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Title: Classification of Weyl and Ricci Tensors

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Bachelor's Degree Project

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Abstract

The theory of General Relativity was formulated by Albert Einstein and introduced a set of equations: Einstein field equations. Since then, many exact solutions of these equations have been found. An important result in the study of exact solutions to Einstein equations is the classification of space-times according to the Weyl tensor (Petrov classification) and the Ricci tensor. Such classifications help to group the set of solutions that share some geometric properties. In this thesis, we review all the basic concepts in differential geometry and tensor calculus, from the definition of smooth manifold to the Riemann curvature tensor. Then we introduce the concept of Lorentz spaces and the tetrad formalism, which is very useful in the field of exact solutions to Einstein equations. After that, the concept of bivector is introduced and some of their main properties are analyzed. Using bivector formalism, we set the algebraic problem that derives in the Petrov classification. Moreover, we study principal null directions, that allow to make an analogous classification. Finally, we study the classification of second order symmetric tensors. From this, we can classify space-times according to the energy tensor.

Contents

Introduction

The theory of General Relativity was developed by Albert Einstein between 1907 and 1915. This theory was born from the necessity of having a relativistic approach to gravitational interactions and thus, it became the next step in the theory of gravitation. In order to establish a solid theory, the author had to use concepts of differential geometry and tensor calculus.

Once the theory was formulated it underwent several validations and was contrasted experimentally. The new formulation of gravitation could explain some observable effects that were unexplained by Newton's gravitation theory, such as minute anomalies in the orbits of Mercury. Moreover, the new theory was able to predict some phenomena that was not discovered until then, like the bending of light beams when they go near a field source (such effect was detected by Arthur Eddington, who organized an expedition to observe a solar eclipse in 1919 in order to measure the deflection of the light by the sun's gravitational field). Other effects predicted by Einstein's gravitation theory were the gravitational time dilatation, the *redshift* of electromagnetic signals emitted by elements such as hydrogen or helium from stellar objects and the existence of gravitational waves. Recently [Phys. Rev. Lett. 116, 061102. 2016] gravitational waves have been successfully detected for first time by the LIGO program. Such gravitational waves were produced by the coalescence of two black holes.

The breaking idea behind the General Relativity theory is that there is no such force as gravity: there is only geometry. The space-time is described as a 4-dimensional manifold with a metric and is a generalization of a Minkowski space-time which results to have a *flat* geometry. Therefore, a space-time in which there are gravitational field sources is not flat anymore according to Einstein and is considered a curved space. Therefore, objects such as light of masses move accordingly to the curvature of the space.

The culmination to General Relativity theory are the *Einstein field equations*, a set of 10 coupled partial differential equations in terms of the metric tensor g_{ab}. These equations are most commonly expressed in tensorial form and are expressed as

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = \frac{8\pi G}{c^4}T_{ab}. \tag{0.1}$$

The left side of the equation has all the necessary information about the *geometry* of the space-time: the Ricci tensor, the scalar curvature and the metric. On the other side of the equation there is the *energy momentum tensor*, which describes the distribution of energy and mass along the space. Thus, a distribution of matter is responsible for changes in the space geometry and at the same time, the very same geometry of the space is the responsible of gravitational interactions.

Albert Einstein, after he formulated these equations, said that there would be very difficult to find exact solutions for them. Even so, the first known exact solution was found by Karl Schwarzschild one year later (1916). This solution represented the geometry of the space-time of the outside of a static star with spherical symmetry in vacuum. Later, Reissner (1916) and Nordström (1918) generalized the Schwazschild solution to stars with electric charge and 45 years later, Roy P. Kerr (1963) found a solution for the field equations with axial symmetry and stationary in vacuum.

From that moment, the theory has been being complemented with huge mathematical background in geometry and tensor calculus in order to find more formal solutions to Einstein equations. Symmetry groups of the equations have been studied exhaustively as well as algebraically special space-times and many generation techniques to obtain new solutions from known ones. All of this caused the discovery of lots of new solutions to the field equations. Today, a hundred years after the formulation of the General Relativity equations, there exists a great number of known exact solutions (most of them without a physical meaning). A good compilation of exact solutions can be found at Kramer [12] (which added a bunch of new exact solutions in its second edition, in 2003).

The study of Einstein spaces (space-times that are solution to the Einstein equations) and their properties is another important field of study in General Relativity. An important result is the classification of the Riemann and Ricci tensors, both of them essential elements in the geometry of the space. These classifications allow to group solutions that share certain geometric properties. The classification of the Riemann tensor (specifically, the Weyl tensor) was first given by Petrov [9] (1954) and has been generalized to spaces of arbitrary dimension and metric. The classification of the Ricci classifies space-times according to the energy-momentum tensor.

The aim of this project is to show in detail all the concepts and formalisms that are used in General Relativity theory that lead to the previously mentioned classifications. Usually, the basics and details are omitted in those publications that show the final results and for this reason this Final Degree Project serves as a walk-through from the basics to the results (in this case, the classifications).

The first chapter is a summary of all necessary concepts of differential geometry to understand the origin of Lorentzian manifolds and the curvature of the space-time.The purpose of this chapter is not to make a deep study on the subjects of geometry, but to define the main concepts and show the most important results. At the beginning, the definition of manifold is given and from there, a series of concepts and results such as tangent space, tensor o metric are shown until the definition of the curvature tensors, the most important objects in General Relativity.

At the beginning of *the second chapter*, we introduce Lorentzian spaces, Lorentz metric and also the tetrad formalism, a very useful choices of vector basis that are related to the metric. We also introduce the Hodge operator and the Lorentz transformations. Later we introduce the bivector formalism, which allows us to simplify the problem of classifying the Weyl tensor. This tensor is defined in the same chapter as a trace-free part of the Riemann curvature tensor. After showin some results regarding this tensor, we proceed to formulate the algebraic problem that allows to classify it. Using the bivector formalism, the Weyl tensor will be regarded as a linear map, and the classification problem is finally reduced to an eigenvalue problem. Finally, another criterion for classifying the Weyl tensor is given: the method of principal null directions. The latter classification will be shown to be equivalent to the Petrov algebraic classification.

Lastly, in *the third and final chapter*, the full classification for second order symmetric tensors (as the Ricci tensor) is given. Again, this classification is given by the eigenvalues of the Ricci tensor regarded as a linear map and then we show that the algebraic type of the Ricci tensor and the energy momentum tensor are the same. At last we take those Einstein spaces that correspond to an electromagnetic field classify them. Finally, we do the same with spaces that are modeled by a fluid.

Chapter 1

Differential Geometry

This chapter aims to be a summary of the basic concepts of differential geometry from the definition of differentiable manifold to the concept of curvature and the curvature tensors. General relativity is built on terms of differential geometry and tensor calculus is the most common tool. First, the concept of smooth manifold will be introduced. On this concept, we will define tangent vector and the tensor space of a manifold, a key concept in geometry, as well as forms, element of the dual space of the tangent space. A generlization of tensors and forms are the tensors, which are the basic objects in general relativity and we will introduce some properties and operations that can be done with them.

The structure of a manifold can be completed with a connection and a metric tensor. From the Levi-Civita connection, which is strongly related to the metric, we define the Riemann curvature tensor and from it, the Ricci tensor. The latter one appears in the Einstein field equations and for this reason is why it is studied here.

The aim of this chapter is not to make an exhaustive study of all the differential geometry, but to define the basic concepts and enunciate the most important results that are important to understand the basics of general relativity. All the results shown in this chapter can be found in [5] and [12].

1.1 Differentiable Manifolds

Manifolds are spaces that are locally homeomorphic to $\mathbb{R}^n$ and are the most basic structure in the study of differential geometry and general relativity. We start by defining the concept in a topological way and then it will be possible to introduce the concept of differentiable manifold.

Definition 1.1.1. A *topological manifold* M of dimension n is a topological space with the following properties:

(i) M is Haussdorf: $\forall p, q \in M$, $p \neq q$, $\exists U, V$ neighborhoods of p, q such that $U \cap V = \emptyset$.

(ii) Any point $p \in M$ has a neighborhood U that is homeomorphic to an open set of $\mathbb{R}^n$

(iii) The topology of M has a countable basis, i.e., M satisfies the second countability axiom.

According to this definition, a manifold of dimension 0 must be a countable set equipped with the discrete topology, and a manifold of dimension 2 is locally homeomorphic to an open set of $\mathbb{R}^2$. A sphere is a topological manifold of dimension 2, but a cylinder is not, because it has what we will define as a *boundary*.

Definition 1.1.2. Let us consider the space

$$\mathbb{H}^n = \left\{(x^1, \ldots, x^n) \in \mathbb{R}^n | x^n \geq 0\right\}.$$

A *topological manifold with boundary* is a Hausdorff topological space M with a countable basis of open sets, such that $\forall p \in M$, $\exists V$ neighborhood of p that is homeomorphic to an open subset of $\mathbb{H}^n \setminus \partial\mathbb{H}^n$ or to an open subset of $\mathbb{H}^n$, with p mapped to a point in $\partial\mathbb{H}^n$ by the corresponding homeomorphism. In this case, p is said to be a *boundary point*. The set of boundary points of M is ∂M and is a manifold of dimension $(n-1)$.

A manifold M is characterized as locally homeomorphic to an open subset of $\mathbb{R}^n$. For each open set $U \subset M$ there is a homeomorphism $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$. Each pair (U, φ) is called a *coordinate system* or *chart*. The inverse map φ^{-1} is called a *parametrization* of M. When two neighborhoods of M overlap, we can define a coordinate change between the two coordinate systems (Figure 1.1). In a differentiable manifold, these coordinate changes are differentiable.

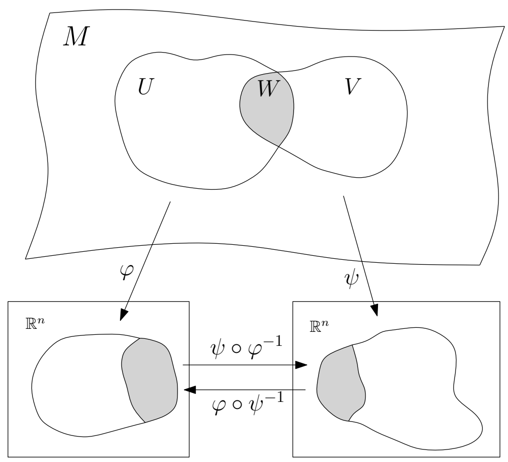


Figure 1.1: Representation of a coordinate change

Definition 1.1.3. An *n-dimensional differentiable or smooth manifold* M is a topological manifold and a family of coordinate systems $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, with $U_\alpha \subset M$ such that:

(i) All the neighborhoods U_α cover M, i.e., $\bigcup\limits_\alpha U_\alpha = M$.

(ii) All the coordinate change maps are $\mathcal{C}^\infty$, i.e., let $W = U_\alpha \cap U_\beta$, then:

$$\begin{aligned} \Phi \quad &= \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(W) \to \varphi_\beta(W) \\ \Phi^{-1} \quad &= \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(W) \to \varphi_\alpha(W) \end{aligned}$$

are both $\mathcal{C}^\infty$.

(iii) The family $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ is maximal with respect to (i) and (ii). Any family $\mathcal{A}\{(U_\alpha, \varphi_\alpha)\}$ that satisfies (i) and (ii) is called an *atlas*.

Now that differentiable manifolds have been defined, we can define functions on these structures and require them to be differentiable. From now on, *differentiable* will denote $\mathcal{C}^\infty$-differentiability.

Definition 1.1.4. Let M and N be two differentiable manifolds of dimension m and n respectively. A function $f : M \to N$ is said to be *locally differentiable at $p \in M$* if there are coordinate charts (U, φ) of M and (V, ψ) of N, with $p \in U$ and $f(U) \subset V$ such that

$$\hat{f} := \psi \circ f \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^m \to \mathbb{R}^n$$

is differentiable. The map $\hat{f}$ is the local representation of f at p.

The function is said to be *globally differentiable* if it is locally differentiable for all $p \in M$.

A map f is said to be a *diffeomorphism* if it is bijective and its inverse is also differentiable. Therefore, two manifolds are considered the same if they are diffeomorphic.

1.2 The tangent space

In this section we will define tangent vectors on a smooth manifold. The definition of tangent vector on a smooth manifold is just a generalization of tangent vector on a surface of $\mathbb{R}^n$.

We start by defining tangent vector on a manifold.

Definition 1.2.1. Let $c : (-\epsilon, \epsilon) \subset \mathbb{R} \to M$ be a differentiable curve on a manifold M. Consider all the functions of $\mathcal{C}^\infty(p)$, $f : M \to \mathbb{R}$ that are differentiable at $c(0) = p$. We say that the *tangent vector to the curve c at p* is the operator $\dot{c}(0) : \mathcal{C}^\infty(p) \to \mathbb{R}$ defined by

$$\dot{c}(0)(f) = \frac{d(f \circ c)}{dt}(0).$$

A tangent vector to M at p is a tangent vector to some differentiable curve c in M such that $c(0) = p$. The *tangent space* at p is the space of all tangent vectors at p and is denoted by T_pM.

The tangent vector $\dot{c}(0)$ evaluated at f represents the directional derivative of f in the direction of c at p. We will denote the tangent vector $\dot{c}(0)$ as $\boldsymbol{v}$. Since it is a derivation, a tangent vector fulfills the following properties:

(i) $\boldsymbol{v}(f + h) = \boldsymbol{v}(f) + \boldsymbol{v}(h)$,

(ii) $\boldsymbol{v}(fh) = h\boldsymbol{v}(f) + f\boldsymbol{v}(h)$,

(iii) $\boldsymbol{v}(cf) = c\boldsymbol{v}(f)$,

where $f, h \in \mathcal{C}^\infty(p)$ and c is a constant.

Given a map $f \in \mathcal{C}^\infty(p)$, its local representation is $\hat{f} = f \circ \varphi^{-1}$. The local representation of a curve c is $\hat{c} = \varphi \circ c = (x^1(t), \dots, x^n(t))$. We can express the operator $\dot{c}(0)$ in terms of local coordinates:

$$\begin{aligned}\hat{c}(0)(f) &= \frac{d(f \circ c)}{dt}(0) = \frac{d}{dt}\left((f \circ \varphi^{-1})(\varphi \circ c)\right)(0) = \frac{d}{dt}\left(\hat{f}\left(x^1(t), \dots, x^n(t)\right)\right)_{|t=0} \\ &= \frac{dx^i}{dt}(0)\frac{\partial \hat{f}}{\partial x^i}(\hat{c}(0)) = \left(\dot{x}^i(0)\left(\frac{\partial}{\partial x^i}\right)_{\varphi(p)}\right)(\hat{f}).\end{aligned}$$

Here the *Einstein summation convention* has been used. When the same index appears in a lower and upper position, it is assumed that there is a sum on this index.

The tangent vector can be written as $\dot{c}(0) = \dot{x}^i(0)\left(\frac{\partial}{\partial x^i}\right)_p$. The term $\left(\frac{\partial}{\partial x^i}\right)_p$ is the tangent vector associated to the coordinate curve c_i at p:

$$c_i(t) := (x^1, \dots, x^{i-1}, x^i + t, x^{i+1}, \dots, x^n), \text{ with } c_i(0) = p.$$

Proposition 1.2.2. *The tangent space of a smooth manifold M at p is a vectorial space of dimension n, and the operators $\left(\frac{\partial}{\partial x^i}\right)_p$ form a basis for this space.*

The basis $\left\{\left(\frac{\partial}{\partial x^i}\right)_p\right\}_{i \in \{1,\dots,n\}}$ is determined by the coordinate chart at p and is called *associated basis* to that chart or *holonomic frame*. We can take a general basis $\{\boldsymbol{e}_i\}$ that is not associated to the previous chart.

The disjoint union of all tangent spaces T_pM of M at all points is called the *tangent bundle* and is denoted by $TM = \bigcup_p T_pM$. We can define functions on TM that give a tangent vector for each $p \in M$.

Definition 1.2.3. Consider a smooth manifold M and its tangent bundle TM. A *vector field* X is defined as a map $X : M \to TM$ that assigns a tangent vector to a point p, i.e. $X(p) := X_p \in T_pM$. The vector field is *differentiable* if this map is differentiable. The set of all vector fields on M is denoted by $\mathfrak{X}(M)$.

Proposition 1.2.4. *Let U be a neighborhood of $p \in M$ and a vector field $X \in \mathfrak{X}(M)$. Let $x : U \to \mathbb{R}^n$ be a coordinate chart. Then, the restriction of X to the open set U is differentiable if and only if $X_p = X^i(p)\left(\frac{\partial}{\partial x^i}\right)_p$ for some differentiable functions $X^i : U \to \mathbb{R}$.*

Definition 1.2.5. Consider a differentiable function $f : M \to \mathbb{R}$ and a vector field X on M. The *directional derivative of f along X* is the function $X \cdot f : M \to \mathbb{R}$ such that $X \cdot f(p) = X_p \cdot f := X_p(f)$. We can consider a vector field as an operator $X : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$.

Consider two vector fields X and Y on M. Then the compositions $X \circ Y$ and $Y \circ X$ are not vector fields in general, but $X \circ Y - Y \circ X$ is a vector field, and it is unique.

Definition 1.2.6. Consider two vector fields $X, Y \in \mathfrak{X}$. The *Lie bracket* or *commutator* of X and Y is the vector field

$$[X, Y] = X \circ Y - Y \circ X.$$

Considering a chart $x : U \subset M \to \mathbb{R}^n$, the vector fields X and Y have the expressions $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^i \frac{\partial}{\partial x^i}$. Computing the expression of the Lie bracket in coordinates yields to the result:

$$[X, Y] = (X \cdot Y^i - Y \cdot X^i)\frac{\partial}{\partial x^i}.$$

The commutator has the following properties: given $X, Y, Z \in \mathfrak{X}(M)$,

(i) Bilinearity: for $\alpha, \beta \in \mathbb{R}$, $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$ and $[Z, \alpha X + \beta Y] = = \alpha[Z, X] + \beta[Z, Y]$.

(ii) Antisymmetry: $[X, Y] = -[Y, X]$.

(iii) *Jacobi identity*: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

(iv) *Leibnitz rule*: for any $f, g \in \mathcal{C}^\infty$, $[fX, gY] = fg[X, Y] + f(X \cdot g)Y - g(Y \cdot f)X$.

Two vector fields *commute* if their commutator is zero, i.e., $[X, Y] = 0$. For a given basis $\{\boldsymbol{e}_i\}$, we define the commutators coefficients D^k_{ij} as:

$$[\boldsymbol{e}_i, \boldsymbol{e}_j] = D^k_{ij}\boldsymbol{e}_k.$$

Since the commutator is antisymmetric, we have $[\boldsymbol{e}_i, \boldsymbol{e}_j] = -[\boldsymbol{e}_j, \boldsymbol{e}_i]$, and hence $D^k_{ij} = -D^k_{ji}$. From the Jacobi identity, we can deduce $D^m_{l[i}D^l_{jk]} = 0$, where the brackets in the subindices denote the anti-symmetrization on those indices, i.e.

$$v_{[a}u_{b]} = \frac{1}{2}(v_a u_b - v_b u_a).$$

We have defined the tangent space of a smooth manifold. Now it is possible to define the *derivative of a differentiable map* $f : M \to N$ between smooth manifolds.

Definition 1.2.7. Let $f : M \to N$ be a differentiable function between manifolds. At a point $p \in M$, the *derivative* or *differential* of f is defined as

$$\begin{aligned} (df)_p :\ T_pM &\longrightarrow T_{f(p)}N, \\ v &\longmapsto \frac{d(f \circ c)}{dt}(0), \end{aligned}$$

where c is a differentiable curve on M such that $c(0) = p$ and $\dot{c}(0) = v$.

The definition of $(df)_p$ does not depend on the choice of the curve c. Given a parametrization $\varphi : U \subset \mathbb{R}^n \to M$, and a vector $\boldsymbol{v} \in \mathbb{R}^n$, its image by $(d\varphi)$ is

$$(d\varphi)_{\varphi^{-1}(p)}(\boldsymbol{v}) = \frac{\partial x^i}{\partial x^j} v^j \frac{\partial}{\partial x^i} = v^i \frac{\partial}{\partial x^i}.$$

The coordinates of $d\varphi(v)$ in the basis $\frac{\partial}{\partial x^i}$ are the same as the components of $\boldsymbol{v}$ in $\mathbb{R}^n$.

Now we will see the concept of integral curve of a manifold given a vector field.

Definition 1.2.8. Let M be a smooth manifold and $X \in \mathfrak{X}$ a vector field. A curve $c : (-\epsilon, \epsilon) \to M$ is said to be an *integral curve* if $\dot{c}(t) = X_{c(t)}$.

The local representation of a vector field $X \in \mathfrak{X}(M)$ is $\hat{X} = d\varphi^{-1} \circ X \circ \varphi$, where φ is a parametrization of M. It maps points of $\mathbb{R}^n$ to vectors in $\mathbb{R}^n$. Then, in local coordinates, an integral curve c satisfies $\dot{\hat{c}}(t) = \hat{X}(\hat{c}(t))$. For each coordinate, this defines an ordinary differential equation and, the whole system with n equations with the initial value $c(0) = p$ determines an integral curve at p.

We shall now define the orientability of a manifold. It is important to distinguish between the orientable manifolds and the not orientable ones. Consider two basis of a vector space V, named $\beta = \{\boldsymbol{u}_i\}_{i=1}^n$ and $\beta' = \{\boldsymbol{v}_i\}_{i=1}^n$. It exists a linear map S such that $S\boldsymbol{u}_i = \boldsymbol{v}_i$ for all $i = 1, \ldots, n$. We say that these two basis are *equivalent* if $\det(S) > 0$. An *orientation* of V is the assignment of positive sign to all the basis of the equivalence class defined by this equivalence relation.

An *orientation* of a smooth manifold M is a choice of orientations for all tangent spaces T_pM. These orientations must fit together smoothly: for each point $p \in M$, it exists a neighborhood U and a chart φ such that the map $(d\varphi)_p$ preserves orientation at every point of U. That means that a positive oriented basis of T_pM is mapped to a positive (or negative) oriented basis of $\mathbb{R}^n$. A manifold is said to be *orientable* if it admits an orientation.

1.3 The tensor space

We have defined the tangent space of a manifold, which is a vector space. In this section we will define its dual space, T_p^*M and a generalization of the tangent space and its dual: the tensor space. Tensor calculus is a very important tool in general relativity and will be introduced in this section.

Definition 1.3.1. Let M be a smooth manifold and T_pM its tangent space at a point $p \in M$. The *dual space of* T_pM or *cotangent space*, T_p^*M, is the set $\mathcal{L}(T_pM, \mathbb{R})$ of all linear maps $\boldsymbol{\sigma} : T_pM \to \mathbb{R}$. Elements of the dual space are called *forms*.

The cotangent space is also a vector space of the same dimension of T_pM. If $\{\boldsymbol{e}_i\}$ is a basis of T_pM, we define its *dual basis* as $\left\{\boldsymbol{\omega}^i \in T_p^*M \text{ such that } \boldsymbol{\omega}^i(\boldsymbol{e}_j) = \delta^i_j\right\}$.

Consider a function on a smooth manifold $f : M \to \mathbb{R}$. Then, the *differential of* f is a form $(df)_p : T_pM \to \mathbb{R}$. Given $\boldsymbol{v} \in T_pM$, $(df)_p(\boldsymbol{v}) = \boldsymbol{v} \cdot f$. If $\boldsymbol{v} = v^i \frac{\partial}{\partial x^i}$, then $(df)_p(\boldsymbol{v}) = v^i \frac{\partial \hat{f}}{\partial x^i}(x(p))$, where $x : M \to \mathbb{R}^n$ is a coordinate chart. Let us take the coordinate functions $x^i : M \to \mathbb{R}$, then: $(dx^i)_p \left(\frac{\partial}{\partial x^j}\right) = \delta^i_j$. Thus, $\{dx^i\}_{i=1}^n$ is the dual basis of $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^n$.

The differential of the previous function f can be expressed on terms of this dual basis: $(df)_p = \left(\frac{\partial}{\partial x^i}\right)_p (f) dx^i = f_{,i} dx^i$.

Definition 1.3.2. Consider a smooth manifold M and its tangent space T_pM on a point $p \in M$. A *k-covariant tensor* on M is a multilinear function $\boldsymbol{T} : (T_pM)^k \to \mathbb{R}$. The set of all k-tensors is a linear space, $\mathcal{T}^k(T_p^*M)$.

Note that 1-tensors are forms from the dual space. Consider now the tensors $T \in \mathcal{T}^r(T_p^*M)$ and $S \in \mathcal{T}^s(T_p^*M)$. Then we can define the *tensor product* as the $(r+s)$-tensor:

$$\boldsymbol{T} \otimes \boldsymbol{S}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r; \boldsymbol{v}_{r+1}, \ldots, \boldsymbol{v}_{r+s}) = \boldsymbol{T}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r) \cdot \boldsymbol{S}(\boldsymbol{v}_{r+1}, \ldots, \boldsymbol{v}_{r+s}).$$

Tensors can be defined over the dual space of T_pM: a *k-contravariant tensor* is a multilinear map $\boldsymbol{T} : (T_p^*M)^k :\to \mathbb{R}$. The space of all cotravariant tensors is also a linear space and is denoted by $\mathcal{T}_k(T_pM)$. The space $\mathcal{T}_1(T_pM)$ is $(T_p^*M)^* \simeq T_pM$. We can construct the space of *mixed* $\binom{k}{m}$*-tensors*, i.e., multilinear functions $\boldsymbol{T} : (T_pM)^k \times (T_p^*M)^m \to \mathbb{R}$. The space of $\binom{k}{m}$-tensors is denoted by $\mathcal{T}^k_m(T_p^*M, T_pM)$.

Proposition 1.3.3. *Consider the basis* $\{\boldsymbol{\omega}^i\}_{i=1}^n$ *of* T_p^*M *and* $\{\boldsymbol{e}_i\}_{i=1}^n$ *of* T_pM*. Then, the set*

$$\left\{\boldsymbol{\omega}^{i_1} \otimes \cdots \otimes \boldsymbol{\omega}^{i_k} \otimes \boldsymbol{e}_{j_1} \otimes \cdots \otimes \boldsymbol{e}_{j_m} \; | 1 \le i_1, \ldots i_k, j_1, \ldots, j_m \le n\right\}$$

is a basis of the space $\mathcal{T}^k_m(T_p^*M, T_pM)$.

Definition 1.3.4. Consider a tensor $\boldsymbol{T} \in \mathcal{T}^k(T_p^*M)$. Then, $\boldsymbol{T}$ is said to be *alternating* if $\boldsymbol{T}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_j, \ldots, \boldsymbol{v}_k) = -\boldsymbol{T}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_j, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_k)$. The space of these alternating tensors is denoted $\Lambda^k(T_p^*M)$

Given tensor $\boldsymbol{T} \in \mathcal{T}^k(T_p^*M)$, we can build an *alternate* tensor:

$$\mathrm{Alt}(\boldsymbol{T}) := \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}(\mathrm{sgn}(\sigma))(\boldsymbol{T}\circ\sigma),$$

where $\mathfrak{S}_k$ is the symmetric group of all permutations of k elements. If $\boldsymbol{T}$ is alternating, $\mathrm{Alt}(\boldsymbol{T}) = \boldsymbol{T}$.

Definition 1.3.5. Consider two tensors $\boldsymbol{T} \in \Lambda^k(T_p^*M)$ and $\boldsymbol{S} \in \Lambda^r(T_p^*M)$. The *wedge product* of $\boldsymbol{T}$ and $\boldsymbol{S}$ is:

$$\boldsymbol{T}\wedge\boldsymbol{S} := \frac{(k+r)!}{k!r!}\mathrm{Alt}(\boldsymbol{T}\otimes\boldsymbol{S}) \in \Lambda^{k+r}(T_p^*M).$$

If $\boldsymbol{T} \in \Lambda^k(T_p^*M)$ and $\boldsymbol{S} \in \Lambda^r(T_p^*M)$, then $\boldsymbol{T}\wedge\boldsymbol{S} = (-1)^{kr}\boldsymbol{S}\wedge\boldsymbol{T}$. We saw in proposition 1.3.3 how to build a basis of a tensor space. Te following proposition shows how to build basis for alternating spaces:

Proposition 1.3.6. *Consider a smooth manifold M of dimension n and a point $p \in M$. Let $\{\boldsymbol{\omega}^i\}_{i=1}^n$ be a basis for $\mathcal{T}^1(T_p^*M) = T_p^*M$. Then, the set*

$$\left\{\boldsymbol{\omega}^{i_1}\wedge\cdots\wedge\boldsymbol{\omega}^{i_k} \ : \ 1 \le i_1 \le \cdots \le i_k \le n\right\}$$

is a basis for $\Lambda^k(T_p^*M)$. As a consequence, $\dim\left(\Lambda^k(T_p^*M)\right) = \binom{n}{k}$.

As shown in definition 1.2.3, we can define *tensor field* as a map $\boldsymbol{T} : M \to \mathcal{T}^k_m(T^*M, TM)$ from a manifold M to its *tensor bundle*. In local coordinates, this tensor field can be expressed as:

$$\boldsymbol{T}(p) = \boldsymbol{T}_p = T^{j_1\ldots j_m}_{i_1\ldots i_k}(p)(dx^{i_1})_p\otimes\cdots\otimes(dx^{i_k})_p\otimes\left(\frac{\partial}{\partial x^{j_1}}\right)_p\otimes\cdots\otimes\left(\frac{\partial}{\partial x^{j_m}}\right)_p,$$

$$1 \le i_1,\ldots i_k, j_1\ldots j_m \le n.$$

Remark 1.1. For simplicity, many times the tensor field $\boldsymbol{T}$ is denoted only by its components, $T^{j_1\ldots j_m}_{i_1\ldots i_k}$. A $(\,{}^{k}_{m}\,)$-tensor can be considered as a map $\boldsymbol{T} : (T_pM)^k \to \mathcal{T}_m(T_pM)$. If the covariant part of $\boldsymbol{T}$ is expressed in terms of the dual basis of T_pM, then the image of k tangent vectors $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k$ is the tensor

$$\boldsymbol{T}(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k) = T^{j_1\ldots j_m}_{i_1\ldots i_k}\cdot v_1^{i_1}\cdots v_k^{i_k}.$$

Definition 1.3.7. Given a $(\,{}^{s}_{r}\,)$-tensor $\boldsymbol{T}$, the *contraction of the ith contravariant index with the jth covariant index,* c_i^j, is a map $\mathcal{T}^s_r(T_p^*M, T_pM) \to \mathcal{T}^{s-1}_{r-1}(T_p^*M, T_pM)$ that sends the tensor $\boldsymbol{T}$ to $c_i^j(\boldsymbol{T})$ such that (in local coordinates $(x^1,\ldots,x^n)$):

$$\begin{aligned}&c_i^j(\boldsymbol{T})(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{s-1},\boldsymbol{\omega}_1,\ldots,\boldsymbol{\omega}_{r-1}) = \\ &\quad = \sum_{k=1}^n \boldsymbol{T}(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{j-1},\frac{\partial}{\partial x^k},\boldsymbol{v}_j,\ldots,\boldsymbol{v}_{s-1},\boldsymbol{\omega}_1,\ldots,\boldsymbol{\omega}_{i-1},dx^k,\boldsymbol{\omega}_i,\ldots,\boldsymbol{\omega}_{r-1})\end{aligned}$$

for all tangent vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{s-1} \in T_pM$ and forms $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_{r-1} \in T_p^*M$. The coordinates of $c_i^j(\boldsymbol{T})$ are

$$c_i^j(\boldsymbol{T})^{j_1\cdots j_{s-1}}{}_{i_1\cdots i_{r-1}} = T^{j_1\cdots j_{j-1}kj_j\cdots j_{s-1}}{}_{i_1\cdots i_{i-1}ki_i\cdots i_{r-1}}\,.$$

There is an important operation on covariant tensors called the *pull-back* by a smooth map.

Definition 1.3.8. Let $f : M \to N$ be a smooth map between manifolds. A differentiable k-covariant tensor field $\boldsymbol{T}$ on N defines a k-covariant tensor $f^*\boldsymbol{T}$ on M in the following way:

$$(f^*\boldsymbol{T})_p(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) = \boldsymbol{T}_{f(p)}((df)_p\boldsymbol{v}_1, \ldots, (df)_p\boldsymbol{v}_k),$$

for $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in T_pM$.

Now we will introduce the term of differential form. We have previously seen that the differential of a function is a form on a manifold.

Definition 1.3.9. Let M be a smooth manifold. A *k-form* on M is a field of alternating tensors $\boldsymbol{\omega}$ such that $\boldsymbol{\omega}_p \in \Lambda^k(T_p^*M)$.

Given a chart x, a k-form can be expressed as $\boldsymbol{\omega}_p = \omega_{i_1\ldots i_k}(p)(dx^{i_1})_p \wedge \cdots \wedge (dx^{i_k})_p$ for some functions $\omega_{i_1\ldots i_k} : M \to \mathbb{R}$ and $1 \leq i_1 < \cdots < i_k \leq n$. We define the *exterior derivative* of $\boldsymbol{\omega}$ as:

$$d\boldsymbol{\omega} = d\omega_{i_1\ldots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Expressing the differential of a function in coordinates, it yields:

$$d\boldsymbol{\omega} = \omega_{i_1\ldots i_k,i}dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where the comma in the subindex denotes directional derivative of the function in the direction of $\frac{\partial}{\partial x^i}$. An important property of differential forms is that $d(d\boldsymbol{\omega}) = 0$ for any form $\boldsymbol{\omega}$.

Definition 1.3.10. Let M be a smooth manifold of dimension n. A *volume form* or *volume element* of M is a differentiable n-form $\boldsymbol{\omega}$ such that $\boldsymbol{\omega}_p \neq 0 \ \forall p \in M$.

If there exists a volume form, then the manifold is orientable.

1.4 Riemannian Manifolds

In order to define all the metric properties on a smooth manifold such as angle between vectors or distance, we have to add more structure to the manifold defined in definition 1.1.1 and definition 1.1.3, that is, the metric tensor. We will define differentiation of vector fields and geodesics.

Definition 1.4.1. Consider a tensor $\boldsymbol{g} \in \mathcal{T}^2(T_p^*M)$ and two vector $\boldsymbol{v}, \boldsymbol{u} \in T_pM$ on a smooth manifold M. The tensor $\boldsymbol{g}$ is:

(i) *symmetric* if $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{u}) = \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v})$.

(ii) *non degenerate* if $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{u}) = 0$ for all $\boldsymbol{u}$ implies $\boldsymbol{v} = 0$.

(iii) *positive definite* if $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) > 0$ for all $\boldsymbol{v} \neq 0$.

For every point $p \in M$ and a local chart $x : V \to \mathbb{R}^n$, a 2-tensor $\boldsymbol{g}$ can be written

$$\boldsymbol{g} = g_{ij} dx^i \otimes dx^j,$$

where $g_{ij} = \boldsymbol{g}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$. The tensor $\boldsymbol{g}$ fulfills the properties defined in definition 1.4.1 if the matrix (g_{ij}) also has these properties.

Definition 1.4.2. A 2-covariant tensor field $\boldsymbol{g}$ on a smooth manifold $M$is said to be a *metric tensor* if it is symmetric and positive definite. A smooth manifold M equipped with a metric is a *Riemannian manifold*, and is denoted by $(M, \boldsymbol{g})$.

Definition 1.4.3. A metric tensor $\boldsymbol{g}$ on a smooth manifold M defines the *scalar product* of two tangent vectors $\boldsymbol{u}, \boldsymbol{v}$:

$$\boldsymbol{u} \cdot \boldsymbol{v} := \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = g_{ij} u^i v^j.$$

Since the metric is non degenerate, we can define the *inverse of the metric* $\boldsymbol{g}$, $\boldsymbol{g}^{-1}$, a 2-contravariant tensor:

$$\boldsymbol{g}^{-1} = g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$$

with $g_{ik} g^{kj} = \delta_i^j$ and $g_{ij} g^{ij} = n$. We use the metric to define the *metric dual* of a tangent vector $\boldsymbol{v} = v^i \frac{\partial}{\partial x^i}$.

Definition 1.4.4. Given a Riemannian manifold $(M, \boldsymbol{g})$. The *metric dual* of a tangent vector $\boldsymbol{v} \in T_pM$ is a 1-form $\boldsymbol{\omega} \in T_p^*M$ such that

$$\boldsymbol{\omega}(\boldsymbol{u}) = \boldsymbol{g}(\boldsymbol{v}, \boldsymbol{u})$$

for any $\boldsymbol{u} \in T_pM$. In local coordinates $(x^1, \ldots, x^n)$, the metric dual of $\boldsymbol{v}$ is expressed as

$$\boldsymbol{\omega} = g_{ij} v^i dx^j.$$

To compute the metric dual of a vector is denoted as *lowering indices*. If we have a tangent vector $v^i \in T_pM$ (we denote the vector only by its components in local coordinates), then we say that its metric dual is the form v_i, with $v_i = g_{ij} v^j$. Using this notation, we can write the scalar product of two tangent vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ as

$$\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = g_{ij} u^i v^j = u_j v^j = u^i v_i.$$

We can define the metric dual of a 1-form $\boldsymbol{\omega}$ as well. Using the expression of the inverse of the metric in local coordinates, we can *raise the index* of the form $\omega_i \in T_p^*M$ to obtain a tangent vector $\omega^i = g^{ij}\omega_j$.

Raising and lowering indices operations can be performed on general $\binom{s}{r}$ tensors, e.g.if we raise the last index of the $\binom{s}{r}$ tensor $\boldsymbol{T} = T^{i_1\cdots i_s}{}_{j_1\cdots j_r}$ we obtain the tensor

$$T^{i_1\cdots i_s}{}_{j_1\cdots j_{r-1}}{}^{k} = g^{kj_r}T^{i_1\cdots i_s}{}_{j_1\cdots j_r}\,.$$

An important element in differential geometry is the derivative of a vector field Y with respect to another vector field X: $\nabla_X Y$. First, we have to define an extra structure on the manifold: the connection.

Definition 1.4.5. An *affine connection* on a smooth manifold M is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that:

(i) $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$,

(ii) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$,

(iii) $\nabla_X(fY) = (X\cdot f)Y + f\nabla_X Y$,

with $X, Y, Z \in \mathfrak{X}(M)$ and $f, g \in \mathcal{C}^\infty(M,\mathbb{R})$.

We say that $\nabla_X Y$ is the *covariant derivative* of Y along X. We can generalize the definition of the covariant derivative operator ∇ to tensors. This operator maps $\binom{s}{r}$ tensors to $\binom{s+1}{r}$ tensors, i.e.

$$\nabla : \mathcal{T}_r^s(T_pM, T_p^*M) \to \mathcal{T}_r^{s+1}(T_pM, T_p^*M).$$

The image of a tensor $\boldsymbol{T}$ by this operator has some coordinates that have to be determined. If $x : W \to \mathbb{R}^n$ are local coordinates on some open set $W \subset M$, then

$$\nabla\boldsymbol{T} = T^{i_1\cdots i_r}{}_{j_1\cdots j_s;k}\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_s}\otimes dx^k. \tag{1.1}$$

The directional covariant derivative of a $\binom{s}{r}$ tensor $\boldsymbol{T}$ along a vector field X is

$$\nabla_X\boldsymbol{T} := \nabla(\boldsymbol{T}, X) = T^{i_1\cdots i_r}{}_{j_1\cdots j_s;k}X^k\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_s}.$$

Proposition 1.4.6. *Let ∇ be an affine connection on M, let $X, Y \in \mathfrak{X}(M)$ and $p \in M$. Then, $(\nabla_X Y)_p$ only depends on X_p and on the values of Y along the curve tangent to X at p. If $x : W \to \mathbb{R}^n$ are local coordinates on an open set $W \subset M$ and*

$$X = X^i\frac{\partial}{\partial x^i} \quad Y = Y^i\frac{\partial}{\partial x^i},$$

then, we can build an expression for the covariant derivative of Y along X:

$$\nabla_X Y = \left(X \cdot Y^i + \Gamma^i_{jk} X^j Y^k \right) \frac{\partial}{\partial x^i},$$

where we have defined n^3 differentiable functions, $\Gamma^i_{jk} : W \to \mathbb{R}$, called Christoffel symbols*, defined by:*

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} = \Gamma^i_{jk} \frac{\partial}{\partial x^i}, \qquad \nabla_{\frac{\partial}{\partial x^j}} dx^i = -\Gamma^i_{jk} dx^k$$

For simplicity, the covariant derivative along the ith coordinate direction $\frac{\partial}{\partial x^i}$ will be denoted by ∇_i.

If we take $\boldsymbol{T}$ to be a tangent vector in eq. (1.1), $\boldsymbol{T} = T^i \frac{\partial}{\partial x^i}$, we can determine now the covariant derivative along a coordinate direction $\frac{\partial}{\partial x^k}$ (this result follows from the previous proposition):

$$T^i_{\ ;j} = T^i_{\ ,k} + \Gamma^i_{jk} T^k.$$

Locally, an affine connection is uniquely determined by specifying the Christoffel symbols on a coordinate neighborhood. The choices of Christoffel symbols on different charts are not independent, since they have to be the same on the overlapping points.

A *vector field along a differentiable curve* $c : I \to M$ is a differentiable map $V : I \to TM$ such that $V(t) \in T_{c(t)}M$, $\forall t \in I$. If V is a vector field defined along a differentiable curve c, with $\dot{c} \neq 0$, the *covariant derivative of V along c* is a vector field defined along c given by

$$\frac{DV}{dt}(t) := \nabla_{\dot{c}(t)} V = (\nabla_X Y)_{c(t)}$$

for any vector fields X, Y such that $X_{c(t)} = \dot{c}(t)$ and $Y_{c(s)} = V(s)$ with $s \in (t - \epsilon, t + \epsilon)$ for some $\epsilon > 0$.

Definition 1.4.7. A vector field V defined along a curve $c : I \to M$ is said to be *parallel along c* if

$$\frac{DV}{dt} = 0, \quad \forall t \in I.$$

The curve c is called a *geodesic* of the connection ∇ if $\dot{c}$ is parallel along c, i.e.

$$\frac{D\dot{c}}{dt}(t) = 0, \quad \forall t \in I.$$

In local coordinates $x : W \to \mathbb{R}^n$, the condition for a vector field V to be parallel along c is

$$\dot{V}^i + \Gamma^i_{jk} \dot{x}^j V^k = 0, \quad (i = 1, \ldots, n).$$

By the Picard theorem, given a curve $c : I \to M$, a point $p \in c(I)$, a vector $\boldsymbol{v} \in T_pM$, there exists a unique vector field $V : I \to TM$ parallel along c such that $V(0) = \boldsymbol{v}$. This vector field is called the *parallel transport* of $\boldsymbol{v}$ along c.

Definition 1.4.8. Given an affine connection ∇ on M, the *torsion operator* T is a map $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = \left(\Gamma^i_{jk} - \Gamma^i_{kj}\right) X^j Y^k \frac{\partial}{\partial x^i},$$

for $X, Y \in \mathfrak{X}(M)$ at $W \subset M$ and given a local chart $x : W \to \mathbb{R}^n$. The connection is said to be *symmetric* if $T = 0$.

This map is bilinear and is defined by a $(1, 2)$-tensor field on M which has the form, in local coordinates,

$$T = \left(\Gamma^i_{jk} - \Gamma^i_{kj}\right) dx^j dx^k \frac{\partial}{\partial x^i}.$$

The condition for the connection to be symmetric in local coordinates is

$$\Gamma^i_{jk} = \Gamma^i_{kj}, \quad (i, j, k = 1, \ldots, n).$$

We said previously that the choice of Christoffel symbols determines the affine connection. In Riemannian manifolds, there is a special choice of connection called the *Levi-Civita connection*, with special geometric properties.

Definition 1.4.9. A connection ∇ in a Riemannian manifold $(M, \boldsymbol{g})$ is said to be *compatible* with the metric if

$$X \cdot \boldsymbol{g}(Y, Z) = \boldsymbol{g}(\nabla_X Y, Z) + \boldsymbol{g}(Y, \nabla_X Z)$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Theorem 1.4.10. *If $(M, \boldsymbol{g})$ is a Riemannian manifold, then there exists a unique connection ∇ on M which is symmetric and compatible with $\boldsymbol{g}$. In local coordinates $(x^1, \ldots, x^n)$, the Christoffel symbols for this connection are*

$$\Gamma^i_{jk} = \frac{1}{2} \sum_{l=1}^{n} g^{il} \left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right),$$

where g^{ij} are the coordinates of the inverse metric.

1.5 Curvature

One of the most important characteristics of a Riemannian manifold is the notion of *curvature*. One way to describe the curvature of a Riemannian manifold is the *Riemann curvature tensor*, defined from an affine connection.

Definition 1.5.1. The *curvature operator* R of a connection ∇ is the association of two vector fields $X, Y \in \mathfrak{X}(M)$ to the map $R(X, Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by:

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The curvature operator can be seen as a way of measuring the non-commutativity of the connection. This operator defines a $\binom{3}{1}$-tensor, the *Riemann curvature tensor* R,

$$R = R_{ijk}{}^l dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^l},$$

where coefficient $R_{ijk}{}^l$ is the l-th coordinate of the vector $R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^k}$ and are found to be

$$R_{ijk}{}^l = \Gamma^l_{jk,i} - \Gamma^l_{ik,j} + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jk}. \tag{1.2}$$

From now we assume that $(M, \boldsymbol{g})$ is a Riemannian manifold and ∇ is the Levi-Civita connection. From the Riemann curvature tensor, we can define a new 4-covariant tensor by *lowering* the last index of R with the metric $\boldsymbol{g}$:

$$R(X, Y, Z, W) = \boldsymbol{g}(R(X, Y)Z, W).$$

In a coordinate system, the coordinates of this 4-covariant Riemann curvature tensor are R_{ijkl} with

$$R_{ijkl} = R_{ijk}{}^m g_{ml}.$$

The Riemann curvature tensor has the following symmetries:

(i) *Bianchi identity.* $R_{ijkl} + R_{jkil} + R_{kijl} = 0$,

(ii) $R_{ijkl} = -R_{jikl}$,

(iii) $R_{ijkl} = -R_{ijlk}$,

(iv) $R_{ijkl} = R_{klij}$.

Another geometric object, which is very important in general relativity is the *Ricci tensor*.

Definition 1.5.2. The *Ricci curvature tensor* is the 2-covariant tensor locally defined as

$$Ric(X, Y) := \sum_{k=1}^{n} dx^k \left(R \left(X, \frac{\partial}{\partial x^k} \right) Y \right).$$

Locally, the Ricci tensor takes the form

$$Ric = R_{ij} dx^i \otimes dx^j, \qquad R_{ij} = R_{ikj}{}^k.$$

An important concept in Riemannian manifolds is the one of *conformal transformation*. It is important in general relativity, since conformally related spaces share some properties.

Definition 1.5.3. Let $(M, \boldsymbol{g})$ and $(N, \boldsymbol{h})$ be two Riemannian manifolds. A *conformal transformation* between these two manifolds is a smooth map $f : M \to N$ such that $f^*\boldsymbol{h} = \mathrm{e}^{2U}\boldsymbol{g}$, with $U \in \mathcal{C}^\infty(M)$. In detail, it means that

$$\boldsymbol{h}_p((df)_p\boldsymbol{u}, (df)_p\boldsymbol{v}) = \mathrm{e}^{2U(p)}\boldsymbol{g}_p(\boldsymbol{u}, \boldsymbol{v})$$

for all tangent vectors $\boldsymbol{u}, \boldsymbol{v} \in T_pM$. If such conformal map exists, we say that those two Riemannian manifolds are *conformally related*.

Given a Riemannian manifold $(M, \boldsymbol{g})$, a conformally related metric $\overline{\boldsymbol{g}}$ in M is a metric tensor such that

$$\overline{\boldsymbol{g}} = \mathrm{e}^{2U}\boldsymbol{g}, \qquad U \in \mathcal{C}^\infty(M).$$

All the elements of the Riemannian space $(M, \overline{\boldsymbol{g}})$ such as the Levi-Civita connection and thus the Riemann curvature tensor are locally related to the ones in the Riemannian space $(M, \boldsymbol{g})$ in the following way:

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j U_{,k} + \delta^i_k U_{,j} - g_{ab}g^{cd}U_{,d} \tag{1.3}$$

Chapter 2

The Weyl Tensor and Petrov Classification

General relativity consists on the study of the set of all Einstein spacetimes, i.e. 4-dimensional *Lorentziant manifolds* that are solution to the Einstein's field equations. These equations (eq. (0.1)) state a relation between the geometry of the space and the energy and mass. Therefore, gravitational fields are directly related to geometric structures such as the curvature of a manifold.

An important way of study of all Einstein spacetimes is to classify the curvature tensor R_{abcd}. In this chapter we will study a classical classification of spacetimes, named the *Petrov Classification.* This particular classification was studied in first place by A.Z. Petrov in 1954 and has been later generalized to manifolds with arbitrary dimension and signature.

The study of the curvature is achieved by decomposing the Riemann curvature tensor (RC tensor) intro a 0-trace part and a non 0-trace tensor. Petrov classification consists on an algebraic study the the 0-trace part of the RC tensor that is called *Weyl tensor.* As it will be seen, this tensor has the same symmetries than the RC tensor and it can be treated as a linear endomorphism of $\Lambda^2(T_pM)$ for each point in the Lorentzian manifold M. The study of different types of manifolds can be reduced to an eigenvalues problem if the correct tools are introduced.

There have been multiple approaches to the Petrov classification, but the most important have been the *spinor* and *bivector* calculus. The latter one is the method that will be used here, while spinor calculus is a powerful tool in general relativity and geometry (see [8]), but it is omitted in this project.

In chis chapter will be introduced the *complexified tangent space* of a manifold, where the components of a tangent vector can be complex numbers, and the concept of *null tetrad.* A tetrad is just a basis of tangent vectors chosen to make some calculus easier. For this reason, the classification of spacetimes must be invariant under Lorentz transformations and, as we will see, the eigenvalues of the Weyl map will depend only on invariants of curvature.

Another important thing of this classification is that is invariant under conformal transformations. Simply by looking at the Weyl tensor is possible, for instance, to distinguish spacetimes that are conformally flat.

Most of the results shown in this chapter can be found at [6], [12] and [16].

2.1 Lorentz spaces and tetrad formalism

In general relativity, *Einstein spaces*, i.e., smooth manifolds with a metric that are solution to the Einstein field equations are *pseudo-Riemannian* manifolds.

Definition 2.1.1. A *pseudo-Riemannian* manifold of dimension n is a pair $(M, \boldsymbol{g})$, where M is a smooth manifold of dimension n and $\boldsymbol{g}$ is a metric tensor that is not positive-definite.

The *signature* of a pseudo-Riemannian manifold is (p, q), and is given by any orthogonal basis $\{\boldsymbol{e}_i\}_{i=1}^n$ as follows: p is the number of vector of the basis such that $\boldsymbol{g}(\boldsymbol{e}_i, \boldsymbol{e}_i) < 0$, and q is the number of vectors such that $\boldsymbol{g}(\boldsymbol{e}_i, \boldsymbol{e}_i) > 0$. The signature can also be given by a number $s = q - p$.

An important case in general relativity are *Lorentzian manifolds*, that are pseudo-Riemannian manifolds with signature $(1, n-1)$. In the most common case in which the manifold is $\mathbb{R}^4$, the signature of this Lorentzian manifold is $(1, 3)$ or $s = 2$.

From now on we will assume $(M, \boldsymbol{g})$ to be 4-dimensional Lorentz space with signature $s = 2$, unless otherwise stated. The definition of Levi-Civita connection is the same for pseudo-Riemannian manifolds and all the tensors related to the curvature, like the RC tensor or the torsion are also defined in the same way. We may classify the subspaces of the tangent space of the manifold depending on the scalar product.

Definition 2.1.2. Let $(M, \boldsymbol{g})$ a Lorentzian manifold and $p \in M$ a point. A vector $\boldsymbol{v} \in T_pM$ is said to be:

(i) *space-like* if $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) > 0$,

(ii) *time-like* if $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) < 0$,

(iii) *null* if $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) = 0$.

A smooth curve $c : I \subset \mathbb{R} \to M$ is space-like, time-like or null if its tangent vector $\dot{c}(t) \in T_{c(t)}M$ is space-like, time-like or null for each $t \in I$.

Remark 2.1. This classification has no sense on a Riemannian manifold, since all tangent vectors are *space-like* due to the positive definiteness of a Riemannian metric.

A 1-dimensional subspace V of T_pM is space-like, time-like or null if the tangent vector that spans V is space-like, time-like or null. The set of such subspaces are named S_1, T_1 and N_1 respectively. The set of all 1-dimensional spaces of T_pM is the disjoint union $S_1 \cup T_1 \cup N_1$. A 1-dimensional subspace is also called *direction*.

We can classify the set of all 2-dimensional subspaces in a similar way than it is done with simple directions.

Definition 2.1.3. Let $(M, \boldsymbol{g})$ be a Lorentzian manifold and consider the tangent space to a point $p \in M$. A 2-dimensional subspace of T_pM is a vector space spanned by two linearly independent tangent vectors. The set of all 2-dimensional subspaces can be divided in 3 sets:

(i) S_2, *space-like* 2-dimensional subspaces are those in which there are no null vectors.

(ii) N_2, *null* 2-dimensional subspaces. All the null vectors of $V \in N_2$ are in the same direction, named *null direction*.

(iii) T_2, *time-like* 2-dimensional subspaces. These subspaces have 2 different null directions.

Each 2-dimensional subspace of T_pM is in either S_2, N_2 or T_2.

Proposition 2.1.4. *Consider a Lorentzian manifold. The set of all 2-dimensional subspaces of T_pM is the disjoint union $S_2 \cup N_2 \cup T_2$.*

Proof. To see this, it is enough to prove that there are no 2-dimensional subspaces with 3 or more different null directions. Consider a 2-subspace V and $\boldsymbol{u}, \boldsymbol{v} \in V$ independent null vectors. Suppose now that it exists a vector $\boldsymbol{w}$ that is null and different from the other two. Then, one can write $\boldsymbol{w} = \lambda \boldsymbol{u} + \mu \boldsymbol{v}$, $\lambda, \mu \neq 0$.

Since $\boldsymbol{w}$ is null, we have: $\boldsymbol{w} \cdot \boldsymbol{w} = 0$. Expanding the expression of $\boldsymbol{w}$, and using that $\boldsymbol{u}$ and $\boldsymbol{v}$ are null, we obtain

$$\boldsymbol{w} \cdot \boldsymbol{w} = (\lambda \boldsymbol{u} + \mu \boldsymbol{v}) \cdot (\lambda \boldsymbol{u} + \mu \boldsymbol{v}) = 2\lambda\mu \boldsymbol{u} \cdot \boldsymbol{v} = 0.$$

It is easy to see that if two null vectors are orthogonal, then one is parallel to the other. Consider an orthonormal basis $\{\boldsymbol{e}_i\}_{i=1}^n$ such that the matrix form of metric tensor in terms of this basis is $g_{ab} = \mathrm{diag}(-1, 1, \dots, 1)$. Consider now a null vector $\boldsymbol{n} = \boldsymbol{e}_0 + \boldsymbol{e}_1$ and a general vector $\boldsymbol{x} = x^i \boldsymbol{e}_i$ that is orthogonal to $\boldsymbol{n}$. Then, expanding the expression of the scalar product we obtain: $\boldsymbol{n} \cdot \boldsymbol{x} = -x^0 + x^1 = 0$, implying $x^0 = x^1$. If we now impose that $\boldsymbol{x}$ is null, it yields $\boldsymbol{x} \cdot \boldsymbol{x} = -(x^0)^2 + (x^1)^2 + \dots + (x^n)^2 = (x^2)^2 + \cdots (x^n)^2 = 0$. Since we are in a real vector space, it is clear that $x^2, \dots x^n = 0$ and $\boldsymbol{x} = x^0 \boldsymbol{e}_0 + x^0 \boldsymbol{e}_1$, i.e. it is parallel to $\boldsymbol{n}$.

We have seen then, that $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel, and this is a contradiction because we assumed them to be different. Therefore, the vector $\boldsymbol{w}$ can not be null and not parallel to $\boldsymbol{u}$ or $\boldsymbol{v}$ and the maximum number of different null directions in a 2-dimensional subspace is 2. $\square$

Given definition 2.1.3, we can characterize the different types of subspaces of dimension 2 by the type of vectors that are in the subspace.

Proposition 2.1.5. *Consider a 2-dimensional subspace $V \subset T_pM$ in a Lorentz manifold of dimension 4. Then:*

(i) If $V \in S_2$, all the vectors in V are space-like.

(ii) If $V \in N_2$, then V contains a null direction $\boldsymbol{k}$ and all the other vectors are space-like and orthogonal to the null vector.

(iii) If $V \in T_2$, i contains 2 different null directions and also contains space-like and time-like vectors.

Proof. To prove (i), we take a basis $\{\boldsymbol{x}, \boldsymbol{t}\}$ of V such that $\boldsymbol{x} \cdot \boldsymbol{x} = -\boldsymbol{t} \cdot \boldsymbol{t} = 1$ and $\boldsymbol{x} \cdot \boldsymbol{t} = 0$. Then, the vector $\boldsymbol{x}+\boldsymbol{t}$ is null and thus, V can not be space-like according to its definition. Therefore, all the vectors of V are space-like.

If $V \in N_2$ and it contains a time-like vector $\boldsymbol{t}$, we can consider a basis $\{\boldsymbol{n}, \boldsymbol{t}\}$ such that $\boldsymbol{n} \cdot \boldsymbol{n} = 0$, $\boldsymbol{t} \cdot \boldsymbol{t} = \boldsymbol{t} \cdot \boldsymbol{n} = -1$. Then, the vector $\lambda\boldsymbol{n} - 2\lambda\boldsymbol{t}$ is a null vector different from $\boldsymbol{n}$:

$$(\lambda\boldsymbol{n} - 2\lambda\boldsymbol{t}) \cdot (\lambda\boldsymbol{n} - 2\lambda\boldsymbol{t}) = \lambda^2\boldsymbol{n} \cdot \boldsymbol{n} + 4\lambda^2\boldsymbol{t} \cdot \boldsymbol{t} - 4\lambda^2\boldsymbol{n} \cdot \boldsymbol{t} = 0.$$

Now we have to prove that every space-like vector is orthogonal no $\boldsymbol{n}$. If we take a space-like vector $\boldsymbol{x}$ such that $\boldsymbol{x} \cdot \boldsymbol{x} = \boldsymbol{x} \cdot \boldsymbol{n} = 1$, we see that the vector $\boldsymbol{x} - \boldsymbol{n}$ is time-like:

$$(\boldsymbol{x} - \boldsymbol{n}) \cdot (\boldsymbol{x} - \boldsymbol{n}) = \boldsymbol{x} \cdot \boldsymbol{x} + \boldsymbol{n} \cdot \boldsymbol{n} - 2\boldsymbol{x} \cdot \boldsymbol{n} = 1 - 2 = -1.$$

Therefore, it can't exist a space-like vector that is not orthogonal to the null direction $\boldsymbol{n}$.

A subspace $V \in T_2$ has a basis $\{\boldsymbol{n}, \boldsymbol{l}\}$ of null vectors. We saw before than these two null vectors can't be orthogonal. Thus we can build two vectors $\boldsymbol{n} + \boldsymbol{l}$ and $\boldsymbol{n} - \boldsymbol{l}$, one being space-like and the other time-like. □

Definition 2.1.6. Let $(M, \boldsymbol{g})$ be a Lorentzian manifold of dimension 4 and T_pM its tangent space at $p \in M$. Consider a 3-dimensional subspace $V \subset T_pM$. We say that V is

(i) *space-like* if its orthogonal space $V^\perp$ is a time-like direction,

(ii) *null* if $V^\perp$ is a null direction or

(iii) *time-like* if $V^\perp$ is space-like.

The collection of all space-like, null and time-like subspaces of dimension 3 are denoted by S_3, N_3 and T_3 respectively.

It is clear that the set of all 3-dimensional subspaces is the disjoint union $S_3 \cup N_3 \cup T_3$.

At given point $p \in M$, one can define an special basis for T_pM depending on their scalar products. It is interesting to choose a basis of vectors that have always the same inner products, because future calculations will be simplified if the correct coordinate system is taken.

Definition 2.1.7. Let $(M, \boldsymbol{g})$ be a 4-dimensional Lorentzian manifold and consider a point $p \in M$. An *orthonormal tetrad* of T_pM is a set of independent vectors $\{\boldsymbol{t}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$ such that $\boldsymbol{g}(\boldsymbol{t}, \boldsymbol{t}) = -1$, $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{y}, \boldsymbol{y}) = \boldsymbol{g}(\boldsymbol{z}, \boldsymbol{z}) = 1$ and all the other scalar products vanish. The matrix form of the metric tensor in terms of an orthonormal tetrad is $\mathrm{diag}(-1, 1, 1, 1)$.

A *real null tetrad* is a set of independent vectors $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{x}, \boldsymbol{y}\} \subset T_pM$ such that $\boldsymbol{g}(\boldsymbol{k}, \boldsymbol{l}) = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{y}, \boldsymbol{y}) = 1$ and other inner products vanish.

We may express the metric tensor $\boldsymbol{g}$ in terms of an orthonormal tetrad as follows:

$$g_{ab} = -t_a t_b + x_a x_b + y_a y_b + z_a z_b,$$

where t_a is the *metric dual* of t^a, i.e., the contraction of the vector t^a with the metric g_{ab}: $t_a := g_{ab} t^b$. Also $t^a = g^{ab} t_b$. One-forms x_a, y_a and z_a are defined the same way.

Remark 2.2. The scalar product of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ can be noted in different ways:

$$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x} \cdot \boldsymbol{y} = g_{ab} x^a y^b = x_b y^b = x^a y_a$$

, where x^a and y^a are the coordinates of these vectors in a chosen basis $\{\boldsymbol{e}_a\}$ and $g_{ab} = \boldsymbol{e}_a \cdot \boldsymbol{e}_b$, and x_a and y_a the coordinates of the contractions of x^a and y^a respectively with the metric g_{ab}.

The basis we have previously defined is real and thus, can only have at most 2 null vectors. If we want to construct a tetrad in which every vector is null, then it must be done on the *complexification* of the tangent space.

Definition 2.1.8. Consider a smooth Lorentzian manifold $(M, \boldsymbol{g})$ and a point $p \in M$. The *complex tangent space*, is the vector space $T_pM \otimes \mathbb{C}$. The associated group of this vector space is formed by tangent vectors of M at p and the field is the set of complex numbers $\mathbb{C}$, instead of $\mathbb{R}$.

Example 2.1.9. Consider two tangent vectors $\boldsymbol{u}, \boldsymbol{v} \in T_pM$. A member of $T_pM \otimes \mathbb{C}$ could be the vector $\boldsymbol{w} = \alpha \boldsymbol{u} + \beta \boldsymbol{v}$, with $\alpha, \beta \in \mathbb{C}$.

Remark 2.3. The dimension of $T_pM \otimes \mathbb{C}$ is the same as the dimension of T_pM. A basis $\{\boldsymbol{e}_i\}_{i=1}^n \subset T_pM$ is also a basis for $T_pM \otimes \mathbb{C}$.

We shall build now a special tetrad on the complexified tangent space, in which all the vectors of the basis are null vectors.

Definition 2.1.10. Consider a Lorentzian manifold $(M, \boldsymbol{g})$ of dimension 4 and its complex tangent space $T_pM \times \mathbb{C}$ at a point $p \in M$. A *complex null tetrad* $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \overline{\boldsymbol{m}}\} \subset T_pM \otimes \mathbb{C}$ is a basis of vectors such that the only non-vanishing inner products are $\boldsymbol{k} \cdot \boldsymbol{l} = -1$ and $\boldsymbol{m} \cdot \overline{\boldsymbol{m}} = 1$, where $\boldsymbol{k}$ and $\boldsymbol{l}$ are real null vectors and $\boldsymbol{m}$ and $\overline{\boldsymbol{m}}$ are complex conjugate null vectors.

It is possible to write down the metric tensor in terms of the complex null tetrad: $g_{ab} = 2m_{(a}\overline{m}_{b)} - 2k_{(a}l_{b)}$. Parentheses in the indices represent the symmetrization on those indices, i.e.,

$$v_{(a}u_{b)} = \frac{1}{2}(v_a u_b + v_b u_a).$$

A complex null tetrad may be related to an orthonormal tetrad by

$$\boldsymbol{m} = \frac{1}{\sqrt{2}}(\boldsymbol{x} - i\boldsymbol{y}), \qquad \overline{\boldsymbol{m}} = \frac{1}{\sqrt{2}}(\boldsymbol{x} + i\boldsymbol{y}),$$
$$\boldsymbol{l} = \frac{1}{\sqrt{2}}(\boldsymbol{t} - \boldsymbol{z}), \qquad \boldsymbol{k} = \frac{1}{\sqrt{2}}(\boldsymbol{t} + \boldsymbol{z}).$$

It is easy to see that these relations fulfill the inner product of the null tetrad, e.g.

$$\boldsymbol{m} \cdot \overline{\boldsymbol{m}} = \frac{1}{2}(\boldsymbol{x} - i\boldsymbol{y}) \cdot (\boldsymbol{x} + i\boldsymbol{y}) = \frac{1}{2}(\boldsymbol{x} \cdot \boldsymbol{x} - i^2 \boldsymbol{y} \cdot \boldsymbol{y}) = 1.$$

The same is applied to verify the other products.

Definition 2.1.11. Consider a manifold with a metric tensor $(M, \boldsymbol{g})$ and its tangent vector T_pM at a point $p \in M$. The *Lorentz group* $\mathcal{L}$ of T_pM is the set of linear maps $f : T_pM \to T_pM$ such that $f(\boldsymbol{u}) \cdot f(\boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v}$ for any $\boldsymbol{u}, \boldsymbol{v} \in T_pM$.

The set $\mathcal{L}$ is a group with the composition operator. It is well explained in [6] how to deduce all the properties of the Lorentz group of vector spaces.

We can apply a linear transformation to a tetrad and obtain a new basis that maintains the inner product of its vectors. These maps will be Lorentz transformations and can be reduced to the following [12]:

Proposition 2.1.12. *Let $(M, \boldsymbol{g})$ be a Lorentzian manifold of dimension 4 and $\{\boldsymbol{l}, \boldsymbol{k}, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$ a complex null basis of T_pM, with $p \in M$. Then, the following linear changes of the basis represent Lorentz transformations:*

(i) Rotations around $\boldsymbol{l}$*:*

$$\boldsymbol{l}' = \boldsymbol{l}, \qquad \boldsymbol{m}' = \boldsymbol{m} + E\boldsymbol{l}, \qquad \boldsymbol{k}' = \boldsymbol{k} + E\overline{\boldsymbol{m}} + \overline{E}\boldsymbol{m} + |E|^2\boldsymbol{l}, \qquad E \in \mathbb{C}$$

(ii) Rotations around $\boldsymbol{k}$*:*

$$\boldsymbol{k}' = \boldsymbol{k}, \qquad \boldsymbol{m}' = \boldsymbol{m} + B\boldsymbol{k}, \qquad \boldsymbol{l}' = \boldsymbol{l} + B\overline{\boldsymbol{m}} + \overline{B}\boldsymbol{m} + |B|^2\boldsymbol{k}, \qquad B \in \mathbb{C}$$

(iii) Rotations in $\boldsymbol{m}-\overline{\boldsymbol{m}}$ plane*:*

$$\boldsymbol{m}' = \mathrm{e}^{i\theta}\,\boldsymbol{m}, \qquad \theta \in [0, 2\pi)$$

(iv) Boosts in the $\boldsymbol{k}-\boldsymbol{l}$ plane*:*

$$\boldsymbol{k}' = \mathrm{e}^{\lambda}\,\boldsymbol{k}, \qquad \boldsymbol{l}' = \mathrm{e}^{-\lambda}\,\boldsymbol{l}, \qquad \lambda \in \mathbb{R}$$

It is easy to deduce that these transformations are Lorentz transformations, since the tetrad $\{\boldsymbol{k}', \boldsymbol{l}', \boldsymbol{m}, \overline{\boldsymbol{m}}\}$ is also a complex null tetrad.

2.2 Bivectors

The main goal of this chapter is to classify the RC tensor by doing an algebraic analysis. We will treat the classification problem as a eigenvetor problem and, in order to do that, we have to introduce the *bivectors* and study all their properties.

Definition 2.2.1. Consider a Lorentzian manifold $(M, \boldsymbol{g})$ of dimension n and a point $p \in M$. A *bivector* $\boldsymbol{F}$ is a skew-symmetric contravariant tensor of order 2, $\boldsymbol{F} \in \Lambda_2(T_pM)$. The set of all bivectors of M at p is denoted by $B_p(M)$, or simply B_p.

The alternate space was defined for forms, but the definition can be analogously extended to vector spaces. Then, if $\boldsymbol{F} \in B_p$ and $\{\boldsymbol{e}_a\}_{a=1}^n$ is a basis for T_pM, the components of $\boldsymbol{f}$ in this basis fulfill $F^{ab} = -F^{ba}$. As in proposition 1.3.6, $\dim(B_p) = \binom{n}{2}$. In the case of Lorentzian manifolds of dimension 4, the dimension of B_p is 6.

We can obtain a skew-symmetric 2-form from a bivector by lowering indices using the metric $\boldsymbol{g}$. If $\boldsymbol{F}$ is a bivector of B_p, then $F_{ab} := g_{ac}g_{bd}F^{cd}$. We will treat bivectors and its associated 2-forms as bivectors indistinctly.

Now we will define the *Hodge operator* which will serve as a useful tool to treat with bivectors. The following results regarding the Hodge operator and its properties can be found at [3], [4] and [6].

Definition 2.2.2. Consider a vector space V of dimension n with a non-degenerate metric $\boldsymbol{g}$ with signature s. Given a natural number $p \leq n$, consider the skew-symmetric spaces $\Lambda^p(V)$ and $\Lambda^{n-p}(V)$. The *Hodge dual operator* is a map $^\star : \Lambda^p(V) \to \Lambda^{n-p}(V)$ such that given a p-vector $\boldsymbol{\alpha} \in \Lambda^p(V)$:

$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = (-1)^{\frac{s}{2}} \boldsymbol{g}(\boldsymbol{\beta}, {}^\star\boldsymbol{\alpha})\boldsymbol{\eta}, \qquad \forall \boldsymbol{\beta} \in \Lambda^{n-p}(V).$$

The element $\boldsymbol{\eta} \in \Lambda^n(V)$ is the *preferred n-vector* for a chosen orthonormal basis $\{\boldsymbol{e}_i\}_{i=1}^n$ and can be expressed as $\boldsymbol{\eta} = \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n$. We call ${}^\star\boldsymbol{\alpha}$ the *Hodge dual* of $\boldsymbol{\alpha}$.

Remark 2.4. In the definition above, we define the scalar product of 2 p-vectors as

$$\boldsymbol{g}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \det\left(\boldsymbol{g}(\boldsymbol{\alpha}_i, \boldsymbol{\beta}_j)\right)$$

for $\boldsymbol{\alpha} = \boldsymbol{\alpha}_1 \wedge \cdots \wedge \boldsymbol{\alpha}_p$ and $\boldsymbol{\beta} = \boldsymbol{\beta}_1 \wedge \cdots \wedge \boldsymbol{\beta}_p$. It fulfills the same properties as the metric $\boldsymbol{g}$. Equivalently, given an orthonormal basis $\{\boldsymbol{e}_i\}$:

$$\boldsymbol{g}(\boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_p}, \boldsymbol{e}_{j_1} \wedge \cdots \wedge \boldsymbol{e}_{j_p}) = \left[\prod_{k=1}^{p} \boldsymbol{g}(\boldsymbol{e}_{i_k}, \boldsymbol{e}_{i_k})\right] \delta^{i_1,\dots,i_p}_{j_1,\dots,j_p}.$$

This definition of the Hodge operator determines it completely. Given $\boldsymbol{\alpha} \in \Lambda^p(V)$ and $\boldsymbol{\beta} \in \Lambda^{n-p}(V)$, we have that $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \in \Lambda^n(V)$ and then, $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = f_\alpha(\boldsymbol{\beta})\boldsymbol{\eta}$ for a function f_α. Due to the metric is non-degenerate, it exists a unique $(n-p)$-vector $\boldsymbol{\xi}$ such that $f_\alpha(\boldsymbol{\beta})) = \boldsymbol{g}(\boldsymbol{\beta}, \boldsymbol{\xi})$, $\forall \boldsymbol{\beta} \in \Lambda^{n-p}(V)$. We therefore defined the Hodge operator to be ${}^\star\boldsymbol{\alpha} := (-1)^{\frac{s}{2}}\boldsymbol{\xi}$.

Proposition 2.2.3. *Consider a vector space V of dimension n with a non-degenerate metric $\boldsymbol{g}$ and an orthonormal basis $\{\boldsymbol{e}_i\}_{i=1}^n$. Let $\sigma = (i_1, \dots, i_n) \in \mathfrak{S}_n$ be a permutation of n elements. Consider now a p-vector $\boldsymbol{\alpha} = \boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_p}$. Then, the Hodge operator applied to $\boldsymbol{\alpha}$ is*

$${}^\star\boldsymbol{\alpha} = \operatorname{sgn}(\sigma) g_{i_1 i_1} \cdots g_{i_p i_p} \boldsymbol{e}_{i_{p+1}} \wedge \cdots \wedge \boldsymbol{e}_n.$$

Proof. Take a p-vector $\boldsymbol{\alpha} = \boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_p}$. Now, consider a generic $(n-p)$-vector $\boldsymbol{\beta} = \boldsymbol{e}_{j_1} \wedge \cdots \wedge \boldsymbol{e}_{j_{n-p}}$. According to definition 2.2.2,

$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = (-1)^{\frac{s}{2}} \boldsymbol{g}(\boldsymbol{\beta}, {}^\star\boldsymbol{\alpha})\boldsymbol{\eta}.$$

The left term is 0, unless $i_k \neq j_l$, $\forall 1 \leq k \leq p$, $1 \leq l \leq n-p$. In case that the left term is not 0, then $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = \boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_p} \wedge \boldsymbol{e}_{j_1} \wedge \cdots \wedge \boldsymbol{e}_{j_{n-p}} = \operatorname{sgn}(\sigma)\boldsymbol{\eta}$, with $\sigma = (i_1, \dots, i_p, j_1, \dots j_{n-p})$ being an n-permutation.

Using remark 2.4, we see that the right term is different to 0 if and only if ${}^\star\boldsymbol{\alpha} = c\boldsymbol{\beta}$ for some constant c. Thus, we have the identity $\operatorname{sgn}(\sigma) = (-1)^{\frac{s}{2}} \boldsymbol{g}(\boldsymbol{\beta}, c\boldsymbol{\beta})$. Using that $(-1)^{\frac{s}{2}} = \boldsymbol{g}(\boldsymbol{\eta}, \boldsymbol{\eta}) = \boldsymbol{g}(\boldsymbol{\alpha}, \boldsymbol{\alpha})\boldsymbol{g}(\boldsymbol{\beta}, \boldsymbol{\beta})$, we get

$$c = \frac{\operatorname{sgn}(\sigma)(-1)^{\frac{s}{2}}}{\boldsymbol{g}(\boldsymbol{\beta}, \boldsymbol{\beta})} = \frac{\operatorname{sgn}(\sigma)\boldsymbol{g}(\boldsymbol{\eta}, \boldsymbol{\eta})}{\boldsymbol{g}(\boldsymbol{\beta}, \boldsymbol{\beta})} = \operatorname{sgn}(\sigma)\boldsymbol{g}(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = \operatorname{sgn}(\sigma) g_{i_1 i_1} \cdots g_{i_p i_p}.$$

Therefore ${}^\star\boldsymbol{\alpha} = c\boldsymbol{\beta}$, with c being the constant computed above. ☐

We define the *Levi-Civita tensor* as $\boldsymbol{\epsilon} = (-1)^{\frac{s}{2}} n! \sqrt{|\det \boldsymbol{g}|}\boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n$. The components of this n-vector are $\epsilon^{1\dots n} = (-1)^{\frac{s}{2}} \sqrt{|\det \boldsymbol{g}|}$. Given a p-vector $\boldsymbol{\alpha}$, its Hodge dual is

$$({}^\star\alpha)^{i_1 \dots i_{n-p}} = \frac{1}{p!} \epsilon^{i_1 \dots i_{n-p}}{}_{j_1 \dots j_p} \alpha^{j_1 \dots j_p}.$$

Expressed in orthonormal coordinates, the coefficients of $\boldsymbol{\epsilon}$ are $\epsilon^{1 \cdots n} = (-1)^{\frac{s}{2}}$.

The Levi-Civita tensor fulfills the following property:

$$\epsilon^{i_1 \cdots i_p}{}_{k_{p+1} \cdots k_n} \epsilon_{j_1 \cdots j_p}{}^{k_{p+1} \cdots k_n} = p!(n-p)!(-1)^{\frac{s}{2}} \delta^{i_1}_{[j_1} \cdots \delta^{i_p}_{j_p]} \tag{2.1}$$

We will work on Lorentzian manifolds $(M, \boldsymbol{g})$ of dimension $n = 4$ and signature $s = 2$ and at a point $p \in M$. We previously defined bivector $\boldsymbol{F} = F^{ab}$ on a manifold. Since the tangent space is a vector space and bivectors ar skew-symmetric elements, we can define Hodge dual the Hodge dual of $\boldsymbol{F}$: ${}^{\star}F^{ab} = \frac{1}{2}\epsilon^{ab}{}_{cd}F^{cd}$.

Proposition 2.2.4. *Let $\boldsymbol{F}, \boldsymbol{H}$ be 2 bivectors on the tensor space of a dimension 4 Lorentz manifold. The Hodge dual operator has the following properties:*

(i) ${}^{\star\star}F^{ab} = -F^{ab}$.

(ii) $F_{ab}F^{ab} = -{}^{\star}F_{ab}{}^{\star}F^{ab}$.

(iii) $H^{ac}F_{bc} - {}^{\star}F^{ac\star}H_{bc} = \frac{1}{2}F^{de}H_{de}\delta^a_b$.

Proof. This result follows from the direct application of eq. (2.1). We will prove the point (iii), but the proofs for (i) and (ii) are similar.

$$
\begin{aligned}
&H^{ac}F_{bc} - {}^{\star}F^{ac\star}H_{bc} = H^{ac}F_{bc} - \frac{1}{4}\epsilon^{acde}\epsilon_{bcfg}F_{de}H^{fg} \\
&= H^{ac}F_{bc} - \frac{1}{4}\epsilon^{cade}\epsilon_{cbfg}F_{de}H^{fg} \\
&= H^{ac}F_{bc} + \frac{6}{4}F_{de}H^{fg}\delta^a_{[b}\delta^d_f\delta^e_{g]} \\
&= H^{ac}F_{bc} + \frac{1}{4}F_{de}H^{fg}\left(\delta^a_b\delta^d_f\delta^e_g + \delta^a_g\delta^d_b\delta^e_f + \delta^a_f\delta^d_g\delta^e_b - \delta^a_b\delta^d_g\delta^e_f - \delta^a_g\delta^d_f\delta^e_b - \delta^a_f\delta^d_b\delta^e_g\right) \\
&= H^{ac}F_{bc} + \frac{1}{4}\left(2F_{fg}H^{fg}\delta^a_b - 2F_{bf}H^{af} - 2F_{bg}H^{ag}\right) \\
&= \frac{1}{2}F_{fg}H^{fg}\delta^a_b
\end{aligned}
$$

□

Definition 2.2.5. Given a bivector $\boldsymbol{F} \in \Lambda^2(T^*_pM)$ is said to be *simple* if $\exists \boldsymbol{r}, \boldsymbol{s} \in T^*_pM$ such that $\boldsymbol{F} = \boldsymbol{r} \wedge \boldsymbol{s}$, i.e $F_{ab} = 2r_{[a}s_{b]}$. The 2-subspace spanned by $\boldsymbol{r}$ and $\boldsymbol{s}$ is called the *blade* of $\boldsymbol{F}$. A simple bivector is space-like, time-like or null if its blade is space-like, time-like or null, respectively.

If $\boldsymbol{F}$ is not simple, then we say that $\boldsymbol{F}$ is a *non-simple* bivector.

Remark 2.5. If a simple bivector F_{ab} is thought as a 4×4 matrix, then the rank of this matrix is 2. Since the matrix of a general bivector F_{ab} is skew-symmetric it has even rank, meaning that the rank of F_{ab} can only be 0, 2 or 4.

Proposition 2.2.6. *Let $\boldsymbol{F} \neq 0$ be a bivector and let $k^a \neq 0$ be a tangent vector such that $F_{ab}k^b = 0$. Then $\boldsymbol{F}$ is simple.*

Proof. Thinking the problem in matrix form yield that the matrix F_{ab} is not full rank. Since $\boldsymbol{F} \neq 0$ and using remark 2.5, the rank of F_{ab} must be 2, and then, it is a simple bivector. □

Another property of simple bivectors is that, given $F_{ab} = 2r_{[a}s_{b]}$, ${}^\star F_{ab}r^b = {}^\star F_{ab}s^b = 0$. It can be seen in the following way:

$$
\begin{gathered}
{}^\star F_{ab}r^b = \frac{1}{2}\epsilon_{abcd}r^c s^d r^b - \frac{1}{2}\epsilon_{abcd}r^d s^c r^b = \frac{1}{2}\epsilon_{abcd}r^b r^c s^d + \frac{1}{2}\epsilon_{abcd}r^b r^c s^d = \epsilon_{abcd}r^b r^c s^d \\
\epsilon_{abcd}r^b r^c s^d = -\epsilon_{abcd}r^c r^b s^d = -\epsilon_{abcd}r^b r^c s^d = 0 \implies {}^\star F_{ab}r^b = 0.
\end{gathered}
$$

The same is applied to the contraction with s^b. We deduce from this that the Hodge dual of a simple bivector is also simple and its blade is the orthogonal space to the blade of F_{ab}. A simple bivector is said to be *space-like*, *null* or *time-like* if its blade is space-like, null or time-like, respectively.

Proposition 2.2.7. *Consider a simple bivector F_{ab}. If F_{ab} is space-like, then its Hodge dual is time-like, and vice-versa. On the other hand, if F_{ab} is null, then its Hodge dual is null with the same null direction.*

Proof. If the blade of F_{ab} is space-like, then ${}^\star F_{ab}$ is spanned by the orthogonal space to the blade of F_{ab}. If we build an orthogonormal tetrad $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t}\}$ such that $\boldsymbol{F} = \boldsymbol{x} \wedge \boldsymbol{y}$, then, by proposition 2.2.3, we have that ${}^\star(\boldsymbol{x} \wedge \boldsymbol{y}) = -\boldsymbol{z} \wedge \boldsymbol{t}$. Since it is spanned by a time-like vector, this subspace is time-like. The same argument is used to prove that the Hodge dual of a time-like bivector is space-like.

If F_{ab} is null, it has a unique null direction $\boldsymbol{k}$. We said that the space spanned by the elements of ${}^\star F_{ab}$ is orthogonal to the blade of F_{ab}. Then is clear that $\boldsymbol{k}$ is inside the orthonogal space because $\boldsymbol{k} \cdot \boldsymbol{k} = 0$ and using proposition 2.1.5, we have that $\boldsymbol{k}$ is also orthogonal to all the other elements of the blade of F_{ab}. Now we consider a vector $\boldsymbol{v}$ contained in the blade of ${}^\star\boldsymbol{F}$. Then, $\boldsymbol{v} \cdot \boldsymbol{k} = 0$ and we deduce that $\boldsymbol{v}$ is space-like or proportional to $\boldsymbol{k}$ (refer to the proof of proposition 2.1.4 to see the proof of it). It follows that ${}^\star\boldsymbol{F}$ is also null with null vector $\boldsymbol{k}$. □

Directly from this proposition, we can give a very useful characterization for null bivectors.

Proposition 2.2.8. *Let F_{ab} be a simple bivector. Then, F_{ab} is null $\iff$ it exists a vector $\boldsymbol{k}$ such that $F_{ab}k^b = {}^\star F_{ab}k^b = 0$.*

Proof. The implication ($\Rightarrow$) is obvious given the previous proposition. The implication ($\Leftarrow$) is also deuced from the same proposition: both F_{ab} and its Hodge dual are orthogonal complement, but do not span all T_pM. Then, they both have a common direction, spanned by $\boldsymbol{k}$, which is null, and hence, bivector F_{ab} is null. □

Now, consider an orthonormal tetrad $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t}\}$ and a real null tetrad $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{k}, \boldsymbol{l}\}$ of T_pM. We relate both tetrads through the following relations: $\sqrt{2}\boldsymbol{z} = \boldsymbol{k} + \boldsymbol{l}$ and $\sqrt{2}\boldsymbol{t} = \boldsymbol{k} - \boldsymbol{l}$. From these basis, we can construct two different basis $\{\boldsymbol{F}_i\}$ and $\{\boldsymbol{G}_i\}$ for B_p:

$$
\begin{array}{lll}
\boldsymbol{F}_1 = \boldsymbol{x} \wedge \boldsymbol{y} & \boldsymbol{F}_2 = \boldsymbol{x} \wedge \boldsymbol{z} & \boldsymbol{F}_3 = \boldsymbol{x} \wedge \boldsymbol{t} \\
\boldsymbol{F}_4 = \boldsymbol{y} \wedge \boldsymbol{z} & \boldsymbol{F}_5 = \boldsymbol{y} \wedge \boldsymbol{t} & \boldsymbol{F}_6 = \boldsymbol{z} \wedge \boldsymbol{t}
\end{array} \tag{2.2}
$$

and

$$\begin{array}{lll} \boldsymbol{G}_1 = \boldsymbol{x} \wedge \boldsymbol{y} & \boldsymbol{G}_2 = \boldsymbol{x} \wedge \boldsymbol{k} & \boldsymbol{G}_3 = \boldsymbol{x} \wedge \boldsymbol{l} \\ \boldsymbol{G}_4 = \boldsymbol{y} \wedge \boldsymbol{k} & \boldsymbol{G}_5 = \boldsymbol{y} \wedge \boldsymbol{l} & \boldsymbol{G}_6 = \boldsymbol{k} \wedge \boldsymbol{l} \end{array} \tag{2.3}$$

From proposition 2.2.3 we note that, up to signs, $(\boldsymbol{F}_1, \boldsymbol{F}_6), (\boldsymbol{F}_2, \boldsymbol{F}_5), (\boldsymbol{F}_3, \boldsymbol{F}_4)$ and $(\boldsymbol{G}_1, \boldsymbol{G}_6), (\boldsymbol{G}_3, \boldsymbol{G}_5), (\boldsymbol{G}_2, \boldsymbol{G}_4)$ are dual pairs. Moreover, given the characterization in proposition 2.1.5, it is clear that $\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_4, \boldsymbol{G}_1$ are space-like bivectors; $\boldsymbol{G}_2, \boldsymbol{G}_4$ are nulls with p.n.d. $\boldsymbol{k}$; $\boldsymbol{G}_3, \boldsymbol{G}_5$ are null with null direction $\boldsymbol{l}$ and $\boldsymbol{G}_6, \boldsymbol{F}_3, \boldsymbol{F}_5, \boldsymbol{F}_6$ are time-like. These two basis are real, but we can define bivectors on the complexification of the tangent space as well.

Definition 2.2.9. Consider a Lorentz manifold $(M, \boldsymbol{g})$ and its complexified tangent space at a point $p \in M$, $T_pM \otimes \mathbb{C}$. A *complex bivector* $\boldsymbol{F}$ is an element of $\Lambda^2(T_pM \otimes \mathbb{C})$. Complex bivectors are objects whose real parts and imaginary parts are real bivectors. The space of all complex bivectors at p is a vector space denoted $\mathbb{C}B_p$.

Remark 2.6. It is clear that the dimension of $\mathbb{C}B_p$ is 6, the same as B_p.

The Hodge dual for complex bivectors is defined in the same way and all the other properties are the same. If we look at the first property of proposition 2.2.4, we realize that the only possible eigenvalues of the Hodge operator are $\pm i$. In the space of real bivectors there are no eigenvalues of the Hodge operator, because if $\boldsymbol{F}$ is a real bivector, then ${}^\star\boldsymbol{F}$ can't be a complex bivector. Regarding this, we can define two subspaces of $\mathbb{C}B_p$.

Definition 2.2.10. Let $\mathbb{C}B_p$ be the vector space of complex bivectors. The *self-dual* space S_p^+ is defined as

$$S_p^+ := \{\boldsymbol{F} \in \mathbb{C}B_p \mid {}^\star\boldsymbol{F} = -i\boldsymbol{F}\}.$$

On the other hand, the *anti-self-dual* space S_p^- is:

$$S_p^- := \{\boldsymbol{F} \in \mathbb{C}B_p \mid {}^\star\boldsymbol{F} = i\boldsymbol{F}\}.$$

Lemma 2.2.11. *Consider a complex bivector* $\boldsymbol{F} = \boldsymbol{A} + i\boldsymbol{B}$, $\boldsymbol{A}, \boldsymbol{B} \in B_p$. *Then,* $\boldsymbol{F} \in S_p^+ \implies \boldsymbol{B} = {}^\star\boldsymbol{A}$ *and* $\boldsymbol{F} \in S_p^- \implies \boldsymbol{B} = -{}^\star\boldsymbol{A}$

Proof. The proof is straight-forward from proposition 2.2.4. □

One deduces then that the elements of S_p^+ have the form $\boldsymbol{F} + i{}^\star\boldsymbol{F}$, $\boldsymbol{F} \in B_p$ and the elements of S_p^- have the form $\boldsymbol{F} - i{}^\star\boldsymbol{F}$, $\boldsymbol{F} \in_p$. Therefore, given a real bivector $\boldsymbol{F}$, we can define the *self-dual* of $\boldsymbol{F}$ as $\boldsymbol{F}^+ := \boldsymbol{F} + i{}^\star\boldsymbol{F}$. On the other hand, the *anti-self-dual* of $\boldsymbol{F}$ is defined as $\boldsymbol{F}^- := \boldsymbol{F} - i{}^\star\boldsymbol{F}$.

Remark 2.7. The bivectors $\boldsymbol{F}^+$ and $\boldsymbol{F}^-$ are self-dual and anti-self-dual even if $\boldsymbol{F}$ is a complex bivector.

Proposition 2.2.12. *Consider the self-dual and anti-self-dual spaces* S_p^+ *and* S_p^-. *Then*

$$\mathbb{C}B_p = S_p^+ \oplus S_p^-.$$

Proof. Every complex bivector $\boldsymbol{F} \in \mathbb{C}B_p$ can be written as a sum of a self-dual bivector and an anti-self-dual one:

$$\boldsymbol{F} = \frac{1}{2}(\boldsymbol{F} + i^\star\boldsymbol{F}) + \frac{1}{2}(\boldsymbol{F} - i^\star\boldsymbol{F}).$$

As we stated that *every* element in $S_p^\pm$ has the form $\boldsymbol{F} \pm i^\star\boldsymbol{F}$, we conclude that $\mathbb{C}B_p = S_p^+ \oplus S_p^-$. □

Recall now the basis for B_p defined in 2.3. Every bivector $\boldsymbol{F} \in B_p$ can be written as $\boldsymbol{F} = \alpha^k \boldsymbol{G}_k$ and a self-dual bivector of S_p^+ may be written as $\boldsymbol{F}^+ = \alpha^k(\boldsymbol{G}_k + i^\star\boldsymbol{G}_k)$ and an anti-self-dual bivector can be written as $\boldsymbol{F}^- = \alpha^k(\boldsymbol{G}_k - i^\star\boldsymbol{G}_k)$. From the dual pairs of the basis in 2.3, it is easy to see that the basis of S_p^+ and S_p^- only have 3 elements and thus, the dimension of S_p^+ is 3, the same as the dimension of S_p^-. The direct sum of these two subspaces has dimension $3 + 3 = 6$, i.e., the dimension of $\mathbb{C}B_p$.

Proposition 2.2.13. *A basis for the space S_p^+ is*

$$\boldsymbol{V} := -\frac{1}{\sqrt{2}}(\boldsymbol{G}_3 + i^\star\boldsymbol{G}_3), \qquad \boldsymbol{U} := -\frac{1}{\sqrt{2}}(\boldsymbol{G}_4 + i^\star\boldsymbol{G}_4), \qquad \boldsymbol{W} := (\boldsymbol{G}_6 + i^\star\boldsymbol{G}_6).$$

Proof. We said that a vector of S_p^+ can be written as $\alpha^k(\boldsymbol{G}_k + i^\star\boldsymbol{G}_k)$. Since $\boldsymbol{G}_1$ is the Hodge dual of $\boldsymbol{G}_6$ and vice-versa, we have that $(\boldsymbol{G}_1 + i^\star\boldsymbol{G}_1) = i(\boldsymbol{G}_6 + i^\star\boldsymbol{G}_6)$. Then these two self-dual bivector are proportional. The same for pairs $(\boldsymbol{G}_2, \boldsymbol{G}_4)$ and $(\boldsymbol{G}_3, \boldsymbol{G}_5)$. It proves that the set $\{\boldsymbol{V}, \boldsymbol{U}, \boldsymbol{W}\}$ is a basis for S_p^+. □

We can expand the expressions of $\boldsymbol{V}, \boldsymbol{U}$ and $\boldsymbol{W}$ in terms of the real null tetrad (definition 2.1.7) and then, using definition 2.1.10, we can express this basis in terms of the complex null tetrad as

$$\begin{aligned} U_{ab} &= 2(l_{[a}x_{b]} - il_{[a}y_{b]}) = 2\overline{m}_{[a}l_{b]}, \\ V_{ab} &= 2(x_{[a}k_{b]} + iy_{[a}k_{b]}) = 2k_{[a}m_{b]}, \\ W_{ab} &= 2l_{[a}k_{b]} + 2ix_{[a}y_{b]} = 2l_{[a}k_{b]} + 2m_{[a}\overline{m}_{b]}. \end{aligned} \tag{2.4}$$

Given the basis in the complex null tetrad reference, it is clear that all the contractions of bivectors of the basis vanish except for the following ones:

$$U_{ab}V^{ab} = 2, \qquad\qquad W_{ab}W^{ab} = -4. \tag{2.5}$$

Remark 2.8. Bivectors $\overline{\boldsymbol{V}}, \overline{\boldsymbol{U}}$ and $\overline{\boldsymbol{W}}$ are a basis for the space S_p^-. This basis is orthogonal to the described basis of S_p^+.

We can give a characterization for non-null real bivector.

Proposition 2.2.14. *Let $(M, \boldsymbol{g})$ be a Lorentz manifold and $\boldsymbol{F} \in B_p$ a non-null real bivector at $p \in M$. Then, it exists a real null tetrad $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{k}, \boldsymbol{l}\}$ in which $\boldsymbol{F}$ takes the form*

$$F_{ab} = 2\alpha l_{[a}k_{b]} + 2\beta x_{[a}y_{b]}, \qquad \alpha, \beta \in \mathbb{R}.$$

Proof. Consider a non null real bivector $\boldsymbol{F}$ and take its self-dualisation $\boldsymbol{F}^+ = \boldsymbol{F} + {}^\star\boldsymbol{F}$. Here, $\boldsymbol{F}$ can be expressed in terms of the basis $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}$ (2.4) as

$$F^+_{ab} = F^1 U_{ab} + F^2 V_{av} + F^3 W_{ab}, \qquad F^i \in \mathbb{C}$$

Now we apply a null rotation to the real null tetrad $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{k}, \boldsymbol{l}\}$ (2.1.12) around $\boldsymbol{l}$ and we obtain a new self-dual bivector basis $\{\boldsymbol{U}', \boldsymbol{V}', \boldsymbol{W}'\}$, which is related to the original one as follows:

$$\begin{aligned} \boldsymbol{U}' &= \boldsymbol{U} \\ \boldsymbol{V}' &= \boldsymbol{V} - E\boldsymbol{W} + E^2\boldsymbol{U} \\ \boldsymbol{W}' &= \boldsymbol{W} - 2E\boldsymbol{U}, \end{aligned}$$

and $F^+_{ab} = F'^1 U'_{ab} + F'^2 V'_{av} + F'^3 W'_{ab} = F^1 U_{ab} + F^2 V_{av} + F^3 W_{ab}$. Expanding the first expression for F^+_{ab}, we obtain an expression for the coordinates of $\boldsymbol{F}$ in the transformed basis, F'^1, F'^2 and F'^3.

$$\begin{aligned} F'^1 &= F^1 + 2EF^3 - E^2F^2 \\ F'^2 &= F^2 \\ F'^3 &= F^3 + EF^2 \end{aligned}$$

If $F^2 = 0$, then F'^2 is also 0. Moreover, if F^3 is also 0, it means that $\boldsymbol{F} = F^1\boldsymbol{U}$ and then it would be null. Hence, $F^2 = 0$ implies that $F^3 \neq 0$. In this case, we can apply a null rotation so that F'^1 vanishes. On the other hand, if $F^2 \neq 0$, we can apply the null rotation to find a tetrad in which F'^1 is 0. From that tetrad, we apply a null rotation about vector $\boldsymbol{k}$ and, by the same procedure, we can get $F'^2 = 0$. Therefore, we can always achieve a real null tetrad in which $F'^1 = F'^2 = 0$ and $F'^3 \neq 0$. It exists a tetrad $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{k}, \boldsymbol{l}\}$ in which

$$F^+_{ab} = F^3 W_{ab} = (\alpha + i\beta)(2l_{[a}k_{b]} + 2ix_{[a}y_{b]}), \qquad \alpha, \beta \in \mathbb{R},$$

and since F_{ab} is the real part of F^+_{ab}, we finally get a general expression for a non-null bivector

$$F_{ab} = 2\alpha l_{[a}k_{b]} + 2\beta x_{[a}y_{b]}.$$

□

In the space of vectors we have defined a metric $\boldsymbol{g}$ that allows to define the *metric-duals* of vectors by simply lowering indices of the components: $g_{ab}x^a = x_b$. In the space of bivectors it is possible to define a metric $\boldsymbol{G}$, the *bivector metric*, which is induced from the Lorentz metric and is defined as

$$\boldsymbol{G} = \frac{1}{2}\boldsymbol{g} \wedge \boldsymbol{g}, \qquad G_{abcd} = g_{a[c}g_{d]b}.$$

The tensor $\boldsymbol{G}$ acts as a metric in the bivector space B_p, and for each bivectors $\boldsymbol{F}$ and $\boldsymbol{H}$ the following properties are fulfilled:

(i) Raising and lowering indices: $G_{abcd}F^{cd} = F_{ab}$.

(ii) Inner product: $(\boldsymbol{F}, \boldsymbol{H}) := \boldsymbol{G}(\boldsymbol{F}, \boldsymbol{H}) = G_{abcd}F^{ab}H^{cd} = F^{ab}H_{ab}$.

The second property is direct from the first one. The first property can be demonstrated taking a bivector from the basis of B_p, $H^{ab} = 2x^{[a}y^{b]}$ and see that (i) holds for $\boldsymbol{H}$.

$$G_{abcd}H^{cd} = \frac{1}{2}(g_{ac}g_{bd} - g_{ad}g_{bc})(x^a y^b - x^b y^a) = \frac{1}{2}(x_a y_b - x_b y_a - x_b y_a + x_a y_b) = H_{ab}.$$

Proposition 2.2.15. *The bivector metric $\boldsymbol{G}$ has signature $s = 0$. There exists a basis of B_p such that $\boldsymbol{G}$ has a diagonal form $(1, 1, 1, -1, -1, -1)$.*

Proof. This is shown by taking a basis $\{\boldsymbol{F}_i'\}$ adapted from (2.2), such that $\boldsymbol{F}_i' = \frac{1}{\sqrt{2}}\boldsymbol{F}_i$. The inner products of these bivectors are

$$\begin{aligned} \boldsymbol{G}(\boldsymbol{F}_i', \boldsymbol{F}_i') &= 1, & i \in \{1,2,4\} \\ \boldsymbol{G}(\boldsymbol{F}_i', \boldsymbol{F}_i') &= -1, & i \in \{3,5,6\} \\ \boldsymbol{G}(\boldsymbol{F}_i', \boldsymbol{F}_j') &= 0, & i \neq j \end{aligned}$$

We will proof that the first one is true for $\boldsymbol{F}_1' = \frac{1}{\sqrt{2}}\boldsymbol{x} \wedge \boldsymbol{y} = \sqrt{2}x^{[a}y^{b]}$. The procedure to show the other cases is analogous.

$$\begin{aligned} \boldsymbol{G}(\frac{1}{\sqrt{2}}\boldsymbol{x} \wedge \boldsymbol{y}, \frac{1}{\sqrt{2}}\boldsymbol{x} \wedge \boldsymbol{y}) &= 2G_{abcd}x^{[a}y^{b]}x^{[c}y^{d]} = 2x^{[a}y^{b]}x_{[a}y_{b]} \\ &= \frac{1}{2}(x^a y^b x_a y_b - x^a y^b x_b y_a - x^b y^a x_a y_b + x^b y^a x_b y_b) \\ &= 1 \end{aligned}$$

□

We have seen that the space of bivectors is a vector space of dimension 6. Then it is convenient to label bivectors using only one index, instead of two, like it has been done until now. Consider a basis $\{E_I\}_{I=1}^{6}$ of B_p. Then, a general bivector is expressed in this basis we have defined as follows [15]:

$$\boldsymbol{F} = F^I E_I, \qquad F_{ab} = F^I (E_I)_{ab}, \qquad F_I = \frac{1}{2}F^{ab}(E_I)_{ab}.$$

We can also denote the Hodge dual using this notation and the bivector metric $\boldsymbol{G}$, which allows to raise and lower indices in the bivector space:

$$^\star F_I = \epsilon_{IJ}F^J, \qquad (\boldsymbol{F}, \boldsymbol{H}) = G_{IJ}F^I H^J = F^I H_J.$$

Remark 2.9. A double bivector W_{IJ} is said to be symmetric when $W_{IJ} = W_{JI}$ and skew symmetric if $W_{IJ} = -W_{JI}$. For example, the hodge tensor $\boldsymbol{\epsilon}$ is symmetric.

This developement of this subject from this point of view can be found at J.A. Saez [15]. There is an introduction to bivector spaces, the bivector metric and their geometric properties.

2.3 Weyl Conformal Tensor

The first step in the classification of manifolds is decomposing the RC tensor $R^i{}_{jkl}$ in two different parts: the trace-free and conformally invariat, the *Weyl tensor*, and the *Ricci* part. Further study on algebraic properties of the Weyl tensor will result in a classification of Einstein spaces that is invariant under Lorentz transformations. We start by defining the concept of double bivector.

Definition 2.3.1. Consider a Lorentz manifold $(M, \boldsymbol{g})$ and the space of bivectors at a point $p \in M$, B_p. A *double bivector* or *double 2-form* $\boldsymbol{W}$ is a tensor of the space B_p, $\boldsymbol{W} \in \mathcal{T}^2(B_p)$. A general double bivector has the form $\boldsymbol{W} = W^{IJ} E_I \otimes E_J$, but we will usually use the compact notation, $\boldsymbol{W} = W^{IJ}$.

A double bivector is also an element of the tensor space of the manifold, and thus it can be expressed in a tangent vector basis, $\boldsymbol{W} = W_{abcd}$. This tensor is, by definition, skew-symmetric in the first and second pair of indices, i.e.

$$W_{abcd} = -W_{bacd} = -W_{abdc} = W_{badc}.$$

Definition 2.3.2. Let $(M, \boldsymbol{g})$ be an n-dimensional smooth manifold and $\boldsymbol{R}$ its RC tensor. Consider the basis $\{\boldsymbol{e}_i\}_{i=1}^n$ of T_pM and its dual $\{\boldsymbol{\omega}^i\}_{i=1}^n$. The *Weyl tensor* is a tensor $\boldsymbol{C} \in \mathcal{T}_1^3(T_p^*M, T_pM)$ that is invariant under conformal transformations and is a combination of the RC tensor and its contractions: $\boldsymbol{R}$, the Ricci tensor $Ric(\boldsymbol{R})$ and the scalar curvature R.

Under this definition, the Weyl tensor is uniquely determined as follows.

Proposition 2.3.3. *Given an n-dimensional manifold $(M, \boldsymbol{g})$ and a basis $\{\boldsymbol{e}_a\}_{a=1}^n$ of T_pM and its dual $\{\boldsymbol{\omega}^a\}_{a=1}^n$, the Weyl tensor $\boldsymbol{C}$ is expressed in local coordinates:*

$$C_{abcd} = R_{abcd} + \frac{1}{n-2}(g_{ac}R_{bd} - g_{bc}R_{ad} + g_{bd}R_{ac} - g_{ad}R_{bc}) + \frac{R}{(n-1)(n-2)}(g_{ac}g_{bd} - g_{ad}g_{bc}),$$

where C_{abcd} is the contraction of the first index of the Weyl tensor with the metric g_{ab}, i.e. $C_{abcd} = g_{ak}C^k{}_{bcd}$.

Proof. There are different ways to derive the Weyl tensor. Here, we will assume that $\boldsymbol{C}$ is invariant under conformal transformations and has the same symmetries as $R^a{}_{bcd}$. Consider the Lorentzian manifolds $(M, \boldsymbol{g})$ and $(M, \overline{\boldsymbol{g}})$, and $id : M \to M$ is a conformal map such that for $\boldsymbol{u}, \boldsymbol{v} \in T_pM$, $\overline{\boldsymbol{g}}_p(\boldsymbol{u}, \boldsymbol{v}) = \mathrm{e}\, \epsilon U(p) \boldsymbol{g}_p(\boldsymbol{u}, \boldsymbol{v})$ for a given function $U : M \to \mathbb{R}$.

We will consider infinitesimal changes of the metric [14], thus from now on, the coordinates of the transformed metric will be considered $\overline{g}_{ab} = (1 + \epsilon U)g_{ab} + \mathcal{O}(\epsilon^2)$. All quantities involving ϵ^2 and more will be neglected. The infinitesimal change of $\overline{\boldsymbol{g}}$ is defined as

$$\delta g_{ab} := \overline{g}_{ab} - g_{ab} = \epsilon U g_{ab}.$$

The definition of infinitesimal change of a general tensor is equivalent. Wee are looking for a tensor $C^a{}_{bcd}$ such that $\delta C^a{}_{bcd} = 0$ or, equivalently, $\delta C_{abcd} = \epsilon U C_{abcd}$. We express this tensor in terms of the Riemann curvature tensor and its contractions:

$$C_{abcd} = R_{abcd} + A_{ac}R_{bd} + B_{ad}R_{bc} + D_{bc}R_{ad} + E_{bd}R_{ac} + F_{ab}R_{cd} + G_{cd}R_{ab} + H_{abcd}R.$$

Using that C_{abcd} has the same symmetries as the RC tensor and $R_{ab} = -R_{ba}$, we obtain the following conditions:

$$A_{ab} = -B_{ab}, \quad D_{ab} = -E_{ab}, \quad F_{ab} = -F_{ab} = G_{ab} = -G_{ab} = 0.$$

The tensor H_{abcd} also has the same symmetries as C_{abcd} and R_{abcd}.

To compute δC_{abcd}, first we need to know how the tensors R_{abcd}, R_{ab} and R change under conformal transformations. We saw in eq. (1.3) how Christoffel symbol change. From $\overline{\Gamma}^c_{ab}$ it is possible to derive $\overline{R}_{abcd}$ and its contractions [14]:

$$\begin{aligned}
\overline{R}_{abcd} &= (1-\epsilon U)R_{abcd} - \frac{1}{2}\epsilon g_{ac}U_{,d;b} + \frac{1}{2}\epsilon g_{ad}U_{,c;b} - \frac{1}{2}\epsilon g_{bd}U_{,a;c} + \frac{1}{2}\epsilon g_{bc}U_{,a;d} \\
\overline{R}_{ab} &= R_{ab} - \frac{1}{2}\epsilon g_{ab}g^{ef}U_{,e;f} - \frac{1}{2}(n-2)\epsilon U_{,a;b} \\
\overline{R} &= (1+\epsilon U)R - (n-1)\epsilon g^{ef}U_{,e;f}
\end{aligned}$$

Taking the infinitesimal change of these tensors we obtain δC_{abcd}:

$$\begin{aligned}
\delta C_{abcd} =& \epsilon U R_{abcd} + \frac{1}{2}\epsilon g_{ac}U_{,d;b} - \frac{1}{2}\epsilon g_{ad}U_{,c;b} + \frac{1}{2}\epsilon g_{bd}U_{,a;c} - \frac{1}{2}\epsilon g_{bc}U_{,a;d} + \\
& + R_{bd}\delta A_{ac} + A_{ac}\left(\frac{1}{2}\epsilon g_{bd}g^{ef}U_{,e;f} + \frac{1}{2}(n-2)\epsilon U_{,b;d}\right) + \\
& + \cdots + \\
& + R\delta H_{abcd} + H_{abcd}\left((n-1)\epsilon g^{ef}U_{,e;f} - \epsilon U R\right).
\end{aligned}$$

Any term involving derivatives and covariant derivative of the function U must vanish. Then, tensors A_{ab}, B_{ab}, D_{ab} and E_{ab} are determined:

$$A_{ac}\frac{1}{2}(n-2)\epsilon U_{,b;d} + \frac{1}{2}\epsilon g_{ab}U_{,b;d} = 0$$

$$A_{ac} = -\frac{1}{n-2}g_{ac}, \qquad \delta A_{ac} = -\frac{1}{n-2}\epsilon U g_{ac}.$$

We do the same for tensors $\boldsymbol{B}$, $\boldsymbol{D}$ and $\boldsymbol{E}$:

$$\begin{aligned}
B_{ad} &= \frac{1}{n-2}g_{ad}, & \delta B_{ad} &= \frac{1}{n-2}\epsilon U g_{ab}, \\
D_{bc} &= \frac{1}{n-2}g_{bc}, & \delta D_{bc} &= \frac{1}{n-2}\epsilon U g_{bc}, \\
E_{bd} &= -\frac{1}{n-2}g_{bd}, & \delta E_{bd} &= -\frac{1}{n-2}\epsilon U g_{bd}.
\end{aligned}$$

In order to find H_{abcd}, we require the terms $g^{ef}U_{,e;f}$ vanish. Using the terms A_{ab}, B_{ab}, D_{ab} and E_{ab}, we obtain the relation:

$$(n-1)H_{abcd} - \frac{1}{2(n-2)}g_{ac}g_{bd} + \frac{1}{2(n-2)}g_{ad}g_{bc} + \frac{1}{2(n-2)}g_{bc}g_{ad} - \frac{1}{2(n-2)}g_{bd}g_{ac} = 0$$

$$H_{abcd} = \frac{1}{(n-1)(n-2)}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right), \qquad \delta H_{abcd} = 2\epsilon U H_{abcd}.$$

We immediately see that the requirement $\delta C_{abcd} = \epsilon U C_{abcd}$ is fulfilled, thus:

$$R_{abcd} + \frac{1}{n-2}(g_{ac}R_{bd} - g_{bc}R_{ad} + g_{bd}R_{ac} - g_{ad}R_{bc}) + \frac{R}{(n-1)(n-2)}(g_{ac}g_{bd} - g_{ad}g_{bc}).$$

The 1-contravariant, 3-covariant form of the Weyl tensor is (taking the identity $g^{ae}g_{eb} = \delta^a_b$):

$$\begin{aligned} C^a{}_{bcd} =& R^a{}_{bcd} + \frac{1}{(n-2)}\left(\delta^a_d R_{bc} - \delta^a_c R_{bd} + g_{bc}R^a{}_d - g_{bd}R^a{}_c\right) + \\ &+ \frac{1}{(n-1)(n-2)}\left(\delta^a_c g_{bd} - \delta^a_d g_{bc}\right) R \end{aligned}$$

□

By definition of the Weyl tensor, it has the same symmetries as the RC tensor and is invariant under conformal transformation of the Lorentzian space. Moreover, this tensor is trace-free, i.e., $C^a{}_{bad} = 0$. It is easy to see once we have an expression of $\boldsymbol{C}$ in local coordinates:

$$\begin{aligned} C^a{}_{bad} &= R_{bd} + \frac{1}{n-2}(\delta^a_d R_{ba} - \delta^a_a R_{bd} + g_{ba}R^a_d - g_{bd}R^a_a) + \frac{1}{(n-1)(n-2)}(\delta^a_a g_{bd} - \delta^a_d g_{bc}) \\ &= R_{bd} + \frac{1}{n-2}(R_{bd} - nR_{bd} + R_b - g_{bd}R) + \frac{1}{(n-1)(n-2)}(ng_{bd} - g_{bd})R \\ &= R_{bd} + \frac{1}{n-2}(-(n-2)R_{bd}) - \frac{1}{n-2}g_{bd}R + \frac{1}{(n-1)(n-2)}((n-1)g_{bd})R \\ &= 0 \end{aligned}$$

The RC tensor in a Lorentzian manifold of dimension $n = 4$ can be decomposed in different tensors as follows:

$$R_{abcd} = C_{abcd} + E_{abcd} + \frac{1}{6}RG_{abcd}, \tag{2.6}$$

where $\boldsymbol{C}$ is the Weyl tensor defined before, and $\boldsymbol{E}$ and $\boldsymbol{G}$ are the tensors

$$\begin{aligned} E_{abcd} &:= \frac{1}{2}(g_{ac}S_{bd} + g_{bd}S_{ac} - g_{ad}S_{bc} - g_{bc}S_{ad}), \\ G_{abcd} &:= \frac{1}{2}(g_{ac}g_{bd} - g_{ad}g_{bc}) = \frac{1}{2}g_{abcd}, \\ S_{ab} &:= R_{ab} - \frac{1}{4}Rg_{ab}. \end{aligned}$$

The Weyl tensor is skew-symmetric on the first and second pair of indices. As the RC tensor, it has the following symmetries:

$$C_{abcd} = -C_{bacd} = -C_{abdc} = C_{badc} = C_{cdab},$$

$$C_{abcd} + C_{acdb} + C_{adbc} = 0 \iff C_{a[bcd]} = 0.$$

For this reason, at fixed indices a, b, the tensor C_{abcd} is a symmetric double bivector and we denote this tensor with capital letter indices to indicate it is expressed in a bivector basis, $\boldsymbol{C} = C_{IJ}$. For this reason, we can define the Hodge dual of the Weyl tensor [12].

Definition 2.3.4. Consider the Weyl tensor C_{abcd} of a Lorentzian manifold of dimension $n = 4$. The *left-dual* of C_{abcd} is

$$^{\star}C_{abcd} = \frac{1}{2}\epsilon_{abef}C^{ef}{}_{cd}, \qquad ^{\star}C_{IJ} = \epsilon_{JK}C^{K}{}_{J}.$$

On the other hand, its *right-dual* is

$$C^{\star}_{abcd} = \frac{1}{2}\epsilon_{cdef}C_{ab}{}^{ef}, \qquad C^{\star}_{IJ} = \epsilon_{JK}C_{I}{}^{K}.$$

Given the symmetries of $\boldsymbol{C}$, it is clear that $^{\star}C_{abcd} = C_{cdab}{}^{\star}$. A non-trivial relation between the left-dual and the right-dual of the Weyl tensor is the following:

$$^{\star}C_{abcd} = C_{abcd}{}^{\star} \tag{2.7}$$

To prove it [2], we take an auxiliary tensor $T_{abcd} = {}^{\star}C_{abcd} - C_{abcd}{}^{\star}$, and calculate the contraction $\epsilon^{acef}T_{abcd}$:

$$\begin{aligned} \epsilon^{acef}T_{abcd} &= \epsilon^{acef}\left(\frac{1}{2}\epsilon_{abgh}C^{gh}{}_{cd} - \frac{1}{2}\epsilon_{cdgh}C_{ab}{}^{gh}\right) \\ &= -3(\delta^{c}_{[b}\delta^{e}_{g}\delta^{f}_{h]}C^{gh}{}_{cd} + \delta^{a}_{[d}\delta^{e}_{g}\delta^{f}_{h]}C_{ab}{}^{gh}) \\ &= -3(C^{ef}{}_{bd} + C_{db}{}^{ef}) \\ &= 0. \end{aligned}$$

From this result is straight-forward to see that $T_{abcd} = T_{cbad}$. Using a similar procedure, contracting e and b in the previous expression we obtain that $T_{abcd} + T_{bcad} + T_{cabd} = 0$. Using this and all the other symmetries, we obtain that $\boldsymbol{T}$ vanishes and then it proves that *eq.* (2.7) is true.

Due to the left-dual and right-dual of $\boldsymbol{C}$ are the same, we can define the *Hodge dual* of the Weyl tensor as the left-dual or right-dual indistinctly. As we did with bivectors, we can define the self-dual and anti-self-dual of the Weyl tensor as

$$C^{\pm}_{abcd} = C_{abcd} \pm i{}^{\star}C_{abcd},$$

satisfying $^{\star}C^{\pm}_{abdc} = \mp iC_{abcd}$.

2.4 Petrov classification

The set of all analytical solutions of Einstein's equations is classified in many different ways. One of the most important classifications is the Petrov classification, which is an algebraic classification of the Weyl tensor and is named after the first person who settled it (see [9]). In the previous section we defined the Weyl tensor and listed some of its properties. From this tensor, we can define a linear function $B_p \to B_p$ that maps a bivector to another bivector. Petrov classification is an study of the eigenvalues of this linear map.

Definition 2.4.1. Consider a Lorentz manifold $(M, \boldsymbol{g})$ and a point $p \in M$. Let $\boldsymbol{W} = C_{abcd} \in \mathcal{T}^2(B_p)$ be a double bivector. Then, we can define the linear map:

$$\begin{aligned} \boldsymbol{C}: \quad & B_p \longrightarrow B_p \\ & F_I \longmapsto C_{JI}F^I \\ & F^{ab} \longmapsto \frac{1}{2}C_{cdab}F^{ab} \end{aligned}$$

If $\boldsymbol{C}$ is the Weyl tensor, then is is said to be the *Weyl automorphism.*

Remark 2.10. This definition can be extended to the complexification of bivectors space, $\mathbb{C}B_p$. The image of a complex bivector $\boldsymbol{F} + i\boldsymbol{H}$ by $\boldsymbol{C}$ is simply $\boldsymbol{C}(\boldsymbol{F} + i\boldsymbol{H}) = \boldsymbol{C}(\boldsymbol{F}) + i\boldsymbol{C}(\boldsymbol{H})$, where $\boldsymbol{F}$ and $\boldsymbol{H}$ are real bivectors. We can also define the morphism associated to the Hodge dual of the Weyl tensor, ${}^\star\boldsymbol{C}$, since it is also a double bivector.

From now on we will consider the Weyl tensor as an automorphism. The algebraic study of this map leads to the Petrov classification. Different algebraic types of the Weyl tensor will represent different Petrov types. This algebraic study begins with the eigenvector equation for the Weyl map. We say that $\boldsymbol{F} \in B_p$ is an *eigenbivector* of the automorphism $\boldsymbol{C}$ if it fulfills the equation:

$$\boldsymbol{C}(\boldsymbol{F}) = \lambda \boldsymbol{F} \iff \frac{1}{2}C_{abcd}F^{cd} = \lambda F_{ab},$$

where λ is said to be the *eigenvalue.* In general, there could be complex eigenvalues, but the space of bivectors is real. This is why it makes sense to work in the complexified space. Thus, the Weyl automorphism is naturally extended to the map $\boldsymbol{C} : \mathbb{C}B_p \to \mathbb{C}B_p$, an automorphism of a vector space of dimension 6.

Proposition 2.4.2. *Consider the Weyl automorphism $\boldsymbol{C}$ and a bivector $\boldsymbol{F}$ on the tangent space of a smooth manifold M. Therefore, the following relations apply:*

$$\boldsymbol{C}({}^\star\boldsymbol{F}) = ({}^\star\boldsymbol{C})(\boldsymbol{F}) = {}^\star(\boldsymbol{C}(\boldsymbol{F}))$$

Proof. To prove this we express $\boldsymbol{C}$, $\boldsymbol{C}$ and their Hodge duals in local coordinates.

$$\boldsymbol{C} = C^{ab}{}_{cd} \qquad C^{\star ab}{}_{cd} = \frac{1}{2}\epsilon^{ef}{}_{cd}C^{ab}{}_{ef}$$

$$\boldsymbol{F} = F^{ab} \qquad {}^{\star}F^{ab} = \frac{1}{2}\epsilon^{ab}{}_{ef}F^{ef}$$

Now we compute separately all the terms of the relation and see that they are equal.

$$[\boldsymbol{C}({}^{\star}\boldsymbol{F})]^{ab} = \frac{1}{4}C^{ab}{}_{cd}\epsilon^{cd}{}_{ef}F^{ef}$$
$$[(\boldsymbol{C}^{\star})(\boldsymbol{F})]^{ab} = \frac{1}{4}\epsilon^{ef}{}_{cd}C^{ab}{}_{ef}F^{cd}$$
$$[{}^{\star}\boldsymbol{C}(\boldsymbol{F})]^{ab} = \frac{1}{4}\epsilon^{ab}{}_{cd}C^{cd}{}_{ef}F^{ef}$$

Due to the property of the Weyl tensor ${}^{\star}\boldsymbol{C} = \boldsymbol{C}^{\star}$, we also have that ${}^{\star}\boldsymbol{C}(\boldsymbol{F}) = ({}^{\star}\boldsymbol{C})(\boldsymbol{F})$. □

Using this property of the Weyl map, we can directly see that the image of a self-dual bivector is a self-dual bivector, and same happens for anti-self-dual bivector. Recall that bivectors belonging to $S_p^{\pm}$ have the form $\boldsymbol{F} \pm i^{\star}\boldsymbol{F}$ for any $\boldsymbol{F} \in \mathbb{C}B_p$. Therefore it holds that

$$\boldsymbol{C}(\boldsymbol{F} \pm i^{\star}\boldsymbol{F}) = \boldsymbol{C}(\boldsymbol{F}) \pm i\boldsymbol{C}({}^{\star}\boldsymbol{F}) = \boldsymbol{C}(\boldsymbol{F}) \pm i^{\star}(\boldsymbol{C}(\boldsymbol{F})).$$

The following lemma is direct from these properties and allows us to simplify the algebraic classification of the Weyl map.

Lemma 2.4.3. *The restrictions of the Weyl automorphism into S_p^+ and S_p^- are also automorphism, i.e:*

$$\boldsymbol{C}|_{S_p^{\pm}} : S_p^{\pm} \to S_p^{\pm}.$$

Moreover, given a bivector $\boldsymbol{F}$, its self-dual $\boldsymbol{F}^+$ is an eigenbivector with eigenvalue λ if and only if, the anti-self-dual $\boldsymbol{F}^-$ is an eigenbivector with eigenvalue $\overline{\lambda}$.

From this lemma we deduce that the Jordan form of both restrictions are the same. Thus, the eigenvalue problem is only studied only on one of the restrictions. These restrictions can be related to the self-dual of the Weyl tensor and the anti-self-dual, $\boldsymbol{C}^+$ and $\boldsymbol{C}^-$ respectively using the following observation:

$$\boldsymbol{F} \in S_p^+ \implies \boldsymbol{C}^+(\boldsymbol{F}) = (\boldsymbol{C} + i^{\star}\boldsymbol{C})(\boldsymbol{F}) = \boldsymbol{C} + i({}^{\star}\boldsymbol{C})(\boldsymbol{F}) = \boldsymbol{C}(\boldsymbol{F}) + i\boldsymbol{C}({}^{\star}\boldsymbol{F}) = 2\boldsymbol{C}(\boldsymbol{F})$$
$$\boldsymbol{F} \in S_p^- \implies \boldsymbol{C}^+(\boldsymbol{F}) = (\boldsymbol{C} + i^{\star}\boldsymbol{C})(\boldsymbol{F}) = \boldsymbol{C} + i({}^{\star}\boldsymbol{C})(\boldsymbol{F}) = \boldsymbol{C}(\boldsymbol{F}) + i\boldsymbol{C}({}^{\star}\boldsymbol{F}) = 0$$

The calculation for the anti-self-dual Weyl tensor $\boldsymbol{C}^-$ is the same. From this, we conclude then that

$$\boldsymbol{C}|_{S_p^+} = \frac{1}{2}(\boldsymbol{C} + i^{\star}\boldsymbol{C}), \qquad \boldsymbol{C}|_{S_p^-} = \frac{1}{2}(\boldsymbol{C} - i^{\star}\boldsymbol{C})$$

Since $\boldsymbol{C}^+$ and $\boldsymbol{C}^-$ have the same algebraic structure, we will classify the Weyl tensor of a manifold according to the eigenvalues structure of the map

$$\boldsymbol{C}|_{S_p^+} = \frac{1}{2}\boldsymbol{C}^+ = \frac{1}{2}(\boldsymbol{C} + i^\star\boldsymbol{C}).$$

It is a linear map in a vector space of dimension 3, S_p^+. The image of a self-dual bivector $\boldsymbol{F}$ is:

$$\frac{1}{2}\boldsymbol{C}^+(\boldsymbol{F})_I = \frac{1}{2}C^+_{IJ}F^J, \qquad \frac{1}{2}\boldsymbol{C}^+(\boldsymbol{F})_{ab} = \frac{1}{4}C^+_{abcd}F^{cd}.$$

We will denote the map $\frac{1}{2}\boldsymbol{C}^+$ by $\boldsymbol{\mathcal{W}}^+$. The eigenvalue problem associated with $\boldsymbol{\mathcal{W}}$ is:

$$\boldsymbol{\mathcal{W}}^+(\boldsymbol{F}) = \lambda\boldsymbol{F}, \qquad \frac{1}{2}\mathcal{W}^+_{abcd}F^{cd} = \lambda F_{ab}$$

From this expression we get that $\frac{1}{2}\mathcal{W}^+_{abcd}F^{cd} = \lambda G_{abcd}F^{cd}$, that results in the characteristic equation $|\boldsymbol{\mathcal{W}}^+ - \lambda\boldsymbol{G}| = 0$ ([9], [15]).

Definition 2.4.4. *Petrov Classification.* Consider a smooth Lorentzian manifold $(M, \boldsymbol{g})$ and its associated self-dual Weyl tensor $\boldsymbol{\mathcal{W}}^+$. Then, the *Petrov type* of M depends on the algebraic structure of the linear map defined by $\boldsymbol{\mathcal{W}}^+$

- *Type I.* The Weyl map has 3 different eigenvalues. This is the most general case. The characteristic equation yields

$$(\boldsymbol{\mathcal{W}}^+ - \lambda_1\boldsymbol{G})(\boldsymbol{\mathcal{W}}^+ - \lambda_2\boldsymbol{G})(\boldsymbol{\mathcal{W}}^+ - \lambda_3\boldsymbol{G}) = 0.$$

- *Type II.* The map has 2 different eigenvalues and it does not diagonalize. We have that

$$(\boldsymbol{\mathcal{W}}^+ - \lambda\boldsymbol{G})^2(\boldsymbol{\mathcal{W}}^+ + 2\lambda\boldsymbol{G}) = 0.$$

  The algebraic multiplicity of the eigenvalue λ is 2, but its geometric multiplicity is 1.

- *Type D.* In Petrov D spaces, the Weyl map also has two different eigenvalues. In this case, the algebraic multiplicity is the same as the geometric multiplicity for every eigenvalue.

$$(\boldsymbol{\mathcal{W}}^+ - \lambda\boldsymbol{G})(\boldsymbol{\mathcal{W}}^+ + 2\lambda\boldsymbol{G}) = 0.$$

- *Type III.* For type III, all the eigenvalues are the same and equal to zero, since the trace of $\boldsymbol{\mathcal{W}}^+$ is also zero. The geometric multiplicity of the eigenvalue is 1 and then, the map does not diagonalize:

$$(\boldsymbol{\mathcal{W}}^+)^2 \neq 0,\ (\boldsymbol{\mathcal{W}}^+)^3 = 0.$$

- *Type N.* As in type III, all three eigenvalues are the same, but with geometric multiplicity equal to 2 and thus, it does not diagonalize.

$$\boldsymbol{\mathcal{W}}^+ \neq 0,\ (\boldsymbol{\mathcal{W}}^+)^2 = 0.$$

- *Type O*. The Weyl tensor is equal to zero.

$$\boldsymbol{\mathcal{W}}^+ = 0.$$

Remark 2.11. Due to the tracelessness of the Weyl tensor, if its diagonal form has two different eigenvalues, λ_1 and λ_2, then $2\lambda_1 = -\lambda_2$, where λ_1 is taken to be the eigenvalue with multiplicity 2.

Taking the basis of S_p^+ $\{\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}\}$, we can express the self-dual Weyl map as follows:

$$\begin{aligned}\frac{1}{2}\boldsymbol{C}^+ = \quad & \Psi_0 \boldsymbol{U} \otimes \boldsymbol{U} + \Psi_1 \boldsymbol{U} \otimes \boldsymbol{V} + \Psi_2 \boldsymbol{U} \otimes \boldsymbol{W} \\ & + \Psi_3 \boldsymbol{V} \otimes \boldsymbol{U} + \Psi_4 \boldsymbol{V} \otimes \boldsymbol{V} + \Psi_5 \boldsymbol{V} \otimes \boldsymbol{W} \\ & + \Psi_6 \boldsymbol{W} \otimes \boldsymbol{U} + \Psi_7 \boldsymbol{W} \otimes \boldsymbol{V} + \Psi_8 \boldsymbol{W} \otimes \boldsymbol{W}\end{aligned}$$

It is useful to work with the coordinates of this tensor, C^+_{IJ}, to show the relation between the Ψ coefficients:

(i) Symmetry. Since $C^+_{IJ} = C^+_{JI}$, we have that $\Psi_1 = \Psi_3$, $\Psi_2 = \Psi_6$ and $\Psi_5 = \Psi_7$.

(ii) Tracelessness. The Weyl tensor is trace-free, i.e. $C^I{}_I = 0$. Moreover, using that $U_I V^I = 2$ and $W_I W^I = -4$, we obtain that $\Psi_1 = \Psi_3 = \Psi_8$.

The final expression for the automorphism that will be studied, in terms of a self-dual bivector basis, is the following:

$$\begin{aligned}\mathcal{W}^+_{IJ} := \frac{1}{2}\boldsymbol{C}^+_{IJ} = & \Psi_0 U_I U_J + \Psi_1 (U_I W_J + W_I U_J) + \Psi_2 (U_I V_J + V_I U_J + W_I W_J) \\ & + \Psi_3 (V_I W_J + W_I V_J) + \Psi_4 V_I V_J.\end{aligned} \tag{2.8}$$

The coefficients can be directly computed from the Weyl tensor. Contracting the bivector $\boldsymbol{V}$ twice to the Weyl tensor $\boldsymbol{C}$, and using that S_p^+ and S_p^- are orthogonal spaces and also eq. (2.5), we obtain:

$$\begin{aligned}C_{abcd} V^{ab} V^{cd} &= \frac{1}{2}(C^+_{abcd} + C^-_{abcd}) V^{ab} V^{cd} = \frac{1}{2} C^+_{abcd} V^{ab} V^{cd} = \\ &= \Psi_0 U_{ab} U_{cd} V^{ab} V^{cd} = 4\Psi_0.\end{aligned}$$

Recalling the form of bivectors from eq. (2.4) it is easy to see that $U_{ab} U_{cd} k^a m^b k^c m^d = 1$. Using all of this, we now know an expression for Ψ_0. With similar procedures, it is possible to give a way to calculate all the other coefficients.

$$\begin{aligned}\Psi_0 &= C_{abcd} k^a m^b k^c m^d & \Psi_1 &= C_{abcd} k^a l^b k^c m^d \\ \Psi_2 &= C_{abcd} k^a m^b \overline{m}^c l^d & \Psi_3 &= C_{abcd} k^a l^b \overline{m}^c l^d \\ \Psi_4 &= C_{abcd} \overline{m}^a l^b \overline{m}^c l^d. & &\end{aligned} \tag{2.9}$$

As the Weyl map is a linear map, we can build a matrix in the bivector basis. Using the expression eq. (2.8) and eq. (2.5), we can obtain an expression for the matrix with Ψ coefficients.

$$\begin{aligned}
\boldsymbol{\mathcal{W}}^+(\boldsymbol{U}) = \mathcal{W}^+_{IJ}U^J =& \quad 2\Psi_2 U_I + 2\Psi_4 V_I + 2\Psi_3 W_I \\
\boldsymbol{\mathcal{W}}^+(\boldsymbol{V}) = \mathcal{W}^+_{IJ}V^J =& \quad 2\Psi_0 U_I + 2\Psi_2 V_I + 2\Psi_3 W_I \\
\boldsymbol{\mathcal{W}}^+(\boldsymbol{W}) = \mathcal{W}^+_{IJ}W^J =& -4\Psi_1 U_I - 4\Psi_3 V_I - 4\Psi_2 W_I.
\end{aligned}$$

In terms of a vector basis $\{\boldsymbol{e}_i \otimes \boldsymbol{e}_j\}$, the coefficients are halved. The matrix for the self-dual Weyl automorphism, also denoted by $\boldsymbol{\mathcal{W}}^+$ is traceless, as the Weyl tensor.

$$\boldsymbol{\mathcal{W}}^+ = \begin{pmatrix} \Psi_2 & \Psi_0 & -2\Psi_1 \\ \Psi_4 & \Psi_2 & -2\Psi_3 \\ \Psi_3 & \Psi_1 & -2\Psi_2 \end{pmatrix} \tag{2.10}$$

We reduced the Petrov classification problem to the algebraic classification of a linear map $\boldsymbol{\mathcal{W}}^+ : \mathbb{C}^3 \to \mathbb{C}^3$. The relation between the coefficients will determine whether the Petrov type of $\boldsymbol{\mathcal{W}}^+$ is one or other.

The first step to the diagonalization of the matrix is to solve the characteristic equation for the Weyl matrix, $\det(\boldsymbol{\mathcal{W}}^+ - \lambda\boldsymbol{I}) = 0$. From the matrix form detailed before, we obtain the following equation for the eigenvalues:

$$\begin{aligned}
\det(\boldsymbol{\mathcal{W}}^+ - \lambda\boldsymbol{I}) =& -\lambda^3 + (\Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3\Psi_2^3)\lambda \\
&+ (2\Psi_0\Psi_2\Psi_4 - 2\Psi_0\Psi_3^2 - 6\Psi_1^2\Psi_4 + 4\Psi_1\Psi_2\Psi_3 - 6\Psi_2^3) \\
=& -\lambda^3 + \frac{1}{2}I\lambda + \frac{1}{3}J = 0.
\end{aligned} \tag{2.11}$$

We have included the scalars I and J, which are the trace of $(\boldsymbol{\mathcal{W}}^+)^2$ and $(\boldsymbol{\mathcal{W}}^+)^3$ respectively. These traces are obtained by contraction of indices of the self-dual Weyl tensor and thus, are *invariant* with respect to change of coordinates.

$$\begin{aligned}
I = tr((\boldsymbol{\mathcal{W}}^+)^2) = \frac{1}{16}C^+_{abcd}C^{+abcd} \qquad &= 2\Psi_0\Psi_4 - 8\Psi_1\Psi_3 + 6\Psi_2^2 \\
J = tr((\boldsymbol{\mathcal{W}}^+)^3) = \frac{1}{64}C^+_{abcd}C^{+cd}{}_{ef}C^{+efab} &= 6\Psi_0\Psi_2\Psi_4 - 6\Psi_0\Psi_3^2 - 6\Psi_1^2\Psi_4 \\
&\quad + 12\Psi_1\Psi_2\Psi_3 - 6\Psi_2^3
\end{aligned} \tag{2.12}$$

Since eq. (2.11) is a cubic equation in λ, it is possible to know analytically [1] the three eigenvalues:

$$\begin{aligned}
\lambda_1 =& -\frac{1}{2}(K_+ + K_-) + i\frac{\sqrt{3}}{2}(K_+ + K_-) \\
\lambda_2 =& -\frac{1}{2}(K_+ + K_-) - i\frac{\sqrt{3}}{2}(K_+ + K_-) \\
\lambda_3 =& K_+ + K_-,
\end{aligned} \tag{2.13}$$

where the terms $K_\pm$ are defined as follows:

$$K_\pm := \left(\frac{J}{6} \pm \frac{1}{6}\sqrt{J^2 - \frac{I^3}{6}}\right).$$

By directly looking at eq. (2.13), it is easy to note that if $I^3 \neq 6J^2$, then all three roots are different and the matrix diagonalizes. The following expression is then fulfilled:

$$(\boldsymbol{\mathcal{W}}^+ - \lambda_1 \boldsymbol{I})(\boldsymbol{\mathcal{W}}^+ - \lambda_2 \boldsymbol{I})(\boldsymbol{\mathcal{W}}^+ - \lambda_3 \boldsymbol{I}) = 0, \qquad \lambda_1 \neq \lambda_2 \neq \lambda_3.$$

That means that we found a condition for Petrov type I spaces that only involves the Ψ coefficients and does not depend on changes of coordinates.

The case $I^3 = 6J^2$ is called *algebraically special* and includes Petrov types II, D, III and N. There are at least two equal eigenvectors:

$$\lambda_1 = \lambda_2 = -\sqrt[3]{\frac{J}{6}},$$
$$\lambda_3 = 2\sqrt[3]{\frac{J}{6}}.$$

When $I = J = 0$ then, all three eigenvalues are 0 and thus, the Petrov type of the manifold can be III, N or O. If the Weyl tensor completely vanishes, then its Petrov type is O. If the Weyl tensor is not zero, but $(\boldsymbol{\mathcal{W}}^+)^2 = 0$, then the Petrov type is N. Finally, if we have that $(\boldsymbol{\mathcal{W}}^+)^2 \neq 0$ but $(\boldsymbol{\mathcal{W}}^+)^3 = 0$, then the manifold is classified in the Petrov type III.

For type D and II spaces, we have that $I^3 = 6J^2 \neq 0$. For type D, the Weyl matrix diagonalizes and therefore, the minimum polynomial is $(x - \lambda_1)(x - \lambda_3)$, with $\lambda := \lambda_1 = -\frac{1}{2}\lambda_3$. For this reason we can obtain the following condition for Petrov type D spaces:

$$(\boldsymbol{\mathcal{W}}^+ - \lambda \boldsymbol{I})(\boldsymbol{\mathcal{W}}^+ + 2\lambda \boldsymbol{I}) = 0 \implies (\boldsymbol{\mathcal{W}}^+)^2 - 2\lambda^2 \boldsymbol{I} = -\lambda \boldsymbol{\mathcal{W}}^+. \tag{2.14}$$

If this equality is not fulfilled, then, the Petrov type is II.

We have build, so far, a criterion to determine the Petrov type of an Einstein space. From the Riemann curvature tensor R_{abcd} we get the Weyl conformal tensor C_{abcd} using eq. (2.6). We obtain a complex null tetrad $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$ (definition 2.1.10) and find coefficients $\Psi_0, \ldots, \Psi_4$ using eq. (2.9). With the coefficients we get the matrix of $\boldsymbol{\mathcal{W}}^+$ (2.10). After computing the traces I and J (eq. (2.12)), we can proceed to classify the space.

- *Petrov Type I.* We have the relation $I^3 \neq 6J^2$.
- *Petrov Type D.* In this case, $I^3 = 6J^2$. For this case, the condition 2.14 applies, i.e.

$$(\boldsymbol{\mathcal{W}}^+)^2 - 2\left(\frac{J}{6}\right)^{\frac{2}{3}} \boldsymbol{I} \propto \boldsymbol{\mathcal{W}}^+. \tag{2.15}$$

- *Petrov Type II.* Like in Petrod type D case, we have that $I^3 = 6J^2$, but in this case, the relation 2.14 is not fulfilled.
- *Petrov Type III.* Traces I and J are 0. We have that $(\boldsymbol{\mathcal{W}}^+)^2 \neq 0$.
- *Petrov Type N.* Here I and J are also 0, but in this case $(\boldsymbol{\mathcal{W}}^+)^2 = 0$.
- *Petrov Type O.* The Weyl tensor is 0.

Note that all possible situations have been visited and hence, this classification is complete, i.e. every space-time belongs to one of the defined Petrov types.

2.5 Null principal directions. Debever classification

We have discussed the classification done by Petrov in the previous section. There are other equivalent approaches to the classification of the Weyl tensor. In this section, the method described by Debever starts with a definition of *principal null direction* and results in the same characterization of spaces, but there is no need of calculating eigenvalues and matrices diagonal forms. This method instead will give us the canonical form of the Weyl matrix. A study on principal null directions can be found at [6].

This classification, as the Petrov classification, is defined locally in a neighborhood of a point p in a Lorentz manifold M of dimension 4.

Definition 2.5.1. Let $\boldsymbol{n}$ be a null vector of T_pM. Consider the Weyl tensor $\boldsymbol{C}$ and its self-dual, $\boldsymbol{C}^+$. Suppose that the Weyl tensor at $p \in M$ is not zero, $\boldsymbol{C}_p \neq 0$. Then, the space spanned by $\boldsymbol{n}$ is said to be a *principal null direction* (p.n.d.) of $\boldsymbol{C}$ (or $\boldsymbol{C}^+$) if $\boldsymbol{n}$ satisfies

$$n_{[e}C_{a]bc[d}n_{f]}n^b n^c = 0.$$

One can find a null tetrad $\{\boldsymbol{k}', \boldsymbol{l}', \boldsymbol{m}', \overline{\boldsymbol{m}'}\}$ in which the vector $\boldsymbol{k}'$ or $\boldsymbol{l}'$ spans a p.n.d. In order to study the p.n.d. problem, we will consider all the null rotations (proposition 2.1.12) around the null vector $\boldsymbol{l}$. This change will give us a null tetrad that depends on a factor E. After a few calculations, one obtains an expression for the transformed bivector basis (eq. (2.4)):

$$\begin{aligned} \boldsymbol{U}' &= \boldsymbol{U} \\ \boldsymbol{V}' &= \boldsymbol{V} - E\boldsymbol{W} + E^2\boldsymbol{U} \\ \boldsymbol{W}' &= \boldsymbol{W} - 2E\boldsymbol{U} \end{aligned}$$

We can write down the Weyl tensor in terms of these bivectors and coefficients $\Psi_0' \dots \Psi_4'$ in the same way we did in eq. (2.8). Expanding the expression of $\boldsymbol{C}^{+'}$ and expressing it in terms of $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{W}$, we obtain the following relations between the coefficients Ψ_i and Ψ'_i:

$$\begin{aligned}
\Psi'_0 &= \Psi_0 - 4E\Psi_1 + 6E^2\Psi_2 - 4E^3\Psi_3 + E^4\Psi_4 \\
\Psi'_1 &= \Psi_1 - 3E\Psi_2 + 3E^2\Psi_3 - E^3\Psi_4 \\
\Psi'_2 &= \Psi_2 - 2E\Psi_3 + E^2\Psi_4 \\
\Psi'_3 &= \Psi_3 - E\Psi_4 \\
\Psi'_4 &= \Psi_4
\end{aligned} \tag{2.16}$$

One can compute an equivalent condition for being p.n.d. using eq. (2.8), eq. (2.4) and the definition of null tetrad. A useful result proven by Sachs [10] make this calculation easier:

Proposition 2.5.2. *Consider a null tetrad $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$. Then we have the following equivalences:*

(i) $\boldsymbol{k}$ *is p.n.d* $\iff C_{abcd}V^{ab}V^{cd} = 0.$

(ii) $\boldsymbol{l}$ *is p.n.d* $\iff C_{abcd}U^{ab}U^{cd} = 0.$

From this statement we deduce, using eq. (2.8) and eq. (2.5) that the condition of $\boldsymbol{k}$ is a p.n.d is equivalent to $\Psi_0 = 0$. On the other hand, if the null vector $\boldsymbol{l}$ spans a p.n.d, then one finds that $\Psi_4 = 0$.

An analysis on the number of different p.n.d. will result in a classification that is completely equivalent to the Petrov classification.

Proposition 2.5.3. *There are at least one and at most four p.n.d..*

Proof. Consider a null tetrad $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$. If $\boldsymbol{l}$ is p.n.d. the coefficient Ψ_4 vanishes. Now we apply a Lorentz rotation around this null vector to obtain vectors $\boldsymbol{k}'$ that are p.n.d., e.g., $\Psi'_0 = 0$. Looking at the first equation of 2.16, we obtain a cubic equation in E. For each solution we obtain a p.n.d., and there are at most 3 of them.

On the other hand, if $\boldsymbol{l}$ is not p.n.d., then the coefficient Ψ_4 is not zero, and then applying a null rotation and looking for tetrads in which Ψ'_0 vanishes, we obtain a quartic equation in E with at least one and at most 4 solutions (each different value of E yields a p.n.d.). ☐

A p.n.d. can have *multiplicity* greater than 1. If that happens, then there are less than four p.n.d. and one of them is said to be a *double p.n.d.*.

Definition 2.5.4. Consider the Weyl tensor $\boldsymbol{C} \neq 0$ of a Lorentz manifold. A null vector $\boldsymbol{n}$ is said to be a *double* or *repeated p.n.d.* of $\boldsymbol{C}$ if

$$n_{[e}C_{a]bcd}n^b n^c.$$

Remark 2.12. Clearly, a repeated p.n.d. is also a p.n.d..

Given a null tetrad, another result from Sachs [10] can be used to characterize repeated p.n.d.:

$$\begin{aligned} \boldsymbol{k} \text{ is repeated p.n.d.} \quad &\Longleftrightarrow \quad V^{ea} C_{abcd} V^{cd} = 0. \\ \boldsymbol{l} \text{ is repeated p.n.d.} \quad &\Longleftrightarrow \quad U^{ea} C_{abcd} U^{cd} = 0. \end{aligned} \tag{2.17}$$

We will see that the first condition is equivalent to say $\Psi_0 = \Psi_1 = 0$. First, using eq. (2.8) and eq. (2.5) we obtain that

$$C_{abcd} V^{cd} = \frac{1}{2} C^{+}_{abcd} V^{cd} = 2\Psi_0 U_{ab} + 2\Psi_1 W_{ab} + 2\Psi_2 V_{ab}.$$

We compute now the contractions of the bivector $\boldsymbol{V}$ with $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{W}$ using definition 2.1.10:

$$\begin{aligned} V^{ab} U_{bc} &= k^a l_c - m^a \overline{m}_c \\ V^{ab} V_{bc} &= 0 \\ V^{ab} W_{bc} &= -k^a m_c + m^a k_c \end{aligned}$$

Now we compute $V^{ea} C_{abcd} V^{cd}$:

$$\begin{aligned} V^{ea} C_{abcd} V^{cd} &= V^{ea} (2\Psi_0 U_{ab} + 2\Psi_1 W_{ab} + 2\Psi_2 V_{ab}) \\ &= 2\Psi_0 (k^e l_b - m^e \overline{m}_b) + 2\Psi_1 (m^e k_b - k^e m_b) \end{aligned}$$

This expression vanishes only when $\Psi_0 = \Psi_1 = 0$, since all four vectors of the null tetrad are independent and non-zero. An analogous argument is used to prove that $\Psi_3 = \Psi_4 = 0 \iff \boldsymbol{l}$ is p.n.d..

Proposition 2.5.5. *There are at most 2 repeated p.n.d. and possibly none.*

Proof. As we did before, consider the null vectors $\boldsymbol{k}$ and $\boldsymbol{l}$ from a null tetrad. If $\boldsymbol{l}$ is not a repeated p.n.d. we apply a rotation around it to find null vectors $\boldsymbol{k}'$ that are repeated null directions. The condition we impose is that $\Psi_0' = \Psi_1' = 0$ in eq. (2.16). There might be no repeated null directions if all solutions for E in the $\Psi_0' = 0$ equation are different from solutions in $\Psi_1' = 0$ equation.

If there exist a repeated principle null direction, then choose it to be $\boldsymbol{l}$ and then apply a rotation. Since $\boldsymbol{l}$ is a double null direction, we have that $\Psi_3 = \Psi_4 = 0$. The condition for $\boldsymbol{k}'$ (the null vector we obtain with the rotation around $\boldsymbol{l}$) being double null direction is that $\Psi_0' = \Psi_1' = 0$. We impose this condition in eq. (2.16) and find out that there is a linear equation in E in the second equation. Therefore, there are *at most* one more repeated p.n.d. (and two in total). □

Remark 2.13. We will name *simple* p.n.d. to those that are not repeated. If there is a repeated p.n.d., then there are at most two simple p.n.d. more.

Definition 2.5.6. Let $\boldsymbol{C} \neq 0$ be the Weyl tensor of an Einstein space. A null vector $\boldsymbol{n}$ is a *triple principal null direction* of $\boldsymbol{C}$ if

$$C_{abc[d}n_{e]}n^c = 0.$$

The Sachs condition [10] for triple p.n.d. is the following (for a null tetrad $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$):

$$\begin{aligned} \boldsymbol{k} \text{ is triple p.n.d} &\iff C_{abcd}V^{cd} = 0 \\ \boldsymbol{l} \text{ is triple p.n.d} &\iff C_{abcd}U^{cd} = 0 \end{aligned} \tag{2.18}$$

Proceeding in a similar way we did before, we obtain that the condition for $\boldsymbol{k}$ being a triple p.n.d. is equivalent to $\Psi_0 = \Psi_1 = \Psi_2 = 0$. Analogously, if $\boldsymbol{l}$ is a triple p.n.d we have that $\Psi_2 = \Psi_3 = \Psi_4 = 0$.

Proposition 2.5.7. *Given a Weyl tensor $\boldsymbol{C}$, there is at most one triple p.n.d..*

Proof. The proof for this is analogous to the previous ones. Consider a null vector $\boldsymbol{l}$ of a null tetrad. Suppose it to be a triple p.n.d. and apply a null rotation around it. Since $\boldsymbol{l}$ is triple p.n.d, we have that $\Psi_2 = \Psi_3 = \Psi_4 = 0$. Looking at eq. (2.16), we look for E factors that make Ψ'_0, Ψ'_1 and Ψ'_2 vanish. The Ψ'_2 factor is already 0, and if there is another triple p.n.d. coefficients Ψ_0 and Ψ_1 vanish, but it would mean that the Weyl tensor is zero. Therefore, there can be at most one triple principal null direction. $\square$

There are another type of principal null direction, that is the most degenerated one.

Definition 2.5.8. Consider the Weyl tensor $\boldsymbol{C}$ and suppose it is different from zero. A *quadruple principal null direction* of $\boldsymbol{C}$ is a null vector $\boldsymbol{n}$ that satisfies

$$C_{abcd}n^d = 0.$$

Another result from Sachs [10] shows again more simple conditions when the null vectors of a null tetrad are quadruple p.n.d.:

$$\begin{aligned} \boldsymbol{k} \text{ is quadruple p.n.d.} &\iff C_{abcd}V^{ce} = 0 \\ \boldsymbol{l} \text{ is quadruple p.n.d.} &\iff C_{abcd}U^{ce} = 0 \end{aligned} \tag{2.19}$$

Then, if $\boldsymbol{k}$ is a quadruple p.n.d. all Ψ coefficients but Ψ_4 vanish. If $\boldsymbol{l}$ is quadruple p.n.d., then all coefficients but Ψ_0 vanish. If there is a quadruple p.n.d., then there are no other principal null directions.

We have described all the types of principal null directions a Weyl tensor can have. We can relate the number of principal null directions to the algebraic type of the Weyl tensor. Therefore, there is an equivalence between the Petrov classification and the number of principal null directions.

Proposition 2.5.9. *Consider an Einstein space, a Lorentz manifold M of dimension 4, and its Weyl tensor $\boldsymbol{C}$. Then we can classify the space according to the number and type of its principal null directions:*

(i) 4 simple p.n.d. $\implies$ *Petrov type I*

(ii) 2 repeated p.n.d. $\implies$ *Petrov type D*

(iii) 2 simple p.n.d and 1 repeated p.n.d $\implies$ *Petrov type II*

(iv) 1 triple p.n.d. and 1 simple p.n.d. $\implies$ *Petrov type III*

(v) 1 quadruple p.n.d. $\implies$ *Petrov type N*

Proof. The proof for the equivalence with the Petrov classification is quite straight forward from the analysis of the Ψ coefficients done before.

If there is a unique p.n.d. it has to be quadruple. Therefore there is a null tetrad in which the null vector $\boldsymbol{l}$ is this quadruple p.n.d. and thus, $\Psi_0 \neq 0$ and $\Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0$. The matrix form for the Weyl tensor is

$$\mathcal{W}^+ = \begin{pmatrix} 0 & \Psi_0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Both scalars I and J (eq. (2.12)) are zero, and $(\mathcal{W}^+)^2 = 0$. The space is then Petrov type N.

In the case of the existence of a triple p.n.d. (that is not quadruple) and a simple p.n.d., there is a null basis in which $\boldsymbol{l}$ is a triple principal null direction and the vector $\boldsymbol{k}$ is a simple one. It means that $\Psi_1 \neq 0$ and $\Psi_0 = \Psi_2 = \Psi_3 = \Psi_4 = 0$. In this case, the matrix form of the Weyl map is the following:

$$\mathcal{W}^+ = \begin{pmatrix} 0 & 0 & -2\Psi_1 \\ 0 & 0 & 0 \\ 0 & \Psi_1 & 0 \end{pmatrix}.$$

The scalars I and J are zero again, but in this case, $(\mathcal{W}^+)^2 \neq 0$ and $(\mathcal{W}^+)^3 = 0$ and thus, this is a case of Petrov type III space.

If the Weyl tensor has two different double p.n.d. there exist a null tetrad in which $\boldsymbol{k}$ and $\boldsymbol{l}$ are those principal null directions. In this basis the coefficients Ψ_0, Ψ_1, Ψ_3 and Ψ_4 vanish, and the matrix of the Weyl map has the form

$$\mathcal{W}^+ = \begin{pmatrix} \Psi_2 & 0 & 0 \\ 0 & \Psi_2 & 0 \\ 0 & 0 & -2\Psi_2 \end{pmatrix}.$$

This matrix is already in its diagonal form and it is clear that in this case the space is Petrov type D.

Now we look at the case in which the Weyl tensor has 3 principal null directions (one of them repeated and two simples). There is a null tetrad in which $\boldsymbol{k}$ is a simple p.n.d. and $\boldsymbol{l}$ is the repeated principal null direction. Coefficients Ψ_0, Ψ_3, Ψ_4 vanish and the matrix form of $\boldsymbol{C}^+$ in this basis is

$$\mathcal{W}^+ = \begin{pmatrix} \Psi_2 & 0 & -2\Psi_1 \\ 0 & \Psi_2 & 0 \\ 0 & \Psi_1 & -2\Psi_2 \end{pmatrix}.$$

If we compute the quantities I and J (eq. (2.12)) we obtain that $I^3 = 6J^2$ and we conclude that this is a degenerate case. These scalars are $I = 6\Psi_2^2$ and $J = -6\Psi_2^3$. In order to determine whether this is a Petrov type II or D space, we must verify if condition 2.15 is fulfilled or not. A straight-forward calculation leads to the result that this is a case of Petrov type II space.

Finally, if we have four different principal null directions, we can choose a null tetrad in which $\boldsymbol{k}$ and $\boldsymbol{l}$ are p.n.d.. In this case, vanishing coefficients are Ψ_0 and Ψ_4. The matrix of the Weyl map is now

$$\mathcal{W}^+ = \begin{pmatrix} \Psi_2 & 0 & -2\Psi_1 \\ 0 & \Psi_2 & -2\Psi_3 \\ \Psi_3 & \Psi_1 & -2\Psi_2 \end{pmatrix}.$$

When we compute the scalars I and J we get that the condition $I^3 = 6J^2$ is only given when

$$\Psi_2^2 = \frac{16}{9}\Psi_1\Psi_3, \tag{2.20}$$

but this can't happen, because then we could find a rotation around $\boldsymbol{l}$ that yields a double p.n.d. $\boldsymbol{k}'$, and we are supposing that there are 4 different simple principal null directions. Imagine we apply a null rotation and look for null vectors $\boldsymbol{k}'$ such that $\Psi_0' = 0$. In the actual tetrad, coefficients Ψ_0 and Ψ_4 are 0, since $\boldsymbol{k}$ and $\boldsymbol{l}$ are simple p.n.d.. The condition we have to fulfill is (from eq. (2.16))

$$\Psi_0' = 0 = 0 - 4\Psi_1 E + 6\Psi_2 E^2 - 4\Psi_3 E^3 + 0, \qquad E \neq 0.$$

We impose the condition 2.20 and divide by E the whole equation. We obtain

$$0 = -\frac{9}{4}\frac{\Psi_2^2}{\Psi_3} + 6\Psi_2 E - 4\Psi_3 E^2.$$

There is only one solution to this quadratic equation with multplicity 2, $E = \frac{3}{4}\frac{\Psi_2}{\Psi_3}$. Then there is only one more p.n.d. different from $\boldsymbol{k}$ and $\boldsymbol{l}$ and thus it must be a repeated p.n.d., since $\boldsymbol{k}$ and $\boldsymbol{l}$ are simple. For this reason, the relation 2.20 is not given and the Petrov type in this case is I.

Therefore, we conclude that in case of having 4 different principal null directions, the space is classified as Petrov type I. □

Since we have covered all the possible situations regarding the number and degeneracies of p.n.d., the reciprocal of this proposition is also valid (e.g. a Petrov type D space has two repeated p.n.d.).

Once we have related principal null directions and Petrov type, we can define a method of classifying spaces by figuring out how many p.n.d it has and identify them. We start by computing the Weyl tensor $\boldsymbol{C}$ in a null tetrad $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$. We compute all Ψ coefficients and then we look for the roots of the equation

$$0 = \Psi_0 - 4E\Psi_1 + 6E^2\Psi_2 - 4E^3\Psi_3 + E^4\Psi_4. \tag{2.21}$$

Each root of E will give a p.n.d. by applying a null rotation around $\boldsymbol{l}$ with the found E. If $\boldsymbol{l}$ is already a p.n.d. the coefficient Ψ_4 will be 0 and this equation will have at most 3 solutions (if $\boldsymbol{l}$ is a repeated p.n.d. it will have at most 2 solutions and so on).

Proposition 2.5.10. *Consider a null tetrad $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$ and coefficients $\Psi_0, \ldots, \Psi_4$ that determine the Weyl tensor in the basis $\{\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}\}$ defined by the null tetrad. A null rotation around the vector $\boldsymbol{l}$ yields a null vector*

$$\boldsymbol{k}' = \boldsymbol{k} + E\overline{\boldsymbol{m}} + \overline{E}\boldsymbol{m} + |E|^2\boldsymbol{l}, \qquad E \in \mathbb{C}.$$

If E is a solution of eq. (2.21) with multiplicity p, then $\boldsymbol{k}'$ is a principal null direction with the same multiplicity.

Proof. We will prove this when eq. (2.21) has a root with multiplicity $p = 2$. For other cases the proof is analogous.

It is obvious that if E_1 is a root of eq. (2.21), then $\boldsymbol{k}'$ is a p.n.d., since $\Psi_0' = 0$, and it is a characterization for principal null directions. Consider E_1 has multiplicity $p = 2$. Then we can write the equation as:

$$0 = (E - E_1)^2(E - E_2)(E - E_3),$$

where E_1 is the root with multiplicity 2 and E_2 and E_3 are simple roots. Expanding this expression we get

$$\begin{aligned} 0 =& E_1^2E_2E_3 + (-2E_1E_3E_3 - E_1^2E_2 - E_1^2E_2)E \\ &+ (E_1^2 + 2E_1E_2 + 2E_1E_3 + E_2E_3)E^2 \\ &+ (-2E_1 - E_2 - E_3)E^3 + E^4, \end{aligned}$$

from where we can get the equivalences:

$$\begin{aligned} \Psi_0 &= E_1^2E_2E_3 \\ \Psi_1 &= \frac{1}{4}(2E_1E_3E_3 + E_1^2E_2 + E_1^2E_2) \\ \Psi_2 &= \frac{1}{6}(E_1^2 + 2E_1E_2 + 2E_1E_3 + E_2E_3) \\ \Psi_3 &= \frac{1}{4}(2E_1 + E_2 + E_3) \\ \Psi_4 &= 1. \end{aligned}$$

We have normalized the equation to $\Psi_4 = 1$. Now we express the coefficient $\Psi_1'(E)$ that results from a null rotation around $\boldsymbol{l}$ (eq. (2.16)):

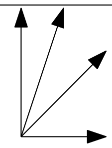
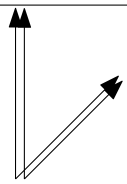
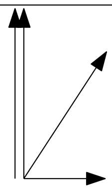
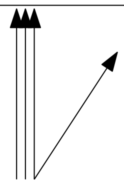
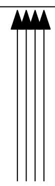
$$\begin{aligned}
\Psi_1' &= \Psi_1 - 3E\Psi_2 + 3E^2\Psi_3 - E^3\Psi_4 \\
&= \frac{1}{4}(2E_1E_3E_3 + E_1^2E_2 + E_1^2E_2) \\
&\quad - \frac{1}{2}(E_1^2 + 2E_1E_2 + 2E_1E_3 + E_2E_3)E \\
&\quad + \frac{3}{4}(2E_1 + E_2 + E_3)E^2 - E^3
\end{aligned}$$

It is easy to check that $\Psi_1'(E_1) = 0$ and thus, the null vector $\boldsymbol{k}'$ resulting from a rotation using E_1 is a repeated principal null direction. It cannot be a triple p.n.d. because E_2 and E_3 give two simple p.n.d., and there can not be a triple p.n.d. and two simple ones.

The proof for cases with multiplicity $p = 1$, $p = 3$ and $p = 4$ is the same. □

Figure 2.1 summarizes all the possibles Petrov types of spaces and their characterizations [12].

Figure 2.1: Relation between principal null directions and Petrov Type

Petrov Type	Number of roots of 2.21	Multiplicities	Principal null directions
I	4	(1,1,1,1)	
D	2	(2,2)	
II	3	(2,1,1)	
III	2	(3,1)	
N	1	(4)	

Chapter 3

Ricci Classification and the Energy-Momentum Tensor

In the previous chapter we decomposed the Riemann-Curvature tensor in different parts (2.6). Then we made an algebraic analysis of one of its parts, the Weyl tensor C_{abcd}, and defined an invariant classification. In this chapter we treat the algebraic classification for the remaining part of the RC tensor, the traceless Ricci tensor S_{ab}. This tensor is defined as

$$S_{ab} = R_{ab} - \frac{1}{4} R g_{ab}$$

and due to the symmetry of the Ricci tensor and the metric, it is also symmetric.

We will see all the possible algebraic types of the Ricci tensor and also the *Energy-Momentum tensor*, since it is a second order symmetric tensor. This will give another criteria of classification for Einstein spaces, the *Segre classification* [11], that has physical meaning. The Petrov classification gives a purely geometric description of the space, whilst the Segre classification gives some information about the physics, since it classifies the manifolds according to the energy tensor.

The results on the classification of second order symmetric tensors can be found on [6] and a discussion on the energy tensor on [7] and [13].

3.1 Algebraic classification of symmetric tensors

We start by defining an eigenvalue equation for the tensor S_{ab} that will lead to an algebraic classification of this tensor, as we did in the previous chapter with the Weyl tensor.

We will consider a smooth Lorentzian manifold $(M, \boldsymbol{g})$ and a point $p \in M$. If $T^a{}_b \in \mathcal{T}^1_1(T^*_p M, T_p M)$ is a second order symmetric tensor, then it defines a linear map

$T_pM \to T_pM$ that maps a vector $\boldsymbol{v}$ to $\boldsymbol{w}$:

$$\boldsymbol{w} = \boldsymbol{T}(\boldsymbol{v}) := T^a{}_b v^b.$$

We state the eigenvalue problem on the tensor S_{ab}, which will define its algebraic type and thus, lead to a further classification of spaces. This is, finding eigenvectors $\boldsymbol{v} \in \mathbb{C}T_pM$ and eigenvalues $\lambda \in \mathbb{C}$. Note that this classification will be local in a neighborhood of p.

$$S_{ab}v^b = \lambda v_a \iff v^b(S_{ab} - \lambda g_{ab}) = 0. \tag{3.1}$$

We have lowered the first index of the tensor $\boldsymbol{S}$ and then we have a map $\boldsymbol{S} : T_pM \to T_p^*M$, but the algebraic structure is the same. We will consider the classification of the Ricci tensor R_{ab}, since it has the same eigenvectors than S_{ab} and the eigenvalues are shifted by $\frac{1}{4}R$, with $R = R^a{}_a$. Suppose that $\boldsymbol{v}$ is an eigenvector with eigenvalue λ of S_{ab}, then:

$$R_{ab}v^b = (S_{ab} + \frac{1}{4}Rg_{ab})v^b = (\lambda + \frac{1}{4}R)v_a.$$

Thus, the Jordan form of R_{ab} and S_{ab} are equivalent.

In a metric space with a definite positive metric, a real symmetric matrix can always be diagonalized in an orthogonal basis. In this case, the metric g_{ab} adds a more complicated algebraic structure. In order to solve this, we can rewrite the eigenvalue problem 3.1 by contracting the expression with the metric. By this procedure, we obtain a standard formulation of the eigenvalue problem:

$$v^a(S_a{}^b - \delta_a^b) = 0, \qquad S_a{}^b = S_{ac}g^{cb} \tag{3.2}$$

Now the tensor $\boldsymbol{S}$ is regarded as a linear map $T_pM \to T_pM$, but it is no longer symmetric, since $S_{ac}g^{cb} \neq S_{bc}g^{ca}$ in general. If we follow this method to classify $\boldsymbol{S}$, we have to introduce the signature of g_{ab} and the symmetry condition afterwards. An introduction to this method is described below.

Remark 3.1. To describe the different algebraic types we will use the *Segré notation*. Between brackets, we write the orders of different Jordan blocks of a matrix. If multiple Jordan blocks correspond to the same eigenvalue, those numbers are written between parentheses. If there are complex conjugates eigenvalues, they are noted as Z and $\overline{Z}$. A general bracket Segré expression for a matrix with k different real eigenvalues and l pairs of complex conjugates eigenvalues would be:

$$\{(p_{11} \dots p_{1r_1}) \dots (p_{k1} \dots p_{kr_k}) Z_1\overline{Z}_1 \dots Z_l\overline{Z}_l\},$$

where the the number of Jordan blocks for the i-th eigenvalue is r_i and theire orders are $p_{i1}, \dots, p_{ir_i}$.

We will see now the different types of a symmetric tensor when all the eigenvalues are real. The method used is to treat with the form $S_a{}^b$ and then reinstate the symmetry condition and the metric signature [6].

Proposition 3.1.1. *Let $(M, \boldsymbol{g})$ be a Lorentz manifold of dimension 4. Let $\boldsymbol{S} \in \mathcal{T}^k(T_p^* M)$ such that $\boldsymbol{S} \neq 0$, be a second order symmetric tensor, $S_{ab} = S_{ba}$. Suppose that all the eigenvalues λ from 3.1 are real numbers. Then, the only possible Segré types (algebraic types) of this tensor resulting from the eigenvalue problem are $\{31\}$, $\{211\}$ and $\{1111\}$ (eigenvalues can be repeated).*

Proof. Consider the linear application defined by the tensor $\boldsymbol{S}$: $\boldsymbol{S}(\boldsymbol{v})^b = S_a{}^b v^a$. The associated matrix is not symmetric and the element (i, j) of this matrix is given by $S_{ac} g^{cb}$. Since all coefficients are real, and we are supposing that all eigenvalues are also real, the canonical form of this matrix can only take the following 5 types: $\{4\}, \{22\}, \{31\}, \{211\}$ and $\{1111\}$. We will proof that the type $\{4\}$ is not possible and, using the same method, that the type $\{31\}$ can be given in a second order symmetric tensor.

If the type of the matrix were $\{4\}$ then, expressed in the Jordan basis, the matrix has the form

$$\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \qquad \lambda \in \mathbb{R}.$$

Now we reinstate the symmetry condition for $\boldsymbol{S}$ by taking an arbitrary symmetric matrix for g_{ab} and imposing $S_{ab} = S_{ba}$, i.e. $S_a{}^c g_{cb} = S_b{}^c g_{ca}$. This is equivalent to say that the product of the matrix $\boldsymbol{S}$ by $\boldsymbol{g}$ is symmetric. The matrix $\boldsymbol{S} \cdot \boldsymbol{g}$ is

$$\begin{pmatrix} \lambda g_{11} + g_{12} & \lambda g_{12} + g_{22} & \lambda g_{13} + g_{23} & \lambda g_{14} + g_{24} \\ \lambda g_{12} + g_{13} & \lambda g_{22} + g_{23} & \lambda g_{23} + g_{33} & \lambda g_{24} + g_{34} \\ \lambda g_{13} + g_{24} & \lambda g_{23} + g_{24} & \lambda g_{33} + g_{34} & \lambda g_{34} + g_{44} \\ \lambda g_{14} & \lambda g_{24} & \lambda g_{34} & \lambda g_{44} \end{pmatrix}.$$

Since it has to be symmetric, we find the relations $g_{13} = g_{22}$, $g_{14} = g_{23}$, $g_{24} = g_{33} = g_{34} = g_{44} = 0$ in the Jordan basis. With this restrictions, the determinant of the metric is $\det(g_{ab}) = (g_{23})^4 \geq 0$. Then, a Lorentz metric with signature -1 is not compatible with the tensor S_{ab} having an algebraic type of $\{4\}$. To show that the type $\{22\}$ is also incompatible the procedure is the same.

For type $\{31\}$, the canonical form of the matrix $S_a{}^b$ is

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}, \qquad \lambda_1, \lambda_2 \in \mathbb{R}.$$

Note that if $\lambda_1 = \lambda_2$, then the Segré type would be expressed as $\{(31)\}$. As we did before, we impose the condition $S_a{}^c g_{bc} = S_b{}^c g_{ba}$ and in this case we obtain the relations $g_{22} = g_{13}$ and $g_{23} = g_{14} = g_{33} = g_{34} = g_{24} = 0$. For $g_{44} > 0$ and $g_{22} > 0$ (or $g_{44}, g_{22} < 0$) we obtain a valid Lorentz metric, since $\det(g_{ab}) = -g_{33} \cdot (g_{22})^3 < 0$. For dimension 4 matrices, it means that there are 1 positive eigenvalue and 3 negative ones, or 1 negative eigenvalue and 3 positive ones. It is then a Lorentz metric with signature $s = -1$. The proof to show that types $\{211\}$ and $\{1111\}$ are also possible are analogous. $\square$

We can use this method to express the Ricci tensor in its canonical form. Regarding the previous proof, if Ricci tensor has type $\{31\}$, then the vector vector $\boldsymbol{l} = (0, 0, 1, 0)$ (expressed in terms of the Jordan basis) is a null eigenvector with eigenvalue λ_1 and the vector $\boldsymbol{x} = (0, 0, 0, 1)$ is a spacelike eigenvector (if $g_{44} > 0$) with eigenvalue λ_2. Moreover, $l_a x^a = 0$. A way to proceed in order to write the canonical expression of $\boldsymbol{S}$ is to build a null tetrad from $\boldsymbol{l}$ and $\boldsymbol{x}' = (\boldsymbol{x} \cdot \boldsymbol{x})^{-\frac{1}{2}} \boldsymbol{x}$ and apply a general null rotation in order to get a general expression for the canonical form of a type $\{31\}$ second order symmetric tensor. Then we repeat the process for other possible algebraic types, even those with complex eigenvalues.

There is another possible approach to the problem in which we deal directly with the symmetric tensor S_{ab}. We start by analyzing how is the structure of 2-invariant planes of the linear application defined by S_{ab}.

Definition 3.1.2. Consider a Lorentz manifold $(M, \boldsymbol{g})$ of dimension 4, and a second order symmetric tensor $S_{ab} \in \mathcal{T}^2(T_p^*M)$, $S_{ab} \neq 0$ at a point $p \in M$. Let $\boldsymbol{S}$ be also the linear application defined by S_{ab}. A plane or 2-space $W \subset T_pM$ is said to be an *invariant 2-space of* $\boldsymbol{S}$ if $\forall \boldsymbol{v} \in W$, $\boldsymbol{S}(\boldsymbol{v}) = v^a S_a{}^b \in W$. We denote the subspace generated by two tensors $\boldsymbol{u}$ and $\boldsymbol{v}$ as $< \boldsymbol{u}, \boldsymbol{v} >$.

We can state some properties of invariant 2-spaces of $\boldsymbol{S}$ that will help in expressing the tensor in its canonical form.

Lemma 3.1.3. *Let M be a Lorentz manifold of dimension 4 and consider a second order symmetric tensor $\boldsymbol{S} \in \mathcal{T}^2(T_p^*M)$, $\boldsymbol{S} \neq 0$ at point $p \in M$. Then:*

(i) *$\boldsymbol{S}$ has at least one invariant 2-space.*

(ii) *If W is an invariant 2-space of $\boldsymbol{S}$, its orthogonal complement is also an invariant plane.*

(iii) *$\boldsymbol{S}$ admits a null invariant 2-space $W \iff \boldsymbol{S}$ admits a real null eigenvector $\boldsymbol{k}$.*

(iv) *$\boldsymbol{S}$ admits a spacelike (or equivalently, timelike) invariant 2-space $\iff \boldsymbol{S}$ has a pair of orthogonal spacelike eigenvectors (and they are included in the invariant plane).*

Proof. (i) In order to prove this, we use the result that we obtained in proposition 3.1.1. If all eigenvalues of $\boldsymbol{S}$ are real, then its algebraic type cannot be $\{4\}$, and thus, there exist at least two real eigenvectors, that clearly span an invariant 2-space. On the other hand, $\boldsymbol{S}$ can have a complex eigenvalue $Z \in \mathbb{C}$ with eigenvector $\boldsymbol{w} \in \mathbb{C}T_pM$, and $\boldsymbol{w} = \boldsymbol{u} + i\boldsymbol{v}$ for some $\boldsymbol{u}, \boldsymbol{v} \in T_pM$. Then, since the tensor $\boldsymbol{S}$ (and the associated matrix) is real, $\boldsymbol{u}$ and $\boldsymbol{v}$ span an invariant 2-space.

In order to prove (ii) to (iv), we write a general expression for $\boldsymbol{S}$ in terms of a real null tetrad $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{k}, \boldsymbol{l}\}$:

$$\begin{aligned} S_{ab} =& 2S^1 x_{(a}y_{b)} + S^2 x_a x_b + S^3 y_a y_b + 2S^4 x_{(a}k_{b)} + 2S^5 x_{(a}l_{b)} + 2S^6 y_{(a}k_{b)} \\ &+ 2S^7 y_{(a}l_{b)} + 2S^8 k_{(a}l_{b)} + S^9 k_a k_b + S^{10} l_a l_b, \end{aligned} \tag{3.3}$$

with $S^i \in \mathbb{R}$.

(ii) Let W be an invariant 2-space. If it is spacelike, there is a basis of two spacelike vectors for W (proposition 2.1.5). For this proof, we take W to be spanned by $\boldsymbol{x}$ and $\boldsymbol{y}$. Then, since W is invariant, coefficients S^4, S^5, S^6 and S^7 must be 0 (otherwise, $S_{ab}x^a$ would have a component in k_b and l_b, which are out of the plane determined by x_a and y_a).

$$\boldsymbol{S}(\alpha\boldsymbol{x} + \beta\boldsymbol{y}) = S_{ab}(\alpha x^b + \beta y^b) = (\beta S^1 + \alpha S^2)x_a + (\alpha S^1 + \beta S^3)y_a \in <\boldsymbol{x}, \boldsymbol{y}> .$$

The orthogonal complement of this plane is spanned by $\boldsymbol{k}$ and $\boldsymbol{l}$, which also determine an invariant 2-space (given that $S^4, \ldots, S^7$ are 0). If W is timelike (2 null vectors), the argument is exactly the inverse. If W is a null 2-space, it has a spacelike vector and a null one. We choose them to be $\boldsymbol{x}$ and $\boldsymbol{k}$ and therefore, coefficients S^1, S^5, S^7 and S^{10} must be 0. The orthogonal complement in this case is spanned by $\boldsymbol{y}$ and $\boldsymbol{k}$ and is also an invariant 2-space and $\boldsymbol{k}$ is an eigenvector.

(iii) If W is a null invariant 2-space, the existence of a null eigenvector is proven at the end of the proof (ii). Conversely, if $\boldsymbol{S}$ admits a null eigenvector, taken as $\boldsymbol{k}$, then we have that $S^5 = S^7 = S^{10} = 0$. If $S^1 = 0$, then $\boldsymbol{x}$ (or $\boldsymbol{y}$) and $\boldsymbol{k}$ would span an invariant null 2-space. If $S^1 \neq 0$, we apply a null rotation i the $\boldsymbol{m} - \overline{\boldsymbol{m}}$ plane (2.1.12) and the new tetrad is

$$\boldsymbol{x}' = \cos(\theta)\boldsymbol{x} + \sin(\theta)\boldsymbol{y}, \qquad \boldsymbol{y}' = -\sin(\theta)\boldsymbol{x} + \cos(\theta)\boldsymbol{y}, \qquad \boldsymbol{k}' = \boldsymbol{k}, \qquad \boldsymbol{l}' = \boldsymbol{l}.$$

Once we apply a rotation, the coefficients $S^{5'}, S^{7'}$ and $S^{10'}$ are still 0, since $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are in the same 2-space than $\boldsymbol{x}$ and $\boldsymbol{y}$. Coefficient $S^{1'}$ is expressed in terms of S^i as follows:

$$S^{1'} = S^1(\cos^2(\theta) - \sin^2(\theta)) + \sin(\theta)\cos(\theta)(S^2 - S^3).$$

We find a value for θ that makes $S^{1'}$ vanish:

$$\frac{\cos^2(\theta) - \sin^2(\theta)}{\sin(\theta)\cos(\theta)} = \frac{S^2 - S^3}{S^1}, \qquad S^1 \neq 0.$$

The solutions to this equation is

$$\theta_\pm = \arctan\left(\frac{C \pm \sqrt{C^2 + 4}}{2}\right), \qquad C = \frac{S^2 - S^3}{S^1}$$

and thus we found a new tetrad $\{\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{k}, \boldsymbol{l}\}$ in which $S^{1'} = S^{5'} = S^{7'} = S^{10'} = 0$. Therefore, $\boldsymbol{x}'$ and $\boldsymbol{k}$ span a null invariant 2-space.

(iv) If W is a spacelike plane we can choose $\boldsymbol{x}$ and $\boldsymbol{y}$ to span W and then $S^4 = S^5 = S^6 = S^7 = 0$. As we did before in (iii), we can find a new tetrad in which the coefficient $S^{1'}$ is 0. This tetrad is achieved by applying a spatial rotation in the $\boldsymbol{x}$ and $\boldsymbol{y}$ plane. If $S^{1'} = S^{4'} = S^{5'} = S^{6'} = S^{7'} = 0$, then both $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are orthogonal spacelike eigenvectors included in W. Conversely, if there are two orthogonal eigenvectors, each of them spans an invariant space and, in particular, both of them span an invariant 2-space. □

Now we can give a full algebraic characterization of S_{ab} at $p \in M$ by giving all the possible canonical forms [6].

Theorem 3.1.4. *Let $(M, \boldsymbol{g})$ be a Lorentz manifold of dimension 4, let $p \in M$ and consider a second order symmetric tensor $\boldsymbol{S} \in \mathcal{T}^2(T_p^*M)$, $\boldsymbol{S} \neq 0$. Then, $\boldsymbol{S}$ takes one of the following canonical forms for some real null tetrad $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{k}, \boldsymbol{k}\}$:*

$$S_{ab} = \lambda_1 x_a x_b + \lambda_2 y_a y_b + 2\lambda_3 k_{(a} l_{b)} + \lambda_4 (k_a k_b + l_a l_b), \tag{3.4}$$

$$S_{ab} = \lambda_1 x_a x_b + \lambda_2 y_a y_b + 2\lambda_3 k_{(a} l_{b)} + \alpha k_a k_b, \tag{3.5}$$

$$S_{ab} = \lambda_1 x_a x_b + \lambda_2 y_a y_b + 2\lambda_3 k_{(a} l_{b)} + 2k_{(a} x_{b)}, \tag{3.6}$$

$$S_{ab} = \lambda_1 x_a x_b + \lambda_2 y_a y_b + 2\lambda_3 k_{(a} l_{b)} + \lambda_4 (k_a k_b - l_a l_b), \tag{3.7}$$

with all $\lambda_i \in \mathbb{R}$, $0 \neq \alpha \in \mathbb{R}$ and $\lambda_4 \neq 0$ in 3.7. The Segré type in 3.4 is $\{111, 1\}$ (or one of its degeneracies), where a comma is used to separate the eigenvalues that correspond to a timelike eigenvector from those that correspond to spacelike eigenvectors. This is the case in which $\boldsymbol{S}$ is diagonalizable over $\mathbb{R}$. The eigenvectors can be taken as $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z} = 2^{-\frac{1}{2}}(\boldsymbol{k} + \boldsymbol{k})$ and $\boldsymbol{t} = 2^{-\frac{1}{2}}(\boldsymbol{k} - \boldsymbol{k})$. Eigenvectors $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ are spacelike and $\boldsymbol{t}$ is timelike and all together form an orthonormal tetrad. The canonical form in 3.4 can be rewritten as

$$S_{ab} = \lambda_1 x_a x_b + \lambda_2 y_a y_b + \lambda_3' z_a z_b + \lambda_4' t_a t_b,$$

with $\lambda_3' = \lambda_3 + \lambda_4$ and $\lambda_4' = \lambda_4 - \lambda_3$.

In 3.5, eigenvectors can be taken as $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{k}$. The null tetrad can be chosen so that $\alpha = \pm 1$, and the Segré type is $\{11, 2\}$ or one of its degeneracies.

In 3.6 the eigenvector can be taken as $\boldsymbol{y}$ and $\boldsymbol{k}$. The Segré type is $\{1, 3\}$ or one of its degeneracies. In the case of 3.7, the tensor have complex eigenvalues and the set of eigenvectors can be taken as $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{k} \pm i\boldsymbol{l}$. The Segré type is $\{11, Z\overline{Z}\}$ or one of its degeneracies.

Proof. First of all we note that the Segré types and eigenvectors can easily be deduced from the canonical expressions 3.4 − 3.7. We will deduce the Segré type and eigenvectors of the form 3.5. Since $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{k}, \boldsymbol{l}\}$ is a null tetrad, we have that $S_{ab} x^b = \lambda_1 x_a$ and we have shown that $\boldsymbol{x}$ is an eigenvector for S_{ab} if it has the form 3.5. The procedure for $\boldsymbol{y}$ and $\boldsymbol{k}$ is the same. The image of $\boldsymbol{l}$ is ${}_{ab}l^b = \alpha k_a + \lambda l_a$. Since $\alpha \neq 0$, the restriction of S_{ab}

to the 2-space $<\boldsymbol{k},\boldsymbol{l}>$ can be expressed with the matrix:

$$\begin{pmatrix}\lambda_3 & \alpha\\ 0 & \lambda_3\end{pmatrix},$$

and then we have that $\boldsymbol{l}$ is a generalized eigenvector with eigenvalue λ_3. The Segré type is then $\{11,2\}$ if $\lambda_1\neq\lambda_2\neq\lambda_3$. In the space spanned by $\boldsymbol{x}$ and $\boldsymbol{y}$ there are no timelike vectors and for this reason, we separate those eigenvalues in the Segré notation with a comma from the eigenvalue corresponding to the 2-space $<\boldsymbol{k},\boldsymbol{l}>$, since it contains timelike vectors. If $\lambda_1=\lambda_2\neq\lambda_3$, then the Segré type is $\{(11),2\}$. The other Segré types deduced from the canonical expression of S_{ab} are $\{1(1,2)\}$ and $\{(11,2)\}$.

The procedure to deduce the eigenvalues and eigenvectors, as well as the Segre type for the canonical forms 3.4, 3.6 and 3.7 is equivalent. Now we are going to show that 3.4 to 3.7 are all the possible canonical forms. We will consider two general cases, when $\boldsymbol{S}$ admits a real null eigenvector and when it does not.

If $\boldsymbol{S}$ has such null eigenvector $\boldsymbol{k}$, we construct a null tetrad $\{\boldsymbol{x},\boldsymbol{y},\boldsymbol{k},\boldsymbol{l}\}$ so that S_{ab} can be written as in 3.3 with $S^5=S^7=S^{10}=0$ (as we saw in the proof of lemma 3.1.3). With a rotation in the $\boldsymbol{x}-\boldsymbol{y}$ plane, we can make S^1 also vanish (proof of lemma 3.1.3). Then we apply a null rotation around $\boldsymbol{k}$ to find a new null tetrad $\{\boldsymbol{x}',\boldsymbol{y}',\boldsymbol{k}',\boldsymbol{l}'\}$ and we obtain the following:

$$\begin{aligned}
&\boldsymbol{x}=\boldsymbol{x}'-\frac{1}{\sqrt{2}}(E+\overline{E})\boldsymbol{k}', && \boldsymbol{y}=\boldsymbol{y}'-\frac{i}{\sqrt{2}}(E-\overline{E})\boldsymbol{k}',\\
&\boldsymbol{k}=\boldsymbol{k}', && \boldsymbol{l}=\boldsymbol{l}'-\frac{1}{\sqrt{2}}(E+\overline{E})\boldsymbol{x}'-\frac{i}{\sqrt{2}}(E-\overline{E})\boldsymbol{y}'+|E|^2\boldsymbol{k}'.
\end{aligned}$$

If we express S_{ab} in terms of the new tetrad we obtain the following primed coefficients:

$$\begin{aligned}
S^{2'}&=S^2,\\
S^{3'}&=S^3,\\
S^{8'}&=S^8,\\
S^{4'}&=S^4+\frac{1}{\sqrt{2}}(E+\overline{E})(S^2-S^8),\\
S^{6'}&=S^6+\frac{i}{\sqrt{2}}(E-\overline{E})(S^3-S^8),
\end{aligned}$$

$$S^{9'}=S^9+|E|^2(S^2+S^3-2S^8)-i\sqrt{2}S^6(E-\overline{E})-\sqrt{2}S^4(E+\overline{E})+\frac{1}{2}(E^2+\overline{E}^2)(S^2-S^3).$$

We will consider 4 different situations regarding the coefficients S^2,S^3 and S^8.

(a) $S^2\neq S^8\neq S^3$. We can choose E so that $S^{4'}=S^{6'}=0$ and S^9 possibly 0. It can be seen that the value of E that makes it possible is:

$$\mathrm{Re}(E)=\frac{S^4}{\sqrt{2}(S^2-S^8)},\qquad \mathrm{Im}(E)=\frac{S^6}{\sqrt{2}(S^3-S^8)}$$

In this case, there exist a tetrad, obtained from a null rotation in which S_{ab} takes the form

$$S_{ab} = S^2 x_a x_b + S^3 y_a y_b + 2S^8 k_{(a} l_{b)} + S^9 k_a k_b. \tag{3.8}$$

If $S^9 \neq 0$ we obtain one of the possible forms of 3.5, where the Segré type is $\{11, 2\}$ of $\{(11), 2\}$ (if $S^2 = S^3$). In the case $S^9 = 0$, we can see that the Segré type is $\{11(1,1)\}$ or $\{(11)(1,1)\}$ since $\boldsymbol{k}$ and $\boldsymbol{l}$ are bot eigenvectors with eigenvalue S^8. If $S^2 = S^3$ then we have the latter Segré type. This is one of the possible canonical forms of 3.4.

(b) $S^2 = S^8 \neq S^3$. If $S^4 = 0$, then we have that $S^1 = S^4 = S^5 = S^7 = S^{10} = 0$. Proceeding in the same way as before, we apply a null rotation around $\boldsymbol{k}$. In this case, $S^{4'} = S^4 = 0$ and we can obtain a tetrad in which S^6 is also 0. In terms of this tetrad, $\boldsymbol{S}$ has the form 3.8. Then, if $S^9 = 0$ we obtain another form of 3.4 with Segré type $\{1(11,1)\}$ (the eigenvalue corresponding to $\boldsymbol{x}$ is the same as the one corresponding to $\boldsymbol{k}$ and $\boldsymbol{l}$). Besides, if $S^9 \neq 0$, we obtain an expression like 3.5 and the Segré type is $\{1(12)\}$. However, if $S^4 \neq 0$, we can find a null rotation in which $S^{6'}$ and $S^{9'}$ are both 0. In this basis, the tensor $\boldsymbol{S}$ takes the form 3.6 with $S^2 \neq S^8$:

$$S_{ab} = S^2 x_a x_b + S^3 y_a y_b + 2S^4 x_{(a} k_{b)} + 2S^8 k_{(a} l_{b)}. \tag{3.9}$$

The only eigenvectors are $\boldsymbol{y}$ with eigenvalue S^3 and $\boldsymbol{k}$ with eigenvalue S^8. The other vectors of the tetrad, $\boldsymbol{x}$ and $\boldsymbol{l}$ are in the invariant 3-space corresponding to the eigenvalue S^8. The Segré type in this case is $\{1, 3\}$.

(c) $S^2 \neq S^8 = S^3$. This case is symmetric to the previous one.

(d) $S^2 = S^8 = S^3$. Here we use the same techniques as before to show that if $S^4 = S^6 = 0$ in the original tetrad, then we have that $\boldsymbol{S}$ has the form 3.8. If $S^9 = 0$, the Segré type is $\{(111,1)\}$ and for $S^9 \neq 0$ the Segré type is $\{(11,2)\}$. But if S^4 and S^6 are not both zero we apply again a null rotation, as before, to make $S^9 = 0$. If $S^4 = 0$ or $S^5 = 0$, then we apply the procedure of (b) and find out that the Segré type is $\{(1,3)\}$. However, if $S^4 \neq 0 \neq S^6$, then not $\boldsymbol{x}$ or $\boldsymbol{y}$ are eigenvectors. In this case, $\boldsymbol{S}$ takes the form

$$S^2 x_a x_b + S^2 y_a y_b + 2S^4 x_{(a} k_{b)} + 2S^6 y_{(a} k_{b)} + 2S^2 k_{(a} l_{b)}. \tag{3.10}$$

The matrix associated to this application is in this case

$$\begin{pmatrix} S^2 & 0 & 0 & S^4 \\ 0 & S^2 & 0 & S^6 \\ S^4 & S^6 & S^2 & 0 \\ 0 & 0 & 0 & S^2 \end{pmatrix},$$

which admits two eigenvectors with the same eigenvalue (S^2): $\boldsymbol{k}$ and $\boldsymbol{z} = \frac{1}{\sqrt{(S^6)^2+(S^4)^2}}(S^6\boldsymbol{x} - S^4\boldsymbol{y})$, which is spacelike. In the canonical form, and using $\boldsymbol{z}$ instead of $\boldsymbol{y}$, $\boldsymbol{S}$ has the form 3.6 and its Segré type is $\{(13)\}$.

Now suppose that $\boldsymbol{S}$ has no null eigenvectors. From lemma 3.1.3 we have that $\boldsymbol{S}$ admits no null invariant 2-spaces. Since $\boldsymbol{S}$ must have some invariant 2-space W, it has to be spacelike or timelike and hence (lemma 3.1.3) it contains a pair of orthogonal spacelike eigenvectors, say $\boldsymbol{x}$ and $\boldsymbol{y}$. We construct a real null tetrad $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{k}, \boldsymbol{l}\}$ around these eigenvectors and from the expression of 3.3 for $\boldsymbol{S}$ we deduce that $S^1 = S^4 = S^5 = S^6 = S^7 = 0$ and $S^9 \neq 0 \neq S^{10}$ (otherwise, there would exist real null eigenvectors). By applying a boost in the $\boldsymbol{k}-\boldsymbol{l}$ plane (2.1.12) we can obtain a tetrad in which $|S^{9'}| = |S^{10'}|$. We have that $\boldsymbol{k} = A\boldsymbol{k}'$, $\boldsymbol{l} = A^{-1}\boldsymbol{l}'$, with $A > 0$. The coefficients for the new tetrad are

$$S^{9'} = A^2 S^9, \qquad S^{10'} = A^{-2} S^{10}.$$

So the desired value of A is $\sqrt[4]{\frac{S^{10}}{S^9}}$. If $S^{9'} = S^{10'}$ we have the expression 3.4 with $\lambda_4 \neq 0$ which leads to Segré types $\{111, 1\}$, $\{(11)1, 1\}$ or $\{(111), 1\}$. On the other hand, if $S^{9'} = -S^{10'}$, we obtain the expression in 3.7 with $\lambda_4 \neq 0$ and has Segré type $\{11, Z\overline{Z}\}$ or $\{(11), Z\overline{Z}\}$

As a remark, in 3.5 we can set α to ± 1 by performing a boost in the $\boldsymbol{k} - \boldsymbol{l}$ plane. The new coefficient α' will be 1 if $\alpha > 0$ and will be -1 if $\alpha < 0$.

We have gone through all the possible situations and checked all the possible canonical forms of $\boldsymbol{S}$. Every form from 3.4 to 3.7 for any combination of λ_i has been proven to be possible. □

3.2 The energy-momentum tensor

The Einstein field equations describe the relation between the geometric structure of a smooth manifold and the distribution of energy and matter. On one side of the equation, we have the Ricci tensor R_{ab} and the metric g_{ab}, which are the terms regarding the curvature of the manifold, the unknown of the problem. On the right side of the equation there is the energy-momentum tensor, that describes the energy density. The latter one is the tensor that defines the physical situation of the problem.

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = \kappa_0 T_{ab} \tag{3.11}$$

These equations define a set of 10 coupled differential equations (because R_{ab}, g_{ab} and T_{ab} are symmetric). Further reading on Einstein's field equations and some of its derivations can be found in [16]. Here we are focusing on the energy-momentum tensor and its possible algebraic types for some physical situations.

Looking at 3.11, we can see that the algebraic type of te Ricci tensor R_{ab} is the same as the energy-momentum one. The eigenvectors that fulfill the eigenvalue equation are the same for both tensors, and all the associated eigenvalues are shifted by the same amount. If $\boldsymbol{v}$ is an eigenvector of R_{ab} with eigenvalue λ, then we have:

$$R_{ab} v^b = \lambda v_a \implies T_{ab} v^b = \frac{1}{\kappa_0} \left(R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} \right) v^b = \frac{1}{\kappa_0} \left(\lambda - \frac{1}{2} R + \Lambda \right) v_a.$$

Both Ricci and energy-momentum tensors are symmetric, and hence they have one of the canonical forms 3.4-3.7. There are some conditions that we must impose to T_{ab} if we want it to represent a reasonable physical condition. They are called the *dominant energy conditions* (see [7]) and are a set of inequalities regarding the energy-momentum tensor contracted with timelike vectors. The local energy density as measured by an observer with 4-velocity $\boldsymbol{u}$ is non-neative and the local energy flow is non-spacelike. If we translate this conditions to a tensorial formulation, we have that for any timeike or null vector $\boldsymbol{u} \in T_pM$ at $p \in M$:

$$T_{ab}u^a u^b \geq 0 \tag{3.12}$$

$$q^a q_a \leq 0, \quad q_a := T_{ab}u^b \tag{3.13}$$

These conditions place restrictions on the Segré type of $\boldsymbol{T}$, as we see below ([7] and [6]).

Proposition 3.2.1. *Let $(M, \boldsymbol{g})$ be a Lorentz manifold and let $\boldsymbol{T}$ be the energy-momentum tensor at $p \in M$.*

(i) If dominant energy conditions are satisfied, then $\boldsymbol{T}$ cannot have Segré type $\{11, Z\overline{Z}\}$ or $\{1, 3\}$ or any of their degeneracies.

(ii) If $\boldsymbol{T}$ has Segré type $\{11, 2\}$ or one of its degeneracies (eq. (3.5)), energy conditions are satisfied if and only if $\alpha > 0$, $\lambda_3 \leq 0$, $\lambda_3 \leq \lambda_1 \leq -\lambda_3$ and $\lambda_3 \leq \lambda_2 \leq -\lambda_3$.

(iii) If $\boldsymbol{T}$ has Segré type $\{111, 1\}$ or one of its degeneracies (eq. (3.4)), energy conditions are satisfied if and only if $\lambda_3 \leq 0$, $\lambda_4 \geq 0$, $\lambda_3 - \lambda_4 \leq \lambda_1 \leq \lambda_4 - \lambda_3$ and $\lambda_3 - \lambda_4 \leq \lambda_2 \leq \lambda_4 - \lambda_3$.

Proof. For (i), we take the null vectors $\boldsymbol{k}$ and $\boldsymbol{l}$ from the canonical tetrad in eq. (3.7) (Segré type $\{11, Z\overline{Z}\}$) and we simply have that $T_{ab}k^a k^b = -T_{ab}l^a l^b = -\lambda_4 \neq 0$ and condition 3.12 fails. Moreover, one of $T_{ab}k^a$ and $T_{ab}l^a$ is spacelike if $\lambda_3 \neq 0$. If $\lambda_3 = 0$, then $T_{ab}(k^b - l^b) = -\lambda_4(k_a + l_a)$ is spacelike since $\lambda_4 \neq 0$ and then, condition 3.13 cannot be achieved neither. For the case it has Segré type $\{1, 3\}$ or one of its degeneracies (eq. (3.6)) we take the null vectors $\boldsymbol{k}' = \boldsymbol{k} - \boldsymbol{l} + \sqrt{2}\boldsymbol{x}$ and $\boldsymbol{l}' = \boldsymbol{k} - \boldsymbol{l} - \sqrt{2}\boldsymbol{x}$, which satisfy $T_{ab}k'^a k'^b = -T_{ab}l'^a l'^b = -2\sqrt{2}$ and condition 3.12 fails. Moreover, $T_{ab}l^b = \lambda_3 l_a + x_a$ is spacelike.

The proof for (ii) and (iii) are similar to the previous one: take a general null vector $(\alpha_1 x_a + \alpha_2 y_a + \alpha_3 k_a + \alpha_4 l_a$, with $\alpha_1^2 + \alpha_2^2 + 2\alpha_3\alpha_4 \leq 0)$. By imposing these relations and the energy conditions we obtain the relations in (ii) and (iii) by going through different cases (e.g. $\alpha_1 = \alpha_2 = 0$ leads clearly to the necessary condition $\alpha > 0$ in (ii)). □

We can determine the Segré type for some characteristic energy-momentum tensor such as the *electromagnetic field tensor* or the energy-momentum tensor for *perfect fluids*.

3.2.1 Electromagnetic fields

The electromagnetic Maxwell equations in tensor form and the derived energy-momentum tensor will not be deduced here, but will be described (it is briefly explained in [13]). An electromagnetic field is described by the current density vector $\boldsymbol{j}$, which represents the electric charge and current distribution.

Definition 3.2.2. Given a Lorentz manifold $(M, \boldsymbol{g})$ of dimension 4, a bivector $\boldsymbol{F} \in B_p$ at a point $p \in M$ is said to be a *Maxwell bivector* if it fulfills the Maxwell equation in tensorial form,

$$F^{ab}{}_{;b} = j^a, \qquad F_{[ab;c]} = 0.$$

The contribution of this tensor to the energy-momentum tensor is given by

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right). \tag{3.14}$$

Using the result (iii) of proposition 2.2.4, we can rewrite 3.14 in terms of $F^+_{ab} = F_{ab} + {}^\star F_{ab}$:

$$T_{ab} = \frac{1}{8\pi}\left(F_{ac}F_b{}^c + {}^\star F_{ac}{}^\star F_b{}^c\right) = \frac{1}{8\pi}F^{+c}_a\overline{F}^+_{bc}. \tag{3.15}$$

The terms of F^+_{ab} can be expanded in terms of a self-dual basis $\{\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}\}$ (2.4) so that $F^+_{ab} = \Phi_0 U_{ab} + \Phi_1 V_{ab} + \psi_2 W_{ab}$, with $\Phi_i \in \mathbb{C}$. If F^+ab is null, the invariant $F^+_{ab}F^{+ab} = \Phi_0\Phi_1 - \Phi_2^2$ is 0.

For some real null tetrad $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{k}, \boldsymbol{l}\}$, a *non-null electromagnetic field* and the corresponding energy-momentum tensor have the form (proposition 2.2.14):

$$\begin{aligned}
F^+_{ab} &= \Phi W_{ab} = 2\Phi(l_{[a}k_{b]} + ix_{[a}y_{b]}) = 2\Phi(m_{[a}\overline{m}_{b]} - k_{[a}l_{[b]}),\\
F_{ab} &= 2\alpha l_{[a}k_{b]} + 2\beta x_{[a}y_{b]},\\
T_{ab} &= -\frac{1}{8\pi}(\alpha^2 + \beta^2)(2k_{(a}l_{b)} - x_a x_b - y_a y_b) = \frac{|\Phi|^2}{4\pi}(m_{(a}\overline{m}_{b)} + k_{(a}l_{b)}),
\end{aligned}$$

where $\Phi = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$. By terms of the orthonormal tetrad $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{u}\}$ (definition 2.1.10), the expression for $\boldsymbol{T}$ takes the form

$$T_{ab} = \frac{|\Phi|^2}{8\pi}(x_a x_b + y_a y_b - z_a z_b + u_a u_b).$$

From this canonical form we can say that the Segré type of the energy-momentum tensor of non-null electromagnetic fields is $\{(11)(1,1)\}$, since $\boldsymbol{x}$ and $\boldsymbol{y}$ are eigenvectors with eigenvalue λ and $\boldsymbol{z}$ and $\boldsymbol{u}$ are eigenvectors associated to the eigenvalue $-\lambda$, with $\lambda = \frac{|\Phi|^2}{8\pi}$. It has the form 3.4.

On the other hand, if $\boldsymbol{F}$ is a *null electromagnetic field*, we can fins a real null tetrad in which F^+_{ab} takes the form

$$F^+_{ab} = \Phi V_{ab}.$$

The procedure to see this is similar to that used to prove proposition 2.2.14. As we did before, the energy momentum tensor takes now the form

$$T_{ab} = |\Phi|^2 k_a k_b.$$

It is a particular case of 3.5 with $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and consequently, the Segré type is $\{(11, 2)\}$.

3.2.2 Fluids

Before we described the energy-momentum tensor when the gravitational field was originated by an electromagnetic source. Now we are going to analyze the energy tensor when the source of the gravitational field is a viscous fluid. Again, a full procedure of how to deduce the equations for fluid sources can be found at [13].

A fluid is described by the coefficients *dynamic viscosity* η and *bulk viscosity* ξ, a unit timelike vector $\boldsymbol{u}$ representing the *fluid flow*, its *energy density* μ with respect to $\boldsymbol{u}$, the *isotropic pressure* p, the *shear tensor* $\boldsymbol{\sigma}$, the *expansion* θ and the *heat flow* vector field $\boldsymbol{q}$. The relation between all these quantities are: $u_a u^a = -1$, $u_a q^a = 0$, $\sigma_{ab} = \sigma_{ba}$, $\sigma^a_a = 0$ and $\sigma_{ab} u^b = 0$. The energy momentum tensor for this field is

$$T_{ab} = (p - \xi\theta + \mu) u_a u_b + (p - \xi\theta) g_{ab} - 2\eta\sigma_{ab} + 2u_{(a} q_{b)}. \tag{3.16}$$

Without any extra assumptions such as the energy conditions (3.12 and 3.13), this energy-momentum tensor is not restricted to any Sergé type [6]. We can, however, analyze some special cases that may be interesting to study for physical reasons.

If the heat flow vector $\boldsymbol{q}$ is zero, then we obtain that $\boldsymbol{u}$ a timelike eigenvector of T_{ab}, since $\sigma_{ab} u^b = 0$. Recalling theorem 3.1.4, the only form that admits timelike eigenvectors is 3.4 and hence, in this case the Segré type would be $\{111, 1\}$ or one of its degeneracies.

Another case is when $\boldsymbol{q} = 0$ and the fluid is non-viscous (i.e. $\xi = \eta = 0$). In this situation we say that this is a *perfect fluid* and the energy-momentum tensor is reduced to

$$T_{ab} = (\mu + p) u_a u_b + p g_{ab}.$$

We assume $\mu \geq 0$ and $p + \mu \geq 0$ if we want T_{ab} to fulfill the energy conditions. If we build an orthonormal tetrad $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{u}\}$ around $\boldsymbol{u}$, it is clear that the spacelike vectors of this tetrad are eigenvectors of $\boldsymbol{T}$ with eigenvalue p, whilst $\boldsymbol{u}$ is an eigenvector with eigenvalue $-(\mu + p)$. Hence, the Segré type of a perfect fluid is $\{(111), 1\}$ or its degeneracy in case $\mu = 0$.

Another special case of the Einstein field equations is when the Ricci tensor is Λ-*term type*. This occurs when $R_{ab} = \Lambda g_{ab}$ and this situation is included in the perfect fluid case, when $\mu + p = 0$. In this case, the Segré type is $\{(111, 1)\}$.

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