Physica D 71 (1994) 285–318
North-Holland

SSDI: 0167-2789(93)E0273-E

PHYSICA D

Weak and strong solutions of the complex Ginzburg–Landau equation

Charles R. Doering[a,1], John D. Gibbon[b,2] and C. David Levermore[c,3]

[a]*Department of Physics and Institute for Nonlinear Studies, Clarkson University, Potsdam, NY 13699-5820, USA*
[b]*Department of Mathematics, Imperial College, London SW7 2BZ, UK*
[c]*Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA*

Received 23 December 1992
Revised manuscript received 30 July 1993
Accepted 5 August 1993
Communicated by H. Flaschka

The generalized complex Ginzburg–Landau equation, $\partial_t A = RA + (1 + i\nu)\,\Delta A - (1 + i\mu)|A|^{2\sigma}A$, has been proposed and studied as a model for "turbulent" dynamics in nonlinear partial differential equations. It is a particularly interesting model in this respect because it is a dissipative version of the Hamiltonian nonlinear Schrödinger equation possessing solutions that form localized singularities in finite time. In this paper we investigate existence and regularity of solutions to this equation subject to periodic boundary conditions in various spatial dimensions. Appropriately defined weak solutions are established globally in time, and unique strong solutions are found locally. A new collection of a priori estimates are presented, and we discuss the relationship of our results for the complex Ginzburg–Landau equation to analogous issues for fluid turbulence described by the incompressible Navier–Stokes equations.

1. Introduction

The problem of turbulence in continuum dynamical systems and its description by nonlinear partial differential equations (PDEs) is one of the central open questions in theoretical physics and applied mathematics. Much work has focused on the study of turbulence in simple fluids, a phenomenon presumably described by the incompressible Navier–Stokes equations [3]. One of the major obstacles in this area is the fact that, as of this writing, the existence of smooth, or *classical*, or *strong*, solutions to the initial value problem in three spatial dimensions has never been demonstrated for arbitrarily large (albeit smooth) initial conditions, or arbitrarily strong (albeit smooth) applied forces [8]. Leray established the existence, although not the uniqueness, of *weak* solutions to the Navier–Stokes equations over half a century ago [15], and interpreted the implied loss of regularity in the general case as an indication of the onset of turbulent behavior. That is, it can be inferred that viscous dissipation in three spatial dimensions is unable to inhibit singularities driven by instabilities in the underlying inviscid system, the Euler equations. There is, however, an obvious logical gap associated with connecting our inability to prove regularity and uniqueness with an actual failure of those properties. Moreover, the necessity of a breakdown of regularity in the Euler equations remains an open area of investigation. Uniqueness of these weak solutions is not established, and this fact carries its own implications for the

[1] E-mail: doering@craft.camp.clarkson.edu
[2] E-mail: j.d.gibbon@ic.ac.uk
[3] E-mail: lvrmr@math.arizona.edu

question of the self consistency of any hydrodynamic description at all! These issues remain today in an unsatisfactory state of nonresolution.

In a recent study [1] it was proposed that some similar mathematical issues could be investigated in the generalized complex Ginzburg–Landau (CGL) equation,

$$\partial_t A = RA + (1 + \mathrm{i}\nu)\Delta A - (1 + \mathrm{i}\mu)|A|^{2\sigma} A . \tag{1.1}$$

This equation, most often considered with a cubic nonlinearity ($\sigma = 1$), has a long history in physics as a generic amplitude equation near the onset of instabilities in fluid mechanical systems, as well as in the theory of phase transitions and superconductivity. For our purpose we remove it from any particular physical context and investigate it as a mathematical model of a variety of possible phenomena in nonlinear PDEs. Considered over the d-dimensional torus $\mathbb{T}^d$, the CGL equation is both theoretically and numerically tractable, and has proved fruitful for illustration of connections between infinite dimensional dynamics and finite dimensional dynamical systems [1,9,21]. Moreover, the analysis in [1] allows for the quantitative investigation of some aspects of turbulent behavior that is not so accessible in, for example, the Navier–Stokes equations. We shall therefore focus our study on the spatially periodic CGL equation (1.1), indicating those results that extend to the whole space $\mathbb{R}^d$.

The key attributes of the CGL equation that allow us to seriously consider it as a worthwhile focus for general studies of turbulent behavior are twofold. First, in the absence of driving ($R = 0$) it has an inviscid limit that is a Hamiltonian dynamical system, namely the nonlinear Schrödinger (NLS) equation

$$\mathrm{i}\partial_t A = -\nu\, \Delta A + \mu |A|^{2\sigma} A . \tag{1.2}$$

Second, this inviscid limit possesses instabilities capable of provoking spatially localized singularities in the solutions, within finite-time, starting from arbitrarily tame initial data. This effect can be thought of as a rather violent cascade process whereby energy concentrated in low wavenumber modes is rapidly transported to high wavenumber modes. These two properties are in direct analogy to the problem of incompressible fluid turbulence, at least so far as the conventional wisdom is concerned. The Euler equations constitute the underlying Hamiltonian dynamics of the dissipative Navier–Stokes equations, and instabilities in the inviscid equations can create a cascade of energy between widely separated length scales. Admittedly the mechanisms for energy transfer from long length scales to short length scales is very different in the NLS equation (self-focusing) and the 3-dimensional Euler equations (vortex stretching), and we make no claims as to a direct physical analogy. Self-focusing blow-up in the NLS equation plays a more direct role in describing strong turbulence in plasmas (the NLS equation is a limit of the Zakharov equations [13,25]), but dissipative mechanisms in plasmas do not directly correspond to the dissipative terms in the CGL equation. Rather, we regard the CGL equation as a convenient nonlinear dissipative PDE displaying intrinsic inviscid instabilities that we may exploit to test and hone our mathematical tools.

There are several distinct advantages to considering the CGL equation rather than, say, the Navier–Stokes equations or the Zakharov equations. First, there is the obvious point that there are simply fewer dependent variables in the CGL equation, so we may in general expect the analysis to be simpler. Indeed, the basic techniques that have been developed for the study of semilinear parabolic equations [24], and the NLS equation over $\mathbb{R}^d$ [7,10,23] and, just recently, over $\mathbb{T}^d$ [5,6], can be applied to the CGL equation. Second, the mechanism of instability and its resulting finite time blow-up for solutions of the NLS equation has been established over $\mathbb{R}^d$ [11,25,26] and over bounded domains for various kinds of boundary conditions (including periodic) [14], and is quite well understood both physically and mathematically (see [17–19] and references therein). Third, from a numerical viewpoint it is much faster and more convenient to run simulations in low spatial dimensions, and the self-focusing blow-up behavior is already present in the 1-dimensional NLS equation for $\sigma \geq 2$. Several numerical studies have recently explored various aspects of the CGL dynamics related to our investigation [12,16,22].

Necessary conditions for the possible formation of finite-time singularities in the NLS equations (1.2) are $\nu\mu < 0$, and $\sigma d \geq 2$. This is simply understood by considering the NLS equation to be a "mean field" quantum mechanical description of locally interacting particles. Taking (WLOG) $\nu > 0$, the potential in (1.2) is $V = \mu |A|^{2\sigma}$. When $\mu > 0$ the interaction is repulsive, and consequently there is no tendency for the particles to concentrate. On the other hand, when $\mu < 0$ the interaction is attractive, and when $\sigma d \geq 2$ this attraction is strong enough that it can induce a collapse of the matter. The term "critical" indicates that the condition for blow-up is satisfied as an equality, $\sigma d = 2$. The terms "supercritical" and "subcritical" indicate $\sigma d > 2$ and $\sigma d < 2$, respectively.

The analytical study in [1] concentrated on the CGL equation in the critical case of the cubic nonlinearity ($\sigma = 1$) in two spatial dimensions ($d = 2$). The notions of "strong" and "weak" turbulence were used in [1] to describe differences in the dynamics under the influence of, respectively, relatively weak dissipation and relatively strong dissipation. A central technical observation in that paper was that Laplacian dissipation in the CGL equation is sufficient to inhibit the blow-up in the critical NLS equation, allowing for global C^∞ solutions. The difference between strong and weak turbulent behavior was then indicated by quantitative differences in estimates of various norms of the solutions and their derivatives, rather than any breakdown of regularity of the solutions. As discussed in [1], global C^∞ solutions also exist for some restricted values of the parameters in the supercritical case of the cubic CGL equation in 3 dimensions, in particular for relatively strong dissipation. Outside that very restricted regime, however, existence, uniqueness and regularity remained an open problem in the supercritical case.

The purpose of this paper, then, is to explore the question of existence and regularity of solutions to various forms of the generalized CGL equations, motivated by the observation that it is a dissipative version of a PDE with self-induced finite-time singularities.

The remainder of this paper is organized as follows. In section 2 the existence of global (in time) weak solutions to the CGL equation is established. The proof is not restricted either by the spatial dimensionality or the degree of the nonlinearity and is similar in spirit to that for Leray's existence theorem for global weak solutions of the Navier–Stokes equations. Also similarly, we cannot demonstrate uniqueness for the weak solutions. In section 3 we establish the existence and uniqueness of local (in time) strong solutions – C^∞ solutions whenever σ is a positive integer. Moreover, we estimate the time interval over which a strong solution exists in terms of L^p norms. Our inability to unconditionally extend the local strong solutions to global strong solutions is a manifestation of the instability in the inviscid dynamics. We cannot generally exclude the possibility of a finite-time blow-up, even in dissipative systems. Section 4 begins the process of establishing global solutions through a priori bounds and does not restrict σ to integer values. Section 5 treats the case where σ is a positive integer and presents a family of differential inequalities for various dimensionally consistent functionals of the solutions, which we refer to as a "functional lattice", and which can be used to translate, or "bootstrap", between estimates of these functionals to develop additional a priori bounds. The concluding section 6 is a summary and discussion of the results.

2. Global weak solutions

The technical setup and notation for this section is as follows. Throughout this section it is more convenient to use $\gamma = \sigma + 1$. We consider the evolution of a complex valued field $A = A(t, x)$ governed by the generalized CGL equation,

$$\partial_t A = RA + (1 + \mathrm{i}\nu)\,\Delta A - (1 + \mathrm{i}\mu)|A|^{2(\gamma-1)}A\,, \tag{2.1a}$$

with initial condition

$$A(0, x) = A^{\mathrm{in}}(x)\,. \tag{2.1b}$$

The spatial domain is taken to be $\mathbb{T}^d$, the d-dimensional torus. The parameter R in (2.1) is assumed to be positive, γ is taken greater than one, while ν and μ can take any real value. This arbitrariness contrasts sharply with the theory of strong solutions where the admissible parameter values of γ, ν, and μ and the dimension d are interrelated.

Without loss of generality the units of length may be chosen so that

$$\int_{\mathbb{T}^d} \mathrm{d}x = 1\,. \tag{2.2}$$

We employ the notation

$$\langle f \rangle = \int_{\mathbb{T}^d} \mathrm{Re}(f)\,\mathrm{d}x\,, \tag{2.3}$$

which by (2.2) can be viewed as denoting a mean value over the torus.

In order to state the main result of this section we must first introduce the principal spaces involved. The classical Lebesgue L^p-space ($p \geq 1$) over $(\mathbb{T}^d, \mathrm{d}x)$ will be denoted $\mathrm{L}^p(\mathbb{T}^d)$, the Sobolev space of functions in $\mathrm{L}^2(\mathbb{T}^d)$ with partial derivatives in $\mathrm{L}^2(\mathbb{T}^d)$ will be denoted $\mathrm{H}^1(\mathbb{T}^d)$. Given any Banach space $\mathbb{X}$ with norm $\|\cdot\|_{\mathbb{X}}$ and $p > 1$, the space of (equivalence classes of) measurable functions $B = B(t)$ from $[0, \infty)$ into $\mathbb{X}$ such that $\|B\|_{\mathbb{X}} \in \mathrm{L}^p([0, T])$ for every $T > 0$ will be denoted $\mathrm{L}^p_{\mathrm{loc}}([0, \infty), \mathbb{X})$. And finally $\mathrm{C}([0, \infty), \text{w-}\mathrm{L}^2(\mathbb{T}^d))$ will denote the space of continuous functions from $[0, \infty)$ into w-$\mathrm{L}^2(\mathbb{T}^d)$ ($\mathrm{L}^2(\mathbb{T}^d)$ equipped with its weak topology). This means that for every $\Psi \in \mathrm{L}^2(\mathbb{T}^d)$ the function $t \mapsto \langle \Psi^* B(t) \rangle$ is in $\mathrm{C}([0, \infty))$ endowed with the usual topology of uniform convergence over compact intervals.

The main result of this section is the following global existence theorem for L^2 initial data; it is the CGL analogue of Leray's result for the Navier–Stokes equation.

Theorem 1.1. Given $A^{\mathrm{in}} \in \mathrm{L}^2(\mathbb{T}^d)$, there exists a function

$$A \in \mathrm{C}([0, \infty), \text{w-}\mathrm{L}^2(\mathbb{T}^d)) \cap \mathrm{L}^2_{\mathrm{loc}}([0, \infty), \mathrm{H}^1(\mathbb{T}^d)) \cap \mathrm{L}^{2\gamma}_{\mathrm{loc}}([0, \infty), \mathrm{L}^{2\gamma}(\mathbb{T}^d))\,, \tag{2.4}$$

that satisfies the initial condition (2.1b) and the weak form of the CGL equation (2.1a),

$$0 = \langle \Psi^* A(t_2) \rangle - \langle \Psi^* A(t_1) \rangle - R \int_{t_1}^{t_2} \langle \Psi^* A \rangle \,\mathrm{d}t' + \int_{t_1}^{t_2} \langle (1 + \mathrm{i}\nu) \nabla \Psi^* \cdot \nabla A \rangle \,\mathrm{d}t'$$
$$+ \int_{t_1}^{t_2} \langle (1 + \mathrm{i}\mu) \Psi^* |A|^{2(\gamma-1)} A \rangle \,\mathrm{d}t' , \tag{2.5}$$

for every $[t_1, t_2] \subset [0, \infty)$ and every test function $\Psi \in \mathrm{C}^\infty(\mathbb{T}^d)$. Moreover, it satisfies the energy relation

$$\tfrac{1}{2} \|A(t)\|_{\mathrm{L}^2}^2 + \int_0^t \|\nabla A\|_{\mathrm{L}^2}^2 \,\mathrm{d}t' + \int_0^t \|A\|_{\mathrm{L}^{2\gamma}}^{2\gamma} \,\mathrm{d}t' \le \tfrac{1}{2} \|A^{\mathrm{in}}\|_{\mathrm{L}^2}^2 + R \int_0^t \|A\|_{\mathrm{L}^2}^2 \,\mathrm{d}t' , \tag{2.6}$$

for every $t \in [0, \infty)$.

Remark. The idea of the proof is similar in spirit to that of Leray for the Navier–Stokes equations. Since we do not limit ourselves to any particular spatial dimension, or to any particular nonlinearity, the results of this section are equally applicable to the supercritical, critical, and subcritical CGL equations.

Proof. The proof proceeds in four distinct steps.

Step 1. Construct a family of approximate solutions A_ε constructed by any method that yields a consistent weak formulation and an energy relation – for example, the Fourier–Galerkin method. Let P_ε denote the L^2-orthogonal projection onto the span of all Fourier modes of wave vectors k with $|k| \le 1/\varepsilon$. Define $A_\varepsilon^{\mathrm{in}} = P_\varepsilon A^{\mathrm{in}}$ and let $A_\varepsilon = A_\varepsilon(t)$ be the unique solution of the ordinary differential initial-value problem,

$$\partial_t A_\varepsilon = R A_\varepsilon + (1 + \mathrm{i}\nu) \Delta A_\varepsilon - (1 + i\mu) P_\varepsilon(|A_\varepsilon|^{2(\gamma-1)} A_\varepsilon) , \tag{2.7a}$$

$$A_\varepsilon(0) = A_\varepsilon^{\mathrm{in}} \in P_\varepsilon \mathrm{L}^2(\mathbb{T}^d) \subset \mathrm{C}^\infty(\mathbb{T}^d) . \tag{2.7b}$$

The regularized initial data $A_\varepsilon^{\mathrm{in}}(x)$ converges to A^{in} strongly in $\mathrm{L}^2(\mathbb{T}^d)$ as ε tends to zero and is chosen so that $\|A_\varepsilon^{\mathrm{in}}\|_{\mathrm{L}^2} \le \|A^{\mathrm{in}}\|_{\mathrm{L}^2}$.

More specifically, these solutions will satisfy the regularized version of the weak form (2.5), given by

$$0 = \langle \Psi^* A_\varepsilon(t_2) \rangle - \langle \Psi^* A_\varepsilon(t_1) \rangle - R \int_{t_1}^{t_2} \langle \Psi^* A_\varepsilon \rangle \,\mathrm{d}t' + \int_{t_1}^{t_2} \langle (1 + \mathrm{i}\nu) \nabla \Psi^* \cdot \nabla A_\varepsilon \rangle \,\mathrm{d}t'$$
$$+ \int_{t_1}^{t_2} \langle (1 + \mathrm{i}\mu) \Psi_\varepsilon^* |A_\varepsilon|^{2(\gamma-1)} A_\varepsilon \rangle \,\mathrm{d}t' , \tag{2.8}$$

for every $[t_1, t_2] \subset [0, \infty)$ and $\Psi \in \mathrm{C}^\infty(\mathbb{T}^d)$. Here $\Psi_\varepsilon \equiv P_\varepsilon \Psi$ will converge to Ψ in C^∞ as ε tends to zero. Moreover, these solutions will satisfy the regularized version of the energy relation (2.6) as the equality

$$\tfrac{1}{2}\|A_\varepsilon(t)\|^2_{L^2} + \int_0^t \|\nabla A_\varepsilon\|^2_{L^2}\,dt' + \int_0^t \|A_\varepsilon\|^{2\gamma}_{L^{2\gamma}}\,dt' = \tfrac{1}{2}\|A^{\text{in}}_\varepsilon\|^2_{L^2} + R\int_0^t \|A_\varepsilon\|^2_{L^2}\,dt' , \tag{2.9}$$

for every $t \in [0, \infty)$.

Step 1 follows from the standard Picard local existence theory for ordinary differential equations applied to (2.7) as posed in the finite dimensional space $P_\varepsilon L^2(\mathbb{T}^d)$, and the fact that (2.9) provides a global L^2 bound on the solutions, ensuring that they are global. Hence, we omit all details of the proof and proceed under the premise of of step 1.

Step 2. Show that the sequence A_ε is a relatively compact set (has compact closure) in

$$C([0,\infty), \text{w-}L^2(\mathbb{T}^d)) \wedge \text{w-}L^2_{\text{loc}}([0,\infty), H^1(\mathbb{T}^d)) \wedge \text{w-}L^{2\gamma}_{\text{loc}}([0,\infty), L^{2\gamma}(\mathbb{T}^d)) . \tag{2.10}$$

Remark. Here the notation " $\wedge$ " indicates the intersection equipped with the weak topology induced by the inclusion maps; this means that a sequence in the intersection is convergent if and only if it converges in each space separately. The sense of convergence for these spaces individually is as follows. We have $B_n \to B$ in $\text{w-}L^2_{\text{loc}}([0,\infty), H^1(\mathbb{T}^d))$ when for every $T > 0$ and $\Psi \in L^2_{\text{loc}}([0,\infty),\ H^1(\mathbb{T}^d))$ we have

$$\int_0^T \langle \Psi^* B_n + \nabla\Psi^* \cdot \nabla B_n \rangle\,dt \to \int_0^T \langle \Psi^* B + \nabla\Psi^* \cdot \nabla B \rangle\,dt . \tag{2.11}$$

Similarly, we have $B_n \to B$ in $\text{w-}L^{2\gamma}_{\text{loc}}([0,\infty),\ L^{2\gamma}(\mathbb{T}^d))$ when for every $T > 0$ and $\Psi \in L^{(2\gamma)^*}_{\text{loc}}([0,\infty),$ $L^{(2\gamma)^*}(\mathbb{T}^d))$ we have

$$\int_0^T \langle \Psi^* B_n \rangle\,dt \to \int_0^T \langle \Psi^* B \rangle\,dt , \tag{2.12}$$

where $(2\gamma)^* = 2\gamma/(2\gamma - 1)$. Finally, we have $B_n \to B$ in $C([0,\infty), \text{w-}L^2(\mathbb{T}^d))$ when for every $\Psi \in L^2(\mathbb{T}^d)$ we have

$$\langle \Psi^* B_n(t) \rangle \to \langle \Psi^* B(t) \rangle , \tag{2.13}$$

uniformly on compact subsets of $[0, \infty)$.

Proof of step 2. Eq. (2.9), along with $\|A^{\text{in}}_\varepsilon\|_{L^2} \le \|A^{\text{in}}\|_{L^2}$, implies that

$$\tfrac{1}{2}\|A_\varepsilon(t)\|^2_{L^2} \le \tfrac{1}{2}\|A^{\text{in}}\|^2_{L^2} + R\int_0^t \|A_\varepsilon\|^2_{L^2}\,dt' , \tag{2.14}$$

from which the Gronwall lemma then gives

$$\|A_\varepsilon(t)\|^2_{L^2} \le \|A^{\text{in}}\|^2_{L^2}\, e^{2Rt} . \tag{2.15}$$

Inserting this into the right side of the regularized energy relation (2.9) yields the explicit bound, uniform in ε,

$$\tfrac{1}{2}\|A_\varepsilon(t)\|_{L^2}^2+\int_0^t \|\nabla A_\varepsilon\|_{L^2}^2\,dt'+\int_0^t \|A_\varepsilon\|_{L^{2\gamma}}^{2\gamma}\,dt'\le \tfrac{1}{2}\|A^{\rm in}\|_{L^2}^2\,e^{2Rt}\,. \tag{2.16}$$

This establishes that the sequence $\{A_\varepsilon\}$ is contained in compact sets of both w-$L^2_{\rm loc}([0,\infty), H^1(\mathbb{T}^d))$ and w-$L^{2\gamma}_{\rm loc}([0,\infty), L^{2\gamma}(\mathbb{T}^d))$, because closed, norm bounded sets are relatively compact in weak-* topologies, which are the same as the weak topologies on these reflexive spaces. For the same reason, the uniform bound (2.15) also shows that $\{A_\varepsilon(t)\}$ is a relatively compact set in w-$L^2(\mathbb{T}^d)$ for every $t\ge 0$.

In order to complete step 2 it must be shown that $\{A_\varepsilon\}$ is a relatively compact set in $C([0,\infty)$, w-$L^2(\mathbb{T}^d))$. Compactness requires more than just boundedness here because of the strong topology over t. We appeal to the Arzela–Ascoli theorem [4,20] which asserts that $\{A_\varepsilon\}$ is a relatively compact set in $C([0,\infty)$, w-$L^2(\mathbb{T}^d))$ if and only if

(i) $\{A_\varepsilon(t)\}$ is a relatively compact set in w-$L^2(\mathbb{T}^d)$ for every $t\ge 0$;

(ii) $\{A_\varepsilon\}$ is equicontinuous in $C([0,\infty)$, w-$L^2(\mathbb{T}^d))$.

As was noted at the end of the last paragaph, condition (i) is satisfied. In order to establish (ii), we must show that for every $\Psi\in L^2(\mathbb{T}^d)$ we have

(ii′) $\{\langle \Psi^* A_\varepsilon\rangle\}$ is equicontinuous in $C([0,\infty))$.

This is done by first establishing (ii′) for Ψ in C^∞ and then using a density argument to extend to the general case of Ψ in $L^2(\mathbb{T}^d)$.

Consider $\Psi\in C^\infty(\mathbb{T}^d)$ and $0<T<\infty$. Then for every $[t_1,t_2]\subset[0,\mathbb{T}]$ use the regularized weak form (2.8) to find

$$\begin{aligned}
&|\langle \Psi^* A_\varepsilon(t_2)\rangle-\langle \Psi^* A_\varepsilon(t_1)\rangle|\\
&=\left|-R\int_{t_1}^{t_2}\langle \Psi^* A_\varepsilon\rangle\,dt'+\int_{t_1}^{t_2}\langle (1+i\nu)\nabla\Psi^*\cdot\nabla A_\varepsilon\rangle\,dt'+\int_{t_1}^{t_2}\langle (1+i\mu)\Psi_\varepsilon^*|A_\varepsilon|^{2(\gamma-1)}A_\varepsilon\rangle\,dt'\right|\\
&\le R\|\Psi\|_\infty\sqrt{\int_{t_1}^{t_2}dt'}\sqrt{\int_{t_1}^{t_1}\|A_\varepsilon\|_{L^2}^2\,dt'}+|1+i\nu|\,\|\nabla\Psi\|_\infty\sqrt{\int_{t_1}^{t_2}dt'}\sqrt{\int_{t_1}^{t_2}\|A_\varepsilon\|_{L^2}^2\,dt'}\\
&\quad+|1+i\mu|\,\|\Psi_\varepsilon\|_\infty\left(\int_{t_1}^{t_2}dt'\right)^{1/2\gamma}\left(\int_{t_1}^{t_2}\|A_\varepsilon\|_{L^{2\gamma}}^{2\gamma}\,dt'\right)^{1-1/2\gamma},
\end{aligned}$$

which is simply

$$\begin{aligned}
|\langle \Psi^* A_\varepsilon(t_2)\rangle-\langle \Psi^* A_\varepsilon(t_1)\rangle|&\le\sqrt{|t_2-t_1|}\,C_{R,\nu,\Psi}\sqrt{\int_0^T\|A_\varepsilon\|_{L^2}^2\,dt'}\\
&\quad+|t_2-t_1|^{1/2\gamma}C_{\mu,\Psi}\left(\int_0^T\|A_\varepsilon\|_{L^{2\gamma}}^{2\gamma}\,dt'\right)^{1-1/2\gamma},
\end{aligned}\tag{2.17}$$

where the constants $C_{R,\nu,\Psi}$ and $C_{\mu,\Psi}$ are

$$C_{R,\nu,\Psi}=R\|\Psi\|_\infty+|1+\mathrm{i}\nu|\,\|\Delta\Psi\|_\infty\,,\qquad C_{\mu,\Psi}=|1+\mathrm{i}\mu|\max_\varepsilon\{\|\Psi_\varepsilon\|_\infty\}\,.$$

Now recalling the uniform bound (2.16) on various forms of A_ε, (2.17) becomes

$$\begin{aligned}|\langle\Psi^*A_\varepsilon(t_2)\rangle-\langle\Psi^*A_\varepsilon(t_1)\rangle|&\le\sqrt{|t_2-t_1|}C_{R,\nu,\Psi}(1/\sqrt{2})\|A^{\mathrm{in}}\|_{\mathrm{L}^2}\,\mathrm{e}^{RT}\\&\quad+|t_2-t_1|^{1/2\gamma}C_{\mu,\Psi}(\tfrac{1}{2}\|A^{\mathrm{in}}\|^2_{\mathrm{L}^2}\,\mathrm{e}^{RT})^{1-1/2\gamma}\,.\end{aligned}\tag{2.18}$$

Because the right side of (2.18) is independent of ε, it shows that the temporal continuity of $\langle\Psi^*A_\varepsilon(t)\rangle$ is uniform in ε, and hence equicontinuous for every $\Psi\in\mathrm{C}^\infty(\mathbb{T}^d)$.

To extend the class of test functions from $\mathrm{C}^\infty(\mathbb{T}^d)$ to $\mathrm{L}^2(\mathbb{T}^d)$ we use a density argument. Let $\eta>0$ be arbitrary. Choose a $\Psi_\eta\in\mathrm{C}^\infty(\mathbb{T}^d)$ with

$$\|\Psi_\eta-\Psi\|_{\mathrm{L}^2}<\frac{\eta}{3}\frac{\mathrm{e}^{-RT}}{\|A^{\mathrm{in}}\|_{\mathrm{L}^2}}\,.$$

By the triangle inequality

$$\begin{aligned}|\langle\Psi^*A_\varepsilon(t_2)\rangle-\langle\Psi^*A_\varepsilon(t_1)\rangle|&=|\langle(\Psi-\Psi_\eta)^*A_\varepsilon(t_2)\rangle-\langle(\Psi-\Psi_\eta)^*A_\varepsilon(t_1)\rangle+\langle\Psi_\eta^*[A_\varepsilon(t_2)-A_\varepsilon(t_1)]\rangle|\\&\le\|\Psi_\eta-\Psi\|_{\mathrm{L}^2}\|A_\varepsilon(t_2)\|_{\mathrm{L}^2}+\|\Psi_\eta-\Psi\|_{\mathrm{L}^2}\|A_\varepsilon(t_1)\|_{\mathrm{L}^2}+|\langle\Psi_\eta^*[A_\varepsilon(t_2)-A_\varepsilon(t_1)]\rangle|\,.\end{aligned}\tag{2.19}$$

Recalling the bound (2.15) on $\|A_\varepsilon(t)\|_{\mathrm{L}^2}$ uniform in both ε and $t<T$, we have

$$|\langle\Psi^*A_\varepsilon(t_2)\rangle-\langle\Psi^*A_\varepsilon(t_1)\rangle|\le\tfrac{1}{3}\eta+\tfrac{1}{3}\eta+|\langle\Psi_\eta^*[A_\varepsilon(t_2)-A_\varepsilon(t_1)]\rangle|\,.\tag{2.20}$$

Applying (2.18) to the C^∞ function Ψ_η, we may choose $|t_2-t_1|$ small enough so that the last term above is less than $\frac{1}{3}\eta$, and hence so that the left side is less than η. This establishes (ii′) and completes the proof of step 2. □

Step 3. Show that the sequence $\{A_\varepsilon\}$ is a relatively compact set in both $\mathrm{L}^2_{\mathrm{loc}}([0,\infty),\mathrm{L}^2(\mathbb{T}^d))$ and $\mathrm{L}^{2\gamma-1}_{\mathrm{loc}}([0,\infty),\mathrm{L}^{2\gamma-1}(\mathbb{T}^d))$ considered with their usual strong topologies.

Remark. This step is necessary because step 2 allows us to assert the existence of a weakly convergent subsequence of $\{A_\varepsilon\}$, say with a limit A, from which we may conclude only that

$$\int_0^T\|A\|^2_{\mathrm{L}^2}\,\mathrm{d}t=\liminf_{\varepsilon\to0}\int_0^T\|A_\varepsilon\|^2_{\mathrm{L}^2}\,\mathrm{d}t\,.\tag{2.21}$$

However, the argument that recovers (2.6) from (2.9) will need

$$\int_0^T\|A\|^2_{\mathrm{L}^2}\,\mathrm{d}t=\lim_{\varepsilon\to0}\int_0^T\|A_\varepsilon\|^2_{\mathrm{L}^2}\,\mathrm{d}t\,,\tag{2.22}$$

which requires strong L^2 convergence. Similarly, the argument that recovers the nonlinear term in (2.5) from that in (2.8) will require strong $\mathrm{L}^{2\gamma-1}$ convergence.

Proof of step 3. The crucial point is to use the results of step 2 along with the following imbedding lemma.

Lemma. The injection

$$C([0,\infty), \text{w-L}^2(\mathbb{T}^d)) \wedge \text{w-L}^2_{\text{loc}}([0,\infty), H^1(\mathbb{T}^d)) \hookrightarrow L^2_{\text{loc}}([0,\infty), L^2(\mathbb{T}^d)) , \tag{2.23}$$

is continuous.

Proof of lemma. It is sufficient to show that if a sequence converges to zero in both $C([0,\infty), \text{w-L}^2(\mathbb{T}^d))$ and $\text{w-L}^2_{\text{loc}}([0,\infty), H^1(\mathbb{T}^d))$ then it also does so in $L^2_{\text{loc}}([0,\infty), L^2(\mathbb{T}^d))$. So, suppose $B_\varepsilon \to 0$ as $\varepsilon \to 0$ in $C([0,\infty), \text{w-L}^2(\mathbb{T}^d))$ and $\text{w-L}^2_{\text{loc}}([0,\infty), H^1(\mathbb{T}^d))$. The key is to note that $B_\varepsilon(t) \to 0$ in $\text{w-L}^2(\mathbb{T}^d)$ implies that $B_\varepsilon(t) \to 0$ in $H^{-1}(\mathbb{T}^d)$. (This is because the Rellich theorem states that $H^1 \hookrightarrow L^2$ compactly, thus $L^2 \hookrightarrow H^{-1}$ compactly and therefore $\text{w-L}^2 \hookrightarrow H^{-1}$ continuously.) Hence $B_\varepsilon \to 0$ in $C([0,\infty), \text{w-L}^2(\mathbb{T}^d))$ implies that $B_\varepsilon \to 0$ in $C([0,\infty), H^{-1}(\mathbb{T}^d))$ and thus also in $L^2_{\text{loc}}[(0,\infty), H^{-1}(\mathbb{T}^d))$.

Now, let $\eta > 0$ be arbitrary. Then

$$\int_0^T \|B_\varepsilon\|^2_{L^2}\, dt \le \frac{\eta}{2} \int_0^T \|B_\varepsilon\|^2_{H^1}\, dt + \frac{1}{2\eta} \int_0^T \|B_\varepsilon\|^2_{H^{-1}}\, dt . \tag{2.24}$$

Take the limit $\varepsilon \to 0$ to find

$$\limsup_{\varepsilon\to 0} \int_0^T \|B_\varepsilon\|^2_{L^2}\, dt \le \frac{\eta}{2} \limsup_{\varepsilon \to 0} \int_0^T \|B_\varepsilon\|^2_{H^1}\, dt . \tag{2.25}$$

Because B_ε converges in $\text{w-L}^2_{\text{loc}}([0,\infty), H^1(\mathbb{T}^d))$, it is therefore norm bounded in $L^2_{\text{loc}}([0,\infty), H^1(\mathbb{T}^d))$, and

$$\limsup_{\varepsilon\to 0} \int_0^T \|B_\varepsilon\|^2_{H^1}\, dt = C < \infty . \tag{2.26}$$

Employing this in (2.25) shows

$$\limsup_{\varepsilon\to 0} \int_0^T \|B_\varepsilon\|^2_{L^2}\, dt \le \tfrac{1}{2}\eta C , \tag{2.27}$$

for arbitrary $\eta > 0$, for which it follows that

$$\lim_{\varepsilon\to 0} \int_0^T \|B_\varepsilon\|^2_{L^2}\, dt = 0 . \tag{2.28}$$

This establishes the lemma.

Step 2 states that $\{A_\varepsilon\}$ is a relatively compact set in both $C([0,\infty), \text{w-L}^2(\mathbb{T}^d))$ and $\text{w-L}^2_{\text{loc}}([0,\infty), H^1(\mathbb{T}^d))$, and because the continuous image of a compact set is compact, it follows that $\{A_\varepsilon\}$ is a relatively compact set in $L^2_{loc}([0,\infty), L^2(\mathbb{T}^d))$. Hence, any subsequence of $\{A_\varepsilon\}$ that converges in both $C([0,\infty), \text{w-}L^2(\mathbb{T}^d))$ and $\text{w-}L^2_{loc}([0,\infty), H^1(\mathbb{T}^d))$ will be strongly convergent in $L^2_{loc}([0,\infty), L^2(\mathbb{T}^d))$.

In order to prove the $L^{2\gamma}$ portion of the assertion of step 3 we consider two cases. First, if $\gamma \le \frac{3}{2}$ (so that $2\gamma - 1 \le 2$) then the injection

$$L^2_{loc}([0,\infty), L^2(\mathbb{T}^d)) \hookrightarrow L^{2\gamma-1}_{loc}([0,\infty), L^{2\gamma-1}(\mathbb{T}^d)) \tag{2.29}$$

is continuous, so that strong convergence in $L^2_{loc}([0,\infty), L^2(\mathbb{T}^d))$ also gives strong convergence in $L^{2\gamma-1}_{loc}([0,\infty), L^{2\gamma-1}(\mathbb{T}^d))$, as was asserted. Second, if $\gamma > \frac{3}{2}$ (so that $2\gamma - 1 > 2$) then the strong convergence in $L^2_{loc}([0,\infty), L^2(\mathbb{T}^d))$ and the weak convergence in $L^{2\gamma}_{loc}([0,\infty), L^{2\gamma}(\mathbb{T}^d))$ combined in the following standard interpolation argument yields the result.

Suppose that $B_\varepsilon \to 0$ in both $L^2_{loc}([0,\infty), L^2(\mathbb{T}^d))$ and $\text{w-}L^{2\gamma}_{loc}([0,\infty), L^{2\gamma}(\mathbb{T}^d))$. Then in particular the sequence is norm bounded in $L^{2\gamma}_{loc}([0,\infty), L^{2\gamma}(\mathbb{T}^d))$, that is,

$$\limsup_{\varepsilon\to 0} \int_0^T \|B_\varepsilon\|_{2\gamma}^{2\gamma}\, dt = C < \infty . \tag{2.30}$$

Now, let $\eta > 0$ be arbitrary, and choose $\delta < \eta/C$. Judicious application of the Young inequality $ab \le a^p/p + b^q/q$, for nonnegative a and b and $p^{-1} + q^{-1} = 1$, gives the pointwise estimate

$$|B_\varepsilon|^{2\gamma-1} \le \delta |B_\varepsilon|^{2\gamma} + \frac{(2\gamma-3)^{(2\gamma-3)}}{(2\gamma-2)^{(2\gamma-2)}}\delta^{-(2\gamma-3)}|B_\varepsilon|^2 . \tag{2.31}$$

Integrating over space and time yields

$$\int_0^T \|B_\varepsilon\|_{2\gamma-1}^{2\gamma-1}\, dt \le \delta \int_0^T \|B_\varepsilon\|_{2\gamma}^{2\gamma}\, dt + \frac{(2\gamma-3)^{(2\gamma-3)}}{(2\gamma-2)^{(2\gamma-2)}}\delta^{-(2\gamma-3)} \int_0^T \|B_\varepsilon\|_2^2\, dt . \tag{2.32}$$

Take $\varepsilon \to 0$ to find

$$\limsup_{\varepsilon\to 0} \int_0^T \|B_\varepsilon\|_{2\gamma-1}^{2\gamma-1}\, dt \le \delta \limsup_{\varepsilon\to 0} \int_0^T \|B_\varepsilon\|_{2\gamma}^{2\gamma}\, dt = \delta C < \eta . \tag{2.33}$$

Because the limit is less than η for arbitrary $\eta > 0$, it must vanish. Hence $B_\varepsilon \to 0$ strongly in $L^{2\gamma-1}_{loc}([0,\infty), L^{2\gamma-1}(\mathbb{T}^d))$. This completes the proof of step 3. □

Step 4. Go to the limit. That is, the weak solution A in theorem 1.1 is identified as the limit of a convergent subsequence of $\{A_\varepsilon\}$. The fact that this subsequence converges in the various function spaces is used to verify the weak form (2.5) and energy relation (2.6).

Proof of step 4. Step 2 ensures that there is a subsequence of $\{A_\varepsilon\}$, which we also refer to as $\{A_\varepsilon\}$, that simultaneously converges to a limit A in $C([0,\infty), \text{w-}L^2(\mathbb{T}^d))$, $\text{w-}L^2_{loc}([0,\infty), H^1(\mathbb{T}^d))$ and $\text{w-}L^{2\gamma}_{loc}([0,\infty), L^{2\gamma}(\mathbb{T}^d))$. Thus, as was asserted in (2.4),

$$A \in C([0,\infty), \text{w-}L^2(\mathbb{T}^d)) \cap L^2_{loc}([0,\infty), H^1(\mathbb{T}^d)) \cap L^{2\gamma}_{loc}([0,\infty), L^{2\gamma}(\mathbb{T}^d)) . \tag{2.34}$$

Step 3 then ensures the strong convergence of A_ε to A in both $L^2_{loc}([0,\infty), L^2(\mathbb{T}^d))$ and $L^{2\gamma-1}_{loc}([0,\infty),$

$L^{2\gamma-1}(\mathbb{T}^d)$). All that remains is to show that the limit A satisfies the weak form of the CGL equation (2.5) as well as the energy relation (2.6). Toward this end we check convergence of each term in the respective regularized versions, (2.8) and (2.9).

For any $\Psi \in C^\infty(\mathbb{T}^d)$, the convergence of A_ε in $C([0,\infty), w\text{-}L^2(\mathbb{T}^d))$ yields

$$\langle \Psi^* A_\varepsilon(t)\rangle \to \langle \Psi^* A(t)\rangle\,, \tag{2.35}$$

for every $t \ge 0$. The convergence of A_ε in $w\text{-}L^2_{\mathrm{loc}}([0,\infty), H^1(\mathbb{T}^d))$ implies

$$\int_{t_1}^{t_2} \langle \Psi^* A_\varepsilon\rangle\, dt' \to \int_{t_1}^{t_2} \langle \Psi^* A\rangle\, dt'\,, \tag{2.36}$$

$$\int_{t_1}^{t_2} \langle (1+i\nu)\nabla\Psi^* \cdot \nabla A_\varepsilon\rangle\, dt' \to \int_{t_1}^{t_2} \langle (1+i\nu)\,\nabla\Psi^* \cdot \nabla A\rangle\, dt'\,. \tag{2.37}$$

The strong convergence of A_ε in $L^{2\gamma-1}_{\mathrm{loc}}([0,\infty), L^{2\gamma-1}(\mathbb{T}^d))$ implies the weak convergence of $|A_\varepsilon|^{2(\gamma-1)}A_\varepsilon$ in $L^1_{\mathrm{loc}}([0,\infty), L^1(\mathbb{T}^d))$. That combined with the fact that Ψ_ε converges to Ψ uniformly leads to

$$\int_{t_1}^{t_2} \langle (1+i\mu)\Psi_\varepsilon^* |A_\varepsilon|^{2(\gamma-1)}A_\varepsilon\rangle\, dt' \to \int_{t_1}^{t_2} \langle (1+i\mu)\Psi^* |A|^{2(\gamma-1)}A\rangle\, dt'\,. \tag{2.38}$$

Thus the limit A satisfies the weak form of the CGL equation (2.5).

Now, to recover the energy relation (2.6) start from its regularized version (2.9),

$$\tfrac{1}{2}\|A_\varepsilon(t)\|^2_{L^2} + \int_0^t \|\nabla A_\varepsilon\|^2_{L^2}\, dt' + \int_0^t \|A_\varepsilon\|^{2\gamma}_{L^{2\gamma}}\, dt' = \tfrac{1}{2}\|A^{\mathrm{in}}_\varepsilon\|^2_{L^2} + R\int_0^t \|A_\varepsilon\|^2_{L^2}\, dt'\,. \tag{2.39}$$

First examining the right side of (2.39), the strong convergence of the initial data in $L^2(\mathbb{T}^d)$ implies

$$\|A^{\mathrm{in}}_\varepsilon\|^2_{L^2} \to \|A^{\mathrm{in}}\|^2_{L^2}\,, \tag{2.40}$$

while the strong convergence of A_ε in $L^2_{\mathrm{loc}}([0,\infty), L^2(\mathbb{T}^d))$ implies

$$\int_0^t \|A_\varepsilon\|^2_{L^2}\, dt' \to \int_0^t \|A\|^2_{L^2}\, dt'\,. \tag{2.41}$$

Now turning to the left side of (2.39), the convergence of A_ε in $C([0,\infty), w\text{-}L^2(\mathbb{T}^d))$, together with the fact that the norm of the weak limit is an eventual lower bound to the norms of the sequence, yields

$$\|A(t)\|^2_{L^2} \le \liminf_{\varepsilon\to 0} \|A_\varepsilon(t)\|^2_{L^2}\,. \tag{2.42}$$

Similarly, the convergence in $w\text{-}L^2_{\mathrm{loc}}([0,\infty), H^1(\mathbb{T}^d))$ and $w\text{-}L^{2\gamma}_{\mathrm{loc}}([0,\infty), L^{2\gamma}(\mathbb{T}^d))$ implies

$$\int_0^t \|A\|^2_{H^1}\, dt' \le \liminf_{\varepsilon\to 0} \int_0^t \|A_\varepsilon\|^2_{H^1}\, dt'\,, \tag{2.43}$$

$$\int_0^t \|A\|_{L^{2\gamma}}^{2\gamma}\,dt' \leq \liminf_{\varepsilon\to 0} \int_0^t \|A_\varepsilon\|_{L^{2\gamma}}^{2\gamma}\,dt' . \tag{2.44}$$

Hence, the energy relation (2.6) is satisfied and theorem 1.1 is proved. □

Remark. Replacing $\mathbb{T}^d$ with $\mathbb{R}^d$ and $C^\infty(\mathbb{T}^d)$ with the compactly supported test functions $C_c^\infty(\mathbb{R}^d)$ everywhere in the statement of theorem 1.1 gives the analogous $\mathbb{R}^d$ result. The strategy for its proof is embodied in the same four steps used above for the periodic case. Indeed, step 2 and step 4 have only superficial differences from those above. The Fourier–Galerkin method used in step 1 to construct approximate classical solutions breaks down over the whole space, but can be replaced by so-called smoothing approximations. Specifically, using contraction mapping arguments such as those in [10], one can construct smooth solutions of the initial-value problem,

$$\partial_t A_\varepsilon = RA_\varepsilon + (1+i\nu)\,\Delta A_\varepsilon - (1+i\mu) j_\varepsilon * (|j_\varepsilon * A_\varepsilon|^{2(\gamma-1)} j_\varepsilon * A_\varepsilon) , \qquad A_\varepsilon(0,x) = A_\varepsilon^{\text{in}}(x) = j_\varepsilon * A^{\text{in}}(x) , \tag{2.45}$$

where the regularization is achieved through convolution with a nonnegative, even smooth mollifier j_ε. Finally, details of step 3 must be modified to reflect differences in the Rellich theorem for unbounded domains. These technical changes are straight forward, and so are omitted.

Remark. One can show the existence of global weak solutions for the defocusing ($\mu\nu > 0$) NLS equation (1.2) with any $\sigma > 0$ for finite-energy initial data using similar arguments, but using the conservation laws of mass and energy in place of the dissipation relation (2.6) to obtain the weak compactness of step 2.

3. Local strong solutions

The solutions found in the last section are too weak to manipulate, and therefore too weak to use in proofs of uniqueness or regularity. In this section we establish the local existence, uniqueness, and regularity results for classical solutions of the generalized complex Ginzburg–Landau equation (1.1). Our objective here is to provide the basic results necessary to justify the formal manipulations used to establish global solutions in the subsequent sections. More general results might be derived by adapting the important new techniques that Bourgain [5,6] has developed for the NLS equation over $\mathbb{T}^d$, however our needs here are less demanding. Indeed, our proofs use standard techniques [24], so only key ideas will be presented.

Local existence and uniqueness of strong solutions of the generalized complex Ginzburg–Landau equation (1.1) can, in some cases, be established within the following general framework. Consider the evolution of $A = A(t)$ in a Banach space $\mathbb{X}$ to be governed by the abstract initial-value problem

$$\partial_t A = LA + N(A) , \quad A(0) = A^{\text{in}} \in \mathbb{X} . \tag{3.1}$$

Here the linear operator L is assumed to be the infinitesimal generator of a strongly continuous semigroup $S(t)$ over $\mathbb{X}$ (so that the linear problem with $N = 0$ is well-posed). The perturbation N is often a nonlinear and noncontinuous map over $\mathbb{X}$.

The well-posedness of the above initial-value problem can be established by a contraction mapping argument after formally recasting (3.1) in its so-called mild formulation,

$$A(t) = S(t)\, A^{\text{in}} + \int_0^t S(t-t')\, N(A(t'))\, \mathrm{d}t' . \tag{3.2}$$

In order to employ the contraction mapping theorem we shall assume that the perturbation N is a locally Lipschitz continuous map from $\mathbb{X}$ into itself. More specifically, this means

(i) $\|N(A)\|_{\mathbb{X}} \le \mathrm{C}_{\mathrm{bd}}(\|A\|_{\mathbb{X}})$ for every $A \in \mathbb{X}$,

(ii) $\|N(A_1) - N(A_2)\|_{\mathbb{X}} \le \mathrm{C}_{\mathrm{Lip}}(\|A_1\|_{\mathbb{X}}, \|A_2\|_{\mathbb{X}})\|A_1 - A_2\|_{\mathbb{X}}$ for every $A_1, A_2 \in \mathbb{X}$,

where $\mathrm{C}_{\mathrm{bc}}(\cdot)$ and $\mathrm{C}_{\mathrm{Lip}}(\cdot,\cdot)$ are nondecreasing functions of their arguments. Given such a perturbation N, one can prove the following basic result.

Proposition (*local existence theorem*). For every $\rho > 0$ there exists a time $T(\rho) > 0$ such that for every initial data $A^{\text{in}} \in \mathbb{X}$ with $\|A^{\text{in}}\|_{\mathbb{X}} \le \rho$ there exists a unique $A \in \mathrm{C}([0, T], \mathbb{X})$ satisfying the mild formulation (3.2). Moreover, A is a locally Lipschitz continuous function of A^{in}.

Such an A is called a mild solution for the initial-value problem (3.1); it provides the starting point for an analysis demonstrating that it is in fact a classical solution.

Remark. If $A \in \mathrm{C}([0, T], \mathbb{X})$ is a mild solution of (3.1) then a direct calculation shows that it is also a weak solution of (3.1) in the sense that

$$\langle \Psi \mid A(t_2)\rangle_{\mathbb{X}} - \langle \Psi \mid A(t_1)\rangle_{\mathbb{X}} = \int_{t_1}^{t_2} \langle L^*\Psi \mid A(t)\rangle_{\mathbb{X}}\, \mathrm{d}t + \int_{t_1}^{t_2} \langle \Psi \mid N(A(t))\rangle_{\mathbb{X}}\, \mathrm{d}t , \tag{3.3}$$

for every $0 \le t_1 \le t_2 \le T$ and $\Psi \in \mathscr{D}(L^*)$. Here $\langle \cdot \mid \cdot \rangle_{\mathbb{X}}$ is the usual bilinear duality between $\mathbb{X}$ and its dual space $\mathbb{X}^*$, while L^* is the usual dual adjoint of L with domain $\mathscr{D}(L^*)$ dense in $\mathbb{X}^*$.

We now apply this general theory to the initial-value problem for the spatially periodic generalized complex Ginzburg–Landau equation (1.1). In that case we choose

$$LA = (1 + \mathrm{i}\nu)\, \Delta A + RA , \qquad N(A) = -(1 + \mathrm{i}\mu)|A|^{2\sigma} A . \tag{3.4}$$

The associated semigroup $S(t)$ acting on $A \in \mathbb{X}$ can be written as a convolution, $S(t)\, A = G_t * A$, with its Green function $G_t = G_t(x)$ given by

$$G_t(x) = \sum_{n \in \mathbb{Z}^d} g_t(x + n) , \quad g_t(x) = \frac{1}{[4\pi(1 + \mathrm{i}\nu)t]^{d/2}} \exp\!\left(-\frac{|x|^2}{4(1 + \mathrm{i}\nu)t} + Rt\right) . \tag{3.5}$$

The integral equation (3.2) recast in terms of this Green function takes the form

$$A(t) = G_t * A^{\text{in}} + \int_0^t G_{t-t'} * N(A(t'))\, \mathrm{d}t' . \tag{3.6}$$

In order to apply the local existence theorem one only need identify the space $\mathbb{X}$ and verify the conditions (i) and (ii).

The Green function (3.5) satisfies the L^1-estimate

$$\|G_t\|_{L^1} \le \sum_{n\in\mathbb{Z}^d} \int_{\mathbb{T}^d} |g_t(x+n)|\,dx = \int_{\mathbb{R}^d} |g_t(x)|\,dx = (1+\nu^2)^{d/4}\,e^{Rt}\,, \tag{3.7}$$

from which it follows that $S(t)$ is a strongly continuous semigroup over $L^p(\mathbb{T}^d)$ for every $1 \le p \le \infty$ with

$$\|S(t)\,A\|_{L^p} = \|G_t * A\|_{L^p} \le \|G_t\|_{L^1}\|A\|_{L^p} = (1+\nu^2)^{d/4}\,e^{Rt}\|A\|_{L^p}\,. \tag{3.8}$$

We first appeal to the local existence theorem with $\mathbb{X} = L^\infty(\mathbb{T}^d)$. The perturbation N given by (3.4) is clearly locally Lipschitz continuous as a map from $L^\infty(\mathbb{T}^d)$ into itself. A direct application of the local existence theorem then yields a unique mild solution $A = A(t)$ of (1.1) over a time interval $[0, T]$ that depends only on the L^∞ norm of A^{in}. This solution is the limit of a sequence $\{A^{(n)}\}$ of successive iterates of (3.6), say that defined by

$$A^{(0)}(t) = G_t * A^{in}\,, \qquad A^{(n+1)}(t) = G_t * A^{in} + \int_0^t G_{t-t'} * N(A^{(n)}(t'))\,dt'\,, \tag{3.9}$$

that converge in $C([0, T], L^\infty(\mathbb{T}^d))$ for some T chosen sufficiently small such that the sequence contracts.

Additional regularity must be demonstrated in order to elevate these mild solutions to classical strong solutions, specifically, so that they are C^2 in space and C^1 in time, hence justifying formal manipulations of this solution. This is done using the following standard bootstrapping argument. First, envoking the regularity of G_t, the gradient of the integral equation (3.6) yields

$$\nabla A(t) = \nabla G_t * A^{in} + \int_0^t \nabla G_{t-t'} * N(A(t'))\,dt'\,. \tag{3.10}$$

Then, use the L^1-estimate

$$\|\nabla G_t\|_{L^1} \le \sum_{n\in\mathbb{Z}^d} \int_{\mathbb{T}^d} |\nabla g_t(x+n)|\,dx = \int_{\mathbb{R}^d} |\nabla g_t(x)|\,dx = C_d(1+\nu^2)^{d/4}\frac{e^{Rt}}{\sqrt{t}}\,, \tag{3.11}$$

where $C_d > 0$ is a constant depending only on the dimension d, applied to the successive iterates (3.9) to show that each $\nabla A^{(n)}$ lies in $C((0, T], C(\mathbb{T}^d))$ and that the sequence converges. It follows that $A \in C((0, T], C^1(\mathbb{T}^d))$ with

$$\begin{aligned}\|\nabla A(t)\|_{L^\infty} &\le \|\nabla G_t\|_{L^1}\|A^{in}\|_{L^\infty} + \int_0^t \|\nabla G_{t-t'}\|_{L^1}\|N(A(t'))\|_{L^\infty}\,dt' \\ &\le C_d(1+\nu^2)^{d/4}\frac{e^{Rt}}{\sqrt{t}}\|A^{in}\|_{L^\infty} + C_d(1+\nu^2)^{d/4}\int_0^t \frac{e^{R(t-t')}}{\sqrt{t-t'}}\|N(A(t'))\|_{L^\infty}\,dt'\,. \end{aligned} \tag{3.12}$$

If it was the case that $A^{in} \in C^1(\mathbb{T}^d)$ then the singularity in this estimate at $t = 0$ disappears and one sees that $A \in C([0, T], C^1(\mathbb{T}^d))$ and that ∇A is a solution of the integral equation

$$\nabla A(t) = G_t * \nabla A^{\text{in}} + \int_0^t G_{t-t'} * [\mathrm{D}N(A(t'))\, \nabla A(t')]\, \mathrm{d}t' , \tag{3.13}$$

where $\mathrm{D}N(A)$ represents the derivative of $N(A)$ with respect to A, which in this case is given explicitly by

$$\mathrm{D}N(A)\nabla A = -(1+\mathrm{i}\mu)[(\sigma+1)|A|^{2\sigma}\nabla A + \sigma|A|^{2\sigma-2}A^2\nabla A^*] . \tag{3.14}$$

A repetition of the above regularity argument starting from (3.13) rather than (3.6) then shows that A is in $\mathrm{C}((0, T], \mathrm{C}^2(\mathbb{T}^d))$. Moreover, because the CGL equation relates the first time derivative to the second space derivative, the solution must also be in $C^1((0, T], \mathrm{C}(\mathbb{T}^d))$ and is therefore a classical strong solution of the CGL equation (1.1) so long as it is a mild solution. Summarizing, we have proved the following result.

Theorem 3.1. (L^∞ – *local strong solutions*). For every $\rho > 0$ there exists a time $T(\rho) > 0$ such that for every initial data $A^{\text{in}} \in \mathrm{L}^\infty(\mathbb{T}^d)$ with $\|A^{\text{in}}\|_{\mathrm{L}^\infty} \le \rho$ there exists a unique

$$A \in \mathrm{C}([0, T], \mathrm{L}^\infty(\mathbb{T}^d)) \cap \mathrm{C}((0, T], \mathrm{C}^2(\mathbb{T}^d)) \cap \mathrm{C}^1((0, T], \mathrm{C}(\mathbb{T}^d)) , \tag{3.15}$$

satisfying the CGL initial-value problem. Moreover, for every initial data $A^{\text{in}} \in \mathrm{C}^2(\mathbb{T}^d)$ one has $A \in \mathrm{C}([0, T], \mathrm{C}^2(\mathbb{T}^d)) \cap \mathrm{C}^1([0, T], \mathrm{C}(\mathbb{T}^d))$.

In this generality one cannot expect these solutions to be in $\mathrm{C}((0, T], \mathrm{C}^3(\mathbb{T}^d))$ because unbounded singularities may be introduced at zeros of A upon further differentiation of (3.14). However, additional regularity can be gained in some cases. For example, when σ is a positive integer the nonlinearity is a polynomial in A and A^* and one can freely differentiate (3.14) without introducing any singularities. Whence, continuing with the bootstrapping argument, it can be shown that for every t in $(0, T]$ the solution has at least one more spatial derivative that it had initially. But then repeating this argument implies that the solution is in $\mathrm{C}((0, T], \mathrm{C}^\infty(\mathbb{T}^d))$. Moreover, because the equation relates temporal derivatives to spatial derivatives, the solution must possess all temporal derivatives too and is therefore a smooth (C^∞) solution of the CGL equation (1.1) so long as it is a mild solution. More precisely, we have proved the following result.

Theorem 3.2 (local smooth solutions). Let $\sigma > 0$ be an integer. Then for every $\rho > 0$ there exists a time $T(\rho) > 0$ such that for every initial data $A^{\text{in}} \in \mathrm{L}^\infty(\mathbb{T}^d)$ with $\|A^{\text{in}}\|_{\mathrm{L}^\infty} \le \rho$ there exists a unique

$$A \in \mathrm{C}([0, T], \mathrm{L}^\infty(\mathbb{T}^d)) \cap \mathrm{C}^\infty((0, T] \times \mathbb{T}^d) , \tag{3.16}$$

satisfying the CGL initial-value problem. Moreover, given smooth initial data $A^{\text{in}} \in \mathrm{C}^\infty(\mathbb{T}^d)$ then $A \in \mathrm{C}^\infty([0, T] \times \mathbb{T}^d)$.

Remark. The above results justifies the formal manipulations we will carry out in section 5 and enables us to focus attention there on establishing global uniform Sobolev bounds on smooth solutions.

Even when σ is not a positive integer, we can still advance the bootstrapping argument so long as the differentiation of (3.14) does not introduce unbounded singularities at the zeros of A. The lowest degree of homogeneity for the factors A and A^* appearing in a term of the $(n+1)$st derivative of $|A|^{2\sigma}A$ will be $2\sigma - n$, and this can be controlled whenever $\sigma \geq \frac{1}{2}n$. In that case the bootstrapping argument will gain an additional n spatial derivatives, showing that the solution is in $\mathrm{C}((0,T], \mathrm{C}^{n+2}(\mathbb{T}^d))$. This observation then leads to the following.

Theorem 3.3 (Local C^k solutions). Let $\sigma \geq \frac{1}{2}n$ for some positive integer n. Then for every $\rho < 0$ there exists a time $T(\rho) > 0$ such that for every initial data $A^{\text{in}} \in \mathrm{L}^\infty(\mathbb{T}^d)$ with $\|A^{\text{in}}\|_{\mathrm{L}^\infty} \leq \rho$ there exists a unique

$$A \in \mathrm{C}([0,T], \mathrm{L}^\infty(\mathbb{T}^d)) \cap \mathrm{C}((0,T], \mathrm{C}^{n+2}(\mathbb{T}^d)) \cap \mathrm{C}^1((0,T], \mathrm{C}^n(\mathbb{T}^d)) , \tag{3.17}$$

satisfying the CGL initial-value problem. Moreover, for every initial data $A^{\text{in}} \in \mathrm{C}^{n+2}(\mathbb{T}^d)$ one has $A \in C([0,T], \mathrm{C}^{n+2}(\mathbb{T}^d)) \cap \mathrm{C}^1([0,T], \mathrm{C}^n(\mathbb{T}^d))$.

Remark. Of course we can readily continue to trade two spatial derivatives for one time derivative. If $n = 2k$ is even this leads to $A \in \mathrm{C}^{k+1}([0,T], \mathrm{C}(\mathbb{T}^d))$, while if $n = 2k+1$ is odd one finds $A \in \mathrm{C}^{k+1}([0,T], \mathrm{C}^1(\mathbb{T}^d))$.

Remark. Theorem 3.3 will be applied in section 4 with $n = 1$ in order to justify some of the formal manipulations there, enabling us to focus attention on the question of global existence.

Refinements of the basic existence argument given above greatly enlarge the class of initial data that evolve into classical strong solutions for finite times. For example, below we establish such solutions for all initial data A^{in} in $\mathrm{L}^p(\mathbb{T}^d)$ provided

$$1 \leq p \quad \text{and} \quad \sigma d < p . \tag{3.18}$$

The ideas found in [24] for semilinear parabolic equations can be applied to obtain this result, but here we present a slightly different approach.

First, a direct L^r estimate of the mild formulation (3.6) for every r that satisfies

$$p \leq r \quad \text{and} \quad 2\sigma + 1 \leq r \tag{3.19}$$

yields

$$\|A(t)\|_{\mathrm{L}^r} \leq \|G_t\|_{\mathrm{L}^q} \|A^{\text{in}}\|_{\mathrm{L}^p} + |1 + \mathrm{i}\mu| \int_0^t \|G_{t-t'}\|_{\mathrm{L}^s} A(t')\|_{\mathrm{L}^r}^{2\sigma+1} \, \mathrm{d}t' , \tag{3.20}$$

where q and s satisfy

$$1 + 1/r = 1/q + 1/p \quad \text{and} \quad 1 + 1/r = 1/s + (2\sigma+1)/r . \tag{3.21}$$

The idea is to recast (3.20) as an inequality for the ratio

$$\|A(t)\|_{\mathrm{L}^r/\mathrm{L}^q} \equiv \frac{\|A(t)\|_{\mathrm{L}^r}}{\|G_t\|_{\mathrm{L}^q}} , \tag{3.22}$$

whence obtaining

$$\|A(t)\|_{\mathrm{L}^r/\mathrm{L}^q_t} \leq \|A^{\mathrm{in}}\|_{\mathrm{L}^p} + \frac{|1+\mathrm{i}\mu|}{\|G_t\|_{\mathrm{L}^q}} \int_0^t \|G_{t-t'}\|_{\mathrm{L}^s} \|G_{t'}\|_{\mathrm{L}^q}^{2\sigma+1} \|A(t')\|_{\mathrm{L}^r/\mathrm{L}^q_t}^{2\sigma+1}\,\mathrm{d}t' . \tag{3.23}$$

Provided that the kernel above satisfies the condition

$$\frac{1}{\|G_t\|_{\mathrm{L}^q}} \int_0^t \|G_{t-t'}\|_{\mathrm{L}^s} \|G_{t'}\|_{\mathrm{L}^q}^{2\sigma+1}\,\mathrm{d}t' \to 0 \quad \text{as } t \to 0 , \tag{3.24}$$

the iterates (3.9) in the proof of the basic local existence theorem contract in the space $\mathrm{E}^{p,r}([0,T],\mathbb{T}^d)$ that is the completion of $\mathrm{C}([0,T],\mathrm{L}^r(\mathbb{T}^d))$ in the norm

$$\|A\|_{\mathrm{E}^{p,r}} \equiv \sup\{\|A(t)\|_{\mathrm{L}^r/\mathrm{L}^q_t} : t \in [0,T]\} , \tag{3.25}$$

for some T sufficiently small.

We can estimate the $\mathrm{L}^q(\mathbb{T}^d)$ norm of G_t by an interpolation argument using the $\mathrm{L}^1(\mathbb{T}^d)$ control in (3.7) and an $\mathrm{L}^\infty(\mathbb{T}^d)$ estimate. Control of the $\mathrm{L}^\infty(\mathbb{T}^d)$ norm of G_t follows from the bound

$$\begin{aligned} |G_t(x)| &\leq \sum_{n\in\mathbb{Z}^d} |g_t(x+n)| \leq \frac{(1+\nu^2)^{d/4}\,\mathrm{e}^{Rt}}{[4\pi(1+\nu^2)t]^{d/2}} \sum_{n\in\mathbb{Z}^d} \exp\left(-\frac{|x+n|^2}{4(1+\nu^2)t}\right) \\ &\leq \frac{(1+\nu^2)^{d/4}\,\mathrm{e}^{Rt}}{[4\pi(1+\nu^2)t]^{d/2}} \{a + b[(1+\nu^2)t]^{d/2}\} , \end{aligned} \tag{3.26}$$

For some positive absolute constants a and b. The $\mathrm{L}^q(\mathbb{T}^d)$ norm of G_t is then estimated by

$$\|G_t\|_{\mathrm{L}^q} \leq \|G_t\|_{\mathrm{L}^\infty}^{1/q^*} \|G_t\|_{\mathrm{L}^1}^{1/q} \leq \frac{(1+\nu^2)^{d/4}\,\mathrm{e}^{Rt}}{[4\pi(1+\nu^2)t]^{d/2q^*}} \{a + b[(1+\nu^2)t]^{d/2}\}^{1/q^*} . \tag{3.27}$$

In particular, one sees that

$$\|G_t\|_{\mathrm{L}^q} = \mathcal{O}(1/t^{d/2q^*}) \quad \text{as } t \to 0 . \tag{3.28}$$

It is easily seen from (3.28) that condition (3.24) holds whenever

$$(d/2q^*)(2\sigma+1) < 1 \quad \text{and} \quad d/2s^* + (d/q^*)\sigma < 1 . \tag{3.29}$$

Because by (3.21)

$$d/2s^* = (d/r)\sigma = (d/p)\sigma - (d/q^*)\sigma , \tag{3.30}$$

the second inequality of (3.29) is seen to be equivalent to the condition $\sigma d < p$. Again using (3.21), the first inequality of (3.29) becomes

$$1/p - 1/(\sigma + \tfrac{1}{2})d < 1/r . \tag{3.31}$$

Now assuming only $\sigma d < p$, one sees that (3.31) is satisfied by choosing $r = (2\sigma+1)p$, thus condition (3.24) is met and the contraction mapping argument then yields a unique solution A in $\mathrm{E}^{p,r}([0,T],\mathbb{T}^d)$ for $r = (2\sigma+1)p$.

This solution is in $\mathrm{C}([0,T],\mathrm{L}^p(\mathbb{T}^d)) \cap \mathrm{C}((0,T],\mathrm{L}^r(\mathbb{T}^d))$. Indeed, it is clear from the definition of the $\mathrm{E}^{p,r}$-norm (3.25) that $\mathrm{E}^{p,r}([0,T],\mathbb{T}^d) \subset \mathrm{C}((0,T],\mathrm{L}^r(\mathbb{T}^d))$. To see that A is also in $\mathrm{C}([0,T],\mathrm{L}^p(T^d))$, one only need check the continuity of A at $t=0$. First subtracting A^{in} from both sides of the mild formulation (3.6), a direct L^p estimate gives

$$\|A(t)-A^{\text{in}}\|_{\mathrm{L}^p} \le \|G_t * A^{\text{in}} - A^{\text{in}}\|_{\mathrm{L}^p} + |1+\mathrm{i}\mu| \int_0^t \|G_{t-t'}\|_{\mathrm{L}^1} \|G_{t'}\|_{\mathrm{L}^q}^{2\sigma+1} \|A(t')\|_{\mathrm{L}^r/\mathrm{L}^q}^{2\sigma+1}\,\mathrm{d}t' , \tag{3.32}$$

where $r=(2\sigma+1)p$ and q is defined in (3.21). The strong L^p-continuity of the linear semigroup implies that the first term on the right side vanishes as t tends to zero. The boundedness of A in $\mathrm{E}^{p,r}$ and the L^1 bound (3.7) of G_t imply that the second term on the right side will also vanish as t tends to zero whenever the singularity of $\|G_{t'}\|_{\mathrm{L}^q}^{2\sigma+1}$ at $t'=0$ is integrable. By (3.28), this will be the case if and only if

$$(d/2q^*)(2\sigma+1) < 1 , \tag{3.33}$$

which was already controlled in (3.29) by the assumption that $\sigma d < p$. Therefore A is continuous at $t=0$, and hence is in $\mathrm{C}([0,T], \mathrm{L}^p(\mathbb{T}^d))$.

This being the case, the following theorem holds.

Theorem 3.4 (L^p-local mild solutions). If p satisfies

$$q \le p \quad \text{and} \quad \sigma d < p , \tag{3.34}$$

then for every $\rho > 0$ there exists a time $T(\rho) > 0$ such that for every initial data $A^{\text{in}} \in \mathrm{L}^p(\mathbb{T}^d)$ with $\|A^{\text{in}}\|_{\mathrm{L}^p} \le \rho$ there exists a unique

$$A \in \mathrm{E}^{p,r}([0,T], \mathbb{T}^d) \cap \mathrm{C}([0,T], \mathrm{L}^p(\mathbb{T}^d)) , \tag{3.35}$$

satisfying the mild formulation (3.6) of the CGL initial-value problem.

Remark. Our proof differs from that found in [24] in that the space $\mathrm{E}^{p,r}([0,T], \mathbb{T}^d)$ used here for the contraction mapping argument is larger than the space used there. This minor modification can also be abstracted in the spirit of [24]. Essentially the same methods lead to similar results for the CGL equation over $\mathbb{R}^d$ [24]. More sophisticated techniques lead to similar results even in the dissipationless case of the NLS equation, both over $\mathbb{R}^d$ [7,10,23] and over $\mathbb{T}^d$ where Bourgain [5,6] has developed an $\mathrm{L}^2(\mathbb{T}^d)$ theory for $\sigma < 2$.

More is true. This mild solution in $\mathrm{E}^{p,r}([0,T], \mathbb{T}^d)$ is a strong solution in $\mathrm{C}((0,T), \mathrm{C}^2(\mathbb{T}^d)) \cap \mathrm{C}^1((0,T], \mathrm{C}(\mathbb{T}^d))$. This follows from theorem 3.1 upon showing it is in $\mathrm{C}((0,T], \mathrm{L}^\infty(\mathbb{T}^d))$. This result is obtained by iterating the argument that initial data in $\mathrm{L}^p(\mathbb{T}^d)$ leads to a solution in $\mathrm{C}((0,T], \mathrm{L}^r(\mathbb{T}^d))$ for every $r > p$ satisfying (3.29) and (3.31). Indeed, if $p > (\sigma + \frac{1}{2})d$ then every $r \in [(2\sigma+1)p, \infty]$ satisfies these criteria, and in particular, $A \in \mathrm{C}((0,T], \mathrm{L}^\infty(\mathbb{T}^d))$. If, on the other hand, $\sigma d < p \le (\sigma + \frac{1}{2})d$ then the solution is at least in $\mathrm{C}((0,T], \mathrm{L}^{p_1}(\mathbb{T}^d))$ for $p_1 = (2\sigma+1)p$. Then when $p_1 = (2\sigma+1)p \ge (\sigma+\frac{1}{2})d$, a second application of the above argument then shows that $A \in \mathrm{C}((0,T], \mathrm{L}^\infty(\mathbb{T}^d))$. More generally, a simple argument shows that whenever

$$p_n = (2\sigma+1)^n p \ge (\sigma + \tfrac{1}{2})d , \quad \text{for some } n = 1,2,3,\ldots , \tag{3.36}$$

after applying the above argument n times it follows that $A \in \mathrm{C}((0,T], \mathrm{L}^{p_n}(\mathbb{T}^d))$, and one more application then shows that $A \in \mathrm{C}((0,T], \mathrm{L}^\infty(\mathbb{T}^d))$.

Combining the above argument with theorems 3.1 and 3.4 yields

Theorem 3.5 (L^p-local strong solutions). If p satisfies

$$1 \le p \quad \text{and} \quad \sigma d < p \,, \tag{3.37}$$

then for every $\rho > 0$ there exists a time $T(\rho) > 0$ such that for every initial data $A^{\mathrm{in}} \in \mathrm{L}^p(\mathbb{T}^d)$ with $\|A^{\mathrm{in}}\|_{\mathrm{L}^p} \le \rho$ there exists a unique

$$A \in \mathrm{C}([0, T], \mathrm{L}^p(\mathbb{T}^d)) \cap \mathrm{C}((0, T], \mathrm{C}^2(\mathbb{T}^d)) \cap \mathrm{C}^1((0, T], \mathrm{C}(\mathbb{T}^d)), \tag{3.38}$$

satisfying the CGL initial-value problem.

Remark. Strictly speaking, we have proved existence and uniqueness in a smaller space than that asserted in (3.38). The uniqueness of such strong solutions can be extended to hold within the class of weak solutions of theorem 1.1. In other words, so long as a weak solution of the CGL equation is strong then it is unique.

Remark. This result does more than extend to L^p the class of initial data that evolve into strong solutions for positive times. More importantly, it estimates the time interval overwhich any strong solution will exist in terms of its L^p norm. This means that in order to prove that any strong solution is global in time it suffices to control its L^p norm, where p satisfies (3.37). In the next section we present temporally global a priori L^p bounds for an interval of ν values depending only on σ and d, but not μ. In that case one has global strong solutions (theorem 4.1).

Remark. An important special case is when $p = 2\sigma + 2$, in which case the above theorem yields local strong solutions whenever

$$\sigma < \begin{cases} \infty & \text{for } d \le 2 \,, \\ 2/(d-2) & \text{for } d > 2 \,. \end{cases} \tag{3.39}$$

In the next section we present temporally global a priori $\mathrm{L}^{2\sigma+2}$ bounds for a region of the (μ, ν)-plane depending on σ, but not d. In that case, whenever (3.39) is satisfied one has global strong solutions (theorem 4.2).

4. Global strong solutions

The task of this section is to show that there are regions of the (μ, ν)-plane where we can elevate the local strong solutions found in the last section to global strong solutions. The values of μ and ν for which we establish this regularity will depend on the spatial dimension d and the degree of nonlinearity σ. The results in the last section were very general – for example, we have the same local existence and regularity results for either sign of the real part of the coefficient of the nonlinear term – and assert substantially little more than the fact that the problem is well-posed. The estimates and results in this section, however, will depend much more delicately on the structure and dynamics of the CGL equation. They rely not only on the signs of the dissipative terms, but on the relative signs of the coefficients of the dispersive terms as well.

As was pointed out in the final remarks of the last section, it suffices to obtain global control of any

L^p norm where p satisfies (3.37). The L^2 norm is the simplest to control; a direct calculation and an application of the Hölder inequality yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |A|^2\,\mathrm{d}x \le R\int |A|^2\,\mathrm{d}x - \int |A|^{2\sigma+2}\,\mathrm{d}x \le R\int |A|^2\,\mathrm{d}x - \left(\int |A|^2\,\mathrm{d}x\right)^{\sigma+1}. \tag{4.1}$$

For $\sigma > 0$ this differential inequality shows the L^2 norm of A to be uniformly bounded in time (see appendix). Whenever

$$\tfrac{1}{2}\sigma d - 1 < 0\,, \tag{4.2}$$

theorem 3.5 then implies that the CGL equation has global strong solutions.

When (4.2) is violated, one must control more than the L^2 norm. Let $p = 2\kappa$ for $\kappa > 1$. Directly computing the evolution of the $L^{2\kappa}$ norm gives

$$\frac{1}{2\kappa}\frac{\mathrm{d}}{\mathrm{d}t}\int |A|^{2\kappa}\,\mathrm{d}x = R\int |A|^{2\kappa}\,\mathrm{d}x - \int |A|^{2\sigma+2\kappa}\,\mathrm{d}x$$
$$-\tfrac{1}{4}\int |A|^{2\kappa-4}\{[1+2(\kappa-1)]|\nabla|A|^2|^2 + 2\nu(\kappa-1)\nabla|A|^2\cdot \mathrm{i}(A^*\nabla A - A\nabla A^*) + |A^*\nabla A - A\nabla A^*|^2\}\,\mathrm{d}x\,. \tag{4.3}$$

Here, as throughout this section, we will make use of the identity

$$|A|^2|\nabla A|^2 = \tfrac{1}{4}|\nabla|A|^2|^2 + \tfrac{1}{4}|A^*\nabla A - A\nabla A^*|^2\,. \tag{4.4}$$

This formula reflects the decomposition of $A^*\nabla A$ into its "mass" gradient $\frac{1}{2}\nabla|A|^2$ and its "current" $\mathrm{i}\frac{1}{2}(A^*\nabla A - A\nabla A^*)$. The integrand in the last term in (4.3) is a quadratic form in these quantities that will be nonnegative provided the matrix

$$\begin{pmatrix} 1+2(\kappa-1) & (k-1)\nu \\ (\kappa-1)\nu & 1 \end{pmatrix} \tag{4.5}$$

is nonnegative definite, i.e., whenever

$$\kappa - 1 \le (\sqrt{1+\nu^2} - 1)^{-1}\,. \tag{4.6}$$

In that case, neglecting the last term of (4.3) and applying the Hölder inequality to the second to the last term leads to

$$\frac{1}{2\kappa}\frac{\mathrm{d}}{\mathrm{d}t}\int |A|^{2\kappa}\,\mathrm{d}x \le R\int |A|^{2\kappa}\,\mathrm{d}x - \int |A|^{2\sigma+2\kappa}\,\mathrm{d}x \le R\int |A|^{2\kappa}\,\mathrm{d}x - \left(\int |A|^{2\kappa}\,\mathrm{d}x\right)^{(\kappa+\sigma)/\kappa}. \tag{4.7}$$

For $\sigma > 0$ this differential inequality shows the $L^{2\kappa}$ norm to be uniformly bounded in time. This then gives a priori control of the $L^{2\kappa}$ norm if $|\nu|$ is small enough that (4.6) is satisfied.

When in addition to (4.6) one also has

$$\sigma d < 2\kappa\,, \tag{4.8}$$

theorem 3.5 then implies that the CGL equation has global strong solutions. We can find a κ satisfying both (4.6) and (4.8) provided (compare with (4.2))

$$\tfrac{1}{2}\sigma d - 1 < (\sqrt{1+\nu^2} - 1)^{-1} . \tag{4.9}$$

We have therefore proved the following.

Theorem 4.1. For $\sigma > 0$, the generalized CGL equation with C^2 initial conditions has unique global strong solutions provided ν satisfies (4.9).

Remark. For both the subcritical and critical cases (i.e., when $\sigma d \le 2$) this places no restrictions on the parameter values of either μ or ν. For the supercritical case ($\sigma d > 2$) this requires ν to lie in the strip around the μ-axis of the (μ, ν)-plane given by

$$|\nu| < 2\sqrt{\sigma d - 1}/(\sigma d - 2) , \tag{4.10}$$

and places no restriction on μ.

Remark. The basic differential inequalities (4.1) on the L^2 norm of A and (4.7) on $L^{2\kappa}$ norm are sharp for the spatially homogeneous solutions of the CGL equation. Moreover, their proofs use essentially the boundedness of the domain $\mathbb{T}^d$ and do not extend to $\mathbb{R}^d$. Hence, we have no global bound on an L^p norm of A over $\mathbb{R}^d$. However, the problem posed in $L^p(\mathbb{R}^d)$ is less interesting than in $L^p(\mathbb{T}^d)$ because $L^p(\mathbb{R}^d)$ does not include the spatially homogeneous and plane-wave solutions that play an important role in physical applications.

An alternative to directly controlling the L^p norm is to rather first directly control the H^1 norm and then control L^p through Sobolev estimates. As the L^2 norm is already controlled by (4.1), it suffices to control the L^2 norm of ∇A. To do so we assume $\sigma \ge \frac{1}{2}$ so that Theorem 3.3 gives enough regularity of the solution to take the gradient of the CGL equation. Once again utilizing the decomposition identity (4.4), a direct calculation shows

$$\begin{aligned}\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla A|^2\,\mathrm{d}x &= R\int |\nabla A|^2\,\mathrm{d}x - \int |\Delta A|^2\,\mathrm{d}x \\ &\quad - \tfrac{1}{4}\int |A|^{2\sigma-2}[(1+2\sigma)|\nabla|A|^2|^2 - 2\mu\sigma\nabla|A|^2\cdot \mathrm{i}(A^*\nabla A - A\nabla A^*) + |A^*\nabla A - A\nabla A^*|^2]\,\mathrm{d}x .\end{aligned} \tag{4.11}$$

The last term above will be nonpositive provided the matrix

$$\begin{pmatrix} 1+2\sigma & -\sigma\mu \\ -\sigma\mu & 1 \end{pmatrix} \tag{4.12}$$

is nonnegative definite, i.e., whenever

$$\sigma \le (\sqrt{1+\mu^2} - 1)^{-1} . \tag{4.13}$$

In that case, neglecting the last term of (4.11) and applying an elementary inequality to the second to the last term (simply integrate by parts and use the Cauchy–Schwarz inequality) leads to

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla A|^2\,\mathrm{d}x \le R\int |\nabla A|^2\,\mathrm{d}x - \int |\Delta A|^2\,\mathrm{d}x \le R\int |\nabla A|^2\,\mathrm{d}x - \Big(\int |\nabla A|^2\,\mathrm{d}x\Big)^2 \Big/ \int |A|^2\,\mathrm{d}x . \tag{4.14}$$

Using the upper bound for the L^2 norm of A provided by (4.1), this differential inequality shows that the L^2 norm of ∇A is uniformly bounded in time. This then gives a priori control of the H^1 norm if $|\mu|$ is small enough that (4.13) is satisfied.

Control of the H^1 norm yields, by a Sobolev inequality, control of the L^p norm for every p satisfying

$$1 \le p < \begin{cases} \infty & \text{for } d = 1 \text{ or } 2\,, \\ 2d/(d-2) & \text{for } d \ge 3\,. \end{cases} \tag{4.15}$$

When, in addition to (4.13) and (4.15), one also has

$$\sigma d < p\,, \tag{4.16}$$

theorem 3.5 then implies that the CGL equation has global strong solutions. Given that μ and σ satisfy (4.13), we can find a p satisfying both (4.15) and (4.16) provided

$$d < 2 + 2/\sigma\,. \tag{4.17}$$

This encompasses all subcritical and critical nonlinearities in every spatial dimension (i.e., $\sigma d \le 2$), the supercritical cubic nonlinearity ($\sigma = 1$) in dimension $d = 3$, and *all* nonlinearities in dimensions $d \le 2$. Condition (4.13) restricts μ to lie in the strip around the ν-axis of the (μ, ν)-plane given by

$$|\mu| \le \sqrt{2\sigma + 1}/\sigma\,, \tag{4.18}$$

and places no restrictions on ν.

Eqs. (4.3) and (4.11) may be combined to gain control on a set that is actually much larger than the union of the strips (4.10) and (4.18) about the axes of the (μ, ν)-plane. Indeed – because the CGL equation is a dissipative perturbation of the NLS equation (disregarding for the moment the linear growth proportional to R), we might expect the NLS energy to give us some control. However, this energy functional is not necessarily positive definite – only in the modulationally stable region of the (μ, ν)-plane, in the first and third quadrants where $\mu\nu \ge 0$, is the energy definite. Here we should expect regularity. Actually, we can prove a bit more; for the CGL equation the boundaries of modulational stability extend a bit into the second and fourth quadrants of the (μ, ν)-plane. This is reflected in the following result.

Theorem 4.2. For $\sigma \ge \frac{1}{2}$ and $d < 2 + 2/\sigma$, the generalized CGL equation with C^2 initial conditions has unique global strong solutions if the dispersive parameters lie in the region of the (μ, ν)-plane, bounded by hyperbolae, that satisfies

$$\frac{-\mu\nu}{|\mu - \nu|} \le \frac{\sqrt{2\sigma + 1}}{\sigma}\,. \tag{4.19}$$

Remark. This region of the (μ, ν)-plane completely contains the first and third quadrants where the underlying NLS equation is modulationally stable, and the strip (4.18) found from the H^1 bound, and a bit more depending on the degree of nonlinearity. However, it does not generally contain the strip (4.10) obtained from the direct L^p bound. This region reduces to that of NLS stability ($-\mu\nu < 0$) in the limit of a very high degree of nonlinearity ($\sigma \to \infty$).

Proof. We will produce bounds on a functional of the form

$$F = \int |\nabla A|^2 + \frac{\beta^2}{\sigma+1} |A|^{2\sigma+2}\, \mathrm{d}x \,, \tag{4.20}$$

where the parameter $\beta > 0$ is to be chosen later. This form has the general structure of the energy function for the NLS equation in the modulationally stable region of the (μ, ν)-plane, in the first and third quadrants where $\mu\nu \geq 0$. Indeed, if $\nu \neq 0$ and $\mu/\nu \geq 0$ then $\beta^2 = \mu/\nu$ would make F the exact NLS energy functional.

Consider the identities

$$\int |\Delta A \mp \beta |A|^{2\sigma} A|^2\, \mathrm{d}x = \int |\Delta A|^2 + \beta^2 |A|^{4\sigma+2}\, \mathrm{d}x \pm \tfrac{1}{2} \int |A|^{2\sigma-2} [(1+2\sigma)\beta |\nabla |A|^2|^2 + \beta |A^* \nabla A - A \nabla A^*|^2]\, \mathrm{d}x \,. \tag{4.21}$$

Taking a linear combination of (4.3) and (4.11) while using (4.21) gives

$$\begin{aligned} \tfrac{1}{2}\frac{\mathrm{d}F}{\mathrm{d}t} = {} & R \int |\nabla A|^2 + \beta^2 |A|^{2\sigma+2}\, \mathrm{d}x - \tfrac{1}{2}(1-\delta) \int |\Delta A - \beta |A|^{2\sigma} A|^2\, \mathrm{d}x - \tfrac{1}{2}(1+\delta) \int |\Delta A + \beta |A|^{2\sigma} A|^2\, \mathrm{d}x \\ & - \tfrac{1}{4} \int |A|^{2\sigma-2} [(1+2\sigma)(1+2\delta\beta+\beta^2) |\nabla |A|^2|^2 + 2\sigma(\nu\beta^2 - \mu) \nabla |A|^2 \cdot i(A^*\nabla A - A\nabla A^*) \\ & + (1+2\delta\beta+\beta^2)|A^*\nabla A - A\nabla A^*|^2]\, \mathrm{d}x \,, \end{aligned} \tag{4.22}$$

for some $-1 \leq \delta \leq 1$ to be chosen. The last term above will be nonpositive provided the matrix

$$\begin{pmatrix} (1+2\sigma)(1+2\delta\beta+\beta^2) & \sigma(\nu\beta^2-\mu) \\ \sigma(\nu\beta^2-\mu) & (1+2\delta\beta+\beta^2) \end{pmatrix} \tag{4.23}$$

is nonnegative definite, i.e., whenever

$$\sigma \leq \frac{1}{\sqrt{1+(\nu\beta^2-\mu)^2/(1+2\delta\beta+\beta^2)^2} - 1} \,. \tag{4.24}$$

In that case, neglecting the last two terms on the right of (4.22) and then applying the Cauchy–Schwarz inequality to the new last term gives

$$\begin{aligned} \tfrac{1}{2}\frac{\mathrm{d}F}{\mathrm{d}t} & \leq R \int |\nabla A|^2 + \beta^2 |A|^{2\sigma+2}\, \mathrm{d}x - \tfrac{1}{2}(1-\delta) \int |\Delta A - \beta |A|^{2\sigma} A|^2\, \mathrm{d}x \\ & \leq R \int |\nabla A|^2 + \beta^2 |A|^{2\sigma+2}\, \mathrm{d}x - \tfrac{1}{2}(1-\delta) \Big(\int |\nabla A|^2 + \beta |A|^{2\sigma+2}\, \mathrm{d}x \Big)^2 \Big/ \int |A|^2\, \mathrm{d}x \,. \end{aligned} \tag{4.25}$$

Using the upper bound for the L^2 norm of A provided by (4.1), this differential inequality shows that F is uniformly bounded in time when $-1 \leq \delta < 1$ and is exponentially bounded in time when $\delta = 1$. This then gives a priori control of F if μ and ν are such that (4.24) is satisfied.

Control of the F obviously gives control of the H^1 norm of A. Arguing as in (4.15)–(4.17), theorem 3.5 implies that the CGL equation has global strong solutions provided (4.17) and (4.24) are satisfied. The following choices for β and δ will maximize the upper bounds (4.24) on σ. If $\nu = 0$, then for any σ and μ we may choose a β large enough to satisfy (4.24) for any fixed value of δ, say $\frac{1}{2}$, and there is no restriction. If $\nu \neq 0$ and $\mu/\nu \geq 0$ then choose $\beta = \sqrt{\mu/\nu}$ and there is again no restriction on σ. If $\nu \neq 0$ and μ and ν have the opposite signs, then choose $\beta = -\mu/\nu$ for any value of δ. The maximum of the upper bounds (4.24) on σ is then obtained by setting $\delta = 1$ and yields (4.19), proving theorem 4.2.

Remark. Inequality (4.25) and its region of validity (4.19) were first derived by one of us (C.D.L.) [2] in a referee report for [1] where it was then applied in the cubic case.

Remark. The basic differential inequalities (4.14) on the L^2 norm of ∇A and (4.25) on F hold over $\mathbb{R}^d$ with no change in their proofs. But without a global bound on the L^2 norm of A over $\mathbb{R}^d$, no global bounds may be inferred.

5. A functional lattice of differential inequalities

The objective of this section is to derive bounds on all derivatives of the global smooth solutions found in the last section that are uniform in time. Throughout this section it will be assumed that σ is a positive integer, so that the local existence of smooth solutions is ensured by theorem 3.2. Here, as in the last section, the estimates and results will depend delicately on the structure and dynamics of the CGL equation.

In [1] a "ladder" structure for the 2D CGL equation, based intuitively on a scaling symmetry, was developed. We assert on dimensional grounds that $|A|^{2\sigma}$ "scales" like Δ in the CGL equation so we consider functionals of the form

$$F_n = \int_{\mathbb{T}^d} |\nabla^n A|^2 + \alpha_n |A|^{2(n\sigma+1)}\,\mathrm{d}x\,, \tag{5.1}$$

where $\alpha_n > 0$ is a dimensionless parameter to be chosen at our convenience. These are higher order generalizations of the Lyapunov functional obtained for $n = 1$ and $\alpha_1 = 1$ in the case $\mu = \nu$, or of the underlying NLS equation's energy functional, proportional to F_1 with $\alpha_1 = \mu/\nu(\sigma + 1)$. They are similar in composition to the infinite set of conserved quantities for the integrable 1D cubic NLS equation. For $n = 0$ they reduce to the L^2 norm. The F_n's are time-dependent scalar quantities when the A's are solutions to the CGL equation, and in [1] it was shown how to derive, for $n \geq 1$, the hierarchy of differential inequalities

$$\frac{\mathrm{d}F_n}{\mathrm{d}t} \leq [2(n\sigma + 1)R + c_n \|A\|_\infty^{2\sigma}]F_n - b_n \frac{F_n^2}{F_{n-1}}\,, \tag{5.2a}$$

where

$$b_n = \min\{1, \alpha_{n-1}\alpha_n\}\,. \tag{5.2b}$$

This ladder may be used to establish a priori bounds on the local smooth solutions, allowing for their extension, in the usual manner, beyond the initial interval of existence. The key to utilizing the ladder is to close it by bounding the $\|A\|_\infty$ term by some power of F_n, which we are able to do by interpolation only for n large enough depending explicitly on the spatial dimension d. Additionally, n must be large enough that the term resulting from the product $\|A\|_\infty^{2\sigma}F_n$ are of lower order in F_n than F_n^2. Once this "rung" of the ladder has been determined, an a priori upper bound on F_{n-1} is required to show that each of the higher F_n's is controlled, providing explicit estimates of the F_n's or, for example, their time-averages. The a priori upper bound on the "bottom rung", F_{n-1}, must be obtained by other more delicate methods, i.e., by returning to the CGL equation and using the detailed structure of the evolution.

In [1] it was shown that for the cubic CGL equation ($\sigma = 1$) in $d = 2$ the bottom rung is F_1, and that its $\limsup_{t\to\infty}$ estimates are controlled by different powers of R depending on the values of the dispersive coefficients μ and ν. The tightest bounds on F_1, which were shown to be sharp, were obtained in and near the region of modulational stability in the (μ, ν)-plane. Outside this region and deep into the modulationally unstable regime ($\mu\nu < 0$ and $|\mu\nu|$ large) the upper bounds are very large, a reflection of the self-focusing finite-time blow-up of the underlying critical NLS equation. The bottom rung for the cubic equation in $d = 3$ is also F_1, but a priori estimates could be obtained only in part of the (μ, ν)-plane, in and near the region of modulational stability.

It is possible to generalize this idea to give a more sensitive demonstration of when and where we can obtain such global estimates over the entire region on the (μ, ν)-plane where local strong solutions become global strong solutions. To do this we construct a wider family of functionals from the scaling of the Laplacian and the nonlinear terms. In particular, for every real $m \geq 1$ we match $|\nabla^n A|^{2m}$ with $|A|^{2m\gamma_n}$, where

$$\gamma_n \equiv n\sigma + 1 . \tag{5.3a}$$

Notice that $\gamma_1 = \sigma + 1 = \gamma$. More specifically, we consider the "lattice" of positive definite functionals

$$F_{n,m} = \int_{\mathbb{T}^d} |\nabla^n A|^{2m} + \alpha_{n,m}|A|^{2m\gamma_n}\,\mathrm{d}x , \tag{5.3b}$$

with $\alpha_{n,m} > 0$ to be chosen at our convenience. Here we consider $\nabla^n A$ to be a symmetric n tensor and define

$$|\nabla^n A|^2 \equiv \nabla^n A^* \cdot \nabla^n A , \tag{5.4}$$

where "$\cdot$" denotes the usual tensor inner product that acts on a symmetric k-tensor and a symmetric l-tensor by contracting over $\min\{k, l\}$ indices to give a symmetric $|k - l|$-tensor. This interpretation of $|\nabla^n A|^2$ is consistent with [1], although there it was not so explicitly stated. We are now interested in obtaining a lattice inequality analogous to (5.1), and in finding the minimum values of n and m which allow us to climb the lattice.

We prove the following.

Theorem 5.1. Given a positive integer σ and real $m \geq 1$ such that

$$\beta_{m,\nu} \equiv 1 - (m-1)(\sqrt{1+\nu^2} - 1) > 0 , \tag{5.5}$$

then for every integer $n \geq 1$ the $F_{n,m}$ satisfies the differential inequality

$$\frac{\mathrm{d}F_{n,m}}{\mathrm{d}t} \leq (2m\gamma_n R + c_{n,m,\mu}\|A\|_\infty^{2\sigma})F_{n,m} - b_{n,m}F_{n,m}^{1+1/m}/F_{n-1,m}^{1/m} , \tag{5.6a}$$

where

$$b_{n,m} = \min\left\{\frac{\beta_{m,\nu}}{[2(m-1)+\sqrt{d}]^2}, \left(\frac{\alpha_{n-1,m}}{\alpha_{n,m}}\right)^{1/m}\right\}. \tag{5.6b}$$

Remark. For given values of the dispersion ν, the condition (5.5) places strong restrictions on the range of values of m for which this lattice holds.

Remark. The values obtained in (5.6b) for the $b_{n,m}$ are far from optimal, although they are sufficient to prove our main result. We can sharpen the dependence of the $b_{n,m}$ on m, d and ν in order to improve the global bounds obtained from (5.6), but in doing so, condition (5.5) remains unchanged.

Proof. This lattice of differential inequalities is defined for the locally existing smooth solutions of the generalized CGL equation, so that the manipulations, differentiations and/or integrations by parts that we will perform are completely valid. We will perform the proof in three steps, but first we define

$$J_{n,m} = \int_{\mathbb{T}^d} |\nabla^n A|^{2m}\, \mathrm{d}x \quad \text{and} \quad G_{n,m} = \int_{\mathbb{T}^d} |A|^{2m\gamma_n}\, \mathrm{d}x\,. \tag{5.7}$$

Step 1. We begin by demonstrating that

$$\frac{1}{2m}\frac{\mathrm{d}J_{n,m}}{\mathrm{d}t} \le RJ_{n,m} + c\|A\|_\infty^{2\sigma} J_{n,m} - \frac{\beta_{m,\nu}}{[2(m-1)+\sqrt{d}]^2}\frac{J_{n,m}^{1+1/m}}{J_{n-1,m}^{1/m}}, \tag{5.8}$$

where c depends on n, m, σ, ν, and μ.

To see this, take the time derivative of $J_{n,m}$, using the fact that A satisfies the CGL equation, to obtain

$$\begin{aligned}\frac{1}{2m}\frac{\mathrm{d}J_{n,m}}{\mathrm{d}t} \le{}& RJ_{n,m} + \mathrm{Re}\Big((1+\mathrm{i}\nu)\int |\nabla^n A|^{2(m-1)}\nabla^n A^* \cdot \nabla^n \Delta A\, \mathrm{d}x\Big)\\ &- \mathrm{Re}\Big((1+\mathrm{i}\mu)\int |\nabla^n A|^{2(m-1)}\nabla^n A^* \cdot \nabla^n(|A|^{2\sigma}A)\, \mathrm{d}x\Big).\end{aligned} \tag{5.9}$$

The second and third terms on the right side above are estimated in the two lemmas below. Note that these estimates depend explicitly on the condition (5.5).

Lemma (i). For every real m satisfying (5.5) and integer $n \ge 1$,

$$I_\nu \equiv \mathrm{Re}\Big((1+\mathrm{i}\nu)\int |\nabla^n A|^{2(m-1)}\nabla^n A^* \cdot \nabla^n \Delta A\, \mathrm{d}x\Big) \le -\frac{\beta_{m,\nu}}{[2(m-1)+\sqrt{d}]^2}\frac{J_{n,m}^{1+1/m}}{J_{n-1,m}^{1/m}}. \tag{5.10}$$

Proof of lemma (i). First, integrate by parts within I_ν to obtain

$$\begin{aligned}I_\nu ={}& -\int |\nabla^n A|^{2(m-1)}|\nabla^{n+1}A|^2\, \mathrm{d}x - (m-1)\int |\nabla^n A|^{2(m-2)}|\nabla^n A^* \cdot \nabla^{n+1} A|^2\, \mathrm{d}x\\ &-(m-1)\mathrm{Re}\Big((1+\mathrm{i}\nu)\int |\nabla^n A|^{2(m-2)}(\nabla^n A^* \cdot \nabla^{n+1}A)\cdot(\nabla^n A^* \cdot \nabla^{n+1}A)\, \mathrm{d}x\Big)\\ \le{}& -\int |\nabla^n A|^{2(m-1)}|\nabla^{n+1}A|^2\, \mathrm{d}x + (m-1)(\sqrt{1+\nu^2}-1)\int |\nabla^n A|^{2(m-2)}|\nabla^n A^* \cdot \nabla^{n+1}A|^2\, \mathrm{d}x\,.\end{aligned} \tag{5.11}$$

The inequality

$$|\nabla^n A^* \cdot \nabla^{n+1} A|^2 \le |\nabla^n A|^2 |\nabla^{n+1} A|^2 , \tag{5.12}$$

applied to (5.11) shows that $I_\nu \le 0$ provided m satisfies (5.5).

Next, integrate by parts within $J_{n,m}$ and apply Cauchy–Schwarz to get

$$\begin{aligned} J_{n,m} &= -\int [|\nabla^n A|^{2(m-1)} \nabla^{n-1} A^* \cdot \nabla^{n-1} \Delta A + (m-1)|\nabla^n A|^{2(m-2)} \nabla |\nabla^n A|^2 \cdot (\nabla^{n-1} A^* \cdot \nabla^n A)]\, \mathrm{d}x \\ &\le \left(\int |\nabla^n A|^{2(m-1)} |\nabla^{n+1} A|^2 \,\mathrm{d}x\right)^{1/2} \left(\int [|\nabla^n A|^{m-1} \nabla^{n-1} \Delta A + (m-1)|\nabla^n A|^{m-3} \nabla |\nabla^n A|^2 \cdot \nabla^n A]^2 \,\mathrm{d}x\right)^{1/2} . \end{aligned} \tag{5.13}$$

Squaring (5.13) and applying the Hölder inequality

$$\int |\nabla^n A|^{2(m-1)} |\nabla^{n-1} A|^2 \,\mathrm{d}x \le J_{n,m}^{(m-1)/m} J_{n-1,m}^{1/m} , \tag{5.14}$$

to the first factor on the right side leads to

$$\begin{aligned} \frac{J_{n,m}^{1+1/m}}{J_{n-1,m}^{1/m}} &\le \int [|\nabla^n A|^{m-1} \nabla^{n-1} \Delta A + (m-1)|\nabla^n A|^{m-3} \nabla |\nabla^n A|^2 \cdot \nabla^n A]^2 \,\mathrm{d}x \\ &\le \int [(1+\lambda)|\nabla^n A|^{2(m-1)} |\nabla^{n-1} \Delta A|^2 + (1+1/\lambda)(m-1)^2 |\nabla^n A|^{2(m-2)} |\nabla |\nabla^n A|^2|^2] \,\mathrm{d}x , \end{aligned} \tag{5.15}$$

for every positive λ. Utilizing the inequalities

$$|\nabla^{n-1} \Delta A|^2 \le d |\nabla^{n+1} A|^2 , \qquad |\nabla |\nabla^n A|^2|^2 \le 4 |\nabla^n A^* \cdot \nabla^{n+1} A|^2 , \tag{5.16}$$

in (5.15) then gives

$$\frac{J_{n,m}^{1+1/m}}{J_{n-1,m}^{1/m}} \le (1+\lambda) d \int [|\nabla^n A|^{2(m-1)} |\nabla^{n-1} \nabla A|^2 + \frac{4(m-1)^2}{\lambda d} |\nabla^n A|^{2(m-2)} |\nabla^n A^* \cdot \nabla^{n+1} A|^2] \,\mathrm{d}x , \tag{5.17}$$

for every positive λ. Comparing (5.11) and (5.17) while recalling (5.12) leads to

$$\frac{J_{n,m}^{1+1/m}}{J_{n-1,m}^{1/m}} \le -\frac{(1+\lambda) d [1 + 4(m-1)^2/\lambda d]}{1 - (m-1)(\sqrt{1+\nu^2} - 1)} I_\nu , \tag{5.18}$$

for every positive λ. Optimizing the (5.18) by picking $\lambda = 2(m-1)/\sqrt{d}$ gives (5.10) and completes the proof of lemma (i).

Lemma (ii). If σ is a positive integer then for every real $m \ge 1$ and integer $n \ge 1$,

$$I_\mu \equiv -\mathrm{Re}\left((1+\mathrm{i}\mu) \int |\nabla^n A|^{2(m-1)} \nabla^n A^* \cdot \nabla^n (|A|^{2\sigma} A) \,\mathrm{d}x\right) \le c_{n,m} |1+\mathrm{i}\mu| \, \|A\|_\infty^{2\sigma} J_{n,m} . \tag{5.19}$$

Proof of lemma (ii). A direct application of the Hölder inequality gives

$$|I_\mu| \le J_{n,m}^{(2m-1)/2m} \|\nabla^n (|A|^{2\sigma} A)\|_{2m} . \tag{5.20}$$

We will demonstrate the calculation for $\sigma = 1$ and indicate how it is generalized to other values of σ. The Leibnitz formula gives

$$\nabla^n(|A|^2 A) = \sum_{\substack{0\le j,k,l\\ j+k+l=n}} \frac{n!}{j!k!l!} \nabla^j A^* \vee \nabla^k A \vee \nabla^l A , \tag{5.21}$$

where '$\vee$" denotes the symmetric tensor outer product that acts on a symmetric k-tensor by symmetrizing the $(k+l)$-tensor that is their usual tensor outer product. From definition (5.4) it is easily checked that

$$|\nabla^n(|A|^2 A)| \le \sum_{\substack{0\le j,k,l\\ j+k+l=n}} \frac{n!}{j!k!l!} |\nabla^j A^* \vee \nabla^k A \vee \nabla^l A| \le \sum_{\substack{0\le j,k,l\\ j+k+l=n}} \frac{n!}{j!k!l!} |\nabla^j A|\, |\nabla^k A|\, |\nabla^l A| . \tag{5.22}$$

Performing a triple Hölder estimate on each term of this sum we have

$$\|\nabla^n(|A|^2 A)\|_{2m} \le \sum_{\substack{0\le j,k,l\\ j+k+l=n}} \frac{n!}{j!k!l!} \|\nabla^j A\|_{2mn/j} \|\nabla^k A\|_{2mn/k} \|\nabla^l A\|_{2mn/l} . \tag{5.23}$$

A Gagliardo–Nirenberg interpolation of each of the norms in this sum shows that

$$\|\nabla^j A\|_{2mn/j} \le c_{n,m} \|A\|_\infty^{(n-j)/n} \|\nabla^n A\|_{2m}^{j/n} , \tag{5.24}$$

for every $j = 0, \ldots, n$. Therefore, combining (5.23) and (5.24) in (5.20) gives

$$|I_\mu| \le J_{n,m}^{(2m-1)/(2m)} \sum_{\substack{0\le j,k,l\\ j+k+l=n}} \frac{n!}{j!k!l!} c_{n,m} \|A\|_\infty^2 \|\nabla^n A\|_{2m} = 3^n c_{n,m} \|A\|_\infty^2 J_{n,m} . \tag{5.25}$$

This calculation has been performed with $\sigma = 1$, however it is clear that whenever σ is a positive integer the Leibnitz formula for $\nabla^n(|A|^{2\sigma} A)$ is similar to (5.21), but simply has more terms. This completes the proof of both lemma (ii) and step 1. □

Step 2. Here we demonstrate that $G_{n,m}$ satisfies the differential inequality

$$\frac{1}{2m\gamma_n} \frac{\mathrm{d}G_{n,m}}{\mathrm{d}t} \le R G_{n,m} - \frac{G_{n,m}^{1+1/m}}{G_{n-1,m}^{1/m}} + c\|A\|_\infty^{2\sigma} F_{n,m} , \tag{5.26}$$

where c depends on n, m, and ν.

To obtain this, we start with the evolution of the $\mathrm{L}^{2\kappa}$ norm (4.3) for $\kappa = m\gamma_n$, then bound the quadratic form in the last term by the extreme eigenvalue of its matrix (4.5) and use identity (4.4) to find that

$$\frac{1}{2m\gamma_n} \frac{\partial G_{n,m}}{\mathrm{d}t} \le R G_{n,m} - \int |A|^{2(m\gamma_n+\sigma)}\, \mathrm{d}x + [(m\gamma_n - 1)\sqrt{1+\nu^2} - m\gamma_n] \int |A|^{2(m\gamma_n-1)} |\nabla A|^2\, \mathrm{d}x . \tag{5.27}$$

The second term on the right side of (5.27) is estimated in the following lemma

Lemma (iii). For every real $m \ge 1$ and integer $n \ge 1$,

$$-\int |A|^{2(m\gamma_n+\sigma)}\,\mathrm{d}x \le -G_{n,m}^{1+1/m}/G_{n-1,m}^{1/m}\,. \tag{5.28}$$

Proom of lemma (iii). Introducing P and Q to be

$$P=\frac{2m(m\gamma_n+\sigma)}{m+1}\,,\qquad Q=\frac{2m(\gamma_n-\sigma)}{m+1}\,, \tag{5.29}$$

so that $P+Q=2m\gamma_n$, an application of the Hölder inequality then gives

$$\begin{aligned}G_{n,m}&=\int |A|^{2m\gamma_n}\,\mathrm{d}x=\int |A|^P|A|^Q\,\mathrm{d}x\le\left(\int |A|^{P(m+1)/m}\,\mathrm{d}x\right)^{m/(m+1)}\left(\int |A|^{Q(m+1)}\,\mathrm{d}x\right)^{1/(m+1)}\\&=\left(\int |A|^{2(m\gamma_n+\sigma)}\,\mathrm{d}x\right)^{m/(m+1)}\left(\int |A|^{2m(\gamma_n-\sigma)}\,\mathrm{d}x\right)^{1/(m+1)}.\end{aligned} \tag{5.30}$$

Recalling the definition (5.3a) of γ_n, (5.30) may be recast as

$$\frac{G_{n,m}^{1+1/m}}{G_{n-1,m}^{1/m}}\le\int |A|^{2(m\gamma_n+\sigma)}\,\mathrm{d}x\,, \tag{5.31}$$

which establishes lemma (iii). □

The last term (5.27) is estimated in

Lemma (iv). For every real $m\ge1$ and integer $n\ge1$,

$$\int |A|^{2(m\gamma_n-1)}|\nabla A|^2\,\mathrm{d}x\le c\|A\|_\infty^{2\sigma}F_{n,m}\,. \tag{5.32}$$

Proof of lemma (iv). By the Hölder and Young inequalities,

$$\begin{aligned}\int |A|^{2(m\gamma_n-1)}|\nabla A|^2\,\mathrm{d}x&\le\|A\|_\infty^{2\sigma}\int |A|^{2(m\gamma_n-\gamma)}|\nabla A|^2\,\mathrm{d}x\\&\le\|A\|_\infty^{2\sigma}\left(\frac{m\gamma_n-\gamma}{m\gamma_n}G_{n,m}+\frac{\gamma}{m\gamma_n}\int |\nabla A|^{2m\gamma_n/\gamma}\,\mathrm{d}x\right).\end{aligned} \tag{5.33}$$

Now we estimate the last integral using the interpolation bound

$$\|\nabla A\|_{2m\gamma_n/\gamma}\le c''\|\nabla^n A\|_{2m}^{1/n}\|A\|_{2m\gamma_n}^{(n-1)/n}\,, \tag{5.34}$$

to obtain

$$\int |\nabla A|^{2m\gamma_n/\gamma}\,\mathrm{d}r\le c'J_{n,m}^{\gamma_n/n\gamma}G_{n,m}^{(n-1)/n\gamma}\le cF_{n,m}^{\gamma_n/n\gamma}F_{n,m}^{(n-1)/n\gamma}\le cF_{n,m}\,. \tag{5.35}$$

The $G_{n,m}$ in (5.33) are also bounded by $F_{n,m}$, proving lemma (iv). □

Step 3. Combine the differential inequalities in (5.8) for $J_{n,m}$ and in (5.26) for $G_{n,m}$ together to form a single differential inequality for $F_{n,m}$.

Elementary applications of the Hölder and Cauchy–Schwarz inequalities complete the proof of theorem 5.1. □

Remark. Theorem 5.1 also holds over $\mathbb{R}^d$. The proof is essentially the same because the Gagliardo–Nirenberg interpolations (5.24) in the proof of lemma (ii) of step 1 and (5.34) in the proof of lemma (iv) of step 2 are valid in that case too.

Application of the lattice to establish a priori bounds relies on our ability to close it so that the evolution of any particular $F_{n,m}$ depends only on itself and "lower" functionals, i.e., $F_{n-1,m}$. In order to do this we must estimate the $\|A\|_\infty$ factor in terms of the $F_{n,m}$'s and this we do via interpolation. A straightforward application of the Gagliardo–Nirenberg and Hölder inequalities yields

$$\|A\|_\infty^2 \le c(F_{n,m}^{2/(2m\gamma_n-\sigma d)} + F_{n,m}^{1/m\gamma_n}) , \tag{5.36}$$

provided

$$n > d/2m . \tag{5.37}$$

Inserting (5.36) into the lattice in (5.6a) we find

$$\frac{\mathrm{d}F_{n,m}}{\mathrm{d}t} \le [2m\gamma_n R + c(F_{n,m}^{2/(2m\gamma_n-\sigma d)} + F_{n,m}^{1/m\gamma_n})^\sigma]F_{n,m} - b_{n,m}\frac{F_{n,m}^{1+1/m}}{F_{n-1,m}^{1/m}} . \tag{5.38}$$

In order to control $F_{n,m}$, the negative definite $F_{n,m}^{1+1/m}$ term must be of higher order than the positive nonlinear terms, which will be the case provided

$$2\sigma/(2m\gamma_n - \sigma d) < 1/m , \tag{5.39}$$

or equivalently, whenever

$$n > 1 + (\sigma d - 2m)/2m\sigma . \tag{5.40}$$

For integer $\sigma > 0$ this requirement is stronger than (5.33). Therefore the "bottom rung" will be $F_{n-1,m}$ provided n and m satisfy (5.5) and (5.40). That is, if we can prove a priori upper bounds on $F_{n-1,m}$, where n and m satisfy (5.5) and (5.40), then all the higher $F_{n,m}$ will become bounded within a fixed and finite amount of time, uniformly in the initial condition, and they will stay bounded uniformly in time from then onwards. In particular, by (5.36) the L^∞ norm of A remains uniformly bounded. By theorem 3.2 this is then sufficient to elevate the local smooth solutions to global smooth solutions. For some values of the dispersive parameters (μ, ν) we are able to do just that.

The only quantity which is known to be uniformly bounded in time for the weak solutions in all dimensions and for all nonlinearities is $F_{0,1} \sim \|A\|_2^2$. Thus, if $F_{0,1}$ is the bottom rung then the weak solutions are in fact strong solutions. In order to have $F_{0,1} \sim \|A\|_2^2$ as the bottom rung, setting $n = 2$ and $m = 1$ in (5.36) shows that we need $\sigma d < 2$. This condition is exactly the *subcritical* case discussed in the introduction: then there is no possibility of a finite-time singularity in the inviscid limit. The 1D cubic equation, with $\sigma = 1$, falls into this category.

The situation $\sigma d \ge 2$ is more interesting because of the possibility of finite-time singularities in the underlying Hamiltonian NLS equation. The question then becomes, for a given spatial dimension and nonlinearity, when can we get control of a bottom rung? We will demonstrate two such bottom rungs, one based on each of the global uniform estimates derived from the differential inequalities (4.7) and the combination of (4.1) with (4.25). This then covers the entire region of the (μ, ν)-plane in which we have established global smooth solutions.

Remark. It is at this point where we will use the fact that we are working over $\mathbb{T}^d$ rather than $\mathbb{R}^d$, the differential inequalities (4.1) and (4.7) not being valid over $\mathbb{R}^d$. Indeed, one can not access a bottom rung of the lattice of Theorem 5.1 without such estimates.

If we attempt to make $n = 1$ the minimum value, so that the bottom rung of the lattice is $F_{0,m} \sim \|A\|_{2m}^{2m}$ for some real $m \geq 1$, we need

$$m - 1 < 1/(\sqrt{1+\nu^2} - 1)\,, \quad \text{and} \quad \sigma d < 2m\,, \tag{5.41}$$

in order to satisfy (5.5) and (5.37), respectively. In that case the global uniform control of $F_{0,m}$ is derived from the differential inequality (4.7) for the L^{2m} norm. Notice that (5.37) implies condition (4.6) is satisfied, which justifies (4.7). We can find an m satisfying (5.41) provided

$$\tfrac{1}{2}\sigma d - 1 < (\sqrt{1+\nu^2} - 1)^{-1}\,. \tag{5.42}$$

But this is exactly condition (4.9) which determines the region of validity of theorem 4.1. We have proved the following.

Theorem 5.2. For integer $\sigma > 0$, when ν satisfies (5.42) the generalized CGL equation with C^∞ initial conditions has unique global smooth solutions that satisfies the differential inequalities (5.6) for those $m \geq 1$ satisfying (5.41).

If, on the other hand, we consider the choice of $n = 2$ and $m = 1$, so that the bottom rung of the lattice is $F_{1,1}$, then (5.5) is automatically satisfied and we need $d < 2 + 2/\sigma$ in order to satisfy (5.40). In that case the global uniform control of $F_{1,1}(=F)$ is derived from the differential inequality (4.25) provided (4.19) is satisfied. We conclude the following.

Theorem 5.3. For integer $\sigma > 0$, when (μ, ν) satisfies (4.19) the generalized CGL equation with C^∞ initial conditions has unique global smooth solutions that satisfies the differential inequalities (5.6) for $m = 1$.

Finally, we remark that although the implications of theorems 5.2 and 5.3 add nothing so far as the existence of global smooth solutions is concerned, the explicit estimates extracted from them through the lattice are useful. Some of these estimates are discussed in detail in [1] for the cubic case $\sigma = 1$ in dimensions $d = 2$ and 3.

6. Summary and discussion

We have studied the existence and regularity of solutions to the generalized CGL equation

$$\partial_t A = RA + (1 + \mathrm{i}\nu)\,\Delta A - (1 + \mathrm{i}\mu)|A|^{2\sigma} A\,, \tag{6.1}$$

with periodic boundary conditions in d spatial dimensions. We proved the existence of global weak solutions in every case, i.e., in all dimensions, for all degrees of nonlinearity $\sigma > 0$, and for all parameter values R, μ, and ν. These weak solutions are square integrable in space, pointwise in time, and their H^1 and $L^{2\sigma+2}$ norms are integrable in time. They are not necessarily unique.

We have also proved the existence of strong (unique, classical) solutions, locally in time, in every case, at least when starting from bounded initial data. Furthermore, we showed that every initial data in L^p for

$$1 \le p \quad \text{and} \quad \sigma d < p \tag{6.2}$$

evolves into a unique solution locally in time that is strong for positive times. Moreover, the time interval overwhich any strong solution exists can be estimated from below in terms of its L^p norm for p satisfying (6.2), thus reducing the question of global existence to control of these norms. When σ is a positive integer we showed that strong solutions are in fact smooth (C^∞) solutions.

We then found conditions where the local strong solutions can be elevated to global strong solutions. In these situations we can assert that the CGL equation does not suffer from the pathologies of the underlying NLS equation. The sufficient conditions that we find for global existence, summarized in table 1 for integer $\sigma > 0$, depend on the dimension d, the degree of nonlinearity σ, and the specific values of the dispersion parameters μ and ν. We can find global smooth solutions in every dimension, but only for a restricted set of nonlinearities and/or values of the dispersive coefficients μ and ν. For integer $\sigma > 0$ we find global strong solutions in dimensions $d \ge 4$ only for small enough values of $|\nu|$.

The hyperbolic boundaries in the (μ, ν)-plane can be understood in terms of the instabilities of the underlying NLS equation. In the first and third quadrants where μ and ν have the same sign, the corresponding NLS equation is modulationally stable. Then we have global existence of strong solutions for the CGL equation. In the second and fourth quadrants, where μ and ν have the opposite signs, the NLS equation is modulationally unstable and in the critical and supercritical cases $\sigma d \ge 2$ forms finite-time singularities. Then we have global existence of strong solutions for the CGL equation only with sufficiently strong damping. It does not seem unlikely that modulational stability of the corresponding NLS equation should in fact be sufficient for the existence of strong solutions to the CGL equation, even in arbitrarily high spatial dimensions. In order to use our method to prove this, though, it will be necessary to derive a priori estimates on some $F_{n,m}$ with $n > 2$ or $m > 1$, probably by methods not unlike those used in the proof of theorem 4.2. This we leave for a future study.

Table 1
Sufficient restrictions for the existence of global smooth solutions to the complex Ginzburg–Landau equation (1.1) for positive integer values of σ. The bounds on $|\nu|$ and the "no restriction" entries are a result of theorem 4.1 while the hyperbolic boundaries arise from theorem 4.2.

	$\sigma = 1$	$\sigma = 2$	$\sigma \ge 3$
$d = 1$	no restriction	no restriction	$\lvert\nu\rvert < \frac{2\sqrt{\sigma-1}}{\sigma-2}$ or $\frac{-\mu\nu}{\lvert\mu-\nu\rvert} < \frac{\sqrt{2\sigma+1}}{\sigma}$
$d = 2$	no restriction	$\lvert\nu\rvert < \sqrt{3}$ or $\frac{-\mu\nu}{\lvert\mu-\nu\rvert} < \frac{1}{2}\sqrt{5}$	$\lvert\nu\rvert < \frac{\sqrt{2\sigma-1}}{\sigma-1}$ or $\frac{-\mu\nu}{\lvert\mu-\nu\rvert} < \frac{\sqrt{2\sigma+1}}{\sigma}$
$d = 3$	$\lvert\nu\rvert < \sqrt{8}$ or $\frac{-\mu\nu}{\lvert\mu-\nu\rvert} < \sqrt{3}$	$\lvert\nu\rvert < \frac{1}{2}\sqrt{5}$	$\lvert\nu\rvert < \frac{2\sqrt{3\sigma-1}}{3\sigma-2}$
$d \ge 4$	$\lvert\nu\rvert < \frac{2\sqrt{d-1}}{d-2}$	$\lvert\nu\rvert < \frac{\sqrt{2d-1}}{d-1}$	$\lvert\nu\rvert < \frac{2\sqrt{\sigma d-1}}{\sigma d-2}$

Acknowledgements

We thank P Constantin for helpful discussions and encouragement and F.B. Weissler for bringing [24] to our attention. C.R.D. was supported in part by NSF Grants PHY-8907755 (PYI Award), PHY-8958506 and PHY-9214715, and by AFOSR grant 89-0510. CDL was supported in part by NSF grant DMS-8914420 and by AFOSR grant F49602-86-C0130, which is associated with the Arizona Center for Mathematical Sciences (ACMS) at the University of Arizona. This work was begun during the authors' visit to the Center for Nonlinear Studies (CNLS) at the Los Alamos National Laboratory.

Appendix

Throughout sections 4 and 5 we infer uniform bounds for any positive quantity $F(t)$ that satisfies a differential inequality of the general form

$$\frac{\mathrm{d}F}{\mathrm{d}t} \leq a(t)\, F - b(t)\, F^{1+s}\,, \tag{A.1}$$

where $s > 0$ and $b(t) > 0$. Following Bernoulli, introduce the variable $Y = F^{-s}$ into (A.1) and obtain the linear differential inequality

$$\frac{\mathrm{d}Y}{\mathrm{d}t} \geq -sa(t)\, Y + sb(t)\,. \tag{A.2}$$

Applying the Gronwall lemma to this gives

$$Y(t) \geq \exp\Big(-\int_0^t sa(t')\,\mathrm{d}t'\Big)\, Y(0) + \int_0^t \exp\Big(-\int_{t'}^t sa(t'')\,\mathrm{d}t''\Big)\, sb(t')\,\mathrm{d}t'\,. \tag{A.3}$$

The uniform upper bounds on $F(t)$ follow directly.

References

[1] M. Bartuccelli, P. Constantin, C.R. Doering , J.D. Gibbon and M. Gisselfält, On the possibility of soft and hard turbulence in the complex Ginzburg–Landau equation, Physica D 44 (1990) 421–444.

[2] M. Bartuccelli, P. Constantin, C.R. Doering, J.D. Gibbon and M. Gisselfält, Errata: Hard turbulence in a finite dimensional dynamical system, Phys. Lett. A 145 (1990) 476–477.

[3] G.K. Batchelor, The Theory of Homogeneous Turbulence (Cambridge Univ. Press, Cambridge, 1953).

[4] N. Bourbaki, Éléments de Mathématique III, Topologie Générale (Hermann, Paris, 1966) Ch. 10.

[5] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, part I: Schrödinger equations, Geom. Funct. Anal. 3 (1993) 107–156.

[6] J. Bourgain, Exponential sums and nonlinear Schrödinger equations, Geom. Funct. Anal. 3 (1993) 157–178.

[7] T. Cazenave and F. Weissler, The Cauchy Problem for the critical nonlinear Schrödinger equations in H^s, Nonlinear Anal. Theor. Meth. Appl. 14 (1990) 807–836.

[8] P. Constantin and C. Foias, The Navier-Stokes Equations (Chicago Univ. Press, Chicago, 1988).

[9] C.R. Doering, J.D. Gibbon, D.D. Holm and B. Nicolaenko, Low dimensional behavior in the complex Ginzburg–Landau equation, Nonlinearity 1 (1988) 279–309.

[10] J. Ginibre and G. Velo, On a class of nonlinear Schrödinger equations I: The Cauchy problem, General case, Funct. Anal. 32 (1979) 1–32.

[11] R. Glassey, On the blowing-up of solutions to the Cauchy problem for the nonlinear Schrödinger equation, J. Math. Phys. 18 (1977) 1794–1797.
[12] D. Goldman, thesis, Brown University (1993).
[13] M. Goldman, Plasma turbulence, Rev. Mod. Phys. 56 (1984) 709–735.
[14] O. Kavian, A remark on the blowing-up of solutions to the Cauchy problem for nonlinear Schrödinger equations, Trans. Am. Math. Soc. 299 (1987) 193–203.
[15] J. Leray, Sur le mouvement d'un fluide visqueux emplissant l'espace, Acta Math. 53 (1934) 193–248.
[16] B. Luce, thesis, Clarkson University (1993).
[17] F. Merle, Limit behavior of saturated approximations of nonlinear Schrödinger equation, Commun. Math. Phys. 146 (1992) 377–414.
[18] F. Merle and Y. Tsutsumi, L^2-concentration of blow-up solutions for the nonlinear Schödinger equation with critical power nonlinearity, J. Diff. Eq. 84 (1990) 205–214.
[19] I. Rasmussen and K. Rypdal, Blow-up in nonlinear Schrödinger equations I – a general review, Phys. Scr. 33 (1987) 498–504.
[20] L. Schwartz, Functional Analysis, Courant Institute Lecture Notes (New York Univ. Press, New York, 1964).
[21] L. Sirovich and J.D. Rodriguez, Coherent structures and chaos: a model problem, Phys. Lett. A 120 (1988) 211–214.
[22] D. Stark, thesis, University of Arizona (1994).
[23] Y. Tsutsumi, L^2-solutions for NLS and nonlinear groups, Funkcialaj Ekvacioj 30 (1987) 115–125.
[24] F.B. Weissler, Local existence for semilinear parabolic equations in L^p, Indiana U. Math J. 29 (1980) 79–102.
[25] V.E. Zakharov, Collapse of Langmuir waves, Sov. Phys. JETP 35 (1972) 908–914.
[26] V.E. Zakharov, V.V. Sobolev and V.S. Synach, Character of the singularity and the stochastic phenomena of self-focusing, Zh. Eksp. Teor. Fiz. Pis'ma Red 14 (1971) 390–393.