Classical Physics: Spacetime and Fields

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Preface

We present a self-contained introduction to the classical theory of spacetime and fields. This exposition is based on the most general principles: the principle of general covariance (relativity) and the principle of least action. The order of the exposition is: 1. Spacetime (principle of general covariance and tensors, affine connection, curvature, metric, tetrad and spin connection, Lorentz group, spinors); 2. Fields (principle of least action, action for gravitational field, matter, symmetries and conservation laws, gravitational field equations, spinor fields, electromagnetic field, action for particles). In this order, a particle is a special case of a field existing in spacetime, and classical mechanics can be derived from field theory.

I dedicate this book to my Parents: Bożenna Popławska and Janusz Popławski. I am also grateful to Chris Cox for inspiring this book.

The Laws of Physics are simple, beautiful, and universal.

Contents

1 Spacetime

1.1 Principle of general covariance and tensors

Physical processes are described in coordinate systems in four-dimensional *spacetime*, called *systems of reference* or *frames of reference*. The *principle of general covariance* or *Einstein's general principle of relativity* states that physical laws do not change their form (are *covariant*) under arbitrary, differentiable (and thereby continuous) coordinate transformations. Equivalently, physical laws have the same form in all admissible frames of reference.

1.1.1 Vectors

Let us consider a coordinate transformation from old (unprimed) to new (primed) *coordinates* in a four-dimensional manifold:

$$x^i \to x'^j(x^i), \tag{1.1.1}$$

where x'^j are differentiable and nondegenerate functions of x^i and the index i (and the other Latin indices) can be 0,1,2, or 3. The corresponding transformation matrix $\frac{\partial x'^j}{\partial x^i}$, which is four-dimensional and square (4×4), has a nonzero determinant $|\frac{\partial x'^j}{\partial x^i}| \neq 0$, thereby that x^i are differentiable and nondegenerate functions of x'^j. The matrix $\frac{\partial x^i}{\partial x'^j}$ is inverse to $\frac{\partial x'^j}{\partial x^i}$:

$$\sum_i \frac{\partial x'^i}{\partial x^j}\frac{\partial x^k}{\partial x'^i} = \delta^k_j, \tag{1.1.2}$$

where

$$\delta^i_k = \left\{ \begin{array}{cc} 1 & i = k \\ 0 & i \neq k \end{array} \right\}. \tag{1.1.3}$$

A *scalar* (invariant) is defined as a quantity that does not change under coordinate transformations:

$$\phi' = \phi. \tag{1.1.4}$$

Accordingly, the differential of a scalar is also a scalar:

$$d\phi' = d\phi. \tag{1.1.5}$$

If $\phi(x^i)$ is a scalar function of the coordinates x^i, then its differential can be expressed as

$$d\phi(x^i) = \sum_i \frac{\partial \phi}{\partial x^i} dx^i. \tag{1.1.6}$$

Coordinate differentials dx^i and partial derivatives $\partial_i = \frac{\partial}{\partial x^i}$ transform according to

$$dx'^j = \sum_i \frac{\partial x'^j}{\partial x^i} dx^i, \tag{1.1.7}$$

$$\frac{\partial}{\partial x'^j} = \sum_i \frac{\partial x^i}{\partial x'^j}\frac{\partial}{\partial x^i}, \quad \partial'_j = \sum_i \frac{\partial x^i}{\partial x'^j}\partial_i. \tag{1.1.8}$$

A *contravariant vector* is defined as a set of quantities that transform like coordinate differentials:

$$A'^j = \sum_i \frac{\partial x'^j}{\partial x^i} A^i. \tag{1.1.9}$$

These quantities are referred to as the *components* of the contravariant vector. A *covariant vector* is defined as a set of quantities that transform like partial derivatives of a scalar:

$$B'_j = \sum_i \frac{\partial x^i}{\partial x'^j} B_i. \tag{1.1.10}$$

These quantities are referred to as the components of the covariant vector. Therefore, coordinate differentials form a contravariant vector and partial derivatives of a scalar form a covariant vector. The coordinates x^i do not form a vector. A linear combination of two scalars is a scalar. A linear combination $aC + bD$ of two contravariant vectors C and D, where a and b are scalars, is a contravariant vector E whose components are $E^i = aC^i + bD^i$. A linear combination $aC + bD$ of two covariant vectors C and D is a covariant vector E whose components are $E_i = aC_i + bD_i$.

An upper index (in a contravariant vector) is called *contravariant*, and a lower index is called *covariant*. The derivative with respect to a quantity with a contravariant index i is a quantity with a covariant index i. Conversely, the derivative with respect to a quantity with a covariant index i is a quantity with a contravariant index i. Henceforth, we adopt the following *Einstein's summation convention*. If the same coordinate index i appears in a given expression twice, as a contravariant index and a covariant index, and we apply the summation $\sum_i$ in this expression, then we do not need to write the summation sign $\sum_i$. Accordingly, we can omit the summation signs in the formulae of this section.

1.1.2 Tensors

A product of several vectors transforms under differentiable coordinate transformations such that each coordinate index transforms separately:

$$A'^i B'^j \dots C'_k D'_l \dots = \frac{\partial x'^i}{\partial x^m}\frac{\partial x'^j}{\partial x^n}\frac{\partial x^p}{\partial x'^k}\frac{\partial x^q}{\partial x'^l} A^m B^n \dots C_p D_q \dots . \tag{1.1.11}$$

A *tensor* is defined as a set of quantities that transform like products of the components of vectors:

$$T'^{ij\dots}{}_{kl\dots} = \frac{\partial x'^i}{\partial x^m}\frac{\partial x'^j}{\partial x^n}\frac{\partial x^p}{\partial x'^k}\frac{\partial x^q}{\partial x'^l} T^{mn\dots}{}_{pq\dots}. \tag{1.1.12}$$

These quantities are referred to as the components of the tensor. A tensor is of rank (k,l) if it has k contravariant and l covariant indices. A scalar is a tensor of rank (0,0), a contravariant vector is a tensor of rank (1,0), and a covariant vector is a tensor of rank (0,1). A linear combination of two tensors of rank (k,l) is a tensor of rank (k,l) such that its components are the same linear combinations of the corresponding components of the tensors. The product of two tensors of ranks (k_1,l_1) and (k_2,l_2) is a tensor of rank (k_1+k_2, l_1+l_2). Tensor indices (all contravariant or all covariant) can be *symmetrized*:

$$T_{(ij\dots k)} = \frac{1}{n!}\sum_{\text{permutations}} T_{\{ij\dots k\}}, \tag{1.1.13}$$

or *antisymmetrized*:

$$T_{[ij\dots k]} = \frac{1}{n!}\sum_{\text{permutations}} T_{\{ij\dots k\}}(-1)^N, \tag{1.1.14}$$

where n is the number of symmetrized or antisymmetrized indices and N is the number of permutations that bring $T_{ij\dots k}$ to $T_{\{ij\dots k\}}$. For example, for two indices: $T_{(ik)} = \frac{1}{2}(T_{ik} + T_{ki})$ and $T_{[ik]} = \frac{1}{2}(T_{ik} - T_{ki})$, and for three indices: $T_{[ijk]} = \frac{1}{3}(T_{ijk} + T_{jki} + T_{kij})$. If $n > 4$ then $T_{[ij\dots k]} = 0$. Symmetrized and antisymmetrized tensors or rank (k,l) are tensors of rank (k,l). Symmetrization of an antisymmetric tensor or antisymmetrization of a symmetric tensor bring these tensors to zero. Any tensor of rank (0,2) is the sum of its symmetric and antisymmetric part,

$$T_{(ik)} + T_{[ik]} = T_{ik}. \tag{1.1.15}$$

The number 0 can be regarded as a tensor of arbitrary rank. Therefore, all covariant equations of classical physics must be represented in the tensor form: $T^{ij\dots}{}_{kl\dots} = 0$.

1.1.3 Densities

The element of volume in four-dimensional spacetime transforms according to

$$d^4x' = \left|\frac{\partial x'^i}{\partial x^k}\right| d^4x. \tag{1.1.16}$$

A *scalar density* is defined as a quantity that transforms such that its product with the element of volume is a scalar, $\mathfrak{s}' d^4x' = \mathfrak{s} d^4x$:

$$\mathfrak{s}' = \left|\frac{\partial x^i}{\partial x'^k}\right| \mathfrak{s}. \tag{1.1.17}$$

A *tensor density*, which includes a contravariant and covariant vector density, is defined as a set of quantities that transform like products of the components of a tensor and a scalar density:

$$\mathfrak{T}'^{ij\dots}_{kl\dots} = \left|\frac{\partial x^i}{\partial x'^k}\right| \frac{\partial x'^i}{\partial x^m} \frac{\partial x'^j}{\partial x^n} \frac{\partial x^p}{\partial x'^k} \frac{\partial x^q}{\partial x'^l} \mathfrak{T}^{mn\dots}_{pq\dots}. \tag{1.1.18}$$

These quantities are referred to as the components of the tensor density. A tensor density is of rank (k, l) if it has k contravariant and l covariant indices. For example, the square root of the determinant of a tensor of rank $(0, 2)$ is a scalar density of weight 1:

$$\sqrt{|T'_{ik}|} = \sqrt{\left|\frac{\partial x^l}{\partial x'^i} \frac{\partial x^m}{\partial x'^k} T_{lm}\right|} = \sqrt{\left|\frac{\partial x^j}{\partial x'^n}\right|^2 |T_{ik}|} = \left|\frac{\partial x^j}{\partial x'^n}\right| \sqrt{|T_{ik}|}. \tag{1.1.19}$$

The above densities are said to be of weight 1. One can generalize this definition of densities by introducing densitites of weight w, which transform according to

$$\mathfrak{s}' = \left|\frac{\partial x^i}{\partial x'^k}\right|^w \mathfrak{s}. \tag{1.1.20}$$

For example, d^4x is a scalar density of weight -1. A linear combination of two densities of rank (k, l) and weight w is a density of rank (k, l) and weight w such that its components are the same linear combinations of the corresponding components of the densities. The product of two densities of weights w_1 and w_2 is a density of weight $w_1 + w_2$. Symmetrized and antisymmetrized densities of weight w are densities of weight w. Densities of weight 1 are simply referred to as densities. Tensors are densities of weight 0.

1.1.4 Contraction

Einstein's summation convention also applies within the same tensor or tensor density, if a given coordinate index i appears twice (as a contravariant and covariant index). Such a tensor or density is said to be *contracted* over index i. A contracted tensor of rank (k, l) transforms like a tensor of rank $(k-1, l-1)$:

$$T'^{ij\dots}_{il\dots} = \frac{\partial x'^i}{\partial x^m} \frac{\partial x'^j}{\partial x^n} \frac{\partial x^p}{\partial x'^i} \frac{\partial x^q}{\partial x'^l} T^{mn\dots}_{pq\dots} = \frac{\partial x'^j}{\partial x^n} \frac{\partial x^q}{\partial x'^l} \delta^p_m T^{mn\dots}_{pq\dots} = \frac{\partial x'^j}{\partial x^n} \frac{\partial x^q}{\partial x'^l} T^{mn\dots}_{mq\dots}. \tag{1.1.21}$$

For example, the contraction of a contravariant and covariant vector $A^i B_i$ is a scalar (*scalar product*). A contracted tensor density of rank (k, l) and weight w transforms like a tensor density of rank $(k-1, l-1)$ and weight w:

$$\begin{aligned}\mathfrak{T}'^{ij\dots}_{il\dots} &= \left|\frac{\partial x^i}{\partial x'^k}\right|^w \frac{\partial x'^i}{\partial x^m} \frac{\partial x'^j}{\partial x^n} \frac{\partial x^p}{\partial x'^i} \frac{\partial x^q}{\partial x'^l} \mathfrak{T}^{mn\dots}_{pq\dots} = \left|\frac{\partial x^i}{\partial x'^k}\right|^w \frac{\partial x'^j}{\partial x^n} \frac{\partial x^q}{\partial x'^l} \delta^p_m \mathfrak{T}^{mn\dots}_{pq\dots} \\ &= \left|\frac{\partial x^i}{\partial x'^k}\right|^w \frac{\partial x'^j}{\partial x^n} \frac{\partial x^q}{\partial x'^l} \mathfrak{T}^{mn\dots}_{mq\dots}.\end{aligned} \tag{1.1.22}$$

Contraction of a symmetric tensor with an antisymmetric tensor (over indices with respect to which these tensors are symmetric or antisymmetric) gives zero. If contraction of two tensors gives zero, these tensors are said to be *orthogonal*. Two orthogonal vectors (one contravariant and one covariant) are said to be *perpendicular*.

1.1.5 Kronecker and Levi-Civita symbols

The *Kronecker symbol* δ^i_k (1.1.3) is a tensor with constant components:

$$\delta'^i_{\ k} = \frac{\partial x'^i}{\partial x^j}\frac{\partial x^l}{\partial x'^k}\delta^j_l = \frac{\partial x'^i}{\partial x^j}\frac{\partial x^j}{\partial x'^k} = \delta^i_k. \tag{1.1.23}$$

A completely antisymmetric tensor of rank $(4,0)$, $T^{ijkl} = T^{[ijkl]}$ has 1 independent component T: $T^{ijkl} = T\epsilon^{ijkl}$, where ϵ^{ijkl} is the completely antisymmetric, contravariant *Levi-Civita permutation symbol*:

$$\epsilon^{0123} = 1,\ \ \epsilon^{ijkl} = \epsilon^{[ijkl]} = (-1)^N, \tag{1.1.24}$$

and N is the number of permutations that bring ϵ^{ijkl} to ϵ^{0123}. The determinant of a square matrix S^i_k, $\det(S^i_k) = |S^i_k|$, is defined through the permutation symbol:

$$|S^r_s|\epsilon^{ijkl} = S^i_m S^j_n S^k_p S^l_q \epsilon^{mnpq}. \tag{1.1.25}$$

Taking $S^i_k = \frac{\partial x'^i}{\partial x^k}$ gives

$$\epsilon^{ijkl} = \left|\frac{\partial x^r}{\partial x'^s}\right|\frac{\partial x'^i}{\partial x^m}\frac{\partial x'^j}{\partial x^n}\frac{\partial x'^k}{\partial x^p}\frac{\partial x'^l}{\partial x^q}\epsilon^{mnpq}. \tag{1.1.26}$$

This equation looks like a transformation law for a tensor density (of weight 1) with constant components: $\epsilon'^{ijkl} = \epsilon^{ijkl}$. Accordingly, T is a scalar density of weight -1. We also introduce the covariant Levi-Civita symbol ε_{ijkl} through:

$$\epsilon^{ijkl}\varepsilon_{mnpq} = -\begin{vmatrix} \delta^i_m & \delta^i_n & \delta^i_p & \delta^i_q \\ \delta^j_m & \delta^j_n & \delta^j_p & \delta^j_q \\ \delta^k_m & \delta^k_n & \delta^k_p & \delta^k_q \\ \delta^l_m & \delta^l_n & \delta^l_p & \delta^l_q \end{vmatrix}. \tag{1.1.27}$$

Therefore, the covariant Levi-Civita symbol is a tensor density of weight -1 and its product with a scalar density is a tensor. The covariant Levi-Civita symbol is given by

$$\varepsilon_{0123} = -1,\ \ \varepsilon_{ijkl} = \varepsilon_{[ijkl]} = (-1)^N, \tag{1.1.28}$$

where N is the number of permutations that bring ε_{ijkl} to ε_{0123}, and satisfies

$$|S^r_s|\varepsilon_{ijkl} = S^m_i S^n_j S^p_k S^q_l \varepsilon_{mnpq}. \tag{1.1.29}$$

Contracting (1.1.27) gives the following relations:

$$\begin{aligned} \epsilon^{ijkl}\varepsilon_{mnpl} &= -\begin{vmatrix} \delta^i_m & \delta^i_n & \delta^i_p \\ \delta^j_m & \delta^j_n & \delta^j_p \\ \delta^k_m & \delta^k_n & \delta^k_p \end{vmatrix}, \\ \epsilon^{ijkl}\varepsilon_{mnkl} &= -2(\delta^i_m\delta^j_n - \delta^i_n\delta^j_m), \\ \epsilon^{ijkl}\varepsilon_{mjkl} &= -6\delta^i_m, \\ \epsilon^{ijkl}\varepsilon_{ijkl} &= -24. \end{aligned} \tag{1.1.30}$$

1.1.6 Dual densities

A contracted product of a covariant tensor and the contravariant Levi-Civita symbol gives a *dual* contravariant tensor density of weight 1:

$$\epsilon^{iklm}A_m = \mathfrak{A}^{ikl},\ \ \epsilon^{iklm}B_{lm} = \mathfrak{B}^{ik},\ \ \epsilon^{iklm}C_{klm} = \mathfrak{C}^i. \tag{1.1.31}$$

A contracted product of a contravariant tensor and the covariant Levi-Civita symbol gives a dual covariant tensor density of weight -1:

$$\varepsilon_{iklm}A^m = \mathfrak{A}_{ikl},\ \ \varepsilon_{iklm}B^{lm} = \mathfrak{B}_{ik},\ \ \varepsilon_{iklm}C^{klm} = \mathfrak{C}_i. \tag{1.1.32}$$

Therefore, there exists an algebraic correspondence between covariant tensors and contravariant densities of weight 1, and between contravariant tensors and covariant densities of weight -1.

1.1.7 Covariant integrals

A covariant line integral is an integral of a tensor contracted with the line differential dx^i: $\int T^{j\dots}_{i\dots}dx^i$. A covariant surface integral is an integral of a tensor contracted with the surface differential $df^{ik} = dx^i dx'^k - dx^k dx'^i$ (which can be geometrically represented as a parallelogram spanned by the vectors dx^i and dx'^i): $\int T^{j\dots}_{ik\dots}df^{ik}$. A covariant hypersurface (volume) integral is an integral of a tensor contracted with the volume differential $dS^{ikl} = \begin{vmatrix} dx^i & dx'^i & dx''^i \\ dx^k & dx'^k & dx''^k \\ dx^l & dx'^l & dx''^l \end{vmatrix}$ (which can be geometrically represented as a parallelepiped spanned by the vectors dx^i, dx'^i) and dx''^i: $\int T^{j\dots}_{ikl\dots}dS^{ikl}$. A covariant four-volume integral is an integral of a tensor contracted with the four-volume differential dS^{ijkl}, defined analogously to dS^{ikl}. The dual density corresponding to the surface element is given by

$$df^\star_{ik} = \frac{1}{2}\varepsilon_{iklm}df^{lm}, \tag{1.1.33}$$

which gives

$$df^{lm} = -\frac{1}{2}\epsilon^{lmik}df^\star_{ik}, \ \ df^{ik}df^\star_{ik} = 0. \tag{1.1.34}$$

The dual density corresponding to the hypersurface element is given by

$$dS_i = -\frac{1}{6}\varepsilon_{iklm}dS^{klm}, \ \ dS^{klm} = -\epsilon^{klmi}dS_i. \tag{1.1.35}$$

The dual density corresponding to the four-volume element is given by

$$d\Omega = \frac{1}{24}\varepsilon_{iklm}dS^{iklm} = dx^0dx^1dx^2dx^3. \tag{1.1.36}$$

Covariant integrands that include the above dual densities of weight -1 must be multiplied by a scalar density, for example, by the square root of the determinant of a tensor of rank $(0,2)$. According to Gauß' and Stokes' theorems, there exists relations between integrals over different elements:

$$dx^i \leftrightarrow df^{ki}\frac{\partial}{\partial x^k}, \tag{1.1.37}$$

$$df^\star_{ik} \leftrightarrow dS_i\frac{\partial}{\partial x^k} - dS_k\frac{\partial}{\partial x^i}, \tag{1.1.38}$$

$$dS_i \leftrightarrow d\Omega\frac{\partial}{\partial x^i}. \tag{1.1.39}$$

1.1.8 Antisymmetric derivatives

A derivative of a covariant vector does not transform like a tensor:

$$\frac{\partial A'_k}{\partial x'^i} = \frac{\partial}{\partial x'^i}\left(\frac{\partial x^m}{\partial x'^k}A_m\right) = \frac{\partial x^m}{\partial x'^k}\frac{\partial A_m}{\partial x'^i} + \frac{\partial^2 x^m}{\partial x'^i\partial x'^k}A_m = \frac{\partial x^l}{\partial x'^i}\frac{\partial x^m}{\partial x'^k}\frac{\partial A_m}{\partial x^l} + \frac{\partial^2 x^m}{\partial x'^i\partial x'^k}A_m, \tag{1.1.40}$$

because of the second term which is linear and homogeneous in A_i, unless x^i are linear functions of x'^j. This term is symmetric in the indices i,k, thereby the antisymmetric part of $\frac{\partial A_k}{\partial x^i}$ with respect to these indices is a tensor:

$$\partial'_{[i}A'_{k]} = \frac{\partial x^l}{\partial x'^{[i}}\frac{\partial x^m}{\partial x'^{k]}}\partial_l A_m = \frac{\partial x^l}{\partial x'^i}\frac{\partial x^m}{\partial x'^k}\partial_{[l}A_{m]}. \tag{1.1.41}$$

The *curl* of a covariant vector A_i is defined as twice the antisymmetric part of $\partial_i A_k$: $\partial_i A_k - \partial_k A_i$, and is a tensor. We will also use $_{,i} = \frac{\partial}{\partial x^i}$ to denote a partial derivative with respect to x^i. Similarly, completely antisymmetrized derivatives of tensors of rank $(0,2)$ and $(0,3)$, $\partial_{[i}B_{kl]}$ and $\partial_{[i}C_{klm]}$, are tensors. If $B_{kl} = A_{[k,l]}$ then $\partial_{[i}B_{kl]} = 0$, or conversely, if $\partial_{[i}B_{kl]} = 0$ then there exists a vector A_i such that $B_{kl} = A_{[k,l]}$. The *divergence* of a tensor (or density) is a contracted derivative of this tensor (density): $\partial_i T^{il\dots}_{jk\dots}$. Because of the correspondence between tensors and dual densities, divergences of (completely antisymmetric if more than 1 index) contravariant densities are densities, dual to completely antisymmetrized derivatives of tensors:

$$\partial_i \mathfrak{C}^i = \epsilon^{iklm}\partial_{[i}C_{klm]},\ \ \partial_k \mathfrak{B}^{ik} = \epsilon^{iklm}\partial_{[k}B_{lm]},\ \ \partial_l \mathfrak{A}^{ikl} = \epsilon^{iklm}\partial_{[l}A_{m]}. \tag{1.1.42}$$

For example, the equations $F_{[ik,l]} = 0$ and $\mathfrak{F}^{ik}{}_{,i} = \mathfrak{j}^k$, that describe Maxwell's electrodynamics (confer (2.7.32) and (2.7.82)), are tensorial.
References: [1, 2].

1.2 Affine connection

1.2.1 Covariant differentiation of tensors

An ordinary derivative of a covariant vector A_i is not a tensor, because its coordinate transformation law (1.1.40) contains an additional noncovariant term, linear and homogeneous in A_i. Such a term vanishes only if x^i are linear functions of x'^j, that is, if the coordinate transformation from old to new coordinates (1.1.1) is linear. Let us consider the expression

$$A_{i;k} = A_{i,k} - \Gamma^{\,l}_{i\,k}A_l, \tag{1.2.1}$$

where the quantity $\Gamma^{\,l}_{i\,k}$ (in the second term which is linear and homogeneous in A_i) transforms such that $A_{i;k}$ is a tensor:

$$A'_{i;k} = \frac{\partial x^l}{\partial x'^i}\frac{\partial x^m}{\partial x'^k}A_{l;m} = \frac{\partial x^l}{\partial x'^i}\frac{\partial x^m}{\partial x'^k}(A_{l,m} - \Gamma^{\,n}_{l\,m}A_n). \tag{1.2.2}$$

Also, (1.1.40) gives

$$A'_{i;k} = A'_{i,k} - \Gamma'^{\,l}_{i\,k}A'_l = \frac{\partial x^m}{\partial x'^k}\frac{\partial x^l}{\partial x'^i}A_{l,m} + \frac{\partial^2 x^n}{\partial x'^k \partial x'^i}A_n - \frac{\partial x^n}{\partial x'^l}\Gamma'^{\,l}_{i\,k}A_n, \tag{1.2.3}$$

so we obtain

$$\frac{\partial x^n}{\partial x'^l}\Gamma'^{\,l}_{i\,k} = \frac{\partial x^l}{\partial x'^i}\frac{\partial x^m}{\partial x'^k}\Gamma^{\,n}_{l\,m} + \frac{\partial^2 x^n}{\partial x'^k \partial x'^i}. \tag{1.2.4}$$

Multiplying this equation by $\frac{\partial x'^j}{\partial x^n}$ gives the transformation law for $\Gamma^{\,l}_{i\,k}$:

$$\Gamma'^{\,j}_{i\,k} = \frac{\partial x'^j}{\partial x^n}\frac{\partial x^l}{\partial x'^i}\frac{\partial x^m}{\partial x'^k}\Gamma^{\,n}_{l\,m} + \frac{\partial x'^j}{\partial x^n}\frac{\partial^2 x^n}{\partial x'^k \partial x'^i}. \tag{1.2.5}$$

The algebraic object $\Gamma^{\,l}_{i\,k}$, which equips spacetime in order to covariantize a derivative of a vector, is referred to as the *affine connection*, affinity or simply connection. The connection has generally 64 independent components. The tensor $A_{i;k}$ is the *covariant derivative* of a vector A_i with respect to x^i. We will also use $\nabla_i = {}_{;i}$ to denote the covariant derivative. The contracted affine connection transforms according to

$$\Gamma'^{\,i}_{i\,k} = \frac{\partial x^m}{\partial x'^k}\Gamma^{\,l}_{l\,m} + \frac{\partial x'^i}{\partial x^n}\frac{\partial^2 x^n}{\partial x'^k \partial x'^i}. \tag{1.2.6}$$

The affine connection is not a tensor because of the second term on the right-hand side of (1.2.5).

A derivative of a scalar is a covariant vector. Therefore, the covariant derivative of a scalar is equal to an ordinary derivative:

$$\phi_{;i} = \phi_{,i}. \tag{1.2.7}$$

If we also assume that the covariant derivative of the product of two tensors obeys the same chain rule as an ordinary derivative:

$$(TU)_{;i} = T_{;i}U + TU_{;i}, \tag{1.2.8}$$

then

$$A_{k,i}B^k + A_k B^k{}_{,i} = (A_k B^k)_{,i} = (A_k B^k)_{;i} = A_{k;i}B^k + A_k B^k{}_{;i} = A_{k,i}B^k - \Gamma^{\,k}_{l\,i}A_k B^l + A_k B^k{}_{;i}. \tag{1.2.9}$$

Therefore, we obtain the covariant derivative of a contravariant vector:

$$B^k{}_{;i} = B^k{}_{,i} + \Gamma^{\,k}_{l\,i} B^l. \tag{1.2.10}$$

The chain rule (1.2.8) also infers that the covariant derivative of a tensor is equal to the sum of the corresponding ordinary derivative of this tensor and terms with the affine connection that covariantize each index:

$$T^{ij\dots}{}_{kl\dots;m} = T^{ij\dots}{}_{kl\dots,m} + \Gamma^{\,i}_{n\,m} T^{nj\dots}{}_{kl\dots} + \Gamma^{\,j}_{n\,m} T^{in\dots}{}_{kl\dots} + \cdots - \Gamma^{\,n}_{k\,m} T^{ij\dots}{}_{nl\dots} - \Gamma^{\,n}_{l\,m} T^{ij\dots}{}_{kn\dots} - \dots . \tag{1.2.11}$$

the covariant derivative of the Kronecker symbol vanishes:

$$\delta^k_{l;i} = \Gamma^{\,k}_{j\,i} \delta^j_l - \Gamma^{\,j}_{l\,i} \delta^k_j = 0. \tag{1.2.12}$$

The second term on the right of (1.2.5) does not depend on the affine connection, but only on the coordinate transformation. Therefore, the difference between two different connections transforms like a tensor of rank (1,2). Consequently, the variation $\delta\Gamma^{\,j}_{i\,k}$, which is an infinitesimal difference between two connections, is a tensor of rank (1,2).

1.2.2 Parallel transport

Let us consider two infinitesimally separated points in spacetime, $P(x^i)$ and $Q(x^i + dx^i)$, and a vector field A which takes the value A^k at P and $A^k + dA^k$ at Q. Because $dA^k = A^k{}_{,i} dx^i$ and $A^k{}_{,i}$ is not a tensor, the difference dA^k between the vectors $A^k + dA^k$ and A^k is not a vector. The differential dA^k is not a vector because it arises from subtracting two vectors which are located at two points with different coordinate transformation laws. The transformation law for dA^k follows from (1.1.40):

$$dA'_k = d\left(\frac{\partial x^m}{\partial x'^k} A_m\right) = \frac{\partial x^m}{\partial x'^k} dA_m + d\left(\frac{\partial x^m}{\partial x'^k}\right) A_m = \frac{\partial x^m}{\partial x'^k} dA_m + \frac{\partial^2 x^m}{\partial x'^i \partial x'^k} A_m dx'^i. \tag{1.2.13}$$

In order to calculate the covariant difference between two vectors at two different points, we must bring these vectors to the same point. Instead of subtracting from the vector $A^k + dA^k$ at Q the vector A^k at P, we must subtract a vector $A^k + \delta A^k$ at Q that corresponds to A^k at P, thereby that the resulting difference (covariant differential) $DA^k = dA^k - \delta A^k$ is a vector. The vector $A^k + \delta A^k$ is the *parallel-transported* or parallel-translated A^k from P to Q. A parallel-transported linear combination of vectors must be equal to the same linear combination of parallel-transported vectors. Therefore, δA^k is a linear and homogeneous function of A^k. It is also on the order of a differential, thus a linear and homogeneous function of dx^i. The most general form of δA^k is

$$\delta A^k = -\Gamma^{\,k}_{l\,i} A^l dx^i, \tag{1.2.14}$$

so

$$DA^k = dA^k + \Gamma^{\,k}_{l\,i} A^l dx^i = A^k{}_{;i} dx^i. \tag{1.2.15}$$

Because δA^k is not a vector, $\Gamma^{\,k}_{l\,i}$ is not a tensor. Because DA^k is a vector, $A^k{}_{;i}$ is a tensor. The expressions for covariant derivatives of a covariant vector and tensors result from

$$\delta\phi = 0, \;\; \delta(TU) = \delta T U + T \delta U. \tag{1.2.16}$$

1.2.3 Torsion tensor

The second term on the right-hand side of (1.2.5) is symmetric in the indices i, k. Antisymmetrizing (1.2.5) with respect to these indices gives

$$S'^j{}_{ik} = \frac{\partial x'^j}{\partial x^n} \frac{\partial x^l}{\partial x'^i} \frac{\partial x^m}{\partial x'^k} S^n{}_{lm}, \tag{1.2.17}$$

where

$$S^j{}_{ik} = \Gamma^{\ \,j}_{[i\,k]} \tag{1.2.18}$$

is the antisymmetric (in the covariant indices) part of the affine connection. Equation (1.2.17) is a transformation formula for a tensor, thereby (1.2.18) is a tensor, called the Cartan *torsion tensor*. The torsion tensor has generally 24 independent components. The contracted torsion tensor,

$$S^k{}_{ik} = S_i, \tag{1.2.19}$$

is called the torsion trace or *torsion vector*.

1.2.4 Covariant differentiation of densities

The differential of the determinant S of a square matrix S is given by

$$d\mathsf{S} = s^k{}_i dS^i{}_k, \tag{1.2.20}$$

where $s^k{}_i$ is the minor corresponding to the component $S^i{}_k$ of the matrix. The components of the matrix S^{-1} inverse to S,

$$S^i{}_j(S^{-1})^j{}_k = (S^{-1})^i{}_j S^j{}_k = \delta^i_k, \tag{1.2.21}$$

are related to the minors of S by

$$(S^{-1})^k{}_i = \frac{s^k{}_i}{\mathsf{S}}. \tag{1.2.22}$$

The differential $d\mathsf{S}$ is therefore equal to

$$d\mathsf{S} = \mathsf{S}(S^{-1})^k{}_i dS^i{}_k = -\mathsf{S}S^k{}_i d(S^{-1})^i{}_k, \tag{1.2.23}$$

which is equivalent to

$$\partial_l\mathsf{S} = \mathsf{S}(S^{-1})^k{}_i \partial_l S^i{}_k = -\mathsf{S}S^k{}_i \partial_l (S^{-1})^i{}_k. \tag{1.2.24}$$

Taking $S^i{}_k = \frac{\partial x^i}{\partial x'^k}$ gives

$$\partial_l\left|\frac{\partial x^r}{\partial x'^s}\right| = \left|\frac{\partial x^r}{\partial x'^s}\right|\frac{\partial x'^n}{\partial x^m}\frac{\partial}{\partial x^l}\frac{\partial x^m}{\partial x'^n}. \tag{1.2.25}$$

A derivative of a scalar density $\mathfrak{s}$ of weight w does not transform like a covariant vector density:

$$\begin{aligned}
\partial'_i\mathfrak{s}' &= \frac{\partial x^l}{\partial x'^i}\partial_l\left(\left|\frac{\partial x^j}{\partial x'^k}\right|^w\mathfrak{s}\right) = \frac{\partial x^l}{\partial x'^i}\left|\frac{\partial x^j}{\partial x'^k}\right|^w\partial_l\mathfrak{s} + w\frac{\partial x^l}{\partial x'^i}\left|\frac{\partial x^j}{\partial x'^k}\right|^{w-1}\partial_l\left|\frac{\partial x^r}{\partial x'^s}\right|\mathfrak{s} \\
&= \frac{\partial x^l}{\partial x'^i}\left|\frac{\partial x^j}{\partial x'^k}\right|^w\partial_l\mathfrak{s} + w\frac{\partial x^l}{\partial x'^i}\left|\frac{\partial x^j}{\partial x'^k}\right|^{w-1}\left|\frac{\partial x^r}{\partial x'^s}\right|\frac{\partial x'^n}{\partial x^m}\frac{\partial}{\partial x^l}\frac{\partial x^m}{\partial x'^n}\mathfrak{s} \\
&= \frac{\partial x^l}{\partial x'^i}\left|\frac{\partial x^j}{\partial x'^k}\right|^w\partial_l\mathfrak{s} + w\left|\frac{\partial x^j}{\partial x'^k}\right|^w\frac{\partial x'^n}{\partial x^m}\frac{\partial^2 x^m}{\partial x'^n\partial x'^i}\mathfrak{s}.
\end{aligned} \tag{1.2.26}$$

Let us consider the expression

$$\mathfrak{s}_{;i} = \mathfrak{s}_{,i} - w\Gamma_i\mathfrak{s}, \tag{1.2.27}$$

where the quantity Γ_i transforms such that $\mathfrak{s}_{;i}$ is a vector density of weight w:

$$\mathfrak{s}'_{;i} = \frac{\partial x^l}{\partial x'^i}\left|\frac{\partial x^j}{\partial x'^k}\right|^w\mathfrak{s}_{;l} = \frac{\partial x^l}{\partial x'^i}\left|\frac{\partial x^j}{\partial x'^k}\right|^w(\mathfrak{s}_{,l} - w\Gamma_l\mathfrak{s}). \tag{1.2.28}$$

Also, (1.2.26) gives

$$\mathfrak{s}'_{;i} = \mathfrak{s}'_{,i} - w\Gamma'_i\mathfrak{s}' = \frac{\partial x^l}{\partial x'^i}\left|\frac{\partial x^j}{\partial x'^k}\right|^w\partial_l\mathfrak{s} + w\left|\frac{\partial x^j}{\partial x'^k}\right|^w\frac{\partial x'^n}{\partial x^m}\frac{\partial^2 x^m}{\partial x'^n\partial x'^i}\mathfrak{s} - w\left|\frac{\partial x^j}{\partial x'^k}\right|^w\Gamma'_i\mathfrak{s}, \tag{1.2.29}$$

so we obtain the transformation law for Γ_i:

$$\Gamma'_i = \frac{\partial x^l}{\partial x'^i}\Gamma_l + \frac{\partial x'^n}{\partial x^m}\frac{\partial^2 x^m}{\partial x'^n \partial x'^i}, \tag{1.2.30}$$

which is the same as the transformation law for $\Gamma^{\,k}_{k\,i}$ (1.2.6). Therefore, the difference $\Gamma_i - \Gamma^{\,k}_{k\,i}$ is some covariant vector V_i.

If we assume that parallel transport of the product of a scalar density of any weight and a tensor obeys the chain rule:

$$\delta(\mathfrak{s}T) = \delta\mathfrak{s}T + \mathfrak{s}\delta T, \tag{1.2.31}$$

so the covariant derivative of such product behaves like an ordinary derivative:

$$(\mathfrak{s}T)_{;i} = \mathfrak{s}_{;i}T + \mathfrak{s}T_{;i}, \tag{1.2.32}$$

then the covariant derivative of a tensor density of weight w is equal to the sum of the corresponding ordinary derivative of this tensor, terms with the affine connection that covariantize each index, and the term with Γ_i:

$$\begin{aligned}&\mathfrak{T}^{ij\dots}_{\quad kl\dots;m} = \mathfrak{T}^{ij\dots}_{\quad kl\dots,m} + \Gamma^{\,i}_{n\,m}\mathfrak{T}^{nj\dots}_{\quad kl\dots} + \Gamma^{\,j}_{n\,m}\mathfrak{T}^{in\dots}_{\quad kl\dots} + \dots \\ &-\Gamma^{\,n}_{k\,m}\mathfrak{T}^{ij\dots}_{\quad nl\dots} - \Gamma^{\,n}_{l\,m}\mathfrak{T}^{ij\dots}_{\quad kn\dots} - \dots - w\Gamma_m\mathfrak{T}^{ij\dots}_{\quad kl\dots}.\end{aligned} \tag{1.2.33}$$

the covariant derivative of the contravariant Levi-Civita density is

$$\epsilon^{ijkl}_{\quad\ ;m} = \Gamma^{\,i}_{n\,m}\epsilon^{njkl} + \Gamma^{\,j}_{n\,m}\epsilon^{inkl} + \Gamma^{\,k}_{n\,m}\epsilon^{ijnl} + \Gamma^{\,l}_{n\,m}\epsilon^{ijkn} - \Gamma_m\epsilon^{ijkl}. \tag{1.2.34}$$

In the summations over n only one term does not vanish for each term on the right-hand side of (1.2.34), thereby

$$\begin{aligned}&\epsilon^{ijkl}_{\quad\ ;m} = \Gamma^{\,i}_{n=i|m}\epsilon^{n=i|jkl} + \Gamma^{\,j}_{n=j|m}\epsilon^{i|n=j|kl} + \Gamma^{\,k}_{n=k|m}\epsilon^{ij|n=k|l} + \Gamma^{\,l}_{n=l|m}\epsilon^{ijk|n=l} - \Gamma_m\epsilon^{ijkl} \\ &= (\Gamma^{\,n}_{n\,m} - \Gamma_m)\epsilon^{ijkl} = -V_m\epsilon^{ijkl}.\end{aligned} \tag{1.2.35}$$

The Levi-Civita symbol is a tensor density with constant components, thereby it does not change under a parallel transport, $\delta\epsilon = 0$. Therefore, we have

$$\epsilon^{ijkl}_{\quad\ ;m} = 0. \tag{1.2.36}$$

By means of (1.1.27), we also have

$$\varepsilon_{ijkl;m} = 0. \tag{1.2.37}$$

Consequently, we obtain $V_i = 0$ and

$$\Gamma_i = \Gamma^{\,k}_{k\,i}. \tag{1.2.38}$$

1.2.5 Antisymmetric covariant derivatives

Completely antisymmetrized ordinary derivatives of tensors, $A_{[i,k]}$, $B_{[ik,l]}$ and $C_{[ikl,m]}$, are tensors because of their antisymmetry. Completely antisymmetrized covariant derivatives of tensors are tensors because ∇_i is a covariant operation, and are given by direct calculation using the definition of the covariant derivative:

$$A_{[i;k]} = A_{[i,k]} - S^l_{\ ik}A_l, \ \ B_{[ik;l]} = B_{[ik,l]} - 2S^m_{\ \ [ik}B_{l]m}. \tag{1.2.39}$$

Divergences of (completely antisymmetric if more than 1 index) contravariant densities, $\mathfrak{C}^i_{\ ,i}$, $\mathfrak{B}^{ik}_{\ \ ,i}$ and $\mathfrak{A}^{ikl}_{\ \ \ ,i}$, are densities because of the correspondence between tensors and dual densities. Covariant divergences of contravariant densities are densities, and are given by direct calculation:

$$\mathfrak{C}^i_{\ ;i} = \mathfrak{C}^i_{\ ,i} + 2S_i\mathfrak{C}^i, \ \ \mathfrak{B}^{ik}_{\ \ ;i} = \mathfrak{B}^{ik}_{\ \ ,i} - S^k_{\ il}\mathfrak{B}^{il} + 2S_i\mathfrak{B}^{ik}. \tag{1.2.40}$$

1.2.6 Partial integration

If the product of two quantities (tensors or densities) TU is a contravariant density $\mathfrak{C}^k$ then

$$\int TU_{;k}d\Omega = \int (TU)_{;k}d\Omega - \int T_{;k}Ud\Omega = \int (TU)_{,k}d\Omega + 2\int S_k TUd\Omega - \int T_{;k}Ud\Omega. \qquad (1.2.41)$$

The first term on the right-hand side can be transformed into a hypersurface integral $\int TUdS_k$. If the region of integration extends to infinity and $\mathfrak{C}^k$ corresponds to some physical quantity then the boundary integral $\int TUdS_k$ vanishes, giving

$$\int TU_{;k}d\Omega = 2\int S_k TUd\Omega - \int T_{;k}Ud\Omega. \qquad (1.2.42)$$

If $T = \delta^k_i$, then $U = \mathfrak{C}^i$ and

$$\int \mathfrak{C}^i{}_{;i}d\Omega = 2\int S_i\mathfrak{C}^i d\Omega. \qquad (1.2.43)$$

Equation (1.2.43) can be written as

$$\int \nabla^*_i\mathfrak{C}^i d\Omega = 0, \qquad (1.2.44)$$

where

$$\nabla^*_i = \nabla_i - 2S_i \qquad (1.2.45)$$

is the *modified covariant derivative.*

1.2.7 Geodesic frame of reference

Let us consider a coordinate transformation

$$x^k = x'^k + \frac{1}{2}a^k{}_{lm}x'^l x'^m, \qquad (1.2.46)$$

where $a^k{}_{lm}$ is symmetric in the indices l, m. Substituting this transformation to (1.2.5) and calculating it at $x^k = x'^k = 0$ gives

$$\frac{\partial x^i}{\partial x'^k} = \delta^i_k \qquad (1.2.47)$$

and

$$\Gamma'^{\,j}_{i\,k} = \Gamma^{\,j}_{i\,k} + a^j{}_{ik}. \qquad (1.2.48)$$

Putting

$$a^j{}_{ik} = -\Gamma^{\,j}_{(i\,k)}|_{x^l=0} \qquad (1.2.49)$$

gives

$$\Gamma'^{\,j}_{(i\,k)} = 0. \qquad (1.2.50)$$

Therefore, there always exists a coordinate frame of reference in which the symmetric part of the connection vanishes locally (at one point). If the affine connection is symmetric in the covariant indices, $\Gamma^{\,j}_{i\,k} = \Gamma^{\,j}_{k\,i}$ (the torsion tensor vanishes), then (1.2.50) gives

$$\Gamma'^{\,j}_{i\,k} = 0. \qquad (1.2.51)$$

The coordinate frame of reference in which the torsionless part of the connection vanishes (locally) is referred to as *geodesic.*

1.2.8 Affine geodesics and four-velocity

Let us consider a point in spacetime $P(x^k)$ and a vector dx^k at this point. We construct a point $P'(x^k + dx^k)$ and find the vector $d'x^k$ which is the parallel-transported dx^k from P to P'. Then construct a point $P''(x^k + dx^k + d'x^k)$ and find the vector $d''x^k$ which is the parallel-transported $d'x^k$ from P' to P''. The next point is $P'''(x^k + dx^k + d'x^k + d''x^k)$ etc. Repeating this step constructs a polygonal line which in the limit $dx^k \to 0$ becomes a curve such that the vector $\frac{dx^k}{d\lambda}$ (where λ is a parameter along the curve) tangent to it at any point, when parallely translated to another point on this curve, coincides with the tangent vector there. Such curve is referred to as an autoparallel curve or *affine geodesic*. Affine geodesics can be attributed with the concept of length, which, for the polygonal curve, is proportional to the number of parallel-transport steps described above.

The condition that parallel transport of a tangent vector be a tangent vector is

$$\frac{dx^i}{d\lambda} + \delta\bigg(\frac{dx^i}{d\lambda}\bigg) = \frac{dx^i}{d\lambda} - \Gamma^{\,i}_{k\,l}\frac{dx^k}{d\lambda}dx^l = M\bigg(\frac{dx^i}{d\lambda} + \frac{d^2x^i}{d\lambda^2}d\lambda\bigg), \tag{1.2.52}$$

where the proportionality factor M is some function of λ, or

$$M\frac{d^2x^i}{d\lambda^2} + \Gamma^{\,i}_{k\,l}\frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda} = \frac{1-M}{d\lambda}\frac{dx^i}{d\lambda}, \tag{1.2.53}$$

from which it follows that M must differ from 1 by the order of $d\lambda$. In the first term on the left-hand side of (1.2.53) we can therefore put $M = 1$, and we denote $1 - M$ by $\phi(\lambda)d\lambda$, thereby

$$\frac{d^2x^i}{d\lambda^2} + \Gamma^{\,i}_{k\,l}\frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda} = \phi(\lambda)\frac{dx^i}{d\lambda}. \tag{1.2.54}$$

If we replace λ by a new variable $s(\lambda)$ then (1.2.54) becomes

$$\frac{d^2x^i}{ds^2} + \Gamma^{\,i}_{k\,l}\frac{dx^k}{ds}\frac{dx^l}{ds} = \frac{\phi s' - s''}{s'^2}\frac{dx^i}{ds}, \tag{1.2.55}$$

where the prime denotes differentiation with respect to λ. Requiring $\phi s' - s'' = 0$, which has a general solution $s = \int^\lambda d\lambda \exp[-\int^\lambda \phi(x)dx]$, brings (1.2.55) to

$$\frac{d^2x^i}{ds^2} + \Gamma^{\,i}_{k\,l}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0, \tag{1.2.56}$$

where the scalar variable s is called the *affine parameter*. The autoparallel equation (1.2.56) is invariant under linear transformations $s \to as + b$ since the two lower limits of integration in the expression for $s(\lambda)$ are arbitrary.

We define the *four-velocity* vector:

$$u^i = \frac{dx^i}{ds}. \tag{1.2.57}$$

This definition brings (1.2.15) to

$$\frac{DA^k}{ds} = A^k_{\ ;i}u^i, \quad \frac{dA^k}{ds} = A^k_{\ ,i}u^i, \tag{1.2.58}$$

thereby

$$\frac{Du^i}{ds} = \frac{du^i}{ds} + \Gamma^{\,i}_{k\,l}u^k u^l = u^i_{\ ;j}u^j = 0. \tag{1.2.59}$$

The relations (1.2.58) can be generalized to any tensor density T:

$$\frac{DT}{ds} = T_{;i}u^i, \quad \frac{dT}{ds} = T_{,i}u^i, \tag{1.2.60}$$

The vector $\frac{dx^i}{ds}|_Q$ is a parallel translation of $\frac{dx^i}{ds}|_P$. Because ds is a scalar, it is invariant under parallel transport, $ds|_Q = ds|_P$. Therefore, the vector $dx^i|_Q$ is a parallel translation of $dx^i|_P$, thereby ds measures the length of an infinitesimal section of an affine geodesic.

Only the symmetric part $\Gamma^{\,i}_{(k\,l)}$ of the connection enters the autoparallel equation (1.2.56) because of the symmetry of $\frac{dx^k}{ds}\frac{dx^l}{ds}$ with respect to the indices k, l; affine geodesics do not depend on torsion. At any point, a coordinate transformation to the geodesic frame (1.2.46) brings all the components $\Gamma^{\,i}_{(k\,l)}$ to zero, thereby the autoparallel equation becomes $\frac{du^i}{ds} = 0$. The autoparallel equation is also invariant under a *projective* transformation

$$\Gamma^{\,i}_{k\,l} \to \Gamma^{\,i}_{k\,l} + \delta^i_k A_l, \tag{1.2.61}$$

where A_i is an arbitrary vector. Substituting this transformation to (1.2.59) gives

$$\frac{du^i}{ds} + \Gamma^{\,i}_{k\,l}u^k u^l = -u^i u^k A_k. \tag{1.2.62}$$

If we replace s by a new variable $\tilde{s}(s)$ then (1.2.62) becomes

$$\frac{dU^i}{d\tilde{s}} + \Gamma^{\,i}_{k\,l}U^k U^l = -\frac{u^k A_k \tilde{s}' + \tilde{s}''}{\tilde{s}'^2}\frac{dx^i}{d\tilde{s}}, \tag{1.2.63}$$

where

$$U^i = \frac{dx^i}{d\tilde{s}} \tag{1.2.64}$$

and the prime denotes differentiation with respect to s. Requiring $u^k A_k \tilde{s}' + \tilde{s}'' = 0$, which has a general solution $\tilde{s} = -\int^s ds \exp[\int^s A_k u^k(x)dx]$, brings (1.2.63) to

$$\frac{dU^i}{d\tilde{s}} + \Gamma^{\,i}_{k\,l}U^k U^l = 0. \tag{1.2.65}$$

1.2.9 Infinitesimal coordinate transformations

Let us consider a coordinate transformation

$$x^i \to x'^i = x^i + \xi^i, \tag{1.2.66}$$

where $\xi^i = \delta x^i$ is an infinitesimal vector (a variation of x^i). For a tensor or density T define

$$\delta T = T'(x'^i) - T(x^i), \tag{1.2.67}$$
$$\bar{\delta}T = T'(x^i) - T(x^i) = \delta T - \xi^k T_{,k}. \tag{1.2.68}$$

For a scalar we find

$$\delta\phi = 0,\ \ \bar{\delta}\phi = -\xi^k \phi_{,k}. \tag{1.2.69}$$

For a covariant vector

$$\delta A_i = \frac{\partial x^k}{\partial x'^i}A_k - A_i \approx -\xi^k_{\ ,i}A_k, \tag{1.2.70}$$
$$\bar{\delta}A_i \approx -\xi^k_{\ ,i}A_k - \xi^k A_{i,k}. \tag{1.2.71}$$

The variation (1.2.70) is not a tensor, but (1.2.71) is:

$$\bar{\delta}A_i = -\xi^k_{\ ;i}A_k - \xi^k A_{i;k} - 2S^j_{\ ik}\xi^k A_j. \tag{1.2.72}$$

We refer to $-\bar{\delta}T$ as the *Lie derivative* of T along the vector ξ^i, $\mathcal{L}_\xi T$. For a contravariant vector

$$\delta B^i = \frac{\partial x'^i}{\partial x^k}B^k - B^i \approx \xi^i_{\ ,k}B^k, \tag{1.2.73}$$
$$\bar{\delta}B^i \approx \xi^i_{\ ,k}B^k - \xi^k B^i_{\ ,k} = \xi^i_{\ ;k}B^k - \xi^k B^i_{\ ;k} + 2S^i_{\ jk}\xi^k B^j. \tag{1.2.74}$$

For a scalar density

$$\delta\mathfrak{s} = \left(\left|\frac{\partial x^i}{\partial x'^i}\right| - 1\right)\mathfrak{s} \approx -\xi^i_{\ ,i}\mathfrak{s}, \tag{1.2.75}$$

$$\bar{\delta}\mathfrak{s} \approx -\xi^i_{\ ,i}\mathfrak{s} - \xi^k\mathfrak{s}_{,k} = -\xi^i_{\ ;i}\mathfrak{s} - \xi^k\mathfrak{s}_{;k} + 2S_i\xi^i\mathfrak{s}. \tag{1.2.76}$$

The chain rule for δ infers that, for a tensor density of weight w (which includes tensors as densities of weight 0), we have

$$\delta\mathfrak{T}^{ij\dots}_{\quad kl\dots} \approx \xi^i_{\ ,m}\mathfrak{T}^{mj\dots}_{\quad kl\dots} + \xi^j_{\ ,m}\mathfrak{T}^{im\dots}_{\quad kl\dots} + \dots - \xi^m_{\ ,k}\mathfrak{T}^{ij\dots}_{\quad ml\dots} - \xi^m_{\ ,l}\mathfrak{T}^{ij\dots}_{\quad km\dots} - \dots$$
$$-w\xi^m_{\ ,m}\mathfrak{T}^{ij\dots}_{\quad kl\dots}, \tag{1.2.77}$$
$$\bar{\delta}\mathfrak{T}^{ij\dots}_{\quad kl\dots} \approx \xi^i_{\ ;m}\mathfrak{T}^{mj\dots}_{\quad kl\dots} + \xi^j_{\ ;m}\mathfrak{T}^{im\dots}_{\quad kl\dots} + \dots - \xi^m_{\ ;k}\mathfrak{T}^{ij\dots}_{\quad ml\dots} - \xi^m_{\ ;l}\mathfrak{T}^{ij\dots}_{\quad km\dots} - \dots$$
$$-w\xi^m_{\ ;m}\mathfrak{T}^{ij\dots}_{\quad kl\dots} - \xi^m\mathfrak{T}^{ij\dots}_{\quad kl\dots;m} + 2S^i_{\ nm}\xi^m\mathfrak{T}^{nj\dots}_{\quad kl\dots} + 2S^j_{\ nm}\xi^m\mathfrak{T}^{in\dots}_{\quad kl\dots} + \dots$$
$$-2S^n_{\ km}\xi^m\mathfrak{T}^{ij\dots}_{\quad nl\dots} - 2S^n_{\ lm}\xi^m\mathfrak{T}^{ij\dots}_{\quad kn\dots} - \dots + 2wS_m\xi^m\mathfrak{T}^{ij\dots}_{\quad kl\dots}. \tag{1.2.78}$$

A Lie derivative of a tensor density of rank (k,l) and weight w is a tensor density of rank (k,l) and weight w.

The formula for the covariant derivative of T can be written as

$$T_{;k} = T_{,k} + \Gamma^{\ j}_{i\ k}\hat{C}^i_j T, \tag{1.2.79}$$

where $\hat{C}$ is an operator acting on tensor densities:

$$\hat{C}^i_j\phi = 0,\ \ \hat{C}^i_j A_k = -\delta^i_k A_j,\ \ \hat{C}^i_j B^k = \delta^k_j B^i,\ \ \hat{C}^i_j\mathfrak{s} = -\delta^i_j\mathfrak{s}, \tag{1.2.80}$$

or generally

$$\hat{C}^m_n\mathfrak{T}^{ij\dots}_{\quad kl\dots} = \delta^i_n\mathfrak{T}^{mj\dots}_{\quad kl\dots} + \delta^j_n\mathfrak{T}^{im\dots}_{\quad kl\dots} + \dots - \delta^m_k\mathfrak{T}^{ij\dots}_{\quad nl\dots} - \delta^m_l\mathfrak{T}^{ij\dots}_{\quad kn\dots} - \dots - w\delta^m_n\mathfrak{T}^{ij\dots}_{\quad kl\dots}. \tag{1.2.81}$$

Such defined operator also enters the formula for δT:

$$\delta T = \hat{C}^k_i T\xi^i_{\ ,k}. \tag{1.2.82}$$

1.2.10 Killing vectors

A covariant vector ζ_i that satisfies

$$\zeta_{(i;k)} = 0 \tag{1.2.83}$$

is referred to as a *Killing vector*. Along an affine geodesic,

$$\frac{D}{ds}(u^i\zeta_i) = u^k(u^i\zeta_i)_{;k} = u^iu^k\zeta_{i;k} + \zeta_iu^ku^i_{\ ;k} = 0. \tag{1.2.84}$$

The first term in the sum in (1.2.84) vanishes because of the definition of ζ_i and the second term vanishes because of the affine geodesic equation. Therefore, to each Killing vector ζ_i there corresponds a quantity $u^i\zeta_i$ which does not change along the affine geodesic:

$$u^i\zeta_i = \text{const.} \tag{1.2.85}$$

References: [1, 2, 3].

1.3 Curvature

1.3.1 Curvature tensor

We define the *commutator* $[A,B]$ of two operators A and B as

$$[A,B]=AB-BA=-[B,A]. \tag{1.3.1}$$

The commutator of covariant derivatives is thus

$$[\nabla_i,\nabla_k]=2\nabla_{[i}\nabla_{k]}. \tag{1.3.2}$$

The commutator of covariant derivatives of a contravariant vector is a tensor:

$$\begin{aligned}
&[\nabla_j,\nabla_k]B^i=2\nabla_{[j}\nabla_{k]}B^i=2\partial_{[j}\nabla_{k]}B^i-2\Gamma^{\,l}_{[k\,j]}\nabla_l B^i+2\Gamma^{\,i}_{l\,[j}\nabla_{k]}B^l\\
&=2\partial_{[j}(\Gamma^{\,i}_{|m|\,k]}B^m)+2S^l_{\ jk}\nabla_l B^i+2\Gamma^{\,i}_{l\,[j}\partial_{k]}B^l+2\Gamma^{\,i}_{l\,[j}\Gamma^{\,l}_{|m|\,k]}B^m\\
&=2(\partial_{[j}\Gamma^{\,i}_{|m|\,k]}+\Gamma^{\,i}_{l\,[j}\Gamma^{\,l}_{|m|\,k]})B^m+2S^l_{\ jk}\nabla_l B^i=R^i_{\ mjk}B^m+2S^l_{\ jk}\nabla_l B^i,
\end{aligned} \tag{1.3.3}$$

where || embraces indices which are excluded from symmetrization or antisymmetrization. Therefore, $R^i_{\ mjk}$, defined as

$$R^i_{\ mjk}=\partial_j\Gamma^{\,i}_{m\,k}-\partial_k\Gamma^{\,i}_{m\,j}+\Gamma^{\,l}_{m\,k}\Gamma^{\,i}_{l\,j}-\Gamma^{\,l}_{m\,j}\Gamma^{\,i}_{l\,k}, \tag{1.3.4}$$

is a tensor, referred to as the *curvature tensor*. The curvature tensor $R^i_{\ mjk}$ is antisymmetric in the indices j,k and has generally 96 independent components. The commutator of covariant derivatives of a covariant vector is

$$[\nabla_j,\nabla_k]A_i=-R^m_{\ ijk}A_m+2S^l_{\ jk}\nabla_l A_i, \tag{1.3.5}$$

and the commutator of covariant derivatives of a tensor is

$$\begin{aligned}
&[\nabla_j,\nabla_k]T^{in\dots}_{\ \ \ lp\dots}=R^i_{\ mjk}T^{mn\dots}_{\ \ \ \ lp\dots}+R^n_{\ mjk}T^{im\dots}_{\ \ \ \ lp\dots}+\dots-R^m_{\ ljk}T^{in\dots}_{\ \ \ mp\dots}-R^m_{\ pjk}T^{in\dots}_{\ \ \ lm\dots}\\
&-\dots+2S^l_{\ jk}\nabla_l T^{in\dots}_{\ \ \ lp\dots}.
\end{aligned} \tag{1.3.6}$$

A change in the connection,

$$\tilde{\Gamma}^{\,i}_{j\,k}=\Gamma^{\,i}_{j\,k}+T^i_{\ jk}, \tag{1.3.7}$$

where $T^i_{\ jk}$ is a tensor, results in the following change of the curvature tensor:

$$\begin{aligned}
&\tilde{R}^i_{\ klm}=\tilde{\Gamma}^{\,i}_{k\,m,l}-\tilde{\Gamma}^{\,i}_{k\,l,m}+\tilde{\Gamma}^{\,j}_{k\,m}\tilde{\Gamma}^{\,i}_{j\,l}-\tilde{\Gamma}^{\,j}_{k\,l}\tilde{\Gamma}^{\,i}_{j\,m}=\Gamma^{\,i}_{k\,m,l}-\Gamma^{\,i}_{k\,l,m}+\Gamma^{\,j}_{k\,m}\Gamma^{\,i}_{j\,l}-\Gamma^{\,j}_{k\,l}\Gamma^{\,i}_{j\,m}\\
&+T^i_{\ km,l}-T^i_{\ kl,m}+\Gamma^{\,j}_{k\,m}T^i_{\ jl}-\Gamma^{\,j}_{k\,l}T^i_{\ jm}+\Gamma^{\,i}_{j\,l}T^j_{\ km}-\Gamma^{\,i}_{j\,m}T^j_{\ kl}+T^j_{\ km}T^i_{\ jl}-T^j_{\ kl}T^i_{\ jm}\\
&=R^i_{\ klm}+T^i_{\ km;l}-T^i_{\ kl;m}+T^j_{\ km}T^i_{\ jl}-T^j_{\ kl}T^i_{\ jm}.
\end{aligned} \tag{1.3.8}$$

For a projective transformation (1.2.61), $T^i_{\ jk}=\delta^i_j A_k$, the curvature tensor changes according to

$$\tilde{R}^i_{\ klm}=R^i_{\ klm}+\delta^i_k(A_{m;l}-A_{l;m}). \tag{1.3.9}$$

The variation of the curvature tensor is

$$\begin{aligned}
&\delta R^i_{\ klm}=(\delta\Gamma^{\,i}_{k\,m})_{,l}-(\delta\Gamma^{\,i}_{k\,l})_{,m}+\delta\Gamma^{\,i}_{j\,l}\Gamma^{\,j}_{k\,m}+\Gamma^{\,i}_{j\,l}\delta\Gamma^{\,j}_{k\,m}-\delta\Gamma^{\,i}_{j\,m}\Gamma^{\,j}_{k\,l}-\Gamma^{\,i}_{j\,m}\delta\Gamma^{\,j}_{k\,l}\\
&=(\delta\Gamma^{\,i}_{k\,m})_{;l}-\Gamma^{\,i}_{j\,l}\delta\Gamma^{\,j}_{k\,m}+\Gamma^{\,j}_{k\,l}\delta\Gamma^{\,i}_{j\,m}+\Gamma^{\,j}_{m\,l}\delta\Gamma^{\,i}_{k\,j}-(\delta\Gamma^{\,i}_{k\,l})_{;m}+\Gamma^{\,i}_{j\,m}\delta\Gamma^{\,j}_{k\,l}-\Gamma^{\,j}_{k\,m}\delta\Gamma^{\,i}_{j\,l}\\
&-\Gamma^{\,j}_{l\,m}\delta\Gamma^{\,i}_{k\,j}+\delta\Gamma^{\,i}_{j\,l}\Gamma^{\,j}_{k\,m}+\Gamma^{\,i}_{j\,l}\delta\Gamma^{\,j}_{k\,m}-\delta\Gamma^{\,i}_{j\,m}\Gamma^{\,j}_{k\,l}-\Gamma^{\,i}_{j\,m}\delta\Gamma^{\,j}_{k\,l}\\
&=(\delta\Gamma^{\,i}_{k\,m})_{;l}-(\delta\Gamma^{\,i}_{k\,l})_{;m}-2S^n_{\ lm}\delta\Gamma^{\,i}_{k\,n}.
\end{aligned} \tag{1.3.10}$$

1.3.2 Integrability of connection

The affine connection is *integrable* if parallel transport of a vector from point P to point Q is independent of a path along which this vector is parallelly translated, or equivalently, parallel transport of a vector around a closed curve does not change this vector. For an integrable connection, we can uniquely translate parallelly a given vector h^i at point P to all points in spacetime:

$$\delta h^i = dh^i, \tag{1.3.11}$$

or

$$h^i_{\ ,k} = -\Gamma^{\ i}_{j\ k} h^j. \tag{1.3.12}$$

Therefore, we have

$$(\Gamma^{\ i}_{j\ k} h^j)_{,l} - (\Gamma^{\ i}_{j\ l} h^j)_{,k} = \Gamma^{\ i}_{j\ k,l} h^j - \Gamma^{\ i}_{j\ k} \Gamma^{\ j}_{m\ l} h^m - \Gamma^{\ i}_{j\ l,k} h^j + \Gamma^{\ i}_{j\ l} \Gamma^{\ j}_{m\ k} h^m = R^i_{\ jlk} h^j = 0, \tag{1.3.13}$$

so, because h^i is arbitrary,

$$R^i_{\ klm} = 0. \tag{1.3.14}$$

Spacetime with a vanishing curvature tensor $R^i_{\ klm} = 0$ is *flat*. Let us consider 4 linearly independent vectors h^i_a, where a is 1,2,3,4, and vectors inverse to h^i_a:

$$\sum_a h^i_a h_{ka} = \delta^i_k. \tag{1.3.15}$$

If the affine connection is integrable then (1.3.12) becomes

$$h^i_{a,k} = -\Gamma^{\ i}_{l\ k} h^l_a. \tag{1.3.16}$$

Multiplying (1.3.16) by h_{ja} gives

$$\Gamma^{\ i}_{j\ k} = -h_{ja} h^i_{a,k} = h_{ja,k} h^i_a. \tag{1.3.17}$$

An integrable connection has thus 16 independent components. If the connection is also symmetric, $S^i_{\ jk} = 0$, then

$$h_{ja,k} - h_{ka,j} = 0, \tag{1.3.18}$$

which is the condition for the independence of the coordinates

$$y_a = \int_P^Q h_{ia} dx^i \tag{1.3.19}$$

of the path of integration PQ. Adopting y_a as the new coordinates (with point $P = (0,0,0,0)$ in the center) gives

$$\frac{\partial y_a}{\partial x^i} = h_{ia}, \quad \frac{\partial x^i}{\partial y_a} = h^i_a, \tag{1.3.20}$$

so (1.3.17) becomes

$$\Gamma^{\ i}_{j\ k}(x^i) = \frac{\partial x^i}{\partial y_a} \frac{\partial^2 y_a}{\partial x^k \partial x^j}. \tag{1.3.21}$$

The transformation law for the connection (1.2.5) gives (with y_a corresponding to x'^j)

$$\Gamma^{\ i}_{j\ k}(y_a) = 0. \tag{1.3.22}$$

A torsionless integrable connection can be thus transformed to zero; one can always find a system of coordinates which is geodesic everywhere. If a connection is symmetric but nonintegrable then a geodesic frame of reference can be constructed only at a given point (or along a given world line).

1.3.3 Parallel transport along closed curve

Let us consider parallel transport of a covariant vector around an infinitesimal closed curve. Such a transport changes this vector, according to Stokes' theorem (1.1.37) by

$$\Delta A_k = \oint \delta A_k = \oint \Gamma^{\,i}_{k\,l} A_i dx^l = \frac{1}{2}\int \Big(\frac{\partial(\Gamma^{\,i}_{k\,m}A_i)}{\partial x^l} - \frac{\partial(\Gamma^{\,i}_{k\,l}A_i)}{\partial x^m}\Big) df^{lm}. \tag{1.3.23}$$

Along the curve, we have $dA_k = \Gamma^{\,i}_{k\,l} A_i dx^l$, which gives $A_{k,l} = \Gamma^{\,i}_{k\,l} A_i$. The last relation is approximately valid, to terms of first order in $\Delta f^{lm} = \int df^{lm}$, inside this curve:

$$\Delta A_k \approx \frac{1}{2}\int \Big[\Big(\frac{\partial\Gamma^{\,i}_{k\,m}}{\partial x^l} - \frac{\partial\Gamma^{\,i}_{k\,l}}{\partial x^m}\Big)A_i + (\Gamma^{\,i}_{k\,m}\Gamma^{\,n}_{i\,l} - \Gamma^{\,i}_{k\,l}\Gamma^{\,n}_{i\,m})A_n\Big] df^{lm} \approx \frac{1}{2}R^i_{klm}A_i\Delta f^{lm}. \tag{1.3.24}$$

The change of a contravariant vector in parallel transport around an infinitesimal closed curve results from $\Delta(A_k B^k) = 0$:

$$\Delta B^k \approx -\frac{1}{2}R^k_{ilm}B^i\Delta f^{lm}, \tag{1.3.25}$$

and the corresponding change of a tensor results from the chain rule for parallel transport:

$$\Delta T^{ik\dots}_{np\dots} \approx -\frac{1}{2}(R^i_{jlm}T^{jk\dots}_{np\dots} + R^k_{jlm}T^{ij\dots}_{np\dots} + \dots - R^j_{nlm}T^{ik\dots}_{jp\dots} - R^j_{plm}T^{ik\dots}_{nj\dots} - \dots)\Delta f^{lm}. \tag{1.3.26}$$

1.3.4 Bianchi identities

Let us consider

$$\nabla_j\nabla_{[k}\nabla_{l]}B^i = \frac{1}{2}\nabla_j(R^i_{mkl}B^m) + \nabla_j(S^m_{kl}\nabla_m B^i) \tag{1.3.27}$$

and

$$\nabla_{[j}\nabla_{k]}\nabla_l B^i = -\frac{1}{2}R^m_{ljk}\nabla_m B^i + \frac{1}{2}R^i_{mjk}\nabla_l B^m + S^m_{jk}\nabla_m\nabla_l B^i = -\frac{1}{2}R^m_{ljk}\nabla_m B^i$$
$$+\frac{1}{2}R^i_{mjk}\nabla_l B^m + S^m_{jk}\nabla_l\nabla_m B^i + S^m_{jk}R^i_{nml}B^n + 2S^m_{jk}S^n_{ml}\nabla_n B^i. \tag{1.3.28}$$

Total antisymmetrization of the indices j, k, l in (1.3.27) and (1.3.28) gives

$$\nabla_{[j}\nabla_k\nabla_{l]}B^i = \frac{1}{2}\nabla_{[j}R^i_{|m|kl]}B^m + \frac{1}{2}R^i_{m[kl}\nabla_{j]}B^m + \nabla_{[j}S^m_{kl]}\nabla_m B^i + S^m_{kl}\nabla_{j]}\nabla_m B^i \tag{1.3.29}$$

and

$$\nabla_{[j}\nabla_k\nabla_{l]}B^i = -\frac{1}{2}R^m_{[ljk]}\nabla_m B^i + \frac{1}{2}R^i_{m[jk}\nabla_{l]}B^m + S^m_{[jk}\nabla_{l]}\nabla_m B^i$$
$$+S^m_{[jk}R^i_{|nm|l]}B^n + 2S^m_{[jk}S^n_{|m|l]}\nabla_n B^i, \tag{1.3.30}$$

so

$$\frac{1}{2}\nabla_{[j}R^i_{|m|kl]}B^m + \nabla_{[j}S^m_{kl]}\nabla_m B^i = -\frac{1}{2}R^m_{[ljk]}\nabla_m B^i + S^m_{[jk}R^i_{|nm|l]}B^n$$
$$+2S^m_{[jk}S^n_{|m|l]}\nabla_n B^i. \tag{1.3.31}$$

Comparing terms in (1.3.31) with B^i gives the second Bianchi identity or the *Bianchi identity*:

$$R^i_{n[jk;l]} = 2R^i_{nm[j}S^m_{kl]}, \tag{1.3.32}$$

while comparing terms with $\nabla_k B^i$ gives the first Bianchi identity or the Ricci *cyclic identity*:

$$R^m_{[jkl]} = -2S^m_{[jk;l]} + 4S^m_{n[j}S^n_{kl]}. \tag{1.3.33}$$

Contracting (1.3.32) and (1.3.33) with respect to one contravariant and one covariant index gives

$$R^i{}_{n[ik;l]} = 2R^i{}_{nm[i}S^m{}_{kl]}, \tag{1.3.34}$$
$$R^k{}_{[jkl]} = -2S^k{}_{[jk;l]} + 4S^k{}_{n[j}S^n{}_{kl]}. \tag{1.3.35}$$

For a symmetric connection, $S^i{}_{jk} = 0$, the Bianchi identity and the cyclic identity reduce to

$$R^i{}_{n[jk;l]} = 0, \tag{1.3.36}$$
$$R^m{}_{[jkl]} = 0. \tag{1.3.37}$$

The cyclic identity (1.3.37) imposes 16 constraints on the curvature tensor, thereby the curvature tensor with a vanishing torsion has 80 independent components.

1.3.5 Ricci tensor

Contraction of the curvature tensor with respect to the contravariant index and the second covariant index gives the *Ricci tensor*:

$$R_{ik} = R^j{}_{ijk} = \Gamma^{\,j}_{i\,k,j} - \Gamma^{\,j}_{i\,j,k} + \Gamma^{\,l}_{i\,k}\Gamma^{\,j}_{l\,j} - \Gamma^{\,l}_{i\,j}\Gamma^{\,j}_{l\,k}. \tag{1.3.38}$$

Contraction of the curvature tensor with respect to the contravariant index and the third covariant index gives the Ricci tensor with the opposite sign because of the antisymmetry of the curvature tensor with respect to its last indices. Contraction of the curvature tensor with respect to the contravariant index and the first covariant index gives the homothetic or *segmental curvature* tensor:

$$Q_{ik} = R^j{}_{jik} = \Gamma^{\,j}_{j\,k,i} - \Gamma^{\,j}_{j\,i,k}, \tag{1.3.39}$$

which is a curl. A change in the connection (1.3.7) results in the following changes of the Ricci tensor and segmental curvature tensor:

$$R_{ik} \to R_{ik} + T^l{}_{ik;l} - T^l{}_{il;k} + T^j{}_{ik}T^l{}_{jl} - T^j{}_{il}T^l{}_{jk}, \tag{1.3.40}$$
$$Q_{ik} \to Q_{ik} + T^j{}_{jk,i} - T^j{}_{ji,k}. \tag{1.3.41}$$

For a projective transformation (1.2.61)

$$R_{ik} \to R_{ik} + A_{k;i} - A_{i;k}, \tag{1.3.42}$$
$$Q_{ik} \to Q_{ik} + 4(A_{k,i} - A_{i,k}). \tag{1.3.43}$$

Therefore, the symmetric part of the Ricci tensor is invariant under projective transformations. The variation of the Ricci tensor is

$$\delta R_{ik} = (\delta\Gamma^{\,l}_{i\,k})_{;l} - (\delta\Gamma^{\,l}_{i\,l})_{;k} - 2S^j{}_{lk}\delta\Gamma^{\,l}_{i\,j}, \tag{1.3.44}$$

while the variation of the segmental curvature tensor is

$$\delta Q_{ik} = (\delta\Gamma^{\,j}_{j\,k})_{,i} - (\delta\Gamma^{\,j}_{j\,i})_{,k}. \tag{1.3.45}$$

1.3.6 Geodesic deviation

Let us consider a family of affine geodesics characterized by the affine parameter s, measured along each curve from its point of intersection with a given hypersurface, and distinguished by a scalar parameter t: $x^i = x^i(s,t)$. We define

$$v^i = \frac{\partial x^i}{\partial t}, \tag{1.3.46}$$

which gives

$$v^i{}_{;k}u^k - u^i{}_{;k}v^k = v^i{}_{,k}u^k - u^i{}_{,k}v^k - 2S^i{}_{kl}u^kv^l = \frac{du^i}{dt} - \frac{dv^i}{ds} - 2S^i{}_{kl}u^kv^l = -2S^i{}_{kl}u^kv^l, \tag{1.3.47}$$

where $u^i = \frac{\partial x^i}{\partial s}$ is the four-velocity along each curve. We therefore have

$$\frac{D^2 v^i}{ds^2} = (v^i{}_{;j}u^j)_{;k}u^k = (u^i{}_{;j}v^j)_{;k}u^k - 2(S^i{}_{kl}u^k v^l)_{;j}u^j$$
$$= u^i{}_{;jk}v^j u^k + u^i{}_{;j}v^j{}_{;k}u^k - 2(S^i{}_{kl}u^k v^l)_{;j}u^j$$
$$= u^i{}_{;kj}v^j u^k - R^i{}_{ljk}u^l v^j u^k - 2S^l{}_{jk}u^i{}_{;l}v^j u^k + u^i{}_{;j}v^j{}_{;k}u^k - 2(S^i{}_{kl}u^k v^l)_{;j}u^j$$
$$= u^i{}_{;kj}v^j u^k - R^i{}_{ljk}u^l v^j u^k - 2S^l{}_{jk}u^i{}_{;l}v^j u^k + u^i{}_{;j}(u^j{}_{;k}v^k - 2S^j{}_{kl}u^k v^l)$$
$$-2(S^i{}_{kl}u^k v^l)_{;j}u^j = (u^i{}_{;k}u^k)_{;j}v^j + R^i{}_{jkl}u^j u^k v^l - 2(S^i{}_{kl}u^k v^l)_{;j}u^j$$
$$= R^i{}_{jkl}u^j u^k v^l - 2\frac{D}{ds}(S^i{}_{kl}u^k v^l), \tag{1.3.48}$$

which can be written as

$$\frac{D}{ds}\left(\frac{Dv^i}{ds} + 2S^i{}_{kl}u^k v^l\right) = R^i{}_{jkl}u^j u^k v^l. \tag{1.3.49}$$

This is the equation of *geodesic deviation.* If we replace affine geodesics by arbitrary curves then $u^i{}_{;k}u^k \neq 0$ and (1.3.49) becomes

$$\frac{D}{ds}\left(\frac{Dv^i}{ds} + 2S^i{}_{kl}u^k v^l\right) = R^i{}_{jkl}u^j u^k v^l + (u^i{}_{;k}u^k)_{;j}v^j. \tag{1.3.50}$$

The separation vector

$$\xi^i = v^i dt \tag{1.3.51}$$

connects points on two infinitely close affine geodesics with t and $t+dt$ for the same s. Multiplying (1.3.48) by dt gives another form of the equation of geodesic deviation,

$$\frac{D^2\xi^i}{ds^2} = R^i{}_{jkl}u^j u^k \xi^l - 2\frac{D}{ds}(S^i{}_{kl}u^k \xi^l). \tag{1.3.52}$$

References: [1, 2, 3, 4].

1.4 Metric

1.4.1 Metric tensor

The affine parameter s is a measure of the length only along an affine geodesic. In order to extend the concept of length to all points in spacetime, we equip spacetime with an algebraic object g_{ik}, referred to as the covariant *metric tensor* and defined as

$$ds^2 = g_{ik}dx^i dx^k. \tag{1.4.1}$$

The quantity ds in (1.4.1) is called the *line element.* The metric tensor is a symmetric tensor of rank (0,2):

$$g_{ik} = g_{ki}. \tag{1.4.2}$$

The affine parameter s, whose differential is given by (1.4.1), is referred to as the *interval.* Because ds does not change under parallel transport along an affine geodesic from point $P(x^i)$ to point $Q(x^i + dx^i)$, $ds|_Q = ds|_P$, and $dx^i|_Q$ is a parallel translation of $dx^i|_P$, $g_{ik}|_Q = g_{ik}|_P + g_{ik,j}dx^j$ is a parallel translation of $g_{ik}|_P$:

$$g_{ik}|_Q = g_{ik}|_P + \delta g_{ik}, \tag{1.4.3}$$

so

$$Dg_{ik} = g_{ik;j}dx^j = dg_{ik} - \delta g_{ik} = g_{ik,j}dx^j - \delta g_{ik} = 0. \tag{1.4.4}$$

Therefore, the covariant derivative of the covariant metric tensor vanishes:

$$g_{ik;j} = 0. \tag{1.4.5}$$

This relation is equivalent to

$$g_{ik,j} - \Gamma^{\,l}_{i\,j} g_{lk} - \Gamma^{\,l}_{k\,j} g_{il} = 0. \tag{1.4.6}$$

The symmetric contravariant metric tensor $g^{ik} = g^{ki}$ is defined as the tensor inverse to g_{ik}:

$$g_{ij} g^{ik} = \delta^k_j. \tag{1.4.7}$$

Since the contravariant metric tensor is a function of the covariant metric tensor only, its covariant derivative also vanishes:

$$g^{ik}{}_{;j} = 0. \tag{1.4.8}$$

The metric tensor allows to associate covariant and contravariant vectors:

$$A^i = g^{ik} A_k, \tag{1.4.9}$$
$$B_i = g_{ik} B^k, \tag{1.4.10}$$

because such association works for the covariant differentials of these vectors which are vectors:

$$DA^i = D(g^{ik} A_k) = g^{ik} DA_i, \;\; DB_i = D(g_{ik} B^k) = g_{ik} DB^k \tag{1.4.11}$$

(raising and lowering of coordinate indices commutes with covariant differentiation with respect to $\Gamma^{\,\rho}_{\mu\,\nu}$). For covariant and contravariant indices of tensors and densities this association is

$$g_{im} \mathfrak{T}^{ij\dots}{}_{kl\dots} = \mathfrak{T}_m{}^{j\dots}{}_{kl\dots}, \tag{1.4.12}$$
$$g^{km} \mathfrak{T}^{ij\dots}{}_{kl\dots} = \mathfrak{T}^{ijm\dots}{}_{l\dots}. \tag{1.4.13}$$

The contravariant and covariant components of a two-dimensional vector are shown in Figure 1. The four-velocity vector (1.2.57) is normalized because of (1.4.1):

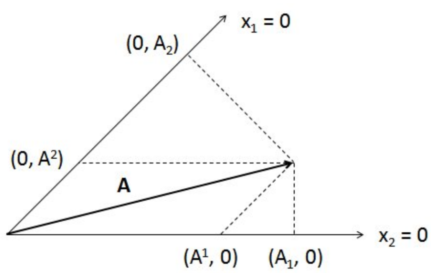

Figure 1: Contravariant and covariant components of a vector.

$$u^i u_i = g_{ik} u^i u^k = \frac{g_{ik} dx^i dx^k}{ds^2} = 1. \tag{1.4.14}$$

This vector thus has 3 independent components.

Let us consider the determinant of the matrix composed from the components of the covariant metric tensor g_{ik},

$$\mathfrak{g} = |g_{ik}|. \tag{1.4.15}$$

The square root of the absolute value of this determinant, $\sqrt{|\mathfrak{g}|}$, is a scalar density of weight 1. We can use it to construct from the Levi-Civita symbols a quantity which behaves like a tensor with respect to continuous coordinate transformations:

$$e_{iklm} = \sqrt{|\mathfrak{g}|} \varepsilon_{iklm}, \tag{1.4.16}$$
$$e^{iklm} = \frac{1}{\sqrt{|\mathfrak{g}|}} \epsilon^{iklm} = g^{in} g^{kp} g^{lq} g^{mr} e_{npqr}. \tag{1.4.17}$$

If we change the sign of 1 or 3 of the coordinates, then the components of e^{iklm} do not change because ϵ^{iklm} and ε_{iklm} have the same components in all coordinate systems, whereas some of the components of a tensor change sign. The components (1.4.16) and (1.4.17) are thus referred to as those of the *completely antisymmetric unit pseudotensor*. The relations (1.1.30) are also valid if we replace ϵ and ε by e.

The differential and derivatives of the determinant of the metric tensor are given, following (1.2.23) and (1.2.24), by

$$d\mathfrak{g} = \mathfrak{g}g^{ik}dg_{ik} = -\mathfrak{g}g_{ik}dg^{ik}, \tag{1.4.18}$$

$$\mathfrak{g}_{,l} = \mathfrak{g}g^{ik}g_{ik,l} = -\mathfrak{g}g_{ik}g^{ik}{}_{,l}. \tag{1.4.19}$$

The variation of the determinant of the metric tensor is thus

$$\delta\mathfrak{g} = \mathfrak{g}g^{ik}\delta g_{ik} = -\mathfrak{g}g_{ik}\delta g^{ik}. \tag{1.4.20}$$

The covariant derivative of the determinant of the metric tensor vanishes:

$$\mathfrak{g}_{;j} = 0. \tag{1.4.21}$$

The relations (1.2.36) and (1.2.37) give thus

$$e^{ijkl}{}_{;m} = 0, \;\; e_{ijkl;m} = 0. \tag{1.4.22}$$

A Lie derivative of the metric tensor is

$$\mathcal{L}_\xi g^{ik} = -2\xi^{(i;k)} - 4S^{(ik)}{}_l\xi^l, \tag{1.4.23}$$

where ${}^{;i} =_{;k} g^{ik}$.

The covariant derivative of the covariant metric tensor defines the *nonmetricity tensor*:

$$N_{jik} = -g_{ik;j}. \tag{1.4.24}$$

The commutator of covariant derivatives (1.3.6) of the metric tensor gives

$$R^{(ij)}{}_{kl} = -N_{[k}{}^{ij}{}_{;l]} - S^m{}_{kl}N_m{}^{ij} = -N_{[k}{}^{ij}{}_{,l]}, \tag{1.4.25}$$

so the segmental curvature tensor (1.3.39) is

$$Q_{kl} = -N_{[k}{}^{ij}{}_{,l]}g_{ij}. \tag{1.4.26}$$

The nonmetricity tensor vanishes because of (1.4.5). Consequently, the curvature tensor is antisymmetric in its first two indices:

$$R_{ijkl} = -R_{jikl}. \tag{1.4.27}$$

Therefore, the segmental curvature tensor also vanishes, and

$$R_{ijkl}g^{jl} = R_{ik}. \tag{1.4.28}$$

Consequently, there is only one independent way to contract the curvature tensor, which gives the Ricci tensor up to a sign.

1.4.2 Christoffel symbols

The condition (1.4.5) is referred to as *metricity* or *metric compatibility* of the affine connection, and imposes 40 constraints on the connection:

$$g_{ik;j} + g_{kj;i} - g_{ji;k} = g_{ik,j} - \Gamma^{\,l}_{i\,j}g_{lk} - \Gamma^{\,l}_{k\,j}g_{il} + g_{kj,i} - \Gamma^{\,l}_{k\,i}g_{lj} - \Gamma^{\,l}_{j\,i}g_{kl} - g_{ji,k} + \Gamma^{\,l}_{j\,k}g_{li}$$
$$+\Gamma^{\,l}_{i\,k}g_{jl} = g_{ik,j} + g_{kj,i} - g_{ji,k} - 2\Gamma^{\,l}_{(i\,j)}g_{kl} - 2S^l{}_{kj}g_{il} - 2S^l{}_{ki}g_{jl} = 0. \tag{1.4.29}$$

Multiplying (1.4.29) by g^{km} gives

$$\Gamma_{(i\,j)}^{\,m} = \{{}^{\,m}_{i\,j}\} + 2S_{(ij)}{}^{m}, \tag{1.4.30}$$

where

$$\{{}^{\,m}_{i\,j}\} = \frac{1}{2}g^{mk}(g_{kj,i} + g_{ki,j} - g_{ij,k}) \tag{1.4.31}$$

are the *Christoffel symbols*. Using (1.4.7), they can be written as

$$\{{}^{\,m}_{i\,j}\} = -\frac{1}{2}(g_{kj}g^{mk}{}_{,i} + g_{ki}g^{mk}{}_{,j} - g^{mk}g_{il}g_{jn}g^{ln}{}_{,k}). \tag{1.4.32}$$

The Christoffel symbols are symmetric in their covariant indices:

$$\{{}^{\,k}_{i\,j}\} = \{{}^{\,k}_{j\,i}\}. \tag{1.4.33}$$

Because $\Gamma_{i\,j}^{\,k} = \Gamma_{(i\,j)}^{\,k} + S^k{}_{ij}$, the metric-compatible affine connection equals

$$\Gamma_{i\,j}^{\,k} = \{{}^{\,k}_{i\,j}\} + C^k{}_{ij}, \tag{1.4.34}$$

where

$$C^k{}_{ij} = 2S_{(ij)}{}^{k} + S^k{}_{ij} \tag{1.4.35}$$

is the *contortion tensor*, antisymmetric in its first two indices:

$$C_{ijk} = -C_{jik}. \tag{1.4.36}$$

The inverse relation between the torsion and contortion tensor is

$$S^i{}_{jk} = C^i{}_{[jk]}. \tag{1.4.37}$$

The Christoffel symbols are the torsionless part of the connection.

The difference between two affine connections is a tensor, thereby the sum of a connection and a tensor of rank (1,2) is a connection. Therefore, the Christoffel symbols form a connection, referred to as the *Levi-Civita connection*. We define the covariant derivative with respect to the Levi-Civita connection analogously to (1.2.11), with $\Gamma_{i\,j}^{\,k}$ replaced by $\{{}^{\,k}_{i\,j}\}$, and denote it ${}_{:i}$ instead of ${}_{;i}$, or $\nabla_i^{\{\}}$ instead of ∇_i. The covariant derivative with respect to the Levi-Civita connection of the metric tensor vanishes, as for that with respect to any connection:

$$g_{ik:j} = g_{ik,j} - \{{}^{\,l}_{i\,j}\}g_{lk} - \{{}^{\,l}_{k\,j}\}g_{il} = 0. \tag{1.4.38}$$

This equation agrees with (1.4.31) and gives the relation between ordinary derivatives of the metric tensor and the Christoffel symbols:

$$g_{ik,j} = \{{}^{\,l}_{i\,j}\}g_{lk} + \{{}^{\,l}_{k\,j}\}g_{il}. \tag{1.4.39}$$

Similarly, we have

$$g^{ik}{}_{:j} = g^{ik}{}_{,j} + \{{}^{\,i}_{l\,j}\}g^{lk} + \{{}^{\,k}_{l\,j}\}g^{il} = 0. \tag{1.4.40}$$

The variation of the Levi-Civita connection is, as for any connection, a tensor:

$$\begin{aligned}
\delta\{{}^{\,k}_{i\,j}\} &= \frac{1}{2}g^{kl}\big((\delta g_{lj})_{,i} + (\delta g_{li})_{,j} - (\delta g_{ij})_{,l}\big) + \frac{1}{2}\delta g^{kl}(g_{lj,i} + g_{li,j} - g_{ij,l}) \\
&= \frac{1}{2}g^{kl}\big((\delta g_{lj})_{:i} + (\delta g_{li})_{:j} - (\delta g_{ij})_{:l}\big) + \frac{1}{2}g^{kl}(\{{}^{\,m}_{l\,i}\}\delta g_{mj} + \{{}^{\,m}_{j\,i}\}\delta g_{lm} + \{{}^{\,m}_{l\,j}\}\delta g_{mi} + \{{}^{\,m}_{i\,j}\}\delta g_{lm} \\
&-\{{}^{\,m}_{i\,l}\}\delta g_{mj} - \{{}^{\,m}_{j\,l}\}\delta g_{im}) + \delta g^{kl}\{{}^{\,m}_{i\,j}\}g_{lm} = \frac{1}{2}g^{kl}\big((\delta g_{lj})_{:i} + (\delta g_{li})_{:j} - (\delta g_{ij})_{:l}\big) \\
&+g^{kl}\{{}^{\,m}_{i\,j}\}\delta g_{lm} + \delta g^{kl}\{{}^{\,m}_{i\,j}\}g_{lm} = \frac{1}{2}g^{kl}\big((\delta g_{lj})_{:i} + (\delta g_{li})_{:j} - (\delta g_{ij})_{:l}\big) + \{{}^{\,m}_{i\,j}\}\delta\delta^k_m \\
&= \frac{1}{2}g^{kl}\big((\delta g_{lj})_{:i} + (\delta g_{li})_{:j} - (\delta g_{ij})_{:l}\big),
\end{aligned} \tag{1.4.41}$$

where we used (1.4.7). The covariant derivative over s of a tensor density with respect to the Levi-Civita connection is, analogously to (1.2.60),

$$\frac{D^{\{\}}T}{ds} = T_{:i}u^i. \tag{1.4.42}$$

The following formulae are satisfied:

$$\{^{\,k}_{k\,i}\} = \frac{1}{2}g^{jk}g_{jk,i} = -\frac{1}{2}g_{jk}g^{jk}{}_{,i} = \frac{1}{2}\frac{\mathfrak{g},i}{\mathfrak{g}} = (\ln\sqrt{|\mathfrak{g}|})_{,i}, \tag{1.4.43}$$

$$\{^{\,k}_{i\,j}\}g^{ij} = -\frac{1}{\sqrt{|\mathfrak{g}|}}(\sqrt{|\mathfrak{g}|}g^{ik})_{,i}, \tag{1.4.44}$$

$$B^i{}_{:i} = \frac{1}{\sqrt{|\mathfrak{g}|}}(\sqrt{|\mathfrak{g}|}B^i)_{,i}, \tag{1.4.45}$$

$$F^{ik}{}_{:i} = \frac{1}{\sqrt{|\mathfrak{g}|}}(\sqrt{|\mathfrak{g}|}F^{ik})_{,i}, \tag{1.4.46}$$

$$A_{i:k} - A_{k:i} = A_{i,k} - A_{k,i}, \tag{1.4.47}$$

$$\oint B^i\sqrt{|\mathfrak{g}|}dS_i = \int B^i{}_{:i}\sqrt{|\mathfrak{g}|}d\Omega, \tag{1.4.48}$$

where $F^{ik} = -F^{ki}$. The Christoffel symbols satisfy all formulae that are satisfied by $\Gamma^{\,k}_{i\,j}$ in which $S^i{}_{jk} = 0$. Because the Levi-Civita connection is a symmetric connection, it can be brought to zero by transforming the coordinates to a geodesic frame of reference. In a geodesic frame, the covariant derivative with respect to the Levi-Civita connection, $\nabla^{\{\}}_i$, coincides with the ordinary derivative ∂_i.

Since the covariant derivatives of the Levi-Civita symbols are equal to zero, according to (1.2.36) and (1.2.37), their covariant derivatives with respect to the Levi-Civita connection vanish:

$$\epsilon^{ijkl}{}_{:m} = 0, \ \ \varepsilon_{ijkl:m} = 0. \tag{1.4.49}$$

The following covariant derivatives with respect to the Levi-Civita connection also vanish:

$$\mathfrak{g}_{:j} = 0, \ \ e^{ijkl}{}_{:m} = 0, \ \ e_{ijkl:m} = 0. \tag{1.4.50}$$

The Lie derivative of the metric tensor (1.4.23) along a vector ξ^i can be written as

$$\mathcal{L}_\xi g^{ik} = -2\xi^{(i:k)}, \ \ \mathcal{L}_\xi g_{ik} = 2\xi_{(i:k)}, \tag{1.4.51}$$

where ${}^{:i} =_{:k} g^{ik}$. A Killing vector (1.2.83) for the Levi-Civita connection satisfies

$$\zeta_{(i:k)} = 0. \tag{1.4.52}$$

It thus becomes a generator of a transformation

$$x'^i = x^i + \epsilon\zeta^i, \tag{1.4.53}$$

where ϵ is an infinitesimal scalar, which coincides with (1.2.66) for

$$\xi^i = \epsilon\zeta^i. \tag{1.4.54}$$

Such transformations are *isometries*: they do not change the metric tensor.

If the nonmetricity tensor does not vanish, the general formula for the affine connection (1.4.34) is

$$\Gamma^{\,k}_{i\,j} = \{^{\,k}_{i\,j}\} + C^k{}_{ij} - \frac{1}{2}N^k{}_{ij} + N_{(i\ \ j)}{}^{k}. \tag{1.4.55}$$

1.4.3 Riemann tensor

The commutator of covariant derivatives with respect to the Levi-Civita connection of a covariant vector is

$$[\nabla^{\{\}}_j, \nabla^{\{\}}_k]A_i = -P^m{}_{ijk}A_m, \tag{1.4.56}$$

analogously to (1.3.5) and without the torsion tensor of this connection that vanishes. The curvature tensor constructed from the Levi-Civita connection is referred to as the Riemannian curvature tensor or the *Riemann tensor*:

$$P^i{}_{mjk} = \partial_j\{{}^{\,i}_{m\,k}\} - \partial_k\{{}^{\,i}_{m\,j}\} + \{{}^{\,i}_{l\,j}\}\{{}^{\,l}_{m\,k}\} - \{{}^{\,i}_{l\,k}\}\{{}^{\,l}_{m\,j}\}. \tag{1.4.57}$$

Similarly, the commutators of covariant derivatives of a contravariant vector and of a tensor are respectively given by (1.3.3) and (1.3.6), in which $R^i{}_{jkl}$ is replaced with $P^i{}_{jkl}$ and $S^i{}_{jk} = 0$. The commutator of covariant derivatives of the metric tensor vanishes:

$$[\nabla^{\{\}}_j, \nabla^{\{\}}_k]g_{lp} = -P^m{}_{ljk}g_{mp} - P^m{}_{pjk}g_{lm} = 0, \tag{1.4.58}$$

so the covariant Riemann tensor P_{imjk} is also antisymmetric in the indices i, m. Substituting (1.4.31) in (1.4.57) gives

$$P_{iklm} = \frac{1}{2}(g_{im,kl} + g_{kl,im} - g_{il,km} - g_{km,il}) + g_{jn}(\{{}^{\,j}_{i\,m}\}\{{}^{\,n}_{k\,l}\} - \{{}^{\,j}_{i\,l}\}\{{}^{\,n}_{k\,m}\}), \tag{1.4.59}$$

which explicitly shows the following symmetry and antisymmetry properties:

$$P_{iklm} = -P_{ikml}, \tag{1.4.60}$$
$$P_{iklm} = -P_{kilm}, \tag{1.4.61}$$
$$P_{iklm} = P_{lmik}. \tag{1.4.62}$$

Accordingly, the *Riemannian Ricci tensor* is symmetric:

$$P_{ik} = P^j{}_{ijk} = P_{ki}. \tag{1.4.63}$$

Substituting (1.4.34) in (1.3.7) and (1.3.8) gives the relation between the curvature and Riemann tensors:

$$R^i{}_{klm} = P^i{}_{klm} + C^i{}_{km:l} - C^i{}_{kl:m} + C^j{}_{km}C^i{}_{jl} - C^j{}_{kl}C^i{}_{jm}. \tag{1.4.64}$$

Contracting (1.4.64) with respect to in the indices i, l gives

$$R_{km} = P_{km} + C^i{}_{km:i} - C^i{}_{ki:m} + C^j{}_{km}C^i{}_{ji} - C^j{}_{ki}C^i{}_{jm}. \tag{1.4.65}$$

Consequently, the Ricci scalar or the *curvature scalar*,

$$R = R_{ik}g^{ik}, \tag{1.4.66}$$

is given by

$$R = P - g^{ik}(2C^l{}_{il:k} + C^j{}_{ij}C^l{}_{kl} - C^l{}_{im}C^m{}_{kl}), \tag{1.4.67}$$

where P is the Riemannian curvature scalar or the *Riemann scalar*:

$$P = P_{ik}g^{ik}. \tag{1.4.68}$$

The variation of the Riemann tensor is, analogously to (1.3.10),

$$\delta P^i{}_{klm} = (\delta\{{}^{\,i}_{k\,m}\})_{:l} - (\delta\{{}^{\,i}_{k\,l}\})_{:m}, \tag{1.4.69}$$

and the variation of the Riemannian Ricci tensor is

$$\delta P_{ik} = (\delta\{{}^{\,l}_{i\,k}\})_{:l} - (\delta\{{}^{\,l}_{i\,l}\})_{:k}. \tag{1.4.70}$$

Contracting (1.3.34) and (1.3.35) with the metric tensor gives

$$R_{nk;l} - R_{nl;k} + R^i{}_{nkl;i} = -2R_{nm}S^m{}_{kl} - 2R^i{}_{nmk}S^m{}_{il} + 2R^i{}_{nml}S^m{}_{ik} \tag{1.4.71}$$

and the *contracted cyclic identity*:

$$R_{jl} - R_{lj} = -2S_{j;l} + 2S_{l;j} - 2S^k{}_{lj;k} + 4S_nS^n{}_{lj}. \tag{1.4.72}$$

Further contraction of (1.4.71) with the metric tensor gives the *contracted Bianchi identity*:

$$R^i{}_{l;i} - \frac{1}{2}R_{;l} = 2R_{km}S^{mk}{}_l - R^{ik}{}_{ml}S^m{}_{ik}. \tag{1.4.73}$$

The Bianchi identity (1.3.36) and the cyclic identity (1.3.37) for the Riemann tensor are

$$P^i{}_{n[jk:l]} = 0, \tag{1.4.74}$$

$$P^m{}_{[jkl]} = 0. \tag{1.4.75}$$

Contracting these equations with the metric tensor gives

$$P_{nk:l} + P^i{}_{nkl:i} - P_{nl:k} = 0, \tag{1.4.76}$$

$$P_{jl} - P_{lj} = 0, \tag{1.4.77}$$

in agreement with (1.4.63). Further contraction of (1.4.76) with the metric tensor gives the contracted Bianchi identity:

$$G^i{}_{k:i} = 0, \tag{1.4.78}$$

for the symmetric *Einstein tensor*, defined as

$$G_{ik} = P_{ik} - \frac{1}{2}Pg_{ik} = G_{ki}. \tag{1.4.79}$$

This identity is a covariant conservation of the Einstein tensor.

1.4.4 Properties of Riemann tensor

In two dimensions there is only 1 independent component of the Riemann tensor, P_{1212}. The Riemann scalar is

$$P = \frac{2P_{1212}}{\mathfrak{s}}, \tag{1.4.80}$$

where $\mathfrak{s}$ is the determinant of the two-dimensional metric tensor γ_{ik}:

$$\mathfrak{s} = |\gamma_{ik}| = \gamma_{11}\gamma_{22} - \gamma_{12}^2. \tag{1.4.81}$$

A surface near point $x = 0, y = 0$ is given by

$$z = \frac{x^2}{2\rho_1} + \frac{y^2}{2\rho_2}, \tag{1.4.82}$$

where ρ_1 and ρ_2 are the radii of curvature. Substituting (1.4.82) to

$$dl^2 = dx^2 + dy^2 + dz^2 = \gamma_{ik}dx^idx^k \tag{1.4.83}$$

gives $\gamma_{ik}(x, y)$, which then gives

$$\left.\frac{P}{2}\right|_{x=y=0} = K = \frac{1}{\rho_1\rho_2}, \tag{1.4.84}$$

where K is the Gauß curvature.

In three dimensions there are 3 independent pairs, 12, 23, and 31, thereby the Riemann tensor has 6 independent components: 3 with identical pairs and $\frac{3\cdot 2}{2} = 3$ with different pairs (the cyclic identity does not reduce the number of independent components). The Ricci tensor has also 6 components, which are related to the components of the Riemann tensor by

$$P_{\alpha\beta\gamma\delta} = P_{\alpha\gamma}\gamma_{\beta\delta} - P_{\alpha\delta}\gamma_{\beta\gamma} + P_{\beta\delta}\gamma_{\alpha\gamma} - P_{\beta\gamma}\gamma_{\alpha\delta} + \frac{P}{2}(\gamma_{\alpha\delta}\gamma_{\beta\gamma} - \gamma_{\alpha\gamma}\gamma_{\beta\delta}). \tag{1.4.85}$$

Choosing the *Cartesian coordinates* at a given point, defined by the condition

$$g_{\alpha\beta} = \mathrm{diag}(1,1,1), \tag{1.4.86}$$

and diagonalizing $P_{\alpha\beta}$, which is equivalent to 3 rotations, brings $P_{\alpha\beta}$ to the canonical form with $6-3=3$ independent components. Consequently, the Riemann tensor in three dimensions has 3 physically independent components. The Gauß curvature of a surface perpendicular to the x^3 axis is given by

$$K = \frac{P_{1212}}{\gamma_{11}\gamma_{22} - \gamma_{12}^2}. \tag{1.4.87}$$

In four dimensions there are 6 independent pairs, 01, 02, 03, 12, 23, and 31, thereby there are 6 components with identical pairs and $\frac{6\cdot5}{2} = 15$ with different pairs. The cyclic identity reduces the number of independent components by 1, thereby the Riemann tensor in four dimensions has generally 20 independent components. Choosing the Cartesian coordinates at a given point and applying 6 rotations brings P_{ijkl} to the canonical form with $20-6=14$ physically independent components.

The *Weyl tensor* is defined as

$$W_{iklm} = P_{iklm} - \frac{1}{2}(P_{il}g_{km} + P_{km}g_{il} - P_{im}g_{kl} - P_{kl}g_{im}) + \frac{1}{6}P(g_{il}g_{km} - g_{im}g_{kl}). \tag{1.4.88}$$

This tensor has all the symmetry and antisymmetry properties of the Riemann tensor, and is also traceless (any contraction of the Weyl tensor vanishes).

1.4.5 Metric geodesics

Let us consider two points in spacetime, P and Q. Among curves that connect these points, one curve has the minimal value of the interval $s = \int ds$, and is referred to as a *metric geodesic*. The equation of a metric geodesic is given by the condition that $\int ds$ be an extremum with the endpoints of the curve fixed:

$$\begin{aligned}
&\delta\int ds = \delta\int(g_{ik}dx^i dx^k)^{1/2} = \int\frac{\delta dx^i g_{ij}dx^j}{ds} + \frac{1}{2}\int\frac{\delta g_{ij}dx^i dx^j}{ds} = \int g_{ij}u^j\delta dx^i \\
&+\frac{1}{2}\int g_{ij,k}\delta x^k u^i u^j ds = \int d(u_i\delta x^i) - \int du_i\delta x^i + \frac{1}{2}\int g_{ij,k}\delta x^k u^i u^j ds \\
&= -\int\frac{du_i}{ds}\delta x^i ds + \frac{1}{2}\int g_{jk,i}\delta x^i u^j u^k ds = 0,
\end{aligned} \tag{1.4.89}$$

where we omit the total differential term $\int d(u_i\delta x^i)$ because $\delta x^i = 0$ at the endpoints. Since δx^i is arbitrary, we obtain

$$\begin{aligned}
&\frac{d}{ds}(g_{ij}u^j) - \frac{1}{2}\int g_{jk,i}u^j u^k ds = g_{ij}\frac{du^j}{ds} + u^k g_{ij,k}u^j - \frac{1}{2}\int g_{jk,i}u^j u^k ds \\
&= g_{ij}\frac{du^j}{ds} + \{^{\,m}_{j\,k}\}g_{im}u^j u^k = 0
\end{aligned} \tag{1.4.90}$$

or, after multiplying (1.4.90) by g^{il}:

$$\frac{D^{\{\}}u^l}{ds} = \frac{du^l}{ds} + \{^{\,l}_{j\,k}\}u^j u^k = u^i u^l{}_{:i} = 0. \tag{1.4.91}$$

The metric geodesic equation (1.4.91) can be written as

$$\frac{d^2x^i}{ds^2} + \{{}_{k\,l}^{\,i}\}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0. \tag{1.4.92}$$

Using (1.4.34) and (1.4.35), the affine geodesic equation (1.2.56) can be written as

$$\frac{d^2x^i}{ds^2} + \{{}_{k\,l}^{\,i}\}\frac{dx^k}{ds}\frac{dx^l}{ds} + 2S_{kl}{}^i\frac{dx^k}{ds}\frac{dx^l}{ds} = 0. \tag{1.4.93}$$

If the torsion tensor is completely antisymmetric then the last term in (1.4.93) vanishes and the affine geodesic equation coincides with the metric geodesic equation. The equation of geodesic deviation with respect to the Levi-Civita connection is, analogously to (1.3.49),

$$\frac{D^{\{\}2}v^i}{ds^2} = P^i_{\ jkl}u^j u^k v^l. \tag{1.4.94}$$

If ζ_i is a Killing vector of the Levi-Civita connection then along a metric geodesic,

$$\frac{D^{\{\}}}{ds}(u^i\zeta_i) = u^k(u^i\zeta_i)_{:k} = u^i u^k\zeta_{i:k} + \zeta_i u^k u^i{}_{:k} = 0. \tag{1.4.95}$$

The first term in the sum in (1.4.95) vanishes because of (1.4.52) and the second term vanishes because of the metric geodesic equation. Therefore, to each Killing vector of the Levi-Civita connection there corresponds a quantity $u^i\zeta_i$ which does not change along the metric geodesic, analogously to (1.2.85):

$$u^i\zeta_i = g_{ik}u^i\zeta^k = \text{const}. \tag{1.4.96}$$

1.4.6 Galilean frame of reference and Minkowski tensor

At a given point, the nondegenerate ($\mathfrak{g} \neq 0$) metric tensor can be brought to a diagonal (canonical) form $g_{ik} = \text{diag}(\pm1, \pm1, \pm1, \pm1)$. *Physical* systems are described by the metric tensor with $\mathfrak{g} < 0$. Without loss of generality, we assume that the canonical form of the metric tensor is

$$g_{ik} = \eta_{ik} = \text{diag}(1, -1, -1, -1), \ \ g^{ik} = \eta^{ik} = \text{diag}(1, -1, -1, -1). \tag{1.4.97}$$

A frame of reference in which g_{ik} has the canonical form is referred to as *Galilean.* The transformation (1.2.46) with (1.2.49) brings a symmetric affine connection, thus the Christoffel symbols, to zero at a given point without changing the components of the metric tensor because of (1.2.47). Therefore, a frame of reference can be locally both geodesic and Galilean. Such a frame is called *inertial.* In this frame, the first derivatives of the metric tensor vanish because of (1.4.39). The corresponding metric tensor (1.4.97) is referred to as the *Minkowski tensor.* The square of the line element for this metric is

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2. \tag{1.4.98}$$

In a locally inertial frame the coordinates x^i, not only the differentials dx^i, are components of a contravariant vector.

In the absence of torsion, spacetime with a vanishing Riemann tensor $P^i_{\ klm} = 0$ is flat. In the new coordinates y_a (1.3.19), (1.3.20) gives

$$g^{ab}(y) = g_{ik}(x)\frac{\partial x^i}{\partial y_a}\frac{\partial x^k}{\partial y_b} = g_{ik}(x)h^{ia}h^{kb} = \eta^{ab}. \tag{1.4.99}$$

Therefore, in a flat spacetime without torsion one can always find a system of coordinates which is Galilean everywhere.

1.4.7 Riemann normal coordinates

If the frame of reference is locally geodesic and Galilean at a given point, taken as the origin of the coordinates, then the metric tensor at a point near the origin depends on the derivatives of the metric at the origin. In this frame, the Christoffel symbols at the origin vanish. We expand the metric tensor up to quadratic terms:

$$g_{ij}(x^k) = g_{ij}(0) + g_{ij,k}(0)x^k + \frac{1}{2}g_{ij,kl}(0)x^k x^l = \eta_{ij} + \frac{1}{2}g_{ij,kl}(0)x^k x^l, \tag{1.4.100}$$

where the metric tensor at the origin is equal to the Minkowski tensor and the first derivatives of the metric tensor at the origin vanish because of (1.4.39). We choose the coordinates such that

$$x^i = a^i s \tag{1.4.101}$$

for every metric geodesic curve passing through the origin and parametrized with the interval s, where a^i is a constant four-vector and $s = 0$ at the origin. Such coordinates are referred to as the *Riemann normal coordinates*. Accordingly, the derivatives of x^i with respect to s are

$$\frac{dx^i}{ds}(0) = a^i, \quad \frac{d^2x^i}{ds^2}(0) = \frac{d^3x^i}{ds^3}(0) = 0. \tag{1.4.102}$$

Consequently, the metric geodesic equation (1.4.92) gives

$$\{{}^{\,i}_{j\,k}\}(0)a^j a^k = 0, \tag{1.4.103}$$

therefore the condition for the geodesic frame of reference (1.2.50) is satisfied:

$$\{{}^{\,i}_{j\,k}\}(0) = 0. \tag{1.4.104}$$

Differentiating (1.4.92) with respect to s gives

$$\frac{d^3x^i}{ds^3} + \frac{d\{{}^{\,i}_{j\,k}\}}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds} + 2\{{}^{\,i}_{j\,k}\}\frac{d^2x^j}{ds^2}\frac{dx^k}{ds} = 0. \tag{1.4.105}$$

At the origin, the relations (1.4.102) reduce this equation to

$$\frac{d\{{}^{\,i}_{j\,k}\}}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds} = \{{}^{\,i}_{j\,k}\}_{,l}\frac{dx^l}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds} = \{{}^{\,i}_{j\,k}\}_{,l}(0)a^l a^j a^k = 0. \tag{1.4.106}$$

Therefore, the Christoffel symbols satisfy

$$\{{}^{\,i}_{(j\,k}\}_{,l)}(0) = 0. \tag{1.4.107}$$

In the geodesic frame of reference, the Riemann tensor (1.4.57) reduces to

$$P^i_{\ jkl} = \{{}^{\,i}_{j\,l}\}_{,k} - \{{}^{\,i}_{j\,k}\}_{,l}. \tag{1.4.108}$$

Consequently, using (1.4.107) gives

$$P^i_{\ jkl} + P^i_{\ kjl} = \{{}^{\,i}_{j\,l}\}_{,k} - \{{}^{\,i}_{j\,k}\}_{,l} + \{{}^{\,i}_{k\,l}\}_{,j} - \{{}^{\,i}_{k\,j}\}_{,l} = -3\{{}^{\,i}_{j\,k}\}_{,l}, \tag{1.4.109}$$

which gives

$$\{{}^{\,i}_{j\,k}\}_{,l}(0) = -\frac{1}{3}(P^i_{\ jkl} + P^i_{\ kjl})(0). \tag{1.4.110}$$

Differentiating (1.4.39) with respect to the coordinates and using vanishing of the first derivatives of the metric tensor at the origin gives

$$g_{ij,kl} = \{{}^{\,m}_{k\,j}\}_{,l}g_{mi} + \{{}^{\,m}_{k\,i}\}_{,l}g_{mj}. \tag{1.4.111}$$

Substituting (1.4.110) into this equation gives

$$g_{ij,kl}(0) = -\frac{1}{3}(P_{ikjl} + P_{ijkl} + P_{jkil} + P_{jikl}) = -\frac{1}{3}(P_{ikjl} - P_{kijl} + P_{jikl}) = -\frac{1}{3}(P_{ikjl} - P_{kjil})$$
$$= -\frac{1}{3}(P_{ikjl} + P_{iljk}). \tag{1.4.112}$$

Consequently, the covariant metric tensor (1.4.100) in the Riemann normal coordinates at a point near the origin, in quadratic approximation, is given by

$$g_{ij}(x^k) = \eta_{ij} - \frac{1}{6}(P_{ikjl} + P_{iljk})(0)x^k x^l = \eta_{ij} - \frac{1}{3}P_{ikjl}(0)x^k x^l. \tag{1.4.113}$$

The deviation of the metric tensor from the Minkowski tensor is proportional to the curvature. The corresponding contravariant metric tensor is given by

$$g^{ij}(x^k) = \eta^{ij} + \frac{1}{3}P^{i\ j}_{\ k\ l}(0)x^k x^l. \tag{1.4.114}$$

Similar calculations lead to the expansion of the covariant metric tensor in quartic approximation:

$$g_{ij}(x^k) = \eta_{ij} - \frac{1}{3}P_{ikjl}(0)x^k x^l - \frac{1}{6}P_{ikjl:m}(0)x^k x^l x^m - \Big(\frac{1}{20}P_{ikjl:mn} - \frac{2}{45}P_{ikl}^{\ \ \ p}P_{jmnp}\Big)(0)x^k x^l x^m x^n. \tag{1.4.115}$$

1.4.8 Intervals, proper time, and distances

The form of the Minkowski tensor distinguishes the coordinate x^0 from the rest of the coordinates x^α, where the index α can be 1,2,3. The *temporal* coordinate x^0 can be written as $x^0 = ct$, where t is referred to as *time* and c is called the *speed of propagation of interaction*. The coordinates x^α are *spatial* and span *space*. The set of 4 coordinates x^i describe an *event* and span *spacetime*. The curve $x^i(\lambda)$, where λ is a parameter, is referred to as a *world line* of a given point. The quantitites

$$v^\alpha = \frac{dx^\alpha}{dt} \tag{1.4.116}$$

are the components of a three-dimensional vector, the *velocity* of this point. An infinitesimal interval ds is *timelike* if $ds^2 > 0$, *spacelike* if $ds^2 < 0$, and *null* if $ds^2 = 0$. In a Galilean frame of reference, the spatial coordinates are Cartesian (1.4.86). In this frame, the square of the line element (interval) between two infinitesimally separated points (events) is

$$ds^2 = \eta_{ik}dx^i dx^k = c^2dt^2 - \sum_\alpha dx^\alpha dx^\alpha, \tag{1.4.117}$$

where dx^i are infinitesimal coordinate differences between the two points. The square of the interval between two finitely separated points is

$$\Delta s^2 = \eta_{ik}\Delta x^i \Delta x^k = c^2\Delta t^2 - \sum_\alpha \Delta x^\alpha \Delta x^\alpha, \tag{1.4.118}$$

where Δx^i are finite coordinate differences between the two points. If Δs is timelike, one can always find a frame of reference in which the two events occur at the same place, $\Delta x^\alpha = 0$. A frame of reference in which $dx^\alpha = 0$, thereby $v^\alpha = 0$, describes a point at *rest* and is referred to as the *rest frame* or the *comoving frame*. In this frame $t = \tau$,

$$ds^2 = c^2d\tau^2, \tag{1.4.119}$$

where τ is the *proper time*. If $dx^\alpha \neq 0$, thereby $v^\alpha \neq 0$, along a world line then the point *moves* or is in *motion*. The proper time for a moving point is equal to the time measured by a clock moving with this point. If Δs is spacelike, one can always find a frame of reference in which the two events occur at the same time (are *synchronous*), $\Delta x^0 = 0$. If $ds = 0$ along a world line, this world line describes the propagation of a signal (*interaction*), with $(\sum_\alpha v^\alpha v^\alpha)^{1/2} = c$. Equations (1.4.117) and (1.4.119) give

$$d\tau^2 = dt^2 - \frac{1}{c^2}\sum_\alpha dx^\alpha dx^\alpha, \tag{1.4.120}$$

so the proper time τ goes more slowly than the coordinate time t. If Δs is timelike, the two events occur at different times: $t_1 \neq t_2$. If $t_2 > t_1$ then t_2 is in the *future* with respect to t_1 and t_1 is in the *past* with respect to t_2. The time of a measurement t_0 is called the *present* time.

All events for which $t < t_0$ form the *absolute past* relative to the event O at the present (events in this region occur *before* O in all systems of reference). All events for which $t > t_0$ form the *absolute future* relative to the event O at the present (events in this region occur *after* O in all systems of reference). Such a division into the absolute past and the absolute future with respect to O is possible only for events for which their intervals with respect to O are timelike, as shown in Figure 2. For $O = (0, 0, 0, 0)$, these events (ct, x, y, z) lie within a cone $(ct)^2 - x^2 - y^2 - z^2 = 0$ which is called the null cone or *light cone*. All events for which their intervals with respect to O are spacelike are *absolutely remote* relative to O. The *principle of causality* states that any event O can be affected only by events in the absolute past relative to O.

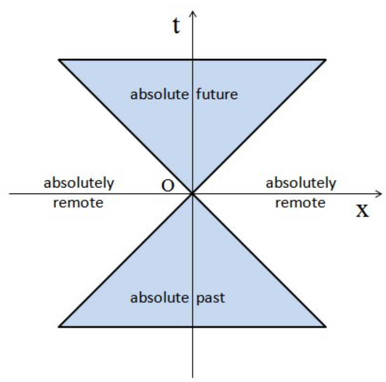

Figure 2: Light cone.

In the rest frame $dx^\alpha = 0$ gives $u^\alpha = 0$. At each point in space, the condition $dx^\alpha = 0$ gives the relation between the proper time and the coordinate time:

$$d\tau = \frac{1}{c}\sqrt{g_{00}}dx^0, \tag{1.4.121}$$

which requires

$$g_{00} \geq 0. \tag{1.4.122}$$

The relation (1.4.14) gives

$$u^0 = (g_{00})^{-1/2}. \tag{1.4.123}$$

The distance between two infinitesimally separated points cannot be obtained by imposing dx^0 because x^0 transforms differently at these points. Instead, we consider a signal that leaves point $B(x^\alpha + dx^\alpha)$ at $x^0 + dx^0_-$, reaching point $A(x^\alpha)$ at x^0 and coming back to point B at $x^0 + dx^0_+$, as shown in Figure 3. Accordingly, we have

$$ds^2 = g_{00}(dx^0)^2 + 2g_{0\alpha}dx^0 dx^\alpha + g_{\alpha\beta}dx^\alpha dx^\beta = 0 \tag{1.4.124}$$

gives

$$dx^0_\pm = \frac{1}{g_{00}}(-g_{0\alpha}dx^\alpha \pm \sqrt{(g_{0\alpha}g_{0\beta} - g_{00}g_{\alpha\beta})dx^\alpha dx^\beta}). \tag{1.4.125}$$

The difference in the time coordinate between emitting and receiving the signal at point B is equal to the difference between dx^0_+ and dx^0_- times $\sqrt{g_{00}}/c$, and the *distance* dl between points A and B is equal to this difference times $c/2$:

$$dl^2 = \gamma_{\alpha\beta}dx^\alpha dx^\beta, \tag{1.4.126}$$

where

$$\gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}} \tag{1.4.127}$$

is the symmetric *spatial metric tensor* of spacetime, that is, the metric tensor of space. The event at point A at x^0 is *synchronized* with the event at point B at the arithmetic mean of the time coordinates of emitting and receiving the signal:

$$x^0 + \frac{1}{2}(dx^0_- + dx^0_+) = x^0 + g_\alpha dx^\alpha, \tag{1.4.128}$$

where

$$g_\alpha = -\frac{g_{0\alpha}}{g_{00}}. \tag{1.4.129}$$

Therefore, we have

$$\delta x^0 = g_\alpha \delta x^\alpha, \tag{1.4.130}$$

which is equivalent to $\delta x_0 = 0$, is the difference in x^0 between two synchronized infinitesimally separated points.

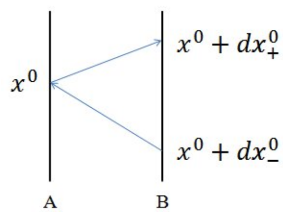

Figure 3: Distance.

In terms of (1.4.126) and (1.4.129), the square of the line element is equal to

$$ds^2 = g_{00}(dx^0 - g_\alpha dx^\alpha)^2 - dl^2, \tag{1.4.131}$$

The three-dimensional velocity (1.4.116),

$$v^\alpha = \frac{dx^\alpha}{d\tau}, \tag{1.4.132}$$

is defined in terms of the synchronized proper time (corresponding to the difference in x^0 between two synchronized infinitesimally separated points (1.4.130)):

$$d\tau = \frac{1}{c}\sqrt{g_{00}}(dx^0 - \delta x^0) = \frac{1}{c}\sqrt{g_{00}}(dx^0 - g_\alpha dx^\alpha). \tag{1.4.133}$$

Therefore, the metric (1.4.131) becomes

$$ds^2 = g_{00}(dx^0 - g_\alpha dx^\alpha)^2\left(1 - \frac{v^2}{c^2}\right), \tag{1.4.134}$$

where v is the *speed*,

$$v = (\gamma_{\alpha\beta}v^\alpha v^\beta)^{1/2}. \tag{1.4.135}$$

Using (1.4.131) in the definition of the four-velocity (1.2.57) gives

$$u^\alpha = \frac{v^\alpha}{c\sqrt{1 - v^2/c^2}}, \;\; u^0 = \frac{1}{\sqrt{g_{00}}\sqrt{1 - v^2/c^2}} + g_\alpha u^\alpha, \tag{1.4.136}$$

from which we also find

$$u_0 = g_{00}u^0 + g_{0\alpha}u^\alpha = \frac{\sqrt{g_{00}}}{\sqrt{1 - v^2/c^2}}. \tag{1.4.137}$$

1.4.9 Spatial vectors

The spatial components of a contravariant four-vector A^i form a three-dimensional, *spatial vector* $\mathbf{A}$:

$$A^i = (A^0, A^\alpha) = (A^0, \mathbf{A}). \tag{1.4.138}$$

The contravariant four-vector index α is also the contravariant spatial-vector index. The covariant components of a spatial vector are related to the contravariant components by the spatial metric tensor (1.4.127) which raises and lowers indices of spatial vectors analogously to the metric tensor acting on four-vectors:

$$A_\alpha = \gamma_{\alpha\beta} A^\beta, \tag{1.4.139}$$

$$B^\alpha = \gamma^{\alpha\beta} B_\beta, \tag{1.4.140}$$

where $\gamma^{\alpha\beta}$ is the inverse of $\gamma_{\alpha\beta}$:

$$\gamma^{\alpha\delta}\gamma_{\beta\delta} = \delta^\alpha_\beta. \tag{1.4.141}$$

A linear combination $a\mathbf{A}+b\mathbf{B}$ of two spatial vectors $\mathbf{A}$ and $\mathbf{B}$, where a and b are scalars, is a spatial vector $\mathbf{C}$ whose components are

$$C^\alpha = aA^\alpha + bB^\alpha, \quad C_\alpha = aA_\alpha + bB_\alpha. \tag{1.4.142}$$

The following formulae are satisfied:

$$\gamma^{\alpha\beta} = -g^{\alpha\beta}, \tag{1.4.143}$$

$$\mathfrak{g} = -g_{00}\mathfrak{s}, \tag{1.4.144}$$

$$g^\alpha = -g^{0\alpha}, \tag{1.4.145}$$

$$g^{00} = \frac{1}{g_{00}} - g_\alpha g^\alpha, \tag{1.4.146}$$

where

$$\mathfrak{s} = \mathrm{det}\gamma_{\alpha\beta}. \tag{1.4.147}$$

For example, contracting (1.4.127) with (1.4.143) gives

$$\begin{aligned}\gamma^{\alpha\delta}\gamma_{\beta\delta} &= g^{\alpha\delta}g_{\beta\delta} - g^{\alpha\delta}g_{0\delta}\frac{g_{0\beta}}{g_{00}} = g^{\alpha i}g_{\beta i} - g^{\alpha 0}g_{\beta 0} - (g^{\alpha i}g_{0i} - g^{\alpha 0}g_{00})\frac{g_{0\beta}}{g_{00}} \\ &= \delta^\alpha_\beta - \delta^\alpha_0\frac{g_{0\beta}}{g_{00}} = \delta^\alpha_\beta,\end{aligned} \tag{1.4.148}$$

in accordance with (1.4.141). The components g^α form a spatial vector $\mathbf{g}$.

The dot product or *scalar product* of two spatial vectors is

$$\mathbf{A}\cdot\mathbf{B} = \gamma_{\alpha\beta}A^\alpha B^\beta. \tag{1.4.149}$$

The *square* of a spatial vector $\mathbf{A}$ is

$$A^2 = \mathbf{A}\cdot\mathbf{A} \tag{1.4.150}$$

and its norm, length, or *magnitude* is

$$A = |\mathbf{A}| = \sqrt{A^2}. \tag{1.4.151}$$

The *angle* between two spatial vectors θ is defined through

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta. \tag{1.4.152}$$

In three-dimensional space, the completely antisymmetric permutation symbols are defined as

$$\epsilon^{123} = \epsilon^{0123} = 1, \;\; \epsilon^{\alpha\beta\gamma} = \epsilon^{[\alpha\beta\gamma]}, \;\; \varepsilon_{123} = -\varepsilon_{0123} = 1, \;\; \varepsilon_{\alpha\beta\gamma} = \varepsilon_{[\alpha\beta\gamma]}. \tag{1.4.153}$$

The spatial analogues of (1.4.16) and (1.4.17) are

$$e_{\alpha\beta\gamma} = \sqrt{\mathfrak{s}}\varepsilon_{\alpha\beta\gamma}, \;\; e^{\alpha\beta\gamma} = \frac{1}{\sqrt{\mathfrak{s}}}\epsilon^{\alpha\beta\gamma}. \tag{1.4.154}$$

The cross product or *vector product* of two spatial vectors $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is defined as the spatial vector density, dual to the antisymmetric tensor

$$C_{\alpha\beta} = A_\alpha B_\beta - A_\beta B_\alpha, \tag{1.4.155}$$

thereby giving

$$C^\alpha = \frac{1}{2}e^{\alpha\beta\gamma}C_{\beta\gamma} = e^{\alpha\beta\gamma}A_\beta B_\gamma, \;\; C_\alpha = \frac{1}{2}e_{\alpha\beta\gamma}C^{\beta\gamma} = e_{\alpha\beta\gamma}A^\beta B^\gamma, \tag{1.4.156}$$

$$C_{\alpha\beta} = e_{\alpha\beta\gamma}C^\gamma, \;\; C^{\alpha\beta} = e^{\alpha\beta\gamma}C_\gamma. \tag{1.4.157}$$

The permutation symbols satisfy

$$\varepsilon_{\alpha\beta\gamma}\varepsilon_{\alpha\delta\zeta} = \delta_{\beta\delta}\delta_{\gamma\zeta} - \delta_{\beta\zeta}\delta_{\gamma\delta}, \tag{1.4.158}$$

$$\varepsilon_{\alpha\beta\gamma}\varepsilon_{\alpha\beta\delta} = 2\delta_{\gamma\delta}, \tag{1.4.159}$$

$$\varepsilon_{\alpha\beta\gamma}\varepsilon_{\alpha\beta\gamma} = 6, \tag{1.4.160}$$

where $\delta_{\alpha\beta}$ is the *Cartesian metric tensor*,

$$\delta_{\alpha\beta} = \delta^{\alpha\beta} = \mathrm{diag}(1,1,1). \tag{1.4.161}$$

The *spatial covariant derivative* ∇_α acts on spatial vectors analogously to the metric covariant derivative acting on four-vectors:

$$\nabla_\alpha A^\beta = \partial_\alpha A^\beta + \{{}^{\;\beta}_{\gamma\,\alpha}\}_\gamma A^\gamma, \tag{1.4.162}$$

$$\nabla_\alpha A_\beta = \partial_\alpha A_\beta - \{{}^{\;\gamma}_{\beta\,\alpha}\}_\gamma A_\gamma, \tag{1.4.163}$$

where $\{{}^{\;\delta}_{\alpha\,\beta}\}_\gamma$ are the three-dimensional, *spatial Christoffel symbols*:

$$\{{}^{\;\delta}_{\alpha\,\beta}\}_\gamma = \frac{1}{2}\gamma^{\delta\gamma}(\gamma_{\gamma\alpha,\beta} + \gamma_{\gamma\beta,\alpha} - \gamma_{\alpha\beta,\gamma}). \tag{1.4.164}$$

The *gradient* operator is given by

$$(\mathrm{grad})^\alpha = (\boldsymbol{\nabla})^\alpha = \gamma^{\alpha\beta}\nabla_\beta. \tag{1.4.165}$$

The spatial components of a covariant-vector operator ∂_i acting on a scalar ϕ form the gradient of ϕ:

$$\partial_i\phi = \left(\frac{\partial\phi}{c\partial t}, \frac{\partial\phi}{\partial x^\alpha}\right) = \left(\frac{\partial\phi}{c\partial t}, \boldsymbol{\nabla}\phi\right). \tag{1.4.166}$$

The divergence of a spatial vector $\mathbf{A}$ is, analogously to (1.4.45),

$$\mathrm{div}\mathbf{A} = \boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{\sqrt{\mathfrak{s}}}\partial_\alpha(\sqrt{\mathfrak{s}}A^\alpha). \tag{1.4.167}$$

The curl of a spatial vector $\mathbf{A}$ is defined as the spatial vector density, dual to the antisymmetric tensor $\partial_\alpha A_\beta - \partial_\beta A_\alpha$:

$$(\mathrm{curl}\mathbf{A})^\alpha = (\boldsymbol{\nabla}\times\mathbf{A})^\alpha = \frac{1}{2}e^{\alpha\beta\gamma}(\partial_\beta A_\gamma - \partial_\gamma A_\beta) = e^{\alpha\beta\gamma}\partial_\beta A_\gamma. \tag{1.4.168}$$

The *Laplace-Beltrami operator* or *Laplacian* is the divergence of the gradient,

$$\triangle = \nabla^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} = \frac{1}{\sqrt{\mathfrak{s}}}\partial_\alpha(\sqrt{\mathfrak{s}}\gamma^{\alpha\beta}\partial_\beta). \tag{1.4.169}$$

The *d'Alembert operator* or *d'Alembertian* is defined as

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \triangle. \tag{1.4.170}$$

The time component of the dual hypersurface element (1.1.35) is equal to the spatial volume element dV:

$$dS_0 = dV. \tag{1.4.171}$$

The spatial analogue of the Gauß-Stokes theorem (1.1.39) is Gauß' theorem:

$$df_\alpha \leftrightarrow dV\frac{\partial}{\partial x^\alpha}, \tag{1.4.172}$$

where

$$df_\alpha = df^\star_{0\alpha} \tag{1.4.173}$$

is the spatial component of the dual surface element (1.1.33), perpendicular to the x^α axis.

In a locally Galilean frame of reference, the covariant and contravariant components of a spatial vector are identical because

$$\gamma_{\alpha\beta} = \delta_{\alpha\beta}. \tag{1.4.174}$$

In this frame, we refer to the Cartesian coordinates x^1, x^2, x^3 as x, y, z. These coordinates form the three-dimensional *radius vector* $\mathbf{x}$. The following formulae are satisfied:

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}, \tag{1.4.175}$$
$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}), \tag{1.4.176}$$
$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}), \tag{1.4.177}$$
$$(\mathbf{A}\cdot\mathbf{B})^2 + (\mathbf{A}\times\mathbf{B})^2 = A^2B^2, \tag{1.4.178}$$
$$\mathrm{curl}\,\mathrm{grad}\phi = 0, \tag{1.4.179}$$
$$\mathrm{div}\,\mathrm{curl}\,\mathbf{A} = 0, \tag{1.4.180}$$
$$\mathrm{grad}(\phi\psi) = \mathrm{grad}\phi\,\psi + \phi\,\mathrm{grad}\psi, \tag{1.4.181}$$
$$\begin{aligned}\mathrm{grad}(\mathbf{A}\cdot\mathbf{B}) &= (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} + \mathbf{A}\times\mathrm{curl}\,\mathbf{B}\\ &+\mathbf{B}\times\mathrm{curl}\,\mathbf{A},\end{aligned} \tag{1.4.182}$$
$$\mathrm{div}(\phi\mathbf{A}) = \mathrm{grad}\phi\cdot\mathbf{A} + \phi\,\mathrm{div}\,\mathbf{A}, \tag{1.4.183}$$
$$\mathrm{curl}(\phi\mathbf{A}) = \mathrm{grad}\phi\times\mathbf{A} + \phi\,\mathrm{curl}\,\mathbf{A}, \tag{1.4.184}$$
$$\mathrm{div}(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\mathrm{curl}\,\mathbf{A} - \mathbf{A}\cdot\mathrm{curl}\,\mathbf{B}, \tag{1.4.185}$$
$$\mathrm{curl}(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + \mathbf{A}\,\mathrm{div}\,\mathbf{B} - \mathbf{B}\,\mathrm{div}\,\mathbf{A}, \tag{1.4.186}$$
$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{A} = \mathrm{grad}\,\mathrm{div}\,\mathbf{A} - \triangle\mathbf{A}, \tag{1.4.187}$$

where

$$(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} = A^\alpha\partial_\alpha\mathbf{B}. \tag{1.4.188}$$

The vector product of two spatial vectors $\mathbf{A}$ and $\mathbf{B}$ satisfies

$$\mathbf{A}\times\mathbf{B} = AB\sin\theta\,\mathbf{n}, \tag{1.4.189}$$

where $\mathbf{n}$ is a unit vector perpendicular to both $\mathbf{A}$ and $\mathbf{B}$, in the direction given by the right-handed corkscrew rule.

1.4.10 Embedded hypersurfaces

A *surface* embedded in a three-dimensional space consists of points whose radius vectors are vector functions of two parameters ξ^α, where the index α can be 1 or 2: $\mathbf{x} = \mathbf{x}(\xi^1, \xi^2)$. A vector

$$\partial_\alpha\mathbf{x} = \frac{\partial\mathbf{x}}{\partial\xi^\alpha} \tag{1.4.190}$$

is *tangent* to the surface. We define the *induced* or *intrinsic metric tensor* on the surface as

$$\gamma_{\alpha\beta} = \partial_\alpha \mathbf{x} \cdot \partial_\beta \mathbf{x}. \tag{1.4.191}$$

The length element dl on the surface is given by the *first fundamental form*:

$$dl^2 = d\mathbf{x} \cdot d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \xi^\alpha} \cdot \frac{\partial \mathbf{x}}{\partial \xi^\beta} d\xi^\alpha d\xi^\beta = \gamma_{\alpha\beta} d\xi^\alpha d\xi^\beta, \tag{1.4.192}$$

and the area element is given by

$$dS = \sqrt{\det \gamma_{\alpha\beta}} d\xi^1 d\xi^2. \tag{1.4.193}$$

The inverse intrinsic metric tensor $\gamma^{\alpha\beta}$ is defined according to

$$\gamma^{\alpha\delta} \gamma_{\beta\delta} = \delta^\alpha_\beta. \tag{1.4.194}$$

We define the unit *normal vector* to a surface as

$$\mathbf{n} = \frac{\partial_1 \mathbf{x} \times \partial_2 \mathbf{x}}{|\partial_1 \mathbf{x} \times \partial_2 \mathbf{x}|}, \quad \mathbf{n} \cdot \mathbf{n} = 1. \tag{1.4.195}$$

This vector is perpendicular to a tangent vector:

$$\partial_\alpha \mathbf{x} \cdot \mathbf{n} = 0. \tag{1.4.196}$$

If the surface is curved, then the normal vectors at two close points on the surface are not parallel. The change of the normal vector is given by the *extrinsic curvature tensor*:

$$K_{\alpha\beta} = \partial_\alpha \partial_\beta \mathbf{x} \cdot \mathbf{n}. \tag{1.4.197}$$

The extrinsic curvature is symmetric,

$$K_{\alpha\beta} = K_{\beta\alpha}. \tag{1.4.198}$$

Differentiating the relation (1.4.196) with respect to ξ^β and using (1.4.197) gives

$$K_{\alpha\beta} = -\partial_\alpha \mathbf{x} \cdot \partial_\beta \mathbf{n}. \tag{1.4.199}$$

The quantity $K_{\alpha\beta} d\xi^\alpha d\xi^\beta$ is the *second fundamental form*. The *intrinsic Christoffel symbols* $\{{}^{\,\alpha}_{\beta\,\gamma}\}$, symmetric in the lower indices, are constructed from the intrinsic metric tensor analogously to the spatial Christoffel symbols (1.4.164) constructed from the spatial metric tensor. They are used to construct the covariant derivative ∇_i acting on the vectors tangent to the surface, analogously to (1.4.162) and (1.4.163).

The covariant derivatives acting on $\mathbf{x}$ and $\mathbf{n}$ are equal to the partial derivatives:

$$\nabla_\alpha \mathbf{x} = \partial_\alpha \mathbf{x}, \;\; \nabla_\alpha \mathbf{n} = \partial_\alpha \mathbf{n}. \tag{1.4.200}$$

The second derivatives of $\mathbf{x}$, which are the first derivatives of the tangent vectors, satisfy the *Gauß equation*:

$$\partial_\alpha \partial_\beta \mathbf{x} = \{{}^{\,\gamma}_{\alpha\,\beta}\} \partial_\gamma \mathbf{x} + K_{\alpha\beta} \mathbf{n}, \tag{1.4.201}$$

which can be written in a covariant form:

$$\nabla_\alpha \nabla_\beta \mathbf{x} = K_{\alpha\beta} \mathbf{n}. \tag{1.4.202}$$

Multiplying this equation by $\mathbf{n}$ gives (1.4.197). The first derivatives of the normal vector satisfy the *Weingarten equation*:

$$\partial_\alpha \mathbf{n} = -K_\alpha{}^\beta \partial_\beta \mathbf{x} = -K_{\alpha\gamma} \gamma^{\beta\gamma} \partial_\beta \mathbf{x}, \tag{1.4.203}$$

which can be written in a covariant form:

$$\nabla_\alpha \mathbf{n} = -K_\alpha{}^\beta \nabla_\beta \mathbf{x}. \tag{1.4.204}$$

Multiplying this equation by $\partial_\gamma \mathbf{x}$ and using (1.4.191) gives (1.4.199). The intrinsic metric tensor and the extrinsic curvature can also be written in a covariant form:

$$\gamma_{\alpha\beta} = \nabla_\alpha \mathbf{x} \cdot \nabla_\beta \mathbf{x}, \tag{1.4.205}$$
$$K_{\alpha\beta} = \nabla_\alpha \nabla_\beta \mathbf{x} \cdot \mathbf{n}. \tag{1.4.206}$$

Using the Gauß equation, the relation

$$\partial_\alpha \partial_\beta \partial_\gamma \mathbf{x} = \partial_\beta \partial_\alpha \partial_\gamma \mathbf{x} \tag{1.4.207}$$

can be written as

$$\partial_\alpha(\{^{\ \delta}_{\beta\,\gamma}\}\partial_\delta \mathbf{x} + K_{\beta\gamma}\mathbf{n}) = \partial_\beta(\{^{\ \delta}_{\alpha\,\gamma}\}\partial_\delta \mathbf{x} + K_{\alpha\gamma}\mathbf{n}). \tag{1.4.208}$$

Effecting the differentiation and using again the Gauß equation gives

$$\begin{aligned} &\partial_\delta \mathbf{x}\partial_\alpha\{^{\ \delta}_{\beta\,\gamma}\} + \{^{\ \delta}_{\beta\,\gamma}\}\{^{\ \epsilon}_{\alpha\,\delta}\}\partial_\epsilon \mathbf{x} + \{^{\ \delta}_{\beta\,\gamma}\}K_{\alpha\delta}\mathbf{n} + \partial_\alpha K_{\beta\gamma}\mathbf{n} + K_{\beta\gamma}\partial_\alpha \mathbf{n} \\ &= \partial_\delta \mathbf{x}\partial_\beta\{^{\ \delta}_{\alpha\,\gamma}\} + \{^{\ \delta}_{\alpha\,\gamma}\}\{^{\ \epsilon}_{\beta\,\delta}\}\partial_\epsilon \mathbf{x} + \{^{\ \delta}_{\alpha\,\gamma}\}K_{\beta\delta}\mathbf{n} + \partial_\beta K_{\alpha\gamma}\mathbf{n} + K_{\alpha\gamma}\partial_j \mathbf{n}. \end{aligned} \tag{1.4.209}$$

Multiplying this equation by $\partial_\zeta \mathbf{x}$ and using (1.4.191), (1.4.196), and (1.4.199) gives

$$\gamma_{\delta\zeta}\partial_\alpha\{^{\ \delta}_{\beta\,\gamma}\} + \gamma_{\epsilon\zeta}\{^{\ \delta}_{\beta\,\gamma}\}\{^{\ \epsilon}_{\alpha\,\delta}\} - K_{\alpha\zeta}K_{\beta\gamma} - \gamma_{\delta\zeta}\partial_\beta\{^{\ \delta}_{\alpha\,\gamma}\} - \gamma_{\epsilon\zeta}\{^{\ \delta}_{\alpha\,\gamma}\}\{^{\ \epsilon}_{\beta\,\delta}\} + K_{\beta\zeta}K_{\alpha\gamma} = 0, \tag{1.4.210}$$

which is equivalent to the *Gauß equation*:

$$r^\epsilon_{\ \gamma\alpha\beta} = K^\epsilon_{\ \alpha}K_{\gamma\beta} - K^\epsilon_{\ \beta}K_{\gamma\alpha}, \tag{1.4.211}$$

where $r^\epsilon_{\ \gamma\alpha\beta}$ is the *intrinsic curvature tensor* constructed from the intrinsic Christoffel symbols analogously to the Riemann tensor (1.4.57) constructed from the Levi-Civita connection. Multiplying (1.4.209) by $\mathbf{n}$ and using (1.4.196) and $\partial_\alpha \mathbf{n} \cdot \mathbf{n} = 0$ gives the *Codazzi-Mainardi-Peterson equation*:

$$\{^{\ \delta}_{\beta\,\gamma}\}K_{\alpha\delta} + \partial_\alpha K_{\beta\gamma} = \{^{\ \delta}_{\alpha\,\gamma}\}K_{\beta\delta} + \partial_\beta K_{\alpha\gamma}, \tag{1.4.212}$$

which can be written in a covariant form:

$$\nabla_\alpha K_{\beta\gamma} = \nabla_\beta K_{\alpha\gamma}. \tag{1.4.213}$$

The *Gauß curvature* is defined as

$$K = \frac{\mathrm{det}K_{\alpha\beta}}{\mathrm{det}\gamma_{\alpha\beta}} = \frac{K_{11}K_{22} - K_{12}K_{21}}{\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21}}. \tag{1.4.214}$$

Using the Gauß equation, it leads to the *Gauß theorem*:

$$K = \frac{r_{1212}}{\mathrm{det}\gamma_{\alpha\beta}}, \tag{1.4.215}$$

which is consistent with (1.4.80) and (1.4.84).

A *curve* on a surface consists of points whose radius vectors depend on a parameter t: $\mathbf{x} = \mathbf{x}(\xi^1(t), \xi^2(t))$. Such a curve is geodesic if it satisfies the metric geodesic equation analogous to (1.4.92):

$$\frac{d^2\xi^\alpha}{dt^2} + \{^{\ \alpha}_{\beta\,\gamma}\}\frac{d\xi^\beta}{dt}\frac{d\xi^\gamma}{dt} = 0. \tag{1.4.216}$$

A geodesic curve also satisfies

$$\frac{d^2\mathbf{x}}{dt^2} \sim \mathbf{n}, \quad \frac{d}{dt}\Big(\frac{d\mathbf{x}}{dt} \cdot \frac{d\mathbf{x}}{dt}\Big) = 0. \tag{1.4.217}$$

A *hypersurface* embedded in a four-dimensional spacetime consists of points whose coordinates are functions of three parameters ξ^α, where the index α can be 1, 2, or 3: $x^i = x^i(\xi^1, \xi^2, \xi^3)$. Equivalently, these coordinates satisfy an equation of constraint:

$$f(x^i) = 0, \tag{1.4.218}$$

where f is a function of the coordinates. The normal vector to this hypersurface is given by

$$n_i = \frac{\partial f}{\partial x^i}. \tag{1.4.219}$$

All infinitesimal displacements dx^i along such a hypersurface satisfy, according to (1.4.219),

$$df = n_i dx^i = 0. \tag{1.4.220}$$

The normal vector (1.4.219) is orthogonal to the hypersurface:

$$n_i n_{j:k} \epsilon^{ijkl} = 0, \tag{1.4.221}$$

where ϵ^{ijkl} is the completely antisymmetric permutation symbol. This condition is equivalent to

$$n_{[i} n_{j:k]} = \frac{1}{6}(n_i n_{j:k} + n_j n_{k:i} + n_k n_{i:j} - n_k n_{j:i} - n_i n_{k:j} - n_j n_{i:k}) = 0. \tag{1.4.222}$$

If the normal vector to a hypersurface is timelike, then the hypersurface is spacelike. Such a normal vector can be normalized:

$$n^i n_i = 1, \tag{1.4.223}$$

which gives

$$n^i n_{i:k} = 0. \tag{1.4.224}$$

In this case, the four-velocity of a point in spacetime can be taken as the normal vector:

$$n^i = u^i. \tag{1.4.225}$$

If the four-velocity has only the time component, then the hypersurface is a hypersurface of constant time and represents a volume in space, in which the point exists at this time. A division of spacetime into such hypersurfaces is referred to as a *foliation* of spacetime.

We consider a spacelike hypersurface. We define the *projection tensor* onto the hypersurface:

$$h^i_{\ j} = \delta^i_j - n^i n_j, \tag{1.4.226}$$

which is orthogonal to n^i:

$$h^i_{\ j} n^j = 0. \tag{1.4.227}$$

The projection tensor satisfies

$$h^i_{\ j} h^j_{\ k} = h^i_{\ k}. \tag{1.4.228}$$

The indices in the projection tensor can be raised or lowered by the metric tensor:

$$h_{ik} = h^j_{\ k} g_{ij} = g_{ik} - n_i n_k = h_{ki}, \quad h^{ik} = h^i_{\ j} g^{jk} = g^{ik} - n^i n^k = h^{ki}. \tag{1.4.229}$$

The tensors h_{ik} and h^{ik} are symmetric and not inverse to one another.

The projection $\perp$ of a tensor T onto a hypersurface is defined as the contraction of the tensor T with the projection tensor through all indices. For example, the projections of vectors are

$$\perp V^i = h^i_{\ k} V^k, \quad \perp V_i = h^k_{\ i} V_k. \tag{1.4.230}$$

These projections are tangent vectors to the hypersurface. The projection of the metric tensor gives

$$\perp g_{ij} = h^k_{\ i} h^l_{\ j} g_{kl} = h_{ij}, \quad \perp g^{ij} = h^i_{\ k} h^j_{\ l} g^{kl} = h^{ij}. \tag{1.4.231}$$

Consequently, the relation

$$h_{ij} \perp V^i \perp V^j = g_{ij} \perp V^i \perp V^j \tag{1.4.232}$$

shows that the tensor h_{ij} is the intrinsic metric tensor γ_{ij} on the hypersurface, analogously to (1.4.191):

$$\gamma_{ij} = h_{ij}. \tag{1.4.233}$$

The inverse intrinsic metric tensor γ^{ij} is defined as in (1.4.194). The projection of the normal vector vanishes:

$$\perp n^i = 0. \tag{1.4.234}$$

If the normal vector to the hypersurface is timelike, then the projections of tensors have only spatial components. Using the tensor $n^i n_j$ instead of $h^i_{\ j}$ projects a tensor onto the direction of the normal vector.

The projection of the covariant derivative (with respect to a torsionless affine connection) of a vector defines the intrinsic covariant derivative of a vector on the hypersurface:

$$D_k V^l =\perp \nabla_k V^l = h^i_{\ k} h^l_{\ j} \nabla_i V^j. \tag{1.4.235}$$

The intrinsic covariant derivative of the intrinsic metric tensor vanishes:

$$D_k \gamma_{ij} =\perp \nabla_k \gamma_{ij} =\perp \nabla_k (g_{ij} - n_i n_j) = -\perp (n_i \nabla_k n_j + n_j \nabla_k n_i) = 0, \tag{1.4.236}$$

which is a consequence of the metric compatibility of the affine connection (1.4.5). Accordingly, the intrinsic covariant derivative is constructed from the intrinsic Christoffel symbols, which are constructed from the intrinsic metric tensor. If ∇_k is related to the Levi-Civita connection of the metric g_{ij}, then D_k is related to the Levi-Civita connection of the intrinsic metric γ_{ij}.

If the parallel transport of the normal vector to a hypersurface along a vector $W^i =\perp V^i$ on the hypersurface does not vanish,

$$W^i \nabla_i n^j \neq 0, \tag{1.4.237}$$

then the hypersurface is curved. Such a hypersurface has a nonzero extrinsic curvature tensor, defined as

$$K_{ij} = -\perp \nabla_i n_j = -h^k_{\ i} h^l_{\ j} \nabla_k n_l. \tag{1.4.238}$$

Using (1.4.224) and (1.4.226), the extrinsic curvature is equal to

$$K_{ij} = -(\delta^k_i - n^k n_i)(\delta^l_j - n^l n_j) n_{l:k} = -n_{j:i} + n_i n^k n_{j:k}. \tag{1.4.239}$$

The extrinsic curvature is a tensor with only spatial components:

$$K_{ij} n^j = 0. \tag{1.4.240}$$

Antisymmetrizing the indices in the extrinsic curvature and using (1.4.239) gives

$$K_{ij} - K_{ji} = n_{i:j} - n_{j:i} + n^k n_i n_{j:k} - n^k n_j n_{i:k}. \tag{1.4.241}$$

The term on the right-hand side is equal to the term in (1.4.222) contracted with n^k, which vanishes. Consequently, the extrinsic curvature is symmetric, as in (1.4.198). This symmetry also results from (1.4.219):

$$K_{ij} = -\perp \nabla_i \partial_j f = -\perp \nabla_j \partial_i f = K_{ji}. \tag{1.4.242}$$

For the Levi-Civita connection, (1.4.51) gives

$$K_{ij} = -\perp n_{j:i} = -\perp n_{(i:j)} = -\frac{1}{2} \perp \mathcal{L}_n g_{ij}, \tag{1.4.243}$$

where $\mathcal{L}_n$ is the Lie derivative of the metric tensor along the vector n^i. If the normal vector is the four-velocity of a point in spacetime, then (1.4.239) gives

$$K_{ij} = -u_{j;i} + u_i \frac{Du_j}{ds}. \tag{1.4.244}$$

The contraction of the extrinsic curvature tensor gives the *extrinsic curvature scalar*:

$$K = K_{ij} \gamma^{ij}. \tag{1.4.245}$$

For a spacelike hypersurface, the spatial coordinates on this hypersurface can be taken as the parameters ξ^α. Differentiating the equation of constraint for a hypersurface $f(x^i(\xi^\alpha)) = 0$ with respect to ξ^α gives

$$\frac{\partial f}{\partial \xi^\alpha} = \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial \xi^\alpha} = n_i \frac{\partial x^i}{\partial \xi^\alpha} = 0. \tag{1.4.246}$$

Differentiating covariantly this equation with respect to ξ^β gives

$$\frac{\nabla n_i}{\partial \xi^\beta}\frac{\partial x^i}{\partial \xi^\alpha} + n_i \frac{\nabla^2 x^i}{\partial \xi^\beta \partial \xi^\alpha} = \frac{\nabla}{\partial x^i}\frac{\partial f}{\partial \xi^\beta}\frac{\partial x^i}{\partial \xi^\alpha} + n_i \nabla_\beta \nabla_\alpha x^i = \frac{\nabla^2 f}{\partial \xi^\alpha \partial \xi^\beta} + n_i \nabla_\beta \nabla_\alpha x^i = \frac{\nabla n_\beta}{\partial \xi^\alpha} + n_i \nabla_\beta \nabla_\alpha x^i = 0, \tag{1.4.247}$$

where the covariant derivatives ∇_α are constructed from the metric tensor $\gamma_{\alpha\beta}$ and the corresponding Levi-Civita connection. Accordingly, using (1.4.238), we obtain

$$K_{\alpha\beta} = -\nabla_\alpha n_\beta = n_i \nabla_\beta \nabla_\alpha x^i, \tag{1.4.248}$$

which is consistent with the extrinsic curvature tensor for a surface (1.4.206).

The intrinsic covariant derivative of a vector $W^i = \perp V^i$ on a hypersurface is

$$\begin{aligned} D_j W_k &= \perp \nabla_j W_k = (\delta^m_j - n^m n_j)(\delta^n_k - n^n n_k)\nabla_m W_n \\ &= \nabla_j W_k - n^m n_j \nabla_m W_k - n^n n_k \nabla_j W_n + n^m n_j n^n n_k \nabla_m W_n \\ &= \nabla_j W_k - n^m n_j \nabla_m W_k + n_k W^n \nabla_j n_n + n^m n_j n^n n_k \nabla_m W_n, \end{aligned} \tag{1.4.249}$$

where we used $n_i W^i = 0$, which gives $n^i \nabla_j W_i = -W^i \nabla_j n_i$. Consequently, the second derivative is

$$\begin{aligned} D_i D_j W_k &= \perp (\nabla_i D_j W_k) = \perp (\nabla_i \perp \nabla_j W_k) \\ &= \perp \nabla_i \nabla_j W_k + \perp \nabla_i(-n^m n_j \nabla_m W_k + n_k W^n \nabla_j n_n + n^m n_j n^n n_k \nabla_m W_n) \\ &= \perp \nabla_i \nabla_j W_k + \perp \nabla_i n_k W^n \nabla_j n_n = \perp \nabla_i \nabla_j W_k + K_{ik} K_{jn} W^n, \end{aligned} \tag{1.4.250}$$

where we used (1.4.234) and (1.4.238). The commutator of intrinsic covariant derivatives gives the intrinsic curvature tensor:

$$[D_i, D_j] W_k = -r^l{}_{kij} W_l, \tag{1.4.251}$$

whereas the commutator of covariant derivatives gives the Riemann tensor, according to (1.4.56). Therefore, antisymmetrizing the indices i, j in (1.4.250) gives

$$r^l{}_{kij} W_l = \perp P^l{}_{kij} W_l - K_{ik} K_{jl} W^l + K_{jk} K_{il} W^l, \tag{1.4.252}$$

which leads to

$$\perp P_{lkij} = r_{lkij} + K_{ik} K_{jl} - K_{jk} K_{il}. \tag{1.4.253}$$

This equation is consistent with (1.4.211) for $P_{lkij} = 0$, satisfied for a curved surface in a flat space.

The projection of $n^i P_{ijkl}$ is given by

$$\begin{aligned} \perp (n^i P_{ijkl}) &= \perp (\nabla_l \nabla_k n_j - \nabla_k \nabla_l n_j) = \perp (\nabla_l (K_{kj} - n_k n^i n_{j:i}) - \nabla_k (K_{lj} - n_l n^i n_{j:i})) \\ &= \perp (\nabla_l K_{kj} - \nabla_k K_{lj} + (\nabla_k n_l - \nabla_l n_k) n^i n_{j:i}) = D_l K_{kj} - D_k K_{lj}, \end{aligned} \tag{1.4.254}$$

where we used (1.4.219) and (1.4.234). This equation is consistent with (1.4.213) for $P_{lkij} = 0$, satisfied for a curved surface in a flat space. Equations (1.4.253) and (1.4.254) are referred to as the *Gauß-Codazzi equations*.

If the normal vector to a hypersurface is spacelike, then the hypersurface is timelike. An example of such a hypersurface is a hypersurface on which a given spatial coordinate is constant. The normal vector can be normalized:

$$n^i n_i = -1. \tag{1.4.255}$$

The projection tensor onto a timelike hypersurface differs from (1.4.226) by a sign:

$$h^i{}_j = \delta^i_j + n^i n_j, \tag{1.4.256}$$

whereas all other definitions are the same as for spacelike hypersurfaces.

If a hypersurface is spacelike or timelike, and forms a boundary between two submanifolds in spacetime, then the intrinsic metric tensor γ_{ij} and the extrinsic curvature tensor K_{ij} are continuous across the hypersurface. Consequently, the first and second fundamental forms are continuous across the hypersurface. These two covariant conditions are referred to as the *Darmois junction conditions*. Equivalently, the metric tensor g_{ij} and its derivatives $g_{ij,k}$ are continuous across the hypersurface. These two conditions are referred to as the *Lichnerowicz junction conditions*.

1.4.11 Event horizon

If the normal vector (1.4.219) to a hypersurface is a null vector,

$$n_i n^i = 0, \tag{1.4.257}$$

then this hypersurface is a null hypersurface. Equations (1.4.220) and (1.4.257) indicate that n^i lies itself on the null hypersurface to which it is normal,

$$dx^i \propto n^i, \tag{1.4.258}$$

which also gives

$$ds^2 = dx_i dx^i \propto n_i n^i = 0. \tag{1.4.259}$$

Therefore, all world lines on a null hypersurface are null. The light cones at the points of such a hypersurface are tangent to this hypersurface. Since all physical world lines must lie within the local light cones, the forward-time motion through a null hypersurface can occur in only one direction. To avoid any discontinuities, this direction is the same for all points on such a hypersurface. A null hypersurface is therefore an *event horizon*: a boundary in spacetime beyond which events cannot affect events on the other side. All laws of classical physics are known to be *time-symmetric*, that is, symmetric under the transformation $t \to -t$. However, the existence of event horizons, which are solutions to these laws and provide boundary conditions for spacetime, violates this symmetry. The unidirectional character of the motion through an event horizon can be used to define the past and future: the *arrow of time*.
References: [1, 2, 3, 4, 5].

1.5 Tetrad and spin connection

1.5.1 Tetrad

In addition to the coordinate systems, at each point in spacetime we can set up four linearly independent vectors e^i_a such that

$$e^i_a e_{ib} = \eta_{ab}, \tag{1.5.1}$$

where $a, b = 0, 1, 2, 3$ are *Lorentz indices* and $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$ is the coordinate-invariant Minkowski metric tensor in a locally geodesic frame of reference at this point. This set of four vectors is referred to as a *tetrad*. The inverse tetrad e^{ai} satisfies

$$e^i_a e^b_i = \delta^b_a, \tag{1.5.2}$$
$$e^i_a e^a_k = \delta^i_k. \tag{1.5.3}$$

The coordinate metric tensors g_{ik} and g^{ik} are related to the Minkowski metric tensor through the tetrad:

$$g_{ik} = e^a_i e^b_k \eta_{ab}, \tag{1.5.4}$$
$$g^{ik} = e^i_a e^k_b \eta^{ab}, \tag{1.5.5}$$

where η^{ab} satisfies

$$\eta_{ac}\eta^{bc} = \delta^b_a. \tag{1.5.6}$$

Any vector V can be specified by its components V^i with respect to the coordinate system or by the coordinate-invariant projections V^a of the vector onto the tetrad field:

$$V^a = e^a_i V^i, \;\; V_a = e^i_a V_i, \tag{1.5.7}$$
$$V^i = e^i_a V^a, \;\; V_i = e^a_i V_a, \tag{1.5.8}$$

and similarly for tensors and densities with more indices. We can use η_{ab} and its inverse η^{ab} to lower and raise Lorentz indices, as we use g_{ik} and its inverse g^{ik} to lower and raise coordinate indices.

Let us consider the determinant of the matrix composed from the components of the tetrad e^a_i,

$$\mathfrak{e} = |e^a_i|. \tag{1.5.9}$$

This determinant is related to the determinant $\mathfrak{g}$ of the metric tensor g_{ik}, using (1.5.4), by

$$\mathfrak{e} = \sqrt{|\mathfrak{g}|}. \tag{1.5.10}$$

The differential and derivatives of the determinant (1.5.9) are given, analogously to (1.4.18) and (1.4.19), by

$$d\mathfrak{e} = \mathfrak{e} e^i_a de^a_i = -\mathfrak{e} e^a_i de^i_a, \tag{1.5.11}$$
$$\mathfrak{e}_{,k} = \mathfrak{e} e^i_a e^a_{i,k} = -\mathfrak{e} e^a_i e^i_{a,k}. \tag{1.5.12}$$

The variation of (1.5.9) is thus, analogously to (1.4.20), equal to

$$\delta\mathfrak{e} = \mathfrak{e} e^i_a \delta e^a_i = -\mathfrak{e} e^a_i \delta e^i_a. \tag{1.5.13}$$

Similarly to (1.4.21), the covariant derivative of (1.5.9) vanishes:

$$\mathfrak{e}_{;j} = 0. \tag{1.5.14}$$

1.5.2 Lorentz transformation

The relation (1.5.4) imposes 10 constraints on the 16 components of the tetrad, leaving 6 components arbitrary. If we change from one tetrad e^i_a to another, $\tilde{e}^i_b$, then the vectors of the new tetrad are linear combinations of the vectors of the old tetrad:

$$\tilde{e}^i_a = \Lambda^b{}_a e^i_b. \tag{1.5.15}$$

The relation (1.5.4) applied to the tetrad field $\tilde{e}^i_b$,

$$g_{ik} = \tilde{e}^a_i \tilde{e}^b_k \eta_{ab}, \tag{1.5.16}$$

imposes on the matrix $\Lambda^b{}_a$ the orthogonality condition:

$$\Lambda^c{}_a \Lambda^d{}_b \eta_{cd} = \eta_{ab}. \tag{1.5.17}$$

We refer to $\Lambda^b{}_a$ as a *Lorentz matrix*, and to a transformation of form (1.5.15) as the *Lorentz transformation.*

1.5.3 Tetrad transport

A natural choice for the zeroth component of a tetrad at a given point is

$$e^i_0 = u^i. \tag{1.5.18}$$

Along a world line this tetrad should be transported such that the zeroth component always coincides with the four-velocity. The *Fermi-Walker transport* of a tetrad is defined as

$$\frac{\nabla e^i_a}{ds} = -u^i e^j_a \frac{Du_j}{ds} + \frac{Du^i}{ds} e^j_a u_j. \tag{1.5.19}$$

Putting $a=0$ in (1.5.19) gives

$$\frac{\nabla u^i}{ds} = \frac{Du^i}{ds}, \tag{1.5.20}$$

so the Fermi-Walker transport of the four-velocity is equivalent to its covariant change and thus (1.5.18) is valid at all points. This transport does not change the orthogonality relation for tetrads (1.5.1) because (1.5.19) gives

$$\frac{\nabla}{ds}(e^i_a e_{ib}) = 0. \tag{1.5.21}$$

1.5.4 Spin connection

We define

$$\omega^i_{\ ak} = e^i_{a;k} = e^i_{a,k} + \Gamma^{\,i}_{j\,k} e^j_a. \tag{1.5.22}$$

The quantities

$$\omega^a_{\ bi} = e^a_k \omega^k_{\ bi} = e^a_k (e^k_{b,i} + \Gamma^{\,k}_{j\,i} e^j_b) \tag{1.5.23}$$

transform like vectors under coordinate transformations. We can extend the notion of covariant differentiation to quantities with Lorentz coordinate-invariant indices by regarding $\omega^{ab}_{\ \ i}$ as a connection, referred to as Lorentz or *spin connection*. For a contravariant Lorentz vector

$$V^a_{\ |i} = V^a_{\ ,i} + \omega^a_{\ bi} V^b, \tag{1.5.24}$$

where $_{|i}$ is the covariant derivative of such a quantity with respect to x^i. The covariant derivative of a scalar $V^a W_a$ coincides with its ordinary derivative:

$$(V^a W_a)_{|i} = (V^a W_a)_{,i}, \tag{1.5.25}$$

which gives the covariant derivative of a covariant Lorentz vector:

$$W_{a|i} = W_{a,i} - \omega^b_{\ ai} W_b. \tag{1.5.26}$$

The chain rule infers that the covariant derivative of a Lorentz tensor is equal to the sum of the corresponding ordinary derivative of this tensor and terms with spin connection corresponding to each Lorentz index:

$$T^{ab\dots}_{\ \ \ \ cd\dots|i} = T^{ab\dots}_{\ \ \ \ cd\dots,i} + \omega^a_{\ ei} T^{eb\dots}_{\ \ \ \ cd\dots} + \omega^b_{\ ei} T^{ae\dots}_{\ \ \ \ cd\dots} + \dots - \omega^e_{\ ci} T^{ab\dots}_{\ \ \ \ ed\dots} - \omega^e_{\ di} T^{ab\dots}_{\ \ \ \ ce\dots} - \dots. \tag{1.5.27}$$

We assume that the covariant derivative $_{|i}$ is total, that is, also recognizes coordinate indices, acting on them like $_{;i}$. For a tensor with both coordinate and Lorentz indices

$$T^{aj\dots}_{\ \ \ \ bk\dots|i} = T^{aj\dots}_{\ \ \ \ bk\dots,i} + \omega^a_{\ ei} T^{ej\dots}_{\ \ \ \ bk\dots} + \Gamma^{\,j}_{l\,i} T^{al\dots}_{\ \ \ \ bk\dots} + \dots - \omega^e_{\ bi} T^{aj\dots}_{\ \ \ \ ek\dots} - \Gamma^{\,l}_{k\,i} T^{aj\dots}_{\ \ \ \ bl\dots} - \dots. \tag{1.5.28}$$

A total covariant derivative of a tetrad is

$$e^i_{a|k} = e^i_{a,k} + \Gamma^{\,i}_{j\,k} e^j_a - \omega^b_{\ ak} e^i_b = 0, \tag{1.5.29}$$

because of (1.5.22). Therefore, total covariant differentiation commutes with converting between coordinate and Lorentz indices. Equation (1.5.29) determines the spin connection $\omega^a_{\ bi}$ in terms of the affine connection, tetrad and its ordinary derivatives, in accordance with (1.5.23). Conversely, the affine connection is determined by the spin connection, tetrad and its derivatives:

$$\Gamma^{\,j}_{i\,k} = \omega^j_{\ ik} + e^a_{i,k} e^j_a. \tag{1.5.30}$$

The torsion tensor is then related to these quantities by

$$S^j_{\ ik} = \omega^j_{\ [ik]} + e^a_{[i,k]} e^j_a, \tag{1.5.31}$$

and the torsion vector is

$$S_i = \omega^k{}_{[ik]} + e^a_{[i,k]}e^k_a. \tag{1.5.32}$$

Metric compatibility of the affine connection leads to

$$g_{ik;j} = g_{ik|j} = e^a_i e^b_k \eta_{ab|j} = -e^a_i e^b_k(\omega^c{}_{aj}\eta_{cb} + \omega^c{}_{bj}\eta_{ac}) = -(\omega_{kij} + \omega_{ikj}) = 0, \tag{1.5.33}$$

so the spin connection is antisymmetric in its first two indices:

$$\omega^a{}_{bi} = -\omega_b{}^a{}_i. \tag{1.5.34}$$

Accordingly, the spin connection has 24 independent components. The contortion tensor is related to the spin connection by

$$C_{ijk} = \omega_{ijk} + \Delta_{ijk}, \tag{1.5.35}$$

where

$$\Delta_{ijk} = e_{ia}e^a_{[j,k]} - e_{ja}e^a_{[i,k]} - e_{ka}e^a_{[i,j]} \tag{1.5.36}$$

are the *Ricci rotation coefficients*. The first term on the right-hand side in (1.5.35) is expected because both the contortion tensor and spin connection are antisymmetric in their first two indices. The quantities

$$\varpi^i{}_{ak} = e^i_{a:k} = e^i_{a,k} + \{{}^{\,i}_{j\,k}\}e^j_a \tag{1.5.37}$$

form the *Levi-Civita spin connection* and are related to the Ricci rotation coefficients by (1.5.35) with $C_{ijk} = 0$,

$$\varpi_{ijk} = -\Delta_{ijk}, \tag{1.5.38}$$

so

$$C_{ijk} = \omega_{ijk} - \varpi_{ijk}. \tag{1.5.39}$$

1.5.5 Tetrad representation of curvature tensor

The commutator of the covariant derivatives of a tetrad with respect to the affine connection is

$$2e^k_{a;[ji]} = R^k{}_{lij}e^\sigma_a + 2S^l{}_{ij}e^k_{a;l}. \tag{1.5.40}$$

This commutator can also be expressed in terms of the spin connection:

$$\begin{aligned} e^k_{a;[ji]} &= \omega^k{}_{a[j;i]} = (e^k_b\omega^b{}_{a[j})_{;i]} = \omega_{ba[j}\omega^{kb}{}_{i]} + \omega^b{}_{a[j;i]}e^k_b \\ &= \omega_{ba[j}\omega^{kb}{}_{i]} + \omega^b{}_{a[j,i]}e^k_b + S^l{}_{ij}\omega^k{}_{al}. \end{aligned} \tag{1.5.41}$$

Consequently, the curvature tensor with two Lorentz and two coordinate indices depends only on the spin connection and its ordinary derivatives:

$$R^a{}_{bij} = \omega^a{}_{bj,i} - \omega^a{}_{bi,j} + \omega^a{}_{ci}\omega^c{}_{bj} - \omega^a{}_{cj}\omega^c{}_{bi}. \tag{1.5.42}$$

Because the spin connection is antisymmetric in its first two indices, the tensor (1.5.42) is antisymmetric in its first two (Lorentz) indices, like the Riemann tensor. The contraction of the curvature tensor (1.5.42) with a tetrad gives the Ricci tensor with one Lorentz and one coordinate index:

$$R_{bj} = R^a{}_{bij}e^i_a. \tag{1.5.43}$$

The contraction of the tensor $R^a{}_i$ with a tetrad gives the Ricci scalar,

$$R = R^a{}_i e^i_a = R^{ab}{}_{ij}e^i_a e^j_b. \tag{1.5.44}$$

The Riemann tensor with two Lorentz and two coordinate indices depends on the Levi-Civita connection (1.5.37) the same way the curvature tensor depends on the affine connection:

$$P^a{}_{bij} = \varpi^a{}_{bj,i} - \varpi^a{}_{bi,j} + \varpi^a{}_{ci}\varpi^c{}_{bj} - \varpi^a{}_{cj}\varpi^c{}_{bi}. \tag{1.5.45}$$

The contraction of (1.5.45) with a tetrad gives the Riemannian Ricci tensor with one Lorentz and one coordinate index:

$$P_{bj} = P^a_{\ bij} e^i_a. \tag{1.5.46}$$

The contraction of the tensor $P^a_{\ i}$ with a tetrad gives the Riemann scalar,

$$P = P^a_{\ i} e^i_a = P^{ab}_{\ \ ij} e^i_a e^j_b. \tag{1.5.47}$$

References: [3, 4, 6, 7, 8].

1.6 Lorentz group

Lorentz transformations relate different tetrads at a given point in spacetime, where the metric tensor can be brought to the Galilean form: $g_{ik} = \eta_{ik}$. Accordingly, we have $\epsilon^{ijkl} = e^{ijkl}$, $\varepsilon_{ijkl} = e_{ijkl}$, $\epsilon^{\alpha\beta\gamma} = e^{\alpha\beta\gamma}$, and $\varepsilon_{\alpha\beta\gamma} = e_{\alpha\beta\gamma}$.

1.6.1 Subgroups of Lorentz group and Einstein principle of relativity

A composition of two Lorentz transformations Λ_1 and Λ_2,

$$\Lambda^a_{\ b} = \Lambda^a_{(1)c} \Lambda^c_{(2)b}, \tag{1.6.1}$$

satisfies (1.5.17), thereby it is a Lorentz transformation. The Kronecker symbol δ^a_b also satisfies (1.5.17), thereby it can be regarded as the identity Lorentz transformation. Therefore, Lorentz transformations form a group, referred to as the *Lorentz group*. Taking the determinant of the relation (1.5.17) gives

$$|\Lambda^a_{\ b}| = \pm 1. \tag{1.6.2}$$

A Lorentz transformation with $|\Lambda^a_{\ b}| = 1$ is *proper* and with $|\Lambda^a_{\ b}| = -1$ is *improper*. Proper Lorentz transformations form a group because the determinant of the product of two proper Lorentz transformations is 1. Improper Lorentz transformations include the *parity* transformation P

$$\Lambda^a_{\ b}(P) = \mathrm{diag}(1, -1, -1, -1), \ \ t \to t, \ \mathbf{x} \to -\mathbf{x}, \tag{1.6.3}$$

and the *time reversal* T

$$\Lambda^a_{\ b}(T) = \mathrm{diag}(-1, 1, 1, 1), \ \ t \to -t, \ \mathbf{x} \to \mathbf{x}. \tag{1.6.4}$$

The relation (1.5.17) gives $\Lambda^0_{\ 0}\Lambda^0_{\ 0} - \Lambda^0_{\ \alpha}\Lambda^0_{\ \alpha} = 1$, thereby

$$|\Lambda^0_{\ 0}| \geq 1. \tag{1.6.5}$$

Lorentz transformations with $\Lambda^0_{\ 0} \geq 1$ are *orthochronous* and form a group. If x^i is a timelike vector, $x^i x_i > 0$, then for an orthochronous transformation $x'^0 = \Lambda^0_{\ 0} x^0 + \Lambda^0_{\ \alpha} x^\alpha$,

$$|\Lambda^0_{\ \alpha} x^\alpha| \leq \sqrt{\Lambda^0_{\ \alpha}\Lambda^0_{\ \alpha} x^\beta x^\beta} < \sqrt{(\Lambda^0_{\ 0})^2 (x^0)^2} = |\Lambda^0_{\ 0} x^0|. \tag{1.6.6}$$

Therefore, the time component of a timelike vector does not change the sign under orthochronous transformations. Einstein's *special principle of relativity* states that physical laws do not change their form under transformations within the orthochronous proper subgroup of the Lorentz group. Equivalently, physical laws have the same form in all admissible inertial frames of reference. The special principle of relativity is a special case of the general principle of relativity, in which arbitrary differentiable coordinate transformations are restricted to linear transformations (orthochronous proper Lorentz transformations) between inertial frames of reference.

Under the parity transformation, the spatial components of contravariant and covariant vectors, which form spatial vectors, change the sign. The permutation symbols do not change under this transformation. Accordingly, the spatial components of dual vector densities, such as the components of a vector product (1.4.157) or a curl (1.4.168), do not change the sign. Such quantities, that transform under proper Lorentz transformations like vectors and do not change the sign in their spatial components under the parity transformation, are referred to as axial vectors or *pseudovectors*. Similarly, the scalar contraction of the Levi-Civita symbol and a tensor changes the sign, while a scalar does not. Quantities that transform under proper Lorentz transformations like scalars and change the sign under the parity transformation are referred to as *pseudoscalars*.

1.6.2 Infinitesimal Lorentz transformations

Let us consider an infinitesimal Lorentz transformation

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon^\mu{}_\nu, \tag{1.6.7}$$

where $\epsilon^\mu{}_\nu$ are infinitesimal quantities. The relation (1.5.17) gives

$$\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}, \tag{1.6.8}$$

where the indices are raised and lowered using the Minkowski metric tensor. Therefore, Lorentz transformations are given by 6 independent antisymmetric parameters $\epsilon_{\mu\nu}$. The corresponding transformation of a contravariant vector A^μ is

$$A'^\mu = A^\mu + \epsilon^\mu{}_\nu A^\nu = A^\mu + \frac{1}{2}\epsilon^{\rho\sigma}(\delta^\mu_\rho \eta_{\sigma\nu} - \delta^\mu_\sigma \eta_{\rho\nu})A^\nu = A^\mu + \frac{1}{2}\epsilon^{\rho\sigma} J^\mu_{\nu\rho\sigma} A^\nu, \tag{1.6.9}$$

where

$$J^\mu_{\nu\rho\sigma} = \delta^\mu_\rho \eta_{\sigma\nu} - \delta^\mu_\sigma \eta_{\rho\nu}. \tag{1.6.10}$$

We define matrices $J_{\rho\sigma}$ such that

$$(J_{\rho\sigma})^\mu_\nu = J^\mu_{\nu\rho\sigma}. \tag{1.6.11}$$

Therefore, in the matrix notation (with A^μ treated as a column),

$$A' = \Big(1 + \frac{1}{2}\epsilon^{\rho\sigma} J_{\rho\sigma}\Big)A. \tag{1.6.12}$$

The 6 matrices $J_{\rho\sigma}$ are the infinitesimal *generators* of the *vector representation* of the Lorentz group. The explicit form of the generators of the Lorentz group in the vector representation is

$$J_{01} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_{02} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$J_{03} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad J_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$J_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad J_{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{1.6.13}$$

1.6.3 Generators and Lie algebra of Lorentz group

The commutator of the generators of the Lorentz group in the vector representation is given, using (1.6.10) and (1.6.11), by

$$[J_{\kappa\tau}, J_{\rho\sigma}]^\mu_\nu = (J_{\kappa\tau})^\mu_\lambda (J_{\rho\sigma})^\lambda_\nu - (J_{\rho\sigma})^\mu_\lambda (J_{\kappa\tau})^\lambda_\nu = (-J_{\kappa\rho}\eta_{\tau\sigma} - J_{\tau\sigma}\eta_{\kappa\rho} + J_{\kappa\sigma}\eta_{\tau\rho} + J_{\tau\rho}\eta_{\kappa\sigma})^\mu_\nu, \tag{1.6.14}$$

so

$$[J_{\kappa\tau}, J_{\rho\sigma}] = -J_{\kappa\rho}\eta_{\tau\sigma} - J_{\tau\sigma}\eta_{\kappa\rho} + J_{\kappa\sigma}\eta_{\tau\rho} + J_{\tau\rho}\eta_{\kappa\sigma}. \tag{1.6.15}$$

The relation (1.6.15) constitutes the *Lie algebra* of the *Lorentz group.* If a set of qantitites ϕ transforms under a Lorentz transformation Λ with a matrix $D(\Lambda)$

$$\phi \to D(\lambda)\phi, \tag{1.6.16}$$

then D is a representation of the Lorentz group if

$$D(I) = I, \;\; D(\Lambda_1\Lambda_2) = D(\Lambda_1)D(\Lambda_2), \tag{1.6.17}$$

where I denotes the identity transformation, and Λ_1 and Λ_2 are two Lorentz transformations. Therefore, we have

$$D(\Lambda^{-1}) = D^{-1}(\Lambda), \tag{1.6.18}$$

where Λ^{-1} is the Lorentz transformation to Λ: $\Lambda\Lambda^{-1} = I$. For an infinitesimal Lorentz transformation in any representation,

$$D(\Lambda) = I + \frac{1}{2}\epsilon^{\rho\sigma}J_{\rho\sigma}, \tag{1.6.19}$$

according to (1.6.12). The relation

$$D(\Lambda_1\Lambda_2\Lambda_1^{-1}) = D(\Lambda_1)D(\Lambda_2)D^{-1}(\Lambda_1) \tag{1.6.20}$$

gives (1.6.15), valid for any representation of the Lorentz group.

If Λ_1 and Λ_2 are two group transformations then $\Lambda_3 = \Lambda_1\Lambda_2\Lambda_1^{-1}$ is a group transformation. If $\Lambda_2 = I + \epsilon_2 G_2$ is an infinitesimal group transformation with generator G_2 then $\Lambda_3 = I + \epsilon_2\Lambda_1 G_2\Lambda_1^{-1}$ is an infinitesimal group transformation with generator $G_3 = \Lambda_1 G_2\Lambda_1^{-1}$. If $\Lambda_1 = I + \epsilon_1 G_1$ is an infinitesimal group transformation with generator G_1 then, neglecting terms in ϵ_1 of higher order, $G_3 = G_2 + \epsilon_1[G_1, G_2]$, thereby $[G_1, G_2]$ is a generator. For a finite number N of linearly independent generators, a general infinitesimal group transformation is $\Lambda = I + \Sigma_{a=1}^N \epsilon_a G_a$. Because $[G_a, G_b]$ is a generator, it is a linear combination of the N generators: $[G_a, G_b] = \Sigma_{c=1}^N f_{abc}G_c$, where f_{abc} are the structure constants of the Lie algebra of the given group. For the Lorentz group, $\epsilon_a G_a = D(\Lambda) - I$, where $D(\Lambda)$ is given by (1.6.19).

1.6.4 Rotations and boosts

Rotations are proper orthochronous Lorentz transformations with

$$\Lambda^0{}_\alpha = \Lambda^\alpha{}_0 = 0, \;\; \Lambda^0{}_0 = 1. \tag{1.6.21}$$

Rotations act only on the spatial coordinates x^α and form a group, referred to as the *rotation group.* *Boosts* are proper orthochronous Lorentz transformations with

$$\Lambda^\alpha{}_\beta = 0. \tag{1.6.22}$$

We define

$$J_\alpha = \frac{1}{2}e_{\alpha\beta\gamma}J^{\beta\gamma}, \tag{1.6.23}$$

$$K_\alpha = J_{0\alpha}, \tag{1.6.24}$$

and

$$\vartheta_\alpha = \frac{1}{2}e_{\alpha\beta\gamma}\epsilon^{\beta\gamma}, \tag{1.6.25}$$

$$\eta_\alpha = \epsilon_{0\alpha}. \tag{1.6.26}$$

The explicit form of the generators of the rotation group J_α in the vector representation is

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \;\; J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \;\; J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.6.27}$$

For an infinitesimal Lorentz transformation (1.6.19)

$$D = I + \boldsymbol{\vartheta} \cdot \mathbf{J} + \boldsymbol{\eta} \cdot \mathbf{K}. \tag{1.6.28}$$

A finite Lorentz transformation can be regarded as a composition of successive identical infinitesimal Lorentz transformations:

$$D = \lim_{n \to \infty}(I + \boldsymbol{\theta} \cdot \mathbf{J}/n + \boldsymbol{\eta} \cdot \mathbf{K}/n)^n = e^{\boldsymbol{\theta} \cdot \mathbf{J} + \boldsymbol{\eta} \cdot \mathbf{K}}. \tag{1.6.29}$$

The finite parameters $\boldsymbol{\theta}$, $\boldsymbol{\eta}$ are the *canonical parameters* for a given Lorentz transformations. For a finite Lorentz transformation, (1.6.19) gives

$$D(\Lambda) = e^{\frac{1}{2}\epsilon^{\rho\sigma} J_{\rho\sigma}}, \tag{1.6.30}$$

so

$$J_{\mu\nu} = \frac{\partial D(\Lambda)}{\partial \epsilon^{\mu\nu}}\Big|_{\Lambda = I}. \tag{1.6.31}$$

The explicit form of a finite Lorentz transformation in the vector representation is

$$R_1 = e^{\theta J_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad R_2 = e^{\theta J_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix},$$
$$R_3 = e^{\theta J_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_1 = e^{\eta K_1} = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$B_2 = e^{\eta K_2} = \begin{pmatrix} \cosh\eta & 0 & \sinh\eta & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\eta & 0 & \cosh\eta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_3 = e^{\eta K_3} = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}, \tag{1.6.32}$$

where R_α denotes a rotation about the x^α axis and B_α denotes a boost along this axis. The canonical parameters $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ are respectively referred to as the *angle of rotation* and *rapidity*. The parameters $\boldsymbol{\vartheta}$ and $\boldsymbol{\eta}$ in (1.6.25) and (1.6.26) are thus respectively infinitesimal values of the angle of rotation and rapidity. A rotation about any axis, say z, by an angle θ turns the two other axes, x and y, into new axes, x' and y', such that the angle between x and x' (or y and y') (1.4.152) is θ. The rotation group is *compact*: $\theta_\alpha \in [0, 2\pi]$ and $\theta = 2\pi \Leftrightarrow \theta = 0$. The explicit form of a finite rotation in the three-dimensional vector representation is

$$R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad R_2(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix},$$
$$R_3(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.6.33}$$

For instance,

$$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \to \begin{pmatrix} V'_x \\ V'_y \\ V'_z \end{pmatrix} = R_3 \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} V_x\cos\theta - V_y\sin\theta \\ V_x\sin\theta + V_y\cos\theta \\ V_z \end{pmatrix}. \tag{1.6.34}$$

The relation (1.6.31) gives

$$J_\alpha = \frac{\partial R_\alpha(\theta)}{\partial \theta}\Big|_{\theta = 0}. \tag{1.6.35}$$

The orthogonality relation (1.5.17) applied to any of the rotation matrices (1.6.33) shows that a rotation matrix R is orthogonal, that is, its transpose R^T is equal to its inverse R^{-1}:

$$R^T_\alpha = R^{-1}_\alpha, \;\; R_\alpha R^T_\alpha = R^T_\alpha R_\alpha = I, \tag{1.6.36}$$

where I is the identity matrix.

The commutation relation (1.6.15) gives

$$[J_\alpha, J_\beta] = e_{\alpha\beta\gamma} J_\gamma, \tag{1.6.37}$$
$$[J_\alpha, K_\beta] = e_{\alpha\beta\gamma} K_\gamma, \tag{1.6.38}$$
$$[K_\alpha, K_\beta] = -e_{\alpha\beta\gamma} J_\gamma. \tag{1.6.39}$$

Therefore, rotations do not commute and form a nonabelian group, rotations and boosts do not commute, and boosts do not commute. Changing the order of two nonparallel boosts is equivalent to applying a rotation, referred to as the *Thomas-Wigner rotation.* The structure constants of the Lie algebra of the rotation group are $f_{abc} = e_{abc}$. Moreover, the square of the generators of rotation,

$$J^2 = J_\alpha J_\alpha, \tag{1.6.40}$$

commutes with J_α:

$$[J^2, J_\beta] = [J_\alpha, J_\beta] J_\alpha + J_\alpha [J_\alpha, J_\beta] = e_{\alpha\beta\gamma}(J_\gamma J_\alpha + J_\alpha J_\gamma) = 0. \tag{1.6.41}$$

Definining

$$\mathbf{L} = \frac{1}{2}(\mathbf{J} + i\mathbf{K}), \tag{1.6.42}$$
$$\mathbf{Q} = \frac{1}{2}(\mathbf{J} - i\mathbf{K}), \tag{1.6.43}$$

gives

$$[L_\alpha, L_\beta] = e_{\alpha\beta\gamma} L_\gamma, \tag{1.6.44}$$
$$[Q_\alpha, Q_\beta] = e_{\alpha\beta\gamma} Q_\gamma, \tag{1.6.45}$$
$$[L_\alpha, Q_\beta] = 0, \tag{1.6.46}$$

so the Lorentz group is isomorphic with the product of two complex rotation groups. Accordingly, the Lorentz group can be regarded as the group of four-dimensional rotations in the Minkowski space, or the group of *tetrad rotations.*

1.6.5 Poincaré group

Under an infinitesimal coordinate transformation (1.2.66) in a locally flat spacetime, (1.4.51) gives

$$\eta_{ik} \to \eta_{ik} - \xi_{i,k} - \xi_{k,i}. \tag{1.6.47}$$

Therefore, the tensor η_{ik} is invariant under (1.2.66) (isometric) if ξ^i is a Killing vector,

$$\xi_{(i,k)} = 0, \tag{1.6.48}$$

which has the solution

$$\xi^i = \epsilon^{ik} x_k + \epsilon^i, \tag{1.6.49}$$

where ϵ^{ik} and ϵ^i are constant. The first term on the right-hand side of (1.6.49) corresponds to a Lorentz rotation described by 6 parameters ϵ^{ik} satisfying (1.6.8). The second term on the right-hand side of (1.6.49) corresponds to a *translation.* A combination of two translations does not change if their order is reversed, thereby translations commute:

$$[T_\mu, T_\nu] = 0, \tag{1.6.50}$$

where T_μ is the generator of translation. The relations (1.6.37) and (1.6.38) mean that J^α and K^α are spatial vectors under rotations. Spatial translations are spatial vectors under rotations, while a time translation is a scalar:

$$[J_\alpha, T_\beta] = e_{\alpha\beta\gamma} T_\gamma, \tag{1.6.51}$$
$$[J_\alpha, T_0] = 0. \tag{1.6.52}$$

The last relation indicates that the generators of rotations, like the generators of spatial translations, correspond to *conserved* quantities, which are quantities that do not change in time. The covariant generalization of (1.6.51) and (1.6.52) is

$$[J_{\mu\nu}, T_\rho] = T_\mu \eta_{\nu\rho} - T_\nu \eta_{\mu\rho}. \tag{1.6.53}$$

The relations (1.6.15), (1.6.50) and (1.6.53) constitute the Lie algebra of the inhomogeneous Lorentz or *Poincaré group*. In particular,

$$[K_\alpha, T_\beta] = -T_0 \delta_{\alpha\beta}, \tag{1.6.54}$$
$$[K_\alpha, T_0] = -T_\alpha. \tag{1.6.55}$$

The last relation indicates that the generators of boosts do not correspond to conserved quantities.

For an infinitesimal rotation about the z axis,

$$\begin{aligned}(I + \vartheta J_z) f(ct, \mathbf{x}) &= D(R_z(\vartheta)) f(ct, \mathbf{x}) = f(ct, R_z(\vartheta)\mathbf{x}) \approx f(ct, x - \vartheta y, \vartheta x + y, z) \\ &= f(ct, \mathbf{x}) - \vartheta y \frac{\partial f}{\partial x} + \vartheta x \frac{\partial f}{\partial y},\end{aligned} \tag{1.6.56}$$

or

$$J_z = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}, \tag{1.6.57}$$

which gives the differential representation of rotations:

$$J_\alpha = e_{\alpha\beta\gamma} x_\beta \partial_\gamma. \tag{1.6.58}$$

For an infinitesimal boost along the z axis,

$$\begin{aligned}(I + \eta K_z) f(ct, \mathbf{x}) &= D(B_z(\eta)) f(ct, \mathbf{x}) = f(B_z(\eta)(ct, \mathbf{x})) \approx f(ct + \eta z, y, z + \eta ct) \\ &= f(ct, \mathbf{x}) + \eta z \frac{\partial f}{c \partial t} + \eta ct \frac{\partial f}{\partial z},\end{aligned} \tag{1.6.59}$$

or

$$K_z = z \frac{\partial}{c\partial t} + ct \frac{\partial}{\partial z}, \tag{1.6.60}$$

which gives the differential representation of boosts:

$$K_\alpha = x_\alpha \frac{\partial}{c\partial t} + ct \frac{\partial}{\partial x^\alpha}. \tag{1.6.61}$$

The relation for an infinitesimal translation, analogous to (1.6.19), is

$$D(t) = I + \epsilon^\mu T_\mu, \tag{1.6.62}$$

so a finite translation is given by

$$D(t) = e^{\epsilon^\mu T_\mu}. \tag{1.6.63}$$

Translation in (1.6.49) can also be written as

$$t_\mu(\epsilon) x^\nu = x^\nu + \epsilon \delta^\nu_\mu. \tag{1.6.64}$$

The relation analogous to (1.6.35) is

$$T_\mu = \frac{\partial t_\mu(\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}. \tag{1.6.65}$$

The differential representation of a translation is thus

$$T_\mu = \frac{\partial}{\partial x^\mu}. \tag{1.6.66}$$

1.6.6 Invariants of Lorentz and Poincaré group

Analogously to (1.6.41),

$$[L^2, L_\beta] = 0, \tag{1.6.67}$$
$$[Q^2, Q_\beta] = 0, \tag{1.6.68}$$

so L^2 and Q^2 commute with all 6 generators of the Lorentz group. Consequently, $J^2 + K^2$ and $\mathbf{J} \cdot \mathbf{K}$ commute with all generators of the Lorentz group, that is, are the invariants or *Casimir operators* of the Lorentz group. The Casimir operators of Lorentz group do not commute with the generators of translation T_μ, thereby they are not the invariants of the Poincaré group. Instead, the *mass operator*

$$m^2 = -T^\mu T_\mu \tag{1.6.69}$$

and

$$W^2 = W^\mu W_\mu, \tag{1.6.70}$$

where W^μ is the *Pauli-Lubański pseudovector*

$$W^\mu = \frac{1}{2} e^{\mu\nu\rho\sigma} J_{\rho\sigma} T_\nu, \tag{1.6.71}$$

commute with all generators of the Poincaré group, thereby they are the Casimir operators of the Poincaré group. The Pauli-Lubański pseudovector obeys the commutation relations

$$[T_\mu, W_\nu] = 0, \tag{1.6.72}$$
$$[J_{\mu\nu}, W_\rho] = W_\mu \eta_{\nu\rho} - W_\nu \eta_{\mu\rho}, \tag{1.6.73}$$
$$[W^\mu, W^\nu] = e^{\mu\nu\rho\sigma} W_\rho T_\sigma. \tag{1.6.74}$$

The relation (1.6.73) is analogous to (1.6.53) because W^μ behaves like a vector under proper Lorentz transformations.

We define the *four-momentum operator*

$$P_\mu = iT_\mu, \tag{1.6.75}$$

whose time component is the *energy operator* $P_0 = iT_0$ and spatial components form the *momentum operator* $P_\alpha = iT_\alpha$. We define the *angular four-momentum operator*

$$M_{\mu\nu} = iJ_{\mu\nu}, \tag{1.6.76}$$

whose spatial components form the *angular momentum operator*

$$M_\alpha = iJ_\alpha. \tag{1.6.77}$$

Therefore, the following relations are satisfied:

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(M_{\mu\rho}\eta_{\nu\sigma} + M_{\nu\sigma}\eta_{\mu\rho} - M_{\mu\sigma}\eta_{\nu\rho} - M_{\nu\rho}\eta_{\mu\sigma}), \tag{1.6.78}$$
$$[P_\mu, P_\nu] = 0, \tag{1.6.79}$$
$$[M_{\mu\nu}, P_\rho] = i(P_\mu \eta_{\nu\rho} - P_\nu \eta_{\mu\rho}), \tag{1.6.80}$$
$$m^2 = P^\mu P_\mu, \tag{1.6.81}$$
$$W^\mu = -\frac{1}{2} e^{\mu\nu\rho\sigma} M_{\rho\sigma} P_\nu, \tag{1.6.82}$$
$$[P_\mu, W_\nu] = 0, \tag{1.6.83}$$
$$[M_{\mu\nu}, W_\rho] = i(W_\mu \eta_{\nu\rho} - W_\nu \eta_{\mu\rho}), \tag{1.6.84}$$
$$[W^\mu, W^\nu] = -ie^{\mu\nu\rho\sigma} W_\rho P_\sigma, \tag{1.6.85}$$
$$[M_\alpha, M_\beta] = ie_{\alpha\beta\gamma} M_\gamma. \tag{1.6.86}$$

1.6.7 Relativistic kinematics

The quantitites v^α (1.4.116) form the three-dimensional vector of velocity:

$$\mathbf{v} = \frac{d\mathbf{x}}{dt}, \tag{1.6.87}$$

where $\mathbf{x}$ is the radius vector. The magnitude of the velocity is equal to the speed (1.4.135):

$$|\mathbf{v}| = v. \tag{1.6.88}$$

Let us consider a boost in the direction of the z axis

$$x'^i = e^{-\eta K_3} x^i, \tag{1.6.89}$$

where x^i and x'^i have a form of a column (4×1 matrix), and $e^{\eta K_3}$ is given by (1.6.32). Therefore, the coordinates in an inertial K-system (unprimed) are related to the coordinates in an inertial K'-system (primed) by

$$\begin{aligned} &ct = ct'\cosh\eta + z'\sinh\eta, \\ &x = x', \;\; y = y', \\ &z = z'\cosh\eta + ct'\sinh\eta. \end{aligned} \tag{1.6.90}$$

Let us consider the origin of the K'-system, $x' = y' = z' = 0$, in the K-system. Therefore, we have

$$\begin{aligned} &ct = ct'\cosh\eta, \\ &z = ct'\sinh\eta, \end{aligned} \tag{1.6.91}$$

which gives the relation between the rapidity η and speed $V = \frac{dz}{dt}$ of K' relative to K:

$$\tanh\eta = \beta, \tag{1.6.92}$$

where

$$\boldsymbol{\beta} = \frac{\mathbf{V}}{c}, \;\; \beta = \frac{V}{c}. \tag{1.6.93}$$

Accordingly, $\cosh\eta = \gamma$ and $\sinh\eta = \beta\gamma$, where

$$\gamma = \Big(1 - \frac{V^2}{c^2}\Big)^{-1/2}. \tag{1.6.94}$$

The relations (1.6.90) become

$$\begin{aligned} &t = \gamma\Big(t' + \frac{V}{c^2}z'\Big), \\ &x = x', \;\; y = y', \\ &z = \gamma(z' + Vt'), \end{aligned} \tag{1.6.95}$$

and are referred to as a *special Lorentz transformation* in the z-direction. The reverse transformation is

$$\begin{aligned} &t' = \gamma\Big(t - \frac{V}{c^2}z\Big), \\ &x' = x, \;\; y' = y, \\ &z' = \gamma(z - Vt). \end{aligned} \tag{1.6.96}$$

For a boost along an arbitrary direction, the spatial vector $\mathbf{x} = (x, y, z)$ transforms such that its component parallel to the velocity $\mathbf{V} = c\boldsymbol{\beta}$ of K' relative to K, $\mathbf{x}_\parallel = (\mathbf{x}\cdot\mathbf{V})\mathbf{V}/V^2$ (similarly for primed), behaves like z in (1.6.95) and its component perpendicular to $\mathbf{V}$, $\mathbf{x}_\perp = \mathbf{x} - \mathbf{x}_\parallel$, behaves like x in (1.6.95):

$$\begin{aligned} t &= \gamma\Big(t' + \frac{\mathbf{V}\cdot\mathbf{x}'}{c^2}\Big), \\ \mathbf{x}_\perp &= \mathbf{x}'_\perp, \\ \mathbf{x}_\parallel &= \gamma(\mathbf{x}'_\parallel + \mathbf{V}t'), \end{aligned} \tag{1.6.97}$$

so

$$\mathbf{x} = \gamma(\mathbf{x}'_\parallel + \mathbf{V}t') + \mathbf{x}'_\perp = \gamma\mathbf{V}t' + \mathbf{x}' + \frac{(\gamma-1)(\mathbf{V}\cdot\mathbf{x}')\mathbf{V}}{V^2}. \tag{1.6.98}$$

Therefore, the transformation law for the coordinates in two inertial frames of reference is

$$\begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta} \\ \gamma\boldsymbol{\beta} & 1 + \frac{(\gamma-1)\boldsymbol{\beta}}{\beta^2}\boldsymbol{\beta} \end{pmatrix} \begin{pmatrix} ct' \\ \mathbf{x}' \end{pmatrix}, \tag{1.6.99}$$

or equivalently

$$\begin{pmatrix} ct' \\ \mathbf{x}' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\boldsymbol{\beta} \\ -\gamma\boldsymbol{\beta} & 1 + \frac{(\gamma-1)\boldsymbol{\beta}}{\beta^2}\boldsymbol{\beta} \end{pmatrix} \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix}. \tag{1.6.100}$$

The matrix in (1.6.100) is called a *boost matrix*. In the local Minkowski spacetime, contravariant vectors transform like x^i, according to (1.6.97) and (1.6.99),

$$\begin{pmatrix} W^0 \\ \mathbf{W} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta} \\ \gamma\boldsymbol{\beta} & 1 + \frac{(\gamma-1)\boldsymbol{\beta}}{\beta^2}\boldsymbol{\beta} \end{pmatrix} \begin{pmatrix} W'^0 \\ \mathbf{W}' \end{pmatrix}, \tag{1.6.101}$$

covariant vectors transform such that they remain related to contravariant vectors by the Minkowski metric tensor, and tensors transform like products of vectors. For example, if $\mathbf{V} = c\beta\hat{z}$ is parallel to the z axis, a tensor of rank (0,2) transforms according to

$$\begin{aligned} T_{00} &= \gamma(T_{00'} + \beta T_{03'}) = \gamma^2(T_{0'0'} + \beta T_{3'0'} + \beta T_{0'3'} + \beta^2 T_{3'3'}), \\ T_{0\perp} &= \gamma(T_{0'\perp'} + \beta T_{3'\perp'}), \\ T_{03} &= \gamma(T_{03'} + \beta T_{00'}) = \gamma^2(T_{0'3'} + \beta T_{3'3'} + \beta T_{0'0'} + \beta^2 T_{3'0'}), \\ T_{\perp\perp} &= T_{\perp'\perp'}, \\ T_{3\perp} &= \gamma(T_{3'\perp'} + \beta T_{0'\perp'}), \\ T_{33} &= \gamma(T_{33'} + \beta T_{30'}) = \gamma^2(T_{3'3'} + \beta T_{0'3'} + \beta T_{3'0'} + \beta^2 T_{0'0'}), \end{aligned} \tag{1.6.102}$$

where the index $\perp$ denotes either 1 or 2, and the transposed components $T^T_{ik} = T_{ki}$ transform like the transpositions of the right-hand sides in (1.6.102). If T_{ik} is antisymmetric then $T_{03} = T_{0'3'}$.

The relations (1.6.95) can be written as

$$\begin{aligned} dt &= \gamma\Big(dt' + \frac{V}{c^2}dz'\Big), \\ dx &= dx',\ \ dy = dy', \\ dz &= \gamma(dz' + V dt'), \end{aligned} \tag{1.6.103}$$

which gives

$$\begin{aligned} v_x &= \frac{v'_x}{\gamma(1 + Vv'_z/c^2)}, \\ v_y &= \frac{v'_y}{\gamma(1 + Vv'_z/c^2)}, \\ v_z &= \frac{v'_z + V}{1 + Vv'_z/c^2}, \end{aligned} \tag{1.6.104}$$

where

$$\mathbf{v} = \frac{d\mathbf{x}}{dt}, \ \ \mathbf{v}' = \frac{d\mathbf{x}'}{dt'}. \tag{1.6.105}$$

Two special Lorentz transformations in the same direction commute because of (1.6.39). If a Lorentz transformation from K' to K has parameters β_1 and γ_1, and a Lorentz transformation from K'' to K' has parameters β_2 and γ_2, then a Lorentz transformation from K'' to K has parameters β_3 and γ_3 such that

$$\beta_3 = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}, \ \ \gamma_3 = \gamma_1\gamma_2(1 + \beta_1\beta_2). \tag{1.6.106}$$

For a boost along an arbitrary direction, (1.6.99) gives the Lorentz transformation of velocities:

$$\mathbf{v} = \frac{\mathbf{v}' + \gamma\mathbf{V} + (\gamma - 1)(\mathbf{v}'\cdot\mathbf{V})\mathbf{V}/V^2}{\gamma(1 + \mathbf{v}'\cdot\mathbf{V}/c^2)}. \tag{1.6.107}$$

If $v' = |\mathbf{V}'| = c$ then $v = |\mathbf{V}| = c$, in agreement with the constancy of the speed of propagation of interaction.

Let us consider two points at rest in an inertial frame of reference K with positions z_1 and z_2, thereby the distance between them is $\Delta z = z_2 - z_1$. In the inertial frame K', moving relative to K in the z-direction with speed V, $z_1 = \gamma(z'_1 + Vt'_1)$ and $z_2 = \gamma(z'_2 + Vt'_2)$, thereby if $t'_1 = t'_2$ is the time at which we measure (simultaneously) the positions of the two points then $\Delta z = \gamma(z'_2 - z'_1) = \gamma\Delta z'$. Therefore, the length of an object in K', whose length in the rest frame K is l (*proper length*), is

$$l' = \frac{l}{\gamma} < l, \tag{1.6.108}$$

which is referred to as the *Lorentz-FitzGerald contraction*. The volume of an object in K', whose volume in the rest frame K is V (*proper volume*), is

$$V' = \frac{V}{\gamma}. \tag{1.6.109}$$

Let us suppose that there are two rods of equal lengths, moving parallel relative to each other. From the point of view of an observer moving with the first rod, the second one is shorter, and from the point of view of an observer moving with the second rod, the first one is shorter. There is no contradiction in this statement because the positions of both ends of a rod must be measured simultaneously and the simultaneity is not invariant: from the transformation law (1.6.95) it follows that if $\delta t = 0$ then $\delta t' \neq 0$ and if $\delta t' = 0$ then $\delta t \neq 0$.

Let us consider a clock (any mechanism with a periodic or evolutionary behavior) at rest in K' with position z'; the time difference between two events with t'_1 and t'_2, as measured by this clock, is $\Delta t' = t'_2 - t'_1$. In the frame K, $t_1 = \gamma(t'_1 + Vz'/c^2)$ and $t_2 = \gamma(t'_2 + Vz'/c^2)$, thereby

$$\Delta t = t_2 - t_1 = \gamma\Delta t' > \Delta t'. \tag{1.6.110}$$

Therefore, the rate of time is slower for moving clocks than those at rest (*time dilation*), in agreement with (1.4.120) and (1.4.126), from which $c^2d\tau^2 = c^2dt^2 - dl^2$ and

$$d\tau = \frac{1}{\gamma}dt. \tag{1.6.111}$$

Let us suppose that there are two clocks linked to the inertial frames K and K', and that when the clock in K passes by the clock in K' the readings of the two clocks coincide. From the point of view of an observer in K clocks in K' go more slowly, and from the point of view of an observer in K' clocks in K go more slowly. There is no contradiction in this statement because to compare the rates of the two clocks in K and K' we must compare the readings of the same moving clock in K' with different clocks in K; we require several clocks in one frame and one in the other, thus the measurement process is not symmetric with respect to the two frames of reference. The clock that goes more slowly is the one which is being compared with different clocks in the other frame. The time interval measured by a clock is equal to the integral

$$\Delta t = \frac{1}{c}\int ds \tag{1.6.112}$$

along its world line. Since the world line is a straight line for a clock at rest and a curved line for a clock moving such that it returns to the starting point, the integral $\int ds$ taken between two world points has its maximum value if it is taken along the straight line connecting these two points.

For a Lorentz transformation with speed $V = |\mathbf{V}|$, (1.6.107) gives

$$\tan\theta = \frac{v'\sin\theta'}{\gamma(v'\cos\theta' + V)}, \tag{1.6.113}$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{V}$, and θ' is the angle between $\mathbf{v}'$ and $\mathbf{V}$. If $v = v' = c$ then

$$\cos\theta = \frac{\cos\theta' + \frac{V}{c}}{1 + \frac{V}{c}\cos\theta'}, \tag{1.6.114}$$

which is referred to as the *aberration* of a signal. Let us suppose that an observer in frame K measures a periodic signal with period T, frequency $\nu = \frac{1}{T}$ and wavelength $\lambda = \frac{c}{\nu}$, propagating in the $-z$ direction; the number of pulses in time dt is $n = \nu dt$. A second observer in frame K', moving in the z direction with speed V relative to the first one, travels a distance $V dt$ and measures $\frac{V dt}{\lambda}$ more pulses: $n' = \nu(1 + \frac{V}{c})dt$. Because the time interval dt with respect to K' is $dt' = \frac{dt}{\gamma}$, the frequency of the signal in K' is $\nu' = \gamma\nu(1 + \frac{V}{c})$ or

$$\nu' = e^{\eta}\nu. \tag{1.6.115}$$

This dependence of the frequency of a signal on a frame of reference is referred to as the *Doppler effect*.

When $c \to \infty$ (at which $\gamma \to 1$) the above formulae, referring to *relativistic kinematics*, reduce to their *nonrelativistic* limit. The Lorentz transformation (1.6.99) reduces to the *Galilei transformation*,

$$\begin{aligned} t &= t', \\ \mathbf{x} &= \mathbf{x}' + \mathbf{V}t', \end{aligned} \tag{1.6.116}$$

so the time is an absolute (invariant) quantity in nonrelativistic (*Newtonian*) physics. Any two Galilei transformations commute. The transformation law for velocities (1.6.107) reduces to the simple addition of vectors,

$$\mathbf{v} = \mathbf{v}' + \mathbf{V}. \tag{1.6.117}$$

1.6.8 Four-acceleration

In the Galilean frame, the line element in (1.4.117) is equal to

$$ds = cdt\Big(1 - \frac{\sum_\alpha v^\alpha v^\alpha}{c^2}\Big)^{1/2} = cdt\Big(1 - \frac{v^2}{c^2}\Big)^{1/2} = \frac{cdt}{\gamma}. \tag{1.6.118}$$

This line element is a special case of that in (1.4.134) for $g_{00} = 1$ and $g_{0\alpha} = 0$. The corresponding differential of the proper time is equal to (1.6.111). In a locally inertial frame of reference, the components of the four-velocity in the Cartesian coordinates are

$$u^0 = \frac{dx^0}{ds} = \frac{cdt}{ds} = \gamma, \;\; u^\alpha = \frac{dx^\alpha}{ds} = \frac{dx^\alpha}{cdt/\gamma} = \frac{\gamma}{c}v^\alpha, \tag{1.6.119}$$

which can be written as

$$u^i = \Big(\gamma, \frac{\gamma}{c}\mathbf{v}\Big), \;\; u_i = \Big(\gamma, -\frac{\gamma}{c}\mathbf{v}\Big), \tag{1.6.120}$$

where $\mathbf{v}$ is the velocity (1.6.87) and $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ (confer (1.6.94)). The spatial components u^α in (1.6.119) coincide with those in (1.4.136), whereas u^0 in (1.6.119) is a special case of that in (1.4.136) for $g_{00} = 1$ and $g_{0\alpha} = 0$.

We define the *four-acceleration*:

$$w^i = \frac{Du^i}{ds} = \frac{D^2x^i}{ds^2} = u^k u^i{}_{;k}. \tag{1.6.121}$$

This vector is orthogonal to u^i because of (1.4.14):

$$w^i u_i = \frac{1}{2}\frac{D}{ds}(u^i u_i) = 0, \tag{1.6.122}$$

thus having 3 independent components. In a locally inertial frame of reference, the four-acceleration is given by

$$w^i = \frac{du^i}{ds} = \frac{d^2x^i}{ds^2} = u^k u^i{}_{,k}. \tag{1.6.123}$$

Its components in the Cartesian coordinates are

$$\begin{aligned} w^0 &= \frac{du^0}{ds} = \frac{d\gamma}{cdt}\frac{dx^0}{ds} = \frac{\gamma}{c}\frac{d\gamma}{dt} = \frac{1}{2c}\frac{d}{dt}(\gamma^2) = \frac{1}{2c}\frac{d}{dt}\Big(\frac{1}{1-\mathbf{v}^2/c^2}\Big) = \Big(1-\frac{\mathbf{v}^2}{c^2}\Big)^{-2}\Big(\frac{\mathbf{v}}{c^3}\cdot\frac{d\mathbf{v}}{dt}\Big) \\ &= \frac{\gamma^4}{c^3}\mathbf{v}\cdot\mathbf{a}, \\ w^\alpha &= \frac{du^\alpha}{ds} = \frac{du^\alpha}{cdt}\frac{dx^0}{ds} = \frac{\gamma}{c^2}\frac{d}{dt}(\gamma v^\alpha) = \frac{\gamma^2}{c^2}a^\alpha + \frac{v^\alpha}{c^2}\gamma\frac{d\gamma}{dt} = \frac{\gamma^2}{c^2}a^\alpha + \frac{\gamma^4}{c^4}(\mathbf{v}\cdot\mathbf{a})v^\alpha, \end{aligned} \tag{1.6.124}$$

which can be written as

$$w^i = \Big(\frac{\gamma^4}{c^3}\mathbf{v}\cdot\mathbf{a}, \frac{\gamma^2}{c^2}\mathbf{a} + \frac{\gamma^4}{c^4}(\mathbf{v}\cdot\mathbf{a})\mathbf{v}\Big),\ \ w_i = \Big(\frac{\gamma^4}{c^3}\mathbf{v}\cdot\mathbf{a}, -\frac{\gamma^2}{c^2}\mathbf{a} - \frac{\gamma^4}{c^4}(\mathbf{v}\cdot\mathbf{a})\mathbf{v}\Big), \tag{1.6.125}$$

where $\mathbf{a}$ is the three-dimensional *acceleration* vector:

$$a^\alpha = \frac{dv^\alpha}{dt} = \frac{d^2x^\alpha}{dt^2},\ \ \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{x}}{dt^2}. \tag{1.6.126}$$

The invariant square of the four-acceleration is thus

$$w^i w_i = \frac{\gamma^8}{c^6}(\mathbf{v}\cdot\mathbf{a})^2 - \Big(\frac{\gamma^2}{c^2}\mathbf{a} + \frac{\gamma^4}{c^4}(\mathbf{v}\cdot\mathbf{a})\mathbf{v}\Big)^2 = -\frac{\gamma^4}{c^4}\Big(\mathbf{a}^2 + \frac{\gamma^2}{c^2}(\mathbf{v}\cdot\mathbf{a})^2\Big). \tag{1.6.127}$$

If $\mathbf{v} = 0$ at a given instant of time, the corresponding frame of reference is referred to as the *instantaneous rest frame*. In this frame

$$w^i w_i = -\frac{a^2}{c^4}, \tag{1.6.128}$$

so

$$a_0 = c^2\sqrt{-w^i w_i} \tag{1.6.129}$$

is the magnitude of the acceleration in the instantaneous rest frame, called the *proper acceleration*. Along an affine geodesic, the four-acceleration with respect to the affine connection (1.6.121) vanishes because of (1.2.59). Along a metric geodesic, the four-acceleration with respect to the Levi-Civita connection (defined by (1.6.121) with colon instead of semicolon) vanishes because of (1.4.91). The equation of geodesic deviation (1.3.52) determines the relative four-acceleration of two bodies moving along two infinitely close affine geodesics.

Let us suppose that a noninertial frame K' moves with velocity $\mathbf{v}$ relative to an inertial frame of reference K. If the velocity of K' changes by $d\mathbf{v}'$ relative to the initial frame K', then it changes by $d\mathbf{v}$ relative to K. In the nonrelativistic limit, the two changes are equal, $d\mathbf{v} = d\mathbf{v}'$, and K' does not rotate with respect to K. In relativistic kinematics, these changes are different because of the Thomas-Wigner rotation. The velocity of K' relative to K after the change, $\mathbf{v}+d\mathbf{v}$, is equal to $d\mathbf{v}'$ boosted by $\mathbf{v}$. Using the Lorentz transformation (1.6.107), in which $\mathbf{v}'$ is replaced with $d\mathbf{v}'$ and $\mathbf{V}$ is replaced with $\mathbf{v}$, we obtain

$$\mathbf{v}+d\mathbf{v}=\frac{d\mathbf{v}'+\gamma\mathbf{v}+(\gamma-1)(d\mathbf{v}'\cdot\mathbf{v})\mathbf{v}/v^2}{\gamma(1+d\mathbf{v}'\cdot\mathbf{v}/c^2)}, \tag{1.6.130}$$

where $\gamma=(1-v^2/c^2)^{-1/2}$. Keeping only terms linear in $d\mathbf{v}'$, we have

$$\mathbf{v}+d\mathbf{v}=\frac{d\mathbf{v}'}{\gamma}+\mathbf{v}(1-d\mathbf{v}'\cdot\mathbf{v}/c^2)+\frac{\gamma-1}{\gamma}(d\mathbf{v}'\cdot\mathbf{v})\frac{\mathbf{v}}{v^2}, \tag{1.6.131}$$

which gives

$$\mathbf{v}\times d\mathbf{v}=\frac{\mathbf{v}\times d\mathbf{v}'}{\gamma}. \tag{1.6.132}$$

The angle of infinitesimal rotation from the nonrelativistic sum $\mathbf{v}+d\mathbf{v}'$ to the relativistic $\mathbf{v}+d\mathbf{v}$ determines the relativistic rotation of K' with respect to K. Using (1.4.189) with $\sin(d\theta)\approx d\theta$ and $d\boldsymbol{\theta}=\mathbf{n}\,d\theta$, where $\mathbf{n}$ is a unit vector parallel to the axis of rotation, gives

$$d\boldsymbol{\theta}=\frac{(\mathbf{v}+d\mathbf{v}')\times(\mathbf{v}+d\mathbf{v})}{v^2}\approx\frac{\mathbf{v}\times(d\mathbf{v}-d\mathbf{v}')}{v^2}=\frac{1-\gamma}{v^2}(\mathbf{v}\times d\mathbf{v}). \tag{1.6.133}$$

Consequently, we obtain the angular velocity of the *Thomas precession*:

$$\boldsymbol{\Omega}=\frac{d\boldsymbol{\theta}}{dt}=\frac{\gamma-1}{v^2}(\mathbf{a}\times\mathbf{v})=\frac{\gamma^2}{\gamma+1}\frac{\mathbf{a}\times\mathbf{v}}{c^2}, \tag{1.6.134}$$

where $\mathbf{a}=d\mathbf{v}/dt$ is the acceleration of K' relative to K. When $v\ll c$,

$$\boldsymbol{\Omega}\approx\frac{\mathbf{a}\times\mathbf{v}}{2c^2}. \tag{1.6.135}$$

References: [2, 3].

1.7 Spinors

1.7.1 Spinor representation of Lorentz group

Let γ^a be the coordinate-invariant 4×4 *Dirac matrices* defined as

$$\gamma^a\gamma^b+\gamma^b\gamma^a=2\eta^{ab}I_4, \tag{1.7.1}$$

where I_4 is the unit 4×4 matrix (4 is the lowest dimension for which (1.7.1) has solutions). Accordingly, the spacetime-dependent Dirac matrices, $\gamma^i=e^i_a\gamma^a$, satisfy

$$\gamma^i\gamma^j+\gamma^j\gamma^i=2g^{ij}I_4. \tag{1.7.2}$$

Under a tetrad rotation, (1.5.15) gives

$$\tilde{\gamma}^a=\Lambda^a{}_b\gamma^b. \tag{1.7.3}$$

Let L be a 4×4 matrix such that

$$\gamma^a=\Lambda^a{}_bL\gamma^bL^{-1}=L\tilde{\gamma}^aL^{-1}, \tag{1.7.4}$$

where L^{-1} is the matrix inverse to L: $LL^{-1}=L^{-1}L=I_4$. The condition (1.7.4) represents the constancy of the Dirac matrices γ^a under the combined tetrad rotation and transformation $\gamma\rightarrow L\gamma L^{-1}$. We refer to L as the *spinor representation* of the Lorentz group. The relation (1.7.4) gives the matrix L as a function of the Lorentz matrix $\Lambda^a{}_b$. For an infinitesimal Lorentz transformation (1.6.7), the solution for L is

$$L = I_4 + \frac{1}{2}\epsilon_{ab}G^{ab}, \;\; L^{-1} = I_4 - \frac{1}{2}\epsilon_{ab}G^{ab}, \tag{1.7.5}$$

where G^{ab} are the *generators* of the spinor representation of the Lorentz group:

$$G^{ab} = \frac{1}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a). \tag{1.7.6}$$

A *spinor* ψ is defined as a quantity that, under tetrad rotations, transforms according to

$$\tilde{\psi} = L\psi. \tag{1.7.7}$$

An *adjoint spinor* $\bar{\psi}$ is defined as a quantity that transforms according to

$$\tilde{\bar{\psi}} = \bar{\psi}L^{-1}, \tag{1.7.8}$$

so the product $\bar{\psi}\psi$ is a scalar:

$$\tilde{\bar{\psi}}\tilde{\psi} = \bar{\psi}\psi. \tag{1.7.9}$$

The indices of the γ^a and L that are implicit in the 4×4 matrix multiplication in (1.7.1), (1.7.2) and (1.7.4) are spinor indices. The relation (1.7.4) shows that the Dirac matrices γ^a can be regarded as quantities that have, in addition to the invariant index a, one spinor index and one adjoint-spinor index. The product $\psi\bar{\psi}$ transforms like the Dirac matrices:

$$\tilde{\psi}\tilde{\bar{\psi}} = L\psi\bar{\psi}L^{-1}. \tag{1.7.10}$$

The spinors ψ and $\bar{\psi}$ can be used to construct tensors. For example, $\bar{\psi}\gamma^a\psi$ transforms like a contravariant Lorentz vector:

$$\bar{\psi}\gamma^a\psi \to \bar{\psi}L^{-1}\Lambda^a{}_bL\gamma^bL^{-1}L\psi = \Lambda^a{}_b\bar{\psi}\gamma^b\psi. \tag{1.7.11}$$

1.7.2 Spinor connection

The derivative of a spinor does not transform like a spinor:

$$\tilde{\psi}_{,i} = L\psi_{,i} + L_{,i}\psi. \tag{1.7.12}$$

If we introduce the *spinor connection* Γ_i that transforms according to

$$\tilde{\Gamma}_i = L\Gamma_iL^{-1} + L_{,i}L^{-1}, \tag{1.7.13}$$

then a *covariant derivative* of a spinor,

$$\psi_{;i} = \psi_{,i} - \Gamma_i\psi, \tag{1.7.14}$$

is a spinor:

$$\tilde{\psi}_{;i} = \tilde{\psi}_{,i} - \tilde{\Gamma}_i\tilde{\psi} = L\psi_{,i} + L_{,i}\psi - (L\Gamma_iL^{-1} + L_{,i}L^{-1})L\psi = L\psi_{;i}. \tag{1.7.15}$$

Because $\bar{\psi}\psi$ is a scalar,

$$(\bar{\psi}\psi)_{;i} = (\bar{\psi}\psi)_{,i}, \tag{1.7.16}$$

the chain rule for covariant differentiation gives the covariant derivative of an adjoint spinor,

$$\bar{\psi}_{;i} = \bar{\psi}_{,i} + \bar{\psi}\Gamma_i. \tag{1.7.17}$$

We also have

$$\psi_{|i} = \psi_{;i}, \;\; \bar{\psi}_{|i} = \bar{\psi}_{;i}. \tag{1.7.18}$$

The Dirac matrices γ^a transform like $\psi\bar{\psi}$, whose covariant derivative is

$$(\psi\bar{\psi})_{;i} = \psi_{;i}\bar{\psi} + \psi\bar{\psi}_{;i} = (\psi\bar{\psi})_{,i} - \Gamma_i\psi\bar{\psi} + \psi\bar{\psi}\Gamma_i = (\psi\bar{\psi})_{,i} - [\Gamma_i, \psi\bar{\psi}]. \tag{1.7.19}$$

Therefore, the covariant derivative of a Dirac matrix is

$$\gamma^a{}_{;i} = \gamma^a{}_{,i} - [\Gamma_i, \gamma^a] = -[\Gamma_i, \gamma^a], \tag{1.7.20}$$

which gives

$$\gamma^j{}_{;i} = \gamma^j{}_{|i} = \gamma^j{}_{,i} + \Gamma^{\,j}_{k\,i}\gamma^k - [\Gamma_i, \gamma^j]. \tag{1.7.21}$$

Accordingly, we obtain

$$\gamma^a{}_{|i} = \omega^a{}_{bi}\gamma^b - [\Gamma_i, \gamma^a]. \tag{1.7.22}$$

The quantity $\bar{\psi}\gamma^i\psi_{|i}$ transforms under Lorentz rotations like a scalar:

$$\bar{\psi}\gamma^i\psi_{|i} \to \bar{\psi}L^{-1}L\gamma^i L^{-1}L\psi_{|i} = \bar{\psi}\gamma^i\psi_{|i}. \tag{1.7.23}$$

The relation $\eta_{ab|i} = 0$ infers that

$$\gamma^a{}_{|i} = 0, \tag{1.7.24}$$

because the Dirac matrices γ^a only depend on η_{ab}. Multiplying both sides of (1.7.22) by γ_a from the left gives

$$\omega_{abi}\gamma^a\gamma^b - \gamma_a\Gamma_i\gamma^a + 4\Gamma_i = 0. \tag{1.7.25}$$

We seek the solution of (1.7.25) in the form

$$\Gamma_i = -\frac{1}{4}\omega_{abi}\gamma^a\gamma^b - A_i, \tag{1.7.26}$$

where A_i is a spinor-tensor quantity with one vector index. Substituting (1.7.26) to (1.7.25), together with the identity $\gamma_c\gamma^a\gamma^b\gamma^c = 4\eta^{ab}$, gives

$$-\gamma_a A_i\gamma^a + 4A_i = 0, \tag{1.7.27}$$

so A_i is an arbitrary vector multiple of I_4. Therefore, the spinor connection Γ_i is given, up to the addition of an arbitrary vector multiple of I_4, by the *Fock-Ivanenko coefficients*:

$$\Gamma_i = -\frac{1}{4}\omega_{abi}\gamma^a\gamma^b = -\frac{1}{2}\omega_{abi}G^{ab}. \tag{1.7.28}$$

Using the definition (1.5.22), we can also write (1.7.28) as

$$\Gamma_i = -\frac{1}{8}e^j_{c;i}[\gamma_j, \gamma^c] = \frac{1}{8}[\gamma^j{}_{;i}, \gamma_j]. \tag{1.7.29}$$

For the Levi-Civita connection, the covariant derivative of a spinor (1.7.14) becomes

$$\psi_{:i} = \psi_{,i} - \Gamma^{\{\}}_i\psi, \tag{1.7.30}$$

and the covariant derivative of an adjoint spinor (1.7.17) becomes

$$\bar{\psi}_{:i} = \bar{\psi}_{,i} + \bar{\psi}\Gamma^{\{\}}_i, \tag{1.7.31}$$

where the *Levi-Civita spinor connection* $\Gamma^{\{\}}_i$ is given, similarly to (1.7.28), by

$$\Gamma^{\{\}}_i = -\frac{1}{4}\varpi_{abi}\gamma^a\gamma^b = -\frac{1}{2}\varpi_{abi}G^{ab}. \tag{1.7.32}$$

Substituting (1.5.39) into (1.7.32) gives

$$\Gamma_i = \Gamma^{\{\}}_i - \frac{1}{4}C_{jki}\gamma^j\gamma^k = \Gamma^{\{\}}_i - \frac{1}{4}C_{jki}\gamma^{[j}\gamma^{k]}. \tag{1.7.33}$$

Accordingly, (1.7.30) and (1.7.31) yield

$$\psi_{;k} = \psi_{:k} + \frac{1}{4}C_{ijk}\gamma^i\gamma^j\psi, \tag{1.7.34}$$

$$\bar{\psi}_{;k} = \bar{\psi}_{:k} - \frac{1}{4}C_{ijk}\bar{\psi}\gamma^i\gamma^j. \tag{1.7.35}$$

1.7.3 Curvature spinor

The commutator of total covariant derivatives of a spinor is

$$\psi_{|ji} - \psi_{|ij} = (\psi_{|j})_{,i} - \Gamma_i\psi_{|j} - \Gamma^{\,k}_{j\,i}\psi_{|k} - (\psi_{|i})_{,j} + \Gamma_j\psi_{|i} + \Gamma^{\,k}_{i\,j}\psi_{|k}$$
$$= -\Gamma_{j,i}\psi + \Gamma_i\Gamma_j\psi + \Gamma_{i,j}\psi - \Gamma_j\Gamma_i\psi + 2S^k_{\ \,ij}\psi_{|k} = K_{ij}\psi + 2S^k_{\ \,ij}\psi_{|k}, \quad (1.7.36)$$

where $K_{ij} = -K_{ji}$ is defined as

$$K_{ij} = \Gamma_{i,j} - \Gamma_{j,i} + [\Gamma_i, \Gamma_j]. \quad (1.7.37)$$

Substituting (1.7.13) to (1.7.37) gives

$$\tilde{K}_{ij} = \tilde{\Gamma}_{i,j} - \tilde{\Gamma}_{j,i} + [\tilde{\Gamma}_i, \tilde{\Gamma}_j] = L(\Gamma_{i,j} - \Gamma_{j,i} + [\Gamma_i, \Gamma_j])L^{-1} = LK_{ij}L^{-1}, \quad (1.7.38)$$

so K_{ij} transforms under tetrad rotations like the Dirac matrices γ^a, that is, K_{ij} is a spinor with one spinor index and one adjoint-spinor index. We refer to K_{ij} as the *curvature spinor*.

The relation (1.7.24) leads to

$$\gamma^k_{\ \,|i} = 0. \quad (1.7.39)$$

Therefore, the commutator of covariant derivatives of the spacetime-dependent Dirac matrices vanishes:

$$2\gamma^k_{\ \,|[ji]} = R^k_{\ \,lij}\gamma^l + 2S^l_{\ \,ij}\gamma^k_{\ \,|l} + [K_{ij}, \gamma^k] = R^k_{\ \,lij}\gamma^l + [K_{ij}, \gamma^k] = 0. \quad (1.7.40)$$

Multiplying both sides of (1.7.40) by γ_k from the left gives

$$R_{klij}\gamma^k\gamma^l + \gamma_k K_{ij}\gamma^k - 4K_{ij} = 0. \quad (1.7.41)$$

We seek the solution of (1.7.41) in the form

$$K_{ij} = \frac{1}{4}R_{klij}\gamma^k\gamma^l + B_{ij}, \quad (1.7.42)$$

where B_{ij} is a spinor-tensor quantity with two vector indices. Substituting (1.7.42) to (1.7.41) gives

$$\gamma_k B_{ij}\gamma^k - 4B_{ij} = 0, \quad (1.7.43)$$

so B_{ij} is an antisymmetric-tensor multiple of I_4. The tensor B_{ij} is related to the vector A_i in (1.7.26) by

$$B_{ij} = A_{j,i} - A_{i,j} + [A_i, A_j]. \quad (1.7.44)$$

Because ψ has no indices other than spinor indices, A_i is a vector and $[A_i, A_j] = 0$. The invariance of (1.7.41) under the addition of an antisymmetric-tensor multiple B_{ij} of the unit matrix to the curvature spinor is related to the invariance of (1.7.25) under the addition of a vector multiple A_i of the unit matrix to the spinor connection. Setting $A_i = 0$, which corresponds to the Fock-Ivanenko spinor connection, gives $B_{ij} = 0$. Therefore, the curvature spinor K_{ij} is given, up to the addition of an arbitrary antisymmetric-tensor multiple of I_4, by

$$K_{ij} = \frac{1}{4}R_{klij}\gamma^k\gamma^l = \frac{1}{2}R_{klij}G^{kl}. \quad (1.7.45)$$

References: [3, 4].

Spacetime is a fabric in which various fields representing matter exist. These fields can be described by vectors, tensors, and spinors. They satisfy the equations derived from two fundamental principles: the principle of relativity and the principle of least action. The physics of fields is referred to as *field theory* and constitutes Chapter 2 (Fields).

2 Fields

2.1 Principle of least action

The most general formulation of the law that governs the dynamics of classical systems is Hamilton's *principle of least action*, according to which every classical system is characterized by a definite scalar-density function $\mathfrak{L}$, and the dynamics of the system is such that a certain condition is satisfied. Let $\phi_A(x^i)$ be a set of physical *fields* (indexed by A), being differentiable functions of the coordinates, and let $\mathfrak{L}$ be a Lorentz covariant quantity constructed from the fields ϕ_A and their derivatives. Let us consider a scalar quantity

$$S = \frac{1}{c}\int \mathfrak{L} d\Omega, \tag{2.1.1}$$

where the integration is over some region in locally Minkowski spacetime. Let $\delta\phi_A$ be arbitrary and independent, small changes of ϕ_A (regarded as a dynamical variable) over the region of integration, which vanish on the boundary. Then the change in S can be written as

$$\delta S = \sum_A \delta_A S, \tag{2.1.2}$$

where

$$\delta_A S = \frac{1}{c}\int F_A \delta\phi_A d\Omega. \tag{2.1.3}$$

The principle of least action states that the dynamics of a physical system is given by the condition the scalar S be a local minimum. Therefore, any infinitesimal change in the dynamics of the system does not alter the value of S:

$$\delta S = 0 \tag{2.1.4}$$

(S is a local extremum). The condition (2.1.4) is referred to as the *principle of stationary action*, which is the necessary part of the principle of least action. If $\mathfrak{L}$ is covariant and ϕ_A transform covariantly under the Lorentz group, the variational condition (2.1.4) gives the Lorentz covariant equations

$$F_A = 0. \tag{2.1.5}$$

These equations are also invariant for any other transformations (internal symmetries) for which $\mathfrak{L}$ is invariant. $\mathfrak{L}$ is referred to as the *Lagrangian density*, S is the *action* functional, $\delta S = 0$ is the principle of least action, and (2.1.5) are the *field equations*. The field equations of a physical system are the result of the action being a local extremum. The condition that the action be a local minimum imposes additional restrictions on possible choices for S. The number of independent field equations for a given system is referred to as the number of the *degrees of freedom* representing this system.

In most physical cases $\mathfrak{L}$ contains only ϕ_A and their first derivatives. A Lagrangian density containing higher derivatives can always be written in terms of first derivatives by increasing the number of the components ϕ_A. Let us consider a physical system in the Galilean frame of reference. If $\mathfrak{L}$ depends only on ϕ and $\partial_i\phi$, $\mathfrak{L} = \mathfrak{L}(\phi, \phi_{,i})$, then

$$\begin{aligned}\delta S &= \frac{1}{c}\int\Big(\frac{\partial\mathfrak{L}}{\partial\phi}\delta\phi + \frac{\partial\mathfrak{L}}{\partial(\phi_{,i})}\delta(\phi_{,i})\Big)d\Omega = \frac{1}{c}\int\Big(\frac{\partial\mathfrak{L}}{\partial\phi}\delta\phi + \frac{\partial\mathfrak{L}}{\partial(\phi_{,i})}(\delta\phi)_{,i}\Big)d\Omega \\ &= \frac{1}{c}\int\Big(\frac{\partial\mathfrak{L}}{\partial\phi}\delta\phi - \partial_i\Big(\frac{\partial\mathfrak{L}}{\partial(\phi_{,i})}\Big)\delta\phi + \partial_i\Big(\frac{\partial\mathfrak{L}}{\partial(\phi_{,i})}\delta\phi\Big)\Big)d\Omega.\end{aligned} \tag{2.1.6}$$

The last term in the integrand in the second line of (2.1.6) is a divergence. Its four-volume integral can be transformed, using the Gauß-Stokes theorem (1.1.39), into a hypersurface integral over the boundary of the integration region. Since $\delta\phi = 0$ on the boundary, this term does not contribute to the variation of the action:

$$\delta S = \frac{1}{c}\int\Big(\frac{\partial\mathfrak{L}}{\partial\phi} - \partial_i\Big(\frac{\partial\mathfrak{L}}{\partial(\phi_{,i})}\Big)\Big)\delta\phi d\Omega + \int\frac{\partial\mathfrak{L}}{\partial(\phi_{,i})}\delta\phi dS_i = \frac{1}{c}\int\Big(\frac{\partial\mathfrak{L}}{\partial\phi} - \partial_i\Big(\frac{\partial\mathfrak{L}}{\partial(\phi_{,i})}\Big)\Big)\delta\phi d\Omega. \tag{2.1.7}$$

If $\delta S = 0$ for arbitrary variations $\delta\phi$ that vanish on the boundary, then

$$\frac{\partial \mathfrak{L}}{\partial \phi} - \partial_i \left(\frac{\partial \mathfrak{L}}{\partial(\phi_{,i})} \right) = 0. \tag{2.1.8}$$

Defining the *variational derivative* of $\mathfrak{L}$ with respect to ϕ,

$$\frac{\delta \mathfrak{L}}{\delta \phi} = \frac{\partial \mathfrak{L}}{\partial \phi} - \partial_i \left(\frac{\partial \mathfrak{L}}{\partial(\phi_{,i})} \right), \tag{2.1.9}$$

we can write (2.1.8) as

$$\frac{\delta \mathfrak{L}}{\delta \phi} = 0. \tag{2.1.10}$$

The set of equations (2.1.8), for each field ϕ_A, is referred to as the *Lagrange equations*.

There is some arbitrariness in the choice of $\mathfrak{L}$; adding to it the divergence of an arbitrary vector density or multiplying it by a constant produces the same field equations. If a system consists of two noninteracting parts A and B, with corresponding Lagrangian densitites $\mathfrak{L}_A(\phi_A, \partial\phi_A)$ and $\mathfrak{L}_B(\phi_B, \partial\phi_B)$, then the Lagrangian density for this system is the sum $\mathfrak{L}_A + \mathfrak{L}_B$. This additivity of the Lagrangian density means that the field equations for either of the two parts do not involve quantities pertaining to the other part. If $\mathfrak{L}_A$ also depends on ϕ_B and/or $\partial\phi_B$, and/or $\mathfrak{L}_B$ depends on ϕ_A and/or $\partial\phi_A$, then the subsystems A and B interact.
References: [1, 2, 3].

2.2 Action for gravitational field

Let us consider a Lagrangian density $\mathfrak{L}$ that depends on the affine (or spin) connection and its first derivatives. Such Lagrangian density can be decomposed into the covariant part $\mathfrak{L}_{\mathrm{g}}$ that contains derivatives of the affine/spin connection, which is referred to as the Lagrangian density for the *gravitational field*, and the covariant part $\mathfrak{L}_{\mathrm{m}}$ that does not contain these derivatives, which is referred to as the Lagrangian density for *matter*:

$$\mathfrak{L} = \mathfrak{L}_{\mathrm{g}} + \mathfrak{L}_{\mathrm{m}}. \tag{2.2.1}$$

The simplest covariant scalar that can be constructed from the affine/spin connection and its first derivatives is the Ricci scalar R. The corresponding Lagrangian density for the gravitational field is proportional to the product of R and the scalar density $\sqrt{-\mathfrak{g}}$:

$$\mathfrak{L}_{\mathrm{g}} = -\frac{1}{2\kappa}\sqrt{-\mathfrak{g}}R = -\frac{1}{2\kappa}\sqrt{-\mathfrak{g}}\Big(P - g^{ik}(2C^l_{\ il:k} + C^j_{\ ij}C^l_{\ kl} - C^l_{\ im}C^m_{\ kl})\Big), \tag{2.2.2}$$

where κ is *Einstein's gravitational constant* and we used (1.4.67). The action for the gravitational field is thus

$$S_{\mathrm{g}} = \frac{1}{c}\int \mathfrak{L}_{\mathrm{g}} d\Omega = -\frac{1}{2\kappa c}\int R\sqrt{-\mathfrak{g}}d\Omega. \tag{2.2.3}$$

The metric tensor and the affine connection are two fundamental quantities describing a gravitational field. Since the affine connection is metric-compatible, given by (1.4.34), it is a function of the metric tensor, its derivatives and the torsion tensor. Accordingly, the metric and torsion tensors are dynamical variables in varying the action. Equivalently, the tetrad and spin connection can be taken as dynamical variables.

Let us consider the Riemannian part of the Lagrangian density for the gravitational field (2.2.2), which is proportional to the Riemann scalar P:

$$\mathfrak{L}_{\mathrm{g}}^{\{\}} = -\frac{1}{2\kappa}\sqrt{-\mathfrak{g}}P. \tag{2.2.4}$$

The scalar density $\sqrt{-\mathfrak{g}}P$ is linear in first derivatives of the Christoffel symbols $\{{}^{\,i}_{k\,l}\}$:

$$\begin{aligned}
&\sqrt{-\mathfrak{g}}P=\sqrt{-\mathfrak{g}}g^{ik}(\{{}^{\,l}_{i\,k}\}_{,l}-\{{}^{\,l}_{i\,l}\}_{,k}+\{{}^{\,m}_{i\,k}\}\{{}^{\,l}_{m\,l}\}-\{{}^{\,m}_{i\,l}\}\{{}^{\,l}_{m\,k}\})\\
&=(\sqrt{-\mathfrak{g}}g^{ik}\{{}^{\,l}_{i\,k}\})_{,l}-\{{}^{\,l}_{i\,k}\}(\sqrt{-\mathfrak{g}}g^{ik})_{,l}-(\sqrt{-\mathfrak{g}}g^{ik}\{{}^{\,l}_{i\,l}\})_{,k}+\{{}^{\,l}_{i\,l}\}(\sqrt{-\mathfrak{g}}g^{ik})_{,k}\\
&+\sqrt{-\mathfrak{g}}g^{ik}(\{{}^{\,m}_{i\,k}\}\{{}^{\,l}_{m\,l}\}-\{{}^{\,m}_{i\,l}\}\{{}^{\,l}_{m\,k}\}).
\end{aligned}\tag{2.2.5}$$

We can therefore subtract from $\sqrt{-\mathfrak{g}}P$ total derivatives without altering the field equations, replacing it by a noncovariant quantity $\mathcal{G}$ that does not contain first derivatives of the Christoffel symbols:

$$\begin{aligned}
&\mathcal{G}=\sqrt{-\mathfrak{g}}P-(\sqrt{-\mathfrak{g}}g^{ik}\{{}^{\,l}_{i\,k}\})_{,l}+(\sqrt{-\mathfrak{g}}g^{ik}\{{}^{\,l}_{i\,l}\})_{,k}\\
&=\{{}^{\,l}_{i\,l}\}(\sqrt{-\mathfrak{g}}g^{ik})_{,k}-\{{}^{\,l}_{i\,k}\}(\sqrt{-\mathfrak{g}}g^{ik})_{,l}+\sqrt{-\mathfrak{g}}g^{ik}(\{{}^{\,m}_{i\,k}\}\{{}^{\,l}_{m\,l}\}-\{{}^{\,m}_{i\,l}\}\{{}^{\,l}_{m\,k}\})\\
&=\{{}^{\,l}_{i\,l}\}\big((\sqrt{-\mathfrak{g}}g^{ik})_{:k}+\{{}^{\,j}_{j\,k}\}\sqrt{-\mathfrak{g}}g^{ik}-\sqrt{-\mathfrak{g}}\{{}^{\,i}_{j\,k}\}g^{jk}-\sqrt{-\mathfrak{g}}\{{}^{\,k}_{j\,k}\}g^{ij}\big)\\
&-\{{}^{\,l}_{i\,k}\}\big((\sqrt{-\mathfrak{g}}g^{ik})_{:l}+\{{}^{\,j}_{j\,l}\}\sqrt{-\mathfrak{g}}g^{ik}-\sqrt{-\mathfrak{g}}\{{}^{\,i}_{j\,l}\}g^{jk}-\sqrt{-\mathfrak{g}}\{{}^{\,k}_{j\,l}\}g^{ij}\big)\\
&+\sqrt{-\mathfrak{g}}g^{ik}(\{{}^{\,m}_{i\,k}\}\{{}^{\,l}_{m\,l}\}-\{{}^{\,m}_{i\,l}\}\{{}^{\,l}_{m\,k}\})=\{{}^{\,l}_{i\,l}\}\big(\{{}^{\,j}_{j\,k}\}\sqrt{-\mathfrak{g}}g^{ik}-\sqrt{-\mathfrak{g}}\{{}^{\,i}_{j\,k}\}g^{jk}\\
&-\sqrt{-\mathfrak{g}}\{{}^{\,k}_{j\,k}\}g^{ij}\big)-\{{}^{\,l}_{i\,k}\}\big(\{{}^{\,j}_{j\,l}\}\sqrt{-\mathfrak{g}}g^{ik}-\sqrt{-\mathfrak{g}}\{{}^{\,i}_{j\,l}\}g^{jk}-\sqrt{-\mathfrak{g}}\{{}^{\,k}_{j\,l}\}g^{ij}\big)\\
&+\sqrt{-\mathfrak{g}}g^{ik}(\{{}^{\,m}_{i\,k}\}\{{}^{\,l}_{m\,l}\}-\{{}^{\,m}_{i\,l}\}\{{}^{\,l}_{m\,k}\})=\sqrt{-\mathfrak{g}}g^{ik}(\{{}^{\,m}_{i\,l}\}\{{}^{\,l}_{m\,k}\}-\{{}^{\,m}_{i\,k}\}\{{}^{\,l}_{m\,l}\}).
\end{aligned}\tag{2.2.6}$$

We also define

$$\mathsf{G}=\frac{\mathcal{G}}{\sqrt{-\mathfrak{g}}}=g^{ik}(\{{}^{\,m}_{i\,l}\}\{{}^{\,l}_{m\,k}\}-\{{}^{\,m}_{i\,k}\}\{{}^{\,l}_{m\,l}\}).\tag{2.2.7}$$

The Riemannian part (2.2.4) of the Lagrangian density for the gravitational field reduces accordingly to

$$\mathfrak{L}_{\mathrm{g}}^{\{\}}=-\frac{1}{2\kappa}\mathcal{G}=-\frac{1}{2\kappa}\sqrt{-\mathfrak{g}}\mathsf{G}.\tag{2.2.8}$$

Any coordinate transformation results in variations of g^{ik}, thereby

$$S_{\mathrm{g}}^{\{\}}=\frac{1}{c}\int\mathfrak{L}_{\mathrm{g}}^{\{\}}d\Omega=-\frac{1}{2\kappa c}\int P\sqrt{-\mathfrak{g}}d\Omega\tag{2.2.9}$$

is not necessarily a minimum with respect to these variations (only an extremum) because not all δg^{ik} correspond to actual variations of the gravitational field. In order to exclude the variations δg^{ik} resulting from changing the coordinates, we must impose on the metric tensor 4 arbitrary constraints. If we choose

$$g_{0\alpha}=0,\ \ |g_{\alpha\beta}|=\mathrm{const},\tag{2.2.10}$$

then G becomes

$$\mathsf{G}=-\frac{1}{4}g^{00}g^{\alpha\beta}g^{\gamma\delta}g_{\alpha\gamma,0}g_{\beta\delta,0}.\tag{2.2.11}$$

In the locally Galilean frame of reference, $g_{\alpha\beta}=-\delta_{\alpha\beta}$, thereby

$$\mathsf{G}=-\frac{1}{4}g^{00}(g_{\alpha\beta,0})^2.\tag{2.2.12}$$

For physical systems, $g^{00}>0$. Therefore, in order for $S_{\mathrm{g}}^{\{\}}$ to have a minimum, κ must be positive, otherwise an arbitrarily rapid change of $g_{\alpha\beta}$ in time would result in an arbitrarily low value of $S_{\mathrm{g}}^{\{\}}$ and there would be no minimum of S.
References: [2, 3].

2.3 Matter

2.3.1 Metric energy-momentum tensor

The variation of the action for matter,

$$S_{\mathrm{m}}=\frac{1}{c}\int\mathfrak{L}_{\mathrm{m}}d\Omega,\tag{2.3.1}$$

with respect to the metric tensor:

$$\delta S_{\mathrm{m}} = \frac{1}{2c}\int \mathcal{T}_{ij}\delta g^{ij}d\Omega = -\frac{1}{2c}\int \mathcal{T}^{ij}\delta g_{ij}d\Omega, \tag{2.3.2}$$

defines the *metric energy-momentum density* $\mathcal{T}_{ij}$. This tensor density is symmetric:

$$\mathcal{T}_{ij} = \mathcal{T}_{ji}. \tag{2.3.3}$$

Equivalently, we have

$$\mathcal{T}_{ij} = 2\frac{\delta\mathfrak{L}_{\mathrm{m}}}{\delta g^{ij}} = 2\frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial g^{ij}} - 2\partial_k\Big(\frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial(g^{ij}{}_{,k})}\Big). \tag{2.3.4}$$

The *metric energy-momentum tensor* is defined as

$$T_{ij} = \frac{\mathcal{T}_{ij}}{\sqrt{-\mathfrak{g}}}. \tag{2.3.5}$$

2.3.2 Tetrad energy-momentum tensor

The variation of the matter action (2.3.1) with respect to the tetrad:

$$\delta S_{\mathrm{m}} = \frac{1}{c}\int \mathfrak{T}_i{}^a\delta e^i_a d\Omega, \tag{2.3.6}$$

defines the *tetrad energy-momentum density* $\mathfrak{T}_i{}^a$. Equivalently

$$\delta\mathfrak{L}_{\mathrm{m}} = \mathfrak{T}_i{}^a\delta e^i_a \tag{2.3.7}$$

or

$$\mathfrak{T}_i{}^a = \frac{\delta\mathfrak{L}_{\mathrm{m}}}{\delta e^i_a}. \tag{2.3.8}$$

The corresponding tensor density with two coordinate indices is

$$\mathfrak{T}_{ij} = e_{aj}\mathfrak{T}_i{}^a. \tag{2.3.9}$$

The *tetrad energy-momentum tensor* is defined as

$$t_{ij} = \frac{\mathfrak{T}_{ij}}{\mathfrak{e}}. \tag{2.3.10}$$

This tensor is generally not symmetric.

2.3.3 Canonical energy-momentum density

A matter Lagrangian density $\mathfrak{L}_{\mathrm{m}}$ can be written as $\mathfrak{L}_{\mathrm{m}} = \mathfrak{e}L$, where L is a scalar. If $\mathfrak{L}$ depends on matter fields ϕ and their covariant derivatives $\phi_{|i}$, then such fields are said to be *minimally coupled* to the affine connection. If these fields are written in terms of Lorentz indices instead of vector indices, then the tetrad appears in L only through a covariant combination $e^i_a\phi_{|i}$. Varying $\mathfrak{L}$ with respect to the tetrad gives, using (1.5.13),

$$\delta\mathfrak{L}_{\mathrm{m}} = \mathfrak{e}\delta L - \mathfrak{e}e^a_i L\delta e^i_a = \mathfrak{e}\frac{\partial L}{\partial\phi_{|a}}\phi_{|i}\delta e^i_a - \mathfrak{L}_{\mathrm{m}}e^a_i\delta e^i_a = \Big(\frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial\phi_{|a}}\phi_{|i} - e^a_i\mathfrak{L}_{\mathrm{m}}\Big)\delta e^i_a. \tag{2.3.11}$$

The last term in (2.3.11),

$$\Theta_i{}^a = \frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial\phi_{|a}}\phi_{|i} - e^a_i\mathfrak{L}_{\mathrm{m}}, \tag{2.3.12}$$

is referred to as the *canonical energy-momentum density*. The corresponding tensor density with two coordinate indices is

$$\Theta_j{}^i = \frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial\phi_{|i}}\phi_{|j} - \delta^i_j\mathfrak{L}_{\mathrm{m}} = \frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial\phi_{,i}}\phi_{|j} - \delta^i_j\mathfrak{L}_{\mathrm{m}}. \tag{2.3.13}$$

Comparing (2.3.11) with (2.3.7) shows that the canonical energy-momentum density is identical with the tetrad energy-momentum density:

$$\Theta_i{}^a = \mathfrak{T}_i{}^a. \tag{2.3.14}$$

2.3.4 Spin tensor

The variation of the matter action (2.3.1) with respect to the spin connection,

$$\delta S_{\mathrm{m}} = \frac{1}{2c}\int \mathfrak{S}_{ab}{}^{i}\delta\omega^{ab}{}_{i}d\Omega, \tag{2.3.15}$$

defines the *spin density* $\mathfrak{S}_{ab}{}^{i}$:

$$\mathfrak{S}_{ab}{}^{i} = 2\frac{\delta\mathfrak{L}_{\mathrm{m}}}{\delta\omega^{ab}{}_{i}} = 2\frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial\omega^{ab}{}_{i}}, \tag{2.3.16}$$

which is antisymmetric in the Lorentz indices:

$$\mathfrak{S}_{ab}{}^{i} = -\mathfrak{S}_{ba}{}^{i}. \tag{2.3.17}$$

The second equality in (2.3.16) is satisfied because a matter Lagrangian density $\mathfrak{L}_{\mathrm{m}}$ may depend on the spin connection but not on its derivatives; a scalar density depending on derivatives of $\omega^{ab}{}_{i}$ is a Lagrangian density for the gravitational field. The variations $\delta\omega^{ab}{}_{i}$ are independent of δe^{i}_{a}, thereby the spin density is independent of the energy-momentum density. The relation (1.5.35) indicates that the spin density with three coordinate indices, which is antisymmetric in the first two indices, is generated by the contortion tensor:

$$\mathfrak{S}_{ij}{}^{k} = -\mathfrak{S}_{ji}{}^{k} = 2\frac{\delta\mathfrak{L}_{\mathrm{m}}}{\delta C^{ij}{}_{k}}. \tag{2.3.18}$$

Accordingly, the variation of $\mathfrak{L}_{\mathrm{m}}$ with respect to the torsion tensor,

$$\tau_{i}{}^{jk} = 2\frac{\delta\mathfrak{L}_{\mathrm{m}}}{\delta S^{i}{}_{jk}}, \tag{2.3.19}$$

is a homogeneous linear function of the spin connection because of (1.4.35):

$$\tau_{ijk} = 2\frac{\delta\mathfrak{L}_{\mathrm{m}}}{\delta S^{ijk}} = 2\frac{\delta\mathfrak{L}_{\mathrm{m}}}{\delta C^{lmn}}\frac{\partial C^{lmn}}{\partial S^{ijk}} = \mathfrak{S}_{lmn}(\delta^{l}_{i}\delta^{m}_{[j}\delta^{n}_{k]} + \delta^{m}_{i}\delta^{n}_{[j}\delta^{l}_{k]} + \delta^{n}_{i}\delta^{m}_{[j}\delta^{l}_{k]})$$
$$= \mathfrak{S}_{ijk} - \mathfrak{S}_{jki} + \mathfrak{S}_{kij}, \tag{2.3.20}$$
$$\mathfrak{S}_{ijk} = \tau_{[ij]k}, \tag{2.3.21}$$

antisymmetric in the last two indices:

$$\tau_{ijk} = -\tau_{ikj}. \tag{2.3.22}$$

The variation of $\mathfrak{L}_{\mathrm{m}}$ with respect to the metric-compatible affine connection in the metric-affine variational formulation of gravity is equivalent to the variation with respect to the torsion (or contortion) tensor.

The spin connection $\omega^{ab}{}_{i}$ appears in $\mathfrak{L}_{\mathrm{m}}$ only through covariant derivatives of ϕ, in a combination $-\frac{\partial\mathfrak{L}}{\partial\phi_{,i}}\Gamma_{i}\phi$, where

$$\Gamma_{i} = -\frac{1}{2}\omega_{abi}G^{ab} \tag{2.3.23}$$

is the connection in the covariant derivative of ϕ:

$$\phi_{|i} = \phi_{,i} - \Gamma_{i}\phi. \tag{2.3.24}$$

Consequently, the spin density $\mathfrak{S}_{ab}{}^{i}$ is identical with

$$\Sigma_{ab}{}^{i} = -\Sigma_{ba}{}^{i} = \frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial\phi_{,i}}G_{ab}\phi, \tag{2.3.25}$$

referred to as the *canonical spin density.* The *spin tensor* is defined as

$$s_{ijk} = \frac{\mathfrak{S}_{ijk}}{\mathfrak{e}}. \tag{2.3.26}$$

2.3.5 Belinfante-Rosenfeld relation

The total variation of the matter action with respect to geometrical variables is either

$$\delta S_{\mathrm{m}} = \frac{1}{c}\int d\Omega \mathfrak{T}_i{}^a \delta e^i_a + \frac{1}{2c}\int d\Omega \mathfrak{S}_{ab}{}^i \delta\omega^{ab}{}_i \tag{2.3.27}$$

or

$$\delta S_{\mathrm{m}} = \frac{1}{2c}\int d\Omega \mathcal{T}_{ik}\delta g^{ik} + \frac{1}{2c}\int d\Omega \tau_j{}^{ik}\delta S^j{}_{ik}. \tag{2.3.28}$$

The relation (1.5.5) gives

$$\frac{1}{2}\int d\Omega \mathcal{T}_{ik}\delta g^{ik} = \frac{1}{2}\int d\Omega \mathcal{T}_{ik}(\delta e^i_a e^k_b + e^i_a \delta e^k_b)\eta^{ab}) = \int d\Omega \mathcal{T}_{ik} e^{ka}\delta e^i_a, \tag{2.3.29}$$

and (1.5.31) gives

$$\begin{aligned}
&\frac{1}{2}\int d\Omega \tau_j{}^{ik}\delta S^j{}_{ik} = \frac{1}{2}\int d\Omega \tau_j{}^{ik}\Big(\delta(e^j_a e_{ib}\omega^{ab}{}_k) + \delta e^a_{i,k}e^j_a + e^a_{i,k}\delta e^j_a\Big) \\
&= \frac{1}{2}\int d\Omega\Big(\tau_j{}^{li}\delta(e^j_a e_{lb})\omega^{ab}{}_i + \tau_{ab}{}^i\delta\omega^{ab}{}_i + (\tau_j{}^{ik}e^j_a\delta e^a_i)_{,k} - (\tau_j{}^{ik}e^j_a)_{,k}\delta e^a_i + \tau_j{}^{ik}e^a_{i,k}\delta e^j_a\Big) \\
&= \frac{1}{2}\int d\Omega\Big(\tau_j{}^{lk}\omega^{cb}{}_k e_{lb}\delta e^j_c + \tau_j{}^{li}\omega^{ab}{}_i e^j_a\delta e_{lb} + \tau_{ab}{}^i\delta\omega^{ab}{}_i - (\tau_j{}^{ik}e^j_a)_{,k}\delta e^a_i + \tau_j{}^{lm}e^b_{l,m}\delta e^j_b\Big) \\
&+\frac{1}{2}\int dS_k\tau_j{}^{ik}e^j_a\delta e^a_i = \frac{1}{2}\int d\Omega\Big(-\tau_j{}^{lk}\omega^{cb}{}_k e_{lb}e^i_c e^j_a\delta e^a_i + \tau_j{}^{il}\omega^b{}_{al}e^j_b\delta e^a_i + \tau_{ab}{}^i\delta\omega^{ab}{}_i \\
&-(\tau_a{}^{ik}{}_{|k} - S^i{}_{jk}\tau_a{}^{jk} - 2S_j\tau_a{}^{ij} + \omega^b{}_{ak}\tau_b{}^{ik})\delta e^a_i - \tau_j{}^{lm}e^b_{l,m}e^i_b e^j_a\delta e^a_i\Big) \\
&= \frac{1}{2}\int d\Omega\Big(\tau_{ab}{}^i\delta\omega^{ab}{}_i - \tau_j{}^{ik}{}_{;k}e^j_a\delta e^a_i + 2S_j\tau_a{}^{ij}\delta e^a_i\Big).
\end{aligned} \tag{2.3.30}$$

Comparing (2.3.27) with (2.3.28) leads to

$$\begin{aligned}
&\int d\Omega \mathfrak{T}_i{}^a\delta e^i_a + \frac{1}{2}\int d\Omega \mathfrak{S}_{ab}{}^i\delta\omega^{ab}{}_i = \frac{1}{2}\int d\Omega \mathcal{T}_{ik}\delta g^{ik} + \frac{1}{2}\int d\Omega\Big(\tau_{ab}{}^i\delta\omega^{ab}{}_i - \tau_j{}^{ik}{}_{;k}e^j_a\delta e^a_i \\
&+2S_j\tau_a{}^{ij}\delta e^a_i\Big) = \int d\Omega \mathcal{T}_{ik}e^{ka}\delta e^i_a + \frac{1}{2}\int d\Omega \tau_{ab}{}^i\delta\omega^{ab}{}_i + \frac{1}{2}\int d\Omega \tau_i{}^{jk}{}_{;k}e^a_j\delta e^i_a \\
&-\int d\Omega S_j\tau_b{}^{kj}e^a_k e^b_i\delta e^i_a.
\end{aligned} \tag{2.3.31}$$

The terms in (2.3.31) with $\delta\omega^{ab}{}_i$ give (2.3.21), while the terms with δe^i_a give

$$\mathfrak{T}_i{}^a = \mathcal{T}_{ik}e^{ka} + \frac{1}{2}\tau_i{}^{jk}{}_{;k}e^a_j - S_j\tau_i{}^{aj} \tag{2.3.32}$$

or

$$\mathcal{T}_{ik} = \mathfrak{T}_{ik} - \frac{1}{2}\nabla_j(\mathfrak{S}_{ik}{}^j - \mathfrak{S}_k{}^j{}_i + \mathfrak{S}^j{}_{ik}) + S_j(\mathfrak{S}_{ik}{}^j - \mathfrak{S}_k{}^j{}_i + \mathfrak{S}^j{}_{ik}). \tag{2.3.33}$$

Equation (2.3.33) is referred to as the *Belinfante-Rosenfeld relation* between the metric and tetrad energy-momentum densites. The Belinfante-Rosenfeld relation can be written, after dividing by $\mathfrak{e}$, as a tensor equation:

$$T_{ik} = t_{ik} - \frac{1}{2}\nabla^*_j(s_{ik}{}^j - s_k{}^j{}_i + s^j{}_{ik}), \tag{2.3.34}$$

where ∇^*_j is given by (1.2.45). In the absence of spin, (2.3.33) and (2.3.34) reduce to

$$\mathfrak{T}_{ik} = \mathcal{T}_{ik}, \ \ t_{ik} = T_{ik}. \tag{2.3.35}$$

References: [2, 3, 4, 7, 9]

2.4 Symmetries and conservation laws

2.4.1 Noether theorem

Let us consider a physical system, described by a Lagrangian density $\mathfrak{L}$ that depends on matter fields ϕ, their first derivatives $\phi_{,i}$, and the coordinates x^i. The change of the Lagrangian density $\delta\mathfrak{L}$ under an infinitesimal coordinate transformation (1.2.66) is thus

$$\delta\mathfrak{L} = \frac{\partial\mathfrak{L}}{\partial\phi}\delta\phi + \frac{\partial\mathfrak{L}}{\partial\phi_{,i}}\delta(\phi_{,i}) + \frac{\bar{\partial}\mathfrak{L}}{\partial x^i}\xi^i, \tag{2.4.1}$$

where the changes $\delta\phi$ and $\delta(\phi_{,i})$ are brought about by the transformation (1.2.66) and $\bar{\partial}$ denotes partial differentiation with respect to x^i at constant ϕ and $\phi_{,i}$. The variation $\delta\mathfrak{L}$ brought about by this transformation is also given by (1.2.75):

$$\delta\mathfrak{L} = -\xi^i_{\ ,i}\mathfrak{L}. \tag{2.4.2}$$

Using the Lagrange equations (2.1.8) and the identities

$$\mathfrak{L}_{,i} = \frac{\bar{\partial}\mathfrak{L}}{\partial x^i} + \frac{\partial\mathfrak{L}}{\partial\phi}\phi_{,i} + \frac{\partial\mathfrak{L}}{\partial\phi_{,j}}\phi_{,ji}, \tag{2.4.3}$$

$$\delta(\phi_{,i}) = (\delta\phi)_{,i} - \xi^j_{\ ,i}\phi_{,j}, \tag{2.4.4}$$

we bring (2.4.1) to

$$\delta\mathfrak{L} = \xi^i\mathfrak{L}_{,i} + \Big(\frac{\partial\mathfrak{L}}{\partial\phi_{,i}}(\delta\phi - \xi^j\phi_{,j})\Big)_{,i}. \tag{2.4.5}$$

Combining (2.4.2) and (2.4.5) gives the conservation law,

$$\mathfrak{J}^i_{\ ,i} = 0, \tag{2.4.6}$$

for the *Noether current*:

$$\mathfrak{J}^i = \xi^i\mathfrak{L} + \frac{\partial\mathfrak{L}}{\partial\phi_{,i}}(\delta\phi - \xi^j\phi_{,j}) = \xi^i\mathfrak{L} + \frac{\partial\mathfrak{L}}{\partial\phi_{,i}}\bar{\delta}\phi. \tag{2.4.7}$$

Equation (2.4.6) represents the *Noether theorem*, which states that to each continuous symmetry of a Lagrangian density there corresponds a conservation law.

2.4.2 Conservation of spin

The Lorentz group is the group of tetrad rotations. Since a physical matter Lagrangian density $\mathfrak{L}_{\mathrm{m}}(\phi, \phi_{,i})$ is invariant under local, proper Lorentz transformations, it is invariant under tetrad rotations:

$$\delta\mathfrak{L}_{\mathrm{m}} = \frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial\phi}\delta\phi + \frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial\phi_{,i}}\delta(\phi_{,i}) + \mathfrak{T}_i^{\ a}\delta e^i_a + \frac{1}{2}\mathfrak{S}_{ab}^{\ \ i}\delta\omega^{ab}_{\ \ i} = 0, \tag{2.4.8}$$

where the changes δ are brought about by a tetrad rotation. Under integration of (2.4.8) over spacetime, the first two terms vanish because of the Lagrange equations for ϕ (2.1.8):

$$\int\Big(\mathfrak{T}_i^{\ a}\delta e^i_a + \frac{1}{2}\mathfrak{S}_{ab}^{\ \ i}\delta\omega^{ab}_{\ \ i}\Big)d\Omega = 0. \tag{2.4.9}$$

For an infinitesimal Lorentz transformation (1.6.7), the tetrad e^a_i changes by

$$\delta e^a_i = \tilde{e}^a_i - e^a_i = \Lambda^a_{\ b}e^b_i - e^a_i = \epsilon^a_{\ i}, \tag{2.4.10}$$

and the tetrad e^i_a, because of the identity $\delta(e^a_i e^j_a) = 0$, according to

$$\delta e^i_a = -\epsilon^i_{\ a}. \tag{2.4.11}$$

The spin connection changes by

$$\delta\omega^{ab}{}_i = \delta(e^a_j\omega^{jb}{}_i) = \epsilon^a{}_j\omega^{jb}{}_i - e^a_j\epsilon^{jb}{}_{;i} = \epsilon^a{}_c\omega^{cb}{}_i - e^a_j\epsilon^{jb}{}_{|i} + \epsilon^a{}_c\omega^{bc}{}_i = -\epsilon^{ab}{}_{|i}. \tag{2.4.12}$$

Substituting (2.4.11) and (2.4.12) to (2.4.9), together with partial integration (1.2.43), gives

$$\begin{aligned}
&-\int\Big(\mathfrak{T}_i{}^a\epsilon^i{}_a + \frac{1}{2}\mathfrak{S}_{ab}{}^i\epsilon^{ab}{}_{|i}\Big)d\Omega = -\int\Big(\mathfrak{T}_{ij}\epsilon^{ij} + \frac{1}{2}\mathfrak{S}_{ij}{}^k\epsilon^{ij}{}_{|k}\Big)d\Omega \\
&= \int\Big(-\mathfrak{T}_{[ij]} - S_k\mathfrak{S}_{ij}{}^k + \frac{1}{2}\mathfrak{S}_{ij}{}^k{}_{;k}\Big)\epsilon^{ij}d\Omega = 0.
\end{aligned} \tag{2.4.13}$$

Since the infinitesimal Lorentz rotation ϵ^{ij} is arbitrary, we obtain the covariant *conservation law for the spin density* (6 equations):

$$\mathfrak{S}_{ij}{}^k{}_{;k} = \mathfrak{T}_{ij} - \mathfrak{T}_{ji} + 2S_k\mathfrak{S}_{ij}{}^k. \tag{2.4.14}$$

Dividing this law by $\mathfrak{e}$ gives the *conservation law for the spin tensor*:

$$\nabla^*_k s_{ij}{}^k = t_{ij} - t_{ji}. \tag{2.4.15}$$

The conservation law (2.4.15) also results from antisymmetrizing the Belinfante-Rosenfeld relation (2.3.34) with respect to the indices i,k. If we use the metric-compatible affine connection $\Gamma^{\,k}_{i\,j}$, which is invariant under tetrad rotations, instead of the spin connection $\omega^{ab}{}_i$ as a variable in $\mathfrak{L}_{\mathrm{m}}$, then we must replace the term with $\delta\omega^{ab}{}_i$ in (2.4.8) by a term with $\delta(e^i_{a,j})$.

2.4.3 Conservation of metric energy-momentum

The metric and torsion tensors can be taken, instead of the tetrad and spin connection, as the dynamical variables describing spacetime. Under an infinitesimal coordinate transformation (1.2.66), the matter Lagrangian density $\mathfrak{L}_{\mathrm{m}}(\phi,\phi_{,i})$ changes according to

$$\begin{aligned}
\delta\mathfrak{L}_{\mathrm{m}} &= \frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial\phi}\delta\phi + \frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial\phi_{,i}}\delta(\phi_{,i}) + \frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial g^{ik}}\delta g^{ik} + \frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial g^{ik}{}_{,l}}\delta(g^{ik}{}_{,l}) \\
&+ \frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial S^j{}_{ik}}\delta S^j{}_{ik} + \frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial S^j{}_{ik,l}}\delta(S^j{}_{ik,l}).
\end{aligned} \tag{2.4.16}$$

The matter action $S_{\mathrm{m}} = \frac{1}{c}\int\mathfrak{L}_{\mathrm{m}}(\phi,\phi_{,i})d\Omega$ is a scalar, thereby it does not change under this transformation:

$$\begin{aligned}
\delta S_{\mathrm{m}} &= \frac{1}{c}\int\Big(\frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial\phi}\delta\phi + \frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial\phi_{,i}}\delta(\phi_{,i}) + \frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial g^{ik}}\delta g^{ik} + \frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial g^{ik}{}_{,l}}\delta(g^{ik}{}_{,l}) \\
&+ \frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial S^j{}_{ik}}\delta S^j{}_{ik} + \frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial S^j{}_{ik,l}}\delta(S^j{}_{ik,l})\Big)d\Omega = 0.
\end{aligned} \tag{2.4.17}$$

The first two terms in (2.4.17) vanish because of the Lagrange equations for ϕ (2.1.8). If the variations δg^{ik} and $S^j{}_{ik}$ vanish on the boundary of the region of integration, then

$$\begin{aligned}
\delta S_{\mathrm{m}} &= \frac{1}{c}\int\Big(\frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial g^{ik}} - \partial_l\frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial g^{ik}{}_{,l}}\Big)\delta g^{ik}d\Omega + \frac{1}{c}\int\Big(\frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial S^j{}_{ik}} - \partial_l\frac{\partial\mathfrak{L}_{\mathrm{m}}}{\partial S^j{}_{ik,l}}\Big)\delta S^j{}_{ik}d\Omega \\
&= \frac{1}{c}\int\frac{\delta\mathfrak{L}_{\mathrm{m}}}{\delta g^{ik}}\delta g^{ik}d\Omega + \frac{1}{c}\int\frac{\delta\mathfrak{L}_{\mathrm{m}}}{\delta S^j{}_{ik}}\delta S^j{}_{ik}d\Omega = \frac{1}{2c}\int\mathcal{T}_{ik}\delta g^{ik}d\Omega + \frac{1}{2c}\int\tau_j{}^{ik}\delta S^j{}_{ik}d\Omega \\
&= -\frac{1}{2c}\int\mathcal{T}^{ik}\delta g_{ik}d\Omega + \frac{1}{2c}\int\tau_j{}^{ik}\delta S^j{}_{ik}d\Omega = 0.
\end{aligned} \tag{2.4.18}$$

The components of the metric tensor change because of an infinitesimal coordinate transformation (1.2.66), thereby the corresponding variation of the metric tensor is given by (1.4.51):

$$\delta g_{ik} = \bar{\delta} g_{ik} = -\mathcal{L}_\xi g_{ik} = -2\xi_{(i:k)}, \tag{2.4.19}$$

and the variation of the torsion tensor is given by a Lie derivative,

$$\delta S^j_{\ ik} = \bar{\delta} S^j_{\ ik} = -\mathcal{L}_\xi S^j_{\ ik} = \xi^j_{\ ,l} S^l_{\ ik} - \xi^l_{\ ,i} S^j_{\ lk} - \xi^l_{\ ,k} S^j_{\ il} - \xi^l S^j_{\ ik,l}. \tag{2.4.20}$$

The variation of the matter action under (1.2.66) is therefore equal to

$$\delta S_{\mathrm{m}} = \bar{\delta} S_{\mathrm{m}} = -\frac{1}{2c}\int \mathcal{T}^{ik}\bar{\delta} g_{ik} d\Omega + \frac{1}{2c}\int \tau_j^{\ ik}\bar{\delta} S^j_{\ ik} d\Omega = 0. \tag{2.4.21}$$

The first term on the right of (2.4.21) is

$$\begin{aligned} -\frac{1}{2c}\int \mathcal{T}^{ik}\bar{\delta} g_{ik} d\Omega &= \frac{1}{c}\int \mathcal{T}^{ik}\xi_{i:k} d\Omega = \frac{1}{c}\int (\mathcal{T}^{ik}\xi_i)_{:k} d\Omega - \frac{1}{c}\int \mathcal{T}^{ik}_{\ \ :k}\xi_i d\Omega \\ = \frac{1}{c}\int (\mathcal{T}^{ik}\xi_i)_{,k} d\Omega - \frac{1}{c}\int \mathcal{T}^{ik}_{\ \ :k}\xi_i d\Omega &= \frac{1}{c}\int \mathcal{T}^{ik}\xi_i dS_k - \frac{1}{c}\int \mathcal{T}_l^{\ k}_{\ :k}\xi^l d\Omega. \end{aligned} \tag{2.4.22}$$

The second term on the right of (2.4.21) is

$$\begin{aligned} &\frac{1}{2c}\int \tau_j^{\ ik}\bar{\delta} S^j_{\ ik} d\Omega = \frac{1}{2c}\int \big((\tau_j^{\ ik}\xi^j S^l_{\ ik})_{,l} - (\tau_j^{\ ik}\xi^l S^j_{\ lk})_{,i} - (\tau_j^{\ ik}\xi^l S^j_{\ il})_{,k}\big) d\Omega \\ &+\frac{1}{2c}\int \big(-(\tau_j^{\ ik} S^l_{\ ik})_{,l}\xi^j + (\tau_j^{\ ik} S^j_{\ lk})_{,i}\xi^l + (\tau_j^{\ ik} S^j_{\ il})_{,k}\xi^l - \tau_j^{\ ik} S^j_{\ ik,l}\xi^l\big) d\Omega \\ &= \frac{1}{2c}\int \tau_j^{\ ik}\xi^j S^l_{\ ik} dS_l - \frac{1}{2c}\int \tau_j^{\ ik}\xi^l S^j_{\ lk} dS_i - \frac{1}{2c}\int \tau_j^{\ ik}\xi^l S^j_{\ il} dS_k \\ &+\frac{1}{2c}\int \big(-(2\tau_j^{\ ik} S^j_{\ li})_{,k} - (\tau_l^{\ ij} S^k_{\ ij})_{,k} - \tau_j^{\ ik} S^j_{\ ik,l}\big)\xi^l d\Omega. \end{aligned} \tag{2.4.23}$$

If the variations ξ^i of the coordinates vanish on the boundary of the region of integration, then (2.4.21) becomes

$$\delta S_{\mathrm{m}} = -\frac{1}{2c}\int \big(2\mathcal{T}_l^{\ k}_{\ :k} + (2\tau_j^{\ ik} S^j_{\ li} + \tau_l^{\ ij} S^k_{\ ij})_{,k} + \tau_j^{\ ik} S^j_{\ ik,l}\big)\xi^l d\Omega = 0. \tag{2.4.24}$$

Since the variations ξ^i are arbitrary, (2.4.24) gives the covariant *conservation law for the metric energy-momentum density* (4 equations):

$$\mathcal{T}_l^{\ k}_{\ :k} + \Big(\tau_j^{\ ik} S^j_{\ li} + \frac{1}{2}\tau_l^{\ ij} S^k_{\ ij}\Big)_{,k} + \frac{1}{2}\tau_j^{\ ik} S^j_{\ ik,l} = 0. \tag{2.4.25}$$

Dividing the conservation law (2.4.25) by $\sqrt{-\mathfrak{g}}$ gives

$$\begin{aligned} &T_l^{\ k}_{\ :k} + \Big(t_j^{\ ik} S^j_{\ li} + \frac{1}{2}t_l^{\ ij} S^k_{\ ij}\Big)_{,k} + \Big(t_j^{\ ik} S^j_{\ li} + \frac{1}{2}t_l^{\ ij} S^k_{\ ij}\Big)\frac{\sqrt{-\mathfrak{g}}_{,k}}{\sqrt{-\mathfrak{g}}} + \frac{1}{2}t_j^{\ ik} S^j_{\ ik,l} \\ &= T_l^{\ k}_{\ :k} + \Big(t_j^{\ ik} S^j_{\ li} + \frac{1}{2}t_l^{\ ij} S^k_{\ ij}\Big)_{:k} - t_j^{\ im} S^j_{\ li}\{^{\,k}_{m\,k}\} + t_j^{\ ik} S^j_{\ mi}\{^{\,m}_{l\,k}\} + \frac{1}{2}t_m^{\ ij} S^k_{\ ij}\{^{\,m}_{l\,k}\} \\ &-\frac{1}{2}t_l^{\ ij} S^m_{\ ij}\{^{\,k}_{m\,k}\} + \Big(t_j^{\ ik} S^j_{\ li} + \frac{1}{2}t_l^{\ ij} S^k_{\ ij}\Big)\{^{\,m}_{m\,k}\} + \frac{1}{2}t_j^{\ ik} S^j_{\ ik:l} - \frac{1}{2}t_j^{\ ik} S^m_{\ ik}\{^{\,j}_{m\,l}\} \\ &+\frac{1}{2}t_j^{\ ik} S^j_{\ mk}\{^{\,m}_{i\,l}\} + \frac{1}{2}t_j^{\ ik} S^j_{\ im}\{^{\,m}_{k\,l}\} = 0, \end{aligned} \tag{2.4.26}$$

where

$$t_{ijk} = \frac{\tau_{ijk}}{\sqrt{-\mathfrak{g}}}. \tag{2.4.27}$$

We therefore obtain the *conservation law for the metric energy-momentum tensor*:

$$\Big(T_l{}^k + t_j{}^{ik}S^j{}_{li} + \frac{1}{2}t_l{}^{ij}S^k{}_{ij}\Big)_{:k} + \frac{1}{2}t_j{}^{ik}S^j{}_{ik:l} = 0. \tag{2.4.28}$$

If the matter Lagrangian density does not depend on the torsion tensor, then $t_{ijk} = 0$ and (2.4.25) reduces to

$$\mathcal{T}^{ik}{}_{:k} = 0. \tag{2.4.29}$$

Equivalently, (2.4.28) reduces to

$$T^{ik}{}_{:k} = 0. \tag{2.4.30}$$

In this case, vanishing of $\int \mathcal{T}^{ij}\bar{\delta}g_{ij}d\Omega$ in (2.4.21) does not imply $\mathcal{T}^{ij} = 0$, because 10 variations $\bar{\delta}g_{ij}$ are not all independent; they are functions of 4 independent variations ξ^i.

2.4.4 Conservation of tetrad energy-momentum

The matter action S_{m} is invariant under infinitesimal translations of the coordinate system (1.2.66). The corresponding changes of the tetrad and spin connection are given by Lie derivatives,

$$\bar{\delta}e^i_a = -\mathcal{L}_\xi e^i_a = \xi^i{}_{,j}e^j_a - \xi^j e^i_{a,j}, \tag{2.4.31}$$

$$\bar{\delta}\omega^{ab}{}_i = -\mathcal{L}_\xi \omega^{ab}{}_i = -\xi^j{}_{,i}\omega^{ab}{}_j - \xi^j\omega^{ab}{}_{i,j}. \tag{2.4.32}$$

Equation (2.4.9) becomes

$$\int\Big(\mathfrak{T}_i{}^a\bar{\delta}e^i_a + \frac{1}{2}\mathfrak{S}_{ab}{}^i\bar{\delta}\omega^{ab}{}_i\Big)d\Omega = 0. \tag{2.4.33}$$

If the variations ξ^i of the coordinates vanish on the boundary of the region of integration, then substituting (2.4.31) and (2.4.32) into (2.4.33) gives

$$\begin{aligned}&\int\Big(\mathfrak{T}_i{}^a\xi^i{}_{,j}e^j_a - \mathfrak{T}_i{}^a\xi^j e^i_{a,j} - \frac{1}{2}\mathfrak{S}_{ab}{}^i\xi^j{}_{,i}\omega^{ab}{}_j - \frac{1}{2}\mathfrak{S}_{ab}{}^i\xi^j\omega^{ab}{}_{i,j}\Big)d\Omega \\ &= \int\Big(-\mathfrak{T}_i{}^j{}_{,j} - \mathfrak{T}_j{}^a e^j_{a,i} + \frac{1}{2}(\mathfrak{S}_{ab}{}^j\omega^{ab}{}_i)_{,j} - \frac{1}{2}\mathfrak{S}_{ab}{}^j\omega^{ab}{}_{j,i}\Big)\xi^i d\Omega = 0.\end{aligned} \tag{2.4.34}$$

This equation is satisfied for an arbitrary vector ξ^i, thereby we obtain

$$\begin{aligned}&\mathfrak{S}_{ab}{}^j{}_{,j}\omega^{ab}{}_i + \mathfrak{S}_{ab}{}^j(\omega^{ab}{}_{i,j} - \omega^{ab}{}_{j,i}) - 2\mathfrak{T}_i{}^j{}_{,j} - 2\mathfrak{T}_j{}^a e^j_{a,i} \\ &= (\mathfrak{S}_{ab}{}^j{}_{|j} - 2S_k\mathfrak{S}_{ab}{}^k + \mathfrak{S}_{cb}{}^j\omega^c{}_{aj} + \mathfrak{S}_{ac}{}^j\omega^c{}_{bj})\omega^{ab}{}_i - 2\mathfrak{T}_i{}^j{}_{,j} - 2\mathfrak{T}_j{}^a e^j_{a,i} \\ &+\mathfrak{S}_{ab}{}^j(-R^{ab}{}_{ij} + \omega^a{}_{ci}\omega^{cb}{}_j - \omega^a{}_{cj}\omega^{cb}{}_i) = 0,\end{aligned} \tag{2.4.35}$$

which reduces to

$$\begin{aligned}&(\mathfrak{S}_{ab}{}^j{}_{|j} - 2S_k\mathfrak{S}_{ab}{}^k)\omega^{ab}{}_i - R^{ab}{}_{ij}\mathfrak{S}_{ab}{}^j - 2\mathfrak{T}_i{}^j{}_{;j} + 4S_j\mathfrak{T}_i^j - 2\mathfrak{T}_{jk}\omega^{jk}{}_i + 4S^{jk}{}_i\mathfrak{T}_{jk} \\ &= (\mathfrak{S}_{jl}{}^k{}_{;k} - 2S_k\mathfrak{S}_{jl}{}^k)\omega^{jl}{}_i - R^{kl}{}_{ij}\mathfrak{S}_{kl}{}^j - 2\mathfrak{T}_i{}^j{}_{;j} + 4S_j\mathfrak{T}_i^j - 2\mathfrak{T}_{jk}\omega^{jk}{}_i + 4S^{jk}{}_i\mathfrak{T}_{jk} \\ &= 0.\end{aligned} \tag{2.4.36}$$

The conservation law for the spin density (2.4.14) brings (2.4.36) to the covariant *conservation law for the tetrad energy-momentum density*:

$$\mathfrak{T}_i{}^j{}_{;j} = 2S_j\mathfrak{T}_i^j + 2S^j{}_{ki}\mathfrak{T}_j{}^k + \frac{1}{2}\mathfrak{S}_{kl}{}^jR^{kl}{}_{ji}, \tag{2.4.37}$$

which is equivalent to

$$\mathfrak{T}^{ij}{}_{:j} = C_{jk}{}^i\mathfrak{T}^{jk} + \frac{1}{2}\mathfrak{S}_{klj}R^{klji}. \tag{2.4.38}$$

This law can be written as the *conservation law for the tetrad energy-momentum tensor*:

$$t^{ij}{}_{:j} = C_{jk}{}^i t^{jk} + \frac{1}{2}s_{klj}R^{klji}. \tag{2.4.39}$$

Equations (2.4.37) and (2.4.39) are equivalent to (2.4.25) and (2.4.28).

2.4.5 Conservation laws for Lorentz group

Let us consider a physical system in which the gravitational field (torsion and curvature) can be neglected, described by a matter Lagrangian density $\mathfrak{L}_\mathrm{m}$. The Lagrangian density $\mathfrak{L}_\mathrm{m}$ therefore depends on the coordinates only through matter fields ϕ and their first derivatives $\phi_{,i}$. Differentiating $\mathfrak{L}_\mathrm{m}$ gives, using the Lagrange equations (2.1.8),

$$\partial_i\mathfrak{L}_\mathrm{m}=\frac{\partial\mathfrak{L}_\mathrm{m}}{\partial\phi}\phi_{,i}+\frac{\partial\mathfrak{L}_\mathrm{m}}{\partial\phi_{,j}}\phi_{,ji}=\partial_j\biggl(\frac{\partial\mathfrak{L}_\mathrm{m}}{\partial\phi_{,j}}\biggr)\phi_{,i}+\frac{\partial\mathfrak{L}_\mathrm{m}}{\partial\phi_{,j}}\phi_{,ji}=\partial_j\biggl(\frac{\partial\mathfrak{L}_\mathrm{m}}{\partial\phi_{,j}}\phi_{,i}\biggr). \tag{2.4.40}$$

This equation can be written as a conservation law:

$$\theta_i{}^j{}_{,j}=0, \tag{2.4.41}$$

for a quantity

$$\theta_i{}^j=\frac{\partial\mathfrak{L}_\mathrm{m}}{\partial\phi_{,j}}\phi_{,i}-\delta^j_i\mathfrak{L}_\mathrm{m}. \tag{2.4.42}$$

The conservation law (2.4.41) is a special case of (2.4.37) in the absence of torsion and curvature, expressed in the Galilean and geodesic frame. The quantity (2.4.42) is a special case of the canonical energy-momentum density (2.3.13) in the absence of torsion, expressed in the Galilean and geodesic frame. The Noether current (2.4.7) can be written as

$$\mathfrak{J}^i=\frac{\partial\mathfrak{L}_\mathrm{m}}{\partial\phi_{,i}}\delta\phi-\theta_j{}^i\xi^j. \tag{2.4.43}$$

If x^i are Cartesian coordinates then for translations, $\xi^i=\epsilon^i=\mathrm{const}$ and $\delta\phi=0$, the current (2.4.7) is

$$\mathfrak{J}^i=\epsilon^i\mathfrak{L}_\mathrm{m}-\frac{\partial\mathfrak{L}_\mathrm{m}}{\partial\phi_{,i}}\epsilon^j\phi_{,j}. \tag{2.4.44}$$

The conservation law (2.4.6) for this current is

$$\epsilon^j\theta_j{}^i{}_{,i}=0, \tag{2.4.45}$$

which gives (2.4.41) because ϵ^i are arbitrary. For Lorentz rotations, $\xi^i=\epsilon^i{}_jx^j$ and $\delta\phi=\frac{1}{2}\epsilon_{ij}G^{ij}\phi$, where G^{ij} are the generators of the Lorentz group, the Noether current (2.4.7) is

$$\mathfrak{J}^i=\epsilon^{ij}x_j\mathfrak{L}_\mathrm{m}+\frac{\partial\mathfrak{L}_\mathrm{m}}{\partial\phi_{,i}}\biggl(\frac{1}{2}\epsilon^{kl}G_{kl}\phi-\epsilon^{jk}x_k\phi_{,j}\biggr)=\epsilon^{kl}\biggl(x_k\frac{\partial\mathfrak{L}_\mathrm{m}}{\partial\phi_{,i}}\phi_{,l}-x_k\delta^i_l\mathfrak{L}_\mathrm{m}+\frac{1}{2}\frac{\partial\mathfrak{L}_\mathrm{m}}{\partial\phi_{,i}}G_{kl}\phi\biggr). \tag{2.4.46}$$

The conservation law (2.4.6) for this current is

$$\epsilon^{kl}\biggl(\frac{\partial\mathfrak{L}_\mathrm{m}}{\partial\phi_{,i}}\phi_{,[l}x_{k]}-\delta^i_{[l}x_{k]}\mathfrak{L}_\mathrm{m}+\frac{1}{2}\frac{\partial\mathfrak{L}_\mathrm{m}}{\partial\phi_{,i}}G_{kl}\phi\biggr)_{,i}=0. \tag{2.4.47}$$

Because ϵ^{kl} are arbitrary, this equation gives the conservation law,

$$\mathfrak{M}_{kl}{}^i{}_{,i}=0, \tag{2.4.48}$$

for the *angular momentum density*:

$$\mathfrak{M}_{kl}{}^i=x_k\theta_l{}^i-x_l\theta_k{}^i+\frac{\partial\mathfrak{L}_\mathrm{m}}{\partial\phi_{,i}}G_{kl}\phi=x_k\theta_l{}^i-x_l\theta_k{}^i+\Sigma_{kl}{}^i. \tag{2.4.49}$$

The angular momentum density is antisymmetric in the first two indices:

$$\mathfrak{M}_{ijk}=-\mathfrak{M}_{jik}. \tag{2.4.50}$$

This quantity is the sum,

$$\mathfrak{M}_{ij}{}^{k} = \Lambda_{ij}{}^{k} + \Sigma_{ij}{}^{k}, \tag{2.4.51}$$

of two tensor densities: the *orbital angular momentum density*,

$$\Lambda_{kl}{}^{i} = x_k\theta_l{}^{i} - x_l\theta_k{}^{i}, \tag{2.4.52}$$

and the *intrinsic angular momentum density* (canonical spin density) (2.3.25).

The conservation law (2.4.48) gives

$$\mathfrak{M}^{kli}{}_{,i} = \delta^k_i\theta^{li} + x^k\theta^{li}{}_{,i} - \delta^l_i\theta^{ki} - x^l\theta^{ki}{}_{,i} + \Sigma^{kli}{}_{,i} = 0, \tag{2.4.53}$$

which reduces, by means of (2.4.41), to

$$\theta_{kl} - \theta_{lk} - \Sigma_{kl}{}^{i}{}_{,i} = 0. \tag{2.4.54}$$

This equation is a special case of the conservation law for the spin density (2.4.14) in the absence of torsion, expressed in the Galilean and geodesic frame. The canonical energy-momentum density θ_{ik} is not symmetric. However, the quantity

$$\tau_{ik} = \theta_{ik} + \partial_j\psi_{ik}{}^{j}, \tag{2.4.55}$$

where

$$\psi_{ik}{}^{j} = -\frac{1}{2}(\Sigma_{ik}{}^{j} - \Sigma_k{}^j{}_i + \Sigma^j{}_{ik}), \tag{2.4.56}$$

is symmetric:

$$\tau_{ik} - \tau_{ki} = \theta_{ik} - \theta_{ki} + \partial_j(\psi_{ik}{}^{j} - \psi_{ki}{}^{j}) = \Sigma_{ik}{}^{j}{}_{,j} + \partial_j(\psi_{ik}{}^{j} - \psi_{ki}{}^{j}) = 0. \tag{2.4.57}$$

Since (2.4.56) is antisymmetric in the last two indices,

$$\psi^{ikj} = -\psi^{ijk}, \tag{2.4.58}$$

the quantity (2.4.55) is also conserved:

$$\tau^{ik}{}_{,k} = \theta^{ik}{}_{,k} + \psi^{ikj}{}_{,jk} = \theta^{ik}{}_{,k} = 0. \tag{2.4.59}$$

The symmetric energy-momentum density τ_{ik} is equal to the metric energy-momentum density (2.3.4), expressed in the Galilean and geodesic frame. Equation (2.4.55) is a special case of the Belinfante-Rosenfeld relation (2.3.34) in the absence of torsion, expressed in the Galilean and geodesic frame.

2.4.6 Momentum four-vector

Integrating the conservation law (2.4.41) for the canonical energy-momentum density (2.4.42), which is satisfied if we neglect torsion and curvature, over a four-volume and using the Gauß-Stokes theorem (1.1.39) gives

$$\int \theta^{ik}{}_{,k}d\Omega = \oint \theta^{ik}dS_k = 0, \tag{2.4.60}$$

where the integral on the right is taken over the closed hypersurface surrounding the four-volume. If the hypersurface represented by the element dS_k is taken as a hyperplane perpendicular to the x^0 axis (volume hypersurface), $dS_k = \delta^0_k dV$, then the closed hypersurface surrounds the four-volume between two hyperplanes at times t_1 and t_2:

$$\oint \theta^{ik}dS_k = \int \theta^{ik}dS_k\Big|_{t_1}^{t_2} = \int \theta^{i0}dV\Big|_{t_1}^{t_2} = 0. \tag{2.4.61}$$

We define the *momentum four-vector* or *four-momentum* of matter in the four-volume as

$$P^i = \frac{1}{c}\int \mathfrak{T}^{ik}dS_k = \frac{1}{c}\int \mathfrak{T}^{i0}dV = \frac{1}{c}\int \Theta^{ik}dS_k = \frac{1}{c}\int \Theta^{i0}dV, \tag{2.4.62}$$

where we used (2.3.14). In the locally Galilean and geodesic frame of reference, the four-momentum (2.4.62) reduces to

$$P^i = \frac{1}{c}\int \theta^{ik} dS_k = \frac{1}{c}\int \theta^{i0} dV, \tag{2.4.63}$$

which is locally conserved because of (2.4.61):

$$P^i|_{t_1} = P^i|_{t_2}, \ \ P^i = \mathrm{const}. \tag{2.4.64}$$

The components $\frac{1}{c}\theta^{i0}$ form the *four-momentum density*. The component θ^{00} is referred to as the *energy density*,

$$W = \theta^{00} = \dot{\phi}\frac{\partial \mathfrak{L}_{\mathrm{m}}}{\partial \dot{\phi}} - \mathfrak{L}_{\mathrm{m}}. \tag{2.4.65}$$

Integrating it over the volume gives the time component P^0 of the four-momentum,

$$cP^0 = \int \theta^{00} dV = \dot{\phi}\frac{\partial L}{\partial \dot{\phi}} - L, \tag{2.4.66}$$

where

$$L = \int \mathfrak{L}_{\mathrm{m}} dV \tag{2.4.67}$$

is the Lagrange function or *Lagrangian*. The covariant time component of cP^i is referred to as the *energy*,

$$E = cP_0. \tag{2.4.68}$$

Hereinafter, a dot above any quantity ϕ denotes the derivative of ϕ with respect to time, $\dot{\phi} = \frac{d\phi}{dt}$, and two dots above ϕ denote second derivative of ϕ with respect to time, $\ddot{\phi} = \frac{d^2\phi}{dt^2}$. Consequently, the action of a physical system is equal to the time integral of the Lagrangian,

$$S = \int L dt. \tag{2.4.69}$$

The components

$$\Pi^\alpha = \frac{1}{c}\theta^{\alpha 0} \tag{2.4.70}$$

form the *momentum density* $\mathbf{\Pi}$. Integrating them over the volume gives the spatial components P^α of the four-momentum,

$$P^\alpha = \frac{1}{c}\int \theta^{\alpha 0} dV, \tag{2.4.71}$$

which form the *momentum* vector $\mathbf{P}$:

$$P^i = (P^0, P^\alpha) = (P^0, \mathbf{P}). \tag{2.4.72}$$

The conservation law (2.4.41) can be written as

$$\frac{1}{c}\frac{\partial \theta^{00}}{\partial t} + \frac{\partial \theta^{0\alpha}}{\partial x^\alpha} = 0, \tag{2.4.73}$$

$$\frac{1}{c}\frac{\partial \theta^{\alpha 0}}{\partial t} + \frac{\partial \theta^{\alpha\beta}}{\partial x^\beta} = 0. \tag{2.4.74}$$

Integrating these equations over a volume and using the Gauß' theorem (1.4.172) gives

$$\frac{\partial}{\partial t}\int \theta^{00} dV = -c\oint \theta^{0\alpha} df_\alpha, \tag{2.4.75}$$

$$\frac{\partial}{\partial t}\int \frac{1}{c}\theta^{\alpha 0} dV = -\oint \theta^{\alpha\beta} df_\beta. \tag{2.4.76}$$

The integral of a three-dimensional vector V^α over a two-dimensional surface represented by the element df_α, $\oint V^\alpha df_\alpha$, is referred to as the *flux* of this vector. Integrating the components of the *energy current* $\mathbf{S}$,

$$S^\alpha = c\theta^{0\alpha}, \tag{2.4.77}$$

over df_α gives the *energy flux* or *power*:

$$P = \oint S^\alpha df_\alpha = -\frac{dE}{dt}. \tag{2.4.78}$$

The last equality results from (2.4.66), (2.4.68) and (2.4.75). Integrating the components $\theta^{\alpha\beta}$, which represent the *momentum current*, over df_α gives the *momentum flux*. The *stress tensor* is defined as

$$\sigma_{\alpha\beta} = -\theta_{\alpha\beta}. \tag{2.4.79}$$

Its integral taken over df_α gives the vector opposite to the momentum flux, called the *surface force* $\mathbf{F}_s$:

$$F_s^\alpha = \oint \sigma^{\alpha\beta} df_\beta. \tag{2.4.80}$$

The relations (2.4.71), (2.4.76), (2.4.79) and (2.4.80) equal the time derivative of the momentum P^α to the surface force F_s^α:

$$\dot{P}^\alpha = F_s^\alpha. \tag{2.4.81}$$

The components of the energy-momentum tensor form the following matrix:

$$\theta^{ik} = \begin{pmatrix} W & \frac{\mathbf{S}}{c} \\ c\mathbf{\Pi} & -\sigma_{\alpha\beta} \end{pmatrix}. \tag{2.4.82}$$

Adding $\psi^{ikj}{}_{,j}$, where (2.4.58) is satisfied, to θ^{ik} preserves the conservation law (2.4.41) and brings θ^{ik} to a symmetric form, τ^{ik}. Using the Gauß-Stokes theorem (1.1.38) gives

$$\int \tau^{ik} dS_k = \int \theta^{ik} dS_k + \int \psi^{ikj}{}_{,j} dS_k = cP^i + \frac{1}{2}\int (\psi^{ikj}{}_{,j} dS_k - \psi^{ikj}{}_{,k} dS_j) = cP^i + \frac{1}{2}\oint \psi^{ikj} df^*_{kj}, \tag{2.4.83}$$

where the last integral is taken over the surface which bounds the hypersurface. If this surface is located in a region, where matter is absent, then the surface integral vanishes. Consequently, replacing θ^{ik} by τ^{ik} does not change the four-momentum (2.4.63):

$$P^i = \frac{1}{c}\int \tau^{ik} dS_k = \frac{1}{c}\int \tau^{i0} dV. \tag{2.4.84}$$

2.4.7 Mass

We define the *mass* of matter in a four-volume as

$$m = \frac{(P^i P_i)^{1/2}}{c}, \tag{2.4.85}$$

where P^i is the four-momentum of the matter (2.4.62). This definition is analogous to the mass operator (1.6.81). If the matter satisfies the dominant energy condition (confer (2.5.10)), then $P^i P_i \geq 0$ and the mass is a real quantity. If $P^i P_i > 0$, then $m > 0$. If $P^i P_i = 0$, then $m = 0$.

2.4.8 Angular momentum four-tensor

Integrating the conservation law (2.4.48) for the angular momentum density (2.4.49), which is satisfied if we neglect torsion and curvature, over a four-volume and using the Gauß-Stokes theorem (1.1.39) gives

$$\int \mathfrak{M}^{ikj}{}_{,j} d\Omega = \oint \mathfrak{M}^{ikj} dS_j = 0, \tag{2.4.86}$$

where the integral on the right is taken over the closed hypersurface surrounding the four-volume. If the hypersurface represented by the element dS_j is taken as a volume hyperplane, then the closed hypersurface surrounds the four-volume between two hyperplanes at times t_1 and t_2:

$$\oint \mathfrak{M}^{ikj} dS_j = \int \mathfrak{M}^{ikj} dS_j \Big|_{t_1}^{t_2} = \int \mathfrak{M}^{ik0} dV \Big|_{t_1}^{t_2} = 0. \tag{2.4.87}$$

The *angular momentum four-tensor* of matter in the four-volume,

$$M^{ik} = \frac{1}{c} \int \mathfrak{M}^{ikj} dS_j = \frac{1}{c} \int \mathfrak{M}^{ik0} dV, \tag{2.4.88}$$

is therefore locally conserved:

$$M^{ik}|_{t_1} = M^{ik}|_{t_2}, \;\; M^{ik} = \text{const}. \tag{2.4.89}$$

The angular momentum four-tensor, following (2.4.50), is antisymmetric:

$$M^{ik} = -M^{ki}. \tag{2.4.90}$$

The components $\frac{1}{c}\mathfrak{M}^{\alpha\beta 0}$ form the spatial *angular momentum density*. Integrating them over the volume gives the components of the spatial *angular momentum tensor*,

$$M^{\alpha\beta} = \frac{1}{c} \int \mathfrak{M}^{\alpha\beta 0} dV. \tag{2.4.91}$$

Since the angular momentum tensor is antisymmetric,

$$M^{\alpha\beta} = -M^{\beta\alpha}, \tag{2.4.92}$$

we can define the *angular momentum pseudovector* $\mathbf{M}$:

$$M^{\alpha} = \frac{1}{2} e^{\alpha\beta\gamma} M_{\beta\gamma}, \;\; M_{\alpha\beta} = e_{\alpha\beta\gamma} M^{\gamma}. \tag{2.4.93}$$

Integrating (2.4.51) over a hypersurface gives

$$M^{ik} = L^{ik} + S^{ik}, \tag{2.4.94}$$

where

$$L^{ik} = \frac{1}{c} \int \Lambda^{ikj} dS_j = \frac{1}{c} \int (x^i \theta^{kj} - x^k \theta^{ij}) dS_j = \frac{1}{c} \int \Lambda^{ik0} dV = \frac{1}{c} \int (x^i \theta^{k0} - x^k \theta^{i0}) dV \tag{2.4.95}$$

is the *orbital angular momentum four-tensor* and

$$S^{ik} = \frac{1}{c} \int \Sigma^{ikj} dS_j = \frac{1}{c} \int \Sigma^{ik0} dV \tag{2.4.96}$$

is the *intrinsic angular momentum four-tensor*. Unlike M^{ik}, these tensors are not separately conserved. These tensors are also antisymmetric:

$$L^{ik} = -L^{ki}, \;\; S^{ik} = -S^{ki}. \tag{2.4.97}$$

Similarly to (2.4.93), we can define the *orbital angular momentum pseudovector* $\mathbf{L}$ and the *intrinsic angular momentum pseudovector* $\mathbf{S}$:

$$L^{\alpha} = \frac{1}{2} e^{\alpha\beta\gamma} L_{\beta\gamma}, \;\; L_{\alpha\beta} = e_{\alpha\beta\gamma} L^{\gamma}, \tag{2.4.98}$$

$$S^{\alpha} = \frac{1}{2} e^{\alpha\beta\gamma} S_{\beta\gamma}, \;\; S_{\alpha\beta} = e_{\alpha\beta\gamma} S^{\gamma}, \tag{2.4.99}$$

$$M^{\alpha} = L^{\alpha} + S^{\alpha}, \;\; \mathbf{M} = \mathbf{L} + \mathbf{S}. \tag{2.4.100}$$

The symmetry of τ^{ik} can be written, using (2.4.59), as

$$\tau^{ki} - \tau^{ik} = \partial_l(x^i\tau^{kl} - x^k\tau^{il}) = 0. \tag{2.4.101}$$

Integrating this equation over a four-volume and using the Gauß-Stokes theorem (1.1.39) gives

$$\oint(x^i\tau^{kl} - x^k\tau^{il})dS_l = \int(x^i\tau^{kl} - x^k\tau^{il})dS_l\Big|_{t_1}^{t_2} = \int(x^i\tau^{k0} - x^k\tau^{i0})dV\Big|_{t_1}^{t_2} = 0, \tag{2.4.102}$$

which shows the conservation of the quantity

$$\tilde{L}^{ik} = \frac{1}{c}\int(x^i\tau^{kl} - x^k\tau^{il})dS_l = \frac{1}{c}\int(x^i\tau^{k0} - x^k\tau^{i0})dV = \text{const.} \tag{2.4.103}$$

The symmetry of a second-rank tensor whose ordinary divergence is zero is therefore related to the local conservation of the orbital angular momentum four-tensor constructed from that tensor. Using (2.3.25), (2.4.55), and the Gauß-Stokes theorem (1.1.38) leads to

$$\begin{aligned} c\tilde{L}^{ik} &= \int\Lambda^{ikl}dS_l + \int(x^i\psi^{klj}{}_{,j} - x^k\tau^{ilj}{}_{,j})dS_l = cL^{ik} + \int\Big((x^i\psi^{klj})_{,j} - (x^k\psi^{ilj})_{,j}\Big)dS_l \\ &- \int(\psi^{kli} - \psi^{ilk})dS_l = cL^{ik} + \frac{1}{2}\Big((x^i\psi^{klj})_{,j}dS_l - (x^i\psi^{klj})_{,l}dS_j - (x^k\psi^{ilj})_{,j}dS_l + (x^k\psi^{ilj})_{,l}dS_j\Big) \\ &+ \int\Sigma^{ikj}dS_j = cM^{ik} + \frac{1}{2}\oint(x^i\psi^{klj} - x^k\psi^{ilj})df^*_{lj}. \end{aligned} \tag{2.4.104}$$

If the integration surface is located in a region, where matter is absent, then the surface integral vanishes. Consequently, the quantity (2.4.103) is equal to the angular momentum four-tensor (2.4.88) if we neglect torsion and curvature. This equality shows that replacing θ^{ik} in (2.4.52) by τ^{ik} changes the values of (2.4.95) and thereby (2.4.88). The angular momentum four-tensor can also be written, using (2.4.63), in terms of P^i:

$$M^{ik} = \int(x^i dP^k - x^k dP^i) + S^{ik}. \tag{2.4.105}$$

In the absence of the intrinsic angular momentum, (2.4.105) reduces to

$$M^{ik} = \int(x^i dP^k - x^k dP^i). \tag{2.4.106}$$

The conservation of $M^{0\alpha}$,

$$M^{\alpha 0} = \frac{1}{c}\Big(\int x^\alpha\theta^{00}dV - x^0\int\theta^{\alpha 0}dV\Big) + S^{\alpha 0} = \frac{1}{c}\int x^\alpha\theta^{00}dV - ctP^\alpha + S^{\alpha 0} = \text{const}, \tag{2.4.107}$$

divided by the conserved P^0 gives

$$X^\alpha = V^\alpha t + \frac{S^{\alpha 0}}{P^0} + \text{const}, \tag{2.4.108}$$

where

$$V^\alpha = \frac{cP^\alpha}{P^0} \tag{2.4.109}$$

and

$$X^\alpha = \frac{\int x^\alpha\theta^{00}dV}{\int\theta^{00}dV}. \tag{2.4.110}$$

If the intrinsic angular momentum is constant, then the relation (2.4.108) describes a uniform motion of the *center of inertia*, whose coordinates are X^α, with velocity V^α. The coordinates of the center of inertia (2.4.110) are not the spatial components of a four-dimensional vector.

2.4.9 Energy-momentum tensor for particles

Let us consider matter which is distributed over a small region in space and consists of points with the coordinates x^i, forming an extended body whose motion is represented by a world tube in spacetime. The motion of the body as a whole is represented by an arbitrary timelike world line γ inside the world tube, which consists of points with the coordinates $X^i(\tau)$, where τ is the proper time on γ. We define

$$\delta x^i = x^i - X^i, \tag{2.4.111}$$

$$u^i = \frac{dX^i}{ds}, \tag{2.4.112}$$

where $ds^2 = g_{ij}dX^i dX^j$. The conservation law for the tetrad energy-momentum density (2.4.39) is

$$\mathfrak{T}^{ji}{}_{,i} + \{{}^{\,j}_{i\,k}\}\mathfrak{T}^{ik} - C_{ik}{}^{j}\mathfrak{T}^{ik} - \frac{1}{2}R_{ikl}{}^{j}\mathfrak{S}^{ikl} = 0. \tag{2.4.113}$$

Integrating (2.4.113) over the volume of the body at a constant time X^0 and using Gauß' theorem to eliminate surface integrals gives

$$\int \mathfrak{T}^{j0}{}_{,0}dV + \int \{{}^{\,j}_{i\,k}\}\mathfrak{T}^{ik}dV - \int C_{ik}{}^{j}\mathfrak{T}^{ik}dV - \frac{1}{2}\int R_{ikl}{}^{j}\mathfrak{S}^{ikl}dV = 0, \tag{2.4.114}$$

where dV is a volume element.

If a body is not spatially extended then it is referred to as a *particle*. In this case, the quantity (2.4.111) satisfies

$$\delta x^i = 0. \tag{2.4.115}$$

The affine connection $\Gamma^{\,i}_{j\,k}$, and consequently $\{{}^{\,i}_{j\,k}\}$, $C^i{}_{jk}$, and the curvature tensor in the integrands in (2.4.114), are therefore equal to their respective values at the point X^i. Consequently, we obtain

$$\int \mathfrak{T}^{j0}{}_{,0}dV + \{{}^{\,j}_{i\,k}\}\int \mathfrak{T}^{ik}dV - C_{ik}{}^{j}\int \mathfrak{T}^{ik}dV - \frac{1}{2}R_{ikl}{}^{j}\int \mathfrak{S}^{ikl}dV = 0. \tag{2.4.116}$$

We define the following integrals:

$$M^{ik} = u^0\int \mathfrak{T}^{ik}dV, \tag{2.4.117}$$

$$N^{ijk} = u^0\int \mathfrak{S}^{ijk}dV. \tag{2.4.118}$$

Since the integration domain is not spatially extended, these quantities are tensors, and can be represented as covariant hypersurface integrals:

$$M^{ik} = u^l\int \mathfrak{T}^{ik}dS_l, \tag{2.4.119}$$

$$N^{ijk} = u^l\int \mathfrak{S}^{ijk}dS_l. \tag{2.4.120}$$

Using these integrals and

$$\int \mathfrak{T}^{j0}{}_{,0}dV = \left(\int \mathfrak{T}^{j0}dV\right)_{,0} = \frac{1}{u^0}\frac{d}{ds}\int \mathfrak{T}^{j0}dV \tag{2.4.121}$$

turns (2.4.116) into

$$\frac{d}{ds}\left(\frac{M^{j0}}{u^0}\right) + \{{}^{\,j}_{i\,k}\}M^{(ik)} - C_{ik}{}^{j}M^{[ik]} - \frac{1}{2}R_{ikl}{}^{j}N^{ikl} = 0. \tag{2.4.122}$$

The conservation law (2.4.113) gives

$$(x^l\mathfrak{T}^{ji})_{,i} = \mathfrak{T}^{jl} - x^l\{^{\,j}_{i\,k}\}\mathfrak{T}^{ik} + x^l C_{ik}^{\ \ j}\mathfrak{T}^{ik} + \frac{1}{2}x^l R_{ikm}^{\ \ \ \ j}\mathfrak{S}^{ikm}, \tag{2.4.123}$$

$$(x^l x^m\mathfrak{T}^{ji})_{,i} = x^m\mathfrak{T}^{jl} + x^l\mathfrak{T}^{jm} - x^l x^m\{^{\,j}_{i\,k}\}\mathfrak{T}^{ik} + x^l x^m C_{ik}^{\ \ j}\mathfrak{T}^{ik}$$
$$+\frac{1}{2}x^l x^m R_{ikn}^{\ \ \ \ j}\mathfrak{S}^{ikn}. \tag{2.4.124}$$

Integrating (2.4.123) over the volume of the body and using Gauß' theorem to eliminate surface integrals gives

$$\int (x^l\mathfrak{T}^{j0})_{,0}dV = \int \mathfrak{T}^{jl}dV - \int x^l\{^{\,j}_{i\,k}\}\mathfrak{T}^{ik}dV + \int x^l C_{ik}^{\ \ j}\mathfrak{T}^{ik}dV + \frac{1}{2}\int x^l R_{ikm}^{\ \ \ \ j}\mathfrak{S}^{ikm}dV. \tag{2.4.125}$$

In this relation, we use $x^i = X^i$, which follows from (2.4.111) and (2.4.115). Substituting (2.4.112) into (2.4.125) and using $X^l_{\ ,0} = u^l/u^0$ gives

$$\frac{u^l}{u^0}\int \mathfrak{T}^{j0}dV + X^l\int \mathfrak{T}^{j0}_{\ \ ,0}dV = \int \mathfrak{T}^{jl}dV - X^l\int \{^{\,j}_{i\,k}\}\mathfrak{T}^{ik}dV + X^l\int C_{ik}^{\ \ j}\mathfrak{T}^{ik}dV$$
$$+\frac{1}{2}X^l\int R_{ikm}^{\ \ \ \ j}\mathfrak{S}^{ikm}dV. \tag{2.4.126}$$

This equation reduces, by means of (2.4.114), to

$$\frac{u^l}{u^0}\int \mathfrak{T}^{j0}dV = \int \mathfrak{T}^{jl}dV. \tag{2.4.127}$$

Using the definition (2.4.117) brings (2.4.127) to

$$\frac{u^l}{u^0}M^{j0} = M^{jl}. \tag{2.4.128}$$

Putting $l = 0$ in (2.4.128) gives the identity. Integrating (2.4.124) over the volume of the body does not introduce new relations. The expressions analogous to (2.4.123) and (2.4.124) with higher multiples of x^i do not introduce new relations as well.

If the spin density vanishes, $\mathfrak{S}^{ijk} = 0$, the particle is *spinless*. The conservation law for the spin density (2.4.14) gives in this case the symmetry of the energy-momentum density, $\mathfrak{T}^{ik} = \mathfrak{T}^{ki}$. The tensor density $\mathfrak{T}^{ij}$ is also equal to the metric energy-momentum density $\mathcal{T}^{ij}$ according to (2.3.35). The quantity (2.4.117) is then symmetric:

$$M^{ik} = M^{ki}. \tag{2.4.129}$$

Putting $j = 0$ in (2.4.128) gives

$$M^{0l} = \frac{u^l}{u^0}M^{00}. \tag{2.4.130}$$

The relation (2.4.128) leads then to

$$M^{jl} = \frac{u^l}{u^0}M^{0j} = \frac{u^j u^l}{(u^0)^2}M^{00}. \tag{2.4.131}$$

The quantity (2.4.117) for a spinless particle is proportional to the product of the components of the four-velocity.

Equations (2.4.62) and (2.4.117) give the four-momentum of a spinless particle:

$$P^i = \frac{1}{c}\int \mathfrak{T}^{i0}dV = \frac{M^{i0}}{cu^0} = \frac{M^{0i}}{cu^0} = \frac{u^i}{c(u^0)^2}M^{00}. \tag{2.4.132}$$

The mass (2.4.85) of the particle is therefore given by

$$m^2 = \frac{P^i P_i}{c^2} = \frac{u^i u_i}{(cu^0)^4}(M^{00})^2 = \left(\frac{M^{00}}{(cu^0)^2}\right)^2, \tag{2.4.133}$$

leading to

$$m = \frac{M^{00}}{(cu^0)^2}. \tag{2.4.134}$$

Consequently, the four-momentum is given by

$$P^i = mcu^i, \tag{2.4.135}$$

and the mass satisfies

$$m = \frac{P^i u_i}{c}. \tag{2.4.136}$$

The four-momentum of a spinless particle is proportional to its four-velocity. The quantity (2.4.131) simplifies to

$$M^{ik} = mc^2 u^i u^k. \tag{2.4.137}$$

This relation gives

$$\int \mathfrak{T}^{ik} dV = mc^2 \frac{u^i u^k}{u^0} \tag{2.4.138}$$

or

$$\mathfrak{T}^{ik}(\mathbf{x}) = mc^2 \boldsymbol{\delta}(\mathbf{x} - \mathbf{x}_0)\frac{u^i u^k}{u^0}, \tag{2.4.139}$$

where $\boldsymbol{\delta}(\mathbf{x} - \mathbf{x}_0)$ is the spatial Dirac delta representing a point mass located at $\mathbf{x}_0$. Contracting (2.4.137) with u^i gives the relations for the four-momentum and mass:

$$P^i = \frac{M^{ik} u_k}{c}, \tag{2.4.140}$$

$$m = \frac{M^{ik} u_i u_k}{c^2}. \tag{2.4.141}$$

Since M^{ik} for a particle is a tensor, P^i is a four-vector and m is a scalar.

In a locally inertial frame of reference, (1.6.120) and (2.4.135) give

$$P^i = (mc\gamma, m\gamma\mathbf{v}) = \left(\frac{E}{c}, \mathbf{p}\right), \tag{2.4.142}$$

so the energy and momentum of the particle are

$$E = mc^2\gamma, \tag{2.4.143}$$

$$\mathbf{P} = m\gamma\mathbf{v}. \tag{2.4.144}$$

Accordingly, (2.4.133) gives

$$E^2 = (\mathbf{P}c)^2 + (mc^2)^2. \tag{2.4.145}$$

In the rest frame of the particle, $\mathbf{P} = 0$, (2.4.145) reduces to Einstein's formula for the *rest energy*,

$$E = mc^2. \tag{2.4.146}$$

The formulae (2.4.143) and (2.4.144) give

$$\mathbf{v} = \frac{\mathbf{P}c^2}{E}. \tag{2.4.147}$$

Taking the differential of (2.4.145) gives $EdE = c^2\mathbf{P} \cdot d\mathbf{P}$, from which we obtain, using (2.4.147),

$$dE = \mathbf{v} \cdot d\mathbf{P}. \tag{2.4.148}$$

If a particle is massless, $m = 0$, then (2.4.145) and (2.4.147) give

$$E = Pc, \ \ v = c. \tag{2.4.149}$$

We define the *mass density* μ such that

$$\mu\sqrt{\mathfrak{s}}dV = dm, \tag{2.4.150}$$

where $\mathfrak{s}$ is given by (1.4.147). The mass density for a particle located at $\mathbf{x}_a$ is

$$\mu(\mathbf{x}) = \frac{m}{\sqrt{\mathfrak{s}}}\boldsymbol{\delta}(\mathbf{x} - \mathbf{x}_a), \tag{2.4.151}$$

so (2.4.139) turns into

$$\mathfrak{T}^{ik} = \mu c^2\sqrt{\mathfrak{s}}\frac{u^i u^k}{u^0}. \tag{2.4.152}$$

Therefore, the energy-momentum tensor for a spinless particle is given by

$$\begin{aligned} T^{ik}(x) &= \mu(\mathbf{x})c^2\frac{u^i u^k}{\sqrt{g_{00}}u^0} = \frac{\mu(\mathbf{x})c}{\sqrt{g_{00}}}\frac{dx^i}{ds}\frac{dx^k}{dt} = mc^2\boldsymbol{\delta}(\mathbf{x} - \mathbf{x}_a)\frac{u^i u^k}{\sqrt{-\mathfrak{g}}u^0} \\ &= mc^2\int\frac{u^i u^k}{\sqrt{-\mathfrak{g}}}\delta(x - x_a(\tau))d\tau, \end{aligned} \tag{2.4.153}$$

where $x_a(\tau)$ is the particle's wordline as a function of its proper time τ. For a system of particles, we have

$$T^{ik}(\mathbf{x}) = \sum_a m_a c^2\boldsymbol{\delta}(\mathbf{x} - \mathbf{x}_a)\frac{u^i u^k}{\sqrt{-\mathfrak{g}}u^0}. \tag{2.4.154}$$

In the absence of torsion and in the locally Galilean frame of reference, the conservation law for the energy-momentum tensor is given by (2.4.41), thereby

$$T_{\alpha\ ,i}^{\ i} = 0. \tag{2.4.155}$$

Let us consider a closed system of particles which carry out a finite motion, in which all quantities vary over finite ranges. We define the average over a certain time interval τ of a function f of these quantities as $\bar{f} = \frac{1}{\tau}\int_0^\tau f dt$. The average of the derivative of a bounded quantity $\bar{\dot{f}} = \frac{1}{\tau}\big(f(\tau) - f(0)\big) \to 0$ as $\tau \to \infty$. Therefore, averaging (2.4.155) over the time gives

$$\bar{T}_{\alpha\ ,\beta}^{\ \beta} = 0. \tag{2.4.156}$$

Multiplying (2.4.156) by x^α and integrating over the volume gives, omitting surface integrals,

$$\int x^\alpha\bar{T}_{\alpha\ ,\beta}^{\ \beta}dV = -\int\bar{T}_\alpha^{\ \alpha}dV = 0. \tag{2.4.157}$$

The average energy of the system (2.4.66) is thus

$$\bar{E} = \int\bar{T}_0^{\ 0}dV = \int\bar{T}_i^{\ i}dV. \tag{2.4.158}$$

Substituting (1.6.120) into (2.4.154) gives

$$T_i^{\ i}(\mathbf{x}) = \sum_a m_a c^2\boldsymbol{\delta}(\mathbf{x} - \mathbf{x}_a)\Big(1 - \frac{v^2}{c^2}\Big)^{1/2}, \tag{2.4.159}$$

so $T_i^{\ i} \geq 0$. Putting (2.4.159) into (2.4.158) gives

$$\bar{E} = \sum_a m_a c^2\overline{\Big(1 - \frac{v^2}{c^2}\Big)^{1/2}}, \tag{2.4.160}$$

which is referred to as the *virial theorem*.

The equation of motion for a spinless particle follows from (2.4.122), which reduces to

$$\frac{d}{ds}\Big(\frac{M^{j0}}{u^0}\Big) + \{^{\,j}_{i\,k}\}M^{ik} = 0. \tag{2.4.161}$$

Substituting (2.4.137) into (2.4.161) gives

$$\frac{d}{ds}(mu^j) + \{^{\,j}_{i\,k}\}mu^i u^k = 0. \tag{2.4.162}$$

Contracting (2.4.162) with u_j yields

$$\frac{dm}{ds} + m\Big(\frac{du^j}{ds}u_j + \{^{\,j}_{i\,k}\}u^i u^k u_j\Big) = 0. \tag{2.4.163}$$

Differentiating $u^l u^i g_{li} = 1$ with respect to s gives

$$2\frac{du^l}{ds}u^i g_{li} + u^l u^i \frac{dg_{li}}{ds} = 2\frac{du^l}{ds}u_l + u^l u^i g_{li,k}u^k = 0. \tag{2.4.164}$$

Using

$$\{^{\,j}_{i\,k}\}u^i u^k u_j = \frac{1}{2}(g_{li,k} + g_{lk,i} - g_{ik,l})u^i u^k u^l = \frac{1}{2}g_{li,k}u^i u^k u^l \tag{2.4.165}$$

turns (2.4.164) into

$$\frac{du^j}{ds}u_j + \{^{\,j}_{i\,k}\}u^i u^k u_j = 0. \tag{2.4.166}$$

Consequently, (2.4.163) reduces to

$$\frac{dm}{ds} = 0, \tag{2.4.167}$$

showing that the mass of a particle is constant. Taking this constancy into account, (2.4.162) reduces to the metric geodesic equation (1.4.91). A spinless particle moves in a gravitational field along a metric geodesic, regardless of its mass. This phenomenon is referred to as the *universality of free fall* or *weak equivalence principle*.

2.4.10 Spin tensor for particles

If the spin density does not vanish, we can use the conservation law for this quantity to determine the spin tensor and the energy-momentum tensor for a system of particles. The conservation law for the spin density (2.4.14) is

$$\mathfrak{S}^{ijk}{}_{,k} - \Gamma^{\,i}_{l\,k}\mathfrak{S}^{jlk} + \Gamma^{\,j}_{l\,k}\mathfrak{S}^{ilk} - 2\mathfrak{T}^{[ij]} = 0. \tag{2.4.168}$$

Integrating (2.4.168) over the volume of the body at a constant time X^0 (analogously to the calculations in the preceding section) and using Gauß' theorem to eliminate surface integrals gives

$$\int \mathfrak{S}^{ij0}{}_{,0}dV - \int \Gamma^{\,i}_{l\,k}\mathfrak{S}^{jlk}dV + \int \Gamma^{\,j}_{l\,k}\mathfrak{S}^{ilk}dV - 2\int \mathfrak{T}^{[ij]}dV = 0. \tag{2.4.169}$$

For a particle, the affine connection $\Gamma^{\,i}_{j\,k}$ in the integrands in (2.4.169) is equal to its value at the point X^i. Consequently, we obtain

$$\int \mathfrak{S}^{ij0}{}_{,0}dV - \Gamma^{\,i}_{l\,k}\int \mathfrak{S}^{jlk}dV + \Gamma^{\,j}_{l\,k}\int \mathfrak{S}^{ilk}dV - 2\int \mathfrak{T}^{[ij]}dV = 0. \tag{2.4.170}$$

Using the integrals (2.4.117) and (2.4.118) turns (2.4.170) into

$$\frac{d}{ds}\Big(\frac{N^{ij0}}{u^0}\Big) - \Gamma^{\,i}_{l\,k}N^{jlk} + \Gamma^{\,j}_{l\,k}N^{ilk} - 2M^{[ij]} = 0. \tag{2.4.171}$$

The conservation law (2.4.168) gives

$$(x^l\mathfrak{S}^{ijk})_{,k}=\mathfrak{S}^{ijl}+x^l\Gamma^{\,i}_{l\,k}\mathfrak{S}^{jlk}-x^l\Gamma^{\,j}_{l\,k}\mathfrak{S}^{ilk}+2x^l\mathfrak{T}^{[ij]}. \tag{2.4.172}$$

Integrating (2.4.172) over the volume of the body and using Gauß' theorem to eliminate surface integrals gives

$$\int(x^l\mathfrak{S}^{ij0})_{,0}dV=\int\mathfrak{S}^{ijl}dV+\int x^l\Gamma^{\,i}_{m\,k}\mathfrak{S}^{jmk}dV-\int x^l\Gamma^{\,j}_{m\,k}\mathfrak{S}^{imk}dV+2\int x^l\mathfrak{T}^{[ij]}dV. \tag{2.4.173}$$

Substituting (2.4.112) into (2.4.173) and using $x^i=X^i$ gives

$$\frac{u^l}{u^0}\int\mathfrak{S}^{ij0}dV+X^l\int\mathfrak{S}^{ij0}{}_{,0}dV=\int\mathfrak{S}^{ijl}dV+X^l\Big(\int\Gamma^{\,i}_{m\,k}\mathfrak{S}^{jmk}dV-\int\Gamma^{\,j}_{m\,k}\mathfrak{S}^{imk}dV$$
$$+2\int\mathfrak{T}^{[ij]}dV\Big), \tag{2.4.174}$$

which reduces, by means of (2.4.169), to

$$\frac{u^l}{u^0}\int\mathfrak{S}^{ij0}dV=\int\mathfrak{S}^{ijl}dV. \tag{2.4.175}$$

Using the definition (2.4.118) brings (2.4.175) to

$$\frac{u^l}{u^0}N^{ij0}=N^{ijl}. \tag{2.4.176}$$

Putting $l=0$ in (2.4.176) gives the identity. The expressions analogous to (2.4.172) with higher multiples of x^i does not introduce new relations.

The relation (2.4.176) infers that the spin tensor for a system of particles satisfies

$$s^{ijl}=s^{ij}u^l, \tag{2.4.177}$$

where

$$s^{ij}=s^{ijl}u_l. \tag{2.4.178}$$

If this tensor is orthogonal to u^i,

$$s^{ij}u_j=0, \tag{2.4.179}$$

then it has 3 independent components. A system satisfying (2.4.177) and (2.4.179) is referred to as a *spin fluid*. The spin tensor (2.4.177) is traceless, because of (2.4.179). In a locally Galilean, rest frame of reference, (2.4.179) becomes

$$s_{0\alpha}=0. \tag{2.4.180}$$

In this frame, the 3 components of s_{ij} are spatial, $s_{\alpha\beta}$, and are equivalent to 3 components of a spatial pseudovector:

$$s^\alpha=\frac{1}{2}e^{\alpha\beta\gamma}s_{\beta\gamma}. \tag{2.4.181}$$

The relation (2.4.128) gives

$$M^{jl}=\frac{u^l}{u^0}M^{0j}+2\frac{u^l}{u^0}M^{[j0]}. \tag{2.4.182}$$

Putting $j=0$ in (2.4.182) gives (2.4.130). The relations (2.4.134) and (2.4.182) lead then to

$$M^{jl}=\frac{u^ju^l}{(u^0)^2}M^{00}+2\frac{u^l}{u^0}M^{[j0]}=mc^2u^ju^l+2\frac{u^l}{u^0}M^{[j0]}. \tag{2.4.183}$$

The relation (2.4.171) gives

$$2M^{[i0]}=\frac{d}{ds}\left(\frac{N^{i00}}{u^0}\right)-\Gamma^{\,i}_{l\,k}N^{0lk}+\Gamma^{\,0}_{l\,k}N^{ilk}. \tag{2.4.184}$$

Substituting this equation into (2.4.183) yields

$$M^{jl} = mc^2 u^j u^l + \frac{u^l}{u^0}\left[\frac{d}{ds}\left(\frac{N^{j00}}{u^0}\right) - \Gamma^{\,j}_{i\,k} N^{0ik} + \Gamma^{\,0}_{i\,k} N^{jik}\right]. \tag{2.4.185}$$

Using (2.4.176) brings (2.4.185) to

$$M^{jl} = mc^2 u^j u^l + \frac{u^l}{u^0}\left[\frac{d}{ds}\left(\frac{N^{j00}}{u^0}\right) + \frac{u^k}{u^0}\Gamma^{\,j}_{i\,k} N^{i00} + \frac{u^k}{u^0}\Gamma^{\,0}_{i\,k} N^{ji0}\right]. \tag{2.4.186}$$

Consequently, the four-momentum (2.4.132) is

$$P^i = \frac{M^{i0}}{cu^0} = mcu^i + \frac{1}{cu^0}\left[\frac{d}{ds}\left(\frac{N^{i00}}{u^0}\right) + \frac{u^k}{u^0}\Gamma^{\,i}_{l\,k} N^{l00} + \frac{u^k}{u^0}\Gamma^{\,0}_{l\,k} N^{il0}\right]. \tag{2.4.187}$$

The four-momentum of a particle with spin is not proportional to its four-velocity.

If the spin density is completely antisymmetric, then (2.4.176) gives $N^{il0} = -\frac{u^l}{u^0} N^{i00}$ and thus

$$N^{ijk} = 0. \tag{2.4.188}$$

Therefore, such a field cannot be represented as a point or a system of points.

2.4.11 Relativistic ideal fluids

In an arbitrary frame of reference, the metric energy-momentum tensor T_{ik} describing *isotropic* spinless matter (without a preferred direction in its rest frame) can be decomposed into several parts. One part is proportional to $u_i u_k$, analogously to (2.4.152). Another part is proportional to the projection tensor (1.4.229):

$$h_{ik} = g_{ik} - u_i u_k, \tag{2.4.189}$$

which is orthogonal to u^i:

$$h_{ik} u^k = 0. \tag{2.4.190}$$

The tensor T_{ik} can also contain terms with covariant derivatives of u^i. Let us assume that T_{ik} does not depend on derivatives of u^i. Therefore, we have

$$T_{ik} = \epsilon u_i u_k - p h_{ik} = (\epsilon + p) u_i u_k - p g_{ik}, \tag{2.4.191}$$

where a scalar ϵ is equal to the energy density W in the locally Galilean rest frame and a scalar p is the *pressure*. In this frame $T^{ik} = \mathrm{diag}(\epsilon, p, p, p)$ and the stress tensor $\sigma_{\alpha\beta} = -p\delta_{\alpha\beta}$, thereby (2.4.80) gives

$$F^\alpha = -\oint p\, df^\alpha = -\oint p\, n^\alpha df. \tag{2.4.192}$$

This equation, referred to as *Pascal's law*, states that the force per unit surface *df* acting on a surface is parallel, with the opposite sign, to the outward normal vector of this surface n^α, $dF^\alpha/df = -pn^\alpha$. The scalars ϵ and p can also be written as

$$\epsilon = T_{ik} u^i u^k, \tag{2.4.193}$$

$$p = -\frac{1}{3} T_{ik} h^{ik}. \tag{2.4.194}$$

Matter described by the tensor (2.4.191) represents an *ideal fluid*. The relation (2.4.191) can be written as

$$T_{ik} = (c\pi_i + pu_i) u_k - p g_{ik}, \tag{2.4.195}$$

where

$$\pi_i = \frac{1}{c} T_{ik} u^k = \frac{\epsilon}{c} u_i \tag{2.4.196}$$

is equal to the four-momentum density in the locally Galilean rest frame:

$$\pi_i = \frac{1}{c}T_{i0}. \tag{2.4.197}$$

We also have

$$\pi_i u^i = \frac{\epsilon}{c}. \tag{2.4.198}$$

The relation between ϵ and p is referred to as the *equation of state*.

In the Galilean frame of reference, combining (1.6.120), (2.4.82), and (2.4.191) gives

$$W = \frac{\epsilon + pv^2/c^2}{1 - v^2/c^2}, \tag{2.4.199}$$

$$\mathbf{S} = \frac{(\epsilon + p)\mathbf{v}}{1 - v^2/c^2}, \tag{2.4.200}$$

$$\sigma_{\alpha\beta} = -\frac{(\epsilon + p)v_\alpha v_\beta}{c^2 - v^2} - p\delta_{\alpha\beta}. \tag{2.4.201}$$

The relation (2.4.191) gives

$$T = T^i_{\ i} = \epsilon - 3p. \tag{2.4.202}$$

The component $T_{00} = \epsilon u_0^2 + p(u_0^2 - g_{00})$ is, by means of $u_0 = (g_{00}dx^0 + g_{0\alpha}dx^\alpha)/ds$, (1.4.126) and (1.4.127), equal to

$$T_{00} = \epsilon u_0^2 + pg_{00}\left(\frac{dl}{ds}\right)^2, \tag{2.4.203}$$

so it is positive under physical conditions $\epsilon > 0$, $p > 0$, and $g_{00} > 0$. If $\mathcal{T}_{ik}$ depends also on derivatives of u^i then matter described by the tensor (2.4.191) with the corresponding additional terms represents a *viscous fluid*.

Comparing (2.4.202) with (2.4.159) gives

$$\epsilon - 3p = \sum_a m_a c^2\left(1 - \frac{v^2}{c^2}\right)^{1/2}, \tag{2.4.204}$$

where the summation extends over all particles in unit volume, thereby $p \leq \epsilon/3$. In the nonrelativistic limit $p \approx 0$, while in the ultrarelativistic limit $(v \rightarrow c)$ $p \rightarrow \epsilon/3$. Let us consider a system of noninteracting identical particles of mass m, which we call an *ideal gas*, with the number of particles in unit volume (*number density* or *concentration*) n, thereby

$$\mu = nm. \tag{2.4.205}$$

Comparing (2.4.191) in the locally Galilean rest frame with (2.4.153) gives the kinetic formulae for ideal gases:

$$\epsilon = nmc^2\bar{\gamma}, \tag{2.4.206}$$

$$p = \frac{nm}{3}\overline{\gamma v^2}. \tag{2.4.207}$$

The covariant conservation (2.4.30) of the metric energy-momentum tensor (2.4.191) gives

$$\big((\epsilon + p)u^k\big)_{:k}u^i + (\epsilon + p)u^k u^i_{\ :k} = p_{,k}g^{ik}. \tag{2.4.208}$$

Multiplying (2.4.208) by u_i gives the *equation of continuity*:

$$\big((\epsilon + p)u^k\big)_{:k} = p_{,k}u^k. \tag{2.4.209}$$

Substituting this equation into (2.4.208) gives the *Euler equation*:

$$(\epsilon + p)\frac{D^{\{\}}u^i}{ds} = p_{,k}h^{ik}. \tag{2.4.210}$$

If $p_{,i} \propto u_i$ (which includes the case $p = \text{const}$) then (2.4.210) reduces to the metric geodesic equation (1.4.91). Defining a quantity n such that

$$\frac{dn}{n} = \frac{d\epsilon}{\epsilon + p} \tag{2.4.211}$$

brings (2.4.209) to the conservation law

$$(nu^i)_{:i} = 0. \tag{2.4.212}$$

The quantity n thus represents the proper (in the rest frame) number density of particles composing the fluid.

The number of particles dN in a volume element dV in the rest frame of reference is equal to

$$dN = n\,dV, \tag{2.4.213}$$

where n is the proper number density. In a frame of reference moving relative to the rest frame with velocity $\mathbf{v}$, the same volume element is given by $dV' = dV\sqrt{1 - v^2/c^2}$ (1.6.109), and the number density is n'. Since $dN' = n'\,dV'$ and $dN' = dN$ is an invariant, we have

$$n' = \frac{n}{\sqrt{1 - v^2/c^2}}. \tag{2.4.214}$$

In an arbitrary frame of reference, the tetrad energy-momentum density $\mathfrak{T}_{ik}$ describing isotropic matter with spin cannot be decomposed, because of its asymmetry, as in (2.4.191). However, it can be decomposed as in (2.4.195):

$$\mathfrak{T}_{ik} = \sqrt{-\mathfrak{g}}\Big((c\mathsf{p}_i + pu_i)u_k - pg_{ik}\Big), \tag{2.4.215}$$

where

$$\mathsf{p}_i = \frac{1}{c\sqrt{-\mathfrak{g}}}\mathfrak{T}_{ik}u^k \tag{2.4.216}$$

is the corresponding four-momentum density in the locally Galilean rest frame. The conservation law for the spin density (2.4.15) gives

$$c(\mathsf{p}_i u_j - \mathsf{p}_j u_i) = \nabla^*_k s_{ij}{}^k. \tag{2.4.217}$$

Defining the energy density analogously to (2.4.198) as

$$\epsilon = c\mathsf{p}_i u^i \tag{2.4.218}$$

and multiplying (2.4.217) by u^j gives

$$\mathsf{p}_i = \frac{\epsilon}{c}u_i + \frac{1}{c}\nabla^*_k s_{ij}{}^k u^j. \tag{2.4.219}$$

Therefore, we obtain

$$\mathfrak{T}_{ij} = \sqrt{-\mathfrak{g}}(\epsilon u_i u_j - ph_{ij} + \nabla^*_k s_{il}{}^k u^l u_j), \tag{2.4.220}$$

which, with the Belinfante-Rosenfeld relation (2.3.34), gives the metric energy-momentum tensor for isotropic matter with spin:

$$T_{ij} = \epsilon u_i u_j - ph_{ij} + \nabla^*_k s_{il}{}^k u^l u_j - \frac{1}{2}\nabla^*_k(s_{ij}{}^k + 2s^k{}_{(ij)}). \tag{2.4.221}$$

Substituting (2.4.219) into (2.4.217) gives the dynamical equation for the spin tensor:

$$\nabla^*_k s_{ij}{}^k - \nabla^*_k s_{il}{}^k u^l u_j + \nabla^*_k s_{jl}{}^k u^l u_i = 0. \tag{2.4.222}$$

If $s_{ijk} = 0$ then (2.4.219) gives $\mathsf{p}_i \propto u_i$. Multiplying (2.4.222) by u^j gives the identity, thereby any 3 components of (2.4.222) are linear combinations of the other components. Therefore, we can impose 3 constraints on $s_{ij}{}^k$.

2.4.12 Multipole expansion of spin tensor

Let us consider matter which is distributed over a small but spatially extended region in space. In this case, the quantity (2.4.111) is small but does not vanish. Since the dimensions of the body are small, integrals with two or more factors δx^i multiplying $\mathfrak{T}^{jk}$ can be neglected. For the consistency of this approximation, we also neglect integrals with one or more factors δx^i multiplying $\mathfrak{S}^{jkl}$ because the corrections to the energy-momentum tensor from intrinsic spin are quadratic in the spin density (confer (2.5.19)). Since the integration is over the volume of the body at a constant time, we also have

$$\delta x^0 = 0. \tag{2.4.223}$$

The conservation law for the spin density is given by (2.4.168). Integrating it over the volume of the body gives (2.4.169). Since the affine connection appears only in terms with the spin density, these terms in this approximation satisfy (2.4.115). Accordingly, the affine connection in the integrands in (2.4.169) is equal to its value at the point X^i, yielding (2.4.170) and (2.4.171). We also have (2.4.172) and (2.4.173). However, we use $x^i = X^i + \delta x^i$ in the term with $\mathfrak{T}^{jk}$. Instead of (2.4.174), we obtain

$$\frac{u^l}{u^0}\int \mathfrak{S}^{ij0}dV + X^l\int \mathfrak{S}^{ij0}{}_{,0}dV = \int \mathfrak{S}^{ijl}dV + X^l\Big(\int \Gamma^{\,i}_{m\,k}\mathfrak{S}^{jmk}dV - \int \Gamma^{\,j}_{m\,k}\mathfrak{S}^{imk}dV$$
$$+2\int \mathfrak{T}^{[ij]}dV\Big) + 2\int \delta x^l\mathfrak{T}^{[ij]}dV. \tag{2.4.224}$$

This equation reduces, by means of (2.4.169), to

$$\frac{u^l}{u^0}\int \mathfrak{S}^{ij0}dV = \int \mathfrak{S}^{ijl}dV + 2\int \delta x^l\mathfrak{T}^{[ij]}dV. \tag{2.4.225}$$

We define the following integral:

$$M^{ijk} = -u^0\int \delta x^i\mathfrak{T}^{jk}dV. \tag{2.4.226}$$

The relation (2.4.223) gives

$$M^{0jk} = 0. \tag{2.4.227}$$

Using the definitions (2.4.118) and (2.4.226) brings (2.4.225) to

$$M^{l[ij]} = -\frac{1}{2}\Big(\frac{u^l}{u^0}N^{ij0} - N^{ijl}\Big). \tag{2.4.228}$$

Putting $l = 0$ in (2.4.228) gives the identity because of (2.4.227). The expressions analogous to (2.4.172) with higher multiples of x^i do not introduce new relations. If the spin density vanishes, the relation (2.4.228) gives

$$M^{ijk} = M^{ikj}. \tag{2.4.229}$$

2.4.13 Multipole expansion of energy-momentum tensor

In the integrated conservation law for the tetrad energy-momentum density (2.4.114), we expand the affine connection, which multiplies $\mathfrak{T}^{ik}$, but not the curvature tensor, which multiplies $\mathfrak{S}^{jkl}$. This expansion in the linear approximation is

$$\Gamma^{\,i}_{j\,k} = \Gamma^{\,i(0)}_{j\,k} + \Gamma^{\,i(0)}_{j\,k,l}\delta x^l, \tag{2.4.230}$$

where the superscript (0) denotes a value at X^i. Accordingly, we have

$$\{^{\,i}_{j\,k}\} = \{^{\,i}_{j\,k}\}^{(0)} + \{^{\,i}_{j\,k}\}^{(0)}_{,l}\delta x^l, \tag{2.4.231}$$
$$C^i_{\ jk} = C^{i(0)}_{\ jk} + C^{i(0)}_{\ jk,l}\delta x^l. \tag{2.4.232}$$

Substituting (2.4.231) and (2.4.232) into (2.4.114) and omitting the superscripts (0) gives

$$\int \mathfrak{T}^{j0}{}_{,0}dV + \{{}^{\,j}_{i\,k}\}\int \mathfrak{T}^{ik}dV + \{{}^{\,j}_{i\,k}\}_{,l}\int \delta x^l\mathfrak{T}^{ik}dV - C_{ik}{}^{j}\int \mathfrak{T}^{ik}dV$$
$$-C_{ik}{}^{j}{}_{,l}\int \delta x^l\mathfrak{T}^{ik}dV - \frac{1}{2}R_{ikl}{}^{j}\int \mathfrak{S}^{ikl}dV = 0. \tag{2.4.233}$$

Using the definitions (2.4.117), (2.4.118), and (2.4.226) turns (2.4.233) into

$$\frac{d}{ds}\left(\frac{M^{j0}}{u^0}\right) + \{{}^{\,j}_{i\,k}\}M^{(ik)} - \{{}^{\,j}_{i\,k}\}_{,l}M^{l(ik)} - C_{ik}{}^{j}M^{[ik]} + C_{ik}{}^{j}{}_{,l}M^{l[ik]} - \frac{1}{2}R_{ikl}{}^{j}N^{ikl} = 0, \tag{2.4.234}$$

which generalizes (2.4.122).

Integrating (2.4.123) over the volume of the body and using Gauß' theorem to eliminate surface integrals gives (2.4.125). Substituting there (2.4.111) and (2.4.112), together with (2.4.231) and (2.4.232), gives

$$\frac{u^l}{u^0}\int \mathfrak{T}^{j0}dV + X^l\int \mathfrak{T}^{j0}{}_{,0}dV + \int(\delta x^l\mathfrak{T}^{j0})_{,0}dV = \int \mathfrak{T}^{jl}dV - X^l\int\{{}^{\,j}_{i\,k}\}\mathfrak{T}^{ik}dV$$
$$-\int \delta x^l\{{}^{\,j}_{i\,k}\}\mathfrak{T}^{ik}dV + X^l\int C_{ik}{}^{j}\mathfrak{T}^{ik}dV + \int \delta x^l C_{ik}{}^{j}\mathfrak{T}^{ik}dV$$
$$+\frac{1}{2}X^l\int R_{ikm}{}^{j}\mathfrak{S}^{ikm}dV. \tag{2.4.235}$$

This equation reduces, by means of (2.4.114), to

$$\frac{u^l}{u^0}\int \mathfrak{T}^{j0}dV + \left(\int(\delta x^l\mathfrak{T}^{j0}dV\right)_{,0} = \int \mathfrak{T}^{jl}dV - \int \delta x^l\{{}^{\,j}_{i\,k}\}\mathfrak{T}^{ik}dV + \int \delta x^l C_{ik}{}^{j}\mathfrak{T}^{ik}dV, \tag{2.4.236}$$

where $\{{}^{\,j}_{i\,k}\}$ and $C_{ik}{}^{j}$ are evaluated at X^i and we omitted the superscripts (0). Using the definitions (2.4.117) and (2.4.226) brings (2.4.236) to

$$\frac{u^l}{u^0}M^{j0} - \frac{d}{ds}\left(\frac{M^{lj0}}{u^0}\right) = M^{jl} + \{{}^{\,j}_{i\,k}\}M^{lik} - C_{ik}{}^{j}M^{lik}, \tag{2.4.237}$$

which generalizes (2.4.128). Putting $l=0$ in (2.4.237) gives the identity because of (2.4.227).

Integrating (2.4.124) over the volume of the body and using Gauß' theorem to eliminate surface integrals gives

$$\int(x^l x^m\mathfrak{T}^{j0})_{,0}dV = \int x^m\mathfrak{T}^{jl}dV + \int x^l\mathfrak{T}^{jm}dV - \int x^l x^m\{{}^{\,j}_{i\,k}\}\mathfrak{T}^{ik}dV$$
$$+\int x^l x^m C_{ik}{}^{j}\mathfrak{T}^{ik}dV + \frac{1}{2}\int x^l x^m R_{ikn}{}^{j}\mathfrak{S}^{ikn}dV. \tag{2.4.238}$$

Substituting (2.4.111) and (2.4.112) into (2.4.238) gives

$$\frac{u^l}{u^0}X^m\int \mathfrak{T}^{j0}dV + \frac{u^l}{u^0}\int \delta x^m\mathfrak{T}^{j0}dV + \frac{u^m}{u^0}X^l\int \mathfrak{T}^{j0}dV + \frac{u^m}{u^0}\int \delta x^l\mathfrak{T}^{j0}dV$$
$$+X^lX^m\int \mathfrak{T}^{j0}{}_{,0}dV + X^l\int \delta x^m\mathfrak{T}^{j0}{}_{,0}dV + X^m\int \delta x^l\mathfrak{T}^{j0}{}_{,0}dV$$
$$= -X^lX^m\Big(\int\{{}^{\,j}_{i\,k}\}\mathfrak{T}^{ik}dV - \int C_{ik}{}^{j}\mathfrak{T}^{ik}dV - \frac{1}{2}\int R_{ikl}{}^{j}\mathfrak{S}^{ikl}dV\Big)$$
$$+X^l\Big(\int \mathfrak{T}^{jm}dV - \int \delta x^m\{{}^{\,j}_{i\,k}\}\mathfrak{T}^{ik}dV + \int \delta x^m C_{ik}{}^{j}\mathfrak{T}^{ik}dV\Big)$$
$$+X^m\Big(\int \mathfrak{T}^{jl}dV - \int \delta x^l\{{}^{\,j}_{i\,k}\}\mathfrak{T}^{ik}dV + \int \delta x^l C_{ik}{}^{j}\mathfrak{T}^{ik}dV\Big)$$
$$+\int \delta x^m\mathfrak{T}^{jl}dV + \int \delta x^l\mathfrak{T}^{jm}dV. \tag{2.4.239}$$

This equation reduces, by means of (2.4.114) and (2.4.236), to

$$\frac{u^l}{u^0}\int \delta x^m \mathfrak{T}^{j0} dV + \frac{u^m}{u^0}\int \delta x^l \mathfrak{T}^{j0} dV = \int \delta x^m \mathfrak{T}^{jl} dV + \int \delta x^l \mathfrak{T}^{jm} dV \tag{2.4.240}$$

or

$$\frac{u^l}{u^0} M^{mi0} + \frac{u^m}{u^0} M^{li0} = M^{mil} + M^{lim}. \tag{2.4.241}$$

The expressions analogous to (2.4.123) and (2.4.124) with higher multiples of x^i do not introduce new relations.

2.4.14 Mathisson-Papapetrou-Dixon equations

We define the following integral:

$$L^{ik} = \frac{1}{c}\int (\delta x^i \mathfrak{T}^{k0} - \delta x^k \mathfrak{T}^{i0}) dV, \tag{2.4.242}$$

which is analogous to the angular momentum four-tensor (2.4.106). This quantity is related to (2.4.226) by

$$L^{ik} = \frac{1}{cu^0}(M^{ki0} - M^{ik0}). \tag{2.4.243}$$

We also define

$$J^{ik} = L^{ik} + \frac{1}{c}\int \mathfrak{S}^{ik0} dV, \tag{2.4.244}$$

which is analogous to the angular momentum four-tensor (2.4.105). This quantity is related to (2.4.118) and (2.4.226) by

$$J^{ik} = \frac{1}{cu^0}(M^{ki0} - M^{ik0} + N^{ik0}). \tag{2.4.245}$$

The relations (2.4.227) and (2.4.243) give

$$M^{j00} = -cu^0 L^{j0}. \tag{2.4.246}$$

Let us consider spinless matter in spacetime without torsion. In this case, the symmetry relations (2.4.129) and (2.4.229) bring (2.4.234) to

$$\frac{d}{ds}\left(\frac{M^{j0}}{u^0}\right) + \{^{\,j}_{i\,k}\} M^{ik} - \{^{\,j}_{i\,k}\}_{,l} M^{lik} = 0. \tag{2.4.247}$$

The relation (2.4.237) reduces to

$$\frac{u^l}{u^0} M^{j0} - \frac{d}{ds}\left(\frac{M^{lj0}}{u^0}\right) = M^{jl} + \{^{\,j}_{i\,k}\} M^{lik}. \tag{2.4.248}$$

Antisymmetrizing (2.4.248) in the indices j, l and using (2.4.129) and (2.4.243) gives

$$\frac{u^j}{cu^0} M^{l0} - \frac{u^l}{cu^0} M^{j0} + \frac{dL^{jl}}{ds} + \frac{1}{c}\{^{\,j}_{i\,k}\} M^{lik} - \frac{1}{c}\{^{\,l}_{i\,k}\} M^{jik} = 0. \tag{2.4.249}$$

In the absence of the external gravitational field and neglecting the gravitational field of the body, we have $\{^{\,j}_{i\,k}\} = 0$. The relation (2.4.247) reduces to

$$\frac{dP^i}{ds} = 0, \tag{2.4.250}$$

where P^i is given by (2.4.132). The integration of this equation gives the conservation of the momentum four-vector along a world line:

$$P^i = \text{const}. \tag{2.4.251}$$

The relation (2.4.249) reduces to

$$\frac{dL^{ik}}{ds} + u^i P^k - u^k P^i = 0, \tag{2.4.252}$$

whose integration gives the conservation of the angular momentum four-tensor along a world line:

$$L^{ik} + X^i P^k - X^k P^i = \text{const.} \tag{2.4.253}$$

The tensor L^{ik} is the intrinsic angular momentum of the body, while the tensor (in the absence of the gravitational field) $X^i P^k - X^k P^i$ is the orbital angular momentum associated with the motion of the body as a whole. If $L^{ik} = 0$ then (2.4.252) gives $P^i \propto u^i$, thereby (2.4.251) is equivalent to $u^i = \text{const}$ and thus X^i is a linear function of the proper time τ. If $L^{ik} \neq 0$ then X^i can be given by 3 arbitrary functions of τ because $u^i u_i = 1$. In the *momentum rest frame*, in which $P^\alpha = 0$, $u^\alpha \neq 0$, thereby the body has an arbitrary internal motion. This arbitrariness is a consequence of 10 equations (2.4.250) and (2.4.252) for 13 independent components of u^i, P^i, and L^{ik}.

Putting $j = 0$ in (2.4.248) gives

$$M^{0l} = M^{l0} = \frac{u^l}{u^0} M^{00} - \frac{d}{ds}\left(\frac{M^{l00}}{u^0}\right) - \{{}^{\,0}_{i\,k}\} M^{lik}. \tag{2.4.254}$$

Substituting (2.4.254) into (2.4.248) gives

$$M^{jl} = \frac{u^l}{u^0}\left[\frac{u^j}{u^0} M^{00} - \frac{d}{ds}\left(\frac{M^{j00}}{u^0}\right) - \{{}^{\,0}_{i\,k}\} M^{jik}\right] - \frac{d}{ds}\left(\frac{M^{lj0}}{u^0}\right) - \{{}^{\,j}_{i\,k}\} M^{lik}. \tag{2.4.255}$$

Antisymmetrizing (2.4.255) in the indices j, l and using (2.4.129), (2.4.243), and (2.4.246) gives

$$\frac{dL^{jl}}{ds} + \frac{u^j}{u^0}\frac{dL^{l0}}{ds} - \frac{u^l}{u^0}\frac{dL^{j0}}{ds} + \frac{1}{c}\left(\{{}^{\,j}_{i\,k}\} - \frac{u^j}{u^0}\{{}^{\,0}_{i\,k}\}\right) M^{lik} - \frac{1}{c}\left(\{{}^{\,l}_{i\,k}\} - \frac{u^l}{u^0}\{{}^{\,0}_{i\,k}\}\right) M^{jik} = 0. \tag{2.4.256}$$

This relation contains the time derivative of the quantity L^{ik} only, thereby it is an equation of motion for this quantity.

The tensor M^{ijk} can be determined from the relation (2.4.241). Taking the cyclic permutations of the indices i, l, m in (2.4.241), adding the first and second of these relations, and subtracting the third gives

$$\frac{u^l}{u^0} M^{[mi]0} + \frac{u^i}{u^0} M^{[ml]0} + \frac{u^m}{u^0} M^{(li)0} = M^{l[im]} + M^{i[lm]} + M^{m(il)}. \tag{2.4.257}$$

This equation, using (2.4.229) and (2.4.243), reduces to

$$cu^l L^{im} + cu^i L^{lm} + 2u^m \frac{M^{(il)0}}{u^0} = 2M^{mil}. \tag{2.4.258}$$

Putting in this relation $l = 0$, and using (2.4.227) and (2.4.243), gives

$$2M^{mio} = cu^0 L^{im} - cu^i L^{m0} - cu^m L^{i0}, \tag{2.4.259}$$

which upon substitution into (2.4.258) gives

$$2M^{mil} = -cu^l L^{mi} - cu^i L^{ml} - \frac{cu^m}{u^0}(u^i L^{l0} + u^l L^{i0}). \tag{2.4.260}$$

Since L^{ik} is a tensor, we also have

$$\frac{D^{\{\}} L^{ik}}{ds} = L^{ik}{}_{:j} u^j = L^{ik}{}_{,j} u^j + \{{}^{\,i}_{l\,j}\} L^{lk} u^j + \{{}^{\,k}_{l\,j}\} L^{il} u^j = \frac{dL^{ik}}{ds} + \{{}^{\,i}_{l\,j}\} L^{lk} u^j + \{{}^{\,k}_{l\,j}\} L^{il} u^j. \tag{2.4.261}$$

Substituting the last two equations into (2.4.256) gives

$$\frac{D^{\{\}} L^{jl}}{ds} + \frac{u^j}{u^0}\frac{D^{\{\}} L^{l0}}{ds} - \frac{u^l}{u^0}\frac{D^{\{\}} L^{j0}}{ds} = 0. \tag{2.4.262}$$

Contracting this equation with u_l gives

$$\frac{1}{u^0}\frac{D^{\{\}}L^{j0}}{ds} = u_k\frac{D^{\{\}}L^{jk}}{ds} + \frac{u^j u_k}{u^0}\frac{D^{\{\}}L^{k0}}{ds}, \tag{2.4.263}$$

which upon substitution into (2.4.262) gives

$$\frac{D^{\{\}}L^{jl}}{ds} - u^j u_k\frac{D^{\{\}}L^{kl}}{ds} - u^l u_k\frac{D^{\{\}}L^{jk}}{ds} = 0. \tag{2.4.264}$$

This equation is a covariant equation of motion for the intrinsic angular momentum four-tensor L^{ik}, confirming that L^{ik} is a tensor. It is analogous to (2.4.222).

For a particle, the quantities M^{ik} and P^i are tensors. For matter distributed over a spatially extended region in space, these quantities are not tensors. In this case, we define the modified four-momentum as

$$\Pi^j = \frac{M^{j0}}{cu^0} + \frac{1}{u^0}\{{}^{\,j}_{i\,k}\}u^i L^{k0}. \tag{2.4.265}$$

Accordingly, we define the modified mass analogously to (2.4.136):

$$m = \frac{\Pi^i u_i}{c}. \tag{2.4.266}$$

Equation (2.4.254) gives, using (2.4.246) and (2.4.260),

$$\begin{aligned} M^{i0} &= \frac{u^i}{u^0}M^{00} + c\frac{dL^{i0}}{ds} + c\{{}^{\,0}_{j\,k}\}\left(u^j L^{ik} + \frac{u^i u^j}{u^0}L^{k0}\right) \\ &= \frac{u^i}{u^0}M^{00} + c\frac{D^{\{\}}L^{i0}}{ds} - cu^k(\{{}^{\,i}_{l\,k}\}L^{l0} + \{{}^{\,0}_{l\,k}\}L^{il}) + c\{{}^{\,0}_{j\,k}\}\left(u^j L^{ik} + \frac{u^i u^j}{u^0}L^{k0}\right). \end{aligned} \tag{2.4.267}$$

Therefore, we obtain

$$M^{i0} + c\{{}^{\,i}_{j\,k}\}L^{j0}u^k = \frac{u^i}{u^0}(M^{00} + c\{{}^{\,0}_{j\,k}\}L^{j0}u^k) + c\frac{D^{\{\}}L^{i0}}{ds}. \tag{2.4.268}$$

Contracting this equation with u_i/u^0, and using (2.4.265) and (2.4.266), gives

$$m = \frac{1}{(cu^0)^2}(M^{00} + c\{{}^{\,0}_{j\,k}\}L^{j0}u^k) + \frac{u_i}{cu^0}\frac{D^{\{\}}L^{i0}}{ds}. \tag{2.4.269}$$

Substituting this equation into (2.4.268) and using (2.4.263) yields

$$\frac{M^{i0}}{u^0} = mc^2u^i - \frac{c}{u^0}\{{}^{\,i}_{j\,k}\}L^{j0}u^k + c\frac{D^{\{\}}L^{ik}}{ds}u_k. \tag{2.4.270}$$

Therefore, the quantity (2.4.265) is equal to

$$\Pi^j = mcu^j + \frac{D^{\{\}}L^{jk}}{ds}u_k. \tag{2.4.271}$$

This equation shows that Π^i is a four-vector, thereby m defined in (2.4.266) is a scalar.

Substituting (2.4.270) into (2.4.247), and using (2.4.248) and (2.4.260), gives

$$\frac{d}{ds}\left(mcu^i + u_k\frac{D^{\{\}}L^{ik}}{ds}\right) + \{{}^{\,i}_{j\,k}\}u^k\left(mcu^j + u_l\frac{D^{\{\}}L^{jl}}{ds}\right) + L^{mk}u^j(\{{}^{\,i}_{j\,k}\}_{,m} + \{{}^{\,l}_{j\,k}\}\{{}^{\,i}_{l\,m}\}) = 0. \tag{2.4.272}$$

Using the Riemann tensor (1.4.57), this equation takes a covariant form:

$$\frac{D^{\{\}}}{ds}\left(mcu^i + u_k\frac{D^{\{\}}L^{ik}}{ds}\right) = -\frac{1}{2}P^i_{\ jmk}u^j L^{mk} \tag{2.4.273}$$

or

$$\frac{D^{\{\}}\Pi^i}{ds} = -\frac{1}{2}P^i_{\ jmk}u^j L^{mk}. \tag{2.4.274}$$

The relation (2.4.271) brings (2.4.264) to

$$\frac{D^{\{\}}L^{lj}}{ds} = u^j\Pi^l - u^l\Pi^j. \tag{2.4.275}$$

Equations (2.4.274) and (2.4.275) are the *Mathisson-Papapetrou-Dixon equations* of motion for a spatially extended body in a gravitational field. For a particle, $L^{ik} = 0$. In this case, (2.4.275) gives Π^i proportional to u^i. Consequently, (2.4.274) reduces to $D^{\{\}}u^i/ds$, which is the metric geodesic equation (1.4.91).

The change of the mass (2.4.266) along a world line is, using (2.4.271) and (2.4.274),

$$\frac{dm}{ds} = \frac{D^{\{\}}m}{ds} = \frac{1}{c}u_j\frac{D^{\{\}}\Pi^j}{ds} + \frac{1}{c}\Pi^j\frac{D^{\{\}}u_j}{ds} = \frac{1}{c}\Pi^j\frac{D^{\{\}}u_j}{ds} = \frac{1}{c}\frac{D^{\{\}}L^{ji}}{ds}u_i\frac{D^{\{\}}u_j}{ds}$$
$$= -\frac{1}{c}L^{ji}\frac{D^{\{\}}u_i}{ds}\frac{D^{\{\}}u_j}{ds} = 0, \tag{2.4.276}$$

showing that the modified mass is constant along the world line. For a spatially extended body with the angular momentum four-tensor (2.4.242), the equation of motion (2.4.273) shows that the body does not move along a metric geodesic. Analogously to the Pauli-Lubański pseudovector (1.6.71), we define a pseudovector:

$$L^i = \frac{1}{2}e^{ijkl}u_j L_{kl}, \tag{2.4.277}$$

which is orthogonal to u^i,

$$L^i u_i = 0. \tag{2.4.278}$$

Differentiating (2.4.277) covariantly with respect to $\{{}^{\,i}_{j\,k}\}$ and using (2.4.264) gives

$$\frac{D^{\{\}}L^i}{ds} = -u^i\frac{D^{\{\}}u^k}{ds}L_k + \frac{D^{\{\}}u^i}{ds}u^k L_k, \tag{2.4.279}$$

thereby the covariant (with respect to the Levi-Civita connection) change of this pseudovector along the world line is equal to the corresponding Fermi-Walker transport. Multiplying (2.4.279) by L_i and using (2.4.278) gives

$$\frac{d(L^i L_i)}{ds} = 0, \tag{2.4.280}$$

thereby the change of the pseudovector L^i along the world line is a four-rotation with a constant value of the magnitude (precession).

For matter with spin in spacetime without torsion, the calculations are similar to those for spinless matter. The modified four-momentum is generalized to

$$\Pi^j = P^j + \frac{1}{cu^0}\{{}^{\,j}_{i\,k}\}(u^i J^{k0} + N^{0ik}) - \frac{1}{2cu^0}C_{ik}^{\ \ j}N^{ik0}, \tag{2.4.281}$$

and the modified mass is generalized to

$$m = \frac{u_j}{c}P^j + \frac{u_j}{c^2u^0}\{{}^{\,j}_{i\,k}\}(u^i J^{k0} + N^{0ik}) - \frac{u_j}{2c^2u^0}C_{ik}^{\ \ j}N^{ik0}. \tag{2.4.282}$$

The equation of motion (2.4.274) becomes

$$\frac{D^{\{\}}\Pi^j}{ds} = -\frac{1}{2c}P^j_{\ imk}u^i J^{mk} - \frac{1}{2c}N_{ikl}C^{ikl:j}, \tag{2.4.283}$$

and the equation of motion (2.4.275) becomes

$$\frac{D^{\{\}}}{ds}J^{lj} = cu^j\Pi^l - cu^l\Pi^j + C^l_{\ ik}N^{jik} + \frac{1}{2}C_{ik}^{\ \ l}N^{ikj} - C^j_{\ ik}N^{lik} - \frac{1}{2}C_{ik}^{\ \ j}N^{ikl}. \tag{2.4.284}$$

For a body with the angular momentum four-tensor (2.4.244), the last equation shows that the body does not move along a metric geodesic.

References: [2, 3, 4, 6, 7, 9, 10].

2.5 Gravitational field equations

2.5.1 Einstein-Cartan action and equations

The metric and torsion tensors are two independent, fundamental variables describing a gravitational field. The action for the gravitational field and matter is equal, following (2.2.3), to

$$S = S_{\mathrm{g}} + S_{\mathrm{m}} = -\frac{1}{2\kappa c}\int R\sqrt{-\mathfrak{g}}d\Omega + S_{\mathrm{m}}. \tag{2.5.1}$$

The action (2.5.1) subjected to varying the metric and torsion tensors is called the *Einstein-Cartan action* for the gravitational field and matter. Using (1.4.67) and applying partial integration (1.4.48) gives

$$\begin{aligned} S_{\mathrm{g}} &= -\frac{1}{2\kappa c}\int\Big(P - g^{ik}(2C^l_{\ il:k} + C^j_{\ ij}C^l_{\ kl} - C^l_{\ im}C^m_{\ kl})\Big)\sqrt{-\mathfrak{g}}d\Omega \\ &= -\frac{1}{2\kappa c}\int\Big(P - g^{ik}(C^j_{\ ij}C^l_{\ kl} - C^l_{\ im}C^m_{\ kl})\Big)\sqrt{-\mathfrak{g}}d\Omega + \frac{1}{\kappa c}\oint C^{lk}_{\ \ l}\sqrt{-\mathfrak{g}}dS_k, \end{aligned} \tag{2.5.2}$$

where dS_i is the element of the closed hypersurface surrounding the integration four-volume. The stationarity of action (2.1.4), which is a part of the principle of least action, is applied with a condition that the variations of the variables at the boundary of integration four-volume vanish. Accordingly, the variation of the hypersurface integral taken over this boundary in (2.5.2) vanishes. This integral therefore does not contribute to the field equations and can be omitted, which reduces (2.5.1) to

$$S = -\frac{1}{2\kappa c}\int\Big(P - g^{ik}(C^j_{\ ij}C^l_{\ kl} - C^l_{\ im}C^m_{\ kl})\Big)\sqrt{-\mathfrak{g}}d\Omega + S_{\mathrm{m}}. \tag{2.5.3}$$

Firstly, we vary (2.5.3) with respect to the metric tensor. Using (2.3.2) and the identity $\delta\sqrt{-\mathfrak{g}} = -\frac{1}{2}\sqrt{-\mathfrak{g}}g_{ik}\delta g^{ik}$, which results from (1.4.20), gives

$$\begin{aligned} \delta_{\mathrm{g}}S = &-\frac{1}{2\kappa c}\int\Big(\delta P_{ik}g^{ik}\sqrt{-\mathfrak{g}} + P_{ik}\delta g^{ik}\sqrt{-\mathfrak{g}} - \frac{1}{2}P\sqrt{-\mathfrak{g}}g_{ik}\delta g^{ik}\Big)d\Omega \\ &-\frac{1}{2\kappa c}\int\Big(-C^j_{\ ij}C^l_{\ kl} + C^l_{\ im}C^m_{\ kl} + \frac{1}{2}g_{ik}(C^{jm}_{\ \ \ j}C^l_{\ ml} - C^{lj}_{\ \ m}C^m_{\ jl})\Big)\sqrt{-\mathfrak{g}}\delta g^{ik}d\Omega \\ &+\frac{1}{2c}\int T_{ik}\sqrt{-\mathfrak{g}}\delta g^{ik}d\Omega. \end{aligned} \tag{2.5.4}$$

We define the contravariant *metric density*,

$$\mathsf{g}^{ik} = \sqrt{-\mathfrak{g}}g^{ik}, \tag{2.5.5}$$

whose covariant derivative with respect to the Christoffel symbols vanishes: $\mathsf{g}^{ik}_{\ \ :l} = 0$. In the first term on the right-hand side of (2.5.4), using (1.4.70) and applying partial integration (1.4.48) brings this term to zero:

$$\int \delta P_{ik}\mathsf{g}^{ik}d\Omega = \int\Big((\delta\{^{\,l}_{i\,k}\})_{:l} - (\delta\{^{\,l}_{i\,l}\})_{:k}\Big)\mathsf{g}^{ik}d\Omega = \oint(\mathsf{g}^{ik}\delta\{^{\,l}_{i\,k}\}dS_l - \mathsf{g}^{ik}\delta\{^{\,l}_{i\,l}\}dS_k) = 0. \tag{2.5.6}$$

We therefore obtain

$$\begin{aligned} \delta_{\mathrm{g}}S = &-\frac{1}{2\kappa c}\int G_{ik}\sqrt{-\mathfrak{g}}\delta g^{ik}d\Omega \\ &-\frac{1}{2\kappa c}\int\Big(-C^j_{\ ij}C^l_{\ kl} + C^l_{\ im}C^m_{\ kl} + \frac{1}{2}g_{ik}(C^{jm}_{\ \ \ j}C^l_{\ ml} - C^{lj}_{\ \ m}C^m_{\ jl})\Big)\sqrt{-\mathfrak{g}}\delta g^{ik}d\Omega \\ &+\frac{1}{2c}\int T_{ik}\sqrt{-\mathfrak{g}}\delta g^{ik}d\Omega, \end{aligned} \tag{2.5.7}$$

where G_{ik} is the Einstein tensor (1.4.79). Secondly, we vary (2.5.3) with respect to the contortion tensor, which is equivalent to varying with respect to the torsion tensor. Using (2.3.18) and (2.3.26) gives

$$\delta_{\mathrm{C}}S = -\frac{1}{\kappa c}\int (C^{kj}_{\ \ \ i} - C^{lj}_{\ \ \ l}\delta^k_i)\sqrt{-\mathfrak{g}}\delta C^i_{\ jk}d\Omega + \frac{1}{2c}\int T_{ik}\sqrt{-\mathfrak{g}}\delta g^{ik}d\Omega$$
$$+\frac{1}{2c}\int s_j^{\ ik}\sqrt{-\mathfrak{g}}\delta C^j_{\ ik}d\Omega. \tag{2.5.8}$$

The total variation of S is then

$$\delta S = \delta_{\mathrm{g}}S + \delta_{\mathrm{C}}S. \tag{2.5.9}$$

Since the variations δg^{ik} and $\delta C^j_{\ ik}$ are independent, the stationarity of action (2.1.4) yields $\delta_{\mathrm{g}}S = \delta_{\mathrm{C}}S = 0$. The condition $\delta_{\mathrm{g}}S = 0$ for an arbitrary δg^{ik} gives the *first Einstein-Cartan equation*:

$$G_{ik} = \kappa(T_{ik} + U_{ik}), \tag{2.5.10}$$

where

$$U_{ik} = \frac{1}{\kappa}\bigg(C^j_{\ ij}C^l_{\ kl} - C^l_{\ ij}C^j_{\ kl} - \frac{1}{2}g_{ik}(C^{jm}_{\ \ \ \ j}C^l_{\ ml} - C^{mjl}C_{ljm})\bigg) \tag{2.5.11}$$

or

$$U_{ik} = \frac{1}{\kappa}\bigg(4S_iS_k - 2S^j_{\ il}S_{jk}^{\ \ \ l} - 2S^j_{\ il}S^l_{\ kj} + S_{ijl}S_k^{\ jl}$$
$$-\frac{1}{2}g_{ik}(4S^jS_j - 2S^l_{\ mn}S^{nm}_{\ \ \ \ l} - S^l_{\ mn}S_l^{\ mn})\bigg). \tag{2.5.12}$$

The first Einstein-Cartan equation (2.5.10) can be written as

$$P_{ik} = \kappa\bigg(T_{ik} + U_{ik} - \frac{1}{2}(T+U)g_{ik}\bigg), \tag{2.5.13}$$

where

$$T = T^i_{\ i},\ \ U = U^i_{\ i}. \tag{2.5.14}$$

The condition $\delta_{\mathrm{C}}S = 0$ for an arbitrary $\delta C^j_{\ ik}$ gives

$$C^k_{\ [ji]} - \delta^k_{[i}C^l_{\ j]l} = \frac{\kappa}{2}s_{ij}^{\ \ k}. \tag{2.5.15}$$

This equation can be written as the Cartan equations or the *second Einstein-Cartan equation*:

$$T^j_{\ ik} = S^j_{\ ik} - S_i\delta^j_k + S_k\delta^j_i = -\frac{\kappa}{2}s_{ik}^{\ \ j}, \tag{2.5.16}$$

where $T^j_{\ ik}$ is the *modified torsion tensor*. The relation (2.5.16) is equivalent to

$$S^k_{\ ij} = -\frac{\kappa}{2}(s_{ij}^{\ \ k} + \delta^k_{[i}s_{j]l}^{\ \ \ l}), \tag{2.5.17}$$
$$C^k_{\ ij} = \frac{\kappa}{2}(s^k_{\ ij} - s_{ij}^{\ \ k} - s_j^{\ k}{}_i - g_{ij}s^{kl}_{\ \ \ l} + \delta^k_j s_{il}^{\ \ l}). \tag{2.5.18}$$

Combining (2.5.11) and (2.5.18) gives

$$U^{ik} = \kappa\bigg(-s^{ij}_{\ \ \ [l}s^{kl}_{\ \ \ j]} - \frac{1}{2}s^{ijl}s^k_{\ jl} + \frac{1}{4}s^{jli}s_{jl}^{\ \ k} + \frac{1}{8}g^{ik}(-4s^l_{\ j[m}s^{jm}_{\ \ \ \ l]} + s^{jlm}s_{jlm})\bigg). \tag{2.5.19}$$

The tensor (2.5.19) represents a correction to the metric energy-momentum tensor from the spin contributions to the geometry of spacetime. It is quadratic in the spin tensor, thereby representing a spin-spin contact interaction. Accordingly, changing the signs of all the components of the spin tensor does not affect this correction. The spin tensor also appears in T_{ik} because $\mathfrak{L}_{\rm m}$ depends on torsion. The first Einstein-Cartan equation (2.5.10), in which U_{ik} is given by (2.5.19), is a field equation with the *combined energy-momentum tensor* $T_{ik}+U_{ik}$ as a source of the curvature. The conservation law (2.4.28) for the metric energy-momentum tensor is, upon substituting (2.5.10) and (2.5.15), equivalent to the the contracted Bianchi identity (1.4.78) for the Einstein tensor. This identity, applied to (2.5.10), gives the Riemannian conservation law for the combined energy-momentum tensor:

$$(T^{ik}+U^{ik})_{:k}=0. \tag{2.5.20}$$

This law is equivalent to (2.4.28) with (2.5.18).

The relation (2.5.17) between the torsion and spin tensors is algebraic. Torsion at a given point in spacetime does not vanish only if matter is present at this point, represented in the Lagrangian density by a function which depends on torsion. If the matter Lagrangian density does not depend on torsion, then the spin tensor vanishes, and so does the torsion tensor. In *vacuum*, which is defined as the absence of matter, $T_{ik}=0$ and $s_{ijk}=0$, the Riemannian Ricci tensor in (2.5.13) also vanishes:

$$P_{ik}=0. \tag{2.5.21}$$

Unlike the metric, which is related to matter through a differential field equation, torsion does not propagate in vacuum. The vanishing of P_{ik} and S_{ijk} at a given point in spacetime is a covariant criterion for the absence of matter at this point. The Riemann tensor P^i_{jkl} at such a point, however, can be different from zero.

2.5.2 Sciama-Kibble action

The tetrad and spin connection, instead of the metric tensor and affine connection, can be regarded as dynamical variables. The action (2.5.1) subjected to varying the tetrad and spin connection is called the *Sciama-Kibble action*. Using (2.3.27) gives

$$\delta S=-\frac{1}{2\kappa c}\int\delta(\mathfrak{e}R)d\Omega+\frac{1}{c}\int\mathfrak{T}_i^{a}\delta e^i_a d\Omega+\frac{1}{2c}\int\mathfrak{S}_{ab}^{i}\delta\omega^{ab}_{i}d\Omega. \tag{2.5.22}$$

The Lagrangian density for the gravitational field is given by (2.2.2), with the curvature scalar R given by (1.5.42) and (1.5.44):

$$\mathfrak{e}R=\mathfrak{e}e^i_a e^{jb}(\omega^a_{bj,i}-\omega^a_{bi,j}+\omega^a_{ci}\omega^c_{bj}-\omega^a_{cj}\omega^c_{bi})=2\mathfrak{e}^{ij}_{ab}(\omega^{ab}_{j,i}+\omega^a_{ci}\omega^{cb}_{j}), \tag{2.5.23}$$

where

$$\mathfrak{e}^{ij}_{ab}=\mathfrak{e}e^{[i}_a e^{j]}_b. \tag{2.5.24}$$

This quantity satisfies $\mathfrak{e}^{ij}_{ab|j}=\mathfrak{e}^{ij}_{ab,j}-\omega^c_{aj}\mathfrak{e}^{ij}_{cb}-\omega^c_{bj}\mathfrak{e}^{ij}_{ac}+\Gamma^{\,i}_{k\,j}\mathfrak{e}^{kj}_{ab}+\Gamma^{\,j}_{k\,j}\mathfrak{e}^{ik}_{ab}-\Gamma^{\,k}_{k\,j}\mathfrak{e}^{ij}_{ab}=0$, which results from (1.5.29). Varying $\mathfrak{e}R$ and omitting total derivatives which lead to hypersurface integrals gives, using (1.5.13),

$$\begin{aligned}\delta(\mathfrak{e}R)&=(2R^a_{i}-Re^a_i)\mathfrak{e}\delta e^i_a+2\mathfrak{e}^{ij}_{ab}\delta(\omega^{ab}_{j,i}+\omega^a_{ci}\omega^{cb}_{j})\\&=(2R^a_{i}-Re^a_i)\mathfrak{e}\delta e^i_a+2(\mathfrak{e}^{ij}_{ab,j}-\omega^c_{aj}\mathfrak{e}^{ij}_{cb}-\omega^c_{bj}\mathfrak{e}^{ij}_{ac})\delta\omega^{ab}_{i}\\&=(2R^a_{i}-Re^a_i)\mathfrak{e}\delta e^i_a-2(S^i_{kj}\mathfrak{e}^{kj}_{ab}+2S_j\mathfrak{e}^{ij}_{ab})\delta\omega^{ab}_{i}.\end{aligned} \tag{2.5.25}$$

The variation (2.5.22) is therefore equal to

$$\begin{aligned}\delta S=&-\frac{1}{\kappa c}\int\Big(R^a_{i}-\frac{1}{2}Re^a_i\Big)\mathfrak{e}\delta e^i_a d\Omega+\frac{1}{\kappa c}\int(S^i_{kj}\mathfrak{e}^{kj}_{ab}+2S_j\mathfrak{e}^{ij}_{ab})\delta\omega^{ab}_{i}d\Omega\\&+\frac{1}{c}\int\mathfrak{T}_i^{a}\delta e^i_a d\Omega+\frac{1}{2c}\int\mathfrak{S}_{ab}^{i}\delta\omega^{ab}_{i}d\Omega.\end{aligned} \tag{2.5.26}$$

The condition $\delta S=0$ for an arbitrary $\delta\omega^{ab}_{i}$ gives

$$S^i_{ab}-S_a e^i_b+S_b e^i_a=-\frac{\kappa}{2\mathfrak{e}}\mathfrak{S}_{ab}^{i}, \tag{2.5.27}$$

which is equivalent to the second Einstein-Cartan equation (2.5.16). The condition $\delta S = 0$ for an arbitrary δe^i_a gives

$$R^a{}_i - \frac{1}{2}Re^a_i = \frac{\kappa}{\mathfrak{e}}\mathfrak{T}_i{}^a, \tag{2.5.28}$$

which is equivalent to

$$R_{ki} - \frac{1}{2}Rg_{ik} = \kappa t_{ik}. \tag{2.5.29}$$

Substituting (2.5.16) and (2.5.29) into the conservation law for the spin tensor (2.4.15) gives

$$-2(S^k{}_{ij;k} - S_{i;j} + S_{j;i}) = R_{ji} - R_{ij} - 4S_k(S^k{}_{ij} - S_i\delta^k_j + S_j\delta^k_i), \tag{2.5.30}$$

which is equivalent to the contracted cyclic identity (1.4.72). Therefore, the contracted cyclic identity imposes the conservation law for the spin density. Substituting (2.5.16) and (2.5.29) into the conservation law for the tetrad energy-momentum tensor (2.4.39) gives

$$R^j{}_{i;j} - \frac{1}{2}R_{;i} = 2S_j\left(R^j{}_i - \frac{1}{2}R\delta^j_i\right) + 2S^j{}_{ki}\left(R^k{}_j - \frac{1}{2}R\delta^k_j\right) - (S^j{}_{kl} - S_k\delta^j_l + S_l\delta^j_k)R^{kl}{}_{ji}, \tag{2.5.31}$$

which is equivalent to the contracted Bianchi identity (1.4.73). Therefore, the contracted Bianchi identity imposes the conservation law for the energy-momentum density. The gravitational field equations therefore contain the equations of motion of matter. Substituting (2.5.16) and (2.5.29) into the Belinfante-Rosenfeld relation (2.3.34) gives

$$\kappa T_{ik} = R_{ki} - \frac{1}{2}Rg_{ik} + \nabla^*_j(S^j{}_{ik} + 2S_k\delta^j_i - 2S_{(ik)}{}^j - 2S^j g_{ik}) = R_{ki} - \frac{1}{2}Rg_{ik}$$
$$+\nabla^*_j(-C^j{}_{ki} + C^l{}_{kl}\delta^j_i - C^{lj}{}_l g_{ik}). \tag{2.5.32}$$

Combining (1.4.65), (1.4.67) and (2.5.32) gives

$$\kappa T_{ik} = P_{ik} - \frac{1}{2}Pg_{ik} + C^l{}_{ki:l} - C^l{}_{kl:i} + C^j{}_{ki}C^l{}_{jl} - C^j{}_{kl}C^l{}_{ji} - \frac{1}{2}g_{ik}(-2C^{lj}{}_{l:j}$$
$$-C^{lj}{}_l C^m{}_{jm} + C^{mjl}C_{ljm}) - C^j{}_{ki:j} - C^j{}_{lj}C^l{}_{ki} + C^l{}_{kj}C^j{}_{li} + C^l{}_{ij}C^j{}_{kl} + C^j{}_{kj:i}$$
$$-C^l{}_{ki}C^j{}_{lj} - g_{ik}(C^{lj}{}_{l:j} + C^j{}_{lj}C^{ml}{}_m) - C_j(-C^j{}_{ki} + C^l{}_{kl}\delta^j_i - C^{lj}{}_l g_{ik}), \tag{2.5.33}$$

which is equivalent to the first Einstein-Cartan equation (2.5.10). Therefore, the relation between the Ricci tensor and the Riemannian Ricci tensor is equivalent to the Belinfante-Rosenfeld relation, whereas (2.5.29) is another form of the first Einstein-Cartan equation. Varying the action for the gravitational field and matter with respect to the metric tensor and the tensorial, antisymmetric part of the affine connection (torsion tensor) constitutes the *metric-affine variational principle of stationary action*.

2.5.3 Einstein-Hilbert action and Einstein equations

In almost all physical situations, the second Einstein-Cartan equation gives a torsion tensor whose squares of the leading components are negligibly small in magnitude relative to the leading components of the Riemann tensor (confer (2.5.11)). In those situations, we can approximate the torsion tensor as zero. In this approximation, the affine connection is equal to the Levi-Civita connection. Varying the action for the gravitational field and matter with respect to the metric tensor, with the affine connection constrained to be equal to the Levi-Civita connection, constitutes the *metric variational principle of stationary action*. If the Lagrangian density for matter does not depend on the affine connection, then the spin density vanishes, and so does the torsion tensor. In this case, the metric-affine field equations reduce to the metric field equations and we can use the metric principle of stationary action.

If the torsion tensor vanishes, the Einstein-Cartan action (2.5.1) reduces to

$$S = -\frac{1}{2\kappa c}\int P\sqrt{-\mathfrak{g}}d\Omega + S_{\mathrm{m}}, \tag{2.5.34}$$

which corresponds to the Lagrangian density (2.2.4). The action (2.5.34) subjected to varying the metric tensor is called the *Einstein-Hilbert action* for the gravitational field and matter. The Einstein-Hilbert action is a special case of the Einstein-Cartan action, where the affine connection is constrained to be symmetric and thus equal to the Levi-Civita connection. Varying (2.5.34) with respect to the metric tensor gives, similarly to (2.5.7),

$$\delta_{\mathrm{g}}S=-\frac{1}{2\kappa c}\int\Big(P_{ik}-\frac{1}{2}Pg_{ik}\Big)\sqrt{-\mathfrak{g}}\delta g^{ik}d\Omega+\frac{1}{2c}\int T_{ik}\sqrt{-\mathfrak{g}}\delta g^{ik}d\Omega. \tag{2.5.35}$$

Applying the stationarity of action $\delta_{\mathrm{g}}S=0$ to (2.5.35) for an arbitrary δg^{ik} gives the *Einstein equations* of the *general theory of relativity*:

$$G_{ik}=P_{ik}-\frac{1}{2}Pg_{ik}=\kappa T_{ik} \tag{2.5.36}$$

or

$$P_{ik}=\kappa\Big(T_{ik}-\frac{1}{2}Tg_{ik}\Big). \tag{2.5.37}$$

Because $\delta\int P\sqrt{-\mathfrak{g}}d\Omega=\delta\int \mathsf{G}\sqrt{-\mathfrak{g}}d\Omega$, where G is the noncovariant quantity (2.2.7), the left-hand side of the Einstein equations is

$$G_{ik}=\frac{1}{\sqrt{-\mathfrak{g}}}\frac{\partial(\sqrt{-\mathfrak{g}}\mathsf{G})}{\partial g^{ik}}. \tag{2.5.38}$$

The covariant conservation of the Einstein tensor (1.4.78) imposes the conservation of the metric energy-momentum tensor (2.4.30). The gravitational field equations therefore contain the equations of motion of matter, like for the Einstein-Cartan action.

The Einstein equations (2.5.36) are 10 second-order partial differential equations for: $10-4=6$ independent components of the metric tensor g_{ik} (the factor 4 is the number of the coordinates which can be chosen arbitrarily), 3 independent components of the four-velocity u^i, and either ϵ or p (which are related to each other by the equation of state). The contracted Bianchi identity (1.4.78) gives the equations of motion of matter. In vacuum, the Einstein equations are $10-4=6$ independent equations (the factor 4 is the number of constraints from the contracted Bianchi identity) for 6 independent components of the metric tensor g_{ik}.

In the Einstein equations (2.5.36), the only second time-derivatives of g_{ik} are the derivatives of the spatial components of the metric tensor, $\ddot{g}_{\alpha\beta}$, and they appear only in the $\alpha\beta$ components of the equations. Therefore, the initial values (at $t=0$) for $g_{\alpha\beta}$ and $\dot{g}_{\alpha\beta}$ can be chosen arbitrarily. The first time-derivatives $\dot{g}_{0\alpha}$ and $\dot{g}_{00}$ appear only in the $\alpha\beta$ components of the field equations (2.5.36). The 0α and 00 components of the field equations (2.5.36) give the initial values for $g_{0\alpha}$ and g_{00}. The undetermined initial values for $\dot{g}_{0\alpha}$ and $\dot{g}_{00}$ correspond to 4 degrees of freedom for a free gravitational field. A general gravitational field has 8 degrees of freedom: 4 degrees of freedom for a free gravitational field, 3 related to the four-velocity, and 1 related to the matter (ϵ or p). The above analysis regarding g_{ik} also applies to the first Einstein-Cartan equation (2.5.10) because the torsion tensor is algebraically related to the matter. If $g_{ik}=\eta_{ik}$, then the general theory of relativity is said to reduce to the *special theory of relativity*.

The Einstein equations (2.5.36) are a special case of the first Einstein-Cartan equation (2.5.10). They are valid when the matter fields do not depend on the affine connection, for which the spin density vanishes and $U_{ik}=0$. They are also an accurate approximation of (2.5.10) when the matter fields depend on the connection but the tensor U_{ik} can be neglected relative to T_{ik}. In the metric-affine variational principle of stationary action, in which the variations $\delta\omega^{ab}{}_i$ are independent of δe^i_a, the spin density is independent of the energy-momentum density. The Einstein-Cartan equations contain the covariant conservation laws for the energy-momentum and spin tensors, which generalize the special-relativistic conservation laws (2.4.41) and (2.4.48). The angular momentum density in (2.4.48) contains both the orbital and intrinsic parts. In the metric variational principle, in which the variations $\delta\omega^{ab}{}_i=\delta\varpi^{ab}{}_i$ are functions of the variations δe^i_a and their derivatives according to (1.5.37), the spin density is a function of the energy-momentum energy-momentum density. The Einstein equations contain only the covariant conservation law for the energy-momentum tensor, which generalizes the special-relativistic conservation law (2.4.41) with a symmetric energy-momentum tensor. The resulting conservation law (2.4.48) contains the angular momentum density only with the orbital part that depends on the energy-momentum density. Accordingly, the metric variational principle does not account for the intrinsic angular momentum (spin) of matter. Consequently, the existence of spin (which does not depend on energy and momentum) requires the metric-affine variational principle.

2.5.4 Utiyama action

The action (2.5.34) subjected to varying the tetrad is called the *Utiyama action*. The Utiyama action is a special case of the Sciama-Kibble action, where the torsion tensor is approximated as zero and the spin connection is constrained to be equal to the Levi-Civita spin connection (1.5.37) which depends on the tetrad. Using (2.3.6) gives

$$\delta S = -\frac{1}{2\kappa c}\int \delta(\mathfrak{e}P)d\Omega + \frac{1}{c}\int \mathfrak{T}_i^{\ a}\delta e^i_a d\Omega. \tag{2.5.39}$$

The Lagrangian density for the gravitational field is given by (2.2.4), with the Riemann scalar P given by (1.5.45) and (1.5.47):

$$\mathfrak{e}P = \mathfrak{e}e^i_a e^{jb}(\varpi^a_{\ bj,i} - \varpi^a_{\ bi,j} + \varpi^a_{\ ci}\varpi^c_{\ bj} - \varpi^a_{\ cj}\varpi^c_{\ bi}) = 2\mathfrak{e}^{ij}_{ab}(\varpi^{ab}_{\ \ j,i} + \varpi^a_{\ ci}\varpi^{cb}_{\ \ j}). \tag{2.5.40}$$

Varying $\mathfrak{e}P$ and omitting total derivatives gives in the absence of torsion, using $\delta\mathfrak{e} = \mathfrak{e}e^i_a\delta e^a_i$ and $\mathfrak{e}^{ij}_{ab|j} = \mathfrak{e}^{ij}_{ab,j} - \varpi^c_{\ aj}\mathfrak{e}^{ij}_{cb} - \varpi^c_{\ bj}\mathfrak{e}^{ij}_{ac} = 0$ (which results from (1.5.29)),

$$\begin{aligned}\delta(\mathfrak{e}P) &= (2P^a_{\ i} - Pe^a_i)\mathfrak{e}\delta e^i_a + 2\mathfrak{e}^{ij}_{ab}\delta(\varpi^{ab}_{\ \ j,i} + \varpi^a_{\ ci}\varpi^{cb}_{\ \ j}) = (2P^a_{\ i} - Pe^a_i)\mathfrak{e}\delta e^i_a \\ &+2(\mathfrak{e}^{ij}_{ab,j} - \varpi^c_{\ aj}\mathfrak{e}^{ij}_{cb} - \varpi^c_{\ bj}\mathfrak{e}^{ij}_{ac})\delta\varpi^{ab}_{\ \ i} = (2P^a_{\ i} - Pe^a_i)\mathfrak{e}\delta e^i_a.\end{aligned} \tag{2.5.41}$$

Equaling $\delta S = 0$ gives

$$P^a_{\ i} - \frac{1}{2}Pe^a_i = \kappa t_i^{\ a}, \tag{2.5.42}$$

which is equivalent to the Einstein equations (2.5.36) because of (2.3.5) and (2.3.34) (in the absence of torsion).

2.5.5 Einstein pseudotensor and principle of equivalence

Since the noncovariant quantity $\mathcal{G}$ (2.2.6) differs from $\sqrt{-\mathfrak{g}}P$ by total divergences, we can use Gauß-Stokes theorem (1.1.39) to write the action for the gravitational field as

$$S_g = -\frac{1}{2\kappa c}\int\Big(\mathcal{G} - \mathsf{g}^{np}(C^i_{\ ni}C^m_{\ \ pm} - C^i_{\ nm}C^m_{\ \ pi})\Big)d\Omega - \frac{1}{2\kappa c}\oint \mathfrak{g}^{ik}\{^{\,l}_{i\,k}\}dS_l + \frac{1}{2\kappa c}\oint \mathfrak{g}^{ik}\{^{\,l}_{i\,l}\}dS_k. \tag{2.5.43}$$

The hypersurface integrals in (2.5.43) do not contribute to the field equations and can be omitted. The action for the gravitational field and matter (2.5.3) thus reduces to

$$S = \frac{1}{c}\int\bigg(-\frac{1}{2\kappa}\Big(\mathcal{G} - \mathsf{g}^{np}(C^i_{\ ni}C^m_{\ \ pm} - C^i_{\ nm}C^m_{\ \ pi})\Big) + \mathfrak{L}_{\mathrm{m}}\bigg)d\Omega. \tag{2.5.44}$$

The condition $\delta_{\mathrm{g}}S = 0$, which is equivalent to

$$\frac{\delta}{\delta g^{jl}}\bigg(-\frac{1}{2\kappa}\Big(\mathcal{G} - \mathsf{g}^{np}(C^i_{\ ni}C^m_{\ \ pm} - C^i_{\ nm}C^m_{\ \ pi})\Big) + \mathfrak{L}_{\mathrm{m}}\bigg) = 0, \tag{2.5.45}$$

gives the first Einstein-Cartan equation (2.5.10). Because $\mathcal{G}$ depends on g^{ij} and its first derivatives $g^{ij}_{\ \ ,k}$, we can construct a canonical energy-momentum density (2.4.42) corresponding to the gravitational field, treating $-\frac{1}{2\kappa}\mathcal{G}$ like $\mathfrak{L}_{\mathrm{m}}$ and g^{ik} like a matter field ϕ:

$$\mathfrak{t}_k^{\ i} = -\frac{1}{2\kappa}\bigg(\frac{\partial\mathcal{G}}{\partial g^{jl}_{\ \ ,i}}g^{jl}_{\ \ ,k} - \delta^i_k\mathcal{G}\bigg). \tag{2.5.46}$$

This quantity is not a tensor density because $\mathcal{G}$ is not a scalar density. Its division by $\sqrt{-\mathfrak{g}}$ defines the *Einstein energy-momentum pseudotensor* for the gravitational field:

$$\frac{\mathfrak{t}_k^{\ i}}{\sqrt{-\mathfrak{g}}} = -\frac{1}{2\kappa}\biggl(\frac{\partial\mathsf{G}}{\partial g^{jl}_{\ \ ,i}}g^{jl}_{\ \ ,k} - \delta^i_k\mathsf{G}\biggr), \tag{2.5.47}$$

where G is the quantity (2.2.7). Differentiating (2.5.46) gives

$$\begin{aligned}2\kappa\mathfrak{t}_{k\ ,i}^{\ i} &= -\partial_i\frac{\partial\mathcal{G}}{\partial g^{jl}_{\ \ ,i}}g^{jl}_{\ \ ,k} - \frac{\partial\mathcal{G}}{\partial g^{jl}_{\ \ ,i}}g^{jl}_{\ \ ,ki} + \mathcal{G}_{,k} = -\partial_i\frac{\partial\mathcal{G}}{\partial g^{jl}_{\ \ ,i}}g^{jl}_{\ \ ,k} - \frac{\partial\mathcal{G}}{\partial g^{jl}_{\ \ ,i}}g^{jl}_{\ \ ,ki} + \frac{\partial\mathcal{G}}{\partial g^{jl}}g^{jl}_{\ \ ,k} \\ &+ \frac{\partial\mathcal{G}}{\partial g^{jl}_{\ \ ,i}}g^{jl}_{\ \ ,ik} = \biggl(\frac{\partial\mathcal{G}}{\partial g^{jl}} - \partial_i\frac{\partial\mathcal{G}}{\partial g^{jl}_{\ \ ,i}}\biggr)g^{jl}_{\ \ ,k} = \frac{\delta\mathcal{G}}{\delta g^{jl}}g^{jl}_{\ \ ,k},\end{aligned} \tag{2.5.48}$$

which, using (2.5.11) and (2.5.45), leads to

$$\mathfrak{t}_{k\ ,i}^{\ i} = \frac{\delta\bigl(\mathfrak{L}_{\mathrm{m}} + \frac{1}{2\kappa}\sqrt{-\mathfrak{g}}g^{np}(C^i_{\ ni}C^m_{\ \ pm} - C^i_{\ nm}C^m_{\ \ pi})\bigr)}{\delta g^{jl}}g^{jl}_{\ \ ,k} = \frac{1}{2}(\mathcal{T}_{jl} + \sqrt{-\mathfrak{g}}U_{jl})g^{jl}_{\ \ ,k}. \tag{2.5.49}$$

The Riemannian conservation law (2.5.20) gives

$$\begin{aligned}(\mathcal{T}_k^{\ i} + \sqrt{-\mathfrak{g}}U_k^{\ i})_{,i} &= \{{}^{\,l}_{k\,i}\}(\mathcal{T}_l^{\ i} + \sqrt{-\mathfrak{g}}U_l^{\ i}) = \frac{1}{2}g^{lm}g_{im,k}(\mathcal{T}_l^{\ i} + \sqrt{-\mathfrak{g}}U_l^{\ i}) \\ &= -\frac{1}{2}g^{lm}_{\ \ \ ,k}(\mathcal{T}_{lm} + \sqrt{-\mathfrak{g}}U_{lm}).\end{aligned} \tag{2.5.50}$$

The total energy-momentum density for the gravitational field and matter is given by

$$\mathfrak{t}_i^{\ k} + \mathcal{T}_i^{\ k} + \sqrt{-\mathfrak{g}}U_i^{\ k} = \mathfrak{t}_i^{\ k} + \frac{\sqrt{-\mathfrak{g}}}{\kappa}G_i^{\ k}. \tag{2.5.51}$$

As a result of adding (2.5.49) and (2.5.50), the ordinary divergence of this quantity vanishes, as that in (2.4.59):

$$(\mathfrak{t}_k^{\ i} + \mathcal{T}_k^{\ i} + \sqrt{-\mathfrak{g}}U_k^{\ i})_{,i} = \biggl(\mathfrak{t}_k^{\ i} + \frac{\sqrt{-\mathfrak{g}}}{\kappa}G_k^{\ i}\biggr)_{,i} = 0. \tag{2.5.52}$$

Integrating (2.5.52) over the four-volume and using the Gauß' theorem (1.1.39) gives

$$\oint(\mathfrak{t}_k^{\ i} + \mathcal{T}_k^{\ i} + \sqrt{-\mathfrak{g}}U_k^{\ i})dS_i = 0. \tag{2.5.53}$$

The corresponding four-momentum (2.4.63) of the gravitational field and matter, which is not a vector (it transforms like a vector only for Lorentz transformations), is therefore conserved:

$$P_i = \frac{1}{c}\int(\mathfrak{t}_i^{\ k} + \mathcal{T}_i^{\ k} + \sqrt{-\mathfrak{g}}U_i^{\ k})dS_k = \mathrm{const.} \tag{2.5.54}$$

Using (1.4.32) and (1.4.43) gives

$$\frac{\partial\{{}^{\,m}_{i\,l}\}}{\partial g^{rs}_{\ \ ,n}} = -\frac{1}{2}(g_{l(r}\delta^m_{s)}\delta^n_i + g_{i(r}\delta^m_{s)}\delta^n_l - g^{mn}g_{i(r}g_{s)l}), \tag{2.5.55}$$

$$\frac{\partial\{{}^{\,l}_{m\,l}\}}{\partial g^{rs}_{\ \ ,n}} = -\frac{1}{2}g_{rs}\delta^n_m. \tag{2.5.56}$$

Consequently, we obtain

$$\begin{aligned}\frac{\partial\mathsf{G}}{\partial g^{rs}_{\ \ ,n}} &= 2g^{ik}\{{}^{\,m}_{i\,l}\}\frac{\partial\{{}^{\,l}_{m\,k}\}}{\partial g^{rs}_{\ \ ,n}} - g^{ik}\{{}^{\,l}_{m\,l}\}\frac{\partial\{{}^{\,m}_{i\,k}\}}{\partial g^{rs}_{\ \ ,n}} - g^{ik}\{{}^{\,m}_{i\,k}\}\frac{\partial\{{}^{\,l}_{m\,l}\}}{\partial g^{rs}_{\ \ ,n}} \\ &= -\{{}^{\,n}_{r\,s}\} + \frac{1}{2}\Bigl(\{{}^{\,l}_{s\,l}\}\delta^n_r + \{{}^{\,l}_{r\,l}\}\delta^n_s - \{{}^{\,l}_{m\,l}\}g^{mn}g_{rs}\Bigr) + \frac{1}{2}\{{}^{\,n}_{j\,l}\}g^{jl}g_{rs},\end{aligned} \tag{2.5.57}$$

which leads to

$$\frac{\partial\mathcal{G}}{\partial g^{rs}{}_{,i}}g^{rs}{}_{,k} = -\sqrt{-\mathfrak{g}}\{{}^{\,i}_{r\,s}\}g^{rs}{}_{,k} + \sqrt{-\mathfrak{g}}\{{}^{\,l}_{r\,l}\}g^{ri}{}_{,k} + \{{}^{\,l}_{m\,l}\}g^{mi}(\sqrt{-\mathfrak{g}})_{,k} - \{{}^{\,i}_{j\,l}\}g^{jl}(\sqrt{-\mathfrak{g}})_{,k}. \qquad (2.5.58)$$

The Einstein pseudotensor (2.5.47) can thus be written as

$$\frac{\mathfrak{t}_k{}^i}{\sqrt{-\mathfrak{g}}} = \frac{1}{2\kappa\sqrt{-\mathfrak{g}}}(\{{}^{\,i}_{l\,m}\}\mathsf{g}^{lm}{}_{,k} - \{{}^{\,l}_{m\,l}\}\mathsf{g}^{mi}{}_{,k} + \delta^i_k\mathcal{G}). \qquad (2.5.59)$$

Accordingly, $\mathfrak{t}_{ik}$ is not symmetric in the indices i, k.

Since the derivatives $\mathsf{g}^{ik}{}_{,j}$ are homogeneous linear functions of the Christoffel symbols, the Einstein pseudotensor (2.5.59) is a homogeneous quadratic function of the Christoffel symbols, so it vanishes in the locally Galilean and geodesic frame of reference. It can also differ from zero in the Minkowski spacetime (in the absence of the gravitational field) if we choose the coordinates such that the Christoffel symbols do not vanish. Therefore, the energy of the gravitational field is not absolutely localized in spacetime; it depends on the choice of the coordinates. A physically meaningful energy-momentum pseudotensor can be constructed if the coordinates are asymptotically (far from the sources of the field) Cartesian, so the Christoffel symbols tend asymptotically to zero. If we neglect torsion, then the gravitational field can be always eliminated locally by transforming the coordinate system to the locally Galilean and geodesic frame of reference in which the Einstein pseudotensor vanishes. This property of the gravitational field is referred to as the *principle of equivalence.*

We can construct from P^i (2.5.54) the angular momentum four-tensor (2.4.106):

$$M^{ik} = \frac{1}{c}\int\big(x^i(\mathfrak{t}^{kl} + \mathcal{T}^{kl} + \sqrt{-\mathfrak{g}}U^{kl}) - x^k(\mathfrak{t}^{il} + \mathcal{T}^{il} + \sqrt{-\mathfrak{g}}U^{il})\big)dS_l, \qquad (2.5.60)$$

which is not a tensor (it transforms like a tensor only for Lorentz transformations). Because the quantity $\mathfrak{t}^{ik}$ is not symmetric, the four-tensor (2.5.60) is not conserved. The conservation law (2.5.52) infers that the quantity (2.5.51) can be written as

$$\mathfrak{t}_k{}^l + \frac{\sqrt{-\mathfrak{g}}}{\kappa}G_k{}^l = \eta_k{}^{li}{}_{,i}, \qquad (2.5.61)$$

where $\eta_k{}^{li}$ satisfies

$$\eta_k{}^{li} = -\eta_k{}^{il}, \qquad (2.5.62)$$

analogously to (2.4.58). The quantity $\eta_i{}^{kl}$ is proportional to $\sqrt{-\mathfrak{g}}$ and linear in first derivatives of g_{ik}. Substituting (2.5.61) into (2.5.54) gives, using the Gauß-Stokes theorem (1.1.38) and (2.5.51),

$$P_i = \frac{1}{c}\int\eta_i{}^{kl}{}_{,l}dS_k = \frac{1}{2c}\int(\eta_i{}^{kl}{}_{,l}dS_k - \eta_i{}^{kl}{}_{,k}dS_l) = \frac{1}{2c}\oint\eta_i{}^{kl}df^*_{kl}, \qquad (2.5.63)$$

where $df^\star_{ik}$ is the element of the closed surface which bounds the hypersurface. If the hypersurface is a volume hypersurface, $dS_k = \delta^0_k dV$, then the four-momentum of the gravitational field and matter (2.5.63) in a given volume can be written as a surface integral:

$$P_i = \frac{1}{2c}\oint\eta_i{}^{0\alpha}df^*_{0\alpha} = \frac{1}{c}\oint\eta_i{}^{0\alpha}df_\alpha, \qquad (2.5.64)$$

where df_α is the element of the closed surface which bounds the volume.

If we neglect torsion, then $U_{ik} = 0$ and the left-hand side of (2.5.51) reduces to $\mathfrak{t}_i{}^k + \mathcal{T}_i{}^k$. The conservation law (2.5.52) reduces to $(\mathfrak{t}_i{}^k + \mathcal{T}_i{}^k)_{,k} = 0$. This quantity can also be written as (2.5.61), thereby the corresponding four-momentum of the gravitational field and matter is also given by (2.5.64).

The quantity $\eta_i{}^{kl}$ is not unique. The relation (2.5.61) is invariant under a transformation

$$\eta_{ik}{}^l \rightarrow \eta_{ik}{}^l + \zeta_i{}^{klm}{}_{,m}, \qquad (2.5.65)$$

where ${\zeta_i}^{klm}$ is a quantity satisfying

$$ {\zeta_i}^{klm} = -{\zeta_i}^{kml}. \tag{2.5.66} $$

We define

$$ \lambda^{iklm} = \frac{1}{2\kappa}(-\mathfrak{g})(g^{ik}g^{lm} - g^{il}g^{km}), \tag{2.5.67} $$

$$ h^{ikl} = {\lambda^{iklm}}_{,m} = -h^{ilk}. \tag{2.5.68} $$

The quantity (2.5.68) is equal, by means of (1.4.40) and (1.4.43), to

$$ h^{ikl} = \frac{1}{2\kappa}(-\mathfrak{g})(\{{}^{\,l}_{j\,m}\}g^{km}g^{ij} + \{{}^{\,k}_{m\,n}\}g^{mn}g^{il} - \{{}^{\,m}_{m\,n}\}g^{kn}g^{il} $$
$$ -\{{}^{\,k}_{j\,m}\}g^{lm}g^{ij} - \{{}^{\,l}_{m\,n}\}g^{mn}g^{ik} + \{{}^{\,m}_{m\,n}\}g^{ln}g^{ik}). \tag{2.5.69} $$

Equations (1.4.40), (1.4.43), (2.2.6) and (2.5.59) infer that

$$ {\eta_i}^{kl} = \frac{1}{\sqrt{-\mathfrak{g}}}{h_i}^{kl} \tag{2.5.70} $$

satisfies (2.5.61). Taking ${\eta_i}^{kl}$ in (2.5.61), which is not related to (2.5.70) by (2.5.65), leads to a different energy-momentum pseudotensor. If we take

$$ {\eta_i}^{kl} = \frac{1}{\sqrt{-\mathfrak{g}}}(2{h_i}^{kl} - \delta^k_i {h_j}^{jl} + \delta^l_i {h_j}^{jk}), \tag{2.5.71} $$

then the right-hand side of (2.5.61) is given by

$$ {{\eta_i}^{kl}}_{,l} = \frac{\sqrt{-\mathfrak{g}}}{\kappa}(g_{in,m} - g_{im,n})g^{km}g^{ln}. \tag{2.5.72} $$

The construction of a conserved four-momentum for the gravitational field and matter is possible because the Lagrangian density for the gravitational field $\mathfrak{L}_{\mathrm{g}}$ (2.2.2) is linear in the curvature tensor. Accordingly, it is linear in second derivatives of the metric tensor, so we can use the noncovariant quantity $\mathcal{G}$ (2.2.6). Another scalar density which is linear in curvature is $\epsilon^{ijkl}R_{ijkl}$. Using (1.4.49), (1.4.64) and (1.4.75), and omitting a total derivative, this parity-violating expression reduces to $-2\epsilon^{iklm}{C^j}_{km}C_{jil}$, which does not depend on the derivatives of the metric tensor and thus does not describe the gravitational field.

2.5.6 Møller pseudotensor

The Riemann scalar P (1.5.47) is linear in first derivatives of the Levi-Civita spin connection ${\varpi^a}_{bi}$:

$$ \mathfrak{e}P = (\mathfrak{e}e^i_a e^j_b {\varpi^{ab}}_j)_{,i} - (\mathfrak{e}e^i_a e^j_b)_{,i}{\varpi^{ab}}_j - (\mathfrak{e}e^i_a e^j_b {\varpi^{ab}}_i)_{,j} + (\mathfrak{e}e^i_a e^j_b)_{,j}{\varpi^{ab}}_i + \mathfrak{e}e^i_a e^j_b {\varpi^{ac}}_i {{\varpi_c}^b}_j $$
$$ -\mathfrak{e}e^i_a e^j_b {\varpi^{ac}}_j {{\varpi_c}^b}_i = 2(\mathfrak{e}e^i_a e^j_b {\varpi^{ab}}_j)_{,i} - 2(\mathfrak{e}e^i_a e^j_b)_{,i}{\varpi^{ab}}_j + \mathfrak{e}e^i_a e^j_b {\varpi^{ac}}_i {{\varpi_c}^b}_j $$
$$ -\mathfrak{e}e^i_a e^j_b {\varpi^{ac}}_j {{\varpi_c}^b}_i, \tag{2.5.73} $$

where we used (1.5.45). We can therefore subtract from $\mathfrak{e}P$ total derivatives without altering the field equations, replacing it by a noncovariant quantity $\mathcal{M}$:

$$ \mathcal{M} = -2(\mathfrak{e}e^i_a e^j_b)_{,i}{\varpi^{ab}}_j + \mathfrak{e}e^i_a e^j_b {\varpi^{ac}}_i {{\varpi_c}^b}_j - \mathfrak{e}e^i_a e^j_b {\varpi^{ac}}_j {{\varpi_c}^b}_i $$
$$ = -2\mathfrak{e}(\{{}^{\,k}_{k\,i}\}{\varpi^{ij}}_j + {\varpi^i}_{ai}{\varpi^{aj}}_j - \{{}^{\,i}_{k\,i}\}{\varpi^{kj}}_j + {\varpi^j}_{bi}{\varpi^{ib}}_j - \{{}^{\,j}_{k\,i}\}{\varpi^{ik}}_j) $$
$$ +\mathfrak{e}({\varpi^{ic}}_i{{\varpi_c}^j}_j - {\varpi^{ic}}_j{{\varpi_c}^j}_i) = \mathfrak{e}({\varpi^{ia}}_i{\varpi^j}_{aj} - {\varpi^{ia}}_j{\varpi^j}_{ai}), \tag{2.5.74} $$

using (1.4.43) and (1.5.37). We also define

$$ \mathsf{M} = \frac{\mathcal{M}}{\mathfrak{e}} = {\varpi^{ia}}_i{\varpi^j}_{aj} - {\varpi^{ia}}_j{\varpi^j}_{ai}. \tag{2.5.75} $$

The Riemannian part (2.2.4) of the Lagrangian density for the gravitational field reduces accordingly to

$$\mathfrak{L}_{\mathrm{g}}^{\{\}} = -\frac{1}{2\kappa}\mathcal{M} = -\frac{1}{2\kappa}\mathfrak{e}\mathsf{M}. \tag{2.5.76}$$

Since the noncovariant quantity $\mathcal{M}$ (2.5.74) differs from $\mathfrak{e}P$ by a total divergence, the action for the gravitational field and matter (2.5.3) is equivalent to

$$S = \frac{1}{c}\int\Big(-\frac{1}{2\kappa}\Big(\mathcal{M} - \mathfrak{e}(C^{ia}_{i}C^{j}_{aj} - C^{ia}_{j}C^{j}_{ai})\Big) + \mathfrak{L}_{\mathrm{m}}\Big)d\Omega. \tag{2.5.77}$$

Using (1.5.39), this action can be written as

$$S = \frac{1}{c}\int\Big(\frac{\mathfrak{e}}{2\kappa}(\omega^{ia}_{i}\omega^{j}_{aj} - \omega^{ia}_{j}\omega^{j}_{ai} - 2\omega^{ia}_{i}\varpi^{j}_{aj} + 2\omega^{ia}_{j}\varpi^{j}_{ai}) + \mathfrak{L}_{\mathrm{m}}\Big)d\Omega. \tag{2.5.78}$$

The condition $\delta_{\mathrm{e}}S = 0$, which is equivalent to

$$\frac{\delta}{\delta e^j_a}\Big(-\frac{1}{2\kappa}\Big(\mathcal{M} - \mathfrak{e}g^{np}(C^{i}_{ni}C^{m}_{pm} - C^{i}_{nm}C^{m}_{pi})\Big) + \mathfrak{L}_{\mathrm{m}}\Big) = 0, \tag{2.5.79}$$

gives the Einstein equations (2.5.28). Because of (1.5.22), $\mathcal{M}$ depends on the tetrad e^i_a and its first derivatives $e^i_{a,j}$. Therefore, analogously to (2.5.46), we can construct a canonical energy-momentum density corresponding to the gravitational field, treating $-\frac{1}{2\kappa}\mathcal{M}$ like $\mathfrak{L}_{\mathrm{m}}$ and e^i_a like a matter field ϕ:

$$\mathfrak{m}_k^{i} = -\frac{1}{2\kappa}\Big(\frac{\partial\mathcal{M}}{\partial e^j_{a,i}}e^j_{a,k} - \delta^i_k\mathcal{M}\Big). \tag{2.5.80}$$

This quantity is not a tensor density because $\mathcal{M}$ is not a scalar density. Its division by $\mathfrak{e}$ defines the *Møller energy-momentum pseudotensor* for the gravitational field:

$$\frac{\mathfrak{m}_k^{i}}{\mathfrak{e}} = -\frac{1}{2\kappa}\Big(\frac{\partial\mathsf{M}}{\partial e^j_{a,i}}e^j_{a,k} - \delta^i_k\mathsf{M}\Big). \tag{2.5.81}$$

Differentiating (2.5.80) gives

$$\begin{aligned}2\kappa\mathfrak{m}_k^{i}{}_{,i} &= -\partial_i\frac{\partial\mathcal{M}}{\partial e^j_{a,i}}e^j_{a,k} - \frac{\partial\mathcal{M}}{\partial e^j_{a,i}}e^j_{a,ki} + \mathcal{M}_{,k} = -\partial_i\frac{\partial\mathcal{M}}{\partial e^j_{a,i}}e^j_{a,k} - \frac{\partial\mathcal{M}}{\partial e^j_{a,i}}e^j_{a,ki} + \frac{\partial\mathcal{M}}{\partial e^j_a}e^j_{a,k} \\ &+\frac{\partial\mathcal{M}}{\partial e^j_{a,i}}e^j_{a,ki} = \Big(\frac{\partial\mathcal{M}}{\partial e^j_a} - \partial_i\frac{\partial\mathcal{M}}{\partial e^j_{a,i}}\Big)e^j_{a,k} = \frac{\delta\mathcal{M}}{\delta e^j_a}e^j_{a,k},\end{aligned} \tag{2.5.82}$$

which, using (2.5.11) and (2.5.79), leads to

$$\mathfrak{m}_k^{i}{}_{,i} = \frac{\delta\big(\mathfrak{L}_{\mathrm{m}} + \frac{1}{2\kappa}\mathfrak{e}g^{np}(C^{i}_{ni}C^{m}_{pm} - C^{i}_{nm}C^{m}_{pi})\big)}{\delta e^j_a}e^j_{a,k} = (\mathfrak{T}_j^{a} + \mathfrak{e}U_j^{a})e^j_{a,k}. \tag{2.5.83}$$

The conservation law (2.5.50) gives, using (1.5.5),

$$(\mathcal{T}_k^{i} + \mathfrak{e}U_k^{i})_{,i} = -\frac{1}{2}g^{lm}_{,k}(\mathcal{T}_{lm} + \mathfrak{e}U_{lm}) = -e^j_{a,k}(\mathcal{T}_j^{a} + \mathfrak{e}U_j^{a}). \tag{2.5.84}$$

Adding (2.5.83) and (2.5.84) leads, using (2.5.28) and (2.5.32), to

$$(\mathfrak{m}_k^{i} + \mathcal{T}_k^{i} + \mathfrak{e}U_k^{i})_{,i} = e^j_{a,k}(\mathfrak{T}_j^{a} - \mathcal{T}_j^{a}) = e^j_{a,k}\frac{\mathfrak{e}}{\kappa}\nabla^*_j(C^{ia}_{j} - 2S^a\delta^i_j + 2S^i e^a_j). \tag{2.5.85}$$

If we neglect torsion, then $U_{ik} = 0$ and the total energy-momentum density for the gravitational field and matter is given by The quantity

$$\mathfrak{m}_i^{k} + \mathcal{T}_i^{k} = \mathfrak{m}_i^{k} + \frac{\mathfrak{e}}{\kappa}G_i^{k}. \tag{2.5.86}$$

The ordinary divergence of this quantity vanishes, as that in (2.5.52):

$$(\mathfrak{m}_k{}^i + \mathcal{T}_k{}^i)_{,i} = 0. \tag{2.5.87}$$

Integrating (2.5.87) over the four-volume and using the Gauß-Stokes theorem gives

$$\oint (\mathfrak{m}_k{}^i + \mathcal{T}_k{}^i) dS_i = 0. \tag{2.5.88}$$

The corresponding four-momentum (2.4.63) of the gravitational field and matter in this approximation, which is not a vector (it transforms like a vector only for Lorentz transformations), is therefore conserved:

$$P_i = \frac{1}{c}\int (\mathfrak{m}_i{}^k + \mathcal{T}_i{}^k) dS_k = \text{const.} \tag{2.5.89}$$

Using (1.4.32), (1.4.43), (1.5.5), (1.5.10) and (1.5.37) gives

$$\frac{\partial \varpi^i{}_{aj}}{\partial e^m_{b,l}} = \frac{1}{2}(\delta^i_m \delta^b_a \delta^l_j - \delta^i_m e^l_a e^b_j - e^{ib} e^l_a g_{jm} - e^{ib} e_{am} \delta^l_j + g^{il} e_{am} e^b_j + g^{il} \delta^b_a g_{jm}), \tag{2.5.90}$$

$$\frac{\partial \varpi^i{}_{ai}}{\partial e^m_{b,l}} = \delta^b_a \delta^l_m - e^l_a e^b_m. \tag{2.5.91}$$

Consequently, we obtain

$$\frac{\partial \mathsf{M}}{\partial e^m_{b,l}} = 2(\delta^l_m \varpi^{jb}{}_j - e^b_m \varpi^{jl}{}_j - \varpi^{lb}{}_m + \varpi_m{}^{bl} + \varpi_m{}^{lb}). \tag{2.5.92}$$

The Møller energy-momentum pseudotensor (2.5.81) can thus be written, using (1.5.37), as

$$\frac{\mathfrak{m}_k{}^i}{\mathfrak{e}} = \frac{1}{\kappa}\left(-\varpi^i{}_{ak}\varpi^{ja}{}_j - \varpi^j{}_{ak}\varpi_j{}^{ai} + \{{}^{\,i}_{l\,k}\}\varpi^{jl}{}_j - \{{}^{\,l}_{l\,k}\}\varpi^{ji}{}_j + \varpi^{lji}g_{kl,j} - \varpi^{ijl}g_{jl,k} + \frac{1}{2}\delta^i_k \mathsf{M}\right). \tag{2.5.93}$$

Accordingly, $\mathfrak{m}_{ik}$ is not symmetric in the indices i, k. We can construct from P^i (2.5.89) the angular momentum four-tensor (2.4.106):

$$M^{ik} = \frac{1}{c}\int \big(x^i(\mathfrak{m}^{kl} + \mathcal{T}^{kl}) - x^k(\mathfrak{m}^{il} + \mathcal{T}^{il})\big) dS_l, \tag{2.5.94}$$

which is not a tensor (it transforms like a tensor only for Lorentz transformations). Because the quantity $\mathfrak{m}^{ik}$ is not symmetric, the four-tensor (2.5.94) is not conserved.

Since the Levi-Civita spin connection and the Christoffel symbols are homogeneous linear functions of the derivatives $e^i_{a,k}$, these derivatives are homogeneous linear functions of the Christoffel symbols. Consequently, the Møller energy-momentum pseudotensor (2.5.93) is a homogeneous quadratic function of the Christoffel symbols, so it vanishes in the locally Galilean and geodesic frame of reference. This pseudotensor depends on the choice of both the coordinates and tetrad. To fix the tetrad, one can impose on it 6 constraints which are covariant under constant Lorentz transformations but not under general Lorentz transformations (otherwise such constraints would not fix the tetrad since Lorentz transformations are tetrad rotations).

2.5.7 Landau-Lifshitz energy-momentum pseudotensor

In the locally Galilean and geodesic frame of reference, the relations (2.5.61) and (2.5.70) give

$$G^{ik} = \kappa h^{ikl}{}_{,l}, \tag{2.5.95}$$

where h^{ikl} is given by (2.5.67) and (2.5.68). The first Einstein-Cartan equation (2.5.10) reduces in this frame to

$$h^{ikl}{}_{,l} = T^{ik} + U^{ik}. \tag{2.5.96}$$

The Riemannian conservation law (2.5.20) reduces to $(T^{ik}+U^{ik})_{,k}=0$, which is consistent with (2.5.96). In an arbitrary frame of reference, (2.5.95) is not valid. We define a quantity $\mathfrak{t}^{ik}$ such that

$$(-\mathfrak{g})(\kappa\mathfrak{t}^{ik}+G^{ik})=h^{ikl}{}_{,l}. \tag{2.5.97}$$

Therefore, we have

$$\big((-\mathfrak{g})(\mathfrak{t}^{ik}+T^{ik}+U^{ik})\big)_{,k}=0. \tag{2.5.98}$$

The corresponding four-momentum of the gravitational field and matter is conserved:

$$P^i=\frac{1}{c}\int(-\mathfrak{g})(\mathfrak{t}^{ik}+T^{ik}+U^{ik})dS_k=\text{const.} \tag{2.5.99}$$

The quantity $\mathfrak{t}^{ik}$ is not a tensor density, thereby the conserved four-momentum P^i (2.5.99) is not a vector. The four-momentum P^i is not a vector even for Lorentz transformations, because of the factor $-\mathfrak{g}$ instead of the weight-1 density $\sqrt{-\mathfrak{g}}$. Dividing P^i by $\sqrt{-\mathfrak{g}}$ at some fixed point (a natural choice is infinity) turns it into a vector with respect to Lorentz transformations. Using (2.5.97) turns (2.5.99), for the hypersurface $dS_0=dV$, into

$$P^i=\frac{1}{c}\int h^{ikl}{}_{,l}dS_k=\frac{1}{2c}\oint h^{ikl}df^*_{kl}=\frac{1}{c}\oint h^{i0\alpha}df_\alpha. \tag{2.5.100}$$

This formula differs from the four-momentum of the gravitational field and matter (2.5.64), constructed from the Einstein energy-momentum pseudotensor, by an additional factor $\sqrt{-\mathfrak{g}}$ in the integrand (confer (2.5.70)). The quantity $\mathfrak{t}^{ik}$ is referred to as the *Landau-Lifshitz energy-momentum pseudotensor* for the gravitational field.

The explicit expression for the Landau-Lifshitz pseudotensor is

$$\begin{aligned}\mathfrak{t}^{ik}=\frac{1}{2\kappa}\Big(&(g^{il}g^{km}-g^{ik}g^{lm})(2\{{}^{\,n}_{l\,m}\}\{{}^{\,p}_{n\,p}\}-\{{}^{\,n}_{l\,p}\}\{{}^{\,p}_{m\,n}\}-\{{}^{\,n}_{l\,n}\}\{{}^{\,p}_{m\,p}\})\\ &+g^{il}g^{mn}(\{{}^{\,k}_{l\,p}\}\{{}^{\,p}_{m\,n}\}+\{{}^{\,k}_{m\,n}\}\{{}^{\,p}_{l\,p}\}-\{{}^{\,k}_{n\,p}\}\{{}^{\,p}_{l\,m}\}-\{{}^{\,k}_{l\,m}\}\{{}^{\,p}_{n\,p}\})\\ &+g^{kl}g^{mn}(\{{}^{\,i}_{l\,p}\}\{{}^{\,p}_{m\,n}\}+\{{}^{\,i}_{m\,n}\}\{{}^{\,p}_{l\,p}\}-\{{}^{\,i}_{n\,p}\}\{{}^{\,p}_{l\,m}\}-\{{}^{\,i}_{l\,m}\}\{{}^{\,p}_{n\,p}\})\\ &+g^{lm}g^{np}(\{{}^{\,i}_{l\,n}\}\{{}^{\,k}_{m\,p}\}-\{{}^{\,i}_{l\,m}\}\{{}^{\,k}_{n\,p}\})\Big)\end{aligned} \tag{2.5.101}$$

or

$$\begin{aligned}(-\mathfrak{g})\mathfrak{t}^{ik}=\frac{1}{2\kappa}\Big(&\mathfrak{g}^{ik}{}_{,l}\mathfrak{g}^{lm}{}_{,m}-\mathfrak{g}^{il}{}_{,l}\mathfrak{g}^{km}{}_{,m}+\frac{1}{2}g^{ik}g_{lm}\mathfrak{g}^{ln}{}_{,p}\mathfrak{g}^{pm}{}_{,n}\\ &-(g^{il}g_{mn}\mathfrak{g}^{kn}{}_{,p}\mathfrak{g}^{mp}{}_{,l}+g^{kl}g_{mn}\mathfrak{g}^{in}{}_{,p}\mathfrak{g}^{mp}{}_{,l})+g_{lm}g^{np}\mathfrak{g}^{il}{}_{,n}\mathfrak{g}^{km}{}_{,p}\\ &+\frac{1}{8}(2g^{il}g^{km}-g^{ik}g^{lm})(2g_{np}g_{qr}-g_{pq}g_{nr})\mathfrak{g}^{nr}{}_{,l}\mathfrak{g}^{pq}{}_{,m}\Big).\end{aligned} \tag{2.5.102}$$

This pseudotensor is symmetric in the indices i,k. The corresponding angular momentum of the gravitational field and matter, constructed from P^i as that in (2.4.106), is therefore conserved:

$$M^{ik}=\frac{1}{c}\int\big(x^i(\mathfrak{t}^{kl}+T^{kl}+U^{kl})-x^k(\mathfrak{t}^{il}+T^{il}+U^{il})\big)(-\mathfrak{g})dS_l=\text{const.} \tag{2.5.103}$$

Dividing M^{ik} by $\sqrt{-\mathfrak{g}}$ at infinity turns it into an antisymmetric tensor under Lorentz transformations. Using (2.5.68) and (2.5.99) turns (2.5.103), for the hypersurface $dS_0=dV$, into

$$\begin{aligned}M^{ik}&=\frac{1}{c}\int(x^i\lambda^{klmn}{}_{,nm}-x^k\lambda^{ilmn}{}_{,nm})dS_l=\frac{1}{2c}\oint(x^i\lambda^{klmn}{}_{,n}-x^k\lambda^{ilmn}{}_{,n})df^*_{lm}\\ &-\frac{1}{c}\oint(\lambda^{klin}-\lambda^{ilkn})_{,n}dS_l=\frac{1}{c}\oint(x^ih^{k0\alpha}-x^kh^{i0\alpha}+\lambda^{i0\alpha k})df_\alpha.\end{aligned} \tag{2.5.104}$$

Choosing the volume hypersurface $dV = dS_0$ gives

$$M^{ik} = \frac{1}{c}\int \big(x^i(\mathfrak{t}^{k0} + T^{k0} + U^{k0}) - x^k(\mathfrak{t}^{i0} + T^{i0} + U^{i0})\big)(-\mathfrak{g})dV = \text{const.} \tag{2.5.105}$$

The conservation of $M^{0\alpha}$ in (2.5.105) divided by the conserved P^0 in (2.5.99) gives a uniform motion (2.4.108) (without the intrinsic angular momentum) of the center of inertia for the gravitational field and matter. The velocity of this motion is given by (2.4.109) and (2.5.99), and the coordinates of the center of inertia are

$$X^\alpha = \frac{\int x^\alpha(\mathfrak{t}^{00} + T^{00} + U^{00})(-\mathfrak{g})dV}{\int(\mathfrak{t}^{00} + T^{00} + U^{00})(-\mathfrak{g})dV}. \tag{2.5.106}$$

These coordinates, like (2.4.110), are not the spatial components of a four-dimensional vector.

The Einstein and Landau-Lifshitz pseudotensors are examples of quantities which in the absence of the gravitational field reduce to $T^{ik}+U^{ik}$, and which upon integration over dS_k give a conservation of some quantity. There exists an infinite number of such pseudotensors, but the Landau-Lifshitz pseudotensor is the only one which contains only the first derivatives of g_{ik} and is also symmetric.

2.5.8 Palatini variation

Instead of varying the action for the gravitational field and matter with respect to the torsion tensor, we can vary it with respect to the affine connection ($\delta\Gamma^{\,\,j}_{i\,k}$ is a tensor) and use the metric compatibility of the connection (1.4.5). Varying S_{g} in (2.5.1) with respect to $\Gamma^{\,\,k}_{i\,j}$ gives, by means of (1.3.44),

$$\delta_\Gamma S_{\mathrm{g}} = -\frac{1}{2\kappa c}\int \delta R_{ik}\mathfrak{g}^{ik}d\Omega = -\frac{1}{2\kappa c}\int\big((\delta\Gamma^{\,\,l}_{i\,k})_{;l} - (\delta\Gamma^{\,\,l}_{i\,l})_{;k} - 2S^j_{\,\,lk}\delta\Gamma^{\,\,l}_{i\,j}\big)g^{ik}\sqrt{-\mathfrak{g}}d\Omega. \tag{2.5.107}$$

Partial integration and omitting total derivatives in (2.5.107) gives, using (1.2.43),

$$\delta_\Gamma S_{\mathrm{g}} = \frac{1}{2\kappa c}\int(\delta\Gamma^{\,\,l}_{i\,k}\mathfrak{g}^{ik}_{\,\,\,\,;l} - 2S_l\delta\Gamma^{\,\,l}_{i\,k}\mathfrak{g}^{ik} - \delta\Gamma^{\,\,l}_{i\,l}\mathfrak{g}^{ik}_{\,\,\,\,;k} + 2S_k\delta\Gamma^{\,\,l}_{i\,l}\mathfrak{g}^{ik} + 2S^j_{\,\,lk}\delta\Gamma^{\,\,l}_{i\,j}\mathfrak{g}^{ik})d\Omega. \tag{2.5.108}$$

The variation of the action is thus

$$\begin{aligned}\delta_\Gamma S = {} & \frac{1}{2\kappa c}\int(\delta\Gamma^{\,\,l}_{i\,k}\mathfrak{g}^{ik}_{\,\,\,\,;l} - 2S_l\delta\Gamma^{\,\,l}_{i\,k}\mathfrak{g}^{ik} - \delta\Gamma^{\,\,l}_{i\,l}\mathfrak{g}^{ik}_{\,\,\,\,;k} + 2S_k\delta\Gamma^{\,\,l}_{i\,l}\mathfrak{g}^{ik} + 2S^j_{\,\,lk}\delta\Gamma^{\,\,l}_{i\,j}\mathfrak{g}^{ik})d\Omega \\ & + \frac{1}{2c}\int \Pi^{i\,\,k}_{\,\,j}\delta\Gamma^{\,\,j}_{i\,k}d\Omega,\end{aligned} \tag{2.5.109}$$

where

$$\Pi^{i\,\,k}_{\,\,j} = 2\frac{\delta\mathfrak{L}_{\mathrm{m}}}{\delta\Gamma^{\,\,j}_{i\,k}}. \tag{2.5.110}$$

Since the connection is metric-compatible, the condition $\delta S = 0$ gives

$$g^{ik}S_j - \delta^k_j S^i + S^{ki}_{\,\,\,\,j} = \frac{\kappa}{2\sqrt{-\mathfrak{g}}}\Pi^{i\,\,k}_{\,\,j}. \tag{2.5.111}$$

Comparing (2.5.111) with the second Einstein-Cartan equation (2.5.27) shows that

$$\Pi^{i\,\,k}_{\,\,j} = -\mathfrak{S}^{i\,\,k}_{\,\,j} = -\Pi_j^{\,\,ik}. \tag{2.5.112}$$

Contracting the indices i, j gives

$$\Pi^{i\,\,k}_{\,\,i} = 0, \tag{2.5.113}$$

which also results from the invariance of the Lagrangian density under a projective transformation (1.2.61) (the symmetric part of the Ricci tensor is invariant under this transformation):

$$\delta\mathfrak{L} = \delta\mathfrak{L}_{\mathrm{m}} = \frac{1}{2}\Pi^{i\,\,k}_{\,\,j}\delta\Gamma^{\,\,j}_{i\,k} = \frac{1}{2}\Pi^{i\,\,k}_{\,\,j}\delta^j_i\delta A_k = 0. \tag{2.5.114}$$

The antisymmetry relation in (2.5.112) algebraically constrains possible forms of matter Lagrangians. Thereby, it is not a conservation law. If the matter Lagrangian density $\mathfrak{L}_{\mathrm{m}}$ does not depend on the affine connection, then the variation of the action with respect to the connection is referred to as the *Palatini variation*. In this case, (2.5.111) turns the torsion tensor into zero, so the connection is formed by the Christoffel symbols and the field equations are the Einstein equations (2.5.36).

2.5.9 Gravitational potential

If the metric tensor g_{ij} is approximately equal to the Minkowski metric tensor η_{ij}, then the corresponding gravitational field is *weak*. We can write

$$g_{00} \approx 1 + \frac{2\phi}{c^2}, \tag{2.5.115}$$

where ϕ is referred to as the *gravitational potential*. Therefore, nonrelativistic gravitational fields, corresponding to the limit $c \to \infty$, are weak. Also $u^0 \approx 1$ and $u^\alpha \approx 0$. In this limit, the leading component of the Levi-Civita connection is

$$\{{}^{\,\alpha}_{0\,0}\} \approx -\frac{1}{2} g^{\alpha\beta} \frac{\partial g_{00}}{\partial x^\beta} = \frac{1}{c^2} \frac{\partial \phi}{\partial x^\alpha}, \tag{2.5.116}$$

so the metric geodesic equation (1.4.91) reduces to

$$\frac{d\mathbf{v}}{dt} = \mathbf{g} = -\boldsymbol{\nabla}\phi, \tag{2.5.117}$$

where $\mathbf{g}$ is the acceleration due to gravity. The quantity G in (2.2.7) reduces to

$$\mathsf{G} = \frac{2}{c^4} (\nabla\phi)^2. \tag{2.5.118}$$

The leading component of the Riemannian Ricci tensor is

$$P_{00} \approx \frac{\partial \{{}^{\,\alpha}_{0\,0}\}}{\partial x^\alpha} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial x^{\alpha 2}} = \frac{1}{c^2} \triangle \phi. \tag{2.5.119}$$

The leading component of the energy-momentum tensor (2.4.153) is

$$T_{00} = \mu c^2. \tag{2.5.120}$$

Therefore, the Einstein equations in the nonrelativistic limit reduce to the *Poisson equation*:

$$\triangle \phi = 4\pi G \mu, \tag{2.5.121}$$

where

$$G = \frac{c^4 \kappa}{8\pi} \tag{2.5.122}$$

is *Newton's gravitational constant*. In vacuum, where $\mu = 0$, the Poisson equation reduces to the Laplace equation:

$$\triangle \phi = 0. \tag{2.5.123}$$

In the nonrelativistic limit of an ideal fluid, we have $c \to \infty$, $\gamma \approx 1$, $u^0 \sim 1$, $u^\alpha \approx \frac{v^\alpha}{c}$, $\epsilon \approx \mu c^2$, and $p \ll \epsilon$. Consequently, the equation of continuity (2.4.209) reduces to

$$\frac{\partial \mu}{\partial t} + \mathrm{div}\,\mathbf{s} = 0, \tag{2.5.124}$$

where

$$\mathbf{s} = \mu \mathbf{v} \tag{2.5.125}$$

is referred to as the *mass current*. Integrating (2.5.124) over the volume gives

$$\frac{\partial}{\partial t}\int \mu dV + \oint \mathbf{s}\cdot d\mathbf{f} = 0. \tag{2.5.126}$$

This equation means that the change in time of the total mass inside a volume, $m = \int \mu dV$, is balanced by the *mass flux* through the surface bounding this volume, representing the conservation of the total mass of a fluid. The Euler equation (2.4.210) reduces in this limit to

$$\mu\bigg(\frac{\partial v^\alpha}{\partial t} + v^\alpha{}_{,\beta}v^\beta\bigg) = \mu\phi_{,\beta}\eta^{\alpha\beta} + p_{,\beta}\eta^{\alpha\beta} \tag{2.5.127}$$

or

$$\mu\frac{d\mathbf{v}}{dt} = \mu\bigg(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\bigg) = -\mu\boldsymbol{\nabla}\phi - \boldsymbol{\nabla}p. \tag{2.5.128}$$

In the nonrelativistic limit, the total momentum (2.4.144) of a fluid is

$$\mathbf{P} = m\mathbf{v} = \int \mu\mathbf{v}dV. \tag{2.5.129}$$

Integrating (2.5.128) over the volume gives its change in time:

$$\frac{d\mathbf{P}}{dt} = -\int \boldsymbol{\nabla}\phi dm - \oint p d\mathbf{f}. \tag{2.5.130}$$

Without pressure gradients, (2.5.128) reduces to (2.5.117).

2.5.10 Raychaudhuri equation

Let us consider a congruence of particles with four-velocity u^i. We define the *expansion scalar* θ, the traceless *shear tensor* σ_{ik}, and the antisymmetric *vorticity tensor* ω_{ik} according to

$$\theta = u^i{}_{:i}, \tag{2.5.131}$$

$$\sigma_{ik} = u_{(i:k)} - \frac{1}{3}\theta h_{ik} - w_{(i}u_{k)}, \tag{2.5.132}$$

$$\omega_{ik} = u_{[i:k]} - w_{[i}u_{k]}, \tag{2.5.133}$$

where w^i is the four-acceleration (1.6.121). Expansion has $\theta > 0$ and contraction has $\theta < 0$. These definitions give

$$u_{i:j}h^j_k = \sigma_{ik} + \omega_{ik} + \frac{1}{3}\theta h_{ik}. \tag{2.5.134}$$

Contracting $u^i{}_{:jk} - u^i{}_{:kj} = P^i_{\ lkj}u^l$ with respect to the indices i, j gives $\theta_{:k}u^k - u^j{}_{:kj}u^k = -P_{kl}u^k u^l$ or

$$-P_{kl}u^k u^l = \theta_{:k}u^k - w^i{}_{:i} + u_{i:k}u^{k:i}. \tag{2.5.135}$$

Defining

$$\sigma^2 = \frac{1}{2}\sigma_{ik}\sigma^{ik},\ \ \omega^2 = \frac{1}{2}\omega_{ik}\omega^{ik} \tag{2.5.136}$$

gives

$$u_{i:k}u^{k:i} = 2(\sigma^2 - \omega^2) + \frac{1}{3}\theta^2, \tag{2.5.137}$$

which brings (2.5.135) to the *Raychaudhuri equation*:

$$\frac{d\theta}{ds} = -2(\sigma^2 - \omega^2) - \frac{1}{3}\theta^2 + w^i{}_{:i} - P_{ik}u^i u^k. \tag{2.5.138}$$

For a perfect fluid, the Einstein equations give

$$P_{ik}u^i u^k = \frac{\kappa}{2}(\epsilon + 3p). \tag{2.5.139}$$

We define four energy conditions. The *null energy condition* is satisfied if

$$T_{ij}k^i k^j \geq 0 \tag{2.5.140}$$

for any null, future-pointing vector k^i. For a perfect fluid, this condition gives

$$\epsilon + p \geq 0. \tag{2.5.141}$$

The *weak energy condition* is satisfied if

$$T_{ij}u^i u^j \geq 0 \tag{2.5.142}$$

for any causal (null or timelike), future-pointing vector u^i. For a perfect fluid, this condition gives

$$\epsilon + p \geq 0, \;\; \epsilon \geq 0. \tag{2.5.143}$$

The *strong energy condition* is satisfied if

$$\Big(T_{ij} - \frac{1}{2}Tg_{ij}\Big)u^i u^j \geq 0 \tag{2.5.144}$$

for any causal, future-pointing vector u^i. For a perfect fluid, this condition gives

$$\epsilon + p \geq 0, \;\; \epsilon + 3p \geq 0. \tag{2.5.145}$$

The *dominant energy condition* is satisfied if the weak condition is satisfied and $T_{ij}u^j$ is a causal, future-pointing vector. For a perfect fluid, this condition gives

$$\epsilon + p \geq 0, \;\; \epsilon \geq |p|. \tag{2.5.146}$$

This condition guarantees that particles in a congruence do not move faster than light. The dominant condition infers the weak condition. The weak condition infers the null condition. The strong condition infers the null condition.

If the strong condition is satisfied and particles in a congruence move without rotation and acceleration ($\omega_{ik} = w^i = 0$), then (2.5.138) and the Einstein equations give

$$\frac{d\theta}{d\tau} \leq -\frac{c}{3}\theta^2. \tag{2.5.147}$$

If θ_0 is θ at $\tau = 0$ then

$$\theta^{-1} \geq \theta_0^{-1} + \frac{c\tau}{3}. \tag{2.5.148}$$

Therefore, if $\theta_0 < 0$ (initial contraction) then θ diverges to a curvature *singularity*, which is a point in spacetime where the density of matter and curvature are infinite, as τ increases: $\theta \to -\infty$. Singularities are unphysical and their appearance in a system indicates that a theory describing such a system is incomplete.

2.5.11 Relativistic spin fluids

For a spin fluid, substituting (2.4.177) into the second Einstein-Cartan equation (2.5.17) and using (2.4.179) gives

$$S_i = 0, \;\; \nabla^*_i = \nabla_i. \tag{2.5.149}$$

Therefore, the corresponding metric energy-momentum tensor (2.4.221) reduces to

$$T_{ij} = \epsilon u_i u_j - p h_{ij} - \nabla_k s^k{}_{(ij)} + \nabla_k s_{il}{}^k u^l u_j - \frac{1}{2}\nabla_k s_{ij}{}^k. \tag{2.5.150}$$

Accordingly, the torsion tensor is

$$S^j{}_{ik} = -\frac{1}{2}\kappa s_{ik} u^j. \tag{2.5.151}$$

The last two terms on the right of the second line of (2.5.150) can be written, using (2.4.217), (2.4.219) and (2.4.179), as

$$\nabla_k s_{il}{}^k u^l u_j - \frac{1}{2}\nabla_k s_{ij}{}^k = c(\mathsf{p}_i u_l - \mathsf{p}_l u_i)u^l u_j - \frac{1}{2}c(\mathsf{p}_i u_j - \mathsf{p}_j u_i)$$
$$= \frac{1}{2}c(\mathsf{p}_i u_j + \mathsf{p}_j u_i) - \epsilon u_i u_j = \frac{1}{2}(\nabla_k s_{il}{}^k u^l u_j + \nabla_k s_{jl}{}^k u^l u_i) = -\nabla_l s_{k(i}{}^l u^k u_{j)}$$
$$= -\nabla_l(s_{k(i}u^l)u^k u_{j)} = -\nabla_l s_{k(i}u^l u^k u_{j)} = -\nabla_l(s_{k(i}u_{j)})u^l u^k$$
$$= -\nabla_l(s^k{}_{(i}u_{j)})u_k u^l. \tag{2.5.152}$$

The metric energy-momentum tensor (2.5.150) is then

$$T^{ij} = \epsilon u^i u^j - p h^{ij} - (\delta^l_k + u_k u^l)\nabla_l(s^{k(i}u^{j)}). \tag{2.5.153}$$

The last term on the right of (2.5.153) can be decomposed according to (1.4.34) into

$$-(\delta^l_k + u_k u^l)\nabla_l(s^{k(i}u^{j)}) = -(\delta^l_k + u_k u^l)\nabla^{\{\}}_l(s^{k(i}u^{j)}) - (\delta^l_k + u_k u^l)(C^k{}_{ml}s^{m(i}u^{j)}$$
$$+C^i{}_{ml}s^{k(m}u^{j)} + C^j{}_{ml}s^{k(i}u^{m)}). \tag{2.5.154}$$

This term reduces, by means of (2.4.179) and (2.5.15), to

$$-(\delta^l_k + u_k u^l)\nabla^{\{\}}_l(s^{k(i}u^{j)}) - \delta^l_k(C^i{}_{ml}s^{k(m}u^{j)} + C^j{}_{ml}s^{k(i}u^{m)}) - u_k u^l C^k{}_{ml}s^{m(i}u^{j)}$$
$$= -(\delta^l_k + u_k u^l)\nabla^{\{\}}_l(s^{k(i}u^{j)}) - C^i{}_{mk}s^{k(m}u^{j)} - C^j{}_{mk}s^{k(i}u^{m)}$$
$$= -(\delta^l_k + u_k u^l)\nabla^{\{\}}_l(s^{k(i}u^{j)}) + \frac{1}{2}\kappa(s_{mk}u^i + s_k{}^i u_m + s_m{}^i u_k)s^{k(m}u^{j)}$$
$$+\frac{1}{2}\kappa(s_{mk}u^j + s_k{}^j u_m + s_m{}^j u_k)s^{k(m}u^{i)}$$
$$= -(\delta^l_k + u_k u^l)\nabla^{\{\}}_l(s^{k(i}u^{j)}) - \frac{1}{2}\kappa(s_{kl}s^{kl}u^i u^j - s^{ik}s^j{}_k). \tag{2.5.155}$$

Therefore, (2.5.153) becomes

$$T^{ij} = \epsilon u^i u^j - p h^{ij} - (\delta^l_k + u_k u^l)\nabla^{\{\}}_l(s^{k(i}u^{j)}) - \kappa s^2 u^i u^j + \frac{1}{2}\kappa s^{ik}s^j{}_k, \tag{2.5.156}$$

where

$$s^2 = \frac{1}{2}s^{ij}s_{ij} > 0. \tag{2.5.157}$$

Substituting (2.4.177) into (2.5.19) and using (2.4.179) gives

$$U^{ij} = \frac{1}{2}\kappa s^2 u^i u^j + \frac{1}{4}\kappa s^2 g^{ij} - \frac{1}{2}\kappa s^{ik}s^j{}_k. \tag{2.5.158}$$

Adding (2.5.156) and (2.5.158) brings the combined energy-momentum tensor $T^{ij} + U^{ij}$ in the first Einstein-Cartan equation (2.5.10) to

$$T^{ij} + U^{ij} = \Big(\epsilon - \frac{1}{4}\kappa s^2\Big)u^i u^j - \Big(p - \frac{1}{4}\kappa s^2\Big)h^{ij} - (\delta^l_k + u_k u^l)\nabla^{\{\}}_l(s^{k(i}u^{j)}). \tag{2.5.159}$$

If the spin orientation of particles in a spin fluid is random then the macroscopic spacetime average of s_{ij} and of its gradients, such as of the last term on the right of (2.5.159), vanish. On the contrary, the terms that are quadratic in the spin tensor do not vanish after averaging. Therefore, the combined energy-momentum tensor of a macroscopic spin fluid describes a perfect fluid with the effective energy density

$$\tilde{\epsilon} = (T_{ij} + U_{ij})u^i u^j = \epsilon - \frac{1}{4}\kappa s^2 \tag{2.5.160}$$

and the effective pressure

$$\tilde{p} = -\frac{1}{3}(T_{ij} + U_{ij})h^{ij} = p - \frac{1}{4}\kappa s^2. \tag{2.5.161}$$

If the spin orientation of particles (confer (2.6.46)) in a spin fluid is not random then the combined energy density of a macroscopic spin fluid is

$$\begin{aligned}\tilde{\epsilon} &= \epsilon - \frac{1}{4}\kappa s^2 - (\delta^l_k + u_k u^l)u^i \nabla^{\{\}}_l s^k{}_i = \epsilon - \frac{1}{4}\kappa s^2 - \nabla^{\{\}}_k s^k{}_i u^i \\ &= \epsilon - \frac{1}{4}\kappa s^2 + s^{ki}\nabla^{\{\}}_{[k} u_{i]} = \epsilon - \frac{1}{4}\kappa s^2 + s^{ki}\partial_{[k} u_{i]}.\end{aligned} \tag{2.5.162}$$

In a locally Galilean frame of reference which is also a rest frame, (2.5.162) becomes

$$\tilde{\epsilon} = \epsilon - \frac{1}{4}\kappa s^2 + \frac{1}{2}\mathbf{s}\cdot \mathrm{curl}\,\mathbf{v}. \tag{2.5.163}$$

where $\mathbf{s}$ is the spatial spin-density pseudovector (2.4.181).

The effective energy density (2.5.160) and pressure (2.5.161) can be negative if the quantity s^2 is sufficiently large. Consequently, a spin fluid could violate the strong energy condition (2.5.145) and thus prevent a singularity.
References: [1, 2, 3, 4, 6, 7, 9, 11].

2.6 Spinor fields

2.6.1 Dirac matrices

The Dirac matrices γ^a defined by (1.7.1) are complex. A particular solution of (1.7.1) is given by the *Dirac representation*:

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},\ \gamma^\alpha = \begin{pmatrix} 0 & \sigma^\alpha \\ -\sigma^\alpha & 0 \end{pmatrix}, \tag{2.6.1}$$

where I_2 is the unit 2×2 matrix and

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.6.2}$$

are the *Pauli matrices* (all indices are coordinate invariant). The Pauli matrices are traceless, $\mathrm{tr}(\sigma^\alpha) = 0$, and Hermitian, $\sigma^{\alpha\dagger} = \sigma^\alpha$ (the Hermitian conjugation of a matrix A is the combination of the complex conjugation and transposition, $A^\dagger = A^{*T}$). They also satisfy

$$\sigma_\alpha \sigma_\beta = \delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma, \tag{2.6.3}$$

and their squares are $(\sigma_\alpha)^2 = I_2$. The identity (2.6.3) gives the anticommutation relation

$$\left[\frac{\sigma_\alpha}{2}, \frac{\sigma_\beta}{2}\right] = i\varepsilon_{\alpha\beta\gamma}\frac{\sigma_\gamma}{2}, \tag{2.6.4}$$

so $\frac{\sigma_\alpha}{2}$ form the lowest, two-dimensional representation of the angular momentum operator M_α (1.6.77). The properties of σ^α infer that the Dirac matrices are traceless, $\mathrm{tr}(\gamma^a) = 0$, and satisfy

$$\gamma^{a\dagger} = \gamma^0\gamma^a\gamma^0,\ \ \gamma^{a*} = \gamma^2\gamma^a\gamma^2, \tag{2.6.5}$$

which gives $\gamma^{0\dagger} = \gamma^0$ and $\gamma^{\alpha\dagger} = -\gamma^\alpha$. Hereinafter, γ^0, γ^1, γ^2 and γ^3 refer to the Dirac matrices with a Lorentz index, γ^a. The relation (1.7.1) yields the total antisymmetry of $\gamma^0\gamma^1\gamma^2\gamma^3$:

$$\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^{[0}\gamma^1\gamma^2\gamma^{3]}. \tag{2.6.6}$$

We define

$$\gamma^5 = -\frac{i}{24}e_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d = i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{2.6.7}$$

which is traceless, $\mathrm{tr}(\gamma^5)=0$, and Hermitian, $\gamma^{5\dagger}=\gamma^5$. It also satisfies

$$\{\gamma^a,\gamma^5\}=0,\ \ (\gamma^5)^2=I_4,\ \ \gamma^5{}_{|i}=0, \tag{2.6.8}$$

where the last relation results from (1.4.22) and (1.7.24). In the Dirac representation,

$$\gamma^5=\gamma^{5*}=\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}. \tag{2.6.9}$$

The anticommutation relation (1.7.1) gives

$$\gamma^a\gamma_a=4I_4, \tag{2.6.10}$$
$$\gamma^a\gamma^b\gamma_a=-2\gamma^b, \tag{2.6.11}$$
$$\gamma^a\gamma^b\gamma^c\gamma_a=4\eta^{bc}I_4, \tag{2.6.12}$$
$$\gamma^a\gamma^b\gamma^c\gamma^d\gamma_a=-2\gamma^d\gamma^c\gamma^b, \tag{2.6.13}$$
$$\gamma^a\gamma^b\gamma^c=\eta^{ab}\gamma^c+\eta^{bc}\gamma^a-\eta^{ac}\gamma^b+ie^{abcd}\gamma_d\gamma^5, \tag{2.6.14}$$
$$\{\gamma^a,\gamma^{[b}\gamma^{c]}\}=2\gamma^{[a}\gamma^b\gamma^{c]}. \tag{2.6.15}$$

The Dirac representation is not unique; the relation (1.7.1) is invariant under a similarity transformation $\gamma^a\rightarrow S\gamma^aS^{-1}$, where S is a nondegenerate ($\det S\neq 0$) matrix. Accordingly, $\psi\rightarrow S\psi$ and $\bar{\psi}\rightarrow\bar{\psi}S^{-1}$. Taking $S=\frac{1}{\sqrt{2}}\begin{pmatrix} I_2 & -I_2 \\ I_2 & I_2 \end{pmatrix}$ turns the Dirac representation into the *chiral* or *Weyl representation*, in which

$$\gamma^0=\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix},\ \ \gamma^\alpha=\begin{pmatrix} 0 & \sigma^\alpha \\ -\sigma^\alpha & 0 \end{pmatrix},\ \ \gamma^5=\begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}. \tag{2.6.16}$$

For an infinitesimal Lorentz transformation (1.6.7), the relations (1.7.5) and (1.7.6) give $L=I_4+\frac{1}{8}\epsilon_{ab}(\gamma^a\gamma^b-\gamma^b\gamma^a)$. Using $(AB)^\dagger=B^\dagger A^\dagger$ gives

$$L^\dagger=I_4+\frac{1}{8}\epsilon_{ab}(\gamma^{b\dagger}\gamma^{a\dagger}-\gamma^{a\dagger}\gamma^{b\dagger}), \tag{2.6.17}$$

which is equal to L^{-1} (so L is unitary) for rotations and equal to L for boosts. The relation (2.6.5) leads thus to

$$L^\dagger\gamma^0=\gamma^0+\frac{1}{8}\epsilon_{ab}(\gamma^{b\dagger}\gamma^{a\dagger}-\gamma^{a\dagger}\gamma^{b\dagger})\gamma^0=\gamma^0-\frac{1}{8}\epsilon_{ab}\gamma^0(\gamma^a\gamma^b-\gamma^b\gamma^a)=\gamma^0L^{-1}. \tag{2.6.18}$$

Therefore, the quantity $\psi^\dagger\gamma^0$ transforms under (1.7.7) like an adjoint spinor:

$$\psi^\dagger\gamma^0\rightarrow\psi^\dagger L^\dagger\gamma^0=\psi^\dagger\gamma^0L^{-1}. \tag{2.6.19}$$

Accordingly, we can associate these two quantities:

$$\bar{\psi}=\psi^\dagger\gamma^0. \tag{2.6.20}$$

The spinors ψ and $\psi^\dagger\gamma^0$ can be used to construct tensors, as in (1.7.11): $\psi^\dagger\gamma^0\psi$ transforms like a scalar, $\psi^\dagger\gamma^0\gamma^i\psi$ is a vector, $\psi^\dagger\gamma^0\gamma^{[i}\gamma^{j]}\psi$ is an antisymmetric tensor, $\psi^\dagger\gamma^0\gamma^5\psi$ is a pseudoscalar, and $\psi^\dagger\gamma^0\gamma^i\gamma^5\psi$ is a pseudovector. Higher-rank tensors constructed from ψ and $\psi^\dagger\gamma^0$ reduce to the above 5 kinds of tensors because of (2.6.14). To show that $\bar{\psi}\gamma^5\psi$ transforms like a pseudoscalar, we substitute (1.7.4) into (2.6.7) and use (2.6.6), which gives

$$\gamma^5=i\Lambda^0{}_a\Lambda^1{}_b\Lambda^2{}_c\Lambda^3{}_dL\gamma^a\gamma^b\gamma^c\gamma^dL^{-1}=i\Lambda^0{}_a\Lambda^1{}_b\Lambda^2{}_c\Lambda^3{}_dL\gamma^{[a}\gamma^b\gamma^c\gamma^{d]}L^{-1}. \tag{2.6.21}$$

Using (1.1.25) and

$$\gamma^{[a}\gamma^b\gamma^c\gamma^{d]}=-ie^{abcd}\gamma^5, \tag{2.6.22}$$

which results from (2.6.7), gives

$$\gamma^5 = e^{abcd}\Lambda^0{}_a\Lambda^1{}_b\Lambda^2{}_c\Lambda^3{}_d L\gamma^5 L^{-1} = \det(\Lambda^a{}_b)L\gamma^5 L^{-1}. \tag{2.6.23}$$

Therefore, we have

$$\bar{\psi}\gamma^5\psi \to \bar{\psi}L^{-1}\det(\Lambda^a{}_b)L\gamma^5 L^{-1}L\psi = \det(\Lambda^a{}_b)\bar{\psi}\gamma^5\psi, \tag{2.6.24}$$

which is the transformation law for a Lorentz scalar density. Similarly,

$$\bar{\psi}\gamma^c\gamma^5\psi \to \bar{\psi}L^{-1}\Lambda^c{}_d\det(\Lambda^a{}_b)L\gamma^d\gamma^5 L^{-1}L\psi = \det(\Lambda^a{}_b)\Lambda^c{}_d\bar{\psi}\gamma^d\gamma^5\psi, \tag{2.6.25}$$

which is the transformation law for a Lorentz vector density.

We define the *chirality projection operators*

$$P_\pm = \frac{I_4 \pm \gamma^5}{2}, \ \ P_+ + P_- = I_4, \ \ P_\pm^2 = I_4, \ \ P_+P_- = P_-P_+ = 0. \tag{2.6.26}$$

They project a spinor ψ into the *right-handed* spinor ψ_R and *left-handed* spinor ψ_L,

$$\psi_R = P_+\psi, \ \ \psi_L = P_-\psi, \ \ \psi = \psi_R + \psi_L. \tag{2.6.27}$$

If we split a spinor ψ into two two-component parts,

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{2.6.28}$$

then, in the chiral representation:

$$\psi_L = \begin{pmatrix} u \\ 0 \end{pmatrix}, \ \ \psi_R = \begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{2.6.29}$$

An infinitesimal rotation is described by a Lorentz matrix (1.6.7) with $\epsilon_{0\alpha} = 0$. The corresponding spinor transformation matrix (1.7.5) in the Dirac representation is, using (1.6.25),

$$L = I_4 + \frac{1}{4}\epsilon_{\alpha\beta}\gamma^\alpha\gamma^\beta = I_4 + \frac{1}{4}e_{\alpha\beta\gamma}\gamma^\alpha\gamma^\beta\vartheta_\gamma = I_4 - \frac{i}{2}\vartheta_\alpha\begin{pmatrix} \sigma^\alpha & 0 \\ 0 & \sigma^\alpha \end{pmatrix}. \tag{2.6.30}$$

Therefore, the (unitary) spinor transformation matrix for a finite rotation by an angle ϑ about an axis parallel to a unit vector $\mathbf{n}$, $\boldsymbol{\vartheta} = \vartheta\mathbf{n}$, is

$$L = \exp\left[-\frac{i}{2}\boldsymbol{\vartheta}\cdot\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}\right] = \cos\frac{\vartheta}{2}I_4 - i\sin\frac{\vartheta}{2}\,\mathbf{n}\cdot\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \tag{2.6.31}$$

where $\boldsymbol{\sigma}$ is a spatial vector composed from the Pauli matrices σ^α. If we split a spinor ψ into two parts u and v (2.6.28), then both u and v transform under rotations according to

$$u \to Su, \ \ v \to Sv, \tag{2.6.32}$$

where

$$S = \cos\frac{\vartheta}{2}I_2 - i\sin\frac{\vartheta}{2}\,\mathbf{n}\cdot\boldsymbol{\sigma}. \tag{2.6.33}$$

A rotation by a full angle 2π changes the sign of a spinor. A rotation by 4π brings a spinor to its original position.

An infinitesimal boost is described by a Lorentz matrix (1.6.7) with $\epsilon_{\alpha\beta} = 0$. The corresponding spinor transformation matrix (1.7.5) in the Dirac representation is, using (1.6.26),

$$L = I_4 + \frac{1}{2}\epsilon_{0\alpha}\gamma^0\gamma^\alpha = I_4 + \frac{1}{2}\eta_\alpha\gamma^0\gamma^\alpha = I_4 + \frac{1}{2}\eta_\alpha\begin{pmatrix} 0 & \sigma^\alpha \\ \sigma^\alpha & 0 \end{pmatrix}. \tag{2.6.34}$$

Therefore, the spinor transformation matrix for a finite boost with a rapidity η along an axis parallel to a unit vector $\mathbf{n}$, $\boldsymbol{\eta} = \eta\mathbf{n}$, is

$$L = \exp\left[\frac{1}{2}\boldsymbol{\eta}\cdot\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}\right] = \cosh\frac{\eta}{2}I_4 + \sinh\frac{\eta}{2}\,\mathbf{n}\cdot\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}. \tag{2.6.35}$$

Using (1.6.92) gives

$$L = \frac{1}{\sqrt{2(1+\gamma)}}\begin{pmatrix} (1+\gamma)I_2 & \gamma\boldsymbol{\beta}\cdot\boldsymbol{\sigma} \\ \gamma\boldsymbol{\beta}\cdot\boldsymbol{\sigma} & (1+\gamma)I_2 \end{pmatrix}. \tag{2.6.36}$$

If this boost transforms a particle of mass m from rest to a motion with momentum $\mathbf{p}$ and energy E then (2.6.36) is equivalent, because of (2.4.143) and (2.4.144), to

$$L = \frac{1}{\sqrt{2mc^2(E+mc^2)}}\begin{pmatrix} (E+mc^2)I_2 & c\boldsymbol{\sigma}\cdot\mathbf{p} \\ c\boldsymbol{\sigma}\cdot\mathbf{p} & (E+mc^2)I_2 \end{pmatrix}. \tag{2.6.37}$$

2.6.2 Lagrangian density and spin tensor for spinor field

A Lagrangian density for dynamical spinor fields must contain first derivatives of spinors and be real. The simplest scalar containing derivatives of spinors is quadratic in ψ, $\bar{\psi}\gamma^i\psi_{;i}$, where $\psi_{;i}$ is the covariant derivative of ψ (1.7.14). Its kinetic part, containing derivatives of spinors, is equal to $\bar{\psi}\gamma^i\psi_{,i}$ and is complex. Since the complex conjugate of this quantity is

$$(\bar{\psi}\gamma^i\psi_{,i})^* = (\bar{\psi}\gamma^i\psi_{,i})^\dagger = \psi^\dagger_{,i}\gamma^{i\dagger}\bar{\psi}^\dagger = \bar{\psi}_{,i}\gamma^0\gamma^{i\dagger}\gamma^0\psi = \bar{\psi}_{,i}\gamma^i\psi, \tag{2.6.38}$$

both scalars $\bar{\psi}\gamma^i\psi_{,i} + \bar{\psi}_{,i}\gamma^i\psi$ and $i(\bar{\psi}\gamma^i\psi_{,i} - \bar{\psi}_{,i}\gamma^i\psi)$ are real. The former scalar is, however, equal to a total divergence $(\bar{\psi}\gamma^i\psi)_{,i}$, thereby a Lagrangian density proportional to such term does not contribute to field equations. Therefore, the simplest kinetic part of a spinor Lagrangian density is proportional to $i(\bar{\psi}\gamma^i\psi_{,i} - \bar{\psi}_{,i}\gamma^i\psi)$. Another scalar that can be used in a spinor Lagrangian density is proportional to $\bar{\psi}\psi$, which is real:

$$(\bar{\psi}\psi)^* = (\bar{\psi}\psi)^\dagger = (\psi^\dagger\gamma^0\psi)^\dagger = \psi^\dagger\gamma^0\psi^{\dagger\dagger} = \bar{\psi}\psi. \tag{2.6.39}$$

The simplest Lagrangian density for a spinor field is thus given by

$$\mathfrak{L}_\psi = \frac{i\mathfrak{e}}{2}(\bar{\psi}\gamma^i\psi_{;i} - \bar{\psi}_{;i}\gamma^i\psi) - m\mathfrak{e}\bar{\psi}\psi = \frac{i\mathfrak{e}}{2}e^i_a(\bar{\psi}\gamma^a\psi_{;i} - \bar{\psi}_{;i}\gamma^a\psi) - m\mathfrak{e}\bar{\psi}\psi, \tag{2.6.40}$$

where m is a real scalar constant, called the *spinor mass.* It is referred to as the *Dirac Lagrangian density.* Putting the definition of the covariant derivative of a spinor (1.7.14) into (2.6.40) gives

$$\mathfrak{L}_\psi = \frac{i\mathfrak{e}}{2}(\bar{\psi}\gamma^i\psi_{,i} - \bar{\psi}_{,i}\gamma^i\psi) - \frac{i\mathfrak{e}}{2}\bar{\psi}\{\gamma^i, \Gamma_i\}\psi - m\mathfrak{e}\bar{\psi}\psi. \tag{2.6.41}$$

Using the Fock-Ivanenko coefficients (1.7.28) as the spinor connection Γ_i turns (2.6.41) into

$$\begin{aligned}\mathfrak{L}_\psi &= \frac{i\mathfrak{e}}{2}(\bar{\psi}\gamma^i\psi_{,i} - \bar{\psi}_{,i}\gamma^i\psi) + \frac{i\mathfrak{e}}{8}\omega_{abi}\bar{\psi}\{\gamma^i, \gamma^a\gamma^b\}\psi - m\mathfrak{e}\bar{\psi}\psi \\ &= \frac{i\mathfrak{e}}{2}(\bar{\psi}\gamma^i\psi_{,i} - \bar{\psi}_{,i}\gamma^i\psi) + \frac{i\mathfrak{e}}{8}\omega_{abi}\bar{\psi}\{\gamma^i, \gamma^{[a}\gamma^{b]}\}\psi - m\mathfrak{e}\bar{\psi}\psi.\end{aligned} \tag{2.6.42}$$

The spin density (2.3.16) corresponding to the Dirac Lagrangian density (2.6.42) is, by means of (2.6.15),

$$\mathfrak{S}^{ijk} = 2\frac{\delta\mathfrak{L}_\psi}{\delta\omega_{ijk}} = \frac{i\mathfrak{e}}{2}\bar{\psi}\gamma^{[i}\gamma^j\gamma^{k]}\psi. \tag{2.6.43}$$

The spin density (2.6.43) is completely antisymmetric,

$$\mathfrak{S}^{ijk} = \mathfrak{S}^{[ijk]}, \tag{2.6.44}$$

and independent of the spinor mass m. The corresponding spin tensor is also completely antisymmetric,

$$s^{ijk} = \frac{i}{2}\bar{\psi}\gamma^{[i}\gamma^{j}\gamma^{k]}\psi = s^{[ijk]} = -e^{ijkl}s_l, \tag{2.6.45}$$

where

$$s^i = \frac{1}{2}\bar{\psi}\gamma^i\gamma^5\psi \tag{2.6.46}$$

is the *spin-density pseudovector*. The second Einstein-Cartan equation (2.5.17) with the spin tensor (2.6.45) gives a completely antisymmetric torsion tensor,

$$S_{ijk} = -\frac{i\kappa}{4}\bar{\psi}\gamma_{[i}\gamma_j\gamma_{k]}\psi, \tag{2.6.47}$$

so $S_i = 0$. Therefore, the contortion tensor is, using (2.6.14),

$$C_{ijk} = S_{ijk} = \frac{\kappa}{4}e_{ijkl}\bar{\psi}\gamma^l\gamma^5\psi = \frac{\kappa}{2}e_{ijkl}s^l. \tag{2.6.48}$$

The torsion tensor is dual to a pseudovector that is proportional to the Dirac spin-density pseudovector (2.6.46). The pseudovector density

$$\mathfrak{j}^i_A = \mathfrak{e}\bar{\psi}\gamma^i\gamma^5\psi = 2\mathfrak{e}s^i \tag{2.6.49}$$

is called the *axial Dirac current*. The axial Dirac current and the pseudovector (2.6.46) are real because

$$(\bar{\psi}\gamma^i\gamma^5\psi)^* = (\psi^\dagger\gamma^0\gamma^i\gamma^5\psi)^\dagger = \psi^\dagger\gamma^5\gamma^{i\dagger}\gamma^0\psi = \psi^\dagger\gamma^5\gamma^0\gamma^i\psi = \psi^\dagger\gamma^0\gamma^i\gamma^5\psi = \bar{\psi}\gamma^i\gamma^5\psi. \tag{2.6.50}$$

Therefore, the spin tensor (2.6.45) is also real, in accordance with the reality of (2.6.40).

For a system of spinor fields, the Lagrangian density is equal to the sum of the Lagrangian densities (2.6.40) for each spinor and a Lagrangian density representing fields that carry the interaction between the spinors. The Lagrangian density for the interaction fields does not depend on derivatives of spinors. Since C_{ijk} appears only in the additive, kinetic term in the Lagrangian density, $\frac{i\mathfrak{e}}{2}e^i_a(\bar{\psi}\gamma^a\psi_{;i} - \bar{\psi}_{;i}\gamma^a\psi)$, the spin tensor for a system of spinor fields is additive. Accordingly, the spin tensor for such a system is also completely antisymmetric.

The complete antisymmetry of the spin density (2.6.44) leads to (2.4.188), which is consistent with (2.6.45) only if $\psi = 0$. Therefore, a spinor field cannot be a point particle or a system of point particles.

2.6.3 Dirac equation

Varying (2.6.41) with respect to $\bar{\psi}$ and omitting total divergences gives

$$\delta\mathfrak{L}_\psi = \frac{i}{2}\delta\bar{\psi}\big(\mathfrak{e}\gamma^k\psi_{,k} + (\mathfrak{e}\gamma^k\psi)_{,k} - \mathfrak{e}\{\Gamma_k, \gamma^k\}\psi\big) - \mathfrak{e}m\delta\bar{\psi}\psi. \tag{2.6.51}$$

The stationarity of the action $\delta S = 0$ under $\delta\bar{\psi}$ therefore gives

$$\frac{i}{2}\big(\mathfrak{e}\gamma^k\psi_{,k} + (\mathfrak{e}\gamma^k\psi)_{,k} - \mathfrak{e}\{\Gamma_k, \gamma^k\}\psi\big) - \mathfrak{e}m\psi = 0. \tag{2.6.52}$$

Substituting

$$(\mathfrak{e}\gamma^k\psi)_{,k} = \mathfrak{e}\gamma^k\psi_{,k} + \mathfrak{e}\gamma^k{}_{;k}\psi - 2\mathfrak{e}S_k\gamma^k\psi = \mathfrak{e}\gamma^k\psi_{,k} + \mathfrak{e}[\Gamma_k, \gamma^k]\psi \tag{2.6.53}$$

into (2.6.52) gives the *Dirac equation*:

$$i\gamma^k\psi_{,k} - i\gamma^k\Gamma_k\psi - m\psi = i\gamma^k\psi_{;k} - m\psi = 0. \tag{2.6.54}$$

Putting (2.6.48) into (1.7.34) yields

$$\gamma^k\psi_{;k} = \gamma^k\psi_{:k} + \frac{\kappa}{16}e_{ijkl}(\bar{\psi}\gamma^l\gamma^5\psi)\gamma^k\gamma^i\gamma^j\psi = \gamma^k\psi_{:k} + \frac{i\kappa}{16}e_{ijkl}(\bar{\psi}\gamma^l\gamma^5\psi)e^{ijkm}\gamma_m\gamma^5\psi$$
$$= \gamma^k\psi_{:k} - \frac{3i\kappa}{8}(\bar{\psi}\gamma^l\gamma^5\psi)\gamma_l\gamma^5\psi. \tag{2.6.55}$$

Therefore, the Dirac equation (2.6.54) becomes

$$i\gamma^k\psi_{:k} + \frac{3\kappa}{8}(\bar{\psi}\gamma_k\gamma^5\psi)\gamma^k\gamma^5\psi = m\psi. \tag{2.6.56}$$

Varying (2.6.41) with respect to ψ and using the stationarity of the action under $\delta\psi$ gives the adjoint Dirac equation:

$$-i\bar{\psi}_{,k}\gamma^k - i\bar{\psi}\Gamma_k\gamma^k - m\bar{\psi} = -i\bar{\psi}_{;k}\gamma^k - m\bar{\psi} = 0, \tag{2.6.57}$$

which is equivalent to the adjoint conjugate of (2.6.56),

$$-i\bar{\psi}_{:k}\gamma^k + \frac{3\kappa}{8}(\bar{\psi}\gamma_k\gamma^5\psi)\bar{\psi}\gamma^k\gamma^5 = m\bar{\psi}. \tag{2.6.58}$$

Equations (2.6.56) and (2.6.58) are cubic in spinor fields. They are also the field equations corresponding to an effective metric Lagrangian density with a quartic, axial-axial spin-spin interaction:

$$\mathfrak{L}_\psi^{\{\}} = \frac{i\mathfrak{e}}{2}(\bar{\psi}\gamma^i\psi_{:i} - \bar{\psi}_{:i}\gamma^i\psi) - m\mathfrak{e}\bar{\psi}\psi + \frac{3\kappa\mathfrak{e}}{16}(\bar{\psi}\gamma_k\gamma^5\psi)(\bar{\psi}\gamma^k\gamma^5\psi). \tag{2.6.59}$$

Subtracting (2.6.57) multiplied by ψ from (2.6.54) multiplied by $\bar{\psi}$ gives, using (1.7.18) and (1.7.39),

$$(\bar{\psi}\gamma^i\psi)_{|i} = (\bar{\psi}\gamma^i\psi)_{:i} = \frac{1}{\mathfrak{e}}(\mathfrak{e}\bar{\psi}\gamma^i\psi)_{,i} = 0. \tag{2.6.60}$$

The vector density

$$\mathrm{j}_V^i = \mathfrak{e}\bar{\psi}\gamma^i\psi, \tag{2.6.61}$$

called the *vector Dirac current*, is thus conserved:

$$\mathrm{j}_{V,i}^i = \frac{\partial \mathrm{j}_V^0}{c\partial t} + \partial_\alpha \mathrm{j}_V^\alpha = 0. \tag{2.6.62}$$

The vector Dirac current is real because

$$(\bar{\psi}\gamma^i\psi)^* = (\psi^\dagger\gamma^0\gamma^i\psi)^\dagger = \psi^\dagger\gamma^{i\dagger}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^i\psi = \bar{\psi}\gamma^i\psi. \tag{2.6.63}$$

Its time component,

$$\mathrm{j}_V^0 = \mathfrak{e}\bar{\psi}\gamma^0\psi = \mathfrak{e}\psi^\dagger\psi, \tag{2.6.64}$$

is thus real and positive.

The Dirac equation (2.6.54) gives

$$-\gamma^j(\gamma^k\psi_{;k})_{|j} = im\gamma^j\psi_{;j} \tag{2.6.65}$$

or, because of (1.7.39),

$$-\gamma^j\gamma^k\psi_{;kj} = -\gamma^{(j}\gamma^{k)}\psi_{;kj} - \gamma^j\gamma^k\psi_{|[kj]} = m^2\psi. \tag{2.6.66}$$

The relations (1.7.36) and (1.7.45) turn (2.6.66) into

$$\psi_{;i}^{\ \ i} + m^2\psi = \frac{1}{2}\gamma^j\gamma^k(K_{kj}\psi + 2S^l_{\ kj}\psi_{;l}) = \frac{1}{8}\gamma^j\gamma^k\gamma^i\gamma^l R_{ilkj}\psi + \gamma^j\gamma^k S^l_{\ kj}\psi_{;l}, \tag{2.6.67}$$

where K_{ij} is the curvature spinor. If a spinor is equal either to its left-handed projection, $\psi = \psi_L$, or right-handed projection, $\psi = \psi_R$, then it is called a *Weyl spinor*. Multiplying (2.6.54) by $P_\pm$ and using (2.6.8) gives

$$iP_\pm \gamma^i \psi_{;i} = i\gamma^i P_\mp \psi_{;i} = mP_\pm \psi \tag{2.6.68}$$

or

$$i\gamma^i \psi^{L(R)}_{;i} = m\psi^{R(L)}. \tag{2.6.69}$$

Therefore, if ψ is a Weyl spinor then $m = 0$.

The nonlinear, cubic terms in (2.6.56) and (2.6.58) represent a spinor self-interaction, corresponding to a spin-spin interaction in the tensor (2.5.19). At densities satisfying $\kappa\bar{\psi}\psi \ll m$, these equations can be approximated by linear equations:

$$i\gamma^k \psi_{:k} = m\psi, \tag{2.6.70}$$

$$-i\bar{\psi}_{:k}\gamma^k = m\bar{\psi}. \tag{2.6.71}$$

Equation (2.6.67) reduces to the *Klein-Gordon-Fock equation*:

$$\psi_{:i}{}^{i} + m^2\psi = \frac{1}{8}\gamma^j\gamma^k\gamma^i\gamma^l P_{ilkj}\psi, \tag{2.6.72}$$

where P_{ijkl} is the Riemann tensor.

2.6.4 Energy-momentum tensor for spinor field

Varying the Lagrangian density (2.6.40) with respect to e^i_a gives the tetrad energy-momentum density for a Dirac spinor field,

$$\mathfrak{T}_i{}^a = \frac{i\mathfrak{e}}{2}(\bar{\psi}\gamma^a\psi_{;i} - \bar{\psi}_{;i}\gamma^a\psi - e^a_i\bar{\psi}\gamma^j\psi_{;j} + e^a_i\bar{\psi}_{;j}\gamma^j\psi) + m\mathfrak{e}e^a_i\bar{\psi}\psi. \tag{2.6.73}$$

The conservation law (2.4.39) applied to the energy-momentum density (2.6.73) gives the Dirac equations (2.6.54) and (2.6.57). We can obtain the metric energy-momentum tensor for such a field using the Belinfante-Rosenfeld relation (2.3.34), the spin tensor (2.6.45), and the torsion tensor in (2.6.48). We can also derive the metric energy-momentum tensor by varying (2.6.40) with respect to g^{ik}:

$$T_{ik} = \frac{2}{\sqrt{-g}}\frac{\delta\mathfrak{L}_\psi}{\delta g^{ik}}. \tag{2.6.74}$$

Using an identity

$$\frac{\delta\gamma^j}{\delta g^{ik}} = \frac{1}{2}\delta^j_{(i}\gamma_{k)}, \tag{2.6.75}$$

which results from the definition of the Dirac matrices (1.7.2), leads to

$$T_{ik} = \frac{i}{2}(\bar{\psi}\gamma_{(i}\psi_{;k)} - \bar{\psi}_{;(i}\gamma_{k)}\psi - g_{ik}\bar{\psi}\gamma^j\psi_{;j} + g_{ik}\bar{\psi}_{;j}\gamma^j\psi) + mg_{ik}\bar{\psi}\psi. \tag{2.6.76}$$

The conservation law (2.4.30) applied to the energy-momentum tensor (2.6.76) gives the Dirac equations (2.6.54) and (2.6.57). Substituting the Dirac equation (2.6.54) into the (2.6.76) gives

$$T_{ik} = \frac{i}{2}(\bar{\psi}\delta^j_{(i}\gamma_{k)}\psi_{;j} - \bar{\psi}_{;j}\delta^j_{(i}\gamma_{k)}\psi). \tag{2.6.77}$$

Putting (1.7.34) and (1.7.35) with (2.6.48) into (2.6.77) yields

$$T_{ik} = \frac{i}{2}(\bar{\psi}\delta^j_{(i}\gamma_{k)}\psi_{:j} - \bar{\psi}_{:j}\delta^j_{(i}\gamma_{k)}\psi) + \frac{\kappa}{2}(-s_i s_k + s^l s_l g_{ik}). \tag{2.6.78}$$

Substituting the completely antisymmetric spin tensor (2.6.45) into (2.5.19) gives

$$U^{ik} = \frac{\kappa}{4}\left(s^{ijl}s^k{}_{jl} - \frac{1}{2}g^{ik}s^{jlm}s_{jlm}\right) = \frac{\kappa}{4}(2s^i s^k + s^l s_l g^{ik}). \tag{2.6.79}$$

The combined energy-momentum tensor for a Dirac field is thus

$$T_{ik}+U_{ik}=\frac{i}{2}(\bar{\psi}\delta^j_{(i}\gamma_{k)}\psi_{:j}-\bar{\psi}_{:j}\delta^j_{(i}\gamma_{k)}\psi)+\frac{3\kappa}{4}s^l s_l g_{ik}. \tag{2.6.80}$$

The tensor (2.6.80) is equal to the energy-momentum tensor for the effective metric Lagrangian density (2.6.59):

$$T_{ik}=\frac{2}{\sqrt{-g}}\frac{\delta\mathfrak{L}^{\{\}}_{\psi}}{\delta g^{ik}}. \tag{2.6.81}$$

The first term on the right of (2.6.80) is the Riemannian part of the energy-momentum tensor for a Dirac field and can be macroscopically averaged as an ideal fluid with the energy density ϵ and pressure p. In the comoving frame of reference, in which $s^0=0$ because of $s_i u^i=0$, the second term on the right of (2.6.80) is equal to $-\frac{3\kappa}{4}\mathbf{s}^2 g_{ik}$, where $\mathbf{s}$ is the spatial spin pseudovector. The average value of $\mathbf{s}^2$ is proportional to n^2, where n is the concentration of spinor particles. The averaged second term on the right of (2.6.80) has therefore a negative contribution to the energy density, $0>\tilde{\epsilon}\propto n^2$, and a positive contribution to the pressure, $\tilde{p}=-\tilde{\epsilon}$.

2.6.5 Discrete symmetries of spinors

The spinor representation of the parity transformation (1.6.3) is given by

$$L_P=C_P\gamma^0, \tag{2.6.82}$$

where $C_P=\mathrm{const}$. Substituting (2.6.82) into (1.7.4) with $L=L_P$ gives, using (2.6.5),

$$\gamma^a=\Lambda^a_{\ b}\gamma^0\gamma^b\gamma^0=\Lambda^a_{\ b}\gamma^{b\dagger}, \tag{2.6.83}$$

which is satisfied if $\Lambda^a_{\ b}=\Lambda^a_{\ b}(P)$. Since the double parity transformation is equivalent to the identity transformation, $L_P^2=I_4$, we have $C_P=\pm1$. The spinor representation of the time-reversal transformation (1.6.4) is given by

$$L_T=C_T\gamma^0\gamma^5, \tag{2.6.84}$$

where $C_T=\mathrm{const}$. Substituting (2.6.84) into (1.7.4) with $L=L_T$ gives, using (2.6.5) and (2.6.8),

$$\gamma^a=\Lambda^a_{\ b}\gamma^0\gamma^5\gamma^b\gamma^5\gamma^0=-\Lambda^a_{\ b}\gamma^{b\dagger}, \tag{2.6.85}$$

which is satisfied if $\Lambda^a_{\ b}=\Lambda^a_{\ b}(T)$. Since the double time-reversal transformation is equivalent to the identity transformation, $L_T^2=I_4$, we have $C_T=\pm i$.

The *charge conjugation* of a spinor ψ is defined as

$$\psi^c=-i\gamma^2\psi^*,\ \ \psi^*=-i\gamma^2\psi^c. \tag{2.6.86}$$

The double charge-conjugation transformation is equivalent to the identity transformation:

$$(\psi^c)^c=-i\gamma^2(\psi^c)^*=-i\gamma^2(-i\gamma^2\psi^*)^*=\gamma^2\gamma^{2*}\psi=\psi. \tag{2.6.87}$$

The charge conjugation of the left-handed projection of a spinor is the right-handed projection of the charge conjugation of the spinor and vice versa (confer (2.6.27)):

$$\begin{aligned}&\Big((I_4\mp\gamma^5)\psi\Big)^c=-i\gamma^2\Big((I_4\mp\gamma^5)\psi\Big)^*=-i\gamma^2(I_4\mp\gamma^5)\psi^*=-i(I_4\pm\gamma^5)\gamma^2\psi^*\\&=(I_4\pm\gamma^5)\gamma^c.\end{aligned} \tag{2.6.88}$$

References: [3, 4, 7, 12].

2.7 Electromagnetic field

2.7.1 Gauge invariance and electromagnetic potential

The Lagrangian density (2.6.40) is a real combination of the complex Dirac matrices γ^i and spinors ψ, $\bar{\psi}$. We define a *gauge transformation of the first type* of the spinor fields,

$$\psi \to \psi' = e^{ie\alpha}\psi, \;\; \bar{\psi} \to \bar{\psi}' = e^{-ie\alpha}\bar{\psi}, \tag{2.7.1}$$

where e is a real scalar constant, called the *spinor electric charge*. If α is a real scalar constant, then (2.6.40) is invariant under (2.7.1). Such a transformation is called *global*. If $\alpha = \alpha(x^i)$ is a scalar function of the coordinates, then (2.6.40) is not invariant under (2.7.1) because

$$\psi'_{;\mu} = e^{ie\alpha}(\psi_{;\mu} + ie\alpha_{,\mu}\psi). \tag{2.7.2}$$

Such a transformation is called *local*. For a local transformation, we must introduce a compensating vector field A_μ, called the *electromagnetic potential*, such that the Weyl *electromagnetic covariant derivative*

$$D_\mu = \nabla_\mu - ieA_\mu \tag{2.7.3}$$

of a spinor ψ,

$$D_\mu\psi = \psi_{;\mu} - ieA_\mu\psi, \tag{2.7.4}$$

transforms under (2.7.1) like ψ:

$$D_\mu\psi' = e^{ie\alpha}D_\mu\psi. \tag{2.7.5}$$

This requirement gives

$$\psi'_{;\mu} - ieA'_\mu\psi' = e^{ie\alpha}(\psi_{;\mu} - ieA_\mu\psi), \tag{2.7.6}$$

which, with (2.7.1) and (2.7.2), yields the transformation law for the electromagnetic potential,

$$A_\mu \to A'_\mu = A_\mu + \alpha_{,\mu}. \tag{2.7.7}$$

This law is called a *gauge transformation of the second type*. The adjoint conjugation of (2.7.4) is

$$D_\mu\bar{\psi} = \bar{\psi}_{;\mu} + ieA^*_\mu\bar{\psi}. \tag{2.7.8}$$

The scalar $\bar{\psi}\psi$ is invariant under (2.7.1), thereby

$$D_\mu\bar{\psi}\psi + \bar{\psi}D_\mu\psi = D_\mu(\bar{\psi}\psi) = \partial_\mu(\bar{\psi}\psi), \tag{2.7.9}$$

which constrains the electromagnetic potential to be real:

$$A^*_\mu = A_\mu. \tag{2.7.10}$$

The time component of A^μ, $\phi = A^0$, is called the *electric potential* and the spatial components A^α form the *magnetic potential* $\mathbf{A}$:

$$A^\mu = (\phi, \mathbf{A}). \tag{2.7.11}$$

The gauge transformation (2.7.7) reads

$$\phi' = \phi + \frac{\partial\alpha}{c\partial t}, \;\; \mathbf{A}' = \mathbf{A} - \boldsymbol{\nabla}\alpha. \tag{2.7.12}$$

In the local Minkowski spacetime, A^μ transforms according to (1.6.101),

$$\left(\begin{array}{c} \phi \\ \mathbf{A} \end{array}\right) = \left(\begin{array}{cc} \gamma & \gamma\boldsymbol{\beta} \\ \gamma\boldsymbol{\beta} & 1 + \frac{(\gamma-1)\boldsymbol{\beta}}{\beta^2}\boldsymbol{\beta} \end{array}\right)\left(\begin{array}{c} \phi' \\ \mathbf{A}' \end{array}\right). \tag{2.7.13}$$

The gauge-invariant modification of the Dirac Lagrangian density (2.6.40) is

$$\mathfrak{L}_\psi = \frac{i\mathfrak{e}}{2}e^i_a(\bar{\psi}\gamma^a D_i\psi - D_i\bar{\psi}\gamma^a\psi) - m\mathfrak{e}\bar{\psi}\psi = \frac{i\mathfrak{e}}{2}e^i_a(\bar{\psi}\gamma^a\psi_{;i} - \bar{\psi}_{;i}\gamma^a\psi) - m\mathfrak{e}\bar{\psi}\psi - eA_i\mathfrak{e}\bar{\psi}\gamma^i\psi. \tag{2.7.14}$$

The spin density corresponding to the Lagrangian density (2.7.14) remains equal to (2.6.43); it is independent of the spinor electric charge e. The electromagnetic potential corresponds, up to the multiplication by an arbitrary constant, to the vector multiple of I_4 in the formula for the spinor connection (1.7.26). The electromagnetic potential is analogous to the affine connection: it modifies a derivative of a spinor so such derivative transforms like a spinor under unitary gauge transformations of the first type, while the connection modifies a derivative of a tensor so such derivative transforms like a tensor under coordinate transformations.

The gauge-invariant modification of the Dirac equation (2.6.54) is

$$i\gamma^k\psi_{;k}+eA_k\gamma^k\psi=m\psi, \tag{2.7.15}$$

whose adjoint conjugate generalizes (2.6.57):

$$-i\bar{\psi}_{;k}\gamma^k+eA_k\bar{\psi}\gamma^k=m\bar{\psi}. \tag{2.7.16}$$

The gauge-invariant modification of the Dirac equation (2.6.56) is

$$i\gamma^k\psi_{:k}+eA_k\gamma^k\psi=m\psi-\frac{3\kappa}{8}(\bar{\psi}\gamma^5\gamma_k\psi)\gamma^5\gamma^k\psi, \tag{2.7.17}$$

whose adjoint conjugate generalizes (2.6.58):

$$-i\bar{\psi}_{:k}\gamma^k+eA_k\bar{\psi}\gamma^k=m\bar{\psi}-\frac{3\kappa}{8}(\bar{\psi}\gamma^5\gamma_k\psi)\bar{\psi}\gamma^5\gamma^k. \tag{2.7.18}$$

Taking the complex conjugate of (2.7.17) gives, using (2.6.50),

$$-i\gamma^{k*}\psi^*_{:k}+eA_k\gamma^{k*}\psi^*=m\psi^*-\frac{3\kappa}{8}(\bar{\psi}\gamma^5\gamma_k\psi)\gamma^{5*}\gamma^{k*}\psi^*. \tag{2.7.19}$$

The relations (1.7.18) and (1.7.24) give

$$\psi^c_{;k}=\psi^c_{|k}=(-i\gamma^2\psi^*)_{|k}=-i\gamma^2\psi^*_{|k}=-i\gamma^2\psi^*_{;k}. \tag{2.7.20}$$

Substituting (2.6.86) and (2.7.20) into (2.7.19) gives, using (2.6.5) and (2.6.9),

$$-i\gamma^2\gamma^k\gamma^2(-i\gamma^2)\psi^c_{:k}+eA_k\gamma^2\gamma^k\gamma^2(-i\gamma^2)\psi^c=m(-i\gamma^2)\psi^c-\frac{3\kappa}{8}(\bar{\psi}\gamma^5\gamma_k\psi)\gamma^5\gamma^2\gamma^k\gamma^2(-i\gamma^2)\psi^c. \tag{2.7.21}$$

Using (2.6.8), the relation (2.7.21) becomes

$$\gamma^2\gamma^k\psi^c_{:k}+ieA_k\gamma^2\gamma^k\psi^c=-im\gamma^2\psi^c+\frac{3i\kappa}{8}(\bar{\psi}\gamma^5\gamma_k\psi)\gamma^2\gamma^5\gamma^k\psi^c. \tag{2.7.22}$$

Multiplying (2.7.22) by $-i\gamma^2$ from the left brings this equation to

$$i\gamma^k\psi^c_{:k}-eA_k\gamma^k\psi^c=m\psi^c-\frac{3\kappa}{8}(\bar{\psi}\gamma^5\gamma_k\psi)\gamma^5\gamma^k\psi^c. \tag{2.7.23}$$

The Hermitian conjugate of (2.6.86) gives

$$\psi^T=\psi^{*\dagger}=i\psi^{c\dagger}\gamma^{2\dagger}=-i\psi^{c\dagger}\gamma^2. \tag{2.7.24}$$

Thus we obtain, using (2.6.50),

$$\begin{aligned}\bar{\psi}\gamma^5\gamma_k\psi&=(\psi^\dagger\gamma^0\gamma^5\gamma_k\psi)^*=\psi^T\gamma^2\gamma^0\gamma^2\gamma^5\gamma^2\gamma_k\gamma^2\psi^*=(-i\psi^{c\dagger}\gamma^2)\gamma^2\gamma^0\gamma^2\gamma^5\gamma^2\gamma_k\gamma^2(-i\gamma^2\psi^c)\\&=-\psi^{c\dagger}\gamma^0\gamma^2\gamma^5\gamma^2\gamma_k\psi^c=-\overline{\psi^c}\gamma^5\gamma_k\psi^c.\end{aligned} \tag{2.7.25}$$

Substituting (2.7.25) into (2.7.23) gives the Dirac equation for the charge-conjugate spinor field ψ^c:

$$i\gamma^k\psi^c_{:k}-eA_k\gamma^k\psi^c=m\psi^c+\frac{3\kappa}{8}(\overline{\psi^c}\gamma^5\gamma_k\psi^c)\gamma^5\gamma^k\psi^c. \tag{2.7.26}$$

Comparing (2.7.26) with (2.7.17) shows that ψ and ψ^c correspond to the same value of the mass m and to the opposite values of the electric charge, e and $-e$. Accordingly, the charge-conjugation transformation does not change the mass of a spinor, but changes the sign of its electric charge. The field equations for ψ and ψ^c are asymmetric because of the opposite signs of the corresponding cubic terms relative to the mass terms. This asymmetry is related to the fact that the scalar $\bar{\psi}\psi$ changes sign under the charge-conjugation transformation:

$$\overline{\psi^c}\psi^c = -\bar{\psi}\psi, \tag{2.7.27}$$

whereas the Lorentz square of $\bar{\psi}\gamma^5\gamma^k\psi$ does not change sign:

$$(\overline{\psi^c}\gamma^5\gamma^k\psi^c)(\overline{\psi^c}\gamma^5\gamma_k\psi^c) = (\bar{\psi}\gamma^5\gamma^k\psi)(\bar{\psi}\gamma^5\gamma_k\psi). \tag{2.7.28}$$

The first two terms in the effective metric Lagrangian density (2.6.59) are thus antisymmetric under charge conjugation, while the last, four-fermion term is symmetric. Therefore, (ψ, m, e) and $(\psi^c, m, -e)$ are asymmetric under the charge-conjugation transformation and do not satisfy the same field equation. Torsion generates an asymmetry between a spinor and its charge conjugate. At densities satisfying $\kappa\bar{\psi}\psi \ll m$, the cubic terms in (2.7.17) and (2.7.26) can be neglected. In this approximation, (ψ, m, e) and $(\psi^c, m, -e)$ are symmetric under the charge-conjugation transformation and satisfy the same field equation.

2.7.2 Electromagnetic field tensor

The commutator of total covariant derivatives of a spinor is given by (1.7.36) with the curvature spinor K_{ij} given by (1.7.42), where the tensor B_{ij} is related to the vector A_i in (1.7.26) by (1.7.44). Therefore, the commutator of the electromagnetic covariant derivatives of a spinor, $[D_i, D_j]\psi$, is given by (1.7.36) with the curvature spinor

$$K_{ij} = \frac{1}{4}R_{klij}\gamma^k\gamma^l + ieF_{ij}I_4, \tag{2.7.29}$$

where the antisymmetric tensor

$$F_{ij} = A_{j,i} - A_{i,j} = A_{j:i} - A_{i:j} \tag{2.7.30}$$

is referred to as the *electromagnetic field tensor*. The electromagnetic field tensor is analogous to the curvature tensor: it appears in the expression for the commutator of electromagnetic covariant derivatives of a spinor, while the curvature tensor appears in the expression for the commutator of coordinate-covariant derivatives of a tensor. Substituting (2.7.7) into (2.7.30) gives

$$F'_{ij} = F_{ij}, \tag{2.7.31}$$

so the electromagnetic field tensor is gauge invariant. The definition (2.7.30) is equivalent to the *first Maxwell-Minkowski equation*

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = F_{ij:k} + F_{jk:i} + F_{ki:j} = 0 \tag{2.7.32}$$

or

$$\epsilon^{ijkl}F_{jk,l} = \epsilon^{ijkl}F_{jk:l} = 0. \tag{2.7.33}$$

We define a spatial vector $\mathbf{E}$ whose covariant components are related to the 0α components of the electromagnetic field tensor (2.7.30):

$$E_\alpha = F_{0\alpha}, \tag{2.7.34}$$

and a spatial tensor $B_{\alpha\beta}$ equal to the spatial part of (2.7.30):

$$B_{\alpha\beta} = F_{\alpha\beta}, \;\; B^\alpha = -\frac{1}{2\sqrt{\mathfrak{s}}}\epsilon^{\alpha\beta\gamma}B_{\beta\gamma}, \;\; B_{\alpha\beta} = -\sqrt{\mathfrak{s}}\varepsilon_{\alpha\beta\gamma}B^\gamma, \tag{2.7.35}$$

where $\mathfrak{s}$ is given by (1.4.147). The component of (2.7.32) with all spatial indices, $B_{\alpha\beta,\gamma}+B_{\beta\gamma,\alpha}+B_{\gamma\alpha,\beta}=0$, gives, using (1.4.167),

$$\operatorname{div}\mathbf{B}=0. \tag{2.7.36}$$

The components of (2.7.32) with one temporal index, $B_{\alpha\beta,0}+E_{\alpha,\beta}-E_{\beta,\alpha}=0$, gives, using (1.4.168),

$$\operatorname{curl}\mathbf{E}=-\frac{1}{c\sqrt{\mathfrak{s}}}\frac{\partial(\sqrt{\mathfrak{s}}\mathbf{B})}{\partial t}. \tag{2.7.37}$$

The spatial vector $\mathbf{E}$ is called the *electric field* and the spatial pseudovector $\mathbf{B}$ is the *magnetic field*.

In the locally geodesic and Galilean frame of reference, these fields depend on the components of the electromagnetic potential (2.7.11) according to (2.7.30):

$$\mathbf{E}=-\frac{\partial\mathbf{A}}{c\partial t}-\boldsymbol{\nabla}\phi, \tag{2.7.38}$$

$$\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}, \tag{2.7.39}$$

and they are invariant under (2.7.12). The tensor F_{ij} is given by

$$F_{ij}=\begin{pmatrix}0 & E_x & E_y & E_z\\ -E_x & 0 & -B_z & B_y\\ -E_y & B_z & 0 & -B_x\\ -E_z & -B_y & B_x & 0\end{pmatrix},\ \ \mathbf{E}=(F_{01},F_{02},F_{03}),\ \ \mathbf{B}=(F_{32},F_{13},F_{21}), \tag{2.7.40}$$

and transforms under Lorentz transformations according to (1.6.102). Therefore, the electric and magnetic fields transform according to

$$\mathbf{E}=\gamma(\mathbf{E}'-\boldsymbol{\beta}\times\mathbf{B}')+\frac{1-\gamma}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{E}')\boldsymbol{\beta}, \tag{2.7.41}$$

$$\mathbf{B}=\gamma(\mathbf{B}'+\boldsymbol{\beta}\times\mathbf{E}')+\frac{1-\gamma}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{B}')\boldsymbol{\beta}. \tag{2.7.42}$$

In this frame, (2.7.36) and (2.7.37) become the *first pair of the Maxwell equations*:

$$\operatorname{div}\mathbf{B}=0, \tag{2.7.43}$$

$$\operatorname{curl}\mathbf{E}=-\frac{\partial\mathbf{B}}{c\partial t}. \tag{2.7.44}$$

Applying the div operator to (2.7.39) gives (2.7.43) and applying the curl operator to (2.7.38) gives (2.7.44). Applying the div operator to (2.7.44) gives (2.7.43). Integrating the first pair of the Maxwell equations over the volume and surface area, respectively, gives

$$\oint\mathbf{B}\cdot d\mathbf{f}=0, \tag{2.7.45}$$

$$\oint\mathbf{E}\cdot d\mathbf{l}=-\frac{\partial}{c\partial t}\Big(\int\mathbf{B}\cdot d\mathbf{f}\Big). \tag{2.7.46}$$

The integral $\oint\mathbf{A}\cdot d\mathbf{f}$ is the flux of a vector $\mathbf{A}$ through the surface $\mathbf{f}$ and the integral $\oint\mathbf{A}\cdot d\mathbf{l}$ is called the *circulation* of $\mathbf{A}$ along the contour $\mathbf{l}$. Therefore, the flux of the magnetic field through a closed surface vanishes and the circulation of the electric field along a contour, which is called the *electromotive force*, is equal to the minus time derivative of the flux of the magnetic field through the surface enclosed by this contour (*Faraday's law*).

In a locally Galilean frame of reference, the simplest invariants (under proper Lorentz transformations) of the electromagnetic field are quadratic in F_{ij}:

$$F_{ij}F^{ij}=2(B^2-E^2)=\text{const},\ \ e^{ijkl}F_{ij}F_{kl}=8\mathbf{E}\cdot\mathbf{B}=\text{const}. \tag{2.7.47}$$

If the vectors $\mathbf{E}$ and $\mathbf{B}$ are mutually perpendicular in frame K, $\mathbf{E}\cdot\mathbf{B}=0$, then they are mutually perpendicular in other inertial frames. If $\mathbf{E}$ and $\mathbf{B}$ are equal in magnitude in K, $B^2-E^2=0$, then they are equal in magnitude in other inertial frames. The transformation laws (2.7.41) and (2.7.42) imply that if $\mathbf{E}' = 0$ in the frame K' then in the frame K

$$\mathbf{E} = -\gamma\boldsymbol{\beta}\times\mathbf{B}' = -\boldsymbol{\beta}\times\mathbf{B}, \tag{2.7.48}$$

and if $\mathbf{B}' = 0$ in the frame K' then in the frame K

$$\mathbf{B} = \gamma\boldsymbol{\beta}\times\mathbf{E}' = \boldsymbol{\beta}\times\mathbf{E}. \tag{2.7.49}$$

If the vectors $\mathbf{E}$ and $\mathbf{B}$ are mutually perpendicular in K, but not equal in magnitude, then there exists a frame K' in which the field is either electric, $\mathbf{B}' = 0$ (if $E > B$), or magnetic, $\mathbf{E}' = 0$ (if $E < B$). The velocity of K' relative to K is perpendicular to $\mathbf{E}$ and $\mathbf{B}$, and it is equal in magnitude to respectively either $c\frac{B}{E}$ or $c\frac{E}{B}$. Equivalently, if one of the vectors $\mathbf{E},\mathbf{B}$ vanishes in one frame of reference then these vectors are mutually perpendicular in other inertial frames. Except for the case where the vectors $\mathbf{E}$ and $\mathbf{B}$ are mutually perpendicular and equal in magnitude, there exist frames in which these vectors are parallel to each other at a given point. These frames move relative to one another with velocities parallel to both vectors. One of such frames, K' (in which $\mathbf{E}' \parallel \mathbf{B}'$), has a velocity $\mathbf{V}$ relative to K which is perpendicular to both vectors $\mathbf{E}$ and $\mathbf{B}$. Substituting the formulae

$$\mathbf{E}' = \gamma(\mathbf{E}+\boldsymbol{\beta}\times\mathbf{B}) + \frac{1-\gamma}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{E})\boldsymbol{\beta}, \tag{2.7.50}$$

$$\mathbf{B}' = \gamma(\mathbf{B}-\boldsymbol{\beta}\times\mathbf{E}) + \frac{1-\gamma}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{B})\boldsymbol{\beta}, \tag{2.7.51}$$

which are inverse to (2.7.41) and (2.7.42), into the condition $\mathbf{E}'\times\mathbf{B}' = 0$ and using $\boldsymbol{\beta} = k\mathbf{E}\times\mathbf{B}$, where k is a constant of proportionality, gives $\Big(\mathbf{E}+k(\mathbf{E}\times\mathbf{B})\times\mathbf{B}\Big)\times\Big(\mathbf{B}-k(\mathbf{E}\times\mathbf{B})\times\mathbf{E}\Big) = 0$ or $k = \frac{1+\beta^2}{E^2+B^2}$, thereby

$$\frac{\frac{\mathbf{V}}{c}}{1+\frac{V^2}{c^2}} = \frac{\mathbf{E}\times\mathbf{B}}{E^2+B^2}. \tag{2.7.52}$$

2.7.3 Lagrangian density for electromagnetic field

The simplest gauge-invariant Lagrangian density representing the electromagnetic field is a linear combination of terms quadratic in F_{ij}: $\sqrt{-\mathfrak{g}}F_{ij}F^{ij}$ and $\epsilon^{ijkl}F_{ij}F_{kl}$, which a locally Galilean frame of reference reduce to (2.7.47). The second, parity-violating term is a total divergence because of (2.7.33):

$$\epsilon^{ijkl}F_{ij}F_{kl} = 2(\epsilon^{ijkl}F_{ij}A_l)_{,k}, \tag{2.7.53}$$

so it does not contribute to the field equations. Therefore, the Lagrangian density for the electromagnetic field is given by

$$\mathfrak{L}_{EM} = -\frac{1}{16\pi}\sqrt{-\mathfrak{g}}F_{ij}F^{ij}, \tag{2.7.54}$$

where the Gaußian factor $\frac{1}{16\pi}$ sets the units of A_i. In the locally geodesic and Galilean frame of reference, (2.7.54) becomes

$$\mathfrak{L}_{EM} = \frac{1}{8\pi}(E^2-B^2). \tag{2.7.55}$$

Therefore, in order for the action S to have a minimum, there must be the minus sign in front of the right-hand side of (2.7.54). Otherwise an arbitrarily rapid change of $\mathbf{A}$ in time would result in an arbitrarily large value of $\mathbf{E}$, according to (2.7.38), and thus an arbitrarily low value of S, thereby the action would have no minimum. A generalization of the tensor (2.7.30) to the covariant derivatives with respect to the affine connection $\Gamma^{\,k}_{i\,j}$, $\tilde{F}_{ij} = A_{j;i} - A_{i;j} = F_{ij} + 2S^k_{\ ij}A_k$, is not gauge invariant, thereby the torsion tensor cannot appear in a gauge-invariant Lagrangian density which is quadratic in F_{ij}. Therefore, the electromagnetic field, unlike spinor fields, does not couple to torsion. Accordingly, the electromagnetic field is not minimally coupled to the affine connection.

2.7.4 Electromagnetic current and electric charge

We define the *electromagnetic current density*

$$\mathsf{j}^i = -\frac{c\delta\mathfrak{L}_\mathrm{m}}{\delta A_i}, \tag{2.7.56}$$

and the *electromagnetic current vector*

$$j^i = \frac{\mathsf{j}^i}{\sqrt{-\mathfrak{g}}}. \tag{2.7.57}$$

The invariance of the action under an arbitrary infinitesimal gauge transformation $\delta A_i = A'_i - A_i = \phi_{,i}$ gives, upon partial integration and omitting a total divergence,

$$\delta S = -\frac{1}{c^2}\int \mathsf{j}^i\delta A_j d\Omega = -\frac{1}{c^2}\int \mathsf{j}^i\phi_{,i}d\Omega = \frac{1}{c^2}\int \mathsf{j}^i{}_{,i}\phi d\Omega = 0, \tag{2.7.58}$$

so the electromagnetic current is conserved,

$$\mathsf{j}^i{}_{,i} = 0,\ \ j^i{}_{:i} = 0. \tag{2.7.59}$$

For spinor matter, we use the gauge-invariant Lagrangian density (2.7.14): $\mathfrak{L}_\mathrm{m} = \mathfrak{L}_\psi$. The electromagnetic current (2.7.56) for the spinor field is thus proportional to the conserved vector Dirac current (2.6.61):

$$\mathsf{j}^i = ec\mathfrak{e}\bar{\psi}\gamma^i\psi = ec\mathsf{j}^i_V. \tag{2.7.60}$$

Let us consider matter which is distributed over a small region in space, as in section 2.4.9. Integrating (2.7.59) over the volume hypersurface and using Gauß' theorem to eliminate surface integrals gives

$$\int \mathsf{j}^0{}_{,0}dV = 0. \tag{2.7.61}$$

The conservation law (2.7.59) also gives

$$(x^k\mathsf{j}^i)_{,i} = x^k{}_{,i}\mathsf{j}^i + x^k\mathsf{j}^i{}_{,i} = \delta^k_i\mathsf{j}^i = \mathsf{j}^k, \tag{2.7.62}$$

which, upon integrating over the volume hypersurface and using Gauß' theorem to eliminate surface integrals, gives

$$\left(\int x^k\mathsf{j}^0 dV\right)_{,0} = \int \mathsf{j}^k dV. \tag{2.7.63}$$

Using (2.4.111) and (2.4.112) turns (2.7.63) into

$$\frac{u^k}{u^0}\int \mathsf{j}^0 dV + \left(\int \delta x^k\mathsf{j}^0 dV\right)_{,0} = \int \mathsf{j}^k dV. \tag{2.7.64}$$

A particle located at $\mathbf{x}_a$ satisfies $\int \delta x^k\mathsf{j}^0 dV = 0$. Consequently, $\mathsf{j}^i(\mathbf{x})$ is proportional to $\boldsymbol{\delta}(\mathbf{x}-\mathbf{x}_a)$, giving

$$\mathsf{j}^k = \frac{u^k}{u^0}\mathsf{j}^0. \tag{2.7.65}$$

The electromagnetic current density of a particle is proportional to its four-velocity, analogously to (2.4.130).

We define the *electric charge density* ρ such that

$$j^0 = \frac{c\rho}{\sqrt{g_{00}}}. \tag{2.7.66}$$

The electric charge density is not a tensor density. We define the *electric charge* e such that

$$\rho\sqrt{\mathfrak{s}}dV = de. \tag{2.7.67}$$

The electric charge density for particles with charges e_a located at $\mathbf{x}_a$ is

$$\rho(\mathbf{x}) = \sum_a \frac{e_a}{\sqrt{\mathfrak{s}}}\boldsymbol{\delta}(\mathbf{x} - \mathbf{x}_a). \tag{2.7.68}$$

A quantity $\int \mathrm{j}^0 dV$, which is equal to $\int \mathrm{j}^i dS_i$ for a volume hypersurface and thereby is a scalar, is thus

$$\int \mathrm{j}^0 dV = \sum_a \int \sqrt{-\mathfrak{g}}\frac{ce_a}{\sqrt{g_{00}}\sqrt{\mathfrak{s}}}\boldsymbol{\delta}(\mathbf{x} - \mathbf{x}_a)dV = c\sum_a e_a. \tag{2.7.69}$$

The electric charge is therefore a scalar. Therefore, the electromagnetic current vector for a system of charged particles is

$$j^k(\mathbf{x}) = \sum_a \frac{cu^k}{u^0}\frac{e_a}{\sqrt{-\mathfrak{g}}}\boldsymbol{\delta}(\mathbf{x} - \mathbf{x}_a), \tag{2.7.70}$$

analogously to (2.4.154). The relation (2.7.61) represents the conservation of the total electric charge of a physical system. For a particle moving along a wordline $x_a(\tau)$, we have

$$j^k(x) = \frac{cu^k}{u^0}\frac{e}{\sqrt{-\mathfrak{g}}}\boldsymbol{\delta}(\mathbf{x} - \mathbf{x}_a) = ec\int \frac{u^k}{\sqrt{-\mathfrak{g}}}\delta(x - x_a(\tau))d\tau. \tag{2.7.71}$$

In the locally geodesic and Galilean frame of reference, $\frac{u^i}{u^0} = (1, \mathbf{v}/c)$, which gives

$$j^i = (c\rho, \mathbf{j}), \tag{2.7.72}$$

where $\mathbf{j}$ is the spatial *electric current density vector*,

$$\mathbf{j} = \rho\mathbf{v}. \tag{2.7.73}$$

The conservation law (2.7.59) in this frame has the form of the equation of continuity, as (2.6.62):

$$j^i_{\ ,i} = \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j} = 0. \tag{2.7.74}$$

For one particle located at $\mathbf{x}_0(t)$, $\rho(\mathbf{x}) = e\boldsymbol{\delta}(\mathbf{x} - \mathbf{x}_0)$, (2.7.74) is explicitly satisfied since

$$\begin{aligned}\frac{\partial\rho}{\partial t} &= e\frac{\partial}{\partial t}\boldsymbol{\delta}(\mathbf{x} - \mathbf{x}_0) = e\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}_0}\boldsymbol{\delta}(\mathbf{x} - \mathbf{x}_0) = -e\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}}\boldsymbol{\delta}(\mathbf{x} - \mathbf{x}_0)\\ &= -\frac{\partial}{\partial\mathbf{x}}\cdot\Big(e\mathbf{v}\boldsymbol{\delta}(\mathbf{x} - \mathbf{x}_0)\Big) = -\boldsymbol{\nabla}\cdot\mathbf{j},\end{aligned} \tag{2.7.75}$$

where $\mathbf{v} = \frac{d\mathbf{x}_0}{dt}$. For a system of charged particles, we also have

$$\int \mathbf{j}dV = \sum_a e_a\mathbf{v}_a, \tag{2.7.76}$$

where $\mathbf{v}_a = \frac{d\mathbf{x}_a}{dt}$ is the velocity of the particle of charge e_a. The equation of continuity (2.7.74) represents, upon integrating over the volume, the conservation of the total electric charge:

$$\frac{\partial}{\partial t}\left(\int \rho dV\right) + \oint \mathbf{j}\cdot d\mathbf{f} = 0. \tag{2.7.77}$$

The relation (2.7.60) gives

$$\rho = e\psi^\dagger\psi,\ \ \mathbf{j} = ec\bar{\psi}\boldsymbol{\gamma}\psi, \tag{2.7.78}$$

where $\boldsymbol{\gamma}$ is a spatial vector composed from the Dirac matrices γ^α. If we identify the electric charge e in (2.7.67) with the spinor electric charge e in (2.7.78), then we find an integral constraint on a Dirac spinor field:

$$\int \psi^\dagger\psi\sqrt{\mathfrak{s}}dV = 1. \tag{2.7.79}$$

2.7.5 Maxwell equations

The Lagrangian density for the electromagnetic field and charged matter is the sum of (2.7.54) and the term $-\sqrt{-\mathfrak{g}}A_i j^i$ arising from (2.7.56):

$$\mathfrak{L}_{\mathrm{EM+q}} = -\frac{1}{16\pi}\sqrt{-\mathfrak{g}}F_{ik}F^{ik} - \frac{1}{c}\sqrt{-\mathfrak{g}}A_k j^k, \tag{2.7.80}$$

where we omitted terms corresponding to the gravitational field and matter fields which do not depend on A_k. Varying (2.7.80) with respect to the electromagnetic potential A_k and omitting a total divergence gives

$$\begin{aligned}\delta_{\mathrm{A}}\mathfrak{L}_{\mathrm{EM+q}} &= -\frac{1}{8\pi}\sqrt{-\mathfrak{g}}F^{ik}\delta F_{ik} - \frac{\mathsf{j}^k}{c}\delta A_k = -\frac{1}{8\pi}\sqrt{-\mathfrak{g}}F^{ik}(\delta A_{k,i} - \delta A_{i,k}) - \frac{\mathsf{j}^k}{c}\delta A_k \\ &= \frac{1}{4\pi}\sqrt{-\mathfrak{g}}F^{ik}\delta A_{k,i} - \frac{\mathsf{j}^k}{c}\delta A_k = \frac{1}{4\pi}(\sqrt{-\mathfrak{g}}F^{ik})_{,i}\delta A_k - \frac{1}{c}\sqrt{-\mathfrak{g}}j^k\delta A_k,\end{aligned} \tag{2.7.81}$$

so the principle of least action $\delta S = 0$ for arbitrary variations δA_k yields the *second Maxwell-Minkowski equation*

$$(\sqrt{-\mathfrak{g}}F^{ik})_{,i} = \frac{4\pi}{c}\mathsf{j}^k \tag{2.7.82}$$

or

$$F^{ik}{}_{:i} = \frac{4\pi}{c}j^k. \tag{2.7.83}$$

The variation (2.7.81) can also be written as

$$\delta_{\mathrm{A}}\mathfrak{L}_{\mathrm{EM+q}} = -\frac{1}{8\pi}\mathfrak{e}F^{ik}(\delta A_{k:i} - \delta A_{i:k}) - \frac{\mathsf{j}^k}{c}\delta A_k = -\frac{1}{4\pi}\mathfrak{e}F^{ik}\delta A_{k:i} - \frac{\mathsf{j}^k}{c}\delta A_k. \tag{2.7.84}$$

Accordingly, we have

$$\frac{\partial\mathfrak{L}_{\mathrm{EM+q}}}{\partial A_{k:i}} = \frac{\partial\mathfrak{L}_{\mathrm{EM+q}}}{\partial A_{k,i}} = -\frac{1}{4\pi}\mathfrak{e}F^{ik}. \tag{2.7.85}$$

The Lagrange equations (2.1.8) with $\mathfrak{L} = \mathfrak{L}_{\mathrm{EM+q}}$ for the field A_k

$$\frac{\partial\mathfrak{L}}{\partial A_k} - \partial_i\left(\frac{\partial\mathfrak{L}}{\partial(A_{k,i})}\right) = 0, \tag{2.7.86}$$

are equivalent to (2.7.82) with (2.7.56). The electromagnetic field equation (2.7.82) infers that j^i is conserved, $\mathsf{j}^i{}_{,i} = 0$, which corresponds to the conservation of the total electric charge, but does not constrain the motion of particles. Therefore, a configuration of charged particles producing the electromagnetic field can be arbitrary, subject only to the condition that the total charge be conserved, unlike a configuration of particles producing the gravitational field which is not arbitrary but constrained by the gravitational field equations.

The Lagrangian density for the electromagnetic field and a charged spinor is

$$\mathfrak{L} = -\frac{\mathfrak{e}}{16\pi}F_{ik}F^{ik} + \frac{i\mathfrak{e}}{2}e^i_a(\bar{\psi}\gamma^a D_i\psi - D_i\bar{\psi}\gamma^a\psi) - m\mathfrak{e}\bar{\psi}\psi. \tag{2.7.87}$$

For an infinitesimal gauge transformation, $\alpha \ll 1$, (2.7.1)) and (2.7.7) give

$$\delta\psi = ie\alpha\psi,\ \ \delta\bar{\psi} = -ie\alpha\bar{\psi},\ \ \delta A_k = \alpha_{,k},\ \ \xi^i = 0. \tag{2.7.88}$$

If this transformation is global, $\alpha = \mathrm{const}$, then the corresponding Noether current (2.4.7) is

$$\begin{aligned}\mathfrak{J}^i &= \frac{\partial\mathfrak{L}}{\partial A_{k,i}}\delta A_k + \frac{\partial\mathfrak{L}}{\partial\psi_{,i}}\delta\psi + \delta\bar{\psi}\frac{\partial\mathfrak{L}}{\partial\bar{\psi}_{,i}} = -\frac{\mathfrak{e}}{4\pi}F^{ik}\delta A_k + \frac{i\mathfrak{e}}{2}\bar{\psi}\gamma^i\delta\psi - \frac{i\mathfrak{e}}{2}\delta\bar{\psi}\gamma^i\psi \\ &= -e\mathfrak{e}\bar{\psi}\gamma^i\psi\alpha = -\frac{\alpha}{c}\mathsf{j}^i,\end{aligned} \tag{2.7.89}$$

where we used (2.7.60). Consequently, the conservation law (2.4.6) gives the conservation (2.7.59) of the electromagnetic current associated with the spinor.

We define

$$D^\alpha = -\sqrt{g_{00}}F^{0\alpha}, \tag{2.7.90}$$

$$H^{\alpha\beta} = \sqrt{g_{00}}F^{\alpha\beta}, \ \ H_\alpha = -\frac{1}{2}\sqrt{\mathfrak{s}}\varepsilon_{\alpha\beta\gamma}H^{\beta\gamma}, \ \ H^{\alpha\beta} = -\frac{1}{\sqrt{\mathfrak{s}}}\epsilon^{\alpha\beta\gamma}H_\gamma. \tag{2.7.91}$$

The relations $F_{0\alpha} = g_{0i}g_{\alpha j}F^{ij}$ and $F^{\alpha\beta} = g^{\alpha i}g^{\beta j}F_{ij}$ give then

$$D_\alpha = \frac{E_\alpha}{\sqrt{g_{00}}} + g^\beta H_{\alpha\beta}, \tag{2.7.92}$$

$$B^{\alpha\beta} = \frac{H^{\alpha\beta}}{\sqrt{g_{00}}} - g^\alpha E^\beta + g^\beta E^\alpha, \tag{2.7.93}$$

or, in the spatial-vector notation,

$$\mathbf{D} = \frac{\mathbf{E}}{\sqrt{g_{00}}} - \mathbf{g}\times\mathbf{H}, \tag{2.7.94}$$

$$\mathbf{B} = \frac{\mathbf{H}}{\sqrt{g_{00}}} + \mathbf{g}\times\mathbf{E}. \tag{2.7.95}$$

Using (1.4.144) brings the temporal component of (2.7.82) to

$$\frac{1}{\sqrt{\mathfrak{s}}}(\sqrt{\mathfrak{s}}D^\alpha)_{,\alpha} = 4\pi\rho \tag{2.7.96}$$

or

$$\mathrm{div}\,\mathbf{D} = 4\pi\rho. \tag{2.7.97}$$

The spatial components of (2.7.82) read

$$\frac{1}{\sqrt{\mathfrak{s}}}(\sqrt{\mathfrak{s}}H^{\alpha\beta})_{,\beta} + \frac{1}{\sqrt{\mathfrak{s}}}(\sqrt{\mathfrak{s}}D^\alpha)_{,0} = -4\pi\rho\frac{dx^\alpha}{dx^0} \tag{2.7.98}$$

or

$$\mathrm{curl}\,\mathbf{H} = \frac{1}{c\sqrt{\mathfrak{s}}}\frac{\partial(\sqrt{\mathfrak{s}}\mathbf{D})}{\partial t} + \frac{4\pi}{c}\mathbf{j}. \tag{2.7.99}$$

The conservation law (2.7.59) reads

$$\frac{1}{\sqrt{\mathfrak{s}}}\frac{\partial(\sqrt{\mathfrak{s}}\rho)}{\partial t} + \mathrm{div}\,\mathbf{j} = 0. \tag{2.7.100}$$

In the locally geodesic and Galilean frame of reference, (2.7.94) and (2.7.95) reduce to

$$\mathbf{D} = \mathbf{E}, \tag{2.7.101}$$

$$\mathbf{B} = \mathbf{H}. \tag{2.7.102}$$

In this frame, (2.7.97) and (2.7.99) become the *second pair of the Maxwell equations*:

$$\mathrm{div}\,\mathbf{E} = 4\pi\rho, \tag{2.7.103}$$

$$\mathrm{curl}\,\mathbf{B} = \frac{\partial\mathbf{E}}{c\partial t} + \frac{4\pi}{c}\mathbf{j}. \tag{2.7.104}$$

Applying the div operator to (2.7.104) and using (2.7.103) gives (2.7.74). Integrating the second pair of the Maxwell equations over the volume and surface area, respectively, gives

$$\oint\mathbf{E}\cdot d\mathbf{f} = 4\pi q, \tag{2.7.105}$$

$$\oint\mathbf{B}\cdot d\mathbf{l} = \frac{\partial}{c\partial t}\Big(\int\mathbf{E}\cdot d\mathbf{f}\Big) + \frac{4\pi}{c}\int\mathbf{j}\cdot d\mathbf{f}. \tag{2.7.106}$$

Therefore, the flux of the electric field through a closed surface is proportional to the total charge inside the volume enclosed by the surface $\mathbf{f}$ (*Gauß' law*) and the circulation of the magnetic field along a contour is equal to the time derivative of the flux of the electric field through the surface enclosed by this contour, called the displacement current, plus the surface integral of the current vector (the *Ampère-Ørsted law*).

The two pairs of the Maxwell equations are linear in the fields $\mathbf{E}$ and $\mathbf{B}$. The sum of any two solutions of the Maxwell equations is also a solution of these equations. Therefore, the electromagnetic field of a system of sources (particles) is the sum of the fields from each source. The additivity of the electromagnetic field is referred to as the *principle of superposition.*

2.7.6 Energy-momentum tensor for electromagnetic field

The variation with respect to the metric tensor (2.3.4) of the Lagrangian density (2.7.54),

$$\delta_{\mathrm{g}}\mathfrak{L}_{\mathrm{EM}} = -\frac{1}{16\pi}F_{ik}F_{lm}\delta(\sqrt{-\mathfrak{g}}g^{il}g^{km}) = \frac{1}{32\pi}\sqrt{-\mathfrak{g}}g_{lm}F_{ik}F^{ik}\delta g^{lm} - \frac{1}{8\pi}\sqrt{-\mathfrak{g}}F_{ik}F_{lm}g^{il}\delta g^{km}$$
$$= \frac{1}{8\pi}\sqrt{-\mathfrak{g}}\left(\frac{1}{4}g_{ik}F_{lm}F^{lm} - F_i^{\ j}F_{kj}\right)\delta g^{ik}, \tag{2.7.107}$$

gives the metric energy-momentum tensor (2.3.5) for the electromagnetic field:

$$T_{ik} = \frac{1}{4\pi}\left(\frac{1}{4}g_{ik}F_{lm}F^{lm} - F_i^{\ j}F_{kj}\right). \tag{2.7.108}$$

The corresponding energy density W, energy current $\mathbf{S}$ called the *Poynting vector*, and stress tensor $\sigma_{\alpha\beta}$ called the *Maxwell stress tensor*, are given in the locally geodesic and Galilean frame of reference, using (2.4.82), by

$$W = \frac{1}{8\pi}(E^2 + B^2), \tag{2.7.109}$$
$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}, \tag{2.7.110}$$
$$\sigma_{\alpha\beta} = \frac{1}{4\pi}\left(E_\alpha E_\beta + B_\alpha B_\beta - \frac{1}{2}\delta_{\alpha\beta}(E^2 + B^2)\right). \tag{2.7.111}$$

Multiplying (2.7.44) by $\mathbf{B}$ and (2.7.104) by $\mathbf{E}$ and adding these scalar products gives

$$\frac{1}{c}\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} + \frac{1}{c}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} = -\frac{4\pi}{c}\mathbf{j}\cdot\mathbf{E} - (\mathbf{B}\cdot\mathrm{curl}\,\mathbf{E} - \mathbf{E}\cdot\mathrm{curl}\,\mathbf{B}), \tag{2.7.112}$$

from which we obtain

$$\frac{1}{2c}\frac{\partial}{\partial t}(E^2 + B^2) = -\frac{4\pi}{c}\mathbf{j}\cdot\mathbf{E} - \mathrm{div}(\mathbf{E}\times\mathbf{B}) \tag{2.7.113}$$

or

$$\frac{\partial W}{\partial t} + \mathbf{j}\cdot\mathbf{E} + \mathrm{div}\,\mathbf{S} = 0. \tag{2.7.114}$$

Under Lorentz transformations, W, $\mathbf{S}$ and $\sigma_{\alpha\beta}$ transform like the corresponding components of a tensor of rank (0,2) (2.4.82), according to (1.6.102).

The energy-momentum tensor for the electromagnetic field is traceless,

$$T_{ik}g^{ik} = 0, \tag{2.7.115}$$

so (2.4.159) and the virial theorem (2.4.160) remain unchanged if the particles interact electromagnetically. The condition (2.7.115) also gives, using (2.4.202),

$$\epsilon = 3p, \tag{2.7.116}$$

so (2.4.204) shows that the free electromagnetic field is ultrarelativistic. In a frame of reference, in which the vectors $\mathbf{E}$ and $\mathbf{B}$ are parallel to one another or one of them vanishes (and the x axis is along the direction of these vectors), the nonzero components of the tensor T_{ik} are

$$T_{00} = -T_{11} = T_{22} = T_{33} = W. \tag{2.7.117}$$

If the vectors $\mathbf{E}$ (along the x axis) and $\mathbf{B}$ (along the y axis) are mutually perpendicular and equal in magnitude then

$$T_{00} = T_{03} = T_{33} = W. \tag{2.7.118}$$

The variation with respect to the tetrad (2.3.8) of the Lagrangian density (2.7.54),

$$\delta_{\mathrm{e}}\mathfrak{L}_{\mathrm{EM}} = -\frac{1}{16\pi}F_{ij}F_{kl}\eta^{ab}\eta^{cd}\delta(\mathfrak{e}e^i_a e^j_b e^k_c e^l_d) = \frac{1}{16\pi}\mathfrak{e}e^a_i F_{jk}F^{jk}\delta e^i_a - \frac{1}{4\pi}\mathfrak{e}F_{ij}F^{aj}\delta e^i_a, \tag{2.7.119}$$

gives the tetrad energy-momentum tensor (2.3.10) for the electromagnetic field:

$$t_i^{\ a} = \frac{1}{4\pi}\biggl(\frac{1}{4}e^a_i F_{lm}F^{lm} - F_{ij}F^{aj}\biggr). \tag{2.7.120}$$

The corresponding tensor t_{ik} is equal to (2.7.108). This equality is a consequence of the reduced Belinfante-Rosenfeld relation (2.3.35), which is valid if the spin tensor (2.3.26) is equal to zero. The spin tensor for the electromagnetic field vanishes because the Lagrangian density (2.7.54) does not depend on the torsion tensor. The canonical energy-momentum density (2.3.13) for the electromagnetic field is given by

$$\Theta_i^{\ j} = \frac{\partial\mathfrak{L}_{\mathrm{EM}}}{\partial A_{k:j}}A_{k:i} - \delta^j_i\mathfrak{L}_{\mathrm{EM}}. \tag{2.7.121}$$

The relation (2.7.85) gives

$$\Theta_i^{\ j} = -\frac{1}{4\pi}\mathfrak{e}F^{jk}A_{k:i} + \frac{1}{16\pi}\mathfrak{e}\delta^j_i F_{kl}F^{kl}. \tag{2.7.122}$$

The canonical energy-momentum density is not identical with the tetrad energy-momentum density $\mathfrak{e}t_i^{\ j}$ corresponding to (2.7.120) because the relation (2.3.14) is valid only for fields that are minimally coupled to the affine connection.

If the Lagrangian density for the electromagnetic field were minimally coupled to the affine connection, $\tilde{\mathfrak{L}}_{\mathrm{EM}} = -\frac{1}{16\pi}\mathfrak{e}\tilde{F}_{ij}\tilde{F}^{ij}$, then $\delta_{\mathrm{A}}\tilde{\mathfrak{L}}_{\mathrm{EM+q}} = -\frac{1}{4\pi}\mathfrak{e}\tilde{F}^{ik}\delta A_{k;i} - \frac{\mathfrak{j}^k}{c}\delta A_k = -\frac{1}{4\pi}\mathfrak{e}(2S_i\tilde{F}^{ik} - \tilde{F}^{ik}_{\ \ \ ;i})\delta A_k - \frac{\mathfrak{j}^k}{c}\delta A_k$, where a total divergence was omitted. The resulting field equation would be $\nabla^*_i\tilde{F}^{ik} = \frac{4\pi}{c}j^k$. The variation $\delta_{\mathrm{C}}\tilde{\mathfrak{L}}_{\mathrm{EM+q}} = -\frac{1}{4\pi}\mathfrak{e}\tilde{F}^{ik}\delta S^j_{\ ik}A_j = -\frac{1}{4\pi}\mathfrak{e}\tilde{F}^{ik}A_j\delta C_{jik}$ gives the spin tensor, $s_{ijk} = -\frac{1}{2\pi}A_{[i}\tilde{F}_{j]k}$. The second Einstein-Cartan equation gives the corresponding torsion tensor, $S_{kij} = \frac{\kappa}{4\pi}A_{[i}\tilde{F}_{j]k} + \frac{\kappa}{8\pi}A^l g_{k[j}F_{i]l}$, leading to $\tilde{F}_{ij} = A_{j,i} - A_{i,j} + \frac{\kappa}{4\pi}A^k A_{[i}\tilde{F}_{j]k}$ and $\tilde{F}^{ik}_{\ \ \ ;i} = \frac{4\pi}{c}j^k + \frac{\kappa}{8\pi}A^l F_{il}F^{ik}$. In the presence of spinors, S_{kij} would have another part (2.6.48), $S_{kij} = \frac{\kappa}{4\pi}A_{[i}\tilde{F}_{j]k} + \frac{\kappa}{8\pi}A^l g_{k[j}F_{i]l} + \frac{\kappa}{2}e_{ijkl}s^l$, leading to $\tilde{F}_{ij} = A_{j,i} - A_{i,j} + \frac{\kappa}{4\pi}A^k A_{[i}\tilde{F}_{j]k} + \kappa e_{ijkl}A^k s^l$.

The metric energy-momentum tensor for $\tilde{\mathfrak{L}}_{\mathrm{EM}}$ is $T_{ik} = \frac{1}{4\pi}(\frac{1}{4}g_{ik}\tilde{F}_{lm}\tilde{F}^{lm} - \tilde{F}_i^{\ j}\tilde{F}_{kj})$. The Belinfante-Rosenfeld relation (2.3.34) gives $t_{ik} = T_{ik} - \frac{1}{4\pi}\nabla^*_j(A_i\tilde{F}_k^{\ j}) = T_{ik} - \frac{1}{4\pi}A_{i;j}\tilde{F}_k^{\ j}$, where the sourceless field equation $\nabla^*_i\tilde{F}^{ik} = 0$ was used. The corresponding canonical energy-momentum density is $\Theta_i^{\ j} = \frac{\partial\tilde{\mathfrak{L}}_{\mathrm{EM}}}{\partial A_{k;j}}A_{k;i} - \delta^j_i\tilde{\mathfrak{L}}_{\mathrm{EM}} = -\frac{1}{4\pi}\mathfrak{e}\tilde{F}^{jk}A_{k;i} + \frac{1}{16\pi}\mathfrak{e}\delta^j_i\tilde{F}_{kl}\tilde{F}^{kl} = \mathfrak{e}t_i^{\ j}$, showing that the relation (2.3.14) is valid.

2.7.7 Lorentz force

Let us consider a charge particle interacting with the electromagnetic field. The total energy-momentum tensor for the particle and electromagnetic field is covariantly conserved, which gives the motion of the particle. The electromagnetic part yields, by means of (2.7.32) and (2.7.83),

$$T_i^{\ k}{}_{:k} = \frac{1}{4\pi}\biggl(\frac{1}{2}F_{lm:i}F^{lm} - F_{il:k}F^{kl} - F_{il}F^{kl}{}_{:k}\biggr) = \frac{1}{4\pi}\biggl(-\frac{1}{2}F_{mi:l}F^{lm} - \frac{1}{2}F_{il:m}F^{lm}$$
$$-F_{il:k}F^{kl} - F_{il}F^{kl}{}_{:k}\biggr) = \frac{1}{4\pi}F_{il}F^{kl}{}_{:k} = -\frac{1}{c}F_{il}j^l. \tag{2.7.123}$$

The particle part gives, using (2.4.153),

$$T_i{}^k{}_{:k} = \left(\mu c^2 \frac{u_i u^k}{\sqrt{g_{00}} u^0}\right)_{:k}, \tag{2.7.124}$$

so we obtain

$$\left(\mu c^2 \frac{u_i u^k}{\sqrt{g_{00}} u^0}\right)_{:k} - \frac{1}{c} F_{il} j^l = 0. \tag{2.7.125}$$

Multiplying (2.7.125) by u^i and using (2.7.65) gives

$$\left(\mu c^2 \frac{u^k}{\sqrt{g_{00}} u^0}\right)_{:k}, \tag{2.7.126}$$

which turns (2.7.125) into

$$\mu c^2 \frac{u^k}{\sqrt{g_{00}} u^0} u_{i:k} = \frac{1}{c} F_{il} \rho \frac{u^l}{\sqrt{g_{00}} u^0} \tag{2.7.127}$$

or

$$mc \frac{D^{\{\}} u^i}{ds} = \frac{e}{c} F^{ij} u_j, \tag{2.7.128}$$

which is the equation of motion of a particle of mass m and charge e in the electromagnetic field F_{ij}. Multiplying (2.7.128) by u_i gives the identity, thereby (2.7.128) has 3 independent components. The right-hand side of (2.7.128) is referred to as the *Lorentz force*.

In the locally geodesic and Galilean frame of reference, we have $\frac{D^{\{\}}}{ds} = \frac{d}{ds} = \frac{u^0}{c}\frac{d}{dt}$ and the four-velocity is given by (1.6.120). The 3 independent spatial components of (2.7.128) are, using (2.4.135),

$$\frac{dP^\alpha}{dt} = mc \frac{du^\alpha}{dt} = eF^{\alpha 0} + \frac{e}{c} F^{\alpha\beta} v_\beta. \tag{2.7.129}$$

In the spatial-vector notation, this equation of motion is given by

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}_{\mathrm{L}} = e\mathbf{E} + \frac{e}{c} \mathbf{v} \times \mathbf{B}, \tag{2.7.130}$$

where $\mathbf{P}$ is the momentum of the particle (2.4.144) and $\mathbf{F}_{\mathrm{L}}$ is the spatial vector of the Lorentz force. The temporal component of (2.7.128) is

$$\frac{dP^0}{dt} = mc \frac{du_0}{dt} = \frac{e}{c} F_{0\alpha} v^\alpha. \tag{2.7.131}$$

In the spatial-vector notation, this equation of motion is given by

$$\frac{dE}{dt} = e\mathbf{v} \cdot \mathbf{E}, \tag{2.7.132}$$

where E is the energy of the particle (2.4.143), which also results from multiplying (2.7.130) by $\mathbf{v}$ and using (2.4.148). Integrating (2.7.114) over the volume gives

$$\frac{\partial}{\partial t} \int W dV + \int \mathbf{j} \cdot \mathbf{E} dV + \oint \mathbf{S} \cdot d\mathbf{f} = 0, \tag{2.7.133}$$

which, with (2.7.76) and (2.7.132), yields the conservation of the total energy (2.4.75) of the electromagnetic field and particles:

$$\frac{\partial}{\partial t}\left(\int W dV + \sum_a E_a\right) + \oint \mathbf{S} \cdot d\mathbf{f} = 0. \tag{2.7.134}$$

References: [2, 3].

2.8 Action for particles

The energy-momentum tensor for a point particle of mass m located at the radius vector $\mathbf{r}_0$ is given by (2.4.154):

$$T^{ik}(\mathbf{r}) = mc^2 \boldsymbol{\delta}(\mathbf{r}-\mathbf{r}_0)\frac{u^i u^k}{\sqrt{-\mathfrak{g}}u^0}, \tag{2.8.1}$$

so the variation of the action with respect to the metric tensor (2.3.2) gives

$$\begin{aligned}\delta S &= -\frac{1}{2c}\int T^{ik}\delta g_{ik}\sqrt{-\mathfrak{g}}d\Omega = -\frac{mc}{2}\int \frac{u^i u^k}{u^0}\delta g_{ik}dx^0 = -\frac{mc}{2}\int u^i u^k \delta g_{ik}ds \\ &= -\frac{mc}{2}\int \frac{\delta g_{ik}dx^i dx^k}{ds} = -mc\int \delta\sqrt{g_{ik}dx^i dx^k} = -mc\delta\int ds. \end{aligned} \tag{2.8.2}$$

Therefore, the action for a *free* (interacting only with the gravitational field) particle is

$$S = -mc\int_1^2 ds, \tag{2.8.3}$$

where 1 and 2 denote the world points corresponding to the arrival of the particle at the initial and final position. The equation of motion of a free particle is thus the metric geodesic equation (1.4.91). The action for a system of noninteracting particles is the sum of the actions corresponding to each particle:

$$S = -\sum_a m_a c\int ds_a. \tag{2.8.4}$$

A particle interacts with other particles through fields that carry the interaction. Consequently, we must add to (2.8.4) the action describing fields that carry the interaction. For the electromagnetic interaction, such a field is described by the electromagnetic potential and couples to the electromagnetic current. The electromagnetic current vector for a point particle of charge e located at the radius vector $\mathbf{r}_0$ is given by (2.7.70):

$$j^k(\mathbf{r}) = \frac{cu^k}{u^0}\frac{e}{\sqrt{-\mathfrak{g}}}\boldsymbol{\delta}(\mathbf{r}-\mathbf{r}_0). \tag{2.8.5}$$

Substituting (2.8.5) into the second term of (2.7.80) gives the action for this coupling:

$$S_{\mathrm{e}} = -\frac{1}{c^2}\int \sqrt{-\mathfrak{g}}A_k j^k dV\,dx^0 = -\frac{e}{c}\int A_k\frac{u^k}{u^0}dx^0 = -\frac{e}{c}\int A_k dx^k, \tag{2.8.6}$$

so the total action for a particle of mass m and charge e interacting with the electromagnetic potential A_i is

$$S = -mc\int ds - \frac{e}{c}\int A_i dx^i. \tag{2.8.7}$$

For a system of particles, the total action is the sum of the actions (2.8.7) for each particle. Under the gauge transformation (2.7.7), the action (2.8.7) changes by the integral of a total differential:

$$S' = -mc\int ds - \frac{e}{c}\int A_i dx^i - \frac{e}{c}\int d\alpha, \tag{2.8.8}$$

so the conditions $\delta S = 0$ and $\delta S' = 0$ are equivalent, and the corresponding equations of motion are gauge invariant.

The variation of (2.8.3) with respect to the coordinates x^i gives, using (1.4.89), (1.4.90) and (1.4.91),

$$\delta S = mc\Big(\int \frac{D^{\{\}}u_i}{ds}\delta x^i ds - \int d(u_i\delta x^i)\Big). \tag{2.8.9}$$

The variation of $\int A_i dx^i$ is

$$\delta\int A_i dx^i = \int \delta A_i dx^i + \int A_i \delta dx^i = \int A_{i,j}\delta x^j dx^i + \int A_i d\delta x^i$$
$$= \int A_{i,j}\delta x^j dx^i + \int d(A_i\delta x^i) - \int dA_i\delta x^i = \int A_{i,j}\delta x^j dx^i + \int d(A_i\delta x^i)$$
$$- \int A_{i,j} dx^j \delta x^i = \int F_{ij} dx^j \delta x^i + \int d(A_i\delta x^i)$$
$$= \int F_{ij} u^j \delta x^i ds + \int d(A_i\delta x^i). \tag{2.8.10}$$

Therefore, the variation of (2.8.7) is, using the four-momentum $p^l = mcu^l$ (2.4.135):

$$\delta S = mc\int \frac{D^{\{\}}u_i}{ds}\delta x^i ds - \frac{e}{c}\int F_{ij}u^j\delta x^i ds - mc\int d(u_i\delta x^i) - \frac{e}{c}\int d(A_i\delta x^i)$$
$$= \int\Big(\frac{D^{\{\}}p_i}{ds} - \frac{e}{c}\int F_{ij}u^j\Big)\delta x^i ds - \int d\Big(p_i\delta x^i + \frac{e}{c}A_i\delta x^i\Big)$$
$$= \int_1^2\Big(\frac{D^{\{\}}p_i}{ds} - \frac{e}{c}\int F_{ij}u^j\Big)\delta x^i ds - \Big(p_i + \frac{e}{c}A_i\Big)\delta x^i\Big|_1^2, \tag{2.8.11}$$

where the limits 1 and 2 denote the endpoints of the particle's world line. The principle of least action $\delta S = 0$ for arbitrary δx^i vanishing at the endpoints gives the Lorentz equation of motion of a particle of mass m and charge e in the electromagnetic field F_{ij} (2.7.128).

The action for a particle (2.8.4) and its generalization (2.8.7) determine the Lagrangian for the particle. This Lagrangian satisfies the Lagrange equations that are analogous to (2.1.8).
References: [2, 3].

A particle is a special case of a field existing in spacetime. The physics of particles and their systems, such as rigid bodies and ideal fluids, is referred to as *mechanics* and will constitute future Chapters 3 (Particles) and 4 (Special Cases). This material is logically presented in *The Course of Theoretical Physics* by L. D. Landau and E. M. Lifshitz [2, 13, 14].

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