Notes on Cordier-Dubuc Paper

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1 Weak formulation

The solution consists of two stages:

(A) Solve for the magnetic scalar potential Ψ, imposing the desired fluxoid quantization condition through the inhomogeneous boundary term B_{QS}

(B) Solve for the shifted vector potential $\mathbf{A}^*$ and the phase θ.

Because both vector- and scalar-valued fields are involved, the finite-element function space will be a direct product of $Q_p(\Omega_{\text{in}} \cup \Omega_{\text{out}})$ (for the scalar fields) and $Q_p^3(\Omega_{\text{in}} \cup \Omega_{\text{out}})$ (for the vector fields).

TODO. Investigate possible issues with convergence related to choice of finite-element spaces and the LBB condition for saddle-point problems. I believe the above (naive) choices should be OK, but there may be better choices / I may have misunderstood the convergence criteria.

Definition 1 (Basis functions)**.** Let

- φ_i denote the scalar basis elements
- $\mathbf{\Phi}_i$ denote the vector basis elements

Definition 2 (System matrices)**.** The following matrices will be used to assemble the linear systems for Items (A) and (B):

$$M_{ij}^{(\text{int})} := \int_{\Omega_{(\text{int})}} \boldsymbol{\nabla}\varphi_i \cdot \boldsymbol{\nabla}\varphi_j \,\mathrm{d}\Omega \tag{0.1}$$

$$M_{ij}^{(\text{ext})} := \int_{\Omega_{(\text{ext})}} \boldsymbol{\nabla}\varphi_i \cdot \boldsymbol{\nabla}\varphi_j \,\mathrm{d}\Omega \tag{0.2}$$

$$C_{ij}^{(\text{int/ext})} := \int_{\Omega_{(\text{int/ext})}} (\boldsymbol{\nabla} \times \boldsymbol{\Phi}_i) \cdot (\boldsymbol{\nabla} \times \boldsymbol{\Phi}_j) \,\mathrm{d}\Omega \tag{0.3}$$

$$D_{ij}^{(\text{int/ext})} := \int_{\Omega_{(\text{int/ext})}} (\boldsymbol{\nabla} \cdot \boldsymbol{\Phi}_i)(\boldsymbol{\nabla} \cdot \boldsymbol{\Phi}_j) \,\mathrm{d}\Omega \tag{0.4}$$

$$S_{ij}^{(\text{int/ext})} := \int_{\Omega_{(\text{int/ext})}} \boldsymbol{\Phi}_i \cdot \boldsymbol{\Phi}_j \,\mathrm{d}\Omega \tag{0.5}$$

$$T_{ij}^{(\text{int/ext})} := \int_{\Omega_{(\text{int/ext})}} \boldsymbol{\Phi}_i \cdot \boldsymbol{\nabla}\varphi_j \,\mathrm{d}\Omega \tag{0.6}$$

$$E_{ij}^{(\text{ext})} := \int_{\Omega_{(\text{ext})}} (\boldsymbol{\nabla} \times \boldsymbol{\Phi}_i) \cdot \boldsymbol{\nabla}\varphi_j \,\mathrm{d}\Omega \tag{0.7}$$

$$H_{i}^{(\text{ext})} := \int_{\Gamma_H} \boldsymbol{\Phi}_i \cdot \mathbf{H}_{(\text{ext})} \,\mathrm{d}\Omega \tag{0.8}$$

$$B_{i}^{(\text{QS})} := \int_{S_a} \varphi_i B_{QS} \,\mathrm{d}S \tag{0.9}$$

The weak formulation of the problem is then

$$M_{ij}^{(\text{ext})} \boldsymbol{\Psi}_j = B_i^{(\text{QS})} \tag{A}$$

$$\begin{cases} \left[C_{ij}^{(\text{int})} + pD_{ij}^{(\text{int})} + \dfrac{1}{\lambda^2} S_{ij}^{(\text{int})}\right] A_j^* = \dfrac{\Phi_0}{2\pi\lambda^2} T_{ij}^{(\text{int})} \theta_i \\ \dfrac{1}{\mu_0 \lambda^2} \widetilde{T}_{ij}^{(\text{int})} A_j^* = \dfrac{\Phi_0}{2\pi\mu_0\lambda^2} M_{ij}^{(\text{int})} \theta_i \\ \left[C_{ij}^{(\text{ext})} + pD_{ij}^{(\text{ext})}\right] A_i^* = -E_{ij}^{(\text{ext})} \Psi_j + \mathbf{H}_i^{(\text{ext})} \end{cases} \tag{B}$$

- Macroscopic complex wavefunction of Cooper pairs (isn't this equivalent to considering a GL equation?):

$$J_s = \frac{1}{\mu_0 \lambda_L^2} \left(-A + \frac{\Phi_0}{2\pi} \boldsymbol{\nabla}\phi \right) \tag{0.10}$$

1.1 Discretization

Following deal.ii's step-46 tutorial, we look for solutions in the following function space:

$$(\Psi, \mathbf{A}^*, \theta) \in \underbrace{[Z^1(\Omega_{\text{int}} \times Q_{p+1}^1(\Omega_{\text{ext}})]}_{\Psi} \times \underbrace{[Q_p^d(\Omega_{\text{int} \cup \Omega_{\text{ext}}})]}_{\mathbf{A}^*} \times \underbrace{[Z^1(\Omega_{\text{ext}}) \times Q_{p+1}^1(\Omega_{\text{int}})]}_{\theta} \tag{0.11}$$

2 References (forward)

- C. Cordier, S Flament, and C. Dubuc, “A 3D finite element formulation for calculating meissner currents in superconductors,” IEEE Trans. Appl. Superconduct., vol. 9, pp. 2–6, Mar. 1999.
  - Only allows calculations with multiply-connected regions if the fluxoid is zero

sectionReferences (backward)
None found. (Searched: ScienceWeb)

3 Questions

1. Values of B_{QS}? (Seems arbitrary except for the requirement that the flux through any “holes” should be the desired value)
2. Boundary conditions on Γ_{12} are apparently enforced implicitly without the need to discretize eqs. 26 - 29. This is very convenient but somewhat mysterious...