Perturbative Gauge Theory As A String Theory In Twistor Space

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Perturbative scattering amplitudes in Yang-Mills theory have many unexpected properties, such as holomorphy of the maximally helicity violating amplitudes. To interpret these results, we Fourier transform the scattering amplitudes from momentum space to twistor space, and argue that the transformed amplitudes are supported on certain holomorphic curves. This in turn is apparently a consequence of an equivalence between the perturbative expansion of $\mathcal{N}=4$ super Yang-Mills theory and the D-instanton expansion of a certain string theory, namely the topological B model whose target space is the Calabi-Yau supermanifold $\mathbb{CP}^{3|4}$.

1. Introduction

The perturbative expansion of Yang-Mills theory has remarkable properties that are not evident upon inspecting the Feynman rules. For example, the tree level scattering amplitudes that are maximally helicity violating (MHV) can be expressed in terms of a simple holomorphic or antiholomorphic function. This was first conjectured by Parke and Taylor based on computations in the first few cases [1]; the general case was proved by Berends and Giele [2]. (Unexpected simplicity and selection rules in Yang-Mills and gravitational helicity amplitudes were first found, as far as I know, by DeWitt [3] for four particle amplitudes.) These unexpected simplifications have echoes, in many cases, in loop amplitudes, especially in the supersymmetric case. For a sampling of one-loop results, see the review [4], and for some recent two-loop results for the theory with maximal or $\mathcal{N}=4$ supersymmetry (which was first constructed in [5]), see [6]. $\mathcal{N}=4$ super Yang-Mills theory is an important test case for perturbative gauge theory, since it is the simplest case and, for example, has the same gluonic tree amplitudes as pure Yang-Mills theory.

In the present paper, we will offer a new perspective on explaining these results. We will study what happens when the usual momentum space scattering amplitudes are Fourier transformed to Penrose's twistor space [7]. We argue that the perturbative amplitudes in twistor space are supported on certain holomorphic curves. Results such as the holomorphy of the tree-level MHV amplitudes, as well as more complicated (and novel) differential equations obeyed by higher order amplitudes, are direct consequences of this.

We interpret these results to mean that the perturbative expansion of $\mathcal{N}=4$ super Yang-Mills theory with U(N) gauge group is equivalent to the instanton expansion of a certain string theory. The instantons in question are D-instantons rather than ordinary worldsheet instantons. The string theory is the topological B model whose target space is the Calabi-Yau supermanifold $\mathbb{CP}^{3|4}$. This is the supersymmetric version of twistor space, as defined [9] and exploited [10] long ago.¹ From the string theory, we recover the tree level MHV amplitudes of gauge theory. They arise from a one-instanton computation that leads to a formalism similar to that suggested by Nair [12].

This representation of weakly coupled $\mathcal{N}=4$ super-Yang-Mills theory as a string theory is an interesting counterpoint to the by now familiar description of the strongly coupled regime of the same theory via Type IIB superstring theory on $\mathrm{AdS}_5 \times \mathbb{S}^5$ [13]. However, many aspects of the B model of $\mathbb{CP}^{3|4}$ remain unclear. One pressing question is

¹ See [11] for the bosonic version of the supersymmetric construction found in [10].

to understand the closed string sector. The closed strings may possibly give some version of $\mathcal{N}=4$ conformal supergravity, in which case the string theory considered here is equivalent to super Yang-Mills theory only in the planar limit (large N with fixed g^2N [14]), in which the closed strings decouple. (If so, as conformal supergravity is generally assumed to have negative energies and ghosts, the B model of $\mathbb{CP}^{3|4}$ may be physically sensible only in the planar limit.)

In twistor theory, it has been a longstanding problem [15,16] to understand how to use twistors to describe perturbative field theory amplitudes. Our proposal in the present paper differs from previous attempts mainly in that we consider families of holomorphic curves in twistor space, and not just products of twistor spaces.

In section 2, we review Yang-Mills helicity amplitudes and their description via spinors [17-24] and the MHV amplitudes. For expositions of this material, see [25,26]. For the use of spinors in relativity, see [27,28]. Then we describe the Fourier transform to twistor space. For general reviews of twistor theory, see [8,15,29-32]; for the twistor transform of the self-dual Einstein and Yang-Mills equations, see respectively [33] and [34], as well as [35,36] for the Euclidean signature case. In section 3, we investigate the behavior of the perturbative Yang-Mills amplitudes in twistor space. We demonstrate (in various examples of tree level and one-loop amplitudes) that they are supported on certain curves in twistor space, by showing that in momentum space they obey certain differential equations that generalize the holomorphy of the MHV amplitudes. In section 4, we argue that the topological B model of super twistor space $\mathbb{CP}^{3|4}$ gives a natural origin for these results.

The analysis in section 3 and the proposal in section 4 are tentative. They represent the best way that has emerged so far to organize and interpret the facts, but much more needs to be understood.

We are left with numerous questions. For example, are there analogous descriptions of perturbative expansions for other field theories with less supersymmetry, at least in their planar limits? What about the $\mathcal{N}=4$ theory on its Coulomb branch, where conformal invariance is spontaneously broken? Can its perturbative expansion be described by the instanton expansion of the string theory we consider in this paper, expanded around a shifted vacuum? In section 3, we also observe that the tree level MHV amplitudes of General Relativity are supported on curves in twistor space. Is this a hint that the perturbative expansion of $\mathcal{N}=8$ supergravity can be described by some string theory? What would be the target space of such a string theory and would its existence imply finiteness of $\mathcal{N}=8$ supergravity?

How do the usual infrared divergences of gauge perturbation theory arise from the twistor point of view? In the one-loop example that we consider, the twistor amplitude is finite; the infrared divergence arises from the Fourier transform back to momentum space. Is this general? In $\mathcal{N}=4$ super Yang-Mills theory, the planar loop amplitudes are known to be simpler than the non-planar ones. Does including the closed string sector of the string theory give a model in which the simplicity persists for non-planar diagrams? Finally, many technical problems need to be addressed in order to properly define the string theory amplitudes and facilitate their computation.

2. Helicity Amplitudes And Twistor Space

2.1. Spinors

Before considering scattering amplitudes, we will review some kinematics in four dimensions. We start out in signature +--, but we sometimes generalize to other signatures. Indeed, this paper is only concerned with perturbation theory, for which the signature is largely irrelevant as the scattering amplitudes are holomorphic functions of the kinematic variables. Some things will be simpler with other signatures or for complex momenta with no signature specified.

First we recall that the Lorentz group in four dimensions, upon complexification, is locally isomorphic to $SL(2) \times SL(2)$, and thus the finite-dimensional representations are classified as (p,q), where p and q are integers or half-integers. The negative and positive chirality spinors transform in the (1/2,0) and (0,1/2) representations, respectively. We write generically λ_a , a=1,2, for a spinor transforming as (1/2,0), and $\tilde{\lambda}_{\dot{a}}$, $\dot{a}=1,2$, for a spinor transforming as (0,1/2).

Spinor indices of type (1/2,0) are raised and lowered with the antisymmetric tensor ϵ_{ab} and its inverse ϵ^{ab} (obeying $\epsilon^{ab}\epsilon_{bc} = \delta^a_c$): $\lambda_a = \epsilon_{ab}\lambda^b$, $\lambda^b = \epsilon^{bc}\lambda_c$. Given two spinors λ_1 , λ_2 both of positive chirality, we can form the Lorentz invariant $\langle \lambda_1, \lambda_2 \rangle = \epsilon_{ab}\lambda_1^a\lambda_2^b$. From the definitions, it follows that $\langle \lambda_1, \lambda_2 \rangle = -\langle \lambda_2, \lambda_1 \rangle = -\epsilon^{ab}\lambda_{1a}\lambda_{2b}$.

Similarly, we raise and lower indices of type (0, 1/2) with the antisymmetric tensor $\epsilon_{\dot{a}\dot{b}}$ and its inverse $\epsilon^{\dot{a}\dot{b}}$, again imposing $\epsilon^{\dot{a}\dot{b}}\epsilon_{\dot{b}\dot{c}}=\delta^{\dot{a}}_{\dot{b}}$. For two spinors $\widetilde{\lambda}_1$, $\widetilde{\lambda}_2$ both of negative chirality, we define $[\widetilde{\lambda}_1, \widetilde{\lambda}_2] = \epsilon_{\dot{a}\dot{b}}\widetilde{\lambda}_1^{\dot{a}}\widetilde{\lambda}_2^{\dot{b}} = -[\widetilde{\lambda}_2, \widetilde{\lambda}_1]$.

The vector representation of SO(3,1) is the (1/2,1/2) representation. Thus, a momentum vector p_{μ} , $\mu = 0, \ldots, 3$, can be represented as a "bi-spinor" $p_{a\dot{a}}$ with one spinor index a or \dot{a} of each chirality. The explicit mapping from p_{μ} to $p_{a\dot{a}}$ can be made using the

chiral part of the Dirac matrices. With signature +---, one can take the Dirac matrices to be

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \overline{\sigma}^{\mu} & 0 \end{pmatrix}, \tag{2.1}$$

where we take $\sigma^{\mu}=(1,\vec{\sigma})$, $\overline{\sigma}^{\mu}=(-1,\vec{\sigma})$, with $\vec{\sigma}$ being the 2×2 Pauli spin matrices. In particular, the upper right hand block of γ^{μ} is a 2×2 matrix $\sigma^{\mu}_{a\dot{a}}$ that maps spinors of one chirality to the other. For any spinor p_{μ} , define

$$p_{a\dot{a}} = \sigma^{\mu}_{a\dot{a}} p_{\mu}. \tag{2.2}$$

Thus, with the above representation of σ^{μ} , we have $p_{a\dot{a}} = p_0 + \vec{\sigma} \cdot \vec{p}$ (where p_0 and \vec{p} are the "time" and "space" parts of p^{μ}), from which it follows that

$$p_{\mu}p^{\mu} = \det(p_{a\dot{a}}). \tag{2.3}$$

Thus a vector p^{μ} is lightlike if and only if the corresponding matrix $p_{a\dot{a}}$ has determinant zero.

Any 2×2 matrix $p_{a\dot{a}}$ has rank at most two, so it can be written $p_{a\dot{a}} = \lambda_a \widetilde{\lambda}_{\dot{a}} + \mu_a \widetilde{\mu}_{\dot{a}}$ for some spinors λ, μ and $\widetilde{\lambda}, \widetilde{\mu}$. The rank of a 2×2 matrix is less than two if and only if its determinant vanishes. So the lightlike vectors p^{μ} are precisely those for which

$$p_{a\dot{a}} = \lambda_a \widetilde{\lambda}_{\dot{a}},\tag{2.4}$$

for some spinors λ_a and $\widetilde{\lambda}_{\dot{a}}$.

If we wish $p_{a\dot{a}}$ to be real with Lorentz signature, we must take $\tilde{\lambda} = \pm \overline{\lambda}$ (where $\overline{\lambda}$ is the complex conjugate of λ). The sign determines whether p^{μ} has positive energy or negative energy.

It will also be convenient to consider other signatures. In signature ++--, λ and $\widetilde{\lambda}$ are independent, real, two-component objects. Indeed, with signature ++--, the Lorentz group SO(2,2) is, without any complexification, locally isomorphic to $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$, so the spinor representations are real. With Euclidean signature ++++, the Lorentz group is locally isomorphic to $SU(2)\times SU(2)$; the spinor representations are pseudoreal. A lightlike vector cannot be real with Euclidean signature.

Obviously, if λ and $\widetilde{\lambda}$ are given, a corresponding lightlike vector p is determined, via (2.4). It is equally clear that if a lightlike vector p is given, this does not suffice to determine λ and $\widetilde{\lambda}$. They can be determined only modulo the scaling

$$\lambda \to u\lambda, \ \widetilde{\lambda} \to u^{-1}\widetilde{\lambda}$$
 (2.5)

for $u \in \mathbb{C}^*$, that is, u is a nonzero complex number. (In signature +---, if p is real, we can restrict to |u|=1. In signature ++--, if λ and $\widetilde{\lambda}$ are real, we can restrict to real u.) Not only is there no natural way to determine λ as a function of p; there is in fact no continuous way to do so, as there is a topological obstruction to this. Consider, for example, massless particles of unit energy; the energy-momentum of such a particle is specified by the momentum three-vector \vec{p} , a unit vector which determines a point in \mathbb{S}^2 . Once \vec{p} is given, the space of possible λ 's is a non-trivial complex line bundle over \mathbb{S}^2 that is known as the Hopf line bundle; non-triviality of this bundle means that one cannot pick λ as a continuously varying function of \vec{p} .

Once p is given, the additional information that is involved in specifying λ (and hence $\widetilde{\lambda}$) is equivalent to a choice of wavefunction for a spin one-half particle with momentum vector p. In fact, the chiral Dirac equation for a spinor ψ^a is

$$i\sigma^{\mu}_{a\dot{a}}\frac{\partial\psi^a}{\partial x^{\mu}} = 0. \tag{2.6}$$

A plane wave $\psi^a = \lambda^a \exp(ip \cdot x)$ (with constant λ^a) obeys this equation if and only if $p_{a\dot{a}}\lambda^a = 0$. This is so if and only if $p_{a\dot{a}}$ can be written as $\lambda_a \widetilde{\lambda}_{\dot{a}}$ for some $\widetilde{\lambda}$.

The formula $p \cdot p = \det(p_{a\dot{a}}) = \epsilon^{ab} \epsilon^{\dot{a}\dot{b}} p_{a\dot{a}} p_{b\dot{b}}$ generalizes for any two vectors p and q to $p \cdot q = \epsilon^{ab} \epsilon^{\dot{a}\dot{b}} p_{a\dot{a}} q_{b\dot{b}}$. Hence if p and q are lightlike vectors, which we write in the form $p_{a\dot{a}} = \lambda_a \widetilde{\lambda}_{\dot{a}}$ and $q_{b\dot{b}} = \mu_b \widetilde{\mu}_{\dot{b}}$, then we have

$$p \cdot q = \langle \lambda, \mu \rangle [\widetilde{\lambda}, \widetilde{\mu}]. \tag{2.7}$$

2.2. Helicity Amplitudes

Now we consider scattering amplitudes of massless particles in four dimensions. We consider n massless particles with momentum vectors p_1, p_2, \ldots, p_n . For scattering of scalar particles, the initial and final states are completely fixed by specifying the momenta. The scattering amplitude is, therefore, a function of the p_i . For example, for n = 4, the only independent Lorentz invariants are the usual Mandelstam variables $s = (p_1 + p_2)^2$ and $t = (p_1 + p_3)^2$, and the scattering amplitude (being dimensionless) is a function of the ratio s/t.

For particles with spin, the scattering amplitude is not merely a function of the momenta. For example, in the case of massless particles of spin one – the main case that we will consider in detail in the present paper – in the conventional description, to each external particle is associated not just a momentum vector p_i^{μ} but also a polarization vector

 ϵ_i^{μ} . The polarization vector obeys the constraint $\epsilon_i \cdot p_i = 0$, and is subject to the gauge invariance

$$\epsilon_i \to \epsilon_i + wp_i,$$
 (2.8)

for any constant w.

The scattering amplitude is most often introduced in textbooks as a function of the p_i and ϵ_i , subject to this constraint and gauge invariance. However, in four dimensions, it is more useful to label external gauge bosons by their helicity, +1 or -1 or simply + or -. If a choice of momentum vector p_i and helicity + or - enabled us to pick for each particle a polarization vector, then the scattering amplitude of gauge bosons would depend only on the momenta and the choices \pm of helicities.

However, given a lightlike momentum vector p and a choice of helicity, there is no natural way to pick a polarization vector with that helicity. (There is not even any continuous way to pick a polarization vector as a function of the momentum; in trying to do so, one runs into a non-trivial complex line bundle which is the square of the Hopf bundle.) Suppose though that instead of being given only a lightlike vector $p_{a\dot{a}}$ one is given a λ , that is a decomposition $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$. Then [22,24] we do have enough information to determine a polarization vector, up to a gauge transformation. To get a negative helicity polarization vector, we pick any positive helicity spinor $\tilde{\mu}_{\dot{a}}$ that is not a multiple of $\tilde{\lambda}$ and set

$$\epsilon_{a\dot{a}} = \frac{\lambda_a \widetilde{\mu}_{\dot{a}}}{[\widetilde{\lambda}, \widetilde{\mu}]}.$$
 (2.9)

This obeys the constraint $0 = \epsilon_{\mu} p^{\mu} = \epsilon_{a\dot{a}} p^{a\dot{a}}$, since $\langle \lambda, \lambda \rangle = 0$. It also is independent of the choice of $\tilde{\mu}$ up to a gauge transformation. To see this, note that since the space of possible $\tilde{\mu}$'s is two-dimensional, any variation of $\tilde{\mu}$ is of the form

$$\widetilde{\mu} \to \widetilde{\mu} + \eta \widetilde{\mu} + \eta' \widetilde{\lambda},$$
 (2.10)

with some complex parameters η , η' . The η term drops out of (2.9), since $\epsilon_{a\dot{a}}$ is invariant under rescaling of $\tilde{\mu}$; the η' term changes $\epsilon_{a\dot{a}}$ by a gauge transformation, a multiple of $\lambda_a \tilde{\lambda}_{\dot{a}}$.

Hopefully, the use of ϵ_{μ} or $\epsilon_{a\dot{a}}$ for a polarization vector, while the Levi-Civita tensors for the spinors are called ϵ_{ab} and $\epsilon_{\dot{a}\dot{b}}$, will not cause confusion. Polarization vectors only appear in the present subsection.

³ In labeling helicities, we consider all particles to be outgoing. In crossing symmetry, an incoming particle of one helicity is equivalent to an outgoing particle of the opposite helicity.

Under $\lambda \to u\lambda$, $\widetilde{\lambda} \to u^{-1}\widetilde{\lambda}$, $\epsilon_{a\dot{a}}$ has the same scaling as λ^2 . This might have been anticipated: since λ carries helicity -1/2 (as we saw above in discussing the Dirac equation), a helicity -1 polarization vector should scale as λ^2 .

To determine more directly the helicity of a massless particle whose polarization vector is $\epsilon_{a\dot{a}}$, we construct the field strength $F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}=-i(p_{\mu}\epsilon_{\nu}-p_{\nu}\epsilon_{\mu})$ and verify that it is selfdual or anti-selfdual. In terms of spinors, the field strength is $F_{ab\dot{a}\dot{b}}=-F_{ba\dot{b}\dot{a}}$ and can be expanded $F_{ab\dot{a}\dot{b}}=\epsilon_{ab}\widetilde{f}_{\dot{a}\dot{b}}+f_{ab}\epsilon_{\dot{a}\dot{b}}$, where f and \tilde{f} are the selfdual and anti-selfdual parts of F. With $p_{a\dot{a}}=\lambda_a\widetilde{\lambda}_{\dot{a}}$ and $\epsilon_{b\dot{b}}$ defined as above, we find that $f_{ab}\sim\lambda_a\lambda_b$ and $\widetilde{f}_{\dot{a}\dot{b}}=0$. So F is selfdual and the photon has negative helicity. ⁴

We can similarly make a polarization vector of positive helicity, introducing an arbitrary negative chirality spinor μ_a that is not a multiple of λ_a and setting

$$\widetilde{\epsilon}_{a\dot{a}} = \frac{\mu_a \widetilde{\lambda}_{\dot{a}}}{\langle \mu, \lambda \rangle}.$$
(2.11)

As one would expect from the above discussion, under $\lambda \to u\lambda$, $\widetilde{\epsilon}$ has the same scaling as λ^{-2} .

Although a scattering amplitude of massless gauge bosons cannot be regarded as a function of the momenta p_i , it can be regarded as a function of the spinors λ_i and $\widetilde{\lambda}_i$, as well as the helicity labels $h_i = \pm 1$, since as we have just seen this data determines the polarization vectors ϵ_i up to a gauge transformation. Thus, instead of writing the amplitude as $\widehat{A}(p_i, \epsilon_i)$, where ϵ_i are the polarization vectors, we write it as $\widehat{A}(\lambda_i, \widetilde{\lambda}_i, h_i)$. When formulated in this way, the amplitude obeys for each i an auxiliary condition

$$\left(\lambda_i^a \frac{\partial}{\partial \lambda_i^a} - \widetilde{\lambda}_i^{\dot{a}} \frac{\partial}{\partial \widetilde{\lambda}_i^{\dot{a}}}\right) \widehat{A}(\lambda_i, \widetilde{\lambda}_i, h_i) = -2h_i \widehat{A}(\lambda_i, \widetilde{\lambda}_i, h_i), \tag{2.12}$$

which reflects the scaling with λ of the polarization vectors. This equation holds for helicity amplitudes for massless particles of any spin.

⁴ In the literature on perturbative QCD, it is conventional that the MHV amplitude, introduced presently, that is a function of λ describes mostly + helicity scattering. In the literature on twistor theory – at least in the mathematical branch of that literature – it is conventional that an instanton is an anti-selfdual gauge field in spacetime and corresponds to a holomorphic vector bundle over twistor space. We will try to follows these two conventions.

The scattering amplitude is of course proportional to a delta function of energy-momentum conservation, $(2\pi)^4 \delta^4(\sum_i p_i)$, or in terms of spinors $(2\pi^4)\delta^4(\sum_i \lambda_i^a \tilde{\lambda}_i^{\dot{a}})$. The general form of the scattering amplitude is thus

$$\widehat{A}(\lambda_i, \widetilde{\lambda}_i, h_i) = i(2\pi)^4 \delta^4 \left(\sum_i \lambda_i^a \widetilde{\lambda}_i^{\dot{a}} \right) A(\lambda_i, \widetilde{\lambda}_i, h_i), \tag{2.13}$$

where the reduced amplitude A obeys the same equation (2.12) as \widehat{A} . (We often write \widehat{A} and A as functions just of λ_i and $\widetilde{\lambda}_i$, with the h_i understood.)

2.3. Maximally Helicity Violating Amplitudes

To make this discussion tangible, let us consider the tree level scattering of n gluons in the simplest configuration. The scattering amplitude with n outgoing gluons all of the same helicity vanishes, as does (for n > 3) the amplitude with n - 1 outgoing gluons of one helicity and one of the opposite helicity.⁵ The "maximally helicity violating" or MHV amplitude is the case with n - 2 gluons of one helicity and 2 of the opposite helicity. To understand the name "maximally helicity violating," recall that in labeling the helicities, we consider all gluons to be outgoing. So after allowing for crossing symmetry, the MHV amplitude describes, for example, a process in which all incoming gluons have one helicity and all but two outgoing gluons – the maximal possible number – have the opposite helicity.

For n=4, the only nonzero tree level amplitude is the MHV amplitude with helicities some permutation of ++--, and similarly for n=5, the nonzero amplitudes are MHV amplitudes such as ++--- or +++--. These amplitudes dominate two-jet and three-jet production in hadron colliders at very high energies, and so are of phenomenological importance. The lowest order non-MHV tree level amplitudes are the n=6 amplitudes such as +++--. They enter, for example, in four-jet production at hadron colliders.

The actual form of the tree-level MHV amplitudes (conjectured by Parke and Taylor based on results for small n [1], and proved by Berends and Giele [2]) is quite remarkable. The reduced amplitude A can be written as a function only of the λ_i or only of the $\widetilde{\lambda}_i$, depending on whether the outgoing helicities are almost all + or almost all -. For real momenta in Minkowski signature, one has $\widetilde{\lambda}_i = \pm \overline{\lambda}_i$, and then the MHV amplitudes are

⁵ The amplitude with n=3 is exceptional and is often omitted, but will be discussed in section 3.2.

holomorphic or antiholomorphic functions of the λ_i , depending on whether the helicities are mostly + or mostly -.

To describe the results more precisely, we take the gauge group to be U(N) (for some sufficiently large N as to avoid accidental equivalences of any traces that we might encounter). We recall that tree level diagrams in Yang-Mills theory are planar, and generate a single-trace interaction [14]. In such a planar diagram, the n gauge bosons are attached to the index loop in a definite cyclic order, as indicated in figure 1. If we number the gauge bosons so that the cyclic order is simply $1, 2, 3, \ldots, n$, then the amplitude includes a group theory factor $I = \text{Tr } T_1 T_2 \ldots T_n$. It suffices to study the amplitude with one given cyclic order; the full amplitude is obtained from this by summing over the possible cyclic orders, to achieve Bose symmetry. Gluon scattering amplitudes considered in this paper are always proportional to the group theory trace I, and this factor is omitted in writing the formulas.

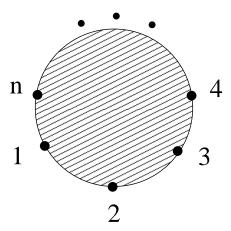


Fig. 1: *n* external gluons cyclically attached to the boundary of a disc, representing the group theory structure of a Yang-Mills tree diagram.

Suppose that gauge bosons r and s ($1 \le r < s \le n$) have negative helicity and the others have positive helicity. The reduced tree level amplitude for this process (with the energy-momentum delta function and the trace I both omitted) is

$$A = g^{n-2} \frac{\langle \lambda_r, \lambda_s \rangle^4}{\prod_{i=1}^n \langle \lambda_i, \lambda_{i+1} \rangle}.$$
 (2.14)

(Here g is the gauge coupling constant, and we set $\lambda_{n+1} = \lambda_1$.) Note that this amplitude has the requisite homogeneity in each variable. It is homogeneous of degree -2 in each λ_i with $i \neq r, s$, since each λ_i appears twice in the denominator in (2.14). But for i = r, s, it

is homogeneous in λ_i with degree +2, since in these cases the numerator is homogeneous of degree four in λ_i . If gauge bosons r and s have positive helicity and the others have negative helicity, the amplitude is instead

$$A = g^{n-2} \frac{[\widetilde{\lambda}_r, \widetilde{\lambda}_s]^4}{\prod_{i=1}^n [\widetilde{\lambda}_i, \widetilde{\lambda}_{i+1}]}.$$
 (2.15)

In order to write such formulas briefly, one often abbreviates $\langle \lambda_i, \lambda_j \rangle$ as $\langle i j \rangle$, and similarly one write $[\widetilde{\lambda}_i, \widetilde{\lambda}_j]$ as [i j]. In this way, (2.14) would be written as

$$A = g^{n-2} \frac{\langle r \ s \rangle^4}{\prod_{i=1}^n \langle i \ i+1 \rangle}.$$
 (2.16)

Specializing to the case of n = 4, we may appear to have a contradiction, since for example the amplitude in which the helicities in cyclic order are --++ is a special case of each of these constructions. Via (2.14), we expect

$$A = \frac{\langle \lambda_1, \lambda_2 \rangle^4}{\prod_{i=1}^4 \langle \lambda_i, \lambda_{i+1} \rangle},\tag{2.17}$$

but via (2.15), we expect

$$A = \frac{\left[\widetilde{\lambda}_3, \widetilde{\lambda}_4\right]^4}{\prod_{i=1}^4 \left[\widetilde{\lambda}_i, \widetilde{\lambda}_{i+1}\right]}.$$
 (2.18)

How can the same function be both holomorphic and antiholomorphic? The resolution is instructive. From momentum conservation, $\sum_i \lambda_i^a \widetilde{\lambda}_i^{\dot{a}} = 0$, it follows, upon taking inner products with λ_y and $\widetilde{\lambda}_z$, that for any y and z we have $\sum_i \langle \lambda_y, \lambda_i \rangle [\widetilde{\lambda}_i, \widetilde{\lambda}_z] = 0$. Setting, for example, y = 1, z = 2, this leads to

$$\frac{\langle \lambda_1, \lambda_3 \rangle}{\langle \lambda_1, \lambda_4 \rangle} = -\frac{[\widetilde{\lambda}_4, \widetilde{\lambda}_2]}{[\widetilde{\lambda}_3, \widetilde{\lambda}_2]}.$$
 (2.19)

By repeated use of such identities, one can show that the two formulas for A are equivalent, when multiplied by a delta function of energy-momentum conservation.

2.4. Conformal Invariance

A useful next step is to verify the conformal invariance of the MHV amplitudes. First we write down the conformal generators in terms of the λ and $\widetilde{\lambda}$ variables. We consider conformal generators for a single massless particle; the corresponding generators for the full n-particle system are obtained simply by summing over the n particles.

Some of the conformal generators are obvious. For example, the Lorentz generators are

$$J_{ab} = \frac{i}{2} \left(\lambda_a \frac{\partial}{\partial \lambda^b} + \lambda_b \frac{\partial}{\partial \lambda^a} \right)$$

$$\widetilde{J}_{\dot{a}\dot{b}} = \frac{i}{2} \left(\widetilde{\lambda}_{\dot{a}} \frac{\partial}{\partial \widetilde{\lambda}^{\dot{b}}} + \widetilde{\lambda}_{\dot{b}} \frac{\partial}{\partial \widetilde{\lambda}^{\dot{a}}} \right).$$
(2.20)

The momentum operator is a multiplication operator,

$$P_{a\dot{a}} = \lambda_a \widetilde{\lambda}_{\dot{a}}.\tag{2.21}$$

As we can see from this formula, it is natural to take λ and $\widetilde{\lambda}$ to have dimension 1/2 (so that $-i[D,\lambda]=\lambda/2$, and similarly for $\widetilde{\lambda}$). This determines the dilatation operator D up to an additive constant k:

$$D = \frac{i}{2} \left(\lambda^a \frac{\partial}{\partial \lambda^a} + \widetilde{\lambda}^{\dot{a}} \frac{\partial}{\partial \widetilde{\lambda}^{\dot{a}}} + k \right). \tag{2.22}$$

What about the special conformal generator $K_{a\dot{a}}$? As it has dimension -1, it cannot be represented by a multiplication operator or a first order differential operator; if we try to represent $K_{a\dot{a}}$ by a second order differential operator, the unique possibility (up to a multiplicative constant) is

$$K_{a\dot{a}} = \frac{\partial^2}{\partial \lambda^a \partial \widetilde{\lambda}^{\dot{a}}}.$$
 (2.23)

To verify that these operators do generate conformal transformations, the non-trivial step is the commutator of $K_{a\dot{a}}$ with $P^{b\dot{b}}$. A short calculation shows that the desired relation $[K_{a\dot{a}}, P^{b\dot{b}}] = -i(\delta_a^b \widetilde{J}_{\dot{a}}^b + \delta_{\dot{a}}^b J_a^b + \delta_a^b \delta_{\dot{a}}^{\dot{b}} D)$ does arise precisely if we fix the constant k so that the dilatation operator becomes

$$D = \frac{i}{2} \left(\lambda^a \frac{\partial}{\partial \lambda^a} + \tilde{\lambda}^{\dot{a}} \frac{\partial}{\partial \tilde{\lambda}^{\dot{a}}} + 2 \right). \tag{2.24}$$

Now let us verify the conformal invariance of the MHV amplitude, which as we recall is

$$\widehat{A} = ig^{n-2} (2\pi)^4 \delta^4 \left(\sum_i \lambda_i^a \widetilde{\lambda}_i^{\dot{a}} \right) \frac{\langle \lambda_s, \lambda_t \rangle^4}{\prod_{i=1}^n \langle \lambda_i, \lambda_{i+1} \rangle}.$$
 (2.25)

Lorentz invariance of this formula is manifest, and momentum conservation is also clear because of the delta function. So we really only have to verify that the amplitude is annihilated by D and by K.

First we consider D. The numerator contains a delta-function of energy-momentum conservation, which in four dimensions has dimension -4, and the factor $\langle \lambda_s, \lambda_t \rangle^4$, of dimension 4. So D commutes with the numerator. We are left acting with D on the remaining factor

$$\frac{1}{\prod_{i=1}^{n} \langle \lambda_i, \lambda_{i+1} \rangle}.$$
 (2.26)

This is homogeneous in each λ_i of degree -2, so allowing for the +2 in the definition (2.24) of D, it is annihilated by D.

Similarly, we can verify that $K_{a\dot{a}}\widehat{A} = 0$. We write $\widehat{A} = ig^{n-2}(2\pi)^4\delta^4(P)A(\lambda)$, where $P^{b\dot{b}} = \sum_{i=1}^n \lambda_i^b \lambda_i^{\dot{b}}$. Using the fact that $\partial A/\partial \widetilde{\lambda} = 0$, we get, on using the chain rule,

$$K_{a\dot{a}}\widehat{A} = \sum_{i} \frac{\partial^{2}}{\partial \lambda_{i}^{a} \partial \widetilde{\lambda}_{i}^{\dot{a}}} \widehat{A}$$

$$= ig^{n-2} (2\pi)^{4} \left(\left(\left(n \frac{\partial}{\partial P^{a\dot{a}}} + P^{b\dot{b}} \frac{\partial^{2}}{\partial P^{a\dot{b}} \partial P^{b\dot{a}}} \right) \delta^{4}(P) \right) A + \left(\frac{\partial}{\partial P^{b\dot{a}}} \delta^{4}(P) \right) \sum_{i} \lambda_{i}^{b} \frac{\partial A}{\partial \lambda_{i}^{a}} \right).$$
(2.27)

Since $J_{ab}A = 0$, we can replace $\sum_i \lambda_i^b \partial A/\partial \lambda_i^a$ by $\frac{1}{2} \delta_b^a \sum_i \lambda_i^c \partial A/\partial \lambda_i^c = -(n-4)\delta_b^a A$. Upon multiplying by a test function and integrating by parts, we find that the distribution

$$P^{b\dot{b}} \frac{\partial^2}{\partial P^{a\dot{b}}\partial P^{b\dot{a}}} \delta^4(P) \tag{2.28}$$

is equal to

$$-4\frac{\partial}{\partial P^{a\dot{a}}}\delta^4(P). \tag{2.29}$$

Combining these statements, the right hand side of (2.27) vanishes.

2.5. Fourier Transform to Twistor Space

The representation of the conformal group that we have encountered above is certainly quite unusual. Some generators are represented by differential operators of degree one, but the momentum operator is a multiplication operator, and the special conformal generators are of degree two.

We can reduce to a more standard representation of the conformal group if (in Penrose's spirit [7]) we make the transformation

$$\widetilde{\lambda}_{\dot{a}} \to i \frac{\partial}{\partial \mu^{\dot{a}}} \\
-i \frac{\partial}{\partial \widetilde{\lambda}^{\dot{a}}} \to \mu_{\dot{a}}.$$
(2.30)

In making this substitution, we have arbitrarily chosen to transform $\tilde{\lambda}$ rather than λ . The choice breaks the symmetry between left and right, and means that henceforth scattering amplitudes with n_1 positive helicity particles and n_2 of negative helicity will be treated completely differently from those with n_1 and n_2 exchanged. Our choice, as we will see in detail in section 3, causes amplitudes with an arbitrary number of + helicities and only a fixed number of - helicities to be treated in a relatively uniform way, while increasing the number of - helicities makes the description more complicated. With the opposite choice for the Fourier transform, the roles of the two helicities would be reversed. Each amplitude can be studied in either of the two formalisms, and hence potentially obeys two different sets of differential equations, as we see in detail in section 3.

Upon making the substitution (2.30), the momentum and special conformal operators become first order operators,

$$P_{a\dot{a}} = i\lambda_a \frac{\partial}{\partial \mu^{\dot{a}}}$$

$$K_{a\dot{a}} = i\mu_{\dot{a}} \frac{\partial}{\partial \lambda^a}.$$
(2.31)

The Lorentz generators are unchanged in form,

$$J_{ab} = \frac{i}{2} \left(\lambda_a \frac{\partial}{\partial \lambda^b} + \lambda_b \frac{\partial}{\partial \lambda^a} \right)$$

$$\widetilde{J}_{\dot{a}\dot{b}} = \frac{i}{2} \left(\mu_{\dot{a}} \frac{\partial}{\partial \mu^{\dot{b}}} + \mu_{\dot{b}} \frac{\partial}{\partial \mu^{\dot{a}}} \right).$$
(2.32)

Finally, the dilatation generator becomes a *homogeneous* first order operator, as the +2 in (2.24) disappears:

$$D = \frac{i}{2} \left(\lambda^a \frac{\partial}{\partial \lambda^a} - \mu^{\dot{a}} \frac{\partial}{\partial \mu^{\dot{a}}} \right). \tag{2.33}$$

This representation of the four-dimensional conformal group is a much more obvious one than the representation that we described above in terms of λ and $\widetilde{\lambda}$. Consider the four-dimensional space \mathbb{T} (called twistor space by Penrose) spanned by λ^a and $\mu^{\dot{a}}$. It is a copy of \mathbb{R}^4 if we are in signature ++- and can consider λ and μ to be real. Otherwise, we must think of \mathbb{T} as a copy of \mathbb{C}^4 . At any rate, the traceless four by four matrices acting on λ and μ generate a group which is $SL(4,\mathbb{R})$ in the real case, or $SL(4,\mathbb{C})$ in the complex case. In fact, the conformal group in four dimensions is a real form of SL(4) (namely $SL(4,\mathbb{R})$, SU(2,2), or SU(4) for signature ++--, +---, or ++++), and the conformal generators found in the last paragraph simply generate the natural action of SL(4) on \mathbb{T} .

The "identity" matrix on \mathbb{T} also has a natural meaning in this framework. In (2.12), we found that the scattering amplitude for a particle of helicity h obeys a condition that in terms of λ and μ becomes

$$\left(\lambda^{a} \frac{\partial}{\partial \lambda^{a}} + \mu^{\dot{a}} \frac{\partial}{\partial \mu^{\dot{a}}}\right) \widetilde{A}(\lambda_{i}, \widetilde{\lambda}_{i}, h_{i}) = (-2h - 2)\widetilde{A}(\lambda_{i}, \widetilde{\lambda}_{i}, h_{i}). \tag{2.34}$$

(We write \widetilde{A} for the scattering amplitude in twistor space; its proper definition will be discussed momentarily. For now, we just formally make the transformation from $\widetilde{\lambda}$ to $\partial/\partial\mu$ in (2.12).) This equation means that the scattering amplitude of a massless particle of helicity h is best interpreted not as a function on \mathbb{T} but as a section of a suitable line bundle \mathcal{L}_h over the projective space \mathbb{PT} whose homogeneous coordinates are λ and μ . \mathbb{PT} is a copy of \mathbb{RP}^3 or \mathbb{CP}^3 depending on whether we consider λ and μ to be real or complex. \mathbb{PT} is called projective twistor space, but when confusion with \mathbb{T} seems unlikely, we will just call it twistor space.

In the complex case, $\mathcal{L}_h = \mathcal{O}(-2h-2)$, which is defined as the line bundle whose sections are functions homogeneous of degree -2h-2 in the homogeneous coordinates. In the real case, we can give a description in real differential geometry. If we let Z^I , $I=1,\ldots,4$, be the coordinates of \mathbb{T} (thus combining together λ^a and $\mu^{\dot{a}}$), then on \mathbb{PT} there is a natural volume form $\Omega = \epsilon_{IJKL}Z^IdZ^JdZ^KdZ^L$ of degree four. Multiplying by Ω converts a function homogeneous of degree -4 into a three-form or measure. So functions homogeneous of degree -2h-2 can be described as $\frac{1}{2}(1+h)$ -densities. The scattering amplitudes, in the case of signature (2,2), can thus be interpreted as fractional densities on \mathbb{PT} of these weights.

In signature ++--, where $\widetilde{\lambda}$ is real, the transformation from scattering amplitudes regarded as functions of $\widetilde{\lambda}$ to scattering amplitudes regarded as functions of μ is made by a simple Fourier transform that is familiar in quantum mechanics. Any function $f(\widetilde{\lambda})$ is transformed to

$$\widetilde{f}(\mu) = \int \frac{d^2 \widetilde{\lambda}}{(2\pi)^2} \exp(i\mu^{\dot{a}}\widetilde{\lambda}_{\dot{a}}) f(\widetilde{\lambda}). \tag{2.35}$$

Starting with the momentum space scattering amplitude $\widehat{A}(\lambda_i, \widetilde{\lambda}_i)$ and making this Fourier transform for each particle, we get the twistor space scattering amplitude $\widetilde{A}(\lambda_i, \mu_i)$.

Analog With Euclidean Signature

For spacetime signatures other than ++--, it is unnatural to interpret λ and $\widetilde{\lambda}$ as real variables. If the variables are complex variables, we can optimistically attempt to

proceed in the same way, interpreting the integral in (2.35) as a contour integral. This will make sense when a suitable contour exists. When the twistor variables are complex, an alternative and more systematic approach to defining the transformation from $\tilde{\lambda}$ to $\tilde{\mu}$ can be used, modeled on Penrose's description [37] of the particle wavefunctions in complex twistor space. This alternative approach uses the weightier mathematical machinery of cohomology (or sheaf cohomology, the alternative most exploited by Penrose). The description of the amplitudes using $\bar{\partial}$ cohomology will not be needed in section 3, but is useful background for the string theory discussion in section 4. I will here describe this approach in a rather naive way, taking the starting point to be Euclidean signature in spacetime. Though we start with Euclidean signature, since we get a description with complex variables λ , $\tilde{\lambda}$ or λ , μ , the result makes sense for computing scattering amplitudes with arbitrary complex momenta. (Hopefully, a reader of this article will be able to give a less naive description, and perhaps a formulation that is more directly related to Lorentz signature.)

In ++++ signature, the spinor representations of the Lorentz group are pseudo-real, so in particular $\widetilde{\lambda}$ and its complex conjugate $\overline{\widetilde{\lambda}}$ transform in the same way. We simply interpret μ as $\overline{\widetilde{\lambda}}$, or equivalently $\widetilde{\lambda}$ as $\overline{\mu}$. Then we write down the same formula as in (2.35) except that we omit to do the integral. For any function $f(\widetilde{\lambda})$, we define

$$\widehat{f} = \frac{d^2 \overline{\mu}}{(2\pi)^2} \exp(i\mu \overline{\mu}) f(\overline{\mu}). \tag{2.36}$$

We interpret this as a (0, 2)-form, which is obviously $\overline{\partial}$ -closed as μ has only two components. We interpret the $\overline{\partial}$ cohomology class represented by this form as the twistor version of the helicity scattering amplitude. Granted this, the scattering amplitude for a massless particle of helicity h can be interpreted in twistor space as an element of the sheaf cohomology group $H^2(\mathbb{PT}', \mathcal{O}(-2h-2))$, where the homogeneity was determined above.

The reason that we have written here \mathbb{PT}' , and not \mathbb{PT} , is that, as with many twistor constructions, one really should not work with all of \mathbb{PT} but with a suitable open set thereof. In fact, $H^2(\mathbb{PT}, \mathcal{O}(-2h-2)) = 0$. For scattering of plane waves, one can take \mathbb{PT}' to be the subspace of \mathbb{PT} in which the λ_a are not both zero. Understanding exactly what

⁶ This statement reflects the fact that plane waves are regular throughout \mathbb{R}^4 , but have a singularity upon conformal compactification to \mathbb{S}^4 . It is standard in twistor theory that omitting the point at infinity in Euclidean spacetime corresponds to omitting the subspace of twistor space in which the λ 's vanish. We will give an illustration of the idea behind this statement at the end of the present subsection, and more information in the appendix.

open set \mathbb{PT}' and what kind of $\overline{\partial}$ cohomology to use in a given physical problem is a large part of more fully understanding the twistor transform with complex variables.

Pairing With External Wavefunctions

Physical particles are not normally in momentum eigenstates. Initial and final states of interest might be arbitrary solutions of the free wave equation. If a momentum space scattering amplitude $\widehat{A}(p_1,\ldots,p_n)$ is known (for simplicity we consider scalar particles, so that the amplitude depends only on the momenta), the amplitude for scattering with initial and final states $\phi_1(x),\ldots,\phi_n(x)$ (each of which obeys the appropriate free wave equation) is obtained by taking a suitable convolution. If $\phi_i(x)$ is expressed in terms of momentum space wavefunctions $a_i(p)$ by $\phi_i(x) = \int d^4p \, e^{ip\cdot x} \delta(p^2) a_i(p)$ (where the factor $\delta(p^2)$ ensures that the momentum space wavefunction is supported at $p^2 = 0$), then the scattering amplitude with external states ϕ_i is

$$A(\phi_1, \dots, \phi_n) = \prod_{i=1}^n \int d^4 p_i \delta(p_i^2) a_i(p_i) \widehat{A}(p_1, \dots, p_n).$$
 (2.37)

In other words, the amplitude with specified external states is obtained from the momentum space scattering amplitude by multiplying by the momentum space wavefunctions and integrating over momentum space.

Similarly, to go from the twistor space amplitude to an amplitude with specified external states, we must multiply by twistor space wavefunctions and integrate over twistor space. To carry this out, we need the twistor description of the initial and final state wavefunctions. This description was originally developed by Penrose [37] for complex twistor space \mathbb{CP}^3 . This is the most useful case for most purposes and is reviewed in the appendix. There is also an analog (explained by Atiyah in [36], section VI.5) for real twistor space \mathbb{RP}^3 . We consider first the real case. (The following discussion is useful background for the computation of tree level MHV amplitudes in section 4.7, but otherwise is not needed for the rest of the paper.)

For signature ++--, a massless field of helicity h corresponds to a $\frac{1}{2}(1-h)$ -density over \mathbf{RP}^3 . As we have seen above, the scattering amplitude for a massless particle of helicity h is a $\frac{1}{2}(1+h)$ density. When these are multiplied, we get a density on \mathbf{RP}^3 , which can be integrated over \mathbf{RP}^3 to get a number. We interpret this number as the scattering amplitude for the given initial and final states. (This process of integration over \mathbf{RP}^3 is analogous to the momentum space integral in (2.37) and has to be carried out separately for each initial and final particle.)

Instead, for the complex case of the twistor transform, the wavefunction of an external particle is, according to the usual complex case of the Penrose transform, an element of the $\overline{\partial}$ cohomology group $H^1(\mathbb{PT}', \mathcal{O}(-2-h))$. Upon taking the cup product of such an element with the scattering amplitude, which as we have argued above is an element of $H^2(\mathbb{PT}', \mathcal{O}(h-2))$, we get an element of $H^3(\mathbb{PT}', \mathcal{O}(-4))$. As $\mathcal{O}(-4)$ is the canonical bundle of \mathbb{CP}^3 , an element of $H^3(\mathbb{PT}', \mathcal{O}(-4))$ can be interpreted as a (3,3)-form on \mathbb{PT}' ; upon integrating this form over \mathbb{PT}' (and repeating this process for each external particle) one gets the desired scattering amplitude for the given initial and final states. This procedure is implemented for MHV tree amplitudes in eqn. (4.54).

Physical Interpretation Of Twistor Space Amplitudes

Finally, let us discuss the physical meaning of the twistor space scattering amplitude $\widetilde{A}(\lambda_i, \mu_i)$. We carry out this discussion in signature ++--, where the definition of \widetilde{A} is more elementary. The scattering amplitude as a function of $p_{a\dot{a}} = \lambda_a \widetilde{\lambda}_{\dot{a}}$ has a clear enough meaning: it is the amplitude for scattering a particle whose wavefunction is $\exp(ip \cdot x)$. What is the wavefunction of a particle whose scattering is described by a function of λ and μ ?

The twistor space scattering amplitude $\widetilde{A}(\lambda,\mu)$ has been obtained from the momentum space version $\widehat{A}(\lambda,\widetilde{\lambda})$ by multiplying by $\exp(i\widetilde{\lambda}_{\dot{a}}\mu^{\dot{a}})$ and integrating over $\widetilde{\lambda}$. This describes a scattering process in which the initial and final states are not plane waves but have the wave function

$$\int \frac{d^2 \widetilde{\lambda}}{(2\pi)^2} \exp(ix^{a\dot{a}}\lambda_a \widetilde{\lambda}_{\dot{a}}) \exp(i\widetilde{\lambda}_{\dot{a}}\mu^a). \tag{2.38}$$

The result of the integral is simply

$$\delta^2(\mu_{\dot{a}} + x_{a\dot{a}}\lambda^a). \tag{2.39}$$

In other words, the scattering amplitude as a function of λ and μ describes the scattering of a particle whose wave function is a delta function supported on the subspace of Minkowski space given by the equation $\mu_{\dot{a}} + x_{a\dot{a}}\lambda^a = 0$. This is the two-dimensional subspace of Minkowski space that is associated by Penrose [7] with the point in \mathbb{PT} whose homogeneous coordinates are λ, μ .

From the equation $\mu_{\dot{a}} + x_{a\dot{a}}\lambda^a$, we see that for $x \to \infty$, λ must vanish. (As λ , μ are homogeneous coordinates, $\lambda \to 0$ is equivalent to $\mu \to \infty$.) This illustrates the idea that not allowing $x \to \infty$ is related to omitting the set in twistor space with $\lambda = 0$.

2.6. Extension With $\mathcal{N}=4$ Supersymmetry

Here, along lines suggested by Nair [12], we will describe the generalization of the MHV amplitudes to $\mathcal{N}=4$ super Yang-Mills theory. Then we will transform to super twistor space. (This discussion is useful background for section 4, but it is not really needed for section 3.)

We describe each external particle by the familiar (commuting) spinors λ_a , $\tilde{\lambda}_{\dot{a}}$, and also by a spinless, anticommuting variable η_A , $A=1,\ldots,4$. η_A will have dimension zero and transforms in the $\overline{4}$ representation of the $SU(4)_R$ symmetry of $\mathcal{N}=4$ Yang-Mills. We take the helicity operator to be

$$h = 1 - \frac{1}{2} \sum_{A} \eta_{A} \frac{\partial}{\partial \eta_{A}}.$$
 (2.40)

Thus, a term in the scattering amplitude that is of k^{th} order in η_{iA} for some i describes a scattering process in which the i^{th} particle has helicity 1-k/2. Notice that in adopting this formalism, we are breaking the symmetry between positive and negative helicity, choosing the term in the scattering amplitude with no η 's to have helicity 1 instead of -1. This choice is well adapted to describing MHV amplitudes in which most external particles have helicity 1; one would make the opposite choice to give a convenient description of the other MHV amplitudes.

The form (2.40) of the helicity operator means that the relation (2.12) between the helicity and the homogeneity of the scattering amplitude \widehat{A} becomes

$$\left(\lambda_i^a \frac{\partial}{\partial \lambda_i^a} - \widetilde{\lambda}_i^{\dot{a}} \frac{\partial}{\partial \widetilde{\lambda}_i^{\dot{a}}} - \eta_{iA} \frac{\partial}{\partial \eta_{iA}} + 2\right) \widehat{A} = 0.$$
 (2.41)

If we let $P_{a\dot{a}} = \sum_{i} \lambda_{ia} \tilde{\lambda}_{i\dot{a}}$, and $\Theta_{bA} = \sum_{i} \lambda_{ib} \eta_{iA}$, then the MHV scattering amplitude (with the gauge theory factor Tr $T_1 T_2 \dots T_n$ suppressed as usual) is

$$\widehat{A} = ig^{n-2}(2\pi)^4 \delta^4(P) \delta^8(\Theta) \prod_{i=1}^n \frac{1}{\langle \lambda_i, \lambda_{i+1} \rangle}.$$
 (2.42)

We recall that for a fermion ψ , one defines $\delta(\psi) = \psi$.

Now let us describe the action of the superconformal group PSU(2,2|4) in this formalism. The conformal group SU(2,2) commutes with η and acts on λ and $\widetilde{\lambda}$ exactly as we have described above. The SU(4) R-symmetry group is generated by

$$\eta_A \frac{\partial}{\partial \eta_B} - \frac{1}{4} \delta_A^B \eta_C \frac{\partial}{\partial \eta_C}.$$
 (2.43)

Note that we consider only "traceless" generators here, as the "trace" generator $\eta_C(\partial/\partial\eta_C)$ is not contained in PSU(2,2|4) and is not a symmetry of $\mathcal{N}=4$ super Yang-Mills theory. Nonetheless, it will play an important role later.

Finally, PSU(2,2|4) has 32 fermionic generators, which act as follows. Half of them are first order differential operators

$$\tilde{\lambda}^{\dot{a}} \frac{\partial}{\partial \eta_A}, \quad \eta_A \frac{\partial}{\partial \tilde{\lambda}^{\dot{a}}}.$$
 (2.44)

One quarter are multiplication operators

$$\lambda^a \eta_A,$$
 (2.45)

and the remainder are second order differential operators

$$\frac{\partial^2}{\partial \lambda^a \partial \eta_A}.\tag{2.46}$$

It is not hard to verify that these generate SU(2,2|4) (but this result is more transparent after the transformation to super twistor space that we make presently).

Next, let us verify the superconformal invariance of the MHV amplitudes. Apart from dilatations, the bosonic and fermion generators that act by first order differential operators are manifest symmetries. The bosonic and fermionic generators that act as multiplication operators are conserved because of the delta functions in the scattering amplitude. Dilatation invariance can be verified exactly as in the previous discussion of the pure Yang-Mills case. Special conformal symmetry follows via closure of the algebra if the scattering amplitude is annihilated by the fermionic second order differential operators (2.46). So we need only check those symmetries, which can be established by a procedure similar to the one that was used to show special conformal invariance of the Yang-Mills amplitudes. For brevity, we set $B = ig^{n-2}(2\pi)^4$, $\delta_1 = \delta^4(P)$, $\delta_2 = \delta^8(\Theta)$, and $S = 1/\prod_{i=1}^n \langle \lambda_i, \lambda_{i+1} \rangle$. So the scattering amplitude is $\widehat{A} = B\delta_1\delta_2S$. Now we compute that

$$B^{-1} \sum_{i} \frac{\partial^{2} \widehat{A}}{\partial \lambda_{i}^{a} \partial \eta_{iA}} = \delta_{1} \frac{\partial \delta_{2}}{\partial \Theta_{A}^{b}} \sum_{i} \lambda_{i}^{b} \frac{\partial S}{\partial \lambda_{i}^{a}} + \delta_{1} \cdot n \frac{\partial \delta_{2}}{\partial \Theta_{A}^{a}} S - \delta_{1} \Theta_{C}^{b} \frac{\partial^{2} \delta_{2}}{\partial \Theta_{A}^{b} \partial \Theta_{C}^{a}} S + P^{ab} \frac{\partial \delta_{1}}{\partial P^{bb}} \frac{\partial \delta_{2}}{\partial \eta_{A}^{a}} S.$$

$$(2.47)$$

This vanishes, upon using the following facts that are analogous to the facts that we used in examining the special conformal symmetries: (i) $\sum_i \lambda_i^a \frac{\partial S}{\partial \lambda_i^b} = -n\delta_b^a S$; (ii) $P^{a\dot{b}}(\partial \delta_1/\partial P^{b\dot{b}}) = -2\delta_b^a \delta_1$; and finally (iii) $\Theta_C^b(\partial^2 \delta_2/\partial \Theta_A^b \partial \Theta_C^a) = -2(\partial \delta_2/\partial \Theta_A^a)$.

Transform To Super-Twistor Space

The next step should not be hard to guess. We make the action of PSU(2,2|4) more transparent by a supersymmetric extension of the same transformation as in the bosonic case:

$$\widetilde{\lambda}_{\dot{a}} \to i \frac{\partial}{\partial \mu^{\dot{a}}}
-i \frac{\partial}{\partial \widetilde{\lambda}^{\dot{a}}} \to \mu_{\dot{a}}
\eta_{A} \to i \frac{\partial}{\partial \psi^{A}}
-i \frac{\partial}{\partial \eta_{A}} \to \psi^{A}.$$
(2.48)

Now all PSU(2,2|4) generators become first order differential operators.

Moreover, the representation of PSU(2,2|4) that we get this way is easy to describe. We introduce a space $\widehat{\mathbb{T}}=\mathbb{C}^{4|4}$ with four bosonic coordinates $Z^I=(\lambda^a,\mu^{\dot{a}})$ and four fermionic coordinates ψ^A . $\widehat{\mathbb{T}}$ is the supersymmetric extension of (nonprojective) twistor space (with $\mathcal{N}=4$ supersymmetry). The full supergroup of linear transformations of $\widehat{\mathbb{T}}$ is called GL(4|4). If we take the quotient of this group by its center – which consists of the nonzero multiples of the identity – we get the quotient group PGL(4|4). This is the symmetry group of the projectivized super-twistor space $\widehat{\mathbb{PT}}$ that has Z^I and ψ^A as homogeneous coordinates. In other words, $\widehat{\mathbb{PT}}$ is parameterized by Z^I and ψ^A subject to the equivalence relation $(Z^I,\psi^A)\sim (tZ^I,t\psi^A)$ for nonzero complex t. If the Z^I are taken to be complex, then $\widehat{\mathbb{PT}}$ is a copy of the supermanifold $\mathbb{CP}^{3|4}$. It also makes sense, and is natural in signature ++--, to take the Z^I to be real. We cannot take the ψ^A to be real, as this would clash with the SU(4) R-symmetry. But we can do the next best thing: we simply do not mention $\overline{\psi}$ and consider only functions that depend only on Z^I and ψ^A . (As functions of ψ^A are automatically polynomials, there are no choices to make of what kind of functions to allow.) This version of $\widehat{\mathbb{PT}}$ might be called $\mathbb{RP}^{3|4}$.

When confusion seems unlikely because the context clearly refers to the supersymmetric case, we will sometimes omit the hats from $\widehat{\mathbb{T}}$ and $\widehat{\mathbb{PT}}$.

If we further introduce the object

$$\Omega_0 = dZ^1 dZ^2 dZ^3 dZ^4 d\psi^1 d\psi^2 d\psi^3 d\psi^4, \tag{2.49}$$

then the subgroup of PGL(4|4) that preserves Ω_0 is called PSL(4|4). Finally, the desired superconformal group is a real form of PSL(4|4). In Lorentz signature, it is PSU(2,2|4).

The object Ω_0 , in contrast to our experience in the bosonic case, is best understood as a measure (in the holomorphic sense) on $\widehat{\mathbb{T}}$, or a section of the Berezinian of the tangent bundle of $\widehat{\mathbb{T}}$, rather than as a differential form.

After the transform to twistor variables, the homogeneity condition for the scattering amplitudes becomes simply

$$\left(Z_i^I \frac{\partial}{\partial Z_i^I} + \psi_i^A \frac{\partial}{\partial \psi_i^A}\right) \widetilde{A} = 0.$$
(2.50)

In other words, the scattering amplitude is homogeneous in the twistor coordinates of each external particle.

In the case of signature ++--, rather as in the bosonic case, the transformation to twistor variables is made by an ordinary Fourier transform from $\widetilde{\lambda}$ to μ together with a Fourier transform from η to ψ . In this signature $\widetilde{\lambda}$ is real (or at least real modulo nilpotents), so the Fourier transform from $\widetilde{\lambda}$ to μ makes sense; fermions are infinitesimal so the Fourier transform from η to ψ is really an algebraic operation that has no analytic difficulties. The twistor transformed scattering amplitude is thus in the signature ++-- case a function on the twistor superspace $\mathbb{RP}^{3|4}$. In contrast to \mathbb{RP}^3 , $\mathbb{RP}^{3|4}$ has a natural measure, associated with the object

$$\Omega = \frac{1}{A!^2} \epsilon_{IJKL} Z^I dZ^J dZ^K dZ^L \epsilon_{ABCD} d\psi^A d\psi^B d\psi^C d\psi^D. \tag{2.51}$$

In the real version of the twistor transform, the external particle wavefunctions are just functions on $\mathbb{RP}^{3|4}$; to compute a scattering amplitude with specified initial and final states, one multiplies the external wavefunctions by the twistor space scattering amplitude and integrates it over twistor space, using the measure (2.51).

For other signatures, one must take twistor space to be the complex manifold $\widehat{\mathbb{PT}} = \mathbb{CP}^{3|4}$. The Fourier transform in the fermions does not introduce any special subtleties, but we must treat the complex bosons just as we did in the absence of supersymmetry. So the transform to twistor variables produces in the complex case a scattering amplitude that for each external particle is an element of the sheaf cohomology group $H^2(\widehat{\mathbb{PT}}', \mathcal{O})$. As before, $\widehat{\mathbb{PT}}'$ is $\widehat{\mathbb{PT}}$ with the set $\lambda^a = 0$ omitted. Also, \mathcal{O} is the trivial line bundle, which is the right one since the twistor wave function according to (2.50) is homogeneous of degree zero in Z^I and ψ^A . The wavefunctions for external particles with specified quantum states are elements of the sheaf cohomology group $H^1(\widehat{\mathbb{PT}}', \mathcal{O})$. To compute a scattering amplitude with specified initial and final states, one takes the cup product of the

external wave functions with the scattering amplitudes, to get for each particle an element of $H^3(\widehat{\mathbb{PT}}',\mathcal{O})$. This can then be integrated, using the section Ω of the Berezinian of the tangent bundle, to get a number, the scattering amplitude.

Even when λ and μ are complex and one is doing $\overline{\partial}$ cohomology, there is no need to introduce a complex conjugate of ψ^A . Though it may be inevitable to consider not necessarily holomorphic functions of λ and μ , there is no need to consider non-holomorphic functions of ψ^A . (In section 4, we will see that the complex conjugate of ψ^A is unavoidable for closed strings, but can be avoided for open strings.)

3. Scattering Amplitudes In Twistor Space

After the Fourier transform to twistor space, each external particle in an n-particle scattering process is labeled by a point P_i in twistor space. The homogeneous coordinates of P_i are $Z_i^I = (\lambda_i^a, \mu_i^{\dot{a}})$. The scattering amplitudes are functions of the P_i , that is, they are functions defined on the product of n copies of twistor space.

In this section, we make an empirical study of scattering amplitudes in twistor space. We work in signature ++--, and therefore in real twistor space \mathbb{RP}^3 . We consider gluon scattering and will (for the most part) make no attempt to make supersymmetry manifest, so we can use \mathbb{RP}^3 rather than its supersymmetric extension $\mathbb{RP}^{3|4}$. The advantage of signature ++- is that the transform to twistor space is an ordinary Fourier transform, and the scattering amplitudes in real twistor space are ordinary functions on $(\mathbb{RP}^3)^n$, one copy of twistor space for each external particle. With other signatures, we would have to use $\overline{\partial}$ cohomology and a weightier mathematical machinery.

The goal is to show that the twistor version of the n particle scattering amplitude is nonzero only if the points P_i are all supported on an algebraic curve in twistor space. This algebraic curve has degree d given by

$$d = q - 1 + l, (3.1)$$

where q is the number of negative helicity gluons in the scattering process, and l is the number of loops. It is not necessarily connected. And its genus g is bounded by the number of loops,

$$g \le l. \tag{3.2}$$

Our goal in the remainder of this section is to explore the hypothesis that twistor amplitudes are supported on the curves described in the last paragraph. We will consider in this light various tree level Yang-Mills scattering amplitudes (which are not sensitive to supersymmetry) and one example of a one-loop scattering amplitude in $\mathcal{N}=4$ super Yang-Mills theory. In each case, we verify our conjecture. (Along the way, a few unexplained properties also appear, showing that there is much more to understand.) These examples, though certainly far short of proving or even determining the general structure, do give a good motivation for seeking a string theory whose instanton expansion might reproduce the perturbation expansion of super Yang-Mills theory and explain the conjecture. We make a proposal for such a string theory in the next section.

We conclude this section with a peek at General Relativity, showing that the tree level MHV amplitudes are again supported on curves. Unfortunately, I do not know of any string theory whose instanton expansion might reproduce the perturbation expansion of General Relativity or supergravity.

A Note On Algebraic Curves

An algebraic curve Σ in \mathbb{RP}^3 is a curve defined as the zero set of a collection of polynomial equations with real coefficients in the homogeneous coordinates Z^I of \mathbb{RP}^3 . The degree and genus of such a curve are defined by complexifying the Z^I , whereupon Σ becomes a Riemann surface in \mathbb{CP}^3 , whose degree and genus are defined in the usual way.

The simplest type of algebraic curve is a "complete intersection," obtained by setting to zero two homogeneous polynomials $F(Z^I)$ and $G(Z^I)$, of degrees (say) d_1 and d_2 .⁷ The degree of such a complete intersection is $d = d_1 d_2$. Curves of degree one and two are of this type, with $(d_1, d_2) = (1, 1)$ and (1, 2), respectively.

3.1. MHV Amplitudes

We begin with the n gluon tree level MHV amplitude with two gluons of negative helicity and n-2 of positive helicity. As we reviewed in section 2.3, it can be written

$$\widehat{A}(\lambda_i, \widetilde{\lambda}_i) = ig^{n-2} (2\pi)^4 \delta^4 \left(\sum_i \lambda_i^a \widetilde{\lambda}_i^a \right) f(\lambda_i), \tag{3.3}$$

 $^{^{-7}}$ In the more general case, one must define a given curve C by the vanishing of more than two polynomials, all of which vanish on C but any two of which vanish on additional branches other than C. Curves of genus zero and degree three give an example; we briefly comment on their role at the end of section 3.4.

where $f(\lambda_i)$ is a function of only the λ 's and not the $\widetilde{\lambda}$'s. The details of $f(\lambda)$ need not concern us here. Using a standard representation of the delta function (used in discussing MHV amplitudes by Nair [12]), we can rewrite the amplitude as

$$\widehat{A}(\lambda_i, \widetilde{\lambda}_i) = ig^{n-2} \int d^4x \, \exp\left(ix_{a\dot{a}} \sum_{i=1}^n \lambda_i^a \widetilde{\lambda}_i^{\dot{a}}\right) f(\lambda_i). \tag{3.4}$$

To transform to twistor space, we simply carry out a Fourier transform with respect to all of the $\widetilde{\lambda}$ variables. The twistor space amplitude is hence

$$\widetilde{A}(\lambda_i, \mu_i) = ig^{n-2} \int d^4x \int \frac{d^2\widetilde{\lambda}_1}{(2\pi)^2} \dots \frac{d^2\widetilde{\lambda}_n}{(2\pi)^2} \exp\left(i\sum_{i=1}^n \mu_{i\dot{a}}\widetilde{\lambda}_i^{\dot{a}}\right) \exp\left(ix_{a\dot{a}}\sum_{i=1}^n \lambda_i^a \widetilde{\lambda}_i^{\dot{a}}\right) f(\lambda_i).$$
(3.5)

The $\widetilde{\lambda}$ integrals can be done trivially, with the result

$$\widetilde{A}(\lambda_i, \mu_i) = ig^{n-2} \int d^4x \prod_{i=1}^n \delta^2(\mu_{i\dot{a}} + x_{a\dot{a}}\lambda_i^a) f(\lambda_i). \tag{3.6}$$

Let us now interpret this result. For every (real) $x_{a\dot{a}}$, the pair of equations

$$\mu_{\dot{a}} + x_{a\dot{a}}\lambda^a = 0, \quad \dot{a} = 1, 2$$
 (3.7)

defines a real algebraic curve C in \mathbb{RP}^3 , or if we complexify the variables, in \mathbb{CP}^3 . This curve is a complete intersection, since it is defined by vanishing of a pair of homogeneous polynomials; it is of degree 1 since the polynomials are linear (thus $d_1 = d_2 = 1$ in the notation used at the end of section 3.1). Moreover, C has genus zero. Indeed, the equations (3.7) can be solved for $\mu_{\dot{a}}$ as a function of λ^a , so the λ^a serve as homogeneous coordinates for C, which therefore is a copy of \mathbb{RP}^1 or \mathbb{CP}^1 depending on whether the variables are real or complex. Conversely, if one allows limiting cases with $x \to \infty$, the degree one, genus zero curves are all of this type.

The integral $\int d^4x$ in (3.6) is thus an integral over the moduli space of real, degree one, genus zero curves in \mathbb{RP}^3 . The delta function means that the amplitude vanishes unless all n points $P_i = (\lambda_i^a, \mu_{i\dot{a}})$ are contained on one of these curves. The MHV amplitudes with mostly plus helicities are thus supported, in twistor space, on configurations of n points that all lie on a curve in \mathbb{RP}^3 of degree one and genus zero. This is a basic example of our proposal.

As long as we are working in the real version of twistor space, which is appropriate to ++-- signature, this curve can be described more intuitively as a straight line. Indeed,

throwing away, for example, the set $\lambda_1 = 0$ in \mathbb{RP}^3 , we can describe the rest of \mathbb{RP}^3 by the affine coordinates $x = \lambda_2/\lambda_1$, $y = \mu_1/\lambda_1$, $z = \mu_2/\lambda_2$. x, y, and z parameterize a copy of \mathbb{R}^3 and the curve C is simply a straight line in \mathbb{R}^3 . Thus (figure 2), the MHV amplitude with these helicities is supported for points P_i that are collinear in \mathbb{R}^3 . The conformal symmetry group (in ++-- signature) is the $SL(4,\mathbb{R})$ symmetry of \mathbb{RP}^3 ; it does not preserve a metric on \mathbb{R}^3 , but it maps straight lines to straight lines.

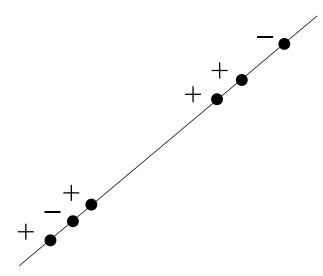


Fig. 2: The MHV amplitude for gluon scattering is associated with a collinear arrangement of points in \mathbb{R}^3 .

Supersymmetric Extension

Here, by Fourier-tranforming the supersymmetric MHV amplitude (2.42), we will obtain the supersymmetric extension of the above result. This is the only example where we will study the twistor transform of a manifestly supersymmetric amplitude.

We write the fermionic delta function in (2.42) as

$$\delta^{8}(\Theta) = \int d^{8}\theta_{a}^{A} \exp\left(i\theta_{a}^{A} \sum_{i} \eta_{iA} \lambda_{i}^{a}\right). \tag{3.8}$$

Using this and the familiar representation of the bosonic delta function in (2.42), the supersymmetric MHV amplitude becomes

$$\widehat{A} = ig^{n-2} \int d^4x \, d^8\theta \, \exp\left(ix_{a\dot{a}} \sum_i \lambda_i^a \widetilde{\lambda}_i^{\dot{a}}\right) \exp\left(i\theta_a^A \sum_i \eta_{iA} \lambda_i^a\right) \prod_{i=1}^n \frac{1}{\langle \lambda_i, \lambda_{i+1} \rangle}. \tag{3.9}$$

The Fourier transform is therefore straightforward. The amplitude in super twistor space is

$$\widetilde{A}(\lambda_i^a, \mu_i^{\dot{a}}, \psi_i^A) = \int \frac{d^2 \widetilde{\lambda}_1 d^4 \eta_1}{(2\pi)^2} \dots \frac{d^2 \widetilde{\lambda}_n d^4 \eta_n}{(2\pi)^2} \exp\left(i \sum_i \mu_i^a \widetilde{\lambda}_{i\,a} + i \sum_i \psi_i^A \eta_{i\,A}\right) \widehat{A}$$

$$= ig^{n-2} \int d^4 x \, d^8 \theta_a^A \prod_{i=1}^n \delta^2 \left(\mu_{i\,\dot{a}} + x_{a\dot{a}} \lambda_i^a\right) \delta^4 \left(\psi_i^A + \theta_a^A \lambda_i^a\right) \prod_{i=1}^n \frac{1}{\langle \lambda_i, \lambda_{i+1} \rangle}.$$
(3.10)

The interpretation is very much as in the bosonic case. For any given x and θ , the equations

$$\mu_{\dot{a}} + x_{a\dot{a}}\lambda^a = 0$$

$$\psi^A + \theta_a^A \lambda^a = 0$$
(3.11)

determine a curve in the supersymmetric extension of twistor space. The equations can be used to solve for μ and ψ in terms of λ , so this curve has the λ^a as homogeneous coordinates and is a copy of \mathbb{CP}^1 . It has degree one; its moduli are x and θ . In the real version of super twistor space, in affine coordinates, this curve is a straight line. The delta functions in (3.10) mean that the supersymmetric MHV amplitudes, when transformed to twistor space, vanish unless all external particles are inserted at collinear points in supertwistor space.

3.2. Degree Minus One And Degree Zero

We considered first the MHV amplitudes with two positive helicity gluons, with the aim of giving the reader a first orientation about our basic conjecture that twistor space scattering amplitudes are supported on the curves described in (3.1). However, there are a few cases to consider that in a sense are more primitive.

The conjecture actually gives a new perspective on the vanishing of the tree level n gluon scattering amplitudes with all or all but one gluons of positive helicity. A tree amplitude with all gluons of positive helicity has q=l=0 in (3.1), leading to d=-1. There are no algebraic curves of degree -1, so such amplitudes vanish. If all gluons but one have positive helicity, we get $q=1,\ l=0$, whence d=0. A curve of degree zero is collapsed to a point, so amplitudes of this type are supported in twistor space by configurations in which all gluons are attached at the same point $P=(\lambda,\mu)$. In particular, $\lambda_i=\lambda_j$ for all particles i and j. For the corresponding momenta p_i and p_j , we have $p_i\cdot p_j=\langle \lambda_i,\lambda_j\rangle[\widetilde{\lambda}_i,\widetilde{\lambda}_j]=0$. This is impossible for a non-trivial scattering amplitude

with $n \geq 4$ particles, since such amplitudes depend on non-trivial kinematic invariants $p_i \cdot p_j$ (such as the Mandelstam variables for n = 4). Hence n gluon tree amplitudes with all but one gluon of positive helicity must vanish for $n \geq 4$.

The n=3 case is exceptional, because here it is true that $p_i \cdot p_j = 0$ for all i and j. Indeed, if $p_1 + p_2 + p_3 = 0$ and $p_i^2 = 0$ for i=1,2,3, then for example $p_1 \cdot p_2 = (p_1 + p_2)^2/2 = p_3^2/2 = 0$. For real momenta in Lorentz signature, the condition $p_i \cdot p_j = 0$ implies that the p_i are collinear, whence the amplitude (written below) vanishes and the phase space also vanishes. Hence this rather degenerate case is often omitted in discussing the Yang-Mills scattering amplitudes. However, the tree level three point function makes sense with other signatures or with complex momenta, so we will consider it here (albeit with some difficulty as will soon appear).

Since $p_i \cdot p_j = \langle \lambda_i, \lambda_j \rangle [\widetilde{\lambda}_i, \widetilde{\lambda}_j]$, it follows that for each i, j, either $\langle \lambda_i, \lambda_j \rangle = 0$ or $[\widetilde{\lambda}_i, \widetilde{\lambda}_j] = 0$. The first condition implies that λ_i and λ_j are proportional, and the second condition implies that $\widetilde{\lambda}_i$ and $\widetilde{\lambda}_j$ are proportional. Since at least one of these conditions is satisfied for each pair i, j, it follows that either all λ_i , i = 1, 2, 3 are proportional, or that all $\widetilde{\lambda}_i$ are proportional.

The momentum space amplitude for -++ is

$$\widehat{A}(\lambda_i, \widetilde{\lambda}_i) = ig\delta^4 \left(\sum_{i=1}^3 \lambda_i^a \widetilde{\lambda}_i^a \right) \frac{[\widetilde{\lambda}_1, \widetilde{\lambda}_2]^4}{\prod_{i=1}^3 [\widetilde{\lambda}_i, \widetilde{\lambda}_{i+1}]}.$$
 (3.12)

This can be read off from the Yang-Mills Lagrangian, or it can be regarded as a special case of the general MHV amplitude (2.16) with two positive helicities and an arbitrary number (which in this case we take to be one) of negative helicities. Since this amplitude vanishes if the $\tilde{\lambda}_i$ are all proportional (for such a configuration, the numerator has a higher order zero than the denominator), it is supported on configurations for which the λ_i , i = 1, 2, 3, are proportional. $SL(4,\mathbb{R})$ invariance then implies that the (λ_i, μ_i) are all proportional, so that the points $P_i \in \mathbb{PT}$ at which the gluons are inserted all coincide, as predicted above.

However, it does not appear possible to justify this statement by Fourier transforming the amplitude (3.12). Actually, trying to define the twistor amplitudes by an ordinary Fourier transform rests on the claim in section 2 that in signature --++, the twistor amplitudes are ordinary functions in the real form of \mathbb{PT} and can be defined by an ordinary Fourier transform without need of $\overline{\partial}$ cohomology. Possibly this claim is too naive in the degenerate case of the -++ helicity amplitude. The string theory proposed in section 4 does lead in the complex form of \mathbb{PT} to a local twistor space three-point function for

the -++ amplitude in terms of $\overline{\partial}$ cohomology, as the argument given above predicts. (It is described in section 4.3.) Apparently, though I do not really know why a problem arises for this one case, the mapping of the $\overline{\partial}$ cohomology classes on complex twistor space to functions (or real densities) on real twistor space does not work for this particular amplitude.

3.3. First Look At Degree Two Curves And Differential Equations

The next case to consider is that the number of negative helicity gluons is three. Amplitudes with many positive helicity gluons and three of negative helicity are quite complicated; no general formula for them is known. We will analyze only the first two cases that the number of positive helicity gluons is two or three.

At any rate, because they are MHV amplitudes, the amplitudes with two negative and three positive helicities are simple, as we reviewed in section 2.3:

$$\widehat{A} = ig^3 (2\pi)^4 \delta^4 \left(\sum_i \lambda_i^a \widetilde{\lambda}_i^a \right) \frac{\left[\widetilde{\lambda}_a, \widetilde{\lambda}_b \right]^4}{\prod_{i=1}^5 \left[\widetilde{\lambda}_i, \widetilde{\lambda}_j \right]}.$$
 (3.13)

The gluons a and b are the ones with negative helicity; up to a cyclic permutation of the five gluons, there are two cases, namely ---++ and --+-+. The two cases need to be treated separately, but the arguments below turn out to work for each.

In principle, we would now like to compute the Fourier transform of this amplitude with respect to the $\tilde{\lambda}$'s and show that it is supported on curves of genus zero and degree two in twistor space. How to actually compute that Fourier transform is not clear. Happily, a short cut is available. The property we want to prove can be proved without actually

computing the Fourier transform. It is equivalent to certain differential equations obeyed by the momentum space amplitudes \widehat{A} .

First, let us describe what genus zero, degree two curves look like. We simply impose a linear and a quadratic equation in the homogeneous coordinates of \mathbb{RP}^3 or \mathbb{CP}^3 :

$$\sum_{I=1}^{4} a_I Z^I = 0$$

$$\sum_{I,J=1}^{4} b_{IJ} Z^I Z^J = 0.$$
(3.14)

Here a_I and b_{IJ} are generic real parameters. The degree of the zero set C of these equations is the product of the degrees of the equations, or $1 \cdot 2 = 2$. The complex Riemann surface in \mathbb{CP}^3 defined by these equations has genus zero. Conversely, any genus zero, degree two curve is of this form. The second equation in (3.14) can be simplified by solving the first for one of the coordinates. For example, generically $a_4 \neq 0$, in which case we can solve the first for Z_4 . Eliminating Z_4 from the second equation gives an equation

$$\sum_{I,J=1}^{3} c_{IJ} Z^{I} Z^{J} = 0 (3.15)$$

with some new coefficients c_{IJ} . Since only three homogeneous coordinates enter, this equation describes a curve in \mathbb{RP}^2 . As we did at the end of section 3.1 for curves of degree one, we can give a more elementary description of the curve defined by this equation if we go to affine coordinates. We throw away the subset of \mathbb{RP}^2 with (for example) $Z_1 = 0$, and introduce affine coordinates $x = Z^2/Z^1$, $y = Z^3/Z^1$ that parameterize a real two-plane \mathbb{R}^2 . Then (3.15) becomes a quadratic equation in x and y, so the solution set, in terminology possibly familiar from elementary algebra, is a conic section of the plane. Thus, we may refer to the degree two curves as conics or conic sections, while the degree one curves, in the same sense, are straight lines.

We would like to prove that the Fourier transform of the amplitude (3.13) is supported on configurations of five points $P_i = (\lambda_i, \mu_i)$ in \mathbb{RP}^3 that obey the equations in (3.14) for some set of the coefficients. Since the zero set of the first equation is an $\mathbb{RP}^2 \subset \mathbb{RP}^3$, we need to prove first of all that the P_i are contained in a common \mathbb{RP}^2 . A priori, we also need to prove that the five points are contained in a common conic section of \mathbb{RP}^2 . However,

for the five point function this is trivial. It is always possible to pick the six coefficients c_{IJ} to obey the five linear equations

$$\sum_{I,J=1}^{3} c_{IJ} Z_i^I Z_i^J = 0, \quad i = 1, \dots, 5.$$
(3.16)

Since the equations are homogeneous, it follows that for generic P_i , they uniquely determine the c_{IJ} up to an overall scaling. Such a scaling does not affect the curve defined in (3.15), so a generic set of five points in \mathbb{RP}^2 (or \mathbb{CP}^2) is contained in a unique conic. So for the five particle amplitudes, we only need to verify that the five points are contained in an \mathbb{RP}^2 .

It may still seem that we need to compute the Fourier transform of the amplitude, but this can be avoided as follows. Consider any four points Q_{σ} in \mathbb{RP}^3 with homogeneous coordinates Z_{σ}^I , $\sigma=1,\ldots,4$. The Q_{σ} are contained in an \mathbb{RP}^2 if and only if (regarding Z_{σ}^I , for fixed σ , as the coordinates of a vector in \mathbb{R}^4) the vectors Z_{σ}^I are linearly dependent. The condition for this is that the matrix Z_{σ}^I of the coefficients of these four vectors has determinant zero. The condition, in other words, is that the twistor space amplitude is supported at K=0, where K is the determinant of the 4×4 matrix with entries Z_{σ}^I :

$$K = \epsilon_{IJKL} Z_1^I Z_2^J Z_3^K Z_4^L. (3.17)$$

If we take the four points Q_{σ} to be the points P_i, P_j, P_k , and P_l associated with four of the external gluons, K becomes a function of the twistor coordinates of the gluons that we will call K_{ijkl} . We want to show that the twistor space amplitude \widehat{A} is supported where each of the functions K_{ijkl} vanishes. In fact we will show that

$$K_{ijkl}\widehat{A} = 0. (3.18)$$

Since the Fourier transform to twistor space is difficult, we want to evaluate this condition in momentum space, where the twistor coordinates $Z^I = (\lambda, \mu)$ are represented as $Z^I = (\lambda^a, -i\partial/\partial \tilde{\lambda}^{\dot{a}})$. So K can be interpreted as a differential operator in λ and $\tilde{\lambda}$. In fact, this operator is homogeneous of degree two in the λ 's and of degree two in the $\tilde{\lambda}$ '; it is

$$K = \frac{1}{4} \sum_{\sigma_1, \dots, \sigma_4} \epsilon_{\sigma_1, \dots, \sigma_4} \langle \lambda_{\sigma_1}, \lambda_{\sigma_2} \rangle \epsilon^{\dot{a}\dot{b}} \frac{\partial^2}{\partial \widetilde{\lambda}_{\sigma_3}^{\dot{a}} \partial \widetilde{\lambda}_{\sigma_4}^{\dot{b}}}.$$
 (3.19)

So we do not need to actually compute the Fourier transform. It suffices to show that the differential operators K annihilate the momentum space scattering amplitudes \widehat{A} of helicites --+++ and -+-++.

This assertion is true, and is simple enough that it can be proved by hand, in contrast with a variety of other statements made later whose verification required computer assistance.

A Preliminary Simplification

In doing so, it is very helpful to make a preliminary simplification that we can describe informally as using conformal invariance to set $P_1 = (1, 0, 0, 0)$ and $P_2 = (0, 1, 0, 0)$. The full procedure is a little more elaborate. We will go through the steps in detail as this procedure will be useful in other examples as well.

First of all, the conformal group $SL(4,\mathbb{R})$ contains an $SL(2,\mathbb{R})$ subgroup that acts on the λ 's. We can use this group plus overall scaling of the homogeneous coordinates to set

$$\lambda_1 = (1,0), \quad \lambda_2 = (0,1).$$
 (3.20)

The conformal group also contains a subgroup which in Minkowski space is the group of translations of the spatial coordinates $x^{a\dot{a}}$, and which on the twistor variables is generated by $\lambda^a \partial/\partial \mu^{\dot{a}}$. It acts as $\mu_{\dot{a}} \to \mu_{\dot{a}} + x_{a\dot{a}}\lambda^a$. We can use this to set

$$\mu_1 = \mu_2 = 0. (3.21)$$

Of course, (3.20) and (3.21) together are equivalent to

$$P_1 = (1, 0, 0, 0), P_2 = (0, 1, 0, 0).$$
 (3.22)

The twistor amplitude with $\mu_1 = \mu_2 = 0$ is

$$\widetilde{A}(\lambda_i, \mu_i) = i \int \frac{d\widetilde{\lambda}_1}{(2\pi)^2} \dots \frac{d\widetilde{\lambda}_n}{(2\pi)^2} \exp\left(i \sum_{j=3}^n \mu_{j\,\dot{a}} \widetilde{\lambda}_j^{\dot{a}}\right) (2\pi)^4 \delta^4 \left(\sum_{i=1}^n \lambda_i^a \widetilde{\lambda}_i^{\dot{a}}\right) A(\lambda_i, \widetilde{\lambda}_i). \quad (3.23)$$

The integrals over $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ can be done with the aid of the delta function. The delta function sets

$$\widetilde{\lambda}_{1}^{\dot{a}} = -\sum_{j=3}^{n} \lambda_{j}^{1} \widetilde{\lambda}_{j}^{\dot{a}}$$

$$\widetilde{\lambda}_{2}^{\dot{a}} = -\sum_{j=3}^{n} \lambda_{j}^{2} \widetilde{\lambda}_{j}^{\dot{a}}.$$
(3.24)

After eliminating $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$, the twistor amplitude reduces to a Fourier transform with respect to the remaining $\tilde{\lambda}$'s:

$$\widetilde{A}'(\lambda_i, \mu_i) = i \int \frac{d\widetilde{\lambda}_3}{(2\pi)^2} \dots \frac{d\widetilde{\lambda}_n}{(2\pi)^2} \exp\left(i \sum_{j=3}^n \mu_{j\,\dot{a}} \widetilde{\lambda}_j^{\dot{a}}\right) A'(\lambda, \widetilde{\lambda}). \tag{3.25}$$

Here the symbol \widetilde{A}' refers to \widetilde{A} with (3.20) and (3.21) imposed, while A' is A with (3.20) and (3.24) imposed.

Now suppose that we want to determine whether the twistor amplitude \widetilde{A} vanishes when multiplied by some polynomial function $X(\lambda,\mu)$. The amplitude \widetilde{A} is $SL(4,\mathbb{R})$ invariant. The one example we have so far of a polynomial that should annihilate a twistor amplitude is the polynomial K defined in (3.17); it is $SL(4,\mathbb{R})$ invariant. In some later examples, we will meet polynomials X that should annihilate an amplitude and are not $SL(4,\mathbb{R})$ invariant. Even when this occurs, the family of functions X_i that should annihilate the amplitude is $SL(4,\mathbb{R})$ -invariant.

To justify a claim that $X\widehat{A} = 0$, where X and \widehat{A} are $SL(4,\mathbb{R})$ -invariant, it suffices to show that this claim is valid when (3.20) and (3.21) are imposed. The same is true for justifying a claim that $X_i\widehat{A} = 0$, where \widehat{A} is $SL(4,\mathbb{R})$ -invariant, and X_i ranges over an $SL(4,\mathbb{R})$ -invariant family of polynomials.

We can use equation (3.25) to evaluate $X\widehat{A}$ (or $X_i\widehat{A}$) on the locus with $P_1=(1,0,0,0)$ and $P_2=(0,1,0,0)$. On the right hand side of (3.25), μ_j , j>2, is equivalent to $-i\partial/\partial\widetilde{\lambda}_j$. So, once we eliminate $\widetilde{\lambda}_i$, i=1,2 by using (3.24), we can convert X to a differential equation X' acting on A' by naively setting $\mu_1=\mu_2=0$ and

$$\mu_j = -i\frac{\partial}{\partial \widetilde{\lambda}_j}, \quad j > 2.$$
 (3.26)

The condition $X\widehat{A} = 0$ for the twistor amplitude is equivalent to the differential equation X'A' = 0 for the reduced and restricted amplitude A' in momentum space.

What have we gained in this process? We have reduced the number of variables, and obtained a simpler differential equation that is equivalent to the original one. The one subtlety in this procedure, and the reason that we have described it at some length, is that to arrive at this result, one must regard $\tilde{\lambda}_i$, i=1,2 as functions of the $\tilde{\lambda}_j$, j>2, via (3.24). This renders more complicated the reduced function A' and the action of the derivatives $\partial/\partial \tilde{\lambda}_j$, j>2.

Verification

We will now verify by the above procedure that (for example) $K_{1234}\widehat{A}=0$. Upon setting $P_1=(1,0,0,0)$ and $P_2=(0,1,0,0)$, K_{1234} reduces to

$$K_{34} = \epsilon^{\dot{a}\dot{b}} \frac{\partial^2}{\partial \tilde{\lambda}_3^{\dot{a}} \partial \tilde{\lambda}_4^{\dot{b}}}.$$
 (3.27)

Since $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ have been eliminated, the reduced momentum space amplitude A' is a function only of the $\widetilde{\lambda}_i$ with $i \geq 3$ (as well as the λ 's). Because of $SL(2,\mathbb{R})$ symmetry acting on the $\widetilde{\lambda}$'s, the dependence on the $\widetilde{\lambda}$'s is only via $a = [\widetilde{\lambda}_3, \widetilde{\lambda}_4], b = [\widetilde{\lambda}_3, \widetilde{\lambda}_5]$, and $c = [\widetilde{\lambda}_4, \widetilde{\lambda}_5]$. Moreover, A' is homogeneous in a, b, and c of degree -1:

$$\left(a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c}\right)A' = -A'. \tag{3.28}$$

This follows directly from the homogeneity of the full momentum space amplitude \widehat{A} in (3.13) as well as the fact that the equations used to solve for $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ are homogeneous.

A short computation using the chain rule shows that acting on any function F(a, b, c),

$$K_{34}F = -2\frac{\partial F}{\partial a} - a\frac{\partial^2 F}{\partial a^2} - b\frac{\partial^2 F}{\partial a\partial b} - c\frac{\partial^2 F}{\partial a\partial c}.$$
 (3.29)

The right hand side can be written

$$-\frac{\partial}{\partial a}\left(a\frac{\partial F}{\partial a} + b\frac{\partial F}{\partial b} + c\frac{\partial F}{\partial c} + F\right),\tag{3.30}$$

and so vanishes for any function that obeys (3.28).

Thus, we have demonstrated that $K_{1234}\widehat{A}=0$. Nothing essentially new is needed to show that $K_{ijkl}\widehat{A}=0$ for all i,j,k,l; one just uses conformal invariance to set (for example) $P_i=(1,0,0,0)$ and $P_j=(0,1,0,0)$, and then proceeds as above.

3.4. The Six Gluon Amplitude With Three Positive And Three Negative Helicities

Continuing our study of tree amplitudes associated with curves of degree two, the next case is the six gluon amplitudes with three positive and three negative helicities. These were first computed by Mangano, Parke, and Xu [38] and by Berends and Giele [22] and are quite complicated. There are three essentially different cases, namely helicities +++---+, or +-+-+-. These amplitudes can all be written

$$A = 8g^{4} \left[\frac{\alpha^{2}}{t_{123}s_{12}s_{23}s_{45}s_{56}} + \frac{\beta^{2}}{t_{234}s_{23}s_{34}s_{56}s_{61}} + \frac{\gamma^{2}}{t_{345}s_{34}s_{45}s_{61}s_{12}} + \frac{t_{123}\beta\gamma + t_{234}\gamma\alpha + t_{345}\alpha\beta}{s_{12}s_{23}s_{34}s_{45}s_{56}s_{61}} \right].$$
(3.31)

with $s_{ij} = (p_i + p_j)^2$, $t_{ijk} = (p_i + p_j + p_k)^2$. The functions α , β , and γ are different for the different helicity orderings. They are presented in the table.

Table 1. Coefficients for six gluon amplitudes with three helicities of each type (table from [38]). The symbol $\langle I|T|J\rangle$ is here short for $[IT]\langle TJ\rangle$; for $T=T_1+T_2+T_3$, a sum over the T_i is understood. The notation $\langle ij\rangle$ is used for $\langle \lambda_i, \lambda_j \rangle$, and [ij] for $[\widetilde{\lambda}_i, \widetilde{\lambda}_j]$.

	1+2+3+4-5-6-	1+2+3-4+5-6-	1+2-3+4-5+6-
	X = 1 + 2 + 3	Y = 1 + 2 + 4	Z = 1 + 3 + 5
α	0	$-[12]\langle 56\rangle\langle 4 Y 3\rangle$	$\boxed{[13]\langle 46\rangle\langle 5 Z 2\rangle}$
β	$[23]\langle 56 \rangle \langle 1 X 4 \rangle$	$[24]\langle 56\rangle\langle 1 Y 3\rangle$	$[51]\langle 24\rangle\langle 3 Z 6\rangle$
γ	$[12]\langle 45\rangle\langle 3 X 6\rangle$	$[12]\langle 35\rangle\langle 4 Y 6\rangle$	$[35]\langle 62\rangle\langle 1 Z 4\rangle$

Our conjecture says again that these amplitudes should be supported on configurations in which all six points P_i labeling the external particles lie on a common genus zero degree two curve or conic in \mathbb{RP}^3 . First of all, to show that the six points are contained in an \mathbb{RP}^2 subspace, we must establish that the amplitudes are annihilated by the differential operator K defined in (3.19), where the Q_{σ} , $\sigma = 1, \ldots, 4$, may be any of the six points P_i . This was verified with some computer assistance, after simplifying the problem as in section 3.3 by using conformal symmetry to set $P_1 = (1, 0, 0, 0)$ and $P_2 = (0, 1, 0, 0)$.

Next, we need to show that the six points are contained not just in an \mathbb{RP}^2 but in a conic section therein. This means that it must be possible to pick the coefficients c_{IJ} in (3.15) so that the equations

$$\sum_{I,J=1}^{3} c_{IJ} Z_i^I Z_i^J = 0, \quad i = 1, \dots, 6$$
(3.32)

are obeyed. In contrast to the five gluon case that we considered in section 3.3, here we have six homogeneous equations for six unknowns, so for a generic set of points P_i , a nonzero solution for the c_{IJ} does not exist. Existence of a nonzero solution is equivalent

to vanishing of the determinant of the 6×6 matrix of coefficients in this equation. With $(Z^1, Z^2, Z^3) = (\lambda^1, \lambda^2, \mu^1)$, this determinant is

$$\widehat{V} = \det \begin{pmatrix} (\lambda_1^1)^2 & \lambda_1^1 \lambda_1^2 & (\lambda_1^2)^2 & \lambda_1^1 \mu_1^1 & \lambda_1^2 \mu_1^1 & (\mu_1^1)^2 \\ (\lambda_2^1)^2 & \lambda_2^1 \lambda_2^2 & (\lambda_2^2)^2 & \lambda_2^1 \mu_2^1 & \lambda_2^2 \mu_2^1 & (\mu_2^1)^2 \\ (\lambda_3^1)^2 & \lambda_3^1 \lambda_3^2 & (\lambda_3^2)^2 & \lambda_3^1 \mu_3^1 & \lambda_3^2 \mu_3^1 & (\mu_3^1)^2 \\ (\lambda_4^1)^2 & \lambda_4^1 \lambda_4^2 & (\lambda_4^2)^2 & \lambda_4^1 \mu_4^1 & \lambda_4^2 \mu_4^1 & (\mu_4^1)^2 \\ (\lambda_5^1)^2 & \lambda_5^1 \lambda_5^2 & (\lambda_5^2)^2 & \lambda_5^1 \mu_5^1 & \lambda_5^2 \mu_5^1 & (\mu_5^1)^2 \\ (\lambda_6^1)^2 & \lambda_6^1 \lambda_6^2 & (\lambda_6^2)^2 & \lambda_6^1 \mu_6^1 & \lambda_6^2 \mu_6^1 & (\mu_6^1)^2 \end{pmatrix}.$$

$$(3.33)$$

(The subscripts $1, \ldots, 6$ label the six gluons, while the superscripts refer to the component a or \dot{a} of λ^a or $\mu^{\dot{a}}$ for each gluon.) Upon interpreting μ as $-i\partial/\partial\tilde{\lambda}$, \hat{V} becomes a fourth order differential operator that should annihilate the six gluon amplitudes with three positive helicities. This statement appears too complicated to check by hand and was verified with computer assistance. A preliminary simplification was again made by using conformal invariance to fix the point P_1 to have coordinates (1,0,0,0) and P_2 to have coordinates (0,1,0,0). Upon doing so, \hat{V} reduces to the determinant of a 4×4 matrix

$$V = \det \begin{pmatrix} \lambda_3^1 \lambda_3^2 & \lambda_3^1 \mu_3^1 & \lambda_3^2 \mu_3^1 & (\mu_3^1)^2 \\ \lambda_4^1 \lambda_4^2 & \lambda_4^1 \mu_4^1 & \lambda_4^2 \mu_4^1 & (\mu_4^1)^2 \\ \lambda_5^1 \lambda_5^2 & \lambda_5^1 \mu_5^1 & \lambda_5^2 \mu_5^1 & (\mu_5^1)^2 \\ \lambda_6^1 \lambda_6^2 & \lambda_6^1 \mu_6^1 & \lambda_6^2 \mu_6^1 & (\mu_6^1)^2 \end{pmatrix}.$$
(3.34)

V again is interpreted via $\mu_j = -i\partial/\partial \widetilde{\lambda}_j$, j > 2, as a fourth order differential operator that should annihilate the reduced momentum space amplitudes A'. As the computer program was unreasonably slow, the vanishing of VA' was verified as a function of $\widetilde{\lambda}_i^1$, $i = 3, \ldots, 6$, with the other variables set to randomly selected values.

Remaining Six Gluon Amplitudes

3.5. ---++ and --+-+ Amplitudes Revisited

By now, we have obtained what may seem like a tidy story for the five and six gluon amplitudes with three negative helicities. However, further examination, motivated by the string theory proposal in section 4 as well as the preliminary examination of one-loop amplitudes that we present in section 3.6, has shown that the full picture is more elaborate and involves disconnected instantons. Here we will re-examine the ---++ and --+-+ tree level amplitudes to consider such contributions. (It would be desireable to similarly re-examine the six gluon amplitudes, but this will not be done here.)

We so far interpreted these five gluon amplitudes in terms of genus zero curves of degree two. In string theory, these curves will be interpreted as instantons. The action of an instanton of degree two is precisely twice the action of a degree one instanton. It therefore has precisely the same action as a pair of separated degree one instantons. Might the --+++ and --+-+ amplitudes receive contributions from configurations with two separated instantons of degree one?

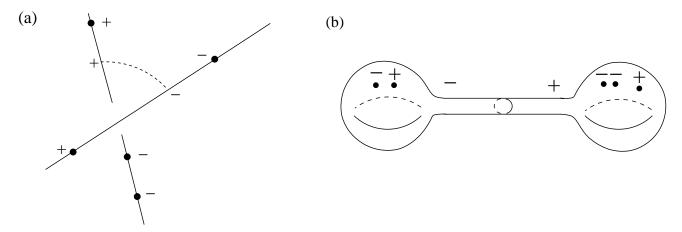


Fig. 3: In part (a), we depict two different straight lines in \mathbb{R}^3 , representing two disjoint curves of genus zero and degree one. A twistor field, represented by a curved dotted line which we call the internal line, is exchanged between them. Various points on the two lines, including the endpoints of the internal line, are labeled by + or - helicity. There are two - helicities on each line. (b) Here we give a complex version of the same picture. The lines of part (a) are replaced by two-spheres, and the internal line becomes a thin tube connecting them. The whole configuration is topologically a two-sphere.

In figure 3, we sketch two different pictures of a configuration with two widely separated instantons of degree one. Figure 3a contains a view of this situation in real twistor

space. The degree one instantons are represented as straight lines. We have attached the five external gluons to the two instantons. In the example sketched, we attached helicities -+ to one side and --+ to the other. Since we are trying to construct a connected amplitude, we also assume that a twistor space field of some kind is exchanged between the two instantons. What it might be will be clearer in section 4. We assume that this field carries negative helicity at one end and positive helicity at the other. (The helicities are reversed between the two ends because we consider all fields attached to either instanton to be outgoing and because crossing symmetry relates an incoming gluon of one helicity to an outgoing gluon of opposite helicity.) Propagation of the twistor space field between the two instantons is shown in figure 3a by connecting them via an internal line (shown as a dotted line in the figure). A degree one instanton must have exactly two - helicities attached to it. In figure 3a, after distributing the external particles between the two instantons, we labeled the ends of the internal line such that this condition is obeyed.

Note that the number of internal lines must be precisely one or we would end up with too many — helicities on one instanton or the other. In figure 3b, we try to give another explanation of what this means. Here we consider the instantons as curves in complex twistor space, so a degree one curve of genus zero is represented as a \mathbb{CP}^1 , or topologically a two-sphere. We also assume that the internal line connecting the two \mathbb{CP}^1 's represents a collapsed limit of a cylinder (exchange of a closed string — presumably a D-string in the proposal of section 4). So in figure 3b, we have drawn two \mathbb{S}^2 's connected by a single narrow tube, corresponding to the internal line in figure 3a. Two two-spheres joined this way make a surface of genus zero, and we regard this as a degenerate case of a Riemann surface of genus zero that can contribute to a tree level scattering amplitude. If we connect the two instantons with more than one narrow tube, we get a Riemann surface of genus one or higher, which should be considered as a contribution to a one-loop scattering amplitude.

The reasoning in the last two paragraphs is certainly not meant to be rigorous, but hopefully it will encourage the reader to follow along with us in contemplating differential equations that the ---++ and --+-+ amplitudes might obey reflecting their support on configurations like that in figure 3a. The salient aspect of figure 3a is that three of the external particles are attached to the same straight line, or degree one curve. In other words, of the five twistor points representing the external particles, three are collinear.

In figure 3, we assume that the field represented by the internal line transforms in the adjoint representation of the gauge group. To get a single trace amplitude whose group theory factor is Tr $T_1T_2T_3T_4T_5$, the particles must be divided between the two instantons in a way that preserves the cyclic order – for example, 12 on one side and 345 on the other or 34 on one side and 512 on the other. The color flow is then as shown in figure 4.

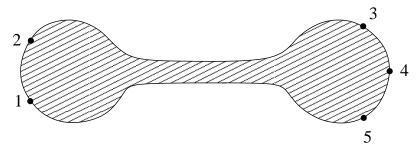


Fig. 4: This diagram is intended to show the color flow in figure 3. We suppose that in a dual Yang-Mills or open string description, each line (or genus zero, degree one curve) in figure 3 corresponds to a disc with gluons cyclically attached on the boundary. Moreover, the internal line becomes a small strip connecting the two discs. The overall figure is thus topologically a disc. It can contribute to a single trace amplitude (such as a Yang-Mills tree amplitude) if the internal line is in the adjoint representation of the gauge group. The amplitude is proportional to $\operatorname{Tr} T_1 T_2 \dots T_n$, where T_i is a U(N) generator of the i^{th} particle, and the particles attached to either disc are consecutive with respect to the cyclic order.

We will now construct a differential operator F_{ijk} that annihilates amplitudes in which the points P_i , P_j , and P_k are collinear. Once this is done, we can construct an operator that annihilates amplitudes in which *any* three consecutive points are collinear by simply forming the product

$$\widehat{F} = F_{123}F_{234}F_{345}F_{451}F_{512}. (3.35)$$

Thus, the operator \widehat{F} should annihilate any amplitude that arises from the sort of configuration sketched in figure 3.

If Q_{σ} , $\sigma = 1, 2, 3$ are three points in twistor space with homogeneous coordinates Z_{σ}^{I} , then the condition that the Q_{σ} are collinear is that⁸

$$\epsilon_{IJKL} Z_1^I Z_2^J Z_3^K = 0, \quad L = 1, \dots, 4.$$
 (3.36)

Setting $Z = (\lambda, \mu) = (\lambda, -i\partial/\partial \tilde{\lambda})$, the expressions on the left hand side of (3.36) become, in the usual way, differential operators that act on the momentum space amplitudes. For example, if we set L = 4, then we get an operator

$$F_{ijk} = \langle \lambda_i, \lambda_j \rangle \frac{\partial}{\partial \widetilde{\lambda}_k^1} + \langle \lambda_k, \lambda_i \rangle \frac{\partial}{\partial \widetilde{\lambda}_i^1} + \langle \lambda_j, \lambda_k \rangle \frac{\partial}{\partial \widetilde{\lambda}_i^1}$$
(3.37)

⁸ For example, if we set L=4 and introduce affine coordinates $x^I=Z^I/Z^4$, I=1,2,3, then three points $Q_{\sigma} \in \mathbb{R}^3$, $\sigma=1,2,3$, with coordinates x^I_{σ} , are collinear if and only if $\epsilon_{IJK}(x_1-x_2)^I(x_2-x_3)^J=0$. This can be rewritten as $\epsilon_{IJK}(x_1^Ix_2^J+x_2^Ix_3^J+x_3^Ix_1^J)=0$, and in that form is readily compared to the following equation in homogeneous coordinates.

that annihilates amplitudes in which the points P_i , P_j , and P_k are collinear. Inserting this definition of F_{ijk} in (3.35), we get a differential operator \hat{F} that should annihilate amplitudes in which any three consecutive points are collinear.

Interpretation

At this point, we really should pause to discuss the interpretation of these results. We have found that the same five gluon amplitudes are annihilated both by an operator K associated with connected curves of degree two and by an operator \widehat{F} associated with disconnected pairs of degree one curves. Does this indicate a duality wherein the same amplitude can be computed either using connected degree two curves or using the disconnected pairs?

I believe that actually the amplitude is a sum of the two types of contribution, and that the reality is more mundane. The amplitudes \widehat{A} that we are exploring have various singularities, and hence one should consider the possibility of delta function contributions in $K\widehat{A}$ and $\widehat{F}\widehat{A}$. I suspect that $K\widehat{A}$ is not quite zero but contains delta functions that are annihilated by \widehat{F} , and vice-versa. I will not try to justify this statement here, but I will give a simple example of a singularity that leads to a delta function contribution. Consider the expression

$$\epsilon^{\dot{a}\dot{b}} \frac{\partial^2}{\partial \tilde{\lambda}_i^{\dot{a}} \partial \tilde{\lambda}_i^{\dot{b}}} \left(\frac{1}{[\tilde{\lambda}_i, \tilde{\lambda}_j]} \right),$$
 (3.38)

which is a simplified example of the sort of contribution we meet in evaluating $K\widehat{A}$. According to the formulas used at the end of section 3.3 in proving that $K\widehat{A}=0$, this appears to vanish, but actually it is a multiple of $\delta^2(\widetilde{\lambda}_i)\delta^2(\widetilde{\lambda}_j)$. In fact, we can regard the four components of $\widetilde{\lambda}_i^{\dot{a}}$ and $\widetilde{\lambda}_j^{\dot{a}}$ as coordinates of \mathbb{R}^4 . The differential operator $L=\epsilon^{\dot{a}\dot{b}}\frac{\partial^2}{\partial\widetilde{\lambda}_i^{\dot{a}}\partial\widetilde{\lambda}_j^{\dot{b}}}$ is then the Laplacian of \mathbb{R}^4 , endowed with a suitable metric g of signature g. Thus, if we combine together the $\widetilde{\lambda}_i^{\dot{a}}$ and $\widetilde{\lambda}_j^{\dot{a}}$ to coordinates g, g, g, g, g, g, which is the usual propagator or Green's function of the Laplacian of g. So in acting on g, which is the differential operator g produces a delta function supported at the origin. It seems plausible that analogous delta function terms appear in more carefully evaluating g, and g, which is g, and g, where g is a delta function terms appear in more carefully evaluating g is g, and g, and g, are g, are g, and g, are g, are g, and g, are g, are g, and g, are g, are g, and g, are g, are g, and g, are g, are g, a

3.6. A Few One-Loop Amplitudes

To conclude this exploration of some perturbative Yang-Mills amplitudes, we would like to at least glimpse a few of the simplest issues concerning some one-loop amplitudes. The obvious one-loop amplitudes to look at first are the planar MHV amplitudes with precisely two negative helicity gluons in $\mathcal{N}=4$ super Yang-Mills theory. General and relatively simple formulas are known for these amplitudes [40].

Until this point, it has not generally mattered if we contemplate pure Yang-Mills theory or a supersymmetric extension thereof. The reason is that so far we have mainly limited ourselves to tree amplitudes in which the external particles are gluons. In such diagrams, the internal particles (in gauge theory or its supersymmetric extensions) are also gluons and supersymmetry simply does not matter. For loop diagrams, supersymmetry definitely does matter as any particle can propagate in the loop. We will consider the amplitudes with maximal supersymmetry, expecting them to be the most likely ones to lead to a simple theory.

Also, we consider planar amplitudes because computations show that they are simpler; in fact, the analysis in section 4 suggests that to get an equally simple result from non-planar diagrams, one should modify the $\mathcal{N}=4$ theory to include closed string contributions (which are not yet understood).

The formula (3.1) says that a one-loop amplitude with two gluons of negative helicity will be associated with curves of degree two in twistor space, since q = 2, l = 1 leads to d = 2. Moreover, with l = 1, the genus of these curves will be bounded by $g \le 1$. However,

there are no curves in twistor space of genus one and degree two. So these amplitudes will actually come from curves of genus zero and degree two. There are two kinds of curves to consider, both of which we have already encountered in studying tree diagrams:

- (1) There are connected curves of genus zero and degree two, consisting of conics located in some $\mathbb{RP}^2 \subset \mathbb{RP}^3$.
- (2) There are disconnected curves, consisting of a pair of degree one curves or lines. Generically these lines are "skew," not contained in any plane or \mathbb{RP}^2 .

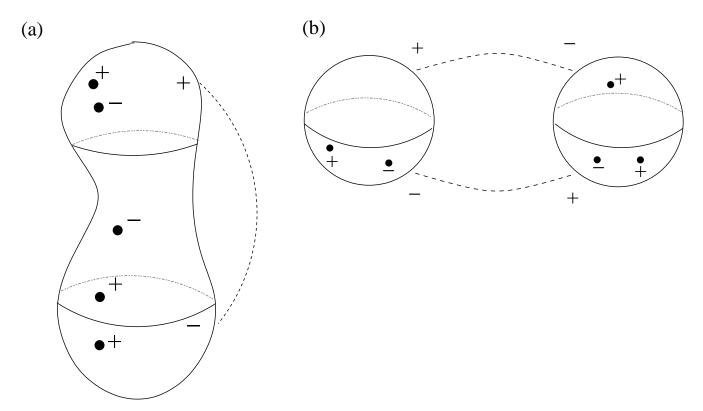


Fig. 5: The two configurations that contribute to the five gluon amplitude in one-loop order. In (a), we have a degree two curve of genus zero – the two bulges are meant merely as a reminder that the degree is two. An internal line – representing propagation of a twistor field – connects the curve to itself. The ends of the internal lines are labeled by + or – helicity, as are the five points at which external gluons are attached. In (b), we consider instead a configuration of two disjoint degree one curves connected by two internal lines, whose ends are again labeled along with the points at which external gluons are attached. If internal lines are replaced by thin tubes, both configurations become topologically equivalent to Riemann surfaces of genus one.

In figure 5, we sketch in what sense these two kinds of curve represent degenerate cases of configurations of genus one. In figure 5a, we consider a degree two curve with gluons of

helicities --++ attached to it, while in addition some twistor space field (of a type that will be clearer in section 4) is exchanged between two points on this curve. This is represented by the "internal line" in the figure, which connects the curve to itself. Labeling the ends of the internal line by helicities + and -, there are a total of three - helicities on the curve in figure 5a. As we have seen, this is the right number for a curve of genus zero and degree two. In addition, if we think of the internal line as representing a very thin tube, then the configuration in figure 5a represents a degenerate case of a Riemann surface of genus one.

In figure 5b, the external particles have been distributed in some way between the two degree one curves, which are also connected by two internal lines. Labeling the ends of the internal lines by + and -, we can ensure that each degree one curve has two - helicities attached to it, which is the correct number. Each degree one curve is topologically \mathbb{S}^2 ; when we interpret the internal lines as two very thin tubes connecting the two \mathbb{S}^2 's, we arrive, again, at a degenerate case of a Riemann surface of genus one. The number of internal lines should be exactly two both to ensure the right number of negative helicity insertions on each side and to get a configuration of genus one.

The simplest one-loop amplitude in the $\mathcal{N}=4$ theory is the four gluon amplitude, inevitably with two positive and two negative helicities (as the other cases vanish). This amplitude, however, is too simple for our purposes. The reason is that any four points are trivially contained pairwise in two skew lines, so we cannot expect to derive from the configurations of figure 5 any differential equation obeyed by the four gluon amplitude at one loop. (However, a string theory, such as the one proposed in section 4, may lead to a new way to calculate these amplitudes.)

We move on, therefore, to the five gluon amplitudes with helicities --+++ or -+-++. (It would be desireable to consider the one-loop MHV amplitudes with any number of positive helicity gluons, but this will not be done in the present paper.) We recall that for these MHV configurations, the tree level amplitudes \widehat{A}_0 are "holomorphic," that is, they are functions only of λ and not $\widetilde{\lambda}$ (times a delta function of energy-momentum conservation). Denoting the corresponding one-loop amplitudes as \widehat{A}_1 , the relation between them is

$$\widehat{A}_1 = g^2 \widehat{A}_0(L_1 + L_2), \tag{3.39}$$

with [41]

$$L_{1} = -\frac{1}{\epsilon^{2}} \sum_{i=1}^{5} \left(\frac{\mu^{2}}{-s_{i,i+1}}\right)^{\epsilon}$$

$$L_{2} = \sum_{i=1}^{5} \ln\left(\frac{-s_{i,i+1}}{-s_{i+1,i+2}}\right) \ln\left(\frac{-s_{i+2,i+3}}{-s_{i-2,i-1}}\right) + \frac{5\pi^{2}}{6}.$$
(3.40)

Here $s_{i,j} = (p_i + p_j)^2$. These amplitudes have been computed with dimensional regularization in $4 - 2\epsilon$ dimensions; the pole in L_1 at $\epsilon = 0$ is the usual infrared divergence of the one-loop diagram.

We have at our disposal the operator K which annihilates any amplitude that has delta function support on configurations sketched in figure 5a, and the operator \widehat{F} which annihilates any amplitude that has delta function support on configurations sketched in figure 5b. Note that K and \widehat{F} trivially commute with \widehat{A}_0 , since they contain derivatives only with respect to $\widetilde{\lambda}$, while \widehat{A}_0 , apart from the delta function, is a function of λ only. Hence K and \widehat{F} will act only on the L's.

By inspection, one can see that the amplitude L_1 is annihilated by \widehat{F} . Indeed, L_1 is a sum of terms each of which depends on the coordinates of two particles only. But \widehat{F} is a product of five operators, each of which contains only derivatives acting on the coordinates of three adjacent particles. Each term in L_1 is thus annihilated by one factor in \widehat{F} , and hence $\widehat{F}L_1=0$. So it is reasonable to interpret L_1 as arising from the configurations of figure 5b with two disconnected curves of degree one.

This observation, together with the fact that the one-loop infrared divergence is contained entirely in L_1 , leads to an interesting thought. If we specify the location in twistor space of five points, three of which are collinear, then the choice of the two degree one curves containing them is uniquely determined; one passes through the three collinear points and the other is the unique straight line through the remaining two points. There are no moduli to integrate over. We do have to integrate over the positions at which the

internal lines in figure 5b are attached. But these integrations run over compact parameter spaces (choices of points on the degree one curves in the figure), and divergences appear quite unlikely. (One can worry about short distance divergences in twistor space, but they would not be expected in the string model considered in section 4.) All of this strongly suggests that at least in this example, the one-loop twistor amplitude is free of infrared divergences, and the usual infrared problem comes from the Fourier transform back to momentum space. Perhaps twistor space amplitudes are free of infrared divergences in general.

If L_1 comes from the disconnected curves, perhaps L_2 is the contribution of the connected curves of degree two. This thought motivates the question of whether K annihilates L_2 . A small amount of computer-based inquiry reveals that $KL_2 \neq 0$, but

$$K^2 L_2 = 0. (3.41)$$

Here we should recall that K is really a collection of operators K_{ijkl} . The statement in (3.41) is that the product of any two of these operators annihilates L_2 .

What does it mean if an amplitude is annihilated not by K (or one of the other differential operators that have appeared in our investigation), but by its square? These operators are all polynomial in the μ 's (which are interpreted in momentum space as $-i\partial/\partial\tilde{\lambda}$). Consider the simplest case of an operator linear in μ . In fact, consider the operator W of multiplication by μ (that is, by one of the components of μ for one of the external particles). W annihilates the distribution $\delta(\mu)$. Now what is annihilated by W^2 but not by W^2 The answer to this question is that multiplication by μ^2 annihilates the distribution $\delta'(\mu)$ which is not annihilated by μ . So a distribution annihilated by W^2 is supported on the same set as a distribution annihilated by W, but in general, rather than delta function support, it has "derivative of a delta function support" in the normal directions. This is the general situation, for any operator that (like all the differential operators we have considered) is in twistor space a multiplication operator by some polynomial $P(\lambda, \mu)$. The distributions annihilated by P^2 are those that have "derivative of a delta function support" on the zero set of P. (We make this reasoning more explicit in discussing General Relativity in section 3.7.)

So that is the meaning of (3.41): the L_2 term is supported on configurations coming from connected curves of degree two, but has "derivative of a delta function support" rather than delta function support on the space of configurations of this type. It is not difficult to prove by hand that no power of K anihilates L_1 . So the full amplitude \widehat{A}_1 is not supported on configurations contained in an \mathbb{RP}^2 . But since $\widehat{F}L_1 = 0$, the amplitude $\widehat{A}_1 = \widehat{A}_0(L_1 + L_2)$ is a sum of contributions supported on the two types of configuration in figure 5. We can write a differential equations that expresses this fact:

$$0 = \widehat{F}K^2\widehat{A}_1. \tag{3.42}$$

(Again, K^2 refers to the product of any two components of K.)

That it is necessary to combine \widehat{F} and K in this way should not come as a surprise – it is what one would guess from the existence of the two configurations of figure 5. What does remain surprising is the rather different result of section 3.5 that, modulo possible delta function terms, the tree level ---++ amplitude is annihilated by \widehat{F} and K separately, while one might have expected it to be annihilated only by the product $\widehat{F}K$. Furthermore, one would like to understand, perhaps using the proposal in section 4, why the one-loop amplitude that we have examined is annihilated precisely by $\widehat{F}K^2$, rather than the minimal operator that annihilates it involving some other powers of \widehat{F} and K. These questions remain open.

A Note On Nonsupersymmetric Amplitudes

It is fascinating to ask whether in some theories with reduced supersymmetry, or no supersymmetry at all, a version of the structure we have found may persist. This question is much too ambitious to be tackled in the present paper. However, we will make one simple observation here.

The formula (3.1) for the degree of a curve from which a given amplitude should be derived implies that at the one loop level, scattering amplitudes with four or more external gluons that all have the same helicity must vanish. Indeed, if we set q = 0, l = 1, we get d = 0, which implies that the support of the amplitude must collapse to a point in twistor space. As we explained in section 3.3, a non-trivial scattering amplitude with at least four external particles cannot have this property.

However, in gauge theories in general, the one-loop diagrams with all gluons of the same helicity are non-zero in general. Indeed, they are given by a simple formula that was conjectured by Bern, Dixon, and Kosower [42] and proved by Mahlon [43], following some early computations in special cases [44-45]. For a useful summary, see [46]. Interestingly, these amplitudes are polynomial (and in fact quartic) in $\tilde{\lambda}$, and hence are supported on a degree one curve of genus zero, by the same argument that we give momentarily for General Relativity. This is not in accord with the most naive extension of our conjecture to non-supersymmetric theories, but it does suggest the possibility of some sort of generalization.

3.7. A Peek At General Relativity

Although we mainly focus on Yang-Mills theory in this paper, it is hard to resist taking a peek at General Relativity. Tree level n graviton amplitudes in General Relativity vanish if more than n-2 gravitons have the same helicity. The maximally helicity violating amplitudes are thus, as in the Yang-Mills case, those with n-2 gravitons of one helicity and two of the opposite helicity. These have been computed by Berends, Giele, and Kuijf [47]; the four particle case was first computed by DeWitt [3]. The salient features are as follows.

If we factor out the delta function of energy-momentum conservation via $\widehat{A}(\lambda_i, \widetilde{\lambda}_i) = i(2\pi)^4 \delta^4 \left(\sum_i \lambda_i^a \widetilde{\lambda}_i^{\dot{a}}\right) A(\lambda_i, \widetilde{\lambda}_i)$, then for Yang-Mills MHV amplitudes, A is actually a function of λ only. In section 3.1, we deduced from this that the twistor transform of those amplitudes is supported on genus zero curves of degree one.

The formulas in [47] show that for tree level MHV scattering in General Relativity, the reduced amplitudes A, although not independent of $\widetilde{\lambda}$, are polynomial in $\widetilde{\lambda}$. This has the following result. In Yang-Mills theory, when we carry out the Fourier transform of MHV amplitudes from momentum space to twistor space, we meet an integral

$$\int \frac{d^2 \widetilde{\lambda}_1}{(2\pi)^2} \dots \frac{d^2 \widetilde{\lambda}_n}{(2\pi)^2} \exp\left(i \sum_i \widetilde{\lambda}_i^{\dot{a}} (\mu_{i\,\dot{a}} + x_{a\dot{a}} \lambda_i^a)\right). \tag{3.43}$$

But in General Relativity, we must instead evaluate integrals of the form

$$\int \frac{d^2 \widetilde{\lambda}_1}{(2\pi)^2} \dots \frac{d^2 \widetilde{\lambda}_n}{(2\pi)^2} \exp\left(i \sum_i \widetilde{\lambda}_i^{\dot{a}} (\mu_{i\,\dot{a}} + x_{a\dot{a}} \lambda_i^a)\right) P(\widetilde{\lambda}_i^{\dot{a}}), \tag{3.44}$$

where P is a polynomial. Since this can be written

$$P(-i\partial/\partial\mu_{i\dot{a}}) \int \frac{d^2\widetilde{\lambda}_1}{(2\pi)^2} \dots \frac{d^2\widetilde{\lambda}_n}{(2\pi)^2} \exp\left(i\sum_i \widetilde{\lambda}_i^{\dot{a}}(\mu_{i\dot{a}} + x_{a\dot{a}}\lambda_i^a)\right), \tag{3.45}$$

the result of the integral is simply

$$P(-i\partial/\partial\mu_{i\dot{a}})\prod_{i=1}^{n}\delta^{2}(\mu_{\dot{a}}+x_{a\dot{a}}\lambda^{a}). \tag{3.46}$$

Thus, the twistor transform of the gravitational MHV amplitudes is supported on the same degree one curves as in the Yang-Mills case, but now with "multiple derivative of a delta function" behavior in the normal directions, roughly as we found in supersymmetric Yang-Mills theory at the one-loop level. It would certainly be interesting to know if such behavior persists for other amplitudes in General Relativity.

4. Interpretation As A String Theory

In this section, we will propose a string theory that gives a natural framework for understanding the results of section 3. This is the topological B model whose target space is the Calabi-Yau supermanifold $\mathbb{CP}^{3|4}$. We begin by outlining how this model is constructed, and then explore its properties, culminating with a computation of the supersymmetric MHV tree amplitudes. As we will have to summarize many things (and because various points are not yet clear), the present section will not be as nearly self-contained as the rest of the paper. The reader will probably find it helpful to have more familiarity that we can convey here with the topological B model and its extension to open strings. For reviews of the B model, see [48,49], for the extension to open strings see [50], and for some recent applications of the B model, see [51,52]. Some of the basics about the Penrose transform that are needed for this analysis are explained in an appendix, but the reader will probably find it helpful to consult [37] or the reviews cited in the introduction.

4.1. Construction Of The Model

To construct the ordinary \mathbb{CP}^{M-1} model in two dimensions, we introduce complex fields Z^I , $I=1,\ldots,M$, with a (constant) hermitian metric $g_{I\overline{J}}$, and a U(1) gauge field B and auxiliary field D. The Z^I all have charge one with respect to B, so their covariant derivative is $DZ^I=dZ^I+iBZ^I$. We work on a two-dimensional surface C with coordinates x^{α} , $\alpha=1,2$ and metric $\gamma_{\alpha\beta}$. The action is taken to be

$$I = \int_{C} d^{2}x \sqrt{\gamma} \left(\gamma^{\alpha\beta} g_{I\overline{J}} \frac{DZ^{I}}{Dx^{\alpha}} \frac{D\overline{Z}^{\overline{J}}}{Dx^{\beta}} + D \left(g_{I\overline{J}} Z^{I} \overline{Z}^{\overline{J}} - r \right) \right), \tag{4.1}$$

where r is a positive constant. The Lagrange multiplier imposes the constraint

$$g_{I\overline{J}}Z^{I}\overline{Z}^{\overline{J}} = r. (4.2)$$

To divide by the gauge group U(1), we must impose the gauge equivalence relation

$$Z^I \to e^{i\alpha} Z^I, \ \alpha \in [0, 2\pi].$$
 (4.3)

Assuming that the hermitian form $g_{I\overline{J}}$ is positive definite, the solution space of (4.2) subject to the equivalence relation (4.3) is a copy of \mathbb{CP}^{M-1} . This statement is not completely trivial, since in section 2 we used a different definition of \mathbb{CP}^{M-1} . According to this definition, \mathbb{CP}^{M-1} is parameterized by M complex variables Z^I , not all zero, subject

to the scaling $Z^I \to tZ^I$, for $t \in \mathbb{C}^*$. Writing $t = \rho e^{i\alpha}$, with ρ real and positive, the scaling by ρ can be used in a unique fashion to obey (4.2), and then the scaling by $e^{i\alpha}$ is the gauge equivalence (4.3).

Since the gauge field B has no kinetic energy in (4.1), it is an "auxiliary field," like the Lagrange multiplier D. We can solve for B in terms of Z using its equation of motion, which says that $g_{I\overline{J}}\overline{Z}^{\overline{J}}D_{\alpha}Z^{I}=0$, leading to $B=-ir^{-1}g_{I\overline{J}}\overline{Z}^{\overline{J}}dZ^{I}$. This formula says that B is the natural U(M)-invariant connection on the Hopf bundle over \mathbb{CP}^{M-1} (or more precisely, on the pullback of this to C via the map $Z:C\to\mathbb{CP}^{M-1}$).

The parameter r determines the Kahler class of \mathbb{CP}^{M-1} . However, as we will ultimately be studying the topological B model, which is independent of the Kahler class, the choice of r will be irrelevant. Likewise, although we have to pick some $g_{I\overline{J}}$ to write the action, the topological B model is independent of the choice of $g_{I\overline{J}}$, and our amplitudes will really be invariant under the complexification $GL(M,\mathbb{C})$ of the unitary group U(M).

To extend this construction to a sigma model in which the target space is a supermanifold $\mathbb{CP}^{M-1|P}$, we make the same construction, except that we replace Z by an extended set of coordinates $\mathcal{Z} = (Z^I, \psi^A)$, $I = 1, \ldots, M$, $A = 1, \ldots, P$, where Z^I are as before and the ψ^A are fermionic and of charge one with respect to the U(1) gauge field B. The components of \mathcal{Z} span a complex supermanifold $\mathbb{C}^{M|P}$. We endow this space with a hermitian form G which we may as well take to be block diagonal:

$$G = \begin{pmatrix} g_{I\overline{J}} & 0\\ 0 & g_{A\overline{B}} \end{pmatrix}. \tag{4.4}$$

It is invariant under a supergroup U(M|P). The action is the obvious extension of (4.1) to include the ψ^A :

$$I = \int_{C} d^{2}x \sqrt{\gamma} \gamma^{\alpha\beta} \left[g_{I\overline{J}} \frac{DZ^{I}}{Dx^{\alpha}} \frac{D\overline{Z}^{\overline{J}}}{Dx^{\beta}} + g_{A\overline{B}} \frac{D\psi^{A}}{Dx^{\alpha}} \frac{D\overline{\psi}^{\overline{B}}}{Dx^{\alpha}} + \frac{1}{2} \gamma_{\alpha\beta} D \left(g_{I\overline{J}} Z^{I} \overline{Z}^{\overline{J}} + g_{A\overline{B}} \psi^{A} \overline{\psi}^{\overline{B}} - r \right) \right]. \tag{4.5}$$

The constraint and gauge equivalence become

$$g_{I\overline{J}}Z^{I}\overline{Z}^{\overline{J}} + g_{A\overline{B}}\psi^{A}\overline{\psi}^{\overline{B}} = r, \tag{4.6}$$

and

$$Z^I \to e^{i\alpha} Z^I, \quad \psi^A \to e^{i\alpha} \psi^A.$$
 (4.7)

The constraint and gauge equivalence turn the model into a sigma model with target space $\mathbb{CP}^{M-1|P}$. Their combined effect is the same as taking the space of all Z^I and ψ^A , with the

 Z^I not all zero, and dividing by $(Z^I, \psi^A) \to (tZ^I, t\psi^A)$, $t \in \mathbb{C}^*$. That was the definition of $\mathbb{CP}^{M-1|P}$ in section 2.6.

World-Sheet Supersymmetry

The next step is to introduce world-sheet supersymmetry. Because $\mathbb{CP}^{M-1|P}$ is a Kahler manifold, a supersymmetric sigma model with this target space (first constructed and studied in [53,54] in the case P=0) will automatically have N=2 worldsheet supersymmetry. We thus replace C with a super Riemann-surface with N=2 supersymmetry, the fermionic coordinates being a complex spinor θ^{α} on C and its complex conjugate $\overline{\theta}^{\alpha}$. The Z^I and ψ^A are promoted to chiral superfields $\widehat{Z}^I(x,\theta)$ and $\widehat{\psi}^A(x,\theta)$. The gauge field B and auxiliary field D combine as part of a vector multiplet in superspace, whose field strength is a "twisted chiral superfield" Σ . This means in particular that Z^I and ψ^A have partners of opposite statistics,

$$\widehat{Z}^{I} = Z^{I} + i\theta^{\alpha}\chi_{\alpha}^{I} + \dots, \ \widehat{\psi}^{A} = \psi^{A} + i\theta^{\alpha}b_{\alpha}^{A} + \dots,$$

$$(4.8)$$

where χ^I_{α} is fermionic and b^A_{α} is bosonic. The superspace action is

$$I = \int d^2x d^2\theta d^2\overline{\theta} \left(g_{I\overline{J}} \overline{Z}^{\overline{J}} Z^I + g_{A\overline{B}} \overline{\psi}^{\overline{B}} \psi^A \right) + \frac{r}{2} \left(\int d^2x d\theta^+ d\overline{\theta}^- \Sigma + \int d^2x d\theta^- d\overline{\theta}^+ \overline{\Sigma} \right). \tag{4.9}$$

For simplicity, we took C to be flat; otherwise, we would need two-dimensional supergravity to make this construction. (4.9) might look mysterious at first sight because of the absence of derivatives with respect to the x^{α} . In sigma models such as this one with four supersymmetries, the derivatives appear [55] upon performing the θ integrals, which convert (4.9) into a supersymmetric extension of the bosonic action (4.5).

The Calabi-Yau Condition

So far, M and P are arbitrary. For reasons that will appear, however, we want to impose a Calabi-Yau condition. The supermanifold $\mathbb{CP}^{M-1|P}$ is a Calabi-Yau supermanifold if and only if M=P. Indeed, the holomorphic measure

$$\Omega_0 = \frac{1}{M!P!} \epsilon_{I_1 I_2 \dots I_M} \epsilon_{A_1 A_2 \dots A_P} dZ^{I_1} dZ^{I_2} \dots dZ^{I_M} d\psi^{A_1} d\psi^{A_2} \dots d\psi^{A_P}$$
(4.10)

on $\mathbb{C}^{M|P}$ is invariant under the U(1) gauge transformation (4.7) if and only if M = P.¹⁰ When it is U(1)-invariant, Ω_0 descends to a holomorphic measure Ω on $\mathbb{CP}^{M-1|P}$, ensuring

¹⁰ In verifying this statement, one must recall that for fermions, ψ and $d\psi$ transform oppositely, so if $\psi \to e^{i\alpha}\psi$, then $d\psi \to e^{-i\alpha}d\psi$. This relation is compatible with the defining property $\int d\psi \, \psi = 1$ of the Berezin integral for fermions.

that $\mathbb{CP}^{M-1|P}$ is a Calabi-Yau manifold for M=P. Of course, the objects Ω_0 and Ω were introduced in section 2.6, for closely related reasons.

As a Calabi-Yau supermanifold, $\mathbb{CP}^{M-1|M}$ should have a Ricci-flat Kahler metric. This in fact is simply the Fubini-Study metric – the one we obtain starting with the flat metric on $\mathbb{C}^{M|M}$ and imposing the constraint and gauge invariance that were described above. It does not take any computation to show the Ricci-flatness. The Ricci tensor of $\mathbb{CP}^{M-1|P}$ is completely determined up to a multiplicative constant by the SU(M|P) symmetry; the multiplicative constant is proportional to the first Chern class of $\mathbb{CP}^{M-1|P}$, which is M-P times a generator of the second cohomology group of this space. For M=P, the Ricci tensor therefore vanishes. (Concretely, the Riemann tensor of $\mathbb{CP}^{M-1|P}$ is non-zero and is given by a natural generalization of what it is in the bosonic case. To construct the Ricci tensor, we must take a supertrace of the Riemann tensor on two of its indices; when we do this, the fermions contribute with opposite sign from the bosons, giving Ricci-flatness for M=P.)

This means that, for M = P, the supersymmetric sigma model with action (4.9) is conformally invariant. More important for our present purposes, the Calabi-Yau condition means that we can introduce a twisted version of the model which is a topological field theory called the B model.

The B Model

The two-dimensional nonlinear sigma model with any Kahler manifold X as the target space has a vector-like R-symmetry which acts on the worldsheet coordinates θ^{α} as $\theta^{\alpha} \to e^{i\gamma}\theta^{\alpha}$; its action on the component fields in (4.8) can be deduced from this. The classical theory also has an axial or parity-violating R-symmetry, acting by $\theta^+ \to e^{i\gamma}\theta^+$, $\theta^- \to e^{-i\gamma}\theta^-$, where θ^+ and θ^- have positive and negative chirality. We write K for the generator of this symmetry. In the quantum theory, K is anomaly-free if and only if X is a Calabi-Yau manifold.

The B model is defined by "twisting" by K, so it can only be defined when X is Calabi-Yau. The twisting operation means, if the theory is formulated on a flat worldsheet $C \cong \mathbb{R}^2$, that one defines a new action of the two-dimensional Poincaré group in which the translation operators P_i are unchanged but the rotation generator J is replaced by J' = J + K/2. This does give a representation of the Poincaré Lie algebra, since $[K, P_i] = 0$. The twisting shifts the spin of every field by K/2. All fermions have integer spin in the twisted theory and two of the supercharges, say Q_1 and Q_2 , have spin zero. They obey

 $Q_1^2 = Q_2^2 = \{Q_1, Q_2\} = 0$, and their cohomology classes are regarded as the physical states of the twisted model. This construction on flat \mathbb{R}^2 can be generalized to an arbitrary curved two-dimensional surface in such a way that Q_1 and Q_2 are still conserved.

When we get to open strings, only one linear combination of Q_1 and Q_2 is conserved; we call this combination Q. Even for closed strings, it will be adequate for our purposes to describe the action of Q.

We will briefly describe the field content and transformation laws of the B-model. Let ϕ^i , $i=1,\ldots,\dim_{\mathbb{C}}X$, be a set of fields representing local complex coordinates on X. (In our example of $\mathbb{CP}^{3|4}$, we can take the ϕ^i to be Z^I/Z^1 , I>1, and θ^A/Z_1 .) The superpartners of the ϕ^i are as follows (as one learns by considering the expansion (4.8) in the twisted theory): $\eta^{\overline{i}}$ is a zero-form on C that transforms¹¹ as a (0,1)-form on X; θ_i is a zero-form on C that transforms as a section of the holomorphic tangent bundle of X; and ρ^i is a one-form on C that transforms as a (1,0)-form on X. The BRST transformation laws of the fields, that is, the transformation laws under the symmetry generated by Q, are

$$\delta \phi^{i} = 0$$

$$\delta \overline{\phi}^{i} = i \alpha \eta^{i}$$

$$\delta \overline{\eta}^{i} = \delta \theta_{i} = 0$$

$$\delta \rho^{i} = -\alpha \, d\phi^{i}.$$

$$(4.11)$$

(α is an infinitesimal anticommuting parameter.) The space of physical states is obtained by taking the cohomology of Q in the space of local functions of these fields (a local function is a functional of the fields and their derivatives up to some finite order, polynomial in the derivatives, evaluated at some given point in C). In fact, the cohomology classes can all be represented by operators that are functions only of ϕ , $\overline{\phi}$, θ , and η without any derivatives. Such operators take the form $V_{\alpha} = \alpha(\phi, \overline{\phi})_{\overline{i_i}\overline{i_2}...\overline{i_p}}^{j_1j_2...j_q}\eta^{\overline{i_1}}\eta^{\overline{i_2}}...\eta^{\overline{i_p}}\theta_{j_1}\theta_{j_2}...\theta_{j_q}$. Upon interpreting $\eta^{\overline{i}}$ as $d\overline{\phi}^{\overline{i}}$, V_{α} can be associated with an object $\alpha = d\overline{\phi}^{\overline{i_1}}d\overline{\phi}^{\overline{i_2}}...d\overline{\phi}^{\overline{i_p}}\alpha_{\overline{i_1}\overline{i_2}...\overline{i_p}}^{j_1j_2...j_q}$ that we interpret as a (0,p)-form on X with values in $\wedge^q TX$, the q^{th} antisymmetric power of the holomorphic tangent bundle TX (or $T^{1,0}X$) of X. With this interpretation, Q can be identified as the $\overline{\partial}$ operator on the space of such forms. The space of physical states is hence the direct sum over p

To be more precise, η is a section of $\phi^*\Omega^{0,1}(X)$, where $\Omega^{0,1}(X)$ is the space of (0,1)-forms on X, and $\phi^*\Omega^{0,1}(X)$ is its pullback to C via the map determined by the fields ϕ^i . A similar remark holds for θ and ρ .

and q of the $\overline{\partial}$ cohomology groups $H^p(X, \wedge^q TX)$. For compact X and more generally for the type of examples familiar in critical string theory, these cohomology groups are finite-dimensional. The richness of twistor theory comes partly from the fact that for X a suitable region in twistor space, the $\overline{\partial}$ cohomology groups are infinite-dimensional and can be identified with solution spaces of wave equations in Minkowski spacetime. For a brief explanation of this, see the appendix.

Other physical quantities in the B model are likewise naturally described in terms of complex geometry of X. For example, for C of genus zero, the B model correlation functions (which for X a Calabi-Yau threefold are important in heterotic and Type II superstring theory) are expressed as follows in terms of the wedge products of classes in $H^p(X, \wedge^q X)$. Let $\alpha_1, \ldots, \alpha_s$ be elements of $H^{p_i}(X, \wedge^{q_i} X)$, with $\sum_i p_i = \sum_j q_j = n$, where $n = \dim_{\mathbb{C}} X$. Each α_i corresponds to a vertex operator V_i , as explained in the last paragraph. The wedge product of the α_i is naturally an element of $H^n(X, \wedge^n TX)$. To define the B model, one must pick a holomorphic n-form Ω on X. By multiplying by Ω^2 , one can map $H^n(X, \wedge^n TX)$ to $H^{n,n}(X)$, the space of (n, n)-forms. Such a form can be integrated over X to obtain the genus zero correlation functions:

$$\langle V_1 \dots V_s \rangle = \int_X \alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_s \Omega^2.$$
 (4.12)

For C of genus greater than zero, B model observables involve more sophisticated invariants of the complex geometry. For genus one, one encounters analytic torsion, and for higher genus one meets less familiar invariants, to whose study powerful methods including mirror symmetry and the holomorphic anomaly have been applied [57].

In our example of $\mathbb{CP}^{M-1|P}$, we want to take M=4, because it is \mathbb{CP}^3 (and its supersymmetric extensions), and not some other \mathbb{CP}^{M-1} , that is related to four-dimensional

¹² In this discussion, we only consider the case of a bosonic Calabi-Yau manifold. The extension to a Calabi-Yau supermanifold involves some technical issues that have not been addressed yet, reflecting the fact that on a supermanifold, what can be integrated is not a differential form but an "integral form." A similar issue would arise for open strings on $\mathbb{CP}^{3|4}$ if we used space-filling branes. That is why we will use branes that are not quite space-filling. See [56] for construction of integral forms on certain complex supermanifolds associated with Yang-Mills theory.

Minkowski spacetime by the Penrose transform.¹³ Once we set M=4, we also need P=4, for the Calabi-Yau condition.

Symmetries Of The B Model

In general, the symmetries of the B model are the transformations of the target space that preserve its complex structure and also act trivially on the holomorphic measure Ω . The reason for this last requirement is visible in (4.12): the correlation functions are proportional to Ω^2 , so a symmetry of X that acts nontrivially on Ω^2 is not a symmetry of the B model. For the open string version that we introduce in section 4.2, the analogous formula (see eqn. (4.15)) is linear in Ω , so symmetries must act trivially on Ω .

The group of symmetries of $\mathbb{CP}^{3|4}$ that act trivially on the holomorphic measure Ω was determined in section 2.6. The group is PSL(4|4), a real form of which is the symmetry group PSU(2,2|4) of $\mathcal{N}=4$ super Yang-Mills theory in Minkowski space.

Among the supersymmetric Yang-Mills theories, the $\mathcal{N}=4$ theory is special, as it has the maximal possible supersymmetry [5]. Among the pure supersymmetric Yang-Mills theories (with only the fields of the super Yang-Mills multiplet), it is also special in being conformally invariant, which makes it a natural candidate for being encoded in twistor space, where the conformal symmetries are built in. We have found here a different explanation for what is special about $\mathcal{N}=4$: it cancels the anomalies in the B model of super twistor space.

Transformations that preserve the complex structure of the target space of the B model but act non-trivially on the holomorphic measure Ω are also interesting. They are not symmetries of B model amplitudes, but they can still be used to constrain these amplitudes in an interesting way. To jump ahead of our story a bit, we will argue that the relation (3.1) between the helicities in a Yang-Mills scattering amplitude and the degree and genus of a holomorphic curve on which its twistor transform is supported arise from such an anomalous symmetry of the B model of $\mathbb{CP}^{3|4}$.

Moreover, the twistor transform does not have a very close analog in Minkowski spacetimes of other dimensions, though some properties can be generalized, as discussed recently in [61]. For example, the conformal symmetry of Minkowski spacetime of n dimensions is SO(2,n) while the symmetry of \mathbb{CP}^{M-1} is SL(M); for one of these to be a real form of the other, we set n=M=4. (The case n=1, M=2 does not seem useful.) The closest analog of twistor space in a different dimension is probably the "mini-twistor space," a complex line bundle over \mathbb{CP}^1 that is used to solve the equations for BPS monopoles in three dimensions [62]. This construction is naturally obtained by dimensional reduction from the twistor correspondence in four dimensions.

Indeed, $\mathbb{CP}^{3|4}$ has a U(1) symmetry which does *not* leave invariant the holomorphic measure Ω . This is the transformation that rotates the fermions by a phase,

$$S: \psi^A \to e^{i\beta} \psi^A, \tag{4.13}$$

while leaving the bosonic coordinates Z^I invariant. Under the transformation S, we have $\Omega_0 \to e^{-4i\beta}\Omega_0$, and (since the transformation commutes with the scaling by which we descend to $\mathbb{CP}^{3|4}$) likewise $\Omega \to e^{-4i\beta}\Omega$. So Ω has S = -4.

4.2. Open String Sector Of The B Model

 $\mathbb{CP}^{3|4}$ is a Calabi-Yau supermanifold whose bosonic reduction is of complex dimension three. Before trying to describe its B model, it is well to begin by recalling the B model on an ordinary Calabi-Yau threefold X.

To define open strings while preserving the topological symmetry of the B model, one needs a boundary condition that preserves a linear combination of the fermionic symmetries of the model. As explained in [50], the simplest boundary conditions that do this are Neumann boundary conditions. We introduce Chan-Paton factors of the gauge group $GL(N,\mathbb{C})$ (a real form of which is U(N)). In modern language, this construction amounts to [63] introducing N space-filling D-branes wrapped on X. The branes are endowed by a vector bundle E with structure group $GL(N,\mathbb{C})$. As is also explained in [50], the only physical open string field in this model is a field A that is the (0,1) part of a connection on E. It is subject to the gauge invariance

$$\delta A = \overline{\partial}\epsilon + [A, \epsilon], \tag{4.14}$$

for any zero-form ϵ with values in the Lie algebra of $GL(N,\mathbb{C})$.

The basic idea of the derivation is that, among the worldsheet fields described in the closed string case in section 4.1, the open string boundary conditions are such that θ and one component of ρ vanish on the boundary of C. Open string vertex operators are local functions of the fields evaluated at a point on the boundary of C, so they depend only on $\phi, \overline{\phi}, \eta$, and the surviving component of ρ . The cohomology of Q in this space can be represented by vertex operators that are functions of just $\phi, \overline{\phi}$, and η . Vertex operators $V_{\alpha} = \eta^{\overline{i}_1} \eta^{\overline{i}_2} \dots \eta^{\overline{i}_p} \alpha_{\overline{i}_1 \overline{i}_2 \dots \overline{i}_p} (\phi, \overline{\phi})$ correspond to (0, p)-forms $\alpha = d\overline{\phi}^{\overline{i}_1} \dots d\overline{\phi}^{\overline{i}_p} \alpha_{\overline{i}_1 \dots \overline{i}_p} (\phi, \overline{\phi})$. The open string B model (for this type of space-filling brane) can thus be described in terms of a set of fields that are (0, p)-forms on X, for $0 \le p \le n$. (These fields are all $N \times N$

matrices because of the Chan-Paton factors.) However, for X a Calabi-Yau threefold, the important such field is the (0,1)-form A. The others can be interpreted in the low energy effective field theory as ghosts that enter in the quantization of A. The BRST operator Q acts as the $\overline{\partial}$ operator on A and the other (0,p)-forms.

A is a complex field (as is the gauge parameter ϵ). Its complex conjugate would be \overline{A} , the (1,0) part of the connection. However, the topological sector of the theory can be described without ever mentioning \overline{A} . Rather, the action is a holomorphic function of A:

$$I = \frac{1}{2} \int_{X} \Omega \wedge \operatorname{Tr} \left(A \overline{\partial} A + \frac{2}{3} A \wedge A \wedge A \right). \tag{4.15}$$

Here Tr $\left(A\overline{\partial}A + \frac{2}{3}A \wedge A \wedge A\right)$ is the Chern-Simon (0,3)-form constructed from A. There is no need to introduce explicitly a string coupling constant in (4.15), as this can be absorbed in a scaling of the holomorphic three-form Ω . The classical equations of motion derived from (4.15) assert simply the vanishing of the curvature (0,2)-form $F = \overline{\partial}A + A \wedge A$. This means that a classical solution defines a holomorphic vector bundle on X. This Chern-Simons action is very special; it is the unique local action that depends only on the complex structure and holomorphic volume-form of X and is invariant under complex gauge transformations of A.

The quantum theory is described by a path integral, which (if for simplicity we omit gauge-fixing and ghosts) is roughly of the form $\int DA \exp(-I)$. Since A is a complex variable and the action is a holomorphic function, one must try to understand this path integral as a contour integral for each mode of A. (See [51] for matrix models based on such contour integrals, and [64] for a thorough discussion of the contours in that context.) In general, the result of the path integral may depend on the choice of contour.

How to make sense of the path integral as a contour integral can be made explicit in perturbation theory. To construct perturbation theory, one must expand around a classical solution, that is, around a field A that defines a holomorphic vector bundle on X. In the case of an isolated and nondegenerate bundle (no zero modes for A), to construct perturbation theory one merely needs to know how to integrate a Gaussian function or a Gaussian times a polynomial. For example, for a single variable ϕ , $\int d\phi \exp(-\lambda \phi^2) = \sqrt{\pi/\lambda}$. One can pick a contour in the complex plane that justifies this Gaussian integral; the same contour will suffice to construct perturbation theory.

More interesting is the case that one is expanding around a moduli space \mathcal{Y} of flat connections. In this case, choosing a contour entails picking a suitable middle-dimensional

real homology cycle in \mathcal{Y} . For example, in a favorable situation, X may have a \mathbb{Z}_2 symmetry τ that reverses its complex structure. Such a symmetry defines what is called a real structure of X. If so, the τ -invariant subspace of \mathcal{Y} , if non-empty, is a suitable real cycle to integrate over. Perturbation theory can be constructed by integrating over this real slice of \mathcal{Y} and constructing perturbation theory in the normal directions. Conceivably, different perturbative series could be constructed using other cycles in \mathcal{Y} .

If $H^3(\mathcal{Y}, \mathbb{R}) \neq 0$, then because of the behavior of the Chern-Simons form under gauge transformations that are not homotopic to the identity, the action integral (4.15) is well-defined only modulo certain periods of the holomorphic volume form Ω . This does not affect perturbation theory, but it is certainly important nonperturbatively. A primary application of holomorphic Chern-Simons theory at the moment is to computing certain chiral amplitudes in physical string theory (for a recent dramatic example, see [51]). In that context, there is a closed string field, the two-form field or B-field, that must be included in establishing invariance under disconnected gauge transformations. In our study below of string theory on $\mathbb{CP}^{3|4}$, this problem does not arise, since $H^3(\mathbb{CP}^{3|4}, \mathbb{R}) = 0$. In any event, in this paper, we will be treating the open string fields perturbatively, in order to compare to perturbative Yang-Mills theory in spacetime.

Since we will be expanding around the trivial solution A = 0, we will not need to construct such a real cycle in a moduli space \mathcal{Y} of bundles on $\mathbb{CP}^{3|4}$. But we will encounter a somewhat analogous moduli space \mathcal{M} of holomorphic curves in $\mathbb{CP}^{3|4}$ (representing D-instanton configurations), and we will have to pick a real cycle in \mathcal{M} , which we will do by using a real structure on \mathbb{CP}^3 .

4.3. Extension to $\mathbb{CP}^{3|4}$

Let us now consider the analog of this in $\mathbb{CP}^{3|4}$. The *D*-branes that we will consider are not quite space-filling.¹⁴ They are defined by the condition that on the boundary of an open string, $\overline{\psi} = 0$, while ψ is free. The analog of this condition would not make sense for bosons; it would not make sense for a complex field Φ to say that $\overline{\Phi} = 0$ on the boundary while Φ is unrestricted. But this condition does make sense for fermions: it merely means that the vertex operators that can be inserted on the boundary are functions of ψ (as

¹⁴ It is conceivable that similar results would arise from space-filling branes, but the necessary formalism appears more complicated as noted in a previous footnote.

well as of the bosonic coordinates Z and \overline{Z}) and not of $\overline{\psi}$. Again we introduce $GL(N,\mathbb{C})$ Chan-Paton factors, or in other words, we consider a stack of N such D-branes.

We write Y for the world-volume of the branes. Y is thus the subspace of $\mathbb{CP}^{3|4}$ parameterized by Z, \overline{Z}, ψ with $\overline{\psi} = 0$.

If we repeat the derivation in [50], we find that the physical states are now described by a field $\mathcal{A} = d\overline{Z}^{\overline{I}}\mathcal{A}_{\overline{I}}$, where the $\mathcal{A}_{\overline{I}}$ depend, of course, on ψ as well as Z and \overline{Z} . (For space-filling branes, we would have had an extra term in the expansion, namely $d\overline{\psi}_A\mathcal{A}^A$, and the functions would all depend on $\overline{\psi}$ as well as the other variables.) We can describe this by saying that \mathcal{A} is a (0,1)-form on \mathbb{CP}^3 that depends on ψ as well as on the coordinates of \mathbb{CP}^3 . We can expand \mathcal{A} in terms of ordinary (0,1)-forms $(A,\chi_A,\phi_{AB},\widetilde{\chi}^A,G)$ with values in various line bundles:

$$\mathcal{A}(Z,\overline{Z},\psi) = d\overline{Z}^{\overline{I}} \left(A_{\overline{I}}(Z,\overline{Z}) + \psi^{A} \chi_{\overline{I}A}(Z,\overline{Z}) + \frac{1}{2!} \psi^{A} \psi^{B} \phi_{\overline{I}AB}(Z,\overline{Z}) + \frac{1}{3!} \epsilon_{ABCD} \psi^{A} \psi^{B} \psi^{C} \widetilde{\chi}_{\overline{I}}^{D}(Z,\overline{Z}) + \frac{1}{4!} \epsilon_{ABCD} \psi^{A} \psi^{B} \psi^{C} \psi^{D} G_{\overline{I}}(Z,\overline{Z}) \right). \tag{4.16}$$

The gauge invariance is

$$\delta \mathcal{A} = \overline{\partial} \epsilon + [\mathcal{A}, \epsilon]. \tag{4.17}$$

where now ϵ depends on ψ as well as Z and \overline{Z} .

Since the fermionic homogeneous coordinates ψ of $\mathbb{CP}^{3|4}$ take values in the holomorphic line bundle $\mathcal{O}(1)$ over \mathbb{CP}^3 , a field that multiplies ψ^k must take values in $\mathcal{O}(-k)$. So $A, \chi, \phi, \widetilde{\chi}$, and G take values respectively in the line bundles $\mathcal{O}, \mathcal{O}(-1), \mathcal{O}(-2), \mathcal{O}(-3)$, and $\mathcal{O}(-4)$. These fields, geometrically, are (0,1)-forms on \mathbb{CP}^3 with values in those line bundles. The fields $(A, \chi, \phi, \widetilde{\chi}, G)$ also have charges (0, -1, -2, -3, -4) for the charge S, defined above, that assigns the value +1 to ψ .

The classical action describing the open strings is the same as (4.15), except that the field $A(Z, \overline{Z})$ must be replaced by $\mathcal{A}(Z, \overline{Z}, \psi)$:

$$I = \frac{1}{2} \int_{Y} \Omega \wedge \operatorname{Tr} \left(A \overline{\partial} A + \frac{2}{3} A \wedge A \wedge A \right). \tag{4.18}$$

Recall that in this supersymmetric case, Ω is a measure for the holomorphic variables Z and ψ , locally taking the form $d^3Zd^4\psi$. The product of Ω with the Chern-Simons (0,3)-form is thus a measure on Y that can be integrated to get the action (4.18). In terms of

the expansion (4.16), the action becomes

$$I = \int_{\mathbb{CP}^{3}} \Omega' \wedge \operatorname{Tr} \left(G \wedge (\overline{\partial} A + A \wedge A) + \widetilde{\chi}^{A} \wedge \overline{D} \chi_{A} + \frac{1}{4} \epsilon^{ABCD} \phi_{AB} \wedge \overline{D} \phi_{CD} + \frac{1}{2} \epsilon^{ABCD} \chi_{A} \wedge \chi_{B} \wedge \phi_{CD} \right).$$

$$(4.19)$$

Here, \overline{D} is the $\overline{\partial}$ operator with respect to the connection A; for any field Φ , $\overline{D}\Phi = \overline{\partial}\Phi + A\Phi$. Also, $\Omega' = \frac{1}{4!} \epsilon_{IJKL} Z^I dZ^J dZ^K dZ^K$ is a (3,0)-form on \mathbb{CP}^3 that is homogeneous of degree 4 and so takes values in the line bundle $\mathcal{O}(4)$. It is obtained by integrating out the ψ^A from the measure Ω on $\mathbb{CP}^{3|4}$. On the other hand, the trace in (4.19) is a (0,3)-form with values in $\mathcal{O}(-4)$. So the product of the two is an ordinary (3,3) form, which can be integrated over \mathbb{CP}^3 to give an action. This action clearly has definite charge S = -4, confirming that the charge S is not a symmetry of the S model in twistor space.

The classical equations of motion obtained by varying the fields in (4.19) are

$$0 = \overline{\partial} \mathcal{A} + \mathcal{A} \wedge \mathcal{A} \tag{4.20}$$

or in components

$$0 = \overline{\partial}A + A \wedge A$$

$$0 = \overline{D}\chi$$

$$0 = \overline{D}\phi_{AB} - \chi_A \wedge \chi_B$$

$$0 = \overline{D}\widetilde{\chi}^A - \frac{1}{2}\epsilon^{ABCD}\left(\chi_B \wedge \phi_{CD} + \phi_{CD} \wedge \chi_B\right)$$

$$0 = \overline{D}G + \chi_A \wedge \widetilde{\chi}^A - \widetilde{\chi}^A \wedge \chi_A + \frac{1}{4}\epsilon^{ABCD}\phi_{AB} \wedge \phi_{CD}.$$

$$(4.21)$$

If we linearize these equations around the trivial solution with $\mathcal{A}=0$, they tell us simply that $0=\overline{\partial}\mathcal{A}$, or in components

$$0 = \overline{\partial}\Phi,\tag{4.22}$$

where Φ is any of $(A, \chi_A, \phi_{AB}, \widetilde{\chi}^B, G)$. Because of the gauge invariance (4.17), which reduces to

$$\delta\Phi = \overline{\partial}\alpha\tag{4.23}$$

for each component Φ , the fields Φ define elements of appropriate cohomology groups. To find the right ones, recall that each field $(A, \chi, \phi, \tilde{\chi}, G)$ has charge S = -k for some k = 0, -1, -2, -3, -4 and is a (0, 1)-form with values in $\mathcal{O}(-k)$. The equations (4.22) and gauge invariance (4.23) mean that such a field determines an element of the sheaf cohomology group $H^1(\mathbb{PT}', \mathcal{O}(-k))$. Here \mathbb{PT}' is whatever portion of twistor space $\mathbb{PT} = \mathbb{CP}^3$ we choose to work with.

Now we come to a central point. According to the Penrose transform [37,29-36], reviewed briefly in the appendix, the sheaf cohomology group $H^1(\mathbb{PT}', \mathcal{O}(-k))$ is equal to the space of solutions of the conformally invariant free massless wave equation for a field of helicity 1 - k/2, on a suitable region \mathbb{U} of complexified and conformally compactified Minkowski space (which depends on the choice of \mathbb{PT}'). These conformally invariant equations are as follows: the anti-selfdual Maxwell equations $F'^+=0$ for helicity 1, where F'^+ is the selfdual part of the field strength F'=dA' of an abelian gauge field A' in spacetime; the massless Dirac equation for helicity 1/2 and -1/2; the conformally coupled Laplace equation for helicity 0; and finally, for helicity -1, the equation dG'=0 where G' is a selfdual two-form.

So in this linearized approximation, the twistor space fields $(A, \chi_B, \phi_{BC}, \tilde{\chi}^C, G)$ correspond to spacetime fields $(A', \chi'_B, \phi'_{BC}, \tilde{\chi}'^C, G')$ which are respectively anti-selfdual gauge fields, positive chirality spinors, scalars, negative chirality spinors, and a selfdual two-form, all in the adjoint representation of $GL(N, \mathbb{C})$. On-shell, they describe particles of helicities (1, 1/2, 0, -1/2, -1), respectively. These are precisely the physical states of $\mathcal{N}=4$ super Yang-Mills theory, with just the familiar $SL(4)_R$ quantum numbers. Of course, this is largely determined by the manifest PSL(4|4) symmetry.

Moreover, the field content is almost recognizable (and will be altogether recognizable to readers familiar with investigations by Siegel of a chiral limit of super Yang-Mills theory [65]). A' is the gauge field of the $\mathcal{N}=4$ theory, while χ' and $\tilde{\chi}'$ are the usual positive and negative chirality fermions, and ϕ' are the usual scalars. We still have to interpret G', as well as the anti-selfduality of A'.

Under the Penrose transform from twistor space fields $(A, \chi, \phi, \tilde{\chi}, G)$ to spacetime fields $(A', \chi', \phi', \tilde{\chi}', G')$, might the action (4.19) magically turn into the standard $\mathcal{N}=4$ action in spacetime, which has the same superconformal symmetry? The answer to this question is "no," for a very instructive reason. The action (4.19), in addition to having the PSL(4|4) symmetry of $\mathcal{N}=4$ super Yang-Mills theory, is homogeneous of degree -4 with respect to the "anomalous" U(1) generator S that assigns the values (0,-1,-2,-3,-4) to $(A,\chi_B,\phi_{BC},\tilde{\chi}^B,G)$. When we linearize around $\mathcal{A}=0$, the Penrose transform is a linear map, so we should assign the same quantum numbers (0,-1,-2,-3,-4) to

 $(A', \chi'_B, \phi'_{BC}, \widetilde{\chi}'^B, G')$. With this assignment, the standard $\mathcal{N}=4$ Yang-Mills action is a sum of terms most of which have S=-4 or S=-8. The S=-4 terms include the kinetic energies $\widetilde{\chi}^{\dot{a}}D_{a\dot{a}}\chi^a$ and $(D_{a\dot{a}}\phi)^2$, as well as the Yukawa coupling $\phi\chi^2$, while the other Yukawa coupling $\phi\widetilde{\chi}^2$ and the ϕ^4 coupling have S=-8. The $\mathcal{N}=4$ theory also has a Yang-Mills kinetic energy $(F')^2$, with $F'=dA'+A'\wedge A'$; it has S=0, but arises, in a description with an auxiliary field, from another term with S=-8, as we explain in section 4.4.

In short, the $\mathcal{N}=4$ action can be described as a sum of terms of S=-4 and S=-8. Our proposal here is that the classical B model of $\mathbb{CP}^{3|4}$ gives the terms of S=-4, while the terms of S=-8 will come from a D-instanton correction that will be introduced later. The S=-4 part of the action is a supersymmetric truncation of $\mathcal{N}=4$ super Yang-Mills theory that has been studied in [65]. According to the second paper in that reference, where the supersymmetry transformations can also be found, the supersymmetric action is

$$I = \int d^4x \operatorname{Tr} \left(\frac{1}{2} G^{ab} F_{ab} + \widetilde{\chi}^{Aa} D_{a\dot{a}} \chi_A^{\dot{a}} + \frac{1}{8} \epsilon^{ABCD} \phi_{AB} D_{a\dot{a}} D^{a\dot{a}} \phi_{CD} + \frac{1}{4} \epsilon^{ABCD} \phi_{AB} \chi_C^{\dot{a}} \chi_{D\dot{a}} \right). \tag{4.24}$$

Since confusion with twistor space fields seems unlikely, we have here omitted the primes from the fields.

The ++- Amplitude And The Twistor Space Propagator

Having identified the twistor fields A and G with spacetime fields of helicities 1 and -1, we can shed a little light on one point from section 3.2. There we predicted, but did not quite find, a local twistor space interaction with helicities ++-. In complex twistor space \mathbb{CP}^3 , this interaction exists; it is simply the AAG interaction that we can read off from (4.19).

Moreover, we can now understand the mysterious "internal lines" that appeared in section 2 in (for example) figures 3 and 5. The fields propagating in these lines are A and G. (For tree level scattering of gluons, which was the main focus of section 2, the $SU(4)_R$ non-singlet fields $\chi, \phi, \tilde{\chi}$ do not contribute.) The kinetic energy of A and G is purely off-diagonal, of the form $G\overline{\partial}A$, so the propagator is also purely off-diagonal. This is why opposite ends of the internal lines are labeled by opposite helicities. It still remains to explain later the Riemann surfaces that the internal lines are attached to.

4.4. The Auxiliary Field G' And The Anti-Selfduality of A'

By now, we have extracted from the twistor theory a spacetime description that is much like conventional $\mathcal{N}=4$ super Yang-Mills theory. The main differences are the appearance of a possibly unexpected field G' and the anti-selfduality of A'.

To elucidate these points, supersymmetry is not really essential, so we will start with a stripped down version with fewer fields. We consider a U(N) gauge field A' in spacetime, and a field G' that is a selfdual antisymmetric tensor with values in the adjoint representation of $GL(N, \mathbb{C})$. We define an S quantum number under which A' and G' have charges 0 and -4. (The peculiar choice for G' is of course motivated by the supersymmetric example described above.) We begin with the action [65,66]

$$I = \int d^4x \,\text{Tr } (G' \wedge F'(A')) = \int d^4x \,\text{Tr } (G' \wedge F'^+(A')). \qquad (4.25)$$

 F'^+ is the selfdual part of F'. The two expressions for the action given in (4.25) are equal, since G' is selfdual. The action has charge S=-4. The classical equations of motion are

$$F'^{+}(A') = 0, \quad \frac{D}{Dx^{\mu}}G'^{\mu\nu} = 0,$$
 (4.26)

where D_{μ} is the covariant derivative including the gauge field A'. The first equation says that the nonzero part of the field strength of A' is the anti-selfdual part F'^- , which of course obeys a Bianchi identity $D_{\mu}F'^{-\mu\nu}=0$ that is rather like the equation of motion for the selfdual field G'.

The fact that G' and the nonzero part of F' are respectively selfdual and anti-selfdual means that they describe particles of opposite helicity. In our conventions, A' describes a particle of helicity 1 and G' describes a particle of helicity -1. The spectrum, at this linearized level, is thus that of conventional Yang-Mills theory, with both helicities present. The interactions, however, are not the standard ones. Indeed, the action (4.25) has an A'A'G' term, describing a vertex of three fields with helicities ++, but in contrast to Yang-Mills theory, it has no -- vertex. Indeed, that term would have S=-8.

To cure this, we add a $(G')^2$ term (as was also discussed in [66]), to get an extended action

$$I_1 = \int d^4x \text{ Tr } \left(G'F' - \frac{\epsilon}{2}(G')^2 \right).$$
 (4.27)

Here ϵ is a small parameter. The term we have added has S = -8. It is nearly the unique term that we can add to (4.25) that is local, gauge invariant, and conformally invariant. The only other possibility is the topological invariant

$$\Delta I = \int \text{Tr } F' \wedge F', \tag{4.28}$$

which has S=0 and is related in a familiar fashion to instantons and the θ angle of four-dimensional quantum gauge theory. As a topological invariant, this interaction has no influence on Yang-Mills perturbation theory, but it is important nonperturbatively.

We can integrate out G' from (4.27) to get an equivalent action for A' only. It is

$$I_2 = \frac{1}{2\epsilon} \int d^4x \operatorname{Tr} (F'^+)^2.$$
 (4.29)

From the point of view of perturbation theory, this is precisely equivalent to conventional Yang-Mills theory. In fact, the topological invariant (4.28) is a multiple of $(F'^+)^2 - (F'^-)^2$. So upon adding it with the right coefficient, ¹⁵ we convert (4.29) to

$$I_3 = \frac{1}{4\epsilon} \int d^4x \, \text{Tr} \, (F')^2.$$
 (4.30)

If desired, we can also add the topological invariant (4.29) with a real coefficient to incorporate the theta angle of quantum gauge theory. This will play no role in the present paper, as we will limit ourselves to trying to reconstruct perturbation theory from twistor space. (We note, however, that the topological invariant (4.28) is mapped by the twistor transform to the second Chern class of the bundle E over twistor space, and so could be represented by a local interaction in twistor space.)

We have obtained our desired result. (4.30) is equivalent to Yang-Mills theory with the usual Yang-Mills coupling g_{YM} being related to ϵ by $g_{YM}^2 = \epsilon$.

What has happened? Clearly, it is possible to take the $g_{YM} \to 0$ limit of Yang-Mills theory in such a way as to arrive at (4.25). We are accustomed to taking the weak coupling limit of Yang-Mills theory in a way that treats the two helicities symmetrically. But it is possible instead to break this symmetry as $g_{YM} \to 0$ and end up with (4.25). Or one could make an opposite choice as $g_{YM} \to 0$ and arrive at the parity conjugate of (4.25). The

The right coefficient is imaginary. For example, if we are in Lorentz signature, the action should be real, but because the selfdual and anti-selfdual conditions are $F = \pm i * F$, ΔI is actually an imaginary multiple of $(F'^+)^2 - (F'^-)^2$.

different choices differ by how the wavefunctions of states of different helicities are scaled as $g_{YM} \to 0$. We make this more explicit momentarily in the context of the $\mathcal{N} = 4$ theory.

Charges For $\mathcal{N}=4$

We can now improve on an assertion that was made in section 4.3. There we described the $\mathcal{N}=4$ super Yang-Mills action as the sum of the $(F')^2$ term, of S=0, plus terms of S=-4 and S=-8. But now we see that, if the auxiliary field G' is included, then the $(F')^2$ term really comes from a $(G')^2$ interaction, which has S=-8. So in this description, the $\mathcal{N}=4$ super Yang-Mills action is of the form

$$I = I_{-4} + \epsilon I_{-8}. \tag{4.31}$$

where I_{-4} is the sum of terms of S = -4 and I_{-8} is the sum of terms with S = -8. I_{-4} comes from the Penrose transform of (4.19) to spacetime, and I_{-8} will arise as a one-instanton contribution. I is the standard $\mathcal{N} = 4$ action [5], and I_{-4} was investigated in [65].

It is also interesting to understand how to express the action in a standard form with all terms proportional to $g_{YM}^{-2} = \epsilon^{-1}$. This is done simply by rescaling every field that has S = -k for some k by a factor of $\epsilon^{-k/4}$. In the new variables, all terms in I are proportional to ϵ^{-1} . After this rescaling and integrating out G, the action becomes manifestly invariant under parity. The different $\epsilon \to 0$ limits of $\mathcal{N} = 4$ super Yang-Mills theory thus arise from different ways to scale the fields $\chi, \phi, \widetilde{\chi}$ (or the corresponding helicity states), as well as the gluon states of one helicity or the other, as $\epsilon \to 0$.

4.5. Relation To Perturbation Theory

Now let us understand how the relation to Yang-Mills perturbation theory must work, to recover the results of section 3. We will then look for an instanton construction that yields the right properties.

For simplicity, we consider only the fields A' and G', with the action (4.27), which takes the general form

$$I \sim G'(dA' + (A')^2) - \epsilon(G')^2.$$
 (4.32)

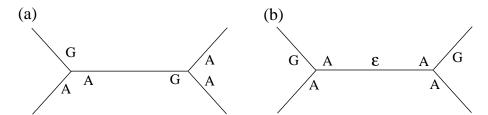


Fig. 6: (a) A tree level Feynman diagram with k vertices of type AAG, connected by AG propagators, leads to an $A^{k+1}G$ interaction, as sketched here for k=2. (b) Replacing an AG propagator by an AA amplitude adds a power of ϵ and replaces an A by a G in the amplitude. For k=2, we generate in order ϵ an ϵA^2G^2 interaction, as sketched here.

We first consider the theory at $\epsilon = 0$. The perturbation theory in this case has already been analyzed in [66]. The only interaction vertex is the G'A'A' vertex, which we identify with a configuration of helicities -++. To form a Feynman diagram, we can start with any number of G'A'A' vertices, and then contract some fields with propagators. Because of the off-diagonal nature of the G'dA' kinetic energy, the propagator in the basis given by (A', G') has the general form

$$\begin{pmatrix} 0 & d^{-1} \\ d^{-1} & 0 \end{pmatrix}. \tag{4.33}$$

The only non-zero matrix element of the propagator is $\langle G'|A'\rangle$. As illustrated in figure 6(a), to make a tree diagram we start with an arbitrary number k of G'A'A' vertices, and connect them by k-1 propagators, in the process "contracting out" k-1 factors of G' and the same number of factors of A'. We are left with an amplitude $G'(A')^{k+1}$ with only one negative helicity field G' and an arbitrary number of positive helicity fields. (These amplitudes actually vanish for k > 1, after summing over diagrams, but this is not very apparent in the present discussion.)

We could have predicted the same result without looking at Feynman diagrams by noting that since (at $\epsilon = 0$) the classical action is homogeneous with S = -4, the tree level S-matrix elements, obtained by integrating out the off-shell degrees of freedom, must have the same property. So they are homogeneous in G' of degree 1.

We can extend that analysis to $\epsilon \neq 0$ simply by assigning charge S=4 to ϵ . Then the whole action is homogeneous with S=-4, so the generating function of scattering amplitudes has the same property. Since the only objects carrying the S-charge are G' with charge -4 and ϵ with charge 4, the generating functional of the tree level scattering matrix elements must have the general form

$$W = f_0(A')G' + \epsilon f_1(A')(G')^2 + \ldots + \epsilon^{r-1} f_{r-1}(A')(G')^r + \ldots$$
(4.34)

Of course, we can also reach the same conclusion by examining Feynman diagrams (as in figure 6(b)). For this, we note that taking $\epsilon \neq 0$ adds no vertices to the Lagrangian, but it does add an additional term $\epsilon(G')^2$ to the kinetic energy. The modification of the propagator is very simple. Upon inverting a 2 × 2 matrix that is schematically

$$\begin{pmatrix} 0 & d \\ d & \epsilon \end{pmatrix} \tag{4.35}$$

in the (A', G') basis, we find that for $\epsilon \neq 0$, $\langle G' G' \rangle$ remains zero, but $\langle A' A' \rangle$ is nonzero and of order ϵ . Every time that we replace the $\langle A' G' \rangle$ propagator in a tree level Feynman diagram by an $\langle A' A' \rangle$ propagator, we multiply the amplitude by a factor of ϵ (from the propagator) and we retain an extra G' field (which is not contracted out). This leads back to the structure found in eqn. (4.34).

We interpret the $e^{r-1}f_{r-1}(A')(G')^r$ term in (4.34) as the generating functional of tree level scattering processes with precisely r gluons of negative helicity. As we also will interpret e as the instanton expansion parameter, it follows that tree amplitudes with precisely r negative helicity gluons must arise from configurations with instanton number r-1. (The instantons in a given configuration may be either connected or disconnected, as we discussed in section 3.) This reasoning was the original motivation for the conjecture that was stated in eqn. (3.1) and explored in section 3.

We can straightforwardly extend this analysis to include loops. After assigning charge 4 to ϵ , the whole action I is of charge -4. If we introduce Planck's constant \hbar with charge -4 and define the rescaled action $I' = I/\hbar$, then I' is invariant under S. On the other hand, an l-loop amplitude is proportional to \hbar^{l-1} . As this factor has charge -4l+4, it must multiply a function of G' and ϵ of total charge 4l-4. The allowed powers are $\epsilon^d(G')^q$ where 4d-4q=4l-4, or

$$d = q - 1 + l. (4.36)$$

This agrees with our basic formula (3.1) when we interpret the power of ϵ as the instanton number and the power of G' as the number of negative helicity gluons in a scattering process.

Of course, we could alternatively have reached this conclusion by counting the powers of ϵ in Feynman diagrams with loops. We leave this to the reader.

4.6. D-Instantons

By now it should be clear that we need to enrich the B model of $\mathbb{CP}^{3|4}$ with instanton contributions that will introduce additional violation of the quantum number S. But what kind of instantons? The most obvious instantons are worldsheet instantons. However, one of the main claims to fame of the B model is that topological amplitudes in this model receive no worldsheet instanton corrections. The A-model does have worldsheet instanton contributions, but otherwise it falls badly short of what we need. For example, its space of physical states is far too trivial, involving ordinary cohomology, which is finite-dimensional for any reasonable subspace \mathbb{PT}' of twistor space. By contrast, the B model leads to the far richer $\overline{\partial}$ cohomology, and, via the Penrose transform, to massless fields in Minkowski spacetime. Somehow, we need a model that combines the virtues of the A model and the B model. Another obvious shortcoming of the A model is that, as it requires no Calabi-Yau condition for the target space, it would not explain the special role of $\mathcal{N}=4$ supersymmetry.

A clue comes by considering duality between heterotic and Type I superstrings. The B model of $\mathbb{CP}^{3|4}$ with U(N) gauge fields incorporated via Chan-Paton factors is a kind of topological version of the Type I model. Suppose that, at least for some values of N, the model has a heterotic string dual. Then we would expect worldsheet instanton contributions to the topological amplitudes. Under duality between heterotic and Type I superstrings, the heterotic string worldsheet instantons turn into Type I D-instantons, which represent submanifolds in the target space $\mathbb{CP}^{3|4}$ on which open strings may end. And accordingly, in physical Type I superstring theory, D-instantons do contribute to chiral amplitudes. All of this suggests that we should incorporate D-instantons in the B model of $\mathbb{CP}^{3|4}$. To preserve the topological symmetry of the B model, these instantons must come from D-branes wrapped on some holomorphic submanifold of $\mathbb{CP}^{3|4}$. These holomorphic submanifolds must be of complex dimension one, since we learned in section 3 that perturbative Yang-Mills theory is related to curves in twistor space.

A D-instanton carries a U(1) gauge field, so whenever we consider a D-instanton wrapped on a curve C, a holomorphic line bundle \mathcal{L} will be part of the discussion. As we discuss presently, if C has genus g, \mathcal{L} should have degree g-1. (In section 3, we did not notice that our curves in twistor space were endowed with line bundles, because they all were of genus zero, so \mathcal{L} had no moduli.) Scattering amplitudes with $g \geq 2$ should also receive contributions from D-instantons of multiplicity k > 1, that is, from a collection of

k D-instantons wrapped on the same curve C. In this case, the gauge group supported on C is $GL(k,\mathbb{C})$, and C will be endowed with a rank k holomorphic vector bundle F. When F is irreducible, the k D-instantons cannot separate; a cluster of k D-instantons with an irreducible vector bundle is a component of instanton moduli space that needs to be included. (But we will not do any computations that are nearly sophisticated enough to see such components.)

The massless modes on the worldvolume of a D-instanton, apart from the gauge fields mentioned in the last paragraph, are just the modes that describe the motion of the D-instanton. So the moduli space \mathcal{M} of D-instanton configurations parameterizes holomorphic curves $C \subset \mathbb{PT}$ endowed with a holomorphic line bundle (or a holomorphic vector bundle in the situation considered in the last paragraph). C may have several disconnected components (possibly with different multiplicities) as in some examples encountered in section 3.

To construct scattering amplitudes, we need, roughly speaking, to integrate over \mathcal{M} . But in the topological B model, as we recalled in section 4.2, the action is a holomorphic function of complex fields, and all path integrals are contour integrals. Thus, the integral will really be taken over a contour in \mathcal{M} , that is, a middle-dimensional real cycle. To be able to integrate over such a contour, \mathcal{M} must be endowed with a holomorphic measure Υ . For an ordinary complex manifold of dimension n, a holomorphic measure would be a holomorphic n-form; for a supermanifold, a holomorphic measure is a holomorphic section of the Berezinian of the tangent bundle.

We will give two choices of contour in \mathcal{M} , using a method explained in section 4.2. We pick a real structure τ on twistor space \mathbb{PT} , that is, a \mathbb{Z}_2 symmetry of \mathbb{PT} that reverses its complex structure.¹⁶ τ automatically acts on \mathcal{M} , and we take the integration contour (for the bosons in \mathcal{M}) to consist of the fixed points of τ .

There are two possible choices of τ . We can define τ to simply act by complex conjugation on each of the homogeneous coordinates Z^I of \mathbb{PT} . This choice is natural for signature --++ in Minkowski spacetime, where the Z^I can all be real. This is the choice we will make in computing MHV amplitudes, because the definition of twistor amplitudes is simplest for that signature. Alternatively, we can consider the symmetry $\widehat{\tau}(Z^1, Z^2, Z^3, Z^4) = (\overline{Z}^2, -\overline{Z}^1, \overline{Z}^4, -\overline{Z}^3)$. This choice is natural for studying Yang-Mills

¹⁶ Since we only need to determine the homology class of the contour, it is enough to have a symmetry that reverses the complex structure of the bosonic reduction of supertwistor space.

theory in signature ++++, because, in acting on complexified Minkowski spacetime \mathbb{M} , $\widehat{\tau}$ leaves fixed a real slice that has Euclidean signature. (Unfortunately, I do not know how to pick a contour that is naturally adapted to Lorentz signature in spacetime.) One hopes that the theories constructed using τ or $\widehat{\tau}$ to pick the contour are equivalent, but it is not clear how to prove this.

Construction Of The Measure

How can we construct the holomorphic measure Υ on the moduli space \mathcal{M} of D-instantons? In the topological B model with target space an ordinary (bosonic) Calabi-Yau manifold, such a measure arises from the determinant of the massless fields on the D-brane (whose zero modes are the moduli). I do not know technically how to do this when the target space is a Calabi-Yau supermanifold, so I will just construct the measure by hand for D-instantons of genus zero and arbitrary degree. As we will see, for these cases \mathcal{M} is a Calabi-Yau supermanifold, and the measure is uniquely determined by holomorphy, up to a multiplicative constant. The choice of this constant for degree one determines the Yang-Mills coupling constant, and the normalization of the measure for higher degrees can be determined by factorization or unitarity. (The normalization given below is presumably compatible with factorization, though this will not be proved.)

To construct a genus zero curve of degree d, we let C_0 be a copy of \mathbb{CP}^1 with homogeneous coordinates (u^1, u^2) . Then we describe a holomorphic map $\Phi : C_0 \to \mathbb{CP}^{3|4}$ that maps the homogeneous coordinates (Z^I, θ^A) of $\mathbb{CP}^{3|4}$ to homogeneous polynomials of degree d in (u^1, u^2) :

$$Z^{I} = c^{I}_{i_{1}...i_{d}} u^{i_{1}} ... u^{i_{d}},$$

$$\psi^{A} = \beta^{A}_{i_{1}...i_{d}} u^{i_{1}} ... u^{i_{d}}.$$
(4.37)

The map Φ is determined by the coefficients $c^I_{i_1...i_d}$ and $\beta^A_{i_1...i_d}$, which are, of course, respectively bosonic and fermionic. The coefficients c and β parameterize a linear space $\mathbb{L} \cong \mathbb{C}^{4d+4|4d+4}$. On \mathbb{L} , there is a natural holomorphic measure,

$$\Upsilon_0 = \prod_{I,\{i_1,\dots,i_d\}} dc^I_{i_1\dots i_d} \prod_{A,\{i_1,\dots,i_d\}} d\beta^A_{i_1\dots i_d}.$$
 (4.38)

The space of maps Φ is parameterized by the c's and β 's modulo the scaling $(c,\beta) \to (tc,t\beta)$, $t \in \mathbb{C}^*$. The measure Υ_0 is invariant under this scaling, since c and β have the same number of coefficients. The space of maps is thus a Calabi-Yau supermanifold $\mathbb{PL} = \mathbb{CP}^{4d+3|4d+4}$.

We really are not interested in maps from C_0 to $\mathbb{CP}^{3|4}$ but in holomorphic curves in $\mathbb{CP}^{3|4}$. Two maps have as their images the same curve if and only if they differ by the action of $SL(2,\mathbb{C})$ on (u^1,u^2) . The moduli space \mathcal{M} of curves in $\mathbb{CP}^{3|4}$ of genus zero and degree d is thus $\mathcal{M} = \mathbb{PL}/SL(2,\mathbb{C})$. As Υ_0 is $SL(2,\mathbb{C})$ -invariant as well as being invariant under scaling, Υ_0 descends to an everywhere nonzero holomorphic volume form Υ on \mathcal{M} . Thus, \mathcal{M} is a Calabi-Yau supermanifold of dimension 4d|4d+4.

For a genus zero instanton of degree 1, Z^I and ψ^A are linear in u^1, u^2 . Writing as usual $Z = (\lambda, \mu)$, we can generically use the $SL(2, \mathbb{C})$ symmetry and scaling to put the map Φ in the form $\lambda^1 = u^1$, $\lambda^2 = u^2$, whereupon the other coordinates $\mu^{\dot{a}}, \psi^A$ become linear functions of λ . After renaming the coefficients, the curve takes the familiar form

$$\mu_{\dot{a}} + x_{a\dot{a}}\lambda^a = 0$$

$$\psi^A + \theta_a^A \lambda^a = 0.$$
(4.39)

The measure Υ reduces to the familiar measure $d^4x d^8\theta$ that we used in section 3.1.

The S Charge Of The D-Instanton Measure

We introduced D-instantons in the hope of finding a new source of violation of the quantum number S whose role in Yang-Mills perturbation theory we have discussed above. Now we can determine if this program has a chance to succeed.

Since the S-charges of (Z, ψ) are (0, 1), the charges of the cofficients (c, β) are likewise (0, 1). The differentials $(dc, d\beta)$ therefore have charges (0, -1), so the S-charge of Υ is -4d-4. So a genus zero instanton of degree d contributes to the effective action a term that violates the S-charge by this amount. This is exactly what we want. Since each negative helicity gluon has S = -4, while positive helicity gluons have S = 0, an amplitude with any number of positive helicity gluons and q gluons of negative helicity has S-charge -4q. So a connected genus zero instanton of degree d, with no other sources of S-charge violation, can contribute to such an amplitude if and only if d and q are related by

$$d = q - 1. (4.40)$$

We recognize the familiar formula (3.1) whose consequences were explored in section 3.

The result is the same for disconnected instantons, of a type that we really have already described in section 3. If we consider r D-instantons, all of genus zero, and of degree d_i , with $d = \sum_i d_i$, then their total S-charge is $\sum_{i=1}^r (-4d_i - 4) = -4d - 4r$. However, for such a configuration to contribute to a connected amplitude, the D-instantons must be

connected by fields propagating in twistor space, as represented by the "internal line" in figure 3. Each such internal line increases the S-charge by +4 (the propagator is the inverse of the kinetic energy, which has charge -4). To make a connected configuration without loops, which we regard as a degenerate case of a configuration of genus zero, the number of internal lines must be r-1, whereupon the total S-charge violation in the amplitude is -4d-4r+4(r-1)=-4d-4, as expected.

To extend the agreement with (3.1) to higher genus, we would like the S-charge of the measure for a connected D-instanton of genus g to be

$$\Delta S = -4(d+1-q). {(4.41)}$$

Though we will not prove this rigorously by properly understanding the appropriate world-volume determinants of the D-instantons, one can give a heuristic explanation of where the formula comes from. For a generic curve C of genus g and degree d in \mathbb{CP}^3 , one expects from the index theorem that $H^0(C, \mathcal{O}(1))$ will be of dimension d+1-g. (Here $\mathcal{O}(1)$ is the usual line bundle over \mathbb{CP}^3 .) The four ψ^A are each sections of $\mathcal{O}(1)$, so C has a total of 4(d+1-g) fermionic moduli, all of S-charge 1, leading to the formula (4.41) for the S-charge of the D-instanton measure. For disconnected D-instantons of any genus, connected by internal lines, the agreement is preserved because of arguments similar to those in the last paragraph.

D1 - D5 Strings

The key ingredient in computing scattering amplitudes, as we will see presently in computing the MHV amplitudes, is the effective action of the D1 - D5 strings.

We consider a D1-brane C located at $\psi^A = 0$; the ψ -dependence will be restored when we integrate over moduli of C. The D5-branes are of course the usual stack of N (almost) space-filling branes. In quantizing the D1 - D5 strings, ψ and its bosonic partners and the bosons and fermions normal to C in \mathbb{CP}^3 have no zero modes, since they obey Dirichlet boundary conditions at one end of the string and Neumann boundary conditions at the other end. Bosons and fermions tangent to C do have zero modes; their quantization leads in the usual way for the B model to the space of (0,q)-forms on C, where in the present problem (as C is of complex dimension one), q = 0, 1. The derivation of this is rather similar to the quantization of D5 - D5 strings, which we briefly explained in section 4.2.

The D1-D5 strings are thus (0,q)-forms α on C, with values in $E_C \otimes \mathcal{L}$, where E_C is the D5-brane gauge bundle restricted to C and \mathcal{L} is a line bundle on C that depends

on the U(1) Chan-Paton gauge field on C. The D5-D1 strings are similarly (0,q)forms β on C, but now with values in $E_C^* \otimes \mathcal{L}'$, where E_C^* is the dual bundle to E_C ,
and \mathcal{L}' is another line bundle. (Dual bundles E_C and E_C^* appear here because D5-D1strings transform in the antifundamental representation of the D5-brane gauge group,
while D1-D5 strings transform in the fundamental representation.) When we want to
make manifest the $GL(N,\mathbb{C})$ quantum numbers of α and β (or the fact that they take
values in E_C and E_C^* , respectively), we write them as α^x , β_x , $x = 1, \ldots, N$. The kinetic
operator for topological strings is the BRST operator Q, which when we reduce to the low
energy modes is the $\overline{\partial}$ operator, or its covariant version \overline{D} to include a background field \mathcal{A} . The effective action for the D1-D5 strings is thus

$$I_{D1-D5} = \int_C dz \, \beta_x \overline{D} \alpha^x. \tag{4.42}$$

Here z is an arbitrary local complex parameter on C. We have incorporated a possible background gauge field \mathcal{A} (which will represent initial and final particles in a scattering amplitude) by using the covariant $\overline{\partial}$ operator $\overline{D} = d\overline{z}(\partial_{\overline{z}} + \mathcal{A}_{\overline{z}})$, where $\mathcal{A}_{\overline{z}}$ is the component of \mathcal{A} along C. For the action to make sense, it must be that $\mathcal{L} \otimes \mathcal{L}' \cong K$, where K is the canonical line bundle of C. (This result should ideally be explained more directly by more carefully quantizing the zero modes.) All choices of \mathcal{L} are allowed, depending on the choice of gauge field on C.

Only the (0,0)-form components of α and β actually appear in this Lagrangian. The (0,1)-form components may possibly play some role in understanding c-number contributions to the measure (at some deeper level that we will not reach in this paper), but they do not couple to the background field A. For the rest of this paper, therefore, we simply take α and β to be (0,0)-forms.

The coupling of A to the D1-D5 strings can be read off from (4.42). It is

$$\Delta I = \int_C \text{Tr } J \mathcal{A}_{\overline{z}} d\overline{z}, \qquad (4.43)$$

where we define $J^x{}_y = \alpha^x \beta_y dz$; we include the factor of dz in the current and interpret J as a (1,0)-form on C that (because of the way α and β transform under a change in local parameter) is independent of the choice of z. J takes values in the Lie algebra of $GL(N,\mathbb{C})$ (acting as endomorphisms of E), and the trace in (4.43) is taken over this Lie algebra.

The model seems to make more sense if we assume that the D1-D5 string fields α and β are fermions. Under appropriate conditions, α and β will have zero modes. If α and

 β are bosons, the zero modes will lead to flat directions which by analogy with phenomena in critical string theory [67,68] will represent the deformation of the D1-brane into a smooth holomorphic bundle on $\mathbb{CP}^{3|4}$ with second Chern class nonzero (and Poincaré dual to C). By the twistor transform of the anti-selfdual Yang-Mills equations, such bundles correspond to instantons in spacetime and thus to nonperturbative contributions in the Yang-Mills theory. However, the D1-branes do not couple to spacetime fields like Yang-Mills instantons; rather, we will argue in section 4.7 that they contribute to perturbative scattering amplitudes.

If α and β are fermions, there is no contradiction, as we would not expect to relate the D1-brane to a spacetime instanton. In the computation that we actually perform in section 4.7, however, the statistics of α and β only affect the overall sign of the single-trace interaction; our computation is not precise enough to determine this sign. In any event, for whatever it is worth, the action (4.42) is more natural for fermions.

In quantizing the D1-branes, one must sum and integrate over the choice of line bundle \mathcal{L} . However, unless \mathcal{L} has degree g-1, where g is the genus of C, there is a non-trivial index because of which α or β have zero modes that are not lifted by the coupling to the external gauge field \mathcal{A} . \mathcal{L} 's of degree other than g-1 hence will not contribute. In the specific computation that we will perform presently, C has genus 0, so we take \mathcal{L} to have degree -1. In genus 0, \mathcal{L} has no moduli. The coupling to \mathcal{L} just means that the fields α and β are ordinary chiral fermions of spin (1/2,0), which is how we will interpret them in section 4.7.

4.7. Computation Of MHV Amplitudes

Now let us discuss how to use D-instantons in twistor space to actually compute a scattering amplitude in spacetime.

We will consider an n-particle scattering amplitude. The i^{th} external particle, for i = 1, ..., n, is represented by a wavefunction that is a $\overline{\partial}$ -closed (0, 1)-form w_i on \mathbb{PT}' (the part of super twistor space $\mathbb{CP}^{3|4}$ with $\lambda \neq 0$). Each w_i takes values in the Lie algebra of $GL(N,\mathbb{C})$ (the gauge group carried by the D5-branes), and so represents a cohomology class that takes values in the tensor product with this Lie algebra of the twistor space cohomology group $H^1(\mathbb{PT}', \mathcal{O})$.

The coupling of w_i to a *D*-instanton wrapped on a Riemann surface *C* is according to (4.43)

$$B_i = \int_C \text{Tr } J \wedge w_i. \tag{4.44}$$

This is found by simply regarding w_i as a contribution to the external gauge field \mathcal{A} in (4.43).

If C had no moduli, its contribution to the scattering amplitude for n particles coupling via B_1, \ldots, B_n would be found by evaluating the corresponding expectation value $\langle B_1 \ldots B_n \rangle$ in the D-instanton worldvolume theory. Concretely, this would be done by integrating over the fields α and β . In actual examples, C is a point in a moduli space \mathcal{M} of holomorphic curves in supertwistor space. We must pick a real cycle $\mathcal{M}_{\mathbb{R}}$ in \mathcal{M} and integrate over it using the holomorphic measure Υ . The scattering amplitude with the given external wavefunctions w_i is consequently

$$A(w_i) = \int_{\mathcal{M}_{\mathbb{P}}} \Upsilon \langle B_1 \dots B_n \rangle. \tag{4.45}$$

Actually, to get the proper power of the Yang-Mills coupling g multiplying a scattering amplitude and a possible multiplicative constant, we need to also include a few additional factors: normalization factors for external wavefunctions and a factor of e^{-I} , with I the D-instanton action. We will omit these factors.

We will now show how to use this formalism to recover the supersymmetric tree level MHV amplitudes, as described in twistor space in eqn. (3.10). The ability to recover these amplitudes gives our most detailed evidence that the B model of $\mathbb{CP}^{3|4}$ is equivalent at least in the planar limit to $\mathcal{N}=4$ super Yang-Mills amplitudes.

For tree level MHV amplitudes, we take C to be a straight line, that is a curve of genus zero and degree one. We recall that the lines in supertwistor space are described by the equations

$$\mu_{\dot{a}} + x_{a\dot{a}}\lambda^{a} = 0$$

$$\psi^{A} + \theta_{a}^{A}\lambda^{a} = 0.$$
(4.46)

Here $x^{a\dot{a}}$ and θ^{aA} are the moduli of C. The measure is the usual superspace measure $\Upsilon = d^4x d^8\theta$. We will use the real slice that is natural for signature ++-- in spacetime; a point in \mathbb{CP}^3 is considered real if λ and μ are real, and the real slice of \mathcal{M} is defined by simply saying that $x^{a\dot{a}}$ is real for $a, \dot{a} = 1, 2$.

The scattering amplitude is therefore

$$A(w_i) = \int d^4x \, d^8\theta \, \langle B_1 \dots B_n \rangle. \tag{4.47}$$

Let us assume that the wavefunctions w_i take the form $w_i = v_i T_i$, where T_i is an element of the Lie algebra of $GL(N, \mathbb{C})$, and v_i is an ordinary (not matrix-valued)

(0,1)-form. The amplitude has a term proportional to $I = \operatorname{Tr} T_1 T_2 \dots T_n$. Let us extract this term. We have to compute the appropriate term in the expectation value of a product of currents $\langle \operatorname{Tr} T_1 J(\lambda_1) \operatorname{Tr} T_2 J(\lambda_2) \dots \operatorname{Tr} T_n J(\lambda_n) \rangle$, or essentially $\langle \operatorname{Tr} T_1 \alpha \beta(\lambda_1) \operatorname{Tr} T_2 \alpha \beta(\lambda_2) \dots \operatorname{Tr} T_n \alpha \beta(\lambda_n) \rangle$. The term proportional to $\operatorname{Tr} T_1 T_2 \dots T_n$ arises from contracting $\beta(\lambda_i)$ with $\alpha(\lambda_{i+1})$ for $i=1,\ldots,n$. The computation is done with free fields on $C=\mathbb{CP}^1$; the result is a function only of the λ_i , since the equations (4.46) that characterize C let us express the other variables in terms of the λ_i . In fact, for doing this computation, we can just think of C as a copy of \mathbb{CP}^1 with homogeneous coordinates λ . The result of computing the free field correlation function is that the desired part of $\langle J(\lambda_1) J(\lambda_2) \dots J(\lambda_n) \rangle$ is

$$\prod_{i=1}^{n} \epsilon_{ab} \lambda_i^a d\lambda_i^b \prod_{i=1}^{n} \frac{1}{\langle \lambda_{i+1}, \lambda_i \rangle}.$$
 (4.48)

This expression is completely determined by the following properties: it is homogeneous of degree zero in each λ_i (so it makes sense), it a (1,0)-form in each variable λ_i (because each current $J(\lambda_i)$ is a (1,0)-form), it is $SL(2,\mathbb{C})$ -invariant, and it has a simple pole at $\lambda_{i+1} = \lambda_i$ because of the contraction of $\beta(\lambda_i)$ with $\alpha(\lambda_{i+1})$. Perhaps the formula (4.48) is more familiar if written in terms of $z_i = \lambda_i^2/\lambda_i^1$. It then takes the form

$$dz_1 dz_2 \dots dz_n \prod_{i=1}^n \frac{1}{z_{i+1} - z_i},$$
(4.49)

where $1/(z_{i+1}-z_i)$ is the usual free-fermion propagator on the complex z-plane. One can calculate this readily by using homogeneity in the λ_i to set $\lambda_i = (1, z_i)$ for all i, whereupon

$$\epsilon_{ab}\lambda^a d\lambda^b = dz$$

$$\langle \lambda_i, \lambda_{i+1} \rangle = z_{i+1} - z_i.$$
(4.50)

The scattering amplitude is thus (with the gauge theory trace Tr $T_1 \dots T_n$ suppressed, as usual)

$$A(v_i) = \int d^4x \, d^8\theta \prod_{i=1}^n \int_C v_i(\lambda_i^a, \mu_i^{\dot{a}}, \psi_i^A) \epsilon_{ab} \lambda_i^a d\lambda_i^b \prod_{i=1}^n \frac{1}{\langle \lambda_i, \lambda_{i+1} \rangle}. \tag{4.51}$$

Here μ_i and ψ_i are functions of λ_i , x, and θ (obeying (4.46)), since v is evaluated on C. What is integrated over C is a (1,1)-form in each variable, since v_i is a (0,1)-form and $\epsilon_{ab}\lambda^a d\lambda^b$ is a (1,0)-form. Clearly, this result is closely related to the desired answer of eqn. (3.10). To finish the derivation, we need to convert the formula from the language of $\overline{\partial}$ cohomology to a formalism more like that used in section 3.

This can be done using the link between $\overline{\partial}$ cohomology and Cech cocycles that is explained at the end of the appendix. In order to carry out the calculation, we again use the homogeneity of the twistor space variables to set $\lambda^1 = 1$ for each particle. We write z for λ^2/λ^1 , as we already did in (4.49), and leave unchanged the names of the rest of the twistor coordinates $\mu^{\dot{a}}$, ψ^A . The homogeneity can be restored at the end of the computation, if one wishes, by multiplying by suitable powers of λ^1 and reversing the steps in (4.50).

As in the derivation of eqn. A.21, we write $z = \sigma + i\tau$, with σ and τ real. We saw in eqn. A.23 that we can pick the external wavefunctions to be

$$v_k = \frac{i}{2} f_k \, d\overline{z}_k \delta(\tau_k), \tag{4.52}$$

where f_k is a holomorphic function (whose singularities are far away from $\tau_k = 0$). Upon inserting this in (4.51), writing $(i/2)dz \wedge d\overline{z} = d\sigma \wedge d\tau$, and doing the τ integrals with the help of the delta functions, we get

$$A(f_i) = \int d^4x \, d^8\theta \prod_{i=1}^n \int_{-\infty}^{\infty} d\sigma_i f_i(\sigma_i, \mu_i^a, \psi_i^A) \prod_{i=1}^n \frac{1}{\sigma_{i+1} - \sigma_i}.$$
 (4.53)

Again, μ_i and ψ_i are functions of σ_i , x, and θ in such a way that the integral runs on the curve C. We can make this explicit:

$$A(f_i) = \prod_{i=1}^{n} \int_{-\infty}^{\infty} d\sigma_i \, d^2 \mu_i^{\dot{a}} \, d^4 \psi_i^A f_i(\sigma_i, \mu_i^{\dot{a}}, \psi_i^A) \widetilde{A}(\sigma_i, \mu_i^{\dot{a}}, \psi_i^A), \tag{4.54}$$

where

$$\widetilde{A}(\sigma_i, \mu_{i\dot{a}}, \psi_i^A) = \int d^4x \, d^8\theta \prod_{i=1}^n \delta^2(\mu_{i\dot{a}} + x_{a\dot{a}}\lambda_i^a) \delta^4(\psi_i^A + \theta_a^A\lambda_i^a) \prod_{i=1}^n \frac{1}{\sigma_{i+1} - \sigma_i}.$$
 (4.55)

The integral in (4.54) is carried out over real twistor space – that is, the integration variables σ_i and μ_i are all real. In \widetilde{A} , we recognize the MHV tree level scattering amplitude of eqn. (3.10), written (with the help of (4.50)) in a coordinate system with $\lambda_i^1 = 1$. The integral in (4.54) is the pairing (described in sections 2.5 and 2.6) by which one integrates over a copy of twistor space for each initial and final particle to go from \widetilde{A} to a scattering amplitude with specified initial and final states.

In our derivation, the integral $d\sigma d^2\mu$ over real twistor space \mathbb{RP}^3 arose in two steps: z became real because of the particular choice of external wavefunctions, and μ became real because, for curves of degree one, with our choice of real slice $\mathcal{M}_{\mathbb{R}}$, z being real leads to μ being real.

The attentive reader might ask why we need not include additional contributions where two external particles join in twistor space (to couple to a quantum A-field that then propagates to the D-instanton), using the $A \wedge A \wedge A$ term in the twistor space effective action. With our gauge choice, this does not occur because the wedge products of the wavefunctions in (4.52) all vanish, as those wavefunctions are all proportional to $d\overline{z}$.

We have obtained the tree level MHV amplitudes in terms of correlation functions of chiral currents on \mathbb{CP}^1 , as suggested by Nair [12]. In Nair's paper, this is an abstract \mathbb{CP}^1 , while in our framework, it is a curve in twistor space. Correlators of chiral currents are what one often gets from heterotic string worldsheet instantons, but we have obtained them from D-instantons.

5. Further Issues

Here we will take a brief survey of a few further issues.

5.1. Closed Strings

The most serious outstanding issue may be to understand the closed strings. In the topological B model in general, closed string modes describe deformations of the complex structure of the target space. In the present problem, the target space is supertwistor space \mathbb{PT}' . Deformations of the complex structure of twistor space describe – according to the original application of twistor theory to nonlinear problems [33] – conformally anti-selfdual deformations in the geometry of Minkowski spacetime. In the case of supertwistor space, one would presumably get some sort of chiral limit (analogous to the GF theory studied in section 4.4 for open strings) of $\mathcal{N}=4$ conformal supergravity, perhaps extended to a more standard theory with the aid of D-instanton contributions. (For some reviews of conformal supergravity, see [69,70].) This remains to be properly understood.

The holomorphic anomaly of the B model [57], which usually obstructs the background independence of the closed string sector of the B model, presents a conundrum. As the closed strings in this problem presumably describe gravitational fluctuations in spacetime, we need to maintain the background independence. Possibly, the anomalous S symmetry,

which eliminates most string loop effects, avoids the holomorphic anomaly in the present context.

There actually is a sign of closed string contributions in the calculation of tree level MHV amplitudes in section 4.7. There, we extracted a single-trace interaction, and found it to agree with the standard tree-level result of Yang-Mills theory. However, the underlying formula (4.47) for the amplitude also gives rise to multi-trace interactions. Where can they come from? The most likely explanation is that they arise from the exchange of closed string states that, being singlets of the $GL(N,\mathbb{C})$ gauge group, naturally produce multi-trace interactions.

To support this idea, we will analyze the four-gluon multi-trace interactions that arise from (4.47). In doing so, we only consider gluons in the $SL(N,\mathbb{C})$ subgroup of $GL(N,\mathbb{C})$; the gluons that gauge the center of $GL(N,\mathbb{C})$ are likely to mix with closed string modes (by analogy with a familiar mechanism for the usual critical string theories), and one would not expect to be able to understand the resulting scattering amplitudes without understanding this mixing. This being so, we assume that Tr $T_i = 0$ for all i. This only allows, up to a permutation of the gluons, one possible group theory factor in a multi-trace four-gluon amplitude; we can assume the group theory factor to be Tr T_1T_2 Tr T_3T_4 . There are two essentially different cases: the helicities may be ++-- or +-+-. Other cases are related to these by the obvious permutation symmetries (exchanging 1 with 2, 3 with 4, or 1,2 with 3,4).

The momenta of the four gluons are denoted as usual $p_i^{a\dot{a}} = \lambda_i^a \tilde{\lambda}_i^{\dot{a}}$. We consider first the ++-- amplitude. The amplitude extracted from (4.47) is

$$A = (2\pi)^4 \delta^4 \left(\sum_i p_i\right) \operatorname{Tr} T_1 T_2 \operatorname{Tr} T_3 T_4 \langle \lambda_3, \lambda_4 \rangle^4 \frac{1}{\langle \lambda_1, \lambda_2 \rangle^2 \langle \lambda_3, \lambda_4 \rangle^2}.$$
 (5.1)

In contrast to our usual practice, we have written the group theory factor, since it is unusual. This amplitude is conformally invariant, by the same analysis as in section 2.4. The derivation of (5.1) goes as follows. The factor $1/\langle \lambda_1, \lambda_2 \rangle^2 \langle \lambda_3, \lambda_4 \rangle^2$ comes from the current correlation function that is needed to get a group theory factor $\operatorname{Tr} T_1 T_2 \operatorname{Tr} T_3 T_4$. (The relevant contribution to $\langle \operatorname{Tr} T_1 J(\lambda_1) \operatorname{Tr} T_2 J(\lambda_2) \dots \operatorname{Tr} T_4 J(\lambda_4) \rangle$ is the disconnected piece $\langle \operatorname{Tr} T_1 J(\lambda_1) \operatorname{Tr} T_2 J(\lambda_2) \rangle \langle \operatorname{Tr} T_3 J(\lambda_3) \operatorname{Tr} T_4 J(\lambda_4) \rangle$, leading to a double contraction in both the λ_1 - λ_2 and λ_3 - λ_4 channels.) The factor $\langle \lambda_3, \lambda_4 \rangle^4$ in the numerator (which is also present in the numerator of the conventional single-trace MHV amplitude (2.14), where it arises in the same way) is one of the terms that comes from the $d^8\theta$ integral. We simply

picked the term associated with the helicity configuration ++--. If we let $k=p_1+p_2$, and observe that $k^2=(p_1+p_2)^2=2p_1\cdot p_2=2\langle\lambda_1,\lambda_2\rangle[\widetilde{\lambda}_1,\widetilde{\lambda}_2]$, we can write

$$A = 4(2\pi)^2 \delta^4(\sum_i p_i) \operatorname{Tr} T_1 T_2 \operatorname{Tr} T_3 T_4 \frac{[\widetilde{\lambda}_1, \widetilde{\lambda}_2]^2 \langle \lambda_3, \lambda_4 \rangle^2}{(k^2)^2}.$$
 (5.2)

We can reproduce this amplitude from tree level exchange of a scalar field ϕ with a propagator $1/(k^2)^2$ (as expected for a scalar field in conformal supergravity, which is a nonunitary theory with higher derivatives) and a coupling ϕ Tr $F_{\mu\nu}F^{\mu\nu}$. Indeed, the matrix element of Tr $F_{\mu\nu}F^{\mu\nu}$ to create two photons of momentum p_1, p_2 and + helicity is $[\tilde{\lambda}_1, \tilde{\lambda}_2]^2$, while the matrix element of the same operator to create two photons of momentum p_3, p_4 and - helicity is $\langle \lambda_3, \lambda_4 \rangle^2$. (We do not need to include scalar exchange in crossed channels, as this produces other group theory factors.) Actually, to avoid generating + + ++ and - - amplitudes that are not present in (4.47), we need a slight elaboration of this mechanism: two scalars ϕ_+ and ϕ_- , with couplings ϕ_+ Tr $(F^+)^2$, ϕ_- Tr $(F^-)^2$ to gluons of one helicity or the other, and a purely off-diagonal propagator $\langle \phi_+\phi_+\rangle = \langle \phi_-\phi_-\rangle = 0$, $\langle \phi_+\phi_-\rangle = 1/k^4$.

The +-+- amplitude can be understood similarly in terms of graviton exchange. The amplitude is read off from (4.47) to be

$$A = (2\pi)^4 \delta^4 \left(\sum_i p_i\right) \operatorname{Tr} T_1 T_2 \operatorname{Tr} T_3 T_4 \langle \lambda_2, \lambda_4 \rangle^4 \frac{1}{\langle \lambda_1, \lambda_2 \rangle^2 \langle \lambda_3, \lambda_4 \rangle^2}.$$
 (5.3)

Using identities such as (2.19), this can be rewritten

$$A = (2\pi)^4 \delta^4 \left(\sum_i p_i\right) \operatorname{Tr} T_1 T_2 \operatorname{Tr} T_3 T_4 \frac{\left[\widetilde{\lambda}_1, \widetilde{\lambda}_3\right]^2 \langle \lambda_2, \lambda_4 \rangle^2}{\langle \lambda_1, \lambda_2 \rangle^2 \left[\widetilde{\lambda}_1, \widetilde{\lambda}_2\right]^2}$$

$$= 4(2\pi)^4 \delta^4 \left(\sum_i p_i\right) \operatorname{Tr} T_1 T_2 \operatorname{Tr} T_3 T_4 \frac{\left[\widetilde{\lambda}_1, \widetilde{\lambda}_3\right]^2 \langle \lambda_2, \lambda_4 \rangle^2}{(k^2)^2}.$$
(5.4)

We now consider a traceless metric fluctuation $h_{\mu\nu}$ that in spinor language is written $h_{ab\dot{a}\dot{b}}$ (symmetric in a, b and in \dot{a}, \dot{b}) with propagator

$$\langle h_{ab\dot{a}\dot{b}}h_{cd\dot{c}\dot{d}}\rangle = \frac{1}{4} \frac{(\epsilon_{ac}\epsilon_{bd} + \epsilon_{ad}\epsilon_{bc})(\epsilon_{\dot{a}\dot{c}}\epsilon_{\dot{b}\dot{d}} + \epsilon_{\dot{a}\dot{d}}\epsilon_{\dot{b}\dot{c}})}{(k^2)^2}.$$
 (5.5)

We assume that h couples to gluons via a coupling $h^{ab\dot{a}\dot{b}}T_{ab\dot{a}\dot{b}}$, where T is the stress tensor. The matrix element of $T_{ab\dot{a}\dot{b}}$ to create two gluons of momenta p_1, p_2 and helicities +, - is $\tilde{\lambda}_{1\dot{a}}\tilde{\lambda}_{1,\dot{b}}\lambda_{2a}\lambda_{2b}$, and similarly with 1, 2 replaced by 3, 4. Combining this matrix element with the propagator in (5.5), we recover the amplitude (5.4).

The tentative conclusion is that the B model of $\mathbb{CP}^{3|4}$ has a closed string sector which describes some sort of $\mathcal{N}=4$ conformal supergravity. If so, this B model describes $\mathcal{N}=4$ Yang-Mills theory only for planar amplitudes, in which the closed strings decouple.

Consideration of anomalies raises numerous puzzles that will not be addressed here. The world-volume determinants of D1-D5 and D1-D1 strings appear potentially anomalous; anomaly cancellation may well involve contributions of closed strings, as in the Green-Schwarz mechanism of heterotic and Type I anomaly cancellation. The c-number conformal anomaly of $\mathcal{N}=4$ super Yang-Mills theory raises the question of how it could possibly be coupled to any version of conformal supergravity. Perhaps there is more to the story.

5.2. Yangian Symmetry

The planar limit of $\mathcal{N}=4$ super Yang-Mills theory seems to have an extended infinite-dimensional symmetry group that can be described as Yangian symmetry. This result was first found in strong coupling in [71] and has also been found for weak coupling [72]. We therefore should look for such symmetry in the present framework.

Along with many two-dimensional models [73], the two-dimensional sigma model with target space \mathbb{CP}^{M-1} has nonlocal symmetries that generate a Yang-Baxter or Yangian algebra, as investigated in [74]. However (in contrast to similar models in which the target space is, for example, a sphere), the quantum version of this model is believed to not be integrable [75]; presumably, the nonlocal symmetries are anomalous, as the local ones appear to be [76].

The supersymmetric \mathbb{CP}^{M-1} model also has Yangian symmetry classically. Quantum mechanically, it is believed to be integrable with a factorizable S-matrix, and anomaly-free Yangian symmetry [77]. Granted this, Yangian symmetry will also hold for $\mathbb{CP}^{M-1|P}$, as the anomalies generated by Feynman diagrams really only depend on M-P. If we set M-P=0, even more anomalies (such as the beta function) cancel. So the $\mathbb{CP}^{3|4}$ model can be expected to have Yangian symmetry at the quantum level.

The Yangian symmetry, like the more obvious PSU(4|4) symmetry, commutes with spacetime supersymmetry. It also has no anomaly with the U(1) R-symmetry current by which we "twist" to make the topological B model. So Yangian symmetry is expected in the B model of $\mathbb{CP}^{3|4}$.

5.3. Other Target Spaces

What models can we make by replacing $\mathbb{CP}^{3|4}$ with another target space?

We can certainly replace \mathbb{CP}^3 , which is the twistor space of Minkowski space, by the twistor space of a more general conformally anti-selfdual four-dimensional spacetime X. (This twistor space is the space of null selfdual complex planes in X, generalizing the α -planes introduced in the appendix.) The topological B model of this twistor space (or rather of its extension with $\mathcal{N}=4$ supersymmetry) will describe $\mathcal{N}=4$ super Yang-Mills theory on X, in the same sense that the topological B model of $\mathbb{CP}^{3|4}$ describes $\mathcal{N}=4$ super Yang-Mills theory in Minkowski space.

A more interesting generalization is to consider the weighted projective space $\mathbb{W} = \mathbb{WCP}^{3|2}(1,1,1,1|1,3)$. This is the projective space with four bosonic homogeneous coordinates Z^I , $I=1,\ldots,4$, of weight one, and two fermionic homogeneous coordinates ψ,χ , of weights one and three. The homogeneous coordinates are subject to the equivalence relation $(Z^I,\psi,\chi)\cong (tZ^I,t\psi,t^3\chi)$, for $t\in\mathbb{C}^*$. \mathbb{W} is a Calabi-Yau supermanifold because the sum of bosonic weights equals the sum of fermionic weights. The holomorphic measure $\Omega_0=dZ^1\ldots dZ^4d\psi d\chi$ is invariant under \mathbb{C}^* , and descends to a holomorphic measure Ω on \mathbb{W} , ensuring that one can define a topological B model with this target space. The supermanifold \mathbb{W} admits $\mathcal{N}=1$ superconformal symmetry SU(4|1), acting on Z^I and ψ . In the presence of N (almost) space-filling D5-branes like those studied in this paper, the spectrum of the model is the U(N) vector multiplet with $\mathcal{N}=1$ supersymmetry, as one can verify by repeating the analysis of section 4.3 for this case.

However, the topological B model with target \mathbb{W} cannot reproduce $\mathcal{N}=1$ super Yang-Mills theory, as one would have hoped, because it has too much symmetry. To have any hope of reproducing $\mathcal{N}=1$ super Yang-Mills theory, one would have to modify the model to deal with two problems: (A) In this model, the SU(4|1) symmetry will persist quantum mechanically, while in $\mathcal{N}=1$ super Yang-Mills symmetry, there is a conformal anomaly that breaks SU(4|1) to a subgroup. (B) The B model with target \mathbb{W} has additional symmetries $\delta\chi=P_3(Z,\psi)$, with P_3 a homogeneous polynomial of degree three. These have no analog in $\mathcal{N}=1$ super Yang-Mills theory.

 $The\ Quadric$

One other possible model is worth mentioning here. First of all, if \mathbb{A} is a copy of \mathbb{CP}^{M-1} with homogeneous coordinates Z^I , then there is a natural "dual" projective space \mathbb{B} whose points parameterize hyperplanes in \mathbb{A} . The equation of a hyperplane is

$$\sum_{i=1}^{N} W_I Z^I = 0, (5.6)$$

for some constants W_I , not all zero. Moreover, an overall scaling of the W_I would give the same hyperplane. So we take W_I as homogeneous coordinates for \mathbb{B} . The relation between \mathbb{A} and \mathbb{B} is clearly symmetric: \mathbb{B} parameterizes hyperplanes in \mathbb{A} , and vice-versa. We can regard the equation (5.6) in one more way: its zero set defines a "quadric" \mathbb{Q} in the product $\mathbb{A} \times \mathbb{B}$.

Now let \mathbb{A} be the complex supermanifold $\mathbb{CP}^{3|3}$, with homogeneous coordinates Z^I , ψ^A , $I = 1, \ldots, 4, A = 1, \ldots, 3$. Let \mathbb{B} be the dual projective supermanifold $\mathbb{CP}^{3|3}$, parametrizing hyperplanes in \mathbb{A} . We write W_I , χ_A , for the homogeneous coordinates of \mathbb{B} . The equation via which \mathbb{B} parameterizes hyperplanes in \mathbb{A} , and vice-versa, is

$$\sum_{I=1}^{4} Z^{I} W_{I} + \sum_{A=1}^{3} \psi^{A} \chi_{A} = 0.$$
 (5.7)

The zero set of this equation is a quadric \mathbb{Q} in $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$.

A and \mathbb{B} are not Calabi-Yau supermanifolds, but \mathbb{Q} is one. (This is so because the first Chern class of $\mathbb{A} \times \mathbb{B}$ is (1,1), which is also the degree of the equation defining \mathbb{Q} .) The topological B model with target \mathbb{Q} therefore exists, and should describe a theory with symmetry group containing SU(4|3), which is the symmetry group of \mathbb{Q} . The only evident four-dimensional field theory with symmetry SU(4|3) is $\mathcal{N}=4$ super Yang-Mills theory, which has the larger symmetry PSU(4|4).

By an analog of the twistor transform [10], a holomorphic vector bundle on a suitable region of \mathbb{Q} corresponds to a solution of the equations of $\mathcal{N}=4$ super Yang-Mills theory on a suitable region of complexified and compactified Minkowski spacetime \mathbb{M} (in a description in which only SU(4|3) is manifest). Essentially the same construction was also obtained in a bosonic language [11]. The equations that arise here are the full Yang-Mills equations, not the selfdual or anti-selfdual version. The intuition behind the construction was that the dependence on Z encodes the gauge fields of one helicity, and the dependence on W encodes the other.

It is therefore plausible that the topological B model of \mathbb{Q} might give another construction of $\mathcal{N}=4$ super Yang-Mills theory. In this model, no D-instanton contributions would be needed, and the mechanism by which perturbative Yang-Mills theory would be reproduced would be completely different from what it is in the case of $\mathbb{CP}^{3|4}$. The main difficulty in making sense of this idea seems to be that it is hard to understand the right measure for the bosonic and fermionic zero modes on a space-filling D-brane on Q. A somewhat similar problem was treated recently by Movshev and Schwarz [56], who showed how to construct an "integral form" that enables one to define a suitable Chern-Simons action on certain complex supermanifolds that are related to super Yang-Mills theory in roughly the same way that \mathbb{Q} is. Their motivation was in part to understand the covariant quantization of the Green-Schwarz superparticle and superstring via pure spinors [78], which in some ways is a cousin of twistor constructions. Many of their examples have nonzero first Chern class and hence no topological B model, but their method of construction of the measure may be relevant to understanding the topological B model of \mathbb{Q} .

Appendix A. A Mini-Introduction To Twistor Theory

Though there are numerous introductions to twistor theory [28,15,29-32], and its applications to Yang-Mills fields [36], we will here offer a mini-introduction to a few facets of the subject, with the aim of making the present paper more accessible. We begin by explaining the twistor transform of the anti-selfdual Yang-Mills equations, following Ward [34], who developed the analog for gauge fields of the Penrose transform [33] of the anti-selfdual Einstein equations. Then we explain the twistor transform of the linear massless wave equations [37].

Self-Dual Yang-Mills Fields

We will describe a one-to-one correspondence between the following two types of object:

(1) A $GL(N, \mathbb{C})$ -valued gauge field $A_{a\dot{a}}(x^{b\dot{b}})$ that obeys the anti-selfdual Yang-Mills equations on complexified (but not compactified) Minkowski spacetime \mathbb{M}' . (A is a connection on a holomorphic vector bundle H over \mathbb{M}' , which is automatically holomorphically trivial as $\mathbb{M}' \cong \mathbb{C}^4$.) Here the $x^{b\dot{b}}$ are complex variables, the $A_{a\dot{a}}$ are entire holomorphic functions of $x^{b\dot{b}}$, and the curvature of A, which we write as $F = dA + A \wedge A$, is anti-selfdual. In spinor notation, with $F_{ab\dot{a}\dot{b}} = [D_{a\dot{a}}, D_{b\dot{b}}]$, anti-selfduality means that

$$F_{ab\dot{a}\dot{b}} = \epsilon_{ab}\Phi_{\dot{a}\dot{b}} \tag{A.1}$$

for some $\Phi_{\dot{a}\dot{b}}$.

(2) A rank N holomorphic vector bundle E over \mathbb{PT}' (defined as the region of $\mathbb{PT} = \mathbb{CP}^3$ with $\lambda^a \neq 0$) such that E is holomorphically trivial when restricted to each genus zero, degree one curve in \mathbb{PT}' .

What is remarkable about this construction is that in (1) we impose a nonlinear differential equation, the anti-selfdual Yang-Mills equation (and the purely holomorphic structure is trivial), but in (2) we only ask for holomorphy. In a sense, therefore, the correspondence solves the anti-selfdual Yang-Mills equations.

This correspondence has numerous analogs and important refinements. One important point that we will omit (referring the reader to standard references such as [36]) is that in ++++ or ++- signature (that is, whenever the anti-selfdual Yang-Mills equations are real), one can impose a reality condition and reduce the gauge group to U(N) rather than $GL(N, \mathbb{C})$ on a real slice of \mathbb{M}' .

We will make the correspondence between (1) and (2) in a computational way, and then explain it more conceptually.

A central role in the correspondence is played by the twistor equation, which by now should be familiar to the reader:

$$\mu_{\dot{a}} + x_{a\dot{a}}\lambda^a = 0. (A.2)$$

This equation can be read in two ways. If x is given, and (A.2) is regarded as an equation for λ and μ , then it defines a curve in twistor space of genus zero and degree one that we will call \mathbb{D}_x . Complexified Minkowski space is the moduli space of such curves, a fact that we have extensively used in this paper.

Alternatively, if λ and μ are given, and (A.2) is regarded as an equation for x, then the solution set \mathbb{K} (or $\mathbb{K}(\lambda,\mu)$) is a two-dimensional complex subspace of complexified Minkowski space \mathbb{M}' . It is completely null (any tangent vector to \mathbb{K} is a null vector) and in a certain sense is selfdual. Penrose calls \mathbb{K} an α -plane. Thus, twistor space is the moduli space of α -planes.

Translations within the α -plane are generated by the operators

$$\partial_{\dot{a}} = \lambda^a \frac{\partial}{\partial x^{a\dot{a}}}, \quad \dot{a} = 1, 2.$$
 (A.3)

These translations take the form

$$x^{a\dot{a}} \to x^{a\dot{a}} + \lambda^a \epsilon^{\dot{a}},$$
 (A.4)

for arbitrary $\epsilon^{\dot{a}}$.

The significance of anti-selfduality for our purposes is that it means that when restricted to an α -plane, the gauge field becomes flat. We can verify this straightforwardly. We define

$$D_{\dot{a}} = \lambda^a \frac{D}{Dx^{a\dot{a}}},\tag{A.5}$$

with $D/Dx^{a\dot{a}}$ the covariant derivative with respect to the anti-selfdual gauge field A. Then

$$[D_{\dot{a}}, D_{\dot{b}}] = \lambda^a \lambda^b \left[\frac{D}{Dx^{a\dot{a}}}, \frac{D}{Dx^{b\dot{b}}} \right] = \lambda^a \lambda^b \epsilon_{ab} \Phi_{\dot{a}\dot{b}} = 0, \tag{A.6}$$

where (A.1) has been used.

Now, let V_1 be the region of \mathbb{PT}' in which $\lambda^1 \neq 0$, and let V_2 be the region with $\lambda^2 \neq 0$. In \mathbb{PT}' , λ^1 and λ^2 are not allowed to both vanish, so V_1 , V_2 give an open cover of \mathbb{PT}' . V_1 and V_2 are both copies of \mathbb{C}^3 (for example, V_1 can be mapped to \mathbb{C}^3 using the coordinates λ^2/λ^1 , μ^1/λ^1 , μ^2/λ^1). So a holomorphic vector bundle on V_1 or V_2 is automatically holomorphically trivial. A holomorphic vector bundle E on \mathbb{PT}' can therefore be defined by giving a "transition function" on $V_{12} = V_1 \cap V_2$. This is a holomorphic function $U: V_{12} \to GL(N, \mathbb{C})$. Explicitly, U is a $GL(N, \mathbb{C})$ -valued holomorphic function $U(\lambda, \mu)$, that is homogeneous in λ and μ of degree zero and singular only if $\lambda^1 = 0$ or $\lambda^2 = 0$. (Given U, the bundle E is defined by using U to glue a trivial rank N complex bundle F_1 on V_1 to a trivial rank N bundle F_2 on V_2 .) Two transition functions U and U' define isomorphic bundles on \mathbb{PT}' if and only if we can write

$$U' = U_1 U U_2^{-1}, (A.7)$$

where $U_1: V_1 \to GL(N, \mathbb{C})$ is holomorphic throughout V_1 and likewise $U_2: V_2 \to GL(N, \mathbb{C})$ is holomorphic throughout V_2 . If this is the case, then U can be converted to U' by making gauge transformations of F_1 and F_2 via U_1 and U_2 , prior to the gluing.

Now as long as $(\lambda, \mu) \in V_1$, the α -plane \mathbb{K} contains a unique point $P(\mathbb{K})$ with $x^{1\dot{a}} = 0$. To prove this, we just observe that if $\lambda^1 \neq 0$, we can use the translations (A.4) to set $x^{1\dot{a}} = 0$ in a unique manner. Likewise, for $(\lambda, \mu) \in V_2$, \mathbb{K} contains a unique point $Q(\mathbb{K})$ with $x^{2\dot{a}} = 0$. For $(\lambda, \mu) \in V_{12}$, $P(\mathbb{K})$ and $Q(\mathbb{K})$ are both defined and vary holomorphically with λ and μ , and we can set

$$U(\lambda, \mu) = P \exp \int_{Q(\mathbb{K})}^{P(\mathbb{K})} A. \tag{A.8}$$

The integral is taken over any contour in \mathbb{K} . The choice of contour does not matter, since the gauge field is flat when restricted to \mathbb{K} . Since U is defined throughout V_{12} and takes values in $GL(N,\mathbb{C})$, we can use U to determine a holomorphic vector bundle E over \mathbb{PT}' .

If in making this construction, we replace A by a gauge-equivalent field, via $(\partial + A) \to Y(\partial + A)Y^{-1}$ for some holomorphic $GL(N,\mathbb{C})$ -valued field Y on spacetime, then U transforms to $\widetilde{U} = Y_P U Y_Q^{-1}$, where Y_P and Y_Q denote the values of Y at $P(\mathbb{K})$ and $Q(\mathbb{K})$. As Y_P is holomorphic and invertible throughout V_1 , and V_Q throughout V_2 , the holomorphic vector bundles defined by \widetilde{U} and U are isomorphic.

We have almost shown how, from an object of type (1), to produce an object of type (2). We still must show that E is holomorphically trivial when restricted to a genus zero, degree one curve in \mathbb{PT}' . These are precisely the curves \mathbb{D}_x for some $x \in \mathbb{M}$, as described by (A.2). To show that E is holomorphically trivial when restricted to \mathbb{D}_x , we first restrict U to \mathbb{D}_x , which is done by regarding $\mu_{\dot{a}}$ as a function of x and λ that obeys the twistor equation: $\mu_{\dot{a}} = -x_{b\dot{a}}\lambda^b$. The transition function of E restricted to \mathbb{D}_x is thus simply $W(\lambda^a, x_{b\dot{b}}) = U(\lambda^a, -x_{b\dot{a}}\lambda^b)$. To show that the restriction of E is trivial, we must show that E can be factored holomorphically as E and E and E is singular only at E is singular only at E and E is singular only at E and E is singular only at E is the s

$$W_1 = P \exp \int_x^{P(\mathbb{K})} A$$

$$W_2 = P \exp \int_x^{Q(\mathbb{K})} A.$$
(A.9)

For any given λ , the contours are taken within the α -plane \mathbb{K} of that given λ which contains x. Clearly, $W = W_1 W_2^{-1}$. The ability to make this factorization depends on choosing x; in general U has no such factorization, but W does.

To establish the converse, we start with a holomorphic bundle E on \mathbb{PT}' that is trivial on each \mathbb{D}_x . We can assume that E is defined by a holomorphic transition function $U(\lambda,\mu):V_{12}\to GL(N,\mathbb{C})$, which is homogeneous of degree zero. Now we reverse the above construction. We define $W(\lambda^a,x_{b\dot{b}})=U(\lambda^a,-x_{b\dot{a}}\lambda^b)$. For any fixed x,W is homogeneous in λ of degree zero. From the definition of $\partial_{\dot{a}}$ and the chain rule, we learn immediately that

$$\partial_{\dot{a}}W = 0. \tag{A.10}$$

The holomorphic triviality of E when restricted to each \mathbb{D}_x means that W can be factored

$$W(\lambda, x) = W_1 W_2^{-1},$$
 (A.11)

where the W_i , i = 1, 2, are singular only at $\lambda^i = 0$. If we plug this factorization into (A.10), we learn that

$$W_1^{-1}\partial_{\dot{a}}W_1 = W_2^{-1}\partial_{\dot{a}}W_2. \tag{A.12}$$

(This is understood as a differential operator, via $W^{-1}\partial W = \partial + W^{-1}[\partial, W]$.) The left hand side of (A.12) can only be singular at $\lambda^1 = 0$. The right hand side can only be singular at $\lambda^2 = 0$. As they are equal, there can be no singularity at all. We define $D_{\dot{a}}$ to equal the left or right hand side of (A.12). It is homogeneous in λ of degree 1, since $\partial_{\dot{a}}$ has this property, and it is clearly of the form $D_{\dot{a}} = \partial_{\dot{a}} + A_{\dot{a}}(\lambda, x)$, where $A_{\dot{a}}$ is some function of λ and x valued in the Lie algebra of $GL(N, \mathbb{C})$. Moreover, as $A_{\dot{a}}$ is homogeneous in λ of degree one and is non-singular, it takes the form $A_{\dot{a}} = \lambda^a A_{a\dot{a}}(x)$, where $A_{a\dot{a}}$ is a function only of x. Hence

$$D_{\dot{a}} = \lambda^a \left(\frac{\partial}{\partial x^{a\dot{a}}} + A_{a\dot{a}}(x) \right). \tag{A.13}$$

Since the $\partial_{\dot{a}}$ commute, and their covariant versions $D_{\dot{a}}$ are conjugate to $\partial_{\dot{a}}$ (via either W_1 or W_2), it follows that the $D_{\dot{a}}$ also commute:

$$[D_{\dot{a}}, D_{\dot{b}}] = 0. (A.14)$$

When this is expanded out using (A.13), we discover that $\lambda^a \lambda^b F_{ab\dot{a}\dot{b}} = 0$, where $F_{ab\dot{a}\dot{b}} = [D_{a\dot{a}}, D_{b\dot{b}}]$. This implies, as promised, that the gauge field $A_{a\dot{a}}(x)$ obeys the anti-selfdual Yang-Mills equations (A.1). We have thus completed the converse step of obtaining an object of type (1) from an object of type (2).

I leave it to the reader to show that if U is replaced by an equivalent transition function $\widetilde{U} = U_1 U U_2^{-1}$ in twistor space, then A is replaced by a gauge-equivalent connection in Minkowski spacetime, and further to show that the two operations that we have defined are indeed inverse to one another.

More Abstract Version

A more conceptual version of the above proof – not strictly needed for the present paper – goes as follows. We start with an anti-selfdual connection A on a $GL(N, \mathbb{C})$ bundle H over spacetime. Given an α -plane \mathbb{K} , we let $E_{\mathbb{K}}$ be the space of covariantly constant sections of H restricted to \mathbb{K} . The $E_{\mathbb{K}}$ vary holomorphically with \mathbb{K} , and fit together, as \mathbb{K} is varied, to the fibers of a holomorphic vector bundle E over \mathbb{PT}' , which parameterizes the space of \mathbb{K} 's.

To prove that the bundle E is holomorphically trivial when restricted to any \mathbb{D}_x , we note that if T passes through x, then E_T can be canonically identified with H_x , the fiber of H at x. Indeed, a covariantly constant section of H over T is uniquely determined by its value at any point $x \in T$; that value can be any element of H_x . So the restriction of E to \mathbb{D}_x is canonically the product of \mathbb{D}_x with the constant vector space H_x . This completes the more abstract explanation of how to construct an object of type (2) from an object of type (1).

Conversely, suppose we are given a rank N holomorphic bundle E over \mathbb{PT}' that is holomorphically trivial on each \mathbb{D}_x . Since it is trivial on \mathbb{D}_x , it has, when restricted to \mathbb{D}_x , an N-dimensional space of holomorphic sections which we call H_x . As x varies, the H_x fit together to a holomorphic vector bundle H over \mathbb{M}' (which is holomorphically trivial as $\mathbb{M}' \cong \mathbb{C}^4$).

We wish to define a connection on H. Suppose x and x' are two points in \mathbb{M}' that are at lightlike separation. Then they are contained in a unique α -plane \mathbb{K} . Once x and \mathbb{K} are given, H_x is canonically isomorphic to $E_{\mathbb{K}}$, the fiber of E at \mathbb{K} . This is so because \mathbb{D}_x can be regarded as the space of all α -planes that pass through x; \mathbb{K} is one of those. An element of H_x is a holomorphic section of the trivial bundle obtained by restricting E to \mathbb{D}_x ; it can be identified with its value at \mathbb{K} . Likewise, when x' and \mathbb{K} are given, $H_{x'}$ is canonically isomorphic to $E_{\mathbb{K}}$. Combining these isomorphisms of H_x and $H_{x'}$ with $E_{\mathbb{K}}$, we get a natural map from H_x to $H_{x'}$ that we interpret as parallel transport from H_x to $H_{x'}$ along the light ray that connects x and x'.

Knowing parallel transport along light rays is enough to uniquely determine a connection A. To show that A obeys the anti-selfdual Yang-Mills equations, it suffices to show flatness on α -planes, which follows from the following: if x, x', and x'' are contained in a common α -plane \mathbb{K} , then parallel transport around a triangle of light rays from x to x' to x'' and back to x gives the identity. This can be readily proved using the above definitions.

Apart from concision and manifest gauge invariance, the advantage of this abstract proof is that it generalizes to regions of complexified, conformally compactified Minkowski space \mathbb{M} other than \mathbb{M}' . Instead of starting with a solution of the anti-selfdual Yang-Mills equations on \mathbb{M}' , we could start with a solution defined on any open set \mathbb{U} in complexified, conformally compactified Minkowski space \mathbb{M} . (Actually, we want a mild restriction on \mathbb{U} : its intersections with α -planes should be connected and simply-connected.) Let \mathbb{G} be the region of \mathbb{PT} that parameterizes α -planes that have a non-empty intersection with \mathbb{U} . Then the conceptual proof of the twistor transform extends immediately to a correspondence

between anti-selfdual Yang-Mills fields on \mathbb{U} and holomorphic vector bundles on \mathbb{G} that are trivial on \mathbb{D}_x for all $x \in \mathbb{U}$. (Likewise, our discussion later of the linear wave equations of helicity h extends to a correspondence between solutions of the wave equations on \mathbb{U} and sheaf cohomology on \mathbb{G} .)

Here is a standard application of this generalization. Yang-Mills instantons on \mathbb{S}^4 are automatically real analytic (since the equation is elliptic) and so extend to a small complex neighborhood \mathbb{U} of \mathbb{S}^4 in \mathbb{M} . Using the fact that every α -plane in \mathbb{M} has a non-empty intersection with \mathbb{U} , one can then show that an instanton on \mathbb{S}^4 corresponds to a holomorphic vector bundle defined on all of \mathbb{PT} , and trivial on the generic \mathbb{D}_x (and on all of the "real" ones that correspond to points $x \in \mathbb{S}^4$), and obeying a certain reality condition assuming that the original instanton is real. For a systematic exposition, see [36].

Free Wave Equations

In most of this paper, more than the nonlinear anti-selfdual Yang-Mills equations, we really need the twistor transform of the linear wave equations for helicity h, for various values of h. According to the Penrose transform, solutions of this wave equation in \mathbb{M}' are equivalent to elements of the sheaf cohomology group $H^1(\mathbb{PT}', \mathcal{O}(2h-2))$. Here, following Penrose [37], we explain this correspondence in the context of Cech cohomology, using the open cover V_1, V_2 of \mathbb{PT}' and a holomorphic cocycle. Then we convert the statement to $\overline{\partial}$ cohomology, used in the rest of the paper.

Concretely, an element of $H^1(\mathbb{PT}', \mathcal{O}(2h-2))$ is given by a "cocycle," a holomorphic function $f(\lambda^a, \mu^{\dot{a}})$ on V_{12} that is homogeneous of degree 2h-2, and so is a section of $\mathcal{O}(2h-2)$. It may have singularities at $\lambda^1=0$ or at $\lambda^2=0$. It is subject to the equivalence relation $f \to f + f_1 - f_2$, where f_1 is holomorphic on V_1 (and so may only be singular at $\lambda^1=0$), and f_2 is holomorphic on V_2 (and so may only be singular at $\lambda^2=0$).

The first step is to define a function of x and λ by setting $g(\lambda^a, x_{b\dot{b}}) = f(\lambda^a, -x_{b\dot{a}}\lambda^b)$. Rather as in the above discussion of the Yang-Mills case, a simple use of the chain rule and definition of $\partial_{\dot{a}}$ gives

$$\partial_{\dot{a}}g(\lambda,x) = 0. \tag{A.15}$$

For fixed x, g can be regarded as a cocycle defining an element of $H^1(\mathbb{D}_x, \mathcal{O}(2h-2))$. We first consider the case that $h \geq 1/2$. For this case, $H^1(\mathbb{D}_x, \mathcal{O}(2h-2)) = 0$, so

$$g(\lambda, x) = g_1(\lambda, x) - g_2(\lambda, x), \tag{A.16}$$

where for $i = 1, 2, g_i$ is nonsingular except perhaps at $\lambda^i = 0$. From (A.15), we have

$$\partial_{\dot{a}}g_1 = \partial_{\dot{a}}g_2. \tag{A.17}$$

The left hand side may be singular only at $\lambda^1 = 0$, and the right hand side only at $\lambda^2 = 0$; so in fact, there are no singularities at all. We write $\phi_{\dot{a}}$ for the left or right hand side.

If h = 1/2, $\phi_{\dot{a}}(\lambda, x)$ is homogeneous in λ of degree zero, and so, being nonsingular, is a function only of x. We claim that it obeys the Dirac equation:

$$\frac{\partial}{\partial x^{a\dot{a}}}\phi^{\dot{a}} = 0. \tag{A.18}$$

In fact, as $\partial_{\dot{a}}\partial^{\dot{a}} = 0$ and $\phi_{\dot{a}} = \partial_{\dot{a}}g_1$, we have $0 = \partial_{\dot{a}}\phi^{\dot{a}}$, so, from the definition of $\partial_{\dot{a}}$, $0 = \lambda^a \partial_{a\dot{a}}\phi^{\dot{a}}$. As $\phi_{\dot{a}}$ is independent of λ , this does imply the Dirac equation. This is the h = 1/2 case of the Penrose correspondence.

If h = 1, then $\phi_{\dot{a}}$, being homogeneous in λ of degree one, is of the form $\phi_{\dot{a}} = \lambda^a C_{a\dot{a}}(x)$, where $C_{a\dot{a}}$ depends on only x and not λ . A linearized version of the same analysis that we gave in discussing the anti-selfdual Yang-Mills equations shows that $C_{a\dot{a}}$ obeys the linear anti-selfdual equation. For h > 1, which we do not need in the present paper, we refer to the literature.

For $h \leq 0$, one instead uses a contour integral method due to Penrose. If h = 0, then g is homogeneous in λ of degree -2. On \mathbb{D}_x , there is the holomorphic differential $\epsilon_{ab}\lambda^a d\lambda^b$ that is homogeneous in λ of degree 2. Letting C be any contour that surrounds one of the two singularities in g, for instance the one at $\lambda^1 = 0$, we define a function $\phi(x)$ by the integral

$$\phi(x) = \frac{1}{2\pi i} \oint_C \left(\epsilon_{ab} \lambda^a d\lambda^b \right) g(\lambda, x). \tag{A.19}$$

The integral makes sense since the form being integrated is homogenous in λ of degree zero. The integral only depends on the cohomology class represented by f; it is invariant under $g \to g + g_1 - g_2$ (where g_1 and g_2 each have only one singularity), since if g is replaced by g_1 or g_2 , the contour can be deformed away, shrinking it to $\lambda^1 = 0$ or $\lambda^2 = 0$. ϕ depends on x only as we have integrated over λ . Finally, a simple use of the chain rule shows that ϕ obeys the scalar wave equation $\partial_{a\dot{a}}\partial^{a\dot{a}}\phi = 0$.

For h < 0, say h = -k, we define a field of helicity h via

$$\phi_{a_1 a_2 \dots a_{2k}} = \frac{1}{2\pi i} \oint_C \left(\epsilon_{cd} \lambda^c d\lambda^d \right) \lambda_{a_1} \dots \lambda_{a_{2k}} g. \tag{A.20}$$

This obeys the free wave equation $\partial_{a\dot{a}}\phi^a{}_{b_1...b_{2k-1}}=0$, as one can again verify by a simple application of the chain rule.

Relation To $\overline{\partial}$ Cohomology

So far we have identified the space of solutions of the massless linear wave equation for helicity h with the sheaf cohomology group $H^1(\mathbb{PT}', \mathcal{O}(2h-2))$ defined in Cech cohomology. In section 4, instead of Cech cohomology, we encountered the $\overline{\partial}$ cohomology group $H^1_{\overline{\partial}}(\mathbb{PT}', \mathcal{O}(2h-2))$. By general arguments in complex geometry, H^1 and $H^1_{\overline{\partial}}$ are naturally isomorphic.

In the present example, because \mathbb{PT}' can be covered by two such simple open sets, we can be completely explicit about this isomorphism. Before doing so, we will make a minor change of notation. This will enable us to obtain formulas that are more convenient in section 4. We make an SL(2) transformation of the λ^a to move the singularities from $\lambda_2/\lambda_1 = \infty$, 0 to $\lambda_2/\lambda_1 = i$, -i. Henceforth, V_1 is the portion of \mathbb{PT}' with $\lambda_2/\lambda_1 \neq i$, V_2 the portion with $\lambda_2/\lambda_1 \neq -i$, and V_{12} remains the intersection.

Let $z = \lambda^2/\lambda^1$, and also let $z = \sigma + i\tau$, where σ and τ are real. Let $\theta(\tau)$ be the function that is 1 if $\tau > 0$ and 0 for $\tau < 0$. We have

$$\overline{\partial}\theta(\tau) = d\overline{z}\frac{\partial}{\partial\overline{z}}\theta(\tau) = \frac{i}{2}d\overline{z}\delta(\tau), \tag{A.21}$$

since $\tau = (z - \overline{z})/2i$, $\partial_{\overline{z}}\overline{z} = 1$, $\partial_{\overline{z}}z = 0$, and $\partial_{\tau}\theta(\tau) = \delta(\tau)$. Consider an element $\omega \in H^1(\mathbb{PT}', \mathcal{O}(k))$, for some k, that is represented by a cocycle f. Thus, f is a section of $\mathcal{O}(k)$ that is holomorphic throughout V_{12} , and subject to the equivalence

$$f \to f + f_1 - f_2, \tag{A.22}$$

where f_i , i = 1, 2, is holomorphic throughout V_i . The $\overline{\partial}$ cohomology class corresponding to ω can be represented by the (0, 1)-form

$$v = f\overline{\partial}\theta(\tau) = \frac{i}{2}f\delta(\tau)d\overline{z}.$$
 (A.23)

The form v is defined globally throughout \mathbb{PT}' , since the singularities of f are disjoint from the delta function. On V_{12} , it can be written $\overline{\partial}(f\theta(\tau))$, and so is trivial, but this representation is not valid everywhere because of the singularity of f at z=i, which is in the support of $\theta(\tau)$. The $\overline{\partial}$ cohomology class of v is invariant under the transformation (A.22), since we have $(f_1 - f_2)\overline{\partial}\theta(\tau) = \overline{\partial}(f_1(\theta(\tau) - 1) - f_2\theta(\tau))$, a formula which is valid

everywhere, as the f_i multiply functions that vanish near their singularities. Thus, we have defined a mapping from H^1 to $H^1_{\overline{\partial}}$. This mapping actually is an isomorphism.

I am indebted to N. Berkovits for numerous helpful discussions of some of these ideas and pointing out a number of significant references, to F. Cachazo for extensive assistance with computer algebra, to L. Dixon for answering many queries about perturbative Yang-Mills theory, and to M. F. Atiyah and R. Penrose for mathematical consultations. This work was supported in part by NSF Grant PHY-0070928.

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