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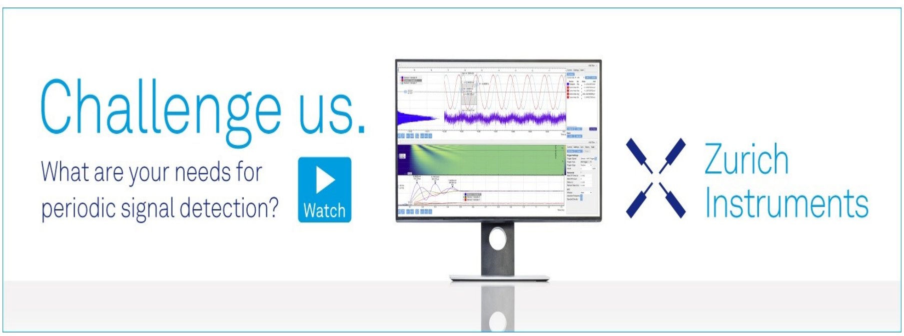

Noncommutative Geometries and Gravity

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Abstract. We briefly review ideas about "noncommutativity of space-time" and approaches toward a corresponding theory of gravity.

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INTRODUCTION

"Noncommutative geometry" (NCG) is a broad framework in which notions of space, symmetry and (differential) geometry can be generalized in various ways. In this short review we will concentrate on aspects related to the concepts of space-time and gravity. Let us recall that spaces can be traded for commutative rings or algebras. Relativists and geometers are familiar with this point of view: in a chart on a manifold (e.g. space-time) one works with the commutative algebra of functions generated by coordinates x^{μ}. (An algebraic formulation of General Relativity has already been proposed in 1972 by R. Geroch [1, 2]. More technically, given a locally compact space M, the set of continuous $\mathbb{C}$-valued functions on it (that vanish "at infinity" if M is not compact) becomes a commutative C^* algebra with the L^∞-norm and f^* the complex conjugate of f. Furthermore, every commutative C^*-algebra $\mathscr{A}$ is isomorphic to the algebra $C(M)$ of continuous functions on some locally compact space M (Gelfand-Naimark theorem, see e.g. [3, 4]). This involves the construction of a space ("Gelfand spectrum") as the set of non-zero characters, i.e. homomorphisms into $\mathbb{C}$. In case of the algebra of continuous functions on a Hausdorff space, one recovers the original space. M is compact if $\mathscr{A}$ is unital).

If a commutative associative algebra thus corresponds to a topological space, a *non*commutative associative (Even *nonassociative* algebras are of interest. In particular nonassociative star products appear in string theory [5]. But we will leave this aside.) algebra $\mathscr{A}$ may be regarded as a "noncommutative space". (C^*-algebras are particularly nice since they admit a faithful representation by bounded operators on a Hilbert space. In quantum physics, a familiar example of a noncommutative C^*-algebra is the (Weyl) algebra of one-parameter unitary groups generated by position and momentum operators, but more flexible are "resolvent algebras" [6]). The analogue of a vector bundle (needed to formulate gauge theory) on such a noncommutative space is then a module over $\mathscr{A}$. (This is based on the equivalence of vector bundles over a compact space M and finitely generated projective modules over $C(M)$ (Serre-Swan theorem). See [4], for instance).

A rule which associates with a "commutative space" some noncommutative space is a kind of quantization, analogous to canonical quantization in physics, which replaces

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an algebra of functions on a phase space with a Heisenberg (Weyl) algebra of operators on a Hilbert space, or deformation quantization [7, 8], which deforms the commutative product of functions to the noncommutative Groenewold-Moyal product [9, 10]. Further examples of "quantized spaces" are provided by quantum groups that are deformations of classical groups reformulated as Hopf algebras (see [11], for example). Several noncommutative spaces do play a role in physical models and theories. The idea of "noncommutative space-time" is more speculative, however. (An example is Snyder's "quantized space-time" which originates from the five-dimensional de Sitter space regarded as "momentum space" of a particle [12]. It preserves Lorentz invariance, but breaks translational invariance (see also [13]). More generally curved momentum spaces correspond to noncommutative configuration spaces, see [14] for the example of a point particle in $(2+1)$-dimensional gravity). Let us discuss critically three arguments that appear in the literature in favor of it. Others will be addressed in the following sections.

1. Before renormalization theory had been developed, quantum field theory (QFT) was plagued by apparently uncontrollable infinite expressions. In those days the idea came up that noncommutativity of coordinates could help to eliminate these (ultraviolet) divergences [12]. Meanwhile the believe is that QFT on noncommutative spaces (with an *infinite* number of degrees of freedom) still requires renormalization [15, 16]. But for a non-renormalizable theory like perturbative Einstein gravity on Minkowski space, improvements (comparable with that of string theory) could perhaps be achieved in such a way. (It should also be noticed that ultraviolet divergences appear in *integrated* expressions and therefore already the introduction of a weaker kind of noncommutativity, namely a noncommutativity between (commuting) functions and differentials can do a good job [17]).

2. At least operationally the concept of space-time underlying General Relativity does not make sense below the length scale given by the Planck length $\ell_P = \sqrt{\hbar\mathscr{G}/c^3}$ (where $\mathscr{G}$ is Newton's gravitational constant). In order to resolve space (-time) with greater accuracy we need more energy. A resolution limit is then obtained when the radius of the ball into which the energy is transmitted becomes smaller than the corresponding Schwarzschild radius, in which case no information can escape from this area (see e.g. [18, 19], and [20] for related arguments). (In string theory a resolution limit is given by the string length). This suggests space-time uncertainty relations, which can be realized [18, 19] by turning coordinate functions into noncommuting self-adjoint operators:

$$[\hat{x}^\mu, \hat{x}^\nu] = i\, Q^{\mu\nu} \ . \tag{1}$$

In a low energy approximation, the operators $Q^{\mu\nu} = -Q^{\nu\mu}$ should be negligible and $\hat{x}^\mu$ become inertial coordinates. Assuming covariance under the Poincaré group, treating $Q^{\mu\nu}$ as a tensor, the analysis in [18, 19] led to conditions for $Q^{\mu\nu}$, which are in particular satisfied if $Q^{\mu\nu}$ is a central element of the algebra, subject to some algebraic constraints. A word of caution is in place, however. In General Relativity coordinates are not regarded as observables, all the information about space-time resides in the metric tensor. Space-time uncertainties may then result from quantization of the metric (on a commutative space). In contrast, the "coordinates" used in [18, 19] are assumed to carry metric information like inertial coordinates in Special Relativity.

3. A kind of space-time noncommutativity appears in string theory in the so-called Seiberg-Witten limit [21]. The bosonic part of the (open) string action in a background metric $g_{\mu\nu}$ and background B-field is

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma g_{\mu\nu}\, \partial_a X^\mu\, \partial^a X^\nu\, \mathrm{d}^2\sigma - \frac{i}{2} \int_\Sigma B_{\mu\nu}\, \mathrm{d}X^\mu \wedge \mathrm{d}X^\nu\,. \tag{2}$$

If $|g_{\mu\nu}| \ll |\alpha' B_{\mu\nu}|$ with *constant* $B_{\mu\nu}$, then $S \approx -(i/2)B_{\mu\nu} \int_{\partial\Sigma} X^\mu\, \partial_t X^\nu$ (with ∂_t tangential to the world sheet boundary $\partial\Sigma$), which upon canonical quantization leads to

$$[\hat{X}^\mu, \hat{X}^\nu] = i\,\theta^{\mu\nu} \qquad \text{on } \partial\Sigma, \tag{3}$$

where $\theta^{\mu\nu} = (B^{-1})^{\mu\nu}$. (More precisely, here we should consider a space-filling D-brane, or a lower-dimensional Dp-brane, then split the set of coordinates accordingly and assume maximal rank of B, see [21]). Thus the embedding functions X^μ restricted to the string end points become noncommuting operators in this limit. This heuristic derivation very much parallels that of noncommutative coordinates in the case of the Landau problem of a quantum particle in a plane with perpendicular strong external magnetic field (see [22, 23] for instance). This does not mean that the classical space-time somehow disappears, but rather that in certain situations physics is (more) effectively described in terms of certain noncommutative "coordinates".

There is an important advantage of noncommutativity (e.g. noncommutative space-time) as compared with discretization ("discrete space-time"). Whereas discretization, i.e. replacing the continuum by a discrete space, typically breaks continuous symmetries, noncommutativity is more flexible. (Introducing noncommutativity can actually restore continuous symmetries which got lost by discretization. Discretizing the sphere by reducing it to a north and a south pole obviously destroys its continuous symmetries. The remaining freedom can be expressed by the set of diagonal 2×2 matrices, on which $SO(3)$ can only act trivially. But if we extend it to the noncommutative space of *all* 2×2 matrices, there is a non-trivial action of $SO(3)$. See also [24] and the fuzzy sphere example).

Example [25]. Let J_a, $a = 1,2,3$, be a Hermitian basis of $su(2)$ such that $[J_a, J_b] = i\,\varepsilon_{abc} J_c$. In the j-dimensional irreducible representation, the value of the Casimir operator is given by $J_1^2 + J_2^2 + J_3^2 = \frac{j^2-1}{4} I$ (with the unit matrix I). Then $x_{(j)a} := 2rJ_a/\sqrt{j^2-1}$, $a = 1,2,3$, with a positive real constant r, satisfy $\vec{x}_{(j)}^2 = r^2 I$, which formally corresponds to the equation defining the two-dimensional sphere in three-dimensional Euclidean space. Since $[x_{(j)a}, x_{(j)b}] = (2ir/\sqrt{j^2-1})\,\varepsilon_{abc}\, x_{(j)c}$, the algebra becomes commutative in the limit $j \to \infty$, and indeed approximates the sphere. $SU(2)$ acts by conjugation (adjoint representation) on its Lie algebra and thus on the *fuzzy sphere* S_j^2 (i.e. the algebra generated by $x_{(j)a}$, $a = 1,2,3$), preserving the "sphere constraint".

An algebra alone is not sufficient to describe a space-time, we need an additional structure which encodes the metric information. There are several (mathematical) ways to implement this, some of which will be considered in later sections.

In the following sections we gather some essentials from several approaches toward a "noncommutative" generalization of the notions of space-time and gravity. It is based on a certain (surely personally based) selection from the existing literature and we regret for not being able to give consideration to all of those who contributed to this field.

Moyal deformation of $\mathbb{R}^n$. In deformation quantization [7, 8], a noncommutative algebra is obtained by replacing the commutative product of functions by the (Groenewold-) Moyal product [9, 10] (so that the Poisson bracket is replaced by the Moyal bracket [10]), defined for functions on $\mathbb{R}^n$ in terms of coordinates x^μ by

$$f \star h := \mathbf{m}_{\mathscr{F}}(f \otimes h), \quad \mathbf{m}_{\mathscr{F}} := \mathbf{m} \circ \mathscr{F}^{-1}, \quad \mathscr{F} := \exp\Big(-\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu\Big). \tag{4}$$

Here $\theta^{\mu\nu}$ are real antisymmetric constants and $\mathbf{m}(f \otimes h) = fh$. In particular, we have $x^\mu \star x^\nu = x^\mu x^\nu + (i/2)\theta^{\mu\nu}$ and thus

$$[x^\mu, x^\nu]_\star := x^\mu \star x^\nu - x^\nu \star x^\mu = i\,\theta^{\mu\nu}, \tag{5}$$

which makes contact with (3). Indeed, the Seiberg-Witten limit of string theory can be described in terms of the Moyal product. Clearly the above relation and also the $\star$-product of two scalars are invariant under constant linear (e.g. Lorentz) transformations if $\theta^{\mu\nu}$ are treated as tensor components. Note that complex conjugation is an involution: $(f \star h)^* = h^* \star f^*$. For Schwartz space functions, $\int f \star h\, \mathrm{d}x^n = \int f h\, \mathrm{d}x^n$.

Kontsevich star-product. There is a covariantization of the Moyal-product and moreover a generalization to the case where $\theta^{\mu\nu}$ is an arbitrary Poisson tensor field [26, 27]. In local coordinates we have

$$\begin{aligned} f \star h \;\; &= \;\; fh + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f\,\partial_\nu h - \frac{1}{8}\theta^{\mu\kappa}\theta^{\nu\lambda}\partial_\mu\partial_\nu f\,\partial_\kappa\partial_\lambda h \\ &\quad - \frac{1}{12}\theta^{\mu\lambda}\partial_\lambda\theta^{\nu\kappa}(\partial_\mu\partial_\nu f\,\partial_\kappa h - \partial_\nu f\,\partial_\mu\partial_\kappa h) + \mathscr{O}(\theta^3) \end{aligned} \tag{6}$$

(see also [28]). This is indeed associative due to the Jacobi identity of the Poisson structure (which is equivalent to $\theta^{\kappa[\lambda}\partial_\kappa\theta^{\mu\nu]} = 0$). Kontsevich found a formal combinatorial expression to all orders in θ [26] (see also [27]). Under a change of coordinates the $\star$-product changes by an equivalence transformation $f \star' h = \mathscr{S}^{-1}(\mathscr{S}f \star \mathscr{S}h)$ with an operator $\mathscr{S}$ (see also [29, 30, 28]). In string theory a non-constant Poisson tensor originates from a non-constant B-field on a D-brane. See [28] for corresponding examples.

Twisted Poincaré symmetry. Let us think of x^μ as (inertial) space-time coordinates. Instead of regarding the parameters $\theta^{\mu\nu}$ in the Moyal-product as tensor components, let us try to treat them as fixed *constant* numbers, so we may restrict to space-space noncommutativity by setting $\theta^{0\mu} = 0$. (In the Lagrangian approach to noncommutative QFT, $\theta^{0\mu} \neq 0$ leads to unitarity violation. See [31], however, for a Hamiltonian approach in which this problem does not show up). But (5) is then obviously *not* invariant under the usual action of the generators $P_\mu, M_{\mu\nu}$ of the Poincaré Lie algebra, which extends to functions via the derivation rule. In Hopf algebra language (Any Lie algebra $\mathfrak{g}$ can be turned into a Hopf algebra by first extending it to the universal enveloping algebra $\mathscr{U}(\mathfrak{g})$. Then $\Delta(Y) := Y \otimes 1 + 1 \otimes Y$ for $Y \in \mathscr{U}(\mathfrak{g})$ defines a homomorphism

$\Delta: \mathscr{U}(\mathfrak{g}) \to \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})$, called the coproduct. Similarly, the antipode S (generalized inverse) and the counit ε are given by $S(Y) = -Y$ and $\varepsilon(Y) = 0$, respectively.) the latter is given by $Y \triangleright (fh) \equiv Y \triangleright \mathbf{m}(f \otimes h) = \mathbf{m} \circ \Delta(Y) \triangleright (f \otimes h)$, using the coproduct $\Delta(Y) = Y \otimes 1 + 1 \otimes Y$. (Writing $\Delta(Y) = Y_{(1)} \otimes Y_{(2)}$ (Sweedler notation), we set $\Delta(Y) \triangleright (f \otimes h) := (Y_{(1)} \triangleright f) \otimes (Y_{(2)} \triangleright h)$). Replacing the coproduct Δ with the *twisted coproduct* $\Delta_{\mathscr{F}} = \mathscr{F} \Delta \mathscr{F}^{-1}$ (with $\mathscr{F}$ defined in (4)), then

$$Y \triangleright (f \star h) = Y \triangleright \mathbf{m}_{\mathscr{F}}(f \otimes h) = \mathbf{m}_{\mathscr{F}} \circ \Delta_{\mathscr{F}}(Y) \triangleright (f \otimes h) \tag{7}$$

restores invariance [32, 33, 28]. See also [34] for further implications.

Twisted (infinitesimal) diffeomorphisms and deformed gravity. The above twist plays a crucial role in a recent formulation of Moyal-deformed differential geometry [35, 36, 37, 38, 39]. In classical differential geometry the action of an infinitesimal coordinate transformation generated by a vector field $\xi = \xi^\mu \partial_\mu$ on a scalar, vector and covector is given by

$$\delta_\xi \phi = -\xi \phi \,, \quad \delta_\xi V^\mu = -\xi V^\mu + (\partial_\nu \xi^\mu) V^\nu \,, \quad \delta_\xi a_\mu = -\xi a_\mu - (\partial_\mu \xi^\nu) a_\nu \,, \tag{8}$$

respectively. In terms of the coproduct $\Delta(\delta_\xi) = \delta_\xi \otimes 1 + 1 \otimes \delta_\xi$, more general tensor transformation laws are recovered by applying $\delta_\xi \circ \mathbf{m} := \mathbf{m} \circ \Delta(\delta_\xi)$ to a tensor product.

One can express the ordinary product of functions in terms of the star product via $fh = \mathbf{m}_{\mathscr{F}} \circ \mathscr{F}(f \otimes h) = X_f \star h =: X_f \triangleright h$ with

$$X_f := \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{2} \right)^n \theta^{\mu_1 \nu_1} \cdots \theta^{\mu_n \nu_n} (\partial_{\mu_1} \cdots \partial_{\mu_n} f) \star \partial_{\nu_1} \cdots \partial_{\nu_n} \,. \tag{9}$$

For a vector field ξ, we then have $\xi f = \xi^\mu \partial_\mu f = X_{\xi^\mu} \triangleright \partial_\mu f =: X_\xi \triangleright f$ with the operator

$$X_\xi = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{2} \right)^n \theta^{\mu_1 \nu_1} \cdots \theta^{\mu_n \nu_n} (\partial_{\mu_1} \cdots \partial_{\mu_n} \xi^\lambda) \star \partial_{\nu_1} \cdots \partial_{\nu_n} \partial_\lambda \tag{10}$$

(and correspondingly for a higher order differential operator replacing ξ). This yields a representation of the classical Lie algebra of vector fields, i.e. $[X_\xi, X_{\xi'}]_\star = X_{[\xi,\xi']}$. Rewriting (8) as

$$\hat{\delta}_\xi \phi = -X_\xi \triangleright \phi \,,\; \hat{\delta}_\xi V^\mu = -X_\xi \triangleright V^\mu + X_{\partial_\nu \xi^\mu} \triangleright V^\nu \,,\; \hat{\delta}_\xi a_\mu = -X_\xi \triangleright a_\mu - X_{\partial_\mu \xi^\nu} \triangleright a_\nu \,, \tag{11}$$

and correspondingly for other tensors, one finds that

$$\hat{\delta}_\xi (S^{\mu_1 \ldots \mu_m}_{\nu_1 \ldots \nu_n} \star T^{\kappa_1 \ldots \kappa_r}_{\lambda_1 \ldots \lambda_s}) = \mathbf{m}_{\mathscr{F}} \circ \Delta_{\mathscr{F}}(\hat{\delta}_\xi)(S^{\mu_1 \ldots \mu_m}_{\nu_1 \ldots \nu_n} \otimes T^{\kappa_1 \ldots \kappa_r}_{\lambda_1 \ldots \lambda_s}) \tag{12}$$

(which generalizes (7)), hence the $\star$-product of tensors is again a tensor [35]. The way toward a noncommutative version of Einstein's equations is now straightforward. A covariant derivative can be introduced in analogy to its action on classical tensors, e.g. $D_\nu V^\mu = \partial_\nu \triangleright V^\mu + \Gamma^\mu_{\lambda\nu} \star V^\lambda$. The connection $\Gamma^\mu_{\lambda\nu}$ has curvature $R^\kappa{}_{\lambda\mu\nu} = \partial_\mu \triangleright \Gamma^\kappa_{\lambda\nu} -$

$\partial_\nu \triangleright \Gamma^\kappa_{\lambda\mu} + \Gamma^\kappa_{\rho\mu} \star \Gamma^\rho_{\lambda\nu} - \Gamma^\kappa_{\rho\nu} \star \Gamma^\rho_{\lambda\mu}$ and Ricci tensor $R_{\mu\nu} = R^\kappa{}_{\mu\kappa\nu}$. A metric should be taken to be a symmetric and $\star$-invertible rank two tensor field $g_{\mu\nu}$, and we can impose vanishing torsion $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ and metric compatibility $D_\lambda g_{\mu\nu} = 0$. As in the classical case these conditions determine the connection in terms of the metric:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}(\partial_\mu \triangleright g_{\nu\kappa} + \partial_\nu \triangleright g_{\mu\kappa} - \partial_\kappa \triangleright g_{\mu\nu}) \star g^{\kappa\lambda}\,, \tag{13}$$

where $g^{\kappa\lambda}$ is the $\star$-inverse of $g_{\mu\nu}$. Because of noncommutativity there are two curvature scalars: $R = g^{\mu\nu} \star R_{\nu\mu}$ and $R' = R_{\nu\mu} \star g^{\mu\nu}$. We refer to [35] for further details and the construction of a deformed Einstein-Hilbert action functional. One can then ask whether this structure shows up in, say, string theory. This seems not to be the case. The above twisted gravity theory does not match the dynamics of closed strings in a constant B-field (beyond Seiberg-Witten approximation) [40]. What we probably should more worry about is the fact that the above formalism apparently distinguishes a class of coordinate systems, namely that with respect to which the twist operator $\mathscr{F}$ is defined (see also [30]).

Yet some other approaches. Keeping $\theta^{\mu\nu}$ constant, and noting that $[x^\mu, f(x)]_\star = i\theta^{\mu\rho}\partial_\rho f(x)$, we find that an infinitesimal coordinate transformation $x^{\mu'} = x^\mu + \xi^\mu(x)$ leaves (5) invariant if $\theta^{\rho[\mu}\partial_\rho \xi^{\nu]} = 0$. This is solved by $\xi^\mu = \theta^{\mu\nu}\partial_\nu f$ with a function $f(x)$. With respect to this *restricted* class of coordinate transformations, one can then develop a formalism of geometry and General Relativity [41, 42].

The standard setup of a gauge theory on a noncommutative space requires that the anticommutator of Lie algebra elements lies in the Lie algebra (If X,Y are elements of a Lie algebra and a,b elements of some algebra $\mathscr{A}$, then $[aX,bY] = \frac{1}{2}\{a,b\}[X,Y] + \frac{1}{2}[a,b]\{X,Y\}$. For this to be Lie-algebra-valued, either $\mathscr{A}$ has to be commutative or the anticommutator $\{X,Y\}$ has to lie in the Lie algebra.), which is the case for a general linear group or $u(N)$ in the fundamental representation. A "gravity theory" formulated as a gauge theory on the Moyal space-time is thus necessarily complexified, see [43, 29, 44, 45, 46, 47]. The problems with diffeomorphism invariance are still present, of course.

DSR AND κ-POINCARÉ SYMMETRY

In "Doubly Special Relativity" (DSR) (see [48, 49], for instance) the basic postulate is that all inertial observers should not only agree about the value of the speed of light, but also on the value of the Planck length ℓ_P. A realization of this idea obviously requires a deformation of the Poincaré symmetry of Special Relativity. This can be achieved by stepping beyond the classical notion of symmetry toward the generalization offered by Hopf algebras. Indeed, a realization is the so-called κ-Poincaré algebra [50, 51] with $\kappa = \ell_P^{-1}$. This is one of the weakest Hopf algebra deformations of the Poincaré Lie algebra. (Hopf algebras comprise generalizations of Lie algebras as well as Lie groups. Correspondingly, there is also a κ-deformation of the Poincaré *group* [52] (as a "matrix quantum group" [53])). It is given by [51]

$$
\begin{aligned}
&[P_\mu, P_\nu] = 0\,, \quad [N_i, P_0] = P_i\,, \quad [N_i, P_j] = \delta_{ij}\left(\frac{\kappa}{2}(1 - e^{-2P_0/\kappa}) + \frac{1}{2\kappa}\vec{P}^2\right) - \frac{1}{\kappa}P_i P_j \\
&[N_i, N_j] = -\varepsilon_{ijk} M_k\,, \quad [M_i, M_j] = \varepsilon_{ijk} M_k\,, \quad [M_i, N_j] = \varepsilon_{ijk} N_k \\
&[M_i, P_j] = \varepsilon_{ijk} P_k\,, \quad [M_i, P_0] = 0\,, \quad [N_i, P_0] = P_i
\end{aligned} \tag{14}
$$

(where $\mu, \nu = 0, 1, 2, 3$ and $i, j = 1, 2, 3$), and

$$
\begin{aligned}
&\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0\,, \quad \Delta(P_i) = P_i \otimes 1 + e^{-P_0/\kappa} \otimes P_i\,, \\
&\Delta(M_i) = M_i \otimes 1 + 1 \otimes M_i\,, \quad \Delta(N_i) = N_i \otimes 1 + e^{-P_0/\kappa} \otimes N_i + \kappa^{-1}\,\varepsilon_{ijk} P_j \otimes M_k\,.
\end{aligned} \tag{15}
$$

In this ("bicrossproduct" [51]) formulation, the Lorentz sector is precisely that of the Poincaré algebra, and

$$
2\,\kappa^2 \cosh(P_0/\kappa) - \vec{P}^2 e^{P_0/\kappa} = 2\,\kappa^2 + P_0^2 - \vec{P}^2 - \kappa^{-1}\vec{P}^2 P_0 + \mathscr{O}(\kappa^{-2}) \tag{16}
$$

lies in the center of the algebra (i.e. commutes with all elements), thus has a fixed value in an irreducible representation. This leads to nonlinear corrections to the classical energy-momentum relations. Unlike the case of a Lie algebra, where only *linear* transformations of the generators are allowed, there is now a huge freedom of *nonlinear* transformations (also involving κ), even if we demand that the limit $\kappa \to 0$ reproduces the standard generators of the Poincaré group. Some additional input (to be expected from a quantum gravity theory), is thus needed to determine the "physical" energy and momentum.

The coproduct is a rule to compose representations and thus to build multi-particle systems. Because of its asymmetry it appears to be difficult to make physical sense of the results in case of the κ-Poincaré algebra. In any case, κ-Poincaré is an interesting example from which we can learn about generalized symmetries (quantum groups) in a physical context. Although quite a lot has been published about DSR, it has by far not reached the status of a physical theory as compared with SR.

κ-Poincaré from three-dimensional quantum gravity. The Lie algebra of the Poincaré group in three space-time dimensions is given by (This is obtained from (14), reduced to $2+1$ dimensions (in which case there is only a single rotation generator $M := J_0$, such that $[M, N_i] = \varepsilon_{ij} N^j$, $[M, P_0] = 0$, $[M, P_i] = \varepsilon_{ij} P^j$), in the limit as $\kappa \to \infty$. We have to identify $N_i = J_{i0}$, $i = 1, 2$. Indices are shifted with $\eta = \mathrm{diag}(-1, 1, 1)$ and we use $\varepsilon_{012} = 1$.)

$$
[J_a, J_b] = \varepsilon_{abc} J^c\,, \quad [J_a, P_b] = \varepsilon_{abc} P^c\,, \quad [P_a, P_b] = 0\,, \tag{17}
$$

where $J^c = \frac{1}{2}\varepsilon^{cab} J_{ab}$, and J_{ab} are generators of $SO(2,1)$. Replacing $[P_a, P_b] = 0$ by

$$
[P_a, P_b] = \mp \ell^{-2}\,\varepsilon_{abc} J^c \tag{18}
$$

where ℓ is a parameter with dimension of length, we have for "-" the Lie algebra of $SO(3,1)$ and for "+" that of $SO(2,2)$. A (dimensionless) connection $A = \omega^a J_a + \theta^a P_a$

then has the field strength

$$F = \mathrm{d}A + A \wedge A = (\mathscr{R}^c \mp \frac{1}{2\ell^2} \varepsilon^c{}_{ab}\, \theta^a \wedge \theta^b)\, J_c + \Theta^c P_c\,, \tag{19}$$

with curvature and torsion

$$\mathscr{R}^c := \mathrm{d}\omega^c + \frac{1}{2}\, \varepsilon^c{}_{ab}\, \omega^a \wedge \omega^b =: \frac{1}{2}\, \varepsilon^{cab}\, \mathscr{R}_{ab}\,, \quad \Theta^c := \mathrm{d}\theta^c + \varepsilon^c{}_{ab}\, \omega^a \wedge \theta^b\,. \tag{20}$$

Using the invariant inner product given by $\langle J_a, P_b \rangle = \ell^{-1} \eta_{ab}$, $\langle J_a, J_b \rangle = 0 = \langle P_a, P_b \rangle$, one can construct a (dimensionless) Chern-Simons form [54, 55, 56]:

$$\langle A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \rangle = \ell^{-1} (\mathscr{R}^{ab} \mp \frac{1}{3\ell^2} \theta^a \wedge \theta^b) \wedge \varepsilon_{abc}\, \theta^c - \mathrm{d}(\ell^{-1} \omega^c \wedge \theta_c)\,, \tag{21}$$

which, up to an exact form, is the three-dimensional Einstein-Cartan Lagrangian with cosmological constant $\Lambda = \pm \ell^{-2}$, if we identify $k = \ell/\ell_P$. (If θ^a, $a = 0,1,2$, form a coframe ("dreibein"), then $g = \eta_{ab} \theta^a \otimes \theta^b$ defines a metric. But here θ^a is a gauge potential which in general does not constitute a coframe. The field equations $F = 0 \Leftrightarrow \{\, \Theta^c = 0 \text{ and } \mathscr{R}^c = 0 \,\}$ even admit exact solutions with $\theta^a = 0$. Allowing a "degenerate dreibein" is crucial for treating three-dimensional gravity as a Chern-Simons gauge theory, but it means a serious departure from the usual understanding of gravity, see also [57]). See also [58, 59] for an analogous relation between higher-dimensional Chern-Simons and generalized gravity actions. Their moduli space is much more complicated than in three dimensions, however). Hence

$$S_{\mathrm{CS}} = \frac{k}{4\pi} \int \langle A \wedge \mathrm{d}A + \frac{2}{3} A^3 \rangle = \frac{k}{4\pi\,\ell} \int (R - 2\Lambda) \sqrt{|\det(g_{ab})|}\, \mathrm{d}^3 x \tag{22}$$

modulo boundary terms. (Global definition of the Chern-Simons action and the fact that $e^{iS_{\mathrm{CS}}}$ is single-valued require that the real constant k has to be "quantized" [57]).

Choosing space-time as $\Sigma \times \mathbb{R}$ with a two-dimensional compact surface Σ of genus g, the Chern-Simons action fixes the rules of canonical quantization. The physical degrees of freedom are that of the moduli space of flat connections (i.e. the space of connections modulo gauge transformations). Considering holonomies of the connection along non-contractable loops, this space can be described as the space of homomorphisms from $\pi_1(\Sigma)$, the fundamental group of Σ, into the global gauge group G (which is $SO(3,1)$ or $SO(2,2)$) [55, 60]. For $U, V \in G$ representing two intersecting loops, one can define invariants, for which the quantization implies commutation relations of the quantum deformation $\mathscr{U}_q(so(3,1))$ of (the universal enveloping algebra of) $so(3,1)$, respectively $\mathscr{U}_q(so(2,2))$, where q is a certain function of k. For positive cosmological constant, one obtains $\ln(q)\ell \approx \ell_P = \kappa^{-1}$ for small ℓ_P/ℓ [61]. On account of this relation, the limit $\ell \to \infty$ (i.e. $\Lambda \to 0$) maps $\mathscr{U}_q(so(3,1))$ to the κ-Poincaré algebra, and correspondingly for $\mathscr{U}_q(so(2,2))$. An essential ingredient in the derivation of this result is the nontrivial holonomy caused by nontrivial topology of the surface Σ, or "punctures" due to the presence of point particles. We refer to [61, 62] for further details, references, and also arguments toward similar results in the $3+1$-dimensional case, under special conditions. In view of new insights [57] into the quantization of three-dimensional gravity, the above arguments may have to be reconsidered, however.

κ-Minkowski space. This is the Hopf algebra with generators x^μ such that (We note that the nontrivial commutation relation of κ-Minkowski space is formally related to that of Klauder's "affine quantum gravity" (see [63] and references therein). We recall the underlying idea. Starting from the canonical commutation relation $[q,p] = i\hbar I$, and multiplying by q, leads to the "affine commutation relation" $[q,y] = i\hbar q$, where $y = (qp+pq)/2$. The news is now that, in contrast to the canonical commutation relation, the affine commutation relation allows that q is selfadjoint with *positive* spectrum. Promoting q to a spatial metric tensor, this would allow to respect metric positivity.) $[x^i,x^0] = \kappa^{-1}x^i$, $[x^i,x^j] = 0$, and $\Delta(x^\mu) = x^\mu \otimes 1 + 1 \otimes x^\mu$. The action of the momenta P_μ is, in the commutative case, given by the partial derivatives with respect to x^μ. Because of the noncommutativity the rule is now $P_\mu \triangleright :f(x^i,x^0): = :\partial_\mu f(x^i,x^0):$, where $:f(x^i,x^0):$ means "normal ordering": all powers of x^0 to the right. As a consequence of the above commutation relations, any analytic function of x^0, x^i can be expressed as a sum of normal ordered functions. The further action of the κ-Poincaré algebra is given by $M_i \triangleright x^j = \varepsilon_{ijk}x^k$, $M_i \triangleright x^0 = 0$, $N_i \triangleright x^j = -\delta_{ij}x^0$, $N_i \triangleright x^0 = -x^i$, and these definitions extend to the whole algebra via the familiar formula (For example, we have $P_0 \triangleright (fh) = (P_0 \triangleright f)\,h + f\,(P_0 \triangleright h)$ and $P_i \triangleright (fh) = (P_i \triangleright f)\,h + (e^{-P_0/\kappa} \triangleright f)\,(P_i \triangleright h)$.) $Y \triangleright (fh) = \mathbf{m} \circ \Delta(Y) \triangleright (f \otimes h) = (Y_{(1)} \triangleright f)(Y_{(2)} \triangleright h)$ for any element Y of the κ-Poincaré algebra (cf. [51]). (The reader should notice that we use the same symbol Δ for different coproducts). It follows that $(x^0)^2 - \vec{x}^2 + 3x^0/\kappa$ is invariant [51].

ELEMENTS OF CONNES' NCG

In this section we sketch some of the main features and results of Connes' framework of "spectral geometry" (see in particular [64, 4, 65, 66, 67]).

Riemannian geometry in terms of the Dirac operator. Let M be an n-dimensional manifold with a pseudo-Riemannian metric $g = \eta_{ab}\,\theta^a \otimes \theta^b$, where θ^a is an orthonormal coframe field. A Dirac spinor field on M has $\mathbb{C}^{2[n/2]}$-valued components ψ with respect to θ^a. With respect to another orthonormal coframe $\theta^{a\prime} = L^a{}_b\,\theta^b$, related to the first by a function L with values in the orthogonal group (invariance group of η), the components of the spinor field are $\psi' = S(L)\psi$, where S is the representation of the orthogonal group determined by $S(L)^{-1}\gamma^a S(L) = L^a{}_b\gamma^b$ with constant matrices γ^a satisfying the Clifford algebra relation $\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}I$. If $\omega^a{}_b$ are the Levi-Civita connection one-forms w.r. to θ^a, we can introduce the covariant derivative and the Dirac operator

$$D_a\psi\,\theta^a := D\psi := \mathrm{d}\psi + \frac{1}{8}\,\omega_{ab}\,[\gamma^a,\gamma^b]\,\psi\,, \qquad \not{D} := \gamma^a D_a\,. \tag{23}$$

In the Riemannian case (i.e. with a positive definite metric), the space of square-integrable spinor fields on M with the inner product $(\psi,\chi) := \int_M \psi^\dagger\,\chi\,\sqrt{|\det(g_{ab})|}\,\mathrm{d}^n x$ provides us with a Hilbert space $\mathscr{H}$. Since S is double-valued, more care is actually needed to define spinor fields. This leads to the notion of a *spin^c structure* and *spin manifold* (see e.g. [4]).

Connes observed that the geodesic distance on a Riemannian space can be recovered as follows from the Dirac operator (see [4] and references therein). Let M be a compact

(If M is not compact, we should restrict $C^\infty(M)$ to functions which vanish sufficiently fast "at infinity".) spin manifold and g a Riemannian metric. The geodesic distance $d(p,q)$ is then equal to

$$\mathrm{dist}(p,q) := \sup\{|p(f)-q(f)|\ ;\ f\in C^\infty(M),\ \|[\not{D},f]\|\leq 1\}\ . \tag{24}$$

Here we regard the points p,q as pure states, so that $p(f)=f(p)$. This suggests the following generalization:

$$\mathrm{dist}(\phi,\phi') := \sup\{|\phi(a)-\phi'(a)|\ ;\ a\in\mathscr{A},\ \|[\mathscr{D},a]\|\leq 1\}\,, \tag{25}$$

where ϕ,ϕ' are states (A state ϕ of a unital C^*-algebra is a normalized ($\phi(1)=1$) and positive ($\phi(a^*a)\geq 0$) linear functional. It is "pure" if it is not a convex combination of other states. For a commutative algebra, pure states coincide with non-zero characters, i.e. homomorphisms into $\mathbb{C}$.) of an algebra $\mathscr{A}$ of operators on a Hilbert space and $\mathscr{D}$ is a suitable analogue of the Dirac operator. (The expression $[\mathscr{D},a]$ plays the role of a differential da. More generally, one-forms are given by $\sum_i a_i[\mathscr{D},b_i]$ with $a_i,b_i\in\mathscr{A}$). The required structure is introduced next.

Spectral triples. A *spectral triple* $(\mathscr{A},\mathscr{H},\mathscr{D})$ consists of an involutive unital algebra $\mathscr{A}$, represented by bounded operators on a Hilbert space $\mathscr{H}$ (so that the antilinear involution * becomes the adjoint and the norm closure of the algebra is a C^*-algebra), and a selfadjoint operator $\mathscr{D}$ with compact (In order to address the case of a noncompact space, thus a non-unital algebra $\mathscr{A}$, one should require instead that the product of the resolvent with any element of $\mathscr{A}$ is a compact operator [64, 68].) resolvent (hence the spectrum consists of countably many real eigenvalues) and such that $[\mathscr{D},a]$ is a bounded operator for each $a\in\mathscr{A}$.

A spectral triple is called *even* if $\mathscr{H}$ is endowed with a $\mathbb{Z}/2$-grading (This generalizes the chirality operator γ_5 of the "commutative" Dirac geometry in four dimensions.), i.e. an operator γ such that $\gamma=\gamma^*$, $\gamma^2=1$, $[\gamma,a]=0$ for all $a\in\mathscr{A}$, and γ anticommutes with $\mathscr{D}$.

A spectral triple is called *real of KO-dimension* (This is actually rather a signature than a "dimension" [69].) $n\in\mathbb{Z}/8$ if there is an antilinear isometry (analogue of charge conjugation operator) $J:\mathscr{H}\to\mathscr{H}$ satisfying $J^2=\varepsilon$, $J\mathscr{D}=\varepsilon'\mathscr{D}J$, and in the even case additionally $J\gamma=\varepsilon''\gamma J$, where

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

Moreover, $[a,Jb^*J^{-1}]=0$ and $[[\mathscr{D},a],Jb^*J^{-1}]=0$ for all $a,b\in\mathscr{A}$. (The first condition has its origin in Tomita-Takesaki theory (cf. [70]) and the second generalizes the property of the classical Dirac operator to be a first order differential operator).

Any compact Riemannian spin manifold M gives rise to a real spectral triple of KO-dimension $n=\dim(M)$ mod 8 with $\mathscr{A}=C^\infty(M)$. Conversely, given a real spectral triple with a *commutative* unital algebra, such that certain additional conditions hold (which we do not list here), a compact Riemannian spin manifold can be constructed from it [71]. Up to unitary equivalence and spin structure preserving diffeomorphisms, compact Riemannian spin manifolds are in one-to-one correspondence with "commutative" real spectral triples subject to the aforementioned additional conditions.

It should be noticed, however, that a Riemannian space cannot be reconstructed from the knowledge of the *spectrum* of its Dirac operator alone. There are non-isometric compact Riemannian spin manifolds with Dirac operators having the same spectrum.

Spectral action. As an analogue of the Einstein action in terms of the Dirac operator, Connes and Chamseddine [72, 66] proposed the *spectral action*

$$S(\not{D}, m) := \mathrm{Tr} f(\not{D}/m)\,, \tag{26}$$

where m is a parameter such that $\not{D}/m$ is dimensionless, and f a positive even function chosen such that the trace exists. Via the heat kernel expansion method (A good review is [73]. See also [74] for heat kernel expansion of the spectral action on some noncommutative spaces like Moyal plane and noncommutative torus.), in four dimensions ($n = 4$) one obtains [66] (Our convention for the Riemann tensor $R^{\kappa}{}_{\lambda\mu\nu} = \partial_\mu \Gamma^{\kappa}_{\lambda\nu} - \ldots$ differs by a minus sign from that e.g. in [66, 67].)

$$\begin{aligned} S(\not{D}, m) &= \frac{1}{16\pi\mathscr{G}} \int_M (-R + 2\Lambda) \sqrt{\det(g)}\, d^4x \\ &\quad + \frac{f(0)}{10\pi^2} \int_M (\frac{11}{6} L_{GB} - 3 C_{\mu\nu\kappa\lambda} C^{\mu\nu\kappa\lambda}) \sqrt{\det(g)}\, d^4x + \mathscr{O}(m^{-2})\,, \end{aligned} \tag{27}$$

where $\mathscr{G} = \pi/(64 m^2 f_2)$, $\Lambda = 6 m^2 f_4/f_2$, $f_2 := \int_0^\infty v f(v) dv$, $f_4 := \int_0^\infty v^3 f(v) dv$. $L_{GB} = \frac{1}{4} \varepsilon^{\mu\nu\kappa\lambda} \varepsilon_{\alpha\beta\gamma\delta} R^{\alpha\beta}{}_{\mu\nu} R^{\gamma\delta}{}_{\kappa\lambda}$ is the Gauss-Bonnet term (which integrates to the Euler-Poincaré characteristic of M, up to some numerical factor) and $C^{\mu}{}_{\nu\kappa\lambda}$ is the Weyl (conformal) tensor of the metric. See [75] for additional boundary terms appearing in case of a manifold with boundary (with boundary conditions consistent with Hermiticity of the Dirac operator). The field theory action obtained from the spectral action has to be regarded as an *effective* theory valid below the energy scale given by m.

Unification. Kaluza-Klein theory attempted to unify all interactions by attaching an "internal space" to each space-time point, such that its isometries yield the gauge group of the standard model of elementary particle physics. In NCG the internal space should be replaced by an associative algebra $\mathscr{A}_i$ chosen in such a way that its group of (inner) automorphisms (An inner automorphism is determined by an invertible element $u \in \mathscr{A}_i$, which moreover has to be *unitary* ($u^* u = 1 = u u^*$), since an automorphism has to commute with the involution.) coincides with this gauge group and it should possess a representation that reproduces the particle content of the standard model. (An advantage of using associative algebras instead of Lie algebras is the more constrained representation theory [66]. See also section 6.1 in [65].)

$$
\begin{array}{ccccc}
\mathrm{Diff}(M) \cong & \mathrm{Aut}(C^\infty(M)) & \times & \mathrm{Aut}(\mathscr{A}_i) & \cong U(1)\times SU(2)\times SU(3) \\
& \uparrow & & \uparrow & \\
& C^\infty(M) & \otimes & \mathscr{A}_i & \\
& \uparrow & & \uparrow & \\
& M & \times & \text{“internal space”} &
\end{array}
$$

Thus we have to extend the four-dimensional space-time algebra $C^\infty(M)$ to a larger algebra $\mathscr{A} = C^\infty(M)\otimes\mathscr{A}_i$ and find an appropriate real spectral triple, with a generalization $\mathscr{D}$ of the ordinary Dirac operator, such that the spectral action, extended by adding a fermionic part $\frac{1}{2}\langle J\psi, \mathscr{D}\psi\rangle$, reproduces the standard model action up to $\mathscr{O}(m^{-2})$. A good candidate for $\mathscr{A}_i$ is $\mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C})$, where $\mathbb{H}$ are the quaternions and $M_3(\mathbb{C})$ the algebra of complex 3×3 matrices. (This has to be considered as a subalgebra of $\mathbb{C}\oplus\mathbb{H}\oplus\mathbb{H}\oplus M_3(\mathbb{C})$ [66, 67]. See also [76] for a variant). A gauge field corresponding to the inner automorphisms of $\mathscr{A}_i$ can then be introduced by adding to the gravitationally coupled Dirac operator a term of the form $A+\varepsilon' JAJ^{-1}$ (which preserves the condition $J\mathscr{D} = \varepsilon'\mathscr{D}J$) with a self-adjoint one-form $A = \sum_i a_i[\mathscr{D}, b_i]$. Another summand of the generalized Dirac operator corresponds to the fermion mass matrix. It turns out that the standard model coupled to gravity is obtained from a real spectral triple of KO-dimension 6. We refer to [66, 67] for details and predictions. So far the focus is still on an “understanding” of the structure of the standard model of elementary particle physics in terms of (spectral) NCG, and the model had to be adapted [69, 66] to more recent findings of particle physics, like neutrino masses. $C^\infty(M)\otimes\mathscr{A}_i$ may well turn out to be a low energy approximation of some other noncommutative algebra.

Comments. Connes' work includes a deep reformulation of *Riemannian* geometry in terms of “spectral geometry”. In many technical points it is restricted to positive definite metrics and their noncommutative analogues (see in particular [65] for some subtleties arising from the use of the Euclidean signature). Since a “Wick rotation” does not make sense for a general gravitational field, this Euclidean point of view cannot be satisfactory. Though ansätze toward a kind of pseudo-Riemannian version of Connes' spectral Riemannian geometry have been proposed [77, 78, 79], a comparable reformulation of Lorentzian geometry, which after all is the physical one, is still out of sight. In particular, it appears to be impossible to define a Lorentzian analogue of the spectral action. Furthermore, the spectral action corresponds to a *classical* field theory, it it not yet quantized. Parameters of the model are thus still subject to renormalization. See also [80] for a critical account of Connes' NCG.

NONCOMMUTATIVE DIFFERENTIAL GEOMETRY

In classical differential geometry, the most basic geometric structure is given by a differentiable manifold, which is a topological space equipped with a “differential structure”. The latter allows to define vector fields (sections of the tangent bundle), and then differential one-forms are introduced as linear maps acting on vector fields. Since vector fields are derivations of the algebra of smooth functions on the manifold, one can think of generalizing them to derivations of an algebra [81]. Though this works for some interesting examples (see [82] and references therein), there are other algebras which do not admit any nontrivial derivation. (For example, the algebra of functions on a finite set admits only the trivial derivation $\delta = 0$). On the other hand, there is a universal generalization of the notion of differential forms.

Let $\mathscr{A}$ be an associative algebra. A *differential calculus* (Ω, d) over $\mathscr{A}$ consists of an $\mathbb{N}_0$-graded algebra $\Omega = \bigoplus_{r \geq 0} \Omega^r$ with $\mathscr{A}$-bimodules Ω^r, $\Omega^0 = \mathscr{A}$, and a linear map $\mathrm{d} : \Omega^r \to \Omega^{r+1}$ with the properties

$$\mathrm{d}^2 = 0\,, \qquad \mathrm{d}(\alpha\,\beta) = (\mathrm{d}\alpha)\,\beta + (-1)^r\,\alpha\,\mathrm{d}\beta \quad \text{(Leibniz rule)} \tag{28}$$

where $\alpha \in \Omega^r$ and $\beta \in \Omega$.

There are, however, *many* differential calculi associated with a given algebra $\mathscr{A}$, the biggest being the "universal differential envelope". What is their significance? If $\mathscr{A}$ is the algebra of functions on a discrete set M, there is a bijective correspondence between (first order) differential calculi and digraphs on M (so that the elements of M are the vertices of the directed graph) [83]. An arrow from one point to another represents a discrete partial derivative component of the exterior derivative d in this "direction". A special example is the oriented hypercubic lattice digraph underlying lattice gauge theory [17, 84]. Thus, in the case of a discrete set, the choice of a differential calculus determines which points are neighbors. Typically no such interpretation exists in case of a calculus on a noncommutative algebra. The choice of a calculus has to be made according to the application one has in mind. Here are some possibilities to select certain calculi:

- A differential calculus can be defined in terms of a more basic structure. In Connes' NCG this is done via a generalized Dirac operator.
- If the algebra admits symmetries, these can be imposed on the calculus. Examples are bicovariant differential calculi on quantum groups [85, 11].
- Demanding the existence of a "classical basis" θ^i of one-forms: $\theta^i a = a\,\theta^i$ for all $a \in \mathscr{A}$ [82, 86]. In many cases there exists an "almost classical basis": $\theta^i a = \phi_i(a)\,\theta^i$ with automorphisms ϕ_i of $\mathscr{A}$ [87, 88].

As "diffeomorphism group" of the "generalized manifold" $(\mathscr{A}, \Omega)$ we should regard the automorphism group $\mathrm{Aut}(\Omega) \subset \mathrm{Aut}(\mathscr{A})$.

Further geometric notions can be built on top of a differential calculus (and will depend on its choice, of course). In the algebraic language, "fields" on a manifold, or sections of a vector bundle, generalize to elements of a left (or right) $\mathscr{A}$-module $\mathscr{M}$.

A *connection* on $\mathscr{M}$ (here we consider a *left* $\mathscr{A}$-module) is a linear map $\nabla : \mathscr{M} \to \Omega^1 \otimes_{\mathscr{A}} \mathscr{M}$ such that

$$\nabla(f\,\psi) = \mathrm{d}f \otimes_{\mathscr{A}} \psi + f\,\nabla\psi \tag{29}$$

for $f \in \mathscr{A}$ and $\psi \in \mathscr{M}$. It extends to a linear map $\nabla : \Omega \otimes_{\mathscr{A}} \mathscr{M} \to \Omega \otimes_{\mathscr{A}} \mathscr{M}$ via

$$\nabla(\alpha \otimes_{\mathscr{A}} \psi) = \mathrm{d}\alpha \otimes_{\mathscr{A}} \psi + (-1)^r\,\alpha\,\nabla\psi \qquad \alpha \in \Omega^r,\ \psi \in \mathscr{M}\,. \tag{30}$$

The *field strenth*, or *curvature*, of the connection ∇ is the map $\mathscr{R} = -\nabla^2$.

If $\mathscr{M} = \Omega^1$, the connection ∇ is a *linear connection* with *torsion* $\Theta = \mathrm{d} \circ \pi - \pi \circ \mathrm{d} : \Omega \otimes_{\mathscr{A}} \Omega^1 \to \Omega$, where π is the projection $\Omega \otimes_{\mathscr{A}} \Omega^1 \to \Omega$.

These are quite natural and universal definitions. If we had also a suitable concept of a *metric* at hand, we could formulate a generalization of Einstein's equations on a "generalized manifold" $(\mathscr{A}, \Omega)$. In particular this would allow to explore deformations of the classical Einstein equations (in general without a concrete expectation of what we could gain in this way). Among the various ways to introduce mathematically (An interpretation in terms of physical measurements has to follow.) a concept of a metric in NCG are the following.

- In Connes' approach a (generalized) *Riemannian* metric is defined in terms of a (generalized) Dirac operator and the spectral action generalizes the Einstein-Hilbert action. But all this is essentially bound to the Euclidean regime.
- The algebraic approach suggests to define a metric as a map $g : \Omega^1 \otimes_{\mathscr{A}} \Omega^1 \to \mathscr{A}$ (see e.g. [82] for some examples), or an element $g \in \Omega^1 \otimes_{\mathscr{A}} \Omega^1$ (with suitable reality and invertibility properties). We mention now that this is not always appropriate.
- In a formulation of pseudo-Riemannian geometry on discrete sets [89, 90, 91] the correct geometric interpretation requires $g = g_{\mu\nu}\, \mathrm{d}x^\mu \otimes_L \mathrm{d}x^\nu$, where $\otimes_L$ is the *left-linear tensor product*, which satisfies $(f\,\alpha) \otimes_L (h\,\beta) = f\,h\,\alpha \otimes_L \beta$ for $f, h \in \mathscr{A}$ and $\alpha, \beta \in \Omega^1$. (The left-linear tensor product does *not* exist for a *non*commutative algebra $\mathscr{A}$).
- If a differential calculus possesses a "classical basis" θ^i (see above), one may postulate it to be orthonormal and introduce in this way a metric $g = \eta_{ij}\, \theta^i \otimes_{\mathscr{A}} \theta^j$ (see e.g. [86]). Note that g has the left-linearity property in this basis.

In particular, a Lorentzian signature can be implemented. We refer to the references cited above for further details of special approaches (and further obstacles to build a deformation or noncommutative analogue of Einstein's theory).

If $\mathscr{A}$ is a deformation of a commutative algebra, say the algebra of (smooth) functions on $\mathbb{R}^n$, there may exist differential calculi over $\mathscr{A}$ which do *not* tend to the classical calculus of differential forms when the deformation vanishes (see [92, 93] for an example). In the following subsection we consider a class of such "noncommutative differential calculi" on $\mathbb{R}^n$ and show how a metric can emerge from it.

A class of noncommutative differential calculi on $\mathbb{R}^n$

Let $\mathscr{A}$ be the algebra of functions generated by commuting objects x^μ, $\mu = 1, \ldots, n$, e.g. coordinate functions on $\mathbb{R}^n$. A class of differential calculi is then determined by

$$[\mathrm{d}x^\mu, x^\nu] = \ell\, C^{\mu\nu}{}_\kappa\, \mathrm{d}x^\kappa\,, \tag{31}$$

where ℓ is a constant with dimension of length and $C^{\mu\nu}{}_\kappa$ are dimensionless functions of the coordinates, which have to satisfy the conditions $C^{\mu\nu}{}_\kappa = C^{\nu\mu}{}_\kappa$ and $C^{\mu\kappa}{}_\lambda\, C^{\nu\lambda}{}_\kappa = C^{\nu\kappa}{}_\lambda\, C^{\mu\lambda}{}_\kappa$ [84, 94, 95]. (In terms of the matrices C^μ with entries $(C^\mu)^\nu{}_\kappa = C^{\mu\nu}{}_\kappa$, the last condition means that they have to commute. The two conditions imply that $x^\mu \bullet x^\nu := C^{\mu\nu}{}_\kappa x^\kappa$ determines a commutative and associative product. Such algebras play a role in a description of topological field theories as lattice models [96, 97]). Thinking of a space-time model, a natural candidate for ℓ would be the Planck length ℓ_P. The above deformation of the classical differential calculus then modifies the kinematical structure of space-time at the Planck scale.

We assume that $\{\mathrm{d}x^\mu\}$ is a basis of Ω^1 as a left- and as a right $\mathscr{A}$-module. Generalized partial (left- and right-) derivatives can then be introduced via

$$\mathrm{d}f = (\partial_{+\mu} f)\,\mathrm{d}x^\mu = \mathrm{d}x^\mu\,(\partial_{-\mu} f)\,. \tag{32}$$

The concrete form of the generalized partial derivatives depends on the structure functions $C^{\mu\nu}{}_\kappa$. A "coordinate transformation" (diffeomorphism) should now be an invertible map $x^\mu \mapsto x^{\mu'}(x^\nu)$ with the property that $\partial_{+\nu} x^{\mu'}$ is invertible. This allows to generalize the notions of manifold and tensors. We find

$$\begin{aligned} [\mathrm{d}x^{\mu'}, x^{\nu'}] &= \partial_{+\kappa} x^{\mu'}\,[\mathrm{d}x^\kappa, x^{\nu'}] = \partial_{+\kappa} x^{\mu'}\,[\mathrm{d}x^{\nu'}, x^\kappa] \\ &= \partial_{+\kappa} x^{\mu'}\,\partial_{+\lambda} x^{\nu'}\,[\mathrm{d}x^\lambda, x^\kappa] = \ell\,\partial_{+\kappa} x^{\mu'}\,\partial_{+\lambda} x^{\nu'}\,C^{\kappa\lambda}{}_\sigma\,\mathrm{d}x^\sigma\,, \end{aligned} \tag{33}$$

using the commutativity of $\mathscr{A}$ and the derivation property of d. Hence $C^{\mu'\nu'}{}_{\kappa'} = \partial_{+\kappa} x^{\mu'}\,\partial_{+\lambda} x^{\nu'}\,C^{\kappa\lambda}{}_\sigma\,\partial_{+\kappa'} x^\sigma$. As a consequence, (This can be written as $g^{\mu\nu} = \mathrm{tr}(C^\mu C^\nu)$.)

$$g^{\mu\nu} := C^{\mu\kappa}{}_\lambda\,C^{\lambda\nu}{}_\kappa \tag{34}$$

(see also [97]) is symmetric and obeys the tensor transformation law $g^{\mu'\nu'} = \partial_{+\kappa} x^{\mu'}\,\partial_{+\lambda} x^{\nu'}\,g^{\kappa\lambda}$. If an inverse $g_{\mu\nu}$ exists (This is the case iff the algebra determined by the $C^{\mu\nu}{}_\kappa$ is semi-simple [97].), then it is also a tensor, i.e. $g_{\mu'\nu'} = \partial_{+\mu'} x^\kappa\,\partial_{+\nu'} x^\lambda\,g_{\kappa\lambda}$.

Example 1. If there are coordinates such that $C^{\mu\nu}{}_\kappa = \delta^\mu_\kappa\,\delta^\nu_\kappa$, then $[\mathrm{d}x^\mu, x^\nu] = \ell\,\delta^{\mu\nu}\,\mathrm{d}x^\nu$, which is the hypercubic lattice differential calculus [17, 84]. In this case the generalized partial derivatives are the left/right discrete derivatives on a lattice with lattice spacing ℓ, and we have the Euclidean metric $g_{\mu\nu} = \delta_{\mu\nu}$, as expected. (With a slight modification one obtains the Minkowski metric).

Example 2. Let $\gamma^{\mu\nu}$ be components of a symmetric tensor field and $\tau = \tau_\mu \mathrm{d}x^\mu$ a one-form on a manifold, such that $\gamma^{\mu\nu}\tau_\nu = 0$ ("generalized Galilei structure"). Then (31) with $C^{\mu\nu}{}_\kappa = \gamma^{\mu\nu}\tau_\kappa$ is invariant under general coordinate transformations and thus extends to the whole manifold. In this case the tensor (34) vanishes. We refer to [98, 92, 93, 99] for appearances of this structure, which in particular makes contact with stochastic calculus on manifolds. See also [100] for related work.

In the limit $\ell \to 0$, where the differential calculus (31) becomes the "classical" one, we should expect that the generalized partial derivatives become ordinary partial derivatives (when acting on smooth functions). In this limit the metric decouples from the differential structure, whereas for $\ell \neq 0$ it is a property of the differential calculus.

FINAL REMARKS

We have briefly reviewed a variety of ideas about "noncommutative space-time" and some ansätze toward corresponding generalizations of General Relativity. Among the most interesting developments is certainly the reformulation of the whole standard model of elementary particle physics including gravity in a concise NCG language by Connes and his disciples. This primarily aims at a better understanding of the quite complicated structure of the standard model. Since elementary particle physics is what tells us about the small scale structure of space-time, this is a promising route toward a deeper unification of space-time, particles and forces, though the lack of a Lorentzian version still presents a serious obstacle.

In NCG a machinery similar to that of quantum physics is already introduced at a "classical" level. So there has to be another, apparently completely different level which introduces a similar machinery on top of the first. This appears to be a major complication and hardly satisfactory. We should rather hope that either both quantizations can be merged to a single one, or one induces the other automatically.

Needless to say, there are many more interesting ideas and facts in the "noncommutative world" related to the notions of space-time and gravity than we touched upon in this short review. For some of them, in particular related to matrix models, we refer to [28].

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