORENTZ TRANSFORMATION GROUP

Abraham A. Ungar

Department of Mathematics
North Dakota State University
Fargo, North Dakota 58105

Received May 14, 1987; revised February 3, 1988

Two successive pure Lorentz transformations are equivalent to a pure Lorentz transformation preceded by a 3×3 space rotation, called a Thomas rotation. When applied to the gyration of the rotation axis of a spinning mass, Thomas rotation gives rise to the well-known *Thomas precession.* A 3×3 parametric, unimodular, orthogonal matrix that represents the Thomas rotation is presented and studied. This matrix representation enables the Lorentz transformation group to be parametrized by two physical observables: the (3-dimensional) relative velocity and orientation between inertial frames. The resulting parametrization of the Lorentz group, in turn, enables the composition of successive Lorentz transformations to be given by parameter composition. This composition is continuously deformed into a corresponding, well-known Galilean transformation composition by letting the speed of light approach infinity. Finally, as an application the Lorentz transformation with given orientation parameter is uniquely expressed in terms of an initial and a final time-like 4-vector.

Key words: Special theory of relativity, Lorentz transformation, parametrization, Thomas rotation.

1. INTRODUCTION

As Thomas pointed out,[1] two successive pure Lorentz transformations, called *boosts* in the jargon, are not equivalent to a pure Lorentz transformation, but to a pure Lorentz transformation preceded, or followed, by a rotation (or precession) of the space coordinates. The relativistic rotation of space coordinates under successive pure Lorentz transformations,[2] called the Thomas rotation, reveals itself in a moving spinning mass as the Thomas precession.[1] It is due to the presence of the Thomas rotation that pure Lorentz transformations, or boosts, do not form a group. Some authors, therefore, refer to the Thomas rotation as the *rotation correction* that one must introduce so that the composition of boosts is a *corrected* boost.[3] Some other authors refer to the Thomas rotation as the Wigner rotation.[4] The term *Thomas precession* is widely used in the literature to describe the Thomas rotation; see, e.g., Møller.[5]

The aim of this article is to present the Thomas rotation matrix and the resulting parametrization of the Lorentz transformation group. The Thomas rotation matrix is presented in a concise canonical form that has not been discovered by earlier investigators since the algebra involved in its calculation is overwhelming. Indeed, in his treatise *Classical Mechanics* Goldstein notes that "The decomposition process [describing successive pure Lorentz transformations as a pure Lorentz transformation preceded, or followed, by a space rotation] can be carried through on the product of two pure Lorentz transformations to obtain explicitly the rotation of the coordinate axes resulting from the two successive boosts [that is, the Thomas rotation]. In general, the algebra involved is *quite forbidding,* more than enough, usually, to discourage any *actual demonstration* of the rotation matrix" (Italics not in the original).[6]

Specifically, Goldstein explains that eq. (11) of Section 4 of the present article can be solved for the Thomas rotation matrix, tom[**u**; **v**], by matrix algebra, as shown in eq. (13). He, however, discourages attempts to simplify the Thomas rotation matrix to the point where its rotation-matrix behavior can be actually demonstrated and applied to various related problems. The algebra involved in the present attempt to express the Thomas rotation matrix in its canonical form is, as expected, overwhelming, and hence will not be presented here. The final result of the simplification of the Thomas rotation matrix is, however, neat and is presented in a canonical rotation matrix form in eqs. (14) and (19) of Section 4. These equations, therefore, embody an important result of the present work.

It is well-known that Lorentz transformations in one time dimension and one space dimension can be represented by complex numbers. Hence, there are attempts in the literature to study Lorentz transformations and their associated Thomas rotations in one time dimension and three space dimensions by looking for hypercomplex numbers or, more generally, for the elements of a Clifford algebra.[7] Exact expressions for the Thomas rotation were presented by several authors; see for instance van Wyk,[8] Ben Menahem[9] Rivas *et al.*[10] and others.[11] Various approximations and, eventually, an exact evaluation of the Thomas rotation were obtained by Salingaros in a series of papers,[3] in which he used the Baker-Campbell-Hausdorf formula which corrects the product of non-commuting exponentials in the Clifford algebra.[12] Salingaros results are discussed by Baylis and Jones, who presented a study of the Thomas rotation by means of the Pauli algebra.[13] An exact expression for a special Wigner angle, studied by Han, Kim and Son,[14] is of particular interest to us since it is contained as a special case in our expression for the general Thomas rotation, as shown in Section 5. Finally, an addition theorem of Wigner rotation matrices was derived by Chen and Pei.[15]

The approach we take for the study of the Thomas rotation is radically different; it is the natural matrix approach, the algebra of which overwhelmed Goldstein, and others.[16] The Herculean task of calculating the Thomas rotation axis and angle by matrix algebra is rewarding: Once the complicated calculations have been accomplished, neat expressions and an interesting mathematical formalism emerge. One of the resulting neat expressions appears in the form of a unit quaternion that represents the Thomas rotation and, hence, called the Thomas quaternion. The Thomas quaternion, presented in eqs. (45) of Section 7, turns out to be an elegant function of the two velocities that generate the underlying Thomas rotation. The simplicity in the resulting expressions for the Thomas rotation enables us to study its properties, some of which turn out to be useful for several applications described in this article. A particularly important application is the use in Sections 8 and 9 of some properties of the Thomas rotation for the parametrization of the Lorentz transformation group, $L\{\mathrm{v}; V\}$, by relative velocity, v, and relative orientation, V, between inertial frames in such a way that composite Lorentz transformations are given by velocity and orientation composition. We identify, in our notation, the Lorentz transformation group with its generic element, $L\{\mathrm{v}; V\}$, which is a homogeneous, proper, orthochronous Lorentz transformation, parametrized by velocity and orientation parameters.

The resulting Lorentz transformation composition, eq. (55), has the form

$$L\{\mathrm{v}_1; V_1\}\, L\{\mathrm{v}_2; V_2\} = L\{\mathrm{v}_1 * V_1\mathrm{v}_2; \mathrm{tom}[\mathrm{v}_1; V_1\mathrm{v}_2]V_1V_2\},$$

which involves the Thomas rotation, *tom,* and the relativistic velocity composition operator, $*$. This transformation composition is continuously deformed into a corresponding Galilean transformation composition, eq. (58), which has the form

$$G\{\mathrm{v}_1; V_1\}\, G\{\mathrm{v}_2; V_2\} = G\{\mathrm{v}_1 + V_1\mathrm{v}_2; V_1V_2\},$$

by letting the speed of light approach infinity. Comparing these two forms, which are fully explained in Section 9, we clearly see that by letting the speed of light tend to infinity, the relativistic velocity composition operator, $*$, reduces to the Galilean velocity composition operator, +, and the relativistic Thomas rotation, *tom,* vanishes. This expected result is not well known! Actually, some investigators believe that a continuous deformation of composite Lorentz transformations into composite Galilean transformations does not exist, due to the presence of the Thomas rotation.[17]

Some preliminary, well-known results are presented in Sections 2 and 3 enabling the study of the Thomas rotation in Sections 4-7. This, in turn, enables us to parametrize the Lorentz transformation group by relative velocity and orientation between inertial frames, in Section 8, and to describe composite Lorentz transformations in terms of velocity and orientation composition, in Section 9. A reduction of the Lorentz transformation in Sections 8 and 9 for the special case when the speed of light approaches infinity, $c \to \infty$, yields the

Galilean transformation, parametrized by relative velocity and orientation between inertial frames.[18] Manipulations of the Lorentz transformation matrix group are, then, illustrated in Section 10 where some identities for the Thomas rotation are discussed. Finally, the capability of the present approach to deal with previously unsolved problems is demonstrated in Section 11 where the unique Lorentz transformation, with given orientation parameter, that links given initial and final vectors is determined. The determination of this Lorentz transformation is an interesting problem to which incomplete solutions are available in the literature.[19–21]

2. *RELATIVISTIC VELOCITY COMPOSITION*

Relationships between inertial frames are measured by velocities and orientations relative to each other: (i) Relative velocities between inertial frames are *admissible,* that is, they are elements of the space $\mathbb{R}_c^3$,

$$\mathbb{R}_c^3 = \left\{ \mathbf{v} \in \mathbb{R}^3 : \ |\mathbf{v}| < c \right\},$$

where $\mathbb{R}^3$ is the 3-dimensional Euclidean space and c is a positive constant representing the speed of light; and (ii) relative orientaions are elements of the space $SO(3)$ of all 3×3 unimodular orthogonal matrices.

Let Σ, Σ' and Σ'' be three inertial frames. For simplicity, the space coordinates of Σ and Σ', as observed by observers moving with Σ or with Σ', are parallel; and, similarly, the space coordinates of Σ' and Σ'', as observed by observers moving with Σ' or with Σ'', are parallel. The frames are depicted in Fig. 1 in which time and one space dimension are suppressed for clarity.

If the velocity of frame Σ'' relative to frame Σ' is $\mathbf{v}$ while the velocity of frame Σ' relative to frame Σ is $\mathbf{u}$, as in Fig. 1(a), then (i) the orientation of Σ'' relative to Σ is determined by the Thomas rotation, shown in Fig. 1(b) and discussed in Section 4; and (ii) the velocity of frame Σ'' relative to frame Σ is the relativistic composition, $\mathbf{u}*\mathbf{v} \in \mathbb{R}_c^3$, of the velocities $\mathbf{u}$ and $\mathbf{v}$, given by the equation

$$\mathbf{u}*\mathbf{v} = \frac{\mathbf{u}+\mathbf{v}}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}} + \frac{1}{c^2}\frac{\gamma_u}{\gamma_u+1}\frac{\mathbf{u}\times(\mathbf{u}\times\mathbf{v})}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}, \qquad \mathbf{u},\mathbf{v} \in \mathbb{R}_c^3, \tag{1}$$

where γ_u is the *Lorentz factor,*

$$\gamma_u = \frac{1}{\sqrt{1-(\frac{\mathbf{u}}{c})^2}} = \frac{1}{\sqrt{1-(\frac{u}{c})^2}}, \tag{2}$$

associated with the velocity $\mathbf{u}$ whose magnitude is u, $u = |\mathbf{u}|$, and where $\cdot$ and $\times$ signify the usual dot (scalar) and cross (vector) product between two vectors. The magnitude of the composite velocity $\mathbf{u}*\mathbf{v}$ is symmetric in $\mathbf{u}$ and $\mathbf{v}$, the square of which is given by the equation

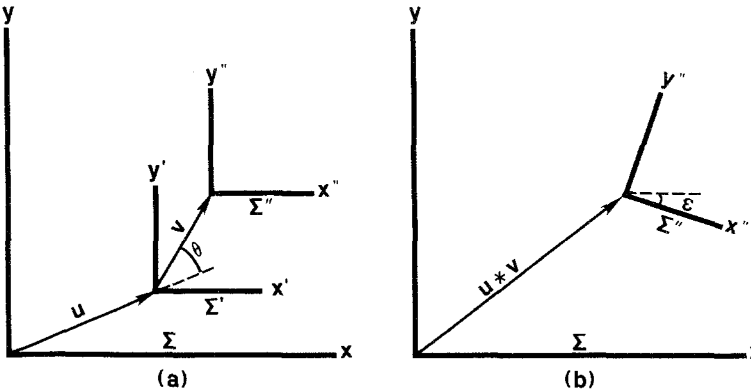

Fig. 1 (a) The axes of both frames Σ and Σ″ have been constructed parallel to those of Σ′ as seen by an observer moving with Σ′. Nevertheless, (b) an observer in Σ sees the axes of Σ″ rotated relative to his own axes by the Thomas angle ε of eqs. (22). If each of the velocities **u** and **v** has magnitude c, the speed of light, then Thomas angle ε equals the angle $-\theta$, $\theta \neq \pi$, between **u** and **v**, that is, $\varepsilon = -\theta$ for $0 \le \theta < 2\pi$ and $\theta \neq \pi$.

$$(\mathbf{u}*\mathbf{v})^2 = \left[\frac{\mathbf{u}+\mathbf{v}}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\right]^2 - \frac{1}{c^2}\left[\frac{\mathbf{u}\times\mathbf{v}}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\right]^2. \tag{3}$$

The Lorentz factor γ_{u*v}, associated with the composite velocity $\mathbf{u}*\mathbf{v}$, is related to the Lorentz factors γ_u and γ_v by the equation

$$\gamma_{u*v} = \gamma_u \gamma_v \left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right). \tag{4}$$

It will be shown in Section 6 that the set $\mathbb{R}^3_c$ of all 3-dimensional admissible velocities, with their composition law (1), forms an interesting noncommutative, nonassociative group. It will be found that the group operation, given by velocity composition, obeys some *weak* commutative and associative laws involving the Thomas rotation.

3. BOOSTS

A pure Lorentz transformation, or a boost, is a Lorentz transformation *without rotation* between time-space coordinates of an event measured in two inertial frames Σ and Σ′ with relative velocity **v**. It is given by the equation

$$\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = B(\mathbf{v}) \begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix}, \tag{5}$$

where $(t,x,y,z)^t$ (the exponent t indicates transposition) and $(t',x',y',z')^t$ are the time-space coordinates of an event measured in Σ and Σ' respectively. $B(\mathbf{v})$ is the boost matrix associated with the velocity $\mathbf{v}$ of Σ' relative to Σ, given by the equation

$$B(\mathbf{v}) = \begin{pmatrix} \gamma_v & c^{-2}\gamma_v v_1 & c^{-2}\gamma_v v_2 & c^{-2}\gamma_v v_3 \\ \gamma_v v_1 & 1+c^{-2}\frac{\gamma_v^2}{\gamma_v+1}v_1^2 & c^{-2}\frac{\gamma_v^2}{\gamma_v+1}v_1v_2 & c^{-2}\frac{\gamma_v^2}{\gamma_v+1}v_1v_3 \\ \gamma_v v_2 & c^{-2}\frac{\gamma_v^2}{\gamma_v+1}v_1v_2 & 1+c^{-2}\frac{\gamma_v^2}{\gamma_v+1}v_2^2 & c^{-2}\frac{\gamma_v^2}{\gamma_v+1}v_2v_3 \\ \gamma_v v_3 & c^{-2}\frac{\gamma_v^2}{\gamma_v+1}v_1v_3 & c^{-2}\frac{\gamma_v^2}{\gamma_v+1}v_2v_3 & 1+c^{-2}\frac{\gamma_v^2}{\gamma_v+1}v_3^2 \end{pmatrix}, \tag{6}$$

where (v_1,v_2,v_3) are the components of the velocity $\mathbf{v}$ of Σ' relative to Σ, measured in Σ.[22] An elegant canonical form for the boost matrix $B(\mathbf{v})$ is given by the equation

$$B(\mathbf{v}) = J + \gamma_v b + \frac{\gamma_v^2}{\gamma_v+1} b^2, \tag{7}$$

where J is the 4×4 identity matrix, and where the matrix $b = b(\mathbf{v})$ is related to the boost velocity parameter $\mathbf{v} = (v_1, v_2, v_3)$ by the equation

$$b(\mathbf{v}) = \begin{pmatrix} 0 & c^{-2}v_1 & c^{-2}v_2 & c^{-2}v_3 \\ v_1 & 0 & 0 & 0 \\ v_2 & 0 & 0 & 0 \\ v_3 & 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

Since the matrix $b(\mathbf{v})$ satisfies the equation

$$b^3 = \frac{\mathbf{v}^2}{c^2} b = \frac{\gamma_v^2 - 1}{\gamma_v^2} b, \tag{9}$$

it is readily seen that $B(\mathbf{v})B(-\mathbf{v}) = J$, that is, the boost inverse to $B(\mathbf{v})$ is $B(-\mathbf{v})$.

4. THE THOMAS ROTATION

A 3×3 space rotation of time-space coordinates $(t',x',y',z')^t$ is represented by the 4×4 unimodular orthogonal block matrix

$$\rho(R) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix} \tag{10}$$

in terms of its effects on the column four-vector $(t',x',y',z')^t$, where $R \in SO(3)$, that is, R is a 3×3 unimodular orthogonal matrix ($RR^t = I$ and $\det R = 1$). The homomorphic mapping $\rho: SO(3) \to SO(4)$, which will be found useful in the description of the Thomas rotation, thus takes a 3×3 unimodular orthogonal matrix into a 4×4 unimodular orthogonal matrix. The Thomas rotation associated with composite boosts $B(\mathbf{u})B(\mathbf{v})$ is represented by the 3×3 unimodular orthogonal matrix, tom[**u**; **v**], satisfying the equation

$$B(\mathbf{u})B(\mathbf{v}) = B(\mathbf{u}*\mathbf{v})\,\mathrm{Tom}[\mathbf{u};\mathbf{v}], \qquad \mathbf{u},\mathbf{v}\in \mathbb{R}_c^3, \tag{11}$$

where we use the notation

$$\mathrm{Tom}[\mathbf{u};\mathbf{v}] = \rho(\mathrm{tom}[\mathbf{u};\mathbf{v}]). \tag{12}$$

Equation (11) describes the composition of two boosts as a boost preceded by a Thomas rotation, Tom[**u**; **v**], generated by the boost velocity parameters **u** and **v**. Eq, (11), thus, expresses the fact that boosting a moving object generates a coordinate rotation. The *lhs* and the *rhs* of eq. (11) are respectively illustrated in Fig. 1(a) and 1(b), where **u** and **v** are velocity parameters that may be given by their components measured in both frames Σ and Σ'[23]. Thomas rotation is referred to as a Wigner rotation by several authors. However, (*i*) Thomas was the first author who drew wide attention to the "forgotten relativistic effect"[24] of the coordinate rotation generated by two successive boosts, previously regarded as a peculiar feature of special relativity with no significant effects on nonrelativistic motions, $v \ll c$; and (*ii*) following Han, Kim and Son,[25] the term *Wigner rotation* is reserved to the Thomas rotation defined in the Lorentz frame in which the boosted particle under consideration is at rest.

Equation (11) can readily be solved for Tom[**u**; **v**],

$$\mathrm{Tom}[\mathbf{u};\mathbf{v}] = B(-\mathbf{u}*\mathbf{v})B(\mathbf{u})B(\mathbf{v}), \tag{13}$$

where $\mathbf{u}*\mathbf{v}$ is the velocity composition of **u** and **v** given by equation (1), and where we use the notation $-\mathbf{u}*\mathbf{v} = -(\mathbf{u}*\mathbf{v}) = (-\mathbf{u})*(-\mathbf{v})$, that is, the negation operator, $-$, distributes with the composition operator, $*$. Equation (13), in turn, can readily be solved for tom[**u**; **v**] by means of eq. (12).[26] The study of eq. (13) for the case when the velocity **u** is taken to be an infinitesimal velocity, $\delta\mathbf{u}$, is well known.[6] In order to simplify the entries of the Thomas rotation matrix tom[**u**; **v**] algebraically to the point where its properties can be actually demonstrated, one must perform the algebraic manipulations that have already overwhelmed previous investigators. Details of the calculations involved in the simplification of the Thomas rotation matrix tom[**u**; **v**] of eq. (13), therefore, cannot be presented here. The final, algebraically simplified form of tom[**u**; **v**] is, however, quite neat! The Thomas rotation matrix, tom[**u**; **v**] of eq. (11), in all its glory turns out to be

$$\mathrm{tom}[\mathbf{u};\mathbf{v}] = I + c_1\Omega + c_2\Omega^2, \qquad \mathbf{u},\mathbf{v}\in \mathbb{R}_c^3, \tag{14}$$

where I is the 3×3 identity matrix, and where the matrix $\Omega=\Omega(\mathbf{u},\mathbf{v})$ and the coefficients $c_1=c_1(\mathbf{u},\mathbf{v})$ and $c_2=c_2(\mathbf{u},\mathbf{v})$ are functions of **u** and **v**, given in eqs. (15) and (16) below.

The matrix $\Omega=\Omega(\mathbf{u},\mathbf{v})$ in eq. (14) is skew symmetric,

$$\Omega(\mathbf{u},\mathbf{v}) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}, \tag{15a}$$

representing the linear transformation of cross product with $\boldsymbol{\omega}$, that is, $\Omega\mathbf{r} = \boldsymbol{\omega}\times\mathbf{r}$ for a 3-vector $\mathbf{r}$. The entries ω_k, $1\le k\le 3$, of the matrix Ω are the components of the vector product $\boldsymbol{\omega}=\mathbf{u}\times\mathbf{v}$ measured in the frame Σ of Fig. 1,

$$\boldsymbol{\omega} = (\omega_1,\omega_2,\omega_3) = \mathbf{u}\times\mathbf{v} = (u_2v_3-u_3v_2,\, u_3v_1-u_1v_3,\, u_1v_2-u_2v_1). \tag{15b}$$

The coefficients $c_1=c_1(\mathbf{u},\mathbf{v})$ and $c_2=c_2(\mathbf{u},\mathbf{v})$ in eq. (14) are given by the equations

$$\begin{aligned} c_1(\mathbf{u},\mathbf{v}) &= -\frac{1}{c^2}\frac{\gamma_u\gamma_v(\gamma_u+\gamma_v+\gamma_{u*v}+1)}{(\gamma_u+1)(\gamma_v+1)(\gamma_{u*v}+1)} \\ c_2(\mathbf{u},\mathbf{v}) &= \frac{1}{c^4}\frac{\gamma_u^2\gamma_v^2}{(\gamma_u+1)(\gamma_v+1)(\gamma_{u*v}+1)}\,. \end{aligned} \tag{16}$$

The verification that tom[$\mathbf{u}$; $\mathbf{v}$] $\varepsilon\, SO(3)$ is immediate if we note that (i) the matrix Ω and the coefficients c_1 and c_2 are related by the equations

$$\begin{aligned} \Omega^3 &= -\boldsymbol{\omega}^2\Omega \\ c_1^2+c_2^2\,\boldsymbol{\omega}^2-2c_2 &= 0\,; \end{aligned} \tag{17}$$

(ii) the transpose of tom[$\mathbf{u}$; $\mathbf{v}$] is tom[$\mathbf{v}$; $\mathbf{u}$],

$$\mathrm{tom}[\mathbf{v};\mathbf{u}] = I - c_1\Omega + c_2\Omega^2\,; \tag{14a}$$

and (iii) tom[$\mathbf{u}$; $\mathbf{v}$] is continuously deformed into the 3×3 identity matrix I by letting $\mathbf{u}$ tend to $\mathbf{v}$.

The verification that the matrix tom[$\mathbf{u}$; $\mathbf{v}$] of eqs. (11) and (12) is indeed the one given explicitly by eq. (14) is a matter of matrix algebra. It is clear from eqs. (14)-(16) that the Thomas rotation vanishes, tom[$\mathbf{u}$; $\mathbf{v}$]$=I$, if and only if either $c\to\infty$ or $\mathbf{u}\times\mathbf{v}=\mathbf{0}$.

There are various attempts in the literature to express the Thomas rotation in terms of its generating velocity parameters, resulting in expressions having different forms.[1-9,11,13] Thus, for example, equations (13) and (14) express the Thomas rotation tom[$\mathbf{u}$; $\mathbf{v}$], generated by the composite boost $B(\mathbf{u})B(\mathbf{v})$, in different forms.[27] The superiority of the expression in eq. (14) over the one in eq. (13) and other existing ones rests on the fact that it appears in a rotation matrix form to which stadard results may be applied.

For the application of standard results to the Thomas rotation tom[$\mathbf{u}$; $\mathbf{v}$], let θ be the angle between the velocity vectors $\mathbf{u}$ and $\mathbf{v}$, $0\le\theta<2\pi$. If $\theta=0$ or $\theta=\pi$ then $\mathbf{u}$ and $\mathbf{v}$ are parallel and $\boldsymbol{\omega}=\mathbf{u}\times\mathbf{v}$ vanishes. Hence, we see from eqs. (14) and (15) that when $\sin\theta=0$ we have $\Omega=0$ and tom[$\mathbf{u}$; $\mathbf{v}$]$=I$, that is, no Thomas rotation takes place. When a Thomas rotation does occur, tom[$\mathbf{u}$; $\mathbf{v}$]$\neq I$, the magnitude $|\boldsymbol{\omega}|$ of $\boldsymbol{\omega}$,

$$|\boldsymbol{\omega}| = |\mathbf{u}|\,|\mathbf{v}|\,|\sin\theta|, \tag{18}$$

is different from zero and, hence, eq. (14) can be written in a canonical form as

$$\mathrm{tom}[\mathbf{u};\mathbf{v}] = \begin{cases} I + \sin\varepsilon\, \dfrac{\Omega(\mathbf{u},\mathbf{v})}{\omega_\theta} + (1-\cos\varepsilon)\, \dfrac{\Omega^2(\mathbf{u},\mathbf{v})}{\omega_\theta^2}, & \omega_\theta \neq 0 \\ I, & \omega_\theta = 0, \end{cases} \tag{19}$$

for $\mathbf{u}, \mathbf{v} \,\varepsilon\, I\!R_c^3$ and $0 \le \varepsilon < 2\pi$, where we use the notation

$$\begin{aligned} \sin\varepsilon &= c_1\omega_\theta \\ 1-\cos\varepsilon &= c_2\omega_\theta^2 \end{aligned} \tag{20a}$$

and

$$\omega_\theta = \pm|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta = c^2\, \frac{\sqrt{\gamma_u^2-1}\,\sqrt{\gamma_v^2-1}}{\gamma_u\,\gamma_v}\sin\theta. \tag{20b}$$

The notation in eqs. (20a) is justified since $(c_1\omega_\theta)^2 + (1-c_2\omega_\theta^2)^2 = 1$. The angle ε associated with the Thomas rotation tom[**u**; **v**] of eq. (19), shown in Fig. 1, is called the Thomas angle.

The *rhs* of the equation in (19) corresponding to $\omega_\theta \neq 0$ appears in a standard rotation-matrix form that, in less modern dress, dates back to Euler.[28] The derivation of the axis and the angle of Thomas rotation, tom[**u**; **v**], from eq. (19) is standard. Owing to the isomorphism between the space of 3×3 skew symmetric matrices and the vector space $I\!R^3$, one may identify the matrix Ω of eq. (15) with the vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$, which is the vector product, $\boldsymbol{\omega} = \mathbf{u}\times\mathbf{v}$, of the velocity vectors **u** and **v**. This vector turns out to be parallel to the rotation axis of the Thomas rotation tom[**u**; **v**]. Accordingly, when $\sin\theta \neq 0$, that is, when a Thomas rotation takes place, the matrix Ω/ω_θ of eq. (19) is identified with a *unit* vector, that is, the vector

$$\mathbf{e} = \frac{\boldsymbol{\omega}}{\omega_\theta} = \pm\frac{\mathbf{u}\times\mathbf{v}}{|\mathbf{u}\times\mathbf{v}|}. \tag{21}$$

Hence, by standard theory, the canonical form (19) of the Thomas rotation matrix tom[**u**; **v**] represents a 3×3 space rotation by the Thomas angle ε in the sense of a right-handed screw propelled in the direction of a positive rotation axis parallel to **e**. Being parallel to $\boldsymbol{\omega} = \mathbf{u}\times\mathbf{v}$, the axis of the Thomas rotation tom[**u**; **v**], generated by two successive boosts of velocities **v** and **u**, is perpendicular to these velocities.

It can readily be shown from eqs. (16) and (20) that

$$\begin{aligned} \cos\varepsilon &= 1 - \frac{(\gamma_u-1)(\gamma_v-1)}{\gamma_{u*v}+1}\sin^2\theta \\ \sin\varepsilon &= -\frac{1}{\gamma_{u*v}+1}\left[\sqrt{\gamma_u^2-1}\,\sqrt{\gamma_v^2-1} + (\gamma_u-1)(\gamma_v-1)\cos\theta\right]\sin\theta \end{aligned} \tag{22}$$

where, from eq. (4),

$$\gamma_{u*v} = \gamma_u \gamma_v + \sqrt{\gamma_u^2 - 1}\sqrt{\gamma_v^2 - 1}\cos\theta. \tag{23}$$

In order to write eqs. (22) in a simpler form, let us define the two positive functions $A = A(\mathbf{u},\mathbf{v})$ and $B = B(\mathbf{u},\mathbf{v})$ by the equations

$$\begin{aligned} A^2 &= \frac{1}{2}\frac{(\gamma_u+1)(\gamma_v+1)}{\gamma_{u*v}+1} \\ B^2 &= \frac{1}{2}\frac{(\gamma_u-1)(\gamma_v-1)}{\gamma_{u*v}+1}, \end{aligned} \tag{24a}$$

or, equivalently, by the equations

$$\begin{aligned} A &= \frac{k}{\sqrt{k^2+2k\cos\theta+1}}, \qquad & \frac{k}{k+1} \le A \le \frac{k}{k-1}, \\ B &= \frac{1}{\sqrt{k^2+2k\cos\theta+1}}, \qquad & \frac{1}{k+1} \le B \le \frac{1}{k-1}, \end{aligned} \tag{24b}$$

where

$$k^2 = \frac{\gamma_u+1}{\gamma_u-1}\frac{\gamma_v+1}{\gamma_v-1}, \qquad k > 1, \qquad \lim_{|\mathbf{u}|,|\mathbf{v}|\to c} k = 1. \tag{24c}$$

It can be shown that the functions A and B are related by the equation

$$A^2 + 2AB\cos\theta + B^2 = 1, \tag{25}$$

and that, in terms of A and B, eqs. (22) take the form

$$\begin{aligned} \cos\varepsilon &= 1 - 2B^2\sin^2\theta \\ \sin\varepsilon &= -2B(A + B\cos\theta)\sin\theta. \end{aligned} \tag{26}$$

We see from eqs. (24) that $A = kB$ and, hence,

$$A \ge B \tag{27}$$

where equality holds if and only if the permissible velocities $\mathbf{u}$ and $\mathbf{v}$ have both magnitude approaching c. Hence, by eq. (25) we have the inequalities

$$2A^2 \ge \frac{1}{1+\cos\theta} \ge 2B^2, \tag{28}$$

where equalities hold if and only if $|\mathbf{u}| = c$ and $|\mathbf{v}| = c$.

The form of the first equation in (26) suggests that eqs. (26) can be simplified if we make use of the trigonometric identity $\cos\varepsilon = 1 - 2\sin^2(\varepsilon/2)$. Indeed, eqs. (26) expressed in terms of half ε take the simpler form,

$$\begin{aligned} \cos\frac{\varepsilon}{2} &= \pm(A + B\cos\theta) \\ \sin\frac{\varepsilon}{2} &= \mp B\sin\theta. \end{aligned} \tag{29}$$

The ambiguous signs in eqs. (29) go together, and the choice of a particular sign is irrelevant since we do not distinguish between the pairs $(\cos\frac{\varepsilon}{2}, \sin\frac{\varepsilon}{2})$ and $(\cos\frac{\varepsilon+2\pi}{2}, \sin\frac{\varepsilon+2\pi}{2}) = -(\cos\frac{\varepsilon}{2}, \sin\frac{\varepsilon}{2})$.

Clearly, the angles ε and θ of eqs. (26) represent oppositely directed rotations about an axis parallel to the vector $\mathbf{u}\times\mathbf{v}$, shown in Fig. 1. Interestingly, $\varepsilon=-\theta$ when the magnitude of both $\mathbf{u}$ and $\mathbf{v}$ approaches the speed of light c, and $\theta\neq\pi$. When $\theta=\pi$ and $\mathbf{u}$ and $\mathbf{v}$ have magnitude c, the Thomas angle ε, as well as the composition of the velocities $\mathbf{u}$ and $\mathbf{v}$, is undefined.

In general, we have the inequality $\cos\varepsilon\geq\cos\theta$ where equality holds if and only if $|\mathbf{u}| = |\mathbf{v}| = c$ and $\theta\neq\pi$. We see this from eqs. (26) and (28),

$$\cos\varepsilon = 1 - 2B^2\sin^2\theta \geq 1 - \frac{\sin^2\theta}{1+\cos\theta} = \cos\theta. \tag{30}$$

When, and only when, both velocities $\mathbf{u}$ and $\mathbf{v}$ have magnitude c, we have the equality $\cos\varepsilon=\cos\theta$ for all θ, $0\leq\theta<2\pi$, with one exception, that is, $\theta\neq\pi$: The case $\theta=\pi$ must be excluded when $|\mathbf{u}| = |\mathbf{v}| = c$ since under this circumstance the coefficients A and B of eqs. (24) and, hence, $\cos\varepsilon$ and $\sin\varepsilon$ of eqs. (26) are undefined. This singularity in the Thomas angle ε is hardly a surprise because it accompanies a corresponding singularity in the velocity composition $\mathbf{u}*\mathbf{v}$ when $\mathbf{u}$ and $\mathbf{v}$ are antiparallel velocities with magnitude c.

We see from eqs. (26) that $\cos\varepsilon$ is an even function of θ and $\sin\varepsilon$ is an odd function of θ so that ε is an odd function of θ. Moreover, for $0\leq\theta\leq\pi$ we have, from eq. (30),

$$|\varepsilon| < \theta \tag{31a}$$

for all permissible velocities $\mathbf{u}$ and $\mathbf{v}$, $|\mathbf{u}|, |\mathbf{v}| < c$, where ε and θ have opposite signs. For, and only for, velocities $\mathbf{u}$ and $\mathbf{v}$ with magnitude c, $|\mathbf{u}| = |\mathbf{v}| = c$, we have

$$\varepsilon = -\theta \tag{31b}$$

for all θ, $0\leq\theta<2\pi$, $\theta\neq\pi$. Hence, the magnitude, $|\varepsilon|$, of the Thomas angle, ε, is smaller than its generating angle, θ, $0\leq\theta\leq\pi$. If $|\mathbf{u}| = |\mathbf{v}| = c$ and $\theta=\pi$ then Thomas angle ε, as well as the composition of $\mathbf{u}$ and $\mathbf{v}$, is undefined. This singularity in the Thomas angle ε is, thus, associated with the fact that there is no rest frame for the photon.

Thomas angle ε and its generating angle θ are shown in Fig. 1. The graphs of $\cos\varepsilon$ and $-\sin\varepsilon$ of eq. (22), viewed as functions of θ, are shown in Fig. 2 for several values of $\beta_u = u/c$ and $\beta_v = v/c$, where u and v are the magnitudes of the velocities $\mathbf{u}$ and $\mathbf{v}$ that generate the Thomas rotation tom[u; v], and where θ is the angle between these two velocities.

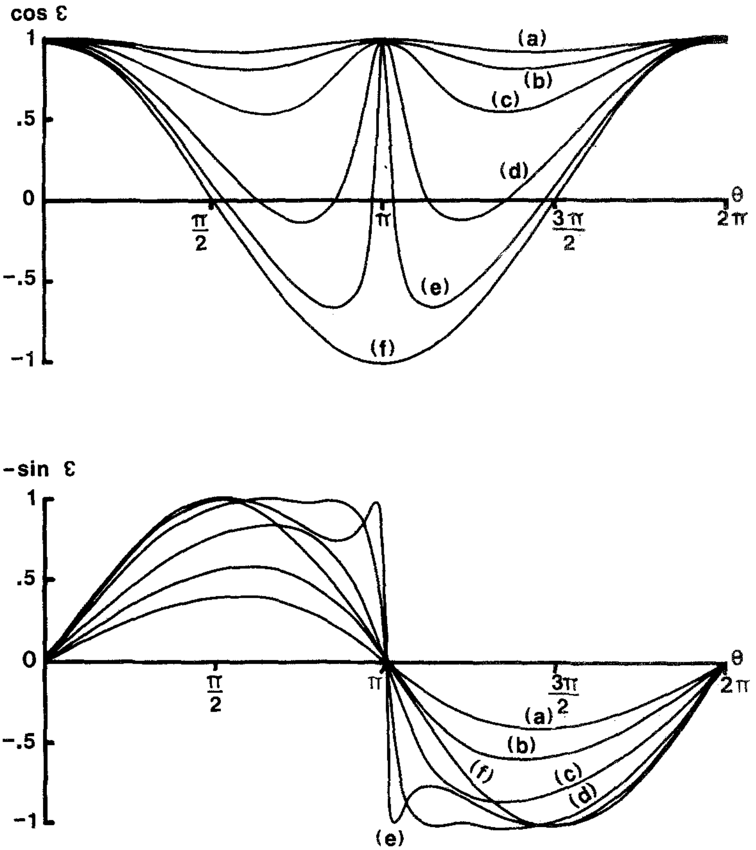

Fig. 2 Graphical presentation of the Thomas angle.

The cosine and the −sine functions of the Thomas angle ε, generated by two successive boosts $B(\mathbf{u})B(\mathbf{v})$, are viewed in Fig. 2 as functions of the angle θ between the two partaking boost velocity parameters $\mathbf{u}$ and $\mathbf{v}$. The graphs of the functions $\cos\varepsilon(\theta)$ and $-\sin\varepsilon(\theta)$ of θ are shown in Fig. 2 for several values of $\beta_u = |\mathbf{u}|/c$ and $\beta_v = |\mathbf{v}|/c$: (*a*) $\beta_u = \beta_v = 0.750$; (*b*) $\beta_u = \beta_v = 0.850$; (*c*) $\beta_u = 0.850$, $\beta_v = 0.990$; (*d*) $\beta_u = \beta_v = 0.990$; (*e*) $\beta_u = \beta_v = 0.999$; and (*f*) $\beta_u = \beta_v = 1$. In graph (*f*) the point $\theta = \pi$, where $\cos\varepsilon$ and $\sin\varepsilon$ are undefined, is excluded. At all other points θ in graph (*f*), $0 \le \theta < 2\pi$, $\cos\varepsilon$ and $-\sin\varepsilon$ are respectively equal to $\cos\theta$ and $\sin\theta$. Graph (*f*), including the point corresponding to $\theta = \pi$, is therefore identical to the graph of $\cos\theta$ and $\sin\theta$ in $[0, 2\pi]$. The graphs (*a*)-(*e*) show that $(\cos\varepsilon, \sin\varepsilon) = (1, 0)$ for all $0 \le \beta_u, \beta_v < 1$ when $\theta = 0$, when $\theta = \pi$ and, again, when $\theta = 2\pi$. The respective approach of $\cos\varepsilon$ and $-\sin\varepsilon$ to $\cos\theta$ and $\sin\theta$, $\theta \ne \pi$, as β_u and β_v approach unity, and the singularity of the limit at $\theta = \pi$ are clearly observed in Fig. 2.

5. SOME COMMENTS ON THE THOMAS ROTATION

In Section 4 the Thomas rotation matrix, tom[**u**; **v**], has been expressed in a standard rotation-matrix form, eq. (19), enabling its axis and angle of rotation to be readily determined from its generating velocities **u** and **v**. As expected, the axis of rotation is parallel to the vector $\mathbf{u}\times\mathbf{v}$ when $\mathbf{u}\times\mathbf{v}\neq 0$. This axis does not exist when $\mathbf{u}\times\mathbf{v}=0$, that is, when the Thomas rotation vanishes, tom[**u**; **v**] $=I$. The rotation angle about this axis, that is, the Thomas angle ε, is given by eqs. (22).

An interesting special case of the Thomas angle has recently been studied by Han, Kim and Son (HKS).[14] It is the case of a Thomas angle generated by two velocities with equal magnitudes, when one of the two velocities is selected to be parallel to a particular coordinate. We will show in the present section that the special result of HKS agrees with our general one.

HKS define the Wigner rotation as the Thomas rotation measured in the frame in which the boosted particle is at rest.[25] The definition of the Wigner angle, θ^*, made by HKS, is therefore slightly different from our definition of the Thomas angle θ:

$$\theta^* = \theta + \varepsilon, \tag{32}$$

where θ and ε are the oppositely directed angles illustrated in Fig. 1: (i) The angle θ is the angle between the velocities **u** and **v**, generating the Thomas rotation, (ii) the angle ε is the Thomas rotation angle, and (iii) the sum of these two angles gives the Wigner angle θ^* as defined by HKS.[14] Expressions representing the Wigner angle, θ^*, can readily be constructed from eqs. (22),

$$\begin{aligned}\cos\theta^* &= \cos(\theta+\varepsilon) = \cos\theta + \frac{\sqrt{\gamma_u^2-1}\sqrt{\gamma_v^2-1}}{\gamma_{u*v}+1}\sin^2\theta \\ \sin\theta^* &= \sin(\theta+\varepsilon) = \frac{\gamma_u+\gamma_v}{\gamma_{u*v}+1}\sin\theta .\end{aligned} \tag{33}$$

The second equation in (33) shows that $\sin\theta^*$ and $\sin\theta$ have equal signs. Since, in addition, θ and ε have opposite signs, eqs. (22), we find that $|\varepsilon|<\theta$ for $0\leq\theta\leq\pi$, thus confirming eq. (31a). In terms of half angles, $\theta^*/2$ and $\theta/2$, eqs. (33) can be simplified. Noting eq. (23), we have from eqs. (33),

$$\begin{aligned}\tan\frac{\theta^*}{2} &= \frac{\sin\theta^*}{1+\cos\theta^*} = \frac{\gamma_u+\gamma_v}{1+\gamma_u\gamma_v+\sqrt{\gamma_u^2-1}\sqrt{\gamma_v^2-1}}\,\frac{\sin\theta}{1+\cos\theta} \\ &= \frac{\gamma_u+\gamma_v}{1+\gamma_u\gamma_v+\sqrt{\gamma_u^2-1}\sqrt{\gamma_v^2-1}}\tan\frac{\theta}{2},\end{aligned} \tag{33a}$$

from which we may calculate θ^* in $0\leq\theta^*,\theta<\pi$:

$$\theta^* = 2\tan^{-1}\left(\frac{\gamma_u+\gamma_v}{1+\gamma_u\gamma_v+\sqrt{\gamma_u^2-1}\sqrt{\gamma_v^2-1}}\tan\frac{\theta}{2}\right). \tag{33b}$$

We see from eqs. (33) that when the magnitude of the two velocities, **u** and **v**, that generate a Thomas rotation approach the speed of light, c, the Thomas angle approaches $-\theta$, and the Wigner angle, as defined by HKS, approaches zero,

$$\lim_{u,v \to c} \varepsilon = -\theta \tag{34}$$

and

$$\lim_{u,v \to c} \theta^* = 0, \tag{35}$$

for $\theta \neq \pi$. The limit in eq. (35), noted by HKS,[14] shows that the Wigner angle, θ^*, generated by velocities with magnitude approaching c, vanishes. The Thomas effect is a relativistic effect *enhanced* by relativistic velocities and accelerations. On the other hand, its associated HKS Wigner angle, θ^*, vanishes when generated by relativistic velocities with magnitude approaching c, regardless of the accelerations involved. We, therefore, do not select θ^* as the angle describing the Thomas rotation effect.

Due to the complexity of the algebra involved in the calculation of $\cos\theta^*$ and $\sin\theta^*$ of eqs. (33), Han, Kim and Son have calculated $\cos\theta^*$ for the simple special case where

$$|\mathbf{u}| = |\mathbf{v}| \qquad \text{and} \qquad \frac{\mathbf{v}}{c} = (0, 0, \alpha), \tag{36}$$

and where, as in eqs. (33), the angle between **u** and **v** is θ. Our aim is to show that the special result of HKS for $\cos\theta^*$ agrees with our general result in eqs. (33).

For the special case (36), $\cos\theta^*$ and $\sin\theta^*$ of eqs. (33) reduce to $\cos\theta_s^*$ and $\sin\theta_s^*$ given by the equations

$$\begin{aligned} \cos\theta_s^* &= \frac{1+\cos\theta - (1-\alpha^2)(1-\cos\theta)}{1+\cos\theta + (1-\alpha^2)(1-\cos\theta)} \\ \sin\theta_s^* &= \frac{2\sqrt{1-\alpha^2}\sin\theta}{1+\cos\theta + (1-\alpha^2)(1-\cos\theta)}\,. \end{aligned} \tag{37a}$$

Results obtained by spinor formulation normally involve half angles. Expressing eqs. (37a) in terms of $\theta/2$ rather than θ we, finally, get

$$\begin{aligned} \cos\theta_s^* &= \frac{1-(1-\alpha^2)\tan^2(\theta/2)}{1+(1-\alpha^2)\tan^2(\theta/2)} \\ \sin\theta_s^* &= \frac{2\sqrt{1-\alpha^2}\tan(\theta/2)}{1+(1-\alpha^2)\tan^2(\theta/2)}\,. \end{aligned} \tag{37b}$$

One of the two equations in (37b), the first one, was presented in an identical form by HKS in their eq. (11).[14] Equations (33) thus generalize the special result (37b) of HKS to general Wigner rotations generated by arbitrary, permissible velocities **u** and **v**. Since HKS did not calculate the second equation in (37b), they did not perform the obvious simplification of their expression for the Wigner angle θ_s^*, that follows in eqs. (38) below.

$$\tan\frac{\theta_s^*}{2} = \frac{\sin\theta_s^*}{1+\cos\theta_s^*} = \sqrt{1-\alpha^2}\tan\frac{\theta}{2}, \tag{38a}$$

that is,

$$\theta_s^* = 2\tan^{-1}\left(\sqrt{1-\alpha^2}\tan\frac{\theta}{2}\right), \tag{38b}$$

in $0 \le \theta_s^*, \theta < \pi$. Had HKS calculated the two equations in (37b) they could have presented the more impressive expression, (38b), for their Wigner angle θ_s^*.

In this section we have shown that eqs. (22) for the Thomas angle, ε, are equivalent to eqs. (33) for the HKS Wigner angle θ^*; and that eqs. (33), in turn, specialize to a result of HKS, obtained by a different method. The particular result of HKS, furtheremore, has been simplified. It is interesting to compare the graphical presentation of the Wigner rotation, given by HKM in their Fig. 2,[14] with our graphical presentation of the Thomas rotation, in Fig. 2 above. Clearly, the graph in Fig. 2 of the *cos/sine* of a rotation angle, ε, as a function of its generating angle, θ, is more revealing than a graph of a rotation angle, ε, as a function of its generating angle, θ.

6. SOME IDENTITIES INVOLVING THE THOMAS ROTATION

A list of several identities most of which involve the Thomas rotation matrix, tom[**u**; **v**], follows. The 3-dimensional velocity vectors **u**, **v**, **w** and **u**∗**v** in $\mathbb{R}_c^3$ are represented by column matrices so that tom[**u**; **v**](**u**∗**v**), for example, is the product of a 3×3 matrix and a 3×1 matrix.

(i)	$\mathbf{u}*\mathbf{v} = \mathrm{tom}[\mathbf{u};\mathbf{v}](\mathbf{v}*\mathbf{u})$	Weak commutative law of velocity composition
(iia)	$\mathbf{u}*(\mathbf{v}*\mathbf{w}) = (\mathbf{u}*\mathbf{v})*\mathrm{tom}[\mathbf{u};\mathbf{v}]\mathbf{w}$	Right weak associative law of velocity composition
(iib)	$(\mathbf{u}*\mathbf{v})*\mathbf{w} = \mathbf{u}*(\mathbf{v}*\mathrm{tom}[\mathbf{v};\mathbf{u}]\mathbf{w})$	Left weak associative law of velocity composition
(iii)	$R\,\mathbf{u}*R\,\mathbf{v} = R\,(\mathbf{u}*\mathbf{v})$	
(iv)	$\mathrm{tom}[\mathbf{u};\mathbf{v}] = I$ for $\mathbf{u}\times\mathbf{v} = \mathbf{0}$	
(v)	$\mathrm{tom}^{-1}[\mathbf{u};\mathbf{v}] = \mathrm{tom}^t[\mathbf{u};\mathbf{v}] = \mathrm{tom}[\mathbf{v};\mathbf{u}]$	
(vi)	$\det \mathrm{tom}[\mathbf{u};\mathbf{v}] = 1$	
(vii)	$\mathrm{tom}[-\mathbf{u};-\mathbf{v}] = \mathrm{tom}[\mathbf{u};\mathbf{v}]$	
$(viii)$	$\mathrm{tom}[\mathbf{u};-\mathbf{v}] = \mathrm{tom}[\mathbf{u}*\mathbf{u};-\mathbf{u}*\mathbf{v}]\mathrm{tom}[\mathbf{u};\mathbf{v}]$	
(ix)	$\mathrm{tom}[-\mathbf{v};\mathbf{v}*\mathbf{u}] = \mathrm{tom}[\mathbf{u};\mathbf{v}]$	
(x)	$R\,\mathrm{tom}[\mathbf{u};\mathbf{v}] = \mathrm{tom}[R\,\mathbf{u};R\,\mathbf{v}]\,R$	
(xi)	$B(\mathbf{u})B(\mathbf{v}) = B(\mathbf{u}*\mathbf{v})\,\mathrm{Tom}[\mathbf{u};\mathbf{v}]$	The boost composition law
(xii)	$B(\mathbf{u}*\mathbf{v})\,\mathrm{Tom}[\mathbf{u};\mathbf{v}] = \mathrm{Tom}[\mathbf{u};\mathbf{v}]\,B(\mathbf{v}*\mathbf{u})$	Boost weak symmetry
$(xiii)$	$\rho(R)B(\mathbf{u}) = B(R\,\mathbf{u})\,\rho(R)$	

The matrices $\mathrm{tom}^{-1}[\mathbf{u};\mathrm{v}]$ and $\mathrm{tom}^{t}[\mathbf{u};\mathrm{v}]$ are respectively the inverse and the transpose of $\mathrm{tom}[\mathbf{u};\mathrm{v}]$, and $\det\mathrm{tom}[\mathbf{u};\mathrm{v}]$ is the determinant of the matrix $\mathrm{tom}[\mathbf{u};\mathrm{v}]$. Equations (*iii*), (*x*) and (*xiii*) are valid for any $R \in SO(3)$. Equations (*iii*) and (*x*) indicate that $\mathbf{u}*\mathrm{v}$ and $\mathrm{tom}[\mathbf{u};\mathrm{v}]$ are cartesian tensors of respective order 1 and 2, as they should.[29] In eq. (*viii*), as in eq. (13), we use the notation $-\mathbf{u}*\mathrm{v} = -(\mathbf{u}*\mathrm{v}) = (-\mathbf{u})*(-\mathrm{v})$. Some equations in the list follow immediately from other equations in the list, e.g., eq. (*xii*) follows from eqs. (*xiii*) and (*i*). A direct proof of eqs. (*i*)-(*xiii*) for the Thomas rotation matrix, $\mathrm{tom}[\mathbf{u};\mathrm{v}]$, of eq. (14) is lengthy and, hence, requires the use of computer algebra.[30] While the matrix representation, $\mathrm{tom}[\mathbf{u};\mathrm{v}]$, of the Thomas rotation in eqs. (14) and (19) is novel, some of the equations listed in eqs. (*i*)-(*xiii*) are well-known. Thus, for example, an equation similar to eqs. (*xi*) and (*xii*) can be found in the form

$$L_1L_2 = RL = L'R \tag{39}$$

where L_1 and L_2 are given boosts and where the boosts L and L' and the rotation matrix R are "easily found by spinor formulation."[8] In fact, if we use the notation $L_1=B(\mathbf{u})$ and $L_2=B(\mathrm{v})$ then, in eq. (39), $R=\mathrm{tom}[\mathbf{u};\mathrm{v}]$ is the Thomas rotation, and $L=B(\mathrm{v}*\mathbf{u})$ and $L'=B(\mathbf{u}*\mathrm{v})$ as we see from eq. (*xii*).

Equations (*i*)–(*ii*) establish a previously overlooked noncommutative, nonassociative group structure for velocities, $(\mathbb{R}_c^3, *)$, which is of interest as much for its own sake as for its bearing on the real world.[31] The group operation is given by relativistic velocity composition, $*$, which is both weakly commutative and weakly associative. The weak commutative law is known in the literature, being used in the determination of the Thomas rotation.[32] The list of equations (*i*)-(*xiii*) clearly indicates the existence of a *mathematical formalism* underlying the Thomas rotation and the relativistic velocity composition that needs to be further explored. We thus see that the Thomas rotation and the relativistic velocity composition, commonly studied as isolated concepts, are actually aspects of a group structure for $\mathbb{R}_c^3$.

For some investigators, the peculiar asymmetry in $\mathbf{u}$ and v of the velocity composition $\mathbf{u}*\mathrm{v}$, eq. (1), seems to introduce difficulties into the theory of special relativity.[33] Equations (*i*) and (*ii*), however, obviate these difficulties tracing the noncommutativity and the nonassociativity of the composition of nonparallel velocities to the presence of the Thomas rotation. Thomas rotation, in turn, reveals itself in a moving spinning mass as the Thomas precession.[1] We, thus, may say that the Thomas rotation results from the lack of commutativity and associativity of velocity composition. Equation (*xi*) in the list is noteworthy, providing the exact result for the product of two boosts as a boost preceded by the Thomas rotation of eq. (14). Other representations of the exact product of two boosts were recently presented by Salingaros,[3] van Wyk,[8] Ben-Menahem,[9] Rivas *et al.*[10] and Baylis and Jones.[13]

As an illustrative example, we present the proof of eq. (*ix*). By eqs. (*xi*) and (*xii*) of the list we have the equation

$$B(\mathbf{u})B(\mathrm{v}) = B(\mathbf{u}*\mathrm{v})\,\mathrm{Tom}[\mathbf{u};\mathrm{v}] = \mathrm{Tom}[\mathbf{u};\mathrm{v}]\,B(\mathrm{v}*\mathbf{u}), \tag{40}$$

implying

$$B(\mathbf{u})B(\mathrm{v})B((-\mathrm{v})*(-\mathbf{u})) = \mathrm{Tom}[\mathbf{u};\mathrm{v}], \tag{41}$$

since $B(-\mathbf{v}*\mathbf{u}) = B((-\mathbf{v})*(-\mathbf{u}))$ is the inverse of $B(\mathbf{v}*\mathbf{u})$. The *lhs* of this equation can be manipulated into the desired expression,

$$\begin{aligned} B(\mathbf{u})B(\mathbf{v})B((-\mathbf{v})*(-\mathbf{u})) &= B(\mathbf{u})B(\mathbf{v}*\{(-\mathbf{v})*(-\mathbf{u})\})\,\mathrm{Tom}[\mathbf{v};(-\mathbf{v})*(-\mathbf{u})] \\ &= B(\mathbf{u})B(-\mathbf{u})\,\mathrm{Tom}[\mathbf{v};(-\mathbf{v})*(-\mathbf{u})] \\ &= \mathrm{Tom}[\mathbf{v};(-\mathbf{v})*(-\mathbf{u})] \\ &= \mathrm{Tom}[-\mathbf{v};\mathbf{v}*\mathbf{u}]\,. \end{aligned} \tag{42}$$

Equations (41) and (42) clearly establish the validity of eq. (*ix*). Another illustrative example, the proof of eq. (*x*), is indicated in Section 10 where some details are left to the reader as an exercise.

7. COMPOSITION OF THOMAS ROTATIONS

The aim of this section is to place the canonical form (19) of the Thomas rotation in the context of a classical result concerning rotations. Since Thomas rotation matrices appear in a canonical form, eq. (19), it would be useful to preserve the form in their compositions. For this sake we rewrite eq. (19) in terms of $\varepsilon/2$ as

$$\mathrm{tom}[\mathbf{u};\mathbf{v}] = I + 2\cos\frac{\varepsilon}{2}\sin\frac{\varepsilon}{2}\,\frac{\Omega}{\omega_\theta} + 2\sin^2\frac{\varepsilon}{2}\,\frac{\Omega^2}{\omega_\theta^2}\,, \tag{43}$$

allowing the representation of tom[$\mathbf{u}$; $\mathbf{v}$] by means of the pair $(\cos\frac{\varepsilon}{2},\ \mathbf{e}\sin\frac{\varepsilon}{2})$ where $\mathbf{e} = \boldsymbol{\omega}/\omega_\theta$ is the unit vector isomorphic to the skew symmetric matrix Ω/ω_θ, as explained in Section 4. The pair, in turn, may be written as a *unit quaternion*,[34]

$$t(\mathbf{u},\mathbf{v}) = \cos\frac{\varepsilon}{2} + \mathbf{e}\sin\frac{\varepsilon}{2}\,, \tag{44}$$

where $\mathbf{e}$ is given by eq. (21) and where the Thomas angle, ε, is a function of $|\mathbf{u}|$ and $|\mathbf{v}|$ and the angle θ between $\mathbf{u}$ and $\mathbf{v}$, given by eqs. (22). Equation (44) provides a quaternion representation, $t(\mathbf{u},\mathbf{v})$, for the Thomas rotation, tom[$\mathbf{u}$; $\mathbf{v}$]; each of the two quaternions $\pm t(\mathbf{u},\mathbf{v})$ represents the Thomas rotation matrix tom[$\mathbf{u}$; $\mathbf{v}$] generated by the composite boost $B(\mathbf{u})B(\mathbf{v})$. The unit quaternion $t(\mathbf{u},\mathbf{v})$ is therefore called the Thomas quaternion. It can readily be shown that a product of Thomas rotation matrices is represented by a corresponding product of Thomas quaternions. One can therefore use quaternion multiplication to describe compositions of Thomas rotations (as well as any 3×3 space rotations). The four components of the normalized quaternion $t(\mathbf{u},\mathbf{v})$ are known as the Euler parameters and form a well known relationship with their corresponding 3×3 unimodular orthogonal matrix.[35]

To express explicitly the Thomas quaternion $t(\mathbf{u},\mathbf{v})$ as a function of its generating velocities $\mathbf{u}$ and $\mathbf{v}$ we substitute $\cos(\varepsilon/2)$ and $\sin(\varepsilon/2)$ from eqs. (29) into eq. (44), noting that $\mathbf{e} = (\mathbf{u}\times\mathbf{v})/(|\mathbf{u}|\,|\mathbf{v}|\sin\theta)$. The resulting Thomas quaternion generated by $\mathbf{u},\mathbf{v} \in \mathbb{R}_c^3$ is neat,

$$t(\mathbf{u},\mathbf{v}) = \begin{cases} A + B\,\dfrac{\mathbf{u}\cdot\mathbf{v} - \mathbf{u}\times\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}, & |\mathbf{u}|\,|\mathbf{v}| \neq 0 \\ 1, & |\mathbf{u}|\,|\mathbf{v}| = 0, \end{cases} \tag{45a}$$

where A and B are positive, defined by eqs. (24). The elegance and the simplicity of the representation of the Thomas rotation, tom[u; v], by its Thomas quaternion, $t(\mathbf{u},\mathbf{v})$, recalls to mind other elegant quaternion expressions arising in the use of the quaternion group in modern physics.[36]

If we denote the unit vector parallel to the vector $\mathbf{u}\times\mathbf{v}$ by $\mathbf{e}$, as in eqs. (21) and (44), and the angle between $\mathbf{u}$ and $\mathbf{v}$ by θ, then eq. (45a) can be written as

$$t(\mathbf{u},\mathbf{v}) = A + B(\cos\theta - \mathbf{e}\sin\theta) \tag{45b}$$

for all permissible velocities $\mathbf{u}$ and $\mathbf{v}$. The quaternion in eq. (45b) is continuous at $|\mathbf{u}|\,|\mathbf{v}| = 0$ since

$$\lim_{|\mathbf{u}|\,|\mathbf{v}|\to 0} A = 1 \qquad \text{and} \qquad \lim_{|\mathbf{u}|\,|\mathbf{v}|\to 0} B = 0,$$

as we see from eqs. (24).

Finally, by means of eqs. (24) we may write the Thomas quaternion of eq. (45b) in another form,

$$t(\mathbf{u},\mathbf{v}) = \frac{k + \cos\theta - \mathbf{e}\sin\theta}{\sqrt{k^2 + 2k\cos\theta + 1}}, \tag{45c}$$

in which the denominator is the quaternionic norm of the quaternion in the numerator (which is different from zero since $k > 1$ for admissible velocities). When $|\mathbf{u}| = |\mathbf{v}| = c$ and $\theta = \pi$ the Thomas quaternion, $t(\mathbf{u},\mathbf{v})$, is undefined. This singularity in the Thomas quaternion, $t(\mathbf{u},\mathbf{v})$, is associated with a corresponding singularity in the composite boost, $B(\mathbf{u})B(\mathbf{v})$, that generates the Thoams rotation represented by the Thomas quaternion, $t(\mathbf{u},\mathbf{v})$. The singularity in the Thomas quaternion is, thus, associated with the fact that there is no rest frame for the photon.

8. PARAMETRIZING THE LORENTZ TRANSFORMATION GROUP

Attempts to parametrize the Lorentz transformation group in (1+3) dimensions exist in the literature. An interesting parametrization, in terms of Eulerian angles and pseudo angles, may be found in a text by Synge.[37] Another interesting parametrization, allowing an elegant parameter composition, was recently presented by Hirshfeld and Metzger.[38] The aim of this section is to parametrize the Lorentz transformation in $(1+n)$ dimensions, $1 \leq n \leq 3$, in a way that gives rise to parameter composition for composite Lorentz transformations, and that reduces naturally to the common velocity-parametrization of the (1+1)-dimensional Lorentz transformation. An extension to $n > 3$ is straightforward, but will not be discussed here. For $n=1$, it is well known that the (1+1)-dimensional Lorentz transformation is parametrized by a 1-dimensional velocity parameter v by means of the one-parameter matrix

$$L(v) = \begin{bmatrix} \gamma & c^{-2}\gamma v \\ \gamma v & \gamma \end{bmatrix}, \qquad -c < v < c, \tag{46}$$

where γ is the Lorentz factor associated with v, $\gamma = [1-(\frac{v}{c})^2]^{-1/2}$. The Lorentz transformation matrix $L(v)$ is the generic element of a *continuous one-parameter matrix group,* called the Lorentz group, for the velocity parameter $v, -c < v < c$. It satisfies the equation

$$L(v_1)L(v_2) = L(v_{12}), \tag{47}$$

where matrix multiplication corresponds to *parameter composition,* and where v_{12} is the composition of v_1 and v_2 given by the equation

$$v_{12} = \frac{v_1+v_2}{1+c^{-2}v_1v_2}. \tag{48}$$

The group operation for the one-parameter matrix group $L(v)$ is matrix multiplication, given by parameter composition. The identity element is $L(0)$ and the inverse of $L(v)$ is $L(-v)$.

Adopting the popular abuse of notation, we identify the Lorentz group with its generic element, $L(v)$. Elements of the Lorentz group $L(v)$ describe (1+1)-dimensional Lorentz transformations in terms of their effects on time-space coordinates (t', x'). The Lorentz transformation, $L(v)$, brings the coordinates of an event measured in a *rocket frame,* (t', x'), into the coordinates of the event measured in a *lab frame,* (t, x), the origin of which coincided at $t=0$ with the origin of the rocket frame, according to the equation

$$\begin{bmatrix} t \\ x \end{bmatrix} = L(v)\begin{bmatrix} t' \\ x' \end{bmatrix} = \begin{bmatrix} \gamma(t'+c^{-2}vx') \\ \gamma(x'+vt') \end{bmatrix}, \tag{49}$$

where v is the velocity of the rocket frame (t', x') relative to the lab frame (t, x). The parameter that parametrizes the Lorentz group $L(v)$ is, thus, a relative velocity between inertial frames; and the parameter composition (48) is Einstein's addition theorem for parallel velocities. The matrix product in eq. (47), therefore, describes two successive Lorentz transformations as a Lorentz transformation. In this sense we say that $L(v)$ of eq. (46) is a *one-parameter matrix group representation* of the (1+1)-dimensional Lorentz transformation group.

Parameter matrices have the capacity for storing information about the composition of their parameters, an example of which is provided by eqs. (47) and (48) for the one-parameter matrix $L(v)$.[39] For the generalization of the matrix representation (46) of the (1+1)–dimensional Lorentz transformation into (1+3) dimensions we, therefore, seek a multi-parameter matrix, involving a (3-dimensional) velocity parameter and an orientation parameter, that stores information about the relativistic composition of velocities and orientations. The latter, in turn, involves Thomas rotations as shown in Fig. 1. Following the presentation of the Thomas rotation matrix and its properties, the straightforward extension of the one-parameter matrix group representation (46) of the (1+1)-dimensional Lorentz transformation into (1+3) dimensions is now possible as we will show in eq. (55) of the next section.

It is well known that a (1+3)-dimensional Lorentz transformation can be broken down into a product of a (1+3)-dimensional pure Lorentz transformation, or a boost, and a 3×3 space rotation. We therefore represent the (1+3)-dimensional Lorentz transformation, $L\{\mathrm{v}; V\}$, by the matrix product representing a boost, $B(\mathrm{v})$, preceded by a 3×3 space rotation, V,

$$L\{\mathrm{v}; V\} = B(\mathrm{v})\rho(V), \qquad \mathrm{v}\in\mathbb{R}^3_c, \quad V\in SO(3), \tag{50}$$

where the boost and the rotation matrices, $B(\mathrm{v})$ and $\rho(V)$, are defined in eqs. (6) and (10). It can readily be shown that the parametrization in eq. (50) is unique: Two Lorentz transformations are equal, $L\{\mathbf{u}; U\}=L\{\mathrm{v}; V\}$, if and only if their corresponding parameters are equal, $\mathbf{u}=\mathrm{v}$ and $U=V$.

The Lorentz transformation represented by the matrix $L\{\mathrm{v}; V\}$ of eq. (50) is *homogeneous, proper* and *orthochronous*. It is homogeneous since it takes the origin of time-space coordinates into an origin of time-space coordinates; it is proper since the determinant of $L\{\mathrm{v}; V\}$ is +1 rather than −1; and it is orthochronous since the Lorentz factor γ_v, eq. (2), involved in the Lorentz ransformation matrix $L\{\mathrm{v}; V\}$, satisfies the inequality $\gamma_v \geq 1$ rather than $\gamma_v \leq -1$. In eq. (50) we thus parametrize the homogeneous, proper, orthochronous Lorentz transformation by a *velocity* parameter v, $\mathrm{v}\in\mathbb{R}^3_c$, and an *orientation* parameter V, $V\in SO(3)$. Since the velocity parameter, v, is represented by a 3-dimensional vector and the orientation parameter, V, is represented by a 3×3 orthogonal matrix with determinant 1, these two nonscalar parameters are equivalent to six scalar ones. The usefulness of the parametrization (50) of the Lorentz transformation by velocity and orientation parameters rests on the parameter composition law that it allows, which will be presented in eq. (55) of the next section.

The matrix $L\{\mathrm{v}; V\}$ describes the Lorentz transformation in terms of its effects on time-space coordinates. It relates the components $(t',x',y',z')^t$ and $(t,x,y,z)^t$ of an event measured in two respective inertial frames Σ' and Σ, by the equation

$$\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = L\{\mathrm{v}; V\}\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix}. \tag{51}$$

The Lorentz transformation, $L\{\mathrm{v}; V\}$, brings the coordinates $(t',x',y',z')^t$ of an event measured in a *rocket frame*, Σ', into the coordinates of the event measured in a *lab frame*, Σ. The origin of the rocket frame, Σ', coincided at time $t=0$ with the origin of the lab frame, Σ; and the velocity and the orientation of the rocket frame, Σ', relative to the lab frame, Σ, are respectively v and V, depicted in Fig. 3.

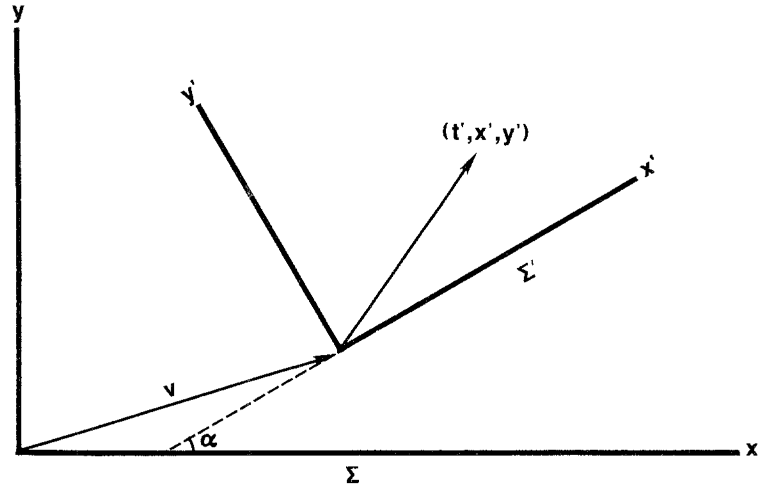

Fig. 3 Velocity v and orientation $V = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ of an inertial frame, Σ', relative to another inertial frame, Σ.

The velocity parameter v and the orientation parameter V, which parametrize the Lorentz transformation $L\{\mathrm{v}; V\}$ of eq. (51), describe the velocity and the orientation of the frame Σ' relative to the frame Σ, as depicted in Fig. 3. In the limit when the speed of light approaches infinity, $c \to \infty$, the Lorentz transformation $L\{\mathrm{v}; V\}$ reduces to a Galilean transformation, $G\{\mathrm{v}; V\} = \lim_{c\to\infty} L\{\mathrm{v}; V\}$, between two inertial frames with relative velocity, v, and orientation, V. From eqs. (50), (6) and (10) we, thus, have the matrix representation,

$$G\{\mathrm{v}; V\} = \lim_{c\to\infty} L\{\mathrm{v}; V\} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ v_1 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0\ 0\ 0 \\ 0 & \\ 0 & V \\ 0 & \end{bmatrix} = \begin{bmatrix} 1 & 0\ 0\ 0 \\ v_1 & \\ v_2 & V \\ v_3 & \end{bmatrix}, \tag{52}$$

of the Galilean transformation, $G\{\mathrm{v}; V\}$, parametrized by relative velocity, $\mathrm{v} \in \mathbb{R}^3$, and orientation, $V \in SO(3)$, between inertial frames, relating the components $(t', x', y', z')^t$ and $(t, x, y, z)^t$ of an event measured in the two respective inertial frames, Σ' and Σ, of Fig. 3.[40]

9. SUCCESSIVE LORENTZ TRANSFORMATIONS

Successive Lorentz transformations involve, in general, Thomas rotations. Since a neat expression for the Thomas rotation is not available in the literature, the standard study of Thomas rotations is limited to eq. (13) and to its implications to infinitesimal velocities.[26] The limited understanding of the Thomas rotation, in turn, restricts the study of successive Lorentz transformations to particularly simple cases. The common restricted study of successive Lorentz transformations is well expressed in a recent excellent text: "When we study Lorentz transformations, especially in physical applications, we often use transformations in the xy coordinate system either only along the x or along the y axis. We seldom discuss Lorentz transformations in the three-dimensional xyz coordinate system. In spite of this simplification, the conventional method of computing velocity additions and successive Lorentz boosts is still complicated."[41]

In this section successive Lorentz transformations will be presented in a form that shares its simplicity with the well-known form of successive Galilei transformations, to which it reduces when $c \to \infty$. Following the construction of a compact expression for the Thomas rotation, in eq. (14) and in eq. (19), and the subsequent discovery of its properties, in Section 6, we can now propose a novel way to describe two successive Lorentz transformations as a Lorentz transformation. It is the way that naturally extends the composition law (47) of (1+1)-dimensional Lorentz transformations into a composition law

$$L\{\mathbf{v}_1; V_1\}\, L\{\mathbf{v}_2; V_2\} = L\{\mathbf{v}_{12}; V_{12}\} \tag{53}$$

of (1+3)-dimensional Lorentz transformations, where matrix multiplication is given by parameter composition. Making consistent use of the Thomas rotation and its properties we will, thus, give the theory of the Lorentz group an extraordinarily elegant form.

The composition law of two successive Lorentz transformations follows (a) from the definition of the Lorentz transformation matrix, eq. (50); (b) from eq. (*xiii*), indicating that boosts are cartesian tensors of order 2; and (c) from the boost composition law in eq. (*xi*), that introduces a Thomas rotation:

$$\begin{aligned} L\{\mathbf{v}_1; V_1\}\, L\{\mathbf{v}_2; V_2\} &= B(\mathbf{v}_1)\, \rho(V_1) B(\mathbf{v}_2)\, \rho(V_2) \\ &= B(\mathbf{v}_1) B\,(V_1\mathbf{v}_2)\, \rho(V_1)\, \rho(V_2) \\ &= B(\mathbf{v}_1 * V_1\mathbf{v}_2)\, \rho(\mathrm{tom}[\mathbf{v}_1; V_1\mathbf{v}_2])\, \rho(V_1)\, \rho(V_2) \\ &= B(\mathbf{v}_1 * V_1\mathbf{v}_2)\, \rho(\mathrm{tom}[\mathbf{v}_1; V_1\mathbf{v}_2] V_1 V_2) \\ &= L\{\mathbf{v}_1 * V_1\mathbf{v}_2; \mathrm{tom}[\mathbf{v}_1; V_1\mathbf{v}_2] V_1 V_2\}. \end{aligned} \tag{54}$$

Equation (54) demonstrates that the Lorentz transformation composition law (53) takes the form

$$L\{\mathbf{v}_1; V_1\}\, L\{\mathbf{v}_2; V_2\} = L\{\mathbf{v}_1 * V_1\mathbf{v}_2; \mathrm{tom}[\mathbf{v}_1; V_1\mathbf{v}_2] V_1 V_2\}, \tag{55}$$

where matrix multiplication corresponds to parameter composition. The Lorentz transformation composition law (55) is, therefore, associative. Moreover, we see from the Lorentz transformation composition law (55) that the identity Lorentz transformation is

$$L\{\mathbf{0}; I\}, \tag{56}$$

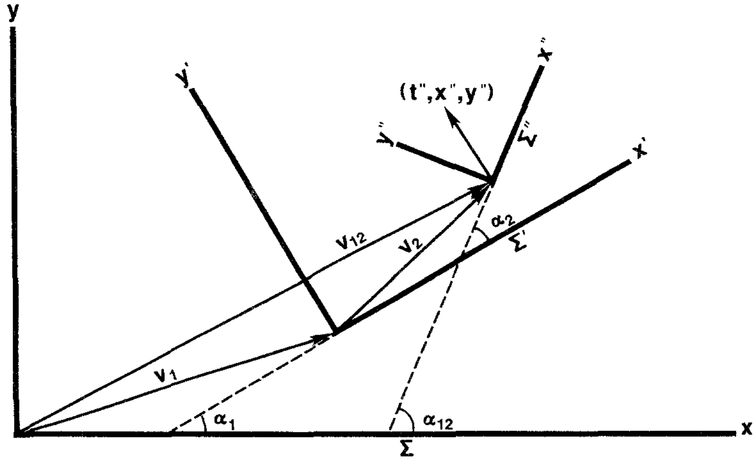

Fig. 4 Two successive (Galilean or Lorentz) transformations between inertial frames are equivalent to a single transformation.

and that the inverse $L^{-1}\{v; V\}$ of a Lorentz transformation $L\{v; V\}$ is

$$L^{-1}\{v; V\} = L\{-V^{-1}v; V^{-1}\}. \tag{57}$$

The Lorentz transformation matrix $L\{v; V\}$ thus forms a *two-parameter transformation group* for the velocity parameter v, $v \varepsilon \mathbb{R}_c^3$, and the orientation parameter V, $V \varepsilon SO(3)$, where the group operation is matrix multiplication given by parameter composition. Adopting the popular abuse of notation we identify the resulting Lorentz group with its generic element, $L\{v; V\}$, describing the generic (1+3)-dimensional Lorentz transformation according to eq. (51).

By letting the speed of light tend to infinity, $c \to \infty$, the composition law (55) of two successive Lorentz transformations is continuously deformed into the composition law

$$G\{v_1; V_1\}\, G\{v_2; V_2\} = G\{v_1 + V_1 v_2; V_1 V_2\}, \tag{58}$$

of two successive Galilean transformations, illustrated in Fig. 4. Equation (58) has an obvious geometrical meaning, and can be directly obtained from eq. (52).[42] The reducibility of the Lorentz composition (55) into its Galilean counterpart (58) by a continuous operation, $c \to \infty$, is expected. Surprisingly, however, it is not well known.[17]

A generic element of the Galilean group, $G\{v; V\}$, is represented by eq. (52) as a translation preceded by a rotation. Since, in addition, the group operation is given by eq. (58), the Galilean group is the *semidirect* product of the translation group, $(\mathbb{R}^3, +)$, and the rotation group, $SO(3)$.[43] The Lorentz group $L\{v; V\}$ with its group operation given by eq. (55) gives rise, similarly, to a novel product of the noncommutative, nonassociative group $(\mathbb{R}_c^3, *)$ and the rotation group $SO(3)$. This novel product of two groups, that we may call the *quasidirect* product, reduces to the semidirect product of two groups, $(\mathbb{R}^3, +)$ and $SO(3)$, when $c \to \infty$.

In the composite transformations of eqs. (55) and (58), each velocity parameter is a velocity vector that may be represented by its components measured in the frame relative to which the velocity is measured. Thus, the velocity parameters $\mathbf{v}_1$ and $V_1\mathbf{v}_2$ in the composition laws (55) and (58) are given by their components measured in the frame Σ of Fig. 4, while the velocity parameter $\mathbf{v}_2$ is given by its components measured in the frame Σ' of Fig. 4. Similarly, the inverse of the velocity parameter $\mathbf{v}_1$ of Fig. 4 is $-V_1^{-1}\mathbf{v}_1$ since $\mathbf{v}_1$ is given by its components measured in the frame Σ while its inverse must be given by its components measured in Σ'. This relationship between a velocity parameter, $\mathbf{v}$, and its inverse velocity parameter, $-V^{-1}\mathbf{v}$, is clearly seen in eq. (57) for the inverse of the Lorentz transformation.

Fig. 4 illustrates the relativistic and the Galilean composition law, eqs. (55) and (58). It depicts three inertial frames, Σ'', Σ' and Σ, in which the time dimension and one space dimension are suppressed for clarity. The velocity and the orientation of Σ'' relative to Σ' are given by the parameters $\mathbf{v}_2$ and $V_2 = \begin{pmatrix} \cos\alpha_2 & -\sin\alpha_2 \\ \sin\alpha_2 & \cos\alpha_2 \end{pmatrix}$, where $\mathbf{v}_2$ is given by its components measured in Σ'. Similarly, the velocity and the orientation of Σ' relative to Σ are given by the parameters $\mathbf{v}_1$ and $V_1 = \begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 \\ \sin\alpha_1 & \cos\alpha_1 \end{pmatrix}$, where $\mathbf{v}_1$ is given by its components measured in Σ. The Galilean velocity addition $\mathbf{v}_{12} = \mathbf{v}_1 + V_1\mathbf{v}_2$ describes the velocity of Σ'' relative to Σ in Galilean relativity, where $\mathbf{v}_{12}$ is given by its components measured in Σ. The Galilean orientation composition, V_{12}, $V_{12} = V_1V_2 = \begin{pmatrix} \cos(\alpha_1+\alpha_2) & -\sin(\alpha_1+\alpha_2) \\ \sin(\alpha_1+\alpha_2) & \cos(\alpha_1+\alpha_2) \end{pmatrix} = \begin{pmatrix} \cos\alpha_{12} & -\sin\alpha_{12} \\ \sin\alpha_{12} & \cos\alpha_{12} \end{pmatrix}$, describes the orientation of Σ'' relative to Σ in Galilean relativity.

To avoid the need for orientation composition in Galilean relativity one may employ only inertial frames which are mutually parallel; Galilei transformations between parallel inertial frames can be parametrized by a single (3-dimensional) velocity parameter. This is however not the case in Einsteinean relativity where orientation composition and velocity composition are *inseparable* due to the presence of the Thomas rotation: As we see from eq. (55), the parameter pair composition $(\mathbf{v}_{12}; V_{12})$ of $(\mathbf{v}_1; V_1)$ and $(\mathbf{v}_2; V_2)$ is given by the equation

$$(\mathbf{v}_1; V_1)*(\mathbf{v}_2; V_2) = (\mathbf{v}_{12}; V_{12}) = (\mathbf{v}_1 * V_1\mathbf{v}_2; \mathrm{tom}[\mathbf{v}_1; V_1\mathbf{v}_2]V_1V_2). \tag{59}$$

Contrary to Galilean orientation composition, eq. (59) shows that the composition of two identity orientations, I, need not result in an identity orientation, I. Hence, when nonparallel velocities are involved, one cannot employ in special relativity only mutually parallel inertial frames. Accordingly, the Lorentz transformation must be parametrized by both a velocity and an orientation parameter.[44]

Two geometrically obvious properties of the Lorentz transformation are presented in the following theorem.

THEOREM Let V_0 be an orientation parameter. Then

$$L\{\mathbf{v}; V\}\begin{bmatrix} t' \\ \mathbf{x}' \end{bmatrix} = L\{\mathbf{v}; V V_0^{-1}\}\begin{bmatrix} t' \\ V_0\mathbf{x}' \end{bmatrix} \tag{60}$$

and

$$L\{\mathbf{v}_1; V_1\} L\{\mathbf{v}_2; V_2\} = L\{\mathbf{v}_1; V_1 V_0^{-1}\} L\{V_0\mathbf{v}_2; V_0 V_2\}. \tag{61}$$

The first equation of the Theorem, (60), follows from the definition, eq. (50), of the Lorentz transformation matrix $L\{\mathbf{v}; V\}$ and from the obvious equation,

$$\rho(V)\begin{bmatrix} t' \\ \mathbf{x}' \end{bmatrix} = \rho(V V_0^{-1})\begin{bmatrix} t' \\ V_0\mathbf{x}' \end{bmatrix}, \tag{62}$$

for the rotation matrix $\rho(V)$ of eq. (10). The validity of the second equation of the Theorem, (61), is verified by an application of the Lorentz transformation composition law (55) to both sides of this equation.

10. MANIPULATIONS OF THE LORENTZ TRANSFORMATION

Before employing the parametrized Lorentz transformation for solving the problem of Section 11, it would be instructive to present some related illustrative manipulations. We see from eq. (55) that the equation

$$L\{\mathbf{u}; I\} L\{\mathbf{v}; V\} = L\{\mathbf{w}; W\}, \tag{63}$$

where I is the 3×3 identity matrix, holds if and only if the two equations

$$\begin{aligned} \mathbf{w} &= \mathbf{u}*\mathbf{v} \\ W &= \mathrm{tom}[\mathbf{u}; \mathbf{v}]V \end{aligned} \tag{64}$$

hold. Solving eq. (63) for $L\{\mathbf{u}; I\}$, we have from eqs. (57) and (55),

$$\begin{aligned} L\{\mathbf{u}; I\} &= L\{\mathbf{w}; W\} L^{-1}\{\mathbf{v}; V\} \\ &= L\{\mathbf{w}; W\} L\{-V^{-1}\mathbf{v}; V^{-1}\} \\ &= L\{\mathbf{w}*(-WV^{-1}\mathbf{v}); \mathrm{tom}[\mathbf{w}; -WV^{-1}\mathbf{v}]WV^{-1}\}. \end{aligned} \tag{65}$$

Substituting (64) into (65) we obtain the identity

$$L\{\mathbf{u}; I\} = L\{(\mathbf{u}*\mathbf{v})*(-\mathrm{tom}[\mathbf{u}; \mathbf{v}]\mathbf{v}); \mathrm{tom}[\mathbf{u}*\mathbf{v}; -\mathrm{tom}[\mathbf{u}; \mathbf{v}]\mathbf{v}]\, \mathrm{tom}[\mathbf{u}; \mathbf{v}]\}. \tag{66}$$

Identity (66) was, thus, derived from eq. (63) by the elimination of $\mathbf{w}$ and W in eq. (65). Since two Lorentz transformations are equal if and only if their corresponding parameters are equal, we have from eq. (66),

$$\mathbf{u} = (\mathbf{u}*\mathbf{v})*(-\mathrm{tom}[\mathbf{u}; \mathbf{v}]\mathbf{v}) \tag{67}$$

and

$$I = \mathrm{tom}[\mathbf{u}*\mathbf{v}; -\mathrm{tom}[\mathbf{u}; \mathbf{v}]\mathbf{v}]\,\mathrm{tom}[\mathbf{u}; \mathbf{v}]. \tag{68}$$

Identity (67) can be established directly from the right weak associative law, eq. (*iia*) of Section 6:

$$(\mathbf{u}*\mathbf{v})*(-\mathrm{tom}[\mathbf{u}; \mathbf{v}]\mathbf{v}) = (\mathbf{u}*\mathbf{v})*\mathrm{tom}[\mathbf{u}; \mathbf{v}](-\mathbf{v}) = \mathbf{u}*(\mathbf{v}*(-\mathbf{v})) = \mathbf{u}*\mathbf{0} = \mathbf{u}. \tag{69}$$

A direct verification of identity (68) is not as immediate as the direct verification of identity (67), but it is instructive and, hence, is presented here. We see from eq. (v) of Section 6 that identity (68) can be written as

$$\mathrm{tom}[\mathbf{v}; \mathbf{u}] = \mathrm{tom}[\mathbf{u}*\mathbf{v}; -\mathrm{tom}[\mathbf{u}; \mathbf{v}]\mathbf{v}] \tag{70}$$

which, in turn, can be written as

$$\mathrm{tom}[\mathbf{v}; \mathbf{u}] = \mathrm{tom}[\mathrm{tom}[\mathbf{u}; \mathbf{v}](\mathbf{v}*\mathbf{u}); -\mathrm{tom}[\mathbf{u},\mathbf{v}]\mathbf{v}] \tag{71}$$

by employing the weak commutative law, eq. (i) of Section 6.

Transposing matrices in eq. (71), by means of eq. (v) of Section 6, we obtain the equation

$$\mathrm{tom}[\mathbf{u}; \mathbf{v}] = \mathrm{tom}[-\mathrm{tom}[\mathbf{u}; \mathbf{v}]\mathbf{v}; \mathrm{tom}[\mathbf{u}; \mathbf{v}](\mathbf{v}*\mathbf{u})]. \tag{72}$$

Since $\mathrm{tom}[\mathbf{u}; \mathbf{v}]$ represents a 3×3 rotation about an axis parallel to the vector $\mathbf{u}\times\mathbf{v}$, eq. (72) is a particular case of the equation

$$\mathrm{tom}[\mathbf{u}; \mathbf{v}] = \mathrm{tom}[-P\mathbf{v}; P(\mathbf{v}*\mathbf{u})]. \tag{73}$$

where P is an arbitrary 3×3 rotation about an axis parallel to the vector $\mathbf{u}\times\mathbf{v}$. Noting from eq. (*iii*) of Section 6 that the rotation P distributes with velocity composition, $P(\mathbf{u}*\mathbf{v}) = P\mathbf{u}*P\mathbf{v}$, eq. (73) can be written as

$$\mathrm{tom}[\mathbf{u}; \mathbf{v}] = \mathrm{tom}[-P\mathbf{v}; P\mathbf{v}*P\mathbf{u}]. \tag{74}$$

This equation, in turn, can be written as

$$\mathrm{tom}[\mathbf{u}; \mathbf{v}] = \mathrm{tom}[P\mathbf{u}; P\mathbf{v}] \tag{75}$$

by means of eq. (*ix*) of Section 6. Equation (75), finally, follows from the fact that rotations P about an axis parallel to the vector $\mathbf{u}\times\mathbf{v}$ do not affect the Thomas rotation $\mathrm{tom}[\mathbf{u}; \mathbf{v}]$: We see from eq. (14) that the Thomas rotation $\mathrm{tom}[\mathbf{u}; \mathbf{v}]$ depends solely on the scalars $|\mathbf{u}|$, $|\mathbf{v}|$ and $\mathbf{u}\cdot\mathbf{v}$ and on the vector $\mathbf{u}\times\mathbf{v}$, all of which are invariant under the application of the rotation P to both $\mathbf{u}$ and $\mathbf{v}$. Reversing the argument one can show that eq. (75) implies the validity of identity (68).

As an exercise in Lorentz matrix manipulation, an interested reader may use the associativity of boost matrix multiplication to establish the identity

$$\mathrm{tom}[\mathbf{u}; \mathbf{v}*\mathbf{w}]\,\mathrm{tom}[\mathbf{v}; \mathbf{w}] = \mathrm{tom}[\mathbf{u}*\mathbf{v}; \mathrm{tom}[\mathbf{u}; \mathbf{v}]\mathbf{w}]\,\mathrm{tom}[\mathbf{u}; \mathbf{v}] \tag{76}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_c^3$. The interested reader may, further, use identity (76) and the associativity of Lorentz matrix multiplication to establish identity (x) of Section 6. The techniques of this article, thus, enable one to calculate a plethora of identities relating Thomas rotations.[31]

11. LORENTZ TRANSFORMATIONS LINKING FOUR-VECTORS

The problem of determining the Lorentz transformation, $L\{\mathbf{u}; U\}$, that links given *initial* and *final* time-like 4-vectors, $(t, \mathbf{x})^t$ and $(\tau, \chi)^t$, according to the equation

$$\begin{bmatrix} \tau \\ \chi \end{bmatrix} = L\{\mathbf{u}; U\} \begin{bmatrix} t \\ \mathbf{x} \end{bmatrix}, \qquad t, \tau > 0, \tag{77}$$

is of interest, but has not yet satisfactorily been solved. A Lorentz transformation connecting an initial four-vector velocity with a final four-vector velocity was displayed by Krause in 1977.[19] Later, in 1982, Fradkin claimed that the Lorentz transformation constructed by Krause lacks sufficient generality and uniqueness. He, therefore, introduced an additional specified four-vector to achieve uniqueness.[20] Finally, van Wyk has recently presented a partial solution, noting that attempts at numerical solution indicate that supplementary information is required.[21]

The problem of constructing the unique Lorentz transformation, $L\{\mathbf{u}; U\}$, with given orientation, U, solely from a set of an initial vector and a final vector is solved here by employing the techniques developed in this article. A unique solution of eq. (77) for the unknown velocity parameter $\mathbf{u}$ and, hence, for the Lorentz transformation $L\{\mathbf{u}; U\}$ is given implicitly by eqs. (85) and (88) below. The difficulty in determining explicitly the velocity parameter $\mathbf{u}$ of eq. (77), reported by van Wyk,[21] is traced in eq. (85) to the presence of the Thomas rotation in the composition of Lorentz transformations. Explicit determination of $\mathbf{u}$ is, finally, given by eq. (92).

Let $(t, \mathbf{x})^t$ and $(\tau, \chi)^t$ be the time-space coordinates of an event measured respectively in an *initial* inertial frame Σ_v and in a *final* inertial frame Σ_w. They are, therefore, connected by a Lorentz transformation $L\{\mathbf{u}; U\}$ satisfying eq. (77). As indicated in eq. (77) and depicted in Fig. 5, the *unknown* velocity of the initial frame Σ_v relative to the final frame Σ_w is $\mathbf{u}$, while the *known* orientation of the initial frame Σ_v relative to the final frame Σ_w is U.

If we use the notation

$$\mathbf{v} = \frac{\mathbf{x}}{t} \qquad \text{and} \qquad \mathbf{w} = \frac{\chi}{\tau}, \qquad \mathbf{v}, \mathbf{w} \in \mathbb{R}_c^3, \tag{78}$$

then the time-like vector $(t, \mathbf{x})^t$ is the first column in the matrix $(t/\gamma_v)L\{\mathbf{v}; V\}$ for *any* orientation parameter V and, similarly, the time-like vector $(\tau, \chi)^t$ is the first column in the matrix $(\tau/\gamma_w)L\{\mathbf{w}; W\}$ for *any* orientation parameter W. Equation (77) is, therefore, identical to the first column equation in the matrix equation

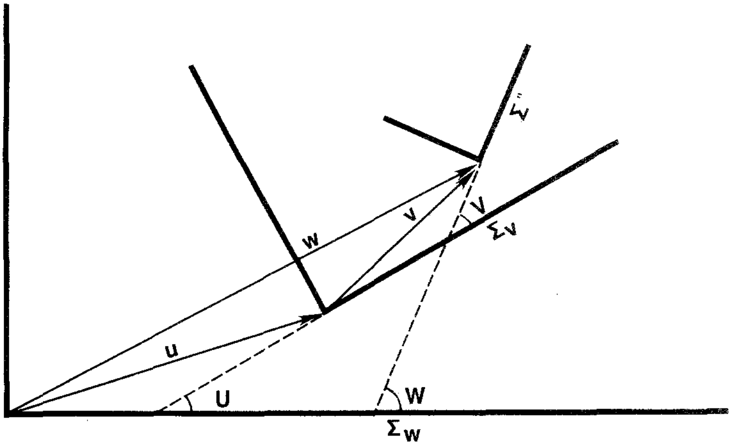

Fig. 5 Two successive Lorentz transformations in the problem of determining the unique Lorentz transformation connecting the components of an event measured in two inertial frames with given relative orientation U and unknown relative velocity **u**.

$$\frac{\tau}{\gamma_w} L\{\mathbf{w}; W\} = L\{\mathbf{u}; U\} \frac{t}{\gamma_v} L\{\mathbf{v}; V\}, \tag{79}$$

where V is an arbitrarily selected auxiliary orientation parameter and where W is a resulting auxiliary orientation parameter determined by eq. (79). The matrix $L\{\mathbf{u}; U\}$ determined by eq. (79), therefore, satisfies eq. (77).

The factors

$$\frac{t}{\gamma_v} = \sqrt{t^2 - \mathbf{x}^2/c^2} \qquad \text{and} \qquad \frac{\tau}{\gamma_w} = \sqrt{\tau^2 - \chi^2/c^2}$$

of eq. (79) are invariant under Lorentz transformations and hence, by eq. (77), are equal. Therefore, eq. (79) can be written as

$$L\{\mathbf{w}; W\} = L\{\mathbf{u}; U\} L\{\mathbf{v}; V\}. \tag{80}$$

The auxiliary orientation parameters V and W of eq. (80) are shown in Fig. 5. Solving eq. (80) for $L\{\mathbf{u}; U\}$ we have

$$\begin{aligned} L\{\mathbf{u}; U\} &= L\{\mathbf{w}; W\} L^{-1}\{\mathbf{v}; V\} \\ &= L\{\mathbf{w}; W\} L\{-V^{-1}\mathbf{v}; V^{-1}\} \\ &= L\{\mathbf{w} * (-W V^{-1}\mathbf{v}); \mathrm{tom}[\mathbf{w}; -W V^{-1}\mathbf{v}] W V^{-1}\}, \end{aligned} \tag{81}$$

where the second and the third equalities are respectively obtained from eqs. (57) and (55).

Figure 5 indicates that the parameter V of eqs. (79)-(81) is the orientation of some *unspecified* auxiliary frame Σ'' relative to the initial frame Σ_v, while the parameter U of eqs. (79)-(81) is the orientation of the initial frame Σ_v relative to the final frame Σ_w. The unspecified auxiliary frame, Σ'', and its auxiliary orientations V and W relative to Σ_v and Σ_w, respectively, were considered in order to extend eq. (77) between two column matrices into the square matrix equation (80). The matrix equation (80), in turn, can be solved for $L\{\mathbf{u}; U\}$ by matrix inversion, eq. (81). Since the solution $L\{\mathbf{u}; U\}$ to eq. (80) is also a solution to eq. (77), we expect the solution $L\{\mathbf{u}; U\}$ of eq. (80) to be *independent* of the unspecified auxiliary orientations, V and W, of the unspecified auxiliary frame Σ'' relative to Σ_v and Σ_w. This is indeed the case as we will see in eq. (83) below.

To manipulate the solution $L\{\mathbf{u}; U\}$, eq. (81), into a form independent of the auxiliary orientations V and W we note that W is the relativistic orientation composition of U and V, Fig. 5. By eq. (59), this composition, W, of U and V is given by the equation

$$W = \mathrm{tom}[\mathbf{u}; U\mathbf{v}]UV, \tag{82}$$

demonstrating that the orientation WV^{-1} is independent of both V and W as expected,

$$WV^{-1} = \mathrm{tom}[\mathbf{u}; U\mathbf{v}]U\,. \tag{83}$$

As we see from eqs. (81), (82) and (83), the Lorentz transformation matrix $L\{\mathbf{u}; U\}$ of eq. (77) is given in terms of $\mathbf{u}$, U, $(t,\mathbf{x})^t$ and $(\tau,\chi)^t$ by the equation

$$L\{\mathbf{u}; U\} = L\{\mathbf{w}*(-\mathrm{tom}[\mathbf{u}; U\mathbf{v}]U\mathbf{v}); \mathrm{tom}[\mathbf{w}; -\mathrm{tom}[\mathbf{u}; U\mathbf{v}]\,U\mathbf{v}]\,\mathrm{tom}[\mathbf{u}; U\mathbf{v}]U\} \tag{84}$$

where $\mathbf{v}$ and $\mathbf{w}$ are related to $(t,\mathbf{x})^t$ and $(\tau,\chi)^t$ by eqs. (78).

Equating the two parameters of the Lorentz transformations on both sides of eq. (84) we obtain the two equations

$$\mathbf{u} = \mathbf{w}*(-\mathrm{tom}[\mathbf{u}; U\mathbf{v}]\,U\mathbf{v}). \tag{85}$$

and

$$U = \mathrm{tom}[\mathbf{w}; -\mathrm{tom}[\mathbf{u}; U\mathbf{v}]U\mathbf{v}]\,\mathrm{tom}[\mathbf{u}; U\mathbf{v}]\,U\,. \tag{86}$$

Equation (86) can be written as

$$I = \mathrm{tom}[\mathbf{w}; -\mathrm{tom}[\mathbf{u}; U\mathbf{v}]U\mathbf{v}]\,\mathrm{tom}[\mathbf{u}; U\mathbf{v}] \tag{87}$$

or, by means of eq. (v) of Section 6, as

$$\mathrm{tom}[-\mathrm{tom}[\mathbf{u}; U\mathbf{v}]U\mathbf{v}; \mathbf{w}] = \mathrm{tom}[\mathbf{u}; U\mathbf{v}]. \tag{88}$$

Equations (85) and (88), where $\mathbf{v}$ and $\mathbf{w}$ are given by eqs. (78), determine implicitly the velocity $\mathbf{u}$ of the initial frame Σ_v relative to the final frame Σ_w as a function of the components of an event measured in these frames, $(t,\mathbf{x})^t$ and $(\tau,\chi)^t$, and the relative orientation, U, between them. For parallel velocities, $\mathbf{u} \,||\, U\mathbf{v}$, Thomas rotations vanish and, hence, the determination of the velocity parameter, $\mathbf{u}$, by eq. (85) becomes explicit,

$$\mathbf{u} = \mathbf{w}*(-U\mathbf{v}). \tag{85a}$$

Similarly, Thomas rotations vanish also in Galilean relativity and, accordingly, the determination of the velocity parameter $\mathbf{u}$ by eq. (85) becomes explicit,

$$\mathbf{u} = \mathbf{w} - U\mathbf{v}, \tag{85b}$$

where $\mathbf{u}$, $\mathbf{w}$ and $U\mathbf{v}$ are vectors given by their components in Σ_w and where $\mathbf{v}$ is a vector given by its components in Σ_v, indicated in Fig. 5.

In general, we determine the velocity $\mathbf{u}$ from eqs. (85) and (88) that can be written as a pair of equations,

$$\begin{aligned} Q &= \mathrm{tom}[\mathbf{u}; U\mathbf{v}] \\ Q &= \mathrm{tom}[-QU\mathbf{v}; \mathbf{w}], \end{aligned} \tag{89}$$

where Q is a new unknown, replacing the unknown $\mathbf{u}$. The unique solution, Q, of the second equation in (89) can be obtained by successive approximations:

$$Q = \lim_{k\to\infty} Q_k \tag{90}$$

where

$$\begin{aligned} Q_0 &= I \\ Q_k &= \mathrm{tom}[-Q_{k-1}U\mathbf{v}; \mathbf{w}], \qquad k = 1, 2, 3, \cdots . \end{aligned} \tag{91}$$

The existence of the limit (90) for $|\mathbf{v}|, |\mathbf{w}| < c$ follows from the behavior of Thomas rotations, as shown in Fig. 1, and the inequality (31a): The sequence $\{Q_k\}_1^\infty$ represents a decreasing sequence of successive, alternating Thomas rotations, say $(-1)^k\varepsilon_k$, of the plane containing the velocity vectors $\mathbf{u}$ and $U\mathbf{v}$. When the decreasing sequence $\{\varepsilon_k\}$ converges to zero, $\varepsilon_k \to 0$, the limit Q exists and represents a rotation of the plane through an angle ε given by the equation $\varepsilon = \sum_{k=1}^{\infty}(-1)^k\varepsilon_k < \infty$. In a forthcoming paper we will show that $Q = \mathrm{tom}[\mathbf{w}; U\mathbf{v}]$.

The unknown velocity $\mathbf{u}$ of eq. (77) can now be expressed in terms of Q,

$$\mathbf{u} = \mathbf{w}*(-QU\mathbf{v}) = \mathbf{w}*(-\mathrm{tom}[\mathbf{w}; U\mathbf{v}]U\mathbf{v}), \tag{92}$$

as we see from eq. (85) and the first equation in (89). Finally, the velocity $\mathbf{u}$ of eq. (92) together with the given orientation U determine uniquely, by eq. (50), the required Lorentz transformation $L\{\mathbf{u}; U\}$ of eq. (77).

12. CONCLUSION

The Thomas rotation, $\mathrm{tom}[\mathbf{u}; \mathbf{v}]$, generated by two successive boosts $B(\mathbf{u})B(\mathbf{v})$, was expressed explicitly in terms of the two partaking boost velocity parameters $\mathbf{u}$ and $\mathbf{v}$, eq. (19), allowing simple expressions for its screw axis direction and for its rotation angle. Properties of Thomas rotations and boosts were presented in Sections 4-8, some of which were found useful in the parametrization of the Lorentz transformation group $L\{\mathbf{v}; V\}$ by means of two parameters, a velocity parameter $\mathbf{v}$ and an orientation parameter V, in such a way that composite Lorentz transformations correspond to parameter compositions, eq. (55),

$$L\{\mathrm{v}_1; V_1\}\, L\{\mathrm{v}_2; V_2\} = L\{\mathrm{v}_1 * V_1\mathrm{v}_2; \mathrm{tom}[\mathrm{v}_1; V_1\mathrm{v}_2]V_1V_2\}.$$

Finally, the techniques developed in this article were employed in Section 11 to determine the unique Lorentz transformation between the coordinates of a given event measured in two inertial frames with *given* relative orientation.

REFERENCES AND NOTES

1. L. H. Thomas, *Nature,* **117,** 514 (1926); *Phil. Mag.* **3,** 1 (1927).
2. Discussed by many authors, see for instance H. Goldstein[6]; W. H. Weihofen, *Am. J. Phys.* **43,** 39 (1975); S. Margulies, *Am. J. Phys.* **50,** 434 (1980); D. E. Fahnline, *Am. J. Phys.* **50,** 818 (1982); C. B. van Wyk, *Am. J. Phys.* **52,** 853 (1984); and A. C. Hirshfeld and F. Metzger, *Am. J. Phys.* **54,** 550 (1986).
3. N. Salingaros, J. Math. Phys. **27,** 157 (1986), and references therein.
4. E. P. Wigner, *Ann. Math.* **40,** 149 (1939). For some more refs. on the Wigner rotation, see for instance E. C. G. Sudarshan and N. Mukunda, *Classical Dynamics: A Modern Perspective,* (Wiley, New York, 1974), S. Gasiorowicz, *Elementary Particle Physics* (Wiley, New York, 1967), and refs. therein; and refs. 8,9,14,15. It seems that the term *Wigner rotation,* used by several authors to describe the rotation that we call *Thomas rotation,* was introduced into the English literature from German literature by Gasiorowicz. An objection to the use of this term to describe the Thomas rotation is expressed in n. 4 of ref. 14.
5. M. C. Møller, *The Theory of Relativity,* pp. 53-56 (Clarendon Press, Oxford, 1952).
6. H. Goldstein, *Classical Mechanics,* pp. 285-288, 2nd edn. (Addison-Wesley, Menlo-Park, California, 1980).
7. D. Hestenes, *Space-Time Algebra* (Gordon & Breach, New York, 1966).
8. C. B. van Wyk, *Am. J. Phys.* **52,** 853 (1984).
9. A. Ben-Menahem, *Am. J. Phys.* **53,** 62 (1985).
10. M. Rivas, M. A. Valle and J. M. Aguirregabiria, *Eur. J. Phys.* **7,** 1 (1986).
11. A. Chakrabarti, *J. Mat. Phys.* **5,** 1747 (1964); V. I. Ritus, *Soviet Phys. JETP* **13,** 240 (1961); H. P. Stapp, *Phys. Rev.* **103,** 425 (1956); and V. Lalan, *C. R. Acad. Sci. (Paris)* **236,** 2297 (1953).
12. V. S. Varadarajan, *Lie Groups, Lie Algebras and their Applications,* (Prentice-Hall, Engelwood Cliffs, 1974).
13. W. E. Baylis and G. Jones, *J. Mat. Phys.* **29,** 57 (1988).
14. D. Han, Y. S. Kim and D. Son, *J. Mat. Phys.* **27,** 2228 (1986).
15. K. Chen and C. Pei, *Chem. Phys. Lett.* **124,** 365 (1985).
16. See for instance, in addition to Goldstein,[6] a remark in the paragraph following eq. (39) in J. T. Cushing, *Am. J. Phys.* **35,** 858 (1967).
17. See, for instance, statement no. (3), 2nd paragraph, in P. S. Farago, *Am. J. Phys.* **35,** 246 (1967), according to which "The resultant of two Lorentz transformations in succession is different from the resultant of two Galilean transformations even in the approximation $v \ll c$." Farago needed this statement to explain why the angular velocity of the Thomas

rotation is not negligible even when it is associated with nonrelativistic velocities. The correct explanation follows from the fact that the angular velocity, ω_T, of the Thomas rotation need not be negligible even when v/c is negligible, due to the high accelerations that may be involved in orbital motions.

18. J. M. Lévy-Leblond, in *Group Theory and its Applications,* Vol. 2, E. M. Loebl ed. (Academic Press, New York, 1971), pp. 221-299, where additional references may be found.
19. J. Krause, *J. Mat. Phys.* **18,** 889 (1977).
20. D. M. Fradkin, *J. Mat. Phys.* **23,** 2520 (1982).
21. C. B. van Wyk, *J. Mat. Phys.* **27,** 1311 (1986).
22. M. C. Møller, *The Theory of Relativity,* p. 42, (Clarendon Press, Oxford, 1952). A simple derivation of the pure Lorentz transformation, in a vector form, may be found in W. Pauli, *Theory of Relativity,* p. 10, (Pergamon Press, New York, 1958). He mentions an earlier writer in whom the boost matrix $B(\mathbf{v})$ can be found: Equation (9) on p. 497 in G. Herglotz, *Ann. Phys. (Leipzig)* **36,** 393 (1911).
23. Calculations of the decomposition in eq. (11) can be found, for instance, in F. R. Halpern, *Special Relativity and Quantum Mechanics* (Prentice-Hall, Englewod Cliffs, NJ, 1968), Appendix 3; and in ref. 2, D. E. Fahnline. See also ref. 6, H. Goldstein, Prob. 13, p. 336 and refs. 8-11,13,14.
24. Citation from G. E. Uhlenbeck, *Phys. Today* **29,** 43 (June 1976).
25. See n. 4 in ref. 14.
26. Equations equivalent to eq. (13) for the Thomas rotation are common in the literature, see for instance, eq. (60) in M. C. Møller, *The Theory of Relativity,* p. 55, (Clarendon Press, Oxford, 1952). For further understanding of composite Lorentz transformations one must study properties of the Thomas rotation, tom[**u**; **v**], that are not readily obtainable from eq. (13).
27. The definition of the Thomas rotation in eq. (11) is identical with the definition of the Wigner rotation made by several authors;[3,7,9,10] see for instance eq. (11) in Rivas *et al.*[10] Objection for this use of the term *Wigner rotation* is expressed by Han, Kim, and Son.[25] Some authors define the Wigner (or Thomas or, simply, space) rotation slightly different, describing a composite boost as a boost *followed,* rather than *preceded,* by a Wigner rotation, as in Fahnline[2] and in Baylis and Jones.[13] This slightly different definitions of the Wigner rotation do not conflict, as seen from eq. (39) or from eq. (*xii*) of Section 6.
28. Elegant derivations of the *rhs* of eq. (19) corresponding to $\omega_0 \neq 0$ can be found in J. Mathew, *Am. J. Phys.* **44,** 1210 (1976), and in J. P. Fillmore, IEEE *Comp. Graph.* **4,** 30 (1984). See also A. E. Fekete, *Real Linear Algebra* (Dekker, New York, 1985) pp. 293 and 347 for a version attributed to N. E. Steenrod.
29. For the theory of Cartesian tensors see, for instance, G. Temple, *Cartesian Tensors* (Wiley, New York, 1960); H. Jeffreys *Cartesian Tensors,* (Cambridge Univ. Press, Cambridge, 1965); and E. C. Young, *Vector and Tensor Analysis* (Dekker, New York, 1978), Chap. 5.

30. See, for instance, R. H. Rand, *Computer Algebra in Applied Mathematics: An Introduction to MACSYMA,* (Pitman, Boston, 1984).
31. A. Ungar, The relativistic noncommutative nonassociative group of velocities and the Thomas rotation, to appear.
32. See, for instance, Ben Menahem(9) and G. P. Fisher, *Am. J. Phys.* **40,** 1772 (1972).
33. C. I. Mocanu, *Rev. Roum. Techn. - Electrotechn. Energ.* **30,** 119 and 367 (1985) and references therein.
34. For the use of quaternions to describe rotations see, for instance, L. Brand, *Vector and Tensor Analysis* (Wiley, New York, 1947), pp. 403-427, and L. A. Pars, *A Treatise on Analytical Dynamics,* (Wiley, New york, 1965), pp. 90-107.
35. J. Wittenburg, *Dynamics of Systems of Rigid Bodies,* (Teubner, Stuttgart, 1977), pp. 23-25.
36. For an excellent demonstration of the applicability of the quaternion group in modern physics and extensive relevant bibliography see P. R. Girard, *Eur. J. Phys.* **5,** 25 (1984).
37. J. L. Synge, *Relativity: The Special Theory,* (North-Holland, Amsterdam, 1967), 2nd ed., p. 79.
38. A. C. Hirshfeld and F. Metzger, *Am. J. Phys.* **54,** 550 (1986).
39. For some other elementary, interesting examples concerning one-parameter matrices see D. Kalman and A. Ungar, *Am. Math. Month.* **94,** 21 (1987), and D. Kalman, *Math. Mag.* **58,** 23 (1982).
40. For such a Galilean transformation in two space dimensions see, for instance, I. M. Yaglom, *A simple Non-Euclidean Geometry and its Physical Basis* (trans. by A. Shenitzer) (Springer, New York, 1979) p. 20 and ref. 18.
41. Y. S. Kim and M. E. Noz, *Theory and Applications of the Poincaré Group* (Reidel, Boston, 1986), p. 215.
42. The composition law in eq. (58) for the homogeneous Galilean transformation may be found, for instance, in eq. (2.8) of ref. 18; in eq. (*I*. 3) of J. M. Lévy-Leblond, *J. Mat. Phys.* **4,** 776 (1963); in Vilenkin(43); in Cornwell(43); and in J. Voisin, *J. Mat. Phys.* **6,** 1519 (1965).
43. See, for instance, N. J. Vilenkin, *Special Functions and the Theory of Group Representations,* (trans. V. N. Singh) (Amer. Math. Soc. Providence, Rhode Island, 1968), p. 197, and J. F. Cornwell, *Group Theory in Physics* (Academic Press, New York, 1984), Vol. I.
44. The need to consider an orientation parameter in addition to the velocity parameter in the parametrization of the Lorentz transformation in 1+3 dimensions is not well known; see for instance R. Skinner, *Relativity for Scientists and Engineers,* Dover, New York, 1982. In his eq. (1.194) and Figure 1.109, pp. 109-110, Skinner presents two successive Lorentz transformations parametrized by nonparallel velocities giving rise to an equivalent Lorentz transformation parametrized by velocity, thus ignoring the coordinate rotation involved.