SPINORS AS DIFFERENTIAL FORMS, AND APPLICATIONS TO ELECTROMAGNETISM AND QUANTUM MECHANICS[1]

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Abstract

In 1928 (the year Dirac discovered his famous equation) Ivanenko and Landau proposed an alternative relativistic equation for the wave function of a spin 1/2 particle. In modern mathematical language Ivanenko-Landau wave function is a section of a particular Clifford bundle (CB) over spacetime. In this paper we show that sections of the CB cannot represent spinor fields. Instead spinors are sections of the so-called Spin-Clifford bundle (SCB). Each section of SCB can be seen as an equivalence class of section of CB. We also write Maxwell and Dirac equations in CB and SCB and show a very surprising relationship between those equations.

Introduction

In 1928 Dirac [1] published a paper introducing his now famous equation which describes the electron (and other particles with spin 1/2). Dirac equation is formulated in terms of the so-called Dirac spinors (which are the objects that carry the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\mathrm{SL}(2,\mathbb{C})$) fields. At the same time, Ivanenko and Landau [2] proposed an alternative description in terms of an equation for a mathematical object represented by a set of skew-symmetric tensor fields. The relationship between those two works and the geometric and algebraic content of Dirac equation have been discussed in 1929 by Fock and Ivanenko [3, 4, 5, 6] (see also [7]). That representation of the electron's wave function in terms of skew-symmetric tensor fields was then rediscovered by Proca [8] and Eddington [9], and then in 1960 by Kähler [10]. In modern mathematical language we say that Ivanenko-Landau wave function is represented by a sum of differential forms of different degrees, or an

[1] Dedicated to the memory of Professor D. D. Ivanenko

inhomogeneous differential form, and the Ivanenko-Landau equation is equivocally called the Dirac-Kähler equation.

The relationship between the approaches of Dirac and Ivanenko-Landau is related to the question of representing spinor fields by means of differential forms (or by multivectors). Several authors [11, 12, 13, 14, 15], besides those quoted above, studied this question. However, most of those representations of spinors by differential forms proposed by those authors are not equivalent to the Dirac spinor. One reason is very simple: those representations and the Dirac spinor have a different number of degrees of freedom. Consider, for example, the work of Benn and Tucker [11]; their representation is given by $\phi = a_0 + \sum_i a_i\, dx^i + \sum_{i<j} a_{ij}\, dx^i \wedge dx^j + \sum_{i<j<k} a_{ijk}\, dx^i \wedge dx^j \wedge dx^k + a_{0123}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$, where the indexes run from 0 to 3, $\wedge$ denotes the exterior product and the a's are complex scalar fields. We immediatly see that ϕ has 32 degrees of freedom, while the Dirac spinor has 8 degrees of freedom. One can see [11, 13] that if spacetime is flat the Ivanenko-Landau-Dirac-Kähler equation for that ϕ decouples into 4 equivalent equations, so that we have a 4-fold degeneracy if a spin 1/2 particle is to be described by that ϕ instead of a Dirac spinor; but, if spacetime is not flat, then in general there is no such decoupling. If we want a representation with the correct number of degrees of freedom we must follow Hestenes [16] and write $\phi = a_0 + \sum_{i<j} dx^i \wedge dx^j + a_{0123}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$, where now the a's are *real* scalar fields. However, there is still another problem.

In modern mathematical language we interpret an expression like $\phi = a_0 + \sum_{i<j} dx^i \wedge dx^j + a_{0123}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ is a section of a bundle called the Clifford bundle of the spacetime. For the structural group of the bundle (in this case the Lorentz group), the elements of the Clifford bundle transforms according to the adjoint representation of the group. On the other hand, we know that the transformation law of spinors is given by multiplication from the left. We see therefore that if a spinor is to be represented by one such ϕ we have a *wrong* transformation law for the action of the Lorentz group. The way of solving this problem [18] is to define an equivalence relation in the Clifford bundle and consider sections of the quotient bundle, the Spin-Clifford bundle.

In this paper we have two objectives. The first (Part A) is to study the theory of the so-called Dirac-Hestenes spinor fields (DHSF), showing how spinors can be represented by differential forms. The second (Part B) uses the DHSF in order to study an old an intriguing question[19, 20, 21, 22]: is there any relationship between Maxwell and Dirac equations?

Part A is divided in two sections. In section A.1 we introduce and compare the concepts of covariant spinors, algebraic spinors and Dirac-Hestenes spinors. This has already been studied before [23, 24, 25, 26, 27, 28] but our present study improves and clarifies those previous presentations by introducing a new and important fact, namely that all kind of spinors refered above must be defined as particular equivalence classes in appropriated Clifford algebras. The hidden geometric meaning of the covariant Dirac spinor will then be disclosed, a fact which will be of fundamental importance in Part B. In section A.2 we study the Clifford bundle of a Riemann-Cartan spacetime and its irreducible module representations. This permit us to define DHSF as certain equivalence classes of even sections of the Clifford bundle. DHSF are then naturally identified with sections of a bundle which we call the Spin-Clifford bundle. The theory of covariant derivative of DHSF has been studied in [18, 29]; in these references we showed how the Clifford and the Spin-Clifford bundle techniques permit us to give a simple presentation of the concept of covariant derivative of DHSF, and that it agrees with the standard one developed for the covariant Dirac spinor fields. We invite the interested reader to those references [18, 29] for a full understanding of the theory used here.

In Part B we use the theory of DHSF developed in Part A in order to find a spinorial representation of Maxwell equations and then to discuss its relationship with Dirac equation. We must observe here that there are several different spinorial representations of Maxwell equations in the literature (see [25] for a study of them). However, those spinorial representations, as a rule, use as components of the spinor field the components of the electromagnetic field, and the deficiency of those representations is obvious: the action of the Lorentz group on the spinor field does not give the correct transformation laws for the electric and magnetic fields. The spinorial representation of Maxwell equations we shall use does not suffer the deficiency indicated above since the electromagnetic field is given by a bilinear expression of DHSF. Once we find that spinorial representation of Maxwell equations we will show that it can be reduced to a form which is identical to Dirac equation. Finally, we discuss some conditions under which that equation can be really interpreted as Dirac equation.

PART A

A.1 Covariant, Algebraic and Dirac-Hestenes Spinors

A.1.1 Some General Features about Clifford Algebras

In this section we fix the notations to be used in this paper and introduce the main ideas concerning the theory of Clifford algebras necessary for the intelligibility of the paper. We follow with minor modifications the conventions used in [26, 27].

A.1.1.1 Formal Definition of the Clifford Algebra $\mathcal{C}\ell(V, Q)$

Let K be a field, char $K \neq 2$,[2] V a vector space of finite dimension n over K, and Q a nondegenerate quadratic form over V. Denote by

$$\mathbf{x} \cdot \mathbf{y} = \tfrac{1}{2}(Q(\mathbf{x}+\mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y})) \tag{1}$$

the associated *symmetric* bilinear form on V and define the *left contraction* $\rfloor$: $\bigwedge V \times \bigwedge V \to \bigwedge V$ and the *right contraction* $\llcorner : \bigwedge V \times \bigwedge V \to \bigwedge V$ by the rules

1. $\mathbf{x} \rfloor \mathbf{y} = \mathbf{x} \cdot \mathbf{y}$
   $\mathbf{x} \llcorner \mathbf{y} = \mathbf{x} \cdot \mathbf{y}$

2. $\mathbf{x} \rfloor (u \wedge v) = (\mathbf{x} \rfloor u) \wedge v + \hat{u} \wedge (\mathbf{x} \rfloor v)$
   $(u \wedge v) \llcorner \mathbf{x} = u \wedge (v \llcorner \mathbf{x}) + (u \llcorner \mathbf{x}) \wedge \hat{v}$

3. $(u \wedge v) \rfloor w = u \rfloor (v \rfloor w)$
   $u \llcorner (v \wedge w) = (u \llcorner v) \llcorner w$

where $\mathbf{x}, \mathbf{y} \in V$, $u, v, w \in \bigwedge V$, and $\hat{}$ is the grade involution in the algebra $\bigwedge V$. The notation $\mathbf{a} \cdot \mathbf{b}$ will be used for contractions when it is clear from the context which factor is the contractor and which factor is being contracted. When just one of the factors is homogeneous, it is understood to be the contractor. When both factors are homogeneous, we agree that the one with lower degree is the contractor, so that for $\mathbf{a} \in \bigwedge^r V$ and $\mathbf{b} \in \bigwedge^s V$, we have $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \rfloor \mathbf{b}$ if $r \leq s$ and $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \llcorner \mathbf{b}$ if $r \geq s$.

Define the (Clifford) product of $\mathbf{x} \in V$ and $u \in \bigwedge V$ by

$$\mathbf{x}u = \mathbf{x} \wedge u + \mathbf{x} \rfloor u \tag{2}$$

and extend this product by linearity and associativity to all of $\bigwedge V$ This provides $\bigwedge V$ with a new product, and provided with this new product $\bigwedge V$ becomes isomorphic to the *Clifford algebra* $\mathcal{C}\ell(V, Q)$.

We recall that $\bigwedge V = T(V)/I$ where $T(V)$ is the tensor algebra of V and $I \subset T(V)$ is the bilateral ideal generated by the elements of the form $\mathbf{x} \otimes \mathbf{x}$, $\mathbf{x} \in V$. It can also be shown that the Clifford algebra of (V, Q) is $\mathcal{C}\ell(V, Q) = T(V)/I_Q$, where I_Q is the bilateral ideal generated by the elements of the form $\mathbf{x} \otimes \mathbf{x} - Q(\mathbf{x})$, $\mathbf{x} \in V$. The Clifford algebra so constructed is an associative algebra with unity. Since K is a field, the space V is naturally imbedded in $\mathcal{C}\ell(V, Q)$

$$\begin{gathered} V \overset{i}{\hookrightarrow} T(V) \overset{j}{\rightarrow} T(V)/I_Q = \mathcal{C}\ell(V, Q) \\ I_Q = j \circ i \quad \text{and} \quad V \equiv i_Q(V) \subset \mathcal{C}\ell(V, Q) \end{gathered} \tag{3}$$

[2]In our applications in this paper, K will be $\mathbb{R}$ or $\mathbb{C}$, respectively the real or complex field. The quaternion ring will be denoted by $\mathbb{H}$.

Let $\mathcal{C}\ell^+(V,Q)$ [resp., $\mathcal{C}\ell^-(V,Q)$] be the j-image of $\oplus_{i=0}^{\infty} T^{2i}(V)$ [resp., $\oplus_{i=0}^{\infty} T^{2i+1}(V)$] in $\mathcal{C}\ell(V,Q)$. The elements of $\mathcal{C}\ell^+(V,Q)$ form a subalgebra of $\mathcal{C}\ell(V,Q)$ called the even subalgebra of $\mathcal{C}\ell(V,Q)$.

$\mathcal{C}\ell(V,Q)$ has the following property: If A is an associative K-algebra with unity then all linear mappings $\rho : V \to A$ such that $(\rho(x))^2 = Q(x)$, $x \in V$, can be extended in a unique way to an algebra homomorphism $\rho : \mathcal{C}\ell(V,Q) \to A$.

In $\mathcal{C}\ell(V,Q)$ there exist three linear mappings which are quite natural. They are extensions of the mappings

Main involution: an automorphism $\hat{}$: $\mathcal{C}\ell(V,Q) \to \mathcal{C}\ell(V,Q)$, extension of $\alpha : V \to T(V)/I_Q, \alpha(x) = -i_Q(x) = -x$, $\forall x \in V$.

Reversion: an antiautomorphism $\tilde{}$: $\mathcal{C}\ell(V,Q) \to \mathcal{C}\ell(V,Q)$, extension of $^t : T^r(V) \to T^r(V)$; $T^r(V) \ni x = x_{i_1} \otimes \ldots \otimes x_{i_r} \mapsto x^t = x_{i_r} \otimes \ldots \otimes x_{i_1}$.

Conjugation: $\bar{}$: $\mathcal{C}\ell(V,Q) \to \mathcal{C}\ell(V,Q)$, defined by the composition of the main involution $\hat{}$ with the reversion $\tilde{}$, i.e., if $x \in \mathcal{C}\ell(V,Q)$ then $\bar{x} = (\hat{x})\tilde{} = (\tilde{x})\hat{}$.

$\mathcal{C}\ell(V,Q)$ can be described through its generators, i.e., if $\Sigma = \{E_i\}$ $(i = 1, 2, \ldots, n)$ is a Q-orthonormal basis of V, then $\mathcal{C}\ell(V,Q)$ is generated by 1 and the E_i's are subjected to the conditions

$$\begin{aligned} &E_i E_i = Q(E_i) \\ &E_i E_j + E_j E_i = 0, \qquad i \neq j;\ i,j = 1, 2, \ldots, n \\ &E_1 E_2 \cdots E_n \neq \pm 1. \end{aligned} \tag{4}$$

A.1.1.2 The Real Clifford Algebra $\mathcal{C}\ell_{p,q}$

Let $\mathbb{R}^{p,q}$ be a real vector space of dimension $n = p+q$ endowed with a nondegenerate metric $g : \mathbb{R}^{p,q} \times \mathbb{R}^{p,q} \to \mathbb{R}$. Let $\Sigma = \{E_i\}$, $(i = 1, 2, \ldots, n)$ be an orthonormal basis of $\mathbb{R}^{p,q}$,

$$g(E_i, E_j) = g_{ij} = g_{ji} = \begin{cases} +1, & i = j = 1, 2, \ldots p \\ -1, & i = j = p+1, \ldots, p+q = n \\ \ \ 0, & i \neq j \end{cases} \tag{5}$$

The Clifford algebra $\mathcal{C}\ell_{p,q} = \mathcal{C}\ell(\mathbb{R}^{p,q}, Q)$ is the Clifford algebra over $\mathbb{R}$, generated by 1 and the $\{E_i\}$, $(i = 1, 2, \ldots, n)$ such that $E_i^2 = Q(E_i) = g(E_i, E_i)$, $E_i E_j = -E_j E_i$ $(i \neq j)$, and $E_1 E_2 \ldots E_n \neq \pm 1$. $\mathcal{C}\ell_{p,q}$ is obviously of dimension 2^n and as a vector space it is the direct sum of vector spaces $\bigwedge^k \mathbb{R}^{p,q}$ of dimensions $\binom{n}{k}$, $0 \leq k \leq n$. The canonical basis of $\bigwedge^k \mathbb{R}^{p,q}$ is given by the elements $e_A = E_{\alpha_1} \ldots E_{\alpha_k}$, $1 \leq \alpha_1 < \ldots < \alpha_k \leq n$. The element $c_J = E_1 \ldots E_n \in \bigwedge^n \mathbb{R}^{p,q}$ commutes (n odd) or anticommutes (n even) with all vectors $E_1, \ldots, E_n \in \bigwedge^1 \mathbb{R}^{p,q} \equiv \mathbb{R}^{p,q}$. The center of $\mathcal{C}\ell_{p,q}$ is $\bigwedge^0 \mathbb{R}^{p,q} \equiv \mathbb{R}$ if n is even and its is the direct sum $\bigwedge^0 \mathbb{R}^{p,q} \oplus \bigwedge^n \mathbb{R}^{p,q}$ if n is odd.

All Clifford algebras are semi-simple. If $p+q=n$ is even, $\mathcal{C}\ell_{p,q}$ is simple and if $p+q=n$ is odd we have the following possibilities:

1. $\mathcal{C}\ell_{p,q}$ is simple $\leftrightarrow c_J^2 = -1 \leftrightarrow p-q \neq 1 \pmod 4 \leftrightarrow$ center of $\mathcal{C}\ell_{p,q}$ is isomorphic to $\mathbb{C}$

2. $\mathcal{C}\ell_{p,q}$ is not simple (but is a direct sum of two simple algebras) $\leftrightarrow c_J^2 = +1 \leftrightarrow p-q = 1 \pmod 4 \leftrightarrow$ center of $\mathcal{C}\ell_{p,q}$ is isomorphic to $\mathbb{R} \oplus \mathbb{R}$.

All these semi-simple algebras are direct sums of two simple algebras.

If A is an associative algebra on the field $K, K \subseteq A$, and if E is a vector space, a homomorphism ρ from A to $\operatorname{End} E$ ($\operatorname{End} E$ is the endomorphism algebra of E) which maps the unit element of A to Id_E is a called a *representation* of A in E. The dimension of E is called the degree of the representation. The addition in E together with the mapping $A \times E \to E$, $(a,x) \mapsto \rho(a)x$ turns E in an A-module, the *representation module.*

Conversely, A being an algebra over K and E being an A-module, E is a vector space over K and if $a \in A$, the mapping $\gamma : a \to \gamma_a$ with $\gamma_a(x) = ax$, $x \in E$, is a homomorphism $A \to \operatorname{End} E$, and so it is a representation of A in E. The study of A modules is then equivalent to the study of the representations of A. A representation ρ is *faithful* if its kernel is zero, i.e., $\rho(a)x = 0$, $\forall x \in E \Rightarrow a = 0$. The kernel of ρ is also known as the annihilator of its module. ρ is said to be *simple* or irreducible if the only invariant subspaces of $\rho(a)$, $\forall a \in A$, are E and $\{0\}$. Then the representation module is also simple, this meaning that it has no proper submodule. ρ is said to be *semi-simple*, if it is the direct sum of simple modules, and in this case E is the direct sum of subspaces which are globally invariant under $\rho(a), \forall a \in A$. When no confusion arises $\rho(a)x$ will be denoted by ax. Two A-modules E and E' (with the exterior multiplication being denoted respectively by $\bullet$ and $*$) are *isomorphic* if there exists a bijection $\varphi : E \to E'$ such that,

$$\begin{aligned} \varphi(x+y) &= \varphi(x) + \varphi(y), \quad \forall x, y \in E, \\ \varphi(a \bullet x) &= a * \varphi(x), \quad \forall a \in A, \end{aligned} \tag{6}$$

and we say that representations ρ and ρ' of A are equivalent if their modules are isomorphic. This implies the existence of a K-linear isomorphism $\varphi : E \to E'$ such that $\varphi \circ \rho(a) = \rho'(a) \circ \varphi$, $\forall a \in A$ or $\rho'(a) = \varphi \circ \rho(a) \circ \varphi^{-1}$ If $\dim E = n$, then $\dim E' = n$. We shall need:

Wedderburn Theorem. *[30] If A is simple algebra then A is equivalent to $F(m)$, where $F(m)$ is a matrix algebra with entries in F, F is a division algebra and m and F are unique (modulo isomorphisms).*

A.1.2 Minimal Left Ideals of $\mathcal{C}\ell_{p,q}$

The minimal left (resp., right) ideals of a semi-simple algebra A are of the type Ae (resp., eA), where e is a primitive idempotent of A, i.e., $e^2 = e$ and e cannot be written as a sum of two non zero annihilating (or orthogonal) idempotents, i.e, $e \neq e_1 + e_2$, where $e_1 e_2 = e_2 e_1 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$ [31].

Theorem. *The maximum number of pairwise annihilating idempotents in $F(m)$ is m.*

The decomposition of $\mathcal{C}\ell_{p,q}$ into minimal ideals is then characterized by a spectral set $\{e_{pq,i}\}$ of idempotents of $\mathcal{C}\ell_{p,q}$ satisfying (i) $\sum_i e_{pq,i} = 1$; (ii) $e_{pq,i}e_{pq,j} = \delta_{ij}e_{pq,i}$; (iii) rank of $e_{pq,i}$ is minimal $\neq 0$, i.e., $e_{pq,i}$ is primitive $(i = 1, 2, \ldots, m)$

By rank of $e_{pq,i}$ we mean the rank of the $\bigwedge \mathbb{R}^{p+q}$-morphism $e_{pq,i} : \psi \mapsto \psi e_{pq,i}$ and $\bigwedge \mathbb{R}^{p,q} = \oplus_{k=0}^{n} \bigwedge^k(\mathbb{R}^{p,q})$ is the exterior algebra of $\mathbb{R}^{p,q}$. Then $\mathcal{C}\ell_{p,q} = \sum_i I^i_{p,q}$, $I^i_{p,q} = \mathcal{C}\ell_{p,q}e_{pq,i}$ and $\psi \in I^i_{p,q}$ is such that $\psi e_{pq,i} = \psi$. Conversely any element $\psi \in I^i_{p,q}$ can be characterized by an idempotent $e_{pq,i}$ of minimal rank $\neq 0$ with $\psi e_{pq,i} = \psi$. We have the following

Theorem. *[31] A minimal left ideal of $\mathcal{C}\ell_{p,q}$ is of the type $I_{p,q} = \mathcal{C}\ell_{p,q}e_{pq}$ where $e_{pq} = \frac{1}{2}(1 + e_{\alpha_1}) \ldots \frac{1}{2}(1 + e_{\alpha_k})$ is a primitive idempotent of $\mathcal{C}\ell_{p,q}$ and are $e_{\alpha_1}, \ldots, e_{\alpha_k}$ commuting elements of the canonical basis of $\mathcal{C}\ell_{p,q}$ such that $(e_{\alpha_i})^2 = 1$, $(i = 1, 2, \ldots, k)$ that generate a group of order 2^k, $k = q - r_{q-p}$ and r_i are the Radon-Hurwitz numbers, defined by the recurrence formula $r_{i+8} = r_i + 4$ and*

i	0	1	2	3	4	5	6	7
r_i	0	1	2	2	3	3	3	3

If we have a linear mapping $L_a : \mathcal{C}\ell_{p,q} \to \mathcal{C}\ell_{p,q}$, $L_a(x) = ax$, $x \in \mathcal{C}\ell_{p,q}$, $a \in \mathcal{C}\ell_{p,q}$, then since $I_{p,q}$ is invariant under left multiplication with arbitrary elements of $\mathcal{C}\ell_{p,q}$ we can consider $L_a|_{I_{p,q}} : I_{p,q} \to I_{p,q}$ and taking into account Wedderburn theorem we have

Theorem. *If $p + q = n$ is even or odd with $p - q \neq 1$ (mod 4) then*

$$\mathcal{C}\ell_{p,q} \simeq \mathrm{End}_F(I_{p,q}) \simeq F(m)$$

where $F = \mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$, $\mathrm{End}_F(I_{p,q})$ is the algebra of linear transformations in $I_{p,q}$ over the field $F, m = \dim_F(I_{p,q})$ and $F \simeq eF(m)e$, e being the representation of e_{pq} in $F(m)$.

If $p+q=n$ is odd, with $p-q=1 \pmod 4$ then

$$\mathcal{C}\ell_{p,q} = \operatorname{End}_F(I_{p,q}) \simeq F(m) \oplus F(m)$$

and $m = \dim_F(I_{p,q})$ and $e_{pq}\mathcal{C}\ell_{p,q}e_{pq} \simeq \mathbb{R} \oplus \mathbb{R}$ or $\mathbb{H} \oplus \mathbb{H}$.

Observe that F is the set

$$F = \{T \in \operatorname{End}_F(I_{p,q}), TL_a = L_a T, \ \forall\, a \in \mathcal{C}\ell_{p,q}\}$$

Periodicity Theorem. *[30] For $n = p+q \geq 0$ there exist the following isomorphisms*

$$\begin{aligned} \mathcal{C}\ell_{n+8,0} &\simeq \mathcal{C}\ell_{n,0} \otimes \mathcal{C}\ell_{8,0} \qquad & \mathcal{C}\ell_{0,n+8} &\simeq \mathcal{C}\ell_{0,n} \otimes \mathcal{C}\ell_{0,8} \\ \mathcal{C}\ell_{p+8,q} &\simeq \mathcal{C}\ell_{p,q} \otimes \mathcal{C}\ell_{8,0} \qquad & \mathcal{C}\ell_{p,q+8} &\simeq \mathcal{C}\ell_{p,q} \otimes \mathcal{C}\ell_{0,8} \end{aligned} \tag{7}$$

We can find, e.g., in [30] tables giving the representations of all algebras $\mathcal{C}\ell_{p,q}$ as matrix algebras. For what follows we need

$$\begin{array}{rl} \text{complex numbers} & \mathcal{C}\ell_{0,1} \simeq \mathbb{C} \\ \text{quarternions} & \mathcal{C}\ell_{0,2} \simeq \mathbb{H} \\ \text{Pauli algebra} & \mathcal{C}\ell_{3,0} \simeq M_2(\mathbb{C}) \\ \text{spacetime algebra} & \mathcal{C}\ell_{1,3} \simeq M_2(\mathbb{H}) \\ \text{Majorana algebra} & \mathcal{C}\ell_{3,1} \simeq M_4(\mathbb{R}) \\ \text{Dirac algebra} & \mathcal{C}\ell_{4,1} \simeq M_4(\mathbb{C}) \end{array} \tag{8}$$

We also need the following

Proposition. $\mathcal{C}\ell^{+}_{p,q} = \mathcal{C}\ell_{q,p-1}$, *for $p > 1$ and $\mathcal{C}\ell^{+}_{p,q} = \mathcal{C}\ell_{p,q-1}$ for $q > 1$.*

From the above proposition we get the following particular results that we shall need later

$$\mathcal{C}\ell^{+}_{1,3} \simeq \mathcal{C}\ell^{+}_{3,1} = \mathcal{C}\ell_{3,0} \qquad \mathcal{C}\ell^{+}_{4,1} \simeq \mathcal{C}\ell_{1,3}, \tag{9}$$

$$\mathcal{C}\ell_{4,1} \simeq \mathbb{C} \otimes \mathcal{C}\ell_{3,1} \qquad \mathcal{C}\ell_{4,1} \simeq \mathbb{C} \otimes \mathcal{C}\ell_{1,3}, \tag{10}$$

which means that the Dirac algebra is the complexification of both the spacetime or the Majorana algebras.

A.1.2.1 Right Linear Structure for $I_{p,q}$

We can give to the ideal $I_{p,q} = \mathcal{C}\ell_{p,q}e$ (resp. $I_{pq} = e\mathcal{C}\ell_{pq}$) a right (resp. left) linear structure over the field $F(\mathcal{C}\ell_{p,q} \simeq F(m)$ or $\mathcal{C}\ell_{p,q} \simeq F(m) \oplus F(m))$. A right linear structure, e.g, consists of an additive group (which is $I_{p,q}$) and the mapping

$$I \times F \to I; \quad (\psi, T) \mapsto \psi T$$

such that the usual axioms of a linear vector space structure are valid, e.g., we have[3] $(\psi T)T' = \psi(TT')$.

From the above discussion it is clear that the minimal (left or right) ideals of $\mathcal{C}\ell_{p,q}$ are representation modules of $\mathcal{C}\ell_{p,q}$. In order to investigate the equivalence of these representations we must introduce some groups that are subsets of $\mathcal{C}\ell_{p,q}$. As we shall see, this is the key for the definition of algebraic and Dirac-Hestenes spinors.

A.1.3 The Groups: $\mathcal{C}\ell^{\star}_{p,q}$, Clifford, Pinor and Spinor

The set of the invertible elements of $\mathcal{C}\ell_{p,q}$ constitutes a non-abelian group which we denote by $\mathcal{C}\ell^{\star}_{p,q}$. It acts naturally on $\mathcal{C}\ell_{p,q}$ as an algebra homomorphism through its adjoint representation

$$\mathrm{Ad} : \mathcal{C}\ell^{\star}_{p,q} \to \mathrm{Aut}(\mathcal{C}\ell_{p,q});\ u \mapsto \mathrm{Ad}_u,\ \text{with } \mathrm{Ad}_u(x) = uxu^{-1}. \tag{11}$$

The Clifford-Lipschitz group is the set

$$\mathbf{\Gamma}_{p,q} = \{u \in \mathcal{C}\ell^{\star}_{p,q} \mid \forall x \in \mathbb{R}^{p,q}, ux\hat{u}^{-1} \in \mathbb{R}^{p,q}\}. \tag{12}$$

The set $\mathbf{\Gamma}^{+}_{p,q} = \mathbf{\Gamma}_{p,q} \cap \mathcal{C}\ell^{+}_{p,q}$ is called special Clifford-Lipschitz group.

Let $N : \mathcal{C}\ell_{p,q} \to \mathcal{C}\ell_{p,q}$, $N(x) = \langle \tilde{x}x \rangle_0$ ($\langle\ \rangle_0$ means the scalar part of the Clifford number). We define further:

The *Pinor group* $\mathbf{Pin}(p,q)$ is the subgroup of $\mathbf{\Gamma}_{p,q}$ such that

$$\mathbf{Pin}(p,q) = \{u \in \mathbf{\Gamma}_{p,q} | N(u) = \pm 1\}. \tag{13}$$

The *Spin group* $\mathbf{Spin}(p,q)$ is the set

$$\mathbf{Spin}(p,q) = \{u \in \mathbf{\Gamma}^{+}_{p,q} | N(u) = \pm 1\}. \tag{14}$$

The $\mathbf{Spin}_{+}(p,q)$ group is the set

$$\mathbf{Spin}_{+}(p,q) = \{u \in \mathbf{\Gamma}^{+}_{p,q} | N(u) = +1\}. \tag{15}$$

[3]For $\mathcal{C}\ell_{3,0}$, $I = \mathcal{C}\ell_{3,0}\frac{1}{2}(1+\sigma_3)$ is a minimal left ideal. In this case it is also possible to give a left linear structure for this ideal. See [23, 24]

Theorem. $\mathrm{Ad}|_{\mathbf{Pin}(p,q)} : \mathbf{Pin}(p,q) \to \mathrm{O}(p,q)$ *is onto with kernel* $\mathbf{Z}_2$. $\mathrm{Ad}|_{\mathbf{Spin}(p,q)} : \mathbf{Spin}(p,q) \to \mathrm{SO}(p,q)$ *is onto with kernel* $\mathbf{Z}_2$.

$\mathrm{O}(p,q)$ is the pseudo-orthogonal group of the vector space $\mathbb{R}^{p,q}$, $\mathrm{SO}(p,q)$ is the special pseudo-orthogonal group of $\mathbb{R}^{p,q}$. We also denote by $\mathrm{SO}_+(p,q)$ the connected component of $\mathrm{SO}(p,q)$. $\mathbf{Spin}_+(p,q)$ is connected for all pairs (p,q) with the exception of $\mathbf{Spin}_+(1,0) \simeq \mathbf{Spin}_+(0,1) \simeq \{\pm 1\}$ and $\mathbf{Spin}_+(1,1)$. We have,

$$\mathrm{O}(p,q) = \frac{\mathbf{Pin}(p,q)}{\mathbf{Z}_2} \quad \mathrm{SO}(p,q) = \frac{\mathbf{Spin}(p,q)}{\mathbf{Z}_2} \quad \mathrm{SO}_+(p,q) = \frac{\mathbf{Spin}_+(p,q)}{\mathbf{Z}_2}.$$

In the following the group homomorphism between $\mathbf{Spin}_+(p,q)$ and $\mathrm{SO}_+(p,q)$ will be denoted

$$\mathcal{H} : \mathbf{Spin}_+(p,q) \to \mathrm{SO}_+(p,q). \tag{16}$$

We also need the important result:

Theorem. *[30] For* $p+q \leq 5$, $\mathbf{Spin}_+(p,q) = \{u \in \mathcal{C}\ell^+_{p,q} | u\tilde{u} = 1\}$.

A.1.3.1 Lie Algebra of $\mathbf{Spin}_+(1,3)$

It can be shown that for each $u \in \mathbf{Spin}_+(1,3)$ it holds

$$u = \pm e^F, \qquad F \in \bigwedge^2 \mathbb{R}^{1,3} \subset \mathcal{C}\ell_{1,3} \tag{17}$$

and F can be chosen in such a way to have a positive sign in Eq. 17, except in the particular case $F^2 = 0$ when $u = -e^F$. From Eq. 17 it follows immediately that the Lie algebra of $\mathbf{Spin}_+(1,3)$ is generated by the bivectors $F \in \bigwedge^2 \mathbb{R}^{1,3} \subset \mathcal{C}\ell_{1,3}$ through the commutator product.

A.1.4 Geometrical and Algebraic Equivalence of the Representation Modules $I_{p,q}$ *of Simple Clifford Algebras* $\mathcal{C}\ell_{p,q}$

Recall that $\mathcal{C}\ell_{p,q}$ is a ring. We already said that the minimal lateral ideals of $\mathcal{C}\ell_{p,q}$ are of the form $I_{p,q} = \mathcal{C}\ell_{p,q} e_{pq}$ (or $e_{pq}\mathcal{C}\ell_{p,q}$) where e_{pq} is a primitive idempotent. Obviously the minimal lateral ideals are modules over the ring $\mathcal{C}\ell_{p,q}$, they are representation modules. According to the discussion of Section A.1.1, given two ideals $I_{p,q} = \mathcal{C}\ell_{p,q} e_{pq}$ and $I'_{p,q} = \mathcal{C}\ell_{p,q} e'_{pq}$ they are by definition isomorphic if there exists a bijection $\varphi : I_{p,q} \to I'_{p,q}$ such that,

$$\varphi(\psi_1 + \psi_2) = \varphi(\psi_1) + \varphi(\psi_2) \ ; \ \varphi(a\psi) = a\varphi(\psi) \ , \ \forall a \in \mathcal{C}\ell_{p,q}, \forall \psi_1, \psi_2 \in I_{p,q} \tag{18}$$

Recalling the Noether-Skolem theorem, which says that all automorphisms of a simple algebra are inner automorphisms, we have:

Theorem. *When $\mathcal{C}\ell_{p,q}$ is simple, its automorphisms are given by inner automorphisms $x \mapsto uxu^{-1}$, $x \in \mathcal{C}\ell_{p,q}$, $u \in \mathcal{C}\ell^{\star}_{p,q}$.*

We also have:

Proposition. *When $\mathcal{C}\ell_{p,q}$ is simple, all its finite-dimensional irreducible representations are equivalent (i.e., isomorphic) under inner automorphisms.*

We quote also the

Theorem. *[17] $I_{p,q}$ and $I'_{p,q}$ are isomorphic if and only if $I'_{p,q} = I_{p,q}X$ for non-zero $X \in I'_{p,q}$.*

We are thus lead to the following definitions:

1. The ideals $I_{p,q} = \mathcal{C}\ell_{p,q}e_{pq}$ and $I'_{p,q} = \mathcal{C}\ell_{p,q}e'_{pq}$ are said to be *geometrically equivalent* if, for some $u \in \mathbf{\Gamma}_{p,q}$,

$$e'_{pq} = ue_{pq}u^{-1}. \tag{19}$$

2. $I_{p,q}$ and $I'_{p,q}$ are said to be *algebraically equivalent* if

$$e'_{pq} = ue_{pq}u^{-1}, \tag{20}$$

for some $u \in \mathcal{C}\ell^{\star}_{p,q}$, but $u \notin \mathbf{\Gamma}_{p,q}$.

It is now time to specialize the above results for $\mathcal{C}\ell_{1,3} \simeq M_2(\mathbb{H})$ and to find a relationship between the Dirac algebra $\mathcal{C}\ell_{4,1} \simeq M_4(\mathbb{C})$ and $\mathcal{C}\ell_{1,3}$ and their respective minimal ideals.

Let $\Sigma_0 = \{E_0, E_1, E_2, E_3\}$ be an orthogonal basis of $\mathbb{R}^{1,3} \subset \mathcal{C}\ell_{1,3}$, $E_\mu E_\nu + E_\nu E_\mu = 2\eta_{\mu\nu}$, $\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$. Then, the elements

$$e = \tfrac{1}{2}(1 + E_0) \qquad e' = \tfrac{1}{2}(1 + E_3E_0) \qquad e'' = \tfrac{1}{2}(1 + E_1E_2E_3), \tag{21}$$

are easily verified to be primitive idempotents of $\mathcal{C}\ell_{1,3}$. The minimal left ideals, $I = \mathcal{C}\ell_{1,3}e$, $I' = \mathcal{C}\ell_{1,3}e'$, $I'' = \mathcal{C}\ell_{1,3}e''$ are right two dimensional linear spaces over the quaternion field (e.g., $\mathbb{H}e = e\mathbb{H} = e\mathcal{C}\ell_{1,3}e$). According to the definition (ii) above these ideals are algebraically equivalent. For example, $e' = ueu^{-1}$, with $u = (1 + E_3) \notin \mathbf{\Gamma}_{1,3}$

The elements $\Phi \in \mathcal{C}\ell_{1,3}\frac{1}{2}(1 + E_0)$ will be called *mother* spinors.[27, 28] We can show[23, 24] that each Φ can be written

$$\Phi = \psi_1 e + \psi_2 E_3E_1 e + \psi_3 E_3E_0 e + \psi_4 E_1E_0 e = \sum_i \psi_i s_i, \tag{22}$$

$$s_1 = e, \quad s_2 = E_3E_1e, \quad s_3 = E_3E_0e, \quad s_4 = E_1E_0e \tag{23}$$

and where the ψ_i are formally complex numbers, i.e., each $\psi_i = (a_i + b_iE_2E_1)$ with $a_i, b_i \in \mathbb{R}$.

We recall that $\mathbf{Pin}(1,3)/\mathbf{Z}_2 \simeq \mathrm{O}(1,3)$, $\mathbf{Spin}(1,3)/\mathbf{Z}_2 \simeq \mathrm{SO}(1,3)$, $\mathbf{Spin}_+(1,3)/\mathbf{Z}_2 \simeq \mathrm{SO}_+(1,3)$, $\mathbf{Spin}_+(1,3) \simeq \mathrm{SL}(2,\mathbb{C})$ the universal covering group of $\mathcal{L}_+^\uparrow \equiv \mathrm{SO}_+(1,3)$, the restrict Lorentz group.

In order to determine the relation between $\mathcal{C}\ell_{4,1}$ and $\mathcal{C}\ell_{1,3}$ we proceed as follows: let $\{F_0, F_1, F_2, F_3, F_4\}$ be an orthogonal basis of $\mathcal{C}\ell_{4,1}$ with $-F_0^2 = F_1^2 = F_2^2 = F_3^2 = F_4^2 = 1$, $F_AF_B = -F_BF_A$ ($A \neq B$; $A, B = 0,1,2,3,4$). Define the pseudoscalar

$$\mathbf{i} = F_0F_1F_2F_3F_4 \qquad \mathbf{i}^2 = -1 \qquad \mathbf{i}F_A = F_A\mathbf{i} \qquad A = 0,1,2,3,4 \tag{24}$$

Define

$$\mathcal{E}_\mu = F_\mu F_4 \tag{25}$$

We can immediately verify that $\mathcal{E}_\mu\mathcal{E}_\nu + \mathcal{E}_\nu\mathcal{E}_\mu = 2\eta_{\mu\nu}$. Taking into account that $\mathcal{C}\ell_{1,3} \simeq \mathcal{C}\ell_{4,1}^+$ we can explicitly exhibit here this isomorphism by considering the map $g : \mathcal{C}\ell_{1,3} \to \mathcal{C}\ell_{4,1}^+$ generated by the linear extension of the map $g^\# : \mathbb{R}^{1,3} \to \mathcal{C}\ell_{4,1}^+$, $g^\#(E_\mu) = \mathcal{E}_\mu = F_\mu F_4$, where E_μ, ($\mu = 0,1,2,3$) is an orthogonal basis of $\mathbb{R}^{1,3}$. Also $g(1_{\mathcal{C}\ell_{1,3}}) = 1_{\mathcal{C}\ell_{4,1}^+}$, where $1_{\mathcal{C}\ell_{1,3}}$ and $1_{\mathcal{C}\ell_{4,1}^+}$ are the identity elements in $\mathcal{C}\ell_{1,3}$ and $\mathcal{C}\ell_{4,1}^+$. Now consider the primitive idempotent of $\mathcal{C}\ell_{1,3} \simeq \mathcal{C}\ell_{4,1}^+$,

$$e_{41} = g(e) = \tfrac{1}{2}(1 + \mathcal{E}_0) \tag{26}$$

and the minimal left ideal $I_{4,1}^+ = \mathcal{C}\ell_{4,1}^+ e_{4,1}$. The elements $Z_{\Sigma_0} \in I_{4,1}^+$ can be written in an analogous way to $\Phi \in \mathcal{C}\ell_{1,3}\frac{1}{2}(1 + E_0)$ (Eq. 22), i.e.,

$$Z_{\Sigma_0} = \Sigma\ z_i\bar{s}_i \tag{27}$$

where

$$\bar{s}_1 = e_{41}, \quad \bar{s}_2 = -\mathcal{E}_1\mathcal{E}_3e_{41}, \quad \bar{s}_3 = \mathcal{E}_3\mathcal{E}_0e_{41}, \quad \bar{s}_4 = \mathcal{E}_1\mathcal{E}_0e_{41}, \tag{28}$$

and

$$z_i = a_i + \mathcal{E}_2\mathcal{E}_1b_i,$$

are formally complex numbers, $a_i, b_i \in \mathbb{R}$.

Consider now the element $f_{\Sigma_0} \in \mathcal{C}\ell_{4,1}$,

$$\begin{aligned} f_{\Sigma_0} &= e_{41}\tfrac{1}{2}(1 + \mathbf{i}\mathcal{E}_1\mathcal{E}_2) \\ &= \tfrac{1}{2}(1 + \mathcal{E}_0)\tfrac{1}{2}(1 + \mathbf{i}\mathcal{E}_1\mathcal{E}_2), \end{aligned} \tag{29}$$

with $\mathbf{i}$ given by Eq. 24.

Since $f_{\Sigma_0}\mathcal{C}\ell_{4,1}f_{\Sigma_0} = \mathbb{C}f_{\Sigma_0} = f_{\Sigma_0}\mathbb{C}$ it follows that f_{Σ_0} is a primitive idempotent of $\mathcal{C}\ell_{4,1}$. We can easily show that each $\Phi_{\Sigma_0} \in I_{\Sigma_0} = \mathcal{C}\ell_{4,1}f_{\Sigma_0}$ can be written

$$\Psi_{\Sigma_0} = \sum_i \psi_i f_i, \quad \psi_i \in \mathbb{C}$$

$$f_1 = f_{\Sigma_0}\ , \ f_2 = -\mathcal{E}_1\mathcal{E}_3 f_{\Sigma_0}\ , \ f_3 = \mathcal{E}_3\mathcal{E}_0 f_{\Sigma_0}, f_4 = \mathcal{E}_1\mathcal{E}_0 f_{\Sigma_0} \tag{30}$$

with the methods described in [23, 24] we find the following representation in $M_4(\mathbb{C})$ for the generators $\mathcal{E}_\mu$ of $\mathcal{C}\ell^+_{4,1} \simeq \mathcal{C}\ell_{1,3}$

$$\mathcal{E}_0 \mapsto \gamma_0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} \leftrightarrow \mathcal{E}_i \mapsto \gamma_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{31}$$

where 1_2 is the unit 2×2 matrix and σ_i, $(i = 1, 2, 3)$ are the standard Pauli matrices. We immediately recognize the γ-matrices in Eq. 31 as the standard ones appearing, e.g., in [32]

The matrix representation of $\Psi_{\Sigma_0} \in I_{\Sigma_0}$ will be denoted by the same letter without the indice, i.e., $\Psi_{\Sigma_0} \mapsto \Psi \in M_4(\mathbb{C})f$, where

$$f = \tfrac{1}{2}(1+\gamma_0)\tfrac{1}{2}(1+i\gamma_1\gamma_2) \qquad i = \sqrt{-1}. \tag{32}$$

We have

$$\Psi = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix} \qquad \psi_i \in \mathbb{C}. \tag{33}$$

Eqs. 22, 27 and (30) are enough to prove that there are bijections between the elements of the ideals $\mathcal{C}\ell_{1,3}\frac{1}{2}(1+E_0)$, $\mathcal{C}\ell^+_{4,1}\frac{1}{2}(1+\mathcal{E}_0)$ and $\mathcal{C}\ell_{4,1}\frac{1}{2}(1+\mathcal{E}_0)\frac{1}{2}(1+\mathbf{i}\mathcal{E}_1\mathcal{E}_2)$.

We can easily find that the following relation exists between $\Psi_{\Sigma_0} \in \mathcal{C}\ell_{4,1}f_{\Sigma_0}$ and $Z_{\Sigma_0} \in \mathcal{C}\ell^+_{4,1}\frac{1}{2}(1+\mathcal{E}_0)$,

$$\Psi_{\Sigma_0} = Z_{\Sigma_0}\tfrac{1}{2}(1+\mathbf{i}\mathcal{E}_1\mathcal{E}_2). \tag{34}$$

Decomposing Z_{Σ_0} into even and odd parts relative to the $\mathbf{Z}_2$-graduation of $\mathcal{C}\ell^+_{4,1} \simeq \mathcal{C}\ell_{1,3}$, $Z_{\Sigma_0} = Z^+_{\Sigma_0} + Z^-_{\Sigma_0}$ we obtain $Z^+_{\Sigma_0} = Z^-_{\Sigma_0}\mathcal{E}_0$ which clearly shows that all information of Z_{Σ_0} is contained in $Z^+_{\Sigma_0}$. Then,

$$\Psi_{\Sigma_0} = Z^+_{\Sigma_0}\tfrac{1}{2}(1+\mathcal{E}_0)\tfrac{1}{2}(1+\mathbf{i}\mathcal{E}_1\mathcal{E}_2). \tag{35}$$

Now, if we take into account[23, 24] that $\mathcal{C}\ell^{++}_{4,1}\frac{1}{2}(1+\mathcal{E}_0) = \mathcal{C}\ell^+_{4,1}\frac{1}{2}(1+\mathcal{E}_0)$ where the symbol $\mathcal{C}\ell^{++}_{4,1}$ means $\mathcal{C}\ell^{++}_{4,1} \simeq \mathcal{C}\ell^+_{1,3} \simeq \mathcal{C}\ell_{3,0}$ we see that each $Z_{\Sigma_0} \in \mathcal{C}\ell^+_{4,1}\frac{1}{2}(1+\mathcal{E}_0)$ can be written

$$Z^-_{\Sigma_0} = \psi_{\Sigma_0}\tfrac{1}{2}(1+\mathcal{E}_0) \qquad \psi_{\Sigma_0} \in (\mathcal{C}\ell^+_{4,1})^+ \simeq \mathcal{C}\ell^+_{1,3}. \tag{36}$$

Then putting $Z^+_{\Sigma_0} = \psi_{\Sigma_0}/2$, Eq. 35 can be written

$$\begin{aligned}\Psi_{\Sigma_0} &= \psi\tfrac{1}{2}(1+\mathcal{E}_0)\tfrac{1}{2}(1+\mathbf{i}\mathcal{E}_1\mathcal{E}_2) \\ &= Z_{\Sigma_0}\tfrac{1}{2}(1+\mathbf{i}\mathcal{E}_1\mathcal{E}_2).\end{aligned} \tag{37}$$

The matrix representations of Z_{Σ_0} and ψ_{Σ_0} in $M_4(\mathbb{C})$ (denoted by the same letter without index) in the spinorial basis given by Eq. 30 are

$$\Psi = \begin{pmatrix} \psi_1 & -\psi_2^* & \psi_3 & \psi_4^* \\ \psi_2 & \psi_1^* & \psi_4 & -\psi_3^* \\ \psi_3 & \psi_4^* & \psi_1 & -\psi_2^* \\ \psi_4 & -\psi_3^* & \psi_2 & \psi_1^* \end{pmatrix}, \quad Z = \begin{pmatrix} \psi_1 & -\psi_2^* & 0 & 0 \\ \psi_2 & \psi_1^* & 0 & 0 \\ \psi_3 & \psi_4^* & 0 & 0 \\ \psi_4 & -\psi_3^* & 0 & 0 \end{pmatrix}. \tag{38}$$

A.1.5 Algebraic Spinors for $\mathbb{R}^{p,q}$

Let $\mathcal{B}_\Sigma = \{\Sigma_0, \dot\Sigma, \ddot\Sigma, \ldots\}$ be the set of all ordered orthonormal basis for $\mathbb{R}^{p,q}$, i.e., each $\Sigma \in \mathcal{B}_\Sigma$ is the set $\Sigma = \{E_1, \ldots, E_p, E_{p+1}, \ldots, E_{p+q}\}$, $E_1^2 = \ldots = E_p^2 = 1$, $E_{p+1}^2 = \ldots = E_{p+q}^2 = -1$, $E_rE_s = -E_sE_r$, $(r \neq s;\ r, s = 1, 2, \ldots, p+q = n)$. Any two basis, say, $\Sigma_0, \dot\Sigma \in \mathcal{B}_\Sigma$ are related by an element of the group $\mathbf{Spin}_+(p,q) \subset \Gamma_{pq}$. We write,

$$\dot\Sigma = u\Sigma_0 u^{-1}, \quad u \in \mathbf{Spin}_+(p,q). \tag{39}$$

A primitive idempotent determined in a given basis $\Sigma \in \mathcal{B}_\Sigma$ will be denoted e_Σ. Then, the idempotents $e_{\Sigma_0}, e_{\dot\Sigma}, e_{\ddot\Sigma}$, etc., such that, e.g.,

$$e_{\dot\Sigma} = ue_{\Sigma_0}u^{-1}, \quad u \in \mathbf{Spin}_+(p,q), \tag{40}$$

define ideals $I_{\Sigma_0}, I_{\dot\Sigma}, I_{\ddot\Sigma}$, etc., that are geometrically equivalent according to the definition given by Eq. 19. We have,

$$I_{\dot\Sigma} = uI_{\Sigma_0}u^{-1} \qquad u \in \mathbf{Spin}_+(p,q) \tag{41}$$

but since $uI_{\Sigma_0} \equiv I_{\Sigma_0}$, Eq. 41 can also be written

$$I_{\dot\Sigma} = I_{\Sigma_0}u^{-1} \tag{42}$$

Eq. 42 defines a new correspondence for the elements of the ideals, $I_{\Sigma_0}, I_{\dot\Sigma}, I_{\ddot\Sigma}$, etc. This suggests the

Definition. *An algebraic spinor for* $\mathbb{R}^{p,q}$ *is an equivalence class of the quotient set* $\{I_\Sigma\}/R$, *where* $\{I_\Sigma\}$ *is the set of all geometrically equivalent ideals, and* $\Psi_{\Sigma_0} \in I_{\Sigma_0}$ *and* $\Psi_{\dot\Sigma} \in I_{\dot\Sigma}$ *are equivalent,* $\Psi_{\dot\Sigma} \simeq \Psi_{\Sigma_0}$ *(mod* R*) if and only if*

$$\Psi_{\dot\Sigma} = \Psi_{\Sigma_0}u^{-1} \tag{43}$$

Ψ_Σ *will be called the representative of the algebraic spinor in the basis* $\Sigma \in \mathcal{B}_\Sigma$. *Recall that* $\dot\Sigma = u\Sigma u^{-1} = L\Sigma$, $u \in \mathbf{Spin}_+(1,3)$, $L \in \mathcal{L}_+^\uparrow$.

A.1.6 What is a Covariant Dirac Spinor (CDS)

As we already know $f_{\Sigma_0} = \frac{1}{2}(1+\mathcal{E}_0)(1+\mathbf{i}\mathcal{E}_1\mathcal{E}_2)$ (Eq. 29) is a primitive idempotent of $\mathcal{C}\ell_{4,1} \simeq M_4(\mathbb{C})$. If $u \in \mathbf{Spin}_+(1,3) \subset \mathbf{Spin}_+(4,1)$ then all ideals $I_{\dot{\Sigma}} = I_{\Sigma_0}u^{-1}$ are geometrically equivalent to I_{Σ_0}. Since $\Sigma_0 = \{\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$ is a basis for $\mathbb{R}^{1,3} \subset \mathcal{C}\ell_{4,1}^+$, the meaning of $\dot{\Sigma} = u\Sigma_0 u^{-1}$ is clear. From Eq. 30 we can write

$$I_{\Sigma_0} \ni \Psi_{\Sigma_0} = \sum \psi_i f_i, \qquad \text{and} \qquad I_{\dot{\Sigma}} \ni \Psi_{\dot{\Sigma}} = \sum \dot{\psi}_i \dot{f}_i, \tag{44}$$

where

$$f_1 = f_{\Sigma_0}, \quad f_2 = -\mathcal{E}_1\mathcal{E}_3 f_{\Sigma_0}, \quad f_3 = \mathcal{E}_3\mathcal{E}_0 f_{\Sigma_0} \quad f_4 = \mathcal{E}_1\mathcal{E}_0 f_{\Sigma_0}$$

and

$$\dot{f}_1 = f_{\dot{\Sigma}}, \quad \dot{f}_2 = -\bar{\dot{\mathcal{E}}}_1\bar{\dot{\mathcal{E}}}_3 f_{\dot{\Sigma}}, \quad \dot{f}_3 = \bar{\dot{\mathcal{E}}}_3\bar{\dot{\mathcal{E}}}_0 f_{\dot{\Sigma}}, \quad \dot{f}_4 = \bar{\dot{\mathcal{E}}}_1\bar{\dot{\mathcal{E}}}_0 f_{\dot{\Sigma}}$$

Since $\Psi_{\dot{\Sigma}} = \Psi_{\Sigma_0}u^{-1}$, we get

$$\Psi_{\dot{\Sigma}} = \sum_i \psi_i u^{-1}\dot{f}_i = \sum_{i,k} S_{ik}(u^{-1})\psi_i \dot{f}_k = \sum_k \dot{\psi}_k \dot{f}_k.$$

Then

$$\dot{\psi}_k = \sum_i S_{ik}(u^{-1})\psi_i, \tag{45}$$

where $S_{ik}(u^{-1})$ are the matrix components of the representation in $M_4(\mathbb{C})$ of $u^{-1} \in \mathbf{Spin}_+(1,3)$. As proved in [23, 24] the matrices $S(u)$ correspond to the representation $D^{(1/2,0)} \oplus D^{(0,1/2)}$ of $SL(2,\mathbb{C}) \simeq \mathbf{Spin}_+(1,3)$.

We remark that all the elements of the set $\{I_\Sigma\}$ of the ideals geometrically equivalent to I_{Σ_0} under the action of $u \in \mathbf{Spin}_+(1,3) \subset \mathbf{Spin}_+(4,1)$ have the same image $I = M_4(\mathbb{C})f$ where f is given by Eq. 32, i.e.,

$$f = \tfrac{1}{2}(1+\gamma_0)(1+i\gamma_1\gamma_2) \qquad i = \sqrt{-1},$$

where γ_μ, $\mu = 0,1,2,3$ are the Dirac matrices given by Eq. 31.

Then, if

$$\begin{array}{rcl} \gamma : \mathcal{C}\ell_{4,1} & \rightarrow & M_4(\mathbb{C}) \equiv \mathrm{End}(M_4(\mathbb{C})f) \\ x & \mapsto & \gamma(x) : M_4(\mathbb{C})f \rightarrow M_4(\mathbb{C})f \end{array} \tag{46}$$

it follows that $\gamma(\mathcal{E}_\mu) = \gamma(\dot{\mathcal{E}}_\mu) = \gamma_\mu$, $\gamma(f_{\Sigma_0}) = \gamma(f_{\dot{\Sigma}}) = f$ for all $\mathcal{E}_\mu, \dot{\mathcal{E}}_\mu$ such that $\dot{\mathcal{E}}_\mu = u\mathcal{E}_\mu u^{-1}$ for some $u \in \mathbf{Spin}_+(1,3)$. Observe that all the information concerning the orthonormal frames $\Sigma_0, \dot{\Sigma}$, etc., disappear in the matrix representation of the ideals $I_{\Sigma_0}, I_{\dot{\Sigma}}, \ldots$ in $M_4(\mathbb{C})$ since all these ideals are mapped in the same ideal $I = M_4(\mathbb{C})f$.

With the above remark and taking into account Eq. 45 we are then lead to the following

Definition. *A Covariant Dirac Spinor (CDS) for* $\mathbb{R}^{1,3}$ *is an equivalent class of triplets* $(\Sigma, S(u), \Psi)$, Σ *being an orthonormal basis of* $\mathbb{R}^{1,3}$, $S(u) \in D^{(1/2,0)} \oplus D^{(0,1/2)}$ *representation of* $\mathbf{Spin}_+(1,3), u \in \mathbf{Spin}_+(1,3)$ *and* $\Psi \in M_4(\mathbb{C})f$ *and*

$$(\Sigma, S(u), \Psi) \sim (\Sigma_0, S(u_0), \Psi_0)$$

if and only if

$$\Psi = S(u)S^{-1}(u_0)\Psi_0, \quad \mathcal{H}(uu_0^{-1}) = L\Sigma_0, \quad L \in \mathcal{L}_+^\uparrow, \quad u \in \mathbf{Spin}_+(1,3). \tag{47}$$

The pair $(\Sigma, S(u))$ is called a spinorial frame. Observe that the CDS just defined depends on the choice of the original spinorial frame (Σ_0, u_0) and obviously, to different possible choices there correspond isomorphic ideals in $M_4(\mathbb{C})$. For simplicity we can fix $u_0 = 1, S(u_0) = 1$.

The definition of CDS just given agrees with that given by Choquet-Bruhat[33] except for the irrelevant fact that Choquet-Bruhat uses as the space of representatives of a CDS the complex four-dimensional vector space $\mathbb{C}^4$ instead of $I = M_4(\mathbb{C})f$. We see that Choquet-Bruhat's definition is well justified from the point of view of the theory of algebraic spinors presented above.

A.1.7 Algebraic Dirac Spinors (ADS) and Dirac-Hestenes Spinors (DHS)

We saw that there is bijection between $\psi_{\Sigma_0} \in \mathcal{C}\ell_{4,1}^{++} \simeq \mathcal{C}\ell_{1,3}^{+}$ and $\Psi_{\Sigma_0} \in I_{\Sigma_0} = \mathcal{C}\ell_{4,1}^{+} f_{\Sigma_0}$, namely (Eq. 37),

$$\Psi_{\Sigma_0} = \psi_{\Sigma_0}\tfrac{1}{2}(1 + \mathcal{E}_0)\tfrac{1}{2}(1 + \mathbf{i}\mathcal{E}_1\mathcal{E}_2)$$

Then, as we already said, all information contained in Ψ_{Σ_0} (that is the representative in the basis Σ_0 of an algebraic spinor for $\mathbb{R}^{1,3}$) is also contained in $\psi_{\Sigma_0} \in \mathcal{C}\ell_{4,1}^{++} \simeq \mathcal{C}\ell_{1,3}^{+}$. We are then lead to the following

Definition. *Consider the quotient set* $\{I_\Sigma\}/\mathcal{R}$ *where* $\{I_\Sigma\}$ *is the set of all geometrically equivalent minimal left ideals of* $\mathcal{C}\ell_{1,3}$ *generated by* $e_{\Sigma_0} = \frac{1}{2}(1 + E_0), \Sigma_0 = (E_0, E_1, E_2, E_3)$ *[i.e.,* $I_{\ddot{\Sigma}}, I_{\dot{\Sigma}} \in \{I_\Sigma\}$ *then* $I_{\ddot{E}} = uI_{\dot{\Sigma}}u^{-1} \equiv I_{\dot{\Sigma}}u^{-1}$ *for some* $u \in \mathbf{Spin}_+(1,3)$*]. An algebraic Dirac Spinor (ADS) is an element of* $\{I_\Sigma\}/\mathcal{R}$*. Then if* $\Phi_{\ddot{\Sigma}} \in I_{\dot{\Sigma}}, \Phi_{\ddot{\Sigma}} \in I_{\ddot{\Sigma}}$*, then* $\Phi_{\ddot{\Sigma}} \simeq \Phi_{\dot{\Sigma}} (\mathrm{mod}\ \mathcal{R})$ *if and only if* $\Phi_{\ddot{\Sigma}} = \Phi_{\dot{\Sigma}}u^{-1}$*, for some* $u \in \mathbf{Spin}_+(1,3)$.

We remark that (see Eq. 36)

$$\Phi_{\dot{\Sigma}} = \psi_{\dot{\Sigma}}e_{\dot{\Sigma}}, \quad \Phi_{\Sigma} = \psi_{\ddot{\Sigma}}e_{\ddot{\Sigma}} \qquad \psi_{\dot{\Sigma}}, \psi_{\ddot{\Sigma}} \in \mathcal{C}\ell_{1,3}^{+}$$

and since $e_{\ddot{\Sigma}} = ue_{\Sigma}u^{-1}$ for some $u \in \mathbf{Spin}_+(1,3)$ we get[4]

$$\psi_{\ddot{\Sigma}} = \psi_{\dot{\Sigma}}u^{-1}. \tag{48}$$

Now, for $p+q \leq 5$, $\mathbf{Spin}_+(p,q) = \{u \in \mathcal{C}\ell^+_{p,q} | u\tilde{u} = 1\}$. Then for all $\psi_\Sigma \in \mathcal{C}\ell^+_{1,3}$ such that $\psi_\Sigma \tilde{\psi}_\Sigma \neq 0$ we obtain immediately the polar form

$$\psi_\Sigma = \rho^{1/2} e^{\beta E_5/2} R_\Sigma, \tag{49}$$

where $\rho \in \mathbb{R}^+, \beta \in \mathbb{R}, R_\Sigma \in \mathbf{Spin}_+(1,3), E_5 = E_0E_1E_2E_3$. With the above remark in mind we present the

Definition. *A Dirac-Hestenes spinor (DHS) is an equivalence class of triplets* (Σ, u, ψ_Σ), *where* Σ *is an oriented orthonormal basis of* $\mathbb{R}^{1,3} \subset \mathcal{C}\ell_{1,3}$, $u \in \mathbf{Spin}_+(1,3)$, *and* $\psi_\Sigma \in \mathcal{C}\ell^+_{1,3})$. *We say that* $(\Sigma, u, \psi_\Sigma) \sim (\Sigma_0, u_0, \psi_{\Sigma_0})$ *if and only if* $\psi_\Sigma = \psi_{\Sigma_0} u_0^{-1} u$, $\mathcal{H}(uu_0^{-1}) = L$, $\Sigma = L\Sigma_0 (\equiv u^{-1}u_0\Sigma_0 u_0^{-1} u)$, $u, u_0 \in \mathbf{Spin}_+(1,3)$, $L \in \mathcal{L}^\uparrow_+$. u_0 *is arbitrary but fixed. A DHS determines a set of vectors* $X_\mu \in \mathbb{R}^{1,3}$, $(\mu = 0,1,2,3)$ *by a given representative* $\psi_{\dot{\Sigma}}$ *of the DHS in the basis* Σ *by*

$$\psi : \dot{\Sigma} \to \mathbb{R}^{1,3}, \quad \psi_{\dot{\Sigma}} \dot{E}_\mu \tilde{\psi}_{\dot{\Sigma}} = X_\mu \quad (\dot{\Sigma} = (\dot{E}_0, \dot{E}_1, \dot{E}_2, \dot{E}_3)). \tag{50}$$

We give yet another equivalent definition of a DHS

Definition. *A Dirac-Hestenes spinor is an element of the quotient set* $\mathcal{C}\ell^+_{1,3}/\mathcal{R}$ *such that given the basis* $\Sigma, \dot{\Sigma}$ *of* $\mathbb{R}^{1,3} \subset \mathcal{C}\ell_{1,3}$, $\psi_\Sigma \in \mathcal{C}\ell^+_{1,3}$, $\psi_{\dot{\Sigma}} \in \mathcal{C}\ell^+_{1,3}$ *then* $\psi_{\dot{\Sigma}} \sim \psi_\Sigma (\mathrm{mod}\mathcal{R})$ *if and only if* $\psi_{\dot{\Sigma}} = \psi_\Sigma u^{-1}$, $\dot{\Sigma} = L\Sigma = u\Sigma u^{-1}$, $\mathcal{H}(u) = L$, $u \in \mathbf{Spin}_+(1,3)$, $L \in \mathcal{L}^\uparrow_+$.

With the canonical form of a DHS given by Eq. 49 some features of the hidden geometrical nature of the Dirac spinors defined above comes to light: Eq. 49 says that when $\psi_\Sigma \tilde{\psi}_\Sigma \neq 0$ the Dirac-Hestenes spinor ψ_Σ is equivalent to a Lorentz rotation followed by a dilation and a duality mixing given by the term $e^{\beta E_5/2}$, where β is the so-called Yvon-Takabayasi angle[34, 35] and the justification for the name duality rotation can be found in [36]. We emphasize that the definition of the Dirac-Hestenes spinors gives above is new. In the past objects $\psi \in \mathcal{C}\ell^+_{1,3}$ satisfying $\psi X \tilde{\psi} = Y$, for $X, Y \in \mathbb{R}^{1,3} \subset \mathcal{C}\ell_{1,3}$ have been called operator spinors (see, e.g., in [37, 27, 28]). DHS have been used as the departure point of many interesting results as, e.g., in [36, 38, 39, 40].

A.2. The Clifford Bundle of Spacetime and their Irreducible Module Representations

[4] In [27, 28] Lounesto calls 2Φ the mother of all the real spinors.

A.2.1 The Clifford Bundle of Spacetime

Let M be a four dimensional, real, connected, paracompact manifold. Let TM $[T^*M]$ be the tangent [cotangent] bundle of M.

Definition. *A Lorentzian manifold is a pair (M,g), where $g \in \sec T^*M \times T^*M$ is a Lorentzian metric of signature (1,3), i.e., for all $x \in M$, $T_xM \simeq T_x^*M \simeq \mathbb{R}^{1,3}$, where $\mathbb{R}^{1,3}$ is the vector Minkowski space.*

Definition. *A spacetime $\mathcal{M}$ is a triple (M,g,∇) where (M,g) is a time oriented and spacetime oriented Lorentzian manifold and ∇ is a linear connection for M such that $\nabla g = 0$. If in addition $\mathbf{T}(\nabla) = 0$ and $\mathbf{R}(\nabla) \neq 0$, where $\mathbf{T}$ and $\mathbf{R}$ are respectively the torsion and curvature tensors, then $\mathcal{M}$ is said to be a Lorentzian spacetime. When $\nabla g = 0$, $\mathbf{T}(\nabla) = 0$, $\mathbf{R}(\nabla) = 0$, $\mathcal{M}$ is called Minkowski spacetime and will be denote by $\mathbb{M}$. When $\nabla g = 0$, $\mathbf{T}(\nabla) \neq 0$ and $\mathbf{R}(\nabla) = 0$ or $\mathbf{R}(\nabla) \neq 0$, $\mathcal{M}$ is said to be a Riemann-Cartan spacetime.*

In what follows $P_{\mathrm{SO}_+(1,3)}(\mathcal{M})$ denotes the principal bundles of oriented Lorentz tetrads.[26, 41] By g^{-1} we denote the "metric" of the cotangent bundle.

It is well known that the natural operations on metric vector spaces, such as, e.g., direct sum, tensor product, exterior power, etc., carry over canonically to vector bundles with metrics. Take, e.g., the cotangent bundle T^*M. If $\pi : T^*M \to M$ is the canonical projection, then in each fiber $\pi^{-1}(x) = T_x^*M \simeq \mathbb{R}^{1,3}$, the "metric" g^{-1} can be used to construct a Clifford algebra $\mathcal{C}\ell(T_x^*M) \simeq \mathcal{C}\ell_{1,3}$. We have the

Definition. *The Clifford bundle of spacetime $\mathcal{M}$ is the bundle of algebras*

$$\mathcal{C}\ell(\mathcal{M}) = \bigcup_{x\in M} \mathcal{C}\ell(T_x^*M) \tag{51}$$

As is well known $\mathcal{C}\ell(\mathcal{M})$ is the quotient bundle

$$\mathcal{C}\ell(\mathcal{M}) = \frac{\tau M}{\mathbf{J}(\mathcal{M})} \tag{52}$$

where $\tau M = \oplus_{r=0}^{\infty} T^{0,r}(M)$ and $T^{(0,r)}(M)$ is the space of r-covariant tensor fields, and $\mathbf{J}(\mathcal{M})$ is the bundle of ideals whose fibers at $x \in M$ are the two side ideals in τM generated by the elements of the form $a \otimes b + b \otimes a - 2g^{-1}(a,b)$ for $a, b \in T^*\mathcal{M}$.

Let $\pi_c : \mathcal{C}\ell(\mathcal{M}) \to M$ be the canonical projection of $\mathcal{C}\ell(\mathcal{M})$ and let $\{U_\alpha\}$ be an open covering of M. From the definition of a fibre bundle[41] we know that there is a trivializing mapping $\varphi\alpha : \pi_c^{-1}(U_\alpha) \to U_\alpha \times \mathcal{C}\ell_{1,3}$ of the form $\varphi\alpha(p) = (\pi_c(p), \overset{\triangle}{\varphi}_\alpha(p))$. If $U_{\alpha\beta} = U_\alpha \cap U_\beta$ and $x \in U_{\alpha\beta}$, $p \in \pi_c^{-1}(x)$, then

$$\overset{\triangle}{\varphi}_\alpha(p) = f_{\alpha\beta}(x)\, \overset{\triangle}{\varphi}_\beta(p) \tag{53}$$

for $f_{\alpha\beta}(x) \in \mathrm{Aut}(\mathcal{C}\ell_{1,3})$, where $f_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Aut}(\mathcal{C}\ell_{1,3})$ are the transition mappings of $\mathcal{C}\ell(\mathcal{M})$. We know that every automorphism of $\mathcal{C}\ell_{1,3}$ is inner and it follows that,

$$f_{\alpha\beta}(x) \overset{\Delta}{\varphi}_\beta (p) = g_{\alpha\beta}(x) \overset{\Delta}{\varphi}_\beta (p) g_{\alpha\beta}(x)^{-1} \tag{54}$$

for some $g_{\alpha\beta}(x) \in \mathcal{C}\ell^{\star}_{1,3}$, the group of invertible elements of $\mathcal{C}\ell_{1,3}$. We can write equivalently instead of Eq. 54,

$$f_{\alpha\beta}(x) \overset{\Delta}{\varphi}_\beta (p) = \overset{\Delta}{\varphi}_\beta (a_{\alpha\beta} p a_{\alpha\beta}^{-1}) \tag{55}$$

for some invertible element $a_{\alpha\beta} \in \mathcal{C}\ell(T^{\star}_x M)$.

Now, the group $\mathrm{SO}_+(1,3)$ has, as we know (Section A.1), a natural extension in the Clifford algebra $\mathcal{C}\ell_{1,3}$. Indeed we know that $\mathcal{C}\ell^{\star}_{1,3}$ acts naturally on $\mathcal{C}\ell_{1,3}$ as an algebra automorphism through its adjoint representation $\mathrm{Ad} : u \mapsto \mathrm{Ad}_u$, $\mathrm{Ad}_u(a) = uau^{-1}$ Also $\mathrm{Ad}\,|_{\mathbf{Spin}_+(1,3)} = \sigma$ defines a group homeomorphism $\sigma : \mathbf{Spin}_+(1,3) \to \mathrm{SO}_+(1,3)$ which is onto with kernel $\mathbf{Z}_2$. It is clear, since Ad_{-1} = identity, that $\mathrm{Ad} : \mathbf{Spin}_+(1,3) \to \mathrm{Aut}(\mathcal{C}\ell_{1,3})$ descends to a representation of $\mathrm{SO}_+(1,3)$. Let us call Ad' this representation, i.e., $\mathrm{Ad}' : \mathrm{SO}_+(1,3) \to \mathrm{Aut}(\mathcal{C}\ell_{1,3})$. Then we can write $\mathrm{Ad}'_{\sigma(u)}\, a = \mathrm{Ad}_u a = uau^{-1}$

From this it is clear that the structure group of the Clifford bundle $\mathcal{C}\ell(\mathcal{M})$ is reducible from $\mathrm{Aut}(\mathcal{C}\ell_{1,3})$ to $\mathrm{SO}_+(1,3)$. This follows immediately from the existence of the Lorentzian structure (M, g) and the fact that $\mathcal{C}\ell(\mathcal{M})$ is the exterior bundle where the fibres are equipped with the Clifford product. Thus the transition maps of the principal bundle of oriented Lorentz tetrads $P_{\mathrm{SO}_+(1,3)}(\mathcal{M})$ can be (through Ad') taken as transition maps for the Clifford bundle. We then have the result[42]

$$\mathcal{C}\ell(\mathcal{M}) = P_{\mathrm{SO}_+(1,3)}(\mathcal{M}) \times_{\mathrm{Ad}'} \mathcal{C}\ell_{1,3} \tag{56}$$

A.2.2 Spinor Bundles

Definition. *[41] A spinor structure for $\mathcal{M}$ consists of a principal fibre bundle $\pi_s : P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \to M$ with group $SL(2,\mathbb{C}) \simeq \mathbf{Spin}_+(1,3)$ and a map*

$$s : P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \to P_{\mathrm{SO}_+(1,3)}(\mathcal{M})$$

satisfying the following conditions

1. $\pi(s(p)) = \pi_s(p)\ \forall p \in P_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$
2. $s(pu) = s(p)\mathcal{H}(u)\ \forall p \in P_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ *and* $\mathcal{H} : SL(2,\mathbb{C}) \to \mathrm{SO}_+(1,3)$.

Now, in Section A.1 we learned that the minimal left (right) ideals of $\mathcal{C}\ell_{p,q}$ are irreducible left (right) module representations of $\mathcal{C}\ell_{p,q}$ and we define a covariant and algebraic Dirac spinors as elements of quotient sets of the type $\{I_\Sigma\}/\mathbb{R}$ in appropriate Clifford algebras. We defined also in Section A.1 the DHS. We are now interested in defining algebraic Dirac spinor fields (ADSF) and also Dirac-Hestenes spinor fields (DHSF).

So, in the spirit of Section A.1 the following question naturally arises: Is it possible to find a vector bundle $\pi_s : S(\mathcal{M}) \to M$ with the property that each fiber over $x \in M$ is an irreducible module over $\mathcal{C}\ell(T_x^*M)$?

The answer to the above question is in general no. Indeed it is now well known[43] that the necessary and sufficient conditions for $S(\mathcal{M})$ to exist is that the Spinor Structure bundle $P_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ exists, which implies the vanishing of the second Stiefel-Whitney class of M, i.e., $\omega_2(M) = 0$. For a spacetime $\mathcal{M}$ this is equivalent, as shown originally by Geroch[44, 45] that $P_{\mathrm{SO}_+(1,3)}(\mathcal{M})$ is a trivial bundle, i.e., that it admits a global section. When $P_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ exists we said that $\mathcal{M}$ is a spin manifold.

Definition. *A real spinor bundle for $\mathcal{M}$ is the vector bundle*

$$S(\mathcal{M}) = P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \times_\mu \mathbf{M} \tag{57}$$

where $\mathbf{M}$ is a left (right) module for $\mathcal{C}\ell_{1,3}$ and where $\mu : P_{\mathbf{Spin}_+(1,3)} \to \mathrm{SO}_+(1,3)$ is a representation given by left (right) multiplication by elements of $\mathbf{Spin}_+(1,3)$.

Definition. *A complex spinor bundle for $\mathcal{M}$ is the vector bundle*

$$S_c(\mathcal{M}) = P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \times_{\mu_c} \mathbf{M}_c \tag{58}$$

where $\mathbf{M}$ is a complex left (right) module for $\mathbb{C} \otimes \mathcal{C}\ell_{1,3} \simeq \mathcal{C}\ell_{4,1} \simeq M_4(\mathbb{C})$, and where $\mu_c : P_{\mathbf{Spin}_+(1,3)} \to \mathrm{SO}_+(1,3)$ is a representation given by left (right) multiplication by elements of $\mathbf{Spin}_+(1,3)$.

Taking, e.g. $\mathbf{M}_c = \mathbb{C}^4$ and μ_c the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\mathbf{Spin}_+(1,3)$ in $\mathrm{End}(\mathbb{C}^4)$, we recognize immediately the usual definition of the covariant spinor bundle of $\mathcal{M}$, as given, e.g., in [33].

Since, besides being right (left) linear spaces over $\mathbb{H}$, the left (right) ideals of $\mathcal{C}\ell_{1,3}$ are representation modules of $\mathcal{C}\ell_{1,3}$, we have the

Definition. *$I(\mathcal{M})$ is a real spinor bundle for $\mathcal{M}$ such that $\mathbf{M}$ in Eq. 57 is I, a minimal left (right) ideal of $\mathcal{C}\ell_{1,3}$.*

In what follows we fix the ideal taking $I = \mathcal{C}\ell_{1,3}\frac{1}{2}(1 + E_0) = \mathcal{C}\ell_{1,3}e$. If $\pi_I : I(\mathcal{M}) \to M$ is the canonical projection and $\{U_\alpha\}$ is an open covering of M we know from the definition of a fibre bundle that there is a trivializing mapping $\chi_\alpha(q) = (\pi_I(q), \overset{\Delta}{\chi}_\alpha(q))$. If $U_{\alpha\beta} = U_\alpha \cap U_\beta$ and $x \in U_{\alpha\beta}$, $q \in \pi_I^{-1}(U_\alpha)$, then

$$\overset{\Delta}{\chi}_\alpha(q) = g_{\alpha\beta}(x)\,\overset{\Delta}{\chi}_\beta(q) \tag{59}$$

for the transition maps in $\mathbf{Spin}_+(1,3)$.[5] Equivalently

$$\overset{\Delta}{\chi}_\alpha(q) = \overset{\Delta}{\chi}_\beta(a_{\alpha\beta}q) \tag{60}$$

for some $a_{\alpha\beta} \in \mathcal{C}\ell(T_x^*\mathcal{M})$. Thus, for the transition maps to be in $\mathbf{Spin}_+(1,3)$ it is equivalent that the right action of $\mathbb{H}e = e\mathbb{H} = e\mathcal{C}\ell_{1,3}e$ be the defined in the bundle, since for $q \in \pi_x^{-1}(x), x \in U_\alpha$ and $a \in \mathbb{H}$ we define qa as the unique element of $\pi_q^{-1}(x)$ such that

$$\overset{\Delta}{\chi}_\alpha(qa) = \overset{\Delta}{\chi}_\alpha(q)a \tag{61}$$

Naturally, for the validity of Eq. 61 to make sense it is necessary that

$$g_{\alpha\beta}(x)(\overset{\Delta}{\chi}_\alpha(q)a) = (g_{\alpha\beta}(x)\,\overset{\Delta}{\chi}_\alpha(q))a \tag{62}$$

and Eq. 62 implies that the transition maps are $\mathbb{H}$-linear.[6]

Let $f_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Aut}(\mathcal{C}\ell_{1,3})$ be the transition functions for $\mathcal{C}\ell(\mathcal{M})$. On the intersection $U_\alpha \cap U_\beta \cap U_\alpha$ it must hold

$$f_{\alpha\beta}f_{\beta\gamma} = f_{\alpha\gamma} \tag{63}$$

We say that a set of *lifts* of the transition functions of $\mathcal{C}\ell(\mathcal{M})$ is a set of elements in $\mathcal{C}\ell_{1,3}^\star$, $\{g_{\alpha\beta}\}$ such that if

$$\begin{aligned} &\mathrm{Ad} : \mathcal{C}\ell_{1,3}^\star \to \mathrm{Aut}(\mathcal{C}\ell_{1,3}) \\ &\mathrm{Ad}(u)X = uXu^{-1}\ ,\ \forall X \in \mathcal{C}\ell_{1,3} \end{aligned}$$

then $\mathrm{Ad}_{g_{\alpha\beta}} = f_{\alpha\beta}$ in all intersections.

Using the theory of the Cěch cohomology[12] it can be shown that any set of lifts can be used to define a characteristic class $\omega(\mathcal{C}\ell(\mathcal{M})) \in \check{H}^2(M, \mathbb{H}^*)$, the second Cěch cohomology group with values in $\mathbb{H}^*$, the space of all non zero $\mathbb{H}$-valued germs of functions in M.

[5]We start with transition maps in $\mathcal{C}\ell_{p,q}^\star$ and then by the bundle reduction process we end with $\mathbf{Spin}_+(1,3)$.

[6]Without the $\mathbb{H}$-linear structure there exists more general bundles of irreducible modules for $\mathcal{C}\ell(\mathcal{M})$.[12]

We say that we can coherently lift the transition maps $\mathcal{C}(\mathcal{M})$ to a set $\{g_{\alpha\beta}\} \in \mathcal{C}\ell^{\star}_{1,3}$ if in the intersection $U_\alpha \cap U_\beta \cap U_\gamma$, $\forall\alpha, \beta, \gamma$, we have

$$g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma} \tag{64}$$

This implies that $\omega(\mathcal{C}\ell(\mathcal{M})) = \mathrm{id}_{(2)}$, i.e., M is Cĕch trivial and the coherent lifts can be classified by an element of the first Cĕch cohomology group $\check{H}^1(M, \mathbb{H}^*)$. Benn and Turcker[12] proved the important result:

Theorem. *There exists a bundle of irreducible representation modules for $\mathcal{C}\ell(\mathcal{M})$ if and only if the transition maps of $\mathcal{C}\ell(\mathcal{M})$ can be coherently lift from* $\mathrm{Aut}(\mathcal{C}\ell_{1,3})$ *to $\mathcal{C}\ell^{\star}_{1,3}$.*

They showed also by defining the concept of equivalence classes of coherent lifts that such classes are in one to one correspondence with the equivalence classes of bundles of irreducible representation modules of $\mathcal{C}\ell(\mathcal{M})$, $I(\mathcal{M})$ and $I'(\mathcal{M})$ being equivalent if there is a bundle isomorphsim $\rho : I(\mathcal{M}) \to I'(\mathcal{M})$ such that

$$\rho(a_x q) = a_x \rho(q), \quad \forall a_x \in \mathcal{C}\ell(T^*_x M), \forall q \in \pi_I^{-1}(x)$$

By defining that a *spin structure* for M is an equivalence class of bundles of irreducible representation modules for $\mathcal{C}\ell(\mathcal{M})$, represented by $I(\mathcal{M})$, Benn and Turcker showed that this agrees with the usual conditions for M to be a spin manifold.

Now, recalling the definition of a vector bundle we see that the prescription for the construction of $I(\mathcal{M})$ is the following. Let $\{U_\alpha\}$ be an open covering of M with $f_{\alpha\beta}$ being the transition functions for $\mathcal{C}\ell(\mathcal{M})$ and let $\{g_{\alpha\beta}\}$ be a coherent lift which is then used to quotient the set $\cup_\alpha U_\alpha \times I$, where e.g., $I = \mathcal{C}\ell_{1,3}\frac{1}{2}(1 + E_0)$ to form the bundle $\cup_\alpha U_\alpha \times I/\mathcal{R}$ where $\mathcal{R}$ is the equivalence relation defined as follows. For each $x \in U_\alpha$ we choose a minimal left ideal $I^\alpha_{\Sigma(x)}$ in $\mathcal{C}\ell(T^*_x M)$ by requiring[7]

$$\overset{\triangle}{\varphi}_\alpha (I^\alpha_{\Sigma(x)}) = I \tag{65}$$

As before we introduce $a_{\alpha\beta} \in \mathcal{C}\ell(T^*_x M)$ such that

$$\overset{\triangle}{\varphi}_\beta (a_{\alpha\beta}) = g_{\alpha\beta}(x) \tag{66}$$

Then for all $X \in \mathcal{C}\ell(T^*_x \mathcal{M}), \overset{\triangle}{\varphi}_\alpha (X) = \overset{\triangle}{\varphi}_\beta (a_{\alpha\beta} X a^{-1}_{\alpha\beta})$. So, if $X \in I^\alpha_{\Sigma(x)}$ then $a_{\alpha\beta} X a^{-1}_{\alpha\beta}$ and also $X a^{-1}_{\alpha\beta} \in I^\beta_{\dot{\Sigma}(x)}$. Putting $Y_\alpha = U_\alpha \times I^\alpha_{\Sigma(x)}$ $Y = \cup_\alpha Y_\alpha$, the equivalence relation $\mathcal{R}$ is defined on Y by $(U_\alpha, x, \psi_\Sigma) \simeq (U_\beta, x, \psi_{\dot{\Sigma}})$ if and only if

$$\psi_{\dot{\Sigma}} = \psi_\Sigma a^{-1}_{\alpha\beta} \tag{67}$$

[7]Recall the notation of Section A.1 where Σ is an orthonormal frame, etc.

Then, $I(\mathcal{M}) = Y/\mathcal{R}$ is a bundle which is an irreducible module representation of $\mathcal{C}(\mathcal{M})$. We see that Eq. 67 captures nicely for $a_{\alpha\beta} \in \mathbf{Spin}_+(1,3) \subset \mathcal{C}\ell^\star_{1,3}$ our discussion of ADS of Section A.1. We then have

Definition. *An algebraic Dirac Spinor Field (ADSF) is a section of $I(\mathcal{M})$ with $a_{\alpha\beta} \in \mathbf{Spin}_+(1,3) \subset \mathcal{C}\ell^\star_{1,3}$ in Eq. 67.*

From the above results we see that ADSF are equivalence classes of sections of $\mathcal{C}\ell(\mathcal{M})$ and it follows that ADSF can locally be represented by a sum of inhomogeneous differential forms that lie in a minimal left ideal of the Clifford algebra $\mathcal{C}\ell_{1,3}$ at each spacetime point.

In Section A.1 we saw that besides the ideal $I = \mathcal{C}\ell_{1,3}\frac{1}{2}(1+E_0)$, other ideals exist for $\mathcal{C}\ell_{1,3}$ that are only algebraically equivalent to this one. In order to capture all possibilities we recall that $\mathcal{C}\ell_{1,3}$ can be considered as a module over itself by left (or right) multiplication by itself. We are thus lead to the

Definition. *The Real Spin-Clifford bundle of $\mathcal{M}$ is the vector bundle*

$$\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) = P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \times_\ell \mathcal{C}\ell_{1,3} \tag{68}$$

It is a "principal $\mathcal{C}\ell_{1,3}$ bundle", i.e., it admits a free action of $\mathcal{C}\ell_{1,3}$ on the right.[25, 42] There is a natural embedding $P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \subset \mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ which comes from the embedding $\mathbf{Spin}_+(1,3) \subset \mathcal{C}\ell^+_{1,3}$. Hence every real spinor bundle for $\mathcal{M}$ can be captured from $\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$. $\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ is different from $\mathcal{C}\ell(\mathcal{M})$. Their relation can be discovered remembering that the representation

$$\mathrm{Ad} : \mathbf{Spin}_+(1,3) \to \mathrm{Aut}(\mathcal{C}\ell_{1,3}) \quad \mathrm{Ad}_u X = uXu^{-1} \qquad u \in \mathbf{Spin}_+(1,3)$$

is such that Ad_{-1} = identity and so Ad descends to a representation Ad' of $\mathrm{SO}_+(1,3)$ which we considered above. It follows that when $P_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ exists

$$\mathcal{C}\ell(\mathcal{M}) = P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \times_{\mathrm{Ad}'} \mathcal{C}\ell_{1,3} \tag{69}$$

From this it is easy to prove that indeed $S(\mathcal{M})$ is a bundle of modules over the bundle of algebras $\mathcal{C}\ell(\mathcal{M})$.[11]

We end this section defining the local Clifford product of $X \in \sec \mathcal{C}\ell(\mathcal{M})$ by a section of $I(\mathcal{M})$ or $\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$. If $\varphi \in I(\mathcal{M})$ we put $X\varphi = \phi \in \sec I(\mathcal{M})$ and the meaning of Eq. 69 is that

$$\phi(x) = X(x)\rho(x) \qquad \forall x \in M \tag{70}$$

where $X(x)\varphi(x)$ is the Clifford product of the Clifford numbers $X(x), \varphi(x) \in \mathcal{C}\ell_{1,3}$.

Analogously if $\psi \in \mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$

$$X\psi = \xi \in \mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \tag{71}$$

and the meaning of Eq. 70 is the same as in Eq. 69.

With the above definition we can "identify" from the algebraically point of view sections of $\mathcal{C}\ell(\mathcal{M})$ with sections of $I(\mathcal{M})$ or $\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$.

A.2.3 Dirac-Hestenes Spinor Fields (DHSF)

The main conclusion of Section A.2.2 is that a given ADSF which is a section of $I(\mathcal{M})$ can locally be represented by a sum of inhomogeneous differential forms in $\mathcal{C}\ell(\mathcal{M})$ that lies in a minimal left ideal of the Clifford algebra $\mathcal{C}\ell_{1,3}$ at each point $x \in M$. Our objective here is to define a DHSF on $\mathcal{M}$. In order to achieve our goal we need to find a vector bundle such that a DHSF is an appropriate section.

In Section A.1.7 we defined a DHS as an element of the quotient set $\mathcal{C}\ell^+_{1,3}/\mathcal{R}$ where $\mathcal{R}$ is the equivalence relation given by Eq. 50. We immediately realize that if it is possible to define globally on M the equivalence relation $\mathcal{R}$, then a DHSF can be defined as an even section of the quotient bundle $\mathcal{C}\ell(\mathcal{M})/\mathcal{R}$.

More precisely, if $\Sigma = \{\gamma^a\}$, $(a = 0, 1, 2, 3)$ and $\dot{\Sigma} = \{\dot{\gamma}^a\}$, $\gamma^a, \dot{\gamma}^a \in \sec \bigwedge^1(T^*M) \subset \mathcal{C}\ell(M)$ are such that $\dot{\gamma}^a = R\gamma^a R^{-1}$, where $R \in \sec \mathcal{C}\ell^+(\mathcal{M})$ is such that $R(x) \in \mathbf{Spin}_+(1,3)$ for all $x \in M$, we say that $\dot{\Sigma} \sim \Sigma$. Then a DHSF is an equivalence class of even sections of $\mathcal{C}\ell(\mathcal{M})$ such that its representatives ψ_Σ and $\psi_{\dot{\Sigma}}$ in the basis Σ and $\dot{\Sigma}$ define a set of 1-form fields $X^a \in \sec \Lambda^1(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M})$ by

$$X^a(x) = \psi_{\dot{\Sigma}}(x)\dot{\gamma}^a(x)\tilde{\psi}_{\dot{\Sigma}}(x) = \psi_\Sigma(x)\gamma^a(x)\tilde{\psi}_\Sigma(x) \tag{72}$$

i.e., ψ_Σ and $\psi_{\dot{\Sigma}}$ are equivalent if and only if

$$\psi_{\dot{\Sigma}} = \psi_\Sigma R^{-1}. \tag{73}$$

Observe that for $\dot{\Sigma} \sim \Sigma$ to be globally defined it is necessary that the 1-form fields $\{\gamma^a\}$ and $\{\dot{\gamma}^a\}$ are globally defined. It follows that $P_{\mathrm{SO}_+(1,3)}(\mathcal{M})$, the principal bundle of orthonormal frames must have a global section, i.e., it must be trivial. This conclusion follows directly from our definitions, and it is a necessary condition for the existence of a DHSF. It is obvious that the condition is also sufficient. This suggests the

Definition. *A spacetime $\mathcal{M}$ admits a spinor structure if and only if it is possible to define a global DHSF on it.*

Then, it follows the

Theorem. *Let $\mathcal{M}$ be a spacetime* ($\dim M = 4$). *Then the necessary and sufficient condition for M to admit a spinor structure is that $P_{\mathrm{SO}_+(1,3)}(\mathcal{M})$ admits a global section.*

In Section A.2.1 we defined the spinor structure as the principal bundle $P_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ and a theorem with the same statement as the above one is known in the literature as Geroch's Theorem.[44] Geroch's deals with the existence of covariant spinor fields on $\mathcal{M}$, but since we already proved, e.g., that covariant Dirac spinors are equivalent to DHS, our theorem and Geroch's one are equivalent. This can be seen more clearly once we verify that

$$\frac{\mathcal{C}\ell(\mathcal{M})}{\mathcal{R}} \equiv \mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \tag{74}$$

where $\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) = P_{\mathbf{Spin}_+(1,3)} \times_\ell \mathcal{C}\ell_{1,3}$ is the Spin-Clifford bundle defined in Section A.2.1. To see this, recall that a DHSF determines through Eq. 70 a set of 1-form fields $X^a \in \sec \bigwedge^1(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M})$. Under an active transformation,

$$X^a \mapsto \dot{X}^a = RX^aR^{-1}, \quad R(x) \in \mathbf{Spin}_+(1,3), \qquad \forall x \in M \tag{75}$$

we obtain the active transformation of a DHSF which in the Σ-frame is given by[8]

$$\psi_\Sigma \mapsto \psi'_\Sigma = R\psi_\Sigma \tag{76}$$

From Eq. 73 it follows that the action of $\mathbf{Spin}_+(1,3)$ on the typical fibre $\mathcal{C}\ell_{1,3}$ of $\mathcal{C}\ell(\mathcal{M})/\mathcal{R}$ must be through left multiplication, i.e. given $u \in \mathbf{Spin}_+(1,3)$ and $X \in \mathcal{C}\ell_{1,3}$, and taking into account that $\mathcal{C}\ell_{1,3}$ is a module over itself we can define $\ell_u \in \mathrm{End}(\mathcal{C}\ell_{1,3})$ by $\ell_u(X) = ux, \forall X \in \mathcal{C}\ell_{1,3}$. In this way we have a representation $\ell : \mathbf{Spin}_+(1,3) \to \mathrm{End}(\mathcal{C}\ell_{1,3}), u \mapsto \ell_u$. Then we can write,

$$\frac{\mathcal{C}\ell(\mathcal{M})}{\mathcal{R}} = P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \times_\ell \mathcal{C}\ell_{1,3}$$

PART B

B.1 Maxwell and Dirac equations in the Clifford bundle

Let $\mathcal{M} = \langle M, g, \nabla \rangle$ be a Riemann-Cartan manifold and let $\mathcal{C}\ell(\mathcal{M}), I(\mathcal{M})$ and $\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ be respectively the Clifford, Real Spinor and Spin Clifford bundles. Let ∇^s be the spinorial connection acting on sections of $I(\mathcal{M})$ or $\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$.

[8] Observe also that in the $\dot{\Sigma}$ we have for the representative of the actively transformed DHSF the relation $\psi'_{\dot{\Sigma}} = R\psi_\Sigma R^{-1}$

Let $<x^\mu>$ be a coordinate chart for $U \subset M, e_a = h_a^\mu \partial_\mu, a = 0,1,2,3$ an orthonormal basis for $TU \subset TM$.[9] Let $\gamma^a \in \sec(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M})$ be the dual basis of $\{e_a\} \equiv \mathcal{B}$. Let $\Sigma = \{\gamma^a\}$ and $\{\gamma_a, a = 0,1,2,3\}$ the reciprocal basis of $\{\gamma^a\}$, i.e., $\gamma^a \cdot \gamma_b = \delta_b^a$ where is the internal product in $\mathcal{C}\ell_{1,3}$. We have $\gamma^a = h_\mu^a dx^\mu$, $\gamma_a = h_a^\mu \eta_{\mu\alpha} dx^\alpha$ When convenient we shall denote the Pfaff derivative [41] by $\partial_a = e_a$.

Definition. *The Dirac operator acting on sections of $\mathcal{C}\ell(\mathcal{M})$ is a canonical first order differential operator $\boldsymbol{\partial} : A \mapsto \boldsymbol{\partial} A$, $A \in \sec \mathcal{C}\ell(\mathcal{M})$, such that*

$$\boldsymbol{\partial} A = (\gamma^a \nabla_{e_a}) A = \gamma^a \cdot (\nabla_{e_a} A) + \gamma^a \wedge (\nabla_{e_a} A) \tag{77}$$

Definition. *The Spin-Dirac operator[10] acting on sections of $I(\mathcal{M})$ or $\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ is a canonical first order differential operator $\mathbf{D} : \Gamma \to \mathbf{D}\Gamma$ ($\Gamma \in \sec\ I(\mathcal{M})$) [or $\Gamma \in \sec \mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$] such that*

$$\begin{aligned} \mathbf{D}\Gamma &= (\gamma^a \nabla^s_{e_a})\Gamma \\ &= \gamma^a \cdot (\nabla^s_{e_a}\Gamma) + \gamma^a \wedge (\nabla^s_{e_a}\Gamma) \end{aligned} \tag{78}$$

where ∇^s is the spinorial connection. The operator $\boldsymbol{\partial}$ is sometimes called the Dirac-Kahler operator when $\mathcal{M}$ is a Lorentzian manifold,[13] i.e., $\mathbf{T}(\nabla) = 0$, $\mathbf{R}(\nabla) = 0$, where $\mathbf{T}$ and $\mathbf{R}$ are respectively the torsion and Riemann tensors. In this case we can show that

$$\boldsymbol{\partial} = d - \delta \tag{79}$$

where d is the differential operator and δ the Hodge codifferential operator. We use the convention that the representative of $\mathbf{D}$ (acting on sections of $\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$) in $\mathcal{C}\ell(\mathcal{M})$ will be also denote by

$$\mathbf{D} = \gamma^a \nabla^s_{e_a} \tag{80}$$

Let F be the Faraday 2-form, that is:

$$F = \frac{1}{2} F_{\mu\nu} \gamma^\mu \wedge \gamma^\nu = \frac{1}{2} F_{\mu\nu} \gamma^\mu \gamma^\nu \ \ (\mu \neq \nu). \tag{81}$$

Maxwell equations are

$$dF = 0, \quad \delta F = -\mathcal{J}, \tag{82}$$

[9] Since M is a spin manifold, $P_{SO_+(1,3)}(\mathcal{M})$ is trivial and $\{e_a\}$, $a = 0,1,2,3$ can be taken as a global tetrad field for the tangent bundle.

[10] In [42] this operator (acting on sections of $I(\mathcal{M})$) is called simply Dirac operator, being the generalization of the operator originally introduced by Dirac. See also [11] for comments on the use of this terminology.

where $\mathcal{J}$ is the current density 1-form. Using the Dirac-Kähler operator Maxwell equations are written as a single equation, namely:

$$\partial F = \mathcal{J} \tag{83}$$

Now let $\Psi \in \sec P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \times_\rho \mathbb{C}^4$ (with ρ the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\mathrm{SL}(2, \mathbb{C}) \simeq \mathbf{Spin}_+(1,3)$). Then, the Dirac equation for the charged fermion field Ψ in interaction with the electromagnetic field A is[32] ($\hbar = c = 1$)

$$\underline{\gamma}^\mu (i\partial_\mu - eA_\mu)\Psi = m\Psi \text{ or } i\mathbf{D}\psi - \underline{\gamma}^\mu A_\mu \Psi = m\Psi \tag{84}$$

where $\underline{\gamma}^\mu \underline{\gamma}^\nu + \underline{\gamma}^\nu \underline{\gamma}^\mu = 2\eta^{\mu\nu}, \underline{\gamma}^\mu$ being the Dirac matrices and

$$A = A_\mu dx^\mu \in \sec \bigwedge^1 (T^*M).$$

As showed, e.g., in [25] this equation is equivalent to the following equation satisfied by $\phi \in \sec I(\mathcal{M})$ $[\phi e_\Sigma = \phi, e_\Sigma = \frac{1}{2}(1 + \gamma^0), \boldsymbol{\gamma}^\mu \boldsymbol{\gamma}^\nu + \boldsymbol{\gamma}^\nu \boldsymbol{\gamma}^\mu = 2\eta^{\mu\nu}, \boldsymbol{\gamma}^\mu \in \sec \mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})]$,

$$\mathbf{D}\phi\boldsymbol{\gamma}^2\gamma^1 - eA\phi = m\phi, \tag{85}$$

where $\mathbf{D}$ is the Dirac operator on $I(\mathcal{M})$ and $A \in \sec \bigwedge^1(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M})$.

Since each ϕ is an equivalence class of sections of $\mathcal{C}\ell(\mathcal{M})$ we can also write an equation equivalent to Eq. 85 for $\phi_\Sigma = \phi_\Sigma e_\Sigma$, $\phi_\Sigma, e_\Sigma \in \sec \mathcal{C}\ell(\mathcal{M})$, $e_\Sigma = \frac{1}{2}(1 + \gamma^0), \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}, \gamma^\mu \in \sec \mathcal{C}\ell(\mathcal{M})$, and $\gamma^\mu = dx^\mu$ for the global coordinate functions $\langle x^\mu \rangle$. In this case the Dirac operator $\boldsymbol{\partial} = \gamma^\mu \nabla_\mu$ is $\partial = \gamma^\mu \partial_\mu$ and we have

$$\partial \phi_\Sigma \gamma^2 \gamma^1 - eA\phi_\Sigma = m\phi_\Sigma \gamma^0 \tag{86}$$

Since each ϕ_Σ can be written $\phi_\Sigma = \psi_\Sigma e_\Sigma$, ($\psi_\Sigma \in \sec\ \mathcal{C}\ell^+(\mathcal{M})$ being the representative of a DHSF) and $\gamma^0 e_\Sigma = e_\Sigma$, we can write the following equation for ψ_Σ that is equivalent to Dirac equation[25, 27, 28]

$$\partial \psi_\Sigma \gamma^2 \gamma^1 - eA\psi_\Sigma = m\psi_\Sigma \gamma^0 \tag{87}$$

which is the so called Dirac-Hestenes equation.[46, 47]

Eq.87 is covariant under passive (and active) Lorentz transformations, in the following sense: consider the change from the Lorentz frame $\Sigma = \{\gamma^\mu = dx^\mu\}$ to the frame $\dot{\Sigma} = \{\dot{\gamma}^\mu = d\dot{x}^\mu\}$ with $\dot{\gamma}^\mu = R^{-1}\gamma^\mu R$ and $R \in \mathbf{Spin}_+(1,3)$ being constant. Then the representative of the Dirac-Hestenes spinor changes from ψ_Σ to $\psi_{\dot{\Sigma}} = \psi_\Sigma R^{-1}$. Then we have $\partial = \gamma^\mu \partial_\mu = \dot{\gamma}^\mu \partial / \partial \dot{x}^\mu$ where $\langle x^\mu \rangle$ and $\langle \dot{x}^\mu \rangle$ are related by a Lorentz transformation and

$$\partial \psi_\Sigma R^{-1} R \gamma^2 R^{-1} R \gamma^1 R^{-1} - eA\psi_\Sigma R^{-1} = m\psi_\Sigma R^{-1} R \gamma^0 R^{-1}, \tag{88}$$

i.e.,

$$\partial\psi_{\dot\Sigma}\dot\gamma^2\dot\gamma^1 - eA\psi_{\dot\Sigma} = m\psi_{\dot\Sigma}\dot\gamma^0 \tag{89}$$

Thus our definition of the Dirac-Hestenes spinor fields as an equivalence class of even sections of $\mathcal{C}\ell(\mathcal{M})$ solves directly the question raised by Parra[48] concerning the covariance of the Dirac-Hestenes equation.

B.2 Spinorial Representation of Maxwell Equations

The spinorial representation of Maxwell equations we shall give in this section is based on the following theorem:

Theorem. *Any electromagnetic field $F \in \bigwedge^2(\mathcal{C}\ell^{1,3})$ can be written in the form*

$$F = \psi\gamma^{21}\tilde{\psi}, \tag{90}$$

where $\psi = \psi_\Sigma$ is the representative of a Dirac-Hestenes spinor field in the basis $\Sigma = \{\gamma^\mu\}$.

The proof of this theorem can be divided in three steps: (i) $F^2 \neq 0$; (ii) $F^2 = 0$, $F \neq 0$; (ii) $F^2 = 0$, $F = 0$. In the first case the proof is based on a theorem by Rainich[49], and reconsidered by Misner and Wheeler[50].

Theorem (Rainich-Misner-Wheeler). *Let an extremal field be an electromagnetic field for which the electric [magnetic] field vanishes and the magnetic [electric] field is parallel to a given spatial direction. Then, at any point of spacetime, any non-null ($F^2 \neq 0$) electromagnetic field F can be transformed in an extremal field by means of a Lorentz transformation and a duality transformation.*

An easy proof of the theorem of Rainich-Misner-Wheeler can be found in [36]. Now consider a Lorentz transformation $F \mapsto F' = LF\tilde{L}$, and a duality transformation $F' \mapsto F'' = \mathrm{e}^{\gamma^5\alpha}F'$ ($\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$). According to the theorem of RMW the electromagnetic field F'' is extremal; let us suppose it of magnetic type along the $\vec{k}$ direction, that is: $F'' = \rho\gamma^{21}$, where ρ is the extremal field intensity. We have therefore that $\rho\gamma^{21} = \mathrm{e}^{\gamma^5\alpha}LF\tilde{L}$. Let us define $R = \tilde{L}$ and $\beta = -\alpha$; then we have that

$$F = \psi\gamma^{21}\tilde{\psi}, \tag{91}$$

where

$$\psi = \sqrt{\rho}\mathrm{e}^{\gamma^5\beta/2}R, \tag{92}$$

which we recognize as the canonical decomposition of Dirac-Hestenes spinor (eq.(49)).

In order to prove our theorem for case (ii) we observe that since $F^2 = 0$ we have $\vec{E}\cdot\vec{H} = 0$ and $\vec{E}^2 = \vec{H}^2$; we can make therefore a spatial rotation $\mathcal{R}$ such that

$E'_1 = H'_1 = 0$ and $H'_3 = \pm E'_2 = \eta_1$ and $H'_2 = \pm E'_3 = \eta_2$; then for $F' = (1/2)(F')^{\mu\nu}\gamma^{\mu\nu}$ we have $F' = (\eta_1 + \gamma^5\eta_2)(1/2)(1 \pm \gamma^{01})\gamma^{21}$. If we take $R = \tilde{\mathcal{R}}$ we have for $F = RF'\tilde{R}$ that $F = (\eta_1 + \gamma^5\eta_2)R(1/2)(1 \pm \gamma^{01})\gamma^{21}\tilde{R}$. Remember that $(1/2)(1 \pm \gamma^{01})$ is an idempotent; defining $\eta_1 = \eta\cos\varphi$ and $\eta_2 = \eta\sin\varphi$ it follows that

$$F = \psi_M \gamma^{21} \tilde{\psi}_M, \tag{93}$$

where

$$\psi_M = \sqrt{\eta} e^{\gamma^5\varphi/2} R \frac{1}{2}(1 \pm \gamma^{01}) = \psi \frac{1}{2}(1 \pm \gamma^{01}), \tag{94}$$

which proves our assertion in this case. ψ_M is a particular type of Dirac-Hestenes spinor known as Majorana spinor[27, 28].

Now for case (iii) $(F = 0)$ we note that $\psi\gamma^{21}\tilde{\psi} = -\psi\gamma^{21}\tilde{\psi} = \gamma^5\psi\gamma^{21}\tilde{\psi}\gamma^5 = \gamma^5\psi\gamma^{21}\gamma^{21}\gamma^{12}\tilde{\psi}\gamma^5$ is satisfied for $\psi = \pm\gamma^5\psi\gamma^{21}$ It follows therefore that

$$F = \psi_W \gamma^{21} \tilde{\psi}_W = 0, \tag{95}$$

where

$$\psi_W = \frac{1}{2}(\psi \pm \gamma^5 \psi \gamma^{21}). \tag{96}$$

This particular kind of Dirac-Hestenes spinor is called a Weyl spinor[27, 28]. We have now proved our theorem. In terms of Dirac spinors eq.(90) gives

$$F_{\mu\nu} = \langle \Psi | \frac{i}{2}[\gamma^\mu, \gamma^\nu] |\Psi\rangle, \tag{97}$$

where γ^μ are Dirac matrices, that is, the matrix representation of 1-forms γ^μ as in eq.(30), $|\Psi\rangle$ is the Dirac spinor and $\langle\Psi|$ its Dirac adjoint.

It remains, of course, the question of the constants in eq.(90) since the units of F are charge $\times$ (length)$^{-2}$ and the units of ψ are (length)$^{-3/2}$ We have now to introduce two postulates: firstly, we suppose that there is a natural unit F_0 of electromagnetic field intensity; secondly, we suppose that there is a natural unit e_0 of electric charge. In this way we have that

$$F = \left(\frac{e_0^3}{F_0}\right)^{\frac{1}{2}} \psi\gamma^{21}\tilde{\psi}. \tag{98}$$

Note that $(e_0/F_0)^{1/2}$ has unit of length. One of the above postulates can be replaced by the one that there is a natural unit L_0 of length, which gives $F_0 = e_0/L_0^2$ and

$$F = e_0 L_0 \psi \gamma^{21} \tilde{\gamma}. \tag{99}$$

In terms of some physical constants, a combination of constants we need can be

$$F = \frac{e\hbar}{2mc}\psi\gamma^{21}\tilde{\psi}, \tag{100}$$

which gives correct units for F and ψ. In this expression the symbols have their usual meaning, that is: e is the elementary electric charge, $\hbar$ is the Planck constant, m is the electron mass and c the velocity of light in vacuum. In what follows we will work, to simplify the notation, with eq.(90) instead of eq.(100).

The idea now is to use $F = \psi\gamma^{21}\tilde{\psi}$ in Maxwell equations and obtain from it an equivalent equation for ψ. Maxwell equations are

$$\partial F = \mathcal{J}, \tag{101}$$

If we use eq.(90) in Maxwell equation (101) we obtain

$$\begin{aligned} \partial(\psi\gamma^{21}\tilde{\psi}) &= \gamma^{\mu}\partial_{\mu}(\psi\gamma^{21}\tilde{\psi}) = \\ &= \gamma^{\mu}(\partial_{\mu}\psi\gamma^{21}\tilde{\psi} + \psi\gamma^{21}\partial_{\mu}\tilde{\psi}) = \mathcal{J}. \end{aligned} \tag{102}$$

But $\psi\gamma^{21}\partial_{\mu}\tilde{\psi} = -(\partial_{\mu}\psi\gamma^{21}\tilde{\psi})\tilde{}$, and since reversion does not change the sign of scalars and of pseudo-scalars (4-forms), we have that

$$2\gamma^{\mu}\langle\partial_{\mu}\psi\gamma^{21}\tilde{\psi}\rangle_2 = \mathcal{J}. \tag{103}$$

There is a more convenient way of rewriting the above equation. Note that

$$\begin{aligned} &\gamma^{\mu}\langle\partial_{\mu}\psi\gamma^{21}\tilde{\psi}\rangle_2 = \\ &= \partial\psi\gamma^{21}\tilde{\psi} - \gamma^{\mu}\langle\partial_{\mu}\psi\gamma^{21}\tilde{\psi}\rangle_0 - \gamma^{\mu}\langle\partial_{\mu}\psi\gamma^{21}\tilde{\psi}\rangle_4, \end{aligned} \tag{104}$$

and if we define the 1-forms

$$j = \gamma^{\mu}\langle\partial_{\mu}\psi\gamma^{21}\tilde{\psi}\rangle_0, \tag{105}$$

$$g = \gamma^{\mu}\langle\partial_{\mu}\psi\gamma^{5}\gamma^{21}\tilde{\psi}\rangle_0, \tag{106}$$

we can rewrite eq.(103) as

$$\partial\psi\gamma^{21}\tilde{\psi} = \left[\frac{1}{2}\mathcal{J} + \left(j + \gamma^5 g\right)\right]. \tag{107}$$

If correct units have been used, that is, if we used eq.(100), then instead of $(1/2)\mathcal{J}$ we would obtained $(mc/e\hbar)\mathcal{J}$. Eq.(107) is the spinorial representation of Maxwell equations we were looking for. In the case where ψ is non-singular (which corresponds to non-null electromagnetic fields) we have

$$\partial\psi\gamma^{21} = \frac{e^{\gamma^5\beta}}{\rho}\left[\frac{1}{2}\mathcal{J} + \left(j + \gamma^5 g\right)\right]\psi. \tag{108}$$

Eq.(108) has been proved [36] to be equivalent to the spinorial representation of Maxwell equations obtained originally by Campolattaro[51, 52] in terms of the usual covariant Dirac spinor.

The spinorial eq.(108) that represents Maxwell ones, as written in that form, does not appear to have any relationship with Dirac equation (87). However, we shall make some modifications on it in such a way to put it in a form that suggests a very interesting and intriguing relationship between them.

Since ψ is supposed to be non-singular (F non-null) we can use the canonical decomposition (92) of ψ and write

$$\partial_\mu\psi = \frac{1}{2}\left(\partial_\mu \ln\rho + \gamma^5\partial_\mu\beta + \Omega_\mu\right)\psi, \tag{109}$$

where we defined

$$\Omega_\mu = 2(\partial_\mu R)\tilde{R}. \tag{110}$$

Using this expression for $\partial_\mu\psi$ into the definitions of the 1-forms j and g (eqs.(105,106)) we obtain that

$$j = \gamma^\mu(\Omega_\mu \cdot S)\rho\cos\beta + \gamma^\mu[\Omega_\mu \cdot (\gamma^5 S)]\rho\sin\beta, \tag{111}$$

$$g = \gamma^\mu[(\Omega_\mu \cdot (\gamma^5 S)]\rho\cos\beta - \gamma^\mu(\Omega_\mu \cdot S)\rho\sin\beta, \tag{112}$$

where we defined the 2-form S by

$$S = \frac{1}{2}\psi\gamma^{21}\psi^{-1} = \frac{1}{2}R\gamma^{21}\tilde{R}. \tag{113}$$

A more convenient expression can be written. Let v be given by $\rho v = J = \psi\gamma^0\tilde{\psi}$, and $v_\mu = v\cdot\gamma^\mu$. Define the 2-form $\Omega = v^\mu\Omega_\mu$ and the scalars Λ and K by

$$\Lambda = \Omega \cdot S, \tag{114}$$

$$K = \Omega \cdot (\gamma^5 S). \tag{115}$$

Using these definitions we have that

$$\Omega_\mu \cdot S = \Lambda v_\mu, \tag{116}$$

$$\Omega_\mu \cdot (\gamma^5 S) = K v_\mu, \tag{117}$$

and for the 1-forms j and g:

$$j = \Lambda v \rho \cos\beta + K v \rho \sin\beta = \lambda \rho v, \tag{118}$$

$$g = K v \rho \cos\beta - \Lambda v \rho \sin\beta = \kappa \rho v, \tag{119}$$

where we defined

$$\lambda = \Lambda \cos\beta + K \sin\beta, \tag{120}$$

$$\kappa = K \cos\beta - \Lambda \sin\beta. \tag{121}$$

The spinorial representation of Maxwell equations are written now as

$$\partial\psi\gamma^{21} = \frac{e^{\gamma^5\beta}}{2\rho}\mathcal{J}\psi + \lambda\psi\gamma^0 + \gamma^5\kappa\psi\gamma^0 \tag{122}$$

If $\mathcal{J} = 0$ (free case) we have that

$$\partial\psi\gamma^{21} = \lambda\psi\gamma^0 + \gamma^5\kappa\psi\gamma^0, \tag{123}$$

which is very similar to Dirac equation.

In order to go a step further into the relationship between those equations, we remember that the electromagnetic field has six degrees of freedom, while a Dirac-Hestenes spinor field has eight degrees of freedom; we are free therefore to impose two constrains on ψ if it is to represent an electromagnetic field[52]. We choose these two "gauge conditions" as

$$\partial \cdot j = 0 \quad \text{and} \quad \partial \cdot g = 0. \tag{124}$$

Using eqs.(118,119) these two constrains become

$$\partial \cdot j = \rho\dot{\lambda} + \lambda\partial \cdot J = 0, \tag{125}$$

$$\partial \cdot g = \rho\dot{\kappa} + \kappa\partial \cdot J = 0, \tag{126}$$

where $J = \rho v$ and $\dot{\lambda} = (v \cdot \partial)\lambda$, $\dot{\kappa} = (v \cdot \partial)\kappa$. These conditions imply that

$$\kappa\dot{\lambda} = \lambda\dot{\kappa}, \tag{127}$$

which gives ($\lambda \neq 0$):

$$\frac{\kappa}{\lambda} = \text{const} = -\tan\beta_0, \tag{128}$$

or from eqs.(120,121):

$$\frac{K}{\Lambda} = \tan(\beta - \beta_0). \tag{129}$$

Now we observe that β is the angle of the duality rotation from F to $F' = e^{\gamma^5\beta}F$. If we perform another duality rotation by β_0 we have $F \mapsto e^{\gamma^5(\beta+\beta_0)}F$, and for the Yvon-Takabayasi angle $\beta \mapsto \beta + \beta_0$. If we work therefore with an electromagnetic field duality rotated by an additional angle β_0, the above relationship becomes

$$\frac{K}{\Lambda} = \tan\beta. \tag{130}$$

This is, of course, just a way to say that we can choose the constant β_0 in eq.(128) to be zero. Now, this expression gives

$$\lambda = \Lambda\cos\beta + \Lambda\tan\beta\sin\beta = \frac{\Lambda}{\cos\beta}, \tag{131}$$

$$\kappa = \Lambda\tan\beta\cos\beta - \Lambda\sin\beta = 0, \tag{132}$$

and the spinorial representation (123) of the free Maxwell equations becomes

$$\partial\psi\gamma^{21} = \lambda\psi\gamma^0. \tag{133}$$

Note that λ is such that

$$\rho\dot{\lambda} = -\lambda\partial\cdot J. \tag{134}$$

The current $J = \psi\gamma^0\tilde{\psi}$ is not conserved unless λ is constant. If we suppose also that

$$\partial\cdot J = 0 \tag{135}$$

we must have

$$\lambda = \text{const.} \tag{136}$$

Now, throughout these calculations we have assumed $\hbar = c = 1$. We observe that in eq.(133) λ has the units of (length)$^{-1}$, and if we introduce the constants $\hbar$ and c we have to introduce another constant with unit of mass. If we denote this constant by m such that

$$\lambda = \frac{mc}{\hbar}, \tag{137}$$

then eq.(133) assumes a form which is identical to Dirac equation:

$$\partial\psi\gamma^{21} = \frac{mc}{\hbar}\psi\gamma^0 \tag{138}$$

It is true that we didn't proved that eq.(138) is really Dirac equation since the constant m has to be identified in this case with the electron's mass. However, we shall make some remarks concerning this identification which are very interesting and intriguing. First, if in analogy to eq.(137) we write

$$\Lambda = \frac{Mc}{\hbar}, \tag{139}$$

then eq.(131) reads

$$m = \frac{M}{\cos\beta}, \tag{140}$$

or

$$M = m\cos\beta = \frac{m}{\sqrt{1+\omega^2/\sigma^2}}, \tag{141}$$

where σ and ω are the invariants of Dirac theory. If m is constant, the above expression defines a variable mass M.

On the other hand, de Broglie introduced in his interpretation of quantum mechanics a variable mass M related to the constant one m by de Broglie-Vigier formula[53] $M = m\sqrt{1+\omega^2/\sigma^2}$, which is very similar the our one. However, the difference in these formulas are unimportant for the free case where we have for the plane wave solutions that[54] $\cos\beta = \pm 1$. Another interesting fact comes from eq.(114). If we write $\psi = \psi_0 \exp(\gamma^{21}\omega t)$, where ψ_0 is a constant spinor (which is the case again for plane waves), then eq.(114) gives $\Lambda = \omega$, or, introducing the constants $\hbar$ and c:

$$Mc^2 = \hbar\omega. \tag{142}$$

The variable mass M now appears to be related to energy of some "internal" vibration, and eq.(142) is just another formula of de Broglie[55], who suggested that mass is related to the frequency of an internal clock supposed associated to a particle.

We can better understand the meaning of eq.(114) after some additional manipulations. We have

$$\Lambda = \Omega \cdot S = v^\mu \langle \Omega_\mu \frac{1}{2}\psi\gamma^{21}\psi^{-1}\rangle_0, \tag{143}$$

and using eq.(109):

$$\begin{aligned}\Lambda = {} & v^\mu \langle \partial_\mu \gamma^{21}\psi^{-1}\rangle_0 - \\ & -v^\mu(\partial_\mu \ln\rho)\langle\psi\gamma^{21}\psi^{-1}\rangle_0 - v^\mu\partial_\mu\beta\langle\gamma^5\psi\gamma^{21}\psi^{-1}\rangle_0.\end{aligned} \tag{144}$$

The second and third terms on the RHS vanish because they are 2-forms; then, using $\psi^{-1} = \tilde{\psi}(\psi\tilde{\psi})^{-1}$,

$$\Lambda = v^\mu\langle\partial_\mu\psi\gamma^{21}\gamma^0\tilde{\psi}(\psi\tilde{\psi})^{-1}\psi\gamma^0\tilde{\psi}(\psi\tilde{\psi})^{-1}\rangle_0 =$$

$$= v^\mu \langle \partial_\mu \psi \gamma^{21} \gamma^0 \tilde{\psi} \frac{e^{-\gamma^5 \beta}}{\rho} \rho v \frac{e^{-\gamma^5 \beta}}{\rho} \rangle_0 =$$
$$= v^\mu \tfrac{1}{\rho} v^\nu \langle \partial_\mu \psi \gamma^{21} \gamma^0 \tilde{\psi} \gamma^\nu \rangle_0. \tag{145}$$

But one can show[54] that the energy-momentum tensor in Dirac theory (Tetrode tensor) in terms of Dirac-Hestenes spinor is given by

$$T_{\mu\nu} = \langle \partial_\mu \psi \gamma^{21} \gamma^0 \tilde{\psi} \gamma^\nu \rangle_0. \tag{146}$$

Since

$$v^\nu T_{\mu\nu} = \rho p_\mu \tag{147}$$

it follows, with correct units, the well-known equation

$$Mc = v \cdot p. \tag{148}$$

We remember that for plane waves $\cos\beta = \pm 1$, which implies from eq.(141) that $p = \pm mcv$, with the plus sign corresponding to the positive energy solution and the minus one to the negative energy solution, and which enable us to look as Feynman and Stueckelberg to electron and positron as particles with opposite momenta.

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