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The Clifford Algebra of Differential Forms

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Abstract. This paper reviews Clifford algebras in mathematics and in theoretical physics. In particular, the little-known differential form realization is constructed in detail for the four-dimensional Minkowski space. This setting is then used to describe spinors as differential forms, and to solve the Klein–Gordon and Kähler–Dirac equations. The approach of this paper, in obtaining the solutions directly in terms of differential forms, is much more elegant and concise than the traditional explicit matrix methods. A theorem given here differentiates between the two real forms of the Dirac algebra by showing that spin can be accommodated in only one of them.

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1. Introduction

Clifford algebras were discovered by W. K. Clifford [1] in 1878 as a generalization of the quaternion algebra. Following a period of little activity, Clifford algebra approaches to vectors in Euclidean and noneuclidean spaces were again initiated in the 1930s in connection with the theory of the electron spin (see [2–7]) and have since been studied separately in physics and mathematics.

Many of the classic problems of physics and their associated equations are simplified using Clifford algebras: Dirac's [8] solution to the relativistic electron equation is a Clifford algebra. The solution by Onsager [9] of the two-dimensional Ising model is perhaps the towering result of our times in statistical mechanics and is essentially Clifford algebraic (see [10–12]). Riesz [13], Hestenes [7], Imaeda [14], and Salingaros [15] employ Clifford algebras in their analysis of Maxwell's equations. In an entirely separate development, Gödel's [16] solution to the Einstein field equations is also a Clifford algebra; for a discussion of this application to general relativity see [17]. Clifford algebras are routinely used to compute particle scattering cross-sections, and form the basic tool for quantum electrodynamics (see [18, 19]). Researchers in space-time symmetries, unification theory, and supersymmetry are all looking at the larger Clifford algebras for an appropriate model. For a sampling of recent work, see [20–30].

After one hundred years, the study of Clifford algebras has come full circle – Clifford algebras are once again being used to describe vector and hypercomplex tensor (spinor) multiplication. We will discuss in particular the Clifford algebra of differential forms.

It should be noted that almost all the physical applications of Clifford algebras were and are performed in terms of specific representation matrices such as the Pauli spin matrices and the Dirac matrices. In this paper we rewrite some of these standard physics techniques in a current, more elegant fashion. The differential form realization was originally discussed by Kähler [31–34] and subsequently developed by one of the authors (see [35–37]). For many years this method was neglected in favor of the explicit matrix representation techniques; exceptions include work by Teitler [38, 39], Hestenes [7, 40], Doria [41], Sobczyk [42], and Greider [43, 44]. There is, however, considerable current activity using Clifford algebras of differential forms (see, for example, [45–55]).

In Section 2, we introduce abstract Clifford algebras and, in Section 3, we present the Clifford algebra of differential forms in Minkowski spacetime. Once the differential form representation of the Dirac spinors is constructed (Section 4), we give, in Section 5, explicit solutions to the Klein–Gordon and Kähler–Dirac equations. This method provides a compact and elegant mathematical description of physical concepts such as spin. We are able to show, using results from ring theory that the spin cannot be accommodated in real spacetime with either metric (1, 1, 1, − 1) or (− 1, − 1, − 1, 1) as has been traditionally assumed: the spin is dependent on the metric signature in real spacetime. In contradistinction, both metrics are allowed in complex spacetime because they correspond to the same complex Clifford algebra (Section 6). We are not aware of a rigorous analysis of this point in the literature.

(Note: We have adopted the physicists' convention of setting both Planck's constant and the speed of light equal to one in Section 5.)

2. The Clifford Algebra of a Vector Space

Let V be an n-dimensional vector space over the field of real numbers, $\mathbb{R}$, with a quadratic form Q: $V \to \mathbb{R}$, where Q is a mapping from V into $\mathbb{R}$ satisfying

$$Q(\alpha v) = \alpha^2 Q(v)$$

for all α in $\mathbb{R}$ and v in V. We will denote by B the associated bilinear form B: $V \times V \to \mathbb{R}$ given by

$$B(x, y) = Q(x + y) - Q(x) - Q(y)$$

for all x, y in V. The bilinear form B is called degenerate if there is some x in V such that $B(x, v) = 0$ for all v in V, and we will say that Q is degenerate if B is. It is well known that every quadratic form can be reduced to a sum of squares by a linear transformation of the variables and, in this reduction, the number of positive squares and the number of negative squares are independent of the method of reduction.

Thus, the quadratic form Q is completely specified by a set of values $Q(e_i)$, $i = 1, 2, \ldots, n$, where the e_i's form a basis of V over $\mathbb{R}$ and satisfy

$$Q\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i^2 Q(e_i).$$

The quadratic form associated with Euclidean n-space is

$$Q\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i^2,$$

where the e_i's are the 'usual' basis of $\mathbb{R}^n$; equivalently, $Q(e_i) = 1$ for each e_i. The Euclidean space $\mathbb{R}^n$ is sometimes said to be a metric described as the n-tuple of ones $(1, 1, \dots, 1)$. A pseudo-Euclidean space is a vector space $\mathbb{R}$ with a real nonpositive-definite quadratic form Q satisfying

$$Q\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n-q} x_i^2 - \sum_{i=n-q+1}^{n} x_i^2,$$

or a metric $(+1, +1, \dots, +1, -1, -1, \dots, -1)$ of q negative ones and of $n-q$ positive ones. The Minkowski space of special relativity is the set of points $v = (x, y, z, t)$ in $\mathbb{R}^4$ with a quadratic form given by

$$Q(v) = t^2 - x^2 - y^2 - z^2,$$

or with metric $(-1, -1, -1, 1)$. The Clifford algebra of Minkowski space forms the basis of description of physical phenomena seen in spacetime, and arises intrinsically both from the geometry of the relativistic transformation group – the Lorentz Lie group – as well as from the solutions to the classical and quantum equations of motion.

We will now present a discussion of abstract Clifford algebras over an arbitrary field Φ. The Clifford algebra (Φ, n, Q), where Φ is a field, n is a positive integer, and Q is a quadratic form defined on the vector space Φ^n, is the quotient algebra of the tensor algebra T of Φ^n modulo the ideal I in T generated by elements $x \otimes x - Q(x) \cdot 1$ for all x in Φ^n. That is, $(\Phi, n, Q) = T/I$.

We observe that several distinct Clifford algebras may be associated with a given vector space Φ^n, depending on which quadratic form Q is defined on Φ^n.

It is a straightforward exercise to verify that the Clifford algebra (Φ, n, Q) is an associative, 2^n-dimensional algebra over Φ (see, for example, [56], pp. 367–370). Since (Φ, n, Q) is finite-dimensional over Φ, its radical $\mathbb{J}$ is nilpotent. Chevalley [5] has worked out in great detail the general structure theory of Clifford algebras. Let $V = \Phi^n$, and V^0 the annihilator of V relative to B, that is,

$$V^0 = \{v \in V \mid B(v, x) = 0 \quad \text{for all } x \in V\}.$$

Let P be the set of all vectors p in V^0 such that $Q(p) = 0$, and let N be the subspace of V such that $V = N \oplus P$. We have, from [5]:

THEOREM 1.1. *The ideal $\mathfrak{p}$ generated by P in (Φ, n, Q) is the radical $\mathbb{J}$ of (Φ, n, Q), and the quotient algebra of (Φ, n, Q) by $\mathfrak{p}$ is isomorphic to the Clifford algebra of the restriction of Q to N.*

We recall that a simple finite-dimensional associative algebra is simply the algebra of $n \times n$ matrices over some division ring Δ. The following theorems tell us that (Φ, n, Q), for Q nondegenerate, is either an $n \times n$ matrix algebra or the (algebra) direct sum of two $n \times n$ matrix algebras.

THEOREM 1.2. *If Q is nondegenerate, then $(\Phi, 2n, Q)$ is a central simple algebra.*

THEOREM 1.3. *Let B be nondengerate and D the discriminant of B. Then $(\Phi, 2n + 1, Q)$ is either simple or the direct sum of two (isomorphic) simple ideals. The center $\mathbb{Z}$ of $(\Phi, 2n + 1, Q)$ is of dimension 2 and is spanned by 1 and an element ω such that $\omega^2 = (-1)^n D$.*

Chevalley goes on to discuss the orthogonal group of (Φ, n, Q). Atiyah *et al.* [57] is the classic reference on modules over Clifford algebras; see also [58, 59].

We now give a naive construction of the Clifford algebra (Φ, n, Q). (Φ, n, Q) is the algebra over Φ, generated by a unit 1 and elements e_i, $i = 1, 2, \ldots, n$, subject to the following rules for multiplication:

$$e_i e_j = -e_j e_i, \quad i \neq j,\ i, j = 1, 2, \ldots, n,$$

$$e_i^2 = Q(e_i), \quad i = 1, 2, \ldots, n,$$

and $Q(e_i)$ is in Φ for each $i = 1, 2, \ldots, n$.

A generalized quaternion algebra G over a field Φ not of characteristic 2 has a basis $\{1, e_1, e_2, e_1e_2\}$ where

$$(e_1)^2 = \alpha, \qquad (e_2)^2 = \beta, \quad e_1 e_2 = -e_2 e_1$$

for α, β nonzero elements of Φ. We observe that any generalized quaternion algebra is a Clifford algebra. A generalized quaternion algebra over $\mathbb{R}$ is either a division ring or the ring of 2×2 matrices over $\mathbb{R}$ (see [60] for necessary and sufficient conditions on α and β for the algebra to be a division algebra). The only generalized quaternion algebra over $\mathbb{C}$ is the ring of 2×2 matrices over $\mathbb{C}$. Likewise, the Clifford algebra of a two-dimensional space with a nondegenerate quadratic form is central simple and it is a generalized quaternion algebra.

The angular momentum algebra illustrates the naive construction. The angular momentum algebra is generated by elements $\{J_1, J_2, J_3\}$ satisfying

$$J_1J_2 - J_2J_1 = iJ_3, \qquad J_2J_3 - J_3J_2 = iJ_1, \qquad J_3J_1 - J_1J_3 = iJ_2,$$

where i is the complex unit $\sqrt{-1}$. A simple, nontrivial representation of this algebra is generated by the Pauli matrices τ_j, $j = 1, 2, 3$ where

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This is the algebra of 2×2 matrices over the complex numbers $\mathbb{C}$; it is the Clifford algebra over $\mathbb{C}$ generated by the elements τ_1, τ_2, since

$$\tau_1\tau_2 = -\tau_2\tau_1, \qquad \tau_1^2 = \tau_2^2 = I_2.$$

The generators of the isospin group satisfy the same relations as the generators of the angular momentum algebra. For further details on algebras of three anticommuting elements and their generalizations, see [61–63].

Motivated by the appearance of many of the Clifford algebras in physical models, we have provided a system of classification in terms of the corresponding finite group structure [35, 36, 64, 65]. A Clifford algebra is the group algebra over $\mathbb{R}$ of some extra-special 2-group [66]. By constructing the finite groups as central products of quaternion and dihedral groups, we are able to recover many of the classic structural results on Clifford algebras [1–5], as well as to obtain other useful information. This approach enables one to identify any particular Clifford algebra arising in a physical model simply by computing the order structure of its 2^n elements [36, 65]. As a result, we were able to identify the Clifford algebras arising in physics, and to clear up some old confusions concerning inequivalent algebras.

For example, the 'spinor' algebras which arise in physics are labelled in [35, 36, 64] as a series of algebras $\mathbb{S}_k$, $k = 1, 2, 3, \ldots$. The first of the series is the Pauli algebra $\mathbb{S}_1$ constructed above; the second spinor algebra is the Dirac algebra $\mathbb{S}_2$. In general, the $\mathbb{S}$ series are all simple algebras which are isomorphic to matrix algebras over $\mathbb{C}$. The exact correspondence is $\mathbb{S}_k \approx \mathbb{C}_{2^k}$.

For completeness, we may look at the field $\mathbb{C}$ as the zeroth member of the $\mathbb{S}$ series. The $\mathbb{S}$ series has two 'normal' real forms, which are labelled $\mathbb{N}_{\text{odd}}$ and $\mathbb{N}_{\text{even}}$. These are simple algebras isomorphic to matrix algebras over $\mathbb{R}$ and $\mathbb{H}$ (ordinary quaternions) respectively. We have:

$$\mathbb{N}_{2k-1} \approx \mathbb{R}_{2^k}, \qquad \mathbb{N}_{2k} \approx \mathbb{H}_{2^{k-1}}.$$

One may regard the zeroth term in the odd $\mathbb{N}$ series as the field $\mathbb{R}$; the quaternion field $\mathbb{H}$ is equal to $\mathbb{N}_2$. There exist two more families of Clifford algebras which are not simple, and are isomorphic to a direct sum of two $\mathbb{N}$ algebras. We have labelled these as $\mathcal{Q}_{\text{odd}}$ and $\mathcal{Q}_{\text{even}}$, where:

$$\mathcal{Q}_k \approx \mathbb{N}_k \oplus \mathbb{N}_k.$$

The zeroth term in the $\mathcal{Q}$ series is not a field, but is the dual numbers $\mathcal{Q} = \mathbb{R} \oplus \mathbb{R}$ [62]. The reader is referred to [20, 35, 36, 59, 64] for the 'triangle classification' of the Clifford algebras according to the metric of the base space of generators. This classification has proved useful in identifying the particular algebras which arise in physics.

To conclude this section, we summarize four very important applications of Clifford algebras which are of current interest. First, we draw attention to the construction of holomorphic fields in a Clifford algebra setting. There have been several related, but distinct, attempts to generalize the theory of analytic functions from the complex plane to higher dimensions. Unlike the usual theory of several (commuting) complex variables, these constructions are based on anticommuting (hyper-complex) bases. Some relevant references are [67–77]. In [15], we propose that the natural generalization of holomorphy in a framework of Clifford algebras leads directly to the field equations of classical electromagnetism. It should be stressed that this is only one of the many possible generalizations of holomorphy from 2 to n dimensions. The construction and logic of [15], however, provides a natural connection between the Cauchy–Riemann equations and the Maxwell equations.

Spinor representation of Lie groups, supersymmetry algebras, and twistors all involve Clifford algebras. In addition to the above cited references [20–30], see [78–90]. Reference [89] is especially relevant to the holomorphic fields discussed in the preceding paragraph.

One of the prime areas of applications of Clifford algebras in mathematics is in 'K-theory'. An excellent exposition is given in the book by Karoubi [58]. The subject is of considerable breadth. One may start from the classic text by Hirzebruch [91] for the beginnings of index theory, then follow the original papers of Atiyah and coworkers [92–102]. Particularly readable are the lectures of R. Bott written up by Kulkarni [102], and the lecture by Singer [101]. Recent mathematical advances in K-theory can be followed from the conference proceedings [103–106].

Intrinsically related to K-theory, yet forming a mathematical branch all of its own is the topic of infinite-dimensional Clifford algebras and representations of canonical anticommutation relations. This is a field of intense activity today. Even though it falls outside the elementary applications which are discussed in this review, the formalism provides a foundation for rigorous field-theoretic results. We refer to [107–120]. The review in [120] is a particularly useful overview of the field.

This concludes the first part of the paper which reviews Clifford algebras and tries to indicate the extremely broad range of their applications both in mathematics and in theoretical physics. The second part of the paper is more specifically concerned with constructing a particular representation of a particular Clifford algebra in order to derive specific results.

3. The Clifford Algebra of Differential Forms

We will now give the explicit construction of the Clifford algebra of four-dimensional Minkowski space; this construction equally applies to spaces of arbitrary dimension.

The following conventions from tensor analysis will be utilized:

(1) A point in Minkowski space will be described by the coordinates
$$x^1 = x, \qquad x^2 = y, \qquad x^3 = z, \qquad x^3 = t.$$
We will use Greek indices $\mu = 1, 2, 3, 4$ to denote the coordinates as x^μ.

(2) Lowered indices will be used as a shorthand to include the metric in the components. Therefore,
$$x_1 = -x^1, \qquad x_2 = -x^2, \qquad x_3 = -x^3, \qquad x_4 = x^4$$

(3) If a Greek index is repeated in a term, then it is understood that a summation with respect to that index over the range 1 to 4 is implied (the Einstein summation convention). Hence,
$$a^\mu b^\mu = \sum_{\mu=1}^{4} a^\mu b^\mu,$$

(4) The inner product of two vectors in four-dimensional Minkowski space will be denoted as:

$$(a, b) = a_\mu b^\mu = a^\mu b_\mu = -\mathbf{a} \cdot \mathbf{b} + a^4 b^4.$$

We will assume that the coordinate axes in Minkowski space are orthogonal. The exterior derivative acting on each coordinate function is simply the differential dx^μ. This differential is a directed quantity, which points in the positive direction of the x^μ axis. The four differentials $\{dx^1, dx^2, dx^3, dx^4\}$ therefore define an orthonormal frame or tetrad in the space. It is convenient to use these as the unit vectors in the Minkowski space (we actually in this manner construct vectors and tensors on the cotangent bundle and not on the tangent bundle).

Define the Grassmann–Cartan exterior (or wedge) product $\wedge$ by

$$\left.\begin{aligned} &dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu, \quad \mu \neq \nu \\ &dx^\mu \wedge dx^\mu = 0 \end{aligned}\right\} \text{ antisymmetry,}$$

and

$$\left.dx^\mu \wedge (dx^\nu \wedge dx^\lambda) \begin{aligned} &= (dx^\mu \wedge dx^\nu) \wedge dx^\lambda \\ &= dx^\mu \wedge dx^\nu \wedge dx^\lambda \end{aligned}\right\} \text{ associativity.}$$

We must have as a consequence,

$$\begin{aligned} &dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} = (-1)^\pi dx^{\lambda_1} \wedge \ldots \wedge dx^{\lambda_p}, \quad \mu_1 \neq \ldots \neq \mu_p \\ &dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} = 0, \quad \text{if any pair of indices is equal.} \end{aligned}$$

Here, $(-1)^\pi$ is the sign of the permutation

$$\pi = \begin{pmatrix} \mu_1 \cdots \mu_p \\ \lambda_1 \cdots \lambda_p \end{pmatrix}.$$

The object $dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3}$ is called a 3-form; in general, the exterior product of p differentials is called a p-form, or a form of rank p. The standard references on differential forms are [121–123].

The dx^μ are identified with the basis vectors of the Minkowski space. The objects $dx^\mu \wedge dx^\nu$ are area elements in the plane defined by the x^μ and x^ν coordinates. The three spatial planes are those with space indices $i, j = 1, 2, 3$, which will always be denoted by Latin superscripts as $dx^i \wedge dx^j$; there are three more planes which determine the three spacetime sheets and have elements $dx^i \wedge dx^4$, $i = 1, 2, 3$, for a total of six planes in all. Similarly, there are four three-dimensional hyperplanes which are characterized by each basis three-form, of which one is the differential volume of ordinary space denoted by $\eta = dx^1 \wedge dx^2 \wedge dx^3$.

Call the unit scalar 1 the 'zero-form', and denote the spacetime volume element as $\omega = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$. Then the 16 basis forms of the exterior algebra are:

1	(one)
dx^μ	(four)

$$
\begin{array}{lll}
dx^\mu \wedge dx^\nu & \mu \neq \nu & \text{(six)} \\
dx^\mu \wedge dx^\nu \wedge dx^\lambda & \mu \neq \nu \neq \lambda \neq \mu & \text{(four)} \\
dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 = \omega & & \text{(one).}
\end{array}
$$

Denote the product in the Clifford algebra generated by the one-forms as $\vee$: this will be referred to as the vee product.

$$\text{(a)} \quad dx^\mu \vee dx^\nu + dx^\nu \vee dx^\mu = 2\delta^{\mu\nu} Q(dx^\mu),$$

$$\text{(b)} \quad (dx^\mu \vee dx^\nu) \vee dx^\lambda = dx^\mu \vee (dx^\nu \vee dx^\lambda).$$

Any expression $dx^{\mu_1} \vee \ldots \vee dx^{\mu_n}$ can be reduced using (a) and (b) to a product of one-forms with an overall sign.

$$(-1)^{\pi_1} \ldots (-1)^{\pi_m} Q(dx^{\lambda_1}) \ldots Q(dx^{\lambda_m})\, dx^{\lambda_{m+1}} \vee \ldots \vee dx^{\lambda_n},$$

where the $\lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_n$ are distinct. We can interpret the $\vee$ product of p different one-forms as

$$dx^{\mu_1} \vee \ldots \vee dx^{\mu_p} = dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}, \quad p \leqslant 4.$$

Hence, we see that the basis elements of the Clifford algebra are simply the 16 basis forms of the exterior algebra. For convenience, we will frequently denote the differentials dx^μ as σ^μ from this point on.

The standard mathematical notation for the Grassmann basis is the following (note that lowered indices here do not incorporate the metric):

$$e_{ij} = e_i e_j = e_i \wedge e_j, \quad i \neq j$$

so that a p-form is compactly labelled as

$$e_{\mu_1 \ldots \mu_p} = e_{\mu_1} \wedge \ldots \wedge e_{\mu_p}, \quad p \leqslant 4.$$

REMARK. The structure theory presented in Section 2 implies that $(\mathbb{R}, 4, (-1, -1, -1, 1))$ is the algebra $\mathbb{H}_2$ of 2×2 matrices over the quaternions $\mathbb{H}$.

In actual applications, it is necessary to have a multiplication table for the elements of the Clifford algebra, especially for antisymmetric tensors expanded on the basis as follows:

$$\alpha = \alpha^{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}, \quad p \leqslant 4 \tag{3.1}$$

where the $\alpha^{\mu_1 \cdots \mu_p}$ are elements of either $\mathbb{R}$ or the complex field $\mathbb{C}$. Obviously in four dimensions, the ranks of such objects can only be zero, one, two, three, or four, corresponding to the rank of the basis forms.

We are motivated in our studies of the objects of the above form (3.1) because of the following observations. The physical world is perceived in terms of tensor fields such as momentum (a rank one tensor, or vector field) or the electromagnetic field (a rank two antisymmetric field), as well as other types of tensor and spinor fields. Spacetime is governed by the transformation rules for the field components under the Lorentz group. The fact that the transformation group of forms in the Clifford algebra is precisely the Lorentz group validates this mathematical formalism as an appropriate model of physical spacetime. For a discussion of the Lorentz group in the context of the Clifford algebra of differential forms, see [124].

The general vee product between a basis p-form and a basis q-form is given as a permutation sum over the indices as follows:

$$\begin{aligned}(\sigma^{\mu_1} \wedge \ldots \wedge \sigma^{\mu_p}) \vee (\sigma^{\nu_1} \wedge \ldots \wedge \sigma^{\nu_q}) \\ = \sum_{k=0}^{\min(p,q)} (-1)^{k(p-k)} \sum_{\pi_p} \sum_{\pi_q} (-1)^{\pi_p} (-1)^{\pi_q} g^{\lambda_1 \rho_1} \ldots g^{\lambda_k \rho_k} \\ \times \sigma^{\lambda_{k+1}} \wedge \ldots \wedge \sigma^{\lambda_p} \wedge \sigma^{\rho_{k+1}} \wedge \ldots \wedge \sigma^{\rho_q}\end{aligned}$$

where the permutations π_p and π_q are given by

$$\pi_p = \begin{pmatrix} \lambda_1 \ldots \lambda_p \\ \mu_1 \ldots \mu_p \end{pmatrix}, \qquad \pi_q = \begin{pmatrix} \rho_1 \ldots \rho_q \\ \nu_1 \ldots \nu_q \end{pmatrix}.$$

This defines the general vee product as a permutation sum of forms with successive pairwise index contractions. Algebraic rules for tensors in Minkowski spacetime are worked out completely in [37], and those in three-dimensional Euclidean space in [7, 125].

We have collected in Table I some algebraic identities which are satisfied by vectors a and b in the Clifford algebra of differential forms in Minkowski spacetime. These rules

Table I. Vector rules for the Clifford algebra of differential forms in Minkowski spacetime

$$a = a^\mu \,\mathrm{d}x^\mu = \mathbf{a} + a^4 \,\mathrm{d}x^4, \qquad b = b^\mu \,\mathrm{d}x^\mu, \quad (a, b) = a^4 b^4 - (\mathbf{a} \cdot \mathbf{b})$$
$$a \vee b = (a, b) + a \wedge b = -(\mathbf{a} \cdot \mathbf{b}) + a^4 b^4 - \eta \vee (\mathbf{a} \times \mathbf{b}) + (b^4 \mathbf{a} - a^4 \mathbf{b}) \vee \mathrm{d}x^4$$
$$a \vee a = (a, a) = -|\mathbf{a}|^2 + (a^4)^2$$

$$\eta \vee \sigma^4 = \omega = -\sigma^4 \vee \eta, \qquad \omega \vee \sigma^4 = \eta = -\sigma^4 \vee \omega,$$
$$\eta \vee \omega = \sigma^4 = -\omega \vee \eta, \qquad \eta \vee \eta = 1, \omega \vee \omega = -1$$

$$\sigma^4 \vee \mathbf{a} = -\mathbf{a} \vee \sigma^4$$
$$\eta \vee \mathbf{a} = \mathbf{a} \vee \eta = -*_3\mathbf{a} = -[a^1 \,\mathrm{d}x^2 \wedge \mathrm{d}x^3 + a^2 \,\mathrm{d}x^3 \wedge \mathrm{d}x^1 + a^3 \,\mathrm{d}x^1 \wedge \mathrm{d}x^2]$$
$$\omega \vee a = -a \vee \omega = *_4 a = a^1 \,\mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 + a^2 \,\mathrm{d}x^3 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^4 + \\ + a^3 \,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^4 + a^4 \eta.$$

are useful in following the manipulations in the following sections. Here, η and ω represent the volume elements in three and four dimensions, respectively. Note, in particular, how the duality in both three and four dimensions reduces to a simple vee product. These simple rules should give an indication of how the vee product works in practice.

4. The Construction of Spinors as Differential Forms

In this section we outline one important application of the Clifford algebra of differential forms in Minkowski space. We describe Dirac spinors which arise as the solutions of the relativistiv electron equation known as the Dirac equation. References for this section include [31–34, 40, 43–47, 53] and especially [81].

We introduce first a Witt decomposition of the Clifford algebra into the two orthogonal subspaces U and W with basis: (here, i is the usual complex unit)

$$u^1 = \frac{1}{\sqrt{2}}(\mathrm{d}x^1 + i\,\mathrm{d}x^2), \qquad u^2 = \frac{1}{\sqrt{2}}(\mathrm{d}x^4 - \mathrm{d}x^3)$$

$$w^1 = \frac{1}{\sqrt{2}}(\mathrm{d}x^1 - i\,\mathrm{d}x^2), \qquad w^2 = \frac{1}{\sqrt{2}}(\mathrm{d}x^3 + \mathrm{d}x^4).$$

Define an element $\xi = w^1 \wedge w^2$, and use it to construct the spinor basis $\{\zeta^\mu\}$ (note differences in sign from reference [81]).

$$\zeta^1 = u^1 \vee u^2 \vee \xi, \qquad \zeta^2 = -2\xi,$$

$$\zeta^3 = -\sqrt{2}u^1 \vee \xi, \qquad \zeta^4 = -\sqrt{2}u^2 \vee \xi.$$

It is necessary in what follows to write the above construction explicitly in terms of the one-form basis $\{\mathrm{d}x^\mu\}$. Using the definition and the vee product rules from the previous section one obtains:

$$\begin{aligned}
\zeta^1 &= 1 - \mathrm{d}x^3 \wedge \mathrm{d}x^4 + i\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 - i\omega,\\
\zeta^2 &= i\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 + \mathrm{d}x^3 \wedge \mathrm{d}x^1 + i\,\mathrm{d}x^2 \wedge \mathrm{d}x^4 - \mathrm{d}x^1 \wedge \mathrm{d}x^4,\\
\zeta^3 &= \mathrm{d}x^3 + \mathrm{d}x^4 + i\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + i\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^4,\\
\zeta^4 &= -i\,\mathrm{d}x^2 + \mathrm{d}x^1 + i\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 + \mathrm{d}x^3 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^4.
\end{aligned} \tag{4.1}$$

In the standard mathematical notation, the ζ basis is more compactly written as:

$$\zeta_1 = 1 - e_{34} + ie_{12} - ie_{1234}, \qquad \zeta_2 = e_{31} - e_{14} + ie_{23} + ie_{24},$$

$$\zeta_3 = e_3 + e_4 + ie_{123} + ie_{124}, \qquad \zeta_4 = e_1 - ie_2 + e_{314} + ie_{234}$$

The above construction is needed to illustrate the following key theorem.

THEOREM 4.1. *$\{\zeta^\mu\}$ is a spinor basis for the Dirac spinor Ψ.*

Proof. The proof follows by displaying an isomorphism between the differential forms and the Dirac gamma matrices which are normally used in relativistic field theory.

The usual Dirac equation is a first-order partial-differential equation written with the derivatives $\partial_\mu = \partial/\partial x^\mu$ and the Dirac gamma matrices γ^μ as follows: $i\,\partial_\mu \gamma^\mu \Psi = m\Psi$.

The object Ψ which is the eigenvector of the Dirac equation is called a Dirac spinor (m is a scalar constant which is the mass of a particle). The spinor Ψ is usually taken to be a column vector in the representation space of the Clifford algebra – the space of gamma matrices, which are 4×4 complex matrices.

REMARK. The Clifford algebra in *real* Minkowski space is $\mathbb{H}_2$, but its complexification is $\mathbb{H}_2 \otimes \mathbb{C} = \mathbb{C}_4$, which is the Dirac algebra (see also Section 6).

Since the spinor Ψ is yet arbitrary, it may be taken to be the first column of a 4×4 matrix (In this context, see [126, 38, 39].)

$$\Psi = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix, by virtue of being in the representation space of the Clifford algebra, may be written in terms of a particular basis of the gamma matrices denoted $\{\zeta^\mu\}$: $\Psi = \psi^\mu \zeta^\mu$.

The ζ^μ will, of course, assume a different form for each distinct matrix representation of the Clifford algebra. For the Kramers representation of the Clifford algebra (see [127]), one has the following result in terms of that particular basis (we leave out a factor of 4):

$$\begin{aligned} \zeta^1 &= 1 + i\gamma^1\gamma^2 - \gamma^3\gamma^4 - i\gamma^1\gamma^2\gamma^3\gamma^4 \\ \zeta^2 &= \gamma^3\gamma^1 + i\gamma^2\gamma^3 - \gamma^1\gamma^4 + i\gamma^2\gamma^4 \\ \zeta^3 &= \gamma^3 + \gamma^4 + i\gamma^1\gamma^2\gamma^3 + i\gamma^1\gamma^2\gamma^4 \\ \zeta^4 &= \gamma^1 - i\gamma^2 + \gamma^3\gamma^1\gamma^4 + i\gamma^2\gamma^3\gamma^4 \end{aligned} \tag{4.2}$$

By inspection, (4.1) is equivalent to (4.2) thus demonstrating the isomorphism and, hence, the spinor representation

$$(\mathrm{d}x^\mu, \vee) \leftrightarrow (\gamma^\mu, \text{matrix product}). \tag{4.3}$$

This is one of the interesting results in this article. One may use the differential form basis and the vee product in order to derive results for the Dirac gamma matrices which are useful in quantum field theory. We describe one practical application here: the derivation of trace identities for quantum electrodynamics.

The invariant quantities in a Clifford algebra will be those combinations which are invariant under the action of the Lie group of automorphisms. In the present case, the intrinsic automorphism (covariance) group is the homogeneous Lorentz group. It is easy to see that the Lorentz invariant combinations are precisely the scalar and type four parts of any vee-product [128].

On the other hand, in a matrix algebra the invariant of a product is obtained as the trace of the product of matrices. If this is a representation of a Clifford algebra, we have an isomorphism, hence the invariants in the spinor representation correspond to scalar and pseudoscalar parts, where:

$$\text{Scalar Part } (\mathrm{d}x^\mu, \vee) = \tfrac{1}{4} \text{ Trace } (\gamma^\mu, \text{matrix product}) \tag{4.4}$$

We can use this result as a starting point to derive the trace identities. The vee product of n vectors $a_k = a_k{}^\mu dx^\mu$ in spacetime can always be written in closed form because of the closure of the Clifford algebra. We have a permutation sum of tensors of rank zero, two and four for n = even. The entirely antisymmetric combination of vector components is the scalar

$$[a_1a_2a_3a_4] = \varepsilon^{\lambda\mu\nu\rho}\, a_1{}^\lambda a_2{}^\mu a_3{}^\nu a_4{}^\rho.$$

The permutation symbols are the same as in the preceding section, and (a_k, a_l) is the inner product in the Minkowski space (Table I) [19].

$$\begin{aligned} &a_1 \vee a_2 \vee \ldots \vee a_n \\ &\quad = \textstyle\sum(-1)^\pi (a_1, a_2)\ldots(a_{n-1}, a_n) + \\ &\quad\quad + \textstyle\sum(-1)^\pi a_1 \wedge a_2(a_3, a_4)\ldots(a_{n-1}, a_n) + \\ &\quad\quad + \omega\textstyle\sum(-1)^\pi [a_1a_2a_3a_4](a_5, a_6)\ldots(a_{n-1}, a_n), \quad n = \text{even}. \end{aligned} \tag{4.5}$$

Similarly, for n = odd we have a permutation sum of a vector and a tensor of rank three [19]

$$\begin{aligned} &a_1 \vee a_2 \vee \ldots \vee a_n \\ &\quad = \textstyle\sum(-1)^\pi a_1(a_2, a_3)\ldots(a_{n-1}, a_n) + \\ &\quad\quad + \textstyle\sum(-1)^\pi a_1 \wedge a_2 \wedge a_3(a_4, a_5)\ldots(a_{n-1}, a_n), \quad n = \text{odd}. \end{aligned} \tag{4.6}$$

From these closed-form expressions, we can take the scalar and pseudoscalar (rank 4) parts which will be independent invariants of the Lorentz group. To obtain the invariants in terms of the traces, we need a dictionary which relates the forms to vectors in the Dirac spinor basis. In the physics literature, a vector type (usually defined with covariant components) is denoted by a slash as $\not{a} = a_\mu \gamma^\mu$. The matrix corresponding to the volume element ω is called $i\gamma^5$. The trace identities that follow from (4.5) and (4.6) with (4.4) are [19]:

$$\mathrm{Tr}(\not{a}_1 \ldots \not{a}_n) = 4\textstyle\sum(-1)^\pi(a_1, a_2)\ldots(a_{n-1}, a_n), \quad n \text{ even}, \tag{4.7a}$$

$$\begin{aligned} &\mathrm{Tr}(i\gamma^5 \not{a}_1 \ldots \not{a}_n) \\ &\quad = -4\textstyle\sum(-1)^\pi [a_1a_2a_3a_4](a_5, a_6)\ldots(a_{n-1}, a_n), \quad n \text{ even}, \end{aligned} \tag{4.7b}$$

$$\mathrm{Tr}(\not{a}_1 \ldots \not{a}_n) = 0, \quad n \text{ odd}, \tag{4.7c}$$

$$\mathrm{Tr}(i\gamma^5 \not{a}_1 \ldots \not{a}_n) = 0, \quad n \text{ odd} \tag{4.7d}$$

Even though formulas (4.7a, c, d) were derived as far back as 1952 by Caianiello and Fubini [129], the standard textbooks on quantum electrodynamics use only the formulas (4.7a, b) for $n \leqslant 4$, and employ recursion relations for higher n (see, for example, [130, 131]). It is easy to obtain the transposes of the products (4.5, 4.6) by switching indices. Then add to the regularly ordered expressions scalar and type four terms to obtain the relations translated into the gamma matrix language [19]:

$$\begin{aligned}
&a\!\!\!/_n \dots a\!\!\!/_1 \\
&\quad = -a\!\!\!/_1 \dots a\!\!\!/_n + \tfrac{1}{2}\mathrm{Tr}(a\!\!\!/_1 \dots a\!\!\!/_n) + \tfrac{1}{2}\gamma^5\mathrm{Tr}(\gamma^5 a\!\!\!/_1 \dots a\!\!\!/_n), \quad n = \text{even}, \\
&a\!\!\!/_n \dots a\!\!\!/_1 \\
&\quad = -a\!\!\!/_1 \dots a\!\!\!/_n + 2\sum(-1)^{\pi} a\!\!\!/_1(a_2, a_3)\dots(a_{n-1}, a_n), \quad n = \text{odd}.
\end{aligned}$$

In addition to references [129], we refer the reader to references [132–134], which discuss general forms of trace identities in the context of Clifford algebras. The book by Caianiello [18] contains an excellent exposition of Clifford algebraic techniques in field theory.

This concludes our discussion of spinors in the traditional sense, and provides a dictionary between the forms and spinors as they usually appear in the physics literature. One may alternately use the Fierz identities for reconstructing a spinor from its components in representation space (see [6, 135, 136]). Nevertheless, that is strictly unnecessary since the fields which describe spin one-half particles such as the electron can be cast directly in terms of differential forms, circumventing the traditional spinor formalism entirely. This work is presented in the following section.

5. Solutions of the Klein–Gordon and Kähler–Dirac Equations

The Kähler–Dirac equation is written in terms of the generalized Dirac operator D and an aggregate of forms Ψ as [31–34, 43, 44]:

$$iD \vee \Psi = m\Psi \tag{5.1}$$

Recall that the generalized Dirac operator factors the Laplace–Beltrami operator in any dimension. In the four-dimensional Minkowski space, the Dirac operator factors the D'Alembert or wave operator [15],

$$D = \partial^\mu \, \mathrm{d}x^\mu \;\Rightarrow\; D \vee D = \Box = (\partial^t)^2 - |\nabla|^2$$

where $\nabla = \sum_{i=1}^{3} \partial_i \, \mathrm{d}x^i$ is the usual gradient operator in three dimensions.

Acting again with D on the Kähler–Dirac equation (5.1) gives the Klein–Gordon equation [130, 131] $\Box\Psi = -m^2\Psi$.

The solutions of the Klein–Gordon equation include the set of solutions of the Kähler–Dirac equation. To obtain a solution, we need the properties of the relativistic momentum form in the Clifford algebra. This is a one-form with the three-space momentum components p^i, $i = 1, 2, 3$, and the fourth component which is the energy E. This form has constant length equal to the mass of the particle

$$p = \sum_{i=1}^{3} p^i \, \mathrm{d}x^i + E \, \mathrm{d}x^4, \quad p \vee p = p_\mu p^\mu = E^2 - |\mathbf{p}|^2 = m^2.$$

(In the rest frame of the particle, there is no space momentum $\mathbf{p}$, so that $E = m$, and therefore, $p = m \, \mathrm{d}x^4$.) Relativistically, the point of p in momentum space defines a surface, called the 'mass shell'. Denote the scalar product in Minkowski space between the momentum p and the spacetime position r as

$$(p, r) = p_\mu x^\mu = Et - (\mathbf{r} \cdot \mathbf{p})$$

Operator D acting on the scalar (p, r) recovers the momentum one-form p;

$$D(p, r) = \mathrm{d}x^\mu\, \partial^\mu p^\nu x_\nu = \mathrm{d}x^\mu p^\nu\, \delta^\mu_\nu = p^\mu\, \mathrm{d}x^\mu = p.$$

One may use the exponentials of (p, r) to give the wave solutions of the Klein–Gordon equation as follows:

$$\Psi = \Psi_0 \exp\ \pm i(p, r)\ \Rightarrow\ \Box\Psi = -m^2\Psi.$$

Here, Ψ_0 may be an entirely arbitrary combination of constant forms. The particular types of Ψ_0 determine the spin of the particle being described. In order to fix the type of Ψ_0, we must solve the first-order equation. For the Kähler–Dirac equation, the solutions are obtained using purely algebraic methods, and the spin is found to be one-half. To begin, choose the positive-energy solution (with a minus sign in the exponential). Substituting the solution of the Klein–Gordon equation into the Kähler–Dirac equation gives:

$$\begin{aligned}\Psi &= \Psi_0 \exp\ -i(p, r)\\ &\Rightarrow iD \vee \Psi = p \vee \Psi_0 \exp\ -i(p, r) = m\Psi_0 \exp\ -i(p, r)\\ &\Rightarrow (m - p) \vee \Psi_0 = 0.\end{aligned}$$

This equation has an algebraic solution as follows, up to a constant factor, $\Psi_0 = m + p$.

This object is a scalar plus a vector type. It may be verified that Ψ_0 is a projection operator onto a state of momentum p and mass m. To be precise, define mutually orthogonal idempotent projection operators by an appropriate normalization:

$$\Lambda_\pm = \frac{m \pm p}{2m} \Rightarrow \Lambda_\pm \vee \Lambda_\pm = \Lambda_\pm, \qquad \Lambda_\pm \vee \Lambda_\mp = 0, \quad m \neq 0.$$

The solution to the Kähler–Dirac equation is constructed from the product of the wave from the Klein–Gordon solution and a projection operator which includes the momentum projector $\Lambda_\pm$. We leave for later a discussion of the other part of the projector in order to discuss the antiparticles. The momentum part of the solution to the Kähler–Dirac equation is:

$$\Psi = \Lambda_+ \exp\ -i(p, r)\ \Rightarrow\ iD \vee \Psi = m\Psi. \tag{5.2}$$

A distinct solution can similarly be constructed from the negative energy wave via the substitution $p \to -p$ in (5.2)

$$\overline{\Psi} = \Lambda_- \exp i(p, r)\ \Rightarrow\ iD \vee \overline{\Psi} = m\overline{\Psi}.$$

This is the profound prediction of Dirac: that for every particle there exists a corresponding antiparticle. The negative energy follows from the fact that a particle must annihilate its own antiparticle.

The solutions to the Kähler–Dirac equation are incomplete without the spin projection operator which defines two states: parallel or antiparallel to a unit vector $\mathbf{s}$, $|\mathbf{s}|^2 = 1$. In our relativistic treatment, we define a four-component one-form n which is identical to $\mathbf{s}$ in the rest frame of the particle, but which picks up additional terms from the Lorentz transformation as shown in Table II. The length of a vector is an invariant of the Minkowski space, therefore: $n \vee n = -|\mathbf{s}|^2 = -1$.

Table. II. Momentum and spin vectors in two different inertial frames

	Rest frame	Laboratory frame
p	$m\,\mathrm{d}x^4$	$\mathbf{p} + E\,\mathrm{d}x^4$
n	$\mathbf{s}$	$\mathbf{s} + \dfrac{(\mathbf{p}\cdot\mathbf{s})\mathbf{p}}{m(E+m)} + \dfrac{(\mathbf{p}\cdot\mathbf{s})}{m}\mathrm{d}x^4 = \mathbf{n} + n^4\,\mathrm{d}x^4$

Invariants $(p,p) = m^2$, $(n,n) = -1$, $(p,n) = 0$, in any frame.

We define the spin projection operator using the volume element $\omega = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4$ as follows:

$$S_\pm = \frac{1 \pm i\omega \vee n}{2} \Rightarrow S_\pm \vee S_\pm = S_\pm, \qquad S_\pm \vee S_\mp = 0.$$

From the Heisenberg theorem, we can measure the momentum and spin simultaneously only if their operators commute. The commutator of the respective projection operators in the Clifford algebra is easily computed to be:

$$[\Lambda, S] = \pm i\frac{(p,n)\omega}{4m} = 0$$

The scalar product (p, n) is the same in any frame. In the rest frame the momentum is $p = m\,\mathrm{d}x^4$ and the spin is $n = \mathbf{s}$. It follows that the inner product is zero $(p, n) = 0$, therefore the momentum is always orthogonal to the spin in the Minkowski metric (but $\mathbf{p}$ is not necessarily orthogonal to $\mathbf{s}$). Therefore, the above commutator vanishes identically, and we can write the general solution to the Kähler–Dirac equation as a product of three commuting parts:

$$\Psi = S_\pm \vee \Lambda_\pm \exp \mp i(p, r).$$

The four-fold degeneracy in the solution corresponds to particle and antiparticle ($\pm$ in the momentum projector Λ and $\mp$ in the exponential), and spin which is polarized parallel or antiparallel to $\mathbf{s}$ ($\pm$ in the spin projector S).

To see what is actually measured, we exhibit the spin eigenvalues. Since the momentum part commutes with the spin part of the solution, we have:

$$S_{\mp} \vee S_{\pm} = 0 \Rightarrow S_{\mp} \vee \Psi[\pm \text{ spin}] = \tfrac{1}{2}(1 \mp i\omega \vee n) \vee \Psi[\pm \text{ spin}] = 0.$$

$$\Rightarrow (\tfrac{1}{2} i\omega \vee n) \vee \Psi[\pm \text{ spin}] = \pm \tfrac{1}{2}\Psi[\pm \text{ spin}]$$

These are the eigenvalues $\pm\frac{1}{2}$ which define the particle to obey 'Fermi–Dirac' statistics. Clearly, the numerical factor $\frac{1}{2}$ is a convention; what is important is that there are *two* distinct spin eigenstates to every solution of the Kähler–Dirac equation. In performing the famous Stern–Gerlach experiment, a beam of neutral silver atoms is split by an inhomogeneous magnetic field into two bands, corresponding to the two spin eigenstates. The deflection of each silver atom is due entirely to the spin $\frac{1}{2}$ of the outside electron. There are other visible manifestations of the spin, such as the splitting of spectral lines when one puts the emitting atom in a magnetic field. This is called the Zeeman effect.

One may write Ψ explicitly in terms of the basis differential forms. Omitting the wave part of the solution, and a numerical factor of $\frac{1}{4}$, one has from $\Psi \sim \Lambda_+ \vee S_+$:

$$\begin{aligned}\Psi \sim 1 &+ \frac{1}{m}p^\mu \, dx^\mu - \frac{i}{m}(p^1n^4 - En^1)\, dx^2 \wedge dx^3 - \frac{i}{m}(p^3n^4 - En^3)\, dx^1 \wedge dx^2 - \\ &- \frac{i}{m}(p^2n^4 - En^2)\, dx^3 \wedge dx^1 + \frac{i}{m}(p^2n^3 - p^3n^2)\, dx^1 \wedge dx^4 + \\ &+ \frac{i}{m}(p^3n^1 - p^1n^3)\, dx^2 \wedge dx^4 + \frac{i}{m}(p^1n^2 - p^2n^1)\, dx^3 \wedge dx^4 + \\ &+ in^1\, dx^2 \wedge dx^3 \wedge dx^4 + in^2\, dx^3 \wedge dx^1 \wedge dx^4 + in^3\, dx^1 \wedge dx^2 \wedge dx^4 + \\ &+ in^4\, dx^1 \wedge dx^2 \wedge dx^3.\end{aligned}$$

Note that of the sixteen basis forms, only the four-form ω does not appear in the above expression (this is because $(p, n) = 0$).

The interested reader may pursue the subject of quantum field theory via the standard references: for example [131]. Note, however, that the physics literature still continues to employ specific matrix representation techniques. This practice leads to confusion between the spacetime components of fields and the matrix components in the space of representation – both dimensions coincidentally being equal to four over $\mathbb{C}$.

6. Spin and the Spacetime Metric

In this section we would like to determine whether either the metric or the underlying field make any difference in the results obtained in the previous section. Specifically, we wish to consider the analogous results for the real forms of the complex Clifford algebra in Minkowski spacetime $(\mathbb{C}, 4, (-1, -1, -1, +1))$ which is isomorphic to the

Dirac algebra $\mathbb{C}_4$. They correspond to the following two algebras:

(i) $(\mathbb{R}, 4, (+1, +1, +1, -1)) \simeq \mathbb{R}_4$

(ii) $(\mathbb{R}, 4, (-1, -1, -1, +1)) \simeq \mathbb{H}_2$

No distinction is usually made between $\mathbb{C}_4$ and its two real forms (i), (ii) in the physics literature. There, the algebra $\mathbb{R}_4$ is referred to as the 'Majorana representation of the Dirac algebra', whereas in fact it is a distinct subalgebra (see [127]). Textbooks and research articles use either metric (i) or (ii) for physical spacetime, according to personal preference. We present here what we feel is an important result that distinguishes between the two real forms, and actually indicates the need for complexification.

RESULT. 'A spin-$\frac{1}{2}$ particle cannot be mathematically accommodated in real Minkowski space with metric $(-1, -1, -1, +1)$'.

This result may be surprising – we have not encountered it elsewhere. The basis for our demonstration is the need for having two distinct pairs of projection operators in order to describe a particle with spin. One is necessarily the momentum projection, and the other defines the spin projection. (This argument will be confined to strictly massive strictly massive particles.) We make crucial use of the fact that the projection operators are idempotents in the algebra and, moreover, that they come in mutually orthogonal pairs.

There is considerable confusion as to the maximal number of pairwise orthogonal idempotents in a $n \times n$ matrix ring over a division ring. The literature does not explicitly address this issue, but, as is shown below, the answer follows quickly from applying the standard concept of rings with minimum condition. (For a general reference, see [137].)

A division ring Δ is a ring in which each nonzero element has a multiplicative inverse. The real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, and the quaternions $\mathbb{H}$ are all division rings. We will denote by Δ_n the ring of $n \times n$ matrices over Δ under the usual matrix operations; the matrix units will be denoted as E_{ij}, $1 \leqslant i, j \leqslant n$.

An element α of a ring is idempotent if $\alpha^2 = \alpha$. Two idempotents α and β are orthogonal if $\alpha\beta = 0 = \beta\alpha$. An idempotent is primitive if it cannot be written as a sum of two nonzero orthogonal idempotents.

Any semisimple ring $\mathscr{R}$ with the minimum condition on left ideals can be written as a direct sum of a finite number of minimal left ideals. The ring Δ_n is simple, hence semisimple with minimal condition on left ideals. We know that all its left ideals are isomorphic as Δ_n spaces. If $\mathscr{R}$ is semisimple, then l is a minimal left ideal if and only if $l = \mathscr{R}\alpha$, for α a primitive idempotent. $\mathscr{R}\alpha$ is simple if and only if $\alpha\mathscr{R}\alpha$ is a division ring [138–140].

THEOREM 6.1. *The maximal number of pairwise orthogonal idempotents in Δ_n is n.*

Proof. $\{E_{ii}\}_{i=1}^{n}$ is a set of n pairwise orthogonal idempotents in Δ_n such that $\Sigma_{i=1}^{n} E_{ii} = 1$. Each E_{ii} is primitive since $E_{ii}\Delta_n E_{ii} = \Delta$. Suppose now that $\{\alpha_j\}_{j=1}^{m}$ is a second set of pairwise orthogonal, primitive idempotents in Δ_n, and $m > n$. We can assume that $\Sigma_{j=1}^{m} \alpha_j = 1$ and, hence, we have the decompositions of Δ_n into sums of minimal left ideals.

$$\Delta_n = \sum_{i=1}^{n} \oplus \mathscr{R}E_{ii} = \sum_{j=1}^{m} \oplus \mathscr{R}\alpha_j,$$

since each of the E_{ii} and α_i is primitive. But the left ideals $\mathscr{R}E_{ii}$ and $\mathscr{R}\alpha_j$ are isomorphic. That is, the dimension of $\Sigma_{i=1}^{n} \oplus \mathscr{R}E_{ii}$ over Δ is less than the dimension of $\Sigma_{j=1}^{m} \mathscr{R}\alpha_j$ over Δ, since $n < m$. This is absurd. Hence $m \leqslant n$. □

The above theorem gives the maximum number of pairwise orthogonal projection operators in the Clifford algebras $\mathbb{C}_4$, $\mathbb{R}_4$, and $\mathbb{H}_2$ as 4, 4, and 2, respectively. Hence, there is no room in $\mathbb{H}_2$ for both momentum and spin projection operators. It follows, therefore, that a massive particle with spin cannot be described in the real Clifford algebra in Minkowski spacetime $(\mathbb{R}, 4, (-1, -1, -1, +1))$.

This result implies that physical spacetime containing particles with spin-$\frac{1}{2}$ is necessarily complex, or is Majorana. This has been a pragmatic procedure in field theory without a rigorous mathematical justification. A closer examination of the projection operators in the Majorana algebra discloses some difficulties with the momentum projector, since, in that particular metric, $p^2 = -m^2$. This is taken care of by complexifying. This point, together with the above result, indicate the intrinsic connection between the spin and the necessity for having the complex form of the Clifford algebra in Minkowski spacetime. Since the complex Clifford algebra corresponds to a real Clifford algebra in five dimensions, the additional fifth dimension can be identified with the spin dimension.

7. Conclusion

In this review, we have attempted to indicate the fundamental connection between Clifford algebras and the physical world.

We have presented the Clifford algebra of differential forms in Minkowski spacetime, and have used it to construct explicit solutions to the Klein–Gordon and Kähler–Dirac equations. We believe that this geometrical picture is more appropriate for formulating these physically important results than the traditional explicit matrix treatment, because it focuses directly on the field observables. The physics is therefore made clearer and is not encumbered by excessive formalism. This is an advantage in calculations and the formulation of models, and is an advantage to the researcher not conversant with the traditional physics formalism.

A result which is not well-known is the explicit dependence of the spin upon the spacetime signature and the underlying field. We were able to provide a rigorous result based on techniques from ring theory. This, we believe, illustrates the intrinsic role of the Clifford algebra of differential forms as an algebraic setting for field theory.

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