On the Laplacian of 1/r

1 author:

On the Laplacian of $1/r$

D V Redžić

Faculty of Physics, University of Belgrade, PO Box 44, 11000 Beograd, Serbia

E-mail: `redzic@ff.bg.ac.rs`

Abstract. A novel definition of the Laplacian of $1/r$ is presented, suitable for advanced undergraduates.

1. Introduction

Discussions of the Laplacian of $1/r$ generally start abruptly, *in medias res*, by stating the relation

$$\nabla^2 \frac{1}{r} = -4\pi\delta^3(\boldsymbol{r}), \tag{1}$$

where r is the magnitude of radius vector $\boldsymbol{r}$ and $\delta^3(\boldsymbol{r})$ is the three-dimensional delta function, which is then proved in various ways, clarifying thus its meaning. A glance at equation (1) reveals, however, that the symbol ∇^2 appearing in it can not have its ordinary, classical meaning of $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$, where, in Cartesian coordinates,

$$\boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

is Hamilton's operator nabla (cf, e.g., Redžić [1]), since the classical expression $\nabla^2(1/r)$ vanishes for $r \neq 0$ and is not defined at $r = 0$. Therefore, instead of the familiar form (1), in this note we will use a less confusing notation [2, 3]

$$\bar{\nabla}^2 \frac{1}{r} = -4\pi\delta^3(\boldsymbol{r}); \tag{2}$$

the expression $\bar{\nabla}^2(1/r)$ we will call the generalized (distributional) Laplacian of $1/r$ and try to fathom its meaning. Let us briefly review some typical proofs of (2).

The well-known way to demonstrate (2) is to regularize $1/r$ in terms of a parameter a so that the regularized function is well-behaved everywhere for $a \neq 0$. Then verification of (2) consists in showing that in the limit $a \to 0$, $-1/4\pi$ times the Laplacian of the regularized function is a representation of the three-dimensional delta function $\delta^3(\boldsymbol{r})$. For example, regularizing $1/r$ as $1/\sqrt{r^2 + a^2}$, Jackson [4] shows that

$$\bar{\nabla}^2 \frac{1}{r} = \lim_{a \to 0} \nabla^2 \frac{1}{\sqrt{r^2 + a^2}} = -4\pi\delta^3(\boldsymbol{r}); \tag{3}$$

the limit here is the weak limit (cf, e.g., [5]). A more sophisticated method of proving
(2) would be to derive first the generalized second-order partial derivatives of $1/r$ with
respect to Cartesian coordinates [3, 6].†

A generalization of (2)

$$\bar\nabla^2 \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = -4\pi\delta^3(\boldsymbol{r}-\boldsymbol{r}'), \tag{4}$$

obtained simply by a different choice of the origin, can be demonstrated either via the
regularization procedure‡ or by employing a well-known electrostatic argument. Namely,
it can be shown that the 'potential'

$$\int \varrho(\boldsymbol{r}') \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \mathrm{d}^3 r', \tag{5}$$

where the 'density' $\varrho(\boldsymbol{r}')$ plays the role of a well-behaved 'test' function, satisfies
Poisson's equation

$$\nabla^2 \int \varrho(\boldsymbol{r}') \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \mathrm{d}^3 r' = \boldsymbol{\nabla}\cdot\int \varrho(\boldsymbol{r}')\boldsymbol{\nabla}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}^3 r'$$
$$= -4\pi\varrho(\boldsymbol{r}), \tag{6}$$

by making use of the divergence theorem and Gauss's theorem from electrostatics [8, 9].
Since the right-hand side of equation (6) can be written as

$$-4\pi\int \varrho(\boldsymbol{r}')\delta^3(\boldsymbol{r}-\boldsymbol{r}')\mathrm{d}^3 r', \tag{7}$$

it follows that the operator ∇^2 can enter an integral of the form (5) under proviso
that 'during entrance' it converts into the generalized operator $\bar\nabla^2$ whose action on the
function $1/|\boldsymbol{r}-\boldsymbol{r}'|$ is defined by equation (4).

On the other hand, the representation formula for a well-behaved scalar function
of position Φ

$$\Phi(\boldsymbol{r}) = -\frac{1}{4\pi}\int_V \frac{\nabla'^2\Phi(\boldsymbol{r}')}{\mathcal{R}}\mathrm{d}^3 r' + \frac{1}{4\pi}\oint_S\left[\frac{1}{\mathcal{R}}\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{\mathcal{R}}\right)\right]\mathrm{d}S', \tag{8}$$

† A typical *informal* derivation of (2) (cf, e.g., [7]) starts from relation

$$\int \nabla^2\frac{1}{r}\mathrm{d}^3 r = \int \boldsymbol{\nabla}\cdot\left(-\frac{\boldsymbol{r}}{r^3}\right)\mathrm{d}^3 r,$$

which, using the divergence theorem, equals $-4\pi = -4\pi\int\delta^3(\boldsymbol{r})\mathrm{d}^3 r$. However, the use of the divergence
theorem is not legitimate here, since the function $-\boldsymbol{r}/r^3$ is singular at $r=0$, and the result

$$\int \nabla^2\frac{1}{r}\mathrm{d}^3 r = -4\pi,$$

is incorrect since the left-hand side of the last equation equals zero.
‡ Jackson [4], in fact, shows the validity of a generalization of (3)

$$\bar\nabla^2 \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \lim_{a\to 0}\nabla^2\frac{1}{\sqrt{(\boldsymbol{r}-\boldsymbol{r}')^2+a^2}} = -4\pi\delta^3(\boldsymbol{r}-\boldsymbol{r}').$$

where the point $\boldsymbol{r}$ is *within* the volume V and $\mathcal{R} \equiv |\boldsymbol{r} - \boldsymbol{r}'|$, is obtained from Green's second identity

$$\int_{V-V_\varepsilon} \left(\frac{1}{\mathcal{R}} \nabla'^2 \Phi(\boldsymbol{r}') - \Phi(\boldsymbol{r}') \nabla'^2 \frac{1}{\mathcal{R}} \right) \mathrm{d}^3 r' = \left(\oint_S + \oint_{S_\varepsilon} \right) \left[\frac{1}{\mathcal{R}} \frac{\partial \Phi}{\partial n'} - \Phi \frac{\partial}{\partial n'} \left(\frac{1}{\mathcal{R}} \right) \right] \mathrm{d}S', \quad (9)$$

where the volume $V - V_\varepsilon$ is obtained by excluding a small ball of radius ε and centre at $\boldsymbol{r}' = \boldsymbol{r}$ from the volume V, and S_ε is the surface of the ball, taking the limit $\varepsilon \to 0$ (cf, e.g. [10]). In this way, the singularity of the function $1/\mathcal{R}$ at $\boldsymbol{r}' = \boldsymbol{r}$ is managed with, making possible the use of Green's second identity. As is well known, this classical procedure provides grounds for replacing the term $\nabla'^2(1/\mathcal{R})$ in (9) by its generalized counterpart $\bar{\nabla}'^2(1/\mathcal{R})$ defined by (4), extending of course at the same time the restricted (singularity-free) volume of integration $V - V_\varepsilon$ to the whole region V and removing the integral over the surface S_ε.§ Thus, employing of the generalized Laplacian of the function $1/|\boldsymbol{r} - \boldsymbol{r}'|$ instead of the classical one in regions that contain the singular point $\boldsymbol{r}' = \boldsymbol{r}$, replaces lengthy procedures based on classical analysis, providing a shortcut to the correct final results.

In this note, an alternative definition of the generalized Laplacian of $1/r$ will be presented. Taking into account the ubiquity and importance of this somewhat tricky concept, the alternative derivation of relation (2) could perhaps be of some pedagogical interest.

2. Integral versus differential definitions of classical operators

A perusal of the literature reveals that a common feature of discussions of the Laplacian of $1/r$ is that, right from the outset, the Laplacian is identified with $\nabla^2 \equiv \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$. That is, in the *primary* definition of the Laplacian of a scalar field, which is the divergence of the gradient of the field, both the divergence and gradient operators are understood according to their differential definitions, $\boldsymbol{\nabla} \cdot$ and $\boldsymbol{\nabla}$, respectively, in accord with common practice of defining familiar operators in vector analysis by differential operations (cf, e.g., [1, 11]). However, as is well known, there exists also an alternative way of defining the classical operators by means of integral operations (cf, e.g., [8, 12, 13]). While the integral and differential definitions are equivalent in the case of differentiable fields, the former definition provides physical insight and computational convenience, and thus should be preferred, as Sommerfeld suggested [12]. Moreover, it appears that the integral definitions of classical operators are more to the point than

§ To put it rigorously, in this case relation

$$\lim_{\varepsilon \to 0} \left\{ \int_{V-V_\varepsilon} \Phi(\boldsymbol{r}') \nabla'^2 \frac{1}{\mathcal{R}} \mathrm{d}^3 r' + \oint_{S_\varepsilon} \left[\frac{1}{\mathcal{R}} \frac{\partial \Phi}{\partial n'} - \Phi \frac{\partial}{\partial n'} \left(\frac{1}{\mathcal{R}} \right) \right] \mathrm{d}S' \right\} = -4\pi \Phi(\boldsymbol{r})$$

serves as a definition of $\bar{\nabla}^2(1/\mathcal{R})$ (cf, e.g., [5]), since

$$-4\pi \Phi(\boldsymbol{r}) = -4\pi \int \Phi(\boldsymbol{r}') \delta^3(\boldsymbol{r}' - \boldsymbol{r}) \mathrm{d}^3 r'.$$

the differential ones when applied to a singular point of the field. Let us examine the last point in some detail.

As a simple illustration, consider the Laplacian of $1/r$ at the singular point $r = 0$. Using the differential definitions, $\nabla^2(1/r)$ is not defined at $r = 0$. On the other hand, using the integral definition of the divergence of a vector field $\boldsymbol{A}$ at a point P,

$$\mathrm{div}\,\boldsymbol{A} = \lim_{\tau \to 0} \frac{1}{\tau} \oint \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{S}, \tag{10}$$

where the flux of $\boldsymbol{A}$ is through a closed surface surrounding P and τ is the enclosed volume, and setting $\boldsymbol{A} = \mathrm{grad}(1/r) = -\boldsymbol{r}/r^3$, we obtain that

$$\mathrm{div}\,\mathrm{grad}(1/r) = -4\pi\infty = -\infty, \tag{11}$$

at $r = 0$. Since infinite value of a function at a point is not permitted in classical analysis, the two definitions seem to be in a dead heat. However, this is not so, as the following argument will show.

Calculate the average value of the Laplacian of $1/r$ over the volume of a ball of radius ε and centre at $r = 0$. Using the differential definition of the Laplacian we obtain

$$\langle \nabla^2(1/r) \rangle = \frac{1}{(4/3)\pi\varepsilon^3} \int_{V_\varepsilon} \nabla^2(1/r)\mathrm{d}^3 r = 0, \tag{12}$$

where $\langle ... \rangle$ stands for the average value, and V_ε is the volume of the ball. Once again, the same result is obtained using the integral definition of the Laplacian,

$$\langle \mathrm{div}\,\mathrm{grad}(1/r) \rangle = \frac{1}{(4/3)\pi\varepsilon^3} \int_{V_\varepsilon} \mathrm{div}(-\boldsymbol{r}/r^3)\mathrm{d}^3 r = 0, \tag{13}$$

since according to both definitions the Laplacian of $1/r$ vanishes for $r \neq 0$, and the value of integral is not affected by a countable number of singular points of the integrand.

Inspection of equation (10) reveals, however, that a reasonable physical definition of the *volume average* of the divergence of a vector field $\boldsymbol{A}$ over a volume τ would be

$$\langle \mathrm{div}\,\boldsymbol{A} \rangle = \frac{1}{\tau} \oint_{S_\tau} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{S}, \tag{14}$$

where S_τ is the surface of the volume τ, rather than the standard definition

$$\langle \mathrm{div}\,\boldsymbol{A} \rangle = \frac{1}{\tau} \int_{\tau} \mathrm{div}\,\boldsymbol{A}\mathrm{d}^3 r. \tag{15}$$

Namely, while both definitions yield the same result when the field $\boldsymbol{A}$ is differentiable over the volume τ, they give different results when $\boldsymbol{A}$ is singular at a point inside τ. For example, if $\boldsymbol{E}$ is the electrostatic field of a point charge q located at the origin, and τ is a volume containing the origin, $\langle \mathrm{div}\,\boldsymbol{E} \rangle$ would be equal to $q/\tau\epsilon_0$ according to definition (14). On the other hand, according to definition (15), $\langle \mathrm{div}\,\boldsymbol{E} \rangle$ would be equal to zero, since $\mathrm{div}\,\boldsymbol{E}$ is zero for $r \neq 0$ and, classically, is not defined at $r = 0$. Keeping in mind that the divergence of a vector field is a measure of the strength of a source or sink of field lines, it is clear that the standard definition (15) would yield the absurd result

that the average strength of a point source over a volume containing the source is zero.‖ Thus, the integral definition (14) of the average divergence not only implies the integral definition (10) of the divergence at a point, but also is adequate for regions that contain singularities (point sources or sinks) of field lines.¶

3. A novel definition of the Laplacian of $1/r$

The above analysis provides the opportunity of introducing an alternative definition of the generalized Laplacian of $1/r$.

As a first step, define the ε-Laplacian of $1/r$, $\mathrm{L}_\varepsilon(1/r)$, in terms of a parameter ε as

$$\mathrm{L}_\varepsilon\frac{1}{r} \equiv \mathrm{div}_\varepsilon\mathrm{grad}\frac{1}{r} \equiv \begin{cases} [\oint_{r=\varepsilon}(-\boldsymbol{r}/r^3)\cdot\mathrm{d}\boldsymbol{S}]/(4/3)\pi\varepsilon^3, & \text{if } r < \varepsilon \\ \mathrm{div}\,\mathrm{grad}(1/r) = 0, & \text{if } r > \varepsilon. \end{cases} \tag{16}$$

Thus, for $r < \varepsilon$, $\mathrm{L}_\varepsilon(1/r)$ is the average divergence (as defined by the integral definition (14)) of the gradient of $1/r$ over the volume of a ball of radius ε and centre at $r = 0$; for $r > \varepsilon$, $\mathrm{L}_\varepsilon(1/r)$ is the classical Laplacian of $1/r$. Definition (16) implies that

$$\mathrm{L}_\varepsilon\frac{1}{r} = -\frac{3}{\varepsilon^3}\Theta(\varepsilon - r) \tag{17}$$

where $\Theta(x)$ is the Heaviside step function,

$$\Theta(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0. \end{cases} \tag{18}$$

When $\varepsilon \to 0$, from equation (17) we obtain

$$\lim_{\varepsilon\to 0}\mathrm{L}_\varepsilon\frac{1}{r} = \begin{cases} -\infty, & \text{if } r = 0 \\ 0, & \text{if } r \neq 0. \end{cases} \tag{19}$$

On the other hand, integrating the product of $\mathrm{L}_\varepsilon(1/r)$ and a well-behaved 'test' function $f(\boldsymbol{r})$,

$$\int \left[\mathrm{L}_\varepsilon\frac{1}{r}\right] f(\boldsymbol{r})\mathrm{d}^3 r = -\int_{r\leq\varepsilon}\frac{3}{\varepsilon^3}f(\boldsymbol{r})\mathrm{d}^3 r, \tag{20}$$

‖ Note that for a given field, a source of field lines and a source of the field need not necessarily coincide. For example, the electric field lines of a point charge which is forever in uniform motion emanate from the present position of the charge, whereas the source of the field is the charge at the retarded positions (cf, e.g.[11, 14]).

¶ Recall that a valid definition, *inter alia*, should be adequate ('*definitio sit adequata*'). Recall also that there are other cases when the average value of a physical quantity can not be defined via an integral of the *classical* local values of the quantity. For example, the average velocity of a particle, is primarily given by $\langle\boldsymbol{v}\rangle = \triangle\boldsymbol{r}/\triangle t$, where $\triangle\boldsymbol{r}$ is a displacement of the particle during a time interval $\triangle t$, and not by $\langle\boldsymbol{v}\rangle = (1/\triangle t)\int_0^{\triangle t}\boldsymbol{v}(t)\mathrm{d}t$. (The two definitions are equivalent only if $\boldsymbol{r}$ is a differentiable function of t.) Similarly, the average charge density over a volume $\triangle V$ is primarily $\langle\varrho\rangle = \triangle Q/\triangle V$, where $\triangle Q$ is a charge inside $\triangle V$, and not $\langle\varrho\rangle = (1/\triangle V)\int_{\triangle V}\varrho(\boldsymbol{r})\mathrm{d}^3 r$, if $\varrho(\boldsymbol{r})$ is the *classical* local charge density. Of course, if $\boldsymbol{v}(t)$ and $\varrho(\boldsymbol{r})$ are described by generalized functions (distributions), such as the Dirac delta function (which is the case when $\boldsymbol{v}(t)$ changes abruptly during $\triangle t$, or there are point charges inside the volume $\triangle V$), no discrepancy arises between the two kinds of definitions.

using expansion of $f(\boldsymbol{r})$ in a Taylor series around $\boldsymbol{r} = 0$, and taking the limit $\varepsilon \to 0$ yields

$$\lim_{\varepsilon \to 0} \int \left[\mathrm{L}_\varepsilon \frac{1}{r} \right] f(\boldsymbol{r}) \mathrm{d}^3 r = -\lim_{\varepsilon \to 0} \int_0^\varepsilon \frac{3}{\varepsilon^3} \left[f(0) + \frac{r^2}{6} \nabla^2 f + ... \right] 4\pi r^2 \mathrm{d}r$$
$$= -4\pi f(0). \tag{21}$$

As is well known, result (21) can be expressed as

$$\bar{\mathrm{L}} \frac{1}{r} = \underset{\varepsilon \to 0}{\mathrm{wlim}} \, \mathrm{L}_\varepsilon \frac{1}{r} = -4\pi \delta^3(\boldsymbol{r}); \tag{22}$$

where a more suitable notation $\bar{\mathrm{L}}$ for the generalized Laplacian is now used instead of $\bar{\nabla}^2$ and wlim stands for the weak limit (cf, e.g., [3, 5]).[+] Equation (22) is tantamount to equation (3), providing another definition of the generalized Laplacian of $1/r$.[*]

    Discuss now briefly a closely related problem of defining analogously the generalized charge density for a *point charge q* located at the origin. Obviously, the corresponding ε-charge density should be defined as

$$\varrho_\varepsilon(\boldsymbol{r}) = \begin{cases} q/(4/3)\pi\varepsilon^3, & \text{if} \quad r < \varepsilon \\ \varrho(\boldsymbol{r}) = 0, & \text{if} \quad r > \varepsilon, \end{cases} \tag{23}$$

which can be recast into

$$\varrho_\varepsilon(\boldsymbol{r}) = \frac{3q}{4\pi\varepsilon^3} \Theta(\varepsilon - r). \tag{24}$$

Passing details, we give the final result

$$\bar{\varrho}(\boldsymbol{r}) = \underset{\varepsilon \to 0}{\mathrm{wlim}} \, \varrho_\varepsilon(\boldsymbol{r}) = q\delta^3(\boldsymbol{r}). \tag{25}$$

where $\bar{\varrho}(\boldsymbol{r})$ is the generalized charge density. Note that $\lim_{\varepsilon \to 0} \int \varrho_\varepsilon(\boldsymbol{r}) \mathrm{d}^3 r = q$, whereas $\int [\lim_{\varepsilon \to 0} \varrho_\varepsilon(\boldsymbol{r})] \mathrm{d}^3 r = 0$. Thus, the volume charge density of a point charge q located at $r = 0$ is naturally described by $\varrho_\varepsilon(\boldsymbol{r})$ and its weak limit $\bar{\varrho}(\boldsymbol{r})$, in perfect analogy with the volume flux density of the flux of $\mathrm{grad}(1/r)$, which is naturally described by $\mathrm{L}_\varepsilon(1/r)$ and its weak limit $\bar{\mathrm{L}}(1/r)$.

[+] As was pointed out in Section 2, the standard practice of using automatically ∇^2 for the classical Laplacian may be misleading.

[*] Recall that the 2D analogue of equation (4) reads

$$\bar{\nabla}^2 \ln |\boldsymbol{s} - \boldsymbol{s}'| = 2\pi\delta^2(\boldsymbol{s} - \boldsymbol{s}'),$$

where $\boldsymbol{s}$ and $\boldsymbol{s}'$ are 2D radius vectors and $\delta^2(\boldsymbol{s} - \boldsymbol{s}')$ is the 2D delta function [4, 8, 15], which setting $\boldsymbol{s}' = 0$ yields

$$\bar{\nabla}^2 \ln s = 2\pi\delta^2(\boldsymbol{s}),$$

which is the 2D analogue of equation (3). To prove the last relation, instead of regularizing $\ln s$ in terms of a parameter a as $\ln \sqrt{s^2 + a^2}$, the alternative 2D definition of the generalized Laplacian of $\ln s$,

$$\bar{\mathrm{L}} \ln s = 2\pi\delta^2(\boldsymbol{s}),$$

can be introduced, following a 2D procedure analogous to the 3D procedure described above.

In the same way we can define the generalized divergence of the electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ of a point charge q that is moving on a specific trajectory $\boldsymbol{r}_q(t)$. Starting from the corresponding ε-divergence

$$\mathrm{div}_\varepsilon \boldsymbol{E}(\boldsymbol{r}, t) \equiv \begin{cases} [\oint_{\xi=\varepsilon} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S}]/(4/3)\pi\varepsilon^3, & \text{if } \xi < \varepsilon \\ \mathrm{div}\boldsymbol{E} = 0, & \text{if } \xi > \varepsilon, \end{cases} \tag{26}$$

where $\xi = |\boldsymbol{r} - \boldsymbol{r}_q(t)|$, using Gauss's law in its integral form, $\oint_{\xi=\varepsilon} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = q/\epsilon_0$, we obtain that $\boldsymbol{E}(\boldsymbol{r}, t)$ must satisfy equation

$$\overline{\mathrm{div}}\boldsymbol{E}(\boldsymbol{r}, t) = \frac{q\delta(\boldsymbol{r} - \boldsymbol{r}_q(t))}{\epsilon_0} = \frac{\bar{\varrho}(\boldsymbol{r}, t)}{\epsilon_0} \tag{27}$$

where

$$\overline{\mathrm{div}}\boldsymbol{E}(\boldsymbol{r}, t) = \underset{\varepsilon \to 0}{\mathrm{wlim}}\, \mathrm{div}_\varepsilon \boldsymbol{E}(\boldsymbol{r}, t) \tag{28}$$

is the generalized divergence of $\boldsymbol{E}(\boldsymbol{r}, t)$. This is consistent with a general *sine qua non* that when the sources of a field are idealized as point, line or surface distributions of charge and/or current, described by generalized functions, generalized operators must be employed in the field or potential equations instead of the classical ones.♯

4. Conclusions

We presented a novel definition of the generalized Laplacian of $1/r$, avoiding regularization of $1/r$ [3, 4], 'electrostatic' procedure [8, 9], or Green's second identity [5]. The definition is constructed employing the integral definition of the divergence instead of the differential one; thus, the usual notation $\bar{\nabla}^2$ is replaced by a less suggestive one $\bar{\mathrm{L}}$. It is shown that the Laplacian of $1/r$ can be naturally construed as the volume flux density of the flux of $\mathrm{grad}(1/r)$, in the same way as the volume charge density of a point charge located at the origin, introducing a reasonable generalized density. We believe that our analysis provides a simple and insightful alternative to the earlier discussions of the concept, clarifying thus its meaning.

Acknowledgments

References

[1] Redžić D V 2001 The operator ∇ in orthogonal curvilinear coordinates *Eur. J. Phys.* **22** 595–9
[2] Estrada R and Kanwal R P 1995 The appearance of nonclassical terms in the analysis of point-source fields *Am. J. Phys.* **63** 278–78
[3] Hnizdo V 2011 Generalized second-order partial derivatives of $1/r$ *Eur. J. Phys.* **32** 287–97

♯ An instructive illustration of the possible pitfalls of using the delta function in the context of solving a classical 2D electrodynamic problem is presented in [16].

[4] Jackson J D 1999 *Classical Electrodynamics* 3rd edn (New York: Wiley)

[5] Vladimirov V S 1979 *Generalized Functions in Mathematical Physics* (Moscow: Mir)

[6] Frahm C P 1983 Some novel delta-function identities *Am. J. Phys.* **51** 826–9

[7] Blinder S M 2003 Delta functions in spherical coordinates and how to avoid losing them: Fields of point charges and dipoles *Am. J. Phys.* **71** 816–8

[8] Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* (New York: McGraw-Hill)

[9] Hnizdo V 2000 On the Laplacian of $1/r$ *Eur. J. Phys.* **21** L1–L3

[10] Stratton J A 1941 *Electromagnetic Theory* (New York: McGraw-Hill)

[11] Griffiths D J 2013 *Introduction to Electrodynamics* 4th edn (Boston: Pearson)

[12] Sommerfeld A 1950 *Mechanics of Deformable Bodies* (New York: Academic) (transl. G Kuerti)

[13] Korn G A and Korn T M 1968 *Mathematical Handbook for Scientists and Engineers* 2nd edn (New York: McGraw-Hill)

[14] French A P 1968 *Special Relativity* (London: Nelson)

[15] Jiménez J L, Campos I and Roa-Neri J A E 2012 Confusing aspects in the calculation of the electrostatic potential function of an infinite line of charge *Eur. J. Phys.* **33** 467–71

[16] Redžić D V and Hnizdo V 2013 Time-dependent fields of a current-carrying wire *Eur. J. Phys.* **34** 495-501; arXiv:1301.1573