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ANALYSIS OF A FINITE ELEMENT METHOD FOR MAXWELL'S EQUATIONS*

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Abstract. The use of finite elements to discretize the time dependent Maxwell equations on a bounded domain in three-dimensional space is analyzed. Energy norm error estimates are provided when general finite element methods are used to discretize the equations in space. In addition, it is shown that if some curl conforming elements due to Nédélec are used, error estimates may also be proved in the L^2 norm.

Key words. Maxwell's equations, finite elements, error estimates

AMS(MOS) subject classifications. primary 65N30; secondary 35L15

1. Introduction. Let Ω be a bounded polygonal domain in $\mathbb{R}^3$ with boundary $\Gamma \equiv \partial\Omega$ and unit outward normal $\boldsymbol{n}$. Let $\epsilon(\boldsymbol{x})$ be the dielectric constant and $\mu(\boldsymbol{x})$ be the magnetic permeability of the material in Ω. In addition, let $\sigma(\boldsymbol{x})$ denote the conductivity of the medium. Then, if $\boldsymbol{E}(\boldsymbol{x},t)$ and $\boldsymbol{H}(\boldsymbol{x},t)$ denote, respectively, the electric and magnetic fields, Maxwell's equations [4] state that

$$\epsilon \boldsymbol{E}_t + \sigma \boldsymbol{E} - \nabla \times \boldsymbol{H} = \boldsymbol{J} \quad \text{in } \Omega \times (0,T), \tag{1}$$

$$\mu \boldsymbol{H}_t + \nabla \times \boldsymbol{E} = 0 \quad \text{in } \Omega \times (0,T), \tag{2}$$

where $\boldsymbol{J} \equiv \boldsymbol{J}(\boldsymbol{x},t)$ is a known function specifying the applied current. For simplicity, in this paper we shall assume that the boundary of Ω is a perfect conductor so that

$$\boldsymbol{n} \times \boldsymbol{E} = 0 \quad \text{on } \Gamma \times (0,T). \tag{3}$$

In addition, the initial conditions for Maxwell's equations are

$$\boldsymbol{E}(\boldsymbol{x},0) = \boldsymbol{E}_0(\boldsymbol{x}) \quad \text{and} \quad \boldsymbol{H}(\boldsymbol{x},0) = \boldsymbol{H}_0(\boldsymbol{x}) \quad \boldsymbol{x} \in \Omega,$$

where $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are given functions. The coefficients ϵ, μ, and σ are bounded $L^\infty(\Omega)$ functions and physically there exist constants ϵ_{min} and μ_{min} such that

$$0 < \epsilon_{\min} \le \epsilon(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \overline{\Omega},$$
$$0 < \mu_{\min} \le \mu(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \overline{\Omega}.$$

Furthermore, the conductivity σ is a nonnegative function on $\overline{\Omega}$.

A common approach to approximating (1)–(2) is to derive a second-order hyperbolic problem for $\boldsymbol{E}(\boldsymbol{x},t)$ [14]. By taking the time derivative of (1) and using (2) we obtain the following electric field equation:

$$\epsilon \boldsymbol{E}_{tt} + \sigma \boldsymbol{E}_t + \nabla \times \left(\frac{1}{\mu} \nabla \times \boldsymbol{E} \right) = \boldsymbol{G} \quad \text{in } \Omega \times (0,T), \tag{4}$$

where $\boldsymbol{G}(\boldsymbol{x},t) = \boldsymbol{J}_t(\boldsymbol{x},t)$. Also using (1) we obtain the initial condition

$$\boldsymbol{E}_t(\boldsymbol{x},0) = \boldsymbol{E}_{t0}(\boldsymbol{x}) \equiv \frac{1}{\epsilon(\boldsymbol{x})} (\boldsymbol{J}(\boldsymbol{x},0) + \nabla \times \boldsymbol{H}_0(\boldsymbol{x}) - \sigma(\boldsymbol{x}) \boldsymbol{E}_0(\boldsymbol{x})).$$

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In order to obtain a weak formulation of (4) we define the space

$$H_0(\mathrm{curl};\Omega) = \{\boldsymbol{v} \in (L^2(\Omega))^3 \mid \nabla \times \boldsymbol{v} \in (L^2(\Omega))^3, \ \ \boldsymbol{n} \times \boldsymbol{v} = 0 \text{ on } \Gamma\}.$$

Multiplying (4) by a function $\boldsymbol{\psi} \in H_0(\mathrm{curl};\Omega)$ and integrating the curl term by parts, we obtain the formulation of Maxwell's equations to be analyzed in this paper. Let $(\cdot,\cdot)$ denote the $(L^2(\Omega))^3$ inner product; then $\boldsymbol{E}(t) \in H_0(\mathrm{curl};\Omega)$ is the unique solution of

$$(5) \qquad (\epsilon \boldsymbol{E}_{tt}, \boldsymbol{\psi}) + (\sigma \boldsymbol{E}_t, \boldsymbol{\psi}) + \left(\frac{1}{\mu}\nabla \times \boldsymbol{E}, \nabla \times \boldsymbol{\psi}\right) = (\boldsymbol{G}, \boldsymbol{\psi}) \quad \forall \boldsymbol{\psi} \in H_0(\mathrm{curl};\Omega),$$

subject to the initial conditions

$$(6) \qquad \boldsymbol{E}(0) = \boldsymbol{E}_0 \quad \text{and} \quad \boldsymbol{E}_t(0) = \boldsymbol{E}_{t0}.$$

Existence and uniqueness for the continuous Maxwell equations have been studied for smooth domains in [4] and for more general domains in [9].

To approximate (5), let $V_h \subset H_0(\mathrm{curl};\Omega)$ be a finite-dimensional space (we shall be more precise about V_h in § 2). Then the semidiscrete electric field problem is to find $\boldsymbol{E}^h(t) \in V_h$ such that

$$(7) \qquad (\epsilon \boldsymbol{E}^h_{tt}, \boldsymbol{\psi}^h) + (\sigma \boldsymbol{E}^h_t, \boldsymbol{\psi}^h) + \left(\frac{1}{\mu}\nabla \times \boldsymbol{E}^h, \nabla \times \boldsymbol{\psi}^h\right) = (\boldsymbol{G}, \boldsymbol{\psi}^h) \quad \forall \boldsymbol{\psi}^h \in V_h,$$

subject to the initial condition

$$(8) \qquad \boldsymbol{E}^h(0) = r_h \boldsymbol{E}_0, \qquad \boldsymbol{E}^h_t(0) = r_h \boldsymbol{E}_{t0},$$

where r_h is an approximation operator associated with V_h (see § 2).

The purpose of this paper is to provide estimates of $(\boldsymbol{E} - \boldsymbol{E}^h)(t)$ in the $H(\mathrm{curl};\Omega)$ and $L^2(\Omega)$ norms. We shall prove an optimal estimate in $H(\mathrm{curl};\Omega)$ for a quite general class of finite element spaces V_h. To prove L^2 error estimates we restrict ourselves to finite element spaces V_h constructed using Nédélec's second curl conforming elements on tetrahedra [11]. Using Nédélec's space we will prove almost optimal estimates of the error in the L^2 norm.

The proofs presented here are based on the techniques of Baker and Bramble [2]. However, in proving the L^2 norm estimates we are faced with the problem that the bilinear form $(\nabla \times \boldsymbol{u}, \ \nabla \times \boldsymbol{v})$ is not coercive on $H_0(\mathrm{curl};\Omega)$. Two techniques might be used to deal with this problem. One might add and subtract the term $(\boldsymbol{E}, \phi)$ to (5) and then work with the coercive bilinear form $(\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}) + (\boldsymbol{u}, \boldsymbol{v})$, or one might decompose $H_0(\mathrm{curl};\Omega)$ and V_h using appropriate orthogonal decompositions and then analyze the problem on each subspace directly. However, even if the first method is used, we are led at a later stage to consider a discrete Helmholtz decomposition of vectors in V_h. Thus we choose to use the second approach from the begining since, in our view, this provides a more natural avenue to understanding the structure of the solution and the choice of discrete initial conditions. The proofs make use of a discrete Helmholtz splitting of $\boldsymbol{E}^h$ [11] and draw heavily on the work of Girault [5].

The idea of using curl conforming elements to discretize Maxwell's equations is not new. Nédélec [10] suggests using his first type elements to discretize $\boldsymbol{E}$ and $\boldsymbol{H}$ via (1)–(2) but provides no error estimates. Kikuchi [8] analyzes the use of Nédélec elements to approximate the eigenvalue problem associated with (5). No order estimates are provided, nor is the time dependent case analyzed. There are a number of works concerned with the approximation of Maxwell's equations in two dimensions but none provide error estimates for the formulation given here (cf. [12], [15], [14], [13], [1]). To our knowledge this is the first analysis of the use of finite elements for the time dependent Maxwell equations in $\mathbb{R}^3$.

On a practical note, we remark that a fully discrete approximation to Maxwell's equations would require the discretization of (7) in time. This is an important aspect of the problem. One possible approach, building on the analysis of the semidiscrete problem, might be to use the methods of [2]. We should also note the connection, pointed out by Kikuchi [8], between the finite element method used here and the finite difference methods of [16], [17], [18].

2. Spaces and preliminary estimates. First we shall describe the notation to be used in this paper. Then we shall present the Nédélec second type curl conforming spaces and summarize some of their properties. Finally, we shall prove error estimates for an operator associated with the zero frequency static Maxwell equations.

Throughout this paper we shall assume that Ω is a bounded, simply connected domain with polyhedral, connected boundary Γ.

We denote the standard Sobolev spaces by

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega) \mid \partial^\alpha v \in L^p(\Omega) \quad \forall |\alpha| \leq m\},$$

equipped with the standard norm

$$\|v\|_{m,p}^p = \sum_{|\alpha| \leq m} \int_\Omega |\partial^\alpha v(\boldsymbol{x})|^p \, d\boldsymbol{x}.$$

When $p = 2$, we drop the subscript p from the norm and denote the space by $H^m(\Omega)$.

In addition we have

$$H_0^1(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma\}.$$

Of central importance in our analysis of Maxwell's equations are two spaces of functions with L^2 curls

$$H(\text{curl}; \Omega) = \{\boldsymbol{v} \in L^2(\Omega)^3 \mid \nabla \times \boldsymbol{v} \in L^2(\Omega)^3\},$$
$$H_0(\text{curl}; \Omega) = \{\boldsymbol{v} \in H(\text{curl}; \Omega) \mid \boldsymbol{n} \times \boldsymbol{v} = 0 \text{ on } \Gamma\},$$

equipped with the graph norm

$$\|\boldsymbol{v}\|_{H^c} = \{\|\boldsymbol{v}\|_0^2 + \|\nabla \times \boldsymbol{v}\|_0^2\}^{1/2}.$$

For a discussion of the properties of the curl spaces see [4], [6].

We shall be discussing time dependent problems and will find it convenient to define the following norm on functions $\boldsymbol{v}(\boldsymbol{x}, t)$,

$$|||\boldsymbol{v}|||_{m,[0,t]} = \int_0^t \|\boldsymbol{v}(s)\|_m \, ds, \tag{9}$$

where $\boldsymbol{v}(s) = \boldsymbol{v}(\cdot, s)$.

Our proof of the L^2 error estimates is based on the use of continuous and discrete Helmholtz splittings of a vector $\boldsymbol{v}$. In the continuous case let

$$M = \{\boldsymbol{v} \mid \boldsymbol{v} = \nabla p, \quad p \in H_0^1(\Omega)\}.$$

Then, if Ω is convex (and satisfies the geometric constraints imposed earlier in this section), the results of Leis [9] and Kikuchi [7] show that

$$(L^2(\Omega))^3 = M \oplus M^\perp, \tag{10}$$

where

$$\begin{aligned} M^\perp &= \{\boldsymbol{v} \in (L^2(\Omega))^3 \mid (\boldsymbol{v}, \nabla q) = 0 \quad \forall\, q \in H_0^1(\Omega)\} \\ &= \overline{\nabla \times H(\text{curl}; \Omega)}. \end{aligned} \tag{11}$$

As far as regularity is concerned, the following result is given in [6].

THEOREM 2.1. *Let Ω be convex. Let $\boldsymbol{\psi} \in H_0(\text{curl}; \Omega)$ be such that*

$$\nabla \cdot \boldsymbol{\psi} = 0 \quad \text{in } \Omega.$$

Then there is an $s > 2$ depending on the angles of Γ such that for each $t \in [2, s]$ we have $\boldsymbol{\psi} \in (W^{1,t}(\Omega))^3$ and

$$\|\boldsymbol{\psi}\|_{1,t} \le C(t) \|\nabla \times \boldsymbol{\psi}\|_{0,t}. \tag{12}$$

Now let us define the finite element spaces to be used in this paper. We shall use the second Nédélec spaces based on tetrahedra [11]. As usual P_k denotes the space of polynomials of total degree at most $k \ge 1$. Let $\tilde{P}_k$ denote the subspace of P_k consisting of homogeneous polynomials of degree exactly k and define

$$\boldsymbol{D}_k = (P_{k-1})^3 \oplus \{p(\boldsymbol{x})\boldsymbol{x} \mid \ p \in \tilde{P}_{k-1}\}.$$

Let K be a nondegenerate tetrahedron with general face f and edge e. Let $\boldsymbol{\tau}$ be a vector in the direction of e and let $\boldsymbol{u} \in (W^{1,t}(K))^3$ for some $t > 2$. Then following Nédélec [11] we define the following three sets of degrees of freedom:

$$M_e(\boldsymbol{u}) = \left\{ \int_e (\boldsymbol{u} \cdot \boldsymbol{\tau}) q \, ds \mid q \in P_k(e) \quad \text{on the six edges } e \text{ of } K \right\} \tag{13}$$

$$M_f(\boldsymbol{u}) = \left\{ \int_f \boldsymbol{u} \cdot \boldsymbol{q} \, dA \mid \boldsymbol{q} \in \boldsymbol{D}_{k-1}(f) \ \text{ tangent to the face } f \right. \\ \left. \text{for the four faces } f \text{ of } K \right\} \tag{14}$$

$$M_K(\boldsymbol{u}) = \left\{ \int_K \boldsymbol{u} \cdot \boldsymbol{q} \, dV \mid \boldsymbol{q} \in \boldsymbol{D}_{k-2}(K) \right\}. \tag{15}$$

Nédélec [11] shows that these degrees of freedom are $(P_k)^3$ unisolvent and $H(\text{curl}; \Omega)$ conforming.

Now suppose Ω is triangulated using a uniformly regular family [6] of tetrahedral meshes denoted $\{\tau_h\}_{h>0}$, where h is the maximum diameter of the elements in τ_h. Suppose in addition that the mesh satisfies the standard finite element geometric restrictions [3]. Let

$$X_h = \{\boldsymbol{v}^h \in H(\text{curl}; \Omega) \mid \boldsymbol{v}^h|_K \in (P_k)^3 \quad \forall K \in \tau_h\},$$

then any function in X_h can be uniquely described by specifying the degrees of freedom (13)–(15) on each $K \in \tau_h$. This implies the existence of an interpolation operator r_h associated with X_h and the degrees of freedom (13)–(15). Precisely, if

$\boldsymbol{u} \in (W^{1,t}(\Omega))^3$, $t > 2$, then $r_h\boldsymbol{u} \in X_h$ is the piecewise polynomial such that $r_h\boldsymbol{u}|_K$ has the same moments (13)–(15) as $\boldsymbol{u}$ on K for each $K \in \tau_h$.

We shall be interested in subspaces of $H_0(\text{curl};\Omega)$, so we define

$$V_h = \{\boldsymbol{v}^h \in X_h \mid \boldsymbol{n} \times \boldsymbol{v}^h = 0 \quad \text{on} \quad \Gamma\}.$$

We shall also need a space corresponding to $H_0^1(\Omega)$, and so we define the standard space of continuous piecewise $k+1$ degree polynomials on Ω:

$$S_h^{k+1} = \{p^h \in C(\overline{\Omega}) \mid p^h|_K \in P_{k+1}\ \forall K \in \tau_h,\ \ p^h|_\Gamma = 0\}.$$

The approximation properties of S_h^{k+1} are well known (cf. [3]). The following approximation and interpolation properties for the Nédélec spaces are given in [11] and [5].

THEOREM 2.2. *Let $\boldsymbol{u} \in (W^{1,t}(\Omega))^3$, $t > 2$. Then:*

(a) *There is a constant $C = C(t)$ such that*

$$\|\boldsymbol{u} - r_h\boldsymbol{u}\|_0 + h\|\boldsymbol{u} - r_h\boldsymbol{u}\|_{H^c} \le Ch\|\boldsymbol{u}\|_{1,t}. \tag{16}$$

(b) *If $\boldsymbol{u} \in H_0(\text{curl};\Omega)$, then $r_h\boldsymbol{u} \in V_h$.*

(c) *If $\nabla \times \boldsymbol{u} = 0$ in Ω, then $\nabla \times r_h\boldsymbol{u} = 0$ in Ω.*

(d) *If $p^h \in S_h^{k+1}$, then $\boldsymbol{u}^h = \nabla p^h \in V_h$, and in addition if $p \in H_0^1(\Omega)$ is such that $\nabla p \in (W^{1,t}(\Omega))^3$, then $r_h\nabla p = \nabla p^h$ for some $p^h \in S_h^{k+1}$.*

(e) *If, in addition $\boldsymbol{u} \in (H^{k+1}(\Omega))^3$, then*

$$\|\boldsymbol{u} - r_h\boldsymbol{u}\|_0 + h\|\boldsymbol{u} - r_h\boldsymbol{u}\|_{H^c} \le C\, h^{k+1}\|u\|_{k+1}. \tag{17}$$

Part (d) of the above theorem guarantees a discrete analogue of the continuous Helmholtz splitting. Let

$$M_h = \{\boldsymbol{u}^h = \nabla p^h \mid p^h \in S_h^{k+1}\}. \tag{18}$$

Then (d) implies that $M_h \subset V_h$ and thus

$$V_h = M_h \oplus M_h^\perp, \tag{19}$$

where

$$M_h^\perp = \{\boldsymbol{v}^h \in V_h | (\boldsymbol{v}^h, \nabla p^h) = 0 \quad \forall p^h \in S_h^{k+1}\}.$$

Unfortunately it is no longer true that $M_h^\perp$ is the curl of another space. However, (d) of the above theorem also shows that

$$r_h : M \cap (W^{1,t}(\Omega))^3 \to M_h.$$

The next lemma is a slight extension of results in [5] and [11].

LEMMA 2.3. *If $\boldsymbol{u} \in H_0(\text{curl};\Omega)$ is such that $\boldsymbol{u} \in (H^1(\Omega))^3$ and $\nabla \times \boldsymbol{u} \in (H^\ell(\Omega))^3$, $1 \le \ell \le k$, then*

$$\|\nabla \times \boldsymbol{u} - \nabla \times r_h\boldsymbol{u}\|_0 \le C\, h^\ell\|\nabla \times \boldsymbol{u}\|_\ell. \tag{20}$$

Proof. Since $\boldsymbol{u}$ and $\nabla \times \boldsymbol{u}$ are in $(H^1(\Omega))^3$, we know that the tangential components of $\boldsymbol{u}|_f$ and the surface curl $\boldsymbol{n}.\nabla \times \boldsymbol{u}|_f$ are both in $(H^{1/2}(f))^2$. Thus by Theorem 2.11 of [6] we know that $\boldsymbol{u} \cdot \boldsymbol{\tau}|_e$ is well defined. Since all the degrees of freedom (13)–(15) are well defined, $r_h \boldsymbol{u}$ exists. Furthermore, the proof of (3.13) of [11] shows that

$$\nabla \times r_h \boldsymbol{u} = \omega_h \nabla \times \boldsymbol{u},$$

where ω_h is the interpolation operator into a divergence conforming space of piecewise $k-1$ degree polynomials on Ω defined by Nédélec [10]. The approximation results for ω_h are proved in [10] and [6]. Using these results

$$\|\nabla \times \boldsymbol{u} - \nabla \times r_h \boldsymbol{u}\|_0 = \|(I - \omega_h)(\nabla \times \boldsymbol{u})\|_0 \le C h^\ell \|\nabla \times \boldsymbol{u}\|_\ell,$$

and the lemma is proved. $\square$

In the last part of this section we shall analyze an equation related to Maxwell's equations for a static electric field. Let $\boldsymbol{\phi} \in H_0(\text{curl};\Omega) \cap M^\perp$, and define $\boldsymbol{\psi}$ to be the solution of

$$\begin{aligned} \nabla \times \nabla \times \boldsymbol{\psi} &= \boldsymbol{\phi} \quad \text{on } \Omega, \\ \nabla \cdot \boldsymbol{\psi} &= 0 \quad \text{on } \Omega, \\ \boldsymbol{n} \times \boldsymbol{\psi} &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Girault [5] shows that the correct variational formulation of this problem is to find $\boldsymbol{\psi} \in H_0(\text{curl};\Omega) \cap M^\perp$ such that

$$(\nabla \times \boldsymbol{\psi}, \nabla \times \boldsymbol{v}) = (\boldsymbol{\phi}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in H_0(\text{curl};\Omega) \cap M^\perp. \tag{21}$$

(Actually Girault's work is for a more general Stokes problem and reduces to this problem when $\boldsymbol{\phi} \in M^\perp$.) If Ω is convex, the regularity result of Girault [5] shows that

$$\|\nabla \times \boldsymbol{\psi}\|_1 \le C \|\boldsymbol{\phi}\|_0.$$

Moreover, since Ω is convex and $\boldsymbol{\psi} \in H_0(\text{curl};\Omega)$ (with $\nabla \cdot \boldsymbol{\psi} = 0$) the result of Kikuchi [7] shows that

$$\|\boldsymbol{\psi}\|_1 \le C \|\nabla \times \boldsymbol{\psi}\|_0,$$

and thus

$$\|\nabla \times \boldsymbol{\psi}\|_1 + \|\boldsymbol{\psi}\|_1 \le C \|\boldsymbol{\phi}\|_0. \tag{22}$$

Let $T : H_0(\text{curl};\Omega) \cap M^\perp \to H_0(\text{curl};\Omega) \cap M^\perp$ be the solution operator for (21). Thus $\boldsymbol{\psi} = T\boldsymbol{\phi}$ solves (21). Now we define a discrete analogue of T by

$$T_h : M^\perp \to M_h^\perp$$

and

$$(\nabla \times T_h \boldsymbol{\phi}, \nabla \times \boldsymbol{v}^h) = (\boldsymbol{\phi}, \boldsymbol{v}^h) \quad \forall \boldsymbol{v}^h \in M_h^\perp. \tag{23}$$

In [11] Nédélec shows that if Ω is convex, then for every $\boldsymbol{w}^h \in M_h^\perp$

$$\|\boldsymbol{w}^h\|_0 \le C \|\nabla \times \boldsymbol{w}^h\|_0, \tag{24}$$

and hence $T_h\phi$ is well defined. Note that T_h is also well defined on $M_h^{\perp}$. We remark that since $\phi \in M^{\perp}$, (21) holds for any $v \in H_0(\text{curl};\Omega)$ and (23) holds for any $v^h \in V_h$. Furthermore, T_h can be extended to an operator on $(L^2(\Omega))^3$ using (23), and in that case T_h is a symmetric and positive semidefinite operator. We have the following error estimate.

THEOREM 2.4. *Let Ω be convex and suppose T and T_h are as defined above. Suppose in addition that $T\phi \in H^{k+1}(\Omega)$. Then for each ϵ in $0 < \epsilon \leq \epsilon_0$ there is a constant $C = C(\epsilon)$ independent of h and ϕ such that*

$$\|\nabla \times T_h\phi\|_0 \leq C\|\phi\|_0. \tag{25}$$

$$\|\nabla \times (T - T_h)\phi\|_0 \leq Ch^{\ell}\|\nabla \times T\phi\|_{\ell}, \qquad 1 \leq \ell \leq k. \tag{26}$$

$$\|(T - T_h)\phi\|_0 \leq Ch^{k+1-\epsilon}\|T\phi\|_{k+1}. \tag{27}$$

Remark. $\epsilon_0 = 3/2 - 3/s$, where $s > 2$ is the index in Theorem (2.1).

Proof. Using (23), the Cauchy–Schwarz inequality and (24) (with w^h replaced by $T_h\phi$)

$$\begin{aligned}(\nabla \times T_h\phi, \nabla \times T_h\phi) &= (\phi, T_h\phi)\\ &\leq \|\phi\|_0 \, \|T_h\phi\|_0\\ &\leq C\|\phi\|_0\|\nabla \times T_h\phi\|_0.\end{aligned}$$

This proves (25). We prove the curl estimate next. Using (21) and (22), together with the fact that $\phi \in M^{\perp}$, we have that

$$(\nabla \times (T\phi - T_h\phi), \nabla \times v^h) = 0 \quad \forall v^h \in V_h. \tag{28}$$

Thus using the Cauchy–Schwarz inequality

$$\begin{aligned}(\nabla \times (T - T_h)\phi, \nabla \times (T - T_h)\phi) &= (\nabla \times (T - T_h)\phi, \nabla \times (T\phi - r_hT\phi))\\ &\leq \|\nabla \times (T - T_h)\phi\|_0 \, \|\nabla \times (T\phi - r_hT\phi)\|_0.\end{aligned}$$

Use of Lemma (2.3) completes the proof of (26).

To prove (27), we follow Girault [5] and set

$$r_hT\phi - T_h\phi = w + \nabla p, \tag{29}$$

where $p \in H_0^1(\Omega)$ is the solution of

$$(\nabla p, \nabla q) = (r_hT\phi - T_h\phi, \nabla q) \quad \forall q \in H_0^1(\Omega),$$

and $w \in H_0(\text{curl};\Omega)$ satisfies

$$\nabla \times w = \nabla \times (r_hT\phi - T_h\phi) \quad \text{in } \Omega \tag{30}$$

$$\nabla \cdot w = 0 \quad \text{in } \Omega. \tag{31}$$

Since $\nabla \times (r_hT\phi - T_h\phi) \in L^{\bar{s}}(\Omega)^3$ for any $\bar{s}$, Theorem (2.1) shows that $w \in (W^{1,t}(\Omega))^3$ for $2 \leq t \leq s$, where s is the index in Theorem (2.1). Hence r_hw is defined, and by part (d) of Theorem (2.2)

$$r_hT\phi - T_h\phi = r_hw + \nabla p^h \tag{32}$$

for some $p^h \in S_h^{k+1}$. We can then estimate $\boldsymbol{w} - r_h\boldsymbol{w}$ using part (a) of Theorem (2.2):

$$\|\boldsymbol{w} - r_h\boldsymbol{w}\|_0 \leq Ch\|\boldsymbol{w}\|_{1,t}.$$

But by Theorem (2.1), $\|\boldsymbol{w}\|_{1,t} \leq C\|\nabla \times \boldsymbol{w}\|_{0,t}$. Then using (30) and standard affine mapping techniques we conclude that

$$\|\boldsymbol{w} - r_h\boldsymbol{w}\|_0 \leq Ch^{1+3/t-3/2}\|\nabla \times (r_hT\boldsymbol{\phi} - T_h\boldsymbol{\phi})\|_0. \tag{33}$$

Finally by Lemma (2.3) and (26)

$$\begin{aligned} \|\nabla \times (r_hT\boldsymbol{\phi} - T_h\boldsymbol{\phi})\|_0 &\leq \|\nabla \times (r_hT\boldsymbol{\phi} - T\boldsymbol{\phi})\|_0 + \|\nabla \times (T\boldsymbol{\phi} - T_h\boldsymbol{\phi})\|_0 \\ &\leq Ch^k\|\nabla \times T\boldsymbol{\phi}\|_k. \end{aligned} \tag{34}$$

Using (34) in (33) shows that

$$\|\boldsymbol{w} - r_h\boldsymbol{w}\|_0 \leq Ch^{k+1-\epsilon}\|\nabla \times T\boldsymbol{\phi}\|_k, \tag{35}$$

where $\epsilon = 3/2 - 3/t$ can be made arbitrarily small by taking t sufficiently close to two. Furthermore, $\epsilon_0 = 3/2 - 3/s$.

To estimate $(T - T_h)\boldsymbol{\phi}$ we proceed as follows:

$$\begin{aligned} ((T - T_h)\boldsymbol{\phi}, (T - T_h)\boldsymbol{\phi}) &= ((T - T_h)\boldsymbol{\phi}, T\boldsymbol{\phi} - r_hT\boldsymbol{\phi}) + ((T - T_h)\boldsymbol{\phi}, r_h\boldsymbol{w} - \boldsymbol{w}) \\ &\quad + ((T - T_h)\boldsymbol{\phi}, \boldsymbol{w}) + ((T - T_h)\boldsymbol{\phi}, \nabla p^h). \end{aligned} \tag{36}$$

The term $((T - T_h)\boldsymbol{\phi}, \nabla p^h)$ vanishes since $T\boldsymbol{\phi} \in M^\perp$ and $T_h\boldsymbol{\phi} \in M_h^\perp$. The first two terms on the right-hand side of (36) are estimated using the Cauchy–Schwarz inequality followed by Theorem (2.2), part (e), and (35), respectively. The remaining term is estimated by duality. Let $\boldsymbol{\psi}$ satisfy

$$\begin{aligned} \nabla \times \nabla \times \boldsymbol{\psi} &= \boldsymbol{w} \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{\psi} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{n} \times \boldsymbol{\psi} &= 0 \quad \text{on } \Gamma. \end{aligned}$$

As discussed earlier this problem is well posed and satisfies the a priori estimate (22) with $\boldsymbol{\phi}$ replaced by $\boldsymbol{w}$. Then integrating by parts and using (28) we have that

$$\begin{aligned} ((T - T_h)\boldsymbol{\phi}, \boldsymbol{w}) &= (\nabla \times (T - T_h)\boldsymbol{\phi}, \nabla \times \boldsymbol{\psi}) \\ &= (\nabla \times (T - T_h)\boldsymbol{\phi}, \nabla \times (\boldsymbol{\psi} - r_h\boldsymbol{\psi})). \end{aligned}$$

But using Lemma (2.3) and (26) we have

$$\begin{aligned} |((T - T_h)\boldsymbol{\phi}, \boldsymbol{w})| &\leq Ch\|\nabla \times (T - T_h)\boldsymbol{\phi}\|_0 \, \|\nabla \times \boldsymbol{\psi}\|_1 \\ &\leq Ch^{k+1}\|\nabla \times T\boldsymbol{\phi}\|_k \, \|\nabla \times \boldsymbol{\psi}\|_1. \end{aligned}$$

Now using (22), the definitions of $\boldsymbol{w}$ and $\boldsymbol{w}_h$, and the estimates in Theorem (2.2), part (e), and (35), we obtain

$$\begin{aligned} |((T - T_h)\boldsymbol{\phi}, \boldsymbol{w})| &\leq Ch^{k+1}\|\nabla \times T\boldsymbol{\phi}\|_k \, \|\boldsymbol{w}\|_0 \\ &\leq Ch^{k+1}\|\nabla \times T\boldsymbol{\phi}\|_k\{\|\boldsymbol{w} - r_h\boldsymbol{w}\|_0 \\ &\qquad + \|r_hT\boldsymbol{\phi} - T\boldsymbol{\phi}\|_0 + \|T\boldsymbol{\phi} - T_h\boldsymbol{\phi}\|_0 + \|\nabla p^h\|_0\}. \\ &\leq Ch^{k+1}\|\nabla \times T\boldsymbol{\phi}\|_k\{h^{k+1-\epsilon}\|T\boldsymbol{\phi}\|_{k+1} + \|(T - T_h)\boldsymbol{\phi}\|_0 + \|\nabla p^h\|_0\}. \end{aligned} \tag{37}$$

It remains to estimate $\|\nabla p^h\|_0$. Since $\boldsymbol{w} \in M^\perp$, (32) implies that if $q^h \in S_h^{k+1}$

$$\begin{aligned}(\nabla p^h, \nabla q^h) &= (r_h T\boldsymbol{\phi} - T_h\boldsymbol{\phi}, \nabla q^h) - (r_h \boldsymbol{w}, \nabla q^h)\\ &= (r_h T\boldsymbol{\phi} - T\boldsymbol{\phi}, \nabla q^h) + ((T - T_h)\boldsymbol{\phi}, \nabla q^h) + (\boldsymbol{w} - r_h\boldsymbol{w}, \nabla q^h).\end{aligned}$$

But $T\boldsymbol{\phi} \in M^\perp$ and $T_h\boldsymbol{\phi} \in M_h^\perp$, and so the second term on the right-hand side of the above expression is zero. Taking $q^h = p^h$ we conclude that

$$\|\nabla p^h\|_0 \le \|r_h T\boldsymbol{\phi} - T\boldsymbol{\phi}\|_0 + \|\boldsymbol{w} - r_h\boldsymbol{w}\|_0.$$

Using Theorem (2.2), part (e), and (35) we obtain

$$\|\nabla p^h\|_0 \le Ch^{k+1-\epsilon}\|T\boldsymbol{\phi}\|_{k+1}. \tag{38}$$

Combining (38), (37) in (36) and using the Cauchy–Schwarz and arithmetic geometric mean inequalities proves the desired estimate. □

3. Energy norm estimates. Our first estimate for $\boldsymbol{E} - \boldsymbol{E}^h$ is a general smooth solution convergence estimate in the $H(\text{curl};\Omega)$ norm. The estimate does not use any special properties of the domain or of the space V_h (except the assumed polygonal nature of the domain).

THEOREM 3.1. *Let $\boldsymbol{E}$ satisfy* (5)–(6) *and let $\boldsymbol{E}^h \in V_h \subset H_0(\text{curl};\Omega)$ satisfy* (7)–(8). *Suppose in addition that $\boldsymbol{E}_t(t), \boldsymbol{E}_{tt}(t) \in H^{k+1}(\Omega)$, $0 \le t \le T$. Then there is a constant $C = C(T)$ such that*

$$\|(\boldsymbol{E} - \boldsymbol{E}^h)(t)\|_{H^c} + \|(\boldsymbol{E} - \mathbf{E}^h)_t(t)\|_0 \le C\Big\{\|(\boldsymbol{E} - \boldsymbol{E}^h)(0)\|_{H^c} + \|(\boldsymbol{E} - \boldsymbol{E}^h)_t(0)\|_0 + h^k\Big\{\max_{0\le s\le t}\|\boldsymbol{E}_t(s)\|_{k+1} + |||\boldsymbol{E}_{tt}|||_{k+1,[0,t]}\Big\}\Big\}.$$

Remarks. (i) As we shall see, this theorem holds for arbitrary $V_h \subset H_0(\text{curl};\Omega)$ provided only that the interpolant in V_h satisfies Theorem (2.2) part (e). Thus this theorem proves convergence for other elements such as

- Nédélec type 1 elements on tetrahedra and cubes [10].
- Nédélec type 2 elements on tetrahedra [11].
- $H^1(\Omega)$ conforming elements such as spaces of continuous k degree piecewise polynomials [3].

This theorem does not suggest any advantages for the Nédélec spaces.

(2) Of course the term $\|(\boldsymbol{E} - \boldsymbol{E}^h)(0)\|_{H^c} + \|(\boldsymbol{E} - \boldsymbol{E}^h)_t(0)\|_0$ may be estimated easily. If discrete initial data is chosen to interpolate the real initial data as suggested in (8), then Theorem (2.2), part (e), shows that

$$\|(\boldsymbol{E} - \boldsymbol{E}^h)(0)\|_{H^c} + \|(\boldsymbol{E} - \boldsymbol{E}^h)_t(0)\|_0 \le Ch^k\left(\|\boldsymbol{E}(0)\|_{k+1} + \|\boldsymbol{E}_t(0)\|_{k+1}\right).$$

The choice of operator for defining the discrete initial data is not critical provided the above estimate holds.

Proof. We shall give the proof in the case $\epsilon = 1$, $\mu = 1$, and $\sigma = 0$. The more general case follows the same lines. The general proof technique used is due to Bramble and Baker [2]. However, for completeness, since arguments of this type will be used for the proof of later theorems, we shall give details here.

Let $\boldsymbol{e} = \boldsymbol{E} - \boldsymbol{E}^h$. Then by subtracting (5) and (7) we have

$$(\boldsymbol{e}_{tt}, \boldsymbol{\phi}^h) + (\nabla \times \boldsymbol{e}, \nabla \times \boldsymbol{\phi}^h) = 0 \quad \forall \boldsymbol{\phi}^h \in V_h. \tag{39}$$

Hence

$$(\boldsymbol{e}_{tt}, \boldsymbol{e}_t) + (\nabla \times \boldsymbol{e}, \nabla \times \boldsymbol{e}_t) = (\boldsymbol{e}_{tt}, (I - r_h)\boldsymbol{E}_t) + (\nabla \times \boldsymbol{e}, \nabla \times (I - r_h)\boldsymbol{E}_t).$$

Integrating this estimate over $[0, t]$, and integrating the term $(\boldsymbol{e}_{tt}, (I - r_h)\boldsymbol{E}_t)$ by parts in time we obtain

$$\begin{aligned}
&\frac{1}{2}\{\|\boldsymbol{e}_t(t)\|_0^2 + \|\nabla \times \boldsymbol{e}(t)\|_0^2\} - \frac{1}{2}\{\|\boldsymbol{e}_t(0)\|_0^2 + \|\nabla \times \boldsymbol{e}(0)\|_0^2\} \\
&\quad = (\boldsymbol{e}_t(s), (I - r_h)\boldsymbol{E}_t(s))|_{s=0}^t + \int_0^t (\nabla \times \boldsymbol{e}, \nabla \times (I - r_h)\boldsymbol{E}_t) ds \\
&\qquad - \int_0^t (\boldsymbol{e}_t, (I - r_h)\boldsymbol{E}_{tt}) ds.
\end{aligned}$$

Using the Cauchy–Schwarz inequality and part (e) of Theorem (2.2), we conclude that, for $0 \leq t \leq t_1 \leq T$,

$$\begin{aligned}
\|\boldsymbol{e}_t(t)\|_0^2 + \|\nabla \times \boldsymbol{e}(t)\|_0^2 \leq C\Big\{ &\|\boldsymbol{e}_t(0)\|_0^2 + \|\nabla \times \boldsymbol{e}(0)\|_0^2 \\
&+ h^{2k} \max_{0 \leq s \leq t_1} (\|\boldsymbol{e}_t(s)\|_0 + \|\nabla \times \boldsymbol{e}(s)\|_0) h^{2k} Q(t_1, \boldsymbol{E})\Big\},
\end{aligned} \tag{40}$$

where

$$Q(t_1, \boldsymbol{E}) = \max_{0 \leq s \leq t_1} \|\boldsymbol{E}_t(s)\|_{k+1} + |||\boldsymbol{E}_t|||_{k+1,[0,t_1]} + |||\boldsymbol{E}_{tt}|||_{k+1,[0,t_1]}.$$

Since estimate (40) holds for each $t \in [0, t_1]$ it must hold for the time at which the maximum error is attained, and hence via the arithmetic geometric mean inequality we conclude that

$$\begin{aligned}
\max_{0 \leq t \leq t_1} (\|\boldsymbol{e}_t(t)\|_0 + \|\nabla \times \boldsymbol{e}(t)\|_0) \leq C\Bigg(&\|\boldsymbol{e}_t(0)\|_0 + \|\nabla \times \boldsymbol{e}(0)\|_0 \\
&+ h^k \left(\max_{0 \leq s \leq t_1} \|\boldsymbol{E}_t(s)\|_{k+1} + |||\boldsymbol{E}_{tt}|||_{k+1,[0,t_1]} \right) \Bigg).
\end{aligned} \tag{41}$$

This proves the desired estimate for $\|\boldsymbol{e}_t\|_0$ and $\|\nabla \times \boldsymbol{e}\|_0$. To prove the estimate for $\|\boldsymbol{e}\|$, we note that

$$\|\boldsymbol{e}(t)\|_0 \leq C\{\|\boldsymbol{e}(0)\|_0 + t \max_{0 \leq s \leq t} \|\boldsymbol{e}_t(s)\|_0\}, \tag{42}$$

and the estimate for $\|\boldsymbol{e}(t)\|_0$ also follows from (41). □

4. L^2 error estimates.

In this section we shall show that if V_h is constructed using Nédélec's curl conforming finite elements as outlined in §2 it is possible to prove L^2 error estimates for $\boldsymbol{E} - \boldsymbol{E}^h$. In this section we shall assume that Ω is convex (in addition to the assumptions outlined in § 2). In addition we shall assume $\epsilon = \mu = 1$ and $\sigma = 0$. Unlike in § 3, the constancy assumptions on ϵ and μ, as well as the assumption that $\sigma \equiv 0$, are essential for the proofs we shall provide.

In order to simplify the exposition, we perform the error analysis by first assuming that $\boldsymbol{G}$ has one of two distinct forms:

(1) $\boldsymbol{G} = \nabla g$ for some $g \in H_0^1(\Omega)$

(2) $\boldsymbol{G} = \nabla \times \boldsymbol{\Phi}$ for some $\boldsymbol{\Phi} \in H(\text{curl};\Omega)$.

This analysis has the advantage of indicating exactly how the solution in these two important special cases is approximated. By virtue of (10) and (11) we can obtain an error estimate for a general vector $\boldsymbol{G}$ by combining the above mentioned results.

THEOREM 4.1. *Suppose $\boldsymbol{E}$ satisfies (5)–(6) and $\boldsymbol{E}^h$ satisfies (7) and (8). Suppose that $\boldsymbol{G} = \nabla g$ for some $g \in H_0^1(\Omega)$ and that $\boldsymbol{E}_0$ and $\boldsymbol{E}_{t0}$ are in $(H^{k+1}(\Omega))^3 \cap M$. If, in addition, $\boldsymbol{E}_t(t), \boldsymbol{E}_{tt}(t) \in (H^{k+1}(\Omega))^3$, and $\boldsymbol{G}(t) \in (H^{k+1}(\Omega))^3$ for $0 \le t \le T$ then there exists a constant $C = C(t)$ such that*

$$\begin{aligned}\|(\boldsymbol{E} - \boldsymbol{E}^h)(t)\|_0 \le Ch^{k+1}\Big\{&\|\boldsymbol{E}_0\|_{k+1} + \max_{0\le s\le t} \|\boldsymbol{E}_t(s)\|_{k+1} \\ &+ |||\boldsymbol{E}_{tt}|||_{k+1,[0,t]} + |||\boldsymbol{G}|||_{k+1,[0,t]}\Big\}.\end{aligned} \tag{43}$$

Remark. Since $\boldsymbol{G} = \nabla g$ and $\boldsymbol{E}(0), \boldsymbol{E}_t(0) \in M$, we know that $\boldsymbol{E} = \nabla p \in M$ for some $p(t) \in H_0^1(\Omega)$, $0 \le t \le T$. Unfortunately, $\boldsymbol{E}^h$ is not in M_h but, instead, $\boldsymbol{E}^h = \overline{\boldsymbol{E}}^h + \nabla p^h$ for some $\overline{\boldsymbol{E}}^h \in M_h^\perp$, $p^h \in S_h^{k+1}$. However, our proof shows that the nonphysical field $\overline{\boldsymbol{E}}^h$ is small in a superconvergent fashion,

$$\|\overline{\boldsymbol{E}}^h(t)\|_{H^c} \le Ch^{k+1}|||\boldsymbol{G}|||_{k+1,[0,t]}.$$

This estimate is one power of h better than expected by approximation theory.

Proof. As noted in the remark, $\boldsymbol{E} = \nabla p$, $p \in H_0^1(\Omega)$, and $\boldsymbol{E}^h = \overline{\boldsymbol{E}}^h + \nabla p^h$ with $p^h \in S_h^{k+1}$ and $\overline{\boldsymbol{E}}^h \in M_h^\perp$. Thus using $\boldsymbol{\phi}^h = \nabla q^h$, $q^h \in S_h^{k+1}$ in (39), we obtain

$$(\nabla(p-p^h)_{tt}, \nabla q^h) = 0 \quad \forall q^h \in S_h^{k+1}.$$

From this orthogonality condition, we conclude that

$$(\nabla(p-p^h)_{tt}, \nabla(p-p^h)_t) = (\nabla(p-p^h)_{tt}, \nabla(p_t - P_h p_t)), \tag{44}$$

where P_h is the $H_0^1(\Omega)$ projection into S_h^{k+1}. Using standard estimates for P_h, integrating (44) over $[0,t_1]$ (integrating the right-hand side by parts in time), and proceeding as in the proof of Theorem (3.1), we have that for $0 \le t \le t_1$

$$\begin{aligned}\|\nabla(p-p^h)_t(t)\|_0^2 \le C\Bigg\{&\|\nabla(p-p^h)_t(0)\|_0^2 \\ &+ h^{2(k+1)}\left(\max_{0\le s\le t_1} \|p_t(s)\|_{k+2}^2 + |||p_{tt}|||_{k+2,[0,t_1]}^2\right)\Bigg\}.\end{aligned} \tag{45}$$

Now we shall estimate $\overline{\boldsymbol{E}}^h$. Taking $\boldsymbol{\phi}^h \in M_h^\perp$ in (7) and using the orthogonality of ∇p^h and $\nabla P_h g$ with $\boldsymbol{\phi}^h \in M_h^\perp$ we have

$$(\overline{\boldsymbol{E}}_{tt}^h, \boldsymbol{\phi}^h) + (\nabla \times \overline{\boldsymbol{E}}^h, \nabla \times \boldsymbol{\phi}^h) = (\nabla(g - P_h g), \boldsymbol{\phi}^h) \quad \forall \boldsymbol{\phi}^h \in M_h^\perp.$$

By selecting $\boldsymbol{\phi}^h = \overline{\boldsymbol{E}}_t^h$ and integrating the equation in time we have that

$$\begin{aligned}\frac{1}{2}\{\|\overline{\boldsymbol{E}}_t^h(t)\|_0^2 + \|\nabla \times \overline{\boldsymbol{E}}^h(t)\|_0^2\} \le{}& \frac{1}{2}\{\|\overline{\boldsymbol{E}}_t^h(0)\|_0^2 + \|\nabla \times \overline{\boldsymbol{E}}^h(0)\|_0^2\} \\ &+ \int_0^t |(\nabla(g - P_h g), \overline{\boldsymbol{E}}_t^h)|\, ds.\end{aligned} \tag{46}$$

But $\boldsymbol{E}^h(0) = r_h\boldsymbol{E}_0$, and by Theorem (2.2), part (d), since $\boldsymbol{E}_0 \in M$, we know that $\boldsymbol{E}^h \in M_h$, and so $\overline{\boldsymbol{E}}^h(0) = 0$. Similarly, since $\boldsymbol{E}_{t0} \in M$ we conclude that $\overline{\boldsymbol{E}}_t^h(0) = 0$. By Theorem (2.2), part (e), this shows that

$$\|\nabla(p - p^h)_t(0)\|_0 = \|\boldsymbol{E}_t(0) - \boldsymbol{E}_t^h(0)\|_0 \le Ch^{k+1}\|\boldsymbol{E}_{t0}\|_{k+1} \tag{47}$$

and completes the estimation of the right-hand side of (45).

Continuing with the estimate of $\overline{\boldsymbol{E}}^h$, if we apply standard estimates for P_h, the Cauchy–Schwarz inequality, and the arithmetic geometric mean inequality to (46), we obtain

$$\|\overline{\boldsymbol{E}}_t^h(t)\|_0 + \|\nabla \times \overline{\boldsymbol{E}}^h(t)\|_0 \le Ch^{k+1}|||g|||_{k+2,[0,t]}. \tag{48}$$

Combining (45), (47), and (48) together with (42) proves the result since

$$\|(\boldsymbol{E}_t - \boldsymbol{E}_t^h)(t)\|_0 \le \|\overline{\boldsymbol{E}}_t^h(t)\|_0 + \|\nabla(p_t - p_t^h)(t)\|_0.$$ □

Our next theorem proves the corresponding result to Theorem (4.1) when $\boldsymbol{G} \in M^\perp$.

THEOREM 4.2. *Suppose* $\boldsymbol{E}$ *satisfies* (5)–(6) *and* $\boldsymbol{E}^h$ *satisfies* (7) *and* (8). *Suppose that* $\boldsymbol{G}(t) \in M^\perp$, $\boldsymbol{E}_0 \in M^\perp$, *and* $\boldsymbol{E}_{t0} \in M^\perp$. *Let*

$$\begin{aligned} Q(t) = {} & |||\boldsymbol{E}_{tt}|||_{k+1,[0,t]} + |||T\boldsymbol{E}_{ttt}|||_{k+1,[0,t]} + |||T\boldsymbol{G}_t|||_{k+1,[0,t]} + \max_{0\le s\le t}\|\boldsymbol{E}_t(s)\|_{k+1} \\ & + \max_{0\le s\le t}\|T\boldsymbol{G}(s)\|_{k+1} + \max_{0\le s\le t}\|T\boldsymbol{E}_{tt}(s)\|_{k+1} + \|\boldsymbol{E}_0\|_{k+1}, \end{aligned} \tag{49}$$

and suppose $Q(t)$ *is bounded for* $0 \le t \le T$. *Then for each* $0 < \epsilon \le \epsilon_0$ *there exists a constant* $C = C(\epsilon, T)$ *such that*

$$\|(\boldsymbol{E} - \boldsymbol{E}^h)(t)\|_0 \le C\, h^{k+1-\epsilon}Q(t).$$

Remark. In this case $\boldsymbol{E}(t) \in M^\perp$ but $\boldsymbol{E}^h \notin M_h^\perp$, due this time to pollution from the initial data. However, this small pollution effect does not destroy the convergence properties of the method. By taking a suitable projection in place of r_h in (8) it is possible to force $\boldsymbol{E}^h \in M_h^\perp$. However, this greatly increases the complexity of the method.

Proof. Let $\boldsymbol{E}^h = \overline{\boldsymbol{E}}^h + \nabla p^h$, where $\overline{\boldsymbol{E}}^h \in M_h^\perp$ and $p^h \in S_h^{k+1}$. Taking $\boldsymbol{\phi}^h = \nabla p_t^h$ in (39) and using the fact that $\boldsymbol{G}, \boldsymbol{E} \in M^\perp$, and $\overline{\boldsymbol{E}}^h \in M_h^\perp$ we find th

$$(\nabla p_{tt}^h, \nabla p_t^h) = 0, \qquad 0 \le t \le T,$$

and hence

$$\|\nabla p_t^h(t)\|_0 = \|\nabla p_t^h(0)\|_0, \qquad 0 \le t \le T.$$

Thus, using (42),

$$\|\nabla p^h(t)\|_0 \le C\{\|\nabla p^h(0)\|_0 + \|\nabla p_t^h(0)\|_0\}. \tag{50}$$

It remains to estimate $\|\nabla p^h(0)\|_0$ and $\|\nabla p_t^h(0)\|_0$. But, since $\overline{\boldsymbol{E}}^h \in M_h^\perp$ and $\boldsymbol{E}_0 \in M^\perp$,

$$(\nabla p^h(0), \nabla p^h(0)) = (\overline{\boldsymbol{E}}^h(0) + \nabla p^h(0), \nabla p^h(0)) \tag{51}$$

$$= (r_h\boldsymbol{E}_0 - \boldsymbol{E}_0, \nabla p^h(0)). \tag{52}$$

Thus $\|\nabla p^h(0)\|_0 \le Ch^{k+1}\|\boldsymbol{E}_0\|_{k+1}$, and similarly $\|\nabla p_t^h(0)\|_0 \le Ch^{k+1}\|\boldsymbol{E}_{t0}\|_{k+1}$. Using these estimates in (50) shows that

$$\|\nabla p_t^h(t)\|_0 + \|\nabla p^h(t)\|_0 \le Ch^{k+1}\{\|\boldsymbol{E}_0\|_{k+1} + \|\boldsymbol{E}_{t0}\|_{k+1}\}. \tag{53}$$

Next we estimate $\boldsymbol{E} - \overline{\boldsymbol{E}}_0^h$ using a Baker–Bramble [2] analysis. $\overline{\boldsymbol{E}}^h$ satisfies

$$(\overline{\boldsymbol{E}}_{tt}^h, \boldsymbol{\phi}^h) + (\nabla \times \overline{\mathbf{E}}^h, \nabla \times \boldsymbol{\phi}^h) = (\boldsymbol{G}, \boldsymbol{\phi}^h) \quad \forall \boldsymbol{\phi}^h \in M_h^\perp.$$

Let $\boldsymbol{\phi}^h = T_h\boldsymbol{\chi}^h$ (see (23)) for $\boldsymbol{\chi}_h \in M_h^\perp$. Then using the symmetry of T_h we have

$$(T_h\overline{\boldsymbol{E}}_{tt}^h, \boldsymbol{\chi}^h) + (\overline{\boldsymbol{E}}^h, \boldsymbol{\chi}^h) = (T_h\boldsymbol{G}, \boldsymbol{\chi}^h) \quad \forall \boldsymbol{\chi}^h \in M_h^\perp. \tag{54}$$

There is a slight difficulty in obtaining a similar result for the continuous problem since $\boldsymbol{\chi}^h \in M_h^\perp$ may not be in $M^\perp$. However, as we have seen, we may write

$$\boldsymbol{\chi}^h = \boldsymbol{w} + \nabla p$$

for $\boldsymbol{w} \in M^\perp$ and $p \in H_0^1(\Omega)$. Then, taking $T\boldsymbol{w}$ to be the test function in (5), we obtain

$$(T\boldsymbol{E}_{tt}, \boldsymbol{w}) + (\boldsymbol{E}, \boldsymbol{w}) = (T\boldsymbol{G}, \boldsymbol{w})$$

but since $\nabla p \in M$ we may conclude that

$$(T\boldsymbol{E}_{tt}, \boldsymbol{\chi}^h) + (\boldsymbol{E}, \boldsymbol{\chi}^h) = (T\boldsymbol{G}, \boldsymbol{\chi}^h) \quad \forall \boldsymbol{\chi}^h \in M_h^\perp. \tag{55}$$

Taking $\boldsymbol{e} = \boldsymbol{E} - \overline{\boldsymbol{E}}^h$ and subtracting (55) from (54) we find that

$$(T_h\boldsymbol{e}_{tt}, \boldsymbol{\chi}^h) + (\boldsymbol{e}, \boldsymbol{\chi}^h) = ((T - T_h)\boldsymbol{G}, \boldsymbol{\chi}^h) + ((T_h - T)\boldsymbol{E}_{tt}, \boldsymbol{\chi}^h) \quad \forall \boldsymbol{\chi}^h \in M_h^\perp. \tag{56}$$

Using (56) we estimate

$$\begin{aligned}(T_h\boldsymbol{e}_{tt}, \boldsymbol{e}_t) + (\boldsymbol{e}, \boldsymbol{e}_t) &= (T_h\boldsymbol{e}_{tt}, \boldsymbol{E}_t - r_h\boldsymbol{E}_t) + (\boldsymbol{e}, \boldsymbol{E}_t - r_h\boldsymbol{E}_t)\\ &\quad + ((T - T_h)\boldsymbol{G}, r_h\boldsymbol{E}_t - \boldsymbol{E}_t^h) + ((T_h - T)\boldsymbol{E}_{tt}, r_h\boldsymbol{E}_t - \boldsymbol{E}_t^h).\end{aligned}$$

Defining $p(t)$ by

$$p(t) = ((T - T_h)\boldsymbol{G}, r_h\boldsymbol{E}_t - \boldsymbol{E}_t) + ((T_h - T)\boldsymbol{E}_{tt}, r_h\boldsymbol{E}_t - \boldsymbol{E}_t) \tag{57}$$

we thus have

$$\begin{aligned}(T_h\boldsymbol{e}_{tt}, \boldsymbol{e}_t) + (\boldsymbol{e}, \boldsymbol{e}_t) &= p(t) + (T_h\boldsymbol{e}_{tt}, \boldsymbol{E}_t - r_h\boldsymbol{E}_t) + (\boldsymbol{e}, \mathbf{E}_t - r_h\boldsymbol{E}_t)\\ &\quad + ((T - T_h)\boldsymbol{G}, \boldsymbol{e}_t) + ((T_h - T)\boldsymbol{E}_{tt}, \boldsymbol{e}_t).\end{aligned}$$

Integrating this expression in time, and integrating the terms in $\boldsymbol{e}_t$ and $\boldsymbol{e}_{tt}$ on the right-hand side by parts we have the estimate

$$\begin{aligned}&\frac{1}{2}\left\{\|T_h^{1/2}\boldsymbol{e}_t(t)\|_0^2 + \|\boldsymbol{e}(t)\|_0^2\right\}\\ &\quad = \frac{1}{2}\left\{\|T_h^{1/2}\boldsymbol{e}_t(0)\|_0^2 + \|\boldsymbol{e}(0)\|_0^2\right\}\\ &\qquad + \int_0^t p(s)ds - \int_0^t (T_h\boldsymbol{e}_t, \boldsymbol{E}_{tt} - r_h\boldsymbol{E}_{tt}) - (\boldsymbol{e}, \boldsymbol{E}_t - r_h\boldsymbol{E}_t)ds\\ &\qquad - \int_0^t ((T - T_h)\boldsymbol{G}_t, \boldsymbol{e}) + ((T_h - T)\boldsymbol{E}_{ttt}, \boldsymbol{e})ds\\ &\qquad + ((T - T_h)\boldsymbol{G}, \boldsymbol{e})|_0^t + ((T_h - T)\boldsymbol{E}_{tt}, \boldsymbol{e})|_0^t + (T_h\boldsymbol{e}_t, \boldsymbol{E}_t - r_h\boldsymbol{E}_t)|_0^t.\end{aligned} \tag{58}$$

Next we estimate each term on the right-hand side of (58). The first term may be estimated using (24) and (25):

$$
\begin{aligned}
(\boldsymbol{e}_t, T_h \boldsymbol{e}_t) &\le \|\boldsymbol{e}_t\|_0 \, \|T_h \boldsymbol{e}_t\|_0 \\
&\le C \|\boldsymbol{e}_t\|_0 \, \|\nabla \times T_h \boldsymbol{e}_t\|_0 \\
&\le C \|\boldsymbol{e}_t\|_0^2 .
\end{aligned} \tag{59}
$$

Obviously the same analysis shows that for $\boldsymbol{v} \in (L^2(\Omega))^3$ the estimate

$$
\|T_h^{1/2} \boldsymbol{v}\|_0 \le C \|\boldsymbol{v}\|_0 \tag{60}
$$

holds, and we shall use this estimate below. Next, using the definition of p in (57) and estimates from Theorems (2.2) and (2.4)

$$
\left| \int_0^t p(s) ds \right| \le C h^{2k+2-\epsilon} \max_{0 \le s \le t} \|\boldsymbol{E}_t(s)\|_{k+1} \left\{ |||T\boldsymbol{G}|||_{k+1,[0,t]} + |||T\boldsymbol{E}_{tt}|||_{k+1,[0,t]} \right\} . \tag{61}
$$

Using estimate (60) as well as Theorem (2.2), part (e),

$$
\begin{aligned}
&\left| \int_0^t (T_h \boldsymbol{e}_t, \boldsymbol{E}_{tt} - r_h \boldsymbol{E}_{tt}) \, ds - \int_0^t (\boldsymbol{e}, \boldsymbol{E}_t - r_h \boldsymbol{E}_t) \, ds \right| \\
&\le C h^{k+1} \left\{ \max_{0 \le s \le t} \|T_h^{1/2} \boldsymbol{e}_t(s)\|_0 \, |||\boldsymbol{E}_{tt}|||_{k+1,[0,t]} + \max_{0 \le s \le t} \|\boldsymbol{e}(s)\|_0 \, |||\boldsymbol{E}_t|||_{k+1,[0,t]} \right\} .
\end{aligned} \tag{62}
$$

In the same way, using Theorem (2.4),

$$
\begin{aligned}
&\left| \int_0^t ((T - T_h)\boldsymbol{G}_t, \boldsymbol{e}) + ((T_h - T)\mathbf{E}_{ttt}, \boldsymbol{e}) \, ds \right| \\
&\qquad \le C h^{k+1-\epsilon} \max_{0 \le s \le t} \|\boldsymbol{e}(s)\|_0 \left\{ |||T\boldsymbol{G}_t|||_{k+1,[0,t]} + |||T\boldsymbol{E}_{ttt}|||_{k+1,[0,t]} \right\} .
\end{aligned} \tag{63}
$$

And using (60), Theorem (2.2), part (e), and Theorem (2.4) we have

$$
\begin{aligned}
|(T_h \boldsymbol{e}_t, \boldsymbol{E}_t - r_h \boldsymbol{E}_t)|_0^t \,| \le C h^{k+1} \{ &\|T_h^{1/2} \boldsymbol{e}_t(t)\|_0 \, \|\boldsymbol{E}_t(t)\|_{k+1} \\
&+ \|T_h^{1/2} \boldsymbol{e}_t(0)\|_0 \, \|\boldsymbol{E}_t(0)\|_{k+1} \}.
\end{aligned} \tag{64}
$$

$$
\begin{aligned}
|((T - T_h)\boldsymbol{G}, \boldsymbol{e})|_0^t \,| \le C h^{k+1-\epsilon} \{ &\|\boldsymbol{e}(t)\|_0 \, \|T\boldsymbol{G}(t)\|_{k+1} \\
&+ \|\boldsymbol{e}(0)\|_0 \, \|T\boldsymbol{G}(0)\|_{k+1} \}.
\end{aligned} \tag{65}
$$

$$
\begin{aligned}
|((T_h - T)\boldsymbol{E}_{tt}, \boldsymbol{e})|_0^t \,| \le C h^{k+1-\epsilon} \{ &\|\boldsymbol{e}(t)\|_0 \, \|T\boldsymbol{E}_{tt}(t)\|_{k+1} \\
&+ \|\boldsymbol{e}(0)\|_0 \, \|T\boldsymbol{E}_{tt}(0)\|_{k+1} \}.
\end{aligned} \tag{66}
$$

Using (59)–(66) in (58) shows that for $0 \le t \le t_1 \le T$

$$
\begin{aligned}
\|T_h^{1/2} \boldsymbol{e}_t(t)\|_0^2 + \|\boldsymbol{e}(t)\|_0^2 \le C \Bigg\{ &\|\boldsymbol{e}_t(0)\|_0^2 + \|\boldsymbol{e}(0)\|_0^2 \\
&+ h^{2(k+1-\epsilon)} \left\{ \max_{0 \le s \le t_1} \|\boldsymbol{e}(s)\|_0 \, A(t_1) \right. \\
&\left. + \max_{0 \le s \le t_1} \|T_h^{1/2} \boldsymbol{e}_t(s)\|_0 \, B(t_1) + (D(t_1))^2 \right\} \Bigg\},
\end{aligned}
$$

where

$$\begin{aligned}
A(t_1) &= |||\boldsymbol{E}_t|||_{k+1,[0,t_1]} + |||T\boldsymbol{E}_{ttt}|||_{k+1,[0,t_1]} + |||T\boldsymbol{G}_t|||_{k+1,[0,t_1]} \\
&\quad + \max_{0\le s\le t_1} \|T\boldsymbol{G}(s)\|_{k+1} + \max_{0\le s\le t_1} \|T\boldsymbol{E}_{tt}(s)\|_{k+1}, \\
B(t_1) &= |||\boldsymbol{E}_{tt}|||_{k+1,[0,t_1]} + \max_{0\le s\le t_1} \|\boldsymbol{E}_t\|_{k+1}, \\
D(t_1) &= |||T\boldsymbol{G}|||_{k+1,[0,t_1]} + |||T\boldsymbol{E}_{tt}|||_{k+1,[0,t_1]} + \max_{0\le s\le t_1} \|\boldsymbol{E}_t(s)\|_{k+1}.
\end{aligned}$$

Now using the same arguments as in the proof of Theorem (3.1), we may conclude that for $0 \le t \le t_1$

$$\|T_h^{1/2}\boldsymbol{e}_t(t)\|_0 + \|\boldsymbol{e}(t)\|_0 \le C\{\|\boldsymbol{e}_t(0)\|_0 + \|\boldsymbol{e}(0)\|_0 + h^{k+1-\epsilon}(A+B+D)\}. \tag{67}$$

It remains to estimate $\boldsymbol{e}(0)$ and $\boldsymbol{e}_t(0)$. But since $\boldsymbol{E} \in M^\perp$ and $\overline{\boldsymbol{E}}^h \in M_h^\perp$, and recalling that $\boldsymbol{e} = \boldsymbol{E} - \overline{\boldsymbol{E}}^h$,

$$\begin{aligned}
(\boldsymbol{e}(0), \boldsymbol{e}(0)) &= (\boldsymbol{E}(0) - \overline{\boldsymbol{E}}^h(0), \boldsymbol{e}(0)) \\
&= (\boldsymbol{E}(0) - \overline{\boldsymbol{E}}^h(0) - \nabla p^h, \boldsymbol{e}(0)) \\
&= ((\boldsymbol{E} - \boldsymbol{E}^h)(0), \boldsymbol{e}(0)),
\end{aligned}$$

and thus

$$\|\boldsymbol{e}(0)\|_0 \le \|\boldsymbol{E}_0 - r_h\boldsymbol{E}_0\|_0 \le Ch^{k+1}\|\boldsymbol{E}_0\|_{k+1}.$$

In the same way

$$\|\boldsymbol{e}_t(0)\|_0 \le Ch^{k+1}\|\boldsymbol{E}_{t0}\|_{k+1}.$$

Using these estimates in (67) and collecting terms proves the theorem. □

COROLLARY 4.3. *Let $\boldsymbol{E}$ satisfy* (5)–(6) *and suppose $\boldsymbol{E}^h$ satisfies* (7) *and* (8) *with a general $\boldsymbol{G}$. Let $\boldsymbol{G} = \nabla \times \boldsymbol{\Phi} + \nabla p$, $p \in H_0^1(\Omega)$. Suppose that ∇p and $\nabla \times \boldsymbol{\Phi}$ satisfy all the assumptions of Theorems* (4.1) *and* (4.2), *respectively. In addition suppose all the regularity assumptions of Theorems* (4.1)–(4.2) *are satisfied. Then for $0 \le t \le T$ and $0 < \epsilon \le \epsilon_0$ there exists a constant $C = C(\epsilon, T)$ such that*

$$\|(\boldsymbol{E} - \boldsymbol{E}^h)(t)\|_0 \le Ch^{k+1-\epsilon},$$

where C also depends on the constants given in Theorems (4.1) *and* (4.2).

Proof. By linearity we can split the estimation problem into two parts, the first covered by Theorem (4.1) and the second by Theorem (4.2). Since $\boldsymbol{G}$ may be written as

$$\boldsymbol{G} = \nabla \times \boldsymbol{\Phi} + \nabla p$$

for $\boldsymbol{\Phi} \in H(\text{curl}; \Omega)$ and $p \in H_0^1(\Omega)$, with similar decompositions for $\boldsymbol{E}_0$ and $\boldsymbol{E}_{t0}$, we may decompose (5)–(6) into two problems with right-hand sides, respectively $\nabla \times \boldsymbol{\Phi}$ and ∇p and initial data in $M^\perp$ and M, respectively. Then using Theorems (4.1) and (4.2) we can estimate the error in each subproblem and hence the overall error. □

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