DISCRETE STRESS-ENERGY TENSOR IN THE LOOP O(n) MODEL

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ABSTRACT. We study the loop O(n) model on the honeycomb lattice. By means of local non-planar deformations of the lattice, we construct a discrete stress-energy tensor. For $n \in [-2,2]$, it gives a new observable satisfying a part of Cauchy-Riemann equations. We conjecture that it is approximately discrete-holomorphic and converges to the stress-energy tensor in the continuum, which is known to be a holomorphic function with the Schwarzian conformal covariance. In support of this conjecture, we prove it for the case of n=1 which corresponds to the Ising model. Moreover, in this case, we show that the correlations of the discrete stress-energy tensor with primary fields converge to their continuous counterparts, which satisfy the OPEs given by the CFT with central charge c=1/2.

Proving the conjecture for other values of n remains a challenge. In particular, this would open a road to establishing the convergence of the interface to the corresponding ${\rm SLE}_{\kappa}$ in the scaling limit.

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1. Introduction

1.1. Loop O(n) model. Definitions, conjectures and known results. The loop O(n)model is one of natural generalisations of the Ising model introduced by Lenz in [38]. His student Ising showed in his thesis [29] that in one dimension the Ising-Lenz model exhibits no phase transition, assuming this to be the case in higher dimensions as well. Though phase transition in dimension $d \geq 2$ was eventually established by Peierls [45], the misunderstanding led to the introduction of other models, including the Heisenberg model [22], with spins taking values on the unit circle. The latter was generalized [51] to the spin O(n) model where the spins take values on the unit sphere in the n-dimensional space, $n \in \mathbb{Z}_+$. Besides the Heisenberg model for n=2, it contains the Ising model for n=1 as well. The spin O(n)model turned out to be very difficult to analyze, the main conjecture being the absence of the phase transition for n > 3, see [36] for the list of known results. On the hexagonal lattice the spin O(n) model was conjectured to be in the same universality class as the loop O(n)model introduced in [14], which is defined for all $n \in \mathbb{R}_+$, with some quantities making sense even for negative values of n. For more details on the relation between the two models and historical remarks see [15], where the case of a large n is considered. We are in position to define the loop O(n) model on the hexagonal lattice Hex_{δ} , where δ is the edge length.

Given a simply-connected domain Ω , we will use the following notation, see Fig. 1:

- $\mathcal{F}_{\delta}(\Omega)$ denotes the set of faces of Hex_{δ} entirely contained in Ω ;
- $\mathcal{E}_{\delta}(\Omega)$ denotes the set of edges contained in boundaries of faces in $\mathcal{F}_{\delta}(\Omega)$;
- $\mathcal{V}_{\delta}(\Omega)$ denotes the set of vertices of Hex_{δ} incident to edges in $\mathcal{E}_{\delta}(\Omega)$;
- $\partial \mathcal{V}_{\delta}(\Omega)$ denotes the set of vertices of $\mathcal{V}_{\delta}(\Omega)$ incident to exactly 2 edges in $\mathcal{E}_{\delta}(\Omega)$;
- Ω_{δ} denotes the subgraph of Hex_{δ} defined by $\mathcal{V}_{\delta}(\Omega)$ and $\mathcal{E}_{\delta}(\Omega)$ plus the half-edges incident to $\partial \mathcal{V}_{\delta}(\Omega)$.

We omit indices δ everywhere before we start speaking about the convergence as δ tends to 0. Consider $\mathfrak{b} = \{b_1, b_2, \dots, b_{2m}\}$ an even subset of boundary half-edges of Ω . The set of configurations $\operatorname{Conf}_{\Omega}(\mathfrak{b})$ is defined as the set of all even degrees subgraphs of Ω containing \mathfrak{b} and not containing any other boundary half-edges of Ω . It is easy to see that each configuration in $\operatorname{Conf}_{\Omega}(\mathfrak{b})$ consists of m paths linking boundary half-edges contained in \mathfrak{b} and possibly several loops. The weight of a configuration $\gamma \in \operatorname{Conf}_{\Omega}(\mathfrak{b})$ and the partition function are defined by

$$\begin{split} \mathbf{w}(\gamma) &= x^{\# \mathrm{edges}(\gamma)} n^{\# \mathrm{loops}(\gamma)}, \\ \mathcal{Z}_{\Omega}^{\mathfrak{b}} &= \sum_{\gamma \in \mathrm{Conf}_{\Omega}(\mathfrak{b})} \mathbf{w}(\gamma), \end{split}$$

where in $\#\text{edges}(\gamma)$ each half-edge contained in γ is counted as one half, parameters n and x are non-negative real numbers which can be chosen independently. We restrict our attention to $n \in [0,2]$ and $x = 1/\sqrt{2 + \sqrt{2 - n}}$ which is conjectured to be the critical value [42]. Two particularly interesting cases of the loop O(n) model are n = 0 and n = 1:

- If n=0 and $\mathfrak{b}=\{b,b'\}$, then configurations of $\mathrm{Conf}_{\Omega}(\mathfrak{b})$ are self-avoiding walks from b to b' in Ω because configurations with loops have zero weights. Main questions for this model remain wide open, one of the biggest conjecture being convergence of the distribution on walks to $\mathrm{SLE}_{8/3}$. The latter is known conditionally on the existence of the limiting measure and its conformal invariance [37]. Recently it was shown [17] that the connective constant of the self-avoiding walk on the hexagonal lattice is $\sqrt{2+\sqrt{2}}$, thus confirming the critical value of x for n=0. See [16, 3, 39] for a survey of known results about the self-avoiding walks.
- If n = 1, then the weight of a configuration depends only on the number of edges in it. This case corresponds to the Ising model, where several important results on the conformal invariance were obtained recently. For details about the correspondence and known results about the Ising model see section 1.3 and references therein.

Nienhuis [43] related the loop O(n) model to the Coulomb gas, see [41] for more details on the subject. This allowed him to predict the exact values of the universal critical exponents. Expressing the central charge corresponding to the loop O(n) model as a function of n, one obtains

$$c = (6 - \kappa)(3\kappa - 8)/2\kappa$$
, where $\kappa = 4\pi/(2\pi - \arccos(-n/2))$. (1.1)

In particular, this description would mean that for Dobrushin boundary conditions, i.e. $\mathfrak{b} = \{b, b'\}$, the interface from b to b' (see Fig. 1) converges to SLE_{κ} in the scaling limit (see [49, 48] for more details).

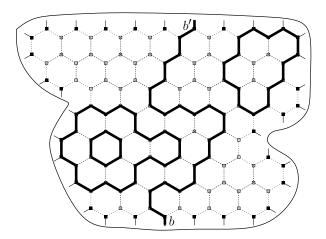


FIGURE 1. Domain Ω and graph Ω_{δ} . All vertices $\mathcal{V}_{\delta}(\Omega)$ are marked by black and white squares, where the black colour corresponds to the boundary vertices $\partial \mathcal{V}_{\delta}(\Omega)$. Edges $\mathcal{E}_{\delta}(\Omega)$ are depicted by dotted and bold lines. Bold edges form a configuration of the loop O(n) model with Dobrushin boundary conditions (two half-edges b and b' on the boundary). These boundary conditions impose the existence of a unique path linking b and b'. We call this path the interface.

The same conjecture can be obtained by considering the parafermionic observables introduced by the third author in the case of the FK cluster models in [47, 49]:

$$F_{\Omega}(b, z; \sigma) = \sum_{\gamma: b \to z} x^{\#\text{edges}(\gamma)} n^{\#\text{loops}(\gamma)} e^{-i\sigma \text{wind}(\gamma)}, \qquad (1.2)$$

where the sum is taken over the configurations of the loop O(n) model containing a path from the boundary half-edge b to a midpoint of an edge z, wind(γ) denotes the full rotation of this path as one goes from b to z and σ is a parameter. One knows the complex phase of the parafermionic observable on the boundary and for $\sigma = 1 - 3\arccos(-n/2)/4\pi$ the observable partially satisfies the discrete Cauchy-Riemann equations. This led the third author [49] to conjecture that the parafermionic observable, when properly normalised, converges to the unique holomorphic solution of the corresponding boundary value problem in the continuous domain:

$$\delta^{-\sigma} \cdot F_{\Omega}(b, z) / \mathcal{Z}_{\Omega}^{b, b'} \to \mathcal{C} \cdot (\varphi'/\varphi)^{\sigma},$$
 (1.3)

where \mathcal{C} is a lattice dependent constant and φ is a conformal map from Ω onto the upper half-plane. Note that the lefthand side in (1.3) is a martingale with respect to the growing interface and $(\varphi'/\varphi)^{\sigma}$ is a martingale with respect to SLE_{κ} for $\kappa = 3/(\sigma + 1/2)$ (one needs to consider slit domains, see [49]). Replacing σ by its expression in terms of n, one gets the same value of κ as the one derived via the Coulomb gas formalism.

The conjecture (1.3) was proven for the Ising model, i.e. n = 1. The work in this direction was originated by the third author [47] who proved the same statement for the FK Ising model. A series of papers [12, 8, 18, 32] by a group of authors led to [9] establishing convergence of the interface to the SLE₃. For other values of n the conjecture remains open, a certain progress being achieved in the case of the self-avoiding walks, i.e. n = 0, where the connective

constant was computed [17] using the parafermionic observable. Both the conjecture and a partial progress for certain models motivate the interest in other combinatorial observables satisfying local relations and having (conjectural) conformally covariant limits, especially in the case of the self-avoiding walks where the conformal invariance remains a big challenge for the probabilistic community.

1.2. Infinitesimal deformations of the discrete complex structure and the discrete stress-energy tensor. In this paper we suggest a new combinatorial observable obtained by evaluating the partition function after a local deformation of the lattice along the integrable line. To the best of our knowledge there is no known integrable way to define the loop O(n)model on a deformed hexagonal lattice by adding non-homogeneous weights on the edges. On the other hand, one can switch to the dual lattice and view the loop O(n) model as a model on the plane tiled with equilateral triangles. Gluing several pairs of adjacent triangles, we get a model on the plane tiled with lozenges with angles $\pi/3$ and $2\pi/3$. In each non-empty lozenge a local configuration is one of 4 types, and a weight of a lozenge depends only on the total length of arcs traversing it (see Fig. 3). Adding one more possible local configuration, Nienhuis discovered [44] a family of integrable weights of lozenges satisfying the Yang-Baxter equation, the weights being parametrized by the angle of a lozenge. Later it was pointed out by Cardy and Ikhlef [28] that for these weights the parafermionic observable still satisfies (as it does on the hexagonal lattice) a part of the Cauchy-Riemann equations; see [2] for a connection between the two approaches. This brings us to a definition of an integrable loop O(n) model on a Riemann surface tiled by rhombi and equilateral triangles and allows to deform each rhombus (see section 2 for details).

In particular, we will be interested in small perturbations of the original triangular tiling dual to the hexagonal lattice. Two examples that we are going to consider are: replacing two adjacent triangles by a rhombus with angles θ and $\pi - \theta$ for θ close to $\pi/3$; inserting a rhombus with angles θ and $\pi - \theta$ for a small θ . First deformation corresponds to the edges of the hexagonal lattice and second is defined for each midline of a hexagon (this is the place where an infinitesimal rhombus is inserted). Both times we calculate the partition function on a new lattice and get $\mathcal{Z}_{\Omega}^{\mathfrak{b}}(e,\theta)$ for an edge e and $\mathcal{Z}_{\Omega}^{\mathfrak{b}}(m,\theta)$ for a midline m of a certain hexagon. The real-valued observables are defined as derivatives of these partition functions:

$$\mathcal{T}_{\text{edge}}^{\mathfrak{b}}(e) := c_{\text{edge}} + \frac{d}{d\theta} \log \mathcal{Z}_{\Omega}^{\mathfrak{b}}(e, \theta) \big|_{\theta = \frac{\pi}{3}} \qquad \mathcal{T}_{\text{mid}}^{\mathfrak{b}}(m) := c_{\text{mid}} + \frac{d}{d\theta} \log \mathcal{Z}_{\Omega}^{\mathfrak{b}}(m, \theta) \big|_{\theta = 0}, \quad (1.4)$$

where constants c_{edge} , c_{mid} come from the infinite-volume limit of the model. For n=1 these constants can be computed explicitly, see eq. (1.5). Note that, alternatively, one can give a combinatorial definition of both observables, staying with the loop O(n) model on the hexagonal lattice and considering sums over the sets $\text{Conf}_{\Omega}^{[e]}(\mathfrak{b})$ and $\text{Conf}_{\Omega}^{[m]}(\mathfrak{b})$ of configurations with defects on e or m, see Fig. 5 and precise definition in Section 2.3. From this definition it is easy to see that for Dobrushin boundary conditions $\mathcal{T}_{\text{edge}}$ and \mathcal{T}_{mid} are martingales with respect to the growing interface.

Remark 1.1. In our definition of the loop O(n) model on a deformed lattice we keep the loop-weight equal to n and change only the edge-weight x. Note that this is consistent with standard considerations of the loop O(n) model on the cylinder, e.g. see [6, 5]. It is probably necessary to deform the loop-weight to understand better the relation with the Coulomb gas following the idea of Nienhuis [42] of describing the full weight of a configuration as product of local weights. This is another interesting question and we do not address it here. We

note that in this case the derivative of the partition function will get an additional term corresponding to varying the loop-weight. This term is closely related to the height function.

The real-valued observables $\mathcal{T}_{\text{edge}}^{\mathfrak{b}}$, $\mathcal{T}_{\text{mid}}^{\mathfrak{b}}$ describe the respond of (the partition function of) the loop O(n) model in a given discrete domain to an infinitesimal change of the discrete complex structure. We construct a complex-valued observable $\mathcal{T}^{\mathfrak{b}}$ which projections on certain directions are given by $\mathcal{T}_{\text{edge}}^{\mathfrak{b}}$ and $\mathcal{T}_{\text{mid}}^{\mathfrak{b}}$. In field theories the corresponding operator is called the stress-energy tensor. Hence, we refer to the observables $\mathcal{T}_{\text{edge}}^{\mathfrak{b}}$, $\mathcal{T}_{\text{mid}}^{\mathfrak{b}}$ as a discrete stress-energy tensor. According to the conformal field theory, the stress-energy tensor has the Schwarzian conformal covariance:

$$\langle T(w)O_1(\varphi(z_1))\dots O_k(\varphi(z_k))\rangle_{\Omega} = \left[(\varphi'(w))^2 \langle T(\varphi(w))O_1(\varphi(z_1))\dots O_k(\varphi(z_k))\rangle_{\Omega'} \right] + \frac{c}{12} \left[\mathcal{S}\varphi\right](w) \langle O_1(\varphi(z_1))\dots O_k(\varphi(z_k))\rangle_{\Omega'} \right] \prod_{j=1}^k |\varphi'(z_j)|^{\Delta_j},$$

where $[S\varphi](w) := \frac{\varphi'''(w)}{\varphi'(w)} - \frac{3}{2} \left[\frac{\varphi''(w)}{\varphi'(w)} \right]^2$ is the Schwarzian derivative of a conformal mapping $\varphi: \Omega \to \Omega'$, and $O_j(z_j)$ stands for a real-valued primary field at $z_j \in \Omega \setminus \{w\}$ with a scaling exponent Δ_j , and c is the central charge of the theory.

As we already mentioned above, the loop O(n) model is conjectured to have a conformally invariant scaling limit which can be described by the CFT with a central charge given by eq. (1.1). Hence, we conjecture that $\mathcal{T}^{b,b'}$, when properly normalized, converges to the unique martingale of SLE_{κ} which has the Schwarzian conformal covariance.

Conjecture T. There exists a lattice dependent constant C such that for $\sigma = 3/\kappa - 1/2$

$$\delta^{-2} \mathcal{T}^{\varnothing} \to \mathcal{C} \cdot \frac{c}{12} [\mathcal{S}\varphi](w) ,$$

$$\delta^{-2} \mathcal{T}^{b,b'} \to \mathcal{C} \cdot \left(\sigma \left[\frac{\varphi'(w)}{\varphi(w)} \right]^2 + \frac{c}{12} [\mathcal{S}\varphi](w) \right) .$$

The function $\mathcal{T}^{b,b'}$ is a martingale with respect to the growing interface from b to b'. Hence, the proof of the above conjecture would imply convergence of the interface to the SLE_{κ} . This is similar to the approach of the parafermionic observables proposed in [49]. As a strong support for this conjecture, we prove it for n=1 (Ising model), see section 1.3. For every $n \in [0,2]$, components of the discrete stress-energy tensor \mathcal{T} satisfy certain linear relations which can be interpreted as a part of the discrete Cauchy-Riemann relations (see section 2).

We note that on the boundary the conjecture for n=0 agrees with the restriction property of the ${\rm SLE}_{8/3}$. Indeed, in this case one has c=0 and Conjecture T means that $\mathcal{T}^{b,b'}$, after a proper rescaling, converges to $\frac{5}{8}|\varphi'/\varphi|^2 \cdot \delta^2$. On the boundary the latter is the probability to intersect the δ -neighbourhood of a boundary point. The boundary values of $\mathcal{T}^{b,b'}$ can be interpreted in a similar way. Indeed, by definition of $\mathcal{T}^{b,b'}$ given in section 2.3, only paths touching a boundary edge contribute to the value of $\mathcal{T}^{b,b'}$ at it.

1.3. Convergence results for the Ising model. Let us now discuss the special case n = 1, which corresponds to the *Ising model* with spins assigned to the *faces* of a discrete domain Ω . This correspondence works as follows: given a configuration $\gamma \in \operatorname{Conf}_{\Omega}(\mathfrak{b})$ with the boundary conditions $\mathfrak{b} = \{b_1, \ldots, b_{2m}\}$, one puts spins ± 1 on faces of Ω (including boundary ones)

so that two spins at adjacent faces are different if and only if their common edge belongs to γ , with boundary spins being +1 everywhere except along the counterclockwise boundary arcs $(b_{2k-1}b_{2k})$ (where, on the contrary, they are fixed to be -1). Below we also use the notation '+' instead of \varnothing for the empty boundary conditions in order to emphasize this choice of the sign of boundary spins. It is easy to see that the partition functions of the loop O(1) model discussed above can be written as

$$\begin{split} \mathcal{Z}_{\Omega}^{\mathfrak{b}} &= \sum_{\sigma \in \operatorname{Conf}_{\Omega}^{\operatorname{spin}}(\mathfrak{b})} x_{\operatorname{crit}}^{\#\{u \sim w: \, \sigma_u \neq \sigma_w\} - \frac{1}{2} |\mathfrak{b}|} \\ &= x_{\operatorname{crit}}^{\frac{1}{2}(|E(\Omega)| - |\mathfrak{b}|)} \sum_{\sigma \in \operatorname{Conf}_{\Omega}^{\operatorname{spin}}(\mathfrak{b})} \exp\left[\, \frac{1}{2} \log x_{\operatorname{crit}} \cdot \sum_{u \sim w} \sigma_u \sigma_w\,\right], \end{split}$$

where $x_{\rm crit} = 1/\sqrt{3}$, we use the notation ${\rm Conf}_{\Omega}^{\rm spin}(\mathfrak{b})$ for the set of all possible *spin* configurations on the faces of Ω with boundary conditions \mathfrak{b} , and $\#\{u \sim w : \sigma_u \neq \sigma_w\}$ is the number of pairs of adjacent faces of Ω carrying opposite spins. The value $x_{\rm crit} = 1/\sqrt{3}$ is known to be critical for the Ising model on faces of the regular honeycomb lattice, see [52].

It is well-known in the theoretical physics context (e.g., see [13] or [41]) that the limit of the critical Ising model as $\delta \to 0$ can be described by the Conformal Field Theory (of a Free Fermionic Field) with the central charge $c=\frac{1}{2}$. Before 2000s, this passage was usually considered in the infinite-volume (or half-infinite-volume) setup. During the last decade new techniques appeared, which led to a number of rigorous convergence results (confirming the existing CFT predictions) for various correlators in the critical Ising model considered on discrete approximations Ω_{δ} to a general planar domain Ω . The first results of this kind were obtained by the third author in [49, 47] for holomorphic observables in the so-called FK-Ising model (also known as the random cluster representation of the Ising model). In a similar manner, one can use the combinatorial definition (1.2) in the case n=1 (and $\sigma=1/2$) to view these observables as solutions to some well-posed discrete Riemann-type boundary value problems in Ω_{δ} . This opens a way for the treatment of their limits as $\delta \to 0$ in general planar domains Ω . Further developments of the techniques proposed in [49, 47] include the convergence results for the basic fermionic observables [8], energy density correlations [25, 23] and spin correlations [11, 10], the latter relying upon the convergence of discrete spinor observables, see Section 3.4 below for their definition.

For the Ising model we use the following notation for the limits from the Conjecture T (substituting c = 1/2):

$$\langle T(w) \rangle_{\Omega}^{+} := \frac{[\mathcal{S}\varphi](w)}{24}, \qquad \langle T(w) \rangle_{\Omega}^{b,b'} := \frac{1}{2} \left[\frac{\varphi'(w)}{\varphi(w)} \right]^{2} + \frac{[\mathcal{S}\varphi](w)}{24},$$

where $\varphi: \Omega \to \mathbb{H}$ is an arbitrary uniformization mapping in the first case and any one of those sending $\{b,b'\}$ to $\{0,\infty\}$ in the second. Moreover, in this case the constants c_{edge} and c_{mid} in the definition (1.4) of $\mathcal{T}_{\text{edge}}$ and \mathcal{T}_{mid} can be computed explicitly and take the following values:

$$c_{\text{edge}} = -\frac{1}{2\pi} + \frac{1}{4\sqrt{3}}, \qquad c_{\text{mid}} = -\frac{1}{\pi} + \frac{1}{3\sqrt{3}}.$$
 (1.5)

We now formulate the first of our convergence results. For technical reasons, in the case of Dobrushin boundary conditions, we assume that the boundary points lie on straight parts of the boundary of Ω which are orthogonal to the edges of the lattice approximation, see also Remark 1.2.

Theorem 1.1. Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain, $w \in \Omega$ and $b, b' \in \partial \Omega$ two points on the boundary. Let Ω_{δ} be a discrete approximation to Ω on the honeycomb grid \mathbb{C}_{δ} with mesh size δ , the boundary (half-)edges b_{δ} and b'_{δ} approximate b and b', and w_{δ} be the face of Ω_{δ} that contains w. Then, for any fixed direction $\tau \in \{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\}$ of edges of \mathbb{C}_{δ} , one has

$$\delta^{-2} \mathcal{T}_{\text{edge}}^{+}(e_{\delta}) \implies \frac{3}{\pi} \operatorname{Re}[\tau^{2} \langle T(w) \rangle_{\Omega}^{+}], \qquad \delta^{-2} \mathcal{T}_{\text{mid}}^{+}(m_{\delta}) \implies \frac{3}{\pi} \operatorname{Re}[(i\tau)^{2} \langle T(w) \rangle_{\Omega}^{+}],$$

$$\delta^{-2} \mathcal{T}_{\text{edge}}^{b_{\delta}, b'_{\delta}}(e_{\delta}) \implies \frac{3}{\pi} \operatorname{Re}[\tau^{2} \langle T(w) \rangle_{\Omega}^{b, b'}], \qquad \delta^{-2} \mathcal{T}_{\text{mid}}^{b_{\delta}, b'_{\delta}}(m_{\delta}) \implies \frac{3}{\pi} \operatorname{Re}[(i\tau)^{2} \langle T(w) \rangle_{\Omega}^{b, b'}],$$

as $\delta \to 0$, where e_{δ} is any of the two edges of w_{δ} oriented in the direction τ , m_{δ} is the midline of w_{δ} orthogonal to e_{δ} , and the convergence is uniform on compact subsets of Ω .

Remark 1.2. The assumption about the orthogonality of the half-edges b_{δ} and b'_{δ} to the straight parts of the boundary $\partial\Omega$ can be relaxed using the results from [8]. Moreover, the techniques developed in [24] allow to relax these technical assumptions even further and prove a similar statement in the case of rough boundaries.

Remark 1.3. We use the following strategy to prove Theorem 1.1. It is not hard to see from the combinatorial definition of $\mathcal{T}^{\varnothing}_{\mathrm{mid}}(m)$ that these quantities can be expressed via the values of discrete fermionic observables defined by (1.2), where n=1 and $\sigma=1/2$, see Section 3.2 for more details. More generally, for boundary conditions \mathfrak{b} with $|\mathfrak{b}|=2m$ one can exploit the fermionic nature of these observables and show that $\mathcal{T}^{\mathfrak{b}}_{\mathrm{mid}}(m)$ admits an expression as a Pfaffian of some $2(m+1)\times 2(m+1)$ matrix with entries given by the values of the fermionic observables; see Section 3.3, where the case of Dobrushin boundary conditions $\mathfrak{b}=\{b,b'\}$ is treated. Therefore, one can use existing discrete complex analysis techniques to derive Theorem 1.1 from the (known in the case n=1) convergence results for the fermionic observables. In continuum, the fact that the stress-energy tensor $T=-\frac{1}{2}:\psi\partial\psi:$ can be expressed via fermions is a particular feature of the corresponding CFT, and there is no surprise that the similar discrete quantities converge to their putative limits as $\delta \to 0$. Nevertheless, it is worth noting that some additional terms appear when working with the geometrical definition given by (1.4). These additional terms have bigger scaling exponents and thus cancel out in the limit $\delta \to 0$, see Remark 3.2 and Proposition 3.7 for more details.

Let us now discuss the limits of correlation functions of the two discrete stress-energy tensors considered at distinct points of Ω . Though these quantities can be introduced in the general loop O(n) model context by considering several spatially separated infinitesimal deformations of the underlying lattice simultaneously, we prefer to choose another way in the special case n=1 corresponding to the Ising model. Namely, in Section 3.1 we show that there exist local fields (polynomials of several nearby spins) $T_{\text{mid}}(m)$ such that

$$\mathcal{T}_{\mathrm{mid}}^{\mathfrak{b}}(m_{\delta}) = \mathbb{E}_{\Omega_{\delta}}^{\mathfrak{b}}[T_{\mathrm{mid}}(m_{\delta})] \tag{1.6}$$

for any boundary conditions \mathfrak{b} and all edges e_{δ} and midlines m_{δ} of a discrete domain Ω_{δ} . Given a face w_{δ} of Ω_{δ} , we then define the complex-valued local field

$$T(w_{\delta}) := -\frac{2}{3} \sum_{\eta \in \{1, e^{\frac{\pi i}{3}}, e^{\frac{2\pi i}{3}}\}} \overline{\eta}^{4} T_{\text{mid}}(w_{\delta}^{[\eta]}),$$

where $w_{\delta}^{[\eta]}$ denotes the midline of w_{δ} that is orthogonal to the direction $\tau = \eta^2$, see Section 3.1 for precise definitions. Note that Theorem 1.1 easily implies the following:

$$\delta^{-2}\mathbb{E}_{\Omega_{\delta}}^{+}[T(w_{\delta})] \ \, \rightrightarrows \ \, \frac{3}{\pi}\langle T(w)\rangle_{\Omega}^{+} \quad \text{and} \quad \delta^{-2}\mathbb{E}_{\Omega_{\delta}}^{b_{\delta},b_{\delta}'}[T(w_{\delta})] \ \, \rightrightarrows \ \, \frac{3}{\pi}\langle T(w)\rangle_{\Omega}^{b,b'} \, .$$

In principle, the techniques discussed in Remark 1.3 allow one to generalize these convergence results to all the multi-point expectations $\mathbb{E}_{\Omega_{\delta}}^{\mathfrak{b}_{\delta}}[T(w_{\delta})T(w_{\delta}')T(w_{\delta}'')\dots]$. For shortness, we do not address such a general question in this paper and restrict ourselves to the convergence of two-point expectations with '+' boundary conditions. Together with the proof of Theorem 1.1, this already contains all necessary ingredients to treat the general situation.

Theorem 1.2. Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain and w, w' be two distinct points of Ω . Let Ω_{δ} be a discrete approximation to Ω on the honeycomb grid \mathbb{C}_{δ} with mesh size δ , and w_{δ}, w'_{δ} be the faces of Ω_{δ} that contain w and w', respectively. Then, one has

$$\delta^{-4}\mathbb{E}_{\Omega_{\delta}}^{+}[T(w_{\delta})T(w_{\delta}')] \implies \frac{9}{\pi^{2}}\langle T(w)T(w')\rangle_{\Omega}^{+}$$

as $\delta \to 0$, where $\langle T(w)T(w')\rangle_{\Omega}^+$ are holomorphic functions of $w, w' \in \Omega$ defined in Section 4.2. The convergence is uniform provided w and w' are separated from each other and from $\partial\Omega$.

Another natural question to ask is the convergence of mixed correlations of $T(w_{\delta})$ and other local fields. The two most important examples are the spin $\sigma(u_{\delta})$ and the energy density

$$\varepsilon(a_{\delta}) := \sigma(a_{\delta}^+)\sigma(a_{\delta}^-) - \frac{2}{3},$$

where a_{δ}^{+} and a_{δ}^{-} denote the two faces adjacent to an edge a_{δ} of Ω_{δ} . Note that the constant $\frac{2}{3}$ in the last definition is lattice dependent and corresponds to the infinite-volume limit of the Ising model (e.g., it should be replaced by $\sqrt{2}/2$ when working on the square lattice, see [25]).

Theorem 1.3. Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain, $w \in \Omega$ and $u, a \in \Omega \setminus \{w\}$. Let Ω_{δ} be a discrete approximation to Ω on the honeycomb grid \mathbb{C}_{δ} with mesh size δ , and w_{δ}, u_{δ} be the faces of Ω_{δ} that contain w and u, respectively, while a_{δ} denotes the closest to the point a edge of Ω_{δ} . Then, one has

$$\delta^{-2} \frac{\mathbb{E}_{\Omega_{\delta}}^{+}[T(w_{\delta})\varepsilon(a_{\delta})]}{\mathbb{E}_{\Omega_{\delta}}^{+}[\varepsilon(a_{\delta})]} \implies \frac{3}{\pi} \cdot \frac{\langle T(w)\varepsilon(a)\rangle_{\Omega}^{+}}{\langle \varepsilon(a)\rangle_{\Omega}^{+}}, \quad \delta^{-2} \frac{\mathbb{E}_{\Omega_{\delta}}^{+}[T(w_{\delta})\sigma(u_{\delta})]}{\mathbb{E}_{\Omega_{\delta}}^{+}[\sigma(u_{\delta})]} \implies \frac{3}{\pi} \cdot \frac{\langle T(w)\sigma(u)\rangle_{\Omega}^{+}}{\langle \sigma(u)\rangle_{\Omega}^{+}}$$

as $\delta \to 0$, where the functions $\langle T(w)\varepsilon(a)\rangle_{\Omega}^+$, $\langle \varepsilon(a)\rangle_{\Omega}^+$, $\langle T(w)\sigma(u)\rangle_{\Omega}^+$ and $\langle \sigma(u)\rangle_{\Omega}^+$ are defined in Sections 4.2 and 4.3. The convergence is uniform provided a, u and w are separated from each other and from $\partial\Omega$.

Remark 1.4. (i) In the square lattice setup, the convergence of the energy density expectations $\delta^{-1}\mathbb{E}_{\Omega_{\delta}}[\varepsilon(a_{\delta})]$ to their CFT limits $\mathcal{C} \cdot \langle \varepsilon(a) \rangle_{\Omega}^{+}$ was proved in [25]. Along the way, we also prove a version of this result with the lattice-dependent constant $\mathcal{C} = \sqrt{3}/\pi$, see Corollary 5.3.

(ii) In the square lattice setup, the convergence of the spin expectations $\delta^{-\frac{1}{8}}\mathbb{E}_{\Omega_{\delta}}[\sigma(u_{\delta})]$ to their CFT limits $\mathcal{C} \cdot \langle \sigma(u) \rangle_{\Omega}^{+}$ was proved in [10]. We do not discuss the possible generalization of this result to the honeycomb lattice in our paper. Nevertheless, let us mention that the proof given in [10] can be adapted to our case in a rather straightforward manner.

The methods developed in [8, 25, 23, 30, 10] and this paper can be used to generalize Theorem 1.3 and to treat the convergence of all mixed correlations of the local fields σ , ε and T, as well as its anti-holomorphic counterpart \overline{T} , to their scaling limits. Note that one can define all these limits without any reference to the CFT means, using the language of solutions to Riemann-type boundary value problems for holomorphic functions in Ω instead. To illustrate this route, we devote Section 4 to a self-contained exposition of a number of such boundary value problems, which allow one to define all the CFT correlation functions $\langle \dots \rangle_{\Omega}^+$ involved

into Theorems 1.1–1.3. Moreover, in Section 4.4 we show that the Schwarzian conformal covariance of the stress-energy tensor expectations $\langle T(w)\rangle_{\Omega}^{\mathfrak{b}}$ and the particular structure of singularities of the correlation functions $\langle T(w)T(w')\rangle_{\Omega}^{+}$, $\langle T(w)\varepsilon(a)\rangle_{\Omega}^{+}$ and $\langle T(w)\sigma(u)\rangle_{\Omega}^{+}$ (which comes from the Operator Product Expansions in the standard CFT approach) can be easily deduced from simple properties of solutions to these boundary value problems.

It is worth noting that one can construct other local fields that (conjecturally) have the CFT stress-energy tensor as a scaling limit. For the critical Ising model on the square lattice, a simple linear combination of spin-spin expectations was suggested for this purpose by Kadanoff and Ceva [31], see also the work of Koo and Saleur [35] for a generalization of this construction to other critical integrable lattice models. A discrete stress-energy tensor for the three-state Potts model was analyzed by Alicea, Clark, Fendley, Lindea and Mong [40], see also a recent preprint [1] where an algebraic approach to topological defects on the lattice is developed.

There is an independent ongoing project by Hongler, Kÿtölä and Viklund, devoted to the construction of the Virasoro algebra on the lattice for the Gaussian Free Field [26] and the Ising model [27]. The discrete stress-energy tensor for the Ising model is under a separate investigation by Benoist and Hongler [4]. There are certain similarities between their definition and the one that we provide in the current paper, as both times the stress-energy tensor is obtained by evaluating the partition function after a particular deformation of the lattice. The difference is that in [4], the whole line of rhombi is deformed, thus preserving the flat structure, whereas in our approach we consider a local non-planar deformation. Finally, there is an ongoing project [21] by Hongler and the second author, where it is argued that the stress-energy tensor on the boundary can be identified with the energy density.

1.4. Organization of the paper. In Section 2 we define the discrete stress-energy \mathcal{T} for the loop O(n) model for $n \in [0,2]$ and show that it satisfies a part of the Cauchy-Riemann equations. We start by defining the integrable loop O(n) model on a Riemann surface tiled by rhombi and equilateral triangles. Then we give two definitions of real components $\mathcal{T}_{\text{edge}}$ and \mathcal{T}_{mid} of the stress-energy tensor: geometrical by means of local deformations of the lattice and combinatorial by summing weights of configurations on the hexagonal lattice with particular coefficients. We show how to combine $\mathcal{T}_{\text{edge}}$ and \mathcal{T}_{mid} in a complex-valued stress-energy tensor satisfying a part of the Cauchy-Riemann equations. We conclude this Section by an informal discussion of separate conical singularities, a linear combination of which gives \mathcal{T} .

In Section 3 we discuss representations of \mathcal{T} for the Ising model (n=1) via discrete fermions and spin expectations. Similar representations are obtained for the correlations of \mathcal{T} and other discrete fields (spin and energy density). On the way we recall the definition of discrete fermions and spinors and discuss their properties.

Section 4 is devoted to the discussion of continuous correlation functions. We show how to construct all of them using continuous fermions and spinors. Thus defined correlations satisfy the same conformal covariance and OPEs as the corresponding correlations from the CFT. In particular the stress-energy tensor has a Schwarzian conformal covariance.

In Section 5 we prove convergence results for the Ising model (n = 1).

Appendix contains computations of coefficients in the combinatorial definition of \mathcal{T} and a construction of the full-plane fermionic observable on the hexagonal lattice in the Ising case.

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2. Definition of the discrete stress-energy tensor in the loop O(n) model

In this section we define the loop O(n) model on a Riemann surface glued from rhombi and equilateral triangles. This allows us to describe two sets of infinitesimal deformations of the honeycomb lattice: for each edge and for each midline of a face. Real-valued observables $\mathcal{T}_{\text{edge}}$ and \mathcal{T}_{mid} are defined as the logarithmic derivatives of the partition function under these deformations. In a sense that will be explained later, these real-valued observables can be regarded as projections of a discrete stress-energy tensor. Then we give an alternative definition of $\mathcal{T}_{\text{edge}}$ and \mathcal{T}_{mid} using only the honeycomb lattice and prove that they satisfy some linear relations that can be interpreted as a part of discrete Cauchy-Riemann equations. Further, we define two candidates \mathcal{T} and \mathcal{T}_{CR} for the discrete holomorphic stress-energy tensor as linear combinations of their "projections" $\mathcal{T}_{\text{edge}}$ and \mathcal{T}_{mid} . For n=1, both functions \mathcal{T} and \mathcal{T}_{CR} have the same limit, also together with correlations, and discrete contour integrals of \mathcal{T}_{CR} around vertices vanish. In the end of the section we argue that the described infinitesimal deformations can be viewed as the insertion of several infinitesimal conical singularities.

2.1. The loop O(n) model on discrete Riemann surfaces. In this subsection we define the loop O(n) model on a Riemann surface glued from rhombi and equilateral triangles (later on, this will allow us to consider infinitesimal deformations of the honeycomb lattice). The definition is chosen in such a way that the partition function is independent of a particular way to tile the surface by rhombi and triangles.

We take any finite connected planar graph G, whose faces (except the exterior one) has either 3 or 4 edges. Then we glue a Riemann surface from G in such a way that all edges have the same length, i.e. all triangles are equilateral and all quadrangles are rhombi. Let us denote this Riemann surface by G_{δ} , where δ is the length of each of the edges. Surface G_{δ} might have some conical singulaties at vertices, but apart from them it is locally flat. A configuration of the loop O(n) model is an ensemble of loops, in each triangle and rhombus looking as depicted in Fig. 2.

Depending on whether a configuration has an arc inside a triangle T, the weight of T is either 1 or x (see Fig. 2). And depending on a configuration inside a rhombus R with angles θ and $\pi - \theta$, the weight of R takes one of the values: $u_1(\theta)$, $u_2(\theta)$, $v(\theta)$, $w_1(\theta)$, $w_2(\theta)$ or 1 (see Fig. 2). For a face f (either a triangle or a quadrangle), we denote this weight by w(f). The weight of a configuration is then defined as the product of local weights times the topoligical factor counting the number of loops:

$$\mathbf{w}(\gamma) = \prod_{f-face} \mathbf{w}(f) \cdot n^{\text{\#loops}}.$$

Note that the weight of a configuration depends on the angles of rhombi chosen when we glue the Riemann surface G_{δ} from the graph G. So for the same graph G we get different models

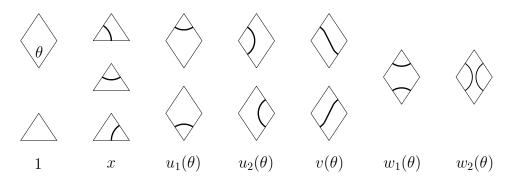


FIGURE 2. Possible local configurations inside the equilateral triangle and a rhombus with angles θ and $\pi - \theta$ and their weights.

depending on our choices of angles when we glue the Riemann surface G_{δ} . Below we will vary one of these angles and look at how the partition function changes.

One can also add into the consideration boundary conditions $\mathfrak{b} = (b_1, b_2, \dots, b_{2k-1}, b_{2k})$, where b_1, \dots, b_{2k} are the midpoints of edges on the boundary of G_{δ} . A configuration of the loop O(n) model with boundary conditions \mathfrak{b} is an ensemble of loops and paths joining b_1 to b_2, \dots, b_{2k-1} to b_{2k} . The partition function $Z^{\mathfrak{b}}(G_{\delta})$ is the sum of weights of all these configurations.

The only missing ingredient to finish the definition of the loop O(n) model on G_{δ} is the weight x of a triangle with an arc and the family of weights $(u_1, u_2, v, w_1, w_2)_{\theta}$ parametrized by the angle θ of a rhombus. From now on, we will consider only the case $n \in [0, 2]$. Let s be such that $n = -2\cos\frac{4\pi}{3}s$. We take $x = \frac{1}{2\cos\frac{\pi}{3}s}$ and consider the following family of weights parametrized by s and the angle θ of a rhombus:

$$u_1 = \frac{1}{t} \cdot \sin[(\pi - \theta)s] \cdot \sin[\frac{2\pi}{3}s]$$
 (2.1)

$$u_2 = \frac{1}{t} \cdot \sin[\theta s] \cdot \sin[\frac{2\pi}{3}s] \tag{2.2}$$

$$v = \frac{1}{t} \cdot \sin[\theta s] \cdot \sin[(\pi - \theta)s] \tag{2.3}$$

$$w_1 = \frac{1}{t} \cdot \sin\left[\left(\frac{2\pi}{3} - \theta\right)s\right] \cdot \sin\left[\left(\pi - \theta\right)s\right] \tag{2.4}$$

$$w_2 = \frac{1}{t} \cdot \sin[(\theta - \frac{\pi}{3})s] \cdot \sin[\theta s], \tag{2.5}$$

where

$$t = \frac{\sin^3[\frac{2\pi}{3}s]}{\sin[\frac{\pi}{3}s]} + \sin[(\theta - \frac{\pi}{3})s] \cdot \sin[(\frac{2\pi}{3} - \theta)s].$$

This family was first discovered by Nienhuis [44] as a solution to the Yang-Baxter equation. Then, it was proved by Cardy and Ikhlef [28], that for these weights the parafermionic observable (1.2) satisfies a half of Cauchy-Riemann equations. There were minor misprints in both works, here we provide a correct version from [20].

Remark 2.1. Note that this definition is preserved under relabelling of the angles of a rhombus, i.e. when θ is changed to $\pi - \theta$. Indeed, the following holds:

$$(u_1, u_2, v, w_1, w_2)_{\theta} = (u_2, u_1, v, w_2, w_1)_{\pi - \theta}$$

Importantly, thus defined loop O(n) model on a Riemann surface does not depend on the way how this surface is tiled by rhombi and equilateral triangles (sometimes there are several possible ways to do that). To be precise, let us introduce the following definition.

Definition 2.1. Assume that there are two different ways to tile G_{δ} , denote these tilings by T and T'. We say that the loop O(n) model is stable under replacing tiling T by tiling T' if the ratio of the partition functions on G_{δ} tiled according to T and according to T' is the same for any boundary conditions.

One can also consider local changes of a tiling. We extend the definition of *stability* of the model to this case in a straightforward way by considering any boundary conditions on the border of a replaced part.

The following local changes are of most importance for the construction of a discrete stress-energy tensor:

- Yang-Baxter transformation. Consider a centrosymmetric equilateral hexagon. There are two different ways to tile it by 3 rhombi (see Fig. 4).
- Pentagonal transformation. Consider a convex pentagon formed by an equilateral triangle and a rhombus. There is a way to tile it by an equilateral triangle and two rhombi (see Fig. 4).
- One can split a rhombus of angle $\frac{\pi}{3}$ into two equilateral triangles (see Fig. 3).

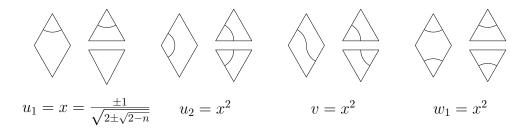


FIGURE 3. The bijection between configurations in a $\frac{\pi}{3}$ -rhombus and two equilateral triangles. The signs in the value of x depend on the choice of s. A rhombus with two $\frac{2\pi}{3}$ -arcs does not occur, i.e. $w_2(\frac{\pi}{3}) = 0$.

Proposition 2.2. The loop O(n) model on a Riemann surface defined by weights (2.1)-(2.5) is stable under the Yang-Baxter and pentagonal transformations and splitting a rhombus into two equilateral triangles. Moreover, the partition function is preserved under the first and the third transformations, and it gets multiplied by $c(\theta)$ when the number of vertices is increased by the pentagonal transformation, where the angle θ is shown in Fig. 4 and

$$c(\theta) = 1 + n \cdot u_2(\theta) u_2(\frac{\pi}{3} - \theta) x. \tag{2.6}$$

Proof. The proof for splitting a $\frac{\pi}{3}$ -rhombus into two triangles is shown in Fig. 3.

The Yang-Baxter equation was investigated in [44]. One can derive it by considering all possible boundary conditions. Each time stability of the model is equivalend to a non-linear equation on the weights. For instance, an example given in Fig. 4 leads to the equation

$$u_2(\phi)w_2(\psi)v(2\pi - \phi - \psi) = v(\phi)u_1(\psi)u_1(2\pi - \phi - \psi) + u_2(\phi)v(\psi)w_1(2\pi - \phi - \psi).$$

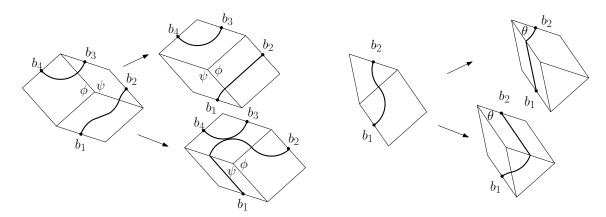


FIGURE 4. Left: A hexagon tiled by 3 rhombi in two different ways and all possible configurations with the boundary conditions $\mathfrak{b} = \{b_1, b_2, b_3, b_4\}$. Right: A pentagon tiled by the equilateral triangle and 2 or 3 rhombi, boundary conditions $\mathfrak{b} = \{b_1, b_2\}$.

These relations also can be derived from the discrete holomorphicity property of the parafermionic observable, see [2]. The latter was established in [28].

The stability under the pentagonal transformation can be checked in the same way. E.g., an example of boundary conditions shown in Fig. 4 leads to the equation

$$u_1(\theta)v(\frac{\pi}{3}-\theta)+v(\theta)u_1(\frac{\pi}{3}-\theta)x=u_1(\frac{\pi}{3}+\theta)x\cdot(1+n\cdot u_2(\theta)u_2(\frac{\pi}{3}-\theta)x).$$

Remark 2.2. 1) To the best of our knowledge, this is the first time when the pentagonal relation on the weights of the loop O(n) model appears in the literature. One can also generalise it to a non-convex pentagon. In this case one of the rhombi will have a negative angle.

- 2) It is natural to conjecture that the model is stable under any change of tiling of the same surface (also allowing negative angles) but we do not address this question here.
- 2.2. Infinitesimal deformations and definition of \mathcal{T}_{edge} and \mathcal{T}_{mid} . In this subsection we describe two sets of infinitesimal deformations of the honeycomb lattice: for each edge and for each midline of a face (i.e. an interval between the midpoints of the opposite edges). Real-valued observables \mathcal{T}_{edge} and \mathcal{T}_{mid} are defined as the logarithmic derivatives of the partition function under these deformations. In a sense that will be explained later, these observables can be regarded as projections of a discrete stress-energy tensor.

Given a simply connected domain Ω on the plane and $\delta > 0$, denote by Ω_{δ} a part (i.e. the union of faces) of the hexagonal lattice $\operatorname{Hex}_{\delta}$ of the mesh-size δ contained in Ω . By $\operatorname{Tri}_{\delta}$ denote the dual triangular lattice to $\operatorname{Hex}_{\delta}$. Consider all triangles of $\operatorname{Tri}_{\delta}$ covering vertices of Ω_{δ} and glue any two adjacent ones if the corresponding vertices of Ω_{δ} . The resulting graph is denoted by $\Omega_{\delta}^{\operatorname{dual}}$; note that two different vertices of $\Omega_{\delta}^{\operatorname{dual}}$ can correspond to the same point on the plane. Below in this subsection we will often omit δ and write just Ω and $\Omega_{\delta}^{\operatorname{dual}}$ instead of Ω_{δ} and $\Omega_{\delta}^{\operatorname{dual}}$, if no confusion arises.

Consider an edge e of Ω . Replace in Ω^{dual} two triangles corresponding to the endpoints of e by a rhombus of angle θ and denote the resulting graph by $\Omega^{\text{dual}}_{\text{edge}}(e;\theta)$. By Proposition 2.2 we know that $Z(\Omega) = Z(\Omega^{\text{dual}}) = Z(\Omega^{\text{dual}}_{\text{edge}}(e;\frac{\pi}{3}))$.

Now consider a midline m of a face of Ω (i.e. an interval between the midpoints of the opposite edges). Clearly, it is contained in two edges of Ω^{dual} . Insert in the place of these two edges in Ω^{dual} a rhombus of angle θ and denote the resulting graph by $\Omega^{\text{dual}}_{\text{mid}}(m;\theta)$. One can easily check that $Z(\Omega) = Z(\Omega^{\text{dual}}) = Z(\Omega^{\text{dual}}_{\text{mid}}(m;0))$.

Definition 2.3. Let e be an edge of Ω and m be a midline of a face of Ω . Choose any boundary conditions \mathfrak{b} for the loop O(n) model. Functions $\mathcal{T}^{\mathfrak{b}}_{\mathrm{edge}}(e)$ and $\mathcal{T}^{\mathfrak{b}}_{\mathrm{mid}}(m)$ are defined as logarithmic derivatives of $Z(\Omega)$:

$$\begin{split} \mathcal{T}_{\text{edge}}^{\mathfrak{b}}(e) &= c_{\text{edge}} + \frac{\partial}{\partial \theta} \left[\log Z(\Omega_{\text{edge}}^{\text{dual}}(e;\theta)) \right]_{\theta = \frac{\pi}{3}} \\ \mathcal{T}_{\text{mid}}^{\mathfrak{b}}(m) &= c_{\text{mid}} + \frac{\partial}{\partial \theta} \left[\log Z(\Omega_{\text{mid}}^{\text{dual}}(m;\theta)) \right]_{\theta = 0} \,, \end{split}$$

where $c_{\rm edge}$ and $c_{\rm mid}$ are constants depending only on n, such that $c_{\rm mid} = 2c_{\rm edge} - nx_c^3u_2'(0)$.

Remark 2.3. We will be especially interested in the empty (i.e. when $\mathfrak{b} = \emptyset$) and Dobrushin boundary conditions (i.e. when $\mathfrak{b} = (b, b')$, where $b, b' \in \partial \Omega$).

2.3. Alternative definition of \mathcal{T}_{edge} and \mathcal{T}_{mid} . In this subsection we define \mathcal{T}_{edge} and \mathcal{T}_{mid} as a sum over all configurations of the loop O(n) model on the honeycomb lattice, where the contribution of a configuration depends only on the way how it looks locally near a particular edge or a midline and on a bit of information about the global loop structure.

The set of considered configurations depends on a particular edge or a midline, so we need to define it properly for each deformation. In fact, each time the set of configurations is in a direct correspondence with the set of configurations on a deformed lattice.

Let e=(u,v) be an edge of Ω . In this case, denote by $\operatorname{Conf}_{\Omega}^{[e]}(\mathfrak{b})$ the set of all subgraphs γ of Ω such that each vertex except for u and v has an even degree in γ , and vertices u and v are allowed to have degrees 0, 2 or 3, see Fig. 5

Let m be a midline of one of the hexagons in Ω and let u and v be its endpoints. Add vertices u and v to Ω (splitting the corresponding edges into half-edges), link them by an edge, and denote thus obtained graph by Ω_m . Then $\operatorname{Conf}_{\Omega}^{[m]}(\mathfrak{b})$ is the set of all subgraphs γ of Ω_m such that each vertex except for u and v has an even degree in γ , and vertices u and v are allowed to have degrees 0, 0 or 0, see Fig. 0.

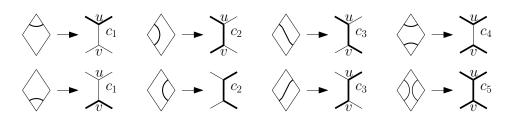


FIGURE 5. Configurations in a rhombus corresponding to the edge e = (u, v) together with their images in $\operatorname{Conf}_{\Omega}^{[e]}(\mathfrak{b})$ and the coefficients from Lemma 2.4.

Now let us describe the correspondence between the configurations on the deformed lattice and the ones in $\operatorname{Conf}_{\Omega}^{[e]}(\mathfrak{b})$ and $\operatorname{Conf}_{\Omega}^{[m]}(\mathfrak{b})$.

We start by deforming two adjacent triangles (Fig. 5). They can be joined into a rhombus R with angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ (this does not affect the O(n) model, see Proposition 2.2). Let e be the edge of Ω corresponding to R and let us change its angles to $\frac{\pi}{3} + \varepsilon$ and $\frac{2\pi}{3} - \varepsilon$. Consider a configuration γ of the O(n) model on this Riemann surface (see Fig. 5). It is clear that, if γ is not of w_2 -type inside R (two long arcs), then it can be mapped in a straightforward way to a subgraph in $\operatorname{Conf}_{\Omega}^{[e]}(\mathfrak{b})$ in which all the vertices have even degrees. Assume now that γ is of w_2 -type inside R. Then it corresponds to a subgraph in $\operatorname{Conf}_{\Omega}^{[e]}(\mathfrak{b})$ in which vertices u and v have degree 3 (this can be thought of as having a "double edge" between u and v).

The other type of the deformations that we are interested in is an insertion of an infinitesimal rhombus (with angles ε and $\pi - \varepsilon$) in the place of two adjacent edges (Fig. 6). Denote these edges by e and f, and the inserted rhombus by R. Consider a configuration γ of the loop O(n) model on this Riemann surface. If γ is of u_1 -type or of w_1 -type inside R or if R is just empty, then the correspondence with a subgraph in $\operatorname{Conf}_{\Omega}^{[m]}(\mathfrak{b})$ is straightforward. If γ is of u_2 -type or of v-type inside R, then γ corresponds to a subgraph in $\operatorname{Conf}_{\Omega}^{[m]}(\mathfrak{b})$ in which u and v are of degree 2 and the edge st is taken. Finally, if γ is of w_2 -type inside R, then it corresponds to a subgraph in $\operatorname{Conf}_{\Omega}^{[m]}(\mathfrak{b})$ in which u and v are of degree 3 (so the edge st is also taken).

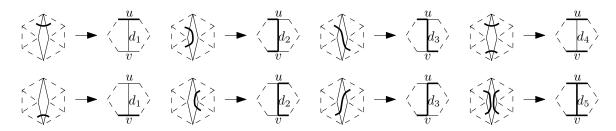


FIGURE 6. Configurations in a rhombus inserted in the place of a midline m=(u,v) together with their images in $\operatorname{Conf}_{\Omega}^{[m]}(\mathfrak{b})$ and the coefficients from Lemma 2.4.

We have described the deformations and graphical representations of deformed configuration on Ω . Now we define the weights of these deformed configurations. Basically, it is still $x^{\text{\#edges}}n^{\text{\#loops}}$, but we need to describe how to count edges near the deformation and loops.

Consider $\gamma \in \operatorname{Conf}_{\Omega}^{[e]}(\mathfrak{b})$ for an edge e = (u, v). If u and v have degree at most 2 in γ , then we count edges and loops in a normal way. Assume that u and v have degree 3 in γ . Then we declare #edges to be the number of edges in $\gamma - e$ decreased by two, and #loops to be the number of loops in $\gamma - e$ decreased (resp., increased) by one if u and v belong to different (resp., the same) loops in $\gamma - e$.

Now consider $\gamma \in \operatorname{Conf}_{\Omega}^{[m]}(\mathfrak{b})$ for a midline m. Let u and v be the endpoints of m. First, m is not counted as an edge, and other edges incident to u and v are counted as half-edges in #edges. Then, if vertices u and v have degree 3 in γ , we define #loops in exactly in the same way as above.

Lemma 2.4. 1) Consider any edge e of Ω . Let u and v be the endpoints of e. Then the following holds:

$$\mathcal{T}_{\mathrm{edge}}^{\mathfrak{b}}(e) = c_{\mathrm{edge}} + \frac{1}{Z^{\mathfrak{b}}} \sum_{\gamma \in \mathrm{Conf}_{\Omega}^{[e]}(\mathfrak{b})} w(\gamma) \cdot t^{[e]}(\gamma) ,$$

where $Z^{\mathfrak{b}} = Z^{\mathfrak{b}}(\Omega)$ is the partition function on the non-deformed graph and $t^{[e]}$ depends on the configuration near e and takes 5 possible values (see Fig. 5):

- c_1 if γ does not contain e but it contains exactly two edges adjacent to e;
- c_2 if γ contains e together with 2 non-parallel edges adjacent to e;
- c_3 if γ contains e together with 2 parallel edges adjacent to e;
- c_4 if γ does not contain e, and both vertices u and v have degree 2 in γ ;
- c_5 if both vertices u and v have degree 3 in γ .
- 2) Let e and f be two opposite edges in a face of Ω , and let u and v be their midpoints. Consider a midline m with the endpoints u and v. Then the following holds:

$$\mathcal{T}^{\mathfrak{b}}_{\mathrm{mid}}(m) = c_{\mathrm{mid}} + \frac{1}{Z^{\mathfrak{b}}} \sum\nolimits_{\gamma \in \mathrm{Conf}_{\Omega}^{[m]}(\mathfrak{b})} \mathrm{w}(\gamma) \cdot t^{[m]}(\gamma) \,,$$

where $Z^{\mathfrak{b}} = Z^{\mathfrak{b}}(\Omega)$ is the partition function (the sum of weights of all normal configurations) and $t^{[m]}(\gamma)$ depends on the configuration near the midline m and takes one of 5 possible values (see Fig. 6):

- d_1 if one of the vertices u and v has degree 2 in γ and the other one has degree 0;
- d_2 if γ contains m and the halves of the edges e and f lying to the same side of m;
- d_3 if γ contains m and the halves of the edges e and f lying on different sides of m;
- d_4 if γ does not contain m and the vertices u and v have degree 2;
- d_5 if the vertices u and v have degree 3 in γ .
- 3) The coefficients c_i and d_i can be computed explicitly as follows:

$$c_{1} = \frac{u'_{1}(\frac{\pi}{3})}{u_{1}(\frac{\pi}{3})} \qquad c_{2} = \frac{u'_{2}(\frac{\pi}{3})}{u_{1}(\frac{\pi}{3})} \qquad c_{3} = \frac{v'(\frac{\pi}{3})}{u_{1}(\frac{\pi}{3})} \qquad c_{4} = \frac{w'_{1}(\frac{\pi}{3})}{w_{1}(\frac{\pi}{3})} \qquad c_{5} = w'_{2}(\frac{\pi}{3}) \qquad (2.7)$$

$$d_{1} = u'_{1}(0) \qquad d_{2} = u'_{2}(0) \qquad d_{3} = v'(0) \qquad d_{4} = w'_{1}(0) \qquad d_{5} = w'_{2}(0) \qquad (2.8)$$

$$d_1 = u'_1(0)$$
 $d_2 = u'_2(0)$ $d_3 = v'(0)$ $d_4 = w'_1(0)$ $d_5 = w'_2(0)$ (2.8)

Proof. This is a straightforward computation, see Lemma A.1 in the Appendix.

Remark 2.4. Let us give the concrete expressions for the coefficients (2.7)-(2.8) in the two special cases: the self-avoiding walk and the Ising model.

For the self-avoiding walk (n = 0), equations (2.7)–(2.8) read as:

$$c_1 = \frac{5}{6\sqrt{3}} \qquad c_2 = -\frac{2}{3\sqrt{3}} \qquad c_3 = -\frac{7}{6\sqrt{3}} \qquad c_4 = -\frac{7}{3\sqrt{3}} \qquad c_5 = \frac{1}{3\sqrt{3}}$$

$$d_1 = -\frac{1}{\sqrt{3}} \qquad d_2 = \frac{1}{2} \qquad d_3 = \frac{1}{\sqrt{3}} \qquad d_4 = -\frac{\sqrt{3}}{2} \qquad d_5 = -\frac{1}{2\sqrt{3}}.$$

For the Ising model (n = 1) equations (2.7)–(2.8) read as:

$$c_1 = -\frac{3}{8} \cdot \sqrt{2} \qquad c_2 = \frac{3}{8} \cdot 2 \qquad c_3 = \frac{3}{8} \qquad c_4 = -\frac{3}{8} \cdot (1 + 2\sqrt{2}) \qquad c_5 = \frac{3}{8} \cdot (\sqrt{2} - 1)$$

$$d_1 = -\frac{3}{8} \cdot (2\sqrt{2} - 1) \qquad d_2 = \frac{3}{8} \cdot \frac{2\sqrt{2}}{\sqrt{2 + \sqrt{2}}} \qquad d_3 = \frac{3}{8} \cdot \sqrt{2} \qquad d_4 = -\frac{3}{8} \cdot 2\sqrt{2} \qquad d_5 = -\frac{3}{8} \cdot (2 - \sqrt{2}).$$

$$\begin{array}{ccc}
& & & \\$$

FIGURE 7. Local relations on observables $\mathcal{T}_{\text{edge}}$ and \mathcal{T}_{mid} .

2.4. Local relations on \mathcal{T}_{edge} and \mathcal{T}_{mid} . In this subsection we prove that $\mathcal{T}_{edge}^{\mathfrak{b}}$ and $\mathcal{T}_{mid}^{\mathfrak{b}}$ satisfy some linear relations that can interpreted as a part of discrete Cauchy-Riemann equations. As boundary conditions \mathfrak{b} play no role in the computations given below, we omit the superscript \mathfrak{b} for shortness.

Lemma 2.5. Let a and b be two adjacent edges in one of the faces of the honeycomb lattice Ω , and let m be the midline in the same hexagon, such that m does not intersect a and b (see Fig. 7). Then one has

$$\mathcal{T}_{\text{mid}}(m) = \mathcal{T}_{\text{edge}}(a) + \mathcal{T}_{\text{edge}}(b)$$
. (2.9)

Proof. Implement simultaneously two infinitesimal deformations described in Subsection 2.1: insert a rhombus of angle ε at the place of m and replace the rhombus of angle $\frac{\pi}{3}$ corresponding to a by a rhombus of angle $\frac{\pi}{3} - \varepsilon$. Since the region of the surface corresponding to a, b and m remains flat, one can apply the pentagonal transformation (see Fig. 4) and equiavalently view this deformation as the change of the angle of the rhombus corresponding to b from $\frac{\pi}{3}$ to $\frac{\pi}{3} + \varepsilon$. Applying the pentagonal relation (see Proposition 2.2) and taking the derivative at $\varepsilon = 0$, we obtain the desired relation (2.9).

Corollary 2.6. Let a, b, c, d, e, f be the edges of a hexagon of Ω (listed in clockwise order). Then the following holds:

$$\mathcal{T}_{\text{edge}}(d) - \mathcal{T}_{\text{edge}}(a) = \mathcal{T}_{\text{edge}}(b) - \mathcal{T}_{\text{edge}}(e) = \mathcal{T}_{\text{edge}}(f) - \mathcal{T}_{\text{edge}}(c)$$
. (2.10)

Proof. One can take a derivative of the Yang-Baxter equation (Proposition 2.2) and derive (2.10) in the same way as in Lemma 2.5, but it is easier to use this lemma directly. Let m be a midline of the same hexagon connecting the midpoints of the edges c and f. Lemma 2.5 implies

$$\mathcal{T}_{\text{edge}}(a) + \mathcal{T}_{\text{edge}}(b) = \mathcal{T}_{\text{mid}}(m) = \mathcal{T}_{\text{edge}}(d) + \mathcal{T}_{\text{edge}}(e)$$
.

Thus the first equation in (2.10) holds true and the second can be proved in a similar way. \Box

Remark 2.5. Lemma 2.5 allows us to express \mathcal{T}_{mid} in terms of $\mathcal{T}_{\text{edge}}$ and vice versa.

Corollary 2.7. Consider midlines m_1 , m_2 , m_3 , m_4 , m_5 and m_6 surrounding this edge as shown in Fig. 7. Then the following relation holds:

$$\mathcal{T}_{\text{mid}}(m_3) + \mathcal{T}_{\text{mid}}(m_5) - \mathcal{T}_{\text{mid}}(m_1) = \mathcal{T}_{\text{mid}}(m_4) + \mathcal{T}_{\text{mid}}(m_6) - \mathcal{T}_{\text{mid}}(m_2). \tag{2.11}$$

Proof. Both left- and right-hand sides are equal to $2\mathcal{T}_{\text{edge}}(e)$ due to Lemma 2.5.

Lemma 2.8. Coefficients $t^{[e]}$ and $t^{[m]}$ provided in Lemma 2.4 are the unique (up to a multiplicative constant) coefficients for which observables $\mathcal{T}_{\text{edge}}$ and \mathcal{T}_{mid} satisfy relations (2.9).

Proof. See Lemma A.1 in the Appendix.
$$\Box$$

The following lemma shows that local realtions on (2.9)–(2.11)) are in fact equivalent to the existence of a function defined on the faces of Ω , for which \mathcal{T}_{edge} and \mathcal{T}_{mid} are the discrete second derivatives.

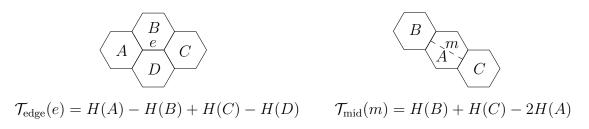


FIGURE 8. Observables $\mathcal{T}_{\text{edge}}$ and \mathcal{T}_{mid} expressed in terms of a function H on faces.

Lemma 2.9. Consider a discrete simply-connected domain Ω on the honeycomb lattice. Then there exists a function H defined on the vertices of Ω^{dual} such that for each edge of Ω and each midline of a face of Ω observables $\mathcal{T}_{\text{edge}}$ and \mathcal{T}_{mid} can be expressed in terms of H in a way shown in Fig. 8. The function H is unique up to a linear term, or, equivalently, up to fixing its value on three arbitrary vertices of Ω^{dual} .

Moreover, for any function H on vertices of Ω^{dual} the two functions on edges

$$F_{\text{edge}}(e) = H(A) - H(B) + H(C) - H(D)$$
 (2.12)

$$F_{\text{mid}}(m) = H(B) + H(C) - 2H(A) \tag{2.13}$$

satisfy the same local relations as \mathcal{T}_{edge} and \mathcal{T}_{mid} , i. e. (2.9)-(2.11).

Proof. See Lemma A.2 in the Appendix.

2.5. Complex-valued observable \mathcal{T} . Take $\rho = e^{i\frac{\pi}{3}}$. We define a complex-valued observable \mathcal{T} on faces and vertices of Ω in the following way.

Definition 2.10. Let a be a face of Ω and denote by m, m' and m'' midlines of a perpendicular to the directions 1, ρ and ρ^2 , respectively. Define the complex-valued observable $\mathcal{T}^{\mathfrak{b}}(a)$ as

$$\mathcal{T}^{\mathfrak{b}}(a) = -\tfrac{2}{3} \left(\mathcal{T}^{\mathfrak{b}}_{\mathrm{mid}}(m) + \bar{\rho}^2 \mathcal{T}^{\mathfrak{b}}_{\mathrm{mid}}(m') + \bar{\rho}^4 \mathcal{T}^{\mathfrak{b}}_{\mathrm{mid}}(m'') \right) \,.$$

Remark 2.6. 1) In the case n=1 (Ising model) we show below that in the limit $\delta \to 0$ the observable $\mathcal{T}^{\mathfrak{b}}$ converges to the holomorphic stress-energy tensor, and observables $\mathcal{T}^{\mathfrak{b}}_{\text{edge}}$ and $\mathcal{T}^{\mathfrak{b}}_{\text{mid}}$ converge to its projections on the corresponding directions.

2) In a similar way one can define the observable $\mathcal{T}^{\mathfrak{b}}$ on vertices of Ω , the value of $\mathcal{T}^{\mathfrak{b}}$ at a vertex v being equal to a linear combination of the values of $\mathcal{T}^{\mathfrak{b}}_{\text{edge}}$ at three edges incident to v multiplied by the corresponding factors 1, $\bar{\rho}^2$, $\bar{\rho}^4$. For the Ising model, all the convergence results proven below for $\mathcal{T}^{\mathfrak{b}}$ remain true for its version on vertices.

Note that the relations (2.9)-(2.11) discussed above are formulated for the observables $\mathcal{T}_{\text{edge}}$ and \mathcal{T}_{mid} . Although to the best of our knowledge observable \mathcal{T} itself does not satisfy any local relations, there exists a linear combination of observables $\mathcal{T}_{\text{edge}}$ (or \mathcal{T}_{mid}) that does satisfy a half of discrete Cauchy-Riemann equations. Moreover, it has the same scaling limit as \mathcal{T} in the case n=1 (Ising model) and we expect this to be the case for other values of n as well, see also Corollary 2.12.

Definition 2.11. Pick any edge e of Ω , and let τ denote the direction of any of two possible orientations of e. Now consider four edges around e, orient them out from e and denote by $e_{\tau\rho}$, $e_{\tau\bar{\rho}}$, $e_{-\tau\rho}$ and $e_{-\tau\bar{\rho}}$ according to their direction. Then we define \mathcal{T}_{CR} on e by

$$\mathcal{T}_{\mathrm{CR}}(e) = \frac{4}{3} \overline{\tau}^2 \left[\mathcal{T}_{\mathrm{edge}}(e) + \frac{\overline{\rho}}{4} \left(\mathcal{T}_{\mathrm{edge}}(e_{\tau \rho}) + \mathcal{T}_{\mathrm{edge}}(e_{-\tau \rho}) \right) + \frac{\rho}{4} \left(\mathcal{T}_{\mathrm{edge}}(e_{\tau \overline{\rho}}) + \mathcal{T}_{\mathrm{edge}}(e_{-\tau \overline{\rho}}) \right) \right] .$$

Note that the definition does not depend on the choice of τ .

Corollary 2.12. 1) The discrete countour integral of \mathcal{T}_{CR} around each hexagon vanishes.

2) Assume that \mathcal{T}_{edge} and \mathcal{T}_{mid} converge to the three projections of a holomorphic function in a way similar to Theorem 1.1. Then \mathcal{T}_{CR} , when properly normalized, converges to the same limit as \mathcal{T} .

Proof. 1) Let a be a hexagon of Ω . Orient the boundary edges of a counterclockwise and denote them by a_{τ} for $\tau = \pm 1, \pm \rho, \pm \rho^2$ according to their direction. Now consider the edges having one endpoint on a, orient them in the direction pointing outside of a and denote them by a_{τ}^{out} for $\tau = \pm 1, \pm \rho, \pm \rho^2$ according to their direction. The values $\mathcal{T}_{\text{CR}}(a_{\tau})$ can be rewritten as

$$\mathcal{T}_{CR}(a_{\tau}) = \frac{4}{3}\overline{\tau}^{2} \left[\mathcal{T}_{edge}(a_{\tau}) + \frac{\overline{\rho}}{4} \left(\mathcal{T}_{edge}(a_{\tau\rho}) + \mathcal{T}_{edge}(a_{\tau\rho}^{out}) \right) + \frac{\rho}{4} \left(\mathcal{T}_{edge}(a_{\tau\overline{\rho}}) + \mathcal{T}_{edge}(a_{\tau\overline{\rho}}^{out}) \right) \right],$$

where we used that the edges around a_{τ} are denoted by $a_{\tau\rho}$, $a_{\tau\bar{\rho}}$, $a_{\tau\bar{\rho}}^{\text{out}}$ and $a_{\tau\bar{\rho}}^{\text{out}}$. We need to show that $\sum_{\tau} \tau \mathcal{T}_{\text{CR}}(a_{\tau}) = 0$, where the sum is taken over $\tau = \pm 1, \pm \rho, \pm \rho^2$. Let us expand this sum:

$$\sum_{\tau} \tau \mathcal{T}_{CR}(a_{\tau}) = \frac{4}{3} \sum_{\tau} \left[\overline{\tau} \mathcal{T}_{edge}(a_{\tau}) + \frac{\overline{\tau}\rho}{4} \mathcal{T}_{edge}(a_{\tau\rho}) + \frac{\overline{\tau}\rho}{4} \mathcal{T}_{edge}(a_{\tau\overline{\rho}}) \right]
+ \frac{1}{3} \sum_{\tau} \left[\overline{\tau}\rho \mathcal{T}_{edge}(a_{-\tau\rho}^{out}) + \overline{\tau}\rho \mathcal{T}_{edge}(a_{\tau\overline{\rho}}^{out}) \right]
= \frac{4}{3} \sum_{\tau} \mathcal{T}_{edge}(a_{\tau}) (\overline{\tau} + \frac{\overline{\tau}}{4} + \frac{\overline{\tau}}{4}) + \frac{1}{3} \sum_{\tau} \mathcal{T}_{edge}(a_{\tau}^{out}) (-\overline{\tau} + \overline{\tau}) = 0,$$

where the second equality is obtained by changing a variable in the summation, and the third follows from (2.10) and the identity $1 + \rho^2 + \rho^4 = 0$.

2) Assuming Theorem 1.1, it follows from the definition of \mathcal{T}_{CR} that

$$\delta^{-2}\mathcal{T}_{\mathrm{CR}}^+(e_\delta) \ \rightrightarrows \ \tfrac{3}{\pi} \cdot \left[\tfrac{4\overline{\tau}^2}{3} \operatorname{Re}[\tau^2 \langle T(w) \rangle_{\Omega}^+] - \tfrac{2(\overline{\tau}\rho)^2}{3} \operatorname{Re}[(\tau\rho)^2 \langle T(w) \rangle_{\Omega}^+] - \tfrac{2(\overline{\tau}\overline{\rho})^2}{3} \operatorname{Re}[(\tau\overline{\rho})^2 \langle T(w) \rangle_{\Omega}^+] \right].$$

It remains to note that the expression in the brackets is equal to $\langle T(w) \rangle_{\Omega}^+$.

Remark 2.7. 1) One can construct another version of \mathcal{T}_{CR} such that it has the same (conjectural) limit as \mathcal{T} and whose discrete contour integrals around inner vertices of Ω vanish.

- 2) Both statements given in Corollary 2.12 remain true in presence of other fields in the correlation function.
- 2.6. Infinitesimal deformations in terms of conical singularities. In this subsection we provide an informal discussion on how one can interpret the infinitesimal deformations defined in Subsection 2.2 can be viewed as insertions of several conformal anomalies (conical singularities). This interpretation agrees with the relations on (2.9) and (2.10) on $\mathcal{T}_{\text{edge}}$ and \mathcal{T}_{mid} .

Let Ω denote a connected union of several faces of the hexagonal lattice. Given the center A of a face in Ω , we informally denote by $\mathfrak{C}(A)$ the operator taking a derivative of the partition function under the insertion of a conical singularity $+\varepsilon$ at A.

Let e be an edge of Ω , and denote by A, B, C and D the centers of hexagons around e (see Fig. 8). Then the deformation corresponding to e is just the insertion of conical singulatities at these points — angles around A and C will be equal to $2\pi - \varepsilon$ and angles around B and D will be equal to $2\pi + \varepsilon$. It is easy to see that $\mathcal{T}_{\text{edge}}(e)$ is a derivative in ε at 0 of the partition function Z. Thus, one can write that

$$\mathcal{T}_{\text{edge}}(e) \approx \mathfrak{C}(A) - \mathfrak{C}(B) + \mathfrak{C}(C) - \mathfrak{C}(D).$$
 (2.14)

For $\mathcal{T}_{\text{mid}}^{\mathfrak{b}}$ the situation is morally the same. Consider any midline m of a hexagon in Ω (see Fig. 8). Let A be the center of this hexagon and let B and C be the centers of the hexagons adjacent to it. The deformation corresponding to m is adding one more vertex (a copy of A) and the insertion of conical singularities — angles around B and C will be equal to $2\pi + \varepsilon$ and angles around each of the copies of A will be equal to $2\pi - \varepsilon$. And $\mathcal{T}_{\text{mid}}(m)$ is a derivative in ε at 0 of the partition function Z. Thus, one can write that

$$\mathcal{T}_{\text{mid}}(m) \approx \mathfrak{C}(B) + \mathfrak{C}(C) - 2\mathfrak{C}(A).$$
 (2.15)

Remark 2.8. 1) In this interpretation, the relations on $\mathcal{T}_{\text{edge}}$ and \mathcal{T}_{mid} from the subsection 2.4 are direct corollaries of the equations (2.14) and (2.15).

- 2) Note that here we do not have access to the constant terms appearing in the definition of \mathcal{T}_{edge} and \mathcal{T}_{mid} .
- 3) The equations (2.14) and (2.15) look exactly as the formulae where we rewrite $\mathcal{T}_{\text{edge}}$ and \mathcal{T}_{mid} in terms of function H from Lemma 2.9.
 - 3. The case n=1: Ising model on faces of the honeycomb lattice

Starting with this section, we focus our attention on the special case n=1 and consider the loop O(1) model defined in discrete domains, by which we mean subsets of the regular honeycomb lattice. Recall that this model considered in a discrete domain Ω with boundary conditions $\mathfrak{b} = \{b_1, \ldots, b_{2m}\}$ can be thought about as the *Ising model* defined on faces of Ω (including boundary ones) or, equivalently, on vertices of Ω^{dual} , with spins of boundary faces being fixed to +1 except along all the (counterclockwise) boundary arcs $(b_{2k-1}b_{2k})$, where they are fixed to be -1. More precisely, given a configuration $\gamma \in \text{Conf}_{\Omega}(\mathfrak{b})$, we put spins ± 1 on faces of Ω so that two spins at adjacent faces are different if and only if their common edge belongs to γ . This procedure defines a two-to-one correspondence between spin configurations and elements of $\text{Conf}_{\Omega}(\mathfrak{b})$, and we kill the ambiguity by fixing the sign in boundary conditions.

Starting with this section, we also use the notation '+' instead of ' \varnothing ' for the boundary conditions in order to emphasize this particular choice of the sign of boundary spins.

3.1. Stress-energy tensor as a local field. The main goal of this section is to show that one can rewrite the geometrical definition of the quantities $\mathcal{T}^{\mathfrak{b}}_{\mathrm{mid}}(m)$ discussed in Section 2.2 in terms of expectations of polynomials of several spins surrounding the midline m, see Definition 3.3 and Proposition 3.4 below. This allows one to think about the observables \mathcal{T} for the loop O(1) model as about expectations of some local field in the corresponding Ising model. Moreover, one can then easily speak about correlations of this local field with other ones (e.g., spin and energy density), which we discuss in Section 3.3 below.

Definition 3.1. For two (inner) half-edges a, e of Ω , we denote by $\operatorname{Conf}_{\Omega}(\mathfrak{b} \cup \{a, e\})$ the set of subgraphs of Ω consisting of several loops and paths whose ends are the half-edges contained in $\mathfrak{b} \cup \{a, e\}$, so that each half-edge of $\mathfrak{b} \cup \{a, e\}$ is the end of exactly one of the paths. The corresponding partition function is given by

$$\mathcal{W}_{\Omega}^{\mathfrak{b}}(a,e) \ := \ \sum\nolimits_{\gamma \in \operatorname{Conf}_{\Omega}(\mathfrak{b} \cup \{a,e\})} \mathcal{w}(\gamma) \,, \qquad \mathcal{w}(\gamma) = x^{|\gamma|} \,,$$

where $x = 1/\sqrt{3}$ and $|\gamma|$ denotes the number of edges in γ (half-edges being counted as $\frac{1}{2}$).

For an oriented midline m of an inner face of Ω , let m_{down} and m_{up} denote the two halfedges oriented in the direction -im, the former starting at the beginning of m and the latter at its endpoint, see Fig. 9. Further, we denote by \overline{m} the same midline oriented in the opposite direction. Slightly abusing the notation, we continue to use the sign $\mathcal{T}_{\text{mid}}(m) = \mathcal{T}_{\text{mid}}(\overline{m})$ for the value of the observable \mathcal{T}_{mid} discussed above (and originally defined for unoriented midlines). The next lemma specifies its definition in the case n = 1.

Lemma 3.2. Let m be an oriented midline of an inner face of Ω . Then, one has

$$\mathcal{T}_{\text{mid}}^{\mathfrak{b}}(m) = c_{\text{mid}} + \frac{1}{2\mathcal{Z}_{\Omega}^{\mathfrak{b}}} \left[W_{\Omega}^{\mathfrak{b}}(m_{\text{down}}, m_{\text{up}}) + W_{\Omega}^{\mathfrak{b}}(\overline{m}_{\text{down}}, \overline{m}_{\text{up}}) \right] \\
+ \frac{1}{\sqrt{3}\mathcal{Z}_{\Omega}^{\mathfrak{b}}} \left[W_{\Omega}^{\mathfrak{b}}(m_{\text{down}}, \overline{m}_{\text{down}}) + W_{\Omega}^{\mathfrak{b}}(m_{\text{up}}, \overline{m}_{\text{up}}) \right] \\
- \frac{1}{\sqrt{3}\mathcal{Z}_{\Omega}^{\mathfrak{b}}} \left[W_{\Omega}^{\mathfrak{b}}(m_{\text{down}}, \overline{m}_{\text{up}}) + W_{\Omega}^{\mathfrak{b}}(m_{\text{up}}, \overline{m}_{\text{down}}) \right],$$
(3.1)

where the constant c_{mid} is given by (1.5).

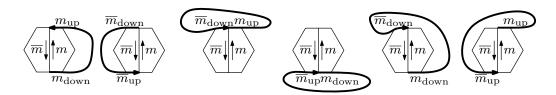


FIGURE 9. A schematic drawing of curves connecting two half-edges adjacent to a midline m, which correspond to the six terms $W_{\Omega}^{\mathfrak{b}}(\cdot,\cdot)$ in the expression (3.1). Due to topological reasons, the winding (total rotation angle) of an oriented curve equals π or -3π in the first two cases and $\pm 2\pi$ in the last four.

Proof. In Lemma 2.4 it is shown that $\mathcal{T}^{\mathfrak{b}}_{\mathrm{mid}}(m)$ can be expressed as a sum over configurations γ contained in the set $\mathrm{Conf}^{[m]}_{\Omega}(\mathfrak{b})$, where each configuration contributes its weight $\mathrm{w}(\gamma)$ times a particular coefficient d_1, \ldots, d_5 depending on the local structure of γ near m, see Fig. 5. Note that for $n \neq 1$ the weight $\mathrm{w}(\gamma)$ depends on the global loop structure of γ but for n = 1 this is irrelevant. Deleting the midline m from γ one obtains a configuration γ' , which is contained in one (weights d_1, d_2, d_3) or two (weights d_4 and d_5) sets of configurations involved into (3.1). In the opposite direction, note that this correspondence is 1-to-1 in the former case, while in the latter each γ' has exactly two preimages. Recall that the coefficients d_1, \ldots, d_5 for n = 1 are computed in Remark 2.4. In particular one has $d_4 + d_5 = 2d_1$, which allows us to consider two edges at the top and at the bottom of m separately when counting the contribution of γ' .

We now introduce some additional notation, which will be particularly convenient later. For simplicity, let us assume that one of the edges of Ω is horizontal, so all others are oriented in directions $1, \rho = -\rho^4, \rho^2, \rho^3 = -1, \rho^4, \rho^5 = -\rho^2$, where $\rho = e^{i\frac{\pi}{3}}$. We fix the choice of square roots of all these directions once forever in the following way:

$$\sqrt{\rho^{2k}} := \rho^k, \quad \sqrt{-\rho^{2k}} := i\rho^k, \text{ where } k = 0, 1, 2,$$

and denote by $\wp := \{1, \rho, \rho^2, i, i\rho, i\rho^2\}$ the set of these six square roots. We also set

$$c_T := \frac{3}{2\pi} - \frac{1}{\sqrt{3}}, \qquad c_R := \frac{7}{3\sqrt{3}} - \frac{4}{\pi}, \qquad c_{\text{mid}} := 2c_T + c_R,$$
 (3.2)

which agrees with (1.5). These explicit values eventually come from the local values (5.2),(5.3) of some full-plane observable and are not important for the combinatorial considerations.

Definition 3.3. For an inner face w of Ω and $\eta \in \wp$, we denote by $w^{[\eta]}$ the midline of w oriented in the direction $i\eta^2$ and by w_{η^2} the face adjacent to w such that the common edge of w and w_{η^2} contains the beginning of $w^{[\eta]}$ (and so the half-edge $w^{[\eta]}_{\text{down}}$), see Fig. 10. Then, we define the following polynomials, which depend only on the spin $\sigma = \sigma(w)$ of the face w and spins $\sigma_{\eta^2} = \sigma(w_{\eta^2})$ of its six adjacent faces:

$$\begin{split} T^{[\eta]}(w) &:= \ c_T + \frac{1}{24\sqrt{3}}(1+\sigma\sigma_{\eta^2})(1+\sigma\sigma_{-\eta^2})(2-\sigma\sigma_{\rho\eta^2})(2-\sigma\sigma_{\rho^2\eta^2}), \\ R^{[\eta]}(w) &:= \ c_R - \frac{1}{4\sqrt{3}}[(1-\sigma\sigma_{\eta^2}) + (1-\sigma\sigma_{-\eta^2})] \\ &\quad + \frac{1}{12\sqrt{3}}(1-\sigma_{\eta^2}\sigma_{-\eta^2})[(2-\sigma\sigma_{\rho\eta^2})(2-\sigma\sigma_{\rho^2\eta^2}) + (2-\sigma\sigma_{-\rho\eta^2})(2-\sigma\sigma_{-\rho^2\eta^2})] \,. \end{split}$$

Finally, we define the local field T(w) as the following linear combination:

$$T(w) \; := \; -\frac{2}{3} \, \sum\nolimits_{\eta=1,\rho,\rho^2} \, \bar{\eta}^4 \left[T^{[\eta]}(w) + T^{[i\eta]}(w) + R^{[\eta]}(w) \right].$$

Remark 3.1. Note that $R^{[\eta]}(w) = R^{[i\eta]}(w)$ by definition, while $T^{[\eta]}(w)$ and $T^{[i\eta]}(w)$ are, in general, unrelated to each other. Also, it is worth mentioning that the notation introduced above could be simplified by using the edge directions $1, \rho, \ldots, \rho^5$ themselves instead of their square roots $\eta \in \wp$. We choose the latter as it is quite useful when rewriting the quantities of interest in terms of fermionic observables (see Sections 3.2–3.4 below) and especially when working with their scaling limits, see Sections 4 and 5.

Proposition 3.4. Let w be an inner face of Ω and $\eta \in \{1, \rho, \rho^2\}$. Then, one has

$$\mathcal{T}^{\mathfrak{b}}_{\mathrm{mid}}(w^{[\eta]}) = \mathbb{E}^{\mathfrak{b}}_{\Omega}[T^{[\eta]}(w) + T^{[i\eta]}(w)] + \mathbb{E}^{\mathfrak{b}}_{\Omega}[R^{[\eta]}(w)],$$

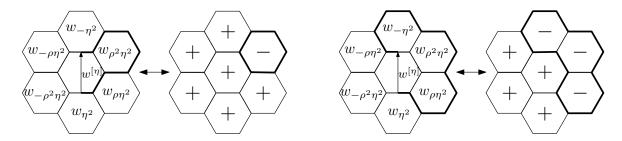


FIGURE 10. Notation for the faces surrounding a midline $w^{[\eta]}$ with $\eta=1$ and two examples of the XOR operation on the right semi-hexagon. Left: A configuration contained in $\mathrm{Conf}_{\Omega}(\{w_{\mathrm{down}}^{[\eta]}, w_{\mathrm{up}}^{[\eta]}\})$ and its image. Right: A configuration contained in $\mathrm{Conf}_{\Omega}(\{w_{\mathrm{down}}^{[\eta]}, w_{\mathrm{down}}^{[\eta]}\})$ and its image.

where the local fields $T^{[\eta]}(w)$ and $R^{[\eta]}(w)$ are given by Definition 3.3.

Remark 3.2. Introducing the notation $T_{\text{mid}}(w^{[\eta]}) := T^{[\eta]}(w) + T^{[i\eta]}(w) + R^{[\eta]}(w)$, one immediately gets (1.6). Above, we prefer to separate the term $R^{[\eta]}(w)$ for the following reason: its expectations have a higher order of decay than δ^2 and hence disappear in the scaling limits considered in Theorem 1.1. Moreover, it has the same effect in Theorems 1.2 and 1.3: the terms containing $R^{[\eta]}(w)$ have higher orders of decay comparing to those containing $T^{[\eta]}(w) + T^{[i\eta]}(w)$ and hence does not contribute to the limits, see Proposition 3.7 and Section 5 for more details.

Proof. Let $m = w^{[\eta]}$ and hence $\overline{m} = w^{[i\eta]}$. Due to Lemma 3.2 it is enough to show that

$$\mathbb{E}_{\Omega}^{\mathfrak{b}}[T^{[\eta]}(w)] = c_T + \frac{1}{2\mathcal{Z}_{\Omega}^{\mathfrak{b}}} \cdot W_{\Omega}^{\mathfrak{b}}(w_{\text{down}}^{[\eta]}, w_{\text{up}}^{[\eta]}), \qquad (3.3)$$

$$\mathbb{E}_{\Omega}^{\mathfrak{b}}[T^{[i\eta]}(w)] = c_T + \frac{1}{2\mathcal{Z}_{\Omega}^{\mathfrak{b}}} \cdot W_{\Omega}^{\mathfrak{b}}(w_{\text{down}}^{[i\eta]}, w_{\text{up}}^{[i\eta]}), \qquad (3.4)$$

$$\mathbb{E}_{\Omega}^{\mathfrak{b}}[R^{[\eta]}(w)] = c_{R} + \frac{1}{\sqrt{3}\mathcal{Z}_{\Omega}^{\mathfrak{b}}} \left[W_{\Omega}^{\mathfrak{b}}(w_{\text{down}}^{[\eta]}, w_{\text{down}}^{[i\eta]}) + W_{\Omega}^{\mathfrak{b}}(w_{\text{up}}^{[\eta]}, w_{\text{up}}^{[i\eta]}) \right] \\
- \frac{1}{\sqrt{3}\mathcal{Z}_{\Omega}^{\mathfrak{b}}} \left[W_{\Omega}^{\mathfrak{b}}(w_{\text{down}}^{[\eta]}, w_{\text{up}}^{[i\eta]}) + W_{\Omega}^{\mathfrak{b}}(w_{\text{up}}^{[\eta]}, w_{\text{down}}^{[i\eta]}) \right] .$$
(3.5)

We start by proving (3.3). Note that all spin configurations that give a nontrivial (i.e. different from the constant c_T) contribution to $\mathbb{E}^{\mathfrak{b}}_{\Omega}[T^{[\eta]}(w)]$ satisfy $\sigma = \sigma_{\eta^2} = \sigma_{-\eta^2}$. This subset of spin configurations is in the natural 1-to-1 correspondence with the set of domain walls representations $\gamma \in \mathrm{Conf}_{\Omega}(\mathfrak{b})$ that do not contain edges passing through the endpoints of $w^{[\eta]}$. On the other hand, it is easy to see that all such γ are in a 1-to-1 correspondence with configurations $\gamma' \in \mathrm{Conf}_{\Omega}(\mathfrak{b} \cup \{w_{\mathrm{up}}^{[\eta]}, w_{\mathrm{down}}^{[\eta]}\})$: given γ , one can perform the XOR operation on the right semi-hexagon as shown in Fig. 10 to obtain γ' and vice versa. Note that $w(\gamma')$ differs from $w(\gamma)$ and their ratio is determined by the number of edges separating σ from $\sigma_{\rho\eta^2}$ and $\sigma_{\rho^2\eta^2}$ that are present in γ . More precisely, one has

$$w(\gamma') = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} (2 - \sigma \sigma_{\rho \eta^2}) \cdot \frac{1}{\sqrt{3}} (2 - \sigma \sigma_{\rho^2 \eta^2}) \cdot w(\gamma),$$

which justifies (3.3). Replacing η by $i\eta$ one easily gets (3.4).

The identity (3.5) can be verified in a similar way. First, note that the two sets of configurations $\operatorname{Conf}_{\Omega}(\mathfrak{b} \cup \{w_{\operatorname{down}}^{[\eta]}, w_{\operatorname{up}}^{[i\eta]}\})$ and $\operatorname{Conf}_{\Omega}(\mathfrak{b} \cup \{w_{\operatorname{up}}^{[\eta]}, w_{\operatorname{down}}^{[i\eta]}\})$ involved in the right-hand side of (3.5) can be directly interpreted as spin configurations such that $\sigma_{\eta^2} \neq \sigma$ and $\sigma_{-\eta^2} \neq \sigma$, respectively. This corresponds to the first polynomial term in the definition of $R^{[\eta]}(w)$. Second, applying the same XOR operation on the right semi-hexagon as above (see Fig. 10), one constructs a bijection of the union of the sets $\operatorname{Conf}_{\Omega}(\mathfrak{b} \cup \{w_{\operatorname{down}}^{[\eta]}, w_{\operatorname{down}}^{[i\eta]}\})$ and $\operatorname{Conf}_{\Omega}(\mathfrak{b} \cup \{w_{\operatorname{up}}^{[\eta]}, w_{\operatorname{up}}^{[i\eta]}\})$ with the set of configurations from $\operatorname{Conf}_{\Omega}(\mathfrak{b})$ that contain exactly one of the two edges separating σ from σ_{η^2} and $\sigma_{-\eta^2}$. Arguing as above, this would lead to the (nonsymmetric with respect to the change of η by $i\eta$) second term

$$\frac{1}{\sqrt{3}} \cdot \frac{1}{2} (1 - \sigma_{\eta^2} \sigma_{-\eta^2}) \cdot \frac{1}{\sqrt{3}} (2 - \sigma \sigma_{\rho\eta^2}) \cdot \frac{1}{\sqrt{3}} (2 - \sigma \sigma_{\rho^2\eta^2}) \tag{3.6}$$

in the definition of $R^{[\eta]}(w)$. In order to make it symmetric, note that similar arguments applied to the XOR operation on the left semi-hexagon lead to the expression

$$\frac{1}{\sqrt{3}} \cdot \frac{1}{2} (1 - \sigma_{\eta^2} \sigma_{-\eta^2}) \cdot \frac{1}{\sqrt{3}} (2 - \sigma \sigma_{-\rho\eta^2}) \cdot \frac{1}{\sqrt{3}} (2 - \sigma \sigma_{-\rho^2\eta^2}). \tag{3.7}$$

Thus we can equivalently use the mean value of (3.6) and (3.7) in the definition of $R^{[\eta]}(w)$.

Remark 3.3. The difference of the local fields (3.6) and (3.7) provides an example of a null-field: a polynomial of several spins that vanishes inside of all correlations with other spatially separated local fields. Another natural example of such a field is given by

$$-1 + \prod_{\nu \in \{\pm \eta^2, \pm \rho \eta^2, \pm \rho^2 \eta^2\}} \left[\frac{1}{\sqrt{3}} (2 - \sigma \sigma_{\nu}) \right],$$

which corresponds to the XOR operation of the full hexagon surrounding σ , or, in other words, describes the conditional distribution of σ given the values of its six neighbors σ_{ν} .

3.2. Stress-energy tensor expectations via fermionic observables. The crucial tool that allows us to prove convergence results for the expectations and correlation functions of the local field T(z) discussed above is a particular combinatorial construction of discrete fermionic observables. It was proposed (see [48, Section 4]) by the third author as a convenient tool to study the scaling limit of the Ising model (and, more generally, loop O(n) models) considered in general planar domains; see also [7, Section 3] for a discussion of their relations with dimers, spin-disorders and Grassmann variables techniques.

For an edge e in a discrete domain Ω , let z_e denote its midpoint. Recall that we identify oriented edges e of Ω with the half-edges emanating from z_e in the same directions, and denote by \overline{e} the oppositely oriented edge e (equivalently, the opposite half-edge emanating from z_e).

Definition 3.5. Let a, e be two distinct half-edges in a discrete domain Ω . For a configuration $\gamma \in \operatorname{Conf}_{\Omega}(a, e)$, we define its complex phase $\phi(\gamma; a, e)$ as

$$\phi(\gamma; a, e) := \exp\left[-\frac{i}{2} \operatorname{wind}(\gamma; a, e)\right], \tag{3.8}$$

where by wind(γ ; a, e) we denote the total rotation angle of the (unique) path in γ linking the half-edges a and e, oriented from a to e (so that it starts in the direction of the half-edge a and ends in the direction opposite to e). Then, the real-valued fermionic observable $F_{\Omega}(a, e)$ is defined as

$$F_{\Omega}(a,e) := \frac{i\overline{\eta}_a \eta_e}{\mathcal{Z}_{\Omega}^+} \sum_{\operatorname{Conf}_{\Omega}(a,e)} \phi(\gamma; a, e) w(\gamma), \qquad (3.9)$$

where η_a and η_b stand for the square roots of directions of the half-edges a and e, respectively. Further, for $z_e \neq z_a$, the complex-valued fermionic observable $F_{\Omega}(a, z_e)$ is defined as

$$F_{\Omega}(a, z_e) := \frac{\overline{\eta}_a}{\mathcal{Z}_{\Omega}^+} \left[\sum_{\text{Conf}_{\Omega}(a, e)} \phi(\gamma; a, e) \mathbf{w}(\gamma) + \sum_{\text{Conf}_{\Omega}(a, \overline{e})} \phi(\gamma; a, \overline{e}) \mathbf{w}(\gamma) \right].$$
(3.10)

Above we set $\operatorname{Conf}_{\Omega}(a, \overline{b}) := \emptyset$ if e = b is a boundary half-edge. We also formally define $F_{\Omega}(a, a) := 0$ and $F_{\Omega}(a, z_a) := -i\overline{\eta}_{\overline{a}}F_{\Omega}(a, \overline{a})$.

Remark 3.4. It is easy to check that $F_{\Omega}(a, e)$ is real and $F_{\Omega}(e, a) = -F_{\Omega}(a, e)$ for all a, e, e, since wind $(\gamma; e, a) = -\text{wind}(\gamma; a, e)$ for all $\gamma \in \text{Conf}_{\Omega}(a, e)$. In particular, the function $F_{\Omega}(a, \cdot)$ satisfies the following boundary conditions:

$$\operatorname{Im}[i\eta_b F_{\Omega}(a, z_b)] = \operatorname{Im}[F_{\Omega}(a, b)] = 0 \tag{3.11}$$

for all boundary half-edges b. Also, one has

$$F_{\Omega}(a, z_e) = -i \cdot [\overline{\eta}_e F_{\Omega}(a, e) + \overline{\eta}_{\overline{e}} F_{\Omega}(a, \overline{e})].$$

Since both $F_{\Omega}(a,e)$ and $F_{\Omega}(a,\overline{e})$ are real and $\eta_{\overline{e}} = \pm i\eta_e$, this implies

$$F_{\Omega}(a,e) = -\operatorname{Im}[\eta_e F_{\Omega}(a,z_e)], \qquad F_{\Omega}(a,\overline{e}) = \mp \operatorname{Re}[\eta_e F_{\Omega}(a,z_e)]. \tag{3.12}$$

Using the Grassmann variables formalism (see [7, Sections 3.2,3.6] for the relation of combinatorial fermionic observables with this notation), for $z_e \neq z_a$, one can write

$$F_{\Omega}(a,e) = \langle \phi_a \phi_e \rangle$$
 and $F_{\Omega}(a,z_e) = \langle \psi(z_e) \phi_a \rangle$,

where $\psi(z_e) := i(\overline{\eta}_e \phi_e + \overline{\eta}_{\overline{e}} \phi_{\overline{e}})$ and the formal correlators $\prec \phi_e \phi_a \succ$ are associated with the classical Pfaffian representation of the Ising model partition function \mathcal{Z}_{Ω}^+ , see [7] for more details. Note that our definitions differ from those discussed in [7] in three respects. First, we drop the additional normalization $t_e = (x_{\text{crit}} + x_{\text{crit}}^{-1})^{1/2}$ in the definition of the complex-valued observable: this factor does not depend on e and thus is irrelevant when working with the homogeneous model. Second, (3.10) does *not* contain the additional complex factor $\exp[-i\frac{\pi}{4}]$, thus it differs by this factor from the definition used, e.g., in [8, 11, 7] and coincides with the one used, e.g., in [25, 23, 10]. Third, our definition of the set $\operatorname{Conf}_{\Omega}(a, \overline{a})$ differs from the set $\mathcal{C}(a, \overline{a})$ used in [7, Theorem 1.2]: one is the complement of the other, thus

$$F_{\Omega}(a, \overline{a}) = 1 + \langle \phi_a \phi_{\overline{a}} \rangle \quad \text{if } \eta_a \in \{1, \rho, \rho^2\}$$

(recall that, according to our choice of square roots, we have $\eta_{\overline{a}} = i\eta_a$ if $\eta_a \in \{1, \rho, \rho^2\}$).

Let us assume that $\mathfrak{b} = \emptyset$. In this case, Lemma 3.2 expresses the quantity $\mathcal{T}_{\text{mid}}(m)$ as a linear combination of sums over the sets $\text{Conf}(m_{\text{down}}, m_{\text{up}})$, $\text{Conf}(m_{\text{down}}, \overline{m}_{\text{up}})$ etc. It is easy to see that the complex phase (3.8) of a configuration γ is constant on each of these sets due to topological reasons. Therefore, one can easily rewrite $\mathcal{T}_{\text{mid}}(m)$ using several values $F_{\Omega}(m_{\text{down}}, m_{\text{up}})$, $F_{\Omega}(m_{\text{down}}, \overline{m}_{\text{up}})$ of the real fermionic observable introduced above. Before doing this, let us introduce some additional notation for discrete derivatives and mean values.

Definition 3.6. For a function F(a,e) defined on half-edges of a discrete domain Ω , an inner face w of Ω and $\eta \in \wp$, we put

$$\begin{aligned} &[\partial_1 F](w^{[\eta]}, e) := \frac{1}{\sqrt{3}} [F(w_{\text{up}}^{[\eta]}, e) - F(w_{\text{down}}^{[\eta]}, e)] \,, \quad [\varsigma_1 F](w^{[\eta]}, e) := \frac{1}{2} [F(w_{\text{up}}^{[\eta]}, e) + F(w_{\text{down}}^{[\eta]}, e)] \,, \\ &[\partial_2 F](a, w^{[\eta]}) := \frac{1}{\sqrt{3}} [F(a, w_{\text{up}}^{[\eta]}) - F(a, w_{\text{down}}^{[\eta]})] \,, \quad [\varsigma_2 F](a, w^{[\eta]}) := \frac{1}{2} [F(a, w_{\text{up}}^{[\eta]}) + F(a, w_{\text{down}}^{[\eta]})] \,, \end{aligned}$$

where $w^{[\eta]}$ denotes the midline of the face w oriented in the direction $i\eta^2$ and $w_{\text{down}}^{[\eta]}, w_{\text{up}}^{[\eta]}$ are the two boundary half-edges of w oriented in the direction η^2 , see Fig. 10.

Proposition 3.7. Let w be an inner face of Ω and $\eta \in \wp$. Then, one has

$$\mathbb{E}_{\Omega}^{+}[T^{[\eta]}(w)] = c_{T} + \frac{\sqrt{3}}{2} [\varsigma_{1} \partial_{2} F_{\Omega}](w^{[\eta]}, w^{[\eta]}),
\mathbb{E}_{\Omega}^{+}[R^{[\eta]}(w)] = c_{R} + \sqrt{3} [\partial_{1} \partial_{2} F_{\Omega}](w^{[\eta]}, w^{[i\eta]}),$$

where we assume $\eta \in \{1, \rho, \rho^2\}$ in the last equation, $T^{[\eta]}(w)$ and $R^{[\eta]}(w)$ are given by Definition 3.3, and the fermionic observable F_{Ω} is defined by (3.9).

Proof. Note that $F(w_{\text{down}}^{[\eta]}, w_{\text{down}}^{[\eta]}) = F(w_{\text{up}}^{[\eta]}, w_{\text{up}}^{[\eta]}) = 0$ and $F(w_{\text{down}}^{[\eta]}, w_{\text{up}}^{[\eta]}) = -F(w_{\text{up}}^{[\eta]}, w_{\text{down}}^{[\eta]})$. Thus,

$$\frac{\sqrt{3}}{2} [\varsigma_1 \partial_2 F_{\Omega}](w^{[\eta]}, w^{[\eta]}) = \frac{\sqrt{3}}{2} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{3}} \left[F_{\Omega}(w^{[\eta]}_{\mathrm{down}}, w^{[\eta]}_{\mathrm{up}}) - F_{\Omega}(w^{[\eta]}_{\mathrm{up}}, w^{[\eta]}_{\mathrm{down}}) \right] = \frac{1}{2} F_{\Omega}(w^{[\eta]}_{\mathrm{down}}, w^{[\eta]}_{\mathrm{up}}) \,.$$

Since $\phi(\gamma; w_{\mathrm{down}}^{[\eta]}, w_{\mathrm{up}}^{[\eta]}) = -i$ for any $\gamma \in \mathrm{Conf}_{\Omega}(w_{\mathrm{down}}^{[\eta]}, w_{\mathrm{up}}^{[\eta]})$, it is easy to see that the right-hand side coincides with the right-hand side of (3.3), see also Fig. 9. The expectation $\mathbb{E}_{\Omega}^{+}[R^{[\eta]}(w)]$ can be treated in the same way.

3.3. Dobrushin boundary conditions and two-point expectations via four-point fermionic observables. The aim of this section is to give expressions similar to Proposition 3.7 in the case of Dobrushin boundary conditions $\mathfrak{b} = \{b, b'\}$, as well as for the two-point expectations $\mathbb{E}_{\Omega}^+[T^{[\eta]}(w)T^{[\mu]}(w')]$ and $\mathbb{E}_{\Omega}^+[T^{[\eta]}(w)\varepsilon(a)]$. Recall that, for an (oriented) edge a of Ω , we define the energy density $\varepsilon(a)$ as

$$\varepsilon(a) := \sigma(a^+)\sigma(a^-) - \frac{2}{3} = \varepsilon(\overline{a}),$$
 (3.13)

where a^+ and a^- are the two adjacent faces to a and the lattice-dependent constant $\frac{2}{3}$ corresponds to the infinite-volume limit of the Ising model on the honeycomb lattice (e.g., it should be replaced by $\sqrt{2}/2$ when working on the square lattice). Since wind($\gamma; a, \overline{a}$) = $\pm 2\pi$ for any configuration $\gamma \in \text{Conf}_{\Omega}(a, \overline{a})$, it easily follows from (3.9) that

$$\mathbb{E}_{\Omega}^{+}[\varepsilon(a)] = 1 - 2(\mathcal{Z}_{\Omega}^{+})^{-1} W_{\Omega}^{+}(a, \overline{a}) - \frac{2}{3} = \frac{1}{3} - 2F_{\Omega}(a, \overline{a}) \quad \text{if} \quad \eta_{a} \in \{1, \rho, \rho^{2}\}.$$
 (3.14)

In order to handle the correlations mentioned above, we need a combinatorial definition of four-point fermionic observables. Given four distinct half-edges a_1, a_2, a_3, a_4 of Ω , let $\operatorname{Conf}_{\Omega}(a_1, a_2 | a_3, a_4)$ denote the set of configurations $\gamma \in \operatorname{Conf}_{\Omega}(a_1, a_2, a_3, a_4)$ containing several loops and two paths linking a_1 with a_2 and a_3 with a_4 . Note that

$$\operatorname{Conf}_{\Omega}(a_1, a_2, a_3, a_4) = \operatorname{Conf}_{\Omega}(a_1, a_2 | a_3, a_4) \sqcup \operatorname{Conf}_{\Omega}(a_1, a_3 | a_2, a_4) \sqcup \operatorname{Conf}_{\Omega}(a_1, a_4 | a_2, a_3)$$

as we are dealing with subsets of a trivalent lattice. The next combinatorial definition of fourpoint fermionic observables can be thought about as a generalization of the definition (3.9). The 2n-point version of this construction was used in [23] to treat the scaling limits of expectations $\mathbb{E}_{\mathcal{O}}^{\mathfrak{g}}[\varepsilon(a_1)...\varepsilon(a_n)]$ on the square lattice, for any boundary conditions $\mathfrak{b} = \{b_1, \ldots, b_{2m}\}$.

Definition 3.8. Let a_1, a_2, a_3, a_4 be four distinct half-edges in a discrete domain Ω . For a configuration $\gamma \in \text{Conf}_{\Omega}(a_1, a_2|a_3, a_4)$, we define its complex phase $\phi(\gamma; a_1, a_2|a_3, a_4)$ as

$$\phi(\gamma; a_1, a_2 | a_3, a_4) := \exp\left[-\frac{i}{2}(\text{wind}(\gamma; a_1, a_2) + \text{wind}(\gamma; a_3, a_4))\right],$$

where wind(γ ; a_1 , a_2) and wind(γ ; a_3 , a_4) stand for the total rotation angles of the (unique) paths in γ linking a_1 with a_2 and a_3 with a_4 , oriented from a_1 to a_2 and from a_3 to a_4 , respectively. Then, we put

$$F_{\Omega}(a_1,a_2|a_3,a_4) \ := \ \frac{(i\overline{\eta}_{a_1}\eta_{a_2})(i\overline{\eta}_{a_3}\eta_{a_4})}{\mathcal{Z}_{\Omega}^+} \sum_{\gamma \in \mathrm{Conf}_{\Omega}(a_1,a_2|a_3,a_4)} \phi(\gamma;a_1,a_2|a_3,a_4) \mathbf{w}(\gamma)$$

and define the real-valued four-point fermionic observable $F_{\Omega}(a_1, a_2, a_3, a_4)$ by

$$F_{\Omega}(a_1, a_2, a_3, a_4) := F_{\Omega}(a_1, a_2|a_3, a_4) - F_{\Omega}(a_1, a_3|a_2, a_4) + F_{\Omega}(a_1, a_4|a_2, a_3).$$

The next lemma reflects the fermionic structure of these combinatorial observables.

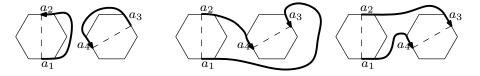


FIGURE 11. Three possible ways to connect four half-edges a_1 , a_2 , a_3 and a_4 (the orientation of curves are chosen as in Lemma 3.9). The complex phases are $\exp\left[-\frac{i}{2}(\frac{\pi}{2}+\frac{\pi}{2})\right]$, $\exp\left[-\frac{i}{2}(\frac{5\pi}{3}-\frac{\pi}{3})\right]$ and $\exp\left[-\frac{i}{2}(-\frac{\pi}{3}-\frac{\pi}{3})\right]$, respectively.

Lemma 3.9. Let a_1, a_2, a_3, a_4 be four distinct half-edges of Ω . Then, the following Pfaffian formula is fulfilled:

$$F_{\Omega}(a_{1},a_{2},a_{3},a_{4}) = F_{\Omega}(a_{1},a_{2})F_{\Omega}(a_{3},a_{4}) - F_{\Omega}(a_{1},a_{3})F_{\Omega}(a_{2},a_{4}) + F_{\Omega}(a_{1},a_{4})F_{\Omega}(a_{2},a_{3}).$$
In particular, $F_{\Omega}(a_{1},a_{2},a_{3},a_{4})$ is an antisymmetric function of its arguments: for any permutation $\varpi \in S_{4}$, one has $F_{\Omega}(a_{\varpi(1)},a_{\varpi(2)},a_{\varpi(3)},a_{\varpi(4)}) = (-1)^{\operatorname{sgn}(\varpi)}F_{\Omega}(a_{1},a_{2},a_{3},a_{4}).$

Proof. A similar result was proved in [23, Proposition 84] in the square lattice setup. This proof is based on a uniqueness theorem for the complex-valued fermionic observables viewed as the so-called s-holomorphic functions defined in a discrete domain Ω ; the same argument can be also found in [11, Section 3.5]. In principle, it can be easily repeated in the honeycomb lattice setup, but we prefer to refer the reader to a general version of the same result provided, e.g., by [7, Theorem 1.2]. In [7], the definition of the observable $F_{\Omega}(a_1, a_2, a_3, a_4)$ given above appears as a combinatorial expansion of the Pfaffian expression of the formal correlator $\langle \phi_{a_1} \phi_{a_2} \phi_{a_3} \phi_{a_4} \rangle$ using the Grassmann variables notation associated with the partition function of the Ising model. The only special case which should be taken care of is when the set $\{a_1, a_2, a_3, a_4\}$ contains a pair of opposite half-edges, since in this case our definition of the set $Conf_{\Omega}(a_1, a_2, a_3, a_4)$ differs from the set $C(a_1, a_2, a_3, a_4)$ appearing in the expansion provided by [7, Theorem 1.2], cf. Remark 3.4. If, say, $a_4 = \overline{a_3}$ and $\eta_{a_3} \in \{1, \rho, \rho^2\}$ (while $a_2 \neq \overline{a_1}$), then $Conf_{\Omega}(a_1, a_2, a_3, \overline{a_3}) = C(a_1, a_2) \setminus C(a_1, a_2, a_3, \overline{a_3})$ and one can write

$$F_{\Omega}(a_1, a_2, a_3, \overline{a}_3) = \langle \phi_{a_1} \phi_{a_2} \rangle + \langle \phi_{a_1} \phi_{a_2} \phi_{a_3} \phi_{\overline{a}_3} \rangle$$

which still implies the correct result since $F_{\Omega}(a_3, \overline{a}_3) = 1 + \langle \phi_{a_3} \phi_{\overline{a}_3} \rangle$. If both $a_1 = \overline{a}_2$ and $a_4 = \overline{a}_3$ with $\eta_{a_1}, \eta_{a_3} \in \{1, \rho, \rho^2\}$, then the inclusion-exlusion formula applied to the sets $\mathcal{C}(\varnothing)$, $\mathcal{C}(a_1, \overline{a}_1)$, $\mathcal{C}(a_3, \overline{a}_3)$ and $\mathcal{C}(a_1, a_2, a_3, a_4)$ to get the set $\mathrm{Conf}_{\Omega}(a_1, a_2, a_3, a_4)$ leads to

$$F_{\Omega}(a_1, \overline{a}_1, a_3, \overline{a}_3) = 1 + \langle \phi_{a_1} \phi_{\overline{a}_1} \rangle + \langle \phi_{a_3} \phi_{\overline{a}_3} \rangle + \langle \phi_{a_1} \phi_{a_2} \phi_{a_3} \phi_{\overline{a}_3} \rangle ,$$

which eventually gives the same Pfaffian formula.

We are now able to give an analogue of Proposition 3.7 for Dobrushin boundary conditions.

Proposition 3.10. Let w be an inner face of Ω and $\eta \in \wp$. Then, for any two boundary edges b, b' of Ω , one has

$$\mathbb{E}_{\Omega}^{b,b'}[T^{[\eta]}(w)] = \mathbb{E}_{\Omega}^{+}[T^{[\eta]}(w)] + \frac{\sqrt{3}}{2} \left[F_{\Omega}(b,b') \right]^{-1} \\ \times \left[[\partial_{1}F_{\Omega}](w^{[\eta]},b) \cdot [\varsigma_{1}F_{\Omega}](w^{[\eta]},b') - [\varsigma_{1}F_{\Omega}](w^{[\eta]},b) \cdot [\partial_{1}F_{\Omega}](w^{[\eta]},b') \right],$$

$$\mathbb{E}_{\Omega}^{b,b'}[R^{[\eta]}(w)] = \mathbb{E}_{\Omega}^{+}[R^{[\eta]}(w)] - \sqrt{3} \left[F_{\Omega}(b,b') \right]^{-1} \\ \times \left[[\partial_{1}F_{\Omega}](w^{[\eta]},b) \cdot [\partial_{1}F_{\Omega}](w^{[i\eta]},b') - [\partial_{1}F_{\Omega}](w^{[i\eta]},b) \cdot [\partial_{1}F_{\Omega}](w^{[\eta]},b') \right],$$

where we assume $\eta \in \{1, \rho, \rho^2\}$ in the last equation, $T^{[\eta]}(w)$ and $R^{[\eta]}(w)$ are given by Definition 3.3, and the fermionic observable F_{Ω} is defined by (3.9).

Proof. Let $\phi^{b,b'}$ denote the common value of $\phi(\gamma; b, b')$ for $\gamma \in \operatorname{Conf}_{\Omega}(\mathfrak{b})$, which is fixed by topological reasons and does not depend on γ . Then, one can easily check that

$$\begin{split} F_{\Omega}(b, w_{\mathrm{down}}^{[\eta]} \mid w_{\mathrm{up}}^{[\eta]}, b') &= \frac{\left(i\overline{\eta}_b \eta\right) \left(i\overline{\eta}\eta_{b'}\right)}{\mathcal{Z}_{\Omega}^{+}} \cdot \left(-i\phi^{b,b'}\right) \cdot \mathbf{W}_{\Omega}(b, w_{\mathrm{up}}^{[\eta]} \mid w_{\mathrm{down}}^{[\eta]}, b') \,, \\ F_{\Omega}(b, w_{\mathrm{up}}^{[\eta]} \mid w_{\mathrm{down}}^{[\eta]}, b') &= \frac{\left(i\overline{\eta}_b \eta\right) \left(i\overline{\eta}\eta_{b'}\right)}{\mathcal{Z}_{\Omega}^{+}} \cdot i\phi^{b,b'} \cdot \mathbf{W}_{\Omega}(b, w_{\mathrm{up}}^{[\eta]} \mid w_{\mathrm{down}}^{[\eta]}, b') \,, \\ F_{\Omega}(b, b' \mid w_{\mathrm{down}}^{[\eta]}, w_{\mathrm{up}}^{[\eta]}) &= \frac{\left(i\overline{\eta}_b \eta_{b'}\right) \left(i\overline{\eta}\eta\right)}{\mathcal{Z}_{\Omega}^{+}} \cdot \left(-i\phi^{b,b'}\right) \cdot \mathbf{W}_{\Omega}(b, b' \mid w_{\mathrm{down}}^{[\eta]}, w_{\mathrm{up}}^{[\eta]}) \,, \end{split}$$

where we are using the notation $W_{\Omega}(a_1, a_2 \mid a_3, a_4) := \sum_{\gamma \in \operatorname{Conf}_{\Omega}(a_1, a_2 \mid a_3, a_4)} w(\gamma)$. Plugging these expressions into the definition of $F_{\Omega}(b, w_{\text{down}}^{[\eta]}, w_{\text{up}}^{[\eta]}, b')$, we get

$$F_{\Omega}(b, w_{\text{down}}^{[\eta]}, w_{\text{up}}^{[\eta]}, b') = \frac{i\overline{\eta}_b \eta_{b'}}{\mathcal{Z}_{\Omega}^+} \cdot \phi^{b,b'} \cdot W_{\Omega}^{b,b'}(w_{\text{down}}^{[\eta]}, w_{\text{up}}^{[\eta]}) = \frac{F_{\Omega}(b, b')}{\mathcal{Z}_{\Omega}^{b,b'}} \cdot W_{\Omega}^{b,b'}(w_{\text{down}}^{[\eta]}, w_{\text{up}}^{[\eta]}),$$

where we used the definition of $F_{\Omega}(b,b')$ to derive the last equality. Together with (3.3) and the definition of $F_{\Omega}(w_{\text{down}}^{[\eta]}, w_{\text{up}}^{[\eta]})$ this implies

$$\begin{split} \mathbb{E}_{\Omega}^{b,b'}[T^{[\eta]}(w)] - \mathbb{E}_{\Omega}^{+}[T^{[\eta]}(w)] &= \frac{1}{2} \left[\frac{1}{\mathcal{Z}_{\Omega}^{b,b'}} \cdot \mathbf{W}_{\Omega}^{[\eta]}(w_{\mathrm{down}}^{[\eta]}, w_{\mathrm{up}}^{[\eta]}) - \frac{1}{\mathcal{Z}_{\Omega}^{+}} \cdot \mathbf{W}_{\Omega}^{+}(w_{\mathrm{down}}^{[\eta]}, w_{\mathrm{up}}^{[\eta]}) \right] \\ &= \frac{1}{2} \left[\frac{F_{\Omega}(b, w_{\mathrm{down}}^{[\eta]}, w_{\mathrm{up}}^{[\eta]}, b')}{F_{\Omega}(b, b')} - F_{\Omega}(w_{\mathrm{down}}^{[\eta]}, w_{\mathrm{up}}^{[\eta]}) \right] \\ &= \frac{1}{2F_{\Omega}(b, b')} \left[F_{\Omega}(b, w_{\mathrm{down}}^{[\eta]}) F_{\Omega}(w_{\mathrm{up}}^{[\eta]}, b') - F_{\Omega}(b, w_{\mathrm{up}}^{[\eta]}) F_{\Omega}(w_{\mathrm{down}}^{[\eta]}, b') \right] \,. \end{split}$$

where the last equality is a consequence of the Pfaffian formula (see Lemma 3.9). The desired expression for $\mathbb{E}_{\Omega}^{b,b'}[T^{[\eta]}(w)]$ follows immediately. The case of $\mathbb{E}_{\Omega}^{b,b'}[R^{[\eta]}(w)]$ is analogous. \square

Exactly the same techniques can be used to derive expressions of the two-point expectations $\mathbb{E}^+_{\Omega}[T^{[\eta]}(w)T^{[\mu]}(w')]$ and $\mathbb{E}^+_{\Omega}[T^{[\eta]}(w)\varepsilon(a)]$ in terms of two-point fermionic observables F_{Ω} .

Proposition 3.11. (i) Let w, w' be two non-adjacent inner faces of Ω and $\eta, \mu \in \wp$. Then, one has

$$\begin{split} \mathbb{E}^{+}_{\Omega}[T^{[\eta]}(w)T^{[\mu]}(w')] &= \mathbb{E}^{+}_{\Omega}[T^{[\eta]}(w)] \cdot \mathbb{E}^{+}_{\Omega}[T^{[\mu]}(w')] \\ &+ \frac{3}{4} \left[[\varsigma_{1}\partial_{2}F_{\Omega}](w^{[\eta]}, w'^{[\mu]}) \cdot [\partial_{1}\varsigma_{2}F_{\Omega}](w^{[\eta]}, w'^{[\mu]}) - [\varsigma_{1}\varsigma_{2}F_{\Omega}](w^{[\eta]}, w'^{[\mu]}) \cdot [\partial_{1}\partial_{2}F_{\Omega}](w^{[\eta]}, w'^{[\mu]}) \right]. \end{split}$$

(ii) Let w be an inner face of Ω , $\eta \in \wp$, and a be an inner, non-adjacent to w, edge of Ω , oriented in one of the directions $1, \rho^2, \rho^4$. Then, one has

$$\begin{split} \mathbb{E}_{\Omega}^{+}[T^{[\eta]}(w)\varepsilon(a)] &= \mathbb{E}_{\Omega}^{+}[T^{[\eta]}(w)] \cdot \mathbb{E}_{\Omega}^{+}\left[\varepsilon(a)\right] \\ &+ \sqrt{3}\left[\left[\varsigma_{1}F_{\Omega}\right](w^{[\eta]},a) \cdot \left[\partial_{1}F_{\Omega}\right](w^{[\eta]},\overline{a}) - \left[\partial_{1}F_{\Omega}\right](w^{[\eta]},a) \cdot \left[\varsigma_{1}F_{\Omega}\right](w^{[\eta]},\overline{a})\right], \end{split}$$

where the local field (energy density) $\varepsilon(a)$ is defined by (3.13).

Remark 3.5. As in Propositions 3.7 and 3.10, one can write down similar expressions for the two-point expectations $\mathbb{E}_{\Omega}^+[R^{[\eta]}(w)T^{[\mu]}(w')]$, $\mathbb{E}_{\Omega}^+[R^{[\eta]}(w)R^{[\mu]}(w')]$ and $\mathbb{E}_{\Omega}^+[R^{[\eta]}(w)\varepsilon(a)]$. They all contain one more discrete derivative operator ∂ instead of ς comparing to the same expectations with the local field $R^{[\eta]}(w)$ replaced by $T^{[\eta]}(w)$, and thus disappear in the scaling limit. We do not include these explicit expressions in Proposition 3.11 for shortness.

Proof. In the same way as in the proofs of Propositions 3.10 and 3.4 one gets

$$\begin{split} \mathbb{E}_{\Omega}^{+}[T^{[\eta]}(w)T^{[\mu]}(w')] - \mathbb{E}_{\Omega}^{+}[T^{[\eta]}(w)] \mathbb{E}_{\Omega}^{+}[T^{[\mu]}(w')] \\ &= \frac{1}{4} \left[F_{\Omega}(w_{\text{down}}^{[\eta]}, w_{\text{down}}^{\prime [\mu]}, w_{\text{up}}^{\prime [\mu]}, w_{\text{up}}^{[\eta]}) - F_{\Omega}(w_{\text{down}}^{[\eta]}, w_{\text{up}}^{[\eta]}) F_{\Omega}(w_{\text{down}}^{\prime [\mu]}, w_{\text{up}}^{\prime [\mu]}) \right] \\ &= \frac{1}{4} \left[F_{\Omega}(w_{\text{down}}^{[\eta]}, w_{\text{down}}^{\prime [\mu]}) F_{\Omega}(w_{\text{up}}^{\prime [\mu]}, w_{\text{up}}^{[\eta]}) - F_{\Omega}(w_{\text{down}}^{[\eta]}, w_{\text{up}}^{\prime [\mu]}) F_{\Omega}(w_{\text{down}}^{\prime [\mu]}, w_{\text{up}}^{[\eta]}) \right]. \end{split}$$

Then the expression for $\mathbb{E}^+_{\Omega}[T^{[\eta]}(w)T^{[\mu]}(w')]$ follows from the identity

$$\begin{split} &\frac{1}{4} \left[F_{\Omega}(w_{\text{down}}^{[\eta]}, w_{\text{down}}^{\prime [\mu]}) F_{\Omega}(w_{\text{up}}^{\prime [\mu]}, w_{\text{up}}^{[\eta]}) - F_{\Omega}(w_{\text{down}}^{[\eta]}, w_{\text{up}}^{\prime [\mu]}) F_{\Omega}(w_{\text{down}}^{\prime [\mu]}, w_{\text{up}}^{[\eta]}) \right] \\ &= \frac{3}{4} \left[[\varsigma_{1} \partial_{2} F_{\Omega}](w^{[\eta]}, w^{\prime [\mu]}) \cdot [\partial_{1} \varsigma_{2} F_{\Omega}](w^{[\eta]}, w^{\prime [\mu]}) - [\varsigma_{1} \varsigma_{2} F_{\Omega}](w^{[\eta]}, w^{\prime [\mu]}) \cdot [\partial_{1} \partial_{2} F_{\Omega}](w^{[\eta]}, w^{\prime [\mu]}) \right]. \end{split}$$

The formula for $\mathbb{E}_{\Omega}^+[T^{[\eta]}(w)\varepsilon(a)]$ can be derived in the same way. One just needs to substitute $w'^{[\mu]}_{\mathrm{down}}, w'^{[\mu]}_{\mathrm{up}}$ with a, \overline{a} and use the identity $\mathbb{E}_{\Omega}^+[\varepsilon(a) - \frac{1}{3}] = -2F_{\Omega}(a, \overline{a})$, see (3.14).

3.4. Spinor observables and correlations with the spin field. In order to handle the two-point expectations involving the spin field $\sigma(u)$, we need a generalization of combinatorial observables discussed in Section 3.2: the so-called spinor observables. This combinatorial definition was proposed in [11] and further in [10] as a tool to prove the convergence (in the square lattice setup) of spin correlations in general planar domains to their scaling limits. We also refer the reader to [7] for a discussion of the relations of this definition with the Grassmann variables formalism and the notion of spin structures.

For a simply connected discrete domain Ω and its inner face $u \in \Omega$, we denote by $[\Omega, u]$ the double-cover of Ω branching around u (and not branching around any other face) and by $\pi : [\Omega, u] \to \Omega$ the corresponding projection mapping. For an (oriented) edge e of $[\Omega, u]$, we denote by $e^* \neq e$ the other (oriented) edge of $[\Omega, u]$ such that $\pi(e^*) = \pi(e)$. One can view the next definition as a generalization of Definition 3.5 in the presence of the branching.

Definition 3.12. Let $[\Omega, u]$ be the double-cover of a simply connected discrete domain Ω branching around its inner face u. For two half-edges a, e of $[\Omega, u]$ such that $\pi(a) \neq \pi(e)$ and a configuration $\gamma \in \mathrm{Conf}_{\Omega}(\pi(a), \pi(e))$, let $\mathrm{loops}_{[u]}(\gamma)$ denote the number of loops in γ that surround u and

$$\phi_{[u]}(\gamma; a, e) := \exp\left[-\frac{i}{2} \operatorname{wind}_{[u]}(\gamma; a, e)\right] \cdot (-1)^{\operatorname{loops}_{[u]}(\gamma)}, \tag{3.15}$$

where wind_[u](γ ; a, e) := wind(γ ; π (a), π (e)) if the path linking the projections π (a) and π (e) in γ lifts to a path linking a and e on the double-cover $[\Omega, u]$, and wind(γ ; π (a), π (e)) + 2π if this path lifts to a path linking a with e^* on $[\Omega, u]$. Then, the real-valued spinor observable $F_{[\Omega, u]}(a, e)$ is defined as

$$F_{[\Omega,u]}(a,e) := \frac{i\overline{\eta}_{\pi(a)}\eta_{\pi(e)}}{\mathbb{E}_{\Omega}^{+}[\sigma(u)] \cdot \mathcal{Z}_{\Omega}^{+}} \sum_{\operatorname{Conf}_{\Omega}(\pi(a),\pi(e))} \phi_{[u]}(\gamma;a,e) \mathbf{w}(\gamma). \tag{3.16}$$

We also formally set $F_{[\Omega,u]}(a,a) = F_{[\Omega,u]}(a,a^*) := 0$ and, for a midedge z_e on $[\Omega,u]$, define the complex-valued spinor observable $F_{[\Omega,u]}(a,z_e)$ by

$$F_{\Omega}(a,z_e) := -i \cdot \left[\overline{\eta}_{\pi(e)} F_{[\Omega,u]}(a,e) + \overline{\eta}_{\pi(\overline{e})} F_{[\Omega,u]}(a,\overline{e}) \right].$$

Remark 3.6. As in the non-branching case, one can easily check that $F_{[\Omega,u]}(a,e)$ is real and anti-symmetric: $F_{[\Omega,u]}(a,e) = -F_{[\Omega,u]}(e,a)$. Moreover, it directly follows form the definition of wind_[u]($\gamma; a, e$) that $F_{[\Omega,u]}(a,e^*) = -F_{[\Omega,u]}(a,e)$ for all a, e in $[\Omega,u]$. Functions obeying this sign-flip symmetry between the two sheets of the double-cover $[\Omega,u]$ are often called *spinors*.

The next proposition is a straightforward analogue of Proposition 3.7. Similarly to Remark 3.5, we do not include a formula for the correlations $\mathbb{E}_{\mathcal{O}}^+[R^{[\eta]}(w)\sigma(u)]$ for shortness.

Proposition 3.13. Let w, u be two non-adjacent inner faces of Ω and $\eta \in \wp$. Then, one has

$$\frac{\mathbb{E}_{\Omega}^{+}[T^{[\eta]}(w)\sigma(u)]}{\mathbb{E}_{\Omega}^{+}[\sigma(u)]} = \frac{3}{\pi} - \frac{2}{\sqrt{3}} + \frac{\sqrt{3}}{2}[\varsigma_{1}\partial_{2}F_{[\Omega,u]}](w^{[\eta]}, w^{[\eta]}),$$

where the spinor observable $F_{[\Omega,u]}$ is defined by (3.16) and the arguments $w_{\mathrm{up}}^{[\eta]}, w_{\mathrm{down}}^{[\eta]}$ of $F_{[\Omega,u]}$ are assumed to be lifted on the same sheet of the double-cover $[\Omega, u]$.

Proof. For each $\gamma \in \operatorname{Conf}_{\Omega}(\pi(w_{\operatorname{down}}^{[\eta]}), \pi(w_{\operatorname{up}}^{[\eta]}))$, denote by γ' the configuration obtained from γ by making the XOR operation on the right semi-hexagon of $\pi(w^{[\eta]})$. Then one has

$$\phi_{[u]}(\gamma; w_{\text{down}}^{[\eta]}, w_{\text{up}}^{[\eta]}) = -i(-1)^{\text{loops}_{[u]}(\gamma')}$$

and $(-1)^{\text{loops}_{[u]}(\gamma')}$ is just the value of $\sigma(u)$ in the spin configuration corresponding to γ' . Using the same arguments as in Proposition 3.4, we get the following analogue of (3.3):

$$\frac{\mathbb{E}_{\Omega}^{+}[T^{[\eta]}(w)\sigma(u)]}{\mathbb{E}_{\Omega}^{+}[\sigma(u)]} - c_{T} = \frac{i\sum_{\gamma \in \text{Conf}_{\Omega}(\pi(w_{\text{down}}^{[\eta]}), \pi(w_{\text{up}}^{[\eta]}))} w(\gamma)\phi_{[u]}(\gamma; w_{\text{down}}^{[\eta]}, w_{\text{up}}^{[\eta]})}{2\mathcal{Z}^{+}\mathbb{E}_{\Omega}^{+}[\sigma(u)]} \\
= \frac{1}{2}F_{[\Omega,u]}(w_{\text{down}}^{[\eta]}, w_{\text{up}}^{[\eta]}) = \frac{\sqrt{3}}{2}[\varsigma_{1}\partial_{2}F_{[\Omega,u]}](w^{[\eta]}, w^{[\eta]}),$$

where the first equality follows from our considerations of $\phi_{[u]}(\gamma; w_{\text{down}}^{[\eta]}, w_{\text{up}}^{[\eta]})$, the second uses the definition of $F_{[\Omega,u]}$ and the third repeats the one in Proposition 3.7.

4. Ising correlation functions in continuum

In this section we briefly discuss correlation functions $\langle T(w)\rangle_{\Omega}^{+}$, $\langle T(w)\rangle_{\Omega}^{b,b'}$, $\langle T(w)T(w')\rangle_{\Omega}^{+}$ etc coming from the (free fermion, central charge $\frac{1}{2}$) Conformal Field Theory, which is known to describe the scaling limit of the 2D Ising model at criticality (e.g., see [41, Sec. 14.2]). It is worth noting that we do not refer to standard CFT tools along this discussion. Instead, we use the language of Riemann-type boundary value problems for holomorphic functions in Ω in order to define all the correlation functions in continuum that we need to formulate Theorems 1.1–1.3. This approach has two particular advantages. First, it does not refer to involved CFT concepts and reduces the discussion below to simple properties of holomorphic functions (of one complex variable) considered in a given planar Ω . Second, it is adapted to our proofs of Theorems 1.1–1.3, which are presented in Section 5 and based on convergence results for solutions to similar discrete boundary value problems. It is worth noting that the same strategy was used in [25, 23] and [11, 10] to prove the convergence of energy-density and spin correlations, respectively, to their scaling limits predicted by the CFT.

4.1. Fermionic correlators as solutions to boundary value problems for holomorphic functions. For a while, let us assume that Ω is a bounded simply connected domain with a smooth boundary $\partial\Omega$, and, for $\zeta \in \partial\Omega$, let $\nu_{\text{out}}(\zeta)$ denote the outward-oriented unit normal vector to $\partial\Omega$ at the point ζ , considered as a complex number.

Definition 4.1. Let Ω be a bounded simply connected domain with a smooth boundary, $a \in \Omega$, and $\eta \in \mathbb{C}$. We define $f_{\Omega}^{[\eta]}(a,\cdot)$ to be the (unique) holomorphic in $\Omega \setminus \{a\}$ and continuous up to $\partial \Omega$ function such that the function $f_{\Omega}^{[\eta]}(a,z) - \eta \cdot (z-a)^{-1}$ is bounded as $z \to a$ and

$$\operatorname{Im}\left[f_{\Omega}^{[\eta]}(a,\zeta)\sqrt{\nu_{\operatorname{out}}(\zeta)}\right] = 0 \quad \text{for all } \zeta \in \partial\Omega. \tag{4.1}$$

Remark 4.1. (i) The uniqueness of solution to this boundary value problem can be easily deduced from the following consideration: the difference f(z) of any two solutions should be holomorphic everywhere in Ω , which implies $\oint_{\partial\Omega} (f(\zeta))^2 d\zeta = 0$ and contradicts to the boundary conditions.

(ii) Let $\varphi: \Omega \to \Omega'$ be a conformal mapping and $\eta' = \eta \cdot (\varphi'(a))^{\frac{1}{2}}$. Then one has

$$f_{\Omega}^{[\eta]}(a,z) = f_{\Omega'}^{[\eta']}(\varphi(a), \varphi(z)) \cdot (\varphi'(z))^{\frac{1}{2}}, \quad a, z \in \Omega.$$
 (4.2)

Indeed, the right-hand side satisfies all the conditions used to define $f_{\Omega}^{[\eta]}(a,z)$, thus this covariance rule follows from the uniqueness property discussed above.

(iii) The existence of $f_{\Omega}^{[\eta]}(a,z)$ can be easily derived by solving the corresponding boundary value problem in the upper half-plane $\mathbb H$ and then applying (4.2) with $\varphi:\Omega\to\mathbb H$, but we prefer to postpone these explicit formulae, taking this existence for granted until Proposition 4.3. Note that the explicit formulae (4.7) and the decomposition (4.4) given below can be also used to define the functions $f_{\Omega}^{[\eta]}(a,z)$ for rough and/or unbounded domains Ω .

Proposition 4.2. Let Ω be a bounded simply connected domain with a smooth boundary, and let $a \in \Omega$. Then, there exists unique pair of holomorphic in $\Omega \setminus \{a\}$ and continuous up to $\partial\Omega$ functions $f_{\Omega}(a,\cdot)$ and $f_{\Omega}^{\dagger}(a,\cdot)$ such that the functions $f_{\Omega}^{\sharp}(a,z) := f_{\Omega}(a,z) - (z-a)^{-1}$ and $f_{\Omega}^{\dagger}(a,z)$ are bounded as $z \to a$ and

$$f_{\Omega}^{\dagger}(a,\zeta) = \overline{i\nu_{\text{out}}(\zeta) \cdot f_{\Omega}(a,\zeta)} \quad \text{for all } \zeta \in \partial\Omega.$$
 (4.3)

Moreover, for any $\eta \in \mathbb{C}$ and $z \in \Omega \setminus \{a\}$, one has

$$f_{\Omega}^{[\eta]}(a,z) = \eta f_{\Omega}(a,z) + i\overline{\eta} f_{\Omega}^{\dagger}(a,z). \tag{4.4}$$

Proof. It is easy to see that, for any $\alpha, \beta \in \mathbb{R}$ and $\eta, \mu \in \mathbb{C}$, one has

$$f_{\Omega}^{[\alpha\eta+\beta\mu]}(a,z) = \alpha f_{\Omega}^{[\eta]}(a,z) + \beta f_{\Omega}^{[\mu]}(a,z),$$

since the right-hand side satisfies all the conditions from Definition 4.1 that determine the function $f_{\Omega}^{[\alpha\eta+\beta\mu]}(a,\cdot)$. In other words, for any $z\in\Omega\setminus\{a\}$, the quantity $f_{\Omega}^{[\eta]}(a,z)$ is a real-linear functional of $\eta\in\mathbb{C}$, and hence identity (4.4) holds true for some $f_{\Omega}(a,z)$ and $f_{\Omega}^{\dagger}(a,z)$. Since these functions can be easily represented as linear combinations of $f_{\Omega}^{[\eta]}(a,z)$, they both are holomorphic in $\Omega\setminus\{a\}$ and continuous up to $\partial\Omega$, and their behavior as $z\to a$ follows immediately. Moreover, boundary conditions (4.1) considered for all $\eta\in\mathbb{C}$ simultaneously are equivalent to boundary conditions (4.3).

Remark 4.2. (i) The functions $f_{\Omega}(a,z)$ and $f_{\Omega}^{\dagger}(a,z)$ have the following covariances, which easily follow from (4.2) and (4.4), under conformal mappings $\varphi: \Omega \to \Omega'$:

$$f_{\Omega}(a,z) = f_{\Omega'}(\varphi(a), \varphi(z)) \cdot (\varphi'(a)\varphi'(z))^{\frac{1}{2}}, \qquad a, z \in \Omega.$$

$$f_{\Omega}^{\dagger}(a,z) = f_{\Omega'}^{\dagger}(\varphi(a), \varphi(z)) \cdot (\overline{\varphi'(a)}\varphi'(z))^{\frac{1}{2}}, \qquad (4.5)$$

(ii) Let $\eta, \mu \in \mathbb{C}$ and $a, z \in \Omega$. Applying the Cauchy residue theorem to the product of the functions $f_{\Omega}^{[\eta]}(a,\cdot)$ and $f_{\Omega}^{[\mu]}(z,\cdot)$ and taking into account boundary conditions (4.1), one obtains

$$0 = 2\pi \cdot \operatorname{Re} \left[\oint_{\partial \Omega} f_{\Omega}^{[\eta]}(a,\zeta) f_{\Omega}^{[\mu]}(z,\zeta) d\zeta \right] = \operatorname{Im} \left[\eta f_{\Omega}^{[\mu]}(z,a) + \mu f_{\Omega}^{[\eta]}(a,z) \right].$$

Since η and μ can be chosen arbitrary, the decomposition (4.4) easily implies the identities

$$f_{\Omega}(z,a) = -f_{\Omega}(a,z), \qquad f_{\Omega}^{\dagger}(z,a) = -\overline{f_{\Omega}^{\dagger}(a,z)}.$$
 (4.6)

In particular, the function $f_{\Omega}(\cdot, z)$ is holomorphic in $\Omega \setminus \{z\}$ while the function $f_{\Omega}^{\dagger}(\cdot, z)$ is anti-holomorphic in Ω .

Proposition 4.3. Let Ω be a bounded simply connected domain with a smooth boundary and $\varphi: \Omega \to \mathbb{H}$ be a conformal mapping from Ω onto the upper half-plane \mathbb{H} . Then, for any $a, z \in \Omega$, one has

$$f_{\Omega}(a,z) = \frac{(\varphi'(a)\varphi'(z))^{\frac{1}{2}}}{\varphi(z) - \varphi(a)}, \qquad f_{\Omega}^{\dagger}(a,z) = \frac{(\overline{\varphi'(a)}\varphi'(z))^{\frac{1}{2}}}{\varphi(z) - \overline{\varphi(a)}}. \tag{4.7}$$

Proof. Note that the functions $f_{\mathbb{H}}(a,z) := (z-a)^{-1}$ and $f_{\mathbb{H}}^{\dagger}(a,z) := (z-\overline{a})^{-1}$ satisfy the conformal covariance rules (4.5) for all Möbius automorphisms $\varphi : \mathbb{H} \to \mathbb{H}$ and solve the boundary value problem from Proposition 4.2 in the upper half-plane \mathbb{H} :

$$f_{\mathbb{H}}^{\dagger}(a,\zeta) = \overline{f_{\mathbb{H}}(a,\zeta)} \quad \text{for} \ \ a \in \mathbb{H} \ \ \text{and} \ \ \zeta \in \partial \mathbb{H} = \mathbb{R}.$$

Therefore, the functions $f_{\Omega}(a,\cdot)$ and $f_{\Omega}^{\dagger}(a,\cdot)$ given by (4.7) do not depend on the choice of the uniformization map $\varphi:\Omega\to\mathbb{H}$, are holomorphic in $\Omega\setminus\{a\}$ and continuous up to $\partial\Omega$, have the correct behavior as $z\to a$, and satisfy boundary conditions (4.3) everywhere on $\partial\Omega$. \square

Remark 4.3. Adopting the colloquial CFT language (e.g., see [19, Sec. 4, pp.20-21] for the discussion of boundary conditions in the upper half-plane), one can write

$$f_{\Omega}(a,z) = \langle \psi(z)\psi(a) \rangle_{\Omega} = -\langle \psi(a)\psi(z) \rangle_{\Omega} ,$$

$$f_{\Omega}^{\dagger}(a,z) = \langle \psi(z)\overline{\psi}(a) \rangle_{\Omega} = -\langle \overline{\psi}(a)\psi(z) \rangle_{\Omega} ,$$

where we use the notation $\prec \ldots \succ_{\Omega}$ for correlators of the free (holomorphic $\psi(z)$ and anti-holomorphic $\overline{\psi}(z)$) fermions in a simply connected domain Ω , coupled along the boundary so that $\overline{\psi}(\zeta) = i\overline{\nu_{\text{out}}(\zeta)\psi(\zeta)}$ for $\zeta \in \partial\Omega$; note that this coupling respects the conformal covariance (4.5) and reads simply as $\overline{\psi}(\zeta) = \overline{\psi(\zeta)}$ if $\Omega = \mathbb{H}$ is the upper half-plane and $\zeta \in \mathbb{R}$. Therefore, the function $f_{\Omega}^{[\eta]}(a,z)$ can be interpreted as the fermionic correlator

$$f_{\Omega}^{[\eta]}(a,z) \ = \ \prec \psi(z) \psi^{[\eta]}(a) \succ_{\Omega} \ , \quad \text{where} \quad \psi^{[\eta]}(a) := \eta \psi(a) + i \eta \overline{\psi}(a) \, .$$

We also refer the interested reader to [7, Section 3.6], where a similar 'Grassmann variables' notation for the discrete combinatorial observables from Section 3 is discussed. Note that the mismatch (comparing to [19] and [7]) of the signs in the boundary conditions for fermions is caused by the fact that in we prefer to drop the additional prefactor $\exp[-i\frac{\pi}{4}]$ in the definition (3.10) of complex fermionic observables, see also Remark 3.4.

4.2. Stress-energy tensor and energy density correlations. In the free fermion CFT framework, it is well-known (e.g., see [41, Sec. 12.3 and 14.2.1]) that both the (holomorphic) stress-energy tensor T and the energy density field ε can be written in terms of $\psi, \overline{\psi}$ as

$$T = -\frac{1}{2} : \psi \partial \psi : \quad \text{and} \quad \varepsilon = i\psi \overline{\psi}$$

(as usual, the colons mean that some regularization is required in the former case). This motivates the following definition, which uses the language of solutions to Riemann-type boundary value problems developed in Section 4.1 instead of the CFT terminology.

Definition 4.4. Given a simply connected domain Ω and $w, a \in \Omega$, we define the quantities $\langle T(w) \rangle_{\Omega}^+$ and $\langle \varepsilon(a) \rangle_{\Omega}^+$ by

$$\langle T(w) \rangle_{\Omega}^{+} := \frac{1}{2} \partial_{z} f_{\Omega}^{\sharp}(w,z) \big|_{z=w} \quad and \quad \langle \varepsilon(a) \rangle_{\Omega}^{+} := i f_{\Omega}^{\dagger}(a,a) \,,$$

where the functions f_{Ω}^{\sharp} and f_{Ω}^{\dagger} are given by Proposition 4.2 (and Definition 4.1).

Note that one can easily derive explicit formulae for the quantities $\langle T(w) \rangle_{\Omega}^{+}$ and $\langle \varepsilon(a) \rangle_{\Omega}^{+}$ using (4.7). Namely, for any uniformization mapping $\varphi : \Omega \to \mathbb{H}$, one has

$$\langle T(w) \rangle_{\Omega}^{+} = \frac{[\mathcal{S}\varphi](w)}{24}, \text{ where } [\mathcal{S}\varphi](w) := \frac{\varphi'''(w)}{\varphi'(w)} - \frac{3}{2} \left[\frac{\varphi''(w)}{\varphi'(w)}\right]^{2}$$
 (4.8)

denotes the Schwarzian derivative of the conformal mapping φ at the point w, and

$$\langle \varepsilon(a) \rangle_{\Omega}^{+} = |\varphi'(a)| \cdot (2\operatorname{Im}\varphi(a))^{-1}. \tag{4.9}$$

In particular, $\langle \varepsilon(a) \rangle_{\Omega}^+ \in \mathbb{R}$ for all $a \in \Omega$ while $\langle T(\cdot) \rangle_{\Omega}^+$ is a holomorphic function in Ω , this is why we prefer to use the letter w (instead of a) for its argument.

The next step is to give a definition of the function $\langle T(w)\rangle_{\Omega}^{b,b'}$ from Theorem 1.1, as well as two-point functions $\langle T(w)\varepsilon(a)\rangle_{\Omega}^+$ and $\langle T(w)T(w')\rangle_{\Omega}^+$ from Theorems 1.2 — 1.3. It is worth mentioning that all these quantities can be alternatively defined via explicit formulae in the upper half-plane and appropriate conformal covariance rules. Nevertheless, we prefer to use

the functions f_{Ω} and f_{Ω}^{\dagger} as the starting point for all our definitions since they provide clear links with both the standard CFT notation on the one hand and with the discrete fermionic observables discussed in Section 3 on the other.

Definition 4.5. Given a simply connected domain Ω , an inner point $w \in \Omega$ and two boundary points (or prime ends) $b, b' \in \partial \Omega$, we define the quantity $\langle T(w) \rangle_{\Omega}^{b,b'}$ by

$$\langle T(w) \rangle_{\Omega}^{b,b'} \ := \ \langle T(w) \rangle_{\Omega}^+ \ + \ \frac{f_{\Omega}(w,b') \partial_w f_{\Omega}(w,b) - f_{\Omega}(w,b) \partial_w f_{\Omega}(w,b')}{2 f_{\Omega}(b,b')} \,,$$

where the one-point function $\langle T(w) \rangle_{\Omega}^+$ is given by Definition 4.4.

Note that for any uniformization mapping $\varphi:\Omega\to\mathbb{H}$ the formula (4.7) yields

$$\langle T(w) \rangle_{\Omega}^{b,b'} = \langle T(w) \rangle_{\Omega}^{+} + \frac{1}{2} \left[\frac{\varphi'(w) \cdot (\varphi(b') - \varphi(b))}{(\varphi(b') - \varphi(w))(\varphi(b) - \varphi(w))} \right]^{2}.$$

Therefore, if $\varphi:\Omega\to\mathbb{H}$ is the (unique up to a scaling) uniformization mapping such that $\varphi(b)=0$ and $\varphi(b')=\infty$ or vice versa, then

$$\langle T(w) \rangle_{\Omega}^{b,b'} = \frac{1}{2} \left[\frac{\varphi'(w)}{\varphi(w)} \right]^2 + \frac{[\mathcal{S}\varphi](w)}{24} \,. \tag{4.10}$$

Remark 4.4. (i) Following the discussion given in Remark 4.3, one can interpret the ratio of partition functions of the Ising model in Ω with Dobrushin and '+' boundary conditions as the (absolute value) of the fermionic correlator $\prec \psi(b')\psi(b) \succ_{\Omega} = f_{\Omega}(b,b')$. Note that this interpretation perfectly agrees with the combinatorial definition of discrete fermionic observables. Therefore, Definition 4.5 can be thought about as a Pfaffian formula for the four-point fermionic correlator $-\frac{1}{2} \prec : \psi(w) \partial \psi(w) \psi(b') \psi(b) : \succ_{\Omega}$ normalized by $\prec \psi(b') \psi(b) \succ_{\Omega}$. (ii) Note that the values of the function $f_{\Omega}(w,\cdot)$ at boundary points b,b' may be ill-defined if $\partial\Omega$ is non-smooth. Nevertheless, the ratio involved in the definition of $\langle T(w) \rangle_{0}^{b,b'}$ is always well-defined: the easiest way to see this is to apply a uniformization map $\varphi: \Omega \to \mathbb{H}$ and to rewrite all the instances of $f_{\Omega}(w,\cdot)$ using (4.7). Along the way, both the numerator and the denominator gain the same ill-defined factor $(\phi'(b)\phi'(b'))^{\frac{1}{2}}$, which just cancels out.

Definition 4.6. Given a simply connected domain Ω and points $w, w', a \in \Omega$, we define the quantities $\langle T(w)\varepsilon(a)\rangle_{\Omega}^+$ and $\langle T(w)T(w')\rangle_{\Omega}^+$ by

$$\langle T(w)T(w')\rangle_{\Omega}^{+} := \langle T(w)\rangle_{\Omega}^{+} \langle T(w')\rangle_{\Omega}^{+} + \frac{1}{4}(\partial_{w}f_{\Omega}(w,w')\partial_{w'}f_{\Omega}(w,w') - f_{\Omega}(w,w')\partial_{w}\partial_{w'}f_{\Omega}(w,w')),$$

$$\langle T(w)\varepsilon(a)\rangle_{\Omega}^{+} := \langle T(w)\rangle_{\Omega}^{+} \langle \varepsilon(a)\rangle_{\Omega}^{+} + \frac{i}{2}(f_{\Omega}(a,w)\partial_{w}f_{\Omega}^{\dagger}(a,w) - f_{\Omega}^{\dagger}(a,w)\partial_{w}f_{\Omega}(a,w)),$$

where the one-point functions $\langle T(w) \rangle_{\Omega}^{+}$ and $\langle \varepsilon(a) \rangle_{\Omega}^{+}$ are given by Definition 4.4.

Remark 4.5. Similarly to Remark 4.4, Definition 4.6 can be understood as Pfaffian formulae for the correlators $\frac{1}{4} \prec : \psi(w) \partial \psi(w) \psi(w') \partial \psi(w') : \succ_{\Omega} \text{ and } -\frac{i}{2} \prec \psi(w) \partial \psi(w) \psi(a) \overline{\psi}(a) \succ_{\Omega}$.

4.3. Correlations with the spin field. The main aim of this section is to define the quantities $\langle \sigma(u) \rangle_{\Omega}^+$ and $\langle T(w)\sigma(u) \rangle_{\Omega}^+$, which appear in the statement of Theorem 1.3. In principle, this can be done by writing explicit formulae for the case $\Omega = \mathbb{H}$ and then applying appropriate covariance rules. Nevertheless, similarly to Section 4.1 we prefer to use the language of Riemann-type boundary value problems for holomorphic spinors branching around a point $u \in \Omega$. Note that such continuous spinors are natural counterparts of discrete branching

observables discussed in Section 3, which are the crucial tool for proving the convergence of discrete correlation functions involving the spin field to their limits, see [11, 10].

Below we need some terminology. Given a simply connected planar domain Ω and a point $u \in \Omega$, we denote by $[\Omega, u]$ the (canonical) double-cover of Ω branching over the point u, and by $\pi: [\Omega, u] \to \Omega$ the corresponding projection. For a point $z \in [\Omega, u] \setminus \{u\}$, let $z^* \in [\Omega, u]$ be such that $\pi(z^*) = \pi(z)$ and $z^* \neq z$. In other words, z^* is the point lying on the other sheet of $[\Omega, u]$ over the same point of Ω as z. We say that a function $f: [\Omega, u] \setminus \{u\} \to \mathbb{C}$ is a holomorphic spinor if $f(z^*) = -f(z)$ for all $z \neq u$ and $f(\cdot)$ is holomorphic on $[\Omega, u] \setminus \{u\}$.

The following description of the one-point function $\langle \sigma(u) \rangle_{\Omega}^+$ in continuum was used in [10], we refer the interested reader to this paper for more details.

Definition 4.7. Let Ω be a bounded simply connected domain with a smooth boundary and $u \in \Omega$. We denote by $g_{[\Omega,u]}(z)$ the (unique) holomorphic spinor defined on $[\Omega,u] \setminus \{u\}$ such that $g_{[\Omega,u]}^{\sharp}(z) := g_{[\Omega,u]}(z) - (z-u)^{-1/2}$ is bounded as $z \to u$ and

$$\operatorname{Im} \left[g_{[\Omega,u]}(\zeta) \sqrt{\nu_{\operatorname{out}}(\zeta)} \right] = 0 \quad \text{for all } \zeta \in \partial[\Omega,u].$$

Then, we define the quantity $\langle \sigma(u) \rangle_{\Omega}^+$ as the (unique up to a multiplicative normalization) real-valued function in Ω satisfying the differential equation

$$d\log\langle\sigma(u)\rangle_{\Omega}^{+} = \operatorname{Re}\mathcal{A}_{\Omega}(u)dx - \operatorname{Im}\mathcal{A}_{\Omega}(u)dy, \qquad (4.11)$$

where u = x + iy and $\mathcal{A}_{\Omega}(u) := \frac{1}{2}(z - u)^{-1/2}g^{\sharp}_{[\Omega,u]}(z)\big|_{z=u}$.

Remark 4.6. (i) The uniqueness of solution $g_{[\Omega,u]}(\cdot)$ to the boundary value problem described above can be easily deduced similarly to Remark 4.1(i). Moreover, one can use this uniqueness property similarly to Remark 4.1(ii) to prove that

$$g_{[\Omega,u]}(z) = g_{[\Omega',u']}(\varphi(z)) \cdot (\varphi'(z))^{\frac{1}{2}}$$
 (4.12)

for any conformal mapping $\varphi:\Omega\to\Omega'$, where $u'=\varphi(u)$. Then, one can use this covariance rule and the explicit formula

$$g_{[\mathbb{H},u]}(z) = (2i \operatorname{Im} u)^{\frac{1}{2}} \cdot ((z-u)(z-\overline{u}))^{-\frac{1}{2}}$$

to prove the existence of $g_{[\Omega,u]}(\cdot)$, and to define this function for rough and/or unbounded Ω .

- (ii) A priori, it is not clear that the differential form (4.11) is exact. The easiest (though not the most conceptual) way to check this fact is to start with the explicit formula given above for $\Omega = \mathbb{H}$ and then to pass to the general case using (4.12).
- (iii) Equation (4.11) defines the function $\langle \sigma(u) \rangle_{\Omega}^+$ up to a multiplicative normalization which, in principle, could depend on Ω . There exists a coherent way to choose this normalization for all planar domains Ω simultaneously, which leads to the following formula:

$$\langle \sigma(u) \rangle_{\Omega}^{+} = (2|\varphi'(u)|)^{\frac{1}{8}} \cdot (\operatorname{Im} \varphi(u))^{-\frac{1}{8}}, \quad u \in \Omega,$$
(4.13)

where φ stands for a uniformization mapping from Ω onto \mathbb{H} , see [10] for more details.

Definition 4.8. Let $[\Omega, u]$ be a double-cover of a bounded simply connected domain Ω with a smooth boundary, branching over the point $u \in \Omega$. Let $a \in [\Omega, u] \setminus \{u\}$ and $\eta \in \mathbb{C}$. We define $f_{[\Omega, u]}^{[\eta]}(a, \cdot)$ to be the (unique) holomorphic spinor in $[\Omega, u] \setminus \{u, a, a^*\}$ such that

$$\operatorname{Im}\left[\,f_{[\Omega,u]}^{[\eta]}(a,\zeta)\sqrt{\nu_{\mathrm{out}}(\zeta)}\,\right]\quad \text{ for all } \,\,\zeta\in\partial[\Omega,u]\,,$$

the spinor $f_{[\Omega,u]}^{[\eta]}(a,z)-ic\cdot(z-u)^{-1/2}$ is bounded as $z\to u$ for some (unknown) constant $c\in\mathbb{R}$, and the function $f_{[\Omega,u]}^{[\eta]}(a,z)-\eta\cdot(z-a)^{-1}$ is bounded as $z\to a$.

Repeating the arguments given in Section 4.1 for holomorphic functions $f_{\Omega}^{[\eta]}(a,z)$, one can prove that the boundary value problem for holomorphic spinors $f_{[\Omega,u]}^{[\eta]}(a,z)$ described in Definition 4.8 has a unique solution which can be represented as

$$f_{[\Omega,u]}^{[\eta]}(a,z)=\eta f_{[\Omega,u]}(a,z)+i\bar{\eta}f_{[\Omega,u]}^{\dagger}(a,z),\quad a,z\in [\Omega,u]\,.$$

Moreover, one easily sees that the function

$$f_{[\Omega,u]}^{\sharp}(a,z) := f_{[\Omega,u]}(a,z) - (z-a)^{-1},$$

is bounded in a vicinity of the point a (note that this function is defined only locally). Further, holomorphic spinors $f_{[\Omega,u]}(a,\cdot)$ and $f_{[\Omega,u]}^{\dagger}(a,\cdot)$ satisfy the same symmetries (4.6) and conformal covariance rules (4.5), and one can use the latter to define these spinors in rough and/or unbounded domains starting with the explicit formulae

$$f_{\left[\mathbb{H},u\right]}(a,z) = \left[\frac{(a-u)(a-\overline{u})}{(z-u)(z-\overline{u})}\right]^{\frac{1}{2}} \cdot \left[\frac{1}{z-a} + \frac{1}{2(a-u)} + \frac{1}{2(a-\overline{u})}\right],$$

$$f_{\left[\mathbb{H},u\right]}^{\dagger}(a,z) = \left[\frac{(\overline{a}-u)(\overline{a}-\overline{u})}{(z-u)(z-\overline{u})}\right]^{\frac{1}{2}} \cdot \left[\frac{1}{z-\overline{a}} + \frac{1}{2(\overline{a}-u)} + \frac{1}{2(\overline{a}-\overline{u})}\right].$$

The next definition mimics Definition 4.4 in the presence of the branching point $u \in \Omega$.

Definition 4.9. Given a simply connected domain Ω and points $w, u \in \Omega$, we set

$$\langle T(w)\sigma(u)\rangle_{\Omega}^{+} := \langle \sigma(u)\rangle_{\Omega}^{+} \cdot \frac{1}{2}\partial_{w}f_{[\Omega,u]}^{\sharp}(w,z)\big|_{z=w},$$

where $\langle \sigma(u) \rangle_{\Omega}^+$ is given by (4.13) and in the right-hand side we assume that w is lifted onto $[\Omega, u]$. Note the choice of $w \in [\Omega, u]$ is irrelevant since $f_{[\Omega, u]}(w, z) = f_{[\Omega, u]}(w^*, z^*)$.

Remark 4.7. It is easy to see that for any fixed $z \in \Omega \setminus \{u\}$ the following holds as $a \to u$:

$$f_{[\Omega,u]}(a,z) \; \sim \; \tfrac{1}{2}(a-u)^{-\frac{1}{2}} \cdot g_{[\Omega,u]}(z) \,, \qquad f_{[\Omega,u]}^\dagger(a,z) \; \sim \; \tfrac{1}{2}(\overline{a}-\overline{u})^{-\frac{1}{2}} \cdot g_{[\Omega,u]}(z) \,.$$

The easiest way to prove these asymptotics is to use the explicit formulae given above for $\Omega = \mathbb{H}$ and then the covariance rules (4.5), (4.12) to treat the general case. One could also derive them directly by analyzing the boundary value problems from Definitions 4.1 and 4.7 but we do not address this question here for shortness.

4.4. Schwarzian conformal covariance and singularities of correlation functions. In this section we mention several CFT formulae for the two-point correlation functions in-

volving the (holomorphic) stress-energy tensor T(w), in which the central charge $c = \frac{1}{2}$ of the theory and the conformal dimensions $\frac{1}{16}$ and $\frac{1}{2}$ of the primary fields ε and σ appear. It should be said that all these claims are perfectly well-known in the CFT context. Nevertheless, let us emphasize that we do *not* use any CFT concepts in our definitions of correlation functions in continuum and all these quantities are constructed starting with natural continuous counterparts of discrete combinatorial observables discussed in Section 3. From this perspective, the Schwarzian nature of the stress-energy tensor (see Proposition 4.10) and the particular coefficients of singular terms in the asymptotics of correlation functions (see Proposition 4.11)

appear as simple corollaries of the conformal covariance rules for the solutions to boundary value problems from Definitions 4.1 4.7 and 4.8.

Proposition 4.10. Let Ω be a simply connected domain, $b, b' \in \partial \Omega$ and $a, u \in \Omega$. Then, each of the four functions

$$\mathcal{F}_{\Omega}(w;\ldots) = \langle T(w) \rangle_{\Omega}^{+}, \ \langle T(w) \rangle_{\Omega}^{b,b'}, \ \frac{\langle T(w)\varepsilon(a) \rangle_{\Omega}^{+}}{\langle \varepsilon(a) \rangle_{\Omega}^{+}}, \ \frac{\langle T(w)\sigma(u) \rangle_{\Omega}^{+}}{\langle \sigma(u) \rangle_{\Omega}^{+}}$$

is holomorphic in $w \in \Omega$ (except at the points a and u for the third and the fourth ones, respectively) and obey the Schwarzian covariance

$$\mathcal{F}_{\Omega}(w;\ldots) = \mathcal{F}_{\Omega'}(\varphi(w);\ldots) \cdot (\varphi'(w))^2 + \frac{1}{24}[\mathcal{S}\varphi](w) \tag{4.14}$$

under conformal mappings $\varphi: \Omega \to \Omega'$.

Proof. The holomorphicity of all these functions directly follows from their definitions and the holomorphicity of the functions $f_{\Omega}(a,\cdot)$ $f_{\Omega}^{\dagger}(a,\cdot)$ and $f_{[\Omega,u]}(a,\cdot)$. The Schwarzian covariance of the first function $\langle T(w)\rangle_{\Omega}^+$ can be easily derived from its definition and the conformal covariance rule (4.5), see also the explicit formula (4.8). Also, exactly the same local computation leads to (4.14) for the fourth function $\langle T(w)\sigma(u)\rangle_{\Omega}^+/\langle\sigma(u)\rangle_{\Omega}^+$. To prove (4.14) for the second and the third functions, it is enough to check that the second terms in their definitions (see Definition 4.5 and Definition 4.4, respectively) are multiplied by $(\varphi'(w))^2$ when applying a conformal mapping $\varphi: \Omega \to \Omega'$. This is straightforward, see also (4.10).

Proposition 4.11. Let Ω be a simply connected domain and w', $a, u \in \Omega$. Then, one has

$$\langle T(w)T(w')\rangle_{\Omega}^{+} = \frac{1}{4} \cdot \frac{1}{(w-w')^{4}} + \frac{2\langle T(w')\rangle_{\Omega}^{+}}{(w-w')^{2}} + \frac{\partial_{w'}\langle T(w')\rangle_{\Omega}^{+}}{w-w'} + O(1) \quad as \quad w \to w', \quad (4.15)$$

$$\langle T(w)\varepsilon(a)\rangle_{\Omega}^{+} = \frac{1}{2} \cdot \frac{\langle \varepsilon(a)\rangle_{\Omega}^{+}}{(w-a)^{2}} + \frac{\partial_{a}\langle \varepsilon(a)\rangle_{\Omega}^{+}}{w-a} + O(1) \quad as \quad w \to a, \tag{4.16}$$

$$\langle T(w)\sigma(u)\rangle_{\Omega}^{+} = \frac{1}{16} \cdot \frac{\langle \sigma(u)\rangle_{\Omega}^{+}}{(w-u)^{2}} + \frac{\partial_{u}\langle \sigma(u)\rangle_{\Omega}^{+}}{w-u} + O(1) \quad as \quad w \to u.$$
 (4.17)

We first prove a simple lemma, which identifies the second coefficients in the expansions of the functions $f_{\Omega}(w,z)$ and $f_{\Omega}^{\dagger}(a,z)$ near the singularity.

Lemma 4.12. Let Ω be a simply connected domain and $w, a \in \Omega$. Then, the following asymptotic expansions hold as $z \to w$ and $z \to a$, respectively:

$$f_{\Omega}(w,z) = \frac{1}{z-w} + 2\langle T(w)\rangle_{\Omega}^{+} \cdot (z-w) + \partial_{w}\langle T(w)\rangle_{\Omega}^{+} \cdot (z-w)^{2} + O((z-w)^{3}),$$

$$if_{\Omega}^{\dagger}(a,z) = \langle \varepsilon(a)\rangle_{\Omega}^{+} + \partial_{a}\langle \varepsilon(a)\rangle_{\Omega}^{+} \cdot (z-a) + O((z-a)^{2}).$$

Proof. Let us start with the second expansion. The definition of $\langle \varepsilon(a) \rangle_{\Omega}^{+}$ yields

$$if_{\Omega}^{\dagger}(a,z) = \langle \varepsilon(a) \rangle_{\Omega}^{+} + c_{1}(a) \cdot (z-a) + O((z-a)^{2}), \quad z \to a,$$

with some (unknown) function $c_1(a)$. Since the function $f_{\Omega}^{\dagger}(a,z)$ is anti-holomorphic in a, one easily concludes that

$$0 = \partial_a f_{\Omega}^{\dagger}(a, z)\big|_{z=a} = \partial_a \langle \varepsilon(a) \rangle_{\Omega}^+ - c_1(a).$$

To derive the first expansion, note that $f_{\Omega}^{\sharp}(w,w) = 0$ since $f_{\Omega}(w,z) = -f_{\Omega}(z,w)$ for all w,z. Therefore, our definition of $\langle T(w) \rangle_{\Omega}^{+}$ yields

$$f_{\mathcal{O}}^{\sharp}(w,z) = 2\langle T(w)\rangle_{\mathcal{O}}^{+} \cdot (z-w) + c_{2}(w) \cdot (z-w)^{2} + O((z-w)^{3}), \quad z \to w,$$

for some $c_2(w)$. Using the symmetry $f_{\Omega}(w,z) = -f_{\Omega}(z,w)$ once again, one easily arrives at

$$0 = \partial_w \partial_z f_{\Omega}^{\sharp}(w, z) \big|_{z=w} = 2 \partial_w \langle T(w) \rangle_{\Omega}^+ - 2c_2(w) ,$$

which gives the result.

Proof of Proposition 4.11. Recall that the two-point functions $\langle T(w)T(w')\rangle_{\Omega}^{+}$ and $\langle T(w)\varepsilon(a)\rangle_{\Omega}^{+}$ are given by Definition 4.6. Thus, one can use the expansions provided by Lemma 4.12 to compute the singular parts of these functions as $w'\to w$ and $w\to a$, respectively. In particular, the asymptotics

$$\begin{split} f_{\Omega}(w',w) &= (w-w')^{-1} + 2\langle T(w')\rangle_{\Omega}^{+} \cdot (w-w') + \partial_{w'}\langle T(w')\rangle_{\Omega}^{+} \cdot (w-w')^{2} + O((w-w')^{3}) \,, \\ \partial_{w}f_{\Omega}(w',w) &= -(w-w')^{-2} + 2\langle T(w')\rangle_{\Omega}^{+} + 2\partial_{w'}\langle T(w')\rangle_{\Omega}^{+} \cdot (w-w') + O((w-w')^{2}) \,, \\ \partial_{w'}f_{\Omega}(w',w) &= (w-w')^{-2} - 2\langle T(w')\rangle_{\Omega}^{+} + O((w-w')^{2}) \,, \\ \partial_{w}\partial_{w'}f_{\Omega}(w',w) &= -2(w-w')^{-3} + O(w-w') \end{split}$$

lead to (4.15) and simpler computations give (4.16). To prove the last asymptotics (4.17), note that Definition 4.9 and the Cauchy residue theorem imply the equality

$$\frac{1}{2\pi i} \oint \frac{(z-u)^{\frac{1}{2}} f_{[\Omega,u]}(w,z) dz}{(z-w)^2} = (w-u)^{\frac{1}{2}} \cdot \left[\frac{2\langle T(w)\sigma(u)\rangle_{\Omega}^+}{\langle \sigma(u)\rangle_{\Omega}^+} - \frac{1}{8(w-u)^2} \right],$$

provided the (fixed) contour of integration surrounds both points w and u. Passing to the limit $w \to u$ and using Remark 4.7 for all z on this contour, one concludes that

$$\frac{1}{2\pi i} \oint \frac{g_{[\Omega,u]}(z)dz}{(z-u)^{3/2}} = 2(w-u) \cdot \left[\frac{2\langle T(w)\sigma(u)\rangle_{\Omega}^{+}}{\langle \sigma(u)\rangle_{\Omega}^{+}} - \frac{1}{8(w-u)^{2}} \right],$$

The formula (4.17) easily follows since the left-hand side is equal to

$$(z-u)^{-\frac{1}{2}}g_{[\Omega,u]}^{\sharp}(z)\big|_{z=u} = 2\mathcal{A}_{\Omega}(u) = 4\partial_{u}\log\langle\sigma(u)\rangle_{\Omega}^{+}$$

due to Definition 4.7.

We conclude this section by mentioning that one can use the language of solutions to boundary value problems for holomorphic functions and spinors discussed above to construct all the correlations of the fields σ, ε and T, as well as the anti-holomorphic counterpart \overline{T} of the stress-energy tensor, for any boundary conditions $\mathfrak{b} = \{b_1, \ldots, b_{2m}\}$. If the spin field is not involved, this simply amounts to writing down a proper Pfaffian formula in the spirit of Definitions 4.5, 4.6, and Remarks 4.4, 4.5; cf. [23] where the multi-point expectations $\langle \varepsilon(a_1) \ldots \varepsilon(a_n) \rangle_{\Omega}^{\mathfrak{b}}$ were considered. In the presence of several spins $\sigma(u_1), \ldots, \sigma(u_k)$, the same techniques can be applied for correlations normalized by $\langle \sigma(u_1) \ldots \sigma(u_k) \rangle_{\Omega}^{\mathfrak{b}}$ similarly to Definition 4.9; cf. [28, Section 5] where the ratios $\langle \sigma(u_1) \ldots \sigma(u_k) \rangle_{\Omega}^{\mathfrak{b}} / \langle \sigma(u_1) \ldots \sigma(u_k) \rangle_{\Omega}^{\mathfrak{b}}$ were discussed. Finally, the spin correlations $\langle \sigma(u_1) \ldots \sigma(u_k) \rangle_{\Omega}^{\mathfrak{b}}$ themselves were treated in [10].

5. Convergence results for the Ising model

In this section we prove convergence results (namely, Theorems 1.1–1.3) for several expectations, involving the discrete stress-energy tensor T(w), in the case n=1 corresponding to the Ising model. Recall that, given a bounded simply connected domain $\Omega \subset \mathbb{C}$, we denote by Ω_{δ} its lattice approximations on the regular honeycomb grid \mathbb{C}_{δ} with the mesh size δ . For simplicity, one can define Ω_{δ} as the (largest connected component of the) union of all grid cells that are contained in Ω , together with appropriately chosen boundary half-edges, or just to assume that Ω_{δ} converges to Ω in the Hausdorff sense as $\delta \to 0$. It is worth mentioning that all results discussed below remain true if one assumes that discrete domains Ω_{δ} approximate Ω in the so-called Carathéodory topology, which is a much weaker notion than the Hausdorff convergence, see [46, Chapter 1.4] and [12, Section 3.2] for definitions.

Let us first describe the general strategy of our proofs of Theorems 1.1–1.3, which is similar to the one used in the series of recent papers [25, 23, 11, 10] devoted to convergence of various correlation functions in the Ising model to their scaling limits. Let w_{δ} denote the face of the discrete domain Ω_{δ} containing w, b_{δ} denote the nearest to $b \in \partial \Omega$ boundary edge of Ω_{δ} etc. As we saw in Section 3, all the expectations $\mathbb{E}_{\Omega_{\delta}}^{+}[T(w_{\delta})]$, $\mathbb{E}_{\Omega_{\delta}}^{b,b'_{\delta}}[T(w_{\delta})]$, $\mathbb{E}_{\Omega_{\delta}}^{+}[T(w_{\delta})T(w'_{\delta})]$ etc can be written in terms of discrete derivatives of the fermionic observables $F_{\Omega_{\delta}}$ (or their spinor analogues). At the same time, their putative scaling limits $\langle T(w) \rangle_{\Omega}^{+}$, $\langle T(w) \rangle_{\Omega}^{b,b'}$, $\langle T(w)T(w') \rangle_{\Omega}^{+}$ etc can be defined in a similar manner using derivatives of the continuous functions f_{Ω} and f_{Ω}^{\dagger} (or their spinor analogues), as discussed in Section 4. Therefore, to prove Theorems 1.1–1.3 it is essentially enough to prove the C^{1} -convergence of (appropriately normalized) discrete fermionic observables $F_{\Omega_{\delta}}$ to their continuous counterparts. The first result of this kind was obtained by the third author in [47] (in a slightly different context of the random-cluster representation of the Ising model aka FK-Ising model). Later, more involved discrete complex analysis techniques treating such convergence problems were developed in [12, 8, 25, 23, 30, 11, 10]. Essentially, we do not need any new machinery in our paper and thus just provide relevant references to the papers mentioned above in Sections 5.1 and 5.2. For the sake of completeness, we include a proof of the similar convergence result for spinor observables in Section 5.3, which also can be viewed as a characteristic example of such proofs.

5.1. S-holomorphicity and convergence of discrete fermionic observables. In this section we briefly discuss the crucial property of complex-valued fermionic observables (3.10): the so-called *s-holomorphicity*, which allows one to think about these observables as about solutions to some discrete boundary value problems and to prove their convergence (as $\delta \to 0$) to holomorphic functions solving the similar boundary value problems in continuum.

Definition 5.1. Let Ω be a discrete domain drawn on the regular hexagonal grid and F be a complex-valued function defined on (a subset of) the set of medial vertices (aka midedges) z_e of Ω . We say that F is an s-holomorphic function, if one has

$$Proj[F(z_e); (u-v)^{-\frac{1}{2}}\mathbb{R}] = Proj[F(z_{e'}); (u-v)^{-\frac{1}{2}}\mathbb{R}]$$

for all pairs of adjacent medial vertices $z_e, z_{e'}$, where v denotes the common vertex of the edges e and e', a point u is the center of the face adjacent to both e and e', and

$$\operatorname{Proj}[F;\tau\mathbb{R}] := \frac{1}{2}[F + (\tau/|\tau|)^2\overline{F}]$$

denotes the orthogonal projection of the complex number F, considered as a point in the plane, onto the line $\tau \mathbb{R} \subset \mathbb{C}$.

Remark 5.1. This property of discrete fermionic observables, which is a stronger version of usual definitions of discrete holomorphic functions, was pointed out by the third author in [49] and used to prove convergence results for similar observables appearing in the random-cluster representation of the Ising model (considered on the square lattice) in [47]. The name s-holomorphicity was suggested in [8, Section 3], where similar convergence results were proved for the spin representation of the Ising model, considered on arbitrary isoradial graphs; see also [48, Section 4], [7, Section 3] and reference therein for a discussion of relations between s-holomorphic functions and other techniques used to study the 2D Ising model (dimers, disorder insertions, etc). Also, it is worth mentioning that there are two versions of this definition appearing in the literature, which differ from each other by the factor $\exp[-i\frac{\pi}{4}]$. The definition given above coincides with those used, e.g., in [25, 23, 10] and differs by this factor from the one used, e.g., in [47, 8, 11, 7], cf. Remark 3.4.

It is well-known that, for any half-edge a_{δ} of Ω_{δ} , the complex-valued fermionic observable $F_{\Omega_{\delta}}(a_{\delta}, \cdot)$ defined by (3.10) satisfies the s-holomorphicity condition everywhere in Ω_{δ} except at the midedge $z_{a_{\delta}}$, we recall this fact in the Appendix. Moreover, the function $F_{\Omega_{\delta}}(a_{\delta}, \cdot)$ satisfies the boundary condition (3.11) for all boundary (half-)edges of Ω_{δ} , which is nothing but the discrete analogue of the boundary condition (4.1). Also, one can explicitly construct a function $F_{\mathbb{C}_{\delta}}(a_{\delta}, \cdot)$, which can be thought about as the infinite-volume limit of discrete fermionic observables $F_{\Omega_{\delta}}(a_{\delta}, \cdot)$, defined on the (medial vertices of the) whole honeycomb lattice \mathbb{C}_{δ} , and such that, for any discrete domain Ω_{δ} , the function

$$F_{\Omega_{\delta}}^{\sharp}(a_{\delta},\,\cdot\,) := F_{\Omega_{\delta}}(a_{\delta},\,\cdot\,) - F_{\mathbb{C}_{\delta}}(a_{\delta},\,\cdot\,)$$

is s-holomorphic everywhere in Ω_{δ} , including the midedge a_{δ} . We give the explicit contruction of the function $F_{\mathbb{C}_{\delta}}(a_{\delta}, \cdot)$ in the Appendix and just mention here two of its properties which are important for the proofs of Theorems 1.1–1.3 given below.

• Asymptotic behavior. For midedges $z_{e_{\delta}}$ such that $|z_{e_{\delta}} - z_{a_{\delta}}|^{-1} = o(\delta^{-1})$, one has

$$\delta^{-1} \cdot F_{\mathbb{C}_{\delta}}(a_{\delta}, z_{e_{\delta}}) = \frac{\sqrt{3} \eta_{a_{\delta}}}{2\pi(z_{e_{\delta}} - z_{a_{\delta}})} + o(1) \quad \text{as} \quad \delta \to 0.$$
 (5.1)

• Local values. One has

$$F_{\mathbb{C}_{\delta}}(a_{\delta}, z_{a_{\delta}}) = -\frac{1}{6} \cdot \overline{\eta}_{a_{\delta}}, \qquad (5.2)$$

$$F_{\mathbb{C}_{\delta}}(a_{\delta}, z_{a_{\delta}}^{\pm}) = \left[(1 - \frac{2\sqrt{3}}{\pi}) \pm i(\frac{3}{\pi} - \frac{2}{\sqrt{3}}) \right] \cdot \overline{\eta}_{a_{\delta}},$$
 (5.3)

where $z_{a_{\delta}}^{\pm} := z_{a_{\delta}} \pm \sqrt{3}\delta \cdot i\eta_{a_{\delta}}^2$ denote the midedges of the two neighboring to a_{δ} edges of \mathbb{C}_{δ} , which are oriented in the same direction $\eta_{a_{\delta}}^2$ as the (half-)edge a_{δ} .

The following convergence theorem for discrete fermionic observables $F_{\Omega_{\delta}}(a_{\delta}, \cdot)$ is a straightforward analogue of [25, Theorem 1.8]. In particular, to obtain this result, one can just repeat the proof given in [25] with minor modifications needed when dealing with the honeycomb grid instead of the square one. Alternatively, one can use a more robust scheme of proving convergence results of this kind, which was proposed in [10, Section 3.4]. In particular, we adopt the latter approach when proving a similar convergence theorem for spinor observables.

Theorem 5.2. Given a bounded simply connected domain $\Omega \subset \mathbb{C}$, two points $a, z \in \Omega$ and $\eta \in \wp$, let Ω_{δ} denote a discrete approximation of Ω on the regular honeycomb grid \mathbb{C}_{δ} ,

 a_{δ} be the closest to $a \in \Omega$ half-edge of Ω_{δ} oriented in the direction η^2 , and $z_{e_{\delta}}$ be the closet to z midedge of Ω_{δ} . Then, the following convergence results hold as $\delta \to 0$:

$$\begin{split} \delta^{-1} \cdot F_{\Omega_{\delta}}(a_{\delta}, z_{e_{\delta}}) & \implies \frac{\sqrt{3}}{2\pi} f_{\Omega}^{[\eta]}(a, z), \quad z \neq a, \\ \delta^{-1} \cdot F_{\Omega_{\delta}}^{\sharp}(a_{\delta}, z_{e_{\delta}}) & \implies \frac{\sqrt{3}}{2\pi} (\eta f_{\Omega}^{\sharp}(a, z) + i \overline{\eta} f_{\Omega}^{\dagger}(a, z)), \end{split}$$

where the function $f_{\Omega}^{[\eta]}(a,z)$ is given by Definition 4.1 and the functions $f_{\Omega}^{\sharp}(a,z)$, $f_{\Omega}^{\dagger}(a,z)$ are defined in Proposition 4.2. Moreover, the first convergence is uniform (with respect to (a,z)) on compact subsets of $(\Omega \times \Omega) \setminus D$, where $D := \{(a,a), a \in \Omega\}$, and the second is uniform on compact subsets of $\Omega \times \Omega$, in particular it holds near the diagonal $D \subset \Omega \times \Omega$.

Proof. See the proof of [25, Theorem 1.8] or the proof of Theorem 5.5 given below. \Box

The following corollary is a straightforward analogue of the main result of [25] for discrete domains drawn on the refining honeycomb grids \mathbb{C}_{δ} (instead of square ones considered in [25]).

Corollary 5.3. Given a bounded simply connected domain $\Omega \subset \mathbb{C}$ and a point $a \in \Omega$, let Ω_{δ} denote a discrete approximation of Ω on the regular honeycomb grid \mathbb{C}_{δ} , and a_{δ} be the closest to $a \in \Omega$ half-edge of Ω_{δ} . Then, the following convergence holds as $\delta \to 0$:

$$\delta^{-1}\mathbb{E}_{\Omega_{\delta}}^{+}[\varepsilon(a_{\delta})] \underset{\delta \to 0}{\longrightarrow} \frac{\sqrt{3}}{\pi} \langle \varepsilon(a) \rangle_{\Omega}^{+},$$

where the energy density $\varepsilon(a_{\delta})$ is defined by (3.13) and the function $\langle \varepsilon(a) \rangle_{\Omega}^{+}$ by Definition 4.4.

Proof. Let η be the square root of the direction of a_{δ} . Since $\varepsilon(\overline{a}_{\delta}) = \varepsilon(a_{\delta})$ does not depend on the orientation of a_{δ} , it is enough to consider the case $\eta \in \{1, \rho, \rho^2\}$. Recall that the following identities hold (see (3.14) and (3.12)):

$$\mathbb{E}_{\Omega_{\delta}}^{+}[\varepsilon(a_{\delta})] = \frac{1}{3} - 2F_{\Omega}(a_{\delta}, \overline{a}_{\delta}) = \frac{1}{3} + 2\operatorname{Re}[\eta F_{\Omega}(a_{\delta}, z_{a_{\delta}})] = 2\operatorname{Re}[\eta F_{\Omega}^{\sharp}(a_{\delta}, z_{a_{\delta}})],$$

where we also used (5.2). Theorem 5.2 now implies

$$\delta^{-1} \mathbb{E}_{\Omega_{\delta}}^{+}[\varepsilon(a_{\delta})] \underset{\delta \to 0}{\to} \frac{\sqrt{3}}{\pi} \operatorname{Re}[\eta^{2} f_{\Omega}^{\sharp}(a, a) + i f_{\Omega}^{\dagger}(a, a)] = \frac{\sqrt{3}}{\pi} \langle \varepsilon(a) \rangle_{\Omega}^{+}.$$

since
$$f_{\Omega}^{\sharp}(a,a) = -f_{\Omega}^{\sharp}(a,a) = 0$$
 and $f_{\Omega}^{\dagger}(a,a) = -i\langle \varepsilon(a) \rangle_{\Omega}^{+}$.

Remark 5.2. To prove the convergence of the stress-energy tensor expectations for Dobrushin boundary conditions, one also needs an extension of Theorem 5.2 to the case when one or both points a, z lie on straight parts of the boundary $\partial\Omega$, oriented in one of the directions $1, \rho, \ldots, \rho^5$ orthogonal to the directions of lattice edges. Note that the function $f_{\Omega}^{[\eta]}(a,\cdot)$ is well-defined on such parts of $\partial\Omega$ due to (4.1) and the Schwarz reflection principle. This passage from the convergence in the bulk of Ω to the convergence on (the straight part of) the boundary $\partial\Omega$ can be found in [8, Section 5.2], where a similar result was proved with no restrictions on the local direction of $\partial\Omega$ (and for general isoradial graphs instead of regular grids). The distilled version of this argument dealing with square lattices and particular orientations of $\partial\Omega$ (in the directions orthogonal to lattice edges) is given in [11, Lemma 4.8]. It can be easily adapted to s-holomorphic functions defined on the refining honeycomb grids \mathbb{C}_{δ} .

5.2. Proofs of Theorems 1.1–1.3. The aim of this section is to deduce the convergence results for various Ising model expectations, involving the local field T(w) discussed in Section 3.1, from the convergence of fermionic observables $F_{\Omega_{\delta}}$ provided by Theorem 5.2. As it was shown in Sections 3.2–3.4, all these expectations can be expressed via discrete *derivatives* of these observables, thus we actually need the stronger C^1 -topology of convergence in Theorem 5.2. The next lemma states that the convergence of discrete derivatives is a straightforward corollary of the convergence of functions themselves, as usual when working with discrete holomorphic or discrete harmonic functions on regular lattices.

Lemma 5.4. Let $\eta \in \wp$, $f: U \to \mathbb{C}$ be a holomorphic function defined in a planar domain U, and $F_{\delta}: U_{\delta} \to \mathbb{C}$ be a family of discrete s-holomorphic functions such that $\delta^{-1}F_{\delta}(z_{e_{\delta}}) \rightrightarrows f(z)$ as $\delta \to 0$ uniformly on compact subsets of U, where $z_{e_{\delta}}$ denotes the closest to $z \in U$ midedge of U_{δ} . Then, one has

$$(\sqrt{3}\delta)^{-1} \cdot (\delta^{-1}F_{\delta}(z_{e_{\delta}}^{\pm}) - \delta^{-1}F_{\delta}(z_{e_{\delta}})) \implies \pm i\eta^{2} \cdot \partial f(z) \quad as \quad \delta \to 0$$

uniformly on compact subsets of U, where η stands for the square root of the direction of e_{δ} and $z_{e_{\delta}}^{\pm} := z_{e_{\delta}} \pm \sqrt{3} \delta \cdot i \eta^2$ are the midpoints of the two neighboring to e_{δ} edges that are oriented in the same direction η^2 as e_{δ} .

Proof. As it is discussed in the Appendix, one can view s-holomorphic functions F_{δ} as collections of six discrete harmonic components F_{δ}^{μ} , each of them being defined on the corners of U_{δ} of type $\mu \in \wp$, which form a piece of a regular triangular lattice with the mesh size $\sqrt{3}\delta$. It is easy to see that the uniform convergence of the functions $\delta^{-1}F_{\delta}$ implies the uniform convergence (to $\text{Proj}[f(z); \mu\mathbb{R}]$) of each of the components $\delta^{-1}F_{\delta}^{\mu}$. Such a convergence of discrete harmonic functions always yields the uniform convergence of their discrete derivatives (e.g., see [12, Proposition 3.3], where a similar result is proved for harmonic functions on general isoradial graphs). Finally, as the discrete derivatives of $\delta^{-1}F_{\delta}$ can be easily reconstructed from the discrete derivatives of their components $\delta^{-1}F_{\delta}^{\mu}$, one easily sees that the uniform convergence of the latter implies the uniform convergence of the former.

We are now in the position to prove Theorems 1.1–1.3.

Proof of Theorem 1.1. Recall that it is enough to prove this theorem for \mathcal{T}_{mid} only as \mathcal{T}_{edge} can be expressed via \mathcal{T}_{mid} (see Lemma 2.5). Let us start with the case of '+' boundary conditions. Due to Proposition 3.4, for each $\eta \in \{1, \rho, \rho^2\}$ we have

$$\mathcal{T}_{\Omega_{\delta}}^{+}(w_{\delta}^{[\eta]}) = \mathbb{E}_{\Omega}^{+}[T^{[\eta]}(w_{\delta}) + T^{[i\eta]}(w_{\delta})] + \mathbb{E}_{\Omega}^{+}[R^{[\eta]}(w_{\delta})]$$

and, for each $\eta \in \wp$, Proposition 3.7 yields

$$\mathbb{E}_{\Omega_{\delta}}^{+}[T^{[\eta]}(w_{\delta})] = c_{T} + \frac{\sqrt{3}}{2} [\varsigma_{1} \partial_{2} F_{\Omega_{\delta}}](w_{\delta}^{[\eta]}, w_{\delta}^{[\eta]}),$$

where the discrete derivative ∂_2 with respect to the second argument and the mean value ς_1 with respect to the first are introduced in Definition 3.6. For any two half-edges a_{δ} and e_{δ} of Ω oriented in the direction η^2 , equation (3.12) gives

$$F_{\Omega_\delta}(a_\delta,e_\delta) \ = \ -\operatorname{Im}[\eta F_{\Omega_\delta}(a_\delta,z_{e_\delta})] \, .$$

Therefore, for both $a_{\delta} \in \{w_{\delta, \text{up}}^{[\eta]}, w_{\delta, \text{down}}^{[\eta]}\}$, we have

$$\frac{\sqrt{3}}{2} [\partial_2 F_{\Omega_{\delta}}](a_{\delta}, w_{\delta}^{[\eta]}) \ = \ -\frac{1}{2} \operatorname{Im} \left[\eta F_{\Omega_{\delta}}(a_{\delta}, z(w_{\delta, \mathrm{up}}^{[\eta]})) - \eta F_{\Omega_{\delta}}(a_{\delta}, z(w_{\delta, \mathrm{down}}^{[\eta]})) \right],$$

where we use the notation z(e) instead of z_e . Moreover, (5.3) implies

$$c_T \ = \ \tfrac{1}{2} \operatorname{Im} \left[\eta F_{\mathbb{C}_{\delta}}(a_{\delta}, z(w_{\delta, \mathrm{up}}^{[\eta]})) - \eta F_{\mathbb{C}_{\delta}}(a_{\delta}, z(w_{\delta, \mathrm{down}}^{[\eta]})) \right],$$

thus we arrive at the equation

$$\mathbb{E}_{\Omega_{\delta}}^{+}[T^{[\eta]}(w_{\delta})] = -\frac{1}{4} \sum_{a_{\delta} \in \{w_{\delta,\mathrm{up}}^{[\eta]}, w_{\delta,\mathrm{down}}^{[\eta]}\}} \operatorname{Im} \left[\eta F_{\Omega_{\delta}}^{\sharp}(a_{\delta}, z(w_{\delta,\mathrm{up}}^{[\eta]})) - \eta F_{\Omega_{\delta}}^{\sharp}(a_{\delta}, z(w_{\delta,\mathrm{down}}^{[\eta]})) \right].$$

For both $a_{\delta} \in \{w_{\delta, \text{up}}^{[\eta]}, w_{\delta, \text{down}}^{[\eta]}\}$, it easily follows from Theorem 5.2 and Lemma 5.4 that

$$\delta^{-2} \left[F_{\Omega_{\delta}}^{\sharp}(a_{\delta}, z(w_{\delta, \text{up}}^{[\eta]})) - F_{\Omega_{\delta}}^{\sharp}(a_{\delta}, z(w_{\delta, \text{down}}^{[\eta]})) \right] \xrightarrow[\delta \to 0]{} i\eta^{2} \cdot \frac{3}{2\pi} \partial_{z} (\eta f_{\Omega}^{\sharp}(w, z) + i\overline{\eta} f_{\Omega}^{\dagger}(w, z)) \big|_{z=w}.$$

This leads to the convergence

$$\delta^{-2} \cdot \mathbb{E}_{\Omega_{\delta}}^{+}[T^{[\eta]}(w_{\delta})] \underset{\delta \to 0}{\rightarrow} -\frac{3}{4\pi} \operatorname{Re}[\, \eta^{4} \partial_{z} f_{\Omega}^{\sharp}(w,z) + i \eta^{2} \partial_{z} f_{\Omega}^{\dagger}(w,z) \,] \big|_{z=w}$$

and hence, for each $\eta \in \{1, \rho, \rho^2\}$, we have

$$\delta^{-2} \cdot \mathbb{E}_{\Omega_{\delta}}^{+}[T^{[\eta]}(w_{\delta}) + T^{[i\eta]}(w_{\delta})] \underset{\delta \to 0}{\longrightarrow} -\frac{3}{2\pi} \operatorname{Re}\left[\eta^{4} \partial_{z} f_{\Omega}^{\sharp}(w, z)\big|_{z=w}\right] = -\frac{3}{\pi} \operatorname{Re}\left[\eta^{4} \langle T(w) \rangle_{\Omega}^{+}\right]. \tag{5.4}$$

Therefore, in order to prove the convergence of the quantities $\delta^{-2}\mathcal{T}_{\Omega_{\delta}}^{+}(w_{\delta}^{[\eta]})$ it is enough to prove that the remainder $\delta^{-2}\mathbb{E}_{\Omega}^{+}[R^{[\eta]}(w)]$ disappers in the limit $\delta \to 0$. Recall that Proposition 3.7 gives

$$\mathbb{E}_{\Omega}^{+}[R^{[\eta]}(w)] = c_R + \sqrt{3} [\partial_1 \partial_2 F_{\Omega}](w^{[\eta]}, w^{[i\eta]}).$$

For both $a_{\delta} \in \{w_{\delta, \text{up}}^{[\eta]}, w_{\delta, \text{down}}^{[\eta]}\}$, it follows from equation (3.12) that

$$\sqrt{3}[\partial_2 F_{\Omega_{\delta}}](a_{\delta}, w_{\delta}^{[i\eta]}) = -\operatorname{Re}\left[\eta F_{\Omega_{\delta}}(a_{\delta}, z(w_{\delta, \text{up}}^{[i\eta]})) - \eta F_{\Omega_{\delta}}(a_{\delta}, z(w_{\delta, \text{down}}^{[i\eta]}))\right]
= -\operatorname{Re}\left[\eta F_{\Omega_{\delta}}(a_{\delta}, z(w_{\delta, \text{down}}^{[\eta]})) - \eta F_{\Omega_{\delta}}(a_{\delta}, z(w_{\delta, \text{up}}^{[\eta]}))\right].$$

At the same time, (5.3) implies

$$\frac{\sqrt{3}}{2} \cdot c_R = \pm \operatorname{Re} \left[\eta F_{\mathbb{C}_{\delta}}(a_{\delta}, z(w_{\delta, \text{down}}^{[\eta]})) - \eta F_{\mathbb{C}_{\delta}}(a_{\delta}, z(w_{\delta, \text{up}}^{[\eta]})) \right],$$

where the '±' sign is '+' if $a_{\delta} = w_{\delta,\text{up}}^{[\eta]}$ and is '-' if $a_{\delta} = w_{\delta,\text{down}}^{[\eta]}$. Therefore, we obtain the equation

$$\begin{split} \mathbb{E}^+_{\Omega}[R^{[\eta]}(w)] \; &= \; \frac{1}{\sqrt{3}} \, \mathrm{Re} \left[- \left[\eta F^{\sharp}_{\Omega_{\delta}}(a_{\delta}, z(w^{[\eta]}_{\delta, \mathrm{down}})) - \eta F^{\sharp}_{\Omega_{\delta}}(a_{\delta}, z(w^{[\eta]}_{\delta, \mathrm{up}})) \right] \, \Big|_{a_{\delta} = w^{[\eta]}_{\delta, \mathrm{up}}} \right. \\ & \left. + \left[\eta F^{\sharp}_{\Omega_{\delta}}(a_{\delta}, z(w^{[\eta]}_{\delta, \mathrm{down}})) - \eta F^{\sharp}_{\Omega_{\delta}}(a_{\delta}, z(w^{[\eta]}_{\delta, \mathrm{up}})) \right] \, \Big|_{a_{\delta} = w^{[\eta]}_{\delta, \mathrm{down}}} \right]. \end{split}$$

Using Theorem 5.2 and Lemma 5.4, it is now easy to conclude that

$$\delta^{-2} \cdot \mathbb{E}_{\Omega_{\epsilon}}^{+}[R^{[\eta]}(w_{\delta})] \to 0 \quad \text{as} \quad \delta \to 0, \tag{5.5}$$

since the two terms (corresponding to $a_{\delta} = w_{\delta, \text{up}}^{[\eta]}$ and $a_{\delta} = w_{\delta, \text{down}}^{[\eta]}$) have the same limits.

Thus we have proved the convergence of the discrete stress-energy expectations with '+' boundary conditions. According to Proposition 3.10, in order to find the scaling limit of the

similar expectations with Dobrushin boundary conditions $\mathfrak{b} = \{b_{\delta}, b'_{\delta}\}$ it is now enough to prove the convergence of quantities

$$\frac{[\varsigma_1 F_{\Omega_{\delta}}](w_{\delta}^{[\eta]}, b_{\delta}')}{F_{\Omega_{\delta}}(b_{\delta}, b_{\delta}')} = \frac{[\varsigma_2 F_{\Omega_{\delta}}](b_{\delta}', w_{\delta}^{[\eta]})}{F_{\Omega_{\delta}}(b_{\delta}', b_{\delta})} \quad \text{and} \quad \delta^{-2} \cdot [\partial_1 F_{\Omega_{\delta}}](w_{\delta}^{[\eta]}, b_{\delta})$$

as $\delta \to 0$ (and similarly with b_{δ} and b'_{δ} interchanged). As discussed in Remark 5.2, this can be viewed as an extension of Theorem 5.2 from the bulk of Ω to straight parts of the boundary $\partial \Omega$. Recall that $F_{\Omega_{\delta}}(b'_{\delta}, b_{\delta}) \in \mathbb{R}$ and (3.11) gives $F_{\Omega_{\delta}}(b'_{\delta}, z(b_{\delta})) = -i\overline{\eta}_{b_{\delta}}F_{\Omega_{\delta}}(b'_{\delta}, b_{\delta})$, where $\eta_{b_{\delta}} = \eta_{b} := [-\nu_{\text{out}}(b)]^{1/2}$ does not depend on δ . Using (3.12), we can write

$$\frac{F_{\Omega_{\delta}}(b_{\delta}', w_{\delta, \star}^{[\eta]})}{F_{\Omega_{\delta}}(b_{\delta}', b_{\delta})} = -\operatorname{Im}\left[\frac{\eta F_{\Omega_{\delta}}(b_{\delta}', z(w_{\delta, \star}^{[\eta]}))}{F_{\Omega_{\delta}}(b_{\delta}', b_{\delta})}\right] = \operatorname{Re}\left[\eta \overline{\eta}_{b} \frac{F_{\Omega_{\delta}}(b_{\delta}', z(w_{\delta, \star}^{[\eta]}))}{F_{\Omega_{\delta}}(b_{\delta}', z(b_{\delta}))}\right],$$

where $w_{\delta,\star}^{[\eta]}$ stands for one of the two half-edges $w_{\delta,\text{up}}^{[\eta]}, w_{\delta,\text{down}}^{[\eta]}$. The convergence theorem

$$\frac{F_{\Omega_{\delta}}(b'_{\delta}, z(w^{[\eta]}_{\delta, \star}))}{F_{\Omega_{\delta}}(b'_{\delta}, z(b_{\delta}))} \xrightarrow[\delta \to 0]{} \frac{f_{\Omega}(b', w)}{f_{\Omega}(b', b)} = \frac{f_{\Omega}(w, b')}{f_{\Omega}(b, b')}$$

for discrete fermionic observables with the source point $b' \in \partial \Omega$, normalized at the other boundary point $b \in \Omega$ lying on a straight part of $\delta \Omega$, was proved in [8, Theorem B]. Hence, we have

$$\frac{[\varsigma_1 F_{\Omega_{\delta}}](w_{\delta}^{[\eta]}, b_{\delta}')}{F_{\Omega_{\delta}}(b_{\delta}, b_{\delta}')} \xrightarrow[\delta \to 0]{} \operatorname{Re} \left[\eta \overline{\eta}_b \frac{f_{\Omega}(w, b')}{f_{\Omega}(b, b')} \right]. \tag{5.6}$$

To find the limit of $\delta^{-2}[\partial_1 F_{\Omega_{\delta}}](w_{\delta}^{[\eta]}, b_{\delta})$ as $\delta \to 0$, note that the extension of Theorem 5.2 to straight boundaries (e.g., see [11, Lemma 4.8]) gives

$$\delta^{-1} F_{\Omega_{\delta}}(w_{\delta,\star}^{[\eta]},b_{\delta}) = i\eta_{b} \cdot \delta^{-1} F_{\Omega_{\delta}}(w_{\delta,\star}^{[\eta]},z(b_{\delta})) \underset{\delta \to 0}{\longrightarrow} i\eta_{b} \cdot \frac{\sqrt{3}}{2\pi} f_{\Omega}^{[\eta]}(w,b),$$

uniformly on compact subsets of Ω . Using a similar convergence result for $\delta^{-1}F_{\Omega_{\delta}}(w_{\delta,\star}^{[i\eta]},b_{\delta})$, we obtain

$$\delta^{-1}F_{\Omega_{\delta}}(b_{\delta}, z(w_{\delta, \star}^{[\eta]})) = \delta^{-1} \left[i\overline{\eta}F_{\Omega_{\delta}}(w_{\delta, \star}^{[\eta]}, b_{\delta}) \pm \overline{\eta}F_{\Omega_{\delta}}(w_{\delta, \star}^{[i\eta]}, b_{\delta}) \right]$$

$$\underset{\delta \to 0}{\longrightarrow} i\eta_{b} \cdot \frac{\sqrt{3}}{2\pi} \left[i\overline{\eta}f_{\Omega}^{[\eta]}(w, b) \pm \overline{\eta}f_{\Omega}^{[\pm i\eta]}(w, b) \right] = -\eta_{b} \cdot \frac{\sqrt{3}}{\pi}f_{\Omega}(w, b),$$

uniformly on compact subsets of Ω . Due to Lemma 5.4, this convergence can be extended to discrete derivatives, thus

$$\delta^{-2} \cdot \frac{1}{\sqrt{3}} \left[F_{\Omega_{\delta}}(b_{\delta}, z(w_{\delta, \text{up}}^{[\eta]})) - F_{\Omega_{\delta}}(b_{\delta}, z(w_{\delta, \text{down}}^{[\eta]})) \right] \xrightarrow[\delta \to 0]{} -i\eta^{2}\eta_{b} \cdot \frac{\sqrt{3}}{\pi} \partial_{w} f_{\Omega}(w, b) .$$

Now one can use (3.12) once again to conclude that

$$F_{\Omega_{\delta}}(w_{\delta,\star}^{[\eta]},b_{\delta}) = -F_{\Omega_{\delta}}(b_{\delta},w_{\delta,\star}^{[\eta]}) = \operatorname{Im}[\eta F_{\Omega_{\delta}}(b_{\delta},z(w_{\delta,\star}^{[\eta]}))]$$

and hence

$$\delta^{-2} \cdot [\partial_1 F_{\Omega_\delta}](w_\delta^{[\eta]}, b_\delta) \underset{\delta \to 0}{\to} -\frac{\sqrt{3}}{\pi} \operatorname{Re} \left[\eta^3 \eta_b \partial_w f_\Omega(w, b) \right]. \tag{5.7}$$

A straightforward substitution of (5.6) and (5.7) into Proposition 3.10(i) together with the identity $\operatorname{Re}[\eta X] \operatorname{Re}[\eta^3 Y] + \operatorname{Re}[i\eta X] \operatorname{Re}[(i\eta)^3 Y] = \operatorname{Re}[\eta^4 X Y]$ lead to

$$\delta^{-2} \cdot \left[\mathbb{E}_{\Omega_{\delta}}^{b_{\delta},b_{\delta}'}[T^{[\eta]}(w_{\delta}) + T^{[i\eta]}(w_{\delta})] - \mathbb{E}_{\Omega_{\delta}}^{+}[T^{[\eta]}(w_{\delta}) + T^{[i\eta]}(w_{\delta})] \right]$$

$$\xrightarrow{\delta \to 0} -\frac{3}{2\pi} \left[\operatorname{Re} \left[\eta \overline{\eta}_{b} \frac{f_{\Omega}(w,b')}{f_{\Omega}(b,b')} \cdot \eta^{3} \eta_{b} \partial_{w} f_{\Omega}(w,b) \right] + \operatorname{Re} \left[\eta \overline{\eta}_{b'} \frac{f_{\Omega}(w,b)}{f_{\Omega}(b',b)} \cdot \eta^{3} \eta_{b'} \partial_{w} f_{\Omega}(w,b') \right] \right]$$

$$= -\frac{3}{\pi} \operatorname{Re} \left[\eta^{4} \cdot \frac{f_{\Omega}(w,b') \partial_{w} f_{\Omega}(w,b) - f_{\Omega}(w,b) \partial_{w} f_{\Omega}(w,b')}{2 f_{\Omega}(b,b')} \right].$$

Taking into account Definition 4.5 and the convergence (5.4), we arrive at

$$\delta^{-2} \cdot \mathbb{E}_{\Omega_{\delta}}^{b_{\delta},b_{\delta}'}[T^{[\eta]}(w_{\delta}) + T^{[i\eta]}(w_{\delta})] \xrightarrow[\delta \to 0]{} -\frac{3}{\pi} \operatorname{Re}[\eta^{4} \langle T(w) \rangle_{\Omega}^{b,b'}].$$

Finally, similarly to the proof of (5.5), one easily sees that $[F_{\Omega_{\delta}}(b_{\delta},b'_{\delta})]^{-1} \cdot [\partial_{1}F_{\Omega_{\delta}}](w_{\delta}^{[i\eta]},b'_{\delta}) \to 0$ since the two terms corresponding to $a_{\delta} = w_{\delta,\text{up}}^{[i\eta]}$ and $a_{\delta} = w_{\delta,\text{down}}^{[i\eta]}$ have the same limits as $\delta \to 0$, cf. the proof of the convergence (5.6). Hence, Proposition 3.10(ii) together with (5.7) and (5.5) imply $\delta^{-2} \cdot \mathbb{E}_{\Omega_{\delta}}^{b_{\delta},b'_{\delta}}[R^{[\eta]}(w_{\delta}^{[\eta]})] \to 0$ as $\delta \to 0$.

Proof of Theorem 1.2. This proof goes along the same lines as the proof of Theorem 1.1 given above. Using Proposition 3.11, we see that it is essentially enough to compute the limits of the following quantities:

$$\delta^{-1} \cdot [\varsigma_1 \varsigma_2 F_{\Omega_{\delta}}](w_{\delta}^{[\eta]}, w_{\delta}'^{[\mu]}), \quad \delta^{-2} \cdot [\partial_1 \varsigma_2 F_{\Omega_{\delta}}](w_{\delta}^{[\eta]}, w_{\delta}'^{[\mu]}) \quad \text{and} \quad \delta^{-3} \cdot [\partial_1 \partial_2 F_{\Omega_{\delta}}](w_{\delta}^{[\eta]}, w_{\delta}'^{[\mu]})$$

as $\delta \to 0$. As above, let $w_{\delta,\star}^{[\eta]}$ (and similarly $w_{\delta,\star}^{'[\mu]}$) denote one of the two half-edges $w_{\delta,\text{up}}^{[\eta]}$, $w_{\delta,\text{down}}^{[\eta]}$. Due to (3.12), we have

$$-\operatorname{Im}[\,\mu F_{\Omega_{\delta}}(w_{\delta,\star}^{[\eta]}\,,z(w_{\delta,\star}'^{[\mu]}))] = F_{\Omega_{\delta}}(w_{\delta,\star}^{[\eta]}\,,w_{\delta,\star}'^{[\mu]}) = -F_{\Omega_{\delta}}(w_{\delta,\star}'^{[\mu]}\,,w_{\delta,\star}^{[\eta]}) = \operatorname{Im}[\,\eta F_{\Omega_{\delta}}(w_{\delta,\star}'^{[\mu]}\,,z(w_{\delta,\star}'^{[\eta]}))]$$

and hence Theorem 5.2 yields the convergence

$$\delta^{-1} \cdot F_{\Omega_{\delta}}(w_{\delta,\star}^{[\eta]}, w_{\delta,\star}^{\prime[\mu]}) \underset{\delta \to 0}{\longrightarrow} -\frac{\sqrt{3}}{2\pi} \operatorname{Im}[\mu f_{\Omega}^{[\eta]}(w, w')] = \frac{\sqrt{3}}{2\pi} \operatorname{Im}[\eta f_{\Omega}^{[\mu]}(w', w)]. \tag{5.8}$$

Moreover, Lemma 5.4 implies the convergence of the discrete derivatives

$$\delta^{-2} \cdot [\partial_2 F_{\Omega_\delta}](w_{\delta,\star}^{[\eta]}, w_\delta^{\prime [\mu]}) \xrightarrow[\delta \to 0]{} -\frac{\sqrt{3}}{2\pi} \operatorname{Re}[\mu^3 \partial_{w'} f_{\Omega}^{[\eta]}(w, w')], \tag{5.9}$$

and similarly

$$\delta^{-2} \cdot [\partial_1 F_{\Omega_\delta}](w_\delta^{[\eta]}, w_{\delta, \star}'^{[\mu]}) \xrightarrow[\delta \to 0]{} \frac{\sqrt{3}}{2\pi} \operatorname{Re}[\eta^3 \partial_w f_\Omega^{[\mu]}(w', w)]. \tag{5.10}$$

In order to handle the mixed discrete derivative $[\partial_1 \partial_2 F_{\Omega_\delta}]$, note that the last convergence (applied to both $w_{\delta,\star}'^{[\mu]}$ and $w_{\delta,\star}'^{[i\mu]}$ as the second argument of F_{Ω_δ}) also implies

$$\begin{split} \delta^{-2} \cdot \left[\partial_1 F_{\Omega_{\delta}} \right] (w_{\delta}^{[\eta]}, z(w_{\delta,\star}^{'[\mu]})) &= -\delta^{-2} \cdot \left(i \overline{\mu} [\partial_1 F_{\Omega_{\delta}}] (w_{\delta}^{[\eta]}, w_{\delta,\star}^{'[\mu]}) + \overline{\mu} [\partial_1 F_{\Omega_{\delta}}] (w_{\delta}^{[\eta]}, w_{\delta,\star}^{'[i\mu]}) \right) \\ & \stackrel{\rightarrow}{\rightarrow} -\frac{\sqrt{3}}{2\pi} \left(i \overline{\mu} \operatorname{Re} \left[\eta^3 \partial_w f_{\Omega}^{[\mu]} (w', w) \right] + \overline{\mu} \operatorname{Re} \left[\eta^3 \partial_w f_{\Omega}^{[i\mu]} (w', w) \right] \right) \\ & = -\frac{\sqrt{3}}{2\pi} \left(i \eta^3 \partial_w f_{\Omega} (w', w) + \overline{\eta^3 \partial_w f_{\Omega}^{\dagger} (w', w)} \right). \end{split}$$

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Applying (3.12) and Lemma 5.4 once again, we obtain the convergence

$$\delta^{-3} \cdot \left[\partial_{1}\partial_{2}F_{\Omega_{\delta}}\right]\left(w_{\delta}^{[\eta]}, w_{\delta}^{'[\mu]}\right) \xrightarrow[\delta \to 0]{} \frac{\sqrt{3}}{2\pi} \operatorname{Im}\left[i\mu^{3}\partial_{w'}\left(i\eta^{3}\partial_{w}f_{\Omega}(w', w) + \overline{\eta^{3}\partial_{w}f_{\Omega}^{\dagger}(w', w)}\right)\right]$$

$$= \frac{\sqrt{3}}{2\pi} \operatorname{Im}\left[\eta^{3}\mu^{3}\partial_{w}\partial_{w'}f_{\Omega}(w, w') - i\overline{\eta}^{3}\mu^{3}\overline{\partial}_{w}\partial_{w'}f_{\Omega}^{\dagger}(w, w')\right]. \quad (5.11)$$

Using Proposition 3.11(i) and formulae (5.8)–(5.11), we are now able to compute the limit

$$\lim_{\delta \to 0} \left[\delta^{-4} \left(\mathbb{E}_{\Omega_{\delta}}^{+} [T^{[\eta]}(w_{\delta}) T^{[\mu]}(w'_{\delta})] - \mathbb{E}_{\Omega_{\delta}}^{+} [T^{[\eta]}(w_{\delta})] \cdot \mathbb{E}_{\Omega_{\delta}}^{+} [T^{[\mu]}(w'_{\delta})] \right) \right] \\
= \frac{9}{16\pi^{2}} \left[-\operatorname{Re}[\eta \mu^{3} \partial_{w'} f_{\Omega}(w, w') + i \overline{\eta} \mu^{3} \partial_{w'} f_{\Omega}^{\dagger}(w, w')] \cdot \operatorname{Re}[\eta^{3} \mu \partial_{w} f_{\Omega}(w', w) + i \eta^{3} \overline{\mu} \partial_{w} f_{\Omega}^{\dagger}(w', w)] \right] \\
+ \operatorname{Im}[\eta \mu f_{\Omega}(w, w') + i \overline{\eta} \mu f_{\Omega}^{\dagger}(w, w')] \cdot \operatorname{Im}[\eta^{3} \mu^{3} \partial_{w} \partial_{w'} f_{\Omega}(w, w') - i \overline{\eta}^{3} \mu^{3} \overline{\partial}_{w} \partial_{w'} f_{\Omega}^{\dagger}(w, w')] \right]$$

and further, using straightforward computations together with (5.4), arrive at the formula

$$\begin{split} &\lim_{\delta \to 0} \left[\delta^{-4} \cdot \mathbb{E}_{\Omega_{\delta}}^{+} [(T^{[\eta]}(w_{\delta}) + T^{[i\eta]}(w_{\delta}))(T^{[\mu]}(w_{\delta}') + T^{[i\mu]}(w_{\delta}'))] \right] \\ &= \frac{9}{\pi^{2}} \left[\operatorname{Re}[\eta^{4} \langle T(w) \rangle_{\Omega}^{+}] \cdot \operatorname{Re}[\mu^{4} \langle T(w') \rangle_{\Omega}^{+}] \right. \\ &+ \frac{1}{8} \operatorname{Re}\left[\eta^{4} \mu^{4} \partial_{w'} f_{\Omega}(w, w') \partial_{w} f_{\Omega}(w, w') + \overline{\eta}^{4} \mu^{4} \partial_{w'} f_{\Omega}^{\dagger}(w, w') \overline{\partial}_{w} f_{\Omega}^{\dagger}(w, w') \right] \\ &- \frac{1}{8} \operatorname{Re}\left[\eta^{4} \mu^{4} f_{\Omega}(w, w') \partial_{w} \partial_{w'} f_{\Omega}(w, w') + \overline{\eta}^{4} \mu^{4} f_{\Omega}^{\dagger}(w, w') \overline{\partial}_{w} \partial_{w'} f_{\Omega}^{\dagger}(w, w') \right] \right]. \end{split}$$

Recall that $T(w_{\delta}) = -\frac{2}{3} \sum_{\eta \in \{1,\rho,\rho^2\}} [T^{[\eta]}(w_{\delta}) + T^{[i\eta]}(w_{\delta}) + R^{[\eta]}(w_{\delta})]$. As in the proof of Theorem 1.1, it is easy to see that the expectations containing the remainders $R^{[\eta]}(w_{\delta})$ and $R^{[\mu]}(w_{\delta}')$ disappear in the scaling limit. Averaging the previous formula over all possible choices of $\eta, \mu \in \{1, \rho, \rho^2\}$, we conclude that

$$\delta^{-4} \cdot \mathbb{E}_{\Omega_{\delta}}^{+}[T(w_{\delta})T(w_{\delta}')] \underset{\delta \to 0}{\longrightarrow} \frac{9}{\pi^{2}} \left[\langle T(w) \rangle_{\Omega}^{+} \langle T(w') \rangle_{\Omega}^{+} + \frac{1}{4} \left[\partial_{w'} f_{\Omega}(w, w') \partial_{w} f_{\Omega}(w, w') - f_{\Omega}(w, w') \partial_{w} \partial_{w'} f_{\Omega}(w, w') \right] \right].$$

The result coincides with $\frac{9}{\pi^2} \langle T(w)T(w')\rangle_{\Omega}^+$, see Definition 4.6.

Proof of Theorem 1.3. In view of Corollary 5.3, to prove the convergence of the ratios $\delta^{-2}\mathbb{E}_{\Omega_{\delta}}^{+}[T(w_{\delta})\varepsilon(a_{\delta})]/\mathbb{E}_{\Omega_{\delta}}^{+}[\varepsilon(a_{\delta})]$ as $\delta \to 0$, it is enough to show that

$$\delta^{-3} \cdot \mathbb{E}_{\Omega_{\delta}}^{+}[T(w)\varepsilon(a)] \underset{\delta \to 0}{\to} \frac{3\sqrt{3}}{\pi^{2}} \langle T(w)\varepsilon(a) \rangle_{\Omega}^{+}. \tag{5.12}$$

This can be easily done using the convergence results obtained in the proof of Theorem 1.2. Let μ denote the square root of the direction of the (oriented) edge a. Since $\varepsilon(a) = \varepsilon(\overline{a})$ by definition, it is enough to consider the case $\mu \in \{1, \rho, \rho^2\}$. Then one can easily combine Proposition 3.11(ii) with (5.8) and (5.10) applied to a_{δ} and \overline{a}_{δ} instead of $w'_{\delta,\star}^{[\mu]}$, and get

$$\delta^{-3} \cdot \left[\mathbb{E}_{\Omega_{\delta}}^{+} [T^{[\eta]}(w_{\delta}) \varepsilon(a_{\delta})] - \mathbb{E}_{\Omega_{\delta}}^{+} [T^{[\eta]}(w_{\delta})] \cdot \mathbb{E}_{\Omega_{\delta}}^{+} [\varepsilon(a_{\delta})] \right]$$

$$\underset{\delta \to 0}{\to} \frac{3\sqrt{3}}{4\pi^{2}} \left[\operatorname{Im}[\eta f_{\Omega}^{[\mu]}(a, w)] \cdot \operatorname{Re}[\eta^{3} \partial_{w} f_{\Omega}^{[i\mu]}(a, w)] - \operatorname{Im}[\eta f_{\Omega}^{[i\mu]}(a, w)] \cdot \operatorname{Re}[\eta^{3} \partial_{w} f_{\Omega}^{[\mu]}(a, w)] \right].$$

Summing this with the similar result for $T^{[i\eta]}(w_{\delta})$, we obtain

$$\delta^{-3} \cdot \left[\mathbb{E}_{\Omega_{\delta}}^{+} [(T^{[\eta]}(w_{\delta}) + T^{[i\eta]}(w_{\delta})) \varepsilon(a_{\delta})] - \mathbb{E}_{\Omega_{\delta}}^{+} [T^{[\eta]}(w_{\delta}) + T^{[i\eta]}(w_{\delta})] \cdot \mathbb{E}_{\Omega_{\delta}}^{+} [\varepsilon(a_{\delta})] \right]$$

$$\xrightarrow{\delta \to 0} \frac{3\sqrt{3}}{4\pi^{2}} \operatorname{Im} \left[\eta^{4} (f_{\Omega}^{[\mu]}(a, w) \partial_{w} f_{\Omega}^{[i\mu]}(a, w) - f_{\Omega}^{[i\mu]}(a, w) \partial_{w} f_{\Omega}^{[\mu]}(a, w)) \right]$$

$$= \frac{3\sqrt{3}}{2\pi^{2}} \operatorname{Im} \left[\eta^{4} (f_{\Omega}(a, w) \partial_{w} f_{\Omega}^{\dagger}(a, w) - f_{\Omega}^{\dagger}(a, w) \partial_{w} f_{\Omega}(a, w)) \right].$$

Taking into account (5.4) and Corollary 5.3, this can be written as

$$\delta^{-3} \cdot \mathbb{E}_{\Omega_{\delta}}^{+}[(T^{[\eta]}(w_{\delta}) + T^{[i\eta]}(w_{\delta}))\varepsilon(a_{\delta})]$$

$$\underset{\delta \to 0}{\to} -\frac{3\sqrt{3}}{\pi^{2}} \Big[\operatorname{Re}[\eta^{4} \langle T(w) \rangle_{\Omega}^{+} \langle \varepsilon(a) \rangle_{\Omega}^{+}]$$

$$-\frac{1}{2} \operatorname{Im}[\eta^{4} (f_{\Omega}(a, w) \partial_{w} f_{\Omega}^{\dagger}(a, w) - f_{\Omega}^{\dagger}(a, w) \partial_{w} f_{\Omega}(a, w))] \Big],$$

which coincides with $-\frac{3\sqrt{3}}{\pi^2} \operatorname{Re}[\eta^4 \langle T(w)\varepsilon(a)\rangle_{\Omega}^+]$, see Definition 4.6. As usual, the similar expectation with $R^{[\eta]}(w_{\delta})$ disappears in the limit and the averaging over $\eta \in \{1, \rho, \rho^2\}$ gives (5.12).

The proof of the convergence of the ratios $\delta^{-2}\mathbb{E}_{\Omega_{\delta}}^{+}[T^{[\eta]}(w_{\delta})\sigma(u_{\delta})]/\mathbb{E}_{\Omega_{\delta}}^{+}[\sigma(u_{\delta})]$ mimics the proof of the convergence of the expectations $\delta^{-2}\mathbb{E}_{\Omega_{\delta}}^{+}[T^{[\eta]}(a_{\delta})]$ in Theorem 1.1 with the following modifications. First, one should use Proposition 3.13 instead of Proposition 3.4 to express these ratios in terms of the local values of spinor observables $F_{[\Omega_{\delta},u_{\delta}]}$. Second, one should use Theorem 5.5 given in the next section instead of Theorem 5.2 to prove the convergence of these discrete spinor observables to their continuous counterparts. Taking into account Definition 4.9, the technical computations needed to derive the convergence of the ratios $\delta^{-2}\mathbb{E}_{\Omega_{\delta}}^{+}[T^{[\eta]}(w_{\delta})\sigma(u_{\delta})]/\mathbb{E}_{\Omega_{\delta}}^{+}[\sigma(u_{\delta})]$ just repeat those in the proof of Theorem 1.1.

5.3. Convergence of the spinor observables. This section is devoted to the proof of an analogue of Theorem 5.2 for spinor observables. For this purpose, we adopt the abstract scheme suggested in [30] (and later implemented, e.g., in [10]) for proving similar convergence results instead of a more constructive approach used in [25] to prove Theorem 5.2 in the square lattice setup. It is worth saying that below we omit a number of technical lemmas (treating particular properties of discrete s-holomorphic functions), using references to the original papers on the subject. Instead, we focus on the logical structure of the arguments, which can be easily adapted for various setups. In particular, a simpler version of these arguments can be used to give an alternative proof of Theorem 5.2.

Recall that the discrete spinor observables $F_{[\Omega_{\delta},u_{\delta}]}(a_{\delta},\cdot)$ are defined in Section 3.4. Similarly to the non-branching case, we denote

$$F^{\sharp}_{[\Omega_{\delta},u_{\delta}]}(a_{\delta},\cdot) \ := \ F_{[\Omega_{\delta},u_{\delta}]}(a_{\delta},\cdot) - F_{\mathbb{C}_{\delta}}(a_{\delta},\cdot) \,.$$

As in the non-branching case, this function is s-holomorphic everywhere near a_{δ} . Note that it is well-defined in a vicinity of a_{δ} only, since the full-plane observable $F_{\mathbb{C}_{\delta}}$ does not respect the spinor property of $F_{[\Omega_{\delta},u_{\delta}]}$.

Theorem 5.5. Given a bounded simply connected domain $\Omega \subset \mathbb{C}$, two distinct points $a, u \in \Omega$ and $\eta \in \wp$, let Ω_{δ} denote a discrete approximation to Ω (converging in the Carathéodory sense as $\delta \to 0$, see [12, Section 3.2] for definitions), u_{δ} be the face of Ω_{δ} containing u, and a_{δ} be the closest to $a \in \Omega$ half-edge of Ω_{δ} oriented in the direction η^2 , which we assume to be lifted

on the discrete double-cover $[\Omega_{\delta}, u_{\delta}]$. For a point $z \in [\Omega, u]$, let $z_{e_{\delta}}$ denote the closest to z midedge of $[\Omega_{\delta}, u_{\delta}]$. Then, the following convergence results hold as $\delta \to 0$:

$$\delta^{-1} \cdot F_{[\Omega_{\delta}, u_{\delta}]}(a_{\delta}, z_{e_{\delta}}) \implies \frac{\sqrt{3}}{2\pi} f_{[\Omega, u]}^{[\eta]}(a, z), \quad z \neq a, a^*, \tag{5.13}$$

$$\delta^{-1} \cdot F^{\sharp}_{[\Omega_{\delta}, u_{\delta}]}(a_{\delta}, z_{e_{\delta}}) \implies \frac{\sqrt{3}}{2\pi} (\eta f^{\sharp}_{[\Omega, u]}(a, z) + i\overline{\eta} f^{\dagger}_{[\Omega, u]}(a, z)), \tag{5.14}$$

where the spinors $f_{\Omega}^{[\eta]}(a,z)$, $f_{\Omega}^{\sharp}(a,z)$ and $f_{\Omega}^{\dagger}(a,z)$ are defined in Section 4.3. Moreover, the first convergence is uniform (with respect to (a,z)) on compact subsets of $([\Omega,u]\times[\Omega,u])\setminus D$, where $D:=\{(a,a),a\in[\Omega,u]\}\cup\{(a,a^*)\in[\Omega,u]\}\cup\{u\}$, and the second is uniform for $a\in[\Omega,u]$ lying in compact subsets of $[\Omega,u]\setminus\{u\}$ and z in a vicinity of a.

Proof. Using standard compactness arguments, it is easy to see that the uniform convergence (5.13) follows from the *pointwise* one, if one is able to prove it for arbitrary sequences of $z_{e_{\delta}}$ and a_{δ} converging to distinct points $z, a \in \Omega \setminus \{u\}$ as $\delta \to 0$. It is well-known that the s-holomorphicity of the discrete spinor observable $F_{[\Omega_{\delta}, u_{\delta}]}(a_{\delta}, \cdot)$ allows one to define the discrete anti-derivative

$$H_{\Omega_{\delta}} := \int \operatorname{Re}\left[(\delta^{-1} F_{[\Omega_{\delta}, u_{\delta}]}(a_{\delta}, z_{\delta}))^{2} dz_{\delta} \right].$$

(we omit u_{δ} and a_{δ} in the left-hand side for shortness), which has the following properties:

- (1) The function $H_{\Omega_{\delta}}$ is defined on both the set of faces of the discrete domain Ω_{δ} and the set of its vertices. Below we denote by $H_{\Omega_{\delta}}^{\circ}$ and $H_{\Omega_{\delta}}^{\bullet}$ the restrictions of $H_{\Omega_{\delta}}$ on these two sets, respectively. Moreover, for each pair (w_{δ}, v_{δ}) of a face w_{δ} and a vertex v_{δ} of Ω_{δ} incident to w_{δ} one has $H_{\Omega_{\delta}}^{\circ}(w_{\delta}) \geq H_{\Omega_{\delta}}^{\bullet}(v_{\delta})$.
- (2) The function $H_{\Omega_{\delta}}^{\circ}$ is discrete subharmonic everywhere in Ω_{δ} except at the face u_{δ} and the two faces adjacent to a_{δ} . The function $H_{\Omega_{\delta}}^{\bullet}$ is discrete superharmonic everywhere in Ω_{δ} except at the two vertices incident to a_{δ} .
- (3) If an a priori bound $H_{\Omega_{\delta}}^{\bullet} \geq -m$ holds true at the three vertices adjacent to some vertex v_{δ} , then $H_{\Omega_{\delta}}^{\bullet}(v_{\delta}) + m \geq \operatorname{const} \cdot (H_{\Omega_{\delta}}^{\circ}(w_{\delta}) + m)$ for each of the three faces w_{δ} incident to v_{δ} , with a constant independent of δ and m.
- (4) The function $H_{\Omega_{\delta}}^{\circ}$ satisfies the Dirichlet boundary condition $H_{\Omega_{\delta}}^{\circ} = 0$ at all boundary faces of Ω_{δ} . Moreover, one can (re-)assign the values $H_{\Omega_{\delta}}^{\bullet} := 0$ along the boundary of Ω_{δ} and modify the conductances of boundary edges so that the function $H_{\Omega_{\delta}}^{\bullet}$ remains discrete superharmonic near the boundary of Ω_{δ} .

The properties (1) and (2) were first pointed out in [49, 47] (in the square lattice setup). The same construction for s-holomorphic functions defined on general isoradial graphs (of which the honeycomb grid is a particular case) was discussed in [8, Proposition 3.6]. The a priori bound (3) and the treatment of the boundary values (4) can be found in [8, Proposition 3.8] and [8, Lemma 3.14], respectively. Note that these papers use a slightly different definition of s-holomorphicity from ours (see Remark 3.4 for a discussion), which causes the interchanging of the roles of $H_{\Omega_{\delta}}^{\circ}$ and $H_{\Omega_{\delta}}^{\bullet}$. The fact that we work with discrete spinors instead of functions is not relevant for (1)–(4) thanks to the square in the definition of $H_{\Omega_{\delta}}$.

In a vicinity of the point $a \in \Omega$ one can similarly define the discrete anti-derivative

$$H_{\Omega_{\delta}}^{\sharp} := \int \operatorname{Re}\left[(\delta^{-1} F_{[\Omega_{\delta}, u_{\delta}]}^{\sharp}(z_{\delta}))^{2} dz_{\delta} \right],$$

which satisfies the same properties (1)–(3). Moreover, the function $F_{\Omega_{\delta}}^{\sharp}$ is s-holomorphic and hence the functions $H_{\Omega_{\delta}}^{\sharp \circ}$ and $H_{\Omega_{\delta}}^{\sharp \circ}$ are discrete subharmonic and superharmonic, respectively, everywhere in a vicinity of a_{δ} , including the two faces and the two vertices incident to a_{δ} .

Let us now assume for a while that the functions $H_{\Omega_{\delta}}$ remain uniformly bounded on compact subsets of $\Omega \setminus \{a, u\}$ as $\delta \to 0$. More precisely, let us assume that the following is fulfilled:

For each
$$r > 0$$
 there exists a constant $C(r) > 0$ independent of δ such that $|H_{\Omega_{\delta}}| \leq C(r)$ everywhere in $\Omega_{\delta}(r) := \Omega_{\delta} \setminus (B_r(a) \cup B_r(u))$. (5.15)

Under this assumption, the following two claims hold true:

(5) The functions $\delta^{-1}F_{\Omega_{\delta}}$ are also uniformly bounded on compact subsets of $\Omega \setminus \{a,u\}$. Moreover, the functions $H_{\Omega_{\delta}}$ and $\delta^{-1}F_{\Omega_{\delta}}$ are equicontinuous away from the points a and u, thus one can use the Arzelà–Ascoli theorem to get subsequential limits

$$H_{\Omega_{\delta}} \rightrightarrows h$$
 and $\delta^{-1}F_{\Omega_{\delta}} \rightrightarrows f$ on compact subsets of $\Omega \setminus \{a, u\}$,

where $h: \Omega \setminus \{a, u\} \to R$ is a real-valued function and $f: [\Omega, u] \setminus \{a, a*\}$ is a complex-valued spinor. Since the spinors $\delta^{-1}F_{\Omega_{\delta}}$ are s-holomorphic, their limit f is holomorphic in $[\Omega, u] \setminus \{a, a^*\}$ and $h = \int \operatorname{Re}[f(z)^2 dz]$ is a harmonic function in $\Omega \setminus \{a, u\}$.

(6) The Dirichlet boundary conditions for the functions $H_{\Omega_{\delta}}$ yields the same boundary condition h = 0 on $\partial\Omega$ for their limit, provided $\Omega_{\delta} \to \Omega$ in the Carathéodory sense.

The claim (5) is a straightforward corollary of [8, Theorem 3.12]. The claim (6) easily follows from the subharmonicity of $H_{\Omega_{\delta}}^{\bullet}$, the superharmonicity of $H_{\Omega_{\delta}}^{\bullet}$, our asumption (5.15) and simple uniform bounds for the discrete harmonic measure known as the weak Beurling-type estimate, e.g. see [12, Proposition 2.11].

Since the functions $\delta^{-1}F_{\mathbb{C}_{\delta}}(a_{\delta},\cdot)$ are uniformly bounded away from the point a (see (5.1)), it follows from the claim (5) that the functions $F_{\Omega_{\delta}}^{\sharp}$ (and hence $H_{\Omega_{\delta}}^{\sharp}$) are also uniformly bounded away from the point a. At the same time, the functions $H_{\Omega_{\delta}}^{\sharp}$ satisfy the discrete maximum principle everywhere near a_{δ} and thus are uniformly bounded everywhere near the point $a \in \Omega$. Applying the same arguments as in the claim (5), we conclude that we have the uniform convergence (at least along subsequences)

$$\delta^{-1}F_{\Omega_{\delta}}^{\sharp} \rightrightarrows f^{\sharp}$$
 in a vicinity of $a \in \Omega$,

where a holomorphic function f^{\sharp} has no singularity at a. On the other hand, it directly follows from the asymptotics (5.1) and the convergence of the functions $\delta^{-1}F_{\Omega_{\delta}}$ themselves that

$$f^{\sharp}(z) = f(z) - \eta \cdot \frac{\sqrt{3}}{2\pi} (z - a)^{-1}$$
 for all $z \neq a$.

Hence, the right-hand side has no singularity at the point $a \in \mathbb{C}$.

Further, recall that the functions $H_{\Omega_{\delta}}^{\bullet}$ are discrete superharmonic everywhere near u_{δ} , and hence are uniformly bounded from below near the point $u \in \Omega$. Therefore, the function h is harmonic and bounded from below in a vicinity of u, which implies

$$h(z) = -c \cdot \log|z - u| + O(1)$$
 and $f(z) = \sqrt{-c} \cdot (z - u)^{-\frac{1}{2}} + O(|z - u|^{\frac{1}{2}}),$ (5.16)

as $z \to u$, for some real positive constant $c \ge 0$. Also, note that the Dirichlet boundary conditions for the anti-derivative $h = \int \text{Re}[(f(z))^2 dz]$ provided by the claim (6) are equivalent to the boundary condition $\text{Im}[f(\zeta)\sqrt{\nu_{\text{out}}(\zeta)}] = 0$ for $\zeta \in \partial\Omega$.

Therefore, any subsequential limit f from the claim (5) solves the boundary value problem described in Definition 4.8. As the solution to this boundary value problem is unique, we have $f(\cdot) = f_{[\Omega,u]}^{[\eta]}(a,\cdot)$ for any such a limit, which gives (5.13). Note that the uniform convergence (5.14) was also derived along the way.

Thus we have proved Theorem 5.5 under assumption (5.15). It remains to show that the scenario opposite to (5.15), when the functions $H_{\Omega_{\delta}}$ blow up as $\delta \to 0$ away from the singularities a and u cannot happen. On the contrary, assume that

$$M_{\delta}(r) := \max_{\Omega_{\delta}(r)} |H_{\Omega_{\delta}}| \to \infty$$

for some r > 0 and along a subsequence $\delta \to 0$. Let us introduce the normalized functions

$$\begin{array}{lcl} \widetilde{H}_{\Omega_{\delta}} \; := \; (M_{\delta}(r))^{-1} \cdot H_{\Omega_{\delta}} \,, & \widetilde{F}_{[\Omega_{\delta}, u_{\delta}]} \; := \; (M_{\delta}(r))^{-1/2} \cdot F_{[\Omega_{\delta}, u_{\delta}]} \,, \\ \widetilde{H}^{\sharp}_{\Omega_{\delta}} \; := \; (M_{\delta}(r))^{-1} \cdot H^{\sharp}_{\Omega_{\delta}} \,, & \widetilde{F}^{\sharp}_{[\Omega_{\delta}, u_{\delta}]} \; := \; (M_{\delta}(r))^{-1/2} \cdot F^{\sharp}_{[\Omega_{\delta}, u_{\delta}]} \,, \end{array}$$

and apply the claim (5) to $\widetilde{H}_{\Omega_{\delta}}$ and $\delta^{-1}\widetilde{F}_{[\Omega_{\delta},u_{\delta}]}$ instead of $H_{\Omega_{\delta}}$ and $\delta^{-1}F_{[\Omega_{\delta},u_{\delta}]}$. By definition, we have $|H_{\Omega_{\delta}}| \leq 1$ in $\Omega_{\delta}(r)$, thus one can find subsequential limits

$$\widetilde{H}_{\Omega_{\delta}} \rightrightarrows \widetilde{h}$$
 and $\delta^{-1}\widetilde{F}_{[\Omega_{\delta},u_{\delta}]} \rightrightarrows \widetilde{f}$ on compact subsets of $\Omega(r) := \Omega \setminus (B_r(a) \cup B_r(u))$.
Moreover, the assumption $M_{\delta}(r) \to \infty$ implies $(M_{\delta}(r))^{-1/2}F_{\mathbb{C}_{\delta}}(a_{\delta},\cdot) \rightrightarrows 0$ and hence $\delta^{-1}\widetilde{F}_{[\Omega_{\delta},u_{\delta}]}^{\sharp} \rightrightarrows \widetilde{f}^{\sharp} = \widetilde{f}$ and $\widetilde{H}_{\Omega_{\delta}}^{\sharp} \rightrightarrows \widetilde{h}^{\sharp} = \widetilde{h}$ on compact subsets of $A_r(a) := B_{2r}(a) \setminus \overline{B_r(a)}$, provided an additive constant in the definition of $\widetilde{H}_{\delta}^{\sharp}$ is chosen so that $\widetilde{H}_{\delta}^{\sharp} = \widetilde{H}_{\Omega}$ at some

provided an additive constant in the definition of $\widetilde{H}_{\Omega_{\delta}}^{\sharp}$ is chosen so that $\widetilde{H}_{\Omega_{\delta}}^{\sharp} = \widetilde{H}_{\Omega_{\delta}}$ at some fixed point lying in the annulus $A_r(a)$. Let us now show that the following claim is fulfilled:

(7) The function h cannot vanish identically in $\Omega(r)$. In particular, this implies that for any r' < r there exists a constant C(r',r) > 0 such that $M_{\delta}(r') \leq C(r',r)M_{\delta}(r)$ uniformly in δ . (Otherwise, the limit of the functions $(M_{\delta}(r'))^{-1}H_{\Omega_{\delta}}$ vanishes everywhere in $\Omega(r)$, and hence everywhere in $\Omega(r')$ because of its harmonicity.)

To prove this claim, assume that $\tilde{h}=0$ everywhere in $\Omega(r)$. Let z_{δ}^{\max} be chosen so that

$$1 \ = \ \max_{\Omega^\delta(r)} |\widetilde{H}_{\Omega_\delta}| \ = \ |\widetilde{H}_{\Omega_\delta}(z_\delta^{\max})|.$$

Passing to a subsequence once again, we can also assume $z_{\delta}^{\max} \to z^{\max} \in \overline{\Omega(r)}$ as $\delta \to 0$. If we had $z^{\max} \in \Omega(r)$, this would immediately imply $|\widetilde{h}(z^{\max})| = 1 \neq 0$. Unfortunately, it easily follows from the discrete maximum principle that $z^{\max} \in \partial B_r(a) \cup \partial B_r(u)$ and we do not know that the functions $H_{\Omega_{\delta}}$ converge to h on the boundary of $\Omega(r)$. Thus some additional arguments are needed. As $H_{\Omega_{\delta}}^{\circ} \geq H_{\Omega_{\delta}}^{\bullet}$, there are three possibilities:

- $\begin{array}{l} \text{(a)} \ \ z^{\max} \in \partial B_r(a); \\ \text{(b)} \ \ z^{\max} \in \partial B_r(u), \ z^{\max}_{\delta} \ \text{is a vertex of} \ \Omega_{\delta} \ \text{and} \ H^{\bullet}_{\Omega_{\delta}}(z_{\delta}) = -1; \\ \text{(c)} \ \ z^{\max} \in \partial B_r(u), \ z^{\max}_{\delta} \ \text{is a face of} \ \Omega_{\delta} \ \text{and} \ H^{\circ}_{\Omega_{\delta}}(z_{\delta}) = 1. \end{array}$

Let us start with the case (a). Recall that the functions $\widetilde{H}_{\Omega_{\delta}}^{\sharp \circ}$ and $\widetilde{H}_{\Omega_{\delta}}^{\sharp \circ}$ are discrete subharmonic and superharmonic, respectively, everywhere in $B_{2r}(a)$. Therefore, the convergence $\widetilde{H}_{\Omega_{\delta}}^{\sharp} \rightrightarrows 0$ in the annulus $A_r(a)$ implies the convergence $\widetilde{H}_{\Omega_{\delta}}^{\sharp} \rightrightarrows 0$ everywhere in the disc $B_{2r}(a)$. This consequently gives $\delta^{-1}\widetilde{F}^{\sharp}_{[\Omega_{\delta},u_{\delta}]} \Rightarrow 0$, $\delta^{-1}\widetilde{F}_{[\Omega_{\delta},u_{\delta}]} \Rightarrow 0$ and $\widetilde{H}_{\Omega_{\delta}} \Rightarrow 0$ in a vicinity of $\partial B_r(a)$, which is a contradiction.

To handle the cases (b) and (c), recall that the functions $\widetilde{H}_{\Omega_{\delta}}^{\bullet}$ are discrete superharmonic everywhere in $B_{2r}(u)$ and $\widetilde{H}_{\Omega_{\delta}}^{\bullet} \Rightarrow 0$ in $\Omega(r)$. Therefore, for any fixed $\varepsilon > 0$ one has $\widetilde{H}_{\Omega_{\delta}}^{\bullet} \geq -\varepsilon$ everywhere in the disc $B_{2r}(u)$ provided δ is small enough. In particular, the case (b) is impossible. In the case (c), the subharmonicity of $\widetilde{H}_{\Omega_{\delta}}^{\circ}$ implies the existence of a nearest-neighbor sequence of faces of Ω_{δ} which starts at z_{δ}^{\max} and ends at u_{δ} such that $\widetilde{H}_{\Omega_{\delta}}^{\circ} \geq 1$ along this sequence. Then one can apply the claim (3) with a (fixed) value $m = \varepsilon$ chosen sufficiently close to 0 and conclude that $\widetilde{H}_{\Omega_{\delta}}^{\bullet} \geq -\varepsilon + \text{const} \cdot (1+\varepsilon) > 0$ at all vertices incident to this sequence of faces. Then the superharmonicity of $\widetilde{H}_{\Omega_{\delta}}^{\bullet}$ and simple harmonic measure estimates yield $\widetilde{H}_{\Omega_{\delta}}^{\bullet} \geq \text{const} > 0$ everywhere in $B_{2r}(u)$, which contradicts to $\widetilde{h} = 0$; see [10, Lemma 3.10] for more details.

The proof of the claim (7) is finished and it is now easy to complete the proof of Theorem 5.5. Using the uniform estimate $M_{\delta}(r') \leq C(r',r)M_{\delta}(r)$ provided by (7) and applying the claim (5) in the domains $\Omega(r')$ with $r' \to 0$, it is easy to see that one can find a subsequence $\delta \to 0$ such that

$$\widetilde{H}_{\Omega_{\delta}} \rightrightarrows \widetilde{h} \quad \text{and} \quad \delta^{-1} \widetilde{F}_{[\Omega_{\delta}, u_{\delta}]} \rightrightarrows \widetilde{f}$$

on all compact subsets of $\Omega \setminus \{a, u\}$ (and not only on subsets of $\Omega(r)$ for some fixed r > 0). As above, it is easy to see that the functions \widetilde{h} and \widetilde{f} satisfy the asymptotics (5.16) near the branching point u and we still have $\widetilde{h} = 0$ on $\partial\Omega$ due to the claim (6). At the same time, the function $\widetilde{f} = \widetilde{f}^{\sharp}$ has no singularity at the point a (recall that this equality is a consequence of the additional multiplicative normalization $(M_{\delta}(r))^{-1} \to 0$ as $\delta \to 0$, which kills the scaling limit of the full-plane observable $\delta^{-1}F_{\mathbb{C}_{\delta}}(a_{\delta}, \cdot)$). This contradicts to the claim (7) since this boundary value problem has only the trivial solution.

A. Appendix

A.1. Coefficients in the definition of $\mathcal{T}_{\text{edge}}$ and \mathcal{T}_{mid} . Define $\mathcal{T}_{\text{edge}}^{\varnothing}$ and $\mathcal{T}_{\text{mid}}^{\varnothing}$ as it is described in Lemma 2.4. Assume that c_i , d_i and x_c are arbitrary numbers.

Lemma A.1. Let a and b be two adjacent edges in one of faces of the honeycomb lattice Ω . And let m be the midline in the same hexagon (see Fig. 7). Then there exists a unique set of coefficients c_i and d_i and a unique value of x such that relation (2.9) is satisfied. The values of c_i and d_i are given by (2.7)–(2.8), and $x = \frac{1}{\sqrt{2+\sqrt{2-n}}}$.

This can be rigorously proved just by considering the first derivative of the equation of the pentagonal equation (Proposition 2.2). And the constant $\frac{-2n}{1+\sqrt{2-n}}$ in the definition of $\mathcal{T}_{\text{mid}}^{\varnothing}$ arises as the derivative of the constant c in the pentagonal equation. But we will show another proof, purely combinatorial.

Proof. The proof is be based on the consideration of certain groups of configurations in $Conf_m(\Omega)$ — the ones that look the same outside the semihexagon formed by a, b and m.

Denote the endpoints of a and b by X, Y and Z, where Y is the common endpoint of a and b. Then, denote by M and N the endpoints of m, where M is closer to X and N is closer to Z. Note that X, Y and Z are vertices of the hexagonal lattice, and M and N are midpoints of its edges.

Consider any configuration $\gamma \in Conf_m(\Omega)$ such that γ does not contain m as an edge. Now, we construct a new configuration γ' by means of XOR on the contour MXYZN, i. e. γ' is equal to γ outside MXYZN and each of the edges and half-edges of this contour is contained exactly in one of the configurations γ and γ' . It is clear, that this defines the bijection between configurations in $Conf_m(\Omega)$ with edge m and the ones without it.

Note that there are some configurations in $Conf_a$ and $Conf_b$ which don't belong to $Conf_m$. These are the ones in which either both vertices X and Y have degree 3, or both vertices Y and Z have degree 3. In the former case, deleting the edge a brings this configuration to $Conf_m$, and in the latter case, deleting the edge does the same.

We will divide all the configurations from $Conf_a \cup Conf_b \cup Conf_m$ in several groups with either 2 or 3 configurations in each of them. In each of these groups the configurations differ only inside on MXYZN.

First consider $\gamma \in Conf_a \setminus Conf_m$. Then the group consists of γ , $\gamma - a$ and $(\gamma - a)'$ (a configuration obtained from $\gamma - a$ by means of XOR on the contour MXYZN).

Then for $\gamma \in Conf_b \setminus Conf_m$ the group consists of γ , $\gamma - b$ and $(\gamma - b)'$.

Finally, for all subgraphs $\gamma \in Conf_m$ such that γ is not put in any group yet, the group consists of γ and γ' .

Assume, that for each such a group we prove that its contribution to $\mathcal{T}_{\text{edge}}^{\varnothing}(a) + \mathcal{T}_{\text{edge}}^{\varnothing}(b)$ is the same as to $\mathcal{T}_{\text{mid}}^{\varnothing}(m)$. Then, summing over all the configurations we get the desired relation.

To prove this fact by checking it for each group. It only matters, how this configuration looks like on MXYZN and which of these edges are in the same loop. Thus, there are not so many different cases. This brings us to the system of the linear equations:

$$c_1 - d_1 - x^2 d_3 = const$$

$$c_2 + c_1 - x d_2 = const$$

$$c_3 + c_1 - d_1 - d_3 = const$$

$$2c_3 - \frac{1}{x} d_2 = const$$

$$c_4 + nc_5 + c_2 - d_1 - nd_3 = const$$

$$c_4 + \frac{1}{n} c_5 + c_2 - d_1 - \frac{1}{n} d_3 = const$$

$$2c_1 - d_4 - nd_5 - xnd_2 = const$$

$$2c_1 - d_4 - \frac{1}{n} d_5 - \frac{x}{n} d_2 = const$$

$$c_2 + c_3 - d_1 - \frac{1}{x^2} d_3 = const$$

$$c_2 + c_4 + nc_5 - d_4 - nd_5 - \frac{n}{x} d_2 = const$$

$$c_2 + c_4 + \frac{1}{n} c_5 - d_4 - nd_5 - \frac{1}{nx} d_2 = const$$

This system of 11 equations with 11 variables has a unique solution — $x = x_c$ and coefficients c_i and d_i are given by (2.7)-(2.8).

Lemma A.2. Consider a simply-connected part Ω of the honeycomb lattice. Then there exists a function H on the vertices of Ω^{dual} , such that for each edge of Ω and each midline of a face of Ω observables $\mathcal{T}_{\text{edge}}$ and \mathcal{T}_{mid} can be expressed in terms of H in a way shown on Fig. 8. This function H is unique up to fixing its value on free arbitrary vertices of Ω^{dual} .

Moreover, for any function H on vertices of Ω^{dual} functions on edges and midlines of Ω defined by (see Fig. 8):

$$F_{\text{edge}}(e) = H(A) - H(B) + H(C) - H(D)$$
(6.1)

$$F_{\text{mid}}(m) = H(B) + H(C) - 2H(A)$$
 (6.2)

will satisfy the same local relations as $\mathcal{T}_{\mathrm{edge}}$ and $\mathcal{T}_{\mathrm{mid}}$, i. e. Eq. (2.9), (2.10) and (2.11).

Proof. The last statement is an easy calculation. We do not give any further details.

In order to express both \mathcal{T}_{edge} and \mathcal{T}_{mid} in terms of a function on faces of Ω , it is enough to express \mathcal{T}_{edge} . The statement about \mathcal{T}_{mid} follows immediately. We start by giving a scetch of the proof Figure 8 describes a linear mapping from the space of all functions on the vertices of Ω^{dual} to the space of functions on the edges of Ω . As we mentioned above, the image is contained in the subspace of all functions on edges of Ω satisfying 2.10 in each hexagon. We calculate the dimensions of the spaces and of the kernel and show that the mapping is surjective on a described subspace.

We start by calculating the dimension of functions defined on edges of Ω that on each hexagon satisfy the relation 2.10. These relations say that the difference between values of the function on the opposite edges in a particular hexagon is independent of a chosen pair of edges. Consider a function f satisfying these relations. For a hexagon a denote by $\partial f(a)$ the difference between the values of f on the opposite edges of a. We say that two boundary edges are paired if they are parallel and the segment joining there midpoints is perpendicular to them and lies entirely inside Ω . It is easy to see that this induces a matching on all boundary edges of Ω . Note that the values of ∂f on each hexagon and the values of f on one edge in each pair of the boundary edges completely define f. Moreover, these for any choice of these values, the obtained function will satisfied the desired relations 2.10 on each hexagon. Thus, the dimension of function on edges of Ω satisfying these relations is equal to:

$$\#\{\text{faces in }\Omega\} + \frac{1}{2}\#\{\text{edges in }\partial\Omega\}.$$
 (6.3)

The dimension of all functions on the vertices of Ω^{dual} is obviously equal to the number of vertices in Ω^{dual} . Clearly, the number of inside vertices of Ω^{dual} is equal to the number of faces of Ω . It is a bit less obvious that the number of boundary vertices of Ω^{dual} is equal to the number of boundary vertices in Ω that have degree 2 in Ω . Indeed, note that when going along the boundary of Ω these vertices mark a switch from one vertex of Ω^{dual} to another. All other vertices in Ω have degree 3. Thus, the dimension of all functions on the vertices of Ω^{dual} is equal to:

$$\#\{\text{faces in }\Omega\} + 3\#\{\text{vertices in }\Omega\} - 2\#\{\text{edges in }\Omega\}.$$
 (6.4)

In fact, dimensions 6.3 and 6.4 differ by 3 and the second is bigger. One just needs to express $\#\{\text{edges in }\partial\Omega\}$ in terms of the number of faces and edges in Ω and use Euler formula.

Now consider a mapping from all functions on the vertices of Ω^{dual} to all functions on edges of Ω given by the formula on figure 8. As we already notices, all obtained functions will satisfy

relation 2.10 on each hexagon. We need to prove that this mapping is a surjective mapping on this subspace. Above we showed that the dimension of all such functions on edges of Ω is equal the dimension of all functions on vertices of Ω^{dual} minus 3. To finish the proof it is enough to show that the dimension of the kernel is not bigger than 3. Suppose that this is not the case. Then one can fix a value of a function in 3 vertices of Ω^{dual} and there will be still a non-trivial way to assign numbers to over vertices of Ω^{dual} , so that the obtained function is in the kernel of the mapping. On the other hand, if we put zeroes in 3 adjacent vertices of Ω^{dual} (hexagons of Ω), and exploring other adjacent vertices one by one, then on all of them the function has to be equal to 0 (otherwise, the value on one of the edges will be not equal to 0).

This contradiction shows that the dimension of the kernel is not bigger than 3. Then the mapping is surjective and the dimension of the kernel is equal to 3.

A.2. S-holomorphic functions on the honeycomb grid and the construction of the full-plane observable. In this part of the appendix we review the results concerning discrete s-holomorphicity (strong holomorphicity) of the fermionic observable, contruct the full-plane fermionic observable on the honeycomb lattice and compute its singularity. We start by defining the notion of discrete s-holomorphicity. Then we formulate the result that the fermionic observable is s-holomorphic everywhere except for the origin. Such a function is unique up to a multiplicative constant. It can be constructed by its projections which are equal to particular linear combinations of the discrete Green's functions (see [34]). Knowing the asymptotic of the Green's function, one can easily compute the asymptotic of the fermionic observable. This gives the constant $\frac{\sqrt{3}}{2\pi}$ in Theorem 5.2.

Definition A.3. Let Ω be a discrete domain drawn on the regular hexagonal grid and F be a complex-valued function defined on (a subset of) the set of medial vertices (aka midedges) z_e of Ω . We say that F is an s-holomorphic function, if one has

$$\operatorname{Proj}[F(z_e); (u-v)^{-\frac{1}{2}}\mathbb{R}] = \operatorname{Proj}[F(z_{e'}); (u-v)^{-\frac{1}{2}}\mathbb{R}]$$
(6.5)

for all pairs of adjacent medial vertices $z_e, z_{e'}$, where v denotes the common vertex of the edges e and e', a point u is the center of the face adjacent to both e and e', and

$$\operatorname{Proj}[F;\tau\mathbb{R}] := \frac{1}{2}[F + (\tau/|\tau|)^2\overline{F}]$$

denotes the orthogonal projection of the complex number F, considered as a point in the plane, onto the line $\tau \mathbb{R} \subset \mathbb{C}$.

For the first time, the notion of s-holomorphic functions appeared in [48]. The next proposition states that the fermionic observable $F_{\Omega_{\delta}}(a,\cdot)$ is s-holomorphic everywhere except for two pairs of edges near a, where it has a defect 1.

Proposition A.4. Let a be a half-edge inside Ω_{δ} . Given two adjacent edges e and e' of Ω_{δ} , denote by z_e and $z_{e'}$ their midpoints. Then $F_{\Omega_{\delta}}(a,\cdot)$ satisfies the following properties:

- If none of e and e' contains a, then the s-holomorphicity condition 6.5 holds for $F_{\Omega_{\delta}}(a, z_e)$ and $F_{\Omega_{\delta}}(a, z_{e'})$;
- If e contains a and e' has a common vertex with a, then the s-holomorphicity condition 6.5 holds for $F_{\Omega_{\delta}}(a, z_e) + 1$ and $F_{\Omega_{\delta}}(a, z_{e'})$;
- If e contains a and e' does not have a common vertex with a, then the s-holomorphicity condition 6.5 holds for $F_{\Omega_{\delta}}(a, z_e)$ and $F_{\Omega_{\delta}}(a, z_{e'})$.

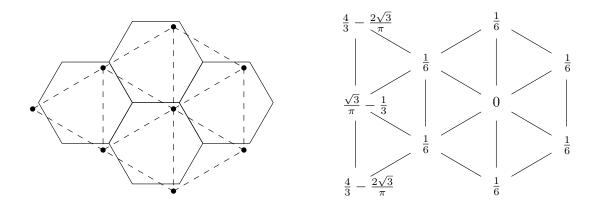


FIGURE 12. Left: One type of corners of the hexagonal lattice of the mesh size δ forms the triangular lattice of the mesh size $\sqrt{3}\delta$. A direction $\eta \in \wp$ associated to the shown type of corners is ρ . Right: Local values of the Green's function (see [34]).

This proposition was proven in [8] (Proposition 2.5) in a general case of the isoradial graphs, and we do not give any details about the proof. See also [47] (Lemma 4.5) for the same result about the fermionic observable in FK-Ising.

In the continuous case, notions of holomorphicity and harmonicity are very well related — the projections of a holomorphic functions are harmonic, and vice versa, if a function has harmonic projections, then it is holomorphic. The next lemma states that the same holds on a discrete level. First we define the corners of the lattice.

Definition A.5. By a corner of a lattice we mean a pair of adjacent edges.

For each corner c = (e, e') we associate the unique number $\eta(c) \in \wp$, such that $\overline{\eta(c)}^2$ has the same argument as u - v, where v denotes the common vertex of the edges e and e', a point u is the center of the face adjacent to both e and e'.

According to their direction, the corners can be divided into 6 types. Each of these types forms the triangular lattice of the mesh size $\sqrt{3}\delta$ (see Fig. 12).

Lemma A.6. Let F be an s-holomorphic function on the midedges of Ω_{δ} . Then one can define six discretely harmonic functions F^{η} , where $\eta \in \wp$, on the vertices of Ω_{δ} , so that for any corner c = (e, e') the following relation holds:

$$F^{\eta(c)}(v) = \operatorname{Proj}[F(z_e); \eta(c)\mathbb{R}] = \operatorname{Proj}[F(z_{e'}); \eta(c)\mathbb{R}],$$

where z_e and $z_{e'}$ denote the midpoints of the edges e and e' and v denotes the common vertex of e and e'.

Given an s-holomorphic function F, we will call the functions F^{η} for $\eta \in \wp$ the projections of F on the corresponding corners.

Remark A.1. Using this terminology, the s-holomorphicity is equivalent to the fact these projections on the corners are well defined, i.e. the projections from the adjacent edges on the corner formed by them are the same.

Our next goal is to construct the so-called full-plane fermionic observable $F_{\mathbb{C}}(a,\cdot)$, i.e. the function on the edges of the whole hexagonal lattice \mathbb{C}_{δ} which is s-holomorphic everywhere except for the neighbourhood of a (in the sense of Proposition A.4). We construct $F_{\mathbb{C}}(a,\cdot)$ by means of its projections. The projections of a holomorphic function are harmonic, so the projections of a holomorphic function with a defect are harmonic functions with some defects. The latter can be obtained as linear combinations of the Green's functions (on the triangular lattice).

$$F_{\mathbb{C}_{\delta}}(\rightarrow,i\delta\sqrt{3}) = -\frac{2\sqrt{3}}{\pi} + 1 + i(\frac{3}{\pi} - \frac{2}{\sqrt{3}})$$

$$F_{\mathbb{C}_{\delta}}(\rightarrow,i\delta\sqrt{3}) = -\frac{2\sqrt{3}}{\pi} + 1 + i(\frac{3}{\pi} - \frac{2}{\sqrt{3}})$$

$$F_{\mathbb{C}_{\delta}}(\rightarrow,0) = -\frac{1}{6}$$

$$F_{\mathbb{C}_{\delta}}(\rightarrow,-i\delta\sqrt{3}) = -\frac{2\sqrt{3}}{\pi} + 1 - i(\frac{3}{\pi} - \frac{2}{\sqrt{3}})$$

$$F_{\mathbb{C}_{\delta}}(\rightarrow,-i\delta\sqrt{3}) = -\frac{2\sqrt{3}}{\pi} + 1 - i(\frac{3}{\pi} - \frac{2}{\sqrt{3}})$$

$$F_{\mathbb{C}_{\delta}}(\rightarrow,-i\delta\sqrt{3}) = -\frac{2\sqrt{3}}{\pi} + 1 - i(\frac{3}{\pi} - \frac{2}{\sqrt{3}})$$

$$F_{\mathbb{C}_{\delta}}(\rightarrow,-i\delta\sqrt{3}) = -\frac{2\sqrt{3}}{\pi} + 1 - i(\frac{3}{\pi} - \frac{2}{\sqrt{3}})$$

$$F_{\mathbb{C}_{\delta}}(\rightarrow,-i\delta\sqrt{3}) = -\frac{2\sqrt{3}}{\pi} + 1 - i(\frac{3}{\pi} - \frac{2}{\sqrt{3}})$$

$$F_{\mathbb{C}_{\delta}}(\rightarrow,-i\delta\sqrt{3}) = -\frac{2\sqrt{3}}{\pi} + 1 - i(\frac{3}{\pi} - \frac{2}{\sqrt{3}})$$

$$F_{\mathbb{C}_{\delta}}(\rightarrow,-i\delta\sqrt{3}) = -\frac{2\sqrt{3}}{\pi} + 1 - i(\frac{3}{\pi} - \frac{2}{\sqrt{3}})$$

$$F_{\mathbb{C}_{\delta}}(\rightarrow,-i\delta\sqrt{3}) = -\frac{2\sqrt{3}}{\pi} + 1 - i(\frac{3}{\pi} - \frac{2}{\sqrt{3}})$$

$$F_{\mathbb{C}_{\delta}}(\rightarrow,-i\delta\sqrt{3}) = -\frac{2\sqrt{3}}{\pi} + 1 - i(\frac{3}{\pi} - \frac{2}{\sqrt{3}})$$

$$\begin{split} F_{\mathbb{C}_{\delta}}^{\rho}(\cdot,\cdot) &= -\frac{5}{2}G_{\sqrt{3}\delta}(\frac{\delta}{2},\cdot) + \frac{1}{2}G_{\sqrt{3}\delta}(-\delta + i\frac{\sqrt{3}}{2},\cdot) + \frac{1}{2}G_{\sqrt{3}\delta}(-\delta - i\frac{\sqrt{3}}{2},\cdot) + \frac{3}{2}G_{\sqrt{3}\delta}(\frac{\delta}{2} + i\sqrt{3}\delta,\cdot) \\ F_{\mathbb{C}_{\delta}}^{\rho^2}(\cdot,\cdot) &= -\frac{5}{2}G_{\sqrt{3}\delta}(\frac{\delta}{2},\cdot) - \frac{1}{2}G_{\sqrt{3}\delta}(-\delta + i\frac{\sqrt{3}}{2},\cdot) - \frac{1}{2}G_{\sqrt{3}\delta}(-\delta - i\frac{\sqrt{3}}{2},\cdot) - \frac{3}{2}G_{\sqrt{3}\delta}(\frac{\delta}{2} - i\sqrt{3}\delta,\cdot) \\ F_{\mathbb{C}_{\delta}}^{i}(\cdot,\cdot) &= \sqrt{3}G_{\sqrt{3}\delta}(\delta + i\frac{\sqrt{3}}{2},\cdot) - \sqrt{3}G_{\sqrt{3}\delta}(\delta - i\frac{\sqrt{3}}{2},\cdot) \\ F_{\mathbb{C}_{\delta}}^{i\rho}(\cdot,\cdot) &= -\frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(-\frac{\delta}{2},\cdot) + \frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(\delta + i\frac{\sqrt{3}}{2},\cdot) + \frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(\delta - i\frac{\sqrt{3}}{2},\cdot) - \frac{3}{2}G_{\sqrt{3}\delta}(-\frac{\delta}{2} - i\sqrt{3}\delta,\cdot) \\ F_{\mathbb{C}_{\delta}}^{i\rho^2}(\cdot,\cdot) &= -\frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(-\frac{\delta}{2},\cdot) + \frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(\delta + i\frac{\sqrt{3}}{2},\cdot) + \frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(\delta - i\frac{\sqrt{3}}{2},\cdot) - \frac{3}{2}G_{\sqrt{3}\delta}(-\frac{\delta}{2} + i\sqrt{3}\delta,\cdot) \\ F_{\mathbb{C}_{\delta}}^{i\rho^2}(\cdot,\cdot) &= -\frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(-\frac{\delta}{2},\cdot) + \frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(\delta + i\frac{\sqrt{3}}{2},\cdot) + \frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(\delta - i\frac{\sqrt{3}}{2},\cdot) - \frac{3}{2}G_{\sqrt{3}\delta}(-\frac{\delta}{2} + i\sqrt{3}\delta,\cdot) \\ F_{\mathbb{C}_{\delta}}^{i\rho^2}(\cdot,\cdot) &= -\frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(-\frac{\delta}{2},\cdot) + \frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(\delta + i\frac{\sqrt{3}}{2},\cdot) + \frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(\delta - i\frac{\sqrt{3}}{2},\cdot) - \frac{3}{2}G_{\sqrt{3}\delta}(-\frac{\delta}{2} + i\sqrt{3}\delta,\cdot) \\ F_{\mathbb{C}_{\delta}}^{i\rho^2}(\cdot,\cdot) &= -\frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(-\frac{\delta}{2},\cdot) + \frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(\delta + i\frac{\sqrt{3}}{2},\cdot) + \frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(\delta - i\frac{\sqrt{3}}{2},\cdot) - \frac{3}{2}G_{\sqrt{3}\delta}(-\frac{\delta}{2} + i\sqrt{3}\delta,\cdot) \\ F_{\mathbb{C}_{\delta}}^{i\rho^2}(\cdot,\cdot) &= -\frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(-\frac{\delta}{2},\cdot) + \frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(\delta + i\frac{\sqrt{3}}{2},\cdot) + \frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(\delta - i\frac{\sqrt{3}}{2},\cdot) - \frac{3}{2}G_{\sqrt{3}\delta}(-\frac{\delta}{2} + i\sqrt{3}\delta,\cdot) \\ F_{\mathbb{C}_{\delta}}^{i\rho^2}(\cdot,\cdot) &= -\frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(-\frac{\delta}{2},\cdot) + \frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(\delta + i\frac{\sqrt{3}}{2},\cdot) + \frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(\delta - i\frac{\sqrt{3}}{2},\cdot) - \frac{3}{2}G_{\sqrt{3}\delta}(-\frac{\delta}{2} + i\sqrt{3}\delta,\cdot) \\ F_{\mathbb{C}_{\delta}}^{i\rho^2}(\cdot,\cdot) &= -\frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(-\frac{\delta}{2},\cdot) + \frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(\delta + i\frac{\sqrt{3}}{2},\cdot) + \frac{\sqrt{3}}{2}G_{\sqrt{3}\delta}(\delta - i\frac{\sqrt{3}}{2},\cdot)$$

FIGURE 13. Full-plane fermionic observable $F_{\mathbb{C}_{\delta}}(\to,\cdot)$, where \to stands for the right half-edge of the horizontal edge which has its midpoint at 0. *Upleft:* Graphical representation of the singularities of the funcitons $F_{\mathbb{C}_{\delta}}^{\eta}$ on the corners of the lattice. The singularities for each type of corners are put in the corresponding corners of this type. *Up-right:* Local values of $F_{\mathbb{C}_{\delta}}(\to,\cdot)$. *Bottom:* Explicit expressions of the functions $F_{\mathbb{C}_{\delta}}^{\eta}$ in terms of the Green's function. The coefficient in front of the Green's function at a given point is prescribed by the value of singularity of $F_{\mathbb{C}_{\delta}}^{\eta}$ at this corner.

Definition A.7. Let v_0 be a vertex of the triangular lattice $\operatorname{Tri}_{\delta}$. The Green's function $G_{\delta}(v_0, \cdot)$ is a (unique) function on vertices of $\operatorname{Tri}_{\delta}$ with the following properties:

- $G_{\delta}(v_0, v_0) = 0$;
- $\Delta G_{\delta}(v_0, v) = \delta_{v, v_0}$
- $G_{\delta}(v_0, v) = O(\log |v_0 v|),$

where by Δ we denote the discrete Laplacian, i.e. $\Delta F(v) = \sum_{v' \sim v} (F(v') - F(v))$, and δ_{v,v_0} is equal to 1 if $v = v_0$ and 0 otherwise.

Remark A.2. It is well known ([33, 50]) that the Green's function $G_{\delta}(v_0, \cdot)$ exists and unique, and that its asymptotic on the triangular lattice looks as follows:

$$G_{\delta}(v_0, v) = \frac{1}{2\pi\sqrt{3}} \log \frac{|v_0 - v|}{\delta} + \frac{\gamma_{Euler}}{2\pi} + O(\frac{\delta}{|v_0 - v|}). \tag{6.6}$$

Note that in [33] the discrete Laplacian is normalized differently. Thus, the additional factor $\frac{1}{\sqrt{3}}$ appears in our case.

Proposition A.8. Let a be a half-edge of the hexagonal lattice. There exists a unique function $F_{\mathbb{C}_{\delta}}(a,\cdot)$ defined on all the edges of the hexagonal lattice which is decreasing on ∞ and which is s-holomorphic everywhere except for the neighbourhood of a, where it has a defect 1 (i.e. satisfies the relations from Proposition A.4).

Moreover, the projections of $F_{\mathbb{C}_{\delta}}(a,\cdot)$ on the corners of the lattice can be expressed in terms of the Green's function and the values of $F_{\mathbb{C}_{\delta}}(a,\cdot)$ in the neighbourhood of a can be computed explicitly (see Fig. 13, where the case of a horizontal edge is treated).

Proof. Assume that a is the right half-edge of a horizontal edge contatining 0 (all other cases can be treated in the same way). Take six functions $F_{\mathbb{C}_{\delta}}^{\eta}(a,\cdot)$, where $\eta \in \wp$, as they are defined in Fig. 13. In order to show the existence of such a function $F_{\mathbb{C}_{\delta}}(a,\cdot)$, it is enough to show that for any edge not containing a its four projections are coherent, and that for the edge containing a the value defined by the projections $F_{\mathbb{C}_{\delta}}^{\rho}(a,-\frac{\delta}{2})$ and $F_{\mathbb{C}_{\delta}}^{\rho^{2}}(a,-\frac{\delta}{2})$, and the value defined by the projections $F_{\mathbb{C}_{\delta}}^{i\rho}(a,-\frac{\delta}{2})$ and $F_{\mathbb{C}_{\delta}}^{i\rho^{2}}(a,-\frac{\delta}{2})$, differ by 1.

We will investigate the case of horizontal edges to obtain a precise value of the singularity at the origin. Other edges can be studied in a similar manner. Denote by $\tilde{F}(z)$ a function on the midpoints of the horizontal edges which projections on the corresponding directions are $F_{\mathbb{C}_{\delta}}^{\rho}(a,z+\frac{\delta}{2})$ and $F_{\mathbb{C}_{\delta}}^{\rho^2}(a,z+\frac{\delta}{2})$. And denote by $\tilde{F}(z)$ a function on the midpoints of the horizontal edges which projections on the corresponding directions are $F_{\mathbb{C}_{\delta}}^{i\rho}(a,z-\frac{\delta}{2})$ and $F_{\mathbb{C}_{\delta}}^{i\rho^2}(a,z-\frac{\delta}{2})$. Our goal is to show that $\tilde{F}(z_e)=\tilde{\tilde{F}}(z_e)$ if $z_e\neq 0$ and that $\tilde{F}(0)=\tilde{\tilde{F}}(0)+1$. It is easy to see that

$$\tilde{F}(z) = \left[F_{\mathbb{C}_{\delta}}^{\rho}(a, z + \frac{\delta}{2}) - F_{\mathbb{C}_{\delta}}^{\rho^{2}}(a, z + \frac{\delta}{2}) \right] + \frac{i}{\sqrt{3}} \left[F_{\mathbb{C}_{\delta}}^{\rho}(a, z + \frac{\delta}{2}) + F_{\mathbb{C}_{\delta}}^{\rho^{2}}(a, z + \frac{\delta}{2}) \right]$$

$$\tilde{\tilde{F}}(z) = -\frac{1}{\sqrt{3}} \left[F_{\mathbb{C}_{\delta}}^{i\rho}(a, z - \frac{\delta}{2}) + F_{\mathbb{C}_{\delta}}^{i\rho^{2}}(a, z - \frac{\delta}{2}) \right] + \left[F_{\mathbb{C}_{\delta}}^{i\rho}(a, z - \frac{\delta}{2}) - F_{\mathbb{C}_{\delta}}^{i\rho^{2}}(a, z - \frac{\delta}{2}) \right].$$
(6.8)

Now we plug in the linear combinations of the Green's functions from Fig. 13 instead of $F_{\mathbb{C}_{\delta}}^{\eta}(a,\cdot)$. Using the translation invariance of the Green's function, we replace everything by the values of the Green's function $G(\cdot) = G_1(0,\cdot) = G_{\sqrt{3}\delta}(0,\cdot\sqrt{3}\delta)$ with the source at 0:

$$\begin{split} \tilde{F}(z) &= \left[-5G(\frac{z}{\sqrt{3}\delta}) + G(\frac{z}{\sqrt{3}\delta} + \lambda - i) + G(\frac{z}{\sqrt{3}\delta} + \lambda) + \frac{3}{2}G(\frac{z}{\sqrt{3}\delta} - i) + \frac{3}{2}G(\frac{z}{\sqrt{3}\delta} + i) \right] \\ &- \frac{i\sqrt{3}}{2} \left[G(\frac{z}{\sqrt{3}\delta} + i) - G(\frac{z}{\sqrt{3}\delta} - i) \right], \\ \tilde{\tilde{F}}(z) &= \left[G(\frac{z}{\sqrt{3}\delta}) - G(\frac{z}{\sqrt{3}\delta} - \lambda) - G(\frac{z}{\sqrt{3}\delta} + i - \lambda) + \frac{1}{2}G(\frac{z}{\sqrt{3}\delta} - i) + \frac{1}{2}G(\frac{z}{\sqrt{3}\delta} + i) \right] \\ &- \frac{i\sqrt{3}}{2} \left[G(\frac{z}{\sqrt{3}\delta} + i) - G(\frac{z}{\sqrt{3}\delta} - i) \right], \end{split}$$

where $\lambda = \frac{\sqrt{3}}{2} + \frac{i}{2}$ denotes a step "right-up" on the triangular lattice (Fig. 12, divided by $\sqrt{3}$).

Substracting one formula from the other, one gets

$$\tilde{F}(z) - \tilde{\tilde{F}}(z) = -6G(\frac{z}{\sqrt{3}\delta}) + G(\frac{z}{\sqrt{3}\delta} + \lambda - i) + G(\frac{z}{\sqrt{3}\delta} + \lambda)$$

$$+ G(\frac{z}{\sqrt{3}\delta} - i) + G(\frac{z}{\sqrt{3}\delta} + i) + G(\frac{z}{\sqrt{3}\delta} - \lambda) + G(\frac{z}{\sqrt{3}\delta} + i - \lambda).$$

$$(6.9)$$

Note that the points where the Green's function is evaluated in Eq. 6.9 are $\frac{z}{\sqrt{3\delta}}$ and its six neighbours. By the definition, the Green's function is harmonic everywhere except 0, where its laplacian is equal to 1. Hence, one gets the desired relation on $\tilde{F}(z)$ and $\tilde{\tilde{F}}(z)$.

Corollary A.9. Given a discrete domain Ω_{δ} on the hexagonal lattice and a half-edge a in Ω_{δ} , the function $F_{\Omega_{\delta}}^{\sharp}(a_{\delta}, \cdot) = F_{\Omega_{\delta}}(a_{\delta}, \cdot) - F_{\mathbb{C}}(a_{\delta}, \cdot)$ is s-holomorphic everywhere in Ω_{δ} .

Proof. It follows from propositions A.4 and A.8 that the s-holomorphicity is satisfied on all pairs of edges not containing a. On the other hand both fermionic observables $F_{\Omega_{\delta}}(a, \cdot)$ and $F_{\mathbb{C}_{\delta}}$ have the same singularity at a. Thus, their difference $F_{\Omega_{\delta}}^{\sharp}(a_{\delta}, \cdot)$ does not have any singularity at a.

Proposition A.10. Given two distinct points $a, z \in \mathbb{C}$ and $\eta \in \wp$, let a_{δ} be the closest to $a \in \Omega$ half-edge of Hex_{δ} oriented in the direction η^2 , and $z_{e_{\delta}}$ be the closet to z midedge of Hex_{δ} . Then, the following convergence results hold as $\delta \to 0$:

$$\delta^{-1} \cdot F_{\mathbb{C}_{\delta}}(a_{\delta}, z_{e_{\delta}}) \implies \frac{\sqrt{3}}{2\pi z}.$$

Moreover, the convergence is uniform (with respect to (a, z)) on compact subsets of $\mathbb{C} \setminus D$, where $D := \{(a, a), a \in \mathbb{C}\}.$

Proof. Assume that a_{δ} is the right half-edge of a horizontal edge containing 0 and z_{δ} is the midpoint of a horizontal edge (all other cases can be treated in the same way). While proving Proposition A.8, we obtained the follwing expression of the full-plane fermionic observable $F_{\mathbb{C}_{\delta}}$ in terms of the values of the Green's function at different points:

$$F(a_{\delta}, z_{\delta}) = \left[-5G(\frac{z}{\sqrt{3}\delta}) + G(\frac{z}{\sqrt{3}\delta} + \lambda - i) + G(\frac{z}{\sqrt{3}\delta} + \lambda) + \frac{3}{2}G(\frac{z}{\sqrt{3}\delta} - i) + \frac{3}{2}G(\frac{z}{\sqrt{3}\delta} + i) \right] - \frac{i\sqrt{3}}{2} \left[G(\frac{z}{\sqrt{3}\delta} + i) - G(\frac{z}{\sqrt{3}\delta} - i) \right],$$

where $G(\,\cdot\,) = G_1(0,\,\cdot\,) = G_{\sqrt{3}\delta}(0,\,\cdot\,\sqrt{3}\delta)$ and $\lambda = \frac{\sqrt{3}}{2} + \frac{i}{2}$. Clearly, the right-hand side can be expressed in terms of the discrete derivatives $\nabla_{\nu}G(\frac{z}{\sqrt{3}\delta}) = G(\frac{z}{\sqrt{3}\delta} + \nu) - G(\frac{z}{\sqrt{3}\delta})$, where ν takes one of the values $\pm \lambda, \, \pm i, \, \pm (\lambda - i)$ (which correspond to the directions of the edges):

$$F(a_{\delta}, z_{\delta}) = \nabla_{\lambda - i} G(\frac{z}{\sqrt{3}\delta}) + \nabla_{\lambda} G(\frac{z}{\sqrt{3}\delta}) + \left(\frac{3}{2} - \frac{i\sqrt{3}}{2}\right) \nabla_{i} G(\frac{z}{\sqrt{3}\delta}) + \left(\frac{3}{2} + \frac{i\sqrt{3}}{2}\right) \nabla_{-i} G(\frac{z}{\sqrt{3}\delta}).$$

It is well-known that discrete derivatives of a convergent sequence of discretely harmonic functions converge to the corresponding derivatives of the limiting function (for instance, one can see Proposition 3.1 in [8]). Using the asymptotic (6.6) of the Green's function, we get that on compact subsets of $\mathbb{C} \setminus \{0\}$

$$\frac{1}{\delta} \nabla_{\nu} G(\frac{z}{\sqrt{3}\delta}) \rightrightarrows \frac{1}{2\pi} \operatorname{Re} \left[\frac{\nu}{z}\right].$$

Hence,

$$\frac{1}{\delta}F(a_{\delta}, z_{\delta}) \rightrightarrows \frac{1}{2\pi} \operatorname{Re} \left[\frac{(\lambda - i) + \lambda + \frac{3}{2}i - \frac{3}{2}i}{z} \right] + \frac{i}{2} \cdot \frac{\sqrt{3}}{2\pi} \operatorname{Re} \left[\frac{-i - i}{z} \right] = \frac{\sqrt{3}}{2\pi z}$$

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