ON SELF-DUAL GAUGE FIELDS

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It is shown how self-dual gauge fields correspond to certain complex vector bundles. This leads to a procedure for generating self-dual solutions of the Yang-Mills field equations.

In this note we describe briefly how the information of self-dual gauge fields may be "coded" into the structure of certain complex vector bundles, and how this information may be extracted, yielding a procedure by which (at least in principle) all self-dual solutions of the Yang-Mills equations may be generated. The construction arose as part of the programme of twistor theory [3]; it is the Yang-Mills analogue of Penrose's "nonlinear graviton" construction [4], which relates to self-dual solutions of Einstein's vacuum equations.

The complex vector bundle. Let G be a Lie group and $\{L_j\}$ a matrix representation of the infinitesimal generators of G. Suppose the L_j are n-by-n matrices. The gauge potential A_a is said to be self-dual if the corresponding gauge field $F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b]$ is self-dual; such a gauge field necessarily satisfies the Yang-Mills field equations. The indices $a, b, \ldots$ are space-time indices, running over the values 0, 1, 2, 3. The underlying space-time is taken to be complexified Minkowski space-time CM. Both real Minkowski space-time and Euclidean 4-space may be regarded as "real slices" of CM, so that both Minkowski Yang-Mills fields and Euclidean "pseudoparticle" solutions [1] can be dealt with under this scheme.

As is well known [2], the gauge potential A_a may be interpreted as a connection on an n-dimensional complex vector bundle X over CM; the "covariant derivative operator" of this connection is given by $D_a = \partial_a + A_a$. Having this connection enables one to parallel propagate vectors around CM: a vector ψ in X is propagated in the direction v^a according to

$$v^a D_a \psi = 0. \tag{1}$$

If we propagate a vector round a closed path, it will in general not return to its original value: in other words, the propagation law (1) is not integrable. But suppose we restrict attention to closed paths which

(a) can be continuously shrunk to a point without crossing singularities of A_a; and

(b) lie in certain complex 2-planes (described below).

Then the propagation law (1) turns out to be integrable, if and only if A_a is self-dual. This characterization of self-dual gauge fields forms the basis for our construction.

The complex 2-planes mentioned above (let us call them β-planes) are defined as follows: Z is a β-plane if

(a) all vectors tangent to Z are null; and

(b) if v^a and w^a are tangent to Z, then $v^a w^b - w^a v^b$ is an anti-self-dual bivector.

The space of β-planes in CM forms a three-dimensional complex manifold PT (the so-called "dual projective twistor space" [3]). The fact that the propagation law (1) is integrable over each β-plane (if A_a is self-dual), means that one can construct a vector bundle K over PT. Furthermore, it turns out that K has all the information about the self-dual gauge field "cd "coded" into its complex structure: given K, we can reconstruct the gauge field. This led to the procedure which follows.

The generation of self-dual fields. It will be convenient to use 2-spinor notation [5], in which the space-time coordinates x^a are arranged in 2-by-2 matrix form:

$$x^{PP'} = 2^{-1/2} \begin{bmatrix} x^0 + x^1 & x^2 + ix^3 \\ x^2 - ix^3 & x^0 - x^1 \end{bmatrix}.$$

The indices P and P′ range over 0, 1 and 0′, 1′ respectively. Write $\partial_{PP'} = \partial/\partial x^{PP'}$.

Suppose that we are given $g(W_\alpha)$, an n-by-n matrix of functions of the four complex variables $W_\alpha = (W_0, W_1, W_2, W_3)$, such that g is complex analytic in some suitable region (which will be described below) and homogeneous of degree zero. The space of W_α's is the dual non-projective twistor space T, and the fact that g is homogeneous of degree zero means that g is defined on its projective version PT. The function g arises as the "patching function" which describes the vector bundle K over PT.

If we write $W_2 = -\mathrm{i}x^{\mathrm{P}0'}W_\mathrm{P}$ and $W_3 = -\mathrm{i}x^{\mathrm{P}1'}W_\mathrm{P}$ and substitute into $g(W_\alpha)$, we obtain a function $G = G(x^{\mathrm{PP}'}, W_\mathrm{P})$ which is homogeneous of degree zero in W_P. (In twistor language, this amounts to putting the twistor W_α through the space-time point $x^{\mathrm{PP}'}$.) We can now specify the region in which we want g (and G) to be analytic. Let R be some region in CM and δ some positive real number, and impose the condition that G be analytic whenever $x^a \in R$ and $1 < |W_1/W_0| < 1+\delta$.

The next step is to "split" G as follows:

$$G(x^{\mathrm{PP}'}, W_\mathrm{P}) = \hat{H}(x^{\mathrm{PP}'}, W_\mathrm{P})H^{-1}(x^{\mathrm{PP}'}, W_\mathrm{P}), \tag{2}$$

where H and $\hat{H}$ are n-by-n matrices of functions, homogeneous of degree zero in W_P and complex analytic for $x^a \in R$, $|W_1/W_0| < 1+\delta$ and for $x^a \in R$, $|W_1/W_0| > 1$, respectively. It is always possible to find such a splitting, provided g is chosen from a suitable class, corresponding to vector bundles with suitable topological properties. Once the splitting has been achieved, it is a simple matter to extract the gauge field: the matrices $H^{-1}W^\mathrm{P}\partial_{\mathrm{PP}'}H$ and $\hat{H}^{-1}W^\mathrm{P}\partial_{\mathrm{PP}'}\hat{H}$ turn out to be equal and to have the form $W^\mathrm{P}A_{\mathrm{PP}'}$, where A_a is a function only of the space-time point x^b. This defines the gauge potential A_a, and it is not difficult to check that A_a is indeed self-dual. The splitting (2) is not unique: the choice of a particular splitting corresponds exactly to a choice of gauge for A_a.

References

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