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A model for superconducting thin films having variable thickness

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A two-dimensional macroscopic model for superconductivity in thin films having variable thickness is derived through an averaging process across the film thickness. The resulting model is similar to the well-known Ginzburg–Landau equations for homogeneous, isotropic materials, except that a function that describes the variations in the thickness of the film now appears in the coefficients of the differential equations. Some results about solutions of the variable thickness model are then given, including existence of solutions and boundedness of the order parameter. It is also shown that the model is consistent in the sense that solutions obtained from the new model are an appropriate limit of a sequence of averages of solutions of the three-dimensional Ginzburg–Landau model as the thickness of the film tends to zero. An application of the variable thickness thin film model to flux pinning is then provided. In particular, the results of numerical calculations are given that show that the vortex-like structures that are present in certain superconductors are attracted to relatively thin regions in a material sample. Finally, extensions of the model to other settings are discussed.

1. Introduction

Type-II superconductors, which include the recently discovered high-temperature superconductors, are characterized by the appearance of vortex-like structures. These structures are set in motion by a variety of mechanisms, including thermal fluctuations and applied voltages and currents. Unfortunately, such vortex motion induces an effective resistance in the material, and thus a loss of superconductivity. For this reason, one is interested in studying mechanisms that can pin the vortices at a fixed location, i.e., prevent their motion. Various such mechanisms have been advanced by physicists, engineers, and material scientists. For example, normal (non-superconducting) impurities in an otherwise superconducting material sample are believed to provide sites at which vortices are pinned. Likewise, regions of the sample that are thin relative to other regions are also believed to provide pinning sites. The latter mechanism, in the setting of thin films, is the subject of this paper. (Pinning by normal impurities is considered in ref. [3].)

The context of our study is the phenomenological model due to Ginzburg and Landau [9] for superconductivity in three-dimensional, isotropic, homogeneous material samples. Thin films of superconducting material are often modeled as two-dimensional objects. The third dimension, i.e., that across the film, is eliminated by an averaging procedure. If the material, viewed as a three-dimensional object, is homogeneous, and the thickness of the film is invariant with position, then the result of the

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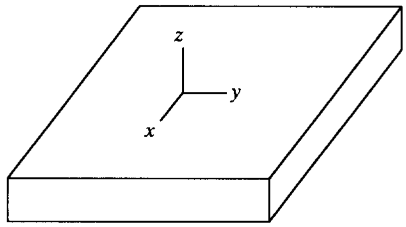

Fig. 1. A thin film of constant thickness.

averaging process will be a two-dimensional model having constant material properties. For example, consider a film such as that sketched in fig. 1 having material properties independent of the spatial coordinates x, y, and z. The extent of the film in the z-direction, i.e., its thickness, is assumed to be much smaller than its extent in the x- and y-directions. After averaging, in some appropriate sense, over z one obtains a model having x and y for independent variables and having material properties that are independent of x and y. This is exactly the type of situation addressed by the great majority of the analyses and approximations extant in the literature.

Now consider a thin film of variable thickness such as that depicted in fig. 2. (In this figure we have suppressed the second coordinate y in the plane of the film.) One would like to develop a two-dimensional model that can account for thickness variations. Any such model would result from some sort of averaging process across the film, i.e., in the z-direction. This averaging process will vary from point-to-point in the plane of the film, and introduce the variable thickness into the coefficients of the resulting two-dimensional model.

The first important consideration of this paper is the derivation of such a two-dimensional, variable thickness, thin film model. This is done in section 3 which follows the brief discussion of section 2 of the Ginzburg–Landau model for superconductivity in three-dimensional, homogeneous, isotropic material samples. Then, in section 4 we give some analytical results concerning the variable thickness model, focusing especially on the question of the consistency of the model. In section 5, we give the results of numerical calculations that show that the vortex-like structures present in type-II superconductors are pinned by relatively thin regions in a material sample. Finally, in section 6, we briefly discuss the extension of the new model to the periodic setting, and simplifications to the model that can be effected in case the Ginzburg–landau parameter is large.

We close this section by introducing some notation that will be used below. Throughout, for any non-negative integer k and domain $\mathscr{D} \subset \mathbb{R}^n$, $n = 2$ or 3, $H^k(\mathscr{D})$ will denote the Sobolev space of real-valued functions having square integrable derivatives of order up to k. The corresponding spaces of complex-valued functions will be denoted by $\mathscr{H}^k(\mathscr{D})$. Corresponding spaces of vector-valued functions, each of whose components belong to $H^k(\mathscr{D})$, will be denoted by $\boldsymbol{H}^k(\mathscr{D})$, i.e., $\boldsymbol{H}^k(\mathscr{D}) = [H^k(\mathscr{D})]^n$. Norms of functions belonging to $H^k(\mathscr{D})$, $\boldsymbol{H}^k(\mathscr{D})$, and $\mathscr{H}^k(\mathscr{D})$ will all be denoted, without any possible

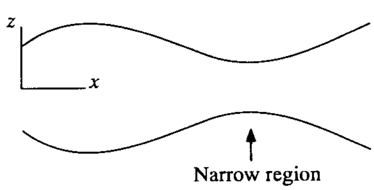

Fig. 2. A thin film of variable thickness.

ambiguity, by $\|\cdot\|_{k,\mathscr{D}}$ or $\|\cdot\|_k$. The latter notation will be used when there is no chance of confusion. For details concerning these spaces, one may consult [2]. We will also use the subspaces

$$\boldsymbol{H}_n^1(\mathscr{D}) = \{\boldsymbol{Q} \in \boldsymbol{H}^1(\mathscr{D}) : \boldsymbol{Q}\cdot\boldsymbol{n} = 0 \text{ on } \partial\mathscr{D}\}\,,$$

where $\partial\mathscr{D}$ denotes the boundary of $\mathscr{D}$, and

$$\boldsymbol{H}_n^1(\mathrm{div};\mathscr{D}) = \{\boldsymbol{Q} \in \boldsymbol{H}_n^1(\mathscr{D}) : \mathrm{div}\, Q = 0 \text{ in } \mathscr{D}\}\,.$$

2. The Ginzburg–Landau model for superconductivity

We begin with a short review of the well-known Ginzburg–Landau model for superconductivity. For details concerning the material of this section, one may consult refs. [1,5–7,10–13].

2.1. The Ginzburg–Landau free energy

Ginzburg and Landau postulated [9] that the Gibbs free energy of a sample of superconducting material is given by

$$\mathscr{G}(\psi, \boldsymbol{A}) = \int_\Omega (f_n + \alpha|\psi|^2 + \tfrac{1}{2}\beta|\psi|^4)\,\mathrm{d}\Omega + \int_\Omega \left[\frac{1}{2m_s}\left|\left(\mathrm{i}\hbar\nabla + \frac{e_s\boldsymbol{A}}{c}\right)\psi\right|^2 + \frac{|\boldsymbol{h}|^2}{8\pi} - \frac{\boldsymbol{h}\cdot\boldsymbol{H}}{4\pi}\right]\mathrm{d}\Omega\,. \tag{2.1}$$

Here, the constant f_n is the free energy of the normal (non-superconducting) state in the absence of magnetic fields, ψ is the (complex-valued) order parameter, $\boldsymbol{A}$ is the magnetic potential, $\boldsymbol{h} = \mathrm{curl}\,\boldsymbol{A}$ is the magnetic field, α and β are constants (with respect to the space variable $\boldsymbol{x}$) whose values depend on the temperature, c is the speed of light, e_s and m_s are the charge and mass, respectively, of the superconducting charge-carriers, and $2\pi\hbar$ is Planck's constant. $\boldsymbol{H}$ denotes the applied field, which throughout this paper will be assumed to be a constant vector, and $\Omega \subset \mathbb{R}^3$ denotes the region occupied by the superconducting sample. We will later make additional assumptions about the domain Ω and its boundary.

The basic thermodynamic postulate of the Ginzburg–Landau theory of superconductivity is that the superconducting sample is in a state such that its Gibbs free energy is a minimum.

The coefficient α in (2.1) changes sign at the critical temperature T_c, with $\alpha < 0$ for $T < T_c$. If the temperature of the sample is lower than T_c, the sample is in the superconducting state; if $T > T_c$, then the sample is in the normal, i.e., non-superconducting, state.

There are two important length scales associated with changes in the order parameter and the magnetic field. These are the *coherence length*

$$\xi = \left(-\frac{\hbar^2}{2m_s\alpha}\right)^{1/2},$$

and the *penetration depth*

$$\lambda = \left(-\frac{\beta m_s c^2}{4\pi\alpha e_s^2}\right)^{1/2},$$

which measure distances over which the order parameter and the magnetic field, respectively, undergo appreciable change. The non-dimensional ratio $\kappa = \lambda/\xi$ is known as the *Ginzburg–Landau parameter*.

2.2. The Ginzburg–Landau equations and boundary conditions

The minimization of $\mathscr{G}$ with respect to variations in ψ and A yields the celebrated Ginzburg-Landau equations

$$\frac{1}{2m_s}\left(\mathrm{i}\hbar\nabla + \frac{e_s A}{c}\right)^2 \psi + \alpha\psi + \beta|\psi|^2\psi = 0 \quad \text{in } \Omega \tag{2.2}$$

and

$$\operatorname{curl}\operatorname{curl} A + \frac{2\pi\mathrm{i}e_s\hbar}{m_s c}(\psi^*\nabla\psi - \psi\nabla\psi^*) + \frac{4\pi e_s^2}{m_s c^2}|\psi|^2 A = \mathbf{0} \quad \text{in } \Omega\,, \tag{2.3}$$

where $(\cdot)^*$ denotes the complex conjugate.

Candidate minimizers of $\mathscr{G}$ are not a priori constrained to satisfy any boundary conditions. Let Γ denote the boundary of Ω and $\boldsymbol{n}$ the unit outer normal vector to Γ. Then, the minimization process also yields the natural boundary conditions

$$\left(\mathrm{i}\hbar\nabla\psi + \frac{e_s}{c}A\psi\right)\cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma \tag{2.4}$$

and

$$\operatorname{curl} A \times \boldsymbol{n} = \boldsymbol{H} \times \boldsymbol{n} \quad \text{on } \Gamma\,. \tag{2.5}$$

More general boundary conditions may also be considered.

We remark that the boundary condition (2.5) implies that the magnetic field in the region exterior to the superconductor is unaffected by the presence of the superconductor, and, in that region, is everywhere equal to the magnetic field at infinity. (In general, one should include in the energy functional the magnetic energy in the region external to the superconductor.) Thus, strictly speaking, our model applies to regimes for which the change in the field in the external region due to the presence of the superconductor is small. One such setting is in the limit of high values of κ; see section 6.

2.3. Non-dimensionalized form

We now introduce the usual non-dimensionalizations. Lengths are non-dimensionalized by λ, magnetic fields by $\sqrt{2}H_c$, where $H_c = \sqrt{4\pi\alpha^2/\beta}$, the magnetic potential by $\sqrt{2}\lambda H_c$, the order parameter by $\sqrt{-\alpha/\beta}$, free energy densities by α^2/β, and the Gibbs free energy by $\alpha^2\lambda^3/\beta$. Denoting the non-dimensionalized variables by the same symbols used for the corresponding dimensional ones, we have that the non-dimensional Gibbs free energy is given by

$$\mathscr{G}(\psi, A) = \int_\Omega \left(f_n - |\psi|^2 + \tfrac{1}{2}|\psi|^4 + \left|\left(\frac{\mathrm{i}}{\kappa}\nabla + A\right)\psi\right|^2 + |\boldsymbol{h}|^2 - 2\boldsymbol{h}\cdot\boldsymbol{H}\right)\mathrm{d}\Omega\,. \tag{2.6}$$

The Ginzburg–Landau functional has an important property, namely, gauge invariance. To be specific, for any $\phi \in H^2(\Omega)$, let the linear transformation G_ϕ be defined by

$$G_\phi(\psi, A) = (\psi\, e^{i\kappa\phi}, A + \nabla\phi) \quad \forall(\psi, A) \in \mathcal{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) .$$

Then, we have:

Definition 2.1. (ψ, A) and (ζ, Q) are said to be gauge equivalent if and only if there exists a function $\phi \in H^2(\Omega)$ such that $(\psi, A) = G_\phi(\zeta, Q)$.

Proposition 2.2. For all $\phi \in H^2(\Omega)$ and $(\psi, A) \in \mathcal{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega)$, $\mathcal{G}(\psi, A) = \mathcal{G}(G_\phi(\psi, A))$, i.e., the Gibbs free energy $\mathcal{G}$ is invariant under the gauge transformation G_ϕ.

Lemma 2.3. Any $(\zeta, Q) \in \mathcal{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ is gauge equivalent to an element of $\mathcal{H}^1(\Omega) \times \boldsymbol{H}^1_n(\mathrm{div};\Omega)$.

For proofs of the above results, and for those for the other results of this section, one may consult ref. [7]. We note that by seeking minimizers belonging to $\mathcal{H}^1(\Omega) \times \boldsymbol{H}^1_n(\mathrm{div};\Omega)$, one is working in the Coulomb gauge.

For convenience, the functional (2.6) is often modified into

$$\mathcal{F}(\psi, A) = \int_\Omega \left(\tfrac{1}{2}(|\psi|^2 - 1)^2 + \left|\left(\frac{i}{\kappa}\nabla + A\right)\psi\right|^2\right) d\Omega + \int_\Omega (|\mathbf{curl}\, A - H|^2 + |\mathrm{div}\, A|^2)\, d\Omega . \tag{2.7}$$

One may then show that:

Theorem 2.4. $\mathcal{G}$ has a least one minimizer belonging to $\mathcal{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ and $\mathcal{F}$ has a least one minimizer belonging to $\mathcal{H}^1(\Omega) \times \boldsymbol{H}^1_n(\Omega)$. Moreover,

$$\min_{\mathcal{H}^1(\Omega)\times \boldsymbol{H}^1(\Omega)} \mathcal{G} = \min_{\mathcal{H}^1(\Omega)\times \boldsymbol{H}^1_n(\Omega)} \mathcal{F} = \min_{\mathcal{H}^1(\Omega)\times \boldsymbol{H}^1_n(\mathrm{div};\Omega)} \mathcal{G} .$$

Thus, the inclusion of the additional last term in $\mathcal{F}$ allows one to look for minimizers in a space of vector potentials that is not required to be solenoidal. This, for example, can greatly simplify numerical calculations.

In the Coulomb gauge, the non-dimensional Ginzburg–Landau equations and boundary conditions are given by

$$\left(\frac{i}{\kappa}\nabla + A\right)^2 \psi - \psi + |\psi|^2\psi = 0 \quad \text{in } \Omega , \tag{2.8}$$

$$\mathrm{curl}\,\mathrm{curl}\, A = -\frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) - |\psi|^2 A \quad \text{in } \Omega , \tag{2.9}$$

$$\mathrm{div}\, A = 0 \quad \text{in } \Omega , \tag{2.10}$$

$$\left(\frac{i}{\kappa}\nabla\psi + A\psi\right)\cdot n = 0 \quad \text{on } \Gamma , \tag{2.11}$$

$$\mathrm{curl}\, A \times n = H \times n \quad \text{on } \Gamma , \tag{2.12}$$

and

$$A \cdot n = 0 \quad \text{on } \Gamma . \tag{2.13}$$

Solutions of the (non-dimensional) Ginzburg–Landau equations satisfy the following maximum principle.

Proposition 2.5. If (ψ, A) is a solution of the Ginzburg–Landau equations (2.8)–(2.13), then $|\psi| \le 1$ almost everywhere.

3. The model for thin films having variable thickness

Starting with the Ginzburg–Landau model for superconductivity in three-dimensional, isotropic, homogeneous material samples, we now derive a two-dimensional model for thin films of variable thickness. We still assume that the thin film, viewed as a three-dimensional object, is made up of isotropic and homogeneous material. Thus, in the dimensional form of the Gibbs free energy (2.1), the parameters α, β, m_s, c, and $\hbar$ are all scalar-valued constants. In the non-dimensionalized forms (2.6) and (2.7), the parameter κ is likewise a scalar-valued constant.

3.1. Derivation of the model

We consider the case of a three-dimensional thin film that is symmetric with respect to the (x,y)-plane. (Non-symmetric thickness variations can be treated in an analogous manner.) Thus, the material sample Ω_ϵ is described by

$$\Omega_\epsilon = \{(x,y,z) \in \mathbb{R}^3 \colon (x,y) \in \Omega_0 \subseteq \mathbb{R}^2,\ z \in (-\epsilon a(x,y), \epsilon a(x,y))\}\ ,$$

where ϵ is a small parameter, $a(x,y)$ is assumed to be smooth, and $a(x,y) \ge a_0 > 0$ for all $(x,y) \in \Omega_0$. Here, we are interested in the Ginzburg–Landau functional (2.7), defined on Ω_ϵ, and its minimizers as $\epsilon \to 0$.

We assume that the constant applied field is directed along the z-axis, i.e., in a direction perpendicular to the symmetry plane of the film. Thus we have that $\boldsymbol{H} = (0,0,H)$, where $H = \text{constant}$.

First, we use averaging techniques to find appropriate limits of various relevant quantities. Let $\tilde{\boldsymbol{V}}$ and $\hat{\boldsymbol{V}}$ denote the projections of a three-dimensional vector $\boldsymbol{V} \in \mathbb{R}^3$ onto the (x,y)-plane and the z-axis so that $\tilde{\boldsymbol{V}} \cdot \hat{\boldsymbol{V}} = 0$ and $\boldsymbol{V} = \tilde{\boldsymbol{V}} + \hat{\boldsymbol{V}}$. For the domain $\Omega_\epsilon \subset \mathbb{R}^3$, define

$$\mathcal{F}_\epsilon(\psi, \boldsymbol{A}) = \int_{\Omega_\epsilon} \left(\tfrac{1}{2}(|\psi|^2 - 1)^2 + \left| \left(\frac{\mathrm{i}}{\kappa} \nabla + \boldsymbol{A} \right) \psi \right|^2 \right) \mathrm{d}\Omega + \int_{\Omega_\epsilon} (|\mathbf{curl}\, \boldsymbol{A} - \boldsymbol{H}|^2 + |\operatorname{div} \boldsymbol{A}|^2)\, \mathrm{d}\Omega\ . \tag{3.1}$$

Clearly, for any $\epsilon > 0$, Theorem 2.4 and proposition 2.5 imply that $\mathcal{F}_\epsilon$ has a minimizer $(\psi_\epsilon, \boldsymbol{A}_\epsilon) \in \mathcal{H}^1(\Omega_\epsilon) \times \boldsymbol{H}^1_n(\Omega_\epsilon)$ and that $|\psi_\epsilon| \le 1$ almost everywhere. We also have the following uniform estimates.

Lemma 3.1. Let $(\psi_\epsilon, \boldsymbol{A}_\epsilon)$ be a minimizer of the functional $\mathcal{F}_\epsilon$ given by (3.1). Then, there exists a constant $C > 0$ such that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \|\psi_\epsilon\|^2_{1,\Omega_\epsilon} \le C \tag{3.2}$$

and

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} \|A_\epsilon\|^2_{1,\Omega_\epsilon} \le C . \tag{3.3}$$

Proof. Let $(\phi, \tilde{Q}) \in \mathcal{H}^1(\Omega_0) \times H^1_n(\Omega_0)$. Then, we have that $(\phi, (\tilde{Q}, (z/a)\, \tilde{Q}\cdot\nabla a)) \in \mathcal{H}^1(\Omega_\epsilon) \times H^1_n(\Omega_\epsilon)$ so that

$$\mathcal{F}_\epsilon(\psi_\epsilon, A_\epsilon) \le \mathcal{F}_\epsilon\Big(\phi, \Big(\tilde{Q}, \frac{z}{a}\tilde{Q}\cdot\nabla a\Big)\Big) \le c\epsilon ,$$

for some constant c, depending on $(\phi, \tilde{Q})$. Then, the estimates (3.2) and (3.3) follow easily from this inequality. □

Formally, let us write

$$\psi(x, y, z) = \sum_{j=0} \psi_j(x, y) z^j \tag{3.4}$$

and

$$A(x, y, z) = \sum_{j=0} A_j(z, y) z^j \tag{3.5}$$

for z small, i.e., ϵ small.

For $A \in H^1_n(\Omega_\epsilon)$, we have on $\partial\Omega_0$

$$\tilde{A}_j \cdot \tilde{n} = 0 \quad \text{for } j = 0, 1, \dots ,$$

where $\tilde{n}$ is the unit normal vector of Ω_0. At $z = \epsilon a(x, y)$ and $z = -\epsilon a(x, y)$, we have

$$\hat{A}_0 = 0 \tag{3.6}$$

and

$$a\hat{A}_j = \tilde{A}_{j-1}\cdot\nabla a \quad \text{for } j = 1, 2, \dots . \tag{3.7}$$

Therefore, for all $|z| \le \epsilon a(x, y)$,

$$\operatorname{div} A = \sum_{j=0} (\operatorname{div} \tilde{A}_j + (j+1)\hat{A}_{j+1}) z^j = \operatorname{div}\tilde{A}_0 + \hat{A}_1 + \mathcal{O}(\epsilon) = \operatorname{div}\tilde{A}_0 + a^{-1}\nabla a\cdot\tilde{A}_0 + \mathcal{O}(\epsilon) ,$$

where we have used (3.7). Similarly, using (3.6), we have

$$\mathbf{curl}\, A = (\{(\nabla\hat{A}_0 - \tilde{A}_1)\times k\}, \operatorname{curl}\tilde{A}_0) + \mathcal{O}(\epsilon) = (\{-\tilde{A}_1 \times k\}, \operatorname{curl}\tilde{A}_0) + \mathcal{O}(\epsilon) ,$$

where k denotes the unit vector in the z-direction. Also,

$$\Big(\frac{\mathrm{i}}{\kappa}\nabla + A\Big)\psi = \Big(\Big(\frac{\mathrm{i}}{\kappa}\tilde{\nabla} + \tilde{A}_0\Big)\psi_0, \frac{\mathrm{i}}{\kappa}\psi_1\Big) + \mathcal{O}(\epsilon)$$

and

$$\tfrac{1}{2}(|\psi|^2 - 1)^2 = \tfrac{1}{2}(|\psi_0|^2 - 1)^2 + \mathcal{O}(\epsilon) .$$

Therefore, if

$$\mathcal{F}_0(\psi, A) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \mathcal{F}_\epsilon(\psi, A) , \tag{3.8}$$

we have that

$$\begin{aligned}\mathcal{F}_0(\psi, A) &= \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \mathcal{F}_\epsilon(\psi, A) \\ &= \int_{\Omega_0} a\Big(\tfrac{1}{2}(|\psi_0|^2 - 1)^2 + \Big|\Big(\frac{\mathrm{i}}{\kappa}\tilde{\nabla} + \tilde{A}_0\Big)\psi_0\Big|^2 + \frac{1}{\kappa^2}|\psi_1|^2\Big)\,\mathrm{d}\Omega \\ &\quad + \int_{\Omega_0} a(|\mathrm{curl}\,\tilde{A}_0 - H|^2 + |\tilde{A}_1|^2 + a^{-2}|\mathrm{div}\,a\tilde{A}_0|^2)\,\mathrm{d}\Omega .\end{aligned}$$

This motivates us to define the functional

$$\begin{aligned}\tilde{\mathcal{F}}_0(\psi_0, \tilde{A}_0) = &\int_{\Omega_0} a\Big(\tfrac{1}{2}(|\psi_0|^2 - 1)^2 + \Big|\Big(\frac{\mathrm{i}}{\kappa}\tilde{\nabla} + \tilde{A}_0\Big)\psi_0\Big|^2\Big)\,\mathrm{d}\Omega \\ &+ \int_{\Omega_0} \Big(a|\mathrm{curl}\,\tilde{A}_0 - H|^2 + a^{-1}|\mathrm{div}\,a\tilde{A}_0|^2\Big)\,\mathrm{d}\Omega .\end{aligned} \tag{3.9}$$

Remark. In the ansatz (3.4) and (3.5), it has been assumed that the coefficients ψ_j and A_j are independent of ϵ. Seemingly, there is no a priori justification for this assumption, its validity being only justified a posteriori by showing that it results in a consistent model; see section 4.1. However, if we do assume, for example, that

$$A_j(x, y; \epsilon) = \sum_{k=0} A_{jk}\epsilon^k$$

and a similar expansion for ψ_j, the only crucial change to the above development is that (3.6) and (3.7) for $j = 1$ are now given by

$$\hat{A}_{00} = 0 \quad \text{and} \quad a\hat{A}_{10} + \hat{A}_{01} = \tilde{A}_{00} \cdot \nabla a ,$$

respectively. We then have that

$$\mathrm{div}\,A = a^{-1}\,\mathrm{div}(a\tilde{A}_{00}) - a^{-1}\hat{A}_{01} + \mathcal{O}(\epsilon)$$

and the last term of the functional $\mathcal{F}_0$ changes accordingly. Then, when minimizing $\mathcal{F}_0$, we could choose $\hat{A}_{01} = \mathrm{div}(a\tilde{A}_{00})$. Then, instead of $\tilde{\mathcal{F}}_0$, we would be motivated to define the functional

$$\mathcal{G}_0(\psi_{00}, \tilde{A}_{00}) = \int_{\Omega_0} a\Big(\tfrac{1}{2}(|\psi_{00}|^2 - 1) + \Big|\Big(\frac{\mathrm{i}}{\kappa}\tilde{\nabla} + \tilde{A}_{00}\Big)\psi_{00}\Big|^2\Big)\,\mathrm{d}\Omega + \int_{\Omega_0} a|\mathrm{curl}\,\tilde{A}_{00} - H|^2\,\mathrm{d}\Omega .$$

Note that this functional is gauge invariant under the transformation

$$\psi_{00} \to \psi_{00}\,\mathrm{e}^{\mathrm{i}\kappa\chi} \quad \text{and} \quad \tilde{A}_{00} \to \tilde{A}_{00} + \nabla\chi$$

for any χ such that $\nabla\chi \cdot \boldsymbol{n} = 0$. Finally, we can fix the gauge by defining the functional

$$\tilde{\mathscr{F}}_0 = \mathscr{G}_0 + \int_{\Omega_0} a^{-1}|\operatorname{div} a\tilde{A}_{00}|^2 \, d\Omega ,$$

which is exactly the functional of (3.9). Thus, if we allow for the coefficients in (3.5) to have a power series dependence on ϵ, we can still arrive at the functional (3.9) and thus the subsequent development given below will remain unchanged.

Now, define ρ to be the positive square root of the function a, i.e.,

$$\rho(x, y) = \sqrt{a(x, y)} \quad \forall (x, y) \in \Omega_0 .$$

Similarly, define σ by

$$\sigma(x, y) = \sqrt{a^{-1}(x, y)} \quad \forall (x, y) \in \Omega_0 .$$

In order to show that $\tilde{\mathscr{F}}_0$ has a minimizer, we first prove the following result.

Lemma 3.2. There exists a constant $c > 0$ such that

$$\|\rho \operatorname{curl} \boldsymbol{A}\| + \|\sigma \operatorname{div}(a\boldsymbol{A})\| \geq c\|\boldsymbol{A}\|_1 \quad \forall \boldsymbol{A} \in \boldsymbol{H}_n^1(\Omega_0) .$$

Proof. Suppose the result is false. Then, we may find a sequence $\{\boldsymbol{A}_k\}$ such that $\|\boldsymbol{A}_k\|_1 = 1$ and

$$\operatorname{curl} \boldsymbol{A}_k \to \boldsymbol{0} \quad \text{and} \quad \operatorname{div}(a\boldsymbol{A}_k) \to 0 \quad \text{strongly in } L^2 .$$

Since $\boldsymbol{H}_n^1(\Omega_0)$ is weakly compact, without loss of generality, we may assume that

$$\boldsymbol{A}_k \to \boldsymbol{A}^* \quad \text{weakly in } \boldsymbol{H}_n^1(\Omega_0) .$$

Hence,

$$\operatorname{curl} \boldsymbol{A}^* = \boldsymbol{0} \quad \text{and} \quad \operatorname{div}(a\boldsymbol{A}^*) = 0 .$$

Since $a(x, y) \geq a_0 > 0$, one easily see that $\boldsymbol{A}^* = \nabla p$ where p is the solution of the second order equation

$$\operatorname{div}(a\nabla p) = 0 ,$$

with a homogeneous Neumann boundary condition. Therefore, p is a constant function. Then, $\boldsymbol{A}^* = \boldsymbol{0}$. By compact imbedding results, we may assume

$$\boldsymbol{A}_k \to \boldsymbol{A}^* \quad \text{strongly in } L^2(\Omega_0) .$$

Since $\boldsymbol{A}^* = \boldsymbol{0}$, we have

$$\nabla a \cdot \boldsymbol{A}_k \to 0 \quad \text{strongly in } L^2(\Omega_0) ,$$

which in turn implies that

$$\operatorname{div} \boldsymbol{A}_k \to 0 \quad \text{strongly in } L^2 .$$

The well-known Korn inequality then yields

$$A_k \to \mathbf{0} \quad \text{strongly in } \boldsymbol{H}_n^1(\Omega_0)\,.$$

But, $\|A_k\|_1 = 1$. This contradiction proves the lemma. □

With the above lemmas, we can use standard variational arguments to prove the following existence result.

Theorem 3.3. $\tilde{\mathscr{F}}_0$ has a minimizer $(\psi_0, \tilde{A}_0)$ in $\mathscr{H}^1(\Omega_0) \times \boldsymbol{H}_n^1(\Omega_0)$. Moreover, every minimizing sequence of $\tilde{\mathscr{F}}_0$ in $\mathscr{H}^1(\Omega_0) \times \boldsymbol{H}_n^1(\Omega_0)$ has a subsequence which converges strongly to a minimizer of $\tilde{\mathscr{F}}_0$ in $\mathscr{H}^1(\Omega_0) \times \boldsymbol{H}_n^1(\Omega_0)$.

3.2. The differential equations for the model

One may then derive the Euler–Lagrange equation for the minimizer of $\tilde{\mathscr{F}}_0$ in $\mathscr{H}^1(\Omega_0) \times \boldsymbol{H}_n^1(\Omega_0)$. These take an appearance similar to the Ginzburg–Landau equations, but with variable coefficients. Specifically, one obtains the differential equations

$$\left(\frac{\mathrm{i}}{\kappa}\tilde{\nabla} + \tilde{A}_0\right) \cdot a\left(\frac{\mathrm{i}}{\kappa}\tilde{\nabla} + \tilde{A}_0\right)\psi_0 + a(|\psi_0|^2 - 1)\psi_0 = 0 \quad \text{in } \Omega_0 \tag{3.10}$$

and

$$\begin{aligned}&\operatorname{curl}(a \operatorname{curl} \tilde{A}_0) + a\tilde{\nabla}(\operatorname{div} \tilde{A}_0) + a\tilde{\nabla}(a^{-1}\tilde{\nabla}a \cdot \tilde{A}_0) \\ &\quad = -\frac{\mathrm{i}}{2\kappa} a(\psi_0^* \tilde{\nabla}\psi_0 - \psi_0 \tilde{\nabla}\psi_0^*) - a|\psi_0|^2 \tilde{A}_0 + \operatorname{curl}(aH) \quad \text{in } \Omega_0\,,\end{aligned} \tag{3.11}$$

and the boundary conditions

$$\left(\frac{\mathrm{i}}{\kappa}\tilde{\nabla}\psi_0 + \tilde{A}_0\psi_0\right) \cdot \tilde{n} = 0 \quad \text{on } \Gamma_0 \tag{3.12}$$

and

$$\operatorname{curl} \tilde{A}_0 = H \quad \text{on } \Gamma_0\,. \tag{3.13}$$

From the definition (3.9) of $\tilde{\mathscr{F}}_0$, we have that the solution is in the gauge

$$\tilde{A}_0 \cdot \tilde{n} = 0 \quad \text{on } \Gamma_0 \tag{3.14}$$

and

$$\operatorname{div}(a\tilde{A}_0) = 0 \quad \text{in } \Omega_0\,. \tag{3.15}$$

With the substitution of (3.15) into (3.11), the latter simplifies to

$$\operatorname{curl}(a \operatorname{curl} \tilde{A}_0) = -\frac{\mathrm{i}}{2\kappa} a(\psi_0^* \tilde{\nabla}\psi_0 - \psi_0 \tilde{\nabla}\psi_0^*) - a|\psi_0|^2 \tilde{A}_0 + \operatorname{curl}(aH) \quad \text{in } \Omega_0\,. \tag{3.16}$$

Note that if one sets a = constant, i.e., the film is of constant thickness, then (3.10)–(3.15) reduce to (2.8)–(2.13). Also, note that the system (3.10), (3.12), (3.13), and (3.16) could have been derived directly without the a priori imposition of the gauge (3.14), (3.15). In particular, (3.10), (3.12), (3.13), and (3.16) result from the minimization of the functional

$$\mathcal{G}_0(\psi_0, \tilde{A}_0) = \int_{\Omega_0} a\left(\tfrac{1}{2}(|\psi_0|^2 - 1)^2 + \left|\left(\frac{\mathrm{i}}{\kappa}\tilde{\nabla} + \tilde{A}_0\right)\psi_0\right|^2\right) \mathrm{d}\Omega + \int_{\Omega_0} a|\mathrm{curl}\,\tilde{A}_0 - H|^2\,\mathrm{d}\Omega$$

over $\mathcal{H}^1(\Omega_0) \times \boldsymbol{H}^1(\Omega_0)$.

4. Some properties of the model

We first give two results about minimizers of $\tilde{\mathcal{F}}_0$ and of solutions of the variable coefficient system (3.10)–(3.15). The proofs of these results are essentially the same as for the analogous results for the constant coefficient Ginzburg–Landau model; the latter proofs may be found in ref. [7].

The first result is similar to proposition 2.5.

Proposition 4.1. If $(\psi_0, \tilde{A}_0)$ is a solution of the variable thickness thin film equations (3.10)–(3.15), then $|\psi_0| \leq 1$ almost everywhere.

The second result states that solutions of the variable thickness thin film equations cannot correspond to a local maximizer of the functional $\tilde{\mathcal{F}}_0$.

Proposition 4.2. A solution of the variable thickness thin film equations (3.10)–(3.15) cannot be a local maximizer of the functional $\tilde{\mathcal{F}}_0$ given in (3.9).

4.1. Consistency of the model

We now study the consistency of the thin film model, i.e., we show that, in an appropriate sense, a minimizer $(\psi_0, \tilde{A}_0)$ of $\tilde{\mathcal{F}}_0$ is the limit of the sequence of minimizers $\{(\psi_\epsilon, A_\epsilon)\}$ of $\mathcal{F}_\epsilon$ as $\epsilon \to 0$.

For given $\epsilon > 0$, let

$$\bar{\psi}_\epsilon(x, y) = \tfrac{1}{2} \int_{-1}^{1} \psi_\epsilon(x, y, \epsilon a t)\,\mathrm{d}t = \frac{1}{2\epsilon a} \int_{-\epsilon a}^{\epsilon a} \psi_\epsilon(x, y, z)\,\mathrm{d}z\,, \quad \forall (x, y) \in \Omega_0\,, \tag{4.1}$$

and

$$\bar{A}_\epsilon(x, y) = \tfrac{1}{2} \int_{-1}^{1} A_\epsilon(x, y, \epsilon a t)\,\mathrm{d}t = \frac{1}{2\epsilon a} \int_{-\epsilon a}^{\epsilon a} A_\epsilon(x, y, z)\,\mathrm{d}z\,, \quad \forall (x, y) \in \Omega_0\,. \tag{4.2}$$

The consistency result depends on the following preliminary results.

Lemma 4.3.

$$\mathcal{F}_\epsilon(\psi_\epsilon, A_\epsilon) \leq \mathcal{F}_\epsilon\left(\psi_0, \left(\tilde{A}_0, \frac{z}{a}\tilde{A}_0 \cdot \nabla a\right)\right) \tag{4.3}$$

and

$$\tilde{\mathcal{F}}_0(\psi_0, \tilde{A}_0) \leq \tilde{\mathcal{F}}_0(\bar{\psi}_\epsilon, \tilde{\bar{A}}_\epsilon)\,, \quad \forall \epsilon > 0\,. \tag{4.4}$$

Proof. Note that $(\tilde{A}_0, (z/a)\,\tilde{A}_0\cdot\nabla a)\in H^1_n(\Omega_\epsilon)$ and $\tilde{\tilde{A}}_\epsilon\in H^1_n(\Omega_0)$, so by the definitions of the minimizers $(\psi_\epsilon, A_\epsilon)$ of $\mathcal{F}_\epsilon$ and $(\psi_0, \tilde{A}_0)$ of $\tilde{\mathcal{F}}_0$, we have (4.3) and (4.4), respectively. □

Lemma 4.4.

$$\mathcal{F}_0\left(\psi_0, \left(\tilde{A}_0, \frac{z}{a}\tilde{A}_0\cdot\nabla a\right)\right) = \tilde{\mathcal{F}}_0(\psi_0, \tilde{A}_0)\,. \tag{4.5}$$

Proof. We have that

$$\begin{aligned}\mathcal{F}_0\left(\psi_0, \left(\tilde{A}_0, \frac{z}{a}\tilde{A}_0\cdot\nabla a\right)\right) &= \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\mathcal{F}_\epsilon\left(\psi_0, \left(\tilde{A}_0, \frac{z}{a}\tilde{A}_0\cdot\nabla a\right)\right) = \tilde{\mathcal{F}}_0(\psi_0, \tilde{A}_0)\\ &\quad + \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{\Omega_\epsilon}\left(\frac{z}{a}\right)^2(|\tilde{A}_0\cdot\nabla a|^2|\psi_0|^2 + |\nabla(\tilde{A}_0\cdot\nabla a)|^2)\,\mathrm{d}\Omega\\ &= \tilde{\mathcal{F}}_0(\psi_0, \tilde{A}_0) + \left(\lim_{\epsilon\to 0}\frac{1}{3\epsilon}\epsilon^3\right)\int_{\Omega_0}[|\tilde{A}_0\cdot\nabla a|^2|\psi_0|^2 + |\nabla(\tilde{A}_0\cdot\nabla a)|^2]\,\mathrm{d}\Omega = \tilde{\mathcal{F}}_0(\psi_0, \tilde{A}_0).\end{aligned}$$ □

Lemma 4.5.

$$\lim_{\epsilon\to 0}\left\{\frac{1}{2\epsilon}\mathcal{F}_\epsilon(\psi_\epsilon, A_\epsilon) - \tilde{\mathcal{F}}_0(\bar{\psi}_\epsilon, \tilde{\tilde{A}}_\epsilon)\right\}\geq 0\,. \tag{4.6}$$

Proof. We prove (4.6) by comparing the two functionals term by term. Throughout, C will denote a positive constant whose value may change with context. First, we have

$$\begin{aligned}\int_{\Omega_0} a^{-1}(|\mathrm{div}(a\tilde{\tilde{A}}_\epsilon)|^2)\,\mathrm{d}\Omega &= \int_{\Omega_0} a^{-1}\left[\frac{1}{4\epsilon^2}\left|\mathrm{div}\left(\int_{-\epsilon a}^{\epsilon a}\tilde{A}_\epsilon(x,y,z)\,\mathrm{d}z\right)\right|^2\right]\mathrm{d}\Omega\\ &= \int_{\Omega_0} a^{-1}\left(\frac{1}{4\epsilon^2}\left|\int_{-\epsilon a}^{\epsilon a}\mathrm{div}\,\tilde{A}_\epsilon(x,y,z)\,\mathrm{d}z + \epsilon\tilde{A}_\epsilon\Big|_{-\epsilon a}^{\epsilon a}\cdot\tilde{\nabla}a\right|^2\right)\mathrm{d}\Omega\\ &= \int_{\Omega_0} a^{-1}\left(\frac{1}{4\epsilon^2}\left|\int_{-\epsilon a}^{\epsilon a}\mathrm{div}\,\tilde{A}_\epsilon\,\mathrm{d}z + \hat{A}_\epsilon\Big|_{-\epsilon a}^{\epsilon a}\right|^2\right)\mathrm{d}\Omega\\ &= \int_{\Omega_0} a^{-1}b\left(\frac{1}{4\epsilon^2}\left|\int_{-\epsilon a}^{\epsilon a}\mathrm{div}\,A_\epsilon\,\mathrm{d}z\right|^2\right)\mathrm{d}\Omega \leq \frac{1}{2\epsilon}\int_{\Omega_\epsilon}|\mathrm{div}\,A_\epsilon|^2\,\mathrm{d}\Omega\,.\end{aligned}$$

Next, we have

$$\begin{aligned}&\left|\int_{\Omega_0}\tfrac{1}{2}a(|\bar{\psi}_\epsilon|^2-1)^2\,\mathrm{d}\Omega - \frac{1}{2\epsilon}\int_{\Omega_\epsilon}\tfrac{1}{2}(|\psi_\epsilon|^2-1)^2\,\mathrm{d}\Omega\right|\\ &\quad= \frac{1}{4\epsilon}\left|\int_{\Omega_\epsilon}(|\bar{\psi}_\epsilon|^2-1)^2 - (|\psi_\epsilon|^2-1)^2\,\mathrm{d}\Omega\right| \leq \frac{C}{4\epsilon}\int_{\Omega_\epsilon}|\bar{\psi}_\epsilon - \psi_\epsilon|\,\mathrm{d}\Omega\\ &\quad\leq \frac{C}{4\epsilon}\int_{\Omega_\epsilon}\frac{1}{2\epsilon a}\left|\int_{-\epsilon a}^{\epsilon a}\int_{z'}^{z}\frac{\partial\psi_\epsilon}{\partial s}(x,y,s)\,\mathrm{d}z'\right|\mathrm{d}\Omega \leq C\int_{\Omega_\epsilon}\left|\frac{\partial\psi_\epsilon}{\partial z}\right|\mathrm{d}\Omega \leq C\epsilon\,.\end{aligned}$$

Next,

$$\int_{\Omega_0} a|\operatorname{curl}\tilde{\tilde{A}}_\epsilon - H|^2\,\mathrm{d}\Omega = \tfrac{1}{4}\int_{\Omega_0} a\left|\operatorname{curl}\Big(\int_{-1}^{1}\tilde{A}_\epsilon(x,y,\epsilon at)\,\mathrm{d}t\Big) - 2H\right|^2\mathrm{d}\Omega$$

$$= \frac{1}{4\epsilon^2}\int_{\Omega_0} a^{-1}\left|\int_{-\epsilon a}^{\epsilon a}\big((\widehat{\mathbf{curl}}\,A_\epsilon)(x,y,z) + \epsilon\nabla a\times\tilde{A}_{\epsilon z}(x,y,z) - H\big)\,\mathrm{d}z\right|^2\mathrm{d}\Omega$$

$$= \frac{1}{4\epsilon^2}\int_{\Omega_0} a^{-1}\left|\int_{-\epsilon a}^{\epsilon a}(\widehat{\mathbf{curl}}\,A_\epsilon + Q - H)\,\mathrm{d}z\right|^2\mathrm{d}\Omega$$

$$\le \frac{1}{4\epsilon^2}\int_{\Omega_0} a^{-1}\left|\int_{-\epsilon a}^{\epsilon a}(\widehat{\mathbf{curl}}\,A_\epsilon - H)\,\mathrm{d}z\right|^2\mathrm{d}\Omega + \frac{1}{4\epsilon^2}\int_{\Omega_0} a^{-1}\left|\int_{-\epsilon a}^{\epsilon a} Q\,\mathrm{d}z\right|^2\mathrm{d}\Omega$$

$$+ \frac{1}{2\epsilon^2}\left[\int_{\Omega_0} a^{-1}\left|\int_{-\epsilon a}^{\epsilon a}(\widehat{\mathbf{curl}}\,A\epsilon - H)\,\mathrm{d}z\right|^2\mathrm{d}\Omega\right]^{1/2}\cdot\left[\int_{\Omega_0} a^{-1}\left|\int_{-\epsilon a}^{\epsilon a} Q\,\mathrm{d}z\right|^2\mathrm{d}\Omega\right]^{1/2}$$

$$\le \frac{1}{2\epsilon}\int_{\Omega_\epsilon}|\widehat{\mathbf{curl}}\,A_\epsilon - H|^2\,\mathrm{d}\Omega + \frac{1}{2\epsilon}\int_{\Omega_\epsilon}|Q|^2\,\mathrm{d}\Omega + \frac{1}{\epsilon}\left[\int_{\Omega_\epsilon}|\widehat{\mathbf{curl}}\,A_\epsilon - H|^2\,\mathrm{d}\Omega\right]^{1/2}\cdot\left[\int_{\Omega_\epsilon}|Q|^2\,\mathrm{d}\Omega\right]^{1/2},$$

where $\tilde{A}_{\epsilon z} = \partial\tilde{A}_\epsilon/\partial z$ and $Q = \epsilon\nabla a\times\tilde{A}_{\epsilon z}$. Then, since $(1/2\epsilon)\int_{\Omega_\epsilon}|Q|^2\,\mathrm{d}\Omega \le C\epsilon^2$, we have

$$\lim_{\epsilon\to 0}\Big(\int_{\Omega_0} a|\mathbf{curl}\,\tilde{\tilde{A}}_\epsilon - H|^2\,\mathrm{d}\Omega - \frac{1}{2\epsilon}\int_{\Omega_\epsilon}|\mathbf{curl}\,A_\epsilon - H|^2\,\mathrm{d}\Omega\Big\} \le 0\,.$$

Finally, we have

$$\int_{\Omega_0} a\left|\Big(\frac{\mathrm{i}}{\kappa}\tilde{\nabla} + \tilde{\tilde{A}}_\epsilon\Big)\bar{\psi}_\epsilon\right|^2\mathrm{d}\Omega \le \tfrac{1}{4}\int_{\Omega_0} a\left|\Big(\frac{\mathrm{i}}{\kappa}\tilde{\nabla} + \tilde{\tilde{A}}_\epsilon\Big)\int_{-1}^{1}\psi_\epsilon(x,y,\epsilon at)\,\mathrm{d}t\right|^2\mathrm{d}\Omega$$

$$= \frac{1}{4}\int_{\Omega_0} a\left|\int_{-1}^{1}\Big(\Big(\frac{\mathrm{i}}{\kappa}\tilde{\nabla} + \tilde{\tilde{A}}_\epsilon\Big)\psi_\epsilon(x,y,\epsilon at) + \frac{\mathrm{i}}{\kappa}\epsilon\nabla a\cdot\frac{\partial\psi_\epsilon}{\partial z}(x,y,\epsilon at)\Big)\,\mathrm{d}t\right|^2\mathrm{d}\Omega$$

$$= \frac{1}{4\epsilon^2}\int_{\Omega_0} a^{-1}\left|\int_{-\epsilon a}^{\epsilon a}\Big(\Big(\frac{\mathrm{i}}{\kappa}\tilde{\nabla} + \tilde{\tilde{A}}_\epsilon\Big)\psi_\epsilon + \frac{\mathrm{i}}{\kappa}\epsilon\frac{\partial\psi_\epsilon}{\partial z}\nabla a\Big)\,\mathrm{d}z\right|^2\mathrm{d}\Omega$$

$$= \frac{1}{4\epsilon^2}\int_{\Omega_0} a^{-1}\left|\int_{-\epsilon a}^{\epsilon a}\Big(\Big(\frac{\mathrm{i}}{\kappa}\tilde{\nabla} + \tilde{A}_\epsilon\Big)\psi_\epsilon + (\tilde{\tilde{A}}_\epsilon - \tilde{A}_\epsilon)\psi_\epsilon + \frac{\mathrm{i}}{\kappa}\epsilon\frac{\partial\psi_\epsilon}{\partial z}\nabla a\Big)\,\mathrm{d}z\right|^2\mathrm{d}\Omega$$

$$= \frac{1}{4\epsilon^2}\int_{\Omega_0} a^{-1}\left|\int_{-\epsilon a}^{\epsilon a}\Big(\Big(\frac{\mathrm{i}}{\kappa}\tilde{\nabla} + \tilde{A}_\epsilon\Big)\psi_\epsilon + Q'\Big)\,\mathrm{d}z\right|^2\mathrm{d}\Omega$$

$$\le \frac{1}{4\epsilon^2}\int_{\Omega_0} a^{-1}\left|\int_{-\epsilon a}^{\epsilon a}\Big(\frac{\mathrm{i}}{\kappa}\tilde{\nabla} + \tilde{A}_\epsilon\Big)\psi_\epsilon\,\mathrm{d}z\right|^2\mathrm{d}\Omega + \frac{1}{4\epsilon^2}\int_{\Omega_0} a^{-1}\left|\int_{-\epsilon a}^{\epsilon a} Q'\,\mathrm{d}z\right|^2\mathrm{d}\Omega$$

$$+\frac{1}{2\epsilon^2}\left[\int_{\Omega_0} a^{-1}\left|\int_{-\epsilon a}^{\epsilon a}\left(\frac{\mathrm{i}}{\kappa}\tilde{\nabla}+\tilde{\boldsymbol{A}}_\epsilon\right)\psi_\epsilon\,\mathrm{d}z\right|^2\mathrm{d}\Omega\right]^{1/2}\cdot\left[\int_{\Omega_0} a^{-1}\left|\int_{-\epsilon a}^{\epsilon a}\boldsymbol{Q}'\,\mathrm{d}z\right|^2\mathrm{d}\Omega\right]^{1/2}$$

$$\leq\frac{1}{2\epsilon}\int_{\Omega_\epsilon}\left|\left(\frac{\mathrm{i}}{\kappa}\tilde{\nabla}+\tilde{\boldsymbol{A}}_\epsilon\right)\psi\epsilon\right|^2\mathrm{d}\Omega+\frac{1}{2\epsilon}\int_{\Omega_\epsilon}|\boldsymbol{Q}'|^2\,\mathrm{d}\Omega$$

$$+\frac{1}{\epsilon}\left[\int_{\Omega_\epsilon}\left|\left(\frac{\mathrm{i}}{\kappa}\tilde{\nabla}+\tilde{\boldsymbol{A}}_\epsilon\right)\psi\epsilon\right|^2\mathrm{d}\Omega\right]^{1/2}\cdot\left[\int_{\Omega_\epsilon}|\boldsymbol{Q}'|^2\,\mathrm{d}\Omega\right]^{1/2},$$

where $\boldsymbol{Q}'=(\tilde{\tilde{\boldsymbol{A}}}_\epsilon-\tilde{\boldsymbol{A}}_\epsilon)\psi_\epsilon-(\mathrm{i}/\kappa)\epsilon(\partial\psi_\epsilon/\partial z)\nabla a$. Then, since

$$|\boldsymbol{Q}'|^2\leq 2|\tilde{\tilde{\boldsymbol{A}}}_\epsilon-\tilde{\boldsymbol{A}}_\epsilon|^2+2\left|\frac{1}{\kappa}\epsilon\frac{\partial\psi_\epsilon}{\partial z}\nabla a\right|^2\leq 2|\tilde{\tilde{\boldsymbol{A}}}_\epsilon-\tilde{\boldsymbol{A}}_\epsilon|^2+C\epsilon^2\left|\frac{\partial\psi_\epsilon}{\partial z}\right|^2$$

and

$$\int_{\Omega_\epsilon}|\tilde{\tilde{\boldsymbol{A}}}_\epsilon-\tilde{\boldsymbol{A}}_\epsilon|^2\,\mathrm{d}\Omega\leq\int_{\Omega_\epsilon}\left|\frac{1}{2\epsilon a}\int_{-\epsilon a}^{\epsilon a}(\tilde{\boldsymbol{A}}_\epsilon(x,y,z')-\tilde{\boldsymbol{A}}_\epsilon(x,y,z))\,\mathrm{d}z'\right|^2\mathrm{d}\Omega$$

$$=\int_{\Omega_0}\int_{-\epsilon a}^{\epsilon a}\left|\frac{1}{2\epsilon a}\int_{-\epsilon a}^{\epsilon a}\int_z^{z'}\frac{\partial\tilde{\boldsymbol{A}}_\epsilon}{\partial s}(x,y,s)\,\mathrm{d}s\,\mathrm{d}z'\right|^1\mathrm{d}z\,\mathrm{d}\Omega$$

$$\leq\int_{\Omega_0}4\epsilon^2a^2\int_{-\epsilon a}^{\epsilon a}\left|\frac{\partial\tilde{\boldsymbol{A}}_\epsilon}{\partial s}(x,y,s)\right|^2\mathrm{d}s\,\mathrm{d}\Omega\leq C\epsilon^2\int_{\Omega_\epsilon}\left|\frac{\partial\tilde{\boldsymbol{A}}_\epsilon}{\partial s}(x,y,s)\right|^2\mathrm{d}\Omega\,,$$

we have

$$\int_{\Omega_0}a\left|\left(\frac{\mathrm{i}}{\kappa}\tilde{\nabla}+\tilde{\tilde{\boldsymbol{A}}}_\epsilon\right)\bar{\psi}_\epsilon\right|^2\mathrm{d}\Omega-\frac{1}{2\epsilon}\int_{\Omega_\epsilon}\left|\left(\frac{\mathrm{i}}{\kappa}\tilde{\nabla}+\tilde{\boldsymbol{A}}_\epsilon\right)\psi_\epsilon\right|^2\mathrm{d}\Omega\leq C\epsilon+C\epsilon^2\,.$$

Therefore,

$$\lim_{\epsilon\to 0}\left(\int_{\Omega_0}a\left|\left(\frac{\mathrm{i}}{\kappa}\tilde{\nabla}+\tilde{\tilde{\boldsymbol{A}}}_\epsilon\right)\bar{\psi}_\epsilon\right|^2\mathrm{d}\Omega-\frac{1}{2\epsilon}\int_{\Omega_\epsilon}\left|\left(\frac{\mathrm{i}}{\kappa}\nabla+\boldsymbol{A}_\epsilon\right)\psi_\epsilon\right|^2\Omega\right)$$

$$\leq\lim_{\epsilon\to 0}\left(\int_{\Omega_0}a\left|\left(\frac{\mathrm{i}}{\kappa}\tilde{\nabla}+\tilde{\tilde{\boldsymbol{A}}}_\epsilon\right)\bar{\psi}_\epsilon\right|^2\mathrm{d}\Omega-\frac{1}{2\epsilon}\int_{\Omega_\epsilon}\left|\left(\frac{\mathrm{i}}{\kappa}\tilde{\nabla}+\tilde{\boldsymbol{A}}_\epsilon\right)\psi_\epsilon\right|^2\mathrm{d}\Omega\right)\leq 0\,.$$

Collecting the above results yields (4.6). □

Lemma 4.6.

$$\lim_{\epsilon\to 0}\tilde{\mathscr{F}}_0(\bar{\psi}_\epsilon,\tilde{\tilde{\boldsymbol{A}}}_\epsilon)=\tilde{\mathscr{F}}_0(\psi_0,\tilde{\boldsymbol{A}}_0)\,. \tag{4.7}$$

Proof. From (4.3), we have

$$\tilde{\tilde{\mathscr{F}}}_0(\bar{\psi}_\epsilon, \tilde{\bar{A}}_\epsilon) - \frac{1}{2\epsilon}\, \mathscr{F}_\epsilon(\psi_\epsilon, A_\epsilon) \tilde{\mathscr{F}}_0(\bar{\psi}_\epsilon, \tilde{\bar{A}}_\epsilon) - \frac{1}{2\epsilon}\, \mathscr{F}_\epsilon\Big(\psi_0, \Big(\tilde{A}_0, \frac{z}{a}\tilde{A}_0 \cdot \nabla a\Big)\Big)$$

so that (4.6), (3.8), and (4.5) then yield

$$0 \geq \lim_{\epsilon \to 0} \tilde{\mathscr{F}}_0(\bar{\psi}_\epsilon, \tilde{\bar{A}}_\epsilon) - \mathscr{F}_0\Big(\psi_0, \Big(\tilde{A}_0, \frac{z}{a}\tilde{A}_0 \cdot \nabla a\Big)\Big) = \lim_{\epsilon \to 0} \tilde{\mathscr{F}}_0(\bar{\psi}_\epsilon, \tilde{\bar{A}}_\epsilon) - \tilde{\mathscr{F}}_0(\psi_0, \tilde{A}_0)\,.$$

This last result and (4.4) then yield (4.7). □

Now we are able to give the consistency result for solutions of the variable thickness thin film model.

Theorem 4.7. Let the functionals $\mathscr{F}_\epsilon$ and $\tilde{\mathscr{F}}_0$ be given by (3.1) and (3.9), respectively. For any $\epsilon > 0$, let $(\psi_\epsilon, A_\epsilon)$ denote a minimizer of $\mathscr{F}_\epsilon$. Let the averages $\bar{\psi}_\epsilon$ and $\bar{A}_\epsilon$ be defined by (4.1) and (4.2), respectively. Then, the sequence $\{(\bar{\psi}_\epsilon, \tilde{\bar{A}}_\epsilon)\}$, $\epsilon \to 0$, converges strongly to a minimizer $(\psi_0, \tilde{A}_0)$ of $\tilde{\mathscr{F}}_0$ in $\mathscr{H}_1(\Omega_0) \times H^1_n(\Omega_0)$.

Proof. The result follows easily from lemma 4.6 and theorem 3.3. □

This result shows that the average across the film of the solution of the isotropic, homogeneous, three-dimensional Ginzburg–Landau equations in the thin film converges, as the thickness parameter ϵ goes to zero, to the solution of the two-dimensional variable thickness thin film model we have derived.

5. An application of the model to flux pinning

We now consider some numerical computations obtained using the model presented in section 3. One of the purposes of these experiments is to show that the model is effective in simulating the pinning of vortices by narrow regions in the film. A more detailed study of this phenomena will be reported on elsewhere.

Approximate solutions are obtained by the finite element method used piecewise continuous biquadratic polynomials based on a subdivision of Ω_0 into a quadrilateral grid. For the particular set of experiments reported on here, we take, in terms of nondimensional variables, $\Omega_0 = (0,1)^2$, i.e., the unique square, $\kappa = 3$, and $H = 1.5$. Various grid sizes where used to ensure that the numerical simulations were properly converged. For the results below, Ω_0 is subdivided into a uniform grid having 12 intervals in each direction, i.e., the grid size is 1/12. For this value of the grid size, the numerical simulation was found to be converged to at least graphical accuracy.

For the first simulation, we solve the original Ginzburg–landau equations for a constant thickness thin film, i.e., we have $a(x, y) = 1$ for all $(x, y) \in (0, 1)^2$. The magnitude of the order parameter is given in the left-hand picture of fig. 3. Four vortices are formed having symmetry with respect to $x = 0.5$ and $y = 0.5$.

In the second simulation, the equations with variable thickness are solved. A narrow region is created by setting $a(x, y) = 0.2$ for $(x, y) \in (0.6, 0.9)^2$, while $a(x, y) = 1$ everywhere else. The magnitude of the order parameter is given in the right-hand picture of fig. 3. There are still four vortices present. However, one of them is now *pinned* at the narrow region, and the rest of the vortices realigned

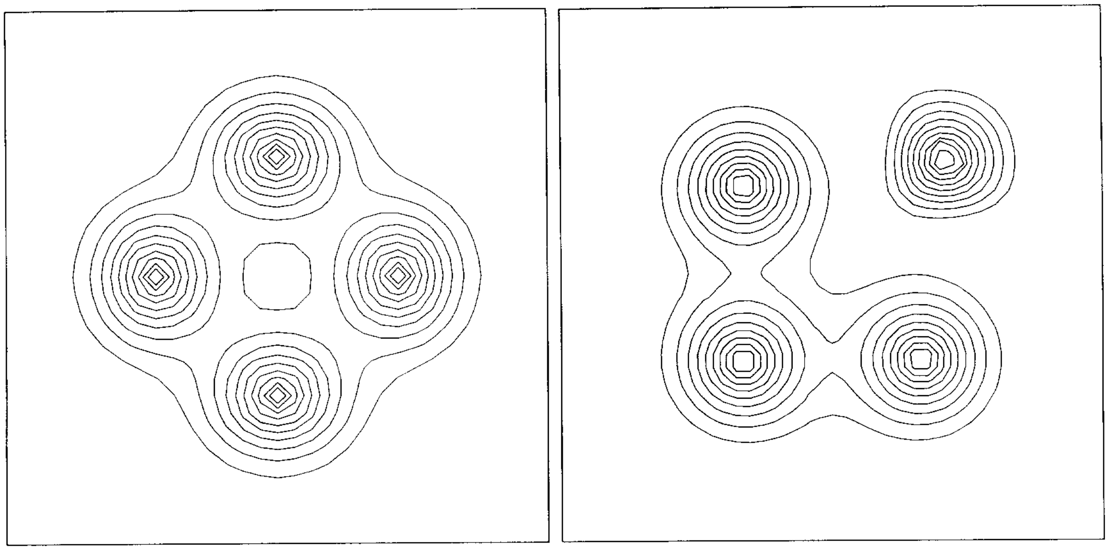

Fig. 3. Contour plots of the magnitude of the order parameter. The left-hand figure is for a constant thickness thin film. The right-hand figures is for a thin film having a narrow region near the upper right corner of the sample.

themself in order that the energy is minimized. Symmetry is now with respect to the diagonal $x = y$ instead of $x = 0.5$ or $y = 0.5$.

The numerical results given in fig. 3 indicate that the model developed in section 3 can indeed be used to simulate flux pinning phenomena in variable thickness thin films.

6. Concluding remarks

The vortex-like structures, such as those illustrated in fig. 3, that are present for type-II superconductors, i.e., those for which $\kappa > 1/\sqrt{2}$, are very closely spaced relative to the linear dimensions of most material samples. (The sample of fig. 3 has sides of length equal to the penetration depth λ which can be as small as a few hundred Ångstroms.) Thus, in most samples, there are a huge number of vortices present. For this reason, there has been much interest in periodic Ginzburg–Landau models wherein the physical properties of the material and of the resultant electromagnetic phenomena are periodic with respect to a given lattice. The variable thickness thin film model presented in section 3 may be easily extended to the periodic case. We now briefly consider this setting. Suppose the thickness function a has a fine periodic structure with respect to a two-dimensional lattice, i.e.,

$$a(\boldsymbol{x} + \boldsymbol{t}_k) = a(\boldsymbol{x}) \quad \text{for } k = 1, 2 \quad \text{and} \quad \forall \boldsymbol{x} \in \mathbb{R}^2 ,$$

where $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ are two vectors that define the lattice geometry. Then, one may expect a similar periodic structure of the physical variables such as the magnetic field, the current, and the density of superconducting charge carriers. In such cases, the procedures of section 3 can be applied to obtain a modified Ginzburg–landau model with *quasi-periodic* boundary conditions similar to the ones discussed in ref. [8]. In fact, the partial differential equations obtained are identical to (3.10) and (3.11); the boundary conditions (3.12) and (3.13) are replaced by quasi-periodic boundary conditions identical to those of ref. [8]. These modified equations may be used to study the structures of the vortex state in the periodic setting.

Another setting of interest is that of large κ and high applied field $H = \kappa H_0$. In this case, one may obtain a simplified model for a thin film having variable thickness using procedures similar to those employed in ref. [4]. If one replaces (x, y) by $(x/\kappa, y/\kappa)$, i.e., if one scales the spatial coordinates by κ, one obtains the simplified model, valid for large κ,

$$(\mathrm{i}\tilde{\nabla} + \tilde{\boldsymbol{A}}_0) \cdot a(\mathrm{i}\tilde{\nabla} + \tilde{\boldsymbol{A}}_0)\psi + a(|\psi|^2 - 1)\psi = 0 \quad \text{in } \Omega_0 \tag{6.1}$$

along with the boundary condition

$$\tilde{\nabla}\psi \cdot \tilde{\boldsymbol{n}} = 0 \quad \text{on } \Gamma_0 . \tag{6.2}$$

Here, the vector potential $\tilde{\boldsymbol{A}}_0$ is determined from

$$\operatorname{div}(a\tilde{\boldsymbol{A}}_0) = 0 \quad \text{in } \Omega_0 , \tag{6.3}$$

$$\operatorname{curl} \tilde{\boldsymbol{A}}_0 = H_0 \quad \text{in } \Omega_0 ,$$

and

$$\tilde{\boldsymbol{A}}_0 \cdot \tilde{\boldsymbol{n}} = 0 \quad \text{on } \Gamma_0 . \tag{6.4}$$

Note that $\tilde{\boldsymbol{A}}_0$ may be determined from (6.3)–(6.5) independently of ψ_0; once $\tilde{\boldsymbol{A}}_0$ is determined, (6.1) and (6.2) may be used to determine ψ_0. In fact, $\tilde{\boldsymbol{A}}_0$ is merely a potential due wholly to the applied field H_0.

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