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Regular Functions of a Quaternionic Variable

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The first author dedicates his work to Luisa, and to Alessandro and Lorenzo.

The second author dedicates her work to Arturo, who shares her passion for mathematics, and to Lisa and Federico, her role models (and greatest supporters).

The third author dedicates his work to his princesses Arianna and Athena, and to Queen Lisa.

Preface

The theory of slice regular functions (originally Cullen-regular functions) was born at George Mason University, in Virginia, where the first and third authors collaborated for a 2-month period in the Fall of 2005. It was originated by the desire to find a new class of quaternionic regular functions that included polynomials and power series. The second author started working on this subject in the Summer of 2006, and her doctoral thesis eventually became the skeleton for this monograph. The theory of slice regular functions has rapidly developed, thanks to a series of visits at Chapman University, in California and to the interest of many mathematicians to whom we are greatly indebted.

We are very grateful to Fabrizio Colombo and Irene Sabadini, who immediately realized that this theory could be applied to create a successful quaternionic functional calculus. They also suggested the extension of these ideas to the case of Clifford algebras and their impulse has greatly contributed to the development of the theory.

We would like to thank Riccardo Ghiloni and Alessandro Perotti, who took an active interest in these developments and introduced a new viewpoint on the theory itself.

We warmly thank Cinzia Bisi, Alberto Damiano, Chiara Della Rocchetta, Giulia Sarfatti, Irene Vignozzi, and Fabio Vlacci for their interest in the subject and for their researches, which contributed to the expansion of the theory presented in this book.

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Special thanks go to Simon Salamon for his role in an unexpected application to the construction and classification of orthogonal complex structures in the quaternionic space.

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July 2012

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Introduction

This book presents a comprehensive treatment of a new theory of quaternionic functions, introduced in 2006 by two of the authors [61]. In this book we will develop this theory in a self-contained fashion, and we will show in which way it offers another important way to generalize the notion of holomorphy to the setting of quaternions.

Quaternions were introduced by Hamilton in 1843, adding a multiplicative structure to $\mathbb{R}^4$. In a modern notation, the vectors $(x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ are represented as

$$q = x_0 + x_1 i + x_2 j + x_3 k,$$

where $\{1, i, j, k\}$ denotes the standard basis. Multiplication is defined on the basis by

$$i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k, jk = -kj = i, ki = -ik = j,$$

and then extended by linearity and distributivity. This defines a (noncommutative) associative real algebra, usually denoted by $\mathbb{H}$. In this algebra, every nonzero element has a multiplicative inverse: defining the *conjugate* of an element $q = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$ as

$$\bar{q} = x_0 - x_1 i - x_2 j - x_3 k$$

and its *norm* as

$$|q| = \sqrt{q\bar{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2},$$

it is not difficult to prove that if $q \neq 0$ then

$$q^{-1} = |q|^{-2}\bar{q}.$$

Hence $\mathbb{H}$ is an associative division algebra (also called a skew field). Hamilton arrived to his construction after attempting (and failing) to build a similar structure on a three-dimensional vector space: only in 1877 Frobenius proved that $\mathbb{R}, \mathbb{C}, \mathbb{H}$ are the only finite dimensional, associative real algebras with division. The construction of $\mathbb{H}$ also shortly preceded the introduction of the modern vector notation, which was later ostracized by Hamilton's followers as a "perverted" version of the quaternionic structure. The birth of quaternions and their algebraic properties are well described in [46]. For general references in noncommutative algebra, see [17, 84, 85, 109].

Since the beginning of last century there have been many attempts to determine a class of quaternion-valued functions of one quaternionic variable playing the same role as the holomorphic functions of one complex variable. In the complex case, a function f is called holomorphic if it admits complex derivative, i.e., a complex number $f'(z)$ such that, for $h \in \mathbb{C}$,

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = f'(z).$$

This is also sometimes expressed by saying that f is complex differentiable at z, namely there exists a complex number $f'(z)$ such that

$$f(z+h) - f(z) = f'(z) \cdot h + o(h).$$

Equivalently f is termed holomorphic if it is complex analytic, i.e., it is represented in a neighborhood of any point of the domain of definition by its Taylor series; a third, equivalent definition calls holomorphic the solutions of the Cauchy–Riemann equation

$$\frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f = 0.$$

These concepts cease to be equivalent in the quaternionic context, and not only because of the lack of commutativity (which implies, for example, that both right and left quotients need to be defined). For instance, asking for a function $f : \mathbb{H} \to \mathbb{H}$ to be *quaternion-differentiable*, i.e., imposing that, for $h \in \mathbb{H}$,

$$\lim_{h \to 0} h^{-1}[f(q+h) - f(q)]$$

exists for all $q \in \mathbb{H}$, implies that f is an affine function of the form $f(q) = a + qb$ for some $a, b \in \mathbb{H}$. This is shown in detail, for example, in [96], and it indicates that the naive approach to quaternionic differentiability is inadequate to replicate the richness of the theory of holomorphic functions of a complex variable. The reader is referred again to [96] for a historical discussion and appropriate references. One may then wonder whether it may be possible to consider a less naive approach to differentiability for functions of quaternionic variables. To realize that this is indeed the case, one needs to observe that complex differentiability is inherently connected with the Cauchy–Riemann operator, and that, in turn, this directly connects with one-dimensional directional derivatives. So, one may suspect that the way to generalize complex differentiability to the quaternionic case may have to go through a suitable generalization of the Cauchy–Riemann system, in the hope that this generalization may give us a hint as to what kind of directional derivatives, so to speak, need to be involved. That this is the case is shown in detail in [96], as well as in [93–95, 99, 116] and in the references contained therein.

Let us thus turn our attention to the quaternionic analogs of the Cauchy–Riemann equations. The most successful of such analogs was developed in the 1920s through the efforts of various mathematicians such as Moisil, Teodorescu, and finally Fueter, to whose work most current literature refers: in [51], he defined a quaternionic function to be *regular* if it solves the equation

$$\frac{\partial f}{\partial \bar{q}} = \frac{1}{4}\left(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}\right) f \equiv 0.$$

It turns out that the operator defined by $\frac{\partial}{\partial \bar{q}}$ is a very good analog for the Cauchy–Riemann operator in the sense that the theory of its zero-solutions enjoys many of the key properties of the theory of holomorphic functions. A full theory of such functions, which includes for instance a Cauchy integral theorem, a Cauchy kernel, and suitable generalizations of the Taylor and Laurent series, is well established (for a first introduction, see [124]). We note, in particular, that pointwise product of two Fueter-regular functions is not Fueter-regular, but that a multiplication preserving regularity can be defined in terms of the Cauchy–Kowalewski extension (see [13, 111, 112, 117]). This theory also has significant applications to physics and engineering and it is the object of a vast literature. Without any pretense of completeness, we mention the monographs [76, 80, 83].

Moreover, and we refer again to [96], it turns out that it is possible to define a two-dimensional differential form with quaternionic coefficients

$$\sigma^{(2)} = i\,dx_2 \wedge dx_3 + j dx_3 \wedge dx_1 + k\,dx_1 \wedge dx_2$$

and a three-dimensional differential form with quaternionic coefficients, which is the analog, in this setting, of the one-dimensional differential form $dz = dx + i\,dy$, namely

$$\sigma^{(3)} = dx_1 \wedge dx_2 \wedge dx_3 - i\,dx_0 \wedge dx_2 \wedge dx_3 + j dx_0 \wedge dx_1 \wedge dx_3 - k\,dx_0 \wedge dx_1 \wedge dx_2.$$

It is through these forms that one can give an appropriate definition of hyperdifferentiability (we use this term, which is standard in the literature, to distinguish it from the expression quaternionic differentiability, which we have reserved for the more naive definition). Specifically, a quaternion-valued function of class C^1 on an open set in $\mathbb{H}$ is said to be *hyperdifferentiable* if there exists a quaternion, denoted $f'(q)$, such that

$$d(\sigma^{(2)} f) = \sigma^{(3)} \cdot f'(q).$$

Finally, just like in the complex case, the derivative can be expressed in terms of the conjugate of the Cauchy–Fueter operator (we refer the reader to [96] for details). The conclusion is that differentiability in the quaternionic setting requires, in order for it to be compatible with the Cauchy–Fueter operator, a quaternionic increment along, so to speak, a three-dimensional manifold. In other words, the generalization of the complex increment moving along a one-dimensional manifold (a curve) is achieved by realizing that the curve is actually a codimension one manifold.

Let us now turn to considering the other equivalent definition of holomorphy, namely *analyticity*. Once again, in a naive analogy with the complex case, one could require the existence at each q_0 of power series expansions

$$f(q) = \sum_{n\in\mathbb{N}} P_n(q - q_0)$$

where $P_n(q)$ is a finite sum of monomials of the type $a_0qa_1 \dots a_{n-1}qa_n$. It turns out, however, that this request would not be helpful: indeed, it would be equivalent to requiring the function f to be analytic in the four real variables $x_0, \dots, x_3$ because

$$x_0 = \frac{1}{4}(q - iqi - jqj - kqk),$$

$$x_1 = \frac{1}{4i}(q - iqi + jqj + kqk),$$

$$x_2 = \frac{1}{4j}(q + iqi - jqj + kqk),$$

$$x_3 = \frac{1}{4k}(q + iqi + jqj - kqk).$$

However, Fueter had discovered already in [52] that Fueter-regular functions could be expanded in power series if one were just to appropriately choose the variables to consider. The three variables that do the trick are now known as the Fueter variables and are the (uniquely defined) Fueter-regular extensions of the three real variables x_1, x_2, x_3 from $\mathbb{R}^3 = \{q \in \mathbb{H} \mid x_0 = 0\}$ to all of $\mathbb{H}$. Such Fueter variables are usually denoted by $\zeta_l = x_l - x_0e_l$, where we have set $e_1 = i, e_2 = j, e_3 = k$. With the use of these variables, one can now represent any Fueter-regular function as a suitable power series. We should note, without entering into much detail, that the story on Taylor expansions for Fueter-regular functions does not end here. Indeed, see, e.g., [5], one can show that new bases can be introduced, whose structures lead to a better understanding of the set of all Fueter-regular functions. While this work actually deals with the more general situation of Clifford analysis and Fueter variables in Clifford analysis (see [13] and references therein), its last section shows how to adapt those ideas to the quaternionic case.

One can conclude from these remarks that the theory of holomorphic functions can indeed be nicely generalized to the quaternionic case, if one replaces the Cauchy–Riemann system by the Cauchy–Fueter system, the notion of complex differentiability by the notion of hyperdifferentiability, and the traditional Taylor series in the complex variable z by the Taylor series in the three Fueter variables $\zeta_1, \zeta_2, \zeta_3$. As a consequence, and over many years, the theory of Fueter-regularity has been developed and generalized in many directions, including now a fairly well-developed study in several quaternionic variables [32] as well as a theory of Clifford-valued regular functions [13].

Despite the richness of the theory of Fueter, some of its features motivated the search for an alternative definition of regularity: for instance, the identity function is not Fueter-regular and the same holds for the quaternionic polynomials in the variable q. These are included, however, in the class of *quaternionic holomorphic* functions, defined by Fueter himself in [50] as solutions of the equation

$$\frac{\partial}{\partial \bar{q}} \Delta f(q) = 0,$$

where Δ denotes the Laplacian in the four real variables $x_0, \dots, x_3$. This study has been recently generalized by Laville and Ramadanoff in [86]. Notice, however, that the class of quaternionic holomorphic functions is extremely large: it includes the class of harmonic functions of four real variables, which strictly includes that of Fueter-regular functions. Polynomials are also contained in the class of functions of the reduced variable $x_0 + x_1 i + x_2 j$ studied by Leutwiler in [87].

A different definition of regularity in the quaternionic context was given in [39] by Cullen, who considered solutions of the equation

$$\left(\frac{\partial}{\partial x_0} + \frac{Im(q)}{r} \frac{\partial}{\partial r} \right) f(q) = 0$$

where $Im(q) = x_1 i + x_2 j + x_3 k$ denotes the *imaginary part* of the variable $q = x_0 + x_1 i + x_2 j + x_3 k$ and $r = |Im(q)| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. The number x_0 is also called the *real part* of q and denoted by $Re(q)$. Gentili and Struppa recently restated and developed Cullen's definition in [61, 62] by means of the algebraic properties of $\mathbb{H}$ that we are about to describe. It is not difficult to realize that, for all q with $Im(q) \neq 0$, the normalization $\frac{Im(q)}{|Im(q)|}$ is an *imaginary unit*, i.e., a quaternion whose square equals -1. Hence every quaternion $q \notin \mathbb{R}$ can be uniquely expressed as $q = x + yI$ for some $x, y \in \mathbb{R}$, $y > 0$ and some I in the set of imaginary units

$$\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\} = \{x_1 i + x_2 j + x_3 k : x_1^2 + x_2^2 + x_3^2 = 1\},$$

(which is a 2-sphere in the 3-space of purely imaginary quaternions). In other words (denoting $\mathbb{R} = \mathbb{R} \cdot 1 \subset \mathbb{H}$), the algebra $\mathbb{H}$ is the union

$$\mathbb{H} = \bigcup_{I \in \mathbb{S}} (\mathbb{R} + \mathbb{R} I)$$

of the complex planes $L_I := \mathbb{R}+\mathbb{R}I$, which are all isomorphic to $\mathbb{C}$ and intersect in the real axis. This decomposition measures quite precisely the lack of commutativity of $\mathbb{H}$: two quaternions commute if, and only if, they belong to the same complex plane L_I; this is always true when one of them is real. To go back to Gentili and Struppa's definition, they originally called a function f *Cullen-regular* if it is complex holomorphic when restricted to each complex plane L_I, i.e., if it solves

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f(x+yI) = 0 \tag{1}$$

for $x, y \in \mathbb{R}$ and for all $I \in \mathbb{S}$ (more details are given in Chap. 1). It turned out that all polynomials and power series of the type

$$\sum_{n\in\mathbb{N}} q^n a_n$$

(with $a_n \in \mathbb{H}$) define Cullen-regular functions on their sets of convergence, which are Euclidean balls centered at the origin of $\mathbb{H}$. Conversely, every Cullen-regular function on such a ball admits a series expansion of this type. These two properties allow us to prove that Cullen-regularity does not imply, nor is implied by Fueter-regularity: the function $q \mapsto q^2$ is an example of Cullen-regular function which is not harmonic, hence not Fueter-regular; the function $x_0 + x_1 i + x_2 j + x_3 k \mapsto x_0 + x_1 i$ is Fueter-regular (hence harmonic and quaternionic holomorphic), but not Cullen-regular. After identifying $\mathbb{H} = (\mathbb{R} + i\mathbb{R}) + (\mathbb{R} + i\mathbb{R})j$ with $\mathbb{C}^2$, the same examples prove that Cullen-regularity does not imply nor is implied by holomorphy in two complex variables.

In the original works [61, 62] the authors develop the foundational material for Cullen-regular functions on balls centered at the origin, and in particular they prove a first version of the Cauchy Formula and its most immediate consequences (Cauchy's inequalities, Liouville Theorem, Morera Theorem, etc.). The fundamental technical tool which is used in this context is the so-called *Splitting Lemma:*

Lemma 1 (Splitting). *Let f be a Cullen-regular function defined on an open set Ω. Then for any $I \in \mathbb{S}$ and any $J \in \mathbb{S}$ with $J \perp I$, there exist two holomorphic functions $F, G : \Omega \cap L_I \to L_I$ such that for every $z = x + yI$ it is*

$$f_I(z) = F(z) + G(z)J.$$

This lemma is, for example, instrumental to prove the Identity Principle and the Maximum Modulus Principle for Cullen-regular functions along the lines of the classical proofs in complex analysis. Different ideas are necessary to obtain more refined results for Cullen-regular functions on open balls centered in the origin. For example, we now understand the structure of the zero sets of Cullen-regular functions [55, 63, 68] and the structure of the poles of the quaternionic analogs of meromorphic functions [119]. We also have suitably modified versions of the

Minimum Modulus Principle and the Open Mapping Theorem [56] as well as [70] some results in the spirit of the celebrated Cartan Fixed Point Theorems [49] and Burns–Krantz Theorem [14]. A comprehensive survey of the state of the art up to this point is given in [60].

A turning point in the theory of these functions came when it became apparent that in order to avoid pathological phenomena (e.g., regular functions which are not even continuous, see Example 1.10), it was necessary to consider a special class of domains of definition for regular functions. The first ideas on the features of these domains appeared in [24, 56, 119]. The subsequent paper [18] presented the definition of the so-called *slice domains*, namely the domains (open connected subsets) of $\mathbb{H}$ that intersect the real axis and have connected intersection with every complex plane L_I, for any $I \in \mathbb{S}$. The implications of this definition were then fully expressed in [19]: it turned out that every function which is regular on a slice domain Ω can be uniquely extended to a larger domain $\widetilde{\Omega}$, the *symmetric completion of* Ω, which is the smallest domain in $\mathbb{H}$ that contains Ω and is symmetric with respect to the real axis. Slice domains of $\mathbb{H}$ that are symmetric with respect to the real axis are called *axially symmetric slice domains*. These domains play, for regular functions, the role that domains of holomorphy play for holomorphic functions. Consequently, and beginning with [18, 19], we and our coauthors began to refer to Cullen-regular functions as *slice regular functions, s-regular functions*, or simply *regular functions* (the expression we will always adopt from now on). Concurrently it was discovered that if f is a regular function on a symmetric slice domain Ω and if $x, y \in \mathbb{R}$ are such that $x + y\mathbb{S} \subset \Omega$, then for all $I, J \in \mathbb{S}$ one has the following *Representation Formula*:

$$f(x + yJ) = \frac{1}{2}\left[f(x + yI) + f(x - yI)\right] + \frac{JI}{2}\left[f(x - yI) - f(x + yI)\right].$$

This formula allowed the definition of a product for regular functions, the $*$-*product*, which extends the classical definition for polynomials on a noncommutative ring and its natural generalization to regular power series. Furthermore, a new and significantly different Cauchy Formula was discovered in [18], which, together with the Representation Formula, quickly became the central tool for a series of far-reaching extensions of known results (e.g., the structure of the zero set of a regular function [57], a more general version of the Open Mapping Theorem [57], and a Pompeiu Formula [37]). Moreover, in this new environment, it is possible to study [58] power series centered at nonreal quaternions and their domains of convergence. This leads to a notion of weak analyticity that turns out to be equivalent to regularity.

In this book we present a unified version of the theory of regular functions by immediately adopting the newer setting of (axially symmetric) slice domains rather than the original setting of balls centered at a real point. The proofs that we offer reflect this orientation, but in the exposition we will always identify those instances in which the original approach already contains all the relevant ideas.

Before concluding, we point out that Gentili and Struppa generalized their definition of regular function to the octonions in [65] and to the Clifford algebra

$Cl(0,3)$ in [64] (both theories are surveyed in [60]). The zeros of octonionic regular functions are studied in [73]. The case of functions defined on $\mathbb{R}^{m+1}$ with values in the Clifford algebra $Cl(0,m)$ has been considered by Colombo, Sabadini and Struppa in [34, 35], where the notion of *slice monogeneity* is introduced. More recently, in [72], Ghiloni and Perotti have made a significant step forward, generalizing the definition to other (finite dimensional) alternative real algebras and endowing all the aforementioned theories with new working tools.

Every time mathematicians define a new object, two questions arise naturally. The first is whether the new object can lead to an interesting theory by itself. We believe that this book will show that the notion of regularity is indeed a very stable notion that yields a rich theory for functions of a quaternionic variable, especially as it includes power series with quaternionic coefficients. At the same time, we identify some crucial differences between the complex and the quaternionic case. The second question is whether the new theory, in addition to its intrinsic value, can also contribute to the solution of some outstanding problem. This is indeed the case. The theory of regular functions has now been applied to develop a new functional calculus in a noncommutative setting [21, 36], to the construction and classification of orthogonal complex structures on dense open sets in $\mathbb{H}$ [54], and to a theory of coherent states in quaternionic quantum mechanics [126].

We now describe in detail the contents of every chapter.

Chapter 1 explains the definition of regularity. It presents the basic results of the theory: the existence of power series expansions in all Euclidean balls centered at the origin, the Identity Principle, and a Representation Formula that is fundamental in the study of regular functions on axially symmetric slice domains. The same formula yields that the axially symmetric slice domains are the quaternionic analogs of the complex domains of holomorphy. Finally, we show that regular functions form a ring with respect to addition and to an appropriately defined regular multiplication.

Chapter 2 explores the possibility of series expansions at points other than the origin. Power series expansions exist at all points of the domain of definition, but their sets of convergence are balls with respect to a non-Euclidean distance σ. This leads to the notion of σ-analyticity, which is equivalent to regularity.

In Chap. 3, we treat the zero sets of regular functions, which exhibit interesting algebraic and topological properties. For instance, the zero set of a regular function consists of isolated points or isolated 2-spheres of a special type. We present the study of nth roots of quaternions and factorizations of regular polynomials, explaining their relations with the zeros and comparing different notions of multiplicity. Finally, we illustrate the Bezout Theorem and the construction of Gröbner bases for quaternionic polynomials.

Chapter 4 contains the Weierstrass Factorization Theorem and all the tools that are necessary for its proof: the study of infinite products of regular functions and of their convergence, based on the construction of the quaternionic logarithm. The formulation of this theorem, which reflects the peculiarities of the structure of the zero sets, is compared to the complex case.

Chapter 5 is devoted to the classification of the singularities of regular functions. We first present the construction of the ring of quotients of regular functions. We then study regular Laurent series and expansions, which are particularly interesting when centered at a point p other than the origin. Laurent expansions allow us to classify the singularities as removable, essential, or as poles. Poles are studied by means of regular quotients, while for essential singularities we present a version of the Casorati–Weierstrass Theorem.

In Chap. 6 we present the regular analogs of many classical integral formulas in complex analysis. We introduce versions of the Cauchy Theorem and of the Morera Theorem. We present results such as the Cauchy Integral Formula, the formula for derivatives, and the Pompeiu Formula, giving new proofs. We illustrate the Cauchy Estimates, the Liouville Theorem, and an integral formula to compute the coefficients of regular Laurent expansions. We conclude this chapter with an argument principle for regular functions.

Chapter 7 presents the Maximum and Minimum Modulus Principles, the Open Mapping Theorem, and the study of the real parts of regular functions. It also contains Principles of Phragmén–Lindelöf type and an Ehrenpreis–Malgrange Lemma for quaternionic polynomials.

In Chap. 8, we present series expansions valid in open subsets of $\mathbb{H}$ and a notion of analyticity which is equivalent to regularity on axially symmetric slice domains. This allows a detailed study of the real differentials of regular functions and of their ranks.

Chapter 9 begins with the Schwarz Lemma, then it presents the transformations of the quaternionic space, unit ball, and Riemann sphere. It overviews rigidity results for regular functions on the unit ball, both of Cartan and of Burns–Krantz type. It then presents the Borel–Carathéodory Theorem and the Bohr Theorem.

Finally, Chap. 10 overviews several generalizations and applications of the theory. Specifically, we describe the theory of regular functions on the space of octonions and on the Clifford algebra $\mathbb{R}_3 = Cl(0, 3)$, as well as the notion of slice monogeneity. We also present a new approach to regularity on alternative real algebras, due to Ghiloni and Perotti. We finally discuss some applications of the theory of regular functions to quaternionic functional calculus and to the construction and classification of orthogonal complex structures on dense open subsets of $\mathbb{H}$.

To conclude, we point out that the bibliographical notes that appear at the end of each chapter are meant to collect the references to material due to our nearest collaborators and us. Instead, references to production by other authors are disseminated in the text.

Chapter 1
Definitions and Basic Results

1.1 Regular Functions

Let Ω be a domain in the space of quaternions $\mathbb{H}$, namely, an open connected subset of $\mathbb{H} = \mathbb{R} + i\mathbb{R} + j\mathbb{R} + k\mathbb{R}$ and let

$$\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}$$

denote the 2-sphere of quaternionic imaginary units. We define the notion of regular function as follows.

Definition 1.1. Let f be a quaternion-valued function defined on a domain Ω. For each $I \in \mathbb{S}$, let $\Omega_I = \Omega \cap L_I$ and let $f_I = f_{|\Omega_I}$ be the restriction of f to Ω_I. The restriction f_I is called *holomorphic* if it has continuous partial derivatives and

$$\bar{\partial}_I f(x + yI) = \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + yI) \equiv 0. \tag{1.1}$$

The function f is called *regular* if, for all $I \in \mathbb{S}$, f_I is holomorphic.

Remark 1.2. It is useful to note that if f is regular in a domain Ω and if r is a real number, then $g(q) := f(q + r)$ is obviously regular on $\Omega - r$. Note, however, that, in general, the composition of two regular functions is not regular.

The following lemma clarifies the relation between quaternionic regularity and complex holomorphy. For each $I \in \mathbb{S}$, let us identify L_I with $\mathbb{C}$. Notice, moreover, that for all $J \in \mathbb{S}$ with $J \perp I$, the following equality holds:

$$\mathbb{H} = L_I + L_I J.$$

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Lemma 1.3 (Splitting). *Let f be a regular function defined on a domain Ω. Then for any $I \in \mathbb{S}$ and any $J \in \mathbb{S}$ with $J \perp I$, there exist two holomorphic functions $F, G : \Omega_I \to L_I$ such that for every $z = x + yI$, it is*

$$f_I(z) = F(z) + G(z)J.$$

The previous Lemma can be reformulated in a way that will be useful in the sequel.

Lemma 1.4. *Let $I \in \mathbb{S}$, let Ω_I be open in L_I, and let $f_I : \Omega_I \to \mathbb{H}$. The function f_I is holomorphic if and only if, for all $J \in \mathbb{S}$ with $J \perp I$ and every $z = x + yI$, it is*

$$f_I(z) = F(z) + G(z)J \tag{1.2}$$

where $F, G : \Omega_I \to L_I$ are complex-valued holomorphic functions of one complex variable.

Let us now review the first examples and the basic properties of regular functions.

Example 1.5. The identity function $q \mapsto q$ is regular in $\mathbb{H}$. The same holds for any polynomial function of the type $q \mapsto a_0 + qa_1 + \ldots + q^n a_n,\ a_l \in \mathbb{H}$ for all l.

This class of examples extends as follows. For all $R \in (0, +\infty]$, let us denote by

$$B(0, R) = \{q \in \mathbb{H} : |q| < R\}$$

the Euclidean ball of radius R centered at 0 in $\mathbb{H}$.

Theorem 1.6 (Abel's Theorem). *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{H}$ and let*

$$R = \frac{1}{\limsup_{n\in\mathbb{N}} |a_n|^{1/n}}. \tag{1.3}$$

If $R > 0$, then the power series

$$\sum_{n\in\mathbb{N}} q^n a_n \tag{1.4}$$

converges absolutely and uniformly on compact sets in $B(0, R)$. Moreover, its sum defines a regular function on $B(0, R)$.

The proof is completely analogous to that of the complex Abel theorem. The converse result holds; in other words all regular functions on $B(0, R)$ can be expressed as power series. In order to prove this, we first introduce an appropriate notion of derivative.

Definition 1.7. Let $f : \Omega \to \mathbb{H}$ be a regular function. For each $I \in \mathbb{S}$, the I-*derivative* of f is defined as

$$\partial_I f(x + yI) = \frac{1}{2}\left(\frac{\partial}{\partial x} - I\frac{\partial}{\partial y}\right) f_I(x + yI) \tag{1.5}$$

on Ω_I. The *slice derivative* of f is the function $f' = \partial_c f : \Omega \to \mathbb{H}$ defined by $\partial_I f$ on Ω_I, for all $I \in \mathbb{S}$.

The definition is well posed because, by direct computation, $\partial_I f = \partial_J f$ in $\Omega_I \cap \Omega_J$ for any choice of $I, J \in \mathbb{S}$. Furthermore, the following can be proven making use of the fact that ∂_I and $\bar{\partial}_I$ commute.

Remark 1.8. For any regular function $f : \Omega \to \mathbb{H}$, the slice derivative f' is regular in Ω.

It is thus possible to iterate the derivation process. Let us denote the nth slice derivative as $f^{(n)}$ for each $n \in \mathbb{N}$. We now come to the announced result.

Theorem 1.9 (Series expansion). *Let $R > 0$ and let $f : B = B(0, R) \to \mathbb{H}$ be a regular function. Then*

$$f(q) = \sum_{n \in \mathbb{N}} q^n \frac{1}{n!} f^{(n)}(0) \tag{1.6}$$

for all $q \in B$. In particular, $f \in C^\infty(B)$.

Proof. Fix $I \in \mathbb{S}$ and identify L_I with $\mathbb{C}$. Choose $J \in \mathbb{S}$ such that $J \perp I$: by the Splitting Lemma 1.3, there exist holomorphic functions $F, G : B_I \to L_I$ such that $f_I = F + GJ$. Notice that, for all $z \in B_I$,

$$f'(z) = \partial_I f(z) = \frac{\partial F}{\partial z}(z) + \frac{\partial G}{\partial z}(z) J$$

and, similarly,

$$f^{(n)}(z) = \frac{\partial^n F}{\partial z^n}(z) + \frac{\partial^n G}{\partial z^n}(z) J.$$

Now observe that the complex series

$$\sum_{n \in \mathbb{N}} z^n \frac{1}{n!} \frac{\partial^n F}{\partial z^n}(0)$$

converges to $F(z)$ for $z \in B_I$ (absolutely and uniformly on its compact subsets). The same can be proved for G, so that for all $z \in B_I$

$$f(z) = F(z) + G(z)J = \sum_{n \in \mathbb{N}} z^n \frac{1}{n!} \frac{\partial^n F}{\partial z^n}(0) + \sum_{n \in \mathbb{N}} z^n \frac{1}{n!} \frac{\partial^n G}{\partial z^n}(0) J =$$

$$= \sum_{n \in \mathbb{N}} z^n \frac{1}{n!} f^{(n)}(0)$$

as desired. The thesis follows from the arbitrariness of $I \in \mathbb{S}$. Finally, $f \in C^\infty(B)$ because each addend $q^n \frac{1}{n!} f^{(n)}(0)$ is clearly in $C^\infty(B)$ and because the convergence is uniform on compact sets. □

Propositions 1.6 and 1.9 are fundamental in the study of regular quaternionic functions on balls $B = B(0, R)$ centered at the origin of $\mathbb{H}$. For instance, they allowed the proof of an identity principle in [62], stating that if, for some $I \in \mathbb{S}$, two regular functions $f, g : B \to \mathbb{H}$ coincide on a subset of B_I having an accumulation point in B_I, then $f = g$ in B. This principle does not hold for an arbitrarily chosen domain in $\mathbb{H}$, as shown by the next example.

Example 1.10. Let $I \in \mathbb{S}$ and let $f : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$ be defined as follows:

$$f(q) = \begin{cases} 0 \text{ if } q \in \mathbb{H} \setminus L_I \\ 1 \text{ if } q \in L_I \setminus \mathbb{R} \end{cases}$$

This function is clearly regular.

The previous example proves that if the domain Ω is not carefully chosen, then a regular function $f : \Omega \to \mathbb{H}$ does not even need to be continuous. It is possible to prevent such pathologies by imposing further conditions on the domain Ω.

Definition 1.11. Let Ω be a domain in $\mathbb{H}$ that intersects the real axis. Ω is called a *slice domain* if, for all $I \in \mathbb{S}$, the intersection Ω_I with the complex plane L_I is a domain of L_I.

The identity principle holds true on all slice domains.

Theorem 1.12 (Identity Principle). *Let f, g be regular functions on a slice domain Ω. If, for some $I \in \mathbb{S}$, f and g coincide on a subset of Ω_I having an accumulation point in Ω_I, then $f = g$ in Ω.*

Proof. The restrictions f_I, g_I are holomorphic functions. Under the hypotheses, f_I and g_I must coincide in Ω_I. In particular, f must coincide with g in $\Omega \cap \mathbb{R}$. For all $K \in \mathbb{S}$, the intersection $\Omega \cap \mathbb{R}$ is a subset of Ω_K that has an accumulation point in Ω_K. Thus, $f_K = g_K$ in Ω_K for all $K \in \mathbb{S}$, and we conclude that $f = g$ in

$$\Omega = \bigcup_{K \in \mathbb{S}} \Omega_K.$$

□

Notice that, in the proof of Theorem 1.12, both properties that define slice domains are essential: the fact that $\Omega \cap \mathbb{R} \neq \emptyset$ and the connectedness of Ω_I for all $I \in \mathbb{S}$. In the next section we will present a symmetry condition for the domains of definition which guarantees continuity and differentiability for regular functions.

1.2 Affine Representation

We now present a very peculiar property of regular functions, which allows to identify the quaternionic analogs of the domains of holomorphy. Consider the following property of quaternionic powers, which is a direct consequence of the (complex) binomial theorem.

Remark 1.13. For each $x, y \in \mathbb{R}$, there exist sequences $\{\alpha_n\}_{n\in\mathbb{N}}, \{\beta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $(x + yI)^n = \alpha_n + \beta_n I$ for all $I \in \mathbb{S}$.

As a consequence, the following formula holds for a regular function $f(q) = \sum_{n\in\mathbb{N}} q^n a_n$:

$$f(x + yI) = \sum_{n\in\mathbb{N}} \alpha_n a_n + I \sum_{n\in\mathbb{N}} \beta_n a_n.$$

This formula has a nice geometric interpretation: the restriction of f to the sphere

$$x + y\mathbb{S} = \{x + yI : I \in \mathbb{S}\}$$

is affine in the imaginary unit I, that is, there exist $b, c \in \mathbb{H}$ such that

$$f(x + yI) = b + Ic \tag{1.7}$$

for all $I \in \mathbb{S}$. This is not only true for power series, but for all regular functions on the slice domains that have the following property.

Definition 1.14. A set $T \subseteq \mathbb{H}$ is called *axially symmetric* if, for all points $x + yI \in T$ with $x, y \in \mathbb{R}$ and $I \in \mathbb{S}$, the set T contains the whole sphere $x + y\mathbb{S}$.

Since no confusion can arise, we will refer to such a set as *symmetric*, tout court. The most general statement is the following.

Theorem 1.15 (Representation Formula). *Let f be a regular function on a symmetric slice domain Ω and let $x + y\mathbb{S} \subset \Omega$. For all $I, J, K \in \mathbb{S}$ with $J \neq K$*

$$\begin{aligned} f(x + yI) = (J - K)^{-1} [Jf(x + yJ) - Kf(x + yK)] + \\ + I(J - K)^{-1} [f(x + yJ) - f(x + yK)] \end{aligned} \tag{1.8}$$

The quaternions $b = (J - K)^{-1} [Jf(x + yJ) - Kf(x + yK)]$ and $c = (J - K)^{-1} [f(x + yJ) - f(x + yK)]$ do not depend on J, K but only on x, y.

Proof. Fix $J, K \in \mathbb{S}$ with $J \neq K$, and set

$$\begin{aligned} \phi(x + yI) &= (J - K)^{-1} [Jf(x + yJ) - Kf(x + yK)] + \\ &+ I(J - K)^{-1} [f(x + yJ) - f(x + yK)] = \\ &= \left[(J - K)^{-1}J + I(J - K)^{-1}\right] f(x + yJ) + \\ &- \left[(J - K)^{-1}K + I(J - K)^{-1}\right] f(x + yK) \end{aligned}$$

for all $I \in \mathbb{S}$ and for all $x, y \in \mathbb{R}$ such that $x + y\mathbb{S} \subset \Omega$. Computing the above formulae for $y = 0$ shows that $\phi(x) = f(x)$ for all $x \in \Omega \cap \mathbb{R}$. If we prove that ϕ is regular in Ω, then we will conclude that $\phi \equiv f$ thanks to the Identity Principle 1.12. The regularity of ϕ is proved computing

$$\frac{\partial \phi(x+yI)}{\partial x} = \left[(J-K)^{-1}J + I(J-K)^{-1}\right]\frac{\partial f(x+yJ)}{\partial x} +$$
$$-\left[(J-K)^{-1}K + I(J-K)^{-1}\right]\frac{\partial f(x+yK)}{\partial x}$$

and

$$I\frac{\partial \phi(x+yI)}{\partial y} = \left[I(J-K)^{-1}J - (J-K)^{-1}\right]\frac{\partial f(x+yJ)}{\partial y} +$$
$$-\left[I(J-K)^{-1}K - (J-K)^{-1}\right]\frac{\partial f(x+yK)}{\partial y} =$$
$$= -\left[I(J-K)^{-1} + (J-K)^{-1}J\right]\frac{\partial f(x+yJ)}{\partial x} +$$
$$+\left[I(J-K)^{-1} + (J-K)^{-1}K\right]\frac{\partial f(x+yK)}{\partial x},$$

so that

$$\frac{\partial \phi(x+yI)}{\partial x} + I\frac{\partial \phi(x+yI)}{\partial y} \equiv 0$$

as desired. The formula is thus proven. The last statement follows because the choice of $J, K \in \mathbb{S}$ is arbitrary. □

The following special case will prove particularly useful in the next chapters.

Corollary 1.16. *Let f be a regular function on a symmetric slice domain Ω and let $x + y\mathbb{S} \subset \Omega$. For all $I, J \in \mathbb{S}$*

$$f(x+yJ) = \frac{1-JI}{2}f(x+yI) + \frac{1+JI}{2}f(x-yI) \tag{1.9}$$
$$= \frac{1}{2}\left[f(x+yI) + f(x-yI)\right] + \frac{JI}{2}\left[f(x-yI) - f(x+yI)\right]$$

We presently deduce an alternative formula, proving that the restriction of f to a sphere $x + y\mathbb{S}$ is actually affine in the variable q.

Corollary 1.17. *Let f be a regular function on a symmetric slice domain Ω and let $S = x + y\mathbb{S} \subset \Omega$. For all $q, q_1, q_2 \in S$ with $q_1 \neq q_2$*

$$f(q) = (q_1 - q_2)^{-1}\left[\bar{q}_2 f(q_2) - \bar{q}_1 f(q_1)\right] + q(q_1 - q_2)^{-1}\left[f(q_1) - f(q_2)\right] \tag{1.10}$$

where $A = (q_1 - q_2)^{-1}[f(q_1) - f(q_2)]$ and $B = (q_1 - q_2)^{-1}[\bar{q}_2 f(q_2) - \bar{q}_1 f(q_1)]$ do not depend on q_1, q_2 but only on S.

Proof. If $q = x + yI, q_1 = x + yJ, q_2 = x + yK$, then we deduce the thesis from (1.8) by direct computation:

$$f(x+yI) = (J-K)^{-1}\,[Jf(x+yJ) - Kf(x+yK)] +$$
$$+I(J-K)^{-1}\,[f(x+yJ) - f(x+yK)] =$$
$$= (q_1-q_2)^{-1}\,[yJf(q_1) - yKf(q_2)] + yI(q_1-q_2)^{-1}\,[f(q_1)-f(q_2)] =$$
$$= (q_1-q_2)^{-1}\,[yJf(q_1) - yKf(q_2)] - x(q_1-q_2)^{-1}\,[f(q_1)-f(q_2)] +$$
$$+(x+yI)(q_1-q_2)^{-1}\,[f(q_1)-f(q_1)] =$$
$$= (q_1-q_2)^{-1}\,[(-x+yJ)f(q_1) + (x-yK)f(q_2)] +$$
$$+q(q_1-q_2)^{-1}\,[f(q_1)-f(q_1)] =$$
$$= (q_1-q_2)^{-1}\,[\bar{q}_2 f(q_2) - \bar{q}_1 f(q_1)] + q(q_1-q_2)^{-1}\,[f(q_1)-f(q_1)]\,.$$

□

The fact that f is affine in each sphere $x+y\mathbb{S}$ justifies the following definition, given in [72] in a more general setting.

Definition 1.18. Let f be a regular function on a symmetric slice domain Ω. The *spherical derivative* of f is defined by the formula

$$\partial_s f(q) = (q-\bar{q})^{-1}[f(q)-f(\bar{q})] = [2Im(q)]^{-1}[f(q)-f(\bar{q})] \tag{1.11}$$

while the *spherical value* is the function

$$v_s f(q) = \frac{1}{2}[f(q)+f(\bar{q})]. \tag{1.12}$$

We conclude by proving a consequence of (1.7), which will be useful in the study of uniform convergence of infinite $*$-products of regular functions, Sect. 4.5.

Proposition 1.19. *Let f be a regular function on a symmetric slice domain $\Omega \subseteq \mathbb{H}$. Let $T \subseteq \Omega$ be a symmetric compact set. For every $I \in \mathbb{S}$, $p \in \mathbb{H}$, and $R > 0$ such that*

$$f_I(T_I) \subseteq B(p,R),$$

we have

$$f(T) \subseteq B(p,2R).$$

Proof. Let $I,\ p,\ R$ be as in the hypothesis. If T is symmetric, then

$$T = \bigcup_{x+yI\in T} x+y\mathbb{S};$$

hence,

$$f(T) = \bigcup_{x+yI\in T} f(x+y\mathbb{S}).$$

It is enough to prove that $f(x + y\mathbb{S}) \subseteq B(p, 2R)$ for all $x, y \in \mathbb{R}$ such that $x + yI \in T$. Let $x, y \in \mathbb{R}$ be such that $x + y\mathbb{S} \subseteq T$. By (1.7) there exist $b, c \in \mathbb{H}$ such that

$$f(x + yJ) = b + Jc$$

for all $J \in \mathbb{S}$. Since $f_I(T_I) \subseteq B(p, R)$, we have that

$$f(x + yI) = b + Ic \ \in B(p, R)$$

and that

$$f(x - yI) = b - Ic \ \in B(p, R).$$

Since two antipodal points (i.e., $b + Ic$ and $b - Ic$) of the 2-sphere $f(x + y\mathbb{S}) = b + \mathbb{S}c$ belong to $B(p, R)$, the center b of $b + \mathbb{S}c$ also belongs to $B(p, R)$ and the radius of $b + \mathbb{S}c$ is less than or equal to R. Hence,

$$f(x + y\mathbb{S}) \subseteq B(p, 2R).$$

□

1.3 Extension Results

The results presented in the previous section show that a regular function on a symmetric slice domain Ω is uniquely determined by its restriction to a slice Ω_I (or to two "half slices" $\Omega_J^+ = \{x + yJ \in \Omega_J : y > 0\}$ and $\Omega_K^+ = \{x + yK \in \Omega_K : y > 0\}$). This suggests the following definition and proposition:

Definition 1.20. The *(axially) symmetric completion* of a set $T \subseteq \mathbb{H}$ is the smallest symmetric set $\widetilde{T}$ that contains T. In other words,

$$\widetilde{T} = \bigcup_{x+yI \in T} (x + y\mathbb{S}). \tag{1.13}$$

Proposition 1.21 (Extension Formula). *Let J, K be distinct imaginary units, let T be a domain in L_J, intersecting the real axis, let $U = \{x + yK : x + yJ \in T\}$, and let Ω be the symmetric completion $\widetilde{T} = \widetilde{U}$. For any choice of holomorphic functions $r : T \to \mathbb{H}, s : U \to \mathbb{H}$ such that $r_{|_{T\cap\mathbb{R}}} = s_{|_{U\cap\mathbb{R}}}$, the function $f : \Omega \to \mathbb{H}$ defined, for all $x + yI \in \Omega$, by*

$$\begin{aligned} f(x + yI) = {} & (J - K)^{-1} \left[Jr(x + yJ) - Ks(x + yK)\right] + \\ & + I(J - K)^{-1} \left[r(x + yJ) - s(x + yK)\right] \end{aligned} \tag{1.14}$$

is the (unique) regular function on Ω such that $f_{|_T} = r$ and $f_{|_U} = s$.

Proof. The function f is proved to be regular in Ω by the same reasoning used for Formula (1.8). Furthermore, $f_{|_T} = r$ by direct computation, since

$$(J-K)^{-1}J + J(J-K)^{-1} = |J-K|^{-2}[(K-J)J + J(K-J)] =$$
$$= [(J-K)(K-J)]^{-1}(2 + JK + KJ) = 1$$

and

$$(J-K)^{-1}K + J(J-K)^{-1} = |J-K|^{-2}[(K-J)K + J(K-J)] =$$
$$= |J-K|^{-2}(-1 - JK + JK + 1) = 0.$$

Similarly, $f_{|_U} = s$. The uniqueness is a consequence of the Identity Principle 1.12. □

The following special case (where $J = I, K = -I$) will be particularly useful in the sequel.

Lemma 1.22. *Let Ω be a symmetric slice domain and let $I \in \mathbb{S}$. If $f_I : \Omega_I \to \mathbb{H}$ is holomorphic, then there exists a unique regular function $g : \Omega \to \mathbb{H}$ such that $g_I = f_I$ in Ω_I.*

The function g will be denoted by $\mathrm{ext}(f_I)$ and called the *regular extension* of f_I. In analogy with what is done in the complex plane, we give the following definition:

Definition 1.23. A slice domain $\Omega \in \mathbb{H}$ is a *domain of regularity* if there exists a regular function on Ω that cannot be extended as a regular function to a larger domain.

It is well known that every domain in $\mathbb{C}$ is a domain of holomorphy. The next theorem shows that this is not the case for $\mathbb{H}$.

Theorem 1.24 (Extension). *Let f be a regular function on a slice domain Ω. There exists a unique regular function $\tilde{f} : \widetilde{\Omega} \to \mathbb{H}$ that extends f to the symmetric completion of Ω.*

Proof. By hypothesis $\Omega \cap \mathbb{R} \neq \emptyset$. Since Ω is open, it is possible to choose a neighborhood D of $\Omega \cap \mathbb{R}$ in $\mathbb{H}$ that is a symmetric slice domain contained in Ω. Now let M be the largest symmetric slice domain with $D \subseteq M \subseteq \widetilde{\Omega}$ to which f extends as a regular function. The domain M cannot be strictly contained in $\widetilde{\Omega}$ because of the following reasoning.

If $\widetilde{\Omega} \setminus M \neq \emptyset$, then there exists a $p \in \partial M \cap \widetilde{\Omega}$. From $p = u + Lv \in \widetilde{\Omega}$, we deduce the existence of $J \in \mathbb{S}$ such that $u + vJ \in \Omega$. Moreover, choosing $K \in \mathbb{S}$ sufficiently near to J, but distinct, there exists an $\varepsilon > 0$ such that Ω contains the discs $\Delta_J = \{z \in L_J : |z-(u+vJ)| < \varepsilon\}$ and $\Delta_K = \{z \in L_K : |z-(u+vK)| < \varepsilon\}$. Now, if $\widetilde{\Delta}_J$ is the symmetric completion of Δ_J, setting

$$g(x+yI) = (J-K)^{-1}[Jf(x+yJ) - Kf(x+yK)] +$$
$$+ I(J-K)^{-1}[f(x+yJ) - f(x+yK)]$$

for all $x+yI \in \widetilde{\Delta}_J$ will define a regular function $g : \widetilde{\Delta}_J \to \mathbb{H}$. The latter coincides with f in $\widetilde{\Delta}_J \cap M$ because formula (1.8) holds in M. Hence, setting

$$\tilde{f} = \begin{cases} f & \text{in } M \\ g & \text{in } \widetilde{\Delta}_J \end{cases}$$

extends f to a regular function $\tilde{f}$ on the symmetric slice domain $M \cup \widetilde{\Delta}_J$, a contradiction with the hypotheses on M. □

Since every domain in $\mathbb{C}$ is a domain of holomorphy, and by Lemma 1.22, it is immediate to see that on every symmetric slice domain, there exists a regular function that cannot be extended. Thus, we proved the following corollary.

Corollary 1.25. *A slice domain $\Omega \subseteq \mathbb{H}$ is a domain of regularity if and only if it is a symmetric slice domain.*

We point out that, as a consequence of Theorem 1.24, considering regular functions on symmetric slice domains is not more restrictive than considering regular functions on slice domains. For this reason, we will often impose the symmetry condition in our presentation.

1.4 Algebraic Structure

In this section we present the algebraic structure of the set of regular functions. It is not hard to see that the class of regular functions is endowed with an addition operation: if f, g are regular functions on Ω, then $f + g$ is regular in Ω, too. The same does not hold for pointwise multiplication: $f \cdot g$ is not regular, except for some special cases. This is easily seen even in the simplest case when $f(q) = qa$ and $g(q) = q$, with $a \in \mathbb{H} \setminus \mathbb{R}$. Then

$$f(q)g(q) = qaq$$

which is clearly not regular. We instead use the multiplicative operation described below, following the classical approach used for polynomials in noncommutative algebra (see, e.g., [84]).

Definition 1.26. Let $f, g : B(0, R) \to \mathbb{H}$ be regular functions and let $f(q) = \sum_{n\in\mathbb{N}} q^n a_n, g(q) = \sum_{n\in\mathbb{N}} q^n b_n$ be their power series expansions. The *regular product* of f and g (sometimes referred to as their $*$-*product*) is the regular function defined by

$$f * g(q) = \sum_{n\in\mathbb{N}} q^n \sum_{k=0}^{n} a_k b_{n-k} \tag{1.15}$$

on the same ball $B(0, R)$.

Notice that if $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, then $f * g(q) = f(q)g(q)$. It turns out that the set of regular functions on a ball $B(0, R)$ is a ring with $+, *$, and that this multiplication can be extended to all regular functions on symmetric slice domains. The definition of regular product in this new setting relies upon the Splitting Lemma 1.3 and upon Lemma 1.22.

Definition 1.27. Let f, g be regular functions on a symmetric slice domain Ω. Choose $I, J \in \mathbb{S}$ with $I \perp J$ and let F, G, H, K be holomorphic functions from Ω_I to L_I such that $f_I = F + GJ$, $g_I = H + KJ$. Consider the holomorphic function defined on Ω_I by

$$f_I * g_I(z) = \left[F(z)H(z) - G(z)\overline{K(\bar{z})}\right] + \left[F(z)K(z) + G(z)\overline{H(\bar{z})}\right] J. \quad (1.16)$$

Its regular extension $\mathrm{ext}(f_I * g_I)$ is called the *regular product* (or $*$-*product*) of f and g, and it is denoted by $f * g$.

It is possible to check directly that this definition is coherent with the previous one in the special case $\Omega = B(0, R)$. Note that Definition 1.27 apparently depends upon the choices of I, J. The next result shows that this is not the case.

Proposition 1.28. *Let Ω be a symmetric slice domain. The definition of regular product is well posed, and the set of regular functions on Ω is a (noncommutative) ring with respect to $+$ and $*$.*

Proof. Let f, g be regular functions on Ω. By hypothesis, Ω intersects the real axis at some $r \in \mathbb{R}$. By possibly substituting $f(q + r)$ for $f(q)$ and $g(q + r)$ for $q(q)$, we may suppose $r = 0$. Then there exists a ball $B = B(0, R) \subseteq \Omega$, with $R > 0$ on which the restrictions $f_{|_B}$ and $g_{|_B}$ are power series. We already observed that, for all $I \in \mathbb{S}$, $\mathrm{ext}(f_I * g_I)$ coincides with $f_{|_B} * g_{|_B}$ in B. In particular, for all $I, J \in \mathbb{S}$, $\mathrm{ext}(f_I * g_I)$ equals $\mathrm{ext}(f_J * g_J)$ in B. By the Identity Principle 1.12,

$$\mathrm{ext}(f_I * g_I) = \mathrm{ext}(f_J * g_J)$$

in Ω. This proves that $f * g$ is well defined on Ω. The operation $*$ is associative: $f * (g * h) = (f * g) * h$ because

$$f_{|_B} * (g_{|_B} * h_{|_B}) = (f_{|_B} * g_{|_B}) * h_{|_B}$$

The distributive law can be proven using the same technique. Finally, $*$ is clearly noncommutative. □

As in the case of power series, the regular product coincides with the pointwise product for a special class of regular functions.

Definition 1.29. A regular function $f : \Omega \to \mathbb{H}$ such that $f(\Omega_I) \subseteq L_I$ for all $I \in \mathbb{S}$ is called a *slice preserving* regular function.

Lemma 1.30. *Let f, g be regular functions on a symmetric slice domain Ω. If f is slice preserving, then fg is a regular function on Ω and $f * g = fg$.*

Proof. For any $I, J \in \mathbb{S}$ with $I \perp J$, let F, G, H, K be holomorphic functions $\Omega_I \to L_I$ such that $f_I = F + GJ, g_I = H + KJ$. If $f(\Omega_I) \subseteq L_I$, then G must vanish identically, so that

$$f_I(z)g_I(z) = F(z)H(z) + F(z)K(z)J.$$

Since FH and FK are holomorphic functions from Ω_I to L_I, $f_I g_I = (fg)_I$ is holomorphic. By the arbitrariness of $I \in \mathbb{S}$, the Splitting Lemma 1.3 implies that $fg : \Omega \to \mathbb{H}$ is regular.

Now fix $I \in \mathbb{S}$. By the equation above, $(fg)_I = FH + FKJ = f_I * g_I$. Hence, fg and $f * g = \mathrm{ext}(f_I * g_I)$ are two regular functions on Ω coinciding in Ω_I. By the Identity Principle 1.12, they must coincide in Ω. □

In the special case when Ω is a ball centered in a real point (which we may assume to be the origin by Remark 1.2), Lemma 1.30 captures what we already observed for power series thanks to the following remark.

Remark 1.31. Let f be a regular function on $\Omega = B(0, R)$ (for some $R > 0$). Then f is slice preserving if, and only if, the power series expansion $f(q) = \sum_{n\in\mathbb{N}} q^n a_n$ has real coefficients $a_n \in \mathbb{R}$.

As we pointed out in Remark 1.2, the composition of two regular functions is not, in general, a regular function. However, the following lemma can be proven by direct computation.

Lemma 1.32. *Let $f : \Omega \to \Omega' \subseteq \mathbb{H}$ and $g : \Omega' \to \mathbb{H}$ be regular functions. If f is a slice preserving function, then the composition $g \circ f$ is regular.*

We can define two additional operations on regular functions. We begin with the case of power series.

Definition 1.33. Let $f : B(0, R) \to \mathbb{H}$ be a regular function and let $f(q) = \sum_{n\in\mathbb{N}} q^n a_n$ be its power series expansion. The *regular conjugate* of f is the regular function defined by

$$f^c(q) = \sum_{n\in\mathbb{N}} q^n \bar{a}_n \tag{1.17}$$

on the same ball $B(0, R)$. The *symmetrization* of f is the function

$$f^s = f * f^c = f^c * f. \tag{1.18}$$

These operations are defined in order to study the zero set, as we will explain in Chap. 3, but they also allow us to construct the ring of quotients of regular functions (see Chap. 5). In the general case of symmetric slice domains, they are defined in the following way.

Definition 1.34. Let f be a regular function on a symmetric slice domain Ω. Choose $I, J \in \mathbb{S}$ with $I \perp J$ and let F, G be holomorphic functions from Ω_I to L_I such that $f_I = F + GJ$. If f_I^c is the holomorphic function defined on Ω_I by

$$f_I^c(z) = \overline{F(\bar{z})} - G(z)J. \tag{1.19}$$

then the *regular conjugate* of f is the regular function defined on Ω as $f^c = \mathrm{ext}(f_I^c)$.

Definition 1.35. Let f be a regular function on a symmetric slice domain Ω. The *symmetrization* of f is the regular function defined on Ω as $f^s = f * f^c = f^c * f$.

Remark 1.36. Using the Splitting Lemma, we can write

$$f_I(z) = F(z) + G(z)J,$$

with $F, G : \Omega_I \to L_I$ holomorphic functions. Therefore,

$$\begin{aligned} f_I^s = f_I * f_I^c = f_I^c * f_I &= (F(z) + G(z)J) * (\overline{F(\bar{z})} - G(z)J) \qquad (1.20) \\ &= [F(z)\overline{F(\bar{z})} + G(z)\overline{G(\bar{z})}] + [-F(z)G(z) + G(z)F(z)]J \\ &= F(z)\overline{F(\bar{z})} + G(z)\overline{G(\bar{z})}. \end{aligned}$$

This shows that $f^s(\Omega_I) \subseteq L_I$ for every $I \in \mathbb{S}$, that is, that f^s is slice preserving.

The previous definitions are well posed and coherent with those given in the special case $\Omega = B(0, R)$ (by direct computation).

We conclude this section by proving a few simple but important consequences of Definition 1.27.

Proposition 1.37. *Let f and g be regular functions on a symmetric slice domain Ω. Then $(f * g)^c = g^c * f^c$.*

Proof. Follows immediately from Definitions 1.27 and 1.34. □

Proposition 1.38. *Let f and g be regular functions on a symmetric slice domain Ω. Then $(f * g)^s = f^s g^s = g^s f^s$.*

Proof. Follows immediately from Definitions 1.27, 1.35, and from the fact that f^s is slice preserving. □

Proposition 1.39. *Let f be a regular function on a symmetric slice domain Ω. If f is slice preserving then $f^c(q) = f(q)$ and $f^s(q) = f(q)^2$ for all $q \in \Omega$.*

Proof. Follows from Definition 1.34 and Lemma 1.30. □

We finally prove the Leibniz formula for slice derivatives.

Proposition 1.40. *Let f and g be regular functions on a symmetric slice domain Ω. Then $(f * g)' = f' * g + f * g'$.*

Proof. Let I be any element of $\mathbb{S}$ and let f_I, g_I be the restrictions of f, g to L_I. Since for every regular function h defined on Ω the equality $(h')_I = (h_I)'$ holds, by the definition of $*$-product and the identity principle, we have

$$(f * g)' = ext\left\{[(f * g)']_I\right\} = ext\left\{[(f * g)_I]'\right\} = ext\left\{(f_I * g_I)'\right\}.$$

Therefore, it is enough to prove that

$$(f_I * g_I)' = f_I' * g_I + f_I * g_I',$$

which can be easily done by applying the Splitting Lemma 1.3 to both f and g, so reducing the problem to the case of the Leibniz rule for holomorphic functions. □

For the sake of completeness, let us also mention the Leibniz formula for the spherical derivative (from [72]).

Proposition 1.41. *Let f and g be regular functions on a symmetric slice domain Ω. Then $\partial_s(f * g) = (\partial_s f)(v_s g) + (v_s f)(\partial_s g)$.*

Bibliographic Notes

Regular functions were introduced (under the name of Cullen regular functions) in [61, 62]. The same articles proved the basic properties presented in Sect. 1.1 (such as the Identity Principle 1.12) for Euclidean balls centered at 0. The definition of slice regular quaternionic function (which requires "slicewise" differentiability instead of global differentiability) was given in [19]. All the aforementioned properties are extended to slice domains in the same paper [19] (except for the Identity Principle 1.12, whose extension was proven in [119]). The notation $\partial_c f$ for the slice derivative of a regular function f derives from the original papers [61, 62] where the subscript c stood for Cullen.

The Representation Formula (1.8) was proven in [19], while (1.9) was proven in [18] (the special case of power series had been considered in [56]). The new formula (1.10) is presented here for the first time, while Proposition 1.19 derives from [69].

The extension results in Sect. 1.3 are all original contributions of [19]. Finally, the algebraic structure presented in Sect. 1.4 was constructed in [55] for power series, and in [19] for regular functions on symmetric slice domains.

Chapter 2
Regular Power Series

As we saw in Chap. 1, regular quaternionic functions on a ball $B(0,R)$ behave as holomorphic complex functions on a disk $\Delta(0,R) = \{z \in \mathbb{C} : |z| < R\}$. Specifically, Propositions 1.6 and 1.9 show that the set of regular functions on $B(0,R)$ consists exactly of the power series $\sum_{n\in\mathbb{N}} q^n a_n$ centered at 0 and having radius of convergence greater than (or equal to) R. The same holds for a ball $B(p,R) = \{q \in \mathbb{H} : |q-p| < R\}$ centered at a point $p \in \mathbb{R}$ of the real axis of $\mathbb{H}$.

We will now see that the situation is much more bizarre if we take a point $p \in \mathbb{H}$ that is not real (i.e., whose imaginary part does not vanish). Indeed, for all such p, a series $\sum_{n\in\mathbb{N}}(q-p)^n a_n$ does not define a regular function on its domain of convergence (except for the trivial case $a_0+(q-p)a_1$) because of the following fact:

Example 2.1. Let $p \in \mathbb{H}\setminus\mathbb{R}$. The function $\mathbb{H}\to\mathbb{H}$ defined by

$$q \mapsto (q-p)^2 = q^2 - qp - pq + p^2$$

is not regular. The same holds for $q \mapsto (q-p)^n$ for all $n \in \mathbb{N}, n > 1$.

Conversely, the possibility of a series expansion of the type $\sum_{n\in\mathbb{N}}(q-p)^n a_n$ is excluded even for polynomials, as shown by the next example.

Example 2.2. Let $p \in \mathbb{H}\setminus\mathbb{R}$. The regular function $q \mapsto q^2$ does not admit an expansion of the type

$$q^2 = (q-p)^2 a_2 + (q-p)a_1 + a_0$$

on $\mathbb{H}$. If it did, then we would have $a_2 = 1$ and

$$0 = -qp - pq + p^2 + (q-p)a_1 + a_0,$$

from which $a_0 = p^2 - pa_1$ and $pq = q(a_1 - p)$ would follow. The latter would imply $a_1 - p = p$ and $pq = qp$, an equality that cannot hold for an arbitrary $q \in \mathbb{H}$. Specifically, it holds if and only of q lies in the same complex plane as p.

G. Gentili et al., *Regular Functions of a Quaternionic Variable*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-33871-7_2,

In both examples, the problem is the lack of commutativity, which makes the monomial pq different from qp and not regular. For this reason, we instead consider the *regular power* $(q-p)^{*n} := (q-p) * (q-p) * \cdots * (q-p)$ (n times) instead of $(q-p)^n$. This immediately solves the issue in the last example, as shown by next lemma.

Lemma 2.3 (Binomial Formula). *For all $p \in \mathbb{H}$ and for all $n \in \mathbb{N}$,*

$$q^n = \sum_{k=0}^{n} (q-p)^{*k} p^{n-k} \binom{n}{k} \tag{2.1}$$

Proof. The result is trivial for $n = 1$. We now suppose it holds for n and prove it for $n+1$:

$$\begin{aligned}
q^{n+1} &= q * q^n = [(q-p)+p] * \sum_{k=0}^{n} (q-p)^{*k} p^{n-k} \binom{n}{k} = \\
&= \sum_{k=0}^{n} (q-p)^{*k+1} p^{n-k} \binom{n}{k} + \sum_{k=0}^{n} (q-p)^{*k} p^{n-k+1} \binom{n}{k} = \\
&= (q-p)^{*n+1} + \sum_{h=1}^{n} (q-p)^{*h} p^{n-h+1} \binom{n}{h-1} + p^{n+1} \\
&\quad + \sum_{k=1}^{n} (q-p)^{*k} p^{n-k+1} \binom{n}{k} = \\
&= p^{n+1} + \sum_{h=1}^{n} (q-p)^{*h} p^{n-h+1} \left[\binom{n}{h-1} + \binom{n}{h} \right] + (q-p)^{*n+1} = \\
&= p^{n+1} + \sum_{h=1}^{n} (q-p)^{*h} p^{n-h+1} \binom{n+1}{h} + (q-p)^{*n+1} = \\
&= \sum_{h=0}^{n+1} (q-p)^{*h} p^{n+1-h} \binom{n+1}{h},
\end{aligned}$$

as desired. □

This result suggests the following definition:

Definition 2.4. For any sequence $\{a_n\}_{n \in \mathbb{N}}$ in $\mathbb{H}$, we call

$$\sum_{n \in \mathbb{N}} (q-p)^{*n} a_n \tag{2.2}$$

the *regular power series centered at p* associated to $\{a_n\}_{n \in \mathbb{N}}$.

In this chapter, we will see that a regular function expands into regular power series at each point p of its domain of definition. The consequent notion of analyticity turns out to be equivalent to regularity.

2.1 The Distance σ

In order to study the convergence of the series in (2.2), we begin by defining the following function:

Definition 2.5. For all $p, q \in \mathbb{H}$

$$\sigma(q,p) = \begin{cases} |q-p| & \text{if } p, q \text{ lie on the same complex line } L_I \\ \omega(q,p) & \text{otherwise} \end{cases} \tag{2.3}$$

where

$$\omega(q,p) = \sqrt{[Re(q) - Re(p)]^2 + [|Im(q)| + |Im(p)|]^2}. \tag{2.4}$$

(See Fig. 2.1)

Lemma 2.6. *Let $p \in L_I$, choose any $q \in \mathbb{H} \setminus L_I$, and let $z, \bar{z}$ be the points of L_I such that $Re(z) = Re(\bar{z}) = Re(q)$ and $|Im(z)| = |Im(\bar{z})| = |Im(q)|$. Then*

$$\sigma(q,p) = \max\{|z-p|, |\bar{z}-p|\}. \tag{2.5}$$

Proof. The proof is by direct computation:

$$\max\{|z-p|, |\bar{z}-p|\}^2 =$$
$$= (Re(z) - Re(p))^2 + \max\{|Im(z) - Im(p)|^2, |-Im(z) - Im(p)|^2\} =$$
$$= (Re(z) - Re(p))^2 + (|Im(z)| + |Im(p)|)^2 =$$
$$= (Re(q) - Re(p))^2 + (|Im(q)| + |Im(p)|)^2 = \sigma(q,p)^2.$$

□

Proposition 2.7. *The function $\sigma : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ is a distance.*

Proof. By definition, $\sigma(q,p) \geq 0$ for all $p, q \in \mathbb{H}$ and $\sigma(q,p) = 0$ if and only if $q = p$. It is also clear that $\sigma(q,p) = \sigma(p,q)$. The triangle inequality can be proven as follows. Let L_I be a complex plane through p. If $q \in L_I$, then for all $v \in \mathbb{H}$,

$$\sigma(q,v) + \sigma(v,p) \geq |q-v| + |v-p| \geq |q-p| = \sigma(q,p)$$

as wanted. Now suppose $q \in \mathbb{H} \setminus L_I$, that is, $q = x + yJ \in L_J$ with $y > 0$, $J \in \mathbb{S} \setminus \{\pm I\}$.

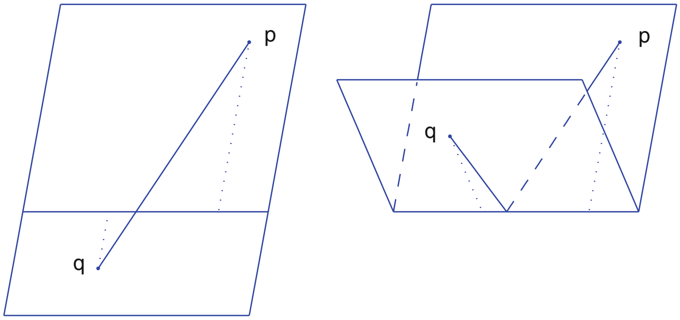

Fig. 2.1 Examples of "how to compute" $\sigma(q, p)$ when p and q lie in the same complex plane and when they do not. *Source*: [58]

(a) For all $v \in L_I$, the previous remark implies

$$\sigma(q, v) + \sigma(v, p) = \max\{|x + yI - v|, |x - yI - v|\} + |v - p| \geq$$
$$\geq \max\{|x + yI - p|, |x - yI - p|\} = \sigma(q, p).$$

(b) For all $v \in L_J$, the triangle inequality follows from (a), reversing the roles of q and p.

(c) Finally, for $v \in \mathbb{H} \setminus (L_I \cup L_J)$, we prove it as follows. Let $z, \bar{z}$ be the two points of L_I having $Re(z) = Re(v) = Re(\bar{z})$ and $|Im(z)| = |Im(v)| = |Im(\bar{z})|$. Without loss of generality, $\max\{|z - p|, |\bar{z} - p|\} = |z - p|$, and the previous remark implies

$$\sigma(v, p) = \max\{|z - p|, |\bar{z} - p|\} = |z - p| = \sigma(z, p)$$

where we used the fact that v and p do not lie in the same complex plane, while z and p do. Furthermore, taking into account that q does not lie in the same complex plane as v nor in the same as z, we compute

$$\sigma(q, v) = \omega(q, v) = \sqrt{[Re(q) - Re(v)]^2 + (|Im(q)| + |Im(v)|)^2} =$$
$$= \sqrt{[Re(q) - Re(z)]^2 + (|Im(q)| + |Im(z)|)^2} = \omega(q, z) = \sigma(q, z).$$

Hence,

$$\sigma(q, v) + \sigma(v, p) = \sigma(q, z) + \sigma(z, p) \geq \sigma(q, p)$$

where the last inequality follows from (a) since z, p lie in the same complex plane L_I.

□

Definition 2.8. The σ-*ball* of radius R centered at p is the set

$$\Sigma(p, R) = \{q \in \mathbb{H} : \sigma(q, p) < R\}. \tag{2.6}$$

Furthermore, we define $\Omega(p, R) = \{q \in \mathbb{H} : \omega(q, p) < R\}$.

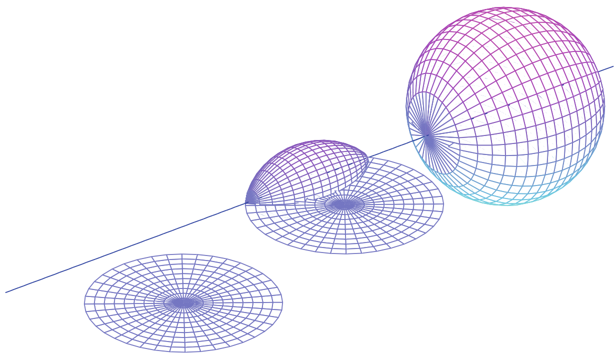

Fig. 2.2 A view in $\mathbb{R} + i\mathbb{R} + j\mathbb{R}$ of σ-balls $\Sigma(p, R)$ centered at points $p \in L_i = \mathbb{R} + i\mathbb{R}$ and having $|Im(p)| \geq R$, $0 < |Im(p)| < R$ and $Im(p) = 0$, respectively. *Source*: [58]

The σ-balls are pictured in Fig. 2.2 and described by next remark.

Remark 2.9. Let $p \in L_I \subset \mathbb{H}$ and $R \geq 0$.

1. If $R \leq |Im(p)|$, then $\Sigma(p, R)$ reduces to the Euclidean disk $\Delta_I(p, R)$ in L_I (and the set $\Omega(p, R)$ is empty).
2. If $R > |Im(p)| > 0$, then $\Sigma(p, R) = \Delta_I(p, R) \cup \Omega(p, R)$, and $\Omega(p, R)$ is the symmetric completion of $\Delta_I(p, R) \cap \Delta_I(\bar{p}, R)$ (see Chap. 1).
3. If $p \in \mathbb{R}$, that is, $Im(p) = 0$, then $\Sigma(p, R)$ coincides with the Euclidean ball $B(p, R)$ (and with $\Omega(p, R)$).

In all three cases, the interior of $\Sigma(p, R)$ with respect to the Euclidean topology is $\Omega(p, R)$, and its Euclidean closure is $\overline{\Sigma(p, R)} = \{q \in \mathbb{H} : \sigma(q, p) \leq R\}$. Notice that $\Omega(p, R)$ is a symmetric slice domain when it is not empty. Furthermore, p is in the interior of $\Sigma(p, R)$, that is, $p \in \Omega(p, R)$, if and only if $2|Im(p)| < R$.

2.2 Convergence of Power Series Centered at p

The modulus of $(q - p)^{*n}$ can be estimated in terms of $\sigma(q, p)$ as follows:

Proposition 2.10. *Fix $p \in \mathbb{H}$. Then*

$$|(q - p)^{*n}| \leq \sigma(q, p)^n \tag{2.7}$$

for all $n \in \mathbb{N}$. Moreover,

$$\lim_{n\to\infty} |(q - p)^{*n}|^{1/n} = \sigma(q, p). \tag{2.8}$$

Proof. Let L_I be the complex plane through p. For all $z = x + yI \in L_I$, z and p commute so that $(q-p)^{*n}$ equals $(z-p)^n$ when computed at $q = z$. For any $q = x + yJ$, Formula (1.9) implies

$$(q-p)^{*n} = \frac{1}{2}\left[(z-p)^n + (\bar{z}-p)^n\right] - \frac{JI}{2}\left[(z-p)^n - (\bar{z}-p)^n\right].$$

Notice that, in vector notation,

$$JI = - < J, I > + J \times I. \tag{2.9}$$

If $< J, I >= \cos\theta$, then $J \times I = (\sin\theta)L$ where $L \in \mathbb{S}$ is orthogonal to I. Hence,

$$|(q-p)^{*n}|^2 = \left|\frac{1}{2}[(z-p)^n + (\bar{z}-p)^n] + \frac{\cos\theta}{2}[(z-p)^n - (\bar{z}-p)^n]\right|^2 +$$

$$+\frac{\sin^2\theta}{4}\,|(z-p)^n - (\bar{z}-p)^n|^2 =$$

$$= \frac{1}{4}\,|(z-p)^n + (\bar{z}-p)^n|^2 + \frac{\cos^2\theta + \sin^2\theta}{4}\,|(z-p)^n - (\bar{z}-p)^n|^2 +$$

$$+\frac{\cos\theta}{2}\,\langle (z-p)^n + (\bar{z}-p)^n, (z-p)^n - (\bar{z}-p)^n\rangle$$

which attains its maximum value when $\cos\theta = 1$ or when $\cos\theta = -1$, in other words when $J = I$ or when $J = -I$; that is, at $q = z$ or at $q = \bar{z}$. Since $|(q-p)^{*n}|$ equals $|z-p|^n$ at $q = z$ and $|\bar{z}-p|^n$ at $q = \bar{z}$,

$$|(q-p)^{*n}| \le \max\{|z-p|^n, |\bar{z}-p|^n\} = \max\{|z-p|, |\bar{z}-p|\}^n = \sigma(q,p)^n$$

for all $q \notin L_I$. If, on the contrary, $q \in L_I$, then $|(q-p)^{*n}| = |q-p|^n = \sigma(q,p)^n$. This proves the first statement.

As for the second, it is trivial in the case $q \in L_I$. When $q \notin L_I$, that is, $y \neq 0$ and $J \neq \pm I$, we prove it as follows. If $|z-p| = |\bar{z}-p|$, then $(y - |Im(p)|)^2 = (-y - |Im(p)|)^2$; hence, $Im(p) = 0$, that is, $p \in \mathbb{R}$, so that $(q-p)^{*n}$ coincides with $(q-p)^n$ and the thesis is trivial. Let us thus suppose $|z-p| \neq |\bar{z}-p|$. Without loss of generality, $|z-p| < |\bar{z}-p|$ and in particular $\sigma(q,p) = \max\{|z-p|, |\bar{z}-p|\} = |\bar{z}-p|$. Since

$$(q-p)^{*n} = \frac{1-JI}{2}(z-p)^n + \frac{1+JI}{2}(\bar{z}-p)^n$$

we have

$$\frac{|(q-p)^{*n}|}{\sigma(q,p)^n} = \left|\frac{1-JI}{2}\left(\frac{z-p}{\bar{z}-p}\right)^n + \frac{1+JI}{2}\right|$$

where $\left|\frac{z-p}{\bar{z}-p}\right|^n \to 0$ as $n \to +\infty$ and $\frac{1+JI}{2} \neq 0$. We easily conclude that

$$\lim_{n\to+\infty} \frac{|(q-p)^{*n}|^{1/n}}{\sigma(q,p)} = 1$$

which is equivalent to our thesis. □

Proposition 2.10 allows to study the convergence of regular power series centered at a generic point $p \in \mathbb{H}$.

Theorem 2.11. *Choose any sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\mathbb{H}$ and let $R \in (0, +\infty]$ be such that $1/R = \limsup_{n\to+\infty} |a_n|^{1/n}$. For all $p \in \mathbb{H}$, the series*

$$f(q) = \sum_{n\in\mathbb{N}} (q-p)^{*n} a_n \tag{2.10}$$

*converges absolutely and uniformly on the compact subsets of $\Sigma(p, R)$, and it does not converge at any point of $\mathbb{H} \setminus \overline{\Sigma(p, R)}$ (we call R the σ-*radius of convergence *of $f(q)$). Furthermore, if $\Omega(p, R) \neq \emptyset$, then the sum of the series defines a regular function $f : \Omega(p, R) \to \mathbb{H}$.*

Proof. In each set $\overline{\Sigma(p,r)}$ with $r < R$, the function series $\sum_{n\in\mathbb{N}}(q-p)^{*n}a_n$ is dominated by the convergent number series $\sum_{n\in\mathbb{N}} r^n|a_n|$ thanks to Proposition 2.10. By the same proposition, $\lim_{n\to+\infty} |(q-p)^{*n}|^{1/n} = \sigma(q, p)$, so that

$$\limsup_{n\to+\infty} |(q-p)^{*n}a_n|^{1/n} = \frac{\sigma(q,p)}{R}.$$

Hence, the series cannot converge at any point q such that $\sigma(q, p) > R$. Now, if $\Omega(p, R) \neq \emptyset$, then each addend $(q-p)^{*n}a_n$ of the series defines a regular function on $\Omega(p, R)$. Since the convergence is uniform on compact sets, we easily deduce that the sum of the series is regular in $\Omega(p, R)$, too. □

2.3 Series Expansion at p and Analyticity

It is now possible to prove an expansion result:

Theorem 2.12. *Let f be a regular function on a domain $\Omega \subseteq \mathbb{H}$ and let $p \in \Omega$. In each σ-ball $\Sigma(p, R)$ contained in Ω, the function f expands as*

$$f(q) = \sum_{n\in\mathbb{N}} (q-p)^{*n} \frac{1}{n!} f^{(n)}(p). \tag{2.11}$$

Proof. Let $p \in L_I$. Since $\Omega \supseteq \Sigma(p, R)$, we conclude that $\Omega_I \supseteq \Sigma(p, R) \cap L_I = \Delta_I(p, R)$. By the properties of holomorphic functions of one complex variable,

$$f_I(z) = \sum_{n \in \mathbb{N}} (z - p)^n \frac{1}{n!} f^{(n)}(p)$$

for all $z \in \Delta_I(p, R)$. By Theorem 2.11, the series in (2.11) must converge in $\Sigma(p, R)$. If $\Omega(p, R)$ is empty, then the assertion is proved. Otherwise, the series in (2.11) defines a regular function g on the symmetric slice domain $\Omega(p, R)$. Since $f_I \equiv g_I$ in $\Delta_I(p, R) \cap \Delta_I(\bar{p}, R)$, the Identity Principle 1.12 allows us to conclude that f and g coincide in $\Omega(p, R)$ (hence in $\Sigma(p, R)$, as desired). □

The previous result inspires the following definition of analyticity. Notice that the distance σ defines a topology on $\mathbb{H}$ that is finer than the Euclidean topology.

Definition 2.13. Let Ω be a σ-open subset of $\mathbb{H}$. A function $f : \Omega \to \mathbb{H}$ is *σ-analytic at $p \in \Omega$* if there exists a regular power series $\sum_{n \in \mathbb{N}} (q - p)^{*n} a_n$ converging in a σ-neighborhood U of p in Ω such that $f(q) = \sum_{n \in \mathbb{N}} (q - p)^{*n} a_n$ for all $q \in U$. We say that f is *σ-analytic* if it is σ-analytic at all $p \in \Omega$.

The subsequent Corollary immediately follows from Theorems 2.11 and 2.12.

Corollary 2.14. *A quaternionic function is regular in a domain if, and only if, it is σ-analytic in the same domain.*

It is also possible to define a different notion of analyticity, which uses the Euclidean topology and which plays an important role in the case of regular functions on slice domains.

Definition 2.15. Let Ω be an open subset of $\mathbb{H}$. A function $f : \Omega \to \mathbb{H}$ is *strongly analytic at $p \in \Omega$* if there exists a regular power series $\sum_{n \in \mathbb{N}} (q - p)^{*n} a_n$ converging in a Euclidean neighborhood U of p in Ω such that $f(q) = \sum_{n \in \mathbb{N}} (q - p)^{*n} a_n$ for all $q \in U$. We say that f is *strongly analytic* if it is strongly analytic at all $p \in \Omega$.

Proposition 2.16. *If $f : \Omega \to \mathbb{H}$ is a regular function then f is strongly analytic at all points of the set*

$$\mathcal{A}(\Omega) = \{p \in \Omega : 2|Im(p)| < R_p\}$$

where $R_p = \sup\{R > 0 : \Sigma(p, R) \subseteq \Omega\}$.

Proof. According to Theorem 2.12, f expands into regular power series at any $p \in \Omega$, and its expansion is valid in $\Sigma(p, R)$ whenever $\Sigma(p, R) \subseteq \Omega$, that is, whenever $R \leq R_p$. As we explained in Sect. 2.1, p is in the Euclidean interior of $\Sigma(p, R)$ if and only if $2|Im(p)| < R$. The thesis immediately follows. □

We will now compute explicitly the set $\mathcal{A}(\Omega)$ in a couple of natural cases.

Remark 2.17. Let $B = B(0, R)$. The set $\mathcal{A}(B) = \{p \in \mathbb{H} : 2|Im(p)| < R - |p|\} \subset B$ is the (open) region bounded by the hypersurface consisting of all points $p \in B$ such that $x = Re(p)$ and $y = |Im(p)|$ verify

$$x^2 - 3\left(y - \frac{2}{3}R\right)^2 + \frac{R^2}{3} = 0. \tag{2.12}$$

In other words, if for any $I \in \mathbb{S}$ we consider the arc of hyperbola $\mathcal{H}(I) = \{x + yI \in L_I : 0 \leq y < R,\ x, y \text{ verify } (2.12)\}$, then the boundary of $\mathcal{A}(B)$ is the hypersurface of revolution generated rotating $\mathcal{H}(I)$ around the real axis as follows:

$$\partial\mathcal{A}(B) = \bigcup_{J \in \mathbb{S}} \mathcal{H}(J).$$

Remark 2.18. Let f be a regular power series centered at $p = x_0 + y_0 I$ and converging in a σ-ball $\Sigma(p, R) = \Omega(p, R) \cup \Delta_I(p, R)$ (such as the one at the center of Fig. 2.2). The set $\mathcal{A}(\Omega(p, R)) \subset \Omega(p, R)$ is the (open) region bounded by the hypersurface of points $s \in \Omega(p, R)$ such that $x = Re(s)$ and $y = |Im(s)|$ solve the equation

$$(x - x_0)^2 - 3\left(y - \frac{y_0 + 2R}{3}\right)^2 + \frac{(2y_0 + R)^2}{3} = 0, \tag{2.13}$$

that is, the hypersurface of revolution $\bigcup_{J \in \mathbb{S}} \mathcal{H}(J)$ where $\mathcal{H}(J)$ is the arc of hyperbola

$$\mathcal{H}(J) = \{x + yJ : 0 \leq y \leq R - y_0,\ x, y \text{ verify } (2.13)\}$$

for all $J \in \mathbb{S}$. Furthermore, f is also strongly analytic in $\mathcal{A}(\Delta_I(p, R))$, which is the plane region in L_I lying between $\mathcal{H}(-I)$ and the arc of hyperbola $\mathcal{K}(I)$ consisting of all points $x + yI$ with $0 \leq y \leq R - y_0$ and

$$(x - x_0)^2 - 3\left(y + \frac{y_0 - 2R}{3}\right)^2 + \frac{(2y_0 - R)^2}{3} = 0. \tag{2.14}$$

Remark 2.19. We notice that $\mathcal{H}(I)$ always lies between $\mathcal{H}(-I)$ and $\mathcal{K}(I)$ in L_I, so that $\mathcal{A}(\Delta_I(p, R))$ always includes $\mathcal{A}(\Omega(p, R)) \cap L_I$ (which is the plane region lying between $\mathcal{H}(-I)$ and $\mathcal{H}(I)$). We also notice that $\mathcal{K}(I)$ always lies in $\Delta_I(p, R) \cap \Delta_I(\bar{p}, R)$, so that $\mathcal{A}(\Delta_I(p, R))$ is always included in $\Omega(p, R)$. Fig. 2.3 portrays the set $\mathcal{A}(\Sigma(p, R)) = \mathcal{A}(\Omega(p, R)) \cup \mathcal{A}(\Delta_I(p, R)) \subset \Omega(p, R)$.

Note that if Ω intersects the real axis, then $\mathcal{A}(\Omega)$ always contains a symmetric slice domain that neighbors $\Omega \cap \mathbb{R}$. However, $\mathcal{A}(\Omega)$ is strictly smaller than Ω, except for the special case $\mathcal{A}(\mathbb{H}) = \mathbb{H}$.

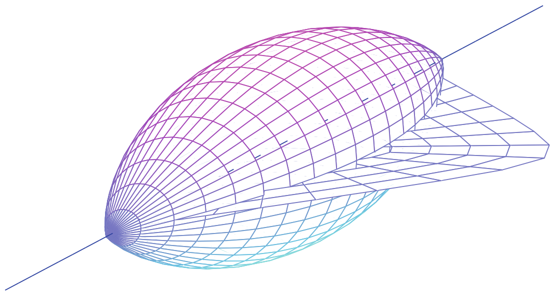

Fig. 2.3 A view in $\mathbb{R} + i\mathbb{R} + j\mathbb{R}$ of $\mathcal{A}(\Sigma(p, R))$, when $p \in \mathbb{R} + i\mathbb{R}$. *Source*: [58]

Corollary 2.20. *The space of quaternionic* entire functions*, that is, of regular functions* $\mathbb{H} \to \mathbb{H}$*, coincides with the space of strongly analytic functions in* $\mathbb{H}$*.*

We conclude this section with an example that proves the sharpness of Proposition 2.16. In the complex case, as explained in [108] Chap. 5 §3, the lacunary series $\sum_{n\in\mathbb{N}} z^{2^n}$ converges in the open unit disk of $\mathbb{C}$, and it does not extend to a holomorphic function near any point of the boundary.

Example 2.21. The sum of the quaternionic series

$$f(q) = \sum_{n\in\mathbb{N}} q^{2^n},$$

which converges in $\mathbb{B} = B(0, 1)$, is strongly analytic in $\mathcal{A}(\mathbb{B})$ only. Indeed, suppose f were strongly analytic at $p \in \mathbb{B} \setminus \mathcal{A}(\mathbb{B})$ (i.e., at $p \in \mathbb{B}$ with $2|Im(p)| \geq 1 - |p|$). There would exist a σ-ball $\Sigma(p, r)$ of radius $r > 2|Im(p)|$ and a regular power series $\sum_{n\in\mathbb{N}}(q - p)^{*n} a_n$ converging in $\Sigma(p, r)$ and coinciding with $f(q)$ for all $q \in \mathbb{B} \cap \Sigma(p, r)$. The coefficients a_n would have to lie in the complex plane L_I through p because $f(\Delta_I(0, 1)) \subseteq L_I$ (using a slight variation of Corollary 2.8 in [62]). Restricting to $L_I \simeq \mathbb{C}$, we would have $f(z) = \sum_{n\in\mathbb{N}}(z - p)^n a_n$ for all $z \in \Delta_I(0, 1) \cap \Delta_I(p, r)$, with $\Delta_I(p, r)$ not contained in the unit disk $\Delta_I(0, 1)$ because $r > 2|Im(p)| \geq 1-|p|$. The sum of the series $\sum_{n\in\mathbb{N}}(z-p)^n a_n$ in $\Delta_I(p, R)$ would thus extend $\sum_{n\in\mathbb{N}} z^{2^n}$ near some point of the boundary of $\Delta_I(0, 1)$, which is impossible.

Bibliographic Notes

The results on power series expansion and analyticity presented in this chapter were proven in [58], with the exception of Theorem 2.10. Indeed, a first estimate of $|(q - p)^{*n}|$ was presented in the same paper, but the optimal estimate $|(q - p)^{*n}| \leq \sigma(q, p)^n$ was attained in [123]. Recent work on analyticity includes [122].

Chapter 3
Zeros

3.1 Basic Properties of the Zeros

We now discuss the properties of the zero sets of regular functions. The first and central algebraic result is the following:

Theorem 3.1. *If f is a regular function on a symmetric slice domain Ω vanishing at a point $x + yI$, then either f vanishes identically in $x + y\mathbb{S}$ or f does not have any other zero in $x + y\mathbb{S}$.*

Proof. This is an immediate consequence of the Representation Formula 1.15. $\square$

Using the same idea, we obtain the following useful lemmas:

Lemma 3.2. *Let f be a regular function on a symmetric slice domain Ω and suppose that $f(\Omega_I) \subseteq L_I$ for a given $I \in \mathbb{S}$. If $f(x + yJ) = 0$ for some $J \in \mathbb{S} \setminus \{\pm I\}$, then $f(x + yI) = 0$ for all $I \in \mathbb{S}$.*

Proof. If $f(x + yI), f(x - yI) \in L_I$, and $f(x + yJ) = 0$ for some $J \in \mathbb{S} \setminus \{\pm I\}$, then, by Formula (1.9), $f(x + yI) + f(x - yI) = 0 = f(x + yI) - f(x - yI)$. The thesis immediately follows. $\square$

Lemma 3.3. *Let f be a slice preserving regular function on a symmetric slice domain Ω. If $f(x + yJ) = 0$ for some $J \in \mathbb{S}$, then $f(x + yI) = 0$ for all $I \in \mathbb{S}$.*

Proof. Since $f(\Omega_I) \subseteq L_I$ for all $I \in \mathbb{S}$, we also have $f(\Omega \cap \mathbb{R}) \subseteq \mathbb{R}$. The holomorphic function $f_J : \Omega_J \to L_J$ fulfills the hypotheses of the (complex) Schwarz Reflection Principle; hence, $f(x+yJ) = \overline{f(x - yJ)}$ for all $x+yJ \in \Omega_J$. Thus, $f(x + yJ) = 0$ implies $f(x - yJ) = 0$, and Theorem 3.1 allows us to conclude. $\square$

Recall that, as pointed out in Remark 1.31, every slice preserving function expands, at any real point, into a power series with real coefficients.

G. Gentili et al., *Regular Functions of a Quaternionic Variable*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-33871-7_3,

Theorem 3.1 and Lemma 3.3, along with the algebraic operations defined in Sect. 1.4, are the tools for a complete study of the zero sets of regular functions.

3.2 Algebraic Properties of the Zeros

Let us start by presenting an alternative expression of the regular product $f * g$. In the special case where $f(q) = \sum_{n\in\mathbb{N}} q^n a_n$ and $g(q) = \sum_{n\in\mathbb{N}} q^n b_n$, we compute

$$f * g(q) = \sum_{n\in\mathbb{N}} q^n \sum_{k=0}^{n} a_k b_{n-k} = \sum_{m\in\mathbb{N}} q^m \left(\sum_{k\in\mathbb{N}} q^k a_k\right) b_m = \sum_{m\in\mathbb{N}} q^m f(q) b_m.$$

Therefore, if $f(q) \neq 0$, we have

$$f * g(q) = f(q) \sum_{m\in\mathbb{N}} f(q)^{-1} q^m f(q) b_m = f(q) g(f(q)^{-1} q f(q)),$$

while clearly $f * g(q) = 0$ if $f(q) = 0$ (but not if $g(q) = 0$). In fact, this result is independent of the power series representation.

Theorem 3.4. *Let f, g be regular functions on a symmetric slice domain Ω. For all $q \in \Omega$, if $f(q) \neq 0$, then*

$$f * g(q) = f(q) g(f(q)^{-1} q f(q)). \tag{3.1}$$

On the other hand, if $f(q) = 0$, then $f * g(q) = 0$.

Proof. Let $\psi : \Omega \to \mathbb{H}$ be the function defined by

$$\psi(q) = \begin{cases} 0 & \text{if } f(q) = 0 \\ f(q) g(f(q)^{-1} q f(q)) & \text{if } f(q) \neq 0. \end{cases} \tag{3.2}$$

By hypothesis, Ω contains a real point, which we may suppose to be the origin without loss of generality. The computations performed for power series show that ψ coincides with $f * g$ in a neighborhood of the origin in Ω. If the function ψ is regular in Ω, then the Identity Principle 1.12 implies the thesis. The regularity of ψ is proven as follows. For each $I \in \mathbb{S}$ and for $z = x + yI \in \Omega_I$ with $f(z) \neq 0$, we have that $f(z)^{-1} I f(z) \in \mathbb{S}$ and Formula (1.9) implies that

$$g(f(z)^{-1} z f(z)) = g(x + y f(z)^{-1} I f(z)) =$$

$$= \frac{1}{2}[g(z) + g(\bar{z})] - f(z)^{-1} I f(z) \frac{I}{2}[g(z) - g(\bar{z})]$$

so that

$$2\psi(z) = 2f(z)g(f(z)^{-1}zf(z)) = f(z)[g(z) + g(\bar{z})] - If(z)I[g(z) - g(\bar{z})].$$

The equality $2\psi(z) = f(z)[g(z) + g(\bar{z})] - If(z)I[g(z) - g(\bar{z})]$ holds at all $z \in \Omega_I$ (also when $f(z) = 0$); hence,

$$2\bar{\partial}_I \psi(z) = \bar{\partial}_I\{f(z)[g(z) + g(\bar{z})]\} - \bar{\partial}_I\{If(z)I[g(z) - g(\bar{z})]\} =$$
$$= \left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f(z) \cdot [g(z) + g(\bar{z})] +$$
$$+ f(z)\frac{\partial}{\partial x}[g(z) + g(\bar{z})] + If(z)\frac{\partial}{\partial y}[g(z) + g(\bar{z})] +$$
$$-I\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f(z) \cdot I[g(z) - g(\bar{z})] +$$
$$-If(z)I\frac{\partial}{\partial x}[g(z) - g(\bar{z})] + f(z)I\frac{\partial}{\partial y}[g(z) - g(\bar{z})] =$$
$$= f(z)\bar{\partial}_I g(z) + f(z)\partial_I g(\bar{z}) - If(z)I\bar{\partial}_I g(z) + If(z)I\partial_I g(\bar{z}) = 0$$

where we first used the fact that $\bar{\partial}_I f(z) \equiv 0$, then the fact that $\bar{\partial}_I g(z) \equiv 0$ and $\partial_I g(\bar{z}) \equiv 0$. □

Corollary 3.5. *If f, g are regular functions on a symmetric slice domain Ω and $q \in \Omega$, then $f * g(q) = 0$ if and only if $f(q) = 0$ or $f(q) \neq 0$ and $g(f(q)^{-1}qf(q)) = 0$.*

Let us denote the zero set of f as Z_f. Notice that $f(q)^{-1}qf(q)$ belongs to the same sphere $x + y\mathbb{S}$ as q. Hence each zero of $f * g$ in $x + y\mathbb{S}$ corresponds to a zero of f or to a zero of g in the same sphere. However, the correspondence between Z_{f*g} and $Z_f \cup Z_g$ need not be one-to-one:

Example 3.6. For all $I \in \mathbb{S}$, the regular product

$$(q - I) * (q + I) = q^2 + 1$$

has $\mathbb{S}$ as its zero set, while $q - I$ only vanishes at I and $q + I$ only vanishes at $-I$.

Example 3.7. Choose $I, J \in \mathbb{S}$ such that $I \neq \pm J$ and consider the regular product

$$(q - I) * (q - J) = q^2 - q(I + J) + IJ.$$

By direct computation, the product vanishes at I, but it has no other zero in $\mathbb{S}$.

It is also possible to study the effect of conjugation and symmetrization on the zero set.

Lemma 3.8. *Let f be a regular function on a symmetric slice domain Ω and let f^s be its symmetrization. Then for each $S = x + y\mathbb{S} \subset \Omega$, either f^s vanishes identically in S or it has no zeros in S.*

Proof. It is an immediate consequence of Remark 1.36. □

Proposition 3.9. *Let f be a regular function on a symmetric slice domain Ω and choose $S = x + y\mathbb{S} \subset \Omega$. The zeros of f in S are in one-to-one correspondence with those of f^c. Furthermore, f^s vanishes identically on S if and only if f^s has a zero in S, if and only if f has a zero in S (if and only if f^c has a zero in S).*

Proof. Let $p = x + yJ$ be a zero of f. Then $f^s = f * f^c$ vanishes at p by Corollary 3.5. Furthermore, the previous lemma implies that $f^s(x+yI) = 0$ for all $I \in \mathbb{S}$. Since $f^s(\bar{p}) = f^s(x-yJ) = 0$, by the same Corollary 3.5 either $f(\bar{p}) = 0$ or $f^c(f(\bar{p})^{-1}\bar{p}f(\bar{p})) = 0$. In the first case f vanishes identically on S, which implies that f^c vanishes on S (by the definition of f^c). In the second case, f^c vanishes at

$$f(\bar{p})^{-1}\bar{p}f(\bar{p}) = x - [f(\bar{p})^{-1}Jf(\bar{p})]y \in S.$$

Thus, if f has a zero in S, then f^s has a zero in S, which leads to the vanishing of f^s on the whole S, which in turn implies the existence of a zero of f^c in S. Since $(f^c)^c = f$, the proof concludes if we exchange the roles of f and f^c. □

We have proven that

$$|Z_f \cap (x + y\mathbb{S}))| = |Z_{f^c} \cap (x + y\mathbb{S})|$$

for all $x + y\mathbb{S} \subset \Omega$ and that

$$Z_{f^s} = \bigcup_{x+yI \in Z_f} (x + y\mathbb{S}).$$

Example 3.10. For any $I \in \mathbb{S}$, let $f(q) = q - I$. By direct computation, $f^c(q) = q + I$ and $f^s(q) = q^2 + 1$. Clearly, $Z_f = \{I\}$ and $Z_{f^c} = \{-I\}$. Furthermore, $Z_{f^s} = \mathbb{S}$.

3.3 Topological Properties of the Zeros

Lemma 3.11. *Let f be a slice preserving regular function on a symmetric slice domain Ω. If $f \not\equiv 0$, then the zero set of f consists of isolated real points and isolated 2-spheres of the type $x + y\mathbb{S}$.*

Proof. The zero set Z_f of such an f consists of real points and 2-spheres of the type $x + y\mathbb{S}$ because of Lemma 3.3. For all $I \in \mathbb{S}$, the intersection $Z_f \cap L_I$ contains all the real zeros of f and exactly two zeros for each sphere $x + y\mathbb{S}$ on which f

vanishes (viz, $x + yI$ and $x - yI$). By the Identity Principle 1.12, the zeros of f in L_I must be isolated if $f \not\equiv 0$. This proves that each real point and each 2-sphere in Z_f is isolated. □

Theorem 3.12 (Structure of the Zero Set). *Let f be a regular function on a symmetric slice domain Ω. If f does not vanish identically, then the zero set of f consists of isolated points or isolated 2-spheres of the form $x + y\mathbb{S}$.*

Proof. By the previous lemma, the zero set of the symmetrization f^s consists of isolated real points or isolated 2-spheres. The real zeros of f and f^s are exactly the same by Proposition 3.9. Moreover, each 2-sphere in Z_{f^s} corresponds either to a 2-sphere or to a single point of Z_f. The thesis immediately follows. □

The previous theorem suggests the following Definition:

Definition 3.13. Let f be a regular function on a symmetric slice domain Ω. A 2-sphere $x + y\mathbb{S}$ of zeros of f is called a *spherical zero* of f. Any point $x + yI$ of such a sphere is called a *generator* of the spherical zero $x + y\mathbb{S}$. Finally, any zero of f that is not a generator of a spherical zero is called an *isolated zero*, a *nonspherical zero*, or simply a *zero* of f.

Corollary 3.14 (Strong Identity Principle). *Let f, g be regular functions on a symmetric slice domain Ω. If there exists $S = x + y\mathbb{S} \subset \Omega$ such that the zeros of $f - g$ in $\Omega \setminus S$ accumulate to a point of S, then $f = g$ in Ω.*

3.4 On the Roots of Quaternions

In this section we study the nice geometric features of the quaternionic solutions of the equation
$$q^m = q_0$$
for a given quaternion q_0. The results we prove are interesting because they highlight further differences between the complex and the quaternionic environment.

Theorem 3.15. *Let $q_0 = x_0 + y_0 I \in L_I$ be a nonzero element of $\mathbb{H}$. For $m \in \mathbb{N}\setminus\{0\}$, the polynomial*
$$P(q) = q^m - q_0 \tag{3.3}$$
has:

1. *m distinct zeros $q_1, \dots, q_m \in L_I \setminus \{0\}$, if q_0 is not real*
2. *p spherical zeros, and a nonspherical real zero, if q_0 is a real number and $m = 2p + 1$*
3. *$p - 1$ spherical zeros, and two distinct nonspherical real zeros, if q_0 is a positive real number and $m = 2p$*
4. *p spherical zeros, if q_0 is a negative real number and $m = 2p$*

Proof. To prove this result, it is enough to apply Lemma 3.2 to the given polynomial and, in view of Theorem 3.1, discuss the geometry of zeros of the complex polynomial P_I. We obtain that $\alpha \in L_I \backslash \mathbb{R}$ is a spherical zero of P if, and only if,

$$\alpha^m = \overline{\alpha}^m = q_0$$

that is, if and only if $\alpha^{2m} = |\alpha|^{2m}$ which corresponds to $(\alpha^m)^2 = (|\alpha|^m)^2$ and to

$$\alpha^m = \pm|\alpha|^m.$$

We conclude that $\alpha \in L_I \backslash \mathbb{R}$ is a spherical zero of P if, and only if, $q_0 \in \mathbb{R}$. The conclusion of the proof is now immediate. □

In the complex case, the map $f(z) = z^m$ has no critical points outside the origin. This is not the case in the quaternionic setting, as the following result shows.

Theorem 3.16. *Let $\mathbb{B} = B(0,1)$ be the open unit ball in $\mathbb{H}$. For any $m \in \mathbb{N}$, $m > 0$, the set of critical values of the differentiable map $f : \partial\mathbb{B} \to \partial\mathbb{B}$ defined by $f(q) = q^m$ consists of:*

1. *The empty set, if $m = 1$*
2. *The point -1, if $m = 2$*
3. *The two points 1 and -1, if $m \geq 3$*

Furthermore, $\deg(f) = m$.

Proof. Any point q in $\partial\mathbb{B}$ can be written as

$$q = \cos\varphi + (\sin\varphi)I,$$

for suitable $I \in \mathbb{S}$ and $\varphi \in [0, 2\pi]$, so that

$$q^m = \cos(m\varphi) + \sin(m\varphi)I.$$

Theorem 3.15 now implies that, for $m \geq 3$, the set $f^{-1}(1)$ contains a sphere of points of the form $\cos(2\pi/m) + \sin(2\pi/m)\mathbb{S}$. Let $q_0 = \cos(2\pi/m) + \sin(2\pi/m)I_0$ be one of these points and let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{S}$ be a differentiable curve such that $\gamma(0) = I_0$ and $\gamma'(0) \neq 0$. Then the differentiable curve $\sigma : (-\varepsilon, \varepsilon) \to \partial\mathbb{B}$ defined by

$$\sigma(t) = \cos(2\pi/m) + \sin(2\pi/m)\gamma(t)$$

is such that $\sigma(0) = q_0$ and $\sigma'(0) = \cos(2\pi/m) + \sin(2\pi/m)\gamma'(0) \neq 0$. It turns out that

$$\frac{d}{dt} f(\sigma(t))_{|_{t=0}} = \frac{d}{dt}[(\sigma(t))^m]_{|_{t=0}} = \frac{d}{dt}1 = 0$$

which, by definition, implies that df_{q_0} is singular, and hence that (q_0 is a critical point and) 1 is a critical value for f when $m \geq 3$. To prove that -1 is a critical value for f when $m \geq 2$, one uses exactly the same argument.

We will now prove that any $u_0 \in \partial\mathbb{B}$, $u_0 \notin \{-1, 1\}$ is a regular value for f. To this purpose, let $u_0 = \cos\vartheta + (\sin\vartheta)I_0$ with $\vartheta \neq k\pi$, for all $k \in \mathbb{Z}$. By Theorem 3.15, we know that $f^{-1}(u_0)$ consists of m distinct points $q_1, \dots, q_m \in \partial\mathbb{B} \cap L_{I_0}$. For any such q_i, there exists $\phi_i \in \mathbb{R}$, $\phi_i \neq k\pi$, for all $k \in \mathbb{Z}$, with $m\phi_i = \vartheta + 2k\pi$ for some $k \in \mathbb{Z}$, so that $q_i = \cos\phi_i + (\sin\phi_i)I_0$. To proceed, we need to consider two different systems of local coordinates for $\partial\mathbb{B}$, respectively defined by

$$\Phi : (0, \pi) \times \mathbb{S} \to \partial\mathbb{B}, \quad \Phi(\phi, I) = \cos\phi + (\sin\phi)I$$

and

$$\Psi : (\pi, 2\pi) \times \mathbb{S} \to \partial\mathbb{B}, \quad \Psi(\psi, J) = \cos\psi + (\sin\psi)J$$

We want to prove that Φ and Ψ induce the same orientation on $\partial\mathbb{B}$. Notice that

$$\cos\phi = \cos\psi \quad \text{and} \quad (\sin\phi)I = (\sin\psi)J$$

imply

$$\psi = 2\pi - \phi \quad \text{and} \quad J = -I$$

and hence that the change of coordinates is expressed as

$$\Psi^{-1} \circ \Phi(\phi, I) = (2\pi - \phi, -I)$$

Since both $\phi \mapsto 2\pi - \phi$ and the antipodal map $I \mapsto -I$ of $\mathbb{S}$ are orientation reversing in our cases, then the change of coordinates $\Psi^{-1} \circ \Phi$ is orientation preserving.

Now, if one chooses a system of local coordinates I for $\mathbb{S}$ in a neighborhood of I_0,

$$(\alpha_1, \alpha_2) \mapsto I(\alpha_1, \alpha_2)$$

then, in a neighborhood of q_i, the map $q \mapsto q^m$ in local coordinates becomes

$$(\phi, \alpha_1, \alpha_2) \mapsto (m\phi - 2k_i\pi, \alpha_1, \alpha_2) \tag{3.4}$$

(for some $k_i \in \mathbb{Z}$); hence, its differential is represented by a diagonal matrix with eigenvalues m and 1. Therefore, all q_i, $i = 1, \dots, m$, are regular points of f, and u_0 is a regular value. Moreover, since the differential of f at each q_i has positive determinant, the local degree of f at q_i is +1, and hence, the degree of f is equal to m.

We are left to prove that, for $m = 2$, the value 1 is a regular value of f, that is, that 1 and -1 are regular points. Let us recall that $\partial\mathbb{B}$ is parallelizable. Now, since any unit vector $J \in \mathbb{S}$ tangent to $\partial\mathbb{B}$ at 1 can be obtained as the tangent vector (for $t = 0$) to the curve $\gamma(t) = \cos t + (\sin t)J$, $0 < t < \varepsilon$, then the vector J can be moved, coherently with the parallelization, to a vector tangent to $\partial\mathbb{B}$ at -1 by means of the differential of the multiplication by -1. The vector J is therefore moved to $-J$, tangent to the curve $-\gamma(t)$ for $t = 0$. To compute $df_{-1}(-J)$, the differential df_{-1} of f at -1 applied to $-J$, it is enough to compute

$$\frac{d}{dt} f(-\cos t - (\sin t)J)_{|_{t=0}} = \frac{d}{dt}(-\cos t - (\sin t)J)^2_{|_{t=0}} = 2J.$$

It follows that, with respect to a basis of the tangent space $T_{-1}(\partial\mathbb{B})$ and a basis of the tangent space $T_1(\partial\mathbb{B})$ which belong to a same parallelization, the representing matrix of the differential df_{-1} is two times the identity matrix. Since the regularity at the point 1 is obvious, the proof is complete. □

3.5 Factorization of Polynomials

Despite the peculiarities of the noncommutative setting, regular polynomials can still be factorized. The relation between roots and factors, however, is more delicate than in the complex case. Among the works on this topic, we wish to mention [104, 107, 115]. We begin with a result concerning regular functions in general.

Proposition 3.17. *Let f be a regular function on a symmetric slice domain Ω. A point $p \in \Omega$ is a zero of f if and only if there exists a regular function $g : \Omega \to \mathbb{H}$ such that*

$$f(q) = (q - p) * g(q). \tag{3.5}$$

Furthermore, f vanishes identically on $x + y\mathbb{S}$ (with $x, y \in \mathbb{R}$, $y \neq 0$) if and only if there exists a regular $h : \Omega \to \mathbb{H}$ such that

$$f(q) = [(q - x)^2 + y^2]h(q). \tag{3.6}$$

Proof. Let p be a zero and let L_I be a complex plane through p. For any $J \in \mathbb{S}$ such that $J \perp I$, let $F, G : \Omega_I \to L_I$ be holomorphic functions such that $f_I = F + GJ$ (see Lemma 1.3). The fact that $f_I(p) = 0$ implies that $F(p) = G(p) = 0$. Since F, G are holomorphic, there exist holomorphic $H, K : \Omega_I \to L_I$ such that $F(z) = (z - p)H(z)$ and $G(z) = (z - p)K(z)$. Setting $g_I = H + KJ$ then defines a holomorphic $g_I : \Omega_I \to \mathbb{H}$ such that

$$f_I(z) = (z - p)g_I(z).$$

If we let $g = \text{ext}(g_I)$ be the regular extension of g_I to Ω, then $(q - p) * g(q)$ coincides with $(z - p)g_I(z) = f_I(z)$ at all $q = z \in \Omega_I$. By the Identity Principle 1.12, $(q - p) * g(q)$ must coincide with $f(q)$ in Ω.

Similarly, if f vanishes identically in a 2-sphere $x + y\mathbb{S}$, then, for any choice of $I \in \mathbb{S}$, the restriction f_I vanishes at both $x + yI$ and $x - yI$ and $f_I(z) = (z - x + yI)(z - x - yI)h_I(z) = [(z - x)^2 + y^2]h_I(z)$ for some holomorphic $h_I : \Omega_I \to \mathbb{H}$. Setting $h = \text{ext}(h_I)$, we conclude $f(q) = [(q - x)^2 + y^2]h(q)$ reasoning as above. □

The quaternionic version of the Fundamental Theorem of Algebra has been proved in several ways, [48, 68, 100], but the following restatement of the proof of [107] is the most natural in our setting.

Theorem 3.18 (Fundamental Theorem of Algebra). *Any polynomial*

$$a_0 + qa_1 + \ldots + q^n a_n \tag{3.7}$$

of degree $n \geq 1$ having coefficients $a_0, \ldots, a_n \in \mathbb{H}$ has at least one root in $\mathbb{H}$.

Proof. If $P(q) = a_0 + qa_1 + \ldots + q^n a_n$, then the symmetrization $P^s(q) = r_0 + qr_1 + \ldots + q^{2n} r_{2n}$ is a polynomial of degree $2n \geq 2$ with real coefficients: indeed, $r_m = \sum_{k=0}^{m} a_k \bar{a}_{m-k} \in \mathbb{R}$. The complex polynomial $r_0 + zr_1 + \ldots + z^{2n} r_{2n}$ has (at least) one root $x + iy \in \mathbb{C}$ by the complex Fundamental Theorem of Algebra. Hence, P^s vanishes identically in $x + y\mathbb{S}$. By Proposition 3.9, P has at least one root in $x + y\mathbb{S}$. □

Theorem 3.18, together with Proposition 3.17, immediately implies the following Corollary:

Corollary 3.19. *If $P(q) = a_0 + qa_1 + \ldots + q^n a_n$ with $a_0, \ldots, a_n \in \mathbb{H}$ and $a_n \neq 0$, then there exist points $p_1, \ldots, p_n \in \mathbb{H}$ such that*

$$P(q) = (q - p_1) * \ldots * (q - p_n)a_n. \tag{3.8}$$

The point p_1 is a root of the polynomial $P(q)$ by the previous proposition, but $p_2, \ldots, p_n$ are not, in general, roots of $P(q)$. In order to explain the relation between factors and roots, it suffices to study the case $n = 2$.

Proposition 3.20. *Let $\alpha, \beta \in \mathbb{H}$ and $P(q) = (q - \alpha) * (q - \beta)$.*

1. *If β does not lie in the same sphere $x + y\mathbb{S}$ as α, then P has two zeros, α and $(\beta - \bar{\alpha})^{-1}\beta(\beta - \bar{\alpha})$.*
2. *If α, β lie in the same sphere $x + y\mathbb{S}$ but $\alpha \neq \bar{\beta}$, then P only vanishes at α.*
3. *Finally, if $\alpha = \bar{\beta} \in x + y\mathbb{S}$, then the zero set of P is $x + y\mathbb{S}$.*

Proof. If $\alpha = \bar{\beta} \in x + y\mathbb{S}$, then

$$P(q) = (q - \alpha) * (q - \beta) = (q - \alpha) * (q - \bar{\alpha}) = q^2 - q(\alpha + \bar{\alpha}) + |\alpha|^2 =$$

$$= q^2 - q2x + x^2 + y^2 = (q - x)^2 + y^2$$

so that the zero set of P is $x + y\mathbb{S}$.

Now let us suppose $\alpha, \beta \in x + y\mathbb{S}$ but $\alpha \neq \bar{\beta}$. Since

$$P^s(q) = (q - \alpha)^s (q - \beta)^s = [(q - x)^2 + y^2]^2,$$

by Proposition 3.9, the zero set of P is contained in $x + y\mathbb{S}$. By Theorem 3.1, either α is the only root of P or P vanishes identically in $x + y\mathbb{S}$. The second possibility is excluded because it would imply $P(q) = [(q-x)^2 + y^2]h(q)$ for some h, which is impossible since $P(q) = q^2 - q(\alpha + \beta) + \alpha\beta$ with $\alpha \neq \bar{\beta}$.

Finally, suppose α, β lie in distinct spheres $x + y\mathbb{S}, u + v\mathbb{S}$. Since

$$P^s(q) = (q-\alpha)^s(q-\beta)^s = [(q-x)^2 + y^2][(q-u)^2 + v^2],$$

P has its roots in $x+y\mathbb{S}$ and $u+v\mathbb{S}$. More precisely, by the same reasoning employed above, P has exactly one root in $x + y\mathbb{S}$, namely, α, and one in $u + v\mathbb{S}$. Let us prove that the latter is $\beta' = (\beta - \bar{\alpha})^{-1}\beta(\beta - \bar{\alpha})$:

$$P(\beta') = \beta'^2 - \beta'(\alpha + \beta) + \alpha\beta = 0$$

because

$$\begin{aligned} &\beta^2(\beta - \bar{\alpha}) - \beta(\beta - \bar{\alpha})(\alpha + \beta) + (\beta - \bar{\alpha})\alpha\beta = \\ &= \beta^3 - \beta^2\bar{\alpha} - \beta^2\alpha - \beta^3 + \beta|\alpha|^2 + \beta\bar{\alpha}\beta + \beta\alpha\beta - |\alpha|^2\beta = \\ &= -\beta^2 2Re(\alpha) + \beta 2Re(\alpha)\beta = 0. \end{aligned}$$

□

Notice that, in the previous proof, the point $(\beta - \bar{\alpha})^{-1}\beta(\beta - \bar{\alpha})$ is a root of P even when α, β lie in the same sphere $x + y\mathbb{S}$. This does not contradict the fact that α is the only root of P, thanks to the following lemma.

Lemma 3.21. *For any two quaternions α, β belonging to a same sphere $x + y\mathbb{S}$, if $\beta \neq \bar{\alpha}$, then $(\beta - \bar{\alpha})^{-1}\beta(\beta - \bar{\alpha}) = \alpha$.*

Proof. If β belongs to the same sphere $x + y\mathbb{S}$ as α, then β must be a root of

$$(q-x)^2 + y^2 = (q - \bar{\alpha}) * (q - \alpha) = (q - \bar{\alpha})[(q - \bar{\alpha})^{-1}q(q - \bar{\alpha}) - \alpha]$$

If $\beta \neq \bar{\alpha}$, then clearly $(\beta - \bar{\alpha})^{-1}\beta(\beta - \bar{\alpha}) = \alpha$. □

The following result gives the general factorization of a polynomial with coefficients in $\mathbb{H}$. If $\alpha = x + yI \in \mathbb{H}$, we will write $S_\alpha = x + y\mathbb{S}$.

Theorem 3.22. *Let $P(q)$ be a regular polynomial of degree m. Then there exist $p, m_1, \dots, m_p \in \mathbb{N}$, and $w_1, \dots, w_p \in \mathbb{H}$, generators of the spherical roots of P, so that*

$$P(q) = (q^2 - 2qRe(w_1) + |w_1|^2)^{m_1} \cdots (q^2 - 2qRe(w_p) + |w_p|^2)^{m_p} Q(q), \quad (3.9)$$

where Q is a regular polynomial with coefficients in $\mathbb{H}$ having no spherical zeros. Moreover, if $n = m - 2(m_1 + \cdots + m_p)$, there exists a constant $c \in \mathbb{H}$, t distinct 2-spheres $S_1 = x_1 + y_1\mathbb{S}, \dots, S_t = x_t + y_t\mathbb{S}$, t integers $n_1, \dots, n_t$ with $n_1 + \cdots + n_t = n$, and (for any $i = 1, \dots, t$) n_i quaternions $\alpha_{ij} \in S_i$, $j = 1, \dots, n_i$, such that

$$Q(q) = \left(\prod_{i=1}^{t}{}^{*} \prod_{j=1}^{n_i}{}^{*} (q - \alpha_{ij}) \right) c. \tag{3.10}$$

Proof. The first part of the theorem, namely, the decomposition (3.9), follows immediately from (3.6). Thus, we only have to prove the decomposition for the quaternionic polynomial Q which has no spherical roots. We can assume Q to be a monic polynomial. If not, there is a constant c, coefficient of the highest degree term, and the process below must be preceded by multiplication by c^{-1} and then followed by multiplication by c. By the Fundamental Theorem of Algebra for quaternionic polynomials, if $\deg(Q) = n > 0$, there is at least one root, say, γ_1. Let us add the root $\overline{\gamma_1}$ to the polynomial $Q(q)$; this can be accomplished by a simple multiplication, and we can now consider the new polynomial

$$\widetilde{Q}(q) = Q(q) * [q - Q(\overline{\gamma_1})^{-1}\overline{\gamma_1}Q(\overline{\gamma_1})].$$

The fact that $\widetilde{Q}(\overline{\gamma_1}) = 0$ is an immediate consequence of Corollary 3.5. We now note that $\widetilde{Q}(q)$ has a spherical zero on S_{γ_1} because it has two roots (γ_1 and $\overline{\gamma_1}$) on that sphere. Thus, by (3.6), one can factor out a spherical root

$$q^2 - 2qRe(\gamma_1) + |\gamma_1|^2,$$

so that

$$Q(q) * [q - Q(\overline{\gamma_1})^{-1}\overline{\gamma_1}Q(\overline{\gamma_1})] = \widetilde{Q}(q) = (q^2 - 2qRe(\gamma_1) + |\gamma_1|^2)Q_{11}(q)$$

for some new polynomial Q_{11}. We set $\delta_{11} = Q(\overline{\gamma_1})^{-1}\overline{\gamma_1}Q(\overline{\gamma_1})$, and we note that $\delta_{11} \in S_{\gamma_1}$. By repeating this same procedure for Q_{11}, if $Q_{11}(\gamma_1) = 0$, we obtain

$$Q_{11}(q) * (q - Q_{11}(\overline{\gamma_1})^{-1}\overline{\gamma_1}Q_{11}(\overline{\gamma_1})) = (q^2 - 2qRe(\gamma_1) + |\gamma_1|^2)Q_{12}(q),$$

and we set $\delta_{12} = Q_{11}(\overline{\gamma_1})^{-1}\overline{\gamma_1}Q_{11}(\overline{\gamma_1})$. We now continue for n_1 steps, generating the quaternion δ_{1j} at step j, until we finally obtain that $Q_{1n_1}(\gamma_1) \neq 0$. If the degree of Q_{1n_1} is still positive, we can find a new isolated root $\gamma_2 \notin S_{\gamma_1}$ (since γ_1 is an isolated root), and we repeat the process once again. Since at each step we decrease the degree of the polynomial, the process necessarily ends after a finite number of steps. We therefore obtain that

$$Q(q) * \prod_{i=1}^{t}{}^{*} \prod_{j=1}^{n_i}{}^{*} (q - \delta_{ij}) = \prod_{i=1}^{t}(q^2 - 2qRe(\gamma_i) + |\gamma_i|^2)^{n_i}.$$

To conclude the proof, we simply multiply both sides of the equality (on the right) by

$$\prod_{i=t}^{1}{}^{*} \prod_{j=n_i}^{1}{}^{*} (q-\overline{\delta}_{ij}).$$

This immediately gives the result with $\alpha_{ij} = \overline{\delta}_{t-i+1,n_{t-i+1}-j+1}$. □

Proposition 3.23. *The polynomial with quaternionic coefficients*

$$P(q) = (q-\alpha_1) * (q-\alpha_2) * \cdots * (q-\alpha_m) \tag{3.11}$$

where $\alpha_i \in S_{\alpha_1}$ for all $i = 1, \ldots, m$ and where $\alpha_{i+1} \neq \overline{\alpha}_i$ for $i = 1, \ldots, m-1$, has a unique root, equal to α_1. Moreover, the factorization (3.11) is the only factorization of the polynomial $P(q)$. Finally the following equality holds

$$P(q) * (q-[P(\overline{\alpha}_1)^{-1}]\overline{\alpha}_1[P(\overline{\alpha}_1)]) = \prod_{i=1}^{m-1}{}^{*} (q-\alpha_i) * (q^2 - 2qRe(\alpha_1) + |\alpha_1|^2) \tag{3.12}$$

Proof. We will prove the first two assertions by induction on the number m of terms of the factorization. If we set $f(q) = (q-\alpha_1)$ and $g(q) = (q-\alpha_2) * \cdots * (q-\alpha_m)$, then Corollary 3.5 establishes that $P(\alpha_1) = 0$. Corollary 3.5 establishes also that $\beta \neq \alpha_1$ is a root of $P(q)$ if and only if $f(\beta)^{-1}\beta f(\beta) = (\beta-\alpha_1)^{-1}\beta(\beta-\alpha_1) \in S_\beta$ is a root of $g(q)$, that is, by Lemma 3.21, if and only if $\overline{\alpha}_1$ is a root of $g(q) = (q-\alpha_2) * \cdots * (q-\alpha_m)$. Since we have that $\alpha_2 \neq \overline{\alpha}_1$, the induction hypothesis leads to the conclusion that no $\beta \neq \alpha_1$ can be a root of $P(q)$.

Suppose now that

$$P(q) = (q-\alpha_1) * (q-\alpha_2) * \cdots * (q-\alpha_m) = (q-\alpha_1') * (q-\alpha_2') \cdots * (q-\alpha_m')$$

are two factorizations of $P(q)$. The fact that α_1 is the only root of $P(q)$ implies that $\alpha_1' = \alpha_1$, which directly yields the equality

$$(q-\alpha_2) * \cdots * (q-\alpha_m) = (q-\alpha_2') \cdots * (q-\alpha_m')$$

and, by the induction hypothesis, the uniqueness of the factorization follows.

To prove equality (3.12), notice at first that Corollary 3.5 and the fact that α_1 is the only root of $P(q)$ imply that $P(q) * (q-[P(\overline{\alpha}_1)]^{-1}\overline{\alpha}_1[P(\overline{\alpha}_1)])$ has two roots on S_{α_1}, namely, α_1 and $\overline{\alpha}_1$. The first assertion of this same theorem now forces the equality $[P(\overline{\alpha}_1)]^{-1}\overline{\alpha}_1[P(\overline{\alpha}_1)] = \overline{\alpha}_m$. Therefore

$$\begin{aligned} P(q) * (q-[P(\overline{\alpha}_1)]^{-1}\overline{\alpha}_1[P(\overline{\alpha}_1)]) &= (q-\alpha_1) * (q-\alpha_2) * \cdots * (q-\alpha_m) * (q-\overline{\alpha}_m) \\ &= (q-\alpha_1) * (q-\alpha_2) * \cdots * (q-\alpha_{m-1}) * (q^2 - 2qRe(\alpha_m) + |\alpha_m|^2). \end{aligned}$$

Since $\alpha_m \in S_{\alpha_1}$, the last assertion of our statement follows. □

Unlike what happens in the complex case, the factorization in (3.10) is not unique. It strictly depends on the order in which the points α_{ij} are taken. What is unique in the factorization is the set of spheres, as well as the numbers n_t. To illustrate this phenomenon, we first recall what follows.

Theorem 3.24. *Let $f(q) = (q-a) * (q-b)$ with a and b lying on different 2-spheres. Then $f(q) = (q-b') * (q-a')$ if and only if $a' = c^{-1}ac$ and $b' = c^{-1}bc$ for $c = \overline{b} - a \neq 0$.*

Using this result, it is easy to show that the polynomial $P(q) = (q-I)*(q-2J)$ can also be factored as

$$P(q) = \left(q - \frac{8I+6J}{5}\right) * \left(q - \frac{4J-3I}{5}\right).$$

Note that this different factorization still has one representative from the sphere $\mathbb{S}$ and one from the sphere $2\mathbb{S}$. This example also shows (as we would expect) that the quaternions α_{ij} are not, in general, roots of the polynomial. In this case, for example, I is a root, and so is $\frac{8I+6J}{5}$, while neither $2J$ nor $\frac{4J-3I}{5}$ are roots of the polynomial P.

For the purpose of the next result, we apply our factorization procedure to the polynomial Q in Theorem 3.22, and we relabel the quaternions α_{ij} which appear in its factorization in lexicographical order so to have a single-index sequence β_k for $k = 1, \dots, n$, so that the factorization can now be written as

$$Q(q) = \prod_{k=1}^{n}{}^{*} (q - \beta_k).$$

A repeated application of Theorem 3.24 immediately demonstrates the next result.

Theorem 3.25. *Let $Q(q)$ be a regular polynomial without spherical zeros, and let*

$$Q(q) = \prod_{k=1}^{n}{}^{*} (q - \beta_k)$$

be one of its factorizations. Then the roots of Q can be obtained from the quaternions β_k as follows: β_1 is a root, β_2 is not a root, but it yields the root $\beta_2^{(1)} = (\overline{\beta_2} - \beta_1)^{-1}\beta_2(\overline{\beta_2} - \beta_1)$. In general if we set, for $r = 1, \dots, n$ and $j = 1, \dots, r-1$,

$$\beta_r^{(j)} = \left(\overline{\beta_r^{(j-1)}} - \beta_{r-j}\right)^{-1} \beta_r^{(j-1)} \left(\overline{\beta_r^{(j-1)}} - \beta_{r-j}\right)$$

we obtain that the roots of Q are given by

$$\beta_r^{(r-1)} = \left(\overline{\beta_r^{(r-2)}} - \beta_1\right)^{-1} \beta_r^{(r-2)} \left(\overline{\beta_r^{(r-2)}} - \beta_1\right).$$

3.6 Multiplicity

The factorization results obtained justify the introduction of several notions of multiplicity.

Definition 3.26. Let f be a regular function on a symmetric slice domain Ω and let $p \in \Omega$. We define the *(classical) multiplicity* of p as a zero of f, and denote by $m_f(p)$, the largest $n \in \mathbb{N}$ such that

$$f(q) = (q - p)^{*n} * g(q) \tag{3.13}$$

for some regular $g : \Omega \to \mathbb{H}$.

Notice that if $p \in L_I$, then $f(q) = (q - p)^{*n} * g(q)$ if and only if $f_I(z) = (z - p)^n g_I(z)$. Hence, the classical multiplicity of p as a zero of f coincides with the multiplicity of p as a zero of the holomorphic function f_I, intended as the largest $n \in \mathbb{N}$ such that $f_I(z) = (z - p)^n g_I(z)$. Notice that if f_I splits as $f_I = F + GJ$ according to Lemma 1.3, then the multiplicity of f_I at p is the minimum between the multiplicity of F and that of G at p. This proves a posteriori that the definition is well posed.

Even though the classical multiplicity is a consistent generalization of complex multiplicity, it does not lead to analogous results for polynomials. At this regard, we point out that this notion of multiplicity had been previously studied, in the special case of polynomials, in [104]. Notice that a polynomial of finite degree can have infinitely many roots with positive multiplicity.

Example 3.27. The polynomial $P(q) = q^2 + 1 = (q - I) * (q + I)$ has multiplicity $m_P(I) = 1$ at all $I \in \mathbb{S}$.

Even if we consider a polynomial having only isolated (hence finitely many) roots, it is still possible for the degree of a polynomial to exceed the sum of the (classical) multiplicities of its zeros.

Example 3.28. Take $I, J \in \mathbb{S}$ with $I \neq \pm J$ and let

$$P(q) = (q - I) * (q - J) = q^2 - q(I + J) + IJ.$$

By Proposition 3.20, the zero set of P is $\{I\}$. Notice that $m_P(I)$, while f has degree 2.

For this reason, alternative notions of multiplicity have been introduced.

Definition 3.29. Let $P(q)$ be a regular quaternionic polynomial and let $x, y \in \mathbb{R}$, $y \neq 0$. We say that P has *spherical multiplicity* $2m$ at $x + y\mathbb{S}$ if m is the largest natural number such that

$$P(q) = [(q - x)^2 + y^2]^m \widetilde{P}(q) \tag{3.14}$$

for some other polynomial $\widetilde{P}(q)$. If $\widetilde{P}(q)$ has a root $p_1 \in x + y\mathbb{S}$, then we say that P has *isolated multiplicity* n at p_1, where n is the largest natural number such that there exist $p_2, \dots, p_n \in x + y\mathbb{S}$ (with $p_i \neq \bar{p}_{i+1}$ for all $i \in \{1, \dots, n-1\}$) and a polynomial $R(q)$ with

$$\widetilde{P}(q) = (q - p_1) * (q - p_2) * \dots * (q - p_n) * R(q). \tag{3.15}$$

Furthermore, if $x \in \mathbb{R}$, then we call *isolated multiplicity* of P at x the largest $k \in \mathbb{N}$ such that

$$P(q) = (q - x)^k \widehat{P}(q) \tag{3.16}$$

for some other polynomial $\widehat{P}(q)$.

Notice that, for a real point x, the notions of isolated multiplicity and classical multiplicity coincide. Now, the degree of a polynomial is nicely related to the spherical and isolated multiplicities of its zeros.

Proposition 3.30. *If $P(q)$ is a regular quaternionic polynomial of degree d, then the sum of the spherical multiplicities and the isolated multiplicities of the zeros of P is d.*

Proof. This result is a direct consequence of Theorem 3.22. □

The classical multiplicities of the zeros of a polynomial P are related to the spherical and isolated multiplicities of the zeros of P as follows.

Remark 3.31. Let $Q(q)$ be a regular quaternionic polynomial and let $p = x + yI \in \mathbb{H}$. Then Q has spherical multiplicity $2\min\{m_Q(p), m_Q(\bar{p})\}$ at $x + y\mathbb{S}$. Moreover, if $m_Q(p) > m_Q(\bar{p})$, then Q has isolated multiplicity $n \geq m_Q(p) - m_Q(\bar{p})$ at p.

The following examples clarify the previous remark.

Example 3.32. The polynomial

$$P(q) = q^2 + 1$$

vanishes on $\mathbb{S}$. We saw that it has classical multiplicity $m_P(I) = 1$ at all $I \in \mathbb{S}$. Moreover, P has spherical multiplicity 2 at $\mathbb{S}$.

Example 3.33. If $I \in \mathbb{S}$, then the polynomial

$$P(q) = (q - I) * (q - I) = (q - I)^{*2}$$

only vanishes at I. The polynomial P has classical multiplicity $m_P(I) = 2$ at I; it has spherical multiplicity 0 at $\mathbb{S}$ and isolated multiplicity 2 at I.

Example 3.34. If $I, J \in \mathbb{S}, I \neq \pm J$, then the polynomial

$$P(q) = (q - I) * (q - J) = q^2 - q(I + J) + IJ$$

only vanishes at I, where it has classical multiplicity $m_P(I) = 1$. The polynomial P has spherical multiplicity 0 at $\mathbb{S}$ and isolated multiplicity 2 at I.

It is possible to combine the three cases presented above to build new examples which prove the sharpness of the preceding remark. Our next result tells us how to build the factorization if we know the roots of the polynomial and their (spherical and isolated) multiplicities.

Theorem 3.35. *The family of all regular polynomials with quaternionic coefficients with assigned spherical roots $x_1 + y_1\mathbb{S}, \dots, x_p + y_p\mathbb{S}$ with multiplicities $2m_1, \dots, 2m_p$, and assigned isolated roots $\gamma_1, \dots, \gamma_t$ with (isolated) multiplicities $n_1, \dots, n_t$ consists of all polynomials P that can be written as*

$$P(q) = [q^2 - 2qx_1 + (x_1^2 + y_1^2)]^{m_1} \cdots [q^2 - 2qx_p + (x_p^2 + y_p^2)]^{m_p} Q(q)c \quad (3.17)$$

with

$$Q(q) = \prod_{i=1}^{t}{}^{*} \prod_{j=1}^{n_i}{}^{*} (q - \alpha_{ij}) = \prod_{i=1}^{t}{}^{*} Q_i(q)$$

where $c \in \mathbb{H}$ is an arbitrary nonzero constant and where $\alpha_{11} = \gamma_1$, the quaternions α_{1j} are arbitrarily chosen in S_{γ_1} for $j = 2, \dots, n_1$ in such a way that $\alpha_{1j+1} \neq \overline{\alpha}_{1j}$ (for $j = 1, \dots, n_1 - 1$), and in general, for $i = 2, \dots, t$,

$$\alpha_{i1} = \left(\left(\prod_{k=1}^{i-1}{}^{*} Q_k\right)(\gamma_i)\right)^{-1} \gamma_i \left(\left(\prod_{k=1}^{i-1}{}^{*} Q_k\right)(\gamma_i)\right)$$

while the remaining α_{ij} are arbitrarily chosen in S_{γ_i}, for $j = 2, \dots, n_i$ in such a way that $\alpha_{ij+1} \neq \overline{\alpha}_{ij}$ (for $j = 1, \dots, n_i - 1$).

Proof. The fact that every polynomial with the assigned roots can be represented in the form (3.17) is an immediate consequence of the proof of Theorem 3.22. Conversely, the fact that if a polynomial can be expressed as in (3.17), then it has the required zeros and multiplicities, is a consequence of Corollary 3.5 and of Proposition 3.23. □

The notions of spherical and isolated multiplicity extend to all regular functions, thanks to the following result.

Theorem 3.36. *Let f be a regular function on a symmetric slice domain Ω, suppose $f \not\equiv 0$, and let $x+y\mathbb{S} \subset \Omega$. There exist $m \in \mathbb{N}, n \in \mathbb{N}$, $p_1, \dots, p_n \in x+y\mathbb{S}$ (with $p_i \neq \bar{p}_{i+1}$ for all $i \in \{1, \dots, n-1\}$) such that*

$$f(q) = [(q-x)^2 + y^2]^m (q - p_1) * (q - p_2) * \dots * (q - p_n) * g(q) \quad (3.18)$$

for some regular function $g : \Omega \to \mathbb{H}$ which does not have zeros in $x + y\mathbb{S}$.

Proof. If $f \not\equiv 0$ in Ω, then there exists an $m \in \mathbb{N}$ such that

$$f(q) = [(q-x)^2 + y^2]^m h(q)$$

for some h which does not vanish identically on $x + y\mathbb{S}$. Suppose indeed it were possible to find, for all $k \in \mathbb{N}$, a function $h^{[k]}(q)$ such that $f(q) = [(q - x)^2 + y^2]^k h^{[k]}(q)$. Then, choosing an $I \in \mathbb{S}$, the holomorphic function f_I would have the factorization

$$f_I(z) = [(z - x)^2 + y^2]^k h_I^{[k]}(z) = [z - (x + yI)]^k [z - (x - yI)]^k h_I^{[k]}(z)$$

for all $k \in \mathbb{N}$. This would imply $f_I \equiv 0$ and, by the Identity Principle 1.12, $f \equiv 0$.

Now let h be a regular function on Ω which does not vanish identically on $x + y\mathbb{S}$. By Proposition 3.9, $g^{[0]} := h$ has at most one zero $p_1 \in x + y\mathbb{S}$. If this is the case, then $h(q) = (q - p_1) * g^{[1]}(q)$ for some function $g^{[1]}$ which does not vanish identically on $x + y\mathbb{S}$. If for all $k \in \mathbb{N}$ there existed a $p_{k+1} \in x + y\mathbb{S}$ and a $g^{[k+1]}$ such that $g^{[k]}(q) = (q - p_{k+1}) * g^{[k+1]}$, then we would have

$$h(q) = (q - p_1) * \ldots * (q - p_k) * g^{[k]}(q)$$

for all $k \in \mathbb{N}$. This would imply, for the symmetrization h^s of h,

$$h^s(q) = [(q - x)^2 + y^2]^k (g^{[k]})^s(q)$$

for all $k \in \mathbb{N}$. By the first part of the proof, this would imply $h^s \equiv 0$. We could then conclude, applying Proposition 3.9, that $h \equiv 0$, a contradiction. Thus there exists an $n \in \mathbb{N}$ such that $g^{[n]}$ does not have zeros in $x + y\mathbb{S}$ and, setting $g = g^{[n]}$, we derive the thesis. □

We can at this point give the definition of multiplicity for the zeros of a generic regular function.

Definition 3.37. Let f be a regular function on a symmetric slice domain Ω and let $x + y\mathbb{S} \subset \Omega$ with $y \neq 0$. Let $m, n \in \mathbb{N}$ and $p_1, \ldots, p_n \in x + y\mathbb{S}$ (with $p_i \neq \bar{p}_{i+1}$ for all $i \in \{1, \ldots, n - 1\}$) be such that (3.18) holds for f and for some regular $g : \Omega \to \mathbb{H}$ which does not have zeros in $x + y\mathbb{S}$. We then say that $2m$ is the *spherical multiplicity* of $x + y\mathbb{S}$ and that n is the *isolated multiplicity* of p_1. On the other hand, if $x \in \mathbb{R}$, then we call *isolated multiplicity* of f at x the number $k \in \mathbb{N}$ such that

$$f(q) = (q - x)^k h(q) \tag{3.19}$$

for some regular $h : \Omega \to \mathbb{H}$ which does not vanish at x.

Remark 3.38. Using the same technique as in the proof of Theorem 3.22, we can derive from (3.18) a factorization of type

$$f(q) = [(q - x)^2 + y^2]^m * \widetilde{g}(q) * (q - \widetilde{p_1}) * (q - \widetilde{p_2}) * \ldots * (q - \widetilde{p_n})$$

for suitable $\widetilde{g}$, and $\widetilde{p_i} \in S_{p_i}$.

3.7 Division Algorithm and Bezout Theorem

We now present a study of common divisors in the ring of regular polynomials $\mathbb{H}[q]$. As in every noncommutative ring, ideals of $\mathbb{H}[q]$ can be left, right, or bilateral. For the sake of simplicity, most of the times we will consider *left* ideals only. Unless otherwise specified, our results on left ideals will translate into the corresponding ones for right ideals in a straightforward manner. Let us recall some basic definitions (see, e.g., [102–104]). If not otherwise stated, all polynomials we consider will be monic.

Definition 3.39. We say that a regular polynomial G divides F on the left (respectively on the right) if there exists $A \in \mathbb{H}[q]$ such that $F = G * A$ (respectively $F = A * G$). Let $F_1, \dots F_n$ be polynomials of $\mathbb{H}[q]$. Their *greatest common left divisor*, $GCLD(F_1, \dots, F_n)$ in short, is the unique monic element $D \in \mathbb{H}[q]$ such that D divides F_i on the left for every i, and such that every other left divisor divides D on the left. Similarly one defines the greatest common right divisor $GCRD(F_1, \dots, F_n)$.

Remark 3.40. Note that the definition of GCLD is well posed since if D_1 and D_2 were both greatest left common divisors, we would have $D_1 = D_2 * A$ and $D_2 = D_1 * B$ which implies $D_2 = D_2 * A * B$. Given that the degree of a $*$-product of polynomials is the sum of the degrees of the factors, we conclude that $A, B \in \mathbb{H}$ and since both D_1 and D_2 are monic, $A = B = 1$.

Example 3.41. In general, it can be difficult to compute the greatest common left divisor of two polynomials via their factorization. Consider, as we did in Sect. 3.5, the polynomials $F(q) = \left(q - \frac{8I+6J}{5}\right) * \left(q - \frac{4J-3I}{5}\right)$ and $G(q) = q^2 + 1$. The latter has a spherical root; hence, it could be factorized in an infinite number of ways: $G(q) = (q - \alpha) * (q - \bar{\alpha})$ for every choice of α in $\mathbb{S}$. Since $\frac{8I+6J}{5}$ is not in $\mathbb{S}$, it may seem that F and G have no common left factor. However, we also have $F(q) = (q - I) * (q - 2J)$ which shows that $D = (q - I)$ is a common left divisor for F and G. Since F and G are quadratic, monic, and $F \neq G$, D is the greatest common divisor.

The previous example shows that in general it can be hard to compute the GCLD of two polynomials due to the fact that one cannot rely on factorization. Performing a full factorization of a polynomial can be very difficult if one does not know its roots, even when the coefficients are chosen in a commutative field. Here we also have the issue of nonuniqueness. We need a better, algorithmic way of calculating the GCLD. As in the commutative case $\mathbb{K}[x]$, where $\mathbb{K}$ is a field, one can use the Euclidean Division Algorithm.

Proposition 3.42 (Euclidean Division). *Let F, G be regular polynomials. Then there exist Q, R, Q' and R' in $\mathbb{H}[q]$, with* $\max(\deg(R), \deg(R')) < \deg(G)$*, such that*

$$F = Q * G + R \qquad \text{and} \qquad F = G * Q' + R'.$$

Moreover, such polynomials are uniquely determined.

Proof. The proof of the existence follows the lines of the commutative case. Note that the two polynomials can always be chosen to be monic, so the division of the terms of F by the leading power of G is unambiguous. The only difference with the commutative case is that in order to get Q and R versus Q' and R', one has to perform multiplications of the partial quotient respectively to the right or to the left by G. Uniqueness follows easily from the fact that $(\mathbb{H}[q], +, *)$ is a domain (i.e., a ring with no zero divisors) and from the fact that the degrees of R and R' are both less than the degree of G. □

Example 3.43. Divide the polynomial $F(q) = q^2 - qj + (k-1)$ on the right by $G(q) = q + i$. Using the classical long division algorithm, with the only difference that divisions and multiplications have to be performed to the right, we can write $q^2 - qj + (k-1) = (q - i - j) * (q + i) - 2$.

Remark 3.44. The left and right remainders are not necessarily equal. In general, they may have different degrees. For instance, consider $F(q) = q^3 + q^2 i - qk + j$ and $G(q) = q^2 + qj - 1$. Dividing F by G on either side, we obtain

$$F(q) = (q + i - j) * G(q) - (2qk - i + j),$$

$$F(q) = G(q) * (q + i - j) + (i - j).$$

Corollary 3.45 (Remainder Theorem). *Let $F(q)$ be a regular polynomial and let $\alpha \in \mathbb{H}$. Then there exists $Q \in \mathbb{H}[q]$ such that $F(q) = (q - \alpha) * Q(q) + F(\alpha)$.*

Notice that the evaluation map $\epsilon_\alpha : \mathbb{H}[q] \to \mathbb{H}$ defined by $\epsilon_\alpha(F) = F(\alpha)$ is not an algebra homomorphism. In particular, the usual Remainder Theorem, which is an immediate consequence of Proposition 3.42, holds only for left division (see also [129]).

Example 3.46. Left Euclidean division can be useful to factorize a regular polynomial once a root is known. Consider $F(q) = q^2 - q(i + 2j) + 2k$. It is clear that $F(i) = 0$, so according to Corollary 3.45, we have that $F(q) = (q - i) * Q(q)$. In order to find Q, we can perform the long division of F to the left by $q - i$ to obtain a factorization of F as $F(q) = (q - i) * (q - 2j)$.

In order to give an algorithm for the calculation of the greatest common divisor using Euclidean division, we introduce the following notation. If $F = Q * G + R$ and $F = G * Q' + R'$ as in Proposition 3.42, we define

$$mod_r(F, G) = R, \quad div_r(F, G) = Q, \quad mod_l(F, G) = R', \quad div_l(F, G) = Q'.$$

Note that the subscripts refer to the side with respect to which division is performed, although Q and Q' are, technically speaking, left and right quotients, respectively.

Algorithm 3.47 (Computation of GCLD). *Let F, G be nonzero regular polynomials. Then the following list of instructions returns their greatest common left divisor in a finite number of steps:*

Input: $F, G \in \mathbb{H}[q] \setminus \{0\}$
Output: *GCLD(F,G)*
Initialization: $a := F$, $b := G$
• **While** $b \neq 0$ **Do**
  $t := b$
  $b := mod_l(a, b)$
  $a := t$
• **Return** a

Proof. The proof is formally the same as in the commutative case, except that one needs to keep all factors in the correct position, either to the left or to the right. Note that termination is guaranteed by the fact that the remainder sequence has strictly decreasing degrees. □

Remark 3.48. In order to compute the greatest common right divisor, the situation is completely symmetric. It is indeed sufficient to replace the second step of the "while" loop in Algorithm 3.47 with $b := mod_r(a, b)$.

A different algorithm based on the use of Gröbner bases will be provided in the next section.

Euclidean division is a very powerful tool that also allows to express the greatest common divisor as an explicit combination of the two polynomials. This is true in every commutative Euclidean domain, and the corresponding algorithm in $\mathbb{H}[q]$ is virtually identical. After completing all the iterated divisions of Algorithm 3.47, one "climbs" back to the top with simple algebraic steps. The only important observation is that quotients of the iterated divisions are *right* multiples, so that one ends up with a *right* linear combination of F and G. The following result appears in [129].

Proposition 3.49. *Let F and G be nonzero regular polynomials. Then there exist $A, B \in \mathbb{H}[q]$ such that $GCLD(F, G) = F * A + G * B$.*

Example 3.50. Consider $F(q) = q * (q - i) * (q - j) = q^3 - q^2(i + j) + qk$ and $G(q) = q * (q - k) = q^2 - qk$. In order to find $mod_l(F, G)$, we need to perform *left* division of F by G. This gives

$$F = G * (q - i - j + k) + q(k - j + i - 1). \tag{3.20}$$

Since the first remainder $R_1 = qc,\ c = (k - j + i - 1)$ is not zero, we need to perform yet another division. We use the remainder to divide G, and we see that

$$G = q * (q - k) = R_1 * c^{-1} * (q - k), \tag{3.21}$$

which ends the iteration since the new remainder is zero. Keeping the common divisor monic, and from (3.20), we have

$$\begin{aligned} q = GCLD(F, G) = R_1 * c^{-1} = \\ = (F - G * (q - i - j + k)) * c^{-1} = \\ = F * c^{-1} - G * (q - i - j + k) * c^{-1}. \end{aligned}$$

Definition 3.51. In $\mathbb{H}[q]$, we define recursively

$$GCLD(F_1, \dots, F_n) = GCLD(F_1, GCLD(F_2, \dots, F_n)).$$

We are now ready to state the following theorem.

Theorem 3.52 (Bezout). *Let $n > 1$ and let $F_1, \dots, F_n$ be nonzero regular quaternionic polynomials. The following facts are equivalent:*

(a) *$F_1, \dots, F_n$ have no common roots in $\mathbb{H}$.*
(b) *$GCLD(F_1, \dots, F_n) = 1$.*
(c) *There exist $A_1, \dots, A_n$ regular polynomials such that*

$$F_1 * A_1 + \cdots + F_n * A_n = 1.$$

Proof. $(a) \Rightarrow (b)$ Let $D := GCLD(F_1, \dots, F_n)$. If $D \notin \mathbb{H}$, then it has at least a root $\alpha \in \mathbb{H}$, so we can write $D = (q - \alpha) * D'$. This implies that α is a common root for the polynomials, which is a contradiction. Hence $D \in \mathbb{H}$, and since it is monic, $D = 1$.

$(b) \Rightarrow (c)$ The case $n = 2$ is an instance of Proposition 3.49. For $n > 2$, consider the $n - 1$ polynomials $GCLD(F_1, F_2)$ and $F_3, \dots, F_n$ which have no common roots. Using induction on n and then again Proposition 3.49, one obtains

$$1 = GCLD(F_1, F_2) * A + F_3 * A_3 + \cdots + F_n * A_n$$

$$= (F_1 * A_1' + F_2 * A_2') * A + F_3 * A_3 + \cdots + F_n * A_n,$$

which is the thesis with $A_i := A_i' * A$, $i = 1, 2$.

$(c) \Rightarrow (a)$ Suppose the polynomials have a common root $\alpha \in \mathbb{H}$. Then, using Corollary 3.45, we have that for every index i, $F_i(q) = (q - \alpha) * F_i'(q)$ for some F_i'. We can write then

$$(q - \alpha) * (F_1' * A_1 + \cdots + F_n' * A_n) = 1$$

which is a contradiction, because the degree of a regular product is the sum of the degrees. □

It is immediate to verify, as a consequence of the previous considerations, that every left or right ideal of $\mathbb{H}[q]$ is principal. We give the statement for left ideals.

Corollary 3.53. *Let $\mathfrak{I}$ be a left ideal in $\mathbb{H}[q]$ and let $F_1, \dots, F_n$ be its generators. Let $D = GCRD(F_1, \dots, F_n)$. Then $\mathfrak{I} = \mathbb{H}[q]\langle D \rangle$.*

Remark 3.54. We observe that Theorem 3.52 cannot be restated using the greatest common right divisor. If we did, only the equivalence of (b) and (c) could be proven. If $GCRD(F_1, \dots, F_n) = 1$, one uses again (right) Euclidean divisions to find a combination $\sum_i A_i * F_i = 1$, while the opposite implication is immediate to prove. However, such conditions are not equivalent to condition (a) because of the noncommutativity of the $*$-product and because the roots of regular polynomials only correspond to left linear factors. The next example will illustrate the situation.

Example 3.55. Consider $F = q^2 - q(i + j) + k$ and $G = q^2 - q(j + k) - i$. The polynomial F vanishes only at $q = i$, while the only root of G is $q = k$. However, performing iterated right divisions, one gets that $GCRD(F, G) = q - j = (k - i)^{-1} * F - (k - i)^{-1} * G$.

3.8 Gröbner Bases for Quaternionic Polynomials

The theory of Gröbner bases provides a powerful tool to perform effective computations in any commutative polynomial ring. A classical reference is [1]. Recently [88, 89], this theory has been extended to a wider collection of algebras, including the so-called *solvable algebras*, or *G-algebras* (see [125]). A G-algebra A is essentially a quotient of a ring of noncommutative polynomials $\mathbb{K}\langle x_1, \ldots, x_n\rangle$ modulo a two-sided ideal of relations. Relations in a G-algebra are of the type

$$x_j x_i = c_{ij} \cdot x_i x_j + d_{ij}, \qquad 1 \le i < j \le n, \tag{3.22}$$

where $c_{ij} \in \mathbb{K}$ and $d_{ij} \in A$ satisfy certain "nondegeneracy" conditions, which are not relevant in this context. In particular, a vector space basis for any G-algebra is given by the standard monomials $x_1^{a_1} \cdots x_n^{a_n}$, a fact which allows one to carry over many concepts and algorithms from the commutative theory of Gröbner bases. A key condition which guarantees the termination of the Buchberger algorithm for G-algebras is the fact that, with respect to a given term ordering on the set of standard monomials, we have that the leading term (LT) of the polynomials d_{ij} satisfies

$$LT(d_{ij}) < x_i x_j, \qquad 1 \le i < j \le n.$$

Examples of G-algebras (and quotients of G-algebras) include the Weyl algebra, the exterior algebra over a finite dimensional vector space, the Clifford algebra, and all universal enveloping algebras associated with simple Lie algebras. The ring of polynomials in $\mathbb{H}$ is also a quotient of a G-algebra, as the following propositions show. We omit their proofs since they are straightforward.

Proposition 3.56. *Let $\mathcal{H}$ be the $\mathbb{R}$-algebra generated by q, i, j, k and satisfying the following relations:*

(1) $qi = iq, qj = jq, qk = kq.$
(2) $ij = -ji, jk = -kj, ik = -ki.$

Then $\mathcal{H}$ is a G-algebra.

Notice that the relations of (2) in the above proposition make i, j, and k into anticommuting variables, while (1) says that q behaves like an indeterminate in a commutative polynomial ring. If we then introduce the relations $i^2 = j^2 = k^2 = -1$, we can state the next result.

Proposition 3.57. *Let $\mathcal{H}$ be as above and let $\mathfrak{I}$ be the two-sided ideal of $\mathcal{H}$ generated by $(i^2+1, j^2+1, k^2+1, ij-k, jk-i, ik+j)$. Then $\mathbb{H}[q] \simeq \mathcal{H}/\mathfrak{I}$.*

We recall the following definition, which we present only for left ideals.

Definition 3.58. Let $\mathfrak{I}$ be a left ideal in a G-algebra A, and let σ be an order relation on the set of standard monomials which is compatible with multiplication (i.e., a *term order*) and such that $t > 1$ for every monomial t (i.e., a *well ordering*). Denote by $LT_\sigma(F)$ the leading term of an element of A with respect to such a relation. A subset $\mathcal{G} \subset A$ is called a *left Gröbner basis* for $\mathfrak{I}$ if $LT(\mathcal{G}) = \{LT_\sigma(g) \mid g \in \mathcal{G}\}$ generates the left monoid $(LT_\sigma(f) \mid f \in \mathfrak{I})$. Moreover, we say that $\mathcal{G}$ is *reduced* if the leading term of every element g of $\mathcal{G}$ does not divide the monomials of $\mathcal{G} \setminus \{g\}$.

Since $\mathbb{H}[q]$ is a quotient of a G-algebra, every left or right ideal admits a unique reduced Gröbner basis with respect, for instance, to the term order given by the extension of $q > i > j > k$ to the set of terms. Corollary 3.53 implies the following result.

Proposition 3.59. *Let $\mathfrak{I} = (F_1, \ldots, F_n)$ be a right (respectively left) ideal of $\mathbb{H}[q]$. Then the reduced right (respectively left) Gröbner basis of any system of generators for $\mathfrak{I}$, with respect to any term ordering, is*

$$\{GCLD(F_1, \ldots, F_n)\} \tag{3.23}$$

(respectively $\{GCRD(F_1, \ldots, F_n)\}$).

Proof. Let us prove the statement for a right ideal. If the generators have no common root, the Bezout Theorem shows that $D = 1$ is their greatest common left divisor and $\mathfrak{I} = (1)$. It is clear that $\{1\}$ is the reduced Gröbner basis of $\mathfrak{I}$. Suppose now that $D := GCLD(F_1, \ldots, F_n)$ is not constant. Since D divides all generators, then $LT(D)$ divides all their leading terms, which shows that $\{D\}$ is a Gröbner basis for $\mathfrak{I}$. The fact that it is reduced is obvious since it only contains one element. □

Remark 3.60. Note that, in particular, this proposition allows to compute greatest common divisors using Gröbner bases, which is an alternative to Algorithm 3.47. If one uses Singular [75], a software package on which Gröbner bases algorithms have been implemented, the algebra $\mathbb{H}[q]$ can be introduced via the sequence of commands

```
ring r=0,(q,i,j,k),dp;
matrix C[4][4]=0,1,1,1,0,0,-1,-1,0,0,0,-1,0,0,0,0;
LIB "nctools.lib";
ncalgebra(C,0);
ideal a=i2+1,j2+1,k2+1,ij-k,jk-i,ik+j;
qring H=twostd(a);
```

Then only *right* greatest common divisors can be computed with the command `std` which returns a *left* Gröbner basis. However, one can transform a *right* Gröbner basis computation into that of a *left* Gröbner basis for the ideal generated by means of the regular conjugate

$$\left(\sum_i q^i a_i\right)^c = \sum_i q^i \bar{a}_i$$

since $(f * g)^c = g^c * f^c$. Therefore,

$$GCLD(f_1, \ldots, f_n) = \left(GCRD(f_1^c, \ldots, f_n^c)\right)^c .$$

We now come to an application of the theory of Gröbner bases, which is classical in the commutative case. Given a (not necessarily commutative) ring R, the question of whether some elements $f_1, \ldots, f_n$ of R satisfy relations of the type

$$a_1 f_1 + \cdots + a_n f_n = 0, \qquad a_i \in R,$$

is highly nontrivial [11]. The n-tuples $(a_1, \ldots, a_n)$ are called (left) *syzygies* of $f_1, \ldots, f_n$, and they clearly form a (left) R-module. When R is the ring of commutative polynomials, a classical algorithm allows to explicitly construct the syzygies of a set of polynomials. We now start with a result on the nature of syzygies of linear polynomials. Let $\{\mathbf{e}_1, \ldots \mathbf{e}_n\}$ be the canonical basis of $\mathbb{H}[q]^n$.

Proposition 3.61. *For every integer $i = 1, \ldots, n$, $n > 1$, let $f_i = q - a_i \in \mathbb{H}[q]$ where a_i are distinct quaternions. Then the $\mathbb{H}[q]$ left module of syzygies of $(f_1, \ldots, f_n)$ is generated by*

$$s_{12} = (q - a_2)(\bar{a}_1 - \bar{a}_2)\mathbf{e_1} + (q - a_1)(\bar{a}_2 - \bar{a}_1)\mathbf{e_2},$$

together with (if $n > 2$) the $n - 2$ polynomials

$$t_i = f_i(a_1 - a_2)^{-1}\mathbf{e_1} + f_i(a_2 - a_1)^{-1}\mathbf{e_2} + \mathbf{e_i}, \qquad 3 \le i \le n.$$

Proof. The fact that the above vectors are syzygies is a straightforward calculation. Notice that the t_i's form a syzygy because $(q-a_1)-(q-a_2) = a_2-a_1$, which implies that $(a_2-a_1)^{-1}(q-a_1)-(a_2-a_1)^{-1}(q-a_2) = 1$. This yields, in particular, that the ideal (f_1, f_2) is actually the whole ring. Hence, we only need to prove the statement for $n = 2$, namely, $Syz(f_1, f_2) = \langle s_{12}\rangle$. Let $f, g \in \mathbb{H}[q]$ be two polynomials such that $f * f_1 + g * f_2 = 0$. Let us consider the linear polynomial $L = f_2 * (\bar{a}_1 - \bar{a}_2)$ and let us divide f to the right by L. We obtain

$$f(q) = Q(q) * L(q) + R,$$

where $R \in \mathbb{H}$. This implies

$$\begin{aligned} f(q) * (q - a_1) &= Q(q) * L(q) * (q - a_1) + R * (q - a_1) \\ &= Q(q) * (q - a_2) * (\bar{a}_1 - \bar{a}_2) * (q - a_1) + R * (q - a_1), \end{aligned}$$

and using the fact that both s_{12} and (f, g) are syzygies of the pair (f_1, f_2), we have

$$-g(q) * (q - a_2) = Q(q) * (q - a_1) * (\bar{a}_1 - \bar{a}_2) * (q - a_2) + R * (q - a_1)$$

from which it follows that

$$-[g(q) + Q(q) * (q - a_1) * (\bar{a}_1 - \bar{a}_2)] * (q - a_2) = R * (q - a_1).$$

The previous equation can only hold if the term $g(q) + Q(q) * (q - a_1) * (\bar{a}_1 - \bar{a}_2)$ is a constant. However, it is easy to see that the only constants $R, S \in \mathbb{H}$ satisfying $S * (q - a_2) = R * (q - a_1)$ are $R = S = 0$, a fact which, combined with (3.24), proves the statement. □

The proof of the previous result only uses Euclidean division. We now present a more general result, whose proof is based on a classical method due to Schreyer for the construction of the syzygies of the generators of an ideal (once a Gröbner basis is given for the ideal).

Consider some nonzero polynomials $F_1, \ldots, F_t \in \mathbb{H}[q]$ having degrees $n_1, \ldots, n_t$, and compute $D = GCRD(F_1, \ldots, F_t)$. We can suppose that F_i be monic and that $n_1 \geq \cdots \geq n_t$, so we can define $d_{ij} := n_i - n_j$ for all $1 \leq i < j \leq t$. Using Corollary 3.53, we can write $F_i = H_i * D$ for all i and find polynomials A_i such that $\sum_{i=1}^{t} A_i * F_i = D$ with the Euclidean Algorithm. For all $1 \leq i < j \leq t$, set $G_{ij} := F_i - q^{d_{ij}} F_j$, and let C_{ij} be such that $G_{ij} = C_{ij} * D$. With this notation, we state the following:

Theorem 3.62. *The module of left syzygies $Syz(F_1, \ldots, F_t)$ is generated by the $\binom{t}{2}$ vectors*

$$v_{ij} := \mathbf{e_i} - q^{d_{ij}} \mathbf{e_j} - C_{ij} * \sum_{k=1}^{t} A_k \mathbf{e_k}, \qquad 1 \leq i < j \leq t, \tag{3.24}$$

where $\mathbf{e_i}$ is the i-th element of the canonical basis of $\mathbb{H}[q]^t$, together with the vectors

$$w_i := \mathbf{e_i} - H_i * \sum_{k=1}^{t} A_k \mathbf{e_k}, \qquad 1 \leq i \leq t. \tag{3.25}$$

Proof. As a consequence of Proposition 3.59, the set $\mathcal{G} = \{F_1, \ldots, F_t, D\}$ is a left Gröbner basis for the ideal $\mathfrak{I} := \mathbb{H}[q](F_1, \ldots, F_t)$, although not a reduced one. We have chosen the Gröbner basis so that it contains the generators of $\mathfrak{I}$, because in this case it is easier to "lift" the syzygies of $\mathcal{G}$ to those of $\mathfrak{I}$. It suffices indeed to calculate the S-polynomials of all pairs in $\mathcal{G}$, express them as a combination of the basis (which amounts to writing them as multiples of D since this is the minimal one), and then read the relations obtained as combinations of $F_1, \ldots, F_t$. Given $i < j$, take

$$S(F_i, F_j) = F_i - q^{d_{ij}} F_j = C_{ij} * D.$$

Since $D = \sum_k A_k F_k$, we can rewrite the previous equality as

$$F_i - q^{d_{ij}} F_j - C_{ij} \sum_{k=1}^{t} A_k * F_k = 0,$$

which says that $v_{ij} \cdot (F_1, \dots, F_t) = 0$. For the second set of syzygies, let d be the degree of D, and remember that D is monic by definition. Therefore, for all $1 \leq i \leq t$, we have

$$S(F_i, D) = F_i - q^{n_i - d} * D = H_i * D - q^{n_i - d} * D = (H_i - q^{n_i - d}) * D.$$

Comparing the second and the last term in the above chain of equalities, and rewriting D, we obtain the following relation:

$$F_i - q^{n_i - d} * D - (H_i - q^{n_i - d}) * D = F_i - H_i * \sum_{k=1}^{n} A_k * F_k,$$

which means that $w_i \cdot (F_1, \dots, F_t) = 0$. Based on the discussion in [89], these vectors generate the module of left syzygies. □

Remark 3.63. The generators presented in the previous theorem may not be minimal. The linear syzygies constructed in Proposition 3.61 are in fact fewer than the ones provided by Theorem 3.62. Consider for instance the case $t = 2$, and $F_i = q - a_i$, with $a_1 \neq a_2 \in \mathbb{H}$. While Proposition 3.61 only gives the syzygy $s_{12} = (q - a_2)(\bar{a}_1 - \bar{a}_2)\mathbf{e_1} + (q - a_1)(\bar{a}_2 - \bar{a}_1)\mathbf{e_2}$, the previous theorem would give three redundant generators. However, since $D = 1$ in this case, we have $H_i = F_i$, $A_i = (-1)^i (a_2 - a_1)^{-1}$ and $C_{12} = (a_2 - a_1)$. Therefore, $v_{12} = 0$ and

$$w_1 = w_2 = ((a_2 - a_1 - F_1)\mathbf{e_1} - F_1 \mathbf{e_2})(a_2 - a_1)^{-1} = |a_2 - a_1|^2 s_{12}.$$

Bibliographic Notes

The properties of the zero sets presented in Sect. 3.1 were proven in [62] on Euclidean balls centered at 0 and generalized to symmetric slice domains in [19, 57].

The algebraic and topological properties of the zero sets in Sects. 3.2 and 3.3 were proven in [55] for power series and extended to all regular functions on symmetric slice domains in [19, 57].

The study of the roots of quaternions in Sect. 3.4 derives from [68]. The factorization properties in Sect. 3.5 were proven in [55, 63].

The notions of multiplicity presented in Sect. 3.6 were introduced for power series in [55] (classical multiplicity) and in [63] (spherical and isolated multiplicities). They were compared in [119] and extended to all regular functions on symmetric slice domains in [57]. Recent work in this area includes [128].

Finally, the results in Sects. 3.7 and 3.8 come from [40].

Chapter 4
Infinite Products

4.1 Infinite Products of Quaternions

We consider an infinite product of quaternions

$$q_0\, q_1 \dots q_i \dots = \prod_{i=0}^{\infty} q_i$$

and, for $n \in \mathbb{N}$, we denote by $Q_n = q_0 q_1 \dots q_n$ the partial products. In analogy with the complex case (see [4]), we give the following definition.

Definition 4.1. The infinite product $\prod_{i=0}^{\infty} q_i$ is said to *converge* if and only if at most a finite number of the factors vanish and if the partial products formed by the nonvanishing factors tend to a nonzero finite limit.

In what follows, we will always refer to an infinite product assuming that only finitely many factors vanish, and we will check its convergence looking at the product of the nonvanishing terms. We point out that in a convergent product we have that $\lim_{i\to\infty} q_i = 1$, since $q_i = Q_{i-1}^{-1} Q_i$. It is therefore preferable to write all infinite products in the form

$$\prod_{i=0}^{\infty} (1 + a_i)$$

so that $\lim_{i\to\infty} a_i = 0$ is a necessary condition for their convergence.

Remark 4.2. The requirement that the partial products of nonvanishing factors of $\prod_{i=0}^{\infty}(1 + a_i)$ tend to a finite limit different from zero is standard in the complex case. In fact, assuming this, an infinite product $\prod_{i=0}^{\infty}(1 + c_i)$ with $c_i \in \mathbb{C}$ converges simultaneously with the series

$$\sum_{i\in\mathbb{N}} \mathrm{Log}(1 + c_i),$$

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whose terms represent the values of the principal branch of the logarithm. The convergence is not necessarily simultaneous if the limit of the infinite product is zero, as the following example shows.

Example 4.3. Let

$$\prod_{m=0}^{\infty}(1-1/m) \tag{4.1}$$

and consider the corresponding series:

$$\sum_{m\in\mathbb{N}} \mathrm{Log}(1-1/m). \tag{4.2}$$

Denoting the nth partial sum of (4.2) by S_n, we have that $Q_n = e^{S_n}$. Since $\lim_{n\to\infty} S_n = -\infty$, then

$$\lim_{n\to\infty} Q_n = \lim_{n\to\infty} e^{S_n} = 0.$$

Therefore the sequence of partial products converges (to zero), while the series (4.2) diverges (to $-\infty$).

4.2 The Quaternionic Logarithm

In order to study the convergence of infinite products in the quaternionic case, we need to introduce a logarithm on $\mathbb{H}$, which inverts the *exponential function* on $\mathbb{H}$ naturally defined as

$$\exp(q) = e^q = \sum_{n\in\mathbb{N}} \frac{q^n}{n!}.$$

Note that e^q coincides with the complex exponential function on any complex plane L_I.

Definition 4.4. Let $\Omega \subseteq \mathbb{H}$ be a connected open set. We define a *branch of the quaternionic logarithm* (or simply a *branch of the logarithm*) on Ω a function $f : \Omega \to \mathbb{H}$ such that for every $q \in \Omega$,

$$e^{f(q)} = q.$$

First of all, since $\exp(q)$ never vanishes, we must suppose that $0 \notin \Omega$. Setting

$$I_q = \begin{cases} Im(q)/|Im(q)| & \text{if q} \in \mathbb{H}\setminus\mathbb{R} \\ \text{any element of } \mathbb{S} \text{ otherwise} \end{cases}$$

we have that for every $q \in \mathbb{H} \setminus \{0\}$, there exists a unique $\theta \in [0, \pi]$ such that $q = |q|e^{\theta I_q}$. Moreover, $\theta = \arccos(Re(q)/|q|)$.

Definition 4.5. The function $\arccos(Re(q)/|q|)$ will be called the *principal quaternionic argument* of q, and it will be denoted by $\mathrm{Arg}_{\mathbb{H}}(q)$ for every $q \in \mathbb{H} \setminus \{0\}$.

We are now ready to define the principal branch of the quaternionic logarithm.

Definition 4.6. Let $\ln(r)$ denote the natural logarithm of a real number $r > 0$. For every $q \in \mathbb{H} \setminus (-\infty, 0]$, we define the *principal branch of the quaternionic logarithm* of q as

$$\mathrm{Log}(q) = \ln|q| + \arccos\left(\frac{Re(q)}{|q|}\right) I_q.$$

This same definition of principal branch of the logarithm was already given in [76], in the setting of Clifford algebras. In fact Definition 4.6 can be obtained by specializing to the case of quaternions that of paravector logarithm introduced in [76], Definition 11.24, page 231. It is also important to state that:

Proposition 4.7. *The principal branch of the logarithm is a continuous function on* $\mathbb{H} \setminus (-\infty, 0]$.

Proof. The function $q \mapsto \ln|q|$ is clearly continuous on $\mathbb{H} \setminus \{0\}$. The function $q \mapsto \arccos\left(\frac{Re(q)}{|q|}\right) \frac{Im(q)}{|Im(q)|}$ is defined and continuous at every $q \in \mathbb{H} \setminus \mathbb{R}$. Moreover, for every strictly positive real r and for every $q \notin \mathbb{R}$, we have

$$\lim_{q \to r} \arccos\left(\frac{Re(q)}{|q|}\right) I_q = 0.$$

Hence $q \mapsto \arccos\left(\frac{Re(q)}{|q|}\right) I_q$ is continuous in $\mathbb{H} \setminus (-\infty, 0]$. □

This result is sharp since the limit

$$\lim_{q \to s} \arccos\left(\frac{Re(q)}{|q|}\right) I_q$$

does not exist if $s \leq 0$.

Remark 4.8. While the principal branch of the logarithm obviously satisfies the relation

$$e^{\mathrm{Log}(q)} = q$$

in $\mathbb{H} \setminus (-\infty, 0]$, the equality

$$\mathrm{Log}(e^q) = q$$

is only valid in the domain $\{q \in \mathbb{H} : |Im(q)| < \pi\}$.

As one may expect, we can prove the following nice result.

Proposition 4.9. *The principal branch of the quaternionic logarithm coincides with the principal branch of the complex logarithm when restricted to any complex plane L_I, with $I \in \mathbb{S}$.*

Proof. Let $I \in \mathbb{S}$. First of all, we notice that, since $L_I = L_{-I}$, every quaternion $q \in L_I \setminus \mathbb{R}$ can be written both as $x + yI$ and as $x - y(-I)$, for suitable $x, y \in \mathbb{R}$. If $(x, y) \in \mathbb{R}^2 \setminus (-\infty, 0]$, then by Definition 4.6, we have that

$$\operatorname{Log}(x + yI) = \ln\left(\sqrt{x^2 + y^2}\right) + \arccos\left(\frac{x}{\sqrt{x^2 + y^2}}\right)\frac{y}{|y|}I$$

and

$$\operatorname{Log}(x - y(-I)) = \ln\left(\sqrt{x^2 + y^2}\right) + \arccos\left(\frac{x}{\sqrt{x^2 + y^2}}\right)\frac{(-y)}{|y|}(-I)$$

coincide.

We now can consider the restriction of the principal branch of the quaternionic logarithm to L_I:

$$\operatorname{Log}(x + yI) = \ln(\sqrt{x^2 + y^2}) + \operatorname{Arg}_{\mathbb{H}}(x + yI)\frac{y}{|y|}I.$$

We set $\phi(x + yI) = \operatorname{Arg}_{\mathbb{H}}(x + yI)\frac{y}{|y|}$. The function $\operatorname{Arg}_{\mathbb{H}}(x + yI)$, with values in $[0, \pi)$, is the amplitude of the angle between the positive real half line and the vector $x + yI$. When $y > 0$, then $x + yI$ belongs to the upper half plane of L_I, and $\phi(x + yI) = \operatorname{Arg}_{\mathbb{H}}(x + yI)$. On the other hand, if $y < 0$, we obtain that $x + yI$ belongs to the lower half plane of L_I, and $\phi(x + yI) = -\operatorname{Arg}_{\mathbb{H}}(x + yI)$. Therefore, $\phi(x + yI)$ coincides with the imaginary part of the principal branch of the complex logarithm of L_I, and the proof is complete. □

In the complex case it is well known that the argument of a product is equal to the sum of the arguments of the factors (up to an integer multiple of 2π). In the quaternionic case we have the following lemma, which will play a key role in the sequel.

Lemma 4.10. *Let $n \in \mathbb{N}$ and let $\theta_1, \ldots, \theta_n \in [0, \pi)$ be such that $\sum_{i=1}^n \theta_i < \pi$. Then for every set $\{I_1, \ldots, I_n\} \subseteq \mathbb{S}$, we have that*

$$\operatorname{Arg}_{\mathbb{H}}(e^{\theta_1 I_1} \cdots e^{\theta_n I_n}) \leq \sum_{i=1}^n \theta_i.$$

Proof. By induction on n. For $n = 1$, the thesis is straightforward. Let $\theta_1, \theta_2, \ldots, \theta_n \in [0, \pi)$ be such that $\sum_{i=1}^n \theta_i < \pi$. Let $\phi \in [0, \pi)$ be the principal quaternionic argument of the product $e^{\theta_1 I_1} \cdots e^{\theta_{n-1} I_{n-1}}$ and let $J \in \mathbb{S}$ be such that $e^{\theta_1 I_1} \cdots e^{\theta_{n-1} I_{n-1}} = e^{\phi J}$. Consider the product $e^{\phi J} e^{\theta_n I_n}$:

$$e^{\phi J} e^{\theta_n I_n} = \cos\phi \cos\theta_n + (\cos\phi \sin\theta_n) I_n + (\sin\phi \cos\theta_n) J + (\sin\phi \sin\theta_n) J I_n.$$

From Formula (2.9), we have that $J I_n = -\langle J, I_n \rangle + J \times I_n$, and hence, we obtain that

$$\cos(\mathrm{Arg}_{\mathbb{H}}(e^{\phi J} e^{\theta_n I_n})) = Re\left(e^{\phi J} e^{\theta_n I_n}\right) = \cos\phi \cos\theta_n - (\sin\phi \sin\theta_n)\langle J, I_n \rangle.$$

Since $\langle J, I_n \rangle \leq |J||I_n| = 1$ and $\sin\phi \sin\theta_n \geq 0$, we obtain the inequality

$$\cos(\mathrm{Arg}_{\mathbb{H}}(e^{\phi J} e^{\theta_n I_n})) \geq \cos\phi \cos\theta_n - \sin\phi \sin\theta_n = \cos(\phi + \theta_n).$$

The function $\cos(x)$ decreases in $[0, \pi]$, and hence, we have

$$\mathrm{Arg}_{\mathbb{H}}(e^{\phi J} e^{\theta_n I_n}) \leq \phi + \theta_n.$$

Thanks to the induction hypothesis we have the thesis. □

If $\{a_i\}_{i \in \mathbb{N}} \subseteq \mathbb{H}$ is a sequence such that the series $\sum_{i \in \mathbb{N}} |a_i|$ converges, we know that the series $\sum_{i \in \mathbb{N}} a_i$ converges too. With this in mind, we can prove the following result.

Theorem 4.11. *Let $\{a_i\}_{i \in \mathbb{N}} \subseteq \mathbb{H}$ be a sequence. If the series $\sum_{i \in \mathbb{N}} |\mathrm{Log}(1 + a_i)|$ converges, then the product $\prod_{i=0}^{\infty}(1 + a_i)$ converges too.*

Proof. The convergence of the series $\sum_{i \in \mathbb{N}} |\mathrm{Log}(1 + a_i)|$ implies that the sequence $\{a_i\}_{i \in \mathbb{N}}$ tends to zero. We can therefore suppose that $1 + a_i \notin (-\infty, 0]$. For every $i \in \mathbb{N}$, let $\theta_i \in [0, \pi)$ and let $I_i \in \mathbb{S}$ be such that $1 + a_i = |1 + a_i| e^{\theta_i I_i}$. Then $\mathrm{Log}(1 + a_i) = \ln|1 + a_i| + \theta_i I_i$, and therefore,

$$|\ln|1 + a_i|| \leq |\mathrm{Log}(1 + a_i)| \tag{4.3}$$

and

$$|\theta_i| = |\theta_i I_i| \leq |\mathrm{Log}(1 + a_i)| \tag{4.4}$$

for every $i \in \mathbb{N}$.

We have to prove that the sequence of the partial products

$$Q_n = \prod_{i=0}^{n}(1 + a_i)$$

tends to a finite limit (which is different from zero). Since real numbers commute with quaternions, for every $n \in \mathbb{N}$, we have that

$$\prod_{i=0}^{n}(1+a_i)=\prod_{i=0}^{n}|1+a_i|\prod_{i=0}^{n}e^{\theta_i I_i}.$$

By hypothesis and by inequality (4.3), the series $\sum_{i\in\mathbb{N}}\ln|1+a_i|$ converges. Hence the infinite product $\prod_{i=0}^{\infty}|1+a_i|$ converges as well (see Remark 4.2). Then it suffices to prove that the sequence $R_n=\prod_{i=0}^{n}e^{\theta_i I_i}\subseteq\partial B(0,1)$ converges. We will show that $\{R_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. For $n>m\in\mathbb{N}$, we have

$$|R_n-R_m|=\left|\prod_{i=0}^{n}e^{\theta_i I_i}-\prod_{i=0}^{m}e^{\theta_i I_i}\right|=\left|\prod_{i=0}^{m}e^{\theta_i I_i}\prod_{i=m+1}^{n}e^{\theta_i I_i}-\prod_{i=0}^{m}e^{\theta_i I_i}\right|=$$
$$=\left|\prod_{i=0}^{m}e^{\theta_i I_i}\right|\left|\prod_{i=m+1}^{n}e^{\theta_i I_i}-1\right|=\left|\prod_{i=m+1}^{n}e^{\theta_i I_i}-1\right|\le\mathrm{Arg}_{\mathbb{H}}\left(\prod_{i=m+1}^{n}e^{\theta_i I_i}\right),$$

where the last inequality holds due to the fact that $\left|\prod_{i=m+1}^{n}e^{\theta_i I_i}\right|=1$. Inequality (4.4) implies that the series $\sum_{i\in\mathbb{N}}\theta_i$ converges. Therefore the sequence $S_n=\sum_{i=0}^{n}\theta_i$ of the partial sums is a Cauchy sequence. This yields that for all $\epsilon>0$, there exists $m_0\in\mathbb{N}$ such that for all $n>m>m_0$,

$$\sum_{i=m+1}^{n}\theta_i<\epsilon.$$

In particular for $\epsilon<\pi$, by Lemma 4.10, we have that

$$\mathrm{Arg}_{\mathbb{H}}\left(\prod_{i=m+1}^{n}e^{\theta_i I_i}\right)\le\sum_{i=m+1}^{n}\theta_i<\epsilon.$$

The assertion follows. □

The following proposition will be useful in the sequel, when studying the convergence of the infinite products.

Proposition 4.12. *Let* Log *be the principal branch of the quaternionic logarithm. Then*

$$\lim_{q\to 0}q^{-1}\mathrm{Log}(1+q)=1. \tag{4.5}$$

Proof. Let $\{q_n\}_{n\in\mathbb{N}}=\{x_n+y_nI_n\}_{n\in\mathbb{N}}$ be a sequence such that $q_n\to 0$ for $n\to\infty$. We can assume without loss of generality that $y_n>0$. We consider the following norm:

$$|(x_n+y_nI_n)^{-1}\mathrm{Log}(1+x_n+y_nI_n)-1|. \tag{4.6}$$

Easy calculations show that both the real part and the norm of the imaginary part of $(x_n+y_nI_n)^{-1}\mathrm{Log}(1+x_n+y_nI_n)$ do not depend on the complex direction I_n. Therefore we can assume $I_n=I_0$ for every $n\in\mathbb{N}$. Hence we consider

$$|(x_n + y_n I_0)^{-1}\mathrm{Log}(1 + x_n + y_n I_0) - 1| \tag{4.7}$$

and we observe that $(x_n + y_n I_0)^{-1}\mathrm{Log}(1 + x_n + y_n I_0) \in L_{I_0}$ for all $n \in \mathbb{N}$. Since we are in the complex plane L_{I_0}, by the same arguments used in the complex case, the sequence (4.7) tends to zero. □

Corollary 4.13. *The series* $\sum_{n\in\mathbb{N}} |\mathrm{Log}(1 + a_n)|$ *converges if and only if the series* $\sum_{n\in\mathbb{N}} |a_n|$ *converges.*

Proof. If either the series $\sum_{n\in\mathbb{N}} |\mathrm{Log}(1+a_n)|$ or the series $\sum_{n\in\mathbb{N}} |a_n|$ converge, we have $\lim_{n\to\infty} a_n = 0$. By Proposition 4.12, for any given $\epsilon > 0$,

$$|a_n|(1-\epsilon) \le |\mathrm{Log}(1 + a_n)| \le |a_n|(1+\epsilon)$$

for all sufficiently large n. It follows immediately that the two series converge simultaneously. □

As a consequence of Theorem 4.11 and Corollary 4.13, we end this section by stating a sufficient condition for the convergence of quaternionic infinite products.

Theorem 4.14. *Let* $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{H}$. *A sufficient condition for the convergence of the product* $\prod_{n=0}^{\infty}(1 + a_n)$ *is the convergence of the series* $\sum_{n\in\mathbb{N}} |a_n|$.

4.3 Infinite Products of Functions Defined on $\mathbb{H}$

Let $\prod_{i=0}^{\infty} g_i(q)$ be an infinite product whose factors are functions of a quaternionic variable defined on an open set $\Omega \subseteq \mathbb{H}$. As in the complex case, we must specify the notion of convergence we refer to for $\prod_{i=0}^{\infty} g_i(q)$. If none of the functions $\{g_i\}_{i\in\mathbb{N}}$ vanish on Ω, then (in accordance with Definition 4.1) we will require the sequence of partial products

$$G_n(q) = \prod_{i=0}^{n} g_i(q)$$

to converge uniformly on compact sets in Ω to a nowhere vanishing function. The possible ambiguity arising from the presence of factors having zeros is avoided with the next definition.

Definition 4.15. Let $\{g_i\}_{i\in\mathbb{N}}$ be a sequence of functions defined on an open set $\Omega \subseteq \mathbb{H}$. We will say that

$$\prod_{i=0}^{\infty} g_i(q)$$

converges compactly in Ω to a function $P : \Omega \to \mathbb{H}$ if, for any compact $K \subseteq \Omega$, the next three properties hold:

1. There exists an integer N_K such that $g_i \neq 0$ on K for all $i \ge N_K$.

2. The *residual product*
$$\prod_{i=N_K}^{\infty} g_i(q)$$
(whose factors do not vanish on K) converges uniformly in K to a nowhere vanishing function P_{N_K}.
3. We have, for all $q \in K$,
$$P(q) = \left[\prod_{i=0}^{N_K-1} g_i(q)\right] P_{N_K}(q).$$

Remark 4.16. If $\prod_{i=0}^{\infty} g_i(q)$ converges compactly in $\Omega \subseteq \mathbb{H}$, by the previous definition, we have that $\lim_{i\to\infty} g_i = 1$ uniformly on compact sets in Ω. Indeed, for any compact $K \subseteq \Omega$, taking any $j > N_K$, setting
$$G_j = \prod_{i=N_K}^{j} g_i(q)$$
and $g_j = G_{j-1}^{-1} G_j$, we have
$$\lim_{j\to\infty} g_j = \lim_{i\to\infty} G_{j-1}^{-1} G_j = \lim_{j\to\infty} G_{j-1}^{-1} \lim_{j\to\infty} G_j = 1.$$

We will thus write any infinite product in the form
$$\prod_{i=0}^{\infty}(1 + f_i(q)),$$
where uniform convergence on compact sets of $\{f_i\}_{i\in\mathbb{N}}$ to the zero function is a necessary condition for the compact convergence of the infinite product itself.

Remark 4.17. Let $\prod_{i=0}^{\infty}(1 + f_i(q))$ be an infinite product. If the sequence of functions $\{f_i\}_{i\in\mathbb{N}}$ converges compactly in $\Omega \subseteq \mathbb{H}$ to the zero function, then condition 1 in Definition 4.15 is automatically fulfilled.

When $\{f_i\}_{i\in\mathbb{N}}$ is a sequence of C^∞ functions, we obtain the following result:

Proposition 4.18. *Let $\{f_i\}_{i\in\mathbb{N}}$ be a sequence of C^∞ functions on an open set $\Omega \subseteq \mathbb{H}$ that converges uniformly to the zero function on compact sets in Ω. If the infinite product $\prod_{i=0}^{\infty}(1 + f_i(q))$ converges compactly in Ω to a function P, then P is C^∞ on Ω.*

Proof. Let $q \in \Omega$ and let $K \subseteq \Omega$ be a compact neighborhood of q. By the assumption on the sequence $\{f_i\}_{i\in\mathbb{N}}$, we have that (see Definition 4.15) there exists an integer N_K such that the residual product

$$\prod_{i=N_K}^{\infty}(1+f_i(q))$$

converges uniformly in K to a nowhere vanishing function, which we call P_{N_K}. Hence P_{N_K} is the limit of a uniformly convergent sequence of C^∞ functions, so that it is C^∞. Then we can write P as a finite product of C^∞ functions on K:

$$P(q)=\left[\prod_{i=0}^{N_K-1}(1+f_i(q))\right]P_{N_K}(q).$$

Therefore P is C^∞ in K. Moreover, in K, the zero set of P coincides with the zero set of $\prod_{i=0}^{N_K-1}(1+f_i(q))$. □

We now consider infinite products

$$\prod_{i=0}^{\infty}(1-qa_i^{-1}) \tag{4.8}$$

with $\{a_i\}_{i\in\mathbb{N}}\subseteq\mathbb{H}\setminus\{0\}$. By Theorem 4.14, for any fixed $q\in\mathbb{H}$, a sufficient condition for the convergence of (4.8) is the convergence of the series $\sum_{i=0}^{\infty}|q|/|a_i| = |q|\sum_{i=0}^{\infty}1/|a_i|$. As a consequence, a sufficient condition for the compact convergence of (4.8) in $\mathbb{H}$ is the convergence of the series $\sum_{i=0}^{\infty}1/|a_i|$. In order to guarantee compact convergence in $\mathbb{H}$ under the weaker hypothesis $\lim_{n\to\infty}|a_n|=\infty$, we will introduce (as in the complex case, see, e.g., [4]) suitable convergence-producing factors. The same arguments used in the complex case lead to the following statement. We will give the proof here for the sake of completeness and to remark a few novelties due to the quaternionic environment.

Theorem 4.19. *Let $\{a_n\}_{n\in\mathbb{N}}\subseteq\mathbb{H}\setminus\{0\}$ be such that $\lim_{n\to\infty}|a_n|=\infty$. Then the infinite product*

$$\prod_{n=0}^{\infty}(1-qa_n^{-1})e^{qa_n^{-1}+\frac{1}{2}(qa_n^{-1})^2+\ldots+\frac{1}{n}(qa_n^{-1})^n} \tag{4.9}$$

converges compactly in $\mathbb{H}$.

Proof. Let $n\in\mathbb{N}$ and let $p_n(q)=q+\frac{1}{2}q^2+\ldots+\frac{1}{n}q^n$. If we prove the compact uniform convergence of the series $\sum_{n\in\mathbb{N}}|r_n|$ with

$$r_n(q)=\operatorname{Log}\left((1-qa_n^{-1})e^{p_n(qa_n^{-1})}\right), \tag{4.10}$$

(where Log is the principal branch of the logarithm so that $r_n(0)=0$), then Theorem 4.11 allows us to conclude. For every $q\in\mathbb{H}$, the quaternions $1-qa_n^{-1}$ and $p_n(qa_n^{-1})$ lie in the same complex plane. Thus, since $r_n(0)=p_n(0)=0$, we can express $r_n(q)$ as

$$r_n(q) = \mathrm{Log}(1 - qa_n^{-1}) + p_n(qa_n^{-1}).$$

Now, for a given $R > 0$, we only consider the terms with $|a_n| > R$. In the ball $B(0, R)$, we have that $|qa_n^{-1}| < 1$, so that $\mathrm{Log}(1 - qa_n^{-1})$ expands as

$$\mathrm{Log}(1 - qa_n^{-1}) = -qa_n^{-1} - \frac{1}{2}(qa_n^{-1})^2 - \frac{1}{3}(qa_n^{-1})^3 - \dots .$$

Then $r_n(q)$ has the representation

$$r_n(q) = -\frac{1}{n+1}(qa_n^{-1})^{n+1} - \frac{1}{n+2}(qa_n^{-1})^{n+2} - \dots$$

and we obtain easily the estimate

$$|r_n(q)| \leq \frac{1}{n+1}\left(\frac{R}{|a_n|}\right)^{n+1}\left(1 - \frac{R}{|a_n|}\right)^{-1}.$$

As in the complex case, we are left with proving the convergence of the series

$$\sum_{n\in\mathbb{N}} \frac{1}{n+1}\left(\frac{R}{|a_n|}\right)^{n+1}\left(1 - \frac{R}{|a_n|}\right)^{-1}. \tag{4.11}$$

The fact that $\lim_{n\to\infty}(1 - R/|a_n|)^{-1} = 1$ implies that the series (4.11) converges if the series

$$\sum_{n\in\mathbb{N}} \frac{1}{n+1}\left(\frac{R}{|a_n|}\right)^{n+1}$$

does. The latter has a majorant geometric series with ratio strictly smaller than 1; hence, it converges. □

A completely analogous proof yields:

Theorem 4.20. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions that converges compactly in $\mathbb{H}$ to the zero function. Then the infinite product*

$$\prod_{n=0}^{\infty}(1 - f_n(q))\, e^{f_n(q)+\frac{1}{2}f_n(q)^2+\dots+\frac{1}{n}f_n(q)^n} \tag{4.12}$$

converges compactly in $\mathbb{H}$.

4.4 Convergence of an Infinite $*$-Product

The results proved in the last section (Theorems 4.19 and 4.20) concern infinite products, which, in general, do not converge to a regular function. In order to study the factorization of zeros of a regular function, we must consider infinite regular products. We will study the convergence of regular products on symmetric slice domains of $\mathbb{H}$. The next lemma (where a regular product $g_1(q) * \ldots * g_n(q)$ is denoted as $\prod_{m=1}^{*\,n} g_m(q)$, as usual) will be useful in the sequel.

Lemma 4.21. *Let $\{f_i\}_{i\in\mathbb{N}}$ be a sequence of regular functions defined on a symmetric slice domain Ω. Let K be a symmetric compact set such that $K \subseteq \Omega$. We suppose that there exists an integer N_K such that $1 + f_i \neq 0$ on K for all $i \geq N_K$. Set*

$$F_{N_K}^m(q) = \mathop{\prod\nolimits^{*}}_{i=N_K}^{m} (1 + f_i(q)).$$

Then for all $m \geq N_K$ and for any $q \in K$,

$$F_{N_K}^m(q) = \prod_{i=N_K}^{m} (1 + f_i(T_i(q))) \neq 0$$

where $T_j(q) = (F_{N_K}^{j-1}(q))^{-1} q F_{N_K}^{j-1}(q)$ for $j > N_K$ and $T_j(q) = q$ for $j = N_K$.

Proof. We will prove the assertion by induction. Let $q \in K$. The assertion is true for $m = N_K$ by hypothesis. Indeed, $F_{N_K}^{N_K}(q) = 1 + f_{N_K}(q) \neq 0$. Suppose the assertion true for $m = N_K, \cdots, n-1$. We can write

$$F_{N_K}^n(q) = F_{N_K}^{n-1}(q) * (1 + f_n(q)). \tag{4.13}$$

Since $Re(T_j(q)) = Re(q)$ and $|Im(T_j(q)| = |Im(q)|$ for all $j \geq N_K$ and since K is a symmetric set, we have that $T_j(q) \in K$ if and only if $q \in K$. By Theorem 3.4 and using the induction hypothesis $F_{N_K}^{n-1}(q) \neq 0$, formula (4.13) becomes

$$F_{N_K}^n(q) = F_{N_K}^{n-1}(q)(1 + f_n(T_n(q))).$$

By hypothesis, the factor $1 + f_n$ has no zeros in K. Since $T_n(q) \in K$, the function $F_{N_K}^n(q)$ never vanishes in K. By the induction hypothesis, we have that $F_{N_K}^{n-1}(q) = \prod_{i=N_K}^{n-1} (1 + f_i(T_i(q)))$. Hence,

$$F_{N_K}^n(q) = \left[\prod_{i=N_K}^{n-1} (1 + f_i(T_i(q)))\right](1 + f_n(T_n(q))) = \prod_{i=N_K}^{n} (1 + f_i(T_i(q))).$$

□

In analogy with Definition 4.15, we set:

Definition 4.22. Let $\{f_i\}_{i\in\mathbb{N}}$ be a sequence of regular functions defined on a symmetric slice domain Ω. The infinite $*$-product

$$\prod_{i=0}^{\infty}{}^{*}(1+f_i(q))$$

is said to *converge compactly* in Ω to a function $F:\Omega\to\mathbb{H}$ if, for any symmetric compact set $K\subseteq\Omega$, the next three conditions are fulfilled:

1. There exists an integer N_K such that $1+f_i\neq 0$ on K for all $i\geq N_K$.
2. The residual product
$$\prod_{i=N_K}^{\infty}{}^{*}(1+f_i(q))$$
converges uniformly in K (i.e., the sequence
$$\left\{F_{N_K}^{m}(q)=\prod_{i=N_K}^{m}{}^{*}(1+f_i(q))\right\}_{m\geq N_K}$$
converges uniformly in K) to a nowhere vanishing function F_{N_K}.
3. We have, for all $q\in K$,
$$F(q)=\left[\prod_{i=0}^{N_K-1}{}^{*}(1+f_i(q))\right]*F_{N_K}(q).$$

We now establish the equivalence between the compact convergence of an infinite regular product and that of the corresponding infinite product.

Theorem 4.23. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of regular functions on a symmetric slice domain Ω. The infinite $*$-product*

$$\prod_{n=0}^{\infty}{}^{*}(1+f_n(q))$$

converges compactly in Ω if and only if the infinite product

$$\prod_{n=0}^{\infty}(1+f_n(q))$$

converges compactly in Ω.

Proof. It is enough to prove the uniform convergence on symmetric compact sets of Ω. Let K be such a compact set. Let N_K be the integer such that for every $n \geq N_K$, the factors $1 + f_n(q)$ have no zeros in K. By Lemma 4.21, the infinite $*$-product

$$\prod_{i=N_K}^{\infty}{}^{*}\,(1 + f_i(q))$$

converges compactly if, and only if, the infinite product

$$\prod_{i=N_K}^{\infty} (1 + f_i(T_i(q)))$$

does. Since $Re(T_j(q)) = Re(q)$ and $|Im(T_j(q)| = |Im(q)|$ for all $j \geq N_K$, we have that $T_j(q) \in K$ if and only if $q \in K$. This concludes the proof, since we are interested in the uniform convergence in K. □

Our next goal is to show that the limit of a compactly convergent infinite regular product is indeed a regular function.

Lemma 4.24. *Let $\{h_n\}_{n\in\mathbb{N}}$ be a sequence of regular functions on a symmetric slice domain $\Omega \subseteq \mathbb{H}$ that converges to a function h uniformly on compact sets in Ω. Then h is regular in Ω.*

Proof. Given any pair of mutually orthogonal vectors I and J in $\mathbb{S}$, we can write $h_I(x + yI) = h(z)$ as

$$h(z) = F(z) + G(z)J$$

with $\Omega_I = \Omega \cap L_I$ and $F, G : \Omega_I \to L_I$. It is now enough to prove that F, G are holomorphic. By the Splitting Lemma 1.3, we have that, for all $n \in \mathbb{N}$, the two functions $F_n, G_n : \Omega_I \to L_I$ such that

$$h_n(z) = F_n(z) + G_n(z)J$$

are holomorphic in Ω_I. Clearly, $F_n \to F$ and $G_n \to G$ uniformly on compact sets in Ω_I, so that F, G must be holomorphic, as desired. □

We are able to prove the announced result on the regularity of a compactly converging infinite regular product.

Proposition 4.25. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of regular functions defined on a symmetric slice domain Ω. If the infinite regular product*

$$\prod_{n=0}^{\infty}{}^{*}\,(1 + f_n(q))$$

converges compactly in Ω to a function F, then F is regular in Ω.

Proof. By Definition 4.22, for any compact set $K \subseteq \mathbb{H}$, there exists an integer N_K such that $1 + f_n \neq 0$ on K if $n \geq N_K$. By Lemma 4.24, the residual $*$-product

$$\prod_{n=N_K}^{\infty}{}^{*} (1 + f_n(q)) \tag{4.14}$$

converges uniformly in K to a regular function F_{N_K} that does not vanish on K. Therefore, the function F can be written as a finite product of regular functions

$$F(q) = \left[\prod_{n=0}^{N_K - 1}{}^{*} (1 + f_n(q)) \right] * F_{N_K}(q)$$

and it is regular on K. □

We point out that, in the previous proof, the zero set of F on K coincides with the zero set of the finite $*$-product $\prod_{n=0}^{N_K-1}{}^{*} (1 + f_n(q))$ by virtue of Theorem 3.4 and Lemma 4.21.

4.5 Convergence-Producing Regular Factors

In the rest of this chapter, we will restrict our study to sequences of entire functions, namely, regular functions $\mathbb{H} \to \mathbb{H}$. In particular, in this section we consider an infinite $*$-product such as

$$\prod_{n=0}^{\infty}{}^{*} (1 - q a_n^{-1}) \tag{4.15}$$

with $\{a_n\}_{n \in \mathbb{N}} \subseteq \mathbb{H} \setminus \{0\}$. We now introduce convergence-producing regular factors and state a technical result (for a proof, see [69]).

Definition 4.26. For any given $a \in L_I$ and $n \in \mathbb{N}$, if

$$g_I(z) = e^{z a^{-1} + \frac{1}{2} z^2 a^{-2} + \ldots + \frac{1}{n} z^n a^{-n}}$$

for all $z \in L_I$, then $g = \mathrm{ext}(g_I) : \mathbb{H} \to \mathbb{H}$ is called the *convergence-producing regular factor* associated to n and a.

Proposition 4.27. *Let $n \in \mathbb{N}$ and $x \in \mathbb{R}$. We set*

$$h_n(x) = (1 - x) e^{x + \frac{1}{2} x^2 + \ldots + \frac{1}{n} x^n}. \tag{4.16}$$

Then there exist $c_k \in \mathbb{R}$ *such that*

$$h_n(x) = 1 - \sum_{k \in \mathbb{N}} c_k x^{k+n+1} \tag{4.17}$$

and

$$0 \leq c_k \leq \frac{1}{n+1}$$

for all $k \in \mathbb{N}$.

We now can prove an analog of Theorem 4.19 for infinite regular products.

Theorem 4.28. *Let* $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{H}\setminus\{0\}$ *be a sequence diverging to* ∞ *and let* $I_n \in \mathbb{S}$ *be such that* $a_n \in L_{I_n}$. *For all* $n \in \mathbb{N}$, *let* g_n *be the convergence-producing regular factor associated to* n *and* a_n. *Then*

$$\prod_{n=0}^{\infty}{}^{*} (1 - qa_n^{-1}) * g_n(q) \tag{4.18}$$

converges compactly in $\mathbb{H}$. *Moreover, for every* $n \in \mathbb{N}$, *the function* g_n *has no zeros in* $\mathbb{H}$.

Proof. For all $n \in \mathbb{N}$, let $\mathcal{V}_n(q) = (1 - qa_n^{-1}) * g_n(q)$. Theorem 4.23 yields that the $*$-product (4.18) converges simultaneously with the product

$$\prod_{n=0}^{\infty} \mathcal{V}_n(q). \tag{4.19}$$

Thanks to Theorem 4.14, the product (4.19) converges compactly in $\mathbb{H}$ if the series

$$\sum_{n\in\mathbb{N}} |1 - \mathcal{V}_n(q)|$$

converges compactly in $\mathbb{H}$. Let $K \subseteq \mathbb{H}$ be a compact set. Let $R > 0$ be such that $K \subseteq B(0, R)$ and let $n_0 \in \mathbb{N}$ be an integer such that $|a_n| > R$ for all $n \geq n_0$. Let $v_n(z)$ be the restriction of $\mathcal{V}_n$ to the complex plane L_{I_n}. Hence

$$v_n(z) = (1 - za_n^{-1})e^{za_n^{-1} + \frac{1}{2}z^2a_n^{-2} + \ldots + \frac{1}{n}z^n a_n^{-n}}$$

and we can estimate the coefficients of the Taylor expansion of v_n at the origin by means of Proposition 4.27. Since the regular extension of v_n is unique by the identity principle, the coefficients of the power series expansion of $\mathcal{V}_n$ are the same as those of v_n. Therefore, by Proposition 4.27, we have that

$$\mathcal{V}_n(q) = 1 - \sum_{k\in\mathbb{N}} c_k q^{k+n+1} a_n^{-(k+n+1)}$$

for every $q \in \mathbb{H}$, where $c_k \in \mathbb{R}$ are such that

$$0 \leq c_k \leq \frac{1}{n+1}$$

for all $k \in \mathbb{N}$. Then

$$|1 - \mathcal{V}_n(q)| = \left| \sum_{k \in \mathbb{N}} c_k q^{k+n+1} a_n^{-(k+n+1)} \right| \leq \sum_{k \in \mathbb{N}} c_k \left(\frac{|q|}{|a_n|} \right)^{k+n+1}$$

and, since $c_k \leq \frac{1}{n+1}$, we have

$$\begin{aligned} |1 - \mathcal{V}_n(q)| &\leq \sum_{k \in \mathbb{N}} \frac{1}{(n+1)} \left(\frac{|q|}{|a_n|} \right)^{n+1+k} \leq \\ &\leq \frac{1}{n+1} \left(\frac{|q|}{|a_n|} \right)^{n+1} \sum_{k \in \mathbb{N}} \left(\frac{|q|}{|a_n|} \right)^{k} . \end{aligned}$$

The factor on the right

$$\sum_{k \in \mathbb{N}} \left(\frac{|q|}{|a_n|} \right)^{k}$$

is a geometric series whose ratio is strictly smaller than 1 when $n \geq n_0$. Hence, the power series converges to the value

$$\left(1 - \frac{|q|}{|a_n|} \right)^{-1}$$

and we obtain

$$|1 - \mathcal{V}_n(q)| \leq \frac{1}{n+1} \left(\frac{|q|}{|a_n|} \right)^{n+1} \left(1 - \frac{|q|}{|a_n|} \right)^{-1} .$$

Since $\lim_{n \to \infty} \left(1 - \frac{|q|}{|a_n|} \right)^{-1} = 1$ and since

$$\sum_{n \in \mathbb{N}} \frac{1}{n+1} \left(\frac{|q|}{|a_n|} \right)^{n+1} \tag{4.20}$$

converges uniformly in K, the series

$$\sum_{n \in \mathbb{N}} |1 - \mathcal{V}_n(q)|$$

converges uniformly in K. To conclude, we remark that, for all $n \in \mathbb{N}$, the function g_n has no zeros. Indeed, g_n is the (unique) regular extension of a holomorphic function $L_{I_n} \to L_{I_n}$ that has no zeros. By Lemma 3.11, the function g_n vanishes nowhere in $\mathbb{H}$. □

We are now able to prove the following result, concerning the uniform convergence of sequences of regular conjugates and symmetrizations.

Proposition 4.29. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of regular functions on a symmetric slice domain $\Omega \subseteq \mathbb{H}$. If the sequence converges compactly in Ω to a regular function f, then*

(*a*) *The sequence of the regular conjugates $\{f_n^c\}_{n\in\mathbb{N}}$ converges compactly in Ω to the regular conjugate f^c.*
(*b*) *The sequence of symmetrizations $\{f_n^s\}_{n\in\mathbb{N}}$ converges compactly in Ω to the symmetrization f^s.*

Proof. Let $K \subseteq \Omega$ be a symmetric compact set. By hypothesis, f_n converges to f uniformly in K. Let $I, J \in \mathbb{S}$ be such that $J \perp I$. By the Splitting Lemma, there exist holomorphic $F, G : L_I \to L_I$ such that

$$f_I(z) = F(z) + G(z)J$$

and, for all $n \in \mathbb{N}$, there exist holomorphic $F_n, G_n : L_I \to L_I$ such that

$$f_{n,I}(z) = F_n(z) + G_n(z)J.$$

Clearly, $F_n \to F$ and $G_n \to G$ uniformly in K_I. Consequently, $\overline{F}_n \to \overline{F}$ and $-G_n \to -G$ uniformly in K_I as well. Hence, by Definition 1.34 $f_{n,I}^c$ converges to f_I^c uniformly in K_I. By Proposition 1.19, we have

$$\sup_K |f_n^c - f^c| \leq 2 \sup_{K_I} |f_{n,I}^c - f_I^c|$$

for all $n \in \mathbb{N}$. Thus, f_n^c converges to f^c uniformly in K, as desired. The proof of (b) is analogous, thanks to the fact that, by Remark 1.36,

$$f_I^s(z) = F(z)\overline{F(\bar{z})} + G(z)\overline{G(\bar{z})}$$

and

$$f_{n,I}^s(z) = F_n(z)\overline{F_n(\bar{z})} + G_n(z)\overline{G_n(\bar{z})}$$

for all $n \in \mathbb{N}$. □

We now go back to our infinite $*$-products.

Proposition 4.30. *Let* $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{H}\setminus\{0\}$ *be such that* $\lim_{n\to\infty}|a_n| = \infty$, *and let*

$$h(q) = \mathop{\boxed{*}}_{n=0}^{\infty} (1 - qa_n^{-1}) * g_n(q)$$

where each g_n *is the convergence-producing regular factor associated to* n *and* a_n. *Then the infinite product*

$$\prod_{n=0}^{\infty}\left(\frac{q^2}{|a_n|^2} - \frac{2qRe(a_n)}{|a_n|^2} + 1\right) e^{q\frac{2Re(a_n)}{|a_n|^2} + \frac{1}{2}q^2\frac{2Re(a_n^2)}{|a_n|^4} + \ldots + \frac{1}{n}q^n\frac{2Re(a_n^n)}{|a_n|^{2n}}} \tag{4.21}$$

converges compactly in $\mathbb{H}$ *to the symmetrization* h^s *of* h.

Proof. For all $k \in \mathbb{N}$, we set

$$h_k(q) = \mathop{\boxed{*}}_{n=0}^{k} (1 - qa_n^{-1}) * g_n(q).$$

By construction, the sequence $\{h_k\}_{k\in\mathbb{N}}$ converges compactly in $\mathbb{H}$ to h. According to Proposition 1.38, the symmetrization of a $*$-product is the product of the symmetrizations of its $*$-factors. Hence,

$$h_k^s(q) = \prod_{n=0}^{k}(1 - qa_n^{-1})^s\ g_n^s(q)$$

for all $k \in \mathbb{N}$. By Proposition 4.29, the sequence $\{h_k^s\}_{k\in\mathbb{N}}$ converges compactly in $\mathbb{H}$ to the symmetrization h^s of h, i.e.,

$$\prod_{n=0}^{\infty}(1 - qa_n^{-1})^s\ g_n^s(q) = \left[\mathop{\boxed{*}}_{n=0}^{\infty} (1 - qa_n^{-1}) * g_n(q)\right]^s. \tag{4.22}$$

We now want to write explicitly the functions g_n^s. For all $n \in \mathbb{N}$, if $a_n \in L_{I_n}$, the function g_n is defined as the unique regular extension of $f_n(z) = e^{za_n^{-1} + \frac{1}{2}z^2a_n^{-2} + \ldots + \frac{1}{n}z^na_n^{-n}}$ (with $z \in L_{I_n}$) to $\mathbb{H}$. Then for all $z \in L_{I_n}$,

$$f_n^s(z) = f_n(z)\ f_n^c(z) =$$

$$e^{za_n^{-1} + \frac{1}{2}z^2a_n^{-2} + \ldots + \frac{1}{n}z^na_n^{-n} + z\bar{a}_n^{-1} + \frac{1}{2}z^2\bar{a}_n^{-2} + \ldots + \frac{1}{n}z^n\bar{a}_n^{-n}} =$$

$$e^{z\frac{2Re(a_n)}{|a_n|^2} + \frac{1}{2}z^2\frac{2Re(a_n^2)}{|a_n|^4} + \ldots + \frac{1}{n}z^n\frac{2Re(a_n^n)}{|a_n|^{2n}}}.$$

Hence we have

$$g_n^s(q) = e^{q\frac{2Re(a_n)}{|a_n|^2} + \frac{1}{2}q^2\frac{2Re(a_n^2)}{|a_n|^4} + \ldots + \frac{1}{n}q^n\frac{2Re(a_n^n)}{|a_n|^{2n}}}$$

for all $q \in \mathbb{H}$. Observing that the symmetrization of $1 - qa_n^{-1}$ is $\frac{q^2}{|a_n|^2} - \frac{2qRe(a_n)}{|a_n|^2} + 1$ yields the thesis. □

4.6 Weierstrass Factorization Theorem

Let f be an entire function having a finite number of zeros. Let m be the isolated multiplicity of 0; let us denote the other real zeros as b_i for $i = 1, \cdots, k$, the spherical zeros as $S_n = x_n + y_n\mathbb{S}$ for $n = 1, \ldots, t$, and the nonreal isolated zeros as $a_j \in \tilde{S}_j = \tilde{x}_j + \tilde{y}_j\mathbb{S}$ for $j = 1, \cdots, r$, all of them repeated according to their respective multiplicities. By Remark 3.38, for every nonreal zero $a_j \in \tilde{S}_j$ of f, there exists $\delta_j \in \tilde{S}_j$ so that

$$f(q) = q^m \prod_{i=1}^{k}(1 - qb_i^{-1}) * \prod_{n=1}^{t}\left(\frac{q^2}{x_n^2 + y_n^2} - \frac{2qx_n}{x_n^2 + y_n^2} + 1\right) * g(q) * \prod_{j=1}^{r}{}^{*}\,(1 - q\delta_j^{-1})$$

where g is a nowhere vanishing entire function. If, instead, f has infinitely many zeros, then the factorization problem is much more delicate (as in the case of entire holomorphic functions). In order to address it, we begin by factorizing the real and spherical zeros.

Theorem 4.31. *Let f be an entire function having infinitely many zeros in $\mathbb{R}$. Let m be the multiplicity of f at 0 and let $\{b_k\}_{k\in\mathbb{N}} \subseteq \mathbb{H} \setminus \{0\}$ be the sequence of the other real zeros of f, repeated according to their isolated multiplicities. Then*

$$f(q) = q^m \prod_{k=0}^{\infty}(1 - qb_k^{-1})e^{qb_k^{-1} + \frac{1}{2}q^2b_k^{-2} + \ldots + \frac{1}{k}q^k b_k^{-k}}\ g(q)$$

where g is an entire function that has no zeros along the real line $\mathbb{R}$.

Proof. First of all, we note that

$$\lim_{k\to\infty} |b_k| = \infty$$

because the real zeros of regular functions are isolated. By Theorem 4.19, the infinite product

$$\mathcal{R}(q) = \prod_{k=0}^{\infty}(1 - qb_k^{-1})e^{qb_k^{-1} + \frac{1}{2}q^2b_k^{-2} + \ldots + \frac{1}{k}q^k b_k^{-k}}$$

converges compactly in $\mathbb{H}$. By Lemma 4.24, $\mathcal{R}$ is an entire function having the same zeros as f in $\mathbb{R} \setminus \{0\}$ (each of them with the same multiplicity). For a fixed $I \in \mathbb{S}$, there exists a nowhere vanishing holomorphic function $g_I : L_I \to \mathbb{H}$ such that

$$f_I(z) = z^m \mathcal{R}_I(z) g_I(z).$$

If we set $g = \text{ext}(g_I)$, then

$$f(q) = q^m \mathcal{R}(q) g(q)$$

where the regular function g vanishes nowhere in $\mathbb{R}$ (if it did, f and $\mathcal{R}$ would have different multiplicities at some point of $\mathbb{R}$). □

Theorem 4.32. *Let f be an entire function, vanishing on an infinite sequence of spheres $\{S_n\}_{n\in\mathbb{N}}$ (each repeated according to its spherical multiplicity). If $\{c_n\}_{n\in\mathbb{N}} \subseteq \mathbb{H} \setminus \{0\}$ is a sequence of generators of the spherical zeros, then*

$$f(q) = \prod_{n=0}^{\infty} \left(\frac{q^2}{|c_n|^2} - \frac{2qRe(c_n)}{|c_n|^2} + 1 \right) e^{q \frac{2Re(c_n)}{|c_n|^2} + \frac{1}{2} q^2 \frac{2Re(c_n^2)}{|c_n|^4} + \ldots + \frac{1}{n} q^n \frac{2Re(c_n^n)}{|c_n|^{2n}}} \quad h(q)$$

where h is an entire function having no spherical zeros.

Proof. We note that $\lim_{n\to\infty} |c_n| = \infty$ because the spherical zeros of regular functions are isolated. By Proposition 4.30, the infinite product

$$\mathcal{S}(q) = \prod_{n=0}^{\infty} \left(\frac{q^2}{|c_n|^2} - \frac{2qRe(c_n)}{|c_n|^2} + 1 \right) e^{q \frac{2Re(c_n)}{|c_n|^2} + \frac{1}{2} q^2 \frac{2Re(c_n^2)}{|c_n|^4} + \ldots + \frac{1}{n} q^n \frac{2Re(c_n^n)}{|c_n|^{2n}}}$$

converges compactly in $\mathbb{H}$. Since $\mathcal{S}$ is an entire function having the same spherical zeros as f, then reasoning as in the previous proof, one can construct an entire function h having no spherical zeros such that

$$f(q) = \mathcal{S}(q)h(q).$$

□

We need one further step to prove the announced factorization result.

Theorem 4.33. *Let f be an entire function, without any real or spherical zero, vanishing on a sequence $\{a_i\}_{i\in\mathbb{N}}$ of (isolated, nonreal) points, each repeated according to its isolated multiplicity. Then, for all $i \in \mathbb{N}$, there exist $\delta_i \in S_{a_i} = Re(a_i) + |Im(a_i)|\mathbb{S}$ such that*

$$f(q) = h(q) * \left(\boxed{\ast}\!\!\prod_{i=0}^{\infty} (1 - q\delta_i^{-1}) * g_i(q) \right)^c \tag{4.23}$$

where h is a nowhere vanishing entire function and g_i is the convergence-producing regular factor associated to i and δ_i.

Proof. First of all, we notice that $f(\overline{a_i}) \neq 0$ for all $i \in \mathbb{N}$ and $\lim_{i\to\infty} |a_i| = \infty$ since the zeros of f are isolated. In analogy with the case of polynomials, we start by "adding" the zero $\overline{a_0}$ to the function f. To this purpose, we consider the function

$$f_1(q) = f(q) * [1 - qf(\overline{a_0})^{-1}\overline{a_0}^{-1} f(\overline{a_0})] * g_0(q)$$

where g_0 is the convergence-producing regular factor associated to 0 and $\delta_0 = f(\overline{a_0})^{-1}\overline{a_0} f(\overline{a_0})$. The fact that $f_1(\overline{a_0}) = 0$ is an immediate consequence of Theorem 3.4. We note that $\delta_0 \in S_{a_0}$. The function f_1 has $\{a_i\}_{i\geq 1}$ as its isolated zeros and S_{a_0} as spherical zero. We stress the fact that every a_i is repeated according to its multiplicity. We can now factor out the spherical zero S_{a_0} to obtain

$$f_1(q) = \left(1 - \frac{2Re(a_0)q}{|a_0|^2} + \frac{q^2}{|a_0|^2}\right) \tilde{f}_1(q)$$

where $\tilde{f}_1$ is an entire function having $\{a_i\}_{i\geq 1}$ as its only zeros. Therefore $\tilde{f}_1(\overline{a_1}) \neq 0$. By repeating this same procedure for f_1, since the polynomial $\left(1 - \frac{2Re(a_0)q}{|a_0|^2} + \frac{q^2}{|a_0|^2}\right)$ has real coefficients, we construct

$$f_2(q) = \left(1 - \frac{2Re(a_0)q}{|a_0|^2} + \frac{q^2}{|a_0|^2}\right) \tilde{f}_1(q) * [1 - q\tilde{f}_1(\overline{a_1})^{-1}\overline{a_1}^{-1} \tilde{f}_1(\overline{a_1})] * g_1(q) =$$
$$= f_1(q) * [1 - q\tilde{f}_1(\overline{a_1})^{-1}\overline{a_1}^{-1} \tilde{f}_1(\overline{a_1})] * g_1(q)$$

where g_1 is the convergence-producing regular factor associated to 1 and $\delta_1 = \tilde{f}_1(\overline{a_1})^{-1}\overline{a_1}\tilde{f}_1(\overline{a_1})$. We note that $\delta_1 \in S_{a_1}$. We also note that f_2 has $\{a_i\}_{i\geq 2}$ as its isolated zeros and S_{a_0} and S_{a_1} as its spherical zeros. Unlike what happens for polynomials, this process has infinitely many steps. This is the reason why we need to add the convergence-producing regular factor at each step. Iterating this process, we get

$$f(q) * \left[\prod_{i=0}^{\infty}{}^{*} (1 - q\delta_i^{-1}) * g_i(q)\right] = k(q)$$

where k is an entire function having $\{\delta_i\}_{i\in\mathbb{N}}$ as the generators of its spherical zeros $\{S_{a_i}\}_{i\in\mathbb{N}}$ and no zeros of other types. We can factor out the spherical zeros of k: according to Formula (4.22), there exists a nowhere vanishing entire function h such that

$$k(q) = \left[\prod_{i=0}^{\infty}{}^{*} (1 - q\delta_i^{-1}) * g_i(q)\right]^s h(q).$$

By $*$-multiplying on the right by $\left[\mathop{\boxed{*}}_{i=0}^{\infty}(1-q\delta_i^{-1})*g_i(q)\right]^c$, we obtain

$$f(q)*\left[\mathop{\boxed{*}}_{i=0}^{\infty}(1-q\delta_i^{-1})*g_i(q)\right]^s=$$

$$=\left[\mathop{\boxed{*}}_{i=0}^{\infty}(1-q\delta_i^{-1})*g_i(q)\right]^s*h(q)*\left(\mathop{\boxed{*}}_{i=0}^{\infty}(1-q\delta_i^{-1})*g_i(q)\right)^c.$$

and

$$f(q)=h(q)*\left(\mathop{\boxed{*}}_{i=0}^{\infty}(1-q\delta_i^{-1})*g_i(q)\right)^c.$$

□

Making use of Theorems 4.31–4.33, we obtain the announced result.

Theorem 4.34 (Weierstrass Factorization Theorem). *Let f be an entire function. Suppose that $m\in\mathbb{N}$ is the multiplicity of f at 0, $\{b_n\}_{n\in\mathbb{N}}\subseteq\mathbb{R}\setminus\{0\}$ is the sequence of the other real zeros of f, $\{S_n\}_{n\in\mathbb{N}}$ is the sequence of the spherical zeros of f, and $\{a_n\}_{n\in\mathbb{N}}\subseteq\mathbb{H}\setminus\mathbb{R}$ is the sequence of the nonreal zeros of f with isolated multiplicity greater than zero. If all the zeros listed above are repeated according to their multiplicities, then there exists a nowhere vanishing entire function h, and, for all $n\in\mathbb{N}$, there exist $c_n\in S_n$ and $\delta_n\in S_{a_n}=Re(a_n)+|Im(a_n)|\mathbb{S}$ such that*

$$f(q)=(q^m\,\mathcal{R}(q)\,\mathcal{S}(q)\,\mathcal{A}(q))*h(q)$$

where

$$\mathcal{R}(q)=\prod_{n=0}^{\infty}(1-qb_n^{-1})e^{qb_n^{-1}+\frac{1}{2}q^2b_n^{-2}+\ldots+\frac{1}{n}q^nb_n^{-n}},$$

$$\mathcal{S}(q)=\prod_{n=0}^{\infty}\left(\frac{q^2}{|c_n|^2}-\frac{2qRe(c_n)}{|c_n|^2}+1\right)e^{q\frac{2Re(c_n)}{|c_n|^2}+\frac{1}{2}q^2\frac{2Re(c_n^2)}{|c_n|^4}+\ldots+\frac{1}{n}q^n\frac{2Re(c_n^n)}{|c_n|^{2n}}},$$

$$\mathcal{A}(q)=\mathop{\boxed{*}}_{n=0}^{\infty}(1-q\delta_n^{-1})*g_n(q)$$

and where, for all $n\in\mathbb{N}$, the function g_n is the (nowhere vanishing) entire function whose restriction to the plane L_{I_n} containing δ_n is

$$z\mapsto e^{z\delta_n^{-1}+\frac{1}{2}z^2\delta_n^{-2}+\ldots+\frac{1}{n}z^n\delta_n^{-n}}.$$

Proof. By Theorems 4.31 and 4.32, there exists an entire function l (without any real or spherical zero) such that

$$f(q) = q^m\ \mathcal{R}(q)\ \mathcal{S}(q)\ l(q).$$

Now we consider the regular conjugate l^c of l. The function l^c is an entire function whose (isolated) zeros on each 2-sphere S are in one-to-one correspondence with those of l (see Proposition 3.9). By Theorem 4.33, for all $i \in \mathbb{N}$, there exist $\delta_i \in S_{a_i}$ and a nowhere vanishing entire function g_i such that

$$l^c(q) = h^c(q) * \left(\prod_{i=0}^{\infty}{}^{*} (1 - q\delta_i^{-1}) * g_i(q) \right)^c$$

where h^c is a nowhere vanishing entire function. Taking the regular conjugate of both members of the previous equality and using Proposition 1.37, we have that

$$l(q) = \left(\prod_{i=0}^{\infty}{}^{*} (1 - q\delta_i^{-1}) * g_i(q) \right) * h(q).$$

Observing that h is also a nowhere vanishing entire function concludes the proof. □

Bibliographic Notes

The results presented in this chapter were originally proven in [69]. They rely upon the ideas of [63] and of [55, 57] presented in Chap. 3.

Chapter 5
Singularities

5.1 Regular Reciprocal and Quotients

In this section we construct the ring of quotients of regular functions. We begin by presenting the definition of regular reciprocal of f, which involves the operations of regular conjugation and symmetrization presented in Sect. 1.4.

Definition 5.1. Let f be a regular function on a symmetric slice domain Ω. If $f \not\equiv 0$, the *regular reciprocal* of f is the function defined on $\Omega \setminus Z_{f^s}$ by

$$f^{-*} = \frac{1}{f^s} f^c. \tag{5.1}$$

In order to prove the regularity of f^{-*}, we will make use of the next lemma.

Lemma 5.2. *Let h be a slice preserving regular function on a symmetric slice domain Ω. If $h \not\equiv 0$, then $\frac{1}{h}$ is a (slice preserving) regular function on $\widehat{\Omega} = \Omega \setminus Z_h$.*

Proof. Set $g(q) = \frac{1}{h(q)}$ for all $q \in \widehat{\Omega} = \Omega \setminus Z_h$. We will prove that g is regular by observing that, for all $I \in \mathbb{S}$, the restriction g_I is holomorphic. Let $I \in \mathbb{S}$. Chosen $J \in \mathbb{S}$ with $J \perp I$, the Splitting Lemma 1.3 guarantees the existence of holomorphic functions $F, G : \Omega_I \to L_I$ such that $h_I = F + GJ$. Since $h(\Omega_I) \subseteq L_I$, we must have $G \equiv 0$ and $h_I = F$. Hence

$$g_I(z) = \frac{1}{h_I(z)} = \frac{1}{F(z)}$$

for all $z \in \widehat{\Omega}_I$, and g_I is a holomorphic function from $\widehat{\Omega}_I$ to L_I. □

G. Gentili et al., *Regular Functions of a Quaternionic Variable*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-33871-7_5,

Theorem 5.3. *Let f be a regular function on a symmetric slice domain Ω. The function f^{-*} is regular in $\Omega \setminus Z_{f^s}$, which is a symmetric slice domain, and*

$$f * f^{-*} = f^{-*} * f = 1 \tag{5.2}$$

in $\Omega \setminus Z_{f^s}$.

Proof. The regularity of $f^{-*} = \frac{1}{f^s} f^c$ is proven as follows: the previous lemma applies to $h = f^s$ thanks to Lemma 3.8; the regularity of $\frac{1}{f^s} f^c$ then follows by Lemma 1.30. Furthermore, $\Omega \setminus Z_{f^s}$ is still a symmetric slice domain since $f \not\equiv 0$ implies that Z_{f^s} consists of isolated real points and isolated 2-spheres $x + y\mathbb{S}$. Finally,

$$f^{-*} * f = \frac{1}{f^s} f^c * f = \frac{1}{f^s} f^s = 1$$

and, similarly, $f * f^{-*} = 1$. □

We now construct the ring of quotients of regular functions. Recall that, in the complex case, the set of quotients $\frac{F}{G}$ of holomorphic functions F, G (with $G \not\equiv 0$) on a disk Δ becomes a field when endowed with the usual operations of addition and multiplication. More precisely, it is the field of quotients of the integral domain (the commutative ring with no zero divisors) obtained by endowing the set of holomorphic functions F on Δ with the natural addition and multiplication. As explained in [109] (see also [17, 85]), the concept of field of quotients of an integral domain can be generalized to the noncommutative case.

Definition 5.4. A *left Ore domain* is a domain (a ring with no zero divisors) $(D, +, \cdot)$ such that $Da \cap Db \neq \{0\}$ for all $a, b \in D \setminus \{0\}$. Similarly, a *right Ore domain* is a domain $(D, +, \cdot)$ such that $aD \cap bD \neq \{0\}$ for all $a, b \in D \setminus \{0\}$.

Theorem 5.5. *If D is a left Ore domain, then the set of formal quotients $L = \{a^{-1}b : a, b \in D\}$ can be endowed with operations $+, \cdot$ such that:*

(i) D is isomorphic to a subring of L (namely $\{1^{-1}a : a \in D\}$).
(ii) L is a skew field (where $(a^{-1}b)^{-1} = b^{-1}a$).

The ring L is called the classical left ring of quotients *of D and, up to isomorphism, it is the only ring having the properties (i) and (ii).*

The *classical right ring of quotients* can be similarly constructed on a right Ore domain. If D is both a left and a right Ore domain, then the two rings of quotients are isomorphic and we speak of the *classical ring of quotients* of D. This happens for the ring of regular functions on a symmetric slice domain, as explained by the next theorem.

Theorem 5.6. *Let Ω be a symmetric slice domain. The set of* regular quotients

$$\mathcal{L}(\Omega) = \{f^{-*} * g : f, g \text{ regular in } \Omega,\ f \not\equiv 0\} \tag{5.3}$$

is a division ring with respect to $+, *$. *Moreover, the ring of regular functions on* Ω *is a left and right Ore domain, and* $\mathcal{L}(\Omega)$ *is its classical ring of quotients.*

Proof. For all regular $f, g, h, k : \Omega \to \mathbb{H}$ with $f, h \not\equiv 0$, the sum and product of $f^{-*} * g$ and $h^{-*} * g$ are regular functions on $\Omega \setminus (Z_{f^s} \cup Z_{h^s})$. They are elements of $\mathcal{L}(\Omega)$, since

$$(f^{-*} * g) + (h^{-*} * k) = \frac{1}{f^s h^s}(h^s f^c * g + f^s h^c * k),$$

$$(f^{-*} * g) * (h^{-*} * k) = \frac{1}{f^s h^s} f^c * g * h^c * k.$$

We easily derive that $\mathcal{L}(\Omega)$ is a ring with respect to $+, *$. Furthermore, $\mathcal{L}(\Omega)$ is a division ring since

$$(f^{-*} * g) * (g^{-*} * f) = \frac{1}{f^s g^s} f^c * g * g^c * f = \frac{1}{f^s g^s} f^c * g^s * f = \frac{1}{f^s g^s} f^s g^s = 1.$$

The ring D of regular functions on Ω is a domain, since $f * g \equiv 0$ if and only if $f \equiv 0$ or $g \equiv 0$. Moreover, D is a left Ore domain: if $f, g \not\equiv 0$, then $(D * f) \cap (D * g)$ contains the nonzero element $f^s g^s = g^s f^s$, which can be obtained as $(g^s * f^c) * f$ or as $(f^s * g^c) * g$. Similarly, D is a right Ore domain. Thus the classical ring of quotients of D is well defined. It must be isomorphic to $\mathcal{L}(\Omega)$ by the uniqueness property: $\mathcal{L}(\Omega)$ is a division ring having D as a subring, and the inclusion $D \to \mathcal{L}(\Omega)$ $f \mapsto f = 1^{-*} * f$ is a ring homomorphism. □

5.2 Laurent Series and Expansion

We will now study regular Laurent series and Laurent expansions for regular functions. A regular Laurent series centered at 0 is an expression of the form

$$\sum_{n \in \mathbb{Z}} q^n a_n = \sum_{n \geq 0} q^n a_n + \sum_{m > 0} q^{-m} a_{-m}.$$

It converges absolutely and uniformly on compact sets in a spherical shell

$$A(0, R_1, R_2) = \{q \in \mathbb{H} : 0 \leq R_1 < |q| < R_2\},$$

where its sum defines a regular function. The shell can be chosen so that the series diverges outside $\overline{A(0, R_1, R_2)}$. We now define regular Laurent series centered at a generic $p \in \mathbb{H}$. From now on, for all $n \in \mathbb{N}$, the expressions $(q - p)^{*(-n)} = (q - p)^{-*n}$ denote the regular reciprocal of $(q - p)^{*n}$.

Definition 5.7. For any $\{a_n\}_{n\in\mathbb{Z}}$ in $\mathbb{H}$, we call

$$\sum_{n\in\mathbb{Z}}(q-p)^{*n}a_n \tag{5.4}$$

the *regular Laurent series centered at* p associated to $\{a_n\}_{n\in\mathbb{Z}}$.

As one may imagine, in order to study convergence we have to estimate the negative regular powers of $q-p$. We begin with the following definition.

Definition 5.8. For all $p, q \in \mathbb{H}$ we define

$$\tau(q,p) = \begin{cases} |q-p| \text{ if } p, q \text{ lie on the same complex line } L_I \\ \sqrt{[Re(q)-Re(p)]^2 + [|Im(q)| - |Im(p)|]^2} \text{ otherwise} \end{cases} \tag{5.5}$$

Equivalently, if $p \in L_I$, then for all $q \in L_I$ we set $\tau(q,p) = |q-p|$; for all $q \in \mathbb{H} \setminus L_I$, we instead set $\tau(q,p) = \min\{|z-p|, |\bar{z}-p|\}$ where $z, \bar{z}$ are the points of L_I having the same real part and the same modulus of the imaginary part as q.

We now come to the desired estimate.

Proposition 5.9. *Fix* $p \in \mathbb{H}$. *Then*

$$|(q-p)^{-*m}| \leq \tau(q,p)^{-m} \tag{5.6}$$

for all $m \in \mathbb{N}$. *Moreover,*

$$\lim_{n\to+\infty} |(q-p)^{-*m}|^{1/m} = \frac{1}{\tau(q,p)}. \tag{5.7}$$

Proof. Let L_I be the complex plane through p and let $m \in \mathbb{N}$. As in the proof of Proposition 2.10, $(q-p)^{-*m}$ equals $(z-p)^{-m} = \frac{1}{(z-p)^m}$ when computed at $q = z = x + yI \in L_I$. For any $q = x + yJ$,

$$(q-p)^{-*m} = \frac{1}{2}\left[\frac{1}{(z-p)^m} + \frac{1}{(\bar{z}-p)^m}\right] - \frac{JI}{2}\left[\frac{1}{(z-p)^m} - \frac{1}{(\bar{z}-p)^m}\right]$$

by Formula (1.9). The same computations used in Proposition 2.10 prove that $|(q-p)^{-*m}|$ attains its maximum value at $J = I$ or at $J = -I$, that is, at $q = z$ or at $q = \bar{z}$. Thus if $q \notin L_I$, then

$$|(q-p)^{-*m}| \leq \max\left\{\frac{1}{|z-p|^m}, \frac{1}{|\bar{z}-p|^m}\right\} =$$

$$= \frac{1}{\min\{|z-p|, |\bar{z}-p|\}^m} = \frac{1}{\tau(q,p)^m}.$$

If, on the contrary, $q \in L_I$ then $|(q-p)^{-*m}| = \frac{1}{|q-p|^m} = \frac{1}{\tau(q,p)^m}$. This proves the first statement.

Let us now prove the second statement. It is trivial in the case $q \in L_I$, so let us suppose $q \notin L_I$. We already know that if $|z-p| = |\bar{z}-p|$, then $p \in \mathbb{R}$, so that $(q-p)^{-*m}$ coincides with $(q-p)^{-m}$ and the thesis is trivial. Let us thus suppose $|z-p| \neq |\bar{z}-p|$. Without loss of generality, $|z-p| > |\bar{z}-p|$ and in particular $\tau(q,p) = \min\{|z-p|, |\bar{z}-p|\} = |\bar{z}-p|$. Since

$$(q-p)^{-*m} = \frac{1-JI}{2}\frac{1}{(z-p)^m} + \frac{1+JI}{2}\frac{1}{(\bar{z}-p)^m}$$

we have

$$\begin{aligned} |(q-p)^{-*m}|\tau(q,p)^m &= \left|\frac{1-JI}{2}\frac{1}{(z-p)^m} + \frac{1+JI}{2}\frac{1}{(\bar{z}-p)^m}\right| |\bar{z}-p|^m \\ &= \left|\frac{1-JI}{2}\left(\frac{\bar{z}-p}{z-p}\right)^m + \frac{1+JI}{2}\right|. \end{aligned}$$

Hence

$$|(q-p)^{-*m}|^{1/m}\tau(q,p) = \left|\frac{1-JI}{2}\left(\frac{\bar{z}-p}{z-p}\right)^m + \frac{1+JI}{2}\right|^{1/m}$$

where $\left(\frac{\bar{z}-p}{z-p}\right)^m \to 0$ as $m \to +\infty$ and $\frac{1+JI}{2} \neq 0$, so that

$$\lim_{m\to+\infty} |(q-p)^{-*m}|^{1/m}\tau(q,p) = 1,$$

which is equivalent to our thesis. □

The function τ is clearly not a distance, since $\tau(q,p) = 0$ for all q lying in the same 2-sphere $x + y\mathbb{S}$ as p except for $\bar{p}$.

Remark 5.10. For $p = x + yI \in \mathbb{H}$ and $R \geq 0$, let us write $T(p,R) := \{q \in \mathbb{H} : \tau(q,p) < R\}$. Then

$$T(p,R) = \bigcup_{J \neq -I} \Delta_J(x + yJ, R).$$

1. If $R \leq |Im(p)|$, then $T(p,R)$ is the symmetric completion of the Euclidean disk $\Delta_I(p,R) \subset L_I$, minus the Euclidean disk $\Delta_I(\bar{p},R) \subset L_I$.
2. If $R > |Im(p)| > 0$, then $T(p,R)$ is the symmetric completion of the Euclidean disk $\Delta_I(p,R) \subset L_I$, minus $\Delta_I(\bar{p},R) \setminus \Delta_I(p,R)$.
3. If $p \in \mathbb{R}$, that is, $Im(p) = 0$, then $T(p,R)$ coincides with the Euclidean ball $B(p,R)$.

In all cases $T(p,R)$ is open in the Euclidean topology. Notice that

$$\{q \in \mathbb{H} : \tau(q,p) \leq R\} = \bigcup_{J \neq -I} \overline{\Delta_J(x+yJ,R)}$$

is not, in general, the closure of $T(p,R)$ with respect to the Euclidean topology. We instead have:

$$\overline{T(p,R)} = \bigcup_{J \in \mathbb{S}} \overline{\Delta_J(x+yJ,R)}.$$

Remark 5.11. With respect to the Euclidean topology, the distance $\sigma : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ (see Definition 2.5) is lower semicontinuous and $\tau : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ is upper semicontinuous.

We now define the sets which turn out to be the sets of convergence of regular Laurent series.

Definition 5.12. For $0 \leq R_1 < R_2 \leq +\infty$, we define

$$\Sigma(p,R_1,R_2) = \{q \in \mathbb{H} : \tau(q,p) > R_1, \sigma(q,p) < R_2\}, \tag{5.8}$$

$$\Omega(p,R_1,R_2) = \Omega(p,R_2) \setminus \overline{T(p,R_1)}, \tag{5.9}$$

where $\Omega(p,R_2) = \{q \in \mathbb{H} : \omega(q,p) < R_2\}$ and σ are defined in Sect. 2.1.

Notice that the interior of $\Sigma(p,R_1,R_2)$ is $\Omega(p,R_1,R_2)$. On the other hand,

$$\overline{\Sigma(p,R_1,R_2)} = \{q \in \mathbb{H} : \tau(q,p) \geq R_1, \sigma(q,p) \leq R_2\}$$

as a consequence of the previous remark. We are now ready to study the convergence of Laurent series.

Theorem 5.13. *Choose any sequence $\{a_n\}_{n\in\mathbb{Z}}$ in $\mathbb{H}$. Let $R_1, R_2 \in [0,+\infty]$ be such that $R_1 = \limsup_{m\to+\infty} |a_{-m}|^{1/m}$, $1/R_2 = \limsup_{n\to+\infty} |a_n|^{1/n}$. For all $p \in \mathbb{H}$ the series*

$$f(q) = \sum_{n\in\mathbb{Z}} (q-p)^{*n} a_n \tag{5.10}$$

converges absolutely and uniformly on the compact subsets of $\Sigma(p,R_1,R_2)$, and it does not converge at any point of $T(p,R_1)$ nor at any point of $\mathbb{H} \setminus \overline{\Sigma(p,R_2)}$. Furthermore, if $\Omega(p,R_1,R_2) \neq \emptyset$ then the sum of the series defines a regular function $f : \Omega(p,R_1,R_2) \to \mathbb{H}$.

Proof. The regular Laurent series converges at all points q with $\tau(q,p) > R_1$ and $\sigma(q,p) < R_2$ by the root test, since

$$\limsup_{n\to+\infty} |(q-p)^{*n} a_n|^{1/n} = \frac{\sigma(q,p)}{R_2},$$

$$\limsup_{m\to+\infty} |(q-p)^{-*m} a_{-m}|^{1/m} = \frac{R_1}{\tau(q,p)}.$$

For the same reason, the series does not converge at any point q such that $\tau(q,p) < R_1$ or $\sigma(q,p) > R_2$. Now let us prove that the convergence is uniform on compact sets. Observe that

$$\Sigma(p, R_1, R_2) = \bigcup_{R_1<r_1<r_2<R_2} \overline{\Sigma(p, r_1, r_2)}$$

where $\overline{\Sigma(p, r_1, r_2)}$ is compact (it is closed by construction, and it is bounded because $r_2 < +\infty$). We already know that, for all $q \in \overline{\Sigma(p, r_1, r_2)}$, $\sigma(q,p) \leq r_2$ and $\tau(q,p) \geq r_1$, so that $\tau(q,p)^{-m} \leq r_1^{-m}$ for all $m > 0$. Hence in $\overline{\Sigma(p, r_1, r_2)}$, the function series $\sum_{n\in\mathbb{N}} (q-p)^{*n} a_n$ is dominated by the convergent number series $\sum_{n\in\mathbb{N}} r_2^n |a_n|$, while $\sum_{m>0} (q-p)^{-*m} a_{-m}$ is dominated by the convergent number series $\sum_{m>0} r_1^{-m} |a_{-m}|$.

Finally, let us prove the regularity of f in $\Omega(p, R_1, R_2)$ when the latter is not empty. Each addend $(q-p)^{*n} a_n$ of the series defines a regular function on $\Omega(p, R_1, R_2)$. Since the convergence is uniform on compact sets, the sum of the series is regular in $\Omega(p, R_1, R_2)$, as desired. □

Conversely, an expansion property holds. Let us begin by observing that, if $p \in L_I$ and if $A_I(p, R_1, R_2)$ denotes the annulus $\{z \in L_I : R_1 < |z-p| < R_2\}$, then

$$\Sigma(p, R_1, R_2) = A_I(p, R_1, R_2) \cup \Omega(p, R_1, R_2).$$

Remark 5.14. Let $p \in \mathbb{H}$, let $R_1, R_2 \in \mathbb{R}$ be such that $0 \leq R_1 < R_2 \leq +\infty$, and let $I \in \mathbb{S}$ be such that $p \in L_I$. If $U = A_I(p, R_1, R_2) \cap A_I(\bar{p}, R_1, R_2)$ denotes the largest open subset of $A_I(p, R_1, R_2)$ that is symmetric with respect to $\mathbb{R}$, then $\Omega(p, R_1, R_2)$ is the symmetric completion $\widetilde{U}$ of U. Since U has at most 2 connected components, if $\Omega(p, R_1, R_2) \neq \emptyset$ then either $\Omega(p, R_1, R_2)$ is a symmetric slice domain or it is the disjoint union of two symmetric slice domains.

We are now ready to prove the expansion property.

Theorem 5.15 (Regular Laurent Expansion). *Let f be a regular function on a domain $\Omega \subseteq \mathbb{H}$ and let $p \in \mathbb{H}$. There exists a sequence $\{a_n\}_{n\in\mathbb{Z}}$ in $\mathbb{H}$ such that, for all $0 \leq R_1 < R_2 \leq +\infty$ with $\Sigma(p, R_1, R_2) \subseteq \Omega$,*

$$f(q) = \sum_{n\in\mathbb{Z}} (q-p)^{*n} a_n \tag{5.11}$$

in $\Sigma(p, R_1, R_2)$.

Proof. Suppose $p \in L_I$: if $\Omega \supseteq \Sigma(p, R_1, R_2)$, then $\Omega_I \supseteq A_I(p, R_1, R_2)$. Choose $J \in \mathbb{S}, J \perp I$, and let $F, G : \Omega_I \to L_I$ be holomorphic functions such that $f_I = F + GJ$. If

$$F(z) = \sum_{n\in\mathbb{Z}} (z-p)^n \alpha_n,$$

$$G(z) = \sum_{n\in\mathbb{Z}} (z-p)^n \beta_n$$

are the Laurent series expansions of F and G in $A_I(p, R_1, R_2)$ then, setting $a_n = \alpha_n + \beta_n J$,

$$f_I(z) = \sum_{n\in\mathbb{Z}} (z-p)^n a_n$$

for all $z \in A_I(p, R_1, R_2)$. If $\sum_{n\in\mathbb{Z}}(z-p)^n a_n$ converges in $A_I(p, R_1, R_2)$, then the series in (5.11) converges in $\Sigma(p, R_1, R_2)$. If $\Omega(p, R_1, R_2) \neq \emptyset$ then the series in (5.11) defines a regular function

$$g : \Omega(p, R_1, R_2) \to \mathbb{H}.$$

Since $f_I \equiv g_I$ in $\Omega(p, R_1, R_2) \cap L_I$, applying the Identity Principle 1.12 to each connected component of $\Omega(p, R_1, R_2)$ allows us to conclude that f and g coincide in $\Omega(p, R_1, R_2)$ (hence in $\Sigma(p, R_1, R_2)$, as desired). □

5.3 Classification of Singularities

We now present the classification of the singularities of regular functions.

Definition 5.16. Let f be a regular function on a symmetric slice domain Ω. We say that a point $p \in \mathbb{H}$ is a *singularity* for f if there exists $R > 0$ such that $\Sigma(p, 0, R) \subseteq \Omega$ (so that the Laurent expansion of f at p, $f(q) = \sum_{n\in\mathbb{Z}}(q - p)^{*n} a_n$, has 0 as its inner radius of convergence, and it has a positive outer radius of convergence).

Remark 5.17. For all $R > 0$ and for all $p = x + yI \in \mathbb{H}$, if $\Sigma(p, R)$ denotes the σ-ball defined in Sect. 2.1 then

$$\Sigma(p, 0, R) = \Sigma(p, R) \setminus [(x + y\mathbb{S}) \setminus \{\bar{p}\}] = [\Sigma(p, R) \setminus (x + y\mathbb{S})] \cup \{\bar{p}\}. \quad (5.12)$$

In other words, $\Sigma(p, 0, R)$ consists of the punctured disk $\Delta_I(p, R) \setminus \{p\}$ and of the symmetric completion $\widetilde{U}$ of $U = [\Delta_I(p, R) \setminus \{p\}] \cap [\Delta_I(\bar{p}, R) \setminus \{\bar{p}\}]$.

We classify the singularities with the next definition.

Definition 5.18. Let p be a singularity for f. We say that p is a *removable singularity* if f extends to a neighborhood of p as a regular function. Otherwise consider the expansion

$$f(q) = \sum_{n\in\mathbb{Z}} (q-p)^{*n} a_n. \tag{5.13}$$

We say that p is a *pole* for f if there exists an $m \geq 0$ such that $a_{-k} = 0$ for all $k > m$; the minimum such m is called the *order* of the pole and denoted by $ord_f(p)$. If p is not a pole, then we call it an *essential singularity* for f and set $ord_f(p) = +\infty$.

We point out that, when $0 < R < |Im(p)|$, $\Sigma(p, 0, R)$ reduces to the punctured disk $\Delta_I(p, R) \setminus \{p\}$ (where $I \in \mathbb{S}$ is chosen so that $p \in L_I$). In this case (5.13) reduces to

$$f_I(z) = \sum_{n\in\mathbb{Z}} (z-p)^n a_n.$$

for all $z \in L_I$. If p is a pole for f_I, that is, if there exists $n \in \mathbb{N}$ such that

$$f_I(z) = (z-p)^{-n} g_I(z)$$

for some holomorphic $g_I : \Delta_I(p, R) \to \mathbb{H}$, it is natural to call order of f_I at p the least such n; that is, the maximum between the order of F and the order of G at p, if f_I splits as $f_I = F + GJ$ according to Lemma 1.3. On the other hand, p is an essential singularity for f_I, that is, it is not a pole for f_I, if and only if it is an essential singularity for F or for G. Finally, p is a removable singularity for f_I, that is, f_I extends as a holomorphic function to a neighborhood of p in L_I, if and only if p is a removable singularity for both F and G. We observe what follows.

Remark 5.19. A point $p \in L_I \subset \mathbb{H}$ is a singularity for a regular function f on a symmetric slice domain if and only if it is an isolated singularity for the holomorphic function f_I. The function f has a pole of order n at p if and only if f_I does. Furthermore, p is an essential singularity for f if and only if it is an essential singularity for f_I.

The same equivalence does not hold for removable singularities: p is a removable singularity for f_I if and only if it is a pole of order 0 for f_I if and only if it is a pole of order 0 for f, but such a pole is not necessarily a removable singularity for f, as proven by the next example.

Example 5.20. Let $I \in \mathbb{S}$ and let $f : \mathbb{H} \setminus \mathbb{S} \to \mathbb{H}$ be the regular function defined by

$$f(q) = (q+I)^{-*} = (q^2+1)^{-1}(q-I).$$

By restricting to the complex plane L_I, we get $f_I(z) = \frac{1}{z+I}$ for all $z \in (L_I) \setminus \{\pm I\}$. The point $-I$ is clearly a pole of order 1 for f_I and f, while I is a pole of order 0. We immediately conclude that I is a removable singularity for f_I. The same is not true for f: each neighborhood U of I in $\mathbb{H}$ includes points $q \in \mathbb{S}$, where $q^2 + 1$ vanishes while $q - I$ does not; this means that $|f|$ is unbounded in $U \setminus \mathbb{S}$; hence, I cannot be removable. Notice that the Laurent series expansion of f at $-I$ converges in

$$\Sigma(-I, 0, +\infty) = \mathbb{H} \setminus (\mathbb{S} \setminus \{I\}) = (\mathbb{H} \setminus \mathbb{S}) \cup \{I\}$$

while

$$\Omega(-I, 0, +\infty) = \mathbb{H} \setminus \mathbb{S}$$

is the maximal domain on which f is defined as a regular function.

The situation presented in this example is quite common, as we will see in the next section. Before proceeding, let us define the analogs of meromorphic functions.

Definition 5.21. A function f is *semiregular* in a symmetric slice domain Ω if it is regular in a symmetric slice domain $\Omega' \subseteq \Omega$ such that every point of $\mathcal{S} = \Omega \setminus \Omega'$ is a pole (or a removable singularity) for f.

Notice that for all $I \in \mathbb{S}$, $\mathcal{S}_I = \mathcal{S} \cap L_I$ is discrete and the restriction $f_I : \Omega_I \setminus \mathcal{S}_I \to \mathbb{H}$ is meromorphic (i.e., it has no essential singularities) in Ω_I. Furthermore, we observe what follows.

Remark 5.22. If f is semiregular in Ω, then the set $\mathcal{S}$ of its nonremovable poles consists of isolated real points or isolated 2-spheres of type $x + y\mathbb{S}$.

5.4 Poles and Quotients

In this section we study the poles and their relation with regular quotients. We begin by proving the semiregularity of the quotients.

Proposition 5.23. *Let f, g be regular functions on a symmetric slice domain Ω and consider the quotient $f^{-*} * g : \Omega \setminus Z_{f^s} \to \mathbb{H}$. Each $p \in Z_{f^s}$ is a pole of order*

$$ord_{f^{-*}*g}(p) \leq m_{f^s}(p) \tag{5.14}$$

for $f^{-*} * g$, where $m_{f^s}(p)$ denotes the classical multiplicity of p as a zero of f^s. As a consequence, $f^{-*} * g$ is semiregular on Ω.

Proof. Suppose $p = x + yI$ and notice that it suffices to prove the thesis for the restriction f_I. If $m_{f^s}(p) = n$ then, as a consequence of Proposition 3.9, $m_{f^s}(\bar{p}) = n$. Therefore, there exists a holomorphic function h_I with $h_I(p) \neq 0$ such that for all $z \in L_I$

$$f_I^s(z) = (z - p)^n (z - \bar{p})^n h_I(z) = \left[(z - x)^2 + y^2\right]^n h_I(z).$$

Since f^s is slice preserving, $h_I(z)$ must be slice preserving, too. Hence $f_I^s(z) = h_I(z)\left[(z - x)^2 + y^2\right]^n = h_I(z)(z - \bar{p})^n (z - p)^n$ and

$$(f^{-*} * g)_I(z) = (f_I^s(z))^{-1}(f^c * g)_I(z) = (z-p)^{-n}(z-\bar{p})^{-n}h_I(z)^{-1}(f^c * g)_I(z)$$

where $(z-\bar{p})^{-n}h_I(z)^{-1}(f^c * g)_I(z)$ is holomorphic in a neighborhood of p in L_I. □

Conversely, all functions that are semiregular can be locally expressed as quotients of regular functions.

Lemma 5.24. *Let f be a semiregular function on a symmetric slice domain Ω. If $p = x+yI \in \Omega$ is a nonremovable singularity for f, then there exists a symmetric slice domain U with $p \in U \subseteq \Omega$ such that f is regular in $U \setminus (x + y\mathbb{S})$.*

Proof. Let $r \in \Omega \cap \mathbb{R}$ be a real point such that f is regular in a neighborhood of r and let $\gamma : [0,1] \to \Omega_I$ be a plane curve connecting p to r. Since the set $\mathcal{S}_I$ of nonremovable poles in Ω_I is discrete, we may suppose γ not to pass through any other nonremovable pole of f_I. Since the image of γ is compact, there exists $\varepsilon > 0$ such that, setting

$$\Gamma = \{z \in L_I : \exists t \in [0,1] \text{ s.t. } |z-\gamma(t)| < \varepsilon\},$$

Γ is contained in Ω_I and f_I is holomorphic in $\Gamma \setminus \{p\}$. Let us choose U as the symmetric completion $\widetilde{\Gamma}$ of Γ, and let $x + y\mathbb{S}$ be the sphere through p. Since the set of nonremovable poles of f is symmetric, f must be regular in $U \setminus (x + y\mathbb{S})$. By construction, U is a symmetric slice domain with $p \in U \subseteq \Omega$. □

Theorem 5.25. *Let f be a semiregular function on a symmetric slice domain Ω. Choose $p = x + yI \in \Omega$ and set $m = ord_f(p), n = ord_f(\bar{p})$ (without loss of generality we assume $m \leq n$). Then there exist a neighborhood U of p in Ω that is a symmetric slice domain and a (unique) regular function $g : U \to \mathbb{H}$ such that*

$$\begin{aligned} f(q) &= [(q-p)^{*m} * (q-\bar{p})^{*n}]^{-*} * g(q) = \\ &= \left[(q-x)^2 + y^2\right]^{-n} (q-p)^{*(n-m)} * g(q) \end{aligned} \tag{5.15}$$

in $U \setminus (x + y\mathbb{S})$. Moreover, if $n > 0$ then neither $g(p)$ nor $g(\bar{p})$ vanish.

Proof. Choose U as in the previous lemma. The restriction f_I is meromorphic in U_I, it is holomorphic in $U_I \setminus \{p, \bar{p}\}$, and it has a pole of order m at p and a pole order of n at $\bar{p}$. Hence there exists a holomorphic $g_I : U_I \to \mathbb{H}$ with

$$f_I(z) = \frac{1}{(z-p)^m(z-\bar{p})^n} g_I(z)$$

for all $z \in U_I \setminus \{p, \bar{p}\}$. Let $g = \mathrm{ext}(g_I)$ and consider the function

$$h(q) = [(q-p)^{*m} * (q-\bar{p})^{*n}]^{-*} * g(q).$$

The function h is regular on its domain of definition, which is $U \setminus (x + y\mathbb{S})$. Furthermore, $h_I(z) = \frac{1}{(z-p)^m (z-\bar{p})^n} g_I(z) = f_I(z)$ for all $z \in U_I \setminus \{p, \bar{p}\}$. The Identity Principle 1.12 allows us to conclude that $f(q) = h(q)$ for all $q \in U \setminus (x + y\mathbb{S})$. The second equality in (5.15) is proven observing that

$$\begin{aligned}[(q-p)^{*m} * (q-\bar{p})^{*n}]^c &= (q-p)^{*n} * (q-\bar{p})^{*m} = \\ &= [(q-x)^2 + y^2]^m (q-p)^{*(n-m)}\end{aligned}$$

and $\{[(q-p)^{*m} * (q-\bar{p})^{*n}]^s\}^{-1} = [(q-x)^2 + y^2]^{-m-n}$, so that

$$[(q-p)^{*m} * (q-\bar{p})^{*n}]^{-*} = \left[(q-x)^2 + y^2\right]^{-n} (q-p)^{*(n-m)}.$$

□

Proposition 5.26. *The set of semiregular functions on a symmetric slice domain Ω is a division ring with respect to $+, *$.*

Proof. Let f, g be two semiregular functions on Ω and let $\mathcal{S}_f, \mathcal{S}_g$ be the sets of nonremovable singularities of f, g (respectively). The sum $f + g$ and the regular product $f * g$ are defined (and regular) on the largest symmetric slice domain in which both f and g are regular, $\Omega \setminus (\mathcal{S}_f \cup \mathcal{S}_g)$. Moreover, if $f \not\equiv 0$ then the regular reciprocal f^{-*} is defined on $\Omega \setminus (\mathcal{S}_f \cup Z_{f^s})$. The functions $f + g, f * g$ are semiregular in Ω since every $p \in \mathcal{S}_f \cup \mathcal{S}_g$ is a pole for $f + g$ and $f * g$. Indeed, by Theorem 5.25, there exists a neighborhood U of p in Ω where f and g can be expressed as quotients of regular functions. By Theorem 5.6, $f + g$ and $f * g$ are quotients of regular functions on U, too. In particular, by Proposition 5.23, $f + g, f * g$ are semiregular in U. Finally, if $f \not\equiv 0$ then the regular reciprocal f^{-*} is semiregular in Ω since every $p \in \mathcal{S}_f \cup Z_{f^s}$ is a pole for f^{-*} by a similar reasoning. □

We can now state and prove the following consequence of Theorem 5.25.

Corollary 5.27. *Let f be a semiregular function on a symmetric slice domain Ω. Choose $p = x + yI \in \Omega$, let $m = ord_f(p), n = ord_f(\bar{p})$, and suppose $m \leq n$. Then there exists a unique semiregular function g on Ω, without poles in $x + y\mathbb{S}$, such that*

$$\begin{aligned} f(q) &= [(q-p)^{*m} * (q-\bar{p})^{*n}]^{-*} * g(q) = \\ &= \left[(q-x)^2 + y^2\right]^{-n} (q-p)^{*(n-m)} * g(q) \end{aligned} \tag{5.16}$$

Furthermore, if $n > 0$ then neither $g(p)$ nor $g(\bar{p})$ vanish.

Proof. By the previous proposition, setting $g(q) = (q - p)^{*m} * (q - \bar{p})^{*n} * f(q)$ defines a function which is semiregular in Ω. The thesis follows by Theorem 5.25. □

The next result explains the distribution of the poles of semiregular functions on each 2-sphere $x + y\mathbb{S}$.

Theorem 5.28. *Let Ω be a symmetric slice domain, and let f be semiregular in Ω. In each sphere $x + y\mathbb{S} \subset \Omega$, all the poles have the same order ord_f with the possible exception of one, which must have lesser order.*

Proof. Choose a sphere $x + y\mathbb{S} \subset \Omega$ and an $I \in \mathbb{S}$ such that $p = x + yI$ and $\bar{p} = x - yI$ have orders m and n with $m > 0$ or $n > 0$. Without loss of generality, $m \leq n$. By the Corollary 5.27, there exists a semiregular function g on Ω which is regular in a neighborhood U of $x + y\mathbb{S}$ such that

$$f(q) = [(q-x)^2 + y^2]^{-n}(q - p)^{*(n-m)} * g(q)$$

and g does not vanish at p nor at $\bar{p}$. If $\tilde{f}(q) = (q - p)^{*(n-m)} * g(q)$, then

$$f(q) = [(q-x)^2 + y^2]^{-n}\tilde{f}(q),$$

$$f_J(z) = [z-(x + yJ)]^{-n}[z-(x - yJ)]^{-n}\tilde{f}_J(z) \tag{5.17}$$

for all $J \in \mathbb{S}$. Now:

1. If $m < n$ then $\tilde{f}(x + yI) = 0$ and $\tilde{f}(x + yJ) \neq 0$ for all $J \in \mathbb{S} \setminus \{I\}$. Equality (5.17) allows us to conclude $ord_f(x + yJ) = n$ for all $J \in \mathbb{S} \setminus \{I\}$. Since we know by hypothesis that $ord_f(x + yI) = ord_f(p) = m < n$, the thesis holds.
2. If $m = n$ then $\tilde{f}(x + yI) \neq 0$. If $\tilde{f}$ does not have zeros in $x + y\mathbb{S}$, then we conclude $ord_f(x + yJ) = n$ for all $J \in \mathbb{S}$. If, on the contrary, $\tilde{f}(x + yK) = 0$ for some $K \in \mathbb{S}$, then we can factor $z-(x + yK)$ out of $\tilde{f}_K(z)$ and conclude that $ord_f(x + yK) < n$ while $ord_f(x + yJ) = n$ for all $J \in \mathbb{S} \setminus \{K\}$, as desired. Notice that, by construction, $\tilde{f}$ cannot have more than one zero in $x + y\mathbb{S}$. □

In the spirit of Sect. 3.6, we state the next result.

Theorem 5.29. *Let f be a semiregular function on a symmetric slice domain Ω, suppose $f \not\equiv 0$ and let $x+y\mathbb{S} \subset \Omega$. There exist $m \in \mathbb{Z}, n \in \mathbb{N}$, $p_1, \ldots, p_n \in x+y\mathbb{S}$ with $p_i \neq \bar{p}_{i+1}$ for all $i \in \{1, \ldots, n-1\}$, so that*

$$f(q) = [(q-x)^2 + y^2]^m(q - p_1) * (q - p_2) * \ldots * (q - p_n) * g(q) \tag{5.18}$$

for some semiregular function g on Ω which does not have poles nor zeros in $x+y\mathbb{S}$.

Proof. The thesis is an immediate consequence of Theorem 3.22 and of Corollary 5.27. □

Definition 5.30. Let f be a semiregular function on a symmetric slice domain, and consider the factorization

$$f(q) = [(q-x)^2 + y^2]^m (q-p_1) * (q-p_2) * \ldots * (q-p_n) * g(q)$$

appearing in Theorem 5.29. If $m \le 0$, then we say that f has *spherical order* $-2m$ at $x + y\mathbb{S}$ and write $ord_f(x + y\mathbb{S}) = -2m$ (even when $y = 0$). Whenever $n > 0$, we say that f has *isolated multiplicity* n at p_1.

As in the case of zeros, the spherical order and isolated multiplicity of f are related to the order ord_f in the following way.

Proposition 5.31. *Let f be a semiregular function on Ω which is not regular at $p = x+yI \in \Omega$. Then f has spherical order $2\max\{ord_f(p), ord_f(\bar{p})\}$ at $x+y\mathbb{S}$. If moreover $ord_f(p) > ord_f(\bar{p})$, then f has isolated multiplicity $n \ge ord_f(p) - ord_f(\bar{p})$ at $\bar{p}$.*

5.5 Casorati–Weierstrass Theorem

In this section we study essential singularities, proving a quaternionic version of the Casorati–Weierstrass Theorem. We begin by showing that the regular quotient $f^{-*} * g(q)$ is nicely related to the pointwise quotient $f(q)^{-1}g(q)$.

Proposition 5.32. *Let f, g be regular functions on a symmetric slice domain Ω. Then, for all $q \in \Omega \setminus Z_{f^s}$,*

$$f^{-*} * g(q) = f(T_f(q))^{-1} g(T_f(q)), \tag{5.19}$$

where $T_f : \Omega \setminus Z_{f^s} \to \Omega \setminus Z_{f^s}$ is defined as $T_f(q) = f^c(q)^{-1} q f^c(q)$. Furthermore, T_f and T_{f^c} are mutual inverses so that T_f is a diffeomorphism.

Proof. According to Proposition 3.9, if $f^s(p) \neq 0$ at some $p \in \Omega$, then $f^c(p) \neq 0$. Hence T_f is well defined on $\Omega \setminus Z_{f^s}$. Recalling Proposition 3.4, we compute

$$f^{-*} * g(q) = f^s(q)^{-1} f^c * g(q) = [f^c * f(q)]^{-1} f^c(q) g(T_f(q)) =$$

$$= \left[f^c(q) f(T_f(q))\right]^{-1} f^c(q) g(T_f(q)) = f(T_f(q))^{-1} f^c(q)^{-1} f^c(q) g(T_f(q)) =$$

$$= f(T_f(q))^{-1} g(T_f(q)).$$

Moreover, $T_f : \Omega \setminus Z_{f^s} \to \mathbb{H}$ maps any sphere $x + y\mathbb{S}$ to itself. In particular, since Z_{f^s} is symmetric by Proposition 3.9, $T_f(\Omega \setminus Z_{f^s}) \subseteq \Omega \setminus Z_{f^s}$. Now, since

$(f^c)^c = f$ we observe that $T_{f^c}(q) = f(q)^{-1}qf(q)$. For all $q \in \Omega \setminus Z_{f^s}$, setting $p = T_f(q)$, we have that

$$T_{f^c} \circ T_f(q) = T_{f^c}(p) = f(p)^{-1}pf(p) =$$
$$= f(p)^{-1}\left[f^c(q)^{-1}qf^c(q)\right]f(p) = [f^c(q)f(p)]^{-1}\,q\,[f^c(q)f(p)]$$

where

$$f^c(q)f(p) = f^c(q)f(f^c(q)^{-1}qf^c(q)) = f^c * f(q) = f^s(q).$$

As explained in Sect. 3.2, $f^s(q)$ and q always lie in the same complex plane L_I. In particular they commute, so that

$$T_{f^c} \circ T_f(q) = f^s(q)^{-1}qf^s(q) = q,$$

as desired. □

Theorem 5.33 (Casorati–Weierstrass). *Let Ω be a symmetric slice domain and let f be a regular function on Ω. If p is an essential singularity for f and if U is a symmetric neighborhood of p in $\mathbb{H}$, then $f(\Omega \cap U)$ is dense in $\mathbb{H}$.*

Proof. Suppose that for some symmetric neighborhood U of p in $\mathbb{H}$, there existed a $v \in \mathbb{H}$ and an $\varepsilon > 0$ such that $f(\Omega \cap U) \cap B(v, \varepsilon) = \emptyset$. Setting $h(q) = f(q) - v$ and $g(q) = h^{-*}(q)$, we would define a function g, semiregular in Ω, with

$$|g(q)| = \frac{1}{|f(T_h(q)) - v|} \leq \sup_{w \in \Omega\cap U} \frac{1}{|f(w) - v|} \leq \frac{1}{\inf_{w\in\Omega\cap U} |f(w) - v|} \leq \frac{1}{\varepsilon}$$

for all $q \in \Omega \cap U$. The function g would then have a removable singularity at p, and the function $f = g^{-*} + v$ would have a pole (or a removable singularity) at p. This is impossible, since we supposed p to be an essential singularity for f. □

We notice that, when $p \in \mathbb{R}$, the situation is completely analogous to the complex case.

Corollary 5.34. *Let f be a regular function on $B(p, R)\setminus\{p\}$ with $p \in \mathbb{R}$, $R > 0$. If f has an essential singularity at p then for each neighborhood U of p in $B(p, R)$, the set $f(U \setminus \{p\})$ is dense in $\mathbb{H}$.*

We end this chapter with an example to prove that, in Theorem 5.33, the symmetry hypothesis on U cannot be removed.

Example 5.35. Let $\Omega = \mathbb{H}\setminus\mathbb{S}$ and let $f : \Omega \to \mathbb{H}$ be the regular function defined as

$$f(q) = \exp((1 + q^2)^{-1}) = \sum_{n\in\mathbb{N}} (1 + q^2)^{-n}\frac{1}{n!}$$

Each imaginary unit $I \in \mathbb{S}$ is an essential singularity for f. If U is a (small enough) nonsymmetric neighborhood of I, then $f(\Omega \cap U)$ is not dense in $\mathbb{H}$. For instance, let

$$U = \bigcup_{J \in C} L_J$$

where C is the spherical cap $C = \{J \in \mathbb{S} : |I - J| < 1/4\}$: since $f(\Omega_J) \subseteq L_J$ for all $J \in C$, we conclude that $f(\Omega \cap U) \subseteq U$ is not dense in $\mathbb{H}$.

Bibliographic Notes

Most results in this chapter were originally proven in [119] in the case of Euclidean balls centered at the origin and later studied on symmetric slice domains in [120, 123] (with a significantly different approach). Exceptions are the study of Laurent series conducted in Sect. 5.2, which derives entirely from [120, 123], and Sect. 5.5. Indeed, the Casorati–Weierstrass Theorem was proven in [121] for the special case of an essential singularity at 0, in [120, 123] in its most general statement.

Chapter 6
Integral Representations

6.1 Cauchy Theorem and Morera Theorem

Regular quaternionic functions inherit a version of the Cauchy Theorem from the holomorphic complex functions. Let us begin with some notations.

Let $\gamma_I : [0,1] \to L_I$ be a rectifiable curve whose support lies in a complex plane L_I for some $I \in \mathbb{S}$, let Γ_I be a neighborhood of γ_I in L_I, and let $f, g : \Gamma_I \to \mathbb{H}$ be continuous functions. If $J \in \mathbb{S}$ is such that $J \perp I$, then $L_I + L_I J = \mathbb{H} = L_I + JL_I$ and there exist continuous functions $F, G, H, K : \Gamma_I \to L_I$ such that $f = F + GJ$ and $g = H + JK$ in Γ_I. We will write

$$\int_{\gamma_I} g(s)ds f(s) := \int_{\gamma_I} H(s)dsF(s) + \int_{\gamma_I} H(s)dsG(s)J +$$
$$+J\int_{\gamma_I} K(s)dsF(s) + J\int_{\gamma_I} K(s)dsG(s)J.$$

Proposition 6.1 (Cauchy Theorem). *Let f be a regular function on a domain Ω. Let $I \in \mathbb{S}$ be such that Ω_I is simply connected and γ_I is a rectifiable closed curve in Ω_I. Then*

$$\int_{\gamma_I} ds f(s) = 0. \tag{6.1}$$

Proof. Choose $J \in \mathbb{S}$ such that $J \perp I$. According to the Splitting Lemma 1.3, there exist holomorphic functions $F, G : \Omega_I \to L_I$ such that $f_I = F + GJ$ and

$$\int_{\gamma_I} ds f(s) = \int_{\gamma_I} dsF(s) + \int_{\gamma_I} dsG(s)J.$$

By the complex Cauchy Theorem, $\int_{\gamma_I} dsF(s) = 0$ and $\int_{\gamma_I} dsG(s) = 0$. The thesis follows. □

We also have a version of the Morera Theorem.

G. Gentili et al., *Regular Functions of a Quaternionic Variable*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-33871-7_6,

Proposition 6.2 (Morera Theorem). *Let Ω be a domain in $\mathbb{H}$ and let $f : \Omega \to \mathbb{H}$. If, for each $I \in \mathbb{S}$, the restriction of f to Ω_I is continuous and it satisfies*

$$\int_{\gamma_I} ds\, f(s) = 0 \tag{6.2}$$

for all rectifiable closed curves $\gamma_I : [0,1] \to \Omega_I$, then f is regular in Ω.

Proof. Let $I, J \in \mathbb{S}$ be such that $J \perp I$ and let F, G be functions on Ω_I such that $f_I = F + GJ$ in Ω_I. Since f_I is continuous, F and G are continuous, too. Moreover, for all rectifiable closed curves $\gamma_I : [0,1] \to \Omega_I$

$$0 = \int_{\gamma_I} ds\, f(s) = \int_{\gamma_I} ds F(s) + \int_{\gamma_I} ds G(s) J,$$

so that $0 = \int_{\gamma_I} ds F(s)$ and $0 = \int_{\gamma_I} ds G(s)$. By the complex Morera Theorem, F and G are holomorphic. Hence, f_I is holomorphic in Ω_I. Since I was arbitrarily chosen, we conclude that f is regular in Ω. □

6.2 Cauchy Integral Formula

We begin with a "slicewise" Cauchy Integral Formula, which reconstructs the values of a regular function f on a slice L_I, by using its values on a closed curve in L_I.

Lemma 6.3. *Let f be a regular function on a symmetric slice domain Ω, let $I \in \mathbb{S}$, and let U_I be a bounded Jordan domain in L_I, with $\overline{U_I} \subset \Omega_I$. If ∂U_I is rectifiable, then*

$$f(z) = \frac{1}{2\pi I} \int_{\partial U_I} \frac{ds}{s - z} f(s) \tag{6.3}$$

for all $z \in U_I$.

Proof. Choose $J \in \mathbb{S}$ such that $J \perp I$. According to the Splitting Lemma 1.3, there exist holomorphic functions $F, G : \Omega_I \to L_I$ such that $f_I = F + GJ$. By the complex Cauchy Integral Formula,

$$F(z) = \frac{1}{2\pi I} \int_{\partial U_I} \frac{ds}{s - z} F(s),$$

$$G(z) = \frac{1}{2\pi I} \int_{\partial U_I} \frac{ds}{s - z} G(s)$$

for all $z \in U_I$. The thesis immediately follows from the fact that $f(z) = F(z) + G(z)J$. □

The next result, on the other hand, is a more powerful Cauchy Formula, which allows the reconstruction of the values of f on the entire open set of definition, by using its values on a slice.

Theorem 6.4 (Cauchy Formula). *Let f be a regular function on a symmetric slice domain Ω. If U is a bounded symmetric open set with $\overline{U} \subset \Omega$, if $I \in \mathbb{S}$, and if ∂U_I is a finite union of disjoint rectifiable Jordan curves, then, for $q \in U$,*

$$f(q) = \frac{1}{2\pi} \int_{\partial U_I} (s-q)^{-*} ds_I \, f(s) \tag{6.4}$$

where $ds_I = -I\,ds$ and $(s-q)^{-*}$ denotes the regular reciprocal of $q \mapsto s-q$, that is,

$$(s-q)^{-*} = (|s|^2 - q2Re(s) + q^2)^{-1}(\bar{s} - q). \tag{6.5}$$

Proof. By Formula (1.9), if $q = x + yJ$ and $z = x + yI$, then

$$f(q) = \frac{1}{2}[f(z) + f(\bar{z})] + \frac{JI}{2}[f(\bar{z}) - f(z)].$$

Since U is a symmetric set containing q, its slice U_I contains both z and $\bar{z}$. We prove that

$$f(z) = \frac{1}{2\pi} \int_{\partial U_I} \frac{1}{s-z} ds_I \, f(s)$$

by applying Proposition 6.1 to each connected component of U_I that does not contain z and the previous Lemma to the (unique) connected component of U_I containing z. Similarly,

$$f(\bar{z}) = \frac{1}{2\pi} \int_{\partial U_I} \frac{1}{s-\bar{z}} ds_I \, f(s).$$

Thus,

$$f(q) =$$

$$= \frac{1}{2\pi} \int_{\partial U_I} \left\{ \frac{1}{2} \left[\frac{1}{s-z} + \frac{1}{s-\bar{z}} \right] + \frac{JI}{2} \left[\frac{1}{s-\bar{z}} - \frac{1}{s-z} \right] \right\} ds_I \, f(s) =$$

$$= \frac{1}{2\pi} \int_{\partial U_I} (s-q)^{-*} ds_I \, f(s)$$

taking into account that

$$(s-q)^{-*} = \frac{1}{2} \left[\frac{1}{s-z} + \frac{1}{s-\bar{z}} \right] + \frac{JI}{2} \left[\frac{1}{s-\bar{z}} - \frac{1}{s-z} \right]$$

by Formula (1.9). □

We should note that the kernel $(s-q)^{-*}$ is the so-called noncommutative Cauchy kernel, which we will recall in more detail in Sect. 10.2.

6.3 Pompeiu Formula

As in the complex case, the Cauchy Formula is a special case of a more general result that goes under the name of Pompeiu Formula.

Let $I \in \mathbb{S}$, let U_I be a bounded open set in L_I, and let $ds_I = -I ds$. Let f be a quaternion-valued function defined almost everywhere in U_I. Let $J \in \mathbb{S}$ be such that $J \perp I$ and let F, G be functions with values in L_I such that $f = F + GJ$. If $\iint_{U_I} F(s) ds_I \wedge d\bar{s}$ and $\iint_{U_I} G(s) ds_I \wedge d\bar{s}$ are defined, then we set

$$\iint_{U_I} f(s) ds_I \wedge d\bar{s} = \iint_{U_I} F(s) ds_I \wedge d\bar{s} + \iint_{U_I} G(s) J ds_I \wedge d\bar{s}.$$

We recall that in Sect. 1.1 we set $\bar{\partial}_I f(x+yI) = \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x+yI)$ for all $I \in \mathbb{S}$.

Theorem 6.5 (Cauchy–Pompeiu Formula). *Let U be a bounded symmetric open set in $\mathbb{H}$. If $I \in \mathbb{S}$, $f_I \in C^1(\overline{U}_I)$, and ∂U_I is a finite union of disjoint rectifiable Jordan curves, then for all $z \in U_I$, the following formula holds:*

$$f(z) = \frac{1}{2\pi}\int_{\partial U_I} \frac{1}{s-z} ds_I f(s) + \frac{1}{2\pi}\iint_{U_I} \frac{1}{s-z}\bar{\partial}_I f(s) ds_I \wedge d\bar{s}. \tag{6.6}$$

Proof. Choose $J \in \mathbb{S}$ such that $J \perp I$ and observe that there exist two C^1 complex-valued functions of one complex variable, F and G, such that $f_I = F + GJ$. By the complex Cauchy–Pompeiu Formula, we have that

$$F(z) = \frac{1}{2\pi}\int_{\partial U_I} \frac{1}{s-z} ds_I F(s) + \frac{1}{2\pi}\iint_{U_I} \frac{1}{s-z}\frac{\partial F(s)}{\partial \bar{s}} ds_I \wedge d\bar{s},$$

$$G(z) = \frac{1}{2\pi}\int_{\partial U_I} \frac{1}{s-z} ds_I G(s) + \frac{1}{2\pi}\iint_{U_I} \frac{1}{s-z}\frac{\partial G(s)}{\partial \bar{s}} ds_I \wedge d\bar{s}.$$

Furthermore,

$$\begin{aligned} ds_I \wedge d\bar{s} J &= -I ds \wedge d\bar{s} J = -IJ d\bar{s} \wedge ds = \\ &= JI d\bar{s} \wedge ds = J(-I ds \wedge d\bar{s}) = J ds_I \wedge d\bar{s} \end{aligned}$$

Hence

$$\begin{aligned} f(z) = F(z) + G(z)J &= \frac{1}{2\pi}\int_{\partial U_I} \frac{1}{s-z} ds_I\, [F(s) + G(s)J] + \\ &+ \frac{1}{2\pi}\iint_{U_I} \frac{1}{s-z}\left[\frac{\partial F(s)}{\partial \bar{s}} + \frac{\partial G(s)}{\partial \bar{s}} J\right] ds_I \wedge d\bar{s} = \\ &= \frac{1}{2\pi}\int_{\partial U_I} \frac{1}{s-z} ds_I f(s) + \frac{1}{2\pi}\iint_{U_I} \frac{1}{s-z}\bar{\partial}_I f(s) ds_I \wedge d\bar{s} \end{aligned}$$

as desired. □

6.4 Derivatives Using the Cauchy Formula

As in the complex case, Cauchy-type formulas allow the computation of the slice derivatives of regular functions.

Lemma 6.6. *Let f be a regular function on a symmetric slice domain Ω, let $I \in \mathbb{S}$, and let U_I be a bounded Jordan domain in L_I, with $\overline{U_I} \subset \Omega_I$. If ∂U_I is rectifiable, then*

$$f^{(n)}(z) = \frac{n!}{2\pi I} \int_{\partial U_I} \frac{ds}{(s-z)^{n+1}} f(s). \tag{6.7}$$

for all $z \in U_I$ and for all $n \in \mathbb{N}$.

Proof. Choose $J \in \mathbb{S}$ with $J \perp I$ and holomorphic F, G such that $f_I = F + GJ$. The result follows from the fact that $f^{(n)}(z) = \frac{\partial^n F}{\partial z^n}(z) + \frac{\partial^n G}{\partial z^n}(z)J$, where

$$\frac{\partial^n F}{\partial z^n}(z) = \frac{n!}{2\pi I} \int_{\partial U_I} \frac{ds}{(s-z)^{n+1}} F(s),$$
$$\frac{\partial^n G}{\partial z^n}(z) = \frac{n!}{2\pi I} \int_{\partial U_I} \frac{ds}{(s-z)^{n+1}} G(s).$$

□

Proposition 6.7. *Let f be a regular function on a symmetric slice domain Ω. If U is a bounded symmetric open set with $\overline{U} \subset \Omega$, if $I \in \mathbb{S}$, and if ∂U_I is a finite union of disjoint rectifiable Jordan curves, then, for $q \in U$,*

$$f^{(n)}(q) = \frac{n!}{2\pi} \int_{\partial U_I} (s-q)^{*(-n-1)} ds_I\, f(s) \tag{6.8}$$

where $ds_I = -I\,ds$ and $(s-q)^{*(-n-1)}$ denotes the regular reciprocal of $q \mapsto (s-q)^{*(n+1)}$, that is,

$$(s-q)^{*(-n-1)} = (|s|^2 - q2Re(s) + q^2)^{-n-1}(\bar{s} - q)^{*(n+1)}. \tag{6.9}$$

Proof. If $q = x + yJ$ and $z = x + yI$, then, by Formula (1.9),

$$f^{(n)}(q) = \frac{1-JI}{2} f^{(n)}(z) + \frac{1+JI}{2} f^{(n)}(\bar{z})$$

and

$$(s-q)^{*(-n-1)} = \frac{1-JI}{2}\frac{1}{(s-z)^{n+1}} + \frac{1+JI}{2}\frac{1}{(s-\bar{z})^{n+1}}.$$

The thesis follows by applying the previous Lemma to calculate both $f^{(n)}(z)$ and $f^{(n)}(\bar{z})$. □

The previous results allow us to estimate the slice derivatives of a regular function.

Proposition 6.8 (Cauchy Estimates). *Let $f : \Omega \to \mathbb{H}$ be a regular function and let $p \in \Omega$. If $p \in L_I$, then for all disks $\Delta_I = \Delta_I(p, R)$ of radius $R > 0$ with $\overline{\Delta}_I \subset \Omega_I$ the following formula holds:*

$$\left|f^{(n)}(p)\right| \leq \frac{n!}{R^n} \max_{\partial \Delta_I} |f|. \tag{6.10}$$

Proof. If $\gamma(t) = p + Re^{2\pi I t}$, then

$$\left|f^{(n)}(p)\right| \leq \frac{n!}{2\pi} \int_\gamma \frac{|f(s)|}{|s-p|^{n+1}} |ds| \leq \frac{n!}{R^n} \max_{\partial \Delta_I} |f|,$$

as desired. □

In the special case of an entire function (a regular function $\mathbb{H} \to \mathbb{H}$), the Cauchy Estimates lead to the Liouville Theorem.

Theorem 6.9 (Liouville). *Let f be a bounded entire function. Then f is constant.*

Proof. Apply the Cauchy Estimates 6.8 at $p = 0$ for $R > 0$. Letting $R \to +\infty$ proves that all the slice derivatives $f^{(n)}(0)$ with $n > 0$ vanish. Hence, $f(q) = \sum_{n \in \mathbb{N}} q^n \frac{1}{n!} f^{(n)}(0) = f(0)$ for all $q \in \mathbb{H}$. □

6.5 Coefficients of the Laurent Series Expansion

In Sect. 5.2, we studied regular Laurent series, whose sets of convergence are of the type

$$\Sigma(p, R_1, R_2) = \{q \in \mathbb{H} : \tau(q, p) > R_1, \sigma(q, p) < R_2\},$$

and we proved that regular functions admit Laurent expansions on these sets. We are now able to compute the coefficients of these expansions explicitly.

Proposition 6.10. *If $p \in L_I \subset \mathbb{H}$, if $0 \leq R_1 < R_2 \leq +\infty$, and if*

$$f(q) = \sum_{n \in \mathbb{Z}} (q-p)^{*n} a_n \tag{6.11}$$

in $\Sigma(p, R_1, R_2)$, then for each $n \in \mathbb{Z}$

$$a_n = \frac{1}{2\pi I} \int_{\partial \Delta_I} \frac{ds}{(s-p)^{n+1}} f(s) \tag{6.12}$$

where $\Delta_I = \Delta_I(p, R)$ is a disk having radius R such that $R_1 < R < R_2$.

Proof. Let $J \perp I$ and let F, G be holomorphic functions such that $f_I = F + GJ$. If $\alpha_n, \beta_n \in L_I$ are such that $a_n = \alpha_n + \beta_n J$ for all $n \in \mathbb{Z}$, then the fact that

$$f_I(z) = \sum_{n \in \mathbb{Z}} (z-p)^n a_n$$

for all z belonging to the annulus $A_I(p, R_1, R_2)$ implies that

$$F(z) = \sum_{n \in \mathbb{Z}} (z-p)^n \alpha_n, \; G(z) = \sum_{n \in \mathbb{Z}} (z-p)^n \beta_n$$

for all $z \in A_I(p, R_1, R_2)$. The thesis follows observing that

$$\alpha_n = \frac{1}{2\pi I} \int_{\partial \Delta_I} \frac{ds}{(s-p)^{n+1}} F(s)$$
$$\beta_n = \frac{1}{2\pi I} \int_{\partial \Delta_I} \frac{ds}{(s-p)^{n+1}} G(s).$$

□

6.6 Argument Principle

In this section we discuss the argument principle for semiregular functions. As in the complex case, this principle has a rather interesting geometrical meaning. We also present, for regular functions, versions of the classical Rouché Theorem and of one of its best known consequences.

Lemma 6.11. *Let f be a regular function on a symmetric slice domain Ω. If both f and its slice derivative f' vanish at a point $p \in \Omega$, then $(f^s)'$ vanishes at p, too.*

Proof. Consider the slice derivative of f^s. Since, from Proposition 1.40, $(f^s)' = f' * f^c + f * (f^c)'$, and the thesis follows. □

This motivates the next definition. As pointed out in Remark 1.36, if f is a regular function on a symmetric slice domain Ω, then f^s is a slice preserving regular function. Therefore, for each $I \in \mathbb{S}$, the function $f_I^s : \Omega_I \to L_I$ is a complex-valued holomorphic function.

Definition 6.12. Consider a regular function f on a symmetric slice domain Ω. Then we define the *logarithmic indicator* of f as

$$\mathcal{L}_f(q) := \frac{(f^s)'(q)}{f^s(q)}.$$

Definition 6.13. Let f be a semiregular function on a symmetric slice domain Ω. For $x, y \in \mathbb{R}$ with $y \neq 0$ and $x + y\mathbb{S} \subset \Omega$, we define the *total multiplicity* of the sphere $x + y\mathbb{S}$ (with respect to f) to be the sum $m_f(x + y\mathbb{S})$ of the spherical multiplicity of $x + y\mathbb{S}$ and of the isolated multiplicities of all $\alpha \in x + y\mathbb{S}$. If, instead, $x \in \Omega$ is a real number, then the *total multiplicity* of x is defined as the isolated multiplicity of x, which coincides in turn with the classical multiplicity $m_f(x)$.

We recall that, according to Theorem 3.36, each sphere $x + y\mathbb{S}$ contains at most one point having a positive isolated multiplicity.

Proposition 6.14. *Let $f \not\equiv 0$ be a regular function on a symmetric slice domain Ω. Let $x, y \in \mathbb{R}$ be such that $x + y\mathbb{S}$ is contained in Ω. Then:*

(*a*) *The total multiplicity of $x + y\mathbb{S}$ with respect to f^s is twice the total multiplicity of $x + y\mathbb{S}$ with respect to f:*

$$m_{f^s}(x + y\mathbb{S}) = 2m_f(x + y\mathbb{S}).$$

(*b*) *If $y \neq 0$, then the function f^s has classical multiplicity $m_{f^s}(x + yI) = m_f(x + y\mathbb{S}) = \frac{1}{2}m_{f^s}(x + y\mathbb{S})$ at $x + yI$ for any $I \in \mathbb{S}$.*

(*c*) *If $y = 0$, then the function f^s has classical (and isolated) multiplicity $m_{f^s}(x) = 2m_f(x)$.*

Proof. If $y \neq 0$, then, by Theorem 3.36, there exist $m \in \mathbb{N}, n \in \mathbb{N}$, $p_1, \ldots, p_n \in x + y\mathbb{S}$ (with $p_i \neq \bar{p}_{i+1}$ for all $i \in \{1, \ldots, n-1\}$) such that

$$f(q) = [(q - x)^2 + y^2]^m (q - p_1) * (q - p_2) * \ldots * (q - p_n) * g(q) \qquad (6.13)$$

for some regular function $g : \Omega \to \mathbb{H}$ vanishing nowhere in $x + y\mathbb{S}$. Since

$$f^c(q) = g^c(q) * (q - \bar{p}_n) * \ldots * (q - \bar{p}_2) * (q - \bar{p}_1) * [(q - x)^2 + y^2]^m$$

by definition of symmetrization of a regular function, we get

$$f^s(q) = f * f^c(q) = [(q - x)^2 + y^2]^{2m+n} * g^s(q)$$

where g^s vanishes nowhere in Ω. As a consequence, $m_{f^s}(x + y\mathbb{S}) = 4m + 2n$, while $m_f(x + y\mathbb{S}) = 2m + n$, which proves (a). When $y = 0$, the proof of (a) is straightforward. To prove (b), it is enough to notice that on Ω_I we can write

$$f_I^s(z) = [z - (x + yI)]^{2m+n}[z - (x - yI)]^{2m+n} g^s(z)$$

where g^s is a nowhere vanishing holomorphic function on Ω_I. The proof of (c) is immediate. ☐

The semiregular function $\mathcal{L}_f(q)$ that we have just introduced plays the role, in the quaternionic setting, of the logarithmic derivative of a holomorphic function. It can be used to prove, with a few further steps, a version of the argument principle for regular functions.

Theorem 6.15. *Let $f \not\equiv 0$ be a regular function defined in a symmetric slice domain Ω and, for $x_0, y_0 \in \mathbb{R}$ and $I_0 \in \mathbb{S}$, let $x_0 + y_0 I_0 \in \Omega$ be a zero of f. There exists $\varepsilon > 0$ such that for all $I \in \mathbb{S}$ and for all $0 < l < \varepsilon$*

$$\frac{1}{2\pi I}\int_{\partial\Delta_I(x_0+y_0I,l)} \mathcal{L}_f(z)\,\mathrm{d}z = \begin{cases} m_f(x_0+y_0\mathbb{S}) \text{ if } y_0 \neq 0 \\ \\ 2m_f(x_0) \qquad\quad \text{if } y_0 = 0 \end{cases} \tag{6.14}$$

where $\Delta_I(x_0 + y_0 I, l)$ is a disk of radius l centered at $x_0 + y_0 I$ contained in Ω_I.

Proof. If $y_0 \neq 0$, we can choose $\varepsilon > 0$ small enough that, for all $I \in \mathbb{S}$, $\Delta_I(x_0 + y_0 I, \varepsilon) \subset \Omega_I$, $\Delta_I(x_0 + y_0 I, \varepsilon) \cap \mathbb{R} = \emptyset$ and the function f never vanishes in the symmetric completion

$$\bigcup_{J\in\mathbb{S}} [\Delta_J(x_0 + y_0 J, \varepsilon) \setminus \{x_0 + y_0 J\}]$$

of $\Delta_I(x_0 + y_0 I, \varepsilon) \setminus \{x_0 + y_0 I\}$. If, instead, $y_0 = 0$, we choose ε small enough that x_0 is the only zero of f contained in the closure of $B(x_0, \varepsilon) \subset \Omega$. Now, in view of Proposition 6.14, an application of the complex argument principle to the holomorphic function f_I^s proves the assertion. □

Notice that in (6.14) the expression on the right-hand side is independent of $I \in \mathbb{S}$.

Lemma 6.16. *Let g and h be regular functions in the symmetric slice domain Ω. Then the semiregular function $f = h^{-*} * g : \Omega \to \mathbb{H}$ is such that $\mathcal{L}_f = \mathcal{L}_g - \mathcal{L}_h$.*

Proof. Thanks to Proposition 5.23, the function f is semiregular in Ω. Since

$$\begin{aligned} f^s(q) &= f * f^c(q) \\ &= h^{-*}(q) * g(q) * g^c(q) * (h^{-*})^c(q) \\ &= h^s(q)^{-1} * h^c(q) * g^s(q) * h(q) * h^s(q)^{-1} \\ &= h^s(q)^{-1} g^s(q) \end{aligned}$$

to compute the slice derivative of the (slice preserving) function f^s we can proceed by computing

$$\begin{aligned} (f^s(q))' &= (h^s(q)^{-1} g^s(q))' \\ &= h^s(q)^{-2}[((g^s)'(q))(h^s(q)) - (g^s(q))((h^s)'(q))] \end{aligned}$$

whence we get

$$\begin{aligned}\mathcal{L}_f(q) &= f^s(q)^{-1}((f^s)'(q))\\ &= g^s(q)^{-1}((g^s)'(q)) - h^s(q)^{-1}((h^s)'(q))\\ &= \mathcal{L}_g(q) - \mathcal{L}_h(q).\end{aligned}$$

□

Recalling the notion of spherical order $ord_f(x_0 + y_0\mathbb{S})$ given in Definition 5.30, we can prove the following result.

Theorem 6.17. *Let $f \not\equiv 0$ be a semiregular function defined in a symmetric slice domain Ω and, for $x_0, y_0 \in \mathbb{R}$ and $I_0 \in \mathbb{S}$, let $x_0 + y_0 I_0 \in \Omega$ be a pole of f. There exists $\varepsilon > 0$ such that for all $I \in \mathbb{S}$ and for all $0 < l < \varepsilon$*

$$\frac{1}{2\pi I}\int_{\Delta_I(x_0+y_0 I, l)} \mathcal{L}_f(z)\,\mathrm{d}z = \begin{cases} m_f(x_0 + y_0\mathbb{S}) - ord_f(x_0 + y_0\mathbb{S}) & \text{if } y \neq 0\\ 2m_f(x_0) - 2ord_f(x_0) & \text{if } y = 0\end{cases}$$

where $\Delta_I(x_0 + y_0 I, l) \subset \Omega_I$.

Proof. If $ord_f(x_0 + y_0\mathbb{S}) = 2n$, then Theorem 5.25 implies that there exist a neighborhood U of $x_0 + y_0\mathbb{S}$ in Ω that is a symmetric slice domain and a (unique) regular function $k : U \to \mathbb{H}$ such that

$$f(q) = [(q - x)^2 + y^2]^{-n} k(q) \tag{6.15}$$

The assertion is now a direct consequence of Lemma 6.16 and Theorem 6.15. □

As a consequence of the previous results, the argument principle can be formulated to "count" the difference between the number of zeros (spherical and isolated, counted with their multiplicities) and the number of poles (spherical and real, counted with their spherical orders) that are contained in a symmetric domain.

Theorem 6.18 (Argument Principle). *Let $f \not\equiv 0$ be a semiregular function on a symmetric slice domain Ω. Let $I \in \mathbb{S}$; let $D_I \subset\subset \Omega_I$ be a domain of L_I, symmetric with respect to the real axis $\mathbb{R}$, whose boundary ∂D_I is the disjoint union of a finite number of rectifiable Jordan curves contained in Ω_I; let D denote the symmetric completion of D_I; and suppose that ∂D does not intersect the zero set nor the set of nonremovable poles of f. If Z denotes the sum of the total multiplicities of all spheres $x + y\mathbb{S}$ and real points x included in D and if P denotes the sum of the spherical orders of all the nonremovable poles of f contained in D, then*

$$\frac{1}{4\pi I}\int_{\partial D_I} \mathcal{L}_f(z)\,\mathrm{d}z = Z - P$$

The result does not depend upon the choice of $I \in \mathbb{S}$.

Proof. Recall that the zeros and poles of f are isolated spheres or isolated points. Since D is relatively compact in Ω, the set W_f of all zeros and (nonremovable) poles of f contains finitely many spheres and isolated points of D. Therefore, we can find p spheres $x_1 + y_1\mathbb{S}, \ldots, x_p + y_p\mathbb{S} \subset D$ (each containing a nonreal zero or a nonreal pole of f) whose union contains the entire set $(W_f \cap D) \setminus \mathbb{R}$. Let $x_{p+1}, \ldots, x_q \in \mathbb{R}$ be all the real zeros and real poles of f in D. For $m = 1, \ldots, p$, consider the symmetric completion P_m of a small open disk $\Delta_I(x_m + y_m I, l_m) \subset D_I \setminus \mathbb{R}$ of radius $l_m > 0$ centered at $x_m + y_m I$. Analogously, for $m = p+1, \ldots, q$, consider the ball $P_m = B(x_m, l_m)$ which is the symmetric completion of a small open disk $\Delta_I(x_m, l_m) \subset \Omega_I$ of radius $l_m > 0$ centered at x_m. The compact set $\overline{C} = \overline{D} \setminus \{P_1, \ldots, P_q\}$ does not contain zeros nor poles of f. Therefore, for $l_1, \ldots, l_q$ small enough to satisfy the hypotheses of Theorem 6.17, we get that if

$$A_I = \left[\bigcup_{m=1}^{p} (\Delta_I(x_m + y_m I, l_m) \cup \Delta_I(x_m - y_m I, l_m))\right] \cup \left[\bigcup_{m=p+1}^{q} \Delta_I(x_m, l_m)\right]$$

then

$$\overline{C_I} = \overline{C} \cap \Omega_I = \overline{D_I} \setminus A_I.$$

Applying the complex argument principle to f_I^s,

$$0 = \frac{1}{4\pi I}\int_{\partial C_I} \mathcal{L}_f(z)\,\mathrm{d}z = \frac{1}{4\pi I}\int_{\partial D_I} \mathcal{L}_f(z)\,\mathrm{d}z - \frac{1}{4\pi I}\int_{\partial A_I} \mathcal{L}_f(z)\,\mathrm{d}z$$

so that

$$\begin{aligned}
&\frac{1}{4\pi I}\int_{\partial D_I} \mathcal{L}_f(z)\,\mathrm{d}z = \frac{1}{4\pi I}\int_{\partial A_I} \mathcal{L}_f(z)\,\mathrm{d}z = \\
&= \sum_{m=1}^{p} \frac{1}{4\pi I}\int_{\partial \Delta_I(x_m + y_m I, l_m)} \mathcal{L}_f(z)\,\mathrm{d}z + \\
&+ \sum_{m=1}^{p} \frac{1}{4\pi I}\int_{\partial \Delta_I(x_m - y_m I, l_m)} \mathcal{L}_f(z)\,\mathrm{d}z + \\
&+ \sum_{m=p+1}^{q} \frac{1}{4\pi I}\int_{\partial \Delta_I(x_m, l_m)} \mathcal{L}_f(z)\,\mathrm{d}z.
\end{aligned}$$

An iterated application of Theorem 6.17 concludes the proof. □

We derive a quaternionic version of the Rouché Theorem.

Corollary 6.19. *Let h and g be two regular functions on $B(0, R)$ and assume that $|h^s - g^s| < |h^s|$ in $\partial B(0, r)$ $(r < R)$. Then h and g have the same number of zeros (counted with spherical and isolated multiplicities) in $B(0, r)$.*

Proof. First, observe that if $|h^s - g^s| < |h^s|$ on $\partial B(0,r)$, then h^s and g^s vanish nowhere in $\partial B(0,r)$. Moreover, the function $f^s = (h^s)^{-1} g^s$ is the symmetrization of the semiregular function $f = h^{-*} * g = (h^s)^{-1} h^c * g$ and it is such that $|f^s - 1| < 1$ on $\partial B(0,r)$. Hence, given any $I \in \mathbb{S}$, the complex Rouché Theorem implies

$$\frac{1}{4\pi I} \int_{\partial(B(0,r)\cap L_I)} \mathcal{L}_f(z)\, \mathrm{d}z = 0$$

and the assertion follows from Theorem 6.17. □

We conclude with a version of the Hurwitz Theorem.

Theorem 6.20. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of regular functions in $B(0,R)$ and assume that $f_n \to f$ as $n \to +\infty$ uniformly on each compact subset of $B(0,R)$. Then either $f \equiv 0$ or every zero of f^s is the limit of a sequence of zeros of $(f_n)^s$.*

Proof. By Lemma 4.24, the limit function f is regular. Furthermore, by Proposition 4.29, if $\{f_n\}_{n\in\mathbb{N}}$ converges to f uniformly on each compact subset of $B(0,R)$, then $\{(f_n)^s\}_{n\in\mathbb{N}}$ converges to f^s uniformly on compact sets in $B(0,R)$. Using the Rouché Theorem as in the complex case (see [38]), we derive that either $f^s \equiv 0$ or every zero of f^s is the limit of a sequence of zeros of $(f_n)^s$. We conclude by recalling that $f \equiv 0$ if and only if $f^s \equiv 0$. □

Bibliographic Notes

The analog of the Cauchy Theorem (Proposition 6.1) first appeared in [120]. The Morera Theorem was proven in [62] for the case of balls centered at 0 and extended to all symmetric slice domains in [120].

The Cauchy Integral Formula and the formulae for slice derivatives were proven in [18] (the same Formula arose in [25] in a different context), while the Pompeiu Formula derives from [37]. We present them all with different proofs (from [120]). The Cauchy Estimates were obtained in [62] (and generalized in [120]). The Liouville Theorem was proven in [62], too.

The coefficients of the Laurent expansion were computed in [120, 123]. The argument principle and its consequences were studied in [127].

Chapter 7
Maximum Modulus Theorem and Applications

7.1 Maximum and Minimum Modulus

The complex Maximum Modulus Principle has a perfect analog for regular functions, proven with the aid of the Splitting Lemma 1.3.

Theorem 7.1 (Maximum Modulus Principle). *Let $\Omega \subseteq \mathbb{H}$ be a slice domain and let $f : \Omega \to \mathbb{H}$ be regular. If $|f|$ has a relative maximum at $p \in \Omega$, then f is constant.*

Proof. If p is a zero of f, then $|f|$ has zero as its maximum value, so that $f \equiv 0$. Now let $f(p) \neq 0$: by possibly multiplying f by $\overline{f(p)}$ on the right side, we may suppose $f(p) \in \mathbb{R}$, $f(p) > 0$. Let $I, J \in \mathbb{S}$ be such that $p \in L_I$ and $I \perp J$; let $F, G : \Omega_I \to L_I$ be holomorphic functions such that $f_I = F + GJ$. Then, for all z in a neighborhood U_I of p in Ω_I,

$$|F(p)|^2 = |f_I(p)|^2 \geq |f_I(z)|^2 = |F(z)|^2 + |G(z)|^2 \geq |F(z)|^2.$$

Hence $|F|$ has a relative maximum at p, and the Maximum Modulus Principle for holomorphic functions of one complex variable allows us to conclude that F is constant. Namely, $F \equiv f(p)$. As a consequence,

$$|G(z)|^2 = |f_I(z)|^2 - |F(z)|^2 = |f_I(z)|^2 - |f_I(p)|^2 \leq |f_I(p)|^2 - |f_I(p)|^2 = 0$$

for all $z \in U_I$. Hence $f_I = F \equiv f(p)$ in U_I. By the Identity Principle 1.12, we conclude that $f \equiv f(p)$ in Ω. □

As in the complex case, Theorem 7.1 implies the following fact: if f is regular on a bounded slice domain Ω and its modulus is bounded by M near the boundary $\partial\Omega$, then $|f| \leq M$ in Ω; the inequality must be strict unless f is constant. We now prove that the inequality $|f| \leq M$ holds even when Ω is not a slice domain.

G. Gentili et al., *Regular Functions of a Quaternionic Variable*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-33871-7_7,

Theorem 7.2. *Let f be a regular function on a bounded domain U. If, for all $q_0 \in \partial U$,*

$$\limsup_{q \to q_0} |f(q)| \leq M \tag{7.1}$$

then $|f| \leq M$ in U. If, moreover, f is not constant and U is a slice domain, then the inequality is strict.

Proof. We already observed that the second statement follows directly from Theorem 7.1. Let us prove the first statement by contradiction. Suppose there exists $p \in U$ such that $|f(p)| > M$. Without loss of generality, $f(p) > 0$. If $p \in L_I$, if $J \in \mathbb{S}$ is such that $J \perp I$ and if $F, G : U_I \to L_I$ are holomorphic functions such that $f_I = F + GJ$, then $f_I(p) = F(p)$, so that $|F(p)| > M$. On the contrary, the complex Maximum Modulus Principle implies that $|F| \leq M$ in U_I, since, for all $z_0 \in \partial U_I \subset \partial U$,

$$\limsup_{z \to z_0} |F(z)| \leq \limsup_{z \to z_0} |f_I(z)| \leq M.$$

This contradiction concludes the proof. □

Let us now state the Minimum Modulus Principle, which is not derived from the complex Minimum Modulus Principle with the same technique used in the previous proofs. We instead use Theorem 7.1 and the nice relation between the regular reciprocal f^{-*} and the pointwise reciprocal $\frac{1}{f}$ established by Proposition 5.32.

Theorem 7.3 (Minimum Modulus Principle). *Let Ω be a symmetric slice domain and let $f : \Omega \to \mathbb{H}$ be a regular function. If $|f|$ has a local minimum point $p \in \Omega$, then either $f(p) = 0$ or f is constant.*

Proof. Suppose $|f|$ to have a minimum point $p = x + yI \in \Omega$ with $f(p) \neq 0$ and let $S = x + y\mathbb{S}$. If f vanished at some point $p' \in S$, then, according to Formula (1.9), $f(S)$ would be a 2-sphere passing through the origin of $\mathbb{H}$. The modulus $|f_{|S}|$ would then have a global minimum at p', a global maximum at some other point, and no other extremal point, in contradiction with the hypothesis on p. Hence f does not have zeros in S nor does f^s. As a consequence, the domain $\Omega' = \Omega \setminus Z_{f^s}$ of the regular reciprocal f^{-*} includes S. By Proposition 5.32,

$$|f^{-*}(q)| = \frac{1}{|f(T_f(q))|}$$

for all $q \in \Omega'$, and if $|f|$ has a minimum at $p \in x + y\mathbb{S} \subseteq \Omega'$, then $|f \circ T_f|$ has a minimum at $T_f^{-1}(p) = T_{f^c}(p) \in \Omega'$. As a consequence, $|f^{-*}|$ has a maximum at $T_{f^c}(p)$. By the Maximum Modulus Principle 7.1, f^{-*} is constant on Ω'. This implies that f is constant in Ω'; hence, in Ω thanks to the Identity Principle 1.12. □

7.2 Open Mapping Theorem

As a first step towards a quaternionic open mapping theorem, we prove that the image of a symmetric open set is always open.

Theorem 7.4. *Let f be a nonconstant regular function on a symmetric slice domain Ω. If U is a symmetric open subset of Ω, then $f(U)$ is open. In particular, the image $f(\Omega)$ is open.*

Proof. Choose $p_0 \in f(U)$ and $q_0 = x_0 + y_0 I \in U$ with $f(q_0) = p_0$, so that $f(q) - p_0$ has a zero in $S = x_0 + y_0\mathbb{S} \subseteq U$. For $r > 0$, consider the symmetric neighborhood V of S defined by $V = \{q \in \mathbb{H} : d(q, S) < r\}$. We may choose $r > 0$ so that $\overline{V} \subseteq U$ and $f(q) - p_0 \neq 0$ for all $q \in \overline{V} \setminus S$. Let $\varepsilon > 0$ be such that $|f(q) - p_0| \geq 3\varepsilon$ for all $q \in \partial V$. For all $p \in B(p_0, \varepsilon)$ and all $q \in \partial V$, we get

$$|f(q) - p| \geq |f(q) - p_0| - |p - p_0| \geq 3\varepsilon - \varepsilon = 2\varepsilon,$$

while at $q_0 \in V$ we have

$$|f(q_0) - p| = |p_0 - p| \leq \varepsilon.$$

Thus, $|f(q) - p|$ must have a local minimum point in $V \subseteq \Omega$. By the Minimum Modulus Principle 7.3, there exists $q \in V$ such that $f(q) - p = 0$. □

Before stating the Open Mapping Theorem, we need to introduce the following definition.

Definition 7.5. Let f be a regular function on a symmetric slice domain Ω. We define the degenerate set of f as the union D_f of the 2-spheres $S = x + y\mathbb{S}$ (with $y \neq 0$) such that $f_{|S}$ is constant.

The topological properties of the degenerate set can be studied by means of Theorem 7.4.

Proposition 7.6. *Let f be a regular function on a symmetric slice domain Ω. If f is not constant, then D_f is closed in $\Omega \setminus \mathbb{R}$, and it has empty interior.*

Proof. By Formula (1.9), there exist C^∞ functions b, c such that $f(x + yI) = b(x, y) + Ic(x, y)$. We observe that $D_f = \Gamma \setminus \mathbb{R}$, where $\Gamma = \{x + y\mathbb{S} : c(x, y) = 0\}$ is the union of all $x + y\mathbb{S}$ such that $c(x, y) = 0$. Γ is clearly closed in Ω, so that D_f is closed in the relative topology of $\Omega \setminus \mathbb{R}$. We are left with proving that the interior of Γ is empty. If it were not, then it would be a symmetric open set having non-open image: indeed, $f(x + yI) = b(x, y)$ for all $x + yI \in \Gamma$ and the image through b of a nonempty subset of $\mathbb{R}^2$ cannot be open in $\mathbb{H}$. By Theorem 7.4, f would have to be constant. By hypothesis this is excluded, and the thesis follows. □

We are now ready for the main result of this section.

Theorem 7.7 (Open Mapping). *Let f be a regular function on a symmetric slice domain Ω and let D_f be its degenerate set. Then $f : \Omega \setminus \overline{D_f} \to \mathbb{H}$ is open.*

Proof. Consider an open subset $U \subseteq \Omega \setminus \overline{D_f}$ and let $p_0 \in f(U)$. Let us show that the image $f(U)$ contains a ball $B(p_0, \varepsilon)$ with $\varepsilon > 0$. Choose $q_0 \in U$ such that $f(q_0) = p_0$. By construction, U does not intersect any degenerate sphere. Thus the point q_0 is an isolated zero of the function $f(q) - p_0$. We may thus choose $r > 0$ such that $\overline{B(q_0, r)} \subseteq U$ and $f(q) - p_0 \neq 0$ for all $q \in \overline{B(q_0, r)} \setminus \{q_0\}$. Let $\varepsilon > 0$ be such that $|f(q) - p_0| \geq 3\varepsilon$ for all q such that $|q - q_0| = r$. For all $p \in B(p_0, \varepsilon)$ and for all $|q - q_0| = r$, we get

$$|f(q)-p| \geq |f(q)-p_0|-|p-p_0| \geq 3\varepsilon-\varepsilon = 2\varepsilon > \varepsilon \geq |p_0-p| = |f(q_0)-p|.$$

Thus $|f(q_0) - p| < \min_{|q-q_0|=r} |f(q) - p|$ and $|f(q) - p|$ must have a local minimum point $q_1 \in B(q_0, r)$. Since $f(q) - p$ is not constant, it must vanish at q_1 by Theorem 7.3. Hence $f(q_1) = p$. This proves that $f(U) \supseteq B(p_0, \varepsilon)$, as desired. □

We conclude this section with an example, which proves that the nondegeneracy hypothesis cannot be removed.

Example 7.8. The quaternionic polynomial $f(q) = q^2+1$ is constant on all spheres $y\mathbb{S} \subset \mathbb{H}$. More precisely,

$$D_f = \{q \in \mathbb{H} : Re(q) = 0\} \setminus \{0\}.$$

By Theorem 7.7, f is open on $\{q \in \mathbb{H} : Re(q) > 0\}$ and on $\{q \in \mathbb{H} : Re(q) < 0\}$. However, $f : \mathbb{H} \to \mathbb{H}$ is not open. Indeed, if $I \in \mathbb{S}$, then the image of the open ball $B = B(I, 1/2)$ centered at I is not open: $0 \in f(B)$ and $f(B) \cap L_J \subseteq \mathbb{R}$ for all $J \in \mathbb{S}$ orthogonal to I. We point out that, nevertheless, $f(\mathbb{H}) = \mathbb{H}$ is an open set, as predicted by Theorem 7.4.

7.3 Real Parts of Regular Functions

In this section we prove a maximum principle and an identity principle, for the real part of a regular function. We then characterize those (real-valued) functions on $\mathbb{H}$ which are real parts of regular functions.

The real part $Ref(q)$ of a regular function f is not (in general) a harmonic function, when considered as a real-valued function of four real variables. Instead, for all $I \in \mathbb{S}$, the real part $Ref_I(x + yI)$ of the restriction f_I of f to the complex plane L_I is a harmonic function of the two variables x, y. As a consequence, phenomena that are impossible for complex holomorphic functions can happen in the setting of quaternionic regular functions. For example, there are nonconstant regular functions whose real part is constant on a slice L_I for some $I \in \mathbb{S}$.

Example 7.9. Set as usual $q = x_0 + x_1 i + x_2 j + x_3 k$. The real part of the regular function $f(q) = qj$ is $-x_2$, which is constant (and in fact equal to zero) on the complex plane L_i. It is interesting to note that in fact the real part of f is constant also on the complex plane L_k and on all complex planes $L_{i\alpha+k\beta}$ with $\alpha^2+\beta^2=1$ generated by 1 and by any real linear combinations of i and k. One may ask what is the minimum number of "independent" slices L_M for $M\in\mathbb{S}$, on which the real part needs to be a constant for the function itself to be constant. The answer is given by the following result.

Theorem 7.10. *Let f be a regular function on* $B(0,r)$*. If there exist three linearly independent vectors* I_1, I_2, I_3 *in* $\mathbb{S}$ *such that* $Re(f)$ *is constant on the three slices* $B(0,r)\cap L_{I_t}, t=1,2,3,$ *then* f *is constant on* $B(0,r)$.

Proof. By the Representation Formula (1.9), for all $x,y\in\mathbb{R}$ such that $x+y\mathbb{S}\in B(0,r)$ there are quaternions b,c such that $f(x+yI)=b+Ic$ for all $I\in\mathbb{S}$. Now we note that since $Re(f(x+yI_t))=Re(b+I_tc)=Re(f(x-yI_t))=Re(b-I_tc)$, we immediately deduce that $Re(I_tc)=0$ for $t=1,2,3$. On the other hand, the real part of I_tc is nothing but the scalar product between c and $-I_t$ as vectors of $\mathbb{R}^4$. Since the I_t's are all purely imaginary, $Re(I_tc)$ coincides with the scalar product between the imaginary part $Im(c)$ of c and $-I_t$ in $\mathbb{R}^3$. As a consequence $Im(c)$ is orthogonal to the three linearly independent vectors I_1, I_2, I_3; hence, it is the zero vector; in particular $Re(Ic)=0$ for any $I\in\mathbb{S}$. This immediately implies that $Re(f)$ is identically equal to $Re(b)$ in $x+y\mathbb{S}$. Since this argument does not depend on the choice of (x,y), the real part $Re(f)$ is constant on the whole of $B(0,r)$. The Open Mapping Theorem 7.7 leads to the conclusion. □

Two holomorphic functions having the same real part coincide. This holds true for regular functions as well, and as it is apparent from the arguments above, we have the following result.

Corollary 7.11. *If two regular functions have real parts that coincide on three slices* $L_{I_1}, L_{I_2}, L_{I_3}$ *where* I_1, I_2, I_3 *are linearly independent, then they coincide.*

In the hypotheses of Theorem 7.10, the number of independent vectors cannot be reduced, as shown by Example 7.9. One may also look at the conditions imposed by requiring that the real part of a regular function has the same value at several points of a 2-sphere. From this point of view, we have the following result:

Theorem 7.12. *Let f be a regular function on* $B(0,r)$*. Consider a 2-sphere* $x+y\mathbb{S}$ *contained in* $B(0,r)$*, and four points* I_1, I_2, I_3, I_4 *in* $\mathbb{S}$ *such that the vectors* I_2-I_1, I_3-I_1, I_4-I_1 *are linearly independent. If* $Re(f)$ *takes the same value on the points* $x+yI_t, t=1,\dots,4,$ *then* $Re(f)$ *is constant on all of* $x+y\mathbb{S}$.

Proof. As we know, given any (x,y), there are quaternions b,c such that $f(x+yI_t)=b+I_tc, t=1,\dots,4$. Then the hypothesis implies that $Re((I_t-I_1)c)=0, t=2,3,4$. The rest of the proof follows with the same argument as in the previous theorem. □

Notwithstanding the peculiarities of the preceding results, when investigating a Maximum Principle for the real part of a regular function f, the statement that one finds is analogous to the classical one. However, the structure of the proof is different because of the peculiarities of the quaternionic setting.

Theorem 7.13 (Maximum Principle). *Let $f : B(0,r) \to \mathbb{H}$ be a regular function on the open ball $B(0,r)$. If $Re(f)$ attains its maximum at a point q_0, then f is constant in $B(0,r)$.*

Proof. Let $q_0 = x_0 + y_0 I_0$. The real part $Re(f_{I_0}(x + yI_0))$ of the restriction f_{I_0} of f to the complex plane L_{I_0} is a harmonic function of the two variables x, y. Thus, $Re(f_{I_0})$ is constant in $L_{I_0} \cap B(0,r)$. In particular $Re(f_{I_0})$ attains its maximum at the point $x_0 \in L_{I_0} \cap \mathbb{R}$, which belongs to $L_I \cap B(0,r)$ for all $I \in \mathbb{S}$. The conclusion follows from Theorem 7.10. □

As a direct consequence of the Maximum Principle for the real part of a regular function, we can state that:

Corollary 7.14. *The function $A_f(r) = \max\{Re(f(q)) : q \leq r\}$ is a strictly increasing function of r, for all r such that f is regular in a neighborhood of $\overline{B(0,r)}$.*

As we mentioned before, the real part $Re(f(q))$ of a regular function f is not (in general) a harmonic function, when considered as a real-valued function of four real variables. In complex analysis, we know that the real part of a holomorphic function is harmonic and, conversely, every harmonic function is the real part of a holomorphic function (which can be found constructing the so-called harmonic conjugate of the real part). It is therefore natural to ask whether one can characterize those functions which are the real parts of regular functions.

Definition 7.15. A harmonic function $u(x, y)$ on the open disk $\Delta(0,r) \subset \mathbb{R}^2$ is said to be x-*simple* if its harmonic conjugate vanishes for $y = 0$, and y-*simple* if its harmonic conjugate vanishes for $x = 0$.

We need the following simple lemma, where $im : \mathbb{C} \to \mathbb{R}$ denotes the complex imaginary part (not to be confused with $Im : \mathbb{H} \to \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$).

Lemma 7.16. *Let u be an* x-*simple harmonic function on $\Delta(0,r)$. Then it can be written as*

$$u(x, y) = \sum_{n \in \mathbb{N}} Re((x + yi)^n) a_n$$

with $a_n \in \mathbb{R}$. Similarly, if u is y-*simple, it can be written as*

$$u(x, y) = \sum_{n \in \mathbb{N}} im((x + yi)^n) b_n,$$

with $b_n \in \mathbb{R}$.

Proof. Consider the harmonic conjugate $\widetilde{u}$ of u and the holomorphic function $F(x+yi) := u(x,y) + \widetilde{u}(x,y)i$. Since u is x-simple, the function F maps the real axis into itself, and therefore it expands into a power series with real coefficients, namely,

$$F(x+yi) = \sum_{n\in\mathbb{N}} (x+yi)^n a_n.$$

Therefore its real part $u(x,y)$ is given exactly by

$$u(x,y) = \sum_{n\in\mathbb{N}} Re((x+yi)^n)a_n. \tag{7.2}$$

Notice that since $|Re((x+yi)^n)| \leq |(x+yi)|^n$ for all n, the series of harmonic functions (7.2) converges on all of $\Delta(0,r)$ to a harmonic function. The case of y-simple functions is dealt with in the same way. □

We are now ready to characterize the real parts of regular functions.

Theorem 7.17. *Let $f : B = B(0,r) \to \mathbb{H}$ be a regular function. Then there is an* x-*simple harmonic function u_0 on $\Delta(0,r)$, and three* y-*simple harmonic functions u_1, u_2, u_3 on $\Delta(0,r)$ such that, for every point $x+yI$ in B, the real part of f can be expressed as:*

$$Re(f(x+yI)) = u_0(x,y) - \langle I, u_1(x,y)i + u_2(x,y)j + u_3(x,y)k\rangle, \tag{7.3}$$

where, as before, $\langle\cdot,\cdot\rangle$ denotes the scalar product of $\mathbb{R}^4$. Specifically, if $f(q) = \sum_{n\in\mathbb{N}} q^n a_n$, with $a_n = a_n^0 + a_n^1 i + a_n^2 j + a_n^3 k$, then the functions $u_t, t = 0,\ldots,3$ can be expressed as

$$u_0(x,y) = \sum_{n\in\mathbb{N}} Re((x+yi)^n)a_n^0$$

$$u_1(x,y) = \sum_{n\in\mathbb{N}} im((x+yi)^n)a_n^1$$

$$u_2(x,y) = \sum_{n\in\mathbb{N}} im((x+yi)^n)a_n^2$$

$$u_3(x,y) = \sum_{n\in\mathbb{N}} im((x+yi)^n)a_n^3.$$

Proof. To prove the assertions, we will use Representation Formula (1.9), according to which, for all $x+yI \in B$, we have:

$$f(x+yI) = \frac{1}{2}\Big[f(x+yi) + f(x-yi)\Big] + I\frac{i}{2}\Big[f(x-yi) - f(x+yi)\Big],$$

whence

$$
\begin{aligned}
&Re(f(x+yI)) = \\
&= Re\left(\frac{1}{2}\Big[f(x+yi)+f(x-yi)\Big]\right) + Re\left(I\frac{i}{2}\Big[f(x-yi)-f(x+yi)\Big]\right) \\
&= Re\left(\frac{1}{2}\Big[f(x+yi)+f(x-yi)\Big]\right) - \left\langle I, \frac{i}{2}\Big[f(x-yi)-f(x+yi)\Big]\right\rangle.
\end{aligned}
$$

Now, since

$$
\begin{aligned}
\frac{1}{2}\Big[f(x+yi)+f(x-yi)\Big] &= \frac{1}{2}\Big[\sum_{n\in\mathbb{N}}(x+yi)^n a_n + \sum_{n\in\mathbb{N}}(x-yi)^n a_n\Big] \\
&= \sum_{n\in\mathbb{N}} \frac{(x+yi)^n+(x-yi)^n}{2} a_n \\
&= \sum_{n\in\mathbb{N}} Re((x+yi)^n)a_n,
\end{aligned}
$$

we can set

$$
\begin{aligned}
&u_0(x,y) := \\
&= Re\left(\frac{1}{2}\Big[f(x+yi)+f(x-yi)\Big]\right) = Re\left(\sum_{n\in\mathbb{N}} Re((x+yi)^n)a_n\right) \\
&= \sum_{n\in\mathbb{N}} Re((x+yi)^n)a_n^0.
\end{aligned}
$$

A completely similar computation shows that

$$
\begin{aligned}
\frac{i}{2}\Big[f(x-yi)-f(x+yi)\Big] &= \sum_{n\in\mathbb{N}} im((x+yi)^n)a_n \\
&= \sum_{n\in\mathbb{N}} im((x+yi)^n)(a_n^0 + a_n^1 i + a_n^2 j + a_n^3 k).
\end{aligned}
$$

Since I is purely imaginary, the real part of the last sum can be omitted when writing

$$
\left\langle I, \frac{i}{2}\Big[f(x-yi)-f(x+yi)\Big]\right\rangle.
$$

Therefore if we define

$$
u_1(x,y) = \sum_{n\in\mathbb{N}} im((x+yi)^n)a_n^1
$$

$$u_2(x, y) = \sum_{n \in \mathbb{N}} im((x + yi)^n) a_n^2$$

$$u_3(x, y) = \sum_{n \in \mathbb{N}} im((x + yi)^n) a_n^3,$$

and if we notice as before that $u_t(x, y), t = 1, 2, 3$ are harmonic real-valued functions, then (7.3) is proved. □

We will now prove the converse of the previous result. This will also clarify in which sense four real-valued simple harmonic functions determine a regular function.

Theorem 7.18. *Let $u_0 : \Delta(0, r) \to \mathbb{R}$ be an* x*-simple harmonic function, and let $u_t : \Delta(0, r) \to \mathbb{R}$, $(t = 1, 2, 3)$, be three* y*-simple harmonic functions. Then there exists a regular function $f : B(0, r) \to \mathbb{H}$, unique up to additive constant, such that, for all $x + yI \in B(0, r)$, the following equality holds:*

$$Re(f(x + yI)) = u_0(x, y) - \langle I, u_1(x, y)i + u_2(x, y)j + u_3(x, y)k \rangle \tag{7.4}$$

Proof. Write the four simple harmonic functions as in Lemma 7.16. Namely, we have

$$u_0(x, y) = \sum_{n \in \mathbb{N}} Re((x + yi)^n) a_n^0$$

$$u_1(x, y) = \sum_{n \in \mathbb{N}} im((x + yi)^n) a_n^1$$

$$u_2(x, y) = \sum_{n \in \mathbb{N}} im((x + yi)^n) a_n^2$$

$$u_3(x, y) = \sum_{n \in \mathbb{N}} im((x + yi)^n) a_n^3,$$

for some sequence of real coefficients $\{a_n^t\}$, $t = 0, 1, 2, 3$. Define

$$f(q) = \sum_{n \in \mathbb{N}} q^n (a_n^0 + a_n^1 i + a_n^2 j + a_n^3 k).$$

The result follows immediately. □

Let f be regular. By the Splitting Lemma 1.3, we can write

$$f_I(x + yI) = F(x + yI) + G(x + yI)J,$$

for some $J \in \mathbb{S}$ orthogonal to I. The knowledge of the four simple harmonic functions we have just described allows us to directly compute F and G by setting

$$Re(F(x + yi)) = u_0(x, y) - u_1(x, y),$$

i.e.,

$$F(x+yi) = \sum_{n\in\mathbb{N}} (x+yi)^n (a_n^0 + a_n^1 i),$$

and

$$Re(G(x+yi)) = \sum_{n\in\mathbb{N}} Re((x+yi)^n) a_n^2 - u_3(x,y),$$

i.e.,

$$G(x+yi) = \sum_{n\in\mathbb{N}} (x+yi)^n (a_n^2 + a_n^3 i).$$

The following result illustrates a special case of Theorem 7.17.

Corollary 7.19. *Let $w : B = B(0,r) \to \mathbb{R}$ be a function such that, for all $I \in \mathbb{S}$, the function $(x,y) \mapsto w_I(x+yI)$ is* x*-simple. Then there is a slice preserving regular function $f : B \to \mathbb{H}$, unique up to an additive constant, whose real part is w. Conversely, the real part of any slice preserving regular function f satisfies the hypotheses above.*

Proof. These hypotheses correspond to the case in which w is the real part of a slice preserving regular function, since we know that, in this case,

$$w_I(x+yI) = \sum_{n\in\mathbb{N}} Re((x+yI)^n) b_n,$$

for some real b_n which do not depend on $I \in \mathbb{S}$. The converse is immediate. $\square$

We conclude with the following remark, which illustrates a second special case.

Remark 7.20. As we stated in Corollary 7.19, the case of a regular function f extending to B a (real-valued) real analytic function defined on $B \cap \mathbb{R}$ is a special case of Theorem 7.17 with $u_1 \equiv u_2 \equiv u_3 \equiv 0$. Instead, the case of a regular function f extending to B a holomorphic function defined on $B \cap L_i$ is a special case of Theorem 7.17 with $u_2 \equiv u_3 \equiv 0$.

7.4 Phragmén–Lindelöf Principles

In this section we present Phragmén–Lindelöf principles for regular functions. These principles generalize Theorem 7.2 to the unbounded case, replicating for the quaternions the classical results of Phragmén, Lindelöf, and Ahlfors of [3, 105, 106] (see [90] Chap. 1, § 14 for a modern presentation).

Definition 7.21. Let $\widehat{\mathbb{H}} = \mathbb{H} \cup \{\infty\}$ denote the Alexandroff compactification of $\mathbb{H}$. The *extended boundary* $\partial_\infty U$ of any $U \subseteq \widehat{\mathbb{H}}$ is the boundary of the closure of U in $\widehat{\mathbb{H}}$. As customary, we denote by $\partial U = \partial_\infty U \setminus \{\infty\}$ the finite boundary of U.

We prove a first version of the Phragmén–Lindelöf principle directly, using techniques (inspired by the complex case) that involve the quaternionic logarithm defined in Sect. 4.2.

Theorem 7.22 (Phragmén–Lindelöf Principle). *Let $\Omega \subset \mathbb{H}$ be a domain whose extended boundary contains the point at infinity; suppose that there exists a real point $t \in \mathbb{R} \cap \Omega$ such that $\Omega \setminus (-\infty, t]$ (or $\Omega \setminus [t, +\infty)$) is a slice domain. If f is a bounded regular function on Ω and $\limsup_{q \to q_0} |f(q)| \leq M$ for all $q_0 \in \partial\Omega$, then $|f(q)| \leq M$ for all $q \in \Omega$.*

Proof. Since $q \mapsto q + t$ and $q \mapsto -q$ are slice preserving regular functions, by Lemma 1.32, we can assume that $t = 0$ and that $\Omega \setminus (-\infty, 0]$ is a slice domain. Choose $r > 0$ such that the closure of $B_r = B(0, r)$ is contained in Ω and let $\omega_r(q) = q^{-1}r$ for $q \neq 0$. Notice that $|\omega_r| < 1$ in $\mathbb{H} \setminus \overline{B_r}$, that $|\omega_r| = 1$ on ∂B_r, and that ω_r is slice preserving.

Recall that the principal branch of the quaternionic logarithm Log is a slice preserving regular function (see Sect. 4.2). By Lemma 1.32, setting $\omega_r^\delta(q) := e^{\delta \mathrm{Log}\omega_r(q)}$ for all $q \in \mathbb{H} \setminus (-\infty, 0]$ defines a regular function. Finally, by Lemma 1.30 the product $\omega_r^\delta f$ is a regular function on $\Omega' = \Omega \setminus (\overline{B_r} \cup (-\infty, r])$, which by hypothesis is a slice domain when r is sufficiently small. The behavior of $|\omega_r^\delta f|$ on the extended boundary $\partial_\infty \Omega' = \{\infty\} \cup \partial\Omega \cup \partial B_r \cup (\Omega \cap (-\infty, r])$ is the following:

1. $\limsup_{q \to \infty} |\omega_r^\delta f(q)| = \limsup_{q \to \infty} |f(q)| \frac{r^\delta}{|q|^\delta} = 0$
2. $\limsup_{q \to q_0} |\omega_r^\delta f(q)| < \limsup_{q \to q_0} |f(q)| \leq M$ for all $q_0 \in \partial\Omega$
3. $\limsup_{q \to q_0} |\omega_r^\delta f(q)| = |f(q_0)| \leq \max_{\partial B_r} |f| =: M_r$ for all $q_0 \in \partial B_r$
4. $\limsup_{q \to q_0} |\omega_r^\delta f(q)| \leq \sup_{\Omega \cap (-\infty, r]} \left(\frac{r^\delta}{|q|^\delta} |f(q)| \right) =: N$ for all $q_0 \in \Omega \cap (-\infty, r]$

Let us prove that N is finite. Choose $\{p_n\}_{n \in \mathbb{N}} \subset \overline{\Omega} \cap (-\infty, r]$ such that $\lim_{n \to \infty} \frac{r^\delta}{|p_n|^\delta} |f(p_n)| = N$. If $N = 0$, there is nothing to prove. Otherwise, by point 1, $\{p_n\}_{n \in \mathbb{N}}$ must be bounded. By possibly extracting a subsequence, we may suppose $\{p_n\}_{n \in \mathbb{N}}$ to converge to some $q_0 \in \overline{\Omega} \cap (-\infty, r]$. If $q_0 \in \partial\Omega$, then $N \leq M$ by hypothesis. Else $q_0 \in \Omega$ and $N = \frac{r^\delta}{|q_0|^\delta} |f(q_0)|$.

As a consequence of points 1–4, $\limsup_{q \to q_0} |\omega_r^\delta f| \leq \max\{M, M_r, N\}$ for all $q_0 \in \partial_\infty \Omega'$, and by an easy application of the Maximum Modulus Principle 7.1, $|\omega_r^\delta f| \leq \max\{M, M_r, N\}$ in Ω'.

Now let us prove that $N \leq \max\{M, M_r\}$. Suppose by contradiction that the opposite inequality holds. In particular $N > M$ and (as we explained above), there exists $q_0 \in \Omega \cap (-\infty, r]$ such that $N = \frac{r^\delta}{|q_0|^\delta} |f(q_0)|$. In a ball $B(q_0, \varepsilon)$ contained in $\Omega \setminus \overline{B_r}$, we define a new branch of logarithm by letting

$$\log(q) = \ln |q| + \left[\arccos\left(\frac{Re(q)}{|q|} \right) - \pi \right] \frac{Im(q)}{|Im(q)|}.$$

As before, the function $g = e^{\delta \log \omega_r} f$ is regular in $B(q_0, \varepsilon)$ and $|g(q)| = \frac{r^\delta}{|q|^\delta}|f(q)|$ for all $q \in B(q_0, \varepsilon)$. As a consequence, $|g(q_0)| \geq |g(q)|$ for all $q \in B(q_0, \varepsilon)$. Indeed:

1. For all $q \in (q_0 - \varepsilon, q_0 + \varepsilon)$, $|g(q)| \leq \sup_{\Omega \cap (-\infty, r]} \left(\frac{r^\delta}{|q|^\delta}|f(q)|\right) = N = |g(q_0)|$.
2. For all $q \in B(q_0, \varepsilon) \setminus (q_0 - \varepsilon, q_0 + \varepsilon)$, we proved $|g(q)| = |\omega_r^\delta(q) f(q)| = \frac{r^\delta}{|q|^\delta}|f(q)| \leq \max\{M, M_r, N\} = N = |g(q_0)|$.

Hence, $|g|$ has a maximum at q_0, and g must be constant. Therefore, $|\omega_r^\delta f| = |g| \equiv N$ in $B(q_0, \varepsilon) \setminus (q_0 - \varepsilon, q_0 + \varepsilon)$. In particular $\omega_r^\delta f$, which is a regular function on the slice domain Ω', has an interior maximum point. As before, the Maximum Modulus Principle 7.1 yields that $\omega_r^\delta f$ must be constant. As a consequence, there exists a constant c such that $f(q) = q^\delta c$ in Ω', a contradiction with the hypothesis that f is bounded.

So far, we proved that $|\omega_r|^\delta |f| \leq \max\{M, M_r\}$ in $\Omega \setminus \overline{B_r}$. We deduce that

$$|f| \leq \frac{\max\{M, M_r\}}{|\omega_r|^\delta}$$

in $\Omega \setminus \overline{B_r}$, and letting $\delta \to 0^+$, we conclude that $|f| \leq \max\{M, M_r\}$ in $\Omega \setminus \overline{B_r}$.

If we let $r \to 0^+$, we obtain $|f(q)| \leq \max\{M, |f(0)|\}$ for all $q \in \Omega \setminus \{0\}$, hence for all $q \in \Omega$. Finally, we prove that $|f(0)| \leq M$: if it were not so, then $|f|$ would have a maximum at 0, a contradiction by the Maximum Modulus Principle. □

The proof we just gave is intrinsic to the quaternionic setting. We now prove variants of Theorem 7.22, employing a "slicewise" approach instead. Let U be a domain whose extended boundary $\partial_\infty U$ contains the point at infinity and assume U_I is simply connected for all $I \in \mathbb{S}$. We will prove that if f is a bounded regular function on U whose modulus is bounded by M near every point of the finite boundary ∂U, then $|f| \leq M$ in U. In the special case where every slice Ω_I is an angle or a strip in L_I, we do not even need to suppose f to be bounded: a weaker hypothesis is sufficient. Moreover, we will prove that the inequality $|f| \leq M$ is strict when U is a slice domain, unless f is constant.

Theorem 7.23 (Phragmén–Lindelöf Principle). *Let U be a domain whose extended boundary $\partial_\infty U$ contains the point at infinity and such that, for all $I \in \mathbb{S}$, U_I is simply connected. If f is a bounded regular function on U and if, for all $q_0 \in \partial U$,*

$$\limsup_{q \to q_0} |f(q)| \leq M \tag{7.5}$$

then $|f| \leq M$ in U. If, moreover, f is not constant and $U \cap \mathbb{R} \neq \emptyset$, then the inequality is strict.

Proof. Suppose that, for some $p \in U$, $|f(p)| > M$. Without loss of generality, we will assume $f(p) > 0$. Let $I \in \mathbb{S}$ be such that $p \in L_I$, choose $J \in \mathbb{S}$ such that $J \perp I$, and let $F, G : U_I \to L_I$ be holomorphic functions such that $f_I = F + GJ$.

Then $f_I(p) = F(p)$. By the complex Phragmén–Lindelöf principle (see [90]), from the fact that $\limsup_{z\to z_0} |F(z)| \le \limsup_{z\to z_0} |f(z)| \le M$ for all $z \in \partial U_I$ and the fact that $|F| \le |f_I|$ is bounded in U_I, we derive $|F| \le M$ in U_I. We then get a contradiction with the assumption $|F(p)| = |f_I(p)| > M$.

Now let us suppose $U \cap \mathbb{R} \neq \emptyset$ so that U is a slice domain. We already know that $|f| \le M$ in U, so if there exists a $p \in U$ such that $|f(p)| = M$, then $|f|$ has a relative maximum at p, and we conclude that f is constant by Theorem 7.1. □

With the same technique, the Phragmén–Lindelöf principle for complex angles (see, e.g., [90]) extends to the quaternionic case. We begin with some definitions.

Definition 7.24. We call a slice domain $\Omega \subset \mathbb{H}$ an *angular domain* if, for all $I \in \mathbb{S}$, the domain $\Omega_I \subseteq L_I$ is an angle $\{re^{I(\zeta_I+\vartheta)} : r > 0, |\vartheta| < \varphi_I/2\}$ for some ζ_I, φ_I with $\zeta_I \in \mathbb{R}, 0 < \varphi_I < 2\pi$. The *opening* of Ω is defined to be $\sup_{I\in\mathbb{S}} |\varphi_I|$.

The class of the angular domains includes all *circular cones*

$$C(\varphi) = \{re^{I\vartheta} : r > 0, |\vartheta| < \varphi/2, I \in \mathbb{S}\}$$

with $0 < \varphi < 2\pi$. Furthermore, the following nice topological property holds.

Proposition 7.25. *Let Ω be an open subset of $\mathbb{H}$ such that, for all $I \in \mathbb{S}$, Ω_I is an angle $\{re^{I(\zeta_I+\vartheta)} : r > 0, |\vartheta| < \varphi_I/2\}$. If $I \mapsto \zeta_I$ is continuous in $\mathbb{S}$, then Ω is automatically a slice domain.*

Proof. In order to prove our assertion, it suffices to show that at least one slice Ω_I contains a real half line. Notice that, for all $I \in \mathbb{S}$, we have the equality $\Omega_I = \Omega_{-I}$. Hence $\zeta_I = 2k\pi - \zeta_{-I}$ for some $k \in \mathbb{Z}$. Since

$$I \mapsto (\zeta_I, 0)$$

is a continuous function from $\mathbb{S} \simeq S^2$ to $\mathbb{R}^2$, by the Borsuk–Ulam Theorem (see Corollary 9.3 in Chap. V of [97]), there exist two antipodal points of $\mathbb{S}$ having the same image. As a consequence, there exists a $J \in \mathbb{S}$ such that $\zeta_J = \zeta_{-J}$, and we conclude that $\zeta_J = k\pi$ for some $k \in \mathbb{Z}$. □

Definition 7.26. If f is a regular function on an angular domain Ω, continuous up to the boundary, setting $M_f(r, \Omega) = \max\{|f(q)| : q \in \overline{\Omega}, |q| = r\}$, the *order* ρ of f is defined as

$$\rho = \limsup_{r\to+\infty} \frac{\ln^+ \ln^+ M_f(r, \Omega)}{\ln r}. \tag{7.6}$$

If f has order ρ, then its *type* σ is defined as

$$\sigma = \limsup_{r\to+\infty} \frac{\ln^+ M_f(r, \Omega)}{r^\rho}. \tag{7.7}$$

Theorem 7.27. *Let Ω be an angular domain of opening $\frac{\pi}{\alpha}$. Let f be a regular function on Ω, continuous up to the boundary and having order $\rho < \alpha$. If $|f| \leq M$ in $\partial\Omega$, then $|f| \leq M$ in Ω. If, moreover, f is not constant, then the inequality is strict.*

Proof. The proof is completely analogous to that of Theorem 7.23, making use of the Phragmén–Lindelöf principle for complex angles (see [90]). □

The hypothesis that $\rho < \alpha$ cannot be weakened, as proven by the following Example.

Example 7.28. Consider the circular cone $\Omega = C\left(\frac{\pi}{\rho}\right)$, which is an angular domain of opening $\frac{\pi}{\rho}$. Set $f(q) = e^{q^\rho}$ where $q^\rho = e^{\rho \mathrm{Log}(q)}$. Then f is a regular function of order $\rho > 0$ on Ω. We notice that, for all $q = re^{I\frac{\pi}{2\rho}} \in \partial\Omega$, $|f(q)| = |\exp(r^\rho e^{I\frac{\pi}{2}})| = |\exp(Ir^\rho)| = 1$, while the function f is unbounded in Ω.

However, when a function f has order ρ in angular domain of opening $\frac{\pi}{\rho}$, we can still control the growth of f in terms of its type.

Theorem 7.29. *Let Ω be an angular domain whose opening is not greater than $\frac{\pi}{\rho}$ so that, for all $I \in \mathbb{S}$, there exists $\varphi_I \leq \frac{\pi}{\rho}$ such that $\Omega_I = \{re^{I(\zeta_I+\vartheta)} : r > 0, |\vartheta| < \varphi_I/2\}$. If f is a regular function of order ρ and type σ on Ω, continuous up to the boundary, and if $|f|$ is bounded by M in $\partial\Omega$, then for all $I \in \mathbb{S}$*

$$|f(re^{I(\zeta_I+\vartheta)})| \leq Me^{\sigma r^\rho \cos(\rho\vartheta)} \tag{7.8}$$

for $r > 0$ and $|\vartheta| < \varphi_I/2$.

Proof. Let $p = re^{I(\zeta_I+\vartheta)} \in \Omega_I$ be such that

$$|f(p)| > Me^{\sigma r^\rho \cos(\rho\vartheta)}.$$

As usual, we may suppose $f(p) > 0$. We choose $J \in \mathbb{S}$ with $J \perp I$ and holomorphic functions $F, G : \Omega_I \to L_I$ such that $f_I = F + GJ$, so that $f_I(p) = F(p)$ and $|F(p)| > Me^{\sigma r^\rho \cos(\rho\vartheta)}$. This is impossible: as explained in Theorem 22 of [90], if $|F| \leq M$ in $\partial\Omega_I$ and if F has order less than or equal to ρ and type less than or equal to σ in Ω_I, then $|F(re^{I(\zeta_I+\vartheta)})| \leq Me^{\sigma r^\rho \cos(\rho\vartheta)}$. □

Let us now state the quaternionic version of the Phragmén–Lindelöf principle for complex strips.

Definition 7.30. A domain $\Omega \subset \mathbb{H}$ is a *strip domain* if, for all $I \in \mathbb{S}$, there exist a line ℓ_I in the plane L_I and a positive real number γ_I such that Ω_I is the strip $\{z \in L_I : d(z, \ell_I) < \gamma_I/2\}$. The number $\sup_{I\in\mathbb{S}} |\gamma_I|$ is called the *width* of Ω.

Theorem 7.31. *Let f be a regular function on a strip domain Ω of width γ, continuous up to the boundary. Suppose that $|f(q)| \leq N \exp(e^{k|q|})$ in Ω for some* *positive constants N and $k < \frac{\pi}{\gamma}$. If there exists an $M \geq 0$ such that $|f| \leq M$ in $\partial\Omega$, then $|f| \leq M$ in Ω. If, moreover, f is not constant and $\Omega \cap \mathbb{R} \neq \emptyset$, then the inequality is strict.*

Proof. The proof is completely analogous to that of Theorem 7.23, making use of the Phragmén–Lindelöf principle for complex strips (see [90]). □

We now characterize those strip domains that are not slice domains.

Proposition 7.32. *Let Ω be a strip domain. If Ω is not a slice domain, then there exist a function $I \mapsto \gamma_I$ on $\mathbb{S}$ and a discontinuous function $\psi : \mathbb{S} \to \mathbb{R}$ such that $\Omega_I = \{x + yI \in L_I : |y - \psi(I)| < \gamma_I/2\}$ for all $I \in \mathbb{S}$.*

Proof. If Ω is not a slice domain, then the strips Ω_I do not intersect $\mathbb{R}$. Hence, they must be parallel to $\mathbb{R}$, i.e., for all $I \in \mathbb{S}$

$$\Omega_I = \{x + yI \in L_I : |y - \psi(I)| < \gamma_I/2\}$$

for some $\psi(I) \in \mathbb{R}, \gamma_I > 0$ such that $|\psi(I)| \geq \gamma_I/2$. Since $\Omega_I = \Omega_{-I}$, we must have $\psi(I) = -\psi(-I)$ for all $I \in \mathbb{S}$. If the function $\psi : \mathbb{S} \to \mathbb{R}$ were continuous, then

$$I \mapsto (\psi(I), 0)$$

would be a continuous function from $\mathbb{S} \simeq S^2$ to $\mathbb{R}^2$. By the Borsuk–Ulam Theorem, there would exist two antipodal points of $\mathbb{S}$ having the same image, i.e., $\pm J \in \mathbb{S}$ such that $\psi(J) = \psi(-J)$. Putting this together with $\psi(J) = -\psi(-J)$, we would conclude that $\psi(J) = 0$, so that Ω_J would contain $\mathbb{R}$, a contradiction with the hypothesis. □

If we extend in the obvious way the definitions of order and type to entire functions, then the Liouville Theorem 6.9 generalizes as follows.

Theorem 7.33. *Let f be an entire function of order $\rho \leq 1$ and type $\sigma = 0$. In other words, for all $\varepsilon > 0$, we suppose $|f(q)| < e^{\varepsilon|q|}$ when $|q|$ is large enough. If, for some $I \in \mathbb{S}$, the plane L_I contains a line on which $|f|$ is bounded, then f is constant.*

Proof. Let $J \in \mathbb{S}$ be orthogonal to I and let $F, G : L_I \to L_I$ be holomorphic functions such that $f_I = F + GJ$. By the corollary to Theorem 22 in [90], F and G are constant. Hence f_I is constant and, by the Identity Principle 1.12, f must be constant too. □

7.5 An Ehrenpreis–Malgrange Lemma

An important topic in the theory of holomorphic functions is the study of bounds for holomorphic quotients of a holomorphic function and a polynomial. These bounds, which have important applications to the theory of differential equations, are usually rooted in some subtle lower bounds for the moduli of polynomials away from their zeros. Among the most important results of this type, one recalls the so-called Ehrenpreis–Malgrange Lemma [47] and the Cartan Minimum Modulus Theorem [15, 90].

We begin here with a simple result, where we find a lower bound on a 3-sphere centered at the origin. As it will become apparent, this is a very special case, since in general bounds exist on 3-dimensional toroidal hypersurfaces.

Theorem 7.34. *Let $P(q)$ be a regular polynomial of degree m, whose leading coefficient we denote by a_m. Let p be the number of distinct spherical zeros of $P(q)$, and let t be the number of distinct isolated zeros. Let $M = p + t$. Given any $R > 0$, we can find a 3-sphere Γ (centered at the origin and having radius $r < R$) on which*

$$|P(q)| \geq |a_m| \left(\frac{R}{2(M+1)} \right)^m .$$

Proof. By Theorem 3.22, we can write $P(q) = S(q)Q(q)a_m$ with

$$S(q) = (q^2 - 2qRe(w_1) + |w_1|^2)^{m_1} \cdots (q^2 - 2qRe(w_p) + |w_p|^2)^{m_p},$$

and

$$Q(q) = \mathop{\boxed{\ast}}\limits_{i=1}^{t} \mathop{\boxed{\ast}}\limits_{j=1}^{n_i} (q - \alpha_{ij}).$$

Since the cardinality of the set $V = \{|q| : q \in \mathbb{H}, P(q) = 0\}$ is less than or equal to M, there exists a subinterval $[a, b]$ of $[0, R]$ of length at least $\frac{R}{M+1}$ which does not contain any element of V. Let Γ be the 3-sphere centered at the origin and with radius $\frac{a+b}{2}$.

We now need to estimate from below the absolute value of $P(q)$ at a generic point $q \in \Gamma$. Since $P(q) = S(q)Q(q)a_m$, we need to estimate the absolute values of both $S(q)$ and $Q(q)$ on Γ. In order to estimate $S(q)$, we recall that for any pair of quaternions q and α, we have

$$|q^2 - 2Re(\alpha)q + |\alpha|^2| = |(q-\alpha)||q - (q-\alpha)^{-1}\overline{\alpha}(q-\alpha)| \geq$$

$$\geq \Big||q| - |\alpha|\Big| \cdot \Big||q| - |(q-\alpha)^{-1}\overline{\alpha}(q-\alpha)|\Big| = \Big||q| - |\alpha|\Big| \cdot \Big||q| - |\overline{\alpha}|\Big| = \Big||q| - |\alpha|\Big|^2.$$

We now conclude that

$$|S(q)| = |(q^2 - 2qRe(w_1) + |w_1|^2)^{m_1} \cdots (q^2 - 2qRe(w_p) + |w_p|^2)^{m_p}| =$$

$$= \Big|q^2 - 2qRe(w_1) + |w_1|^2\Big|^{m_1} \cdots \Big|q^2 - 2qRe(w_p) + |w_p|^2\Big|^{m_p} \geq$$

$$\geq \Big||q| - |w_1|\Big|^{2m_1} \cdots \Big||q| - |w_p|\Big|^{2m_p}.$$

To estimate $Q(q)$, we first note that, for suitable quaternions $\alpha_1, \dots, \alpha_N$, we can rewrite $Q(q)$ as $Q(q) = (q - \alpha_1) * \cdots * (q - \alpha_N)$, and the estimate for $Q(q)$ can be obtained recursively as we now proceed to show. It is obvious that

$$|q - \alpha_N| \geq \Big| |q| - |\alpha_N| \Big|.$$

We now assume that, for some integer $\ell < N-1$, we have established for $h_{\ell+1}(q) = (q - \alpha_{\ell+1}) * \cdots * (q - \alpha_N)$ that

$$|h_{\ell+1}(q)| = |(q - \alpha_{\ell+1}) * \cdots * (q - \alpha_N)| \geq \Big| |q| - |\alpha_{\ell+1}| \Big| \cdots \Big| |q| - |\alpha_N| \Big|, \quad (7.9)$$

and we proceed to estimate

$$|h_\ell(q)| = |(q - \alpha_\ell) * \cdots * (q - \alpha_N)|.$$

To this purpose, we recall Theorem 3.4 which implies that

$$|(q - \alpha_\ell) * (q - \alpha_{\ell+1}) * \cdots * (q - \alpha_N)| = |(q - \alpha_\ell) * ((q - \alpha_{\ell+1}) * \cdots * (q - \alpha_N))| =$$
$$= |(q - \alpha_\ell) \cdot h_{\ell+1}((q - \alpha_\ell)^{-1} q (q - \alpha_\ell))| = |(q - \alpha_\ell)| \cdot |h_{\ell+1}((q - \alpha_\ell)^{-1} q (q - \alpha_\ell))|.$$

Since

$$|(q - \alpha_\ell)^{-1} q (q - \alpha_\ell)| = |q|$$

by (7.9), we obtain

$$|(q - \alpha_\ell) * \cdots * (q - \alpha_N)| \geq \Big| |q| - |\alpha_\ell| \Big| \cdots \Big| |q| - |\alpha_N| \Big|,$$

and therefore,

$$|Q(q)| \geq \Big| |q| - |\alpha_1| \Big| \cdots \Big| |q| - |\alpha_N| \Big|.$$

Since each factor in the decomposition of $P(q)$ is bounded from below by $\frac{R}{2(M+1)}$, the thesis follows. □

Note that our proof actually shows that

$$|P(q)| \geq |a_m| \left(\frac{R}{2(M' + 1)} \right)^m$$

where M' is the cardinality of $V = \{|q| : q \in \mathbb{H}, P(q) = 0\}$. We chose to use M instead because it distinguishes the nature of the various roots. In the statement given, the worst case scenario would occur when there are no spherical roots, and all the other roots are distinct. In that situation $M = m$ and so, for every quaternionic polynomial of degree m, the lower bound on Γ is given by

$$|P(q)| \geq |a_m| \left(\frac{R}{2(m+1)} \right)^m .$$

Remark 7.35. The same estimate holds if we center the sphere Γ at any other real point q_0.

The next step is to understand what happens if one attempts to estimate $|P(q)|$ from below, on spheres centered at points q_0 which are not real. As it turns out, this is a much more delicate issue, and the theorem below shows the appropriate modification.

Theorem 7.36. *Let $P(q)$ be a regular polynomial of degree m having only spherical zeros, i.e., a polynomial of the form*

$$P(q) = (q^2 - 2qRe(w_1) + |w_1|^2)^{m_1} \cdots (q^2 - 2qRe(w_p) + |w_p|^2)^{m_p} a_m,$$

with $w_1, \ldots, w_p, a_m \in \mathbb{H}$. For any $u + v\mathbb{S} \subset \mathbb{H}$ and for any $R > 0$, there exist $r < R$ and a 3-dimensional compact hypersurface

$$\Gamma = \Gamma(u + v\mathbb{S}, r) = \{x + yI : (x-u)^2 + (y-v)^2 = r^2 \ \textit{and} \ I \in \mathbb{S}\},$$

(which is smooth if $r < v$), such that for every $q \in \Gamma$

$$|P(q)| \geq |a_m| \left(\frac{R}{2(m+1)} \right)^m .$$

Proof. Without loss of generality, we assume $a_m = 1$. We now consider the restriction of P_I to a complex plane L_I. This restriction is a complex polynomial having at most m distinct zeros. Let $q_0 = u + vI$ and consider the set $V = \{|z - q_0| : z \in L_I, P_I(z) = 0\}$, which has at most m elements. Then we can find a subinterval $[a, b]$ of $[0, R]$ of length at least $\frac{R}{m+1}$ which does not contain any element of V. Let Γ_I be the circle in L_I centered at q_0 and having radius $r = \frac{a+b}{2}$. Then, for all $z \in L_I$, one has the estimate

$$|z^2 - 2zRe(w) + |w|^2| = |(z-w)(z-\overline{w})| = |z-w||z-\overline{w}| =$$

$$= |(z-q_0) - (w-q_0)||(z-q_0) - (\overline{w}-q_0)| \geq \Big| |z-q_0| - |w-q_0| \Big| \cdot \Big| |z-q_0| - |\overline{w}-q_0| \Big| .$$

Since both w and $\bar{w}$ are roots of P_I, we obtain

$$|z^2 - 2zRe(w) + |w|^2| \geq \left(\frac{R}{2(m+1)} \right)^2$$

for all $z \in \Gamma_I$. Therefore

$$|P_I(z)| \geq \left(\frac{R}{2(m+1)} \right)^m$$

for all $z \in \Gamma_I$. Since I is arbitrary, letting Γ be the symmetric completion of Γ_I, the thesis follows. $\square$

Remark 7.37. If u, v, and r are such that Γ does not intersect the real axis, then Γ is homeomorphic to the Cartesian product $S^1 \times S^2$: hence, it is not singular. If Γ intersects (or is tangent to) the real axis (but $v \neq 0$), then Γ is singular at the points of intersection, and it is not a product of spheres any longer. Finally, if $v = 0$, then Γ is a 3-sphere S^3 centered at the real point u.

In order to prove a more general statement, we need the following simple lemma:

Lemma 7.38. *Let $q_0 = u + vI_0$ be a given point in $\mathbb{H}$ and let $a, b \in \mathbb{R}$. Then, for $w \in a + b\mathbb{S}$, the distance $|w - q_0|$ has $w^0 = a + bI_0$ and $\overline{w}^0 = a - bI_0$ as its extremal points.*

Proof. Without loss of generality we may assume $I_0 = i$ so that $q_0 = u + vi$. The generic element $I \in \mathbb{S}$ can be written as $I = \alpha i + \beta j + \gamma k$ with $\alpha^2 + \beta^2 + \gamma^2 = 1$ so that $w = a + b(\alpha i + \beta j + \gamma k)$. The square of the distance between w and q_0 is therefore given by

$$|w - q_0|^2 = a^2 + u^2 - 2au + b^2\alpha^2 + v^2 - 2b\alpha v + b^2\beta^2 + b^2\gamma^2.$$

Since $\mathbb{S}$ is a compact set, we know that $|w - q_0|^2$ has at least a maximum and a minimum for $w \in a + b\mathbb{S}$, which can be found by using the Lagrange multipliers. A quick computation shows that the maximum and minimum are achieved for $\alpha = \pm 1, \beta = \gamma = 0$. $\square$

Theorem 7.39. *Let $P(q)$ be a regular polynomial of degree m having only isolated zeros, i.e., a polynomial of the form*

$$Q(q) = (q - \alpha_1) * \cdots * (q - \alpha_m)a_m,$$

with $\alpha_1, \ldots, \alpha_m, a_m \in \mathbb{H}$. For any $u + v\mathbb{S} \subset \mathbb{H}$ and for any $R > 0$, there exist $r < R$ and a 3-dimensional compact hypersurface $\Gamma = \Gamma(u + v\mathbb{S}, r)$ (which is smooth if $r < v$), such that for every $q \in \Gamma$ it is

$$|Q(q)| \geq |a_m| \left(\frac{R}{2(2m+1)} \right)^m .$$

Proof. Without loss of generality, we assume that $a_m = 1$. Choose $I_0 \in \mathbb{S}$ and set $q_0 = u + vI_0$. For $t = 1, \ldots m$, define $\alpha_t^0 = Re(\alpha_t) + |Im(\alpha_t)|I_0$ and consider the set

$$V = \{|\alpha_t^0 - q_0|, |\overline{\alpha}_t^0 - q_0| \text{ for } t = 1, \ldots, m\}.$$

Given any $R > 0$, there is at least one subinterval $[c, d]$ of $[0, R]$ which does not contain any element of V and whose length is at least $\frac{R}{2m+1}$. Set $r = \frac{c+d}{2}$ and $\Gamma = \Gamma(u + v\mathbb{S}, r)$. We now estimate, for $q \in \Gamma$, the modulus of $Q(p) = (q - \alpha_1) * \cdots * (q - \alpha_m)$. By Theorem 3.4, there are quaternions $\alpha_t' \in S_{\alpha_t} = Re(\alpha_t) + |Im(\alpha_t)|\mathbb{S}$, for $t = 1, \dots, m$, such that

$$|Q(p)| = |q - \alpha_1'| \cdots |q - \alpha_m'|.$$

We are now able to estimate every single factor. Specifically:

$$|q - \alpha_t'| = |(q - q_0) - (\alpha_t' - q_0)| \geq \Big| |q - q_0| - |\alpha_t' - q_0| \Big|.$$

We point out that $\Big||q-q_0|-|\alpha_t'-q_0|\Big|$ is either $|q-q_0|-|\alpha_t'-q_0|$ or $|\alpha_t'-q_0|-|q-q_0|$. Since the two cases can be treated the same way, we consider

$$\Big| |q - q_0| - |\alpha_t' - q_0| \Big| = |q - q_0| - |\alpha_t' - q_0|.$$

In order to find a lower bound for this expression, we need a lower bound for $|q-q_0|$ and an upper bound for $|\alpha_t' - q_0|$. By Lemma 7.38 and since $\alpha_t' \in S_{\alpha_t}$, we have

$$|q - q_0| - |\alpha_t' - q_0| \geq |q^0 - q_0| - |\widetilde{\alpha}_t^0 - q_0|$$

where $q^0 = Re(q) \pm |Im(q)| I_0$ and $\widetilde{\alpha}_t^0$ is either α_t^0 or $\overline{\alpha}_t^0$. By the definition of the set V, we finally conclude that

$$|q - \alpha_t'| \geq \frac{R}{2(2m+1)}.$$

Therefore,

$$|Q(q)| \geq |a_m| \left(\frac{R}{2(2m+1)} \right)^m.$$

□

We now obtain lower bounds of Ehrenpreis–Malgrange type for general regular polynomials:

Theorem 7.40. *Let $P(q)$ be a regular polynomial of degree m, whose leading coefficient we denote by a_m. Let p be the number of spherical zeros of $P(q)$, and let t be the number of its isolated zeros, all counted according to their multiplicities. For any $u + v\mathbb{S} \subset \mathbb{H}$ and for any $R > 0$, there exist $r < R$ and a 3-dimensional compact hypersurface $\Gamma = \Gamma(u + v\mathbb{S}, r)$ (which is smooth if $r < v$), on which*

$$|P(q)| \geq |a_m| \left(\frac{R}{2(p + 2t + 1)} \right)^m.$$

Proof. The statement is obtained merging the proofs of Theorems 7.36 and 7.39.

□

Theorem 7.40 can be reformulated in a way that does not require the knowledge of the type of the zeros. The estimate that one obtains is naturally not as sharp, but it is the exact analog of the corresponding result in the complex case.

Theorem 7.41. *Let $P(q)$ be a regular polynomial of degree m, whose leading coefficient we denote by a_m. For any $u + v\mathbb{S} \subset \mathbb{H}$ and for any $R > 0$, there exist $r < R$ and a 3-dimensional compact hypersurface $\Gamma = \Gamma(u + v\mathbb{S}, r)$ (which is smooth if $r < v$), on which*

$$|P(q)| \geq |a_m| \left(\frac{R}{2(2m+1)} \right)^m .$$

The following consequence of the last result is a key tool in estimating the regular quotient between a regular function and a polynomial.

Theorem 7.42. *Let $P(q)$ be a regular polynomial of degree m, whose leading coefficient we denote by a_m. For any $R > 0$, there exist a natural number $n \leq m$, n quaternions $q_1 = u_1 + v_1 I_1, \dots, q_n = u_n + v_n I_n$, n strictly positive radii $r_1 < R, \dots, r_n < R$, and n corresponding compact sets*

$$D(q_\ell, r_\ell) = \{x + yI : (x - u_\ell)^2 + (y - v_\ell)^2 \leq r_\ell^2 \ \ and \ \ I \in \mathbb{S}\}$$

($\ell = 1, \dots, n$) bounded, respectively, by the 3-dimensional hypersurfaces $\Gamma(u_\ell + v_\ell\mathbb{S}, r_\ell)$, smooth if $r_\ell < v_\ell$, such that

$$|P(q)| \geq |a_m| \left(\frac{R}{2(2m+1)} \right)^m$$

outside

$$D = \bigcup_{\ell=1}^{n} D(q_\ell, r_\ell).$$

Proof. Given $R > 0$, it is straightforward to find $n \leq m$ points $q_1 = u_1 + v_1 I_1, \dots, q_n = u_n + v_n I_n$ such that $A = \bigcup_{\ell=1}^{n} D(q_\ell, R)$ contains all the roots of the polynomial P. We then apply Theorem 7.41 and find that, for each q_ℓ, there exists $r_\ell < R$ such that

$$|P(q)| \geq |a_m| \left(\frac{R}{2(2m+1)} \right)^m \tag{7.10}$$

on the 3-hypersurface $\Gamma(u_\ell + v_\ell\mathbb{S}, r_\ell)$. As a consequence, inequality (7.10) holds on $B = \bigcup_{\ell=1}^{n} \Gamma(u_\ell + v_\ell\mathbb{S}, r_\ell)$. Then B contains the boundary ∂D of $D = \bigcup_{\ell=1}^{n} D(q_\ell, r_\ell)$, which in turn coincides with the boundary $\partial(\mathbb{H} \setminus D)$ of $\mathbb{H} \setminus D$. Now, since the regular polynomial P has no zeros in $\mathbb{H} \setminus D$ and since $\lim_{q \to +\infty} |P(q)| = +\infty$, we get that inequality (7.10) holds in each (open) connected component of $\mathbb{H} \setminus D$: indeed, if this were not the case, $|P|$ would have a local minimum at some point $q \in \mathbb{H} \setminus D$ with $P(q) \neq 0$, and by the Minimum Modulus Principle applied to P on $\mathbb{H}$, P would be constant. This concludes the proof. □

Theorem 7.43 (Lindelöf Theorem). *Let $f : \Omega \to \mathbb{H}$ be a regular function on a symmetric slice domain, and suppose that there exists a constant $c > 0$ such that*

$$|f(q)| \leq c$$

on Ω. Let P be a polynomial of degree m such that $f * P^{-*} = h$ is regular in Ω. For any symmetric slice domain $\Omega' \subset\subset \Omega$, there exists $R > 0$ such that

$$|h(q)| \leq c \left(\frac{2(2m+1)}{R} \right)^m .$$

in the whole of Ω.

Proof. Apply Theorem 7.42 to P, with $R > 0$ chosen in such a way that $D \subset\subset \Omega$. In this way we obtain that

$$|P(q)| \geq |a_m| \left(\frac{R}{2(2m+1)} \right)^m$$

in

$$\Omega \setminus D = \Omega \setminus \bigcup_{\ell=1}^{m} D(q_\ell, r_\ell).$$

By Theorem 3.4 and Proposition 5.32,

$$|h(q)| = |f * P^{-*}(q)| = |f(q)||P(T(q))|^{-1} \leq c \left(\frac{2(2m+1)}{R} \right)^m \tag{7.11}$$

for all $q \in \Omega \setminus D$ and for an appropriate diffeomorphism T of $\Omega \setminus D$ onto itself. Since $D \subset\subset \Omega$, by the Maximum Modulus Principle, inequality (7.11) holds in the whole of Ω. □

Note that if Ω is unbounded, and we know the growth of f at infinity, the previous theorem naturally implies that the growth of h at infinity is, up to a constant, the same as f.

Bibliographic Notes

The Maximum Modulus Principle for regular functions on $B(0, R)$ was proven in [62] by means of the Cauchy Formula 6.3. Another proof was later developed on the basis of the Splitting Lemma and of the complex Maximum Modulus Principle. The most general statement, which we present here, appeared in [57].

The Minimum Modulus Principle and the Open Mapping Theorem were proven in [56] for the case of Euclidean balls centered at 0 and extended to symmetric slice domains in [57].

The results concerning the real parts of regular function, and in particular the Maximum Principle 7.13, appeared in [66].

The Principles of Phragmén–Lindelöf type were originally proven in [59], but we present them here with more stringent statements. Finally, the Ehrenpreis–Malgrange Lemma was proven in [67].

Chapter 8
Spherical Series and Differential

In Chap. 2 we saw that regularity is equivalent to σ-analyticity, that is, for a function $f : \Omega \to \mathbb{H}$, to the existence of regular power series expansions at each point p of the domain Ω. At the same time, such an expansion is not sufficient to reconstruct the function f in a Euclidean neighborhood of p, because the set of convergence $\Sigma(p, R)$ needs not be open (see Remark 2.9).

In this chapter, we will show that, when the domain of definition Ω of the regular function f is a symmetric slice domain, a different type of series expansion is possible. The series used in this alternative construction involve the powers of a suitable second-degree polynomial, and they converge in symmetric open sets of $\mathbb{H}$. Hence, the corresponding expansion of a regular function f provides full information on the local behavior of f. This makes it possible to study in detail the (real) differential of f and its rank.

8.1 Spherical Series and Expansions

We begin by proving a consequence of Theorem 3.17.

Theorem 8.1. *Let f be a regular function on a symmetric slice domain Ω. For each $q_0 \in \Omega$, let us denote as $R_{q_0} f : \Omega \to \mathbb{H}$ the function such that*

$$f(q) = f(q_0) + (q - q_0) * R_{q_0} f(q).$$

If $q_0 = x_0 + y_0 I$ for some $x_0, y_0 \in \mathbb{R}$, $I \in \mathbb{S}$, then

$$f(q) = f(q_0) + (q - q_0) R_{q_0} f(\bar{q}_0) + [(q - x_0)^2 + y_0^2] R_{\bar{q}_0} R_{q_0} f(q).$$

for all $q \in \Omega$.

G. Gentili et al., *Regular Functions of a Quaternionic Variable*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-33871-7_8,

Proof. The existence of a regular $R_{q_0}f : \Omega \to \mathbb{H}$ such that

$$f(q) = f(q_0) + (q - q_0) * R_{q_0} f(q)$$

is granted by Theorem 3.17, since $f - f(q_0)$ is a regular function on Ω vanishing at q_0. Applying the same procedure to $R_{q_0} f$ at the point $\bar{q}_0$ yields

$$\begin{aligned} f(q) &= f(q_0) + (q - q_0) * \left[R_{q_0} f(\bar{q}_0) + (q - \bar{q}_0) * R_{\bar{q}_0} R_{q_0} f(q)\right] = \\ &= f(q_0) + (q - q_0) R_{q_0} f(\bar{q}_0) + [(q - x_0)^2 + y_0^2] R_{\bar{q}_0} R_{q_0} f(q) \end{aligned}$$

where we have taken into account that

$$(q - q_0) * (q - \bar{q}_0) = (q - x_0)^2 + y_0^2.$$

□

If we repeatedly apply the previous theorem, we get the formal expansion

$$\begin{aligned} &f(q) = f(q_0) + (q - q_0)\, R_{q_0} f(\bar{q}_0) + \\ &+[(q - x_0)^2 + y_0^2]\left[R_{\bar{q}_0} R_{q_0} f(q_0) + (q - q_0) R_{q_0} R_{\bar{q}_0} R_{q_0} f(\bar{q}_0)\right] + \ldots + \\ &+[(q - x_0)^2 + y_0^2]^n \left[(R_{\bar{q}_0} R_{q_0})^n f(q_0) + (q - q_0) R_{q_0} (R_{\bar{q}_0} R_{q_0})^n f(\bar{q}_0)\right] + \ldots \end{aligned}$$

where $(R_{\bar{q}_0} R_{q_0})^n$ denotes the nth iterate of $R_{\bar{q}_0} R_{q_0}$. If $A_{2n} = (R_{\bar{q}_0} R_{q_0})^n f(q_0)$ and $A_{2n+1} = R_{q_0}(R_{\bar{q}_0} R_{q_0})^n f(\bar{q}_0)$ for all $n \in \mathbb{N}$, our new formal expansion reads as

$$f(q) = \sum_{n \in \mathbb{N}} P_n(q) A_n \tag{8.1}$$

where $P_{2n}(q) = [(q - x_0)^2 + y_0^2]^n$ and $P_{2n+1}(q) = [(q - x_0)^2 + y_0^2]^n (q - q_0)$ for all $n \in \mathbb{N}$. Let us study the sets of convergence of function series of this type, which we will sometimes call *spherical series*.

Lemma 8.2. *Let $x_0 \in \mathbb{R}$, $y_0 > 0$, $q \in \mathbb{H}$ and let $r \geq 0$ be such that $|(q - x_0)^2 + y_0^2| = r^2$. If $q_0 = x_0 + y_0 I$ for some $I \in \mathbb{S}$ then*

$$\sqrt{r^2 + y_0^2} - y_0 \leq |q - q_0| \leq \sqrt{r^2 + y_0^2} + y_0$$

Proof. If $r = 0$, that is, $q \in x_0 + y_0\mathbb{S}$, then clearly $0 \leq |q - q_0| \leq 2y_0$. Else $q \notin x_0 + y_0\mathbb{S}$ and

$$|(q - x_0)^2 + y_0^2| = |q - q_0||(q - q_0)^{-1} q (q - q_0) - \bar{q}_0| = |q - q_0||q - \widetilde{q}_0|$$

where $\widetilde{q}_0 = (q - q_0)\bar{q}_0(q - q_0)^{-1} \in x_0 + y_0\mathbb{S}$. If $|q - q_0| > \sqrt{r^2 + y_0^2} + y_0$ then

$$|q-\widetilde{q}_0| \geq |q-q_0| - |q_0-\widetilde{q}_0| \geq |q-q_0| - 2y_0 > \sqrt{r^2+y_0^2} - y_0$$

so that

$$|(q-x_0)^2+y_0^2| = |q-q_0||q-\widetilde{q}_0| > r^2+y_0^2-y_0^2 = r^2$$

a contradiction with the hypothesis. A similar reasoning excludes that $|q-q_0| < \sqrt{r^2+y_0^2} - y_0$. □

Proposition 8.3. *Let $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{H}$ and suppose*

$$\limsup_{n\to+\infty} |a_n|^{1/n} = 1/R \tag{8.2}$$

for some $R > 0$. Let $q_0 = x_0 + y_0 I \in \mathbb{H}$ with $x_0 \in \mathbb{R}, y_0 > 0, I \in \mathbb{S}$ and set $P_{2n}(q) = [(q-x_0)^2+y_0^2]^n$ and $P_{2n+1}(q) = [(q-x_0)^2+y_0^2]^n(q-q_0)$ for all $n \in \mathbb{N}$. Then the function series

$$\sum_{n\in\mathbb{N}} P_n(q)a_n \tag{8.3}$$

converges absolutely and uniformly on compact sets in

$$U(x_0+y_0\mathbb{S}, R) = \{q \in \mathbb{H} : |(q-x_0)^2+y_0^2| < R^2\}, \tag{8.4}$$

where it defines a regular function. Furthermore, the series (8.3) diverges at every $q \in \mathbb{H} \setminus \overline{U(x_0+y_0\mathbb{S}, R)}$.

Proof. If K is a compact subset of $U(x_0+y_0\mathbb{S}, R)$, then there exists $r < R$ such that $|(q-x_0)^2+y_0^2| \leq r^2$ for all $q \in K$. Thus, for all $q \in K$

$$|P_{2n}(q)a_{2n}| = |(q-x_0)^2+y_0^2|^n|a_{2n}| \leq r^{2n}|a_{2n}|$$

while (thanks to the previous lemma)

$$|P_{2n+1}(q)a_{2n+1}| = |(q-x_0)^2+y_0^2|^n|q-q_0||a_{2n+1}| \leq$$
$$\leq r^{2n}\left(\sqrt{r^2+y_0^2}+y_0\right)|a_{2n+1}|.$$

Hence (8.3) is dominated on K by a number series $\sum_{n\in\mathbb{N}} c_n$ with

$$\limsup_{n\to+\infty} |c_n|^{1/n} = r/R < 1.$$

This guarantees absolute and uniform convergence of (8.3) in K. This, in turn, proves the regularity of the sum (since all the addends in (8.3) are regular polynomials).

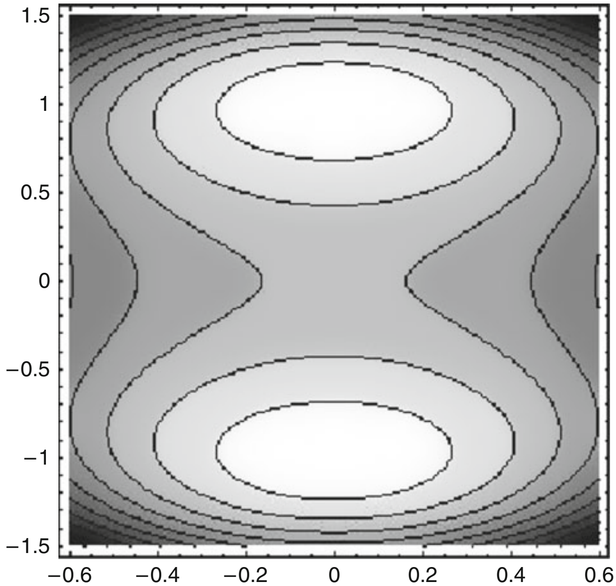

Fig. 8.1 Examples of $U(\mathbb{S}, R)$, intersected with a plane L_I, for different values of R. *Source:* [122]

Finally, if $q \in \mathbb{H} \setminus \overline{U(x_0 + y_0\mathbb{S}, R)}$, then $|(q - x_0)^2 + y_0^2| = r^2$ for some $r > R$. Reasoning as before, we get $|P_{2n}(q)a_{2n}| \geq r^{2n}|a_{2n}|$ and

$$|P_{2n+1}(q)a_{2n+1}| \geq r^{2n}\left(\sqrt{r^2 + y_0^2} - y_0\right)|a_{2n+1}|.$$

Thus, $\sum_{n\in\mathbb{N}} P_n(q)a_n$ dominates a number series $\sum_{n\in\mathbb{N}} C_n$ with

$$\limsup_{n\to+\infty} |C_n|^{1/n} = r/R > 1$$

and it must diverge. □

We notice that the intersection between the hypersurface $|(q - x_0)^2 + y_0^2| = R^2$ bounding $U(x_0 + y_0\mathbb{S}, R)$ and any complex plane L_I is a polynomial lemniscate, connected for $R \geq y_0$ and having two connected components for $R < y_0$. If $y_0 \neq 0$, then at the critical value $R = y_0$, the lemniscate is of figure-eight type (see Fig. 8.1 for some examples).

Remark 8.4. For $R > y_0$ the set $U(x_0 + y_0\mathbb{S}, R)$ is a symmetric slice domain; for $0 < R \leq y_0$ the set $U(x_0 + y_0\mathbb{S}, R)$ is a symmetric domain whose intersection with any complex plane L_I has two connected components.

8.2 Integral Formulas and Cauchy Estimates

After introducing the formal expansion (8.1), we will bound the coefficients A_n of the expansion by means of Cauchy-type integral representations.

Theorem 8.5. *Let f be a regular function on a symmetric slice domain Ω, let $I \in \mathbb{S}$, and let U_I be a symmetric bounded Jordan domain in L_I, with $\overline{U_I} \subset \Omega_I$. If ∂U_I is rectifiable, then for each $z_0 = x_0 + y_0 I \in U_I$ and for all $z \in U_I$*

$$f(z) = f(z_0) + (z - z_0) \cdot \frac{1}{2\pi I} \int_{\partial U_I} \frac{ds}{(s-z)(s-z_0)} f(s). \tag{8.5}$$

Furthermore,

$$(R_{\bar{z}_0} R_{z_0})^n f(z) = \frac{1}{2\pi I} \int_{\partial U_I} \frac{ds}{(s-z)[(s-x_0)^2 + y_0^2]^n} f(s) \tag{8.6}$$

$$R_{z_0}(R_{\bar{z}_0} R_{z_0})^n f(z) = \frac{1}{2\pi I} \int_{\partial U_I} \frac{ds}{(s-z)(s-z_0)[(s-x_0)^2 + y_0^2]^n} f(s) \tag{8.7}$$

for all $n \in \mathbb{N}$ and all $z \in U_I$.

Proof. Lemma 6.3 is equivalent to (8.6) in the special case $n = 0$. That immediately implies

$$R_{z_0} f(z) = (z - z_0)^{-1}[f(z) - f(z_0)] =$$
$$= \frac{1}{2\pi I} \int_{\partial U_I} \frac{1}{z - z_0} \left[\frac{1}{s - z} - \frac{1}{s - z_0} \right] ds f(s) =$$
$$= \frac{1}{2\pi I} \int_{\partial U_I} \frac{ds}{(s-z)(s-z_0)} f(s),$$

which is (8.7) in the special case $n = 0$. This last equality, in turn, implies (8.5). Moreover,

$$R_{z_0} f(\bar{z}_0) = \frac{1}{2\pi I} \int_{\partial U_I} \frac{ds}{(s-x_0)^2 + y_0^2} f(s)$$

and

$$R_{\bar{z}_0} R_{z_0} f(z) = (z - \bar{z}_0)^{-1}[R_{z_0} f(z) - R_{z_0} f(\bar{z}_0)] =$$
$$= \frac{1}{2\pi I} \int_{\partial U_I} \frac{1}{z - \bar{z}_0} \left[\frac{1}{(s-z)(s-z_0)} - \frac{1}{(s-x_0)^2 + y_0^2} \right] ds f(s) =$$
$$= \frac{1}{2\pi I} \int_{\partial U_I} \frac{1}{z - \bar{z}_0} \frac{s - \bar{z}_0 - (s - z)}{(s-z)[(s-x_0)^2 + y_0^2]} ds f(s) =$$
$$= \frac{1}{2\pi I} \int_{\partial U_I} \frac{ds}{(s-z)[(s-x_0)^2 + y_0^2]} f(s),$$

which is (8.6) with $n=1$. An analogous reasoning proves that if (8.6) holds for $n=k$, then (8.7) holds for $n=k$, which in turn implies (8.6) for $n=k+1$ completing the proof. □

We now formulate the estimates for the coefficients A_n appearing in the formal expansion (8.1).

Corollary 8.6. *Let f be a regular function on a symmetric slice domain Ω, let $x_0 + y_0\mathbb{S} \subset \Omega$, and let $q_0 \in x_0 + y_0\mathbb{S}$. Set $A_{2n} = (R_{\bar{q}_0} R_{q_0})^n f(q_0)$ and $A_{2n+1} = R_{q_0}(R_{\bar{q}_0} R_{q_0})^n f(\bar{q}_0)$. For every $U = U(x_0 + y_0\mathbb{S}, R)$ such that $\overline{U} \subset \Omega$, there exists a constant $C > 0$ such that*

$$|A_n| \leq C \cdot \frac{\max_{\partial U} |f|}{R^n} \tag{8.8}$$

for all $n \in \mathbb{N}$.

Proof. If we choose $U = U(x_0 + y_0\mathbb{S}, R)$, then the previous theorem implies

$$|(R_{\bar{q}_0} R_{q_0})^n f(q_0)| = \left| \frac{1}{2\pi I} \int_{\partial U_I} \frac{ds}{(s-z_0)[(s-x_0)^2 + y_0^2]^n} f(s) \right| \leq$$
$$\leq \frac{1}{2\pi} \int_{\partial U_I} \frac{|f(s)|}{\left(\sqrt{R^2 + y_0^2} - y_0\right) R^{2n}} |ds| \leq C \cdot \frac{\max_{\partial U} |f|}{R^{2n}}$$

if we set $C = \frac{\text{length}(\partial U_I)}{2\pi\left(\sqrt{R^2+y_0^2}-y_0\right)}$. On the other hand,

$$|R_{q_0}(R_{\bar{q}_0} R_{q_0})^n f(\bar{q}_0)| = \left| \frac{1}{2\pi I} \int_{\partial U_I} \frac{ds}{[(s-x_0)^2 + y_0^2]^{n+1}} f(s) \right| \leq$$
$$\leq \frac{1}{2\pi} \int_{\partial U_I} \frac{|f(s)|}{R^{2n+2}} d|s| \leq K \cdot \frac{\max_{\partial U} |f|}{R^{2n+1}}$$

where $K = \frac{\text{length}(\partial U_I)}{2\pi R} \leq C$. □

8.3 Symmetric Analyticity

The estimates given in Corollary 8.6 prove the convergence of the formal expansion (8.1). This, in turn, allows the definition of a new notion of analyticity.

Theorem 8.7. *Let f be a regular function on a symmetric slice domain Ω, and let $x_0, y_0 \in \mathbb{R}$ and $R > 0$ be such that $U(x_0 + y_0\mathbb{S}, R) \subseteq \Omega$. For all $q_0 \in x_0 + y_0\mathbb{S}$, setting $A_{2n} = (R_{\bar{q}_0} R_{q_0})^n f(q_0)$ and $A_{2n+1} = R_{q_0}(R_{\bar{q}_0} R_{q_0})^n f(\bar{q}_0)$, we have that*

$$f(q) = \sum_{n \in \mathbb{N}} [(q - x_0)^2 + y_0^2]^n [A_{2n} + (q - q_0) A_{2n+1}] \tag{8.9}$$

for all $q \in U(x_0 + y_0\mathbb{S}, R)$.

Proof. Thanks to Proposition 8.3 and to Corollary 8.6, the function series in (8.9) converges in $U = U(x_0 + y_0\mathbb{S}, R)$, where it defines a regular function. Let us consider the difference

$$g(q) = f(q) - \sum_{n \in \mathbb{N}} [(q - x_0)^2 + y_0^2]^n [A_{2n} + (q - q_0) A_{2n+1}].$$

By construction, $g(q) = [(q - x_0)^2 + y_0^2]^n (R_{\bar{q}_0} R_{q_0})^n f(q)$ for all $n \in \mathbb{N}$. For any choice of $I \in \mathbb{S}$ and for all $z \in U_I$, we derive that

$$g_I(z) = [(z - x_0)^2 + y_0^2]^n (R_{\bar{q}_0} R_{q_0})^n f_I(z) = [z - (x_0 + y_0 I)]^n h_I^{[n]}(z)$$

where $h_I^{[n]}(z) = [z - (x_0 - y_0 I)]^n (R_{\bar{q}_0} R_{q_0})^n f_I(z)$ is holomorphic in U_I. The identity principle for holomorphic functions of one complex variable implies that $g_I \equiv 0$. Since I can be arbitrarily chosen in $\mathbb{S}$, the function g must be identically zero in U. This is equivalent to (8.9). □

Definition 8.8. Let f be a regular function on a symmetric slice domain Ω. We say that f is *symmetrically analytic* if it admits at every $q_0 \in \Omega$ an expansion of type (8.1) valid in a neighborhood of q_0.

The previous theorem, along with Proposition 8.3, proves what follows.

Corollary 8.9. *Let Ω be a symmetric slice domain. A function $f : \Omega \to \mathbb{H}$ is regular if, and only if, it is symmetrically analytic.*

We conclude this section reformulating expansion (8.9) as follows, thanks to Theorem 1.10.

Corollary 8.10. *Let f be a regular function on a symmetric slice domain Ω, and let $x_0, y_0 \in \mathbb{R}$ and $R > 0$ be such that $U(x_0 + y_0\mathbb{S}, R) \subseteq \Omega$. Then, for all $q \in U(x_0 + y_0\mathbb{S}, R)$*

$$f(q) = \sum_{n \in \mathbb{N}} [(q - x_0)^2 + y_0^2]^n [C_{2n} + q C_{2n+1}] \tag{8.10}$$

where the coefficients C_n depend only on f, x_0, and y_0 and can be computed as

$$\begin{aligned} C_{2n} &= (q_2 - q_1)^{-1} \left[\bar{q}_1 (R_{\bar{q}_0} R_{q_0})^n f(q_1) - \bar{q}_2 (R_{\bar{q}_0} R_{q_0})^n f(q_2)\right], \\ C_{2n+1} &= (q_2 - q_1)^{-1} \left[(R_{\bar{q}_0} R_{q_0})^n f(q_2) - (R_{\bar{q}_0} R_{q_0})^n f(q_1)\right] \end{aligned} \tag{8.11}$$

for all $q_0, q_1, q_2 \in x_0 + y_0\mathbb{S}$ with $q_1 \neq q_2$.

Clearly, the odd-indexed coefficients in expansions (8.9) and (8.10) coincide.

As a first application of spherical expansions, we notice that they allow an immediate computation of the spherical multiplicity, and they give some information on the isolated multiplicity (these notions have been introduced in Definition 3.37).

Remark 8.11. Let f be a regular function on a symmetric slice domain Ω, and let $q_0 = x_0 + y_0 I \in \Omega$. If A_{2n} or A_{2n+1} is the first nonvanishing coefficient in the expansion (8.9), then $2n$ is the spherical multiplicity of f at $x_0 + y_0\mathbb{S}$. Moreover, q_0 has positive isolated multiplicity if and only if $A_{2n} = 0$.

Remark 8.12. Let f be a regular function on a symmetric slice domain Ω, and let $x_0 + y_0\mathbb{S} \subset \Omega$. If the first nonvanishing coefficient in the expansion (8.10) is C_{2n} or C_{2n+1} then $2n$ is the spherical multiplicity of f at $x_0 + y_0\mathbb{S}$. Moreover, there exists a point with positive isolated multiplicity in $x_0 + y_0\mathbb{S}$ if, and only if, $C_{2n+1}^{-1} C_{2n} \in x_0 + y_0\mathbb{S}$.

A second application regards the differentiation of regular functions, as we will see in the next section.

8.4 Differentiating Regular Functions

The spherical expansion allows the computation of the real partial derivatives of f.

Theorem 8.13. *Let f be a regular function on a symmetric slice domain Ω, and let $q_0 = x_0 + y_0 I \in \Omega$. For all $v \in \mathbb{H}$, $|v| = 1$, the derivative of f along v can be computed at q_0 as*

$$\lim_{t\to 0} \frac{f(q_0 + tv) - f(q_0)}{t} = vA_1 + (q_0 v - v\bar{q}_0)A_2 \tag{8.12}$$

where $A_1 = R_{q_0} f(\bar{q}_0)$ and $A_2 = R_{\bar{q}_0} R_{q_0} f(q_0)$. In particular, if $e_0, e_1, e_2, e_3 \in \mathbb{H}$ form a basis for $\mathbb{H}$ and if x_0, x_1, x_2, and x_3 denote the corresponding coordinates, then

$$\frac{\partial f}{\partial x_i}(q_0) = e_i R_{q_0} f(\bar{q}_0) + (q_0 e_i - e_i \bar{q}_0) R_{\bar{q}_0} R_{q_0} f(q_0) \tag{8.13}$$

Proof. We observe that

$$(q - x_0)^2 + y_0^2 = (q - q_0) * (q - \bar{q}_0) = q(q - q_0) - (q - q_0)\bar{q}_0$$

and for $q = q_0 + tv$

$$(q - x_0)^2 + y_0^2 = (q_0 + tv)tv - tv\bar{q}_0 = t(tv^2 + q_0 v - v\bar{q}_0)$$

whence

$$f(q_0 + tv) = \sum_{n\in\mathbb{N}} t^n (tv^2 + q_0 v - v\bar{q}_0)^n [A_{2n} + tvA_{2n+1}].$$

Hence,

$$f(q_0 + tv) - f(q_0) = tvA_1 + t(tv^2 + q_0v - v\bar{q}_0)[A_2 + tvA_3] + o(t)$$

and the thesis immediately follows. □

Higher order derivatives can be computed in the same fashion. Furthermore, let us relate $R_{q_0} f$ to the slice derivative $\partial_c f$ (Definition 1.7) and to the spherical derivative $\partial_s f$ (Definition 1.18).

Remark 8.14. Let f be a regular function on a symmetric slice domain Ω, and let $q_0 \in \Omega$. Then $R_{q_0} f(q_0)$ equals $\partial_c f(q_0)$, and $R_{q_0} f(\bar{q}_0)$ equals $\partial_s f(q_0)$.

Indeed, if v lies in the same L_I as q_0, then v commutes with q_0 and

$$\begin{aligned} vR_{q_0} f(\bar{q}_0) + (q_0v - v\bar{q}_0)R_{\bar{q}_0} R_{q_0} f(q_0) &= v[R_{q_0} f(\bar{q}_0) + (q_0 - \bar{q}_0)R_{\bar{q}_0} R_{q_0} f(q_0)] = \\ &= v[R_{q_0} f(\bar{q}_0) + R_{q_0} f(q_0) - R_{q_0} f(\bar{q}_0)] = vR_{q_0} f(q_0). \end{aligned}$$

On the other hand, if $q_0 = x_0 + y_0 I$ with $y_0 \neq 0$ and v is tangent to the 2-sphere $x_0 + y_0\mathbb{S}$ at q_0, then $q_0 v = v\bar{q}_0$ and

$$vR_{q_0} f(\bar{q}_0) + (q_0v - v\bar{q}_0)R_{\bar{q}_0} R_{q_0} f(q_0) = vR_{q_0} f(\bar{q}_0).$$

We conclude by looking at the implications on the real differential f_* of f and at possible complex coordinates.

Remark 8.15. Let f be a regular function on a symmetric slice domain Ω, and let $(f_*)_{q_0}$ denote the real differential of f at $q_0 \in \Omega_I$. If we identify $T_{q_0}\mathbb{H}$ with $\mathbb{H} = L_I \oplus L_I^{\perp}$, then for all $u \in L_I$ and $w \in L_I^{\perp}$,

$$(f_*)_{q_0}(u + w) = u(A_1 + 2Im(q_0)A_2) + wA_1. \tag{8.14}$$

In other words, the differential f_* at q_0 acts by right multiplication by $A_1 + 2Im(q_0)A_2 = R_{q_0} f(q_0) = \partial_c f(q_0)$ on L_I and by right multiplication by $A_1 = R_{q_0} f(\bar{q}_0) = \partial_s f(q_0)$ on $L_I^{\perp}$.

Given $I \in \mathbb{S}$ such that $q_0 \in L_I$, choose $J \in \mathbb{S}$ with $I \perp J$, and set $e_0 = 1, e_1 = I, e_2 = J, and\ e_3 = IJ$. Then $\mathbb{H} = L_I + L_I J$ can be identified with $\mathbb{C}^2$ setting $z_1 = x_0 + x_1 I, z_2 = x_2 + x_3 I, \bar{z}_1 = x_0 - x_1 I, and\ \bar{z}_2 = x_2 - x_3 I$; we may as well split $f = f_1 + f_2 J$ for some $f_1, f_2 : \Omega \to L_I$.

Theorem 8.16. *Let Ω be a symmetric slice domain, let $f : \Omega \to \mathbb{H}$ be a regular function, and let $q_0 \in \Omega$. Choosing $I, J \in \mathbb{S}$ so that $q_0 \in L_I$ and $I \perp J$, let $z_1, z_2, \bar{z}_1$, and $\bar{z}_2$ be the induced coordinates, and let $\partial_1, \partial_2, \bar{\partial}_1$, and $\bar{\partial}_2$ be the corresponding derivations. Then*

$$\left.\begin{pmatrix} \bar{\partial}_1 f_1 & \bar{\partial}_2 f_1 \\ \bar{\partial}_1 f_2 & \bar{\partial}_2 f_2 \end{pmatrix}\right|_{q_0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{8.15}$$

In other words, f is complex differentiable at q_0. Furthermore, if $R_{q_0} f$ splits as $R_{q_0} f = R_1 + R_2 J$ with R_1, R_2 ranging in L_I, then the complex Jacobian of f at q_0 is

$$\begin{pmatrix} \partial_1 f_1 & \partial_2 f_1 \\ \partial_1 f_2 & \partial_2 f_2 \end{pmatrix}\Bigg|_{q_0} = \begin{pmatrix} R_1(q_0) & -\overline{R_2(\bar{q}_0)} \\ R_2(q_0) & \overline{R_1(\bar{q}_0)} \end{pmatrix} \tag{8.16}$$

Proof. If $e_0 = 1, e_1 = I, e_2 = J$, and $e_3 = IJ$ and if x_0, x_1, x_2, and x_3 are the corresponding real coordinates, the derivations $\partial_1, \partial_2, \bar{\partial}_1$, and $\bar{\partial}_2$ are defined by

$$\partial_1 = \frac{1}{2}\left(\frac{\partial}{\partial x_0} - I\frac{\partial}{\partial x_1}\right), \quad \bar{\partial}_1 = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + I\frac{\partial}{\partial x_1}\right)$$
$$\partial_2 = \frac{1}{2}\left(\frac{\partial}{\partial x_2} - I\frac{\partial}{\partial x_3}\right), \quad \bar{\partial}_2 = \frac{1}{2}\left(\frac{\partial}{\partial x_2} + I\frac{\partial}{\partial x_3}\right)$$

By means of the Remark 8.15, we compute

$$\begin{aligned}
\frac{\partial f}{\partial x_0}(q_0) &= R_{q_0} f(q_0) = R_1(q_0) + R_2(q_0)J, \\
\frac{\partial f}{\partial x_1}(q_0) &= IR_{q_0} f(q_0) = IR_1(q_0) + IR_2(q_0)J, \\
\frac{\partial f}{\partial x_2}(q_0) &= JR_{q_0} f(\bar{q}_0) = J(R_1(\bar{q}_0) + R_2(\bar{q}_0)J) = -\overline{R_2(\bar{q}_0)} + \overline{R_1(\bar{q}_0)}J, \\
\frac{\partial f}{\partial x_3}(q_0) &= KR_{q_0} f(\bar{q}_0) = IJ(R_1(\bar{q}_0) + R_2(\bar{q}_0)J) = -I\,\overline{R_2(\bar{q}_0)} + I\,\overline{R_1(\bar{q}_0)}J,
\end{aligned}$$

where we have taken into account that $Jz = \bar{z}J$ for all $z \in L_I$. The thesis follows by direct computation. ☐

In Sect. 10.3, we will see that two complex variables can be a useful tool in the setting of quaternionic regularity. We also point out that in the special case where $q_0 \in \mathbb{R}$, the complex Jacobian has the form

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix},$$

which implies $\lim_{h\to 0} h^{-1}[f(q_0 + h) - f(q_0)] = a + bJ$ and proves what follows.

Corollary 8.17. *Let Ω be a symmetric slice domain. If $f : \Omega \to \mathbb{H}$ is a regular function, then it is quaternion-differentiable at each $q_0 \in \Omega \cap \mathbb{R}$, and* $\lim_{h\to 0} h^{-1}[f(q_0 + h) - f(q_0)] = R_{q_0} f(q_0) = \partial_c f(q_0)$.

8.5 Rank of the Differential

Now that we gained some understanding of the real differential f_* of a regular function f, it is natural to ask ourselves when f_* is invertible and, more in general, how its rank behaves.

Proposition 8.18. *Let f be a regular function on a symmetric slice domain Ω, let $q_0 = x_0 + y_0 I \in \Omega \setminus \mathbb{R}$, and let A_n be the coefficients of the expansion (8.9).*

- *If $A_1 = 0$, then f_* has rank 2 if $A_2 \neq 0$, rank 0 if $A_2 = 0$.*
- *If $A_1 \neq 0$, then f_* is not invertible at q_0 if and only if f_* has rank 2 at q_0; this happens if and only if $1 + 2Im(q_0)A_2A_1^{-1} \in L_I^{\perp}$.*

Finally, for all $x_0 \in \Omega \cap \mathbb{R}$, f_* is invertible at x_0 if and only if its rank is not 0 at q_0 if and only if $A_1 \neq 0$.

Proof. If $A_1 = 0$ then, by (8.14), $(f_*)_{q_0}(u+w) = u\, 2Im(q_0)A_2$ for all $u \in L_I, w \in L_I^{\perp}$. Hence the kernel of $(f_*)_{q_0}$ is $L_I^{\perp}$ if $A_2 \neq 0$, the whole space $\mathbb{H}$ if $A_2 = 0$.

Let us now turn to the case $A_1 \neq 0$. By (8.14),

$$(f_*)_{q_0}(u+w) = \left[u(1 + 2Im(q_0)A_2A_1^{-1}) + w\right]A_1$$

for all $u \in L_I$ and $w \in L_I^{\perp}$. The differential $(f_*)_{q_0}$ is not invertible if and only if $1 + 2Im(q_0)A_2A_1^{-1} = p \in L_I^{\perp}$. In this case, if $p = 0$, then the kernel of $(f_*)_{q_0}$ is L_I; if $p \neq 0$, then the kernel is the 2-plane of vectors $-wp^{-1} + w$ for $w \in L_I^{\perp}$.

The last statement follows from the fact that if $x_0 \in \Omega \cap \mathbb{R}$, then $(f_*)_{x_0} v = vA_1$ for all $v \in \mathbb{H}$. $\square$

Now let us study in detail the set

$$N_f = \{q \in \Omega : f_* \text{ is not invertible at } q\},$$

which we may call the *singular set* of f. In this study, we will make use of the following property (see [7]).

Remark 8.19. For all $q_0 = x_0 + y_0 I \in \mathbb{H} \setminus \mathbb{R}$, setting $\Psi(q) = (q - q_0)(q - \bar{q}_0)^{-1}$ defines a stereographic projection of $x_0 + y_0\mathbb{S}$ onto the plane $L_I^{\perp}$ from the point $\bar{q}_0$.

Indeed, if we choose $J \in \mathbb{S}$ with $J \perp I$ and set $K = IJ$ then for all $q = x_0 + My_0$ with $M = tI + uJ + vK \in \mathbb{S}$, we have $\Psi(q) = (M - I)(M + I)^{-1} = \frac{u+Iv}{1+t}K$ and $L_I K = L_I^{\perp}$. We are now in a position to characterize the points of the singular set in algebraic terms. We recall that the notion of total multiplicity has been introduced in Definition 6.13 and that the degenerate set is presented in Definition 7.5.

Proposition 8.20. *Let f be a regular function on a symmetric slice domain Ω; let $q_0 = x_0 + y_0 I \in \Omega$. Then f_* is not invertible at q_0 if, and only if, there exist $\widetilde{q}_0 \in x_0 + y_0\mathbb{S}$ and a regular function $g : \Omega \to \mathbb{H}$ such that*

$$f(q) = f(q_0) + (q - q_0) * (q - \widetilde{q}_0) * g(q). \tag{8.17}$$

Equivalently, f_* is not invertible at q_0 if, and only if, $f - f(q_0)$ has total multiplicity $n \geq 2$ at $x_0 + y_0\mathbb{S}$. Moreover, q_0 belongs to the degenerate set D_f if, and only if, it belongs to the singular set N_f and there exists a regular $g : \Omega \to \mathbb{H}$ such that (8.17) holds for $\widetilde{q}_0 = \bar{q}_0$. The latter is equivalent to saying that $f - f(q_0)$ has spherical multiplicity $n \geq 2$ at $x_0 + y_0\mathbb{S}$.

Proof. If $q_0 \in \Omega \setminus \mathbb{R}$, then it belongs to D_f if and only if f is constant on the 2-sphere $x_0 + y_0\mathbb{S}$, that is, there exists a regular function $g : \Omega \to \mathbb{H}$ such that

$$f(q) = f(q_0) + [(q - x_0)^2 + y_0^2] * g(q) = f(q_0) + (q - q_0) * (q - \bar{q}_0) * g(q).$$

This happens if and only if the coefficient A_1 in the expansion (8.9) vanishes.

Now let us turn to the case $q_0 \in \Omega \setminus \mathbb{R}$, $q_0 \notin D_f$. By Proposition 8.18, $q_0 \in N_f$ if and only if $1 + 2Im(q_0)A_2A_1^{-1} = p \in L_I^{\perp}$. Thanks to the Remark 8.19, $p \in L_I^{\perp}$ if, and only if, there exists $\widetilde{q}_0 \in (x_0 + y_0\mathbb{S}) \setminus \{\bar{q}_0\}$ such that $p = (\widetilde{q}_0 - q_0)(\widetilde{q}_0 - \bar{q}_0)^{-1}$. The last formula is equivalent to

$$2Im(q_0)A_2A_1^{-1} = (\widetilde{q}_0 - q_0 - \widetilde{q}_0 + \bar{q}_0)(\widetilde{q}_0 - \bar{q}_0)^{-1} = -2Im(q_0)(\widetilde{q}_0 - \bar{q}_0)^{-1},$$

that is, $A_1 = (\bar{q}_0 - \widetilde{q}_0)A_2$. Finally, this last equality is equivalent to

$$\begin{aligned}
f(q) &= A_0 + (q - q_0)(\bar{q}_0 - \widetilde{q}_0)A_2 + \big[(q - x_0)^2 + y_0^2\big]\big[A_2 + (q - q_0) * h(q)\big] \\
&= f(q_0) + (q - q_0) * \big\{\big[\bar{q}_0 - \widetilde{q}_0 + q - \bar{q}_0\big]A_2 + \big[(q - x_0)^2 + y_0^2\big] * h(q)\big\} \\
&= f(q_0) + (q - q_0) * \big[(q - \widetilde{q}_0)A_2 + (q - \widetilde{q}_0) * (q - \overline{\widetilde{q}_0}) * h(q)\big] \\
&= f(q_0) + (q - q_0) * (q - \widetilde{q}_0)\big[A_2 + (q - \overline{\widetilde{q}_0}) * h(q)\big],
\end{aligned}$$

for some regular $h : \Omega \to \mathbb{H}$.

To conclude, we observe that if $x_0 \in \Omega \cap \mathbb{R}$, then $A_1 = 0$ if and only if

$$f(q) = f(q_0) + (q - x_0)^2 * g(q) = f(q_0) + (q - x_0) * (q - x_0) * g(q)$$

for some regular function $g : \Omega \to \mathbb{H}$. □

We conclude with a complete characterization of the singular set of f, proving that it is empty when f is an injective function.

Theorem 8.21. *Let Ω be a symmetric slice domain, and let $f : \Omega \to \mathbb{H}$ be a nonconstant regular function. Then its singular set N_f has empty interior. Moreover, for each $q_0 = x_0 + y_0I \in N_f$, there exist an $n > 1$, a neighborhood U of q_0 and a neighborhood T of $x_0 + y_0\mathbb{S}$ such that for all $q_1 \in U$, the sum of the total* *multiplicities of the zeros of* $f - f(q_1)$ *in* T *equals* n*; in particular, for all* $q_1 \in U \setminus N_f$*, the preimage of* $f(q_1)$ *includes at least two distinct points of* T*.*

Proof. Since f is not constant, by Proposition 7.6 the singular set N_f has empty interior if and only if $N_f \setminus D_f$ does. By Proposition 8.18, the rank of f_* equals 2 at each point of $N_f \setminus D_f$. If $N_f \setminus D_f$ contained a nonempty open set A, its image $f(A)$ could not be open by the constant rank theorem, and the Open Mapping Theorem 7.7 would be contradicted.

In order to prove our second statement, let us introduce the notation $f_{q_1}(q) = f(q) - f(q_1)$ for $q_1 \in \Omega$. By Proposition 8.20, if $q_0 = x_0 + y_0 I \in N_f$, then the total multiplicity n of f_{q_0} at $x_0 + y_0\mathbb{S}$ is strictly greater than 1. We will now make use of the operation of symmetrization (see Definition 1.35). By Proposition 6.14, the sphere $x_0 + y_0\mathbb{S}$ has total multiplicity $2n$ for $f_{q_0}^s$, that is, there exists a regular $h : \Omega \to \mathbb{H}$ having no zeros in $x_0 + y_0\mathbb{S}$ such that $f_{q_0}^s(q) = [(q - x_0)^2 + y_0^2]^n h(q)$ and

$$f_{q_0}^s(z) = \left[(z - x_0)^2 + y_0^2\right]^n h(z) = (z - q_0)^n (z - \bar{q}_0)^n h(z)$$

for all $z \in \Omega_I$. Furthermore, since $f_{q_0}^s$ is slice preserving, the restriction of $f_{q_0}^s$ to Ω_I can be viewed as a complex-valued holomorphic function. Let us choose an open 2-disc $\Delta = \Delta_I(q_0, R)$ centered at q_0 such that $f_{q_0}^s$ has no zeros in $\overline{\Delta} \setminus \{q_0\}$, with $\overline{\Delta}$ included both in Ω_I and in the half-plane of L_I that contains q_0. If we denote by F_{q_1} the restriction of $f_{q_1}^s$ to Δ, then $q_1 \mapsto F_{q_1}$ is continuous in the topology of compact uniform convergence, and F_{q_0} has a zero of multiplicity n at q_0 (and no other zero). We claim that there exists a neighborhood U of q_0 such that for all $q_1 \in U$ the sum of the multiplicities of the zeros of F_{q_1} in Δ is n: if this were not the case, it would be possible to construct a sequence $\{q_k\}_{k\in\mathbb{N}}$ converging to q_0 such that $\{F_{q_k}\}_{k\in\mathbb{N}}$ contradicted Hurwitz's Theorem (in the version of [38]). Now let

$$T = T(x_0 + y_0\mathbb{S}, R) = \{x + yJ : |x - x_0|^2 + |y - y_0|^2 < R^2, J \in \mathbb{S}\}$$

be its symmetric completion. Then for all $q_1 \in U$, $2n$ equals the sum of the spherical multiplicities of the zeros of $f_{q_1}^s$ in T. Hence, there exist points $q_2, \dots, q_n \in \Omega$ and a function $h : \Omega \to \mathbb{H}$ having no zeros in T such that

$$f_{q_1}(q) = (q - q_1) * \cdots * (q - q_n) * h(q).$$

Now let us suppose $q_1 = x_1 + y_1 J \in U \setminus N_f$. Then f_* is invertible at q_1, and, by Proposition 8.20, for every $l \in \{2, \dots, n\}$, the point q_2 cannot belong to the sphere $x_1 + y_1\mathbb{S}$. Hence, $f_{q_1}(q)$ has at least 2 zeros in T, and the preimage of $f(q_1)$ via f intersects T at least twice. □

Corollary 8.22. *Let* Ω *be a symmetric slice domain, and let* $f : \Omega \to \mathbb{H}$ *be a regular function. If* f *is injective, then its singular set* N_f *is empty.*

Bibliographic Notes

The construction of spherical series and expansions was undertaken in [122], which also presents the applications to the computation of multiplicities and to differentiation. The study of the rank of the real differential was accomplished in [54].

Chapter 9
Fractional Transformations and the Unit Ball

9.1 Transformations of the Quaternionic Space

Our description of the regular transformations of the quaternionic space begins by considering those affine transformations of $\mathbb{H}$ which are regular.

Definition 9.1. To every pair $(a, b) \in \mathbb{H}^2$ with $a \neq 0$, we associate the *regular affine transformation* $l := l_{(a,b)} : \mathbb{H} \to \mathbb{H}$, $l(q) = qa + b$.

All regular affine transformations are regular and bijective. Moreover, it is easy to prove that the composition of two regular affine transformations is still a regular affine transformation and that the inverse of $l(q) = qa+b$ is $l^{-1}(q) = qa^{-1} - ba^{-1}$.

Remark 9.2. The set $\mathcal{A}$ of regular affine transformations is a group with respect to the composition operation.

We can prove that $\mathcal{A}$ coincides with the group

$$Aut(\mathbb{H}) = \{f : \mathbb{H} \to \mathbb{H} \text{ biregular}\}$$

of regular functions $\mathbb{H} \to \mathbb{H}$ having regular inverse. The proof makes use of Corollary 5.34 and of the following property of polynomials. This property is not as obvious as it is in the complex case, since a polynomial of degree $n > 1$ does not necessarily have more than one root (see Example 3.28).

Lemma 9.3. *Let $f(q) = q^n a_n + \ldots + qa_1 + a_0$ be a polynomial of degree n. If f is injective, then $n = 1$.*

Proof. Suppose $n > 1$. Since $f(0) = a_0$ and f is injective, f does not equal a_0 at any point other than 0. In other words, $f(q) - a_0 = q^n a_n + \ldots + qa_1 = q$ $(q^{n-1}a_n + \ldots + qa_2 + a_1)$ only vanishes at $q = 0$. Thus $q^{n-1}a_n + \ldots + a_1$ can only vanish at 0. For $n > 1$, by the Fundamental Theorem of Algebra 3.18, $q^{n-1}a_n + \ldots + qa_2 + a_1$ must have a root. Thus it vanishes at 0, and we conclude $a_1 = 0$. Iterating this process shows that $a_j = 0$ for $1 \leq j \leq n-1$, so that f reduces

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Springer Monographs in Mathematics, DOI 10.1007/978-3-642-33871-7_9,

to $f(q) = q^n a_n + a_0$. We then find a contradiction: for $n > 1$, the monomial q^n is clearly not injective and neither is $q^n a_n + a_0$. □

Theorem 9.4. *A function $f : \mathbb{H} \to \mathbb{H}$ is regular and injective if and only if there exist $a, b \in \mathbb{H}$ with $a \neq 0$ such that $f(q) = qa + b$ for all $q \in \mathbb{H}$.*

Proof. Let $f : \mathbb{H} \to \mathbb{H}$ be an injective regular function and define $g : \mathbb{H} \setminus \{0\} \to \mathbb{H}$ by $g(q) = f(q^{-1})$. By Lemma 1.32, the function g is regular. Recall that $\mathbb{B}$ denotes the unit ball $B(0, 1)$. If 0 were an essential singularity for g then, by Corollary 5.34, $g(\mathbb{B} \setminus \{0\})$ would be dense in $\mathbb{H}$. By Theorem 7.4, $f(\mathbb{B})$ is an open subset of $\mathbb{H}$. We would then have

$$\emptyset \neq g(\mathbb{B} \setminus \{0\}) \cap f(\mathbb{B}) = f(\mathbb{H} \setminus \mathbb{B}) \cap f(\mathbb{B}),$$

and f would not be injective, a contradiction with the hypothesis. Therefore, the origin must be a pole for g, and f is a polynomial. By the previous lemma, f has degree 1 as desired. □

Corollary 9.5. *The set $Aut(\mathbb{H})$ of biregular functions on $\mathbb{H}$ is a group, and it coincides with the group of regular affine transformations.*

9.2 Regular Fractional Transformations

As in the complex case, adding the *reciprocal function* $\rho(q) = q^{-1}$ to the group of regular affine transformations generates a group of transformations of $\widehat{\mathbb{H}} = \mathbb{H} \cup \{\infty\} \cong \mathbb{HP}^1$, which we also call the *quaternionic Riemann sphere*. Let us begin by recalling that $GL(2, \mathbb{H})$ denotes the set of the invertible 2×2 quaternionic matrices, that is

$$GL(2, \mathbb{H}) = \left\{ A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} : a, b, c, d \in \mathbb{H}, det_{\mathbb{H}}(A) \neq 0 \right\},$$

were $det_{\mathbb{H}}$ denotes the *Dieudonné determinant*

$$det_{\mathbb{H}}(A) = \sqrt{|a|^2|d|^2 + |c|^2|b|^2 - 2Re(c\overline{a}b\overline{d})}$$

(see [53]).

Definition 9.6. For every $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in GL(2, \mathbb{H})$, the continuous function $F_A : \widehat{\mathbb{H}} \to \widehat{\mathbb{H}}$ defined by

$$F_A(q) = (qc + d)^{-1}(qa + b) \tag{9.1}$$

is called the *linear fractional transformation* associated to A. Let us set $\mathbb{G} = \{F_A : A \in GL(2, \mathbb{H})\}$.

The next result is classical (for a proof, see [7]).

Theorem 9.7. *The set $\mathbb{G}$ is a group with respect to the composition operation $\circ$, and the map*

$$\begin{aligned} \Phi : GL(2,\mathbb{H}) &\to \mathbb{G} \\ A &\mapsto F_A \end{aligned} \tag{9.2}$$

is a surjective group antihomomorphism. If $\mathbb{I}$ denotes the 2×2 identity matrix, then $Ker(\Phi) = \{t\mathbb{I} : t \in \mathbb{R} \setminus \{0\}\}$. The restriction of Φ to the special linear group $SL(2,\mathbb{H}) = \{A \in GL(2,\mathbb{H}) : det_{\mathbb{H}}(A) = 1\}$ is still surjective, and it has kernel $\{\pm\mathbb{I}\}$.

It is not difficult to deduce from Theorem 9.7 that every linear fractional transformation is a homeomorphism from $\widehat{\mathbb{H}}$ to itself. On the other hand, despite the regularity of the generators of $\mathbb{G}$, not all linear fractional transformations are regular. Indeed, given a regular function f and a regular affine transformation l, say $l(q) = q\alpha + \beta$, the composition $l \circ f = f\alpha + \beta$ is still regular, but we cannot say the same for the composition $F \circ f$ of a linear fractional transformation F with a regular function f. For instance, the linear fractional transformation $q \mapsto c^{-1}q^{-1} = (qc)^{-1}$ is not regular if $c \in \mathbb{H} \setminus \mathbb{R}$, even though $q \mapsto qc$ is. This motivates the following construction:

Definition 9.8. For any $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in GL(2,\mathbb{H})$, we define the *regular fractional transformation* $\mathcal{F}_A$ associated to A as

$$\mathcal{F}_A(q) = (qc + d)^{-*} * (qa + b), \tag{9.3}$$

and we set $\mathcal{G} = \{\mathcal{F}_A : A \in GL(2,\mathbb{H})\}$.

Recall that, in Theorem 5.6, we defined the ring of quotients of regular quaternionic functions on a symmetric slice domain Ω

$$\mathcal{L}(\Omega) = \{f^{-*} * g : f, g \text{ regular in } \Omega,\ f \not\equiv 0\}.$$

Theorem 9.9. *For all $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in GL(2,\mathbb{H})$ and for all $f \in \mathcal{L}(\Omega)$, we can define a right action of $GL(2,\mathbb{H})$ on $\mathcal{L}(\Omega)$ by setting*

$$f.A = (fc + d)^{-*} * (fa + b). \tag{9.4}$$

The normal subgroup

$$N = \{t\mathbb{I} : t \in \mathbb{R} \setminus \{0\}\} \tag{9.5}$$

of $GL(2,\mathbb{H})$ is included in the stabilizer of any element of $\mathcal{L}(\Omega)$, and the action of $GL(2,\mathbb{H})/N \cong SL(2,\mathbb{H})/\{\pm\mathbb{I}\} \cong \mathbb{G}$ is faithful.

Proof. If $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and $B = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$, then

$$AB = \begin{pmatrix} a\alpha + c\beta & a\gamma + c\delta \\ b\alpha + d\beta & b\gamma + d\delta \end{pmatrix}.$$

We compute:

$$(f.A).B = \{(f.A)\gamma + \delta\}^{-*} * \{(f.A)\alpha + \beta\} =$$
$$= \left\{(fc+d)^{-*} * [(fa+b)\gamma + (fc+d)\delta]\right\}^{-*} *$$
$$* \left\{(fc+d)^{-*} * [(fa+b)\alpha + (fc+d)\beta]\right\} =$$
$$= [(fa+b)\gamma + (fc+d)\delta]^{-*} * (fc+d) * (fc+d)^{-*} * [(fa+b)\alpha + (fc+d)\beta] =$$
$$= [f(a\gamma + c\delta) + b\gamma + d\delta]^{-*} * [f(a\alpha + c\beta) + b\alpha + d\beta] = f.AB$$

Moreover, $f.\mathbb{I} = (0+1)^{-*} * (f1+0) = 1^{-1}f = f$. The normal subgroup $N = \{t\mathbb{I} : t \in \mathbb{R} \setminus \{0\}\}$ is included in the stabilizer of any element $f \in \mathcal{L}(\Omega)$ because

$$(fc+d)^{-*} * (fa+b) = f,$$

if and only if

$$fa + b = (fc+d) * f,$$

if and only if

$$fa + b = fc * f + d * f,$$

and the last equality holds whenever $b = c = 0, d = a = t \in \mathbb{R}$. Conversely, the equality $fa + b = fc * f + d * f$ holds for all $f \in \mathcal{L}(\Omega)$ if, and only if, $b = c = 0, d = a = t \in \mathbb{R}$. This proves that the action of $GL(2,\mathbb{H})/N$ is faithful.

□

We point out that not all stabilizers coincide with $N = \{t\mathbb{I} : t \in \mathbb{R} \setminus \{0\}\}$. For instance, the stabilizer of the identity function is the subgroup

$$\{a\mathbb{I} : a \in \mathbb{H} \setminus \{0\}\}$$

of $GL(2,\mathbb{H})$, since

$$(qc+d)^{-*} * (qa+b) = q,$$

if and only if

$$qa + b = q^2c + qd,$$

if and only if

$$b = c = 0, d = a.$$

In particular, the faithful action of $GL(2,\mathbb{H})/N$ is not free.

Proposition 9.10. *The set $\mathcal{G}$ of regular fractional transformations is the orbit of the identity function in $\mathcal{L}(\mathbb{H})$ under the right action of $GL(2,\mathbb{H})$. Two elements $\mathcal{F}_A, \mathcal{F}_B \in \mathcal{G}$ coincide if, and only if, there exists $c \in \mathbb{H} \setminus \{0\}$ such that $B = cA$. In particular, for all $\mathcal{F} \in \mathcal{G}$, either $\mathcal{F}$ is a regular affine transformation or there exist* *(unique) $a, b, p \in \mathbb{H}$ such that*

$$\mathcal{F}(q) = (q - p)^{-*} * (qa + b). \tag{9.6}$$

Proof. By definition, $\mathcal{F}_A = id.A$. Thus $\mathcal{G}$ is the orbit of $id(q) = q$ under the action of $GL(2, \mathbb{H})$. We already observed that $\{c\mathbb{I} : c \in \mathbb{H} \setminus \{0\}\}$ is the stabilizer of id; hence, $\mathcal{F}_A = \mathcal{F}_B$ if and only if $id = \mathcal{F}_{BA^{-1}}$ if and only if $BA^{-1} = c\mathbb{I}$ for some $c \neq 0$ if and only if $B = cA$ for some $c \neq 0$. Now consider any $\mathcal{F} = \mathcal{F}_M$ with

$$M = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \in GL(2, \mathbb{H}).$$

If $\mathcal{F}(q) = (q\gamma + \delta)^{-*} * (q\alpha + \beta)$ is not a regular affine transformation, then $\gamma \neq 0$ and $\mathcal{F} = \mathcal{F}_{\gamma^{-1}M}$ with

$$\gamma^{-1}M = \begin{pmatrix} \gamma^{-1}\alpha & 1 \\ \gamma^{-1}\beta & \gamma^{-1}\delta \end{pmatrix} = \begin{pmatrix} a & 1 \\ b & -p \end{pmatrix}$$

where $a = \gamma^{-1}\alpha$, $b = \gamma^{-1}\beta$ and $p = -\gamma^{-1}\delta$. The thesis immediately follows. $\square$

We now observe that Proposition 5.32 relates $f.A = (fc + d)^{-*} * (fa + b)$ to $F_A \circ f = (fc + d)^{-1}(fa + b)$ as follows:

Lemma 9.11. *Let $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in GL(2, \mathbb{H})$ and let $f \in \mathcal{L}(\Omega)$. If $T = T_{fc+d}$ then*

$$f.A = F_A \circ f \circ T \tag{9.7}$$

wherever $(fc + d)^s$ does not vanish.

Proof. Set $g(q) = f(q)c + d$ and $h(q) = f(q)a + b$ and notice that, by definition, $f.A = g^{-*} * h$. Now, according to Proposition 5.32, $g^{-*} * h(q) = [g \circ T_g(q)]^{-1}[h \circ T_g(q)]$ with $T_g(q) = g^c(q)^{-1}qg^c(q)$. Moreover, $g \circ T_g(q) = f(T_g(q))c + d$ and $h \circ T_g(q) = f(T_g(q))a + b$ so that

$$f.A = g^{-*} * h = [(f \circ T_g)c + d]^{-1}[(f \circ T_g)a + b] = F_A \circ f \circ T_g$$

as desired. $\square$

9.3 Transformations of the Quaternionic Riemann Sphere

In the previous section we proposed the definition of regular fractional transformation without discussing the domains of definition of such functions. Clearly, a regular affine transformation $q \mapsto qa + b$ defines a homeomorphism from $\widehat{\mathbb{H}}$ onto itself. As for all other regular fractional transformations, Proposition 9.10 tells us that they are of the form $\mathcal{F}(q) = (q - p)^{-*} * (qa + b)$ for some $a, b, p \in \mathbb{H}$.

Theorem 9.12. *Let $\mathcal{F}$ be a regular fractional transformation of the type $\mathcal{F}(q) = (q - p)^{-*} * (qa + b)$ for some $a, b, p \in \mathbb{H}$ and let $F(q) = (q - p)^{-1}(qa + b)$. If $p = x \in \mathbb{R}$, then $\mathcal{F} = F$ and in particular $\mathcal{F}$ is a homeomorphism from $\widehat{\mathbb{H}}$ onto itself. If, on the contrary, $p = x + yI$ with $y \neq 0$, then $\mathcal{F}$ is a homeomorphism of $\widehat{\mathbb{H}} \setminus (x + y\mathbb{S})$ onto $\widehat{\mathbb{H}} \setminus F(x + y\mathbb{S})$ (where $F(x + y\mathbb{S})$ is a 2-plane) and $\mathcal{F}$ does not extend to $\widehat{\mathbb{H}}$ as a continuous function. Indeed, $\lim_{q \to x+yJ} \mathcal{F}(q) = \infty$ for all $J \neq -I$ and $\lim_{q \to \bar{p}} \mathcal{F}(q)$ is not defined.*

Proof. If $p = x \in \mathbb{R}$, then by direct computation $\mathcal{F}(q) = (q - x)^{-*} * (qa + b) = (q - x)^{-1}(qa + b) = F(q)$. Since we already observed that $F : \widehat{\mathbb{H}} \to \widehat{\mathbb{H}}$ is a homeomorphism, the first statement is proven. As for the second one, thanks to Lemma 9.11, $\mathcal{F}$ is related to the linear fractional transformation F by the formula $\mathcal{F} = F \circ T$, where

$$T(q) = (q - \bar{p})^{-1} q (q - \bar{p})$$

is a homeomorphism from $\widehat{\mathbb{H}} \setminus (x + y\mathbb{S})$ onto itself. By Lemma 3.21, T maps all points of $(x + y\mathbb{S}) \setminus \{\bar{p}\}$ to p. On the other hand, the restriction of T to the complex line $\mathbb{R} + I\mathbb{R}$ through p and $\bar{p}$ coincides with the identity function of $\mathbb{R} + I\mathbb{R}$, so that

$$\lim_{z \to \bar{p}, (z \in L_I)} T(z) = \bar{p}.$$

We conclude that T does not extend to $\widehat{\mathbb{H}}$ as a continuous function and, as a consequence, neither does $\mathcal{F} = F \circ T$. Finally, since T maps $\widehat{\mathbb{H}} \setminus (x + y\mathbb{S})$ onto itself and $\mathcal{F} = F \circ T$, we observe that $\mathcal{F}$ maps $\widehat{\mathbb{H}} \setminus (x + y\mathbb{S})$ onto $\widehat{\mathbb{H}} \setminus F(x + y\mathbb{S})$. According to [7], $F(x + y\mathbb{S})$ is either a 2-sphere or a plane. Since $F(x + yI) = \infty$, we conclude that $F(x + y\mathbb{S})$ is a plane. □

Now let us present our characterization of the regular transformations of $\widehat{\mathbb{H}}$.

Theorem 9.13. *Let $f : \widehat{\mathbb{H}} \to \widehat{\mathbb{H}}$ be a continuous, injective function. If f is semiregular in $\mathbb{H}$, then either f is a regular affine transformation or it is a regular fractional transformation with $f(x) = \infty$ at a real point $x \in \mathbb{R}$.*

Proof. If $f(\mathbb{H}) \subseteq \mathbb{H}$, then $f_{|\mathbb{H}} : \mathbb{H} \to \mathbb{H}$ is an injective regular map. By Theorem 9.4, f must be a regular affine transformation.

If, on the contrary, f equals ∞ at some point of $\mathbb{H}$, then such point is unique by the injectivity of f. By the results presented in Sect. 5.4, this can only happen if the point is real, say $x \in \mathbb{R}$. Setting $\tilde{f}(q) = f(q^{-1} + x)$ defines an injective function $\tilde{f} : \widehat{\mathbb{H}} \to \widehat{\mathbb{H}}$ which is semiregular in $\mathbb{H}$. Since $\tilde{f}(\infty) = f(x) = \infty$ and $\tilde{f}$ is injective, $\tilde{f}$ cannot have a pole in $\mathbb{H} = \widehat{\mathbb{H}} \setminus \{\infty\}$. By the first part of the proof, $\tilde{f}$ is a regular affine transformation, say $\tilde{f}(q) = qa + b$ for some $a, b \in \mathbb{H}, a \neq 0$. Hence $f(q) = \tilde{f}((q - x)^{-1}) = (q - x)^{-1} a + b = (q - x)^{-1}(qb + a - xb) = (q - x)^{-*} * (qb + a - xb)$. □

The two theorems above yield:

Corollary 9.14. *Let f be semiregular in $\mathbb{H}$. Then the following properties are equivalent:*

1. *f extends to a homeomorphism from $\widehat{\mathbb{H}}$ onto itself.*
2. *f extends to a continuous, injective function on $\widehat{\mathbb{H}}$.*
3. *f is a regular fractional transformation having its pole in $\widehat{\mathbb{R}}$.*
4. *f is both a regular fractional transformation and a linear fractional transformation.*

Proof. If f extends to a homeomorphism from $\widehat{\mathbb{H}}$ onto itself, then obviously f extends to a continuous, injective function on $\widehat{\mathbb{H}}$. By the previous theorem, if $f : \widehat{\mathbb{H}} \to \widehat{\mathbb{H}}$ is continuous and injective and it is semiregular in $\mathbb{H}$, then it is a regular fractional transformation with pole at ∞ or in $\mathbb{R}$. Such regular fractional transformations are exactly those elements $\mathcal{F}$ of $\mathcal{G}$ which can be written in the form $\mathcal{F}(q) = (qr + t)^{-*} * (qa + b)$ for some $r, t \in \mathbb{R}$. In this case we compute $\mathcal{F}(q) = (qr + t)^{-*} * (qa + b) = (qr + t)^{-1}(qa + b)$ so that $\mathcal{F}$ is also a linear fractional transformation, that is, an element of $\mathbb{G}$. Finally, if $\mathcal{F} \in \mathcal{G} \cap \mathbb{G}$, then $\mathcal{F}$ is a linear fractional transformation and in particular it is a homeomorphism from $\widehat{\mathbb{H}}$ onto itself. □

9.4 Schwarz Lemma and Transformations of the Unit Ball

As in the complex case, the basic tool in the study of the open unit ball $\mathbb{B} = B(0, 1)$ is the Schwarz Lemma. Recall that f' denotes the slice derivative of a regular function f (see Sect. 1.1).

Theorem 9.15 (Schwarz Lemma). *Let $f : \mathbb{B} \to \mathbb{B}$ be a regular function. If $f(0)=0$, then*

$$|f(q)| \leq |q| \tag{9.8}$$

for all $q \in \mathbb{B}$ and

$$|f'(0)| \leq 1. \tag{9.9}$$

Both inequalities are strict (except at $q = 0$) unless $f(q) = qu$ for some $u \in \partial\mathbb{B}$.

Proof. Since $f(0) = 0$, there exists a regular $g : \mathbb{B} \to \mathbb{H}$ such that $f(q) = q * g(q) = qg(q)$. Whenever $0 < R < 1$, we have for all $|q| = R$

$$|g(q)| = \frac{|f(q)|}{|q|} \leq \frac{1}{|q|} = \frac{1}{R}.$$

By Corollary 7.2, $|g(q)| \leq \frac{1}{R}$ for all $q \in B(0, R)$. Letting R tend to 1^-, we conclude that $|g(q)| \leq 1$ for all $q \in \mathbb{B}$. This yields both inequalities since $g(q) = q^{-1} f(q)$ for all $q \neq 0$ and $g(0) = f'(0)$. If one of the inequalities is not strict, then $|g|$

has a relative maximum point and g must be constant by the Maximum Modulus Principle. The thesis immediately follows. $\square$

We now study the regular transformations of $\mathbb{B}$. Recall the notations $\mathbb{G}, \mathcal{G}$ from the previous section, let

$$Sp(1,1) = \{C \in GL(2,\mathbb{H}) : \overline{C}^t HC = H\}$$

with $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and let $\mathbb{M} = \{F \in \mathbb{G} : F(\mathbb{B}) = \mathbb{B}\}$.

Theorem 9.16. *For all $A \in SL(2,\mathbb{H})$, the linear fractional transformation F_A maps $\mathbb{B}$ onto itself if and only if $A \in Sp(1,1)$, and this holds if and only if there exist $u, v \in \partial\mathbb{B}, a \in \mathbb{B}$ such that*

$$F_A(q) = v^{-1}(1 - q\bar{a})^{-1}(q - a)u \tag{9.10}$$

for all $q \in \mathbb{B}$. In particular the antihomomorphism $\Phi(A) = F_A$ can be restricted to a surjective group antihomomorphism $\Phi : Sp(1,1) \to \mathbb{M}$ with kernel $Ker(\Phi) = \{\pm\mathbb{I}\}$.

For a proof, see [7]. An analogous result holds for regular fractional transformations.

Corollary 9.17. *For all $A \in SL(2,\mathbb{H})$, the regular fractional transformation $\mathcal{F}_A$ maps $\mathbb{B}$ onto itself if and only if $A \in Sp(1,1)$, and this holds if and only if there exist (unique) $u \in \partial\mathbb{B}, a \in \mathbb{B}$ such that*

$$\mathcal{F}_A(q) = (1 - q\bar{a})^{-*} * (q - a)u \tag{9.11}$$

for all $q \in \mathbb{B}$. In particular, the set $\mathcal{M} = \{f \in \mathcal{G} : f(\mathbb{B}) = \mathbb{B}\}$ of regular Möbius transformations of $\mathbb{B}$ *is the orbit of the identity function under the action of $Sp(1,1)$.*

Proof. $\mathcal{F}_A$ maps $\mathbb{B}$ onto $\mathbb{B}$ if and only of F_A does (indeed, by Lemma 9.11, $\mathcal{F}_A = F_A \circ T$, where T maps $\mathbb{B}$ onto itself). Moreover, $F_A(q) = v^{-1}(1-q\bar{a})^{-1}(q-a)u = (v - q\bar{a}v)^{-1}(q - a)u$ implies

$$\mathcal{F}_A(q) = (v - q\bar{a}v)^{-*} * (q - a)u =$$

$$= (1 - q\bar{v}\bar{a}v)^{-*} * \bar{v} * (q - a)u = (1 - q\bar{v}\bar{a}v)^{-*} * (q\bar{v} - \bar{v}a)u =$$
$$= (1 - q\bar{v}\bar{a}v)^{-*} * (q - \bar{v}av)\bar{v}u = (1 - q\bar{\alpha})^{-*} * (q - \alpha)\nu$$

with $\alpha = \bar{v}av \in \mathbb{B}$ and $\nu = \bar{v}u \in \partial\mathbb{B}$. $\square$

Finally, we prove that all regular bijections $\mathbb{B} \to \mathbb{B}$ are regular Möbius transformations. Two preliminary steps are required.

Lemma 9.18. *If $f : \mathbb{B} \to \mathbb{B}$ is regular and bijective, then for all $v \in \partial\mathbb{B}$,* $\lim_{q\to v} |f(q)| = 1$.

Proof. Fix $v \in \partial\mathbb{B}$ and let $\{q_n\}_{n\in\mathbb{N}}$ be a sequence converging to v. The sequence $\{|f(q_n)|\}_{n\in\mathbb{N}}$ must have a limit point since it is bounded. Let us show, by contradiction, that every such limit point is 1.

If $\{|f(q_n)|\}_{n\in\mathbb{N}}$ had a limit point $t < 1$, there would exist a subsequence converging to t and, by relative compactness, $\{f(q_n)\}_{n\in\mathbb{N}}$ would admit a subsequence $\{f(q_{n_k})\}_{k\in\mathbb{N}}$ converging to a point μ with $|\mu| = t < 1$. By the surjectivity of f, there would exist $q \in \mathbb{B}$ such that $f(q) = \mu$. Choosing an open neighborhood U of q in $\mathbb{B}$ such that $v \notin \overline{U}$, there would exist an $N \in \mathbb{N}$ such that $q_n \notin U$ for all $n \geq N$. By the injectivity of f, we could conclude that $f(q_n) \notin f(U)$ for $n \geq N$. But according to the Open Mapping Theorem 7.7, $f(U)$ would be an open neighborhood of μ. We would then have a contradiction with the fact that $f(q_{n_k}) \to \mu$ as $k \to +\infty$. □

Lemma 9.19. *If $f : \mathbb{B} \to \mathbb{B}$ is regular and bijective and if $f(0) = 0$, then there exists $u \in \partial\mathbb{B}$ such that $f(q) = qu$ for all $q \in \mathbb{B}$.*

Proof. There exist $n \in \mathbb{N}$ and a regular function g on $\mathbb{B}$ such that $f(q) = q^n * g(q) = q^n g(q)$ and $g(0) \neq 0$. Applying Lemma 9.18 to f proves that, for all $v \in \partial\mathbb{B}$,

$$\lim_{q\to v} |g(q)| = \lim_{q\to v} \frac{|f(q)|}{|q|^n} = 1.$$

By Corollary 7.2, $|g| \leq 1$ and $g \equiv u$ for some $u \in \partial\mathbb{B}$ if the inequality is not strict. We prove that it is not strict reasoning as follows: if there existed a point $q \in \mathbb{B}$ with $|g(q)| < 1$, then g would have a minimum modulus point $p \in \mathbb{B}$; by the Minimum Modulus Principle, $g(p) = 0$; this would imply $f(p) = 0 = f(0)$ (with $p \neq 0$ since $g(0) \neq 0$ by construction); we would then find a contradiction with the injectivity assumption for f. Hence there exists $u \in \partial\mathbb{B}$ such that $g \equiv u$ and $f(q) = q^n u$. By the injectivity of f and by Lemma 9.3, $n = 1$. □

We now come to the main result of this section.

Theorem 9.20. *If $f : \mathbb{B} \to \mathbb{B}$ is regular and bijective, then there exist $a \in \mathbb{B}, u \in \partial\mathbb{B}$ such that $f(q) = (1 - q\bar{a})^{-*} * (q - a)u$ for all $q \in \mathbb{B}$.*

Proof. Let $c = f(0) \in \mathbb{B}$, $C = \begin{pmatrix} 1 & -\bar{c} \\ -c & 1 \end{pmatrix}$ and

$$\tilde{f} = f.C = (1 - f\bar{c})^{-*} * (f - c).$$

The function $\tilde{f}$ is regular in $\mathbb{B}$ because $1 - f\bar{c}$ does not have zeros in $\mathbb{B}$ (since $|c| < 1$ and $|f(q)| < 1$ for $|q| < 1$). Now recall that by Lemma 9.11,

$$\tilde{f} = f.C = F_C \circ f \circ T,$$

where $T = T_{1-f\bar{c}}$ is a bijection of $\mathbb{B}$ onto $\mathbb{B}$. Since $f : \mathbb{B} \to \mathbb{B}$ is bijective and $F_C(q) = (1 - q\bar{c})^{-1}(q - c)$ is a Möbius transformation of $\mathbb{B}$, the function $\tilde{f}$ is a bijection of $\mathbb{B}$ onto $\mathbb{B}$, too. Moreover,

$$\tilde{f}(0) = F_C \circ f \circ T(0) = F_C \circ f(0) = F_C(c) = 0.$$

By Theorem 9.19, we conclude that $\tilde{f}(q) = qu$ for some $u \in \partial\mathbb{B}$. Now, $f = (f.C).C^{-1} = \tilde{f}.C^{-1}$, where C^{-1} coincides, up to a real multiplicative constant, with the matrix $\begin{pmatrix} 1 & \bar{c} \\ c & 1 \end{pmatrix}$. Thus

$$f(q) = (1 + \tilde{f}\bar{c})^{-*} * (\tilde{f} + c) = (1 + qu\bar{c})^{-*} * (qu + c) =$$

$$= (1 + q\overline{c\bar{u}})^{-*} * (q + c\bar{u})u,$$

as desired. □

Additionally, we are able to establish the following property:

Proposition 9.21. *For all $a, b \in \mathbb{B}$, there exists a transformation $\mathcal{F} \in \mathcal{M}$ mapping a to b.*

Proof. For $b = 0$, it suffices to set $\mathcal{F} = \mathcal{F}_A$ with

$$A = \begin{pmatrix} 1 & -\bar{a} \\ -a & 1 \end{pmatrix}.$$

Indeed, by direct computation, $\mathcal{F}_A(q) = (1 - q\bar{a})^{-*} * (q - a)$ maps a to 0. Otherwise, we set $\mathcal{F} = \mathcal{F}_{ABM}$ with the same matrix A and

$$B = \begin{pmatrix} 1 & |b| \\ |b| & 1 \end{pmatrix}, \ M = \begin{pmatrix} \frac{b}{|b|} & 0 \\ 0 & 1 \end{pmatrix}.$$

Indeed,

$$\mathcal{F}_{AB} = (\mathcal{F}_A).B = F_B \circ \mathcal{F}_A \circ T$$

where $T(q) = (1 + \mathcal{F}_A^c(q)|b|)^{-1} q (1 + \mathcal{F}_A^c(q)|b|)$. Now, $1 + \mathcal{F}_A^c(a)|b|$ belongs to the same complex line as a; hence, $T(a) = a$. We conclude that $\mathcal{F}_A \circ T$ maps a to 0. Furthermore,

$$F_B(q) = (1 + q|b|)^{-1}(q + |b|)$$

clearly maps 0 to $|b|$. Hence $\mathcal{F}_{AB}$ maps a to $|b|$. Finally, $\mathcal{F}_{ABM} = \mathcal{F}_{AB}\frac{b}{|b|}$ maps a to b. □

9.5 Rigidity and a Boundary Schwarz Lemma

Theorem 9.15 extends the classical Schwarz Lemma to regular functions. In this section we investigate the possible extensions of the classical Cartan Uniqueness Theorem. We begin with the following rigidity-type generalization of Theorem 9.15:

Theorem 9.22. *Assume that $f : \mathbb{B} \to \mathbb{B}$ is a regular function and there exists $q_0 \in \mathbb{B}$ such that $f(q_0) = q_0$ and $f'(q_0) = 1$. Then $f(q) = q$ for every $q \in \mathbb{B}$.*

Proof. Let $I \in \mathbb{S}$ be such that $q_0 \in L_I$ and let us split f_I as

$$f_I(z) = F(z) + G(z)J$$

where $J \perp I$ and F and G are holomorphic self-maps of $\mathbb{B}_I$. Then

$$|f_I(z)|^2 = |F(z)|^2 + |G(z)|^2 \tag{9.12}$$

and

$$f_I' = F' + G'J. \tag{9.13}$$

Since $q_0 \in L_I$, we have $F(q_0) = q_0$, $G(q_0) = 0$, $F'(q_0) = 1$ and $G'(q_0) = 0$. Therefore, by the complex Cartan Uniqueness Theorem, $F(z) \equiv z$. Moreover, equality (9.12) and the Maximum Modulus Principle 7.1 imply that $G(z) \equiv 0$, so that $f_I(z) \equiv z$ on $\mathbb{B}_I$. By the Identity Principle 1.12, $f(q) = q$ for all $q \in \mathbb{B}$. □

The term "rigidity" usually describes "germ properties" showed by holomorphic functions f on a complex domain D with additional geometric assumptions on $f(D)$. Specifically, for certain classes of such holomorphic functions, it is sometimes possible to describe—in terms of the first coefficients of the Taylor expansions at a given point of D—what are the minimal *local* conditions which guarantee *global* information on the functions. Typically, one aims at proving that the function considered is the identity or that it is constant. As a further example, we present a regular analog of a result that is classical in the complex setting (see [110]).

Theorem 9.23. *Let f be a regular self-map of the unit ball $\mathbb{B}$ of $\mathbb{H}$, and suppose that there exists $r \in \mathbb{B} \cap \mathbb{R}$ such that $f(r) = r$. Then either f has no other fixed points in $\mathbb{B}$ or f is the identity function.*

Proof. If f has another fixed point in $\mathbb{B}$, say $q_0 \in L_I$, let us split f_I as

$$f_I(z) = F(z) + G(z)J$$

where $J \perp I$ and F, G are holomorphic self-maps of $\mathbb{B}_I$. Since $f(r) = r$ and $f(q_0) = q_0$ imply that $F(r) = r$ and $F(q_0) = q_0$, the complex Schwarz–Pick Lemma (see [4]) yields that $F(z) = z$. From (9.12) and by the Maximum Modulus Principle 7.1, $G \equiv 0$. Hence, $f_I(z) = z$ for all $z \in \mathbb{B}_I$. By the Identity Principle 1.12, $f(q) = q$ for all $q \in \mathbb{B}$. □

A very interesting, but less known, consequence of the Schwarz Lemma is the Herzig Theorem (see [81]), which we generalize to the setting of regular functions.

Theorem 9.24. *If $f : \mathbb{B} \to \mathbb{B}$ is a regular function such that $f(0) = 0$ and if there exists $q_0 \in \partial\mathbb{B}$ with*

$$\lim_{t \to 1^-,\ (t \in \mathbb{R})} f(tq_0) = q_0, \tag{9.14}$$

then

$$\lim_{t \to 1^-,\ (t \in \mathbb{R})} f'(tq_0) > 1, \tag{9.15}$$

unless $f(q) = q$ for all $q \in \mathbb{B}$.

Proof. Take the restriction $f_I = F + GJ$ (with $J \perp I$) of f along the plane L_I that contains q_0. In view of the Schwarz Lemma 9.15, the assumptions on f and f' are clearly "inherited" by F and by F'. By the (complex) Herzig Theorem, F turns out to be the identity on L_I. So, as in Theorem 9.22, the thesis follows from the Maximum Modulus Principle 7.1 and the Identity Principle 1.12. □

We now want to consider and prove "boundary" generalizations of some Schwarz–Cartan-type theorems in the setting of regular functions. Specifically, we prove the following quaternionic version of the renowned Burns–Krantz Theorem [14].

Theorem 9.25. *Let f be a regular self-map of $\mathbb{B}$. Assume that there exists $q_0 \in \partial\mathbb{B}$ such that*

$$f(q) = q + o(|q - q_0|^3)$$

as $q \to q_0$. Then f is the identity function.

Proof. Assume now that for a self-map f, regular in $\mathbb{B}$, there exists $q_0 \in \partial\mathbb{B}$ such that

$$f(q) = q + o(|q - q_0|^3)$$

as $q \to q_0$. Then take the restriction $f_I = F + GJ$ (with $J \perp I$) of f along the plane L_I containing q_0; clearly on L_I, one obtains $F(z) = z + o(|z - w_0|^3)$ as $z \to w_0$ in $\mathbb{B}_I$, and then, by the Burns–Krantz Theorem, $F(z) = z$ for all $z \in \mathbb{B}_I$. By the Maximum Modulus Principle 7.1 and the Identity Principle 1.12, we conclude once again that f is the identity function. □

We end this section with some remarks concerning the quaternionic half space

$$\mathbb{H}^+ = \{q \in \mathbb{H} : Re(q) > 0\}.$$

Remark 9.26. The *Cayley transformation* $\psi : \mathbb{B} \to \mathbb{H}^+$, defined as

$$\psi(q) = (1 - q)^{-1}(1 + q), \tag{9.16}$$

has $\psi^{-1}(q) = (q - 1)(q + 1)^{-1}$ as its inverse function. Hence, ψ is biregular.

There is a version of the Burns–Krantz Theorem adapted to the quaternionic half-space from a complex result due to Migliorini and Vlacci [98].

Theorem 9.27. *Let* $f : \mathbb{B} \to \mathbb{H}^+$ *be a regular function such that* $f(q) = o(|1+q|)$ *as* $q \to -1$. *Then* $f \equiv 0$.

Proof. If, for any $I \in \mathbb{S}$, we consider the splitting $f_I = F + GJ$, then $F(z) = o(|1+z|)$ as $z \to -1$ in L_I and $Re(F(z)) > 0$ for any $z \in L_I$. According to [98], $F \equiv 0$ in $\mathbb{B}_I$. Therefore, $f_I(z) = G(z)J$. In other words, f_I is a holomorphic map in $\mathbb{B}_I \to \mathbb{H}^+$ such that $f_I(\mathbb{B}_I) \subset \partial\mathbb{H}^+$. On the other hand, if G expands as $G(z) = \sum_{n\in\mathbb{N}} z^n a_n$ (with $a_n \in L_I$ for all $n \in \mathbb{N}$), then

$$f(q) = \sum_{n\in\mathbb{N}} q^n a_n J.$$

Now, if $a_n = s_n + t_n I \in L_I$, then

$$a_n J = (s_n + t_n I)J = s_n J - t_n J I,$$

since $I \perp J$. Therefore, the restriction f_J splits as

$$f_J(q) = \sum_{n\in\mathbb{N}} q^n s_n J - \left(\sum_{n\in\mathbb{N}} q^n J t_n\right) I.$$

Thus, by the same argument used above for F, one concludes that

$$f_J(q) = -\left(\sum_{n\in\mathbb{N}} q^n J t_n\right) I$$

or, if $K = IJ$,

$$f_J(q) = \sum_{n\in\mathbb{N}} q^n t_n K.$$

Again, from the uniqueness of the series expansion for regular maps,

$$f_K(q) = \sum_{n\in\mathbb{N}} q^n t_n K.$$

But now f_K is nothing but a holomorphic function from $\mathbb{B}_K$ to $\mathbb{H}^+_K$ to which the result of [98] applies again. So $f_K \equiv 0$ and $f \equiv 0$. □

9.6 Borel–Carathéodory Theorem

This section presents a regular analog of the well-known Borel–Carathéodory Theorem. (For a Clifford-analytic version, see [78].)

Theorem 9.28 (Borel–Carathéodory). *Let $q_0 \in \mathbb{R}$, let $B = B(q_0, r)$ and let f be a regular function on (a neighborhood of) $\overline{B(q_0, r)}$. Set*

$$A = \max_{|q-q_0|=r} |Re(f(q))|$$

and $f(q_0) = \beta + \gamma I$ with $\beta, \gamma \in \mathbb{R}$ and $I \in \mathbb{S}$. If $\varrho \in \mathbb{R}$ is such that $0 < \varrho < r$, then

$$|f(q)| \le |\gamma| + |\beta| \frac{r+\varrho}{r-\varrho} + 2A \frac{\varrho}{r-\varrho}$$

for all $q \in B$ such that $|q - q_0| \le \varrho$.

Proof. The thesis is obviously true if f is constant. If it is not, then its real part $Re(f(q))$ is not constant either, in view of Theorem 7.4. Let $J \in \mathbb{S}$ be such that $Re(f(q))$ is not constant on L_J. By construction, $\beta = Re(f(q_0)) \le A$, and in fact $\beta < A$. Indeed, suppose that $\beta = A$. By the Splitting Lemma 1.3, there exist $F, G : B_J \to L_J$ holomorphic functions, and $K \in \mathbb{S}$, K orthogonal to J, such that

$$f(z) = F(z) + G(z)K$$

for every $z \in B_J$. Then $Re(F(q_0)) = Re(f(q_0)) = A$. Since F is holomorphic, then $Re(F)$ is harmonic in B_J. Hence, if it attains its maximum at an interior point, it must be constant, and F is constant as well. As a consequence, $Re(f)$ is constant in B_J, a contradiction with our assumption. Hence, $\beta < A$.

Set $w_0 = f(q_0) - A$ and let $h(q) = f(q) - A$. Consider the function

$$H(q) = (f(q) - A + \overline{w_0})^{-*} * (f(q) - A - w_0). \tag{9.17}$$

The function H is regular for $|q - q_0| \le r$. Indeed, if the symmetrization $(f(q) - A + \overline{w_0})^s$ vanishes somewhere in the same ball, then $f(q) - A + \overline{w_0} = 0$ for some $q \in B(q_0, r)$. In particular its real part vanishes, namely,

$$0 = Re(f(q)) - A + Re(\overline{w_0}) \le A - A + \beta - A = \beta - A.$$

This implies $\beta \ge A$, a contradiction.

By Lemma 9.11, we can express $H(q)$ as

$$H(q) = g \circ h \circ T(q),$$

where

$$g(q) = (q + \overline{w_0})^{-1}(q - w_0)$$

and

$$T(q) = (h^c(q) + w_0)^{-1} q (h^c(q) + w_0).$$

Since $q_0 \in \mathbb{R}$ and since $g(h(q_0)) = g(w_0) = 0$, we easily get that $H(q_0) = 0$.

Moreover, $|H(q)| \leq 1$ in B. In order to prove this claim, we begin by noting that g maps the 3-space $\{q \in \mathbb{H} : Re(q) = 0\}$ onto the unit sphere $\mathbb{S}^3$. We recall that any linear fractional transformation maps the family $\mathcal{F}_3$ of 3-spheres and affine 3-subspaces onto itself (see [7] for details). Now, since $H(q_0) = 0$ and $Re(h(T(q)) \leq 0$ for all $q \in B$, we get that

$$|H(q)| = |g(h(T(q)))| \leq 1 \tag{9.18}$$

for all $q \in B$.

Furthermore, for all q such that $0 < |q - q_0| \leq \varrho < r$, the following inequality holds:

$$|H(q)| \leq \frac{\varrho}{r}. \tag{9.19}$$

In fact, consider the function $(q-q_0)^{-*} * H(q)$. Since $H(q_0) = 0$ and $q_0 \in \mathbb{R}$, this is a regular function and $(q-q_0)^{-*} * H(q) = (q-q_0)^{-1} H(q)$. Thus, by Theorem 7.2 and by inequality (9.18), we have

$$|(q - q_0)^{-*} * H(q)| = \frac{|H(q)|}{|q - q_0|} \leq \frac{1}{r} \tag{9.20}$$

for all $q \in B$. By Theorem 7.2, we also have that for all $q \in B(q_o, \varrho)$

$$\frac{|H(q)|}{|q - q_0|} \leq \max_{|q-q_0|=\varrho} \frac{|H(q)|}{|q - q_0|} = \max_{|q-q_0|=\varrho} \frac{|H(q)|}{\varrho}. \tag{9.21}$$

Since inequality (9.20) holds for all $q \in B$, we get that in $B(q_o, \varrho)$

$$|H(q)| \leq \frac{\varrho}{r}.$$

Now, we want to use this inequality to estimate the modulus $|f|$. Recall that

$$H(q) = (f(T(q)) - A + \overline{w_0})^{-1}(f(T(q)) - A - w_0).$$

Notice that if $q + \overline{w_0} \neq 0$, then

$$(q + \overline{w_0})^{-1}(q - w_0) = (q - w_0)(q + \overline{w_0})^{-1}$$

if, and only if,

$$(q - w_0)(q + \overline{w_0}) = (q + \overline{w_0})(q - w_0),$$

that is,

$$q^2 - w_0 q + q\overline{w_0} - |w_0|^2 = q^2 + \overline{w_0}q - qw_0 - |w_0|^2.$$

This is equivalent to

$$-2Re(w_0)q = q(-2Re(w_0)),$$

which is obviously true. Therefore, we can write

$$H(q) = (f(T(q)) - A - w_0)(f(T(q)) - A + \overline{w_0})^{-1}.$$

Hence,

$$H(q)(f(T(q)) - A + \overline{w_0}) = f(T(q)) - A - w_0,$$

which yields

$$(H(q) - 1)f(T(q)) = H(q)(A - \overline{w_0}) - A - w_0.$$

Recalling the definition of w_0, we get

$$\begin{aligned}
|f(T(q))| &= \left|(H(q)-1)^{-1}\left(H(q)(-\overline{f(q_0)} + 2A) - f(q_0)\right)\right| \\
&= \left|(H(q)-1)^{-1}\left[(H(q)-1)f(q_0) + H(q)\left(-\overline{f(q_0)} + 2A - f(q_0)\right)\right]\right| \\
&= \left|\beta + \gamma I + (H(q)-1)^{-1}H(q)(-\beta + \gamma I + 2A - \beta - \gamma I)\right| \\
&= \left|\beta + \gamma I + 2(H(q)-1)^{-1}H(q)(A - \beta)\right|.
\end{aligned}$$

Using the triangle inequality and inequality (9.19), we have that for $0 \le |q - q_0| \le \varrho \le r$

$$\begin{aligned}
|f(T(q))| &\le |\beta| + |\gamma| + 2\frac{(A + |\beta|)|H(q)|}{|H(q) - 1|} \\
&\le |\beta| + |\gamma| + 2\frac{(A + |\beta|)\frac{\varrho}{r}}{1 - \frac{\varrho}{r}} \\
&= |\gamma| + |\beta|\left(1 + 2\frac{\frac{\varrho}{r}}{1 - \frac{\varrho}{r}}\right) + 2\frac{A\frac{\varrho}{r}}{1 - \frac{\varrho}{r}} \\
&= |\gamma| + |\beta|\frac{r + \varrho}{r - \varrho} + A\frac{2\varrho}{r - \varrho}.
\end{aligned} \tag{9.22}$$

Since $h(q) + \overline{w_0} \neq 0$ for all $q \in B$, the transformation $T = T_{h(q)+\overline{w_0}}$ is a diffeomorphism of B onto itself. Hence, inequality (9.22) implies that

$$|f(q)| \le |\gamma| + |\beta|\frac{r + \varrho}{r - \varrho} + 2A\frac{\varrho}{r - \varrho}$$

for all q such that $|q - q_0| \le \varrho$. □

Historically, the Borel–Carathéodory Theorem was applied to prove minimum modulus results of Ehrenpreis–Malgrange and Cartan type; the paper [67] surveyed in Sect. 7.5 is a first foray on this topic in the framework of regular functions, and its authors are currently investigating applications of Theorem 9.28 to this topic. In the complex setting, the Borel–Carathéodory Theorem was also useful in the proof of a first version of the Bohr Theorem. A sharp version of this theorem was then proven with a different technique (see [12]). Regular analogs of both versions were proven in [42].

9.7 Bohr Theorem

In this section we prove the analog, for regular functions, of the Bohr Theorem in its sharp formulation. We wish to mention that Bohr-type phenomena are the subject of recent research also in Clifford analysis (see, e.g., [77, 79]).

Theorem 9.29 (Bohr). *Let $f(q) = \sum_{n\geq 0} q^n a_n$ be a regular function on $\mathbb{B}$, continuous on the closure $\overline{\mathbb{B}}$, such that $|f(q)| < 1$ when $|q| \leq 1$. Then*

$$\sum_{n\geq 0} |q^n a_n| < 1$$

for all $|q| \leq \frac{1}{3}$. Moreover, $\frac{1}{3}$ is the largest radius for which the statement is true.

Proof. By possibly multiplying f by the constant $\frac{\bar{a}_0}{|a_0|}$ on the right, we may assume $a_0 \geq 0$. We will show that $|a_n| < 1 - a_0^2$ for all $n \geq 1$. Let us first treat the case $n = 1$. Consider the function

$$H(q) = (1 - f(q)a_0)^{-*} * (q^{-1}(f(q) - a_0)).$$

By hypothesis, $f : \mathbb{B} \to \mathbb{H}$ is continuous up to the boundary, $f(0) = a_0 < 1$, and $|f(q)| < 1$ on $\overline{\mathbb{B}}$. Hence, H is regular on $\mathbb{B}$ and continuous up to the boundary. Moreover, if we set $T(q) = (1 - \bar{a}_0 * f^c(q))^{-1} q (1 - \bar{a}_0 * f^c(q))$, then by Proposition 5.32, we can write

$$\begin{aligned} H(q) &= (1 - f(q)a_0)^{-*} * (q^{-1} \sum_{n\geq 1} q^n a_n) = (1 - f(q)a_0)^{-*} * \sum_{n\geq 1} q^{n-1} a_n \\ &= (1 - f(T(q))a_0)^{-1} \sum_{n\geq 1} T(q)^{n-1} a_n. \end{aligned}$$

Furthermore, $T(0) = 0$ implies that $H(0) = (1 - a_0^2)^{-1} a_1$, and since $|T(q)| = |q|$, by Theorem 7.2, we get that for all $q \in \mathbb{B}$

$$\begin{aligned}|H(q)| &\le \max_{|q|=1} |H(q)| \\ &= \max_{|q|=1} |1 - f(T(q))a_0|^{-1} |T(q)|^{-1} |f(T(q)) - a_0| \\ &= \max_{|q|=1} |1 - f(T(q))a_0|^{-1} |f(T(q)) - a_0|.\end{aligned} \tag{9.23}$$

Notice that $(1-qa_0)^{-1}(q-a_0)$ is a Möbius transformation of $\mathbb{B}$. Since $|f(T(q))| < 1$, and taking into account (9.23), we conclude that $|H(q)| < 1$ for all $q \in \mathbb{B}$. In particular, $|H(0)| < 1$, that is,

$$a_1 < 1 - a_0^2.$$

For the case $n > 1$, we construct a function having the same properties as f, whose first degree coefficient is a_n. Let ω be a quaternionic primitive nth root of unity (see Theorem 3.15) and let I be such that $\omega \in L_I$. Consider the function defined on $\mathbb{B}_I$ by the formula

$$g_I(z) = f_I(z) + f_I(z\omega) + \cdots + f_I(z\omega^{n-1}),$$

where f_I is the restriction of f to the plane L_I. The function g_I is holomorphic in $\mathbb{B}_I$ and continuous in $\overline{\mathbb{B}_I}$. Moreover,

$$|g_I(z)| \le |f_I(z)| + |f_I(z\omega)| + \cdots + |f_I(z\omega^{n-1})| < n.$$

Its power series expansion is

$$\begin{aligned}g_I(z) &= \sum_{m\in\mathbb{N}} z^m a_m + \sum_{m\in\mathbb{N}} z^m \omega^m a_m + \cdots + \sum_{m\in\mathbb{N}} z^m \omega^{(n-1)m} a_m \\ &= \sum_{m\in\mathbb{N}} z^m \sum_{k=0}^{n-1} \omega^{km} a_m.\end{aligned}$$

Now, if n divides m, then $\omega^m = 1$ and

$$\sum_{k=0}^{n-1} \omega^{km} = n.$$

Otherwise ω^m is an nth root of unity other than 1, hence a root of the polynomial $z^{n-1} + \cdots + z + 1$. This yields

$$\sum_{k=0}^{n-1} \omega^{km} = 0.$$

Therefore,

$$g_I(z) = \sum_{m\in\mathbb{N}} z^{nm} n a_{nm}.$$

Let us consider the regular extension $g = \text{ext}(g_I) : \mathbb{B} \to \mathbb{H}$, namely,

$$g(q) = \sum_{m \geq 0} q^{nm} n a_{nm}.$$

For all $J \in \mathbb{S}$ and for all $z \in \mathbb{B}_J$

$$g_J(z) = f_J(z) + f_J(z\omega_J) + \cdots + f_J(z\omega_J^{n-1})$$

for some nth root of unity in L_J, ω_J. Hence, $|g_J(z)| \leq n$ for all $z \in \mathbb{B}_J$ and $|g(q)| < n$ for all $q \in \mathbb{B}$. Setting

$$\Phi(q) = \sum_{m \in \mathbb{N}} q^m a_{mn} = a_0 + q a_n + q^2 a_{2n} + \ldots,$$

(so that $\Phi(q^n) = \frac{g(q)}{n}$), we obtain a regular function $\Phi : \mathbb{B} \to \mathbb{B}$, continuous up to the boundary $\partial\mathbb{B}$. Since the coefficient of the first degree term of Φ is a_n, the same argument used to prove that $|a_1| < 1 - a_0^2$ implies that $|a_n| < 1 - a_0^2$. Since $0 \leq a_0 < 1$, we conclude that

$$|a_n| < (1 + a_0)(1 - a_0) < 2(1 - a_0).$$

Therefore, for $|q| = \frac{1}{3}$,

$$\sum_{n \in \mathbb{N}} |q^n a_n| = \sum_{n \in \mathbb{N}} \frac{1}{3^n} |a_n| < a_0 + 2(1 - a_0) \sum_{n \geq 1} \frac{1}{3^n}$$
$$= a_0 + 2(1 - a_0)\frac{1}{2} = a_0 + 1 - a_0 = 1.$$

In order to prove that the statement is sharp, we will proceed as follows: For any point $q_0 \in \mathbb{B}$ such that $|q_0| > \frac{1}{3}$, we will find a regular function $g_{q_0} : \mathbb{B} \to \mathbb{B}$ continuous up to the boundary, such that $g_{q_0}(q) = \sum_{n \in \mathbb{N}} q^n b_n$ and $\sum_{n \in \mathbb{N}} |q_0^n b_n| > 1$. To start with, take $a \in (0, 1)$ and consider the function

$$\varphi(q) = (1 - qa)^{-*} * (1 - q) = (1 - qa)^{-1}(1 - q).$$

Since $a < 1$, φ is regular in $\mathbb{B}$, continuous in $\overline{\mathbb{B}}$, and slice preserving. By Theorem 7.2

$$\max_{q \leq 1} |\varphi(q)| = \max_{|q|=1} |\varphi(q)|,$$

and for all $I \in \mathbb{S}$,

$$\max_{|q|=1} |\varphi(q)| = \max_{|z|=1} |\varphi_I(z)| = \frac{2}{1 + a} > 1.$$

We now compute the power series expansion $\varphi(q) = \sum_{n\in\mathbb{N}} q^n b_n$:

$$\varphi(q) = (1-qa)^{-1}(1-q) = \left(\sum_{n\in\mathbb{N}} q^n a^n\right)(1-q) = 1 + q(a-1)\sum_{n\in\mathbb{N}} q^n a^n.$$

Hence,

$$\sum_{n\in\mathbb{N}} |q^n b_n| = 1 + |q|(1-a)\sum_{n\in\mathbb{N}} |q|^n a^n = 1 + \frac{|q|(1-a)}{1-|q|a}.$$

In particular,

$$\sum_{n\in\mathbb{N}} |q^n b_n| > \frac{2}{1+a} \quad \text{if and only if} \quad 1 + \frac{|q|(1-a)}{1-|q|a} > \frac{2}{1+a},$$

which holds if and only if

$$|q| > \frac{1}{1+2a}.$$

Now fix $q_0 \in \mathbb{B}$ such that $|q_0| > \frac{1}{3}$. Then we can take $a \in (0,1)$ such that $|q_0| > \frac{1}{1+2a} > \frac{1}{3}$. The corresponding φ is such that

$$\sum_{n\geq 0} |q_0|^n |b_n| > \frac{2}{1+a}.$$

Let us consider the function

$$\varphi_c(q) = c\varphi(q) = c(1-qa)^{-1}(1-q),$$

where $c \in (0,1)$. Then φ_c is regular on $\mathbb{B}$ and continuous on $\overline{\mathbb{B}}$, and its maximum modulus is

$$\max_{|q|=1} |\varphi_c(q)| = \frac{2c}{1+a}.$$

Moreover, its power series expansion is obtained multiplying by c that of φ. The computations done for φ implies that

$$c\sum_{n\geq 0} |q^n b_n| > \frac{2c}{1+a}$$

if and only if

$$|q| > \frac{1}{1+2a}.$$

Hence,

$$c\sum_{n\geq 0} |q_0^n b_n| > \frac{2c}{1+a}.$$

To conclude, we notice that we can choose $c \in (0, 1)$ such that

$$c \sum_{n \geq 0} |q_0^n b_n| > 1 > \frac{2c}{1+a}$$

and we set $g_{q_0} = \varphi_c$. $\square$

Bibliographic Notes

The study of affine, fractional, and Möbius transformations in the first sections of this chapter was conducted in [121]. The Schwarz Lemma and the biregularity of the Cayley transform were proven in [62]. On the other hand, the rigidity results and the boundary Schwarz Lemma in Sect. 9.5 were proven in [70]. The Borel–Carathéodory Theorem and the Bohr Theorem were proven in [42]. Recent developments in the study of the geometry of the quaternionic unit ball include [8–10, 41, 71].

Chapter 10
Generalizations and Applications

10.1 Slice Regularity in Algebras Other than $\mathbb{H}$

In this section we will discuss several generalizations of the notion of (slice) regularity to the case of algebras other than $\mathbb{H}$. We will begin with the algebra of octonions $\mathbb{O}$, where the foundational results mimic very closely those described in this book. We will then consider the case of functions defined on the Clifford algebra $\mathbb{R}_3$ (sometimes referred to as $Cl(0,3)$ in the literature) with values in that same algebra. We will see that in this case, it is not possible to fully reconstruct an analogous theory of regularity due to some peculiarities in the algebraic structure of $\mathbb{R}_3$. The basic theory of regularity over $\mathbb{H}, \mathbb{O}$, and $\mathbb{R}_3$ is surveyed in [60]. It was partly because of the difficulties encountered in the study of functions defined on a Clifford algebra that the next significant generalization that we will discuss is the notion of (slice) monogeneity, which is defined for functions on the Euclidean space $\mathbb{R}^{m+1}$ and with values in $\mathbb{R}_m = Cl(0,m)$. The theory of such functions is fully discussed in the recent [36] and will be presented here only for completeness. Finally, we will mention the new and exciting general approach due to Ghiloni and Perotti, [72], which allows a powerful generalization of some of our ideas to the setting of real alternative algebras.

10.1.1 The Case of Octonions

Let us begin with the nonassociative, alternative division algebra $\mathbb{O}$ of real Cayley numbers (also known as octonions). The study of this algebra is not simply a curiosity but has recently assumed increasing relevance; see for example [6].

G. Gentili et al., *Regular Functions of a Quaternionic Variable*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-33871-7_10,

The algebra of octonions can be constructed by considering a basis $\mathcal{E} = \{e_0 = 1, e_1, \dots, e_6, e_7\}$ of $\mathbb{R}^8$ and relations

$$e_\alpha e_\beta = -\delta_{\alpha\beta} + \psi_{\alpha\beta\gamma} e_\gamma, \quad \alpha, \beta, \gamma = 1, 2, \dots, 7;$$

here $\delta_{\alpha\beta}$ is the Kronecker delta, and $\psi_{\alpha\beta\gamma}$ equals 1 if (α, β, γ) is one of the seven combinations in the following set

$$\sigma = \{(1,2,3), (1,4,5), (2,4,6), (3,4,7), (2,5,7), (1,6,7), (5,3,6)\},$$

is totally antisymmetric in α, β, γ and it equals 0 in the remaining cases. This implies that every octonion can be written as $w = x_0 + \sum_{k=1}^{7} x_k e_k$. One can then define in a natural fashion its square norm $|w|^2 = \sum_{k=0}^{7} x_k^2$.

The set $(1, e_1, e_2, e_1 e_2)$ is a basis for a subalgebra of $\mathbb{O}$ isomorphic to the algebra $\mathbb{H}$ of quaternions. Moreover, one can easily show that every Cayley number can be thought of as four complex numbers (each one in $\mathbb{C} = \mathbb{R} + \mathbb{R}e_1$) or as two quaternions (each one in $\mathbb{C} + \mathbb{C}e_2$). We have therefore the decomposition

$$\begin{aligned} \mathbb{O} &= (\mathbb{R} + \mathbb{R}e_1) + (\mathbb{R} + \mathbb{R}e_1)e_2 + [(\mathbb{R} + \mathbb{R}e_1) + (\mathbb{R} + \mathbb{R}e_1)e_2]e_4 = \\ &= \mathbb{C} + \mathbb{C}e_2 + (\mathbb{C} + \mathbb{C}e_2)e_4 = \mathbb{H} + \mathbb{H}e_4. \end{aligned} \tag{10.1}$$

A theory of analyticity for $\mathbb{O}$-valued functions defined on $\mathbb{O}$ that follows the ideas of Fueter is available and, like the theory of Fueter for quaternions, it has been very successful. Specifically, one can define a Cauchy–Riemann-like operator on functions on $\mathbb{O}$, and a very rich theory for its null solutions can be found, for example, in [33, 43, 91, 92, 113].

We are interested, however, in using the decomposition of $\mathbb{O}$ described above to offer a different definition of regularity that employs complex slices of $\mathbb{O}$. As the reader will see, the central ideas that are necessary are not very dissimilar to those described in the first chapter of this book. Let therefore $\mathbb{S}$ be the unit sphere of purely imaginary Cayley numbers, that is, $\mathbb{S} = \{w = \sum_{k=1}^{7} x_k e_k : \sum_{k=1}^{7} x_k^2 = 1\}$.

In the pioneering paper [65] regularity on octonions was defined as follows:

Definition 10.1. Let Ω be a domain in $\mathbb{O}$. A real differentiable function $f : \Omega \to \mathbb{O}$ is said to be *regular* if, for every $I \in \mathbb{S}$, its restriction f_I to the complex line $L_I = \mathbb{R} + \mathbb{R}I$ passing through the origin and containing 1 and I is holomorphic on $\Omega_I = \Omega \cap L_I$. We also call regular those functions defined on an open subset of a quaternionic subspace $\mathcal{H}$ of $\mathbb{O}$ and whose restriction to L_I is holomorphic for every $I \in \mathbb{S} \cap \mathcal{H}$.

As in the case of regular functions on $\mathbb{H}$, such functions admit a notion of derivative.

Definition 10.2. Let Ω be a domain in $\mathbb{O}$ and let $f : \Omega \to \mathbb{O}$ be a real differentiable function. For any $I \in \mathbb{S}$ and any point $w = x + yI$ in Ω (x and y are real numbers here), we define the *I-derivative* of f at w by the formula

$$\partial_I f(x+yI) := \frac{1}{2}\left(\frac{\partial}{\partial x} - I\frac{\partial}{\partial y}\right) f_I(x+yI)$$

If, moreover, f is regular, then its *(slice) derivative*, ∂f, is defined as follows:

$$\partial(f)(w) = \begin{cases} \partial_I(f)(w) & \text{if } w = x+yI \text{ with } y \neq 0 \\ \frac{\partial f}{\partial x}(x) & \text{if } w = x \text{ is real} \end{cases}$$

In the quaternionic case, the key technical tool for the study of regularity over the octonions is a splitting lemma, which allows to think of a regular function as a pair of holomorphic functions. To do this in the octonionic setting is somewhat harder, because it is not immediately obvious how to find an appropriate basis of imaginary units. Still, as shown in [65], this can be done:

Lemma 10.3. *If f is a regular function on a domain Ω, then for every $I_1 \in \mathbb{S}$, we can find I_2 and I_4 in $\mathbb{S}$, such that there are four holomorphic functions $F_1, F_2, G_1, G_2 : \Omega_{I_1} \to L_{I_1}$ such that for any $z = x + yI_1$,*

$$f_{I_1}(z) = F_1(z) + F_2(z)I_2 + (G_1(z) + G_2(z)I_2)I_4.$$

Corollary 10.4. *If f is a regular function on a domain Ω, then for every $I_1 \in \mathbb{S}$, we can find I_2 and I_4 in $\mathbb{S}$, such that if $\mathcal{H}$ is the subspace of $\mathbb{O}$ generated by $(1, I_1, I_2, I_1I_2)$, then there are two functions $F : \Omega \cap \mathcal{H} \to \mathcal{H}$ and $G : \Omega \cap \mathcal{H} \to \mathcal{H}I_4$, regular on $\Omega \cap \mathcal{H}$ according to Definition 10.1 and such that for any $q \in \Omega \cap \mathcal{H}$,*

$$f(q) = F(q) + G(q).$$

Note that, because of the lack of associativity in $\mathbb{O}$, the function $G_1 + G_2I_2$ is not, by itself, regular. An interesting consequence of this last result is the fact that one can represent regular functions on $\mathbb{O}$ either as pairs of regular functions or as four-tuples of holomorphic functions, consistently with decomposition (10.1).

A consequence of this splitting lemma is now the fact that every regular function on a ball $B(0, R) = \{w \in \mathbb{O} : |w| < R\}$ centered at the origin can be represented as a convergent power series.

Theorem 10.5. *If $f : B = B(0, R) \to \mathbb{O}$ is regular, then it has a series expansion of the form*

$$f(w) = \sum_{n=0}^{\infty} w^n \frac{1}{n!} \frac{\partial^n f}{\partial x^n}(0)$$

in B.

We will not further describe this generalization, except to point out that most of the foundational results of regular functions in $\mathbb{H}$ can be replicated in this context. It is clear that, as in the case of quaternions, the notion of regularity is naturally expressed for functions defined on the eight-dimensional analogs of axially symmetric slice domains (see Definitions 1.11 and 1.14). Standard theorems such as the Identity Principle, the Maximum Modulus Principle, the Morera and Liouville theorems, and the "slicewise" Cauchy Formula hold, in a suitable form, for regular functions on $\mathbb{O}$.

Finally, we conclude this subsection by pointing out that these techniques allow to study in detail the zero sets of octonionic power series. In accordance with what we had done for quaternions, we say that an octonion $w_0 = x_0 + y_0 I$ is a *spherical zero* for a regular function f if every point of the 6-sphere $x_0 + y_0\mathbb{S}$ is a zero for f. We then have the following characterization of the zeros of octonionic power series.

Proposition 10.6. *If f is a series $f(w) = \sum_{n\in\mathbb{N}} w^n a_n$ with real coefficients a_n, then every real zero x_0 is isolated, and if $x_0 + y_0 I$ is a nonreal zero (i.e., $y_0 \neq 0$), then it is a spherical zero. In particular, if $f \not\equiv 0$, then the zero set of f consists of isolated zeroes (lying on $\mathbb{R}$) or isolated six-spheres.*

The recent article [73] presents many properties of the zero sets of regular octonionic functions, in the spirit of Chap. 3. The peculiarities of the nonassociative setting produce a completely new phenomenon, called the *camshaft effect*: an isolated zero p of a regular function f is not necessarily a zero for the regular product $f * g$ of f with another regular function g; nevertheless, in such a case $f * g$ vanishes at some point p' having the same real part and modulus of the imaginary part as p.

10.1.2 The Case of $\mathbb{R}_3$

In this subsection, we will consider the next natural setting for a theory of slice regularity, after quaternions and octonions. Namely, we will consider the case of functions defined on the Clifford algebra $\mathbb{R}_3$ with three generators. It will become immediately clear that this algebraic setting requires a different kind of approach and that new phenomena occur. In fact, one of the most interesting features of this new theory, developed in [64], is the existence of a natural boundary for these functions. Indeed, it turns out that a notion of regularity can only be defined outside a codimension 3 closed set of $\mathbb{R}_3$, in contrast with the simpler situation that occurs in the case of Hamilton and Cayley numbers.

Let $\mathbb{R}_m$ denote the *real Clifford algebra of signature* $(0, m)$. This algebra can be introduced as follows (see [13, 32, 76] for this and other related definitions): let $E = \{e_1, e_2, \ldots, e_m\}$ be the canonical orthonormal basis for $\mathbb{R}^m$ with defining relations $e_i e_j + e_j e_i = -2\delta_{ij}$. An element of the Clifford algebra $\mathbb{R}_m$ can be written in a unique way as

$$x = x_0 + x_1e_1 + x_2e_2 + \ldots + x_me_m + x_{12}e_1e_2 + x_{13}e_1e_3 + x_{23}e_2e_3+$$

$$+ \ldots + x_{m-1,m}e_{m-1}e_m + \ldots + x_{12,\ldots,m}e_1e_2\ldots e_m,$$

where the coefficients $x_i, x_{ij}, x_{ijk}, \ldots$ are real numbers. Thus, in the case $m = 3$, we get an eight-dimensional real space, whose elements can be uniquely written as

$$x = x_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_{12}e_1e_2 + x_{13}e_1e_3 + x_{23}e_2e_3 + x_{123}e_1e_2e_3.$$

The space $\mathbb{R}_3$ is endowed with a natural multiplicative structure. Note that the square of each e_i or e_ie_j (with $i \neq j$) is -1, while the square of $e_1e_2e_3$ equals 1. For this reason, the element $e_1e_2e_3$ is often referred to as a pseudoscalar. Therefore we can decompose any element x of $\mathbb{R}_3$ as the sum of its *real part* $Re(x)$ and its *imaginary part* $Im(x)$, respectively, elements of the set of Clifford real numbers $Re(\mathbb{R}_3) = \{x = x_0 + x_{123}e_1e_2e_3\}$ and of the set of Clifford imaginary numbers as $Im(\mathbb{R}_3) = \{x = x_1e_1 + x_2e_2 + x_3e_3 + x_{12}e_1e_2 + x_{13}e_1e_3 + x_{23}e_2e_3\}$.

Let $\mathbb{K}$ denote any of the algebras $\mathbb{C}, \mathbb{H}, \mathbb{O}, \mathbb{R}_3$. For each of these algebras, we can define the set of roots of -1 as $\mathbb{S}_{\mathbb{K}} = \{w \in \mathbb{K} : w^2 = -1\}$ and the unit imaginary sphere as $\mathbb{U}_{\mathbb{K}} = \{w \in \mathbb{K} : Re(w) = 0, |Im(w)| = 1\}$. The theory developed in this book as well as in the previous subsection on octonions exploits the identity between $\mathbb{S}_{\mathbb{K}}$ and $\mathbb{U}_{\mathbb{K}}$ which holds for $\mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{O}$. A first interesting phenomenon that occurs in the Clifford algebra case is the fact that $\mathbb{S}_{\mathbb{R}_3}$ is properly contained in $\mathbb{U}_{\mathbb{R}_3}$. Indeed the following proposition holds (for a proof, see [64]).

Proposition 10.7. *An element $x = x_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_{12}e_1e_2 + x_{13}e_1e_3 + x_{23}e_2e_3 + x_{123}e_1e_2e_3$ in $\mathbb{R}_3$ belongs to $\mathbb{S} = \mathbb{S}_{\mathbb{R}_3}$ if and only if it belongs to $\mathbb{U} = \mathbb{U}_{\mathbb{R}_3}$ and its coordinates satisfy*

$$x_1x_{23} - x_2x_{13} + x_3x_{12} = 0.$$

In order to mimic the construction of slice regularity accomplished for $\mathbb{H}$ and $\mathbb{O}$, one needs to express the variable x as

$$x = (\alpha + \beta e_1e_2e_3) + I(\gamma + \delta e_1e_2e_3)$$

where α, β, γ, and δ are real numbers and $I \in \mathbb{S}$.

This is not possible for all $x \in \mathbb{R}_3$, but the set of exceptions is sufficiently small; in fact it has codimension 3 in $\mathbb{R}_3 \cong \mathbb{R}^8$, and the following property holds.

Proposition 10.8. *Given an element $w = x_1e_1 + x_2e_2 + x_3e_3 + x_{12}e_1e_2 + x_{13}e_1e_3 + x_{23}e_2e_3$ in $Im(\mathbb{R}_3)$, w is invertible if and only if there are two distinct real numbers a and b, and an element $I \in \mathbb{S}$ such that $w = I(a + be_1e_2e_3)$ if and only if $(x_1, x_2, x_3) \neq \pm(x_{23}, -x_{13}, x_{12})$.*

Actually, for a given element $w = x_1e_1 + x_2e_2 + x_3e_3 + x_{12}e_1e_2 + x_{13}e_1e_3 + x_{23}e_2e_3$ in $Im(\mathbb{R}_3)$, with the condition $(x_1, x_2, x_3) \neq \pm(x_{23}, -x_{13}, x_{12})$, there are (up to a sign) two imaginary units I and J and two possible pairs (a, b) satisfying the conclusion of the previous propositions. Indeed if $w = I(a + be_1e_2e_3)$, then it is also true that there exists $J \in \mathbb{S}$ such that $w = I(a + be_1e_2e_3) = J(b + ae_1e_2e_3)$. The computations show that if $I = i_1e_1 + i_2e_2 + i_3e_3 + i_{12}e_1e_2 + i_{13}e_1e_3 + i_{23}e_2e_3$

and $J = j_1e_1 + j_2e_2 + j_3e_3 + j_{12}e_1e_2 + j_{13}e_1e_3 + j_{23}e_2e_3$, then I and J are orthogonal and their coordinates are related by the two systems

$$\begin{cases} i_1 = -j_{23} \\ i_2 = j_{13} \\ i_3 = -j_{12} \end{cases} \qquad \begin{cases} i_{12} = -j_3 \\ i_{13} = j_2 \\ i_{23} = -j_1. \end{cases}$$

In fact, it is easy to show that $J = Ie_1e_2e_3$ and that therefore

$$I(a + be_1e_2e_3) = Ie_1e_2e_3e_1e_2e_3(a + be_1e_2e_3) = J(b + ae_1e_2e_3).$$

One further algebraic property is required to develop a function theory on $\mathbb{R}_3$. Given $I \in \mathbb{S}$, it is possible to start with $1, e_1e_2e_3, I, Ie_1e_2e_3$, and to choose $J \in \mathbb{S}$ so that the vectors $J, Je_1e_2e_3, IJ, IJe_1e_2e_3$ complete a basis for $\mathbb{R}_3$. This will allow us immediately to show that every element $x = x_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_{12}e_1e_2 + x_{13}e_1e_3 + x_{23}e_2e_3 + x_{123}e_1e_2e_3 \in \mathbb{R}_3$ can be represented as

$$X_1 + X_2e_1e_2e_3 + X_3I + X_4Ie_1e_2e_3 + X_5J + X_6Je_1e_2e_3 + X_7IJ + X_8IJe_1e_2e_3$$

and, since $Je_1e_2e_3 = e_1e_2e_3J$, as

$$(X_1 + X_2e_1e_2e_3) + I(X_3 + X_4e_1e_2e_3) + [(X_5 + X_6e_1e_2e_3) + I(X_7 + X_8e_1e_2e_3)]J.$$

Proposition 10.9. *Given $I \in \mathbb{S}$, it is possible to choose $J \in \mathbb{S}$ such that J is perpendicular to I and to $Ie_1e_2e_3$. Moreover, for such a J, the set*

$$\mathcal{B} = \{1, e_1e_2e_3, I, Ie_1e_2e_3, J, Je_1e_2e_3, IJ, IJe_1e_2e_3\}$$

is a basis for $\mathbb{R}_3$, whose elements (except for 1 and $e_1e_2e_3$) are in $\mathbb{S}$.

We now have all the tools we need to construct a notion of regularity for functions defined on the set

$$U = \{x \in \mathbb{R}_3 : x = 0 \text{ or } (x_1, x_2, x_3) \neq \pm(x_{23}, -x_{13}, x_{12})\}.$$

As we have already noticed, by construction, $\mathbb{R}_3 \setminus U$ has codimension 3 in $\mathbb{R}_3 \cong \mathbb{R}^8$.

Definition 10.10. Let Ω be a domain in U. A real differentiable function $f : \Omega \to \mathbb{R}_3$ is said to be *regular* if, for every $I \in \mathbb{S}$, its restriction f_I to the four-dimensional Clifford plane $L_I = Re(\mathbb{R}_3) + IRe(\mathbb{R}_3) = \{(t_1 + t_2e_1e_2e_3) + I(t_3 + t_4e_1e_2e_3)\}$ passing through the origin and containing 1 and I satisfies, on $\Omega \cap L_I$, the system

$$2D_I f_I = (d_{12} + Id_{34}) f_I = 0, \tag{10.2}$$

where $d_{ij} = \frac{\partial^2}{\partial t_i \partial t_j}$.

It turned out that Definition 10.10 is equivalent to the following four two-dimensional Cauchy–Riemann systems:

$$\begin{cases} d_{12} f_{00} = d_{34} f_{10} \\ d_{12} f_{10} = -d_{34} f_{00} \end{cases}$$

$$\begin{cases} d_{12} f_{01} = d_{34} f_{11} \\ d_{12} f_{11} = -d_{34} f_{01} \end{cases}$$

$$\begin{cases} d_{12} f_{00} = d_{43} f_{11} \\ d_{12} f_{11} = -d_{43} f_{00} \end{cases}$$

$$\begin{cases} d_{12} f_{01} = d_{43} f_{10} \\ d_{12} f_{10} = -d_{43} f_{01}. \end{cases}$$

The solutions of this system have a very nice geometrical interpretation in terms of a basis $\{1, e_1e_2e_3, I, Ie_1e_2e_3, J, Je_1e_2e_3, IJ, IJe_1e_2e_3\}$ constructed as in Proposition 10.9.

Theorem 10.11. *Let Ω be a domain in U, let $f : \Omega \to \mathbb{R}_3$ be (real) differentiable, suppose that, for each $I \in \mathbb{S}$ and for $K = Ie_1e_2e_3$, $f_I = f_K$ splits as*

$$f_I = F_0 + F_1e_1e_2e_3 + (G_0 + G_1e_1e_2e_3)J = M_0 + M_1e_1e_2e_3 + (N_0 + N_1e_1e_2e_3)J = f_K$$

in $\Omega \cap L_I = \Omega \cap L_K$, and let

$$z_1 = t_1 + It_3, z_2 = t_2 + It_4, w_1 = t_1 + Kt_4, w_2 = t_2 + Kt_3.$$

Then f is regular in Ω if and only if F_0, F_1, G_0, G_1 are holomorphic in (z_1, z_2) and M_0, M_1, N_0, N_1 are holomorphic in (w_1, w_2).

A peculiar property of this class of functions is their expansion into power series.

Theorem 10.12. *Let f be regular function in U. For an arbitrary point $w = z_1 + z_2e_1e_2e_3$ in U,*

$$f(z_1 + z_2e_1e_2e_3) = \sum_{m,n \in \mathbb{N}} z_1^m z_2^n a_{mn}$$

where the coefficients are the Clifford numbers

$$a_{mn} = \frac{\partial^{m+n} f(0)}{\partial z_1^m \partial z_2^n}.$$

Moreover, the series that appear in this last theorem always define regular functions on the open subsets of U where they converge.

10.1.3 The Monogenic Case

The previous subsection has shown how the study of Clifford-valued functions defined on open subsets of a Clifford algebra $\mathbb{R}_m$ is not the most natural setting.

Historically, the study of Clifford-valued functions is undertaken on Euclidean spaces. Indeed, in analogy with the theory of Fueter-regular functions, one can study an interesting class of functions from $\mathbb{R}^{m+1}$ to $\mathbb{R}_m$, called *monogenic*. The theory of such functions is extremely well developed, and some recent references include, for example, [16, 101, 118]. The theory has, not surprisingly, many similarities with the theory of Fueter-regular functions (and some of these similarities are exploited, for example, in [32] in order to study the case of several vector variables). Among such similarities is the fact that the theory does not allow to consider power series in the vector variable $x \in \mathbb{R}^{m+1}$.

Shortly after the introduction of slice regular functions on quaternions, it became apparent that the same idea could be used to define the class of *slice monogenic* functions. The idea is simple, and it is described in great detail (along with significant applications) in [36]. In order to present it, let us introduce some basic notations. An element $(x_0, x_1, \ldots, x_m)$ of the Euclidean space $\mathbb{R}^{m+1}$ is identified with an element $x = x_0 + x_1 e_1 + \ldots + x_m e_m = x_0 + \underline{x}$ of the Clifford algebra $\mathbb{R}_m$. The scalar x_0 corresponds to the real part $Re(x)$ of x. In the sequel, we will denote

$$\mathbb{S} = \{\underline{x} = e_1 x_1 + \ldots + e_m x_m \mid x_1^2 + \ldots + x_m^2 = 1\},$$

which is a sphere of real dimension $m - 1$. In analogy with the quaternionic case, one defines (see [34]):

Definition 10.13. Let $U \subseteq \mathbb{R}^{m+1}$ be an open set and let $f : U \to \mathbb{R}_m$ be a function. Let $I \in \mathbb{S}$ and let f_I be the restriction of f to the complex plane $L_I := \mathbb{R} + \mathbb{R}I$ passing through 1 and I and denote by $u + vI$ an element on L_I. We say that f is a *left slice monogenic* function if for every $I \in \mathbb{S}$,

$$\overline{\partial}_I f = \frac{1}{2}\left(\frac{\partial}{\partial u} + I\frac{\partial}{\partial v}\right) f_I(u + vI) = 0.$$

As shown in [36] (but see also [25, 27]), it is possible to fully develop a theory for such functions, including expansions into power series centered at the origin, a characterization of the zero sets, a Cauchy Formula (both in the "slicewise" and general forms), a Pompeiu Formula, as well as duality theorems. In particular, one can prove the following Representation Formula (where the notions of axial symmetry and slice domain are analogous to the quaternionic ones).

Lemma 10.14. *Let $U \subseteq \mathbb{R}^{m+1}$ be an axially symmetric slice domain. Let $f : U \to \mathbb{R}_m$ be a slice monogenic function. For every $x = x_0 + I_x|\underline{x}| \in U$, the following formula holds:*

$$f(x) = \frac{1}{2}\Big[1 - I_x I\Big] f(x_0 + I|\underline{x}|) + \frac{1}{2}\Big[1 + I_x I\Big] f(x_0 - I|\underline{x}|).$$

This last formula is the basis for the subsequent generalization of the theory of slice regularity, due to Ghiloni and Perotti [72]. They constructed an interesting class of functions over a large number of (finite-dimensional) alternative real algebras, including $\mathbb{H}$, $\mathbb{O}$, and $\mathbb{R}_m$ for all $m \geq 1$. In the first two cases, the class substantially coincides with that of slice regular functions, while in the Clifford case, it contains that of slice monogenic functions. In all real alternative algebras, the functions in the class considered are still called *slice regular*, and they enjoy properties that are analogous to those described in Chap. 1 of this book. The article [72] also presents results on the zero sets in the spirit of Chap. 3, some of them requiring more stringent hypotheses. The recent [74] presents integral formulas, generalizing the material of Chap. 6.

The theory of slice regularity in the case of Clifford algebras and in the more general context of real alternative algebras is currently a very active field of research.

10.2 Quaternionic Functional Calculus

The most successful application of the theory of regular functions is, without any doubt, the creation of a brand new quaternionic functional calculus. This new functional calculus is described in great detail in [36] and in [20, 22–24, 26, 28–31]. The development of this theory, which is motivated to a great extent by the interest in quaternionic quantum field theory (see, e.g., [2]), is continuing at a very fast pace.

As is well known, if X is a complex Banach space, and $T : X \to X$ is a bounded $\mathbb{C}$-linear operator, the purpose of the classical (Dunford–Riesz) functional calculus is to give sense to the operator $f(T)$, when f is a suitable analytic function. The reader is referred to [45] for an accurate and detailed description of the classical functional calculus and for a sense of the importance of this theory in mathematics. If $\mathcal{I} : X \to X$ is the identity, one defines the *resolvent set* of T as the set

$$\rho(T) = \{\Lambda \in \mathbb{C} : (\Lambda\mathcal{I} - T)^{-1} \text{is a bounded linear operator}\}$$

and the *spectrum* of T as $\sigma(T) = \mathbb{C} \setminus \rho(T)$. Then, for every function f, analytic in a neighborhood of $\sigma(T)$, one can take advantage of the classical Cauchy Formula for holomorphic functions to naturally define the operator $f(T)$ by setting

$$f(T) = \frac{1}{2\pi i}\int_\gamma (\Lambda\mathcal{I} - T)^{-1} f(\Lambda) d\Lambda$$

where γ is a closed piecewise differentiable curve encircling $\sigma(T)$ clockwise. The expression functional calculus usually refers to the study of the operator $f(T)$ and of its general properties. For example, it is known that if f admits a series representation $f(\Lambda) = \sum a_n \Lambda^n$, then $f(T)$ also has a series representation as $f(T) = \sum a_n T^n$. One should note here that a similar, more complicated, complicated, theory can be designed for unbounded operators (see [45]).

Even without entering into any details of the theory, it is apparent that in order to consider an analog in the quaternionic case (i.e., in the case in which T is an operator on a quaternionic Banach space), one needs an appropriate quaternionic generalization of the theory of holomorphic functions, including power series and a Cauchy Formula with an appropriate Cauchy kernel.

To begin with, we will then consider a two-sided quaternionic Banach space V, and we will denote by $\mathcal{B}(V)$ the two-sided vector space of linear bounded operators (one should note here that we can speak of both right and left linear operators and that this is an important distinction to be considered when studying the details of the theory). If T is an element of $\mathcal{B}(V)$, and if f is a suitable "analytic" function, one wants to find a way to appropriately define $f(T)$. Because of the widespread use and importance of the notion of Fueter-regularity (as we discussed it already in the introduction to this volume), it is natural to imagine a functional calculus in which we define $f(T)$ for Fueter-regular functions f. Since a Cauchy Formula exists for Fueter-regular functions, it is indeed possible to do so; (see for instance [82] and the rather complete discussion in the last chapter of [36]), but with severe limitations. In particular, we have already observed that even a simple polynomial such as q^2 is not Fueter-regular, so that any quaternionic functional calculus based on the notion of Fueter-regularity will be unable to offer a definition for T^2. As we already remarked, there are Fueter versions of polynomials, based on the first-degree Fueter polynomials

$$P_1 = x_1 - ix_0, P_2 = x_2 - jx_0, P_3 = x_3 - kx_0,$$

and so we can expect that if the operator T is decomposed in its quaternionic components as $T = T_0 + iT_1 + jT_2 + kT_3$, then we can define, for example, $P_1(T) = T_0 + iT_1$ as well as the nth degree homogeneous polynomials

$$\sum_{\nu \in \sigma_n} p_\nu(T)$$

where

$$\sigma_n = \{(n_1, n_2, n_3) \in \mathbb{N}^3 : n_1 + n_2 + n_3 = n\}$$

and where, for all $\nu = (n_1, n_2, n_3) \in \sigma_n$, the polynomial $n!\, p_\nu$ is the sum of all possible products involving n_1 factors equal to P_1, n_2 factors equal to P_2, and n_3 factors equal to P_3.

On the other hand, we now have the notion of slice regularity, which naturally includes polynomials and power series, and there is a Cauchy Formula which can be used to develop a functional calculus. We recall that Theorem 6.4 states that if f is a regular function on U, then for every $q \in U$ and $I \in \mathbb{S}$, we have

$$f(q) = \frac{1}{2\pi} \int_{\partial U_I} (s-q)^{-*} ds_I\, f(s)$$

where $ds_I = -dsI$ and where $(s-q)^{-*}$ is the Cauchy kernel

$$(s-q)^{-*} = -(q^2 - 2Re(s)q + |s|^2)^{-1}(q - \bar{s}) = \sum_{n\in\mathbb{N}} q^n s^{-1-n}.$$

Now, let V be a two-sided Banach space over $\mathbb{H}$ and let $T \in \mathcal{B}(V)$. The norm of T is

$$||T|| := \sup_{v\in V} \frac{||T(v)||}{||v||},$$

and, whenever $||T|| < s$, we can define the *left S-resolvent* by using the Cauchy kernel series as

$$S_L^{-1}(s,T) = \sum_{n\geq 0} T^n s^{-1-n}.$$

One has that for $||T|| < s$,

$$S_L^{-1}(s,T) = \sum_{n\geq 0} T^n s^{-1-n} = -(T^2 - 2Re(s)T + |s|^2\mathcal{I})^{-1}(T - \bar{s}\mathcal{I}).$$

One can see that even though the equality only holds for $||T|| < s$, the right-hand side is well defined on a larger set, and we can therefore define the *left S-resolvent operator* as

$$S_L^{-1}(s,T) = -(T^2 - 2Re(s)T + |s|^2\mathcal{I})^{-1}(T - \bar{s}\mathcal{I}).$$

This allows us to define the *S-spectrum* of T as the set

$$\sigma_S(T) = \{s \in \mathbb{H} : T^2 - 2Re(s)T + |s|^2\mathcal{I} \text{ is not invertible}\}.$$

As in the classical Riesz–Dunford case, it turns out that $\sigma_S(T)$ is a compact nonempty set, and the functional calculus is now defined by setting, for $T \in \mathcal{B}(V)$, and f regular in a neighborhood U of $\sigma_S(T)$,

$$f(T) = \frac{1}{2\pi} \int_{\partial U_I} S_L^{-1}(s,T)\, ds_I\, f(s).$$

As shown in [36] and references therein, this functional calculus can be extended to unbounded operators and, using the theory of slice monogenic functions described in the previous subsection, to m-tuples of noncommuting operators.

10.3 Orthogonal Complex Structures Induced by Regular Functions

As we have already mentioned, the aim of the current research on regular functions is not only to extend the theory to Clifford algebras and to other real alternative algebras but also to apply it to the solution of challenging open problems. In addition to the introduction of new versions of functional calculus described in Sect. 10.2, we will illustrate an application of the theory of regular functions to the construction and classification of orthogonal complex structures on certain domains in the Euclidean 4-space. Experts in complex differential geometry have been studying this last problem since the early 1990s.

It is well known that orthogonal complex structures on the four-dimensional Euclidean space are better described using quaternions. Specifically, certain dense open subsets of $\mathbb{R}^4 = \mathbb{H}$ possess complex structures that are induced from algebraic surfaces in the twistor space $\mathbb{CP}^3$ that fibers over $S^4 = \mathbb{HP}^1$. Let us recall the general definition of this class of structures.

Definition 10.15. Let (M^4, g) be a four-dimensional oriented Riemannian manifold. An *almost complex structure* is an endomorphism $\mathcal{J}$ of TM satisfying $\mathcal{J}^2 = -\mathcal{I}$. It is said to be *orthogonal* if it is an orthogonal transformation, that is, $g(\mathcal{J}v, \mathcal{J}w) = g(v, w)$ for every $v, w \in T_pM$. In addition, we shall assume that such a $\mathcal{J}$ preserves the orientation of M. An orthogonal almost complex structure is said to be an *orthogonal complex structure*, abbreviated to *OCS*, if $\mathcal{J}$ is integrable.

Every imaginary unit $I \in \mathbb{S}$ can be regarded as a complex structure on $\mathbb{H}$ in a tautological fashion as follows. After identifying each tangent space $T_p\mathbb{H}$ with $\mathbb{H}$ itself, we define the complex structure by left multiplication by I, that is, $\mathcal{J}_p v = I v$ for all $v \in T_p\mathbb{H} \cong \mathbb{H}$. Such a complex structure is called *constant,* and it is clearly orthogonal: if we choose $J \in \mathbb{S}$ such that $I \perp J$, then $\mathcal{J}_p$ is associated to the matrix

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

with respect to the orthogonal basis $1, I, J, IJ$. It was proven in [130] that every OCS defined globally on $\mathbb{H}$ is constant. More generally, it was proven in [114] that

Theorem 10.16. *Let $\mathcal{J}$ be an OCS of class C^1 on $\mathbb{R}^4 \setminus \Lambda$, where Λ is a closed set whose one-dimensional Hausdorff measure vanishes. Then either $\mathcal{J}$ is constant or $\mathcal{J}$ can be maximally extended to the complement $\mathbb{R}^4 \setminus \{p\}$ of a point. In both cases, $\mathcal{J}$ is the push-forward of the standard OCS on $\mathbb{R}^4$ under a conformal transformation.*

If the real axis $\mathbb{R}$ is removed from $\mathbb{H}$, then a totally different structure $\mathbb{J}$ can be constructed, setting

$$\mathbb{J}_q v = I_q v$$

for all $q = x + yI_q \in \mathbb{H} \setminus \mathbb{R}$ and for all $v \in T_q\mathbb{H} \cong \mathbb{H}$. Observe that $\mathbb{J}$ is integrable. Indeed, when we express

$$\mathbb{H} \setminus \mathbb{R} \;=\; \mathbb{R} + \mathbb{S} \cdot \mathbb{R}^+ \;\cong\; \mathbb{CP}^1 \times \mathbb{C}^+, \tag{10.3}$$

as the product of the Riemann sphere and the upper half-plane $\mathbb{C}^+ = \{x + iy \in \mathbb{C} : y > 0\}$, then $\mathbb{J}$ is the product complex structure on the two factors. Furthermore, $\mathbb{J}$ is orthogonal and it arises from a quadric in the twistor space $\mathbb{CP}^3$. For details, see [114], where the next result is also proven.

Theorem 10.17. *Let $\mathcal{J}$ be an OCS of class C^1 on $\mathbb{R}^4 \setminus \Lambda$, where Λ is a round circle or a straight line, and assume that $\mathcal{J}$ is not conformally equivalent to a constant OCS. Then $\mathcal{J}$ is unique up to sign, and $\mathbb{R}^4 \setminus \Lambda$ is a maximal domain of definition for $\mathcal{J}$.*

As a consequence, $\mathbb{J}$ and $-\mathbb{J}$ are the only nonconstant OCS's on $\mathbb{H} \setminus \mathbb{R}$.

For a closed subset Λ of $\mathbb{H}$ other than those described in the previous theorems, the existence and the classification of OCS's on $\mathbb{H} \setminus \Lambda$ are, in general, open problems. No systematic methods were known for addressing this problem, other than attempting to identify graphs in the twistor space $\mathbb{CP}^3$ manufactured from some obvious algebraic subsets.

The paper [54] addresses such a problem working directly on $\mathbb{H}$, and it establishes a rather surprising connection with the class of regular functions. The idea is simple, namely, that a regular function maps $\mathbb{J}$ locally to another OCS. Indeed, we saw in Corollary 8.22 that the real differential f_* of an injective regular function f is invertible at all points. This allows to push-forward the complex structure $\mathbb{J}$ (defined on $\mathbb{H} \setminus \mathbb{R}$).

Definition 10.18. Let Ω be a symmetric slice domain and let $f : \Omega \to \mathbb{H}$ be an injective regular function. The *induced structure* on $f(\Omega \setminus \mathbb{R})$ is the push-forward

$$\mathbb{J}^f = f_* \, \mathbb{J} \, (f_*)^{-1}.$$

Let us formulate an explicit expression for this induced structure.

Theorem 10.19. *Let Ω be a symmetric slice domain and let $f : \Omega \to \mathbb{H}$ be an injective regular function. Then*

$$\mathbb{J}^f_{f(q)} v = I_q v \tag{10.4}$$

for all $q \in \Omega \setminus \mathbb{R}$ and $v \in T_{f(q)} f(\Omega \setminus \mathbb{R}) \cong \mathbb{H}$. As a consequence, $\mathbb{J}^f$ is an OCS on $f(\Omega \setminus \mathbb{R})$.

This new approach can be illustrated describing OCS's induced by mapping the real axis of $\mathbb{H}$ onto a parabola γ by means of the function $f(q) = q^2 + qi$. Proposition 3.20 allows to study the roots of the quadratic polynomial $q^2 + qi + c$

over $\mathbb{H}$, and it turns out that f is injective when restricted either to the open right half-space

$$\mathbb{H}^+ = \{x_0 + ix_1 + jx_2 + kx_3 : x_0 > 0\}$$

or to the open left half-space $\mathbb{H}^-$. It is easily seen that their boundary $i\mathbb{R} + j\mathbb{R} + k\mathbb{R}$ maps onto the (three-dimensional) solid paraboloid

$$\mathfrak{G} = \left\{x_0 + jx_2 + kx_3 : x_0, x_2, x_3 \in \mathbb{R},\ x_0 \le 1/4 - (x_2^2 + x_3^2)\right\}$$

and that $f(\mathbb{H}^+) = \mathbb{H} \setminus \mathfrak{G} = f(\mathbb{H}^-)$. This allowed to define single-valued complex structures on a dense open subset of $\mathbb{H} \setminus \gamma$:

Theorem 10.20. *Let $f(q) = q^2 + qi$. Let $\mathbb{J}^+$ denote the complex structure induced by the restriction $f : \mathbb{H}^+ \to \mathbb{H}$ on $\mathbb{H} \setminus (\gamma \cup \mathfrak{G})$ and let $\mathbb{J}^-$ be induced by $f : \mathbb{H}^- \to \mathbb{H}$. Then $\mathbb{H} \setminus (\gamma \cup \mathfrak{G})$ is the maximal open domain of definition for both $\mathbb{J}^+$ and $\mathbb{J}^-$. Indeed, both $\mathbb{J}^+$ and $\mathbb{J}^-$ extend continuously to the boundary Γ of $\mathfrak{G}$ in the 3-space $\mathbb{R} + j\mathbb{R} + k\mathbb{R}$, but neither of them extends continuously to any point of $\mathfrak{G} \setminus \Gamma$.*

The induced complex structures $\mathbb{J}^+, \mathbb{J}^-$ are studied in detail in [54], and it turns out that they arise from a (singular) quartic surface $\mathscr{K}$ (a scroll) in $\mathbb{CP}^3$. Furthermore, it turned out that it is not possible to define an OCS on the complement $\mathbb{H} \setminus \gamma$ of the parabola. The proof of this fact follows the line of [114, Theorem 14.4 of v1], which also gives an idea of the reason why it is necessary to remove a solid paraboloid $\mathfrak{G}$ (in addition to γ) in order to define an OCS. Indeed, the theorem just mentioned states that a (non-singular) algebraic subvariety of $\mathbb{CP}^3$ of degree $d > 2$ cannot define a single-valued OCS unless one removes a set of real dimension at least 3.

It seems natural to expect that the treatment of OCSs on analogous subsets of $\mathbb{H}$ will become possible with the aid of regular functions, possibly involving other quartic scrolls described in [44].

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