The Theory of Vector Spherical Harmonics

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whipped up, longer waves were only generated while the storm was at its peak. Hence the short-period side of the isolated band, marked with arrows in the spectra, yielded the most accurate estimate of the position of the storm center. Plotting this period as a function of time, the smoothed values were: at 0001 on July 1 period 17.4 seconds, and at subsequent 12-hour intervals 15.8, 14.3, and 12.9 seconds. These figures will be seen to locate the storm quite accurately.

ACKNOWLEDGMENT

Figure 4 is based on the results quoted by Barber and Ursell, and is published by kind permission.

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The Theory of Vector Spherical Harmonics

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Differential equations involving vector fields expressed in terms of spherical polar coordinates are readily manipulated by the use of vector spherical harmonics. The explicit definitions of these functions and a sufficient set of formulas for easy calculation are given.

1. INTRODUCTION

THE manipulation of the partial differential equations of physics involving vector quantities is a matter of some complexity when spherical polar coordinates are employed. Fortunately it is possible in the most important cases to express the vector fields in terms of scalar potential functions, and then to deal with the latter by the usual theory of spherical harmonics. For example, electromagnetic field problems associated with a conducting sphere can be formulated in terms of two scalar potentials.[1]

During the last several years a number of procedures have been developed for the direct treatment of vector fields by means of vector functions which are analogous to the scalar spherical harmonics, to which we shall refer as *vector spherical harmonics*.[2] Different attacks on this problem have met with varying degrees of success in the establishment of a suitable set of orthonormal basis functions. The most direct method is that which has arisen in applications to radiation problems for atomic nuclei and to meson fields,[3] but which is by no means restricted to these cases. This scheme has its roots in the modern algebraic theory of unitary linear vector spaces, which has become one of the pillars of quantum mechanics, and possesses the prime advantage that the conditions of orthogonality and normalization of the basis functions are an automatic feature of the analysis. Furthermore, it is possible to generalize it to spinor and tensor fields in Euclidean spaces of any number of dimensions.

Calculations with vector spherical harmonics are much facilitated by the possession of a reasonably complete table of their properties. This paper provides such a compendium of formulas. Detailed proofs are not given, but since these can be constructed by direct analysis they will be left to the interested reader. Our discussion is self-contained but can be considered as an ex-

[1] W. R. Smythe, *Static and Dynamic Electricity* (McGraw-Hill Book Company, Inc., New York, 1950), second edition, p. 487. J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), Chap. 7.

[2] W. W. Hansen, Phys. Rev. **47**, 139 (1935).

[3] W. Heitler, Proc. Cambridge Phil. Soc. **32**, 112 (1936); S. M. Dancoff and P. Morrison, Phys. Rev. **55**, 122 (1936); H. C. Corben and J. Schwinger, Phys. Rev. **58**, 953 (1936)

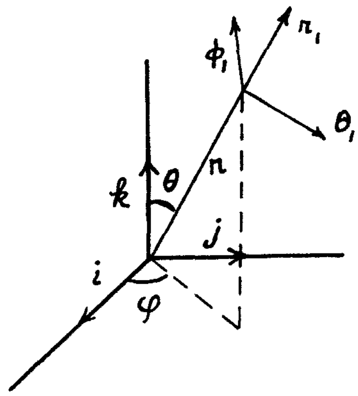

FIG. 1. Geometrical relations for spherical polar coordinates.

tension of that of Blatt and Weisskopf,[4] who have given a minimum set of formulas for use in nuclear problems.

2. BASIC DEFINITIONS

Let a fixed right-handed Euclidean coordinate system be defined by a set of unit vectors $(\mathbf{i},\mathbf{j},\mathbf{k})$. In terms of this we introduce a spherical polar coordinate system (r,θ,φ) with its associated unit vector system $(\mathbf{r}_1,\boldsymbol{\theta}_1,\boldsymbol{\phi}_1)$, as is shown in Fig. 1. The equations connecting these unit vector systems are

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{i}\sin\theta\cos\varphi+\mathbf{j}\sin\theta\sin\varphi+\mathbf{k}\cos\theta, \\ \boldsymbol{\theta}_1 &= \mathbf{i}\cos\theta\cos\varphi+\mathbf{j}\cos\theta\sin\varphi-\mathbf{k}\sin\theta, \\ \boldsymbol{\phi}_1 &= -\mathbf{i}\sin\varphi+\mathbf{j}\cos\varphi. \end{aligned} \tag{1}$$

The formal definition of the scalar spherical harmonics which we adopt is that now most widely accepted in the literature of quantum mechanics. The basic expressions are given in Appendix A and an extensive set of formulas is given in the paper by Hill and Landshoff.[5] With these preliminaries we proceed to give the definitions of the three types of vector spherical harmonics.

$$\mathbf{V}_{lm} \equiv \mathbf{r}_1\left\{-\left(\frac{l+1}{2l+1}\right)^{\frac{1}{2}} Y_l^m\right\} + \boldsymbol{\theta}_1\left\{\frac{1}{[(l+1)(2l+1)]^{\frac{1}{2}}}\frac{\partial Y_l^m}{\partial\theta}\right\} + \boldsymbol{\phi}_1\left\{\frac{imY_l^m}{[(l+1)(2l+1)]^{\frac{1}{2}}\sin\theta}\right\}, \tag{2a}$$

$$\mathbf{X}_{lm} \equiv \boldsymbol{\theta}_1\left\{\frac{-mY_l^m}{[l(l+1)]^{\frac{1}{2}}\sin\theta}\right\} + \boldsymbol{\phi}_1\left\{\frac{-i}{[l(l+1)]^{\frac{1}{2}}}\frac{\partial Y_l^m}{\partial\theta}\right\}, \tag{2b}$$

$$\mathbf{W}_{lm} \equiv \mathbf{r}_1\left\{\left(\frac{l}{2l+1}\right)^{\frac{1}{2}} Y_l^m\right\} + \boldsymbol{\theta}_1\left\{\frac{1}{[l(2l+1)]^{\frac{1}{2}}}\frac{\partial Y_l^m}{\partial\theta}\right\} + \boldsymbol{\phi}_1\left\{\frac{imY_l^m}{[l(2l+1)]^{\frac{1}{2}}\sin\theta}\right\}. \tag{2c}$$

The index l takes all nonnegative integral values $(l=0, 1, 2, \cdots)$ while the index m takes all integral values compatible with the restriction $|m|\leqslant l$.

The following conditions of orthogonality and normalization can be proved by direct evaluation of the integrals:

$$\int \mathbf{V}_{lm}\cdot(\mathbf{V}_{l'm'})^* d\omega = \int \mathbf{X}_{lm}\cdot(\mathbf{X}_{l'm'})^* d\omega = \int \mathbf{W}_{lm}\cdot(\mathbf{W}_{l'm'})^* d\omega = \delta_{ll'}\delta_{mm'}, \tag{3a}$$

$$\int \mathbf{V}_{lm}\cdot(\mathbf{X}_{l'm'})^* d\omega = \int \mathbf{X}_{lm}\cdot(\mathbf{W}_{l'm'})^* d\omega = \int \mathbf{W}_{lm}\cdot(V_{l'm'})^* d\omega = 0. \tag{3b}$$

The range of integration is $(0\leqslant\theta\leqslant\pi,\ 0\leqslant\varphi<2\pi)$ with $d\omega=\sin\theta d\theta d\varphi$. The asterisk * indicates the complex conjugate function and the dot · indicates the scalar product.

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[4] J. M. Blatt and V. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), Appendix B and chapter 12. The correlation between our notation and that of Blatt and Weisskopf is

$$\mathbf{V}_{lm}=\mathbf{Y}_{l,\,l+1,\,1}{}^m,\quad \mathbf{X}_{lm}=\mathbf{Y}_{l,\,l,\,1}{}^m,\quad \mathbf{W}_{lm}=\mathbf{Y}_{l,\,l-1,\,1}{}^m.$$

[5] E. L. Hill and R. Landshoff, Revs. Modern Phys. **10**, 87 (1938); E. U. Condon and G. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, New York, 1932), Chap. 3; H. Bateman, *Partial Differential Equations* (Dover Press, New York, 1951), Chap. 6.

3. SPECIAL PROPERTIES

It is of particular interest that the vector harmonics $\mathbf{X}_{lm}$ have no radial component and that $\operatorname{div}\mathbf{X}_{lm}=0$.

The connections between the harmonics of indices $+m$ and $-m$ are the following:

$$\mathbf{V}_{l,-m}(\theta,\varphi)=(-)^m[\mathbf{V}_{lm}(\theta,\varphi)]^*, \tag{4a}$$

$$\mathbf{X}_{l,-m}(\theta,\varphi)=(-)^{m+1}[\mathbf{X}_{lm}(\theta,\varphi)]^*, \tag{4b}$$

$$\mathbf{W}_{l,-m}(\theta,\varphi)=(-)^m[\mathbf{W}_{lm}(\theta,\varphi)]^*. \tag{4c}$$

The operation of reflection in the origin of coordinates is called the *parity operation* in quantum-mechanical literature. It corresponds to the geometrical transformation of coordinates

$$\theta'=\pi-\theta,\quad \varphi'=\pi+\varphi. \tag{5}$$

Owing to the symmetry of 3-dimensional space under this operation the differential equations of physics are such as to allow a classification of their solutions into sets which are respectively even and odd functions under the parity operation. The parity properties of the vector spherical harmonics are the following:[6]

$$\mathbf{V}_{lm}(\pi-\theta,\ \varphi+\pi)=(-)^{l+1}\mathbf{V}_{lm}(\theta,\varphi), \tag{6a}$$

$$\mathbf{X}_{lm}(\pi-\theta,\ \varphi+\pi)=(-)^{l}\mathbf{X}_{lm}(\theta,\varphi), \tag{6b}$$

$$\mathbf{W}_{lm}(\pi-\theta,\ \varphi+\pi)=(-)^{l+1}\mathbf{W}_{lm}(\theta,\varphi). \tag{6c}$$

The vector harmonics $\mathbf{V}_{lm}$ and $\mathbf{W}_{lm}$ have the same parity, which is odd or even according as the index l is even or odd, but they are of opposite parity to the harmonic $\mathbf{X}_{lm}$.

The classification according to parity is of particular help in the study of sets of differential equations, such as the electromagnetic field equations, if one keeps in mind that the vector differential operators grad, div, and curl are of odd parity, and so alter the parity of a function on which they operate. This point has been stressed by Blatt and Weisskopf[4] in the application to nuclear radiation.

[6] In verifying these relations it is to be noted that the unit vectors $\mathbf{r}_1$ and $\boldsymbol{\phi}_1$ are to be treated as of odd parity, while $\boldsymbol{\theta}_1$ is of even parity.

APPENDIX A. THE SCALAR SPHERICAL HARMONICS

More complete tabulations of the properties of these functions will be found in the literature quoted in reference 5. The variables θ and φ are the usual spherical polar angles on the surface of a sphere with center at the origin, and $\mu=\cos\theta$.

$$P_l^m(\mu)=\frac{(1-\mu^2)^{m/2}}{2^l\cdot l!}\frac{d^{l+m}}{d\mu^{l+m}}(\mu^2-1)^l, \tag{A-1}$$

$$\Theta_l^m(\theta)=(-)^m\left[\frac{2l+1}{2}\cdot\frac{(l-m)!}{(l+m)!}\right]^{\frac{1}{2}}P_l^m(\mu). \tag{A-2}$$

$$Y_l^m(\theta,\varphi)=\Theta_l^m(\theta)\cdot\frac{e^{im\varphi}}{(2\pi)^{\frac{1}{2}}}, \tag{A-3}$$

$$Y_l^{-m}(\theta,\varphi)=(-)^m[Y_l^m(\theta,\varphi)]^*, \tag{A-4}$$

$$Y_l^m(\pi-\theta,\ \varphi+\pi)=(-)^lY_l^m(\theta,\varphi), \tag{A-5}$$

$$\int Y_l^m(\theta,\varphi)\cdot[Y_{l'}^{m'}(\theta,\varphi)]^*d\omega=\delta_{ll'}\delta_{mm'}. \tag{A-6}$$

Special Functions

$$Y_0^0=\frac{1}{(4\pi)^{\frac{1}{2}}} \tag{A-7}$$

$$Y_1^0=\left(\frac{3}{4\pi}\right)^{\frac{1}{2}}\cos\theta,\ Y_1^{\pm1}=\mp\left(\frac{3}{8\pi}\right)^{\frac{1}{2}}\sin\theta e^{\pm i\varphi} \tag{A-8}$$

$$Y_2^0=\left(\frac{5}{16\pi}\right)^{\frac{1}{2}}(3\cos^2\theta-1), \tag{A-9}$$

$$Y_2^{\pm1}=\mp\left(\frac{15}{8\pi}\right)^{\frac{1}{2}}\cos\theta\sin\theta\cdot e^{\pm i\varphi} \tag{A-10}$$

$$Y_2^{\pm2}=\left(\frac{15}{32\pi}\right)^{\frac{1}{2}}\sin^2\theta\cdot e^{\pm i2\varphi}. \tag{A-11}$$

APPENDIX B. THE VECTOR SPHERICAL HARMONICS

The formulas of this table are direct consequences of the definitions (2a,b,c). Here $F(r)$ is any scalar function which does not depend on the polar angles; in particular, it can be taken to be constant. The vector differential relations for the spherical polar unit vectors are given for convenience in calculations.

$$\operatorname{div}\mathbf{r}_1=\frac{2}{r},\quad \operatorname{div}\boldsymbol{\theta}_1=\frac{\cot\theta}{r},\qquad \operatorname{div}\boldsymbol{\phi}_1=0, \tag{B-1}$$

$$\operatorname{curl}\mathbf{r}_1=0,\quad \operatorname{curl}\boldsymbol{\theta}_1=\frac{\boldsymbol{\phi}_1}{r},\qquad \operatorname{curl}\boldsymbol{\phi}_1=\frac{\mathbf{r}_1\cot\theta}{r}-\frac{\boldsymbol{\theta}_1}{r}, \tag{B-2}$$

$$\mathbf{r}_1\times\mathbf{V}_{lm}=i\left(\frac{l}{2l+1}\right)^{\frac{1}{2}}\mathbf{X}_{lm},\qquad \mathbf{r}_1\times\mathbf{W}_{lm}=i\left(\frac{l+1}{2l+1}\right)^{\frac{1}{2}}\mathbf{X}_{lm}, \tag{B-3}$$

$$\mathbf{r}_1\times\mathbf{X}_{lm}=i\left(\frac{l}{2l+1}\right)^{\frac{1}{2}}\mathbf{V}_{lm}+i\left(\frac{l+1}{2l+1}\right)^{\frac{1}{2}}\mathbf{W}_{lm},$$

$$\operatorname{div}[F(r)\mathbf{V}_{lm}]=-\left(\frac{l+1}{2l+1}\right)^{\frac{1}{2}}\left[\frac{dF}{dr}+\frac{l+2}{r}F\right]Y_l^m, \tag{B-4a}$$

$$\operatorname{div}[F(r)\mathbf{X}_{lm}]=0, \tag{B-4b}$$

$$\operatorname{div}[F(r)\mathbf{W}_{lm}]=\left(\frac{l}{2l+1}\right)^{\frac{1}{2}}\left[\frac{dF}{dr}-\frac{l-1}{r}F\right]Y_l^m, \tag{B-4c}$$

$$\operatorname{curl}[F(r)\mathbf{V}_{lm}]=i\left(\frac{l}{2l+1}\right)^{\frac{1}{2}}\left[\frac{dF}{dr}+\frac{l+2}{r}F\right]\mathbf{X}_{lm}, \tag{B-5a}$$

$$\operatorname{curl}[F(r)\mathbf{X}_{lm}]=i\left(\frac{l}{2l+1}\right)^{\frac{1}{2}}\left[\frac{dF}{dr}-\frac{l}{r}F\right]\mathbf{V}_{lm}+i\left(\frac{l+1}{2l+1}\right)^{\frac{1}{2}}\left[\frac{dF}{dr}+\frac{l+1}{r}F\right]\mathbf{W}_{lm}, \tag{B-5b}$$

$$\operatorname{curl}[F(r)\mathbf{W}_{lm}]=i\left(\frac{l+1}{2l+1}\right)^{\frac{1}{2}}\left[\frac{dF}{dr}-\frac{l-1}{r}F\right]\mathbf{X}_{lm}, \tag{B-5c}$$

$$\nabla^2[F(r)\mathbf{V}_{lm}]=L_{l+1}(F)\mathbf{V}_{lm}, \tag{B-6a}$$

$$\nabla^2[F(r)\mathbf{X}_{lm}]=L_l(F)\mathbf{X}_{lm}, \tag{B-6b}$$

$$\nabla^2[F(r)\mathbf{W}_{lm}]=L_{l-1}(F)\mathbf{W}_{lm}, \tag{B-6c}$$

with

$$L_l\equiv\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\frac{l(l+1)}{r^2}. \tag{B-7}$$

$$\operatorname{grad}Y_l^m=\frac{l}{r}\left(\frac{l+1}{2l+1}\right)^{\frac{1}{2}}\mathbf{V}_{lm}+\frac{l+1}{r}\left(\frac{l}{2l+1}\right)^{\frac{1}{2}}\mathbf{W}_{lm}. \tag{B-8}$$

$$\mathbf{r}_1Y_l^m=-\left(\frac{l+1}{2l+1}\right)^{\frac{1}{2}}\mathbf{V}_{lm}+\left(\frac{l}{2l+1}\right)^{\frac{1}{2}}\mathbf{W}_{lm}. \tag{B-9}$$

Special Functions

$$\mathbf{V}_{0,0}=-\mathbf{r}_1/(4\pi)^{\frac{1}{2}},\quad \mathbf{X}_{0,0}=0,\quad \mathbf{W}_{0,0}=0, \tag{B-10}$$

$$\mathbf{V}_{1,0}=\mathbf{r}_1\left\{\frac{-\cos\theta}{(2\pi)^{\frac{1}{2}}}\right\}+\boldsymbol{\theta}_1\left\{\frac{-\sin\theta}{(8\pi)^{\frac{1}{2}}}\right\}, \tag{B-11a}$$

$$\mathbf{V}_{1,\pm1}=\mathbf{r}_1\left\{\frac{\pm\sin\theta e^{\pm i\varphi}}{(4\pi)^{\frac{1}{2}}}\right\}+\boldsymbol{\theta}_1\left\{\frac{\mp\cos\theta e^{\pm i\varphi}}{(16\pi)^{\frac{1}{2}}}\right\}+\boldsymbol{\phi}_1\left\{\frac{-ie^{\pm i\varphi}}{(16\pi)^{\frac{1}{2}}}\right\}, \tag{B-11b}$$

$$\mathbf{X}_{1,0}=\boldsymbol{\phi}_1\left\{i\left(\frac{3}{8\pi}\right)^{\frac{1}{2}}\sin\theta\right\} \tag{B-12a}$$

$$\mathbf{X}_{1,\pm1}=\boldsymbol{\theta}_1\left\{\left(\frac{3}{16\pi}\right)^{\frac{1}{2}}e^{\pm i\varphi}\right\}+\boldsymbol{\phi}_1\left\{\pm i\left(\frac{3}{16\pi}\right)^{\frac{1}{2}}\cos\theta e^{\pm i\varphi}\right\}, \tag{B-12b}$$

$$\mathbf{W}_{1,0}=\mathbf{r}_1\left\{\frac{\cos\theta}{(4\pi)^{\frac{1}{2}}}\right\}+\boldsymbol{\theta}_1\left\{\frac{-\sin\theta}{(4\pi)^{\frac{1}{2}}}\right\}, \tag{B-13a}$$

$$\mathbf{W}_{1,\pm1}=\mathbf{r}_1\left\{\frac{\mp\sin\theta e^{\pm i\varphi}}{(8\pi)^{\frac{1}{2}}}\right\}+\boldsymbol{\theta}_1\left\{\frac{\mp\cos\theta e^{\pm i\varphi}}{(8\pi)^{\frac{1}{2}}}\right\}+\boldsymbol{\phi}_1\left\{\frac{-ie^{\pm i\varphi}}{(8\pi)^{\frac{1}{2}}}\right\}. \tag{B-13b}$$