# TRENDS IN TOPOLOGICAL COMBINATORICS

DMITRY N. KOZLOV

Habilitationsschrift zur Vorlage an der philosophisch-naturwissenschaftlichen Fakultät der Universität Bern

Dezember 2001

I was the shadow of the waxwing slain By the false azure in the windowpane; I was the smudge of ashen fluff - and I Lived on, flew on, in the reflected sky.

-Vladimir Nabokov, Pale Fire

## **PREFACE**

This thesis is thought to be reflective of the transformation, mistily rendered in the citation on the previous page, which has taken place in me since I have handed in my Ph.D. thesis in the spring of 1996. If this transformation, not unnatural taking into account that I was 23 at the time, was to the better or to the worse, I leave for the reader to judge.

Now, before indulging into mathematics, I would like to thank my coauthors Eva-Maria Feichtner and Eric Babson, without whom the second, resp. the fifth, chapters would not exist.

In recent years I was supported in various forms by a number of research and educational establishments. I therefore express my graditude to:

University of Bern,

Swiss National Science Foundation,

Forschungsinstitut für Mathematik at ETH Zürich,

The Institute for Advanced Study at Princeton,

Massachusetts Institute of Technology at Cambridge,

Mathematical Science Research Institute at Berkeley,

Royal Institute of Technology at Stockholm,

Swedish Natural Science Research Council.

Finally, I would like to mention again Eva-Maria Feichtner without whose unrequiring support this work would have been a mere Gedankenspiel.

I dedicate this thesis to her.

Bern December 14, 2001 DMITRY N. KOZLOV

## SUMMARY

This thesis opens with an introductory discussion, where the reader is gently led to the world of topological combinatorics, and, where the results of this Habilitationsschrift are portrayed against the backdrop of the broader philosophy of the subject. That introduction is followed by 5 chapters, where the main body of research is presented, and 4 appendices, where various standard tools and notations, which we use throughout the text, are collected.

The main purpose of the first chapter is to introduce a new category, which we call resonance category, whose combinatorics reflects that of canonical stratifications of n-fold symmetric smash products. The study of the stratifications can then be abstracted to the study of functors, which we name resonance functors, satisfying a certain set of axioms.

One frequently studied stratification is that of the set of all degree n polynomials, defined by fixing the allowed multiplicities of roots. We describe how our abstract combinatorial framework helps to yield new information on the homology groups of the strata.

In the second chapter we introduce notions of combinatorial blowups, building sets, and nested sets for arbitrary meet-semilattices. This gives a common abstract framework for the incidence combinatorics occurring in the context of De Concini-Procesi models of subspace arrangements and resolutions of singularities in toric varieties. Our main theorem states that a sequence of combinatorial blowups, prescribed by a building set in linear extension compatible order, gives the face poset of the corresponding simplicial complex of nested sets. As applications we trace the incidence combinatorics through every step of the De Concini-Procesi model construction, and we introduce the notions of building sets and nested sets to the context of toric varieties.

In the third chapter we study rational homology groups of one-point compactifications of spaces of complex monic polynomials with multiple roots. These spaces are indexed by number partitions. A standard reformulation in terms of quotients of orbit arrangements reduces the problem to studying certain triangulated spaces  $X_{\lambda,\mu}$ .

viii Summary

We present a combinatorial description of the cell structure of  $X_{\lambda,\mu}$  using the language of marked forests. This allows us to perform a complete combinatorial analysis of the topological properties of  $X_{\lambda,\mu}$  for several specific cases. As applications we prove that  $\Delta(\Pi_n)/\mathcal{S}_n$  is collapsible, we obtain a new proof of a theorem of Arnol'd, and we find a counterexample to a conjecture of Sundaram and Welker, along with a few other smaller results.

To every directed graph G one can associate a complex  $\Delta(G)$  consisting of directed subforests. This construction, suggested to us by R. Stanley, is especially important in the case of a complete double directed graph  $G_n$ , where it leads to studying certain representations of the symmetric group and relates (via Stanley-Reisner correspondence) to an interesting quotient ring. In the fourth chapter we study complexes  $\Delta(G)$  and associated quotient complexes.

One of our results states that  $\Delta(G_n)$  is shellable, in particular Cohen-Macaulay, which can be further translated to say that the Stanley-Reisner ring of  $\Delta(G_n)$  is Cohen-Macaulay. Besides that, by computing the homology groups of  $\Delta(G)$  for the cases when G is essentially a tree, and when G is a double directed cycle, we touch upon the general question of the interaction of combinatorial properties of a graph and topological properties of the associated complex.

We then continue with the study of the  $S_n$ -quotient of the complex of directed forests, denoted  $\Delta(G_n)/S_n$ . It is a simplicial complex whose cell structure is defined combinatorially. We make use of the machinery of spectral sequences to analyze these complexes. In particular, we prove that the integral homology groups of  $\Delta(G_n)/S_n$  may have torsion.

In the fifth chapter we study quotients of posets by group actions. In order to define the quotient correctly we enlarge the considered class of categories from posets to loopfree categories: categories without nontrivial automorphisms and inverses. We view group actions as certain functors and define the quotients as colimits of these functors. The advantage of this definition over studying the quotient poset (which in our language is the colimit in the poset category) is that the realization of the quotient loopfree category is more often homeomorphic to the quotient of the realization of the original poset. We give conditions under which the quotient commutes with the nerve functor, as well as conditions which guarantee that the quotient is again a poset.

# ZUSAMMENFASSUNG

Wir beginnen mit einer einleitenden Diskussion, die den Leser in die Welt der topologischen Kombinatorik einführt und die Ergebnisse dieser Habilitationsschrift vor den breiteren Hintergrund der das Gebiet bestimmenden Gedankenwelt stellt. Dieser Einleitung folgen fünf Kapitel, in denen wir die Hauptergebnisse entwickeln, und vier Appendizes, in denen wir Standardwerkzeuge und Notationen zusammenstellen, wie sie im Laufe der Arbeit verwendet werden.

Ziel des ersten Kapitels ist es, eine neue Kategorie, die Resonanzkategorie, einzuführen, deren Kombinatorik diejenige kanonischer Stratifizierungen n-facher symmetrischer Smash-Produkte widerspiegelt. Das Studium dieser Stratifizierungen kann so zum Studium sogenannter Resonanzfunktoren abstrahiert werden, die den Axiomen eines bestimmten Axiomensystems genügen.

Eine häufig studierte Stratifikation ist die der Menge von Polynomen festen Grades n nach den unter den Wurzeln auftretenden Vielfachheiten. Wir zeigen, wie wir dank unseres abstrakten kombinatorischen Systems zu neuen Informationen über die Homologie der Strata gelangen.

Im zweiten Kapitel führen wir für beliebige Durchschnittshalbverbände die Begriffe kombinatorischer Aufblasungen, Bausatzmengen und vernetzter Mengen ein. Dies liefert einen gemeinsamen abstrakten Rahmen für Inzidenzen, wie sie einerseits im Zusammenhang mit De Concini-Procesi Modellen von Arrangements, andererseits bei Auflösungen von Singularitäten in torischen Varietäten eine Rolle spielen. Unser Hauptsatz besagt, dass eine Sequenz kombinatorischer Aufblasungen, bestimmt durch eine mittels linearer Erweiterung geordnete Bausatzmenge, auf die Seitenhalbordnung des zugehörigen Simplizialkomplexes vernetzter Mengen führt. Als Anwendung verfolgen wir die Inzidenzen in der schrittweisen Modellkonstruktion nach De Concini und Procesi und führen die Begriffe von Bausatzmengen und vernetzten Mengen in den Zusammenhang torischer Varietäten ein.

Im dritten Kapitel studieren wir rationale Homologiegruppen der 1-Punkt-Kompaktifizierungen von Räumen komplexer normierter Polynome mit mehrfachen Nullstellen. Eine Umformulierung für Quotienten von Bahnenarrangements reduziert das Problem auf das Studium gewisser triangulierter Räume  $X_{\lambda,\mu}$ .

x Summary

Wir geben eine kombinatorische Beschreibung der Zellstruktur von  $X_{\lambda,\mu}$ , indem wir uns der Terminologie sogenannter markierter Bäume bedienen. Dies erlaubt uns eine vollständige Analyse topologischer Eigenschaften von  $X_{\lambda,\mu}$  in verschiedenen Spezialfällen. Als Anwendung beweisen wir einerseits die Kollabierbarkeit von  $\Delta(\Pi_n)/\mathcal{S}_n$ , andererseits erhalten wir, neben anderen Ergebnissen, einen neuen Beweis für einen Satz von Arnol'd und geben ein Gegenbeispiel für eine Vermutung von Sundaram und Welker.

Jedem gerichteten Graphen G kann man einen Komplex  $\Delta(G)$  bestehend aus gerichteten Teilwäldern zuordnen. Diese Konstruktion, die auf einen Vorschlag von R. Stanley zurückgeht, ist besonders wichtig für vollständige, doppelt gerichtete Graphen  $G_n$ . Für solche führt sie auf das Studium bestimmter Darstellungen der symmetrischen Gruppe und entspricht via Stanley-Reisner Korrespondenz interessanten Quotientenringen. Im vierten Kapitel studieren wir die Komplexe  $\Delta(G)$  und zugehörige Quotientenkomplexe. Eines unserer Resultate besagt, dass  $\Delta(G)$  schälbar ist, damit insbesondere Cohen-Macaulay, was wiederum übersetzt werden kann in die Tatsache, dass der Stanley-Reisner-Ring von  $\Delta(G)$  die Cohen-Macaulay Eigenschaft besitzt. Darüberhinaus rühren wir durch Homologieberechnungen von  $\Delta(G)$  in Fällen, in denen G im wesentlichen ein Baum oder ein doppelt gerichteter Kreis ist, an die allgemeinere Frage des Zusammenspiels von kombinatorischen Eigenschaften des Graphen und topologischen Eigenschaften des zugehörigen Komplexes.

Darauffolgend studieren wir den  $S_n$ -Quotienten des Komplexes gerichteter Wälder,  $\Delta(G_n)/S_n$ . Es handelt sich um einen Simplizialkomplex mit kombinatorisch definierter Zellstruktur. Wir benutzen Spektralsequenztechniken, um diese Komplexe zu studieren. Insbesondere zeigen wir, dass Torsion auftreten kann in der ganzzahligen Homologie von  $\Delta(G_n)/S_n$ .

Im fünften Kapitel studieren wir Quotienten von Halbordnungen nach Gruppenoperationen. Um Quotienten sauber definieren zu können, erweitern wir die betrachtete Klasse von Kategorien von Halbordnungen zu schleifenfreien Kategorien: Kategorien mit nicht-trivialen Automorphismen und Inversen. Wir fassen Gruppenaktionen als Funktoren auf und definieren Quotienten als Kolimiten dieser Funktoren. Der Vorteil dieser Definition gegenüber Quotientenhalbordnungen (in unserer Sprache die Kolimiten in der Kategorie der Halbordnungen) ist, dass die Realisierungen von Quotienten schleifenfreier Kategorien häufiger homöomorph sind zu den Quotienten von Realisierungen der ursprünglichen Halbordnung. Wir geben Bedingungen dafür an, dass die Quotientenbildung mit dem Nerv-Funktor vertauschbar ist, und formulieren darüberhinaus Bedingungen, die garantieren, dass der Quotient wiederum eine Halbordnung ist.

# **CONTENTS**

Pr	eface			V
Su	ımma	ry		vii
Zι	ısamı	nenfass	sung	ix
In	trodu	ction		1
PA	ART I	. Comb	inatorial Structures in Topology and Geometry	7
1	The	Resona	ance Category	9
	1.1	Canon	nical stratifications of symmetric smash products	9
		1.1.1	Combinatorial stratifications of topological spaces	9
		1.1.2	The idea of the resonance category and resonance functors	10
	1.2	The re	esonance category	11
		1.2.1	Resonances and their symbolic notation	11
		1.2.2	Acting on cuts with ordered set partitions	13
		1.2.3	The definition of the resonance category and the terminol-	
			ogy for its morphisms	14
	1.3	Structi	ures related to the resonance category	15
		1.3.1	Relative resonances	15
		1.3.2	Direct products of relative resonances	17
		1.3.3	Resonance functors	18
	1.4	First a	pplications	19
		1.4.1	Resonance compatible stratifications	19
		1.4.2	Resonances $(a^k, 1^l)$	20
		1.4.3	Resonances $(a^k, b^l)$	23
	1.5	Seque	ntial resonances	26
		1.5.1	The structure theory of strata associated to sequential res-	
			onances	26
		1.5.2	Resonances $(a^k, b^l, 1^m)$	29
		1.5.3	Resonances consisting of powers	30
	1.6	Comp	lexity of resonances	34

xii Contents

2	Inci	dence Combinatorics of Resolutions	37
	2.1	The motivation for the abstract framework	37
	2.2	Building sets and nested sets of meet-semilattices	38
		2.2.1 Irreducible elements in posets	38
		2.2.2 Building sets	39
		2.2.3 Nested sets	41
	2.3	Sequences of combinatorial blowups	44
		2.3.1 Combinatorial blowups	44
		2.3.2 Blowing up building sets	45
	2.4	De Concini-Procesi models of subspace arrangements	46
		2.4.1 Building sets for subspace arrangements	46
		2.4.2 Local subspace arrangements	48
		2.4.3 Intersection stratification of local arrangements after blowup	50
		2.4.4 Tracing incidence structure during arrangement model	
		construction	51
	2.5	Simplicial resolutions of toric varieties	56
PA	ART I	I. Complexes of Trees and Quotient Constructions	59
3	Spac	ces of Complex Monic Polynomials and Complexes of Forests	61
	3.1	The stratification by root multiplicities of the space of complex	
		monic polynomials	61
	3.2	Orbit arrangements and spaces $X_{\lambda,\mu}$	62
		3.2.1 Reformulation in the language of orbit arrangements	62
		3.2.2 Applying Sundaram-Welker formula	63
		3.2.3 Spaces $X_{\lambda,\mu}$ and their properties	64
	3.3	The cell structure of $X_{\lambda,\mu}$ and marked forests	65
		3.3.1 The terminology of marked forests	65
		3.3.2 The main theorem	67
		3.3.3 Remarks	68
	3.4	A new proof of a theorem of Arnol'd	69
		3.4.1 Formulation of the main theorem and its corollaries	69
		3.4.2 Auxiliary propositions	70
		3.4.3 Proof of Theorem 3.4.2	72
	3.5	On a conjecture of Sundaram and Welker	73
		3.5.1 A counterexample to the general conjecture	73
		3.5.2 Verification of the conjecture in a special case	74
4	Con	plexes of Directed Trees and Their Quotients	75
	4.1	The objects of study and the main questions	75
		The sojets of story with the main questions	

Contents

		4.2.1	Conventions	76
		4.2.2	First examples	76
		4.2.3	Elementary properties	77
	4.3	Graph	s with complete source	77
		4.3.1	Shellability of complexes of directed trees	77
		4.3.2	Algebraic consequences of the shelling	79
		4.3.3	A few words on the related $S_n$ -representations	79
	4.4	Comp	utations for other classes of graphs	79
		4.4.1	Graphs which are essentially trees	79
		4.4.2	Double directed strings	81
		4.4.3	Cycles	82
	4.5	$\mathcal{S}_n$ -que	otients of complexes of directed forests	84
		4.5.1	A combinatorial description for the cell structure of $X_n$ .	84
		4.5.2	Filtration and description of the $E^1$ tableau	86
		4.5.3	$\mathbb{Q}$ coefficients	87
		4.5.4	$\mathbb{Z}$ coefficients	
		4.5.5	Homology groups of $X_n$ for $n = 2, 3, 4, 5, 6 \dots$	90
5	Gro	up Acti	ons on Posets	93
	5.1	Pream	ble	93
	5.2	Forma	dization of group actions and the main question	93
		5.2.1	Preliminaries	93
		5.2.2	Definition of the quotient and formulation of the main	0.4
	<i>5</i> 2	C 1:	problem	
	5.3		tions on group actions	95
		5.3.1	Outline of the results and surjectiveness of the canonical	95
		522	map	
		5.3.2 5.3.3	Conditions for the enterprise to be closed under taking	97
		5.5.5	Conditions for the categories to be closed under taking quotients	99
			quotients	
A	Con	binato	rial Tools	103
	<b>A</b> .1	Numb	er and set partitions	103
	A.2	Graph	s	104
В	Pose	ets and	Related Topological Constructions	105
	B.1	Basic	notions	105
	B.2		complexes of posets	
	B.3		bility	

xiv	Contents

C	Subs	space Arrangements	107
	C.1	Definition and related constructions	107
	C.2	Goresky-MacPherson theorem	107
D	Торо	ological Tools	109
	D.1	Operations on topological spaces	109
	D.2	Spectral sequences	110
Bil	bliogr	aphy	113

#### **Topological Combinatorics - a chapter of Discrete Mathematics**

For the sake of brevity, the theme of this thesis could be defined as **discrete mathematics** or **combinatorics** and its interactions with pure mathematics, in particular with algebraic topology. This subject has long gone either without a name, or subordinated under other titles, which did not fully reflect its philosophy. Lately, the term **topological combinatorics** has been used with rising frequency, so, for the matter of presentation, allow me to extend the use of this name to the present exposition.

Despite some striking applications, see e.g. [Lo78], and the subsequent explosive development, topological combinatorics is still a comparatively young subject. Its main problems (*Fragestellungen*), as well as its methods are still in their forming stages, although a lot of consolidating work has been undertaken in recent years, see e.g. [Bj95]. Let me highlight here two major venues of research.

#### Combinatorial structures in topology and geometry

Succinctly, the philosophy of the subject might be described as follows. The first step consists of singling out a structure, with combinatorial flavor, occurring naturally in topology or geometry. Then, one tries to find a suitable, purely combinatorial definition of this structure. This usually results in combining known, well-studied objects in a way, which would be neither very natural, nor apparent from the original context, in which these objects were defined.

Once this is done, there are two major possibilities to continue. The first option is to develop better structural understanding of the new combinatorial theory, which has just been distiled. That understanding may then be used to refine our knowledge of the original object.

The second option is to seek out the occurrences of this combinatorial object in other topological or geometric situations, where the connections to the original situation are not clear, and at the moment the link is only provided on the abstract combinatorial level. Hopefully, one can then use this new combinatorial connection to better understand the topological objects in question. A few instances of the developments in this spirit are: matroid bundles and combinatorial differential

manifolds of MacPherson, [An99, GeM92, Mac93], complex matroids, [Zi93], or, even such a general concept as that of oriented matroids, [BLSWZ].

The first part of this thesis can be classified into this category of research. In the first chapter we study a circle of problems arising from topology, with some specific instances appearing in, among others, singularity theory. The goal is to describe in a functorial way the combinatorial structure of canonical stratifications of n-fold symmetric smash products. By canonical stratifications we mean those implied by point coincidences, with strata being indexed by number partitions of n.

The approach, which we choose to achieve that objective, is to introduce a new combinatorial object, a new category, which we call *resonance category*. The objects of this category are not number partitions themselves, but rather certain equivalence classes consisting of number partitions, which we call n-cuts. The introduction of these equivalence classes, in a characteristic for the ideology of this subject way, is motivated by the topological fact that strata, which are indexed by number partitions belonging to the same n-cut, are homeomorphic.

Furthermore, substrata of a stratum can be glued so as to obtain a new stratum, and also various strata can be included into each other. To reflect these topological maps combinatorially, we define the morphisms of the resonance category to be compositions of two types of elementary morphisms, called "gluings" and "inclusions," which are defined purely in terms of n-cuts.

The original problem of studying stratifications can be viewed as studying certain functors from the resonance category to the category of pointed topological spaces, **Top**\*. This way we split the problem into two parts: the combinatorial one, encoded by the resonance category, and the topological one, encoded by such a functor.

After that, we go on with the structural study of the resonance category. Another, somewhat technical notion, which we need to introduce is that of *a relative resonance*. The main idea is then to study direct product decomposition properties of relative resonances. The general theory is rather rich, and we have succeeded to understand just a small part of it, but already that was sufficient to perform various concrete computations.

For example, for a certain class of resonances we can produce combinatorial models to explicitly compute algebraic invariants of the corresponding strata. More specifically, we prove that these strata are homotopy equivalent to wedges of spheres. Then, we go ahead to construct weighted directed graphs, whose longest directed paths enumerate these spheres, and such that the total weights of the longest paths give the dimensions of the spheres in question. This provides an explicit algorithm to compute the homotopy type of strata for this class of resonances.

In the second chapter we turn to a somewhat different topic. In [DP95], De Concini & Procesi have constructed certain models for complements of subspace arrangements. The idea of their construction is to turn any given subspace arrangement into a divisor with normal crossings, by means of a sequence of blowups. To use blowups to resolve singularities is a standard idea of algebraic geometry, yet the performance of such a resolution in specific cases often leads to intricate constructions.

One aspect of the De Concini & Procesi (and of the earlier Fulton & MacPherson, [FM94]) construction, which became apparent early on, was the fact that the combinatorics of the situation was far from being trivial or standard material. To deal with the mounting technicalities, De Concini & Procesi generalized the notions of building sets and nested sets, originally coined by Fulton & MacPherson in the more specific context of combinatorics of configuration spaces, to cover the case of subspace arrangements.

In [DP95] these notions were still defined in geometric terms, relying on the subspace arrangements themselves, rather than on their intersection semilattices. In the research, presented in the second chapter, we generalize the notions of building sets and nested sets to the purely combinatorial context of semilattices. Moreover, we introduce a new combinatorial construction, which we call a "combinatorial blowup."

These three notions can then be combined to yield a lattice-theoretic theorem (Theorem 2.3.4), which parallels in statement one of the main theorems of [DP95]. As a result, we obtain a consistent, nontrivial combinatorial theory, which canonically depends on the underlying combinatorial data, and not on the geometric structures which it encodes, with Theorem 2.3.4 as the main structural result.

Now, turning back to geometry, we are able to step-by-step trace the construction of De Concini & Procesi, by using the concepts which we introduced. Furthermore, following the ideas outlined in the beginning of the discussion, we go on to describe how these combinatorial notions also appear in the context of toric varieties.

#### Combinatorially defined algebraic invariants of topological spaces

A somewhat different direction of research is to work on determining explicitly algebraic invariants of topological spaces given by some combinatorial construction. This area has many aspects. One is to derive general formulae for a class of objects. An example of that is provided by the Goresky & MacPherson formula, see Appendix C. Among further instances one could mention the Orlik-Solomon algebra, which describes the cohomology algebra of the complement of a complex hyperplane arrangement, [OS80], or the Yuzvinsky basis for the cohomology algebra of De Concini-Procesi compactifications, [Yu97].

Another example is described in the third chapter, where we give a combinatorial formula to compute the rational Betti numbers of spaces of complex monic polynomials with roots of fixed multiplicities. These spaces can be also viewed as strata in the natural stratification of the n-fold symmetric smash product of  $S^2$ , which were previously described in this exposition. To give this combinatorial description we need to introduce cell complexes, which we call  $X_{\lambda,\mu}$ , which are parameterized by pairs of number partitions  $(\lambda,\mu)$ , such that  $\lambda \vdash \mu$ .

The study of these complexes yields a variety of applications. To start with, we prove collapsibility of the regular CW complex  $\Delta(\Pi_n)/\mathcal{S}_n$ . The partition lattice  $\Pi_n$ , and its order complex  $\Delta(\Pi_n)$ , are of fundamental importance for encoding the combinatorics of configuration spaces, and for computing cohomology of the colored braid group, which motivates the appearance of the complex  $\Delta(\Pi_n)/\mathcal{S}_n$  as a basic object of study.

We also obtain a new, conceptual proof of the Arnol'd Finiteness Theorem, which boils down to a combinatorial fact about marked forests. Furthermore, we are able to furnish a counterexample to a conjecture of Sundaram & Welker concerning the multiplicity of the trivial character in certain  $S_n$ -representations, as well as to compute this multiplicity for  $S_n$ -representations previously studied by R. Stanley, [St82], and P. Hanlon, [Ha83].

In the fourth chapter we turn to a related aspect of this area, namely, the combinatorial analysis of simplicial complexes with explicit combinatorial description of the simplices. Such study facilitates the richness of the theory, and provides a reasonable testing field for the computational methods of topological combinatorics. Some instances of such investigations are various forms of lexicographic shellability, developed by Björner et al. [Bj80, Bj95, BWa83, BWa96, BWa97, Ko97], and the study of topological properties of complexes of not *k*-connected graphs, with implications in knot theory, [BBLSW].

In our case, the focus is on studying complexes of directed subforests of a fixed graph, a problem suggested to us by R. Stanley, [St97]. For the complete graph  $G_n$  we are able to prove that  $\Delta(G_n)$  is shellable, implying that it is homotopy equivalent to a wedge of  $(n-1)^{n-1}$  spheres of dimension n-2. Furthermore, we develop machinery to analyze topological properties of  $\Delta(G)$  for a class of directed graphs, called essentially trees, and perform complete computations in the special cases of cycles and double directed strings.

In the last section of the fourth chapter, we study the quotient complexes  $\Delta(G_n)/\mathcal{S}_n$ . These turn out to be rather complex and, after giving the combinatorial description of the simplicial structure of  $\Delta(G_n)/\mathcal{S}_n$  in terms of marked forests, we perform their analysis by means of the apparatus of spectral sequences. In particular, we prove that the homology groups of  $\Delta(G_n)/\mathcal{S}_n$  with integer coefficients are *not* torsion-free in general.

Finally, in the fifth chapter, we take a look at various quotient constructions which appear in the context of group actions on posets. The quotients appear in the third and the fourth chapters, so a short and abstract study of these constructions is commanded by the natural yearning for completeness.

Since an action of a group G is simply a functor from the one-object-category associated to G, it is natural to ask the quotient to be the colimit of this functor. The three main questions of study, which we concentrate on, are: when are the morphisms of P/G given by the G-orbits of the morphisms of P (a property called "regularity"), when does taking the quotient commute with Quillen's nerve functor  $\Delta$ , and, finally, which classes of categories we generally may get as quotients.

We are able to give simple combinatorial conditions on the group actions, which are equivalent to the desired properties. These conditions are, in some sense, local, which implies that they are frequently verifiable in specific situations, boding well for them being useful in the future.

# PART I

# COMBINATORIAL STRUCTURES IN TOPOLOGY AND GEOMETRY

Now entertain conjecture of a time When creeping murmur and the poring dark Fills the wide vessel of the universe.

-William Shakespeare, Henry V

#### CHAPTER 1

## THE RESONANCE CATEGORY

#### 1.1 CANONICAL STRATIFICATIONS OF SYMMETRIC SMASH PRODUCTS

#### 1.1.1 Combinatorial stratifications of topological spaces

Complicated combinatorial problems often arise when one studies the homological properties of strata in some topological space with a given natural stratification. The examples of such stratifications are numerous. A very simple one is provided by taking the n-fold direct product of a topological space (possibly also taking the quotient with respect to the  $S_n$ -action), stratified by point coincidences. The strata are indexed by set partitions (or number partitions), and the biggest open stratum is a configuration space (ordered or unordered), whose topological properties have been widely studied, see e.g., [FH01, FZ00].

Another example is the stratification of a vector space induced by a subspace arrangement. The strata are all possible intersections of subspaces, they are indexed by the intersection semilattice of the arrangement. The biggest open stratum is the complement of the subspace arrangement, whose topological properties have also been of quite some interest, see e.g., [Bj94, Bj95, GoM88, OS80, Vas94, Zi92, ZZ93].

Of course, in both examples above, the main objective is to study the open stratum, which is the complement of the largest closed stratum. However, it was suggested by Arnol'd in a much more general context, see for example [Ar70a], that in situations of this kind one should study the problem for all closed strata. The main argument in support of this point of view is that there is usually no immediate natural structure on the largest open stratum, while there is one on its complement, also known as discriminant. The structure is simply given by stratification. To put it in philosophical terms: "There is only one way for the point in the stratified space to be good, but there are many different ways for it to be bad". Once some information has been obtained about the closed strata, one can try to find out something about the open stratum by means of some kind of duality.

After this general introduction we would like to describe the specific example which will be of particular importance for this chapter. Let X be a pointed

topological space (we refer to the chosen point as a point at  $\infty$ ), and denote

$$X^{(n)} = X \wedge X \wedge \dots \wedge X / \mathcal{S}_n,$$

where  $\wedge$  is the smash product of pointed spaces. In other words,  $X^{(n)}$  is the set of all unordered collections of n points on X with the infinity point attached in the appropriate way.  $X^{(n)}$  is naturally stratified by point coincidences, and the strata are indexed by the number partitions of n. Note that we consider the closed strata, so, for example, the stratum indexed  $(1,1,\ldots,1)$  is the whole space  $X^{(n)}$ .

If one specifies  $X = S^1$ , resp.  $X = \overset{n}{S^2}$ , one obtains as strata the spaces of all monic real hyperbolic, resp. monic complex, polynomials of degree n with specified root multiplicities. These spaces naturally appear in singularity theory, [AGV85]. Homological invariants of several of these strata were in particular computed by Arnol'd, Shapiro, Sundaram, Welker, Vassiliev, and the author, see [Ar70a, Ko99a, Ko00b, ShW98, SuW97, Vas98].

#### 1.1.2 The idea of the resonance category and resonance functors

The purpose of the research presented in this chapter is to take a different, more abstract look at this set of problems. More specifically, the idea is to introduce a new canonical combinatorial object, independent of the topology of the particular space X, where the combinatorial aspects of these stratifications would be fully reflected. This object is a certain category, which we name **resonance category**. The word resonance stands for certain linear identities valid among parts of the indexing number partition for the particular stratum. The usage of this word was suggested to the author by B. Shapiro, [Sh00].

Having this canonically combinatorially defined category at hand, one then can, for each specific topological space X, view the natural stratification of  $X^{(n)}$  as a certain functor from the resonance category to  $\mathbf{Top}^*$ . These functors satisfy a system of axioms, which we take as a definition of **resonance functors**. The combinatorial structures in the resonance category will then project to the corresponding structures in each specific  $X^{(n)}$ . This opens the door to develop the general combinatorial theory of the resonance category, and then prove facts valid for all resonance functors satisfying some further conditions, such as for example acyclicity of certain spaces.

As the main technical tools to unearth the combinatorial structures in the resonance category, we put forward the notions of relative resonances and direct products (most importantly of a resonance and a relative resonance). Intuitively one can think of the relative resonance as a stratum with a substratum shrunk to the infinity point.

As mentioned above, to illustrate a possible appearance of this abstract framework we choose to use a class of topological spaces which come in particular from singularity theory, and whose topological properties have been studied: spaces of polynomials (real or complex) with prescribed root multiplicities. In particular, in case of strata  $(k^m, 1^t)$ , which were studied in [Ar70a, Ko99a, SuW97] for the complex case, and in [Ko00b, ShW98] for the real case, we demonstrate how the inherent combinatorial structure of the resonance category makes this particular resonance especially "reducible."

Here is the brief summary of the contents of this chapter.

**Section 1.2.** We introduce the notion of resonance category, and describe the structure of its set of morphisms.

**Section 1.3.** We introduce the notions of relative resonances, direct products of relative resonances, and resonance functors.

**Section 1.4.** We formulate the problem of Arnol'd and Shapiro which motivated this research as that concerning a specific resonance functor. Then, we analyze the combinatorial structure of resonances  $(a^k, b^l)$ , which leads to the complete determination of homotopy types of the corresponding strata for  $X = S^1$ .

Section 1.5. We analyze the combinatorial structure of sequential resonances. For  $X = S^1$ , this leads to a complete computation of homotopy types of the strata corresponding to resonances  $(a^k, b^l, 1^m)$ , such that  $a - bl \le m$ , as well as resonances consisting of powers of some number. In the case of the latter, the strata always have the homotopy type of a bouquet of spheres. We describe a combinatorial model to enumerate these spheres as paths in a certain weighted directed graph, with dimensions of the spheres being given by the total weights of the paths.

**Section 1.6.** We introduce the notion of a complexity of a resonance and give a series of examples of resonances having arbitrarily high complexity.

#### 1.2 The resonance category

#### 1.2.1 Resonances and their symbolic notation

For every positive integer n, let  $\{-1,0,1\}^n$  denote the set of all points in  $\mathbb{R}^n$  with coordinates in the set  $\{-1,0,1\}$ . We say that a subset  $S \subseteq \{-1,0,1\}^n$  is **span-closed** if span  $(S) \cap \{-1,0,1\}^n = S$ , where span (S) is the linear subspace spanned by the origin and points in S. Of course the origin lies in every span-closed set. For  $x = (x_1, \dots, x_n) \in \{-1,0,1\}^n$ , we use the notations Plus  $(x) = \{i \in [n] \mid x_i = 1\}$  and Minus  $(x) = \{i \in [n] \mid x_i = -1\}$ .

#### **Definition 1.2.1**

- (1) A subset  $S \subseteq \{-1,0,1\}^n$  is called an n-cut if it is span-closed and for every  $x \in S \setminus \{origin\}$  we have  $Plus(x) \neq \emptyset$  and  $Minus(x) \neq \emptyset$ . We denote the set of all n-cuts by  $\mathcal{R}_n$ .
- (2)  $S_n$  acts on  $\{-1,0,1\}^n$  by permuting coordinates, which in turn induces  $S_n$ -action on  $R_n$ . The n-resonances are defined to be the orbits of the latter  $S_n$ -action. We let [S] denote the n-resonance represented by the n-cut S.

The cut or the resonance consisting of origin only is called *trivial*.

#### **Examples 1.2.2** n-resonances for small values of n.

- (1) There are no nontrivial 1-resonances.
- (2) There is one nontrivial 2-resonance:  $[\{(0,0),(1,-1),(-1,1)\}]$ .
- (3) There are four nontrivial 3-resonances:

$$[\{(0,0,0),(1,-1,0),(-1,1,0)\}],$$
 
$$[\{(0,0,0),(1,-1,0),(-1,1,0),(1,0,-1),(-1,0,1),(0,1,-1),(0,-1,1)\}],$$
 
$$[\{(0,0,0),(1,-1,-1),(-1,1,1)\}],$$
 
$$[\{(0,0,0),(1,-1,-1),(-1,1,1),(0,1,-1),(0,-1,1)\}].$$

(4) Here is an example of a nontrivial 6-resonance:

$$\{origin, \pm(1, 1, 0, -1, -1, 0), \pm(0, 1, 1, 0, -1, -1), \pm(1, 0, -1, -1, 0, 1)\}$$

**Symbolic notation.** To describe an n-resonance, rather than to list all of the elements of one of its representatives, it is more convenient to use the following symbolic notation: we write a sequence of n linear expressions in some number (between 1 and n) of parameters, the order in which the expressions are written is inessential.

Here is how to get from such a symbolic expression to the n-resonance: choose an order on the n linear expressions and observe that now they parameterize some linear subspace of  $\mathbb{R}^n$ , which we denote by A. The n-resonance is now the orbit of  $A^{\perp} \cap \{-1,0,1\}^n$ .

Reversely, to go from an n-resonance to a symbolic expression: choose a representative n-cut S, the symbolic expression can now be obtained as a linear parameterization of span  $(S)^{\perp}$ .

For example the 6 nontrivial resonances listed in the Example 1.2.2 are (in the same order):

$$(a, a), (a, a, b), (a, a, a), (a+b, a, b), (2a, a, a), (a+b, b+c, a+d, b+d, c+d, 2d).$$

#### 1.2.2 Acting on cuts with ordered set partitions

From now on we assume known the terminology and notations of set partitions and ordered set partitions, as described in the Appendix A.

**Definition 1.2.3** Given  $\pi = (\pi_1, \dots, \pi_k)$  an ordered set partition of [m] with k parts, and  $\nu = (\nu_1, \dots, \nu_m)$  an ordered set partition of [n] with m parts, their **composition**  $\pi \circ \nu$  is an ordered set partition of [n] with k parts, defined by  $\pi \circ \nu = (\mu_1, \dots, \mu_k)$ ,  $\mu_i = \bigcup_{j \in \pi_i} \nu_j$ , for  $i = 1, \dots, k$ .

Analogously, we can define  $\pi \circ \nu$  for an ordered set partition  $\nu$  and a set partition  $\pi$ , in which case  $\pi \circ \nu$  is a set partition without any specified order on the blocks.

In particular, when m=n, and  $|\pi_i|=1$ , for  $i=1,\ldots,n$ , we can identify  $\pi=(\pi_1,\ldots,\pi_n)$  with the corresponding permutation of [n]. The composition of two such ordered set partitions corresponds to the multiplication of corresponding permutations, and we denote the ordered set partition  $(\{1\},\ldots,\{n\})$  by  $\mathrm{id}_n$ , or just id.

**Definition 1.2.4** For  $A \subseteq B$ , let  $p_{B,A} : P(B) \to P(A)$  denote map induced by the restriction from B to A. For two disjoint set A and B, and  $\Pi \subseteq P(A)$ ,  $\Lambda \subseteq P(B)$ , we define  $\Pi \times \Lambda = \{\pi \in P(A \cup B) \mid p_{A \cup B, A}(\pi) \in \Pi, p_{A \cup B, B}(\pi) \in \Lambda\}$ .

The following definition provides the combinatorial constructions necessary to describe the morphisms of the resonance category, as well as to define the relative resonances.

#### **Definition 1.2.5** Assume S is an n-cut.

- (1) For an ordered set partition of [n],  $\pi = (\pi_1, \dots, \pi_m)$ , we define  $\pi S \in \mathcal{R}_m$  to be the set of all m-tuples  $(t_1, \dots, t_m) \in \{-1, 0, 1\}^m$ , for which there exists  $(s_1, \dots, s_n) \in S$ , such that for all  $j \in [m]$ , and  $i \in \pi_j$ , we have  $s_i = t_j$ .
- (2) For an unordered set partition of [n],  $\pi = (\pi_1, \ldots, \pi_m)$ , we define  $S^{\pi} \in \mathcal{R}_n$  to be the subset of S consisting of all  $(s_1, \ldots, s_n)$  such that if  $i, j \in \pi_k, i \neq j$ , for some  $k = 1, \ldots, m$ , then  $s_i = s_j \neq 0$ ; in other words if  $k = 1, \ldots, m$  is such that  $|\pi_k| \geq 2$ , then either  $\pi_k \subseteq Plus(s_1, \ldots, s_n)$  or  $\pi_k \subseteq Minus(s_1, \ldots, s_n)$ .

Clearly idS = S, and one can see that  $(\pi \circ \nu)S = \pi(\nu S)$ .

Verification of  $(\pi \circ \nu)S = \pi(\nu S)$ .

By definition we have

$$(\pi \circ \nu)S = \{(t_1, \dots, t_k) \mid \exists (s_1, \dots, s_n) \in S \text{ s.t. } \forall j \in [k], i \in \mu_j : s_i = t_j \},$$
$$\nu S = \{(x_1, \dots, x_m) \mid \exists (s_1, \dots, s_n) \in S \text{ s.t. } \forall q \in [m], i \in \nu_q : s_i = x_q \},$$

$$\pi(\nu S) = \{ (t_1, \dots, t_k) \mid \exists (x_1, \dots, x_m) \in \nu S, \\ \text{such that } \forall j \in [k], q \in \pi_j, i \in \nu_q : s_i = t_j \}.$$

The identity  $(\pi \circ \nu)S = \pi(\nu S)$  follows now from the equality  $\mu_i = \bigcup_{q \in \pi_i} \nu_q$ .

There are many different ways to formulate the Definition 1.2.5. We chose the ad hoc combinatorial language, but it is also possible to put it in the linear-algebraic terms. An ordered set partition of [n],  $\pi = (\pi_1, \ldots, \pi_m)$ , defines an inclusion map  $\phi: \mathbb{R}^m \to \mathbb{R}^n$  by  $\phi(e_i) = \sum_{j \in \pi_i} \tilde{e}_j$ , where  $\{e_1, \ldots, e_m\}$ , resp.  $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$ , is the standard orthonormal basis of  $\mathbb{R}^m$ , resp.  $\mathbb{R}^n$ . Given  $S \in \mathcal{R}_n$ ,  $\pi S$  can then be defined as  $\phi^{-1}(\operatorname{Im} \phi \cap S)$ . Furthermore,  $S^{\pi} = \operatorname{Im} \phi|_Z \cap S$ , where Z is the set of all  $(z_1, \ldots, z_m) \in \mathbb{R}^m$ , such that if  $|\pi_k| \geq 2$ , for some  $k = 1, \ldots, m$ , then  $z_k \neq 0$ .

1.2.3 The definition of the resonance category and the terminology for its morphisms

We are now ready to give the definition of the central notion of this chapter.

**Definition 1.2.6** *The* **resonance category**, *denoted* R, *is defined as follows:* 

- (1) The set of objects is the set of all n-cuts, for all positive integers n,  $\mathcal{O}(\mathcal{R}) = \bigcup_{n=1}^{\infty} \mathcal{R}_n$ .
- (2) The set of morphisms is indexed by triples  $(S, T, \pi)$ , where  $S \in \mathcal{R}_m$ ,  $T \in \mathcal{R}_n$ , and  $\pi$  is an ordered set partition of [n] with m parts, such that  $S \subseteq \pi T$ . For the reasons which will become clear later we denote the morphism indexed with  $(S, T, \pi)$  by  $S \to \pi T \stackrel{\pi}{\hookrightarrow} T$ .

As the notation suggests, the initial object of the morphism  $S \to \pi T \stackrel{\pi}{\hookrightarrow} T$  is S and terminal object is T. The composition rule is defined by

$$(S \twoheadrightarrow \pi T \stackrel{\pi}{\hookrightarrow} T) \circ (T \twoheadrightarrow \nu Q \stackrel{\nu}{\hookrightarrow} Q) = S \twoheadrightarrow \pi \nu Q \stackrel{\pi \nu}{\hookrightarrow} Q,$$

where  $S \in \mathcal{R}_k$ ,  $T \in \mathcal{R}_m$ ,  $Q \in \mathcal{R}_n$ ,  $\pi$  is an ordered set partition of [m] with k parts, and  $\nu$  is an ordered set partition of [n] with m parts.

An alert reader will notice that the resonances themselves did not appear explicitly in the definition of the resonance category. In fact, it is not difficult to notice that resonances are isomorphism classes of objects of  $\mathcal{R}$ . Let us now look at the set of morphisms of  $\mathcal{R}$  in some more detail.

- (1) For  $S \in \mathcal{R}_n$ , the identity morphism of S is  $S \twoheadrightarrow S \stackrel{\text{id}}{\hookrightarrow} S$ .
- (2) Let us introduce short hand notations:  $S \to T$  for  $S \to T \stackrel{\text{id}}{\hookrightarrow} T$ , and  $\pi T \stackrel{\pi}{\hookrightarrow} T$  for  $\pi T \to \pi T \stackrel{\pi}{\hookrightarrow} T$ . Then we have

$$S \twoheadrightarrow \pi T \stackrel{\pi}{\hookrightarrow} T = (S \twoheadrightarrow \pi T) \circ (\pi T \stackrel{\pi}{\hookrightarrow} T).$$

Note also that  $S woheadrightarrow S = S \overset{\text{id}}{\hookrightarrow} S$ .

(3) The associativity of the composition rule can be derived from the commutation relation

$$(\pi S \stackrel{\pi}{\hookrightarrow} S) \circ (S \twoheadrightarrow T) = (\pi S \twoheadrightarrow \pi T) \circ (\pi T \stackrel{\pi}{\hookrightarrow} T)$$

as follows:

$$(S \twoheadrightarrow \pi T \hookrightarrow T) \circ (T \twoheadrightarrow \nu Q \hookrightarrow Q) \circ (Q \twoheadrightarrow \rho X \hookrightarrow X) =$$

$$(S \twoheadrightarrow \pi T) \circ (\pi T \hookrightarrow T) \circ (T \twoheadrightarrow \nu Q) \circ (\nu Q \hookrightarrow Q) \circ (Q \twoheadrightarrow \rho X) \circ (\rho X \hookrightarrow X) =$$

$$(S \twoheadrightarrow \pi T) \circ (\pi T \twoheadrightarrow \pi \nu Q) \circ (\pi \nu Q \twoheadrightarrow \pi \nu \rho X) \circ$$

$$(\pi \nu \rho X \hookrightarrow \nu \rho X) \circ (\nu \rho X \hookrightarrow \rho X) \circ (\rho X \hookrightarrow X).$$

(4) We shall use the following names: morphisms S woheadrightarrow T are called **gluings** (or n-gluings, if it is specified that  $S, T \in \mathcal{R}_n$ ); morphisms  $\pi T \overset{\pi}{\hookrightarrow} T$  are called **inclusions** (or (n, m)-inclusions, if it is specified that  $T \in \mathcal{R}_n$ ,  $\pi T \in \mathcal{R}_m$ ), the inclusions are called **symmetries** if  $\pi$  is a permutation. As observed above, the symmetries are the only isomorphisms in  $\mathcal{R}$ . Here are two examples of inclusions:

$$\{(0,0), (1,-1), (-1,1)\} \overset{(\{1\},\{2,3\})}{\hookrightarrow} \{(0,0,0), (1,-1,-1), (-1,1,1)\},$$

$$\{(0,0), (1,-1), (-1,1)\} \overset{(\{1\},\{2,3\})}{\hookrightarrow} \{(0,0,0), \pm (1,-1,-1), \pm (0,1,-1)\}.$$

#### 1.3 STRUCTURES RELATED TO THE RESONANCE CATEGORY

#### 1.3.1 Relative resonances

Let A(n) denote the set of all collections of non-empty multisubsets of [n], and let  $P(n) \subseteq A(n)$  be the set of all partitions of [n]. For every  $S \in \mathcal{R}_n$  let us define a closure operation on A(n), resp. on P(n).

**Definition 1.3.1** *Let*  $A \in A(n)$ . We define  $A \Downarrow S \subseteq A(n)$  to be the minimal set satisfying the following conditions:

- (1)  $A \in A \Downarrow S$ ;
- (2) if  $\{B_1, B_2, \dots, B_m\} \in A \Downarrow S$ , then  $\{B_1 \cup B_2, B_3, \dots, B_m\} \in A \Downarrow S$ ;
- (3) if  $\{B_1, B_2, \dots, B_m\} \in \mathcal{A} \Downarrow S$ , and there exists  $x \in S$ , such that Plus  $(x) \subseteq B_1$ , then  $\{(B_1 \setminus Plus(x)) \cup Minus(x), B_2, \dots, B_m\} \in \mathcal{A} \Downarrow S$ .

For  $\pi \in P(n)$ , we define  $\pi \downarrow S \subseteq P(n)$  as  $\pi \downarrow S = (\pi \Downarrow S) \cap P(n)$ . For a set  $\Pi \subseteq P(n)$  we define  $\Pi \downarrow S = \bigcup_{\pi \in \Pi} \pi \downarrow S$ . We say that  $\Pi$  is S-closed if  $\Pi \downarrow S = \Pi$ .

The idea behind this definition comes from the context of the standard stratification of the n-fold symmetric product. Given a stratum X indexed by a number partition of n with m parts, let us fix some order on the parts. A substratum Y is obtained by choosing some partition  $\pi$  of [m] and summing the numbers within the blocks of  $\pi$ . Since the order of the parts of the number partition indexing X is fixed, X gives rise to a unique m-cut S. The set  $\pi \downarrow S$  describes all partitions  $\nu$  of [m] such that if the numbers within the blocks of  $\nu$  are summed then the obtained stratum Z satisfies  $Z \subseteq Y$ . In particular, if Y is shrunk to a point, then so is Z. The two following examples illustrate how the different parts of the Definition 1.3.1 might be needed.

**Example 1.3.2** The equivalences of type (2) from the Definition 1.3.1 are needed. Let the stratum X be indexed by (3,2,1,1,1) (fix this order of the parts), and let  $\pi = \{1\}\{23\}\{4\}\{5\}$ . Then, the stratum Y is indexed by (3,3,1,1). Clearly, the stratum Z, which is indexed by (3,3,2), lies inside Y, hence  $\{1\}\{2\}\{345\} \in \pi \downarrow S$ , where S is the cut corresponding to (3,2,1,1,1). However, if one starts from the partition  $\pi$  and uses equivalences of type (3) from the Definition 1.3.1, the only other partitions one can obtain are  $\{1\}\{24\}\{3\}\{5\}$ , and  $\{1\}\{25\}\{3\}\{4\}$ . None of them refines  $\{1\}\{2\}\{345\}$ , hence it would not be enough in the Definition 1.3.1 to just take the partitions which can be obtained via the equivalences of type (3) and then take  $\pi \downarrow S$  to be the set of all the partitions which are refined by these.

**Example 1.3.3** It is necessary to view the equivalence relation on the larger set A(n). This time, let the stratum X be indexed by (a+b,b+c,a+d,b+d,c+d,2d) (fix this order of the parts, and assume as usual that there are no linear relations on the parts other than those induced by the algebraic identities on the variables a, b, c, and d). Furthermore, let  $\pi = \{16\}\{23\}\{45\}$ . Then the stratum Y is indexed by (a+b+2d,a+b+c+d,b+c+2d). Clearly, we have  $\{34\}\{15\}\{26\} \in \pi \downarrow S$ , where S is the cut corresponding to (a+b,b+c,a+d,b+d,c+d,2d).

A natural idea for the Definition 1.3.1 could have been to define the equivalence relation directly on the set P(n) and use "swaps" instead of the equivalences of type (3), i.e., to replace the condition (3) by:

```
if \{B_1, B_2, \ldots, B_m\} \in \mathcal{A} \downarrow S, and there exists x \in S, such that Plus(x) \subseteq B_1, and Minus(x) \subseteq B_2, then \{(B_1 \setminus Plus(x)) \cup Minus(x), (B_2 \setminus Minus(x)) \cup Plus(x), B_3, \ldots, B_m\} \in \mathcal{A} \downarrow S.
```

However, this would not have been sufficient as the Example 1.3.3 shows, since no swaps would be possible on  $\pi = \{16\}\{23\}\{45\}$ .

**Definition 1.3.4** Let S be an n-cut,  $\Pi \subseteq P(n)$  an S-closed set of partitions. We define

$$S \setminus \Pi = S \setminus \bigcup_{\pi \in \Pi} S^{\pi}.$$

In the next definition we give a combinatorial analog of viewing a stratum relative to a substratum.

#### **Definition 1.3.5**

(1) A relative n-cut is a pair  $(S,\Pi)$ , where  $S \subseteq \{-1,0,1\}^n$ ,  $\Pi \subseteq P(n)$ , such that the following two conditions are satisfied:

- $(span S) \setminus \Pi = S$ ;
- $\Pi$  is (span S)-closed.

(2) The permutation  $S_n$ -action on  $\{-1,0,1\}^n$  induces an  $S_n$ -action on the relative n-cuts by  $(S,\Pi) \stackrel{\sigma}{\mapsto} (\sigma S, \Pi \sigma^{-1})$ , for  $\sigma \in S_n$ . The **relative** n-resonances are defined to be the orbits of this  $S_n$ -action. We let  $[S,\Pi]$  denote the relative n-resonance represented by the relative n-cut  $(S,\Pi)$ .

When  $S \in \mathcal{R}_n$  and  $\Pi \subseteq P(n)$ ,  $\Pi$  is S-closed, it is convenient to use the notation  $Q(S,\Pi)$  to denote the relative cut  $(S \setminus \Pi,\Pi)$ . Clearly we have  $(S,\Pi) = Q(\operatorname{span} S,\Pi)$ . Analogously,  $[Q(S,\Pi)]$  denotes the relative resonance  $[S \setminus \Pi,\Pi]$ . We use these two notations interchangeably depending on which one is more natural in the current context.

The special case of the particular importance for our computations in the later sections is that of  $Q(S, \pi \downarrow S)$ , where  $\pi$  is a partition of [n] with m parts. In this case, we call  $(S \setminus (\pi \downarrow S), \pi \downarrow S)$  the relative (n, m)-cut associated to S and  $\pi$ .

By the Definition 1.3.5, the relative cut  $(S,\Pi) = ((\operatorname{span} S) \setminus \Pi,\Pi)$  consists of two parts. We intuitively think of  $(\operatorname{span} S) \setminus \Pi$  as the set of all resonances which survive the shrinking of the strata associated to the elements of  $\Pi$ , so it is natural to call them *surviving elements*. We also think of  $\Pi$  as the set of all partitions whose associated strata are shrunk to the infinity point, so, accordingly, we call them *partitions at infinity*.

#### 1.3.2 Direct products of relative resonances

**Definition 1.3.6** For relative resonances  $(S, \Pi)$  and  $(T, \Lambda)$  we define

$$(S,\Pi) \times (T,\Lambda) = (S \times T, (\Pi \times P(m)) \cup (P(n) \times \Lambda)).$$

Clearly the orbit  $[(S,\Pi)\times(T,\Lambda)]$  does not depend on the choice of representatives of the orbits  $[S,\Pi]$  and  $[T,\Lambda]$ , so we may define  $[S,\Pi]\times[T,\Lambda]$  to be  $[(S,\Pi)\times(T,\Lambda)]$ .

The following special cases are of particular importance for our computation:

#### (1) A direct product of two resonances.

For an *m*-cut *S*, and an *n*-cut *T*, we have 
$$S \times T = \{(x_1, ..., x_m, y_1, ..., y_n) | (x_1, ..., x_m) \in S, (y_1, ..., y_n) \in T\} \in \mathcal{R}_{m+n}$$
, and  $[S] \times [T] = [S \times T]$ .

#### (2) A direct product of a relative resonance and a resonance.

For  $S \in \mathcal{R}_n$ ,  $\Pi \subseteq P(n)$  an S-closed set of partitions, and  $T \in \mathcal{R}_k$ , we have  $Q(S,\Pi) \times T = Q(S \times T,\widetilde{\Pi})$ , where  $\widetilde{\Pi} = \Pi \times P(\{n+1,n+2,\ldots,n+k\})$ , and  $[Q(S,\Pi)] \times [T] = [Q(S,\Pi) \times T]$ .

#### **Example 1.3.7**

$$Q(\{(0,0,0),\pm(1,-1,-1),\pm(0,1,-1)\},\{1\}\{23\}) = \{(0)\} \times Q(\{(0,0),\pm(1,-1)\},\{12\}).$$

**Remark 1.3.8** One can define a category, called **relative resonance category**, whose set of objects is the set of all relative n-cuts. A new structure which it has in comparison to  $\mathcal{R}$  is provided by "shrinking morphisms":  $(S,\Pi) \leadsto (T,\Lambda)$ , for  $S,T \subseteq \{-1,0,1\}^n$ ,  $P(n) \supseteq \Lambda \supseteq \Pi$ , such that  $(\operatorname{span} S) \setminus \Lambda = T$ . They correspond to shrinking strata to infinity. The full definition with relations on morphisms and the corresponding combinatorial analysis, will appear in [Ko01c].

#### 1.3.3 Resonance functors

Given a functor  $\mathcal{F}: \mathcal{R} \longrightarrow \mathbf{Top}^*$ , we introduce the following notation:

$$\mathcal{F}(Q(S,\Pi)) = \mathcal{F}(S) \bigg/ \bigcup_{\text{un } (\pi) \in \Pi} \text{Im } \mathcal{F}(\pi S \overset{\pi}{\hookrightarrow} S).$$

**Definition 1.3.9** A functor  $\mathcal{F}: \mathcal{R} \longrightarrow \mathbf{Top}^*$  is called a **resonance functor** if it satisfies the following axioms:

#### (A1) Inclusion axiom.

If  $S \in \mathcal{R}_n$ , and  $\pi \in OP(n)$ , then  $\mathcal{F}(\pi S \xrightarrow{\pi} S)$  is an inclusion map, and  $\mathcal{F}(S)/\operatorname{Im} \mathcal{F}(\pi S \xrightarrow{\pi} S) \simeq \mathcal{F}(Q(S, \pi \downarrow S))$ .

#### (A2) Relative resonance axiom.

If, for some  $S, T \in \mathcal{R}_n$ , and  $\Pi, \Lambda \subseteq P(n)$ ,  $[Q(S, \Pi)] = [Q(T, \Lambda)]$ , then  $\mathcal{F}(Q(S, \Pi)) \simeq \mathcal{F}(Q(T, \Lambda))$ .

#### (A3) Direct product axiom.

For two relative n-cuts  $(S,\Pi)$  and  $(T,\Lambda)$  we have

$$\mathcal{F}(S,\Pi) \times \mathcal{F}(T,\Lambda) \simeq \mathcal{F}((S,\Pi) \times (T,\Lambda)).$$

Given  $S \in \mathcal{R}_n$ , and  $\pi \in OP(n)$ , let  $i_{S,\pi}$  denote the inclusion map  $\mathcal{F}(\pi S \stackrel{\pi}{\hookrightarrow} S)$ . There is a canonical homology long exact sequence associated to the triple

$$\mathcal{F}(\pi S) \stackrel{i_{S,\pi}}{\hookrightarrow} \mathcal{F}(S) \stackrel{p}{\longrightarrow} \mathcal{F}(Q(S, \pi \downarrow S)),$$
 (1.3.1)

namely

$$\dots \xrightarrow{\partial_*} \widetilde{H}_n(\mathcal{F}(\pi S)) \xrightarrow{(i_{S,\pi})_*} \widetilde{H}_n(\mathcal{F}(S)) \xrightarrow{p_*} \widetilde{H}_n(\mathcal{F}(Q(S,\pi \downarrow S))) \xrightarrow{\partial_*} \cdots \widetilde{H}_{n-1}(\mathcal{F}(\pi S)) \xrightarrow{(i_{S,\pi})_*} \dots (1.3.2)$$

We call (1.3.1), resp. (1.3.2), the *standard triple*, resp. the *standard long exact sequence* associated to the morphism  $\pi S \stackrel{\pi}{\hookrightarrow} S$  and the functor  $\mathcal{F}$  (usually  $\mathcal{F}$  is fixed, so its mentioning is omitted).

#### 1.4 FIRST APPLICATIONS

#### 1.4.1 Resonance compatible stratifications

As mentioned in the Section 1.1.1 we shall now look at the natural strata of the spaces  $X^{(n)}$ . The strata are defined by point coincidences and are indexed by number partitions of n. Let  $\Sigma_{\lambda}^{X}$  denote the stratum indexed by  $\lambda$ .

Let  $\lambda$  be a number partition of n and let  $\lambda \in OP(n)$  be  $\lambda$  with some fixed order on the parts. Then  $\tilde{\lambda}$  can be thought of as a vector with positive integer coordinates in  $\mathbb{R}^n$ . Let  $S_{\tilde{\lambda}}$  be the set  $\{x \in \{-1,0,1\}^n \mid \langle x,\tilde{\lambda}\rangle = 0\}$ . Obviously,  $S_{\tilde{\lambda}}$  is an n-cut, and the n-resonance  $S_{\lambda}$ , which it defines, does not depend on the choice of  $\tilde{\lambda}$ , but only on the number partition  $\lambda$ .

The crucial topological observation is that if  $\nu$  is another partition of n, such that  $S_{\lambda} = S_{\nu}$ , then the spaces  $\Sigma_{\lambda}^{X}$  and  $\Sigma_{\nu}^{X}$  are homeomorphic. This is precisely the fact which lead us to introduce resonances and the surrounding combinatorial framework and to forget about the number partitions themselves.

That observation allows us to introduce a functor  $\mathcal{F}$  mapping  $S_{\tilde{\lambda}}$  to  $\Sigma_{\lambda}^{X}$ ; the morphisms map accordingly. Clearly,  $\mathcal{F}(1^{l}) = X^{(l)}$ . One can detect in this example the justification for the names which we chose for the morphisms of  $\mathcal{R}$ : "inclusions" and "gluings". Furthermore, it is easy, in this case, to verify the axioms of the Definition 1.3.9, and hence to conclude that  $\mathcal{F}$  is a resonance functor.

The only nontrivial point is the verification of the second part of (A1), which we do in the next proposition.

**Proposition 1.4.1** *Let* S *be an* n-*cut and*  $\pi \in OP(n)$ . Then  $\mathcal{F}(\nu S) \subseteq \mathcal{F}(\pi S)$  if and only if  $un(\nu) \in un(\pi) \downarrow S$ .

**Proof.** It is obvious that all the steps of the definition of un  $(\pi) \downarrow S$  which change the partition preserve the property  $\mathcal{F}(\nu S) \subseteq \mathcal{F}(\pi S)$ , hence the *if* direction follows.

Assume now  $\mathcal{F}(\nu S)\subseteq \mathcal{F}(\pi S)$ . This means that there exists  $\tau\in OP(m)$ , where m is the number of parts of  $\pi$ , such that  $\mathcal{F}(\tau\pi S)=\mathcal{F}(\nu S)$ . By definition,  $\tau\circ\pi\in\mathrm{un}\,(\pi)\downarrow S$ . Now, we can reach  $\mathrm{un}\,(\nu)$  from  $\mathrm{un}\,(\tau\circ\pi)$  by moves of type (3) from the definition of the relative resonances.

Indeed, if  $\mathcal{F}(\tau\pi S) = \mathcal{F}(\nu S) = \Sigma_{\lambda}^{X}$ , then the sizes of the resulting blocks after gluing along  $\tau \circ \pi$  and along  $\nu$  are the same. For every block b of  $\lambda$  we can go, by means of moves of type (3), from the block of un  $(\tau \circ \pi)$  which glues to b to the block of un  $(\nu)$  which glues to b. Since we can do it for any block of  $\lambda$ , we can go from un  $(\tau \circ \pi)$  to un  $(\nu)$ , and hence un  $(\nu) \in \text{un }(\pi) \downarrow S$ .  $\square$ 

In the context of this stratification the following central question arises.

**The Main Problem.** (Arnol'd, Shapiro, [Sh00]). Describe an algorithm which, for a given resonance  $\lambda$ , would compute the Betti numbers of  $\Sigma_{\lambda}^{S^1}$ , or  $\Sigma_{\lambda}^{S^2}$ .

The case of the strata  $\Sigma_{\lambda}^{S^1}$  is simpler, essentially because of the following elementary, but important property of smash products: if X and Y are pointed spaces and X is contractible, then  $X \wedge Y$  is also contractible.

In the subsequent subsections we shall look at a few interesting special cases, and also will be able to say a few things about the general problem.

# 1.4.2 Resonances $(a^k, 1^l)$

Let a,k,l be positive integers such that  $a\geq 2$ . Let S be the (l+k)-cut consisting of all the elements of  $\{-1,0,1\}^{l+k}$ , which are orthogonal to the vector  $(\underbrace{1,\ldots,1}_{k},\underbrace{a,\ldots,a}_{k})$ . Clearly, the (l+k)-resonance [S] is equal to  $(a^k,1^l)$ . The

case l < a is not very interesting, since then  $(a^k, 1^l) = (1^k) \times (1^l)$ . Therefore we may assume that  $l \ge a$ .

We would like to understand the topological properties of the space  $\mathcal{F}(a^k,1^l)$ . In general, this is rather hard. However, as the following theorem shows, it is possible under some additional conditions on  $\mathcal{F}$ .

**Theorem 1.4.2** Let  $\mathcal{F}: \mathcal{R} \longrightarrow \mathbf{Top}^*$  be a resonance functor, such that  $\mathcal{F}(1^l)$  is contractible for  $l \geq 2$ . Let  $l = am + \epsilon$ , where  $0 \leq \epsilon \leq a - 1$ .

- (a) If  $k \neq 1$ , or  $\epsilon \geq 2$ , then  $\mathcal{F}(a^k, 1^l)$  is contractible.
- (b) If k = 1, and  $\epsilon \in \{0, 1\}$ , then

$$\mathcal{F}(a^k, 1^l) \simeq susp^m(\mathcal{F}(1)^{m+\epsilon+1}),$$
 (1.4.1)

where  $\mathcal{F}(1)^{m+\epsilon+1}$  denotes the  $(m+\epsilon+1)$ -fold smash product.

Since for the resonance functor  $\mathcal{F}$  described in the subsection 1.4.1 we have  $\mathcal{F}(1^l) = X^{(l)}$ , we have the following corollary.

**Corollary 1.4.3** If  $X^{(l)}$  is contractible for  $l \geq 2$ , then

- (a) If  $k \neq 1$ , or  $\epsilon \geq 2$  (again  $l = am + \epsilon$ ), then  $\Sigma_{(a^k, 1^l)}^X$  is contractible.
- (b) If k=1, and  $\epsilon\in\{0,1\}$ , then  $\Sigma_{(a^k,1^l)}^X\simeq susp^m(X^{m+\epsilon+1})$ , where  $X^{m+\epsilon+1}$  denotes the  $(m+\epsilon+1)$ -fold smash product.

**Remark 1.4.4** Clearly,  $(S^1)^{(l)}$  is contractible for  $l \ge 2$ , so the Corollary 1.4.3 is valid. In this situation, the case k > 1 was proved in [Ko00b], and the case k = 1 in [BWa97, ShW98].

Before we proceed with proving the Theorem 1.4.2 we need a crucial lemma. Let  $\pi \in P(k+l)$  be  $(\{1,\ldots,a\},\{a+1\},\{a+2\},\ldots,\{k+l\})$ . It is immediate that  $[\tilde{\pi}S]=(a^{k+1},1^{l-a})$ , if un  $(\tilde{\pi})=\pi$ .

**Lemma 1.4.5** Let S be as above,  $T \in \mathcal{R}_l$ , such that  $[T] = (1^l)$ , and let  $\nu$  be the partition  $(\{1, \ldots, a\}, \{a+1\}, \{a+2\}, \ldots, \{l\})$ , then we have

$$[Q(S, \pi \downarrow S)] = [Q(T, \nu \downarrow T)] \times (1^k). \tag{1.4.2}$$

**Remark 1.4.6** Lemma 1.4.5 is a special case of the Lemma 1.4.8, however we choose to include a separate proof for it for two reasons: firstly, it is the first, still not too technical example of investigating the combinatorial structure of the resonance category, which is a new object; secondly, the particular case of  $(a^k, 1^l)$  resonances was a subject of substantial previous attention.

#### Proof of the Lemma 1.4.5.

Recall that by the definition of the direct product,

$$[Q(T, \nu \downarrow T)] \times (1^k) = [Q(T \times U, (\nu \downarrow T) \times P(\{l+1, \dots, l+k\}))],$$

where  $U \in \mathcal{R}_k$  and  $[U] = (1^k)$ . Clearly,  $(\nu \downarrow T) \times P(\{l+1, \ldots, l+k\}) = \pi \downarrow S$ , hence we just need to show that  $S \setminus (\pi \downarrow S) = (T \times U) \setminus ((\nu \downarrow T) \times P(\{l+1, \ldots, l+k\}))$ . Note that  $(T \times U) \setminus ((\nu \downarrow T) \times P(\{l+1, \ldots, l+k\})) = (T \setminus (\nu \downarrow T)) \times U$ . Furthermore,

$$S = \left\{ (x_1, \dots, x_{l+k}) \in \{-1, 0, 1\}^{l+k} \mid \sum_{j=l+1}^{l+k} x_j + a \sum_{i=1}^{l} x_i = 0 \right\},\,$$

and

$$\bigcup_{\tau \in \pi \downarrow S} S^{\tau} = \left\{ (x_1, \dots, x_{l+k}) \in \{-1, 0, 1\}^{l+k} \mid \sum_{j=l+1}^{l+k} x_j + a \sum_{i=1}^{l} x_i = 0, \\ \max(|\text{Plus}(x_1, \dots, x_l)|, |\text{Minus}(x_1, \dots, x_l)|) \ge a \right\}.$$

Therefore, by the definition of the relative resonances, we have

$$S \setminus (\pi \downarrow S) = \left\{ (x_1, \dots, x_{l+k}) \in \{-1, 0, 1\}^{l+k} \mid \sum_{i=1}^{l} x_i = 0, \sum_{j=l+1}^{l+k} x_j = 0, |\operatorname{Plus}(x_1, \dots, x_l)| < a \right\}.$$

On the other hand,  $(1^k) = [\{(y_1, \dots, y_k) \in \{-1, 0, 1\}^k \mid \sum_{j=1}^k y_j = 0\}]$ , and

$$T\setminus (\nu \downarrow T) = \left\{ (z_1, \dots, z_l) \in \{-1, 0, 1\}^l \mid \sum_{i=1}^l z_i = 0, |\text{Plus}(z_1, \dots, z_l)| < a \right\},$$

which proves (1.4.2).  $\square$ 

#### Proof of the Theorem 1.4.2.

(a) We use induction on l. The case l < a can be taken as an induction base, since then  $(a^k, 1^l) = (1^k) \times (1^l)$ , hence, by the axiom (A3),  $\mathcal{F}(a^k, 1^l) = \mathcal{F}(1^k) \wedge \mathcal{F}(1^l)$ , which is contractible, since  $\mathcal{F}(1^k)$  is. Thus we assume that  $l \geq a$ , and  $\mathcal{F}(a^k, 1^{l'})$  is contractible for all l' < l.

Let S and  $\pi$  be as in the Lemma 1.4.5. The standard triple associated to the morphism  $\pi S \stackrel{\pi}{\hookrightarrow} S$  is  $\mathcal{F}(a^{k+1},1^{l-a}) \hookrightarrow \mathcal{F}(a^k,1^l) \to \mathcal{F}(a^k,1^l)/\mathcal{F}(a^{k+1},1^{l-a})$ .

Since, by the induction assumption,  $\mathcal{F}(a^{k+1}, 1^{l-a})$  is contractible, we conclude that  $\mathcal{F}(a^k, 1^l) \simeq \mathcal{F}(a^k, 1^l) / \mathcal{F}(a^{k+1}, 1^{l-a})$ .

Basically by the definition, we have  $\mathcal{F}(a^k,1^l)/\mathcal{F}(a^{k+1},1^{l-a})=\mathcal{F}(Q(S,\pi\downarrow S))$ . On the other hand, we have proved in the Lemma 1.4.5 that  $[Q(S,\pi\downarrow S)]=Q(T,\nu\downarrow T)\times(1^k)$ , where T and  $\nu$  are described in the formulation of that lemma. By axioms (A2) and (A3) we get that  $\mathcal{F}(Q(S,\pi\downarrow S))\simeq\mathcal{F}(Q(T,\nu\downarrow T))\wedge\mathcal{F}(1^k)$ , which is contractible, since  $\mathcal{F}(1^k)$  is. Therefore,  $\mathcal{F}(a^k,1^l)$  is also contractible.

(b) The argument is very similar to (a). We again assume  $l \geq a$ , which implies  $l \geq 2$ . By the using the same ordered set partition  $\pi$  as in (a), we get that  $\mathcal{F}(a,1^l) \simeq \mathcal{F}(a,1^l)/\mathcal{F}(a^2,1^{l-a})$ . Further, by Lemma 1.4.5 and the axioms (A2) and (A3) we conclude that  $\mathcal{F}(a,1^l) \simeq \mathcal{F}(1) \wedge (\mathcal{F}(1^l)/\mathcal{F}(a,1^{l-a}))$ . Since  $\mathcal{F}(1^l)$  is contractible, we get

$$\mathcal{F}(a, 1^l) \simeq \mathcal{F}(1) \wedge \operatorname{susp} \mathcal{F}(a, 1^{l-a}). \tag{1.4.3}$$

Since  $\mathcal{F}(a) = \mathcal{F}(1)$ ,  $\mathcal{F}(a,1) = \mathcal{F}(1) \wedge \mathcal{F}(1)$ , and  $\mathcal{F}(a,1^l)$  is contractible if  $2 \leq l < a$ , we obtain (1.4.1) by the repeated usage of (1.4.3).  $\square$ 

# 1.4.3 Resonances $(a^k, b^l)$

The algebraic invariants of these strata have not been computed before, not even in the case  $X=S^1$ , and  $\mathcal{F}$  - the standard resonance functor associated to the stratification of  $X^{(n)}$ .

We would like to apply a technique similar to the one used in the subsection 1.4.2. A problem is that, once one starts to "glue" a's, one cannot get b's in the same way as one could in the previous section from 1's. Thus, we are forced to consider a more general case of resonances, namely  $(g^m, a^k, b^l)$ , where g is the least common multiple of a and b. Assume  $g = a \cdot \bar{a} = b \cdot \bar{b}$ , and  $b > a \ge 2$ . Analogously with the Theorem 1.4.2 we have the following result.

**Theorem 1.4.7** Let  $\mathcal{F}$  be as in the Theorem 1.4.2. Let furthermore  $k = x \cdot \bar{a} + \epsilon_1$ ,  $l = y \cdot \bar{b} + \epsilon_2$ , where  $0 \le \epsilon_1 < \bar{a}$ ,  $0 \le \epsilon_2 < \bar{b}$ . Then

$$\mathcal{F}(g^m, a^k, b^l) \simeq \begin{cases} susp^{x+y+m-1}(\mathcal{F}(1)^{x+y+m+\epsilon_1+\epsilon_2}), & \text{if } m, \epsilon_1, \epsilon_2 \in \{0, 1\}; \\ point, & \text{otherwise.} \end{cases}$$

$$(1.4.4)$$

Just as in the subsection 1.4.2 (Corollary 1.4.3), the Theorem 1.4.7 is true if one replaces  $\mathcal{F}(\lambda)$  with  $\Sigma_{\lambda}^{S^1}$ .

The proof of the Theorem 1.4.7 follows the same general scheme as that of the Theorem 1.4.2, but the technical details are more numerous. Again there is a crucial combinatorial lemma.

Let S be an (m+k+l)-cut consisting of all the elements of  $\{-1,0,1\}^{m+k+l}$  which are orthogonal to the vector  $(\underline{a,\ldots,a},\underline{b,\ldots,b},\underline{g,\ldots,g})$ . Assume  $k\geq \bar{a}$ , and let an unordered set partition  $\pi$  be equal to  $(\{1,\ldots,\bar{a}\},\{\bar{a}+1\},\ldots,\{k+l+m\})$ . We see that  $[S]=(g^m,a^k,b^l)$ , and  $[\tilde{\pi}S]=(g^{m+1},a^{k-\bar{a}},b^l)$ , if  $\pi=\mathrm{un}\,(\tilde{\pi})$ . Lemma 1.4.8 Let  $T\in\mathcal{R}_k$ , such that  $[T]=(1^k)$ , and  $\nu=(\{1,\ldots,\bar{a}\},\{\bar{a}+1\},\ldots,\{k\})$ , then

$$[Q(S, \pi \downarrow S)] = [Q(T, \nu \downarrow T)] \times (\bar{b}^m, 1^l). \tag{1.4.5}$$

**Proof.** Again, it is easy to see that the sets of the partitions at infinity on both sides of (1.4.5) coincide. Indeed,

$$[Q(T,\nu)] imes (\bar{b}^m,1^l) = [Q(T imes U,(\nu \downarrow T) imes P(\{k+1,\ldots,k+m+l\}))],$$
 where  $U \in \mathcal{R}_{m+l}$ , such that  $[U] = (\bar{b}^m,1^l)$ , and  $(\nu \downarrow T) imes P(\{k+1,\ldots,k+m+l\}) = \pi \downarrow S$ . Also, we again have the equality

$$(T \times U) \setminus ((\nu \downarrow T) \times P(\{k+1, \dots, k+m+l\})) = T \setminus (\nu \downarrow T) \times U,$$

which greatly helps to prove that the sets if the surviving elements on the two sides of (1.4.5) coincide.

By the definition

$$S = \left\{ (x_1, \dots, x_{k+l+m}) \in \{-1, 0, 1\}^{k+l+m} \mid a \sum_{i=1}^k x_i + b \sum_{i=k+1}^{k+l} x_i + g \sum_{i=k+l+1}^{k+l+m} x_i = 0 \right\},$$

and

$$\bigcup_{\tau \in \pi \downarrow S} S^{\tau} = \left\{ (x_1, \dots, x_{k+l+m}) \in \{-1, 0, 1\}^{k+l+m} \mid a \sum_{i=1}^k x_i + b \sum_{i=k+1}^{k+l} x_i + g \sum_{i=k+l+1}^{k+l+m} x_i = 0, \max(|\operatorname{Plus}(x_1, \dots, x_k)|, |\operatorname{Minus}(x_1, \dots, x_k)|) \ge \bar{a} \right\}.$$

By the definition of the relative resonances and some elementary number theory we conclude that

$$S \setminus (\pi \downarrow S) = \left\{ (x_1, \dots, x_{k+l+m}) \in \{-1, 0, 1\}^{k+l+m} \mid |\text{Plus}(x_1, \dots, x_k)| < \bar{a}, \\ \sum_{i=1}^k x_i = 0, \quad b \sum_{i=k+1}^{k+l} x_i + g \sum_{i=k+l+1}^{k+l+m} x_i = 0 \right\}.$$

The number theory argument which we need is that if ax + by + lcm(a, b)z = 0, then  $\bar{a} \mid x$ , where  $\bar{a} \cdot a = \text{lcm}(a, b)$ . This can be seen by, for example, noticing that if ax + by + lcm(a, b)z = 0, then  $b \mid ax$ , but since also  $a \mid ax$ , we have  $\text{lcm}(a, b) \mid ax$ , hence  $\bar{a} \mid x$ .

The equation (1.4.5) follows now from the earlier observations together with the equalities

$$T \setminus (\nu \downarrow T) = \left\{ (x_1, \dots, x_k) \in \{-1, 0, 1\}^k \mid |\text{Plus}(x_1, \dots, x_k)| < \bar{a}, \sum_{i=1}^k x_i = 0 \right\}$$

and

$$(\bar{b}^m, 1^l) = \left[ \left\{ (y_1, \dots, y_{m+l}) \in \{-1, 0, 1\}^{m+l} \mid \sum_{i=1}^l y_i + \bar{b} \sum_{i=l+1}^{l+m} x_i = 0 \right\} \right]. \square$$

**Proof of the Theorem 1.4.7.** The cases  $k < \bar{a}$  and  $l < \bar{b}$  are easily reduced to the Theorem 1.4.2. Assume therefore that  $k \geq \bar{a}$  and  $l \geq \bar{b}$ . Recall also that  $b > a \geq 2$ , and hence  $\bar{a} \geq 2$ .

Let S and  $\pi$  be as in the formulation of the Lemma 1.4.8. The standard triple associated to the morphism  $\pi S \stackrel{\pi}{\hookrightarrow} S$  is

$$\mathcal{F}(q^{m+1}, a^{k-\bar{a}}, b^l) \hookrightarrow \mathcal{F}(q^m, a^k, b^l) \to \mathcal{F}(q^m, a^k, b^l) / \mathcal{F}(q^{m+1}, a^{k-\bar{a}}, b^l).$$
 (1.4.6)

We break the rest of the proof into 3 cases.

Case  $m \geq 2$ . Again, we prove that  $\mathcal{F}(g^m, a^k, b^l)$  is contractible by induction on k. This is clear if  $k < \bar{a}$ . If  $k \geq \bar{a}$ , it follows from (1.4.6) that  $\mathcal{F}(g^m, a^k, b^l) \simeq \mathcal{F}(g^m, a^k, b^l)/\mathcal{F}(g^{m+1}, a^{k-\bar{a}}, b^l) = \mathcal{F}(Q(S, \pi \downarrow S))$ . By Lemma 1.4.8 we conclude that  $\mathcal{F}(g^m, a^k, b^l) \simeq \mathcal{F}(Q(T, \nu \downarrow T)) \wedge \mathcal{F}(\bar{b}^m, 1^l)$ . By the Theorem 1.4.2,  $\mathcal{F}(\bar{b}^m, 1^l)$  is contractible, hence so is  $\mathcal{F}(g^m, a^k, b^l)$ .

**Case** m=0. By Lemma 1.4.8 we get that  $\mathcal{F}(Q(S,\pi\downarrow S))\simeq \mathcal{F}(Q(T,\nu\downarrow T))\wedge \mathcal{F}(1^l)$ . Since  $l\geq 2$ , we have that  $\mathcal{F}(1^l)$  is contractible, hence so is  $\mathcal{F}(Q(S,\pi\downarrow S))=\mathcal{F}(a^k,b^l)/\mathcal{F}(g,a^{k-\bar{a}},b^l)$ . Therefore, by (1.4.6)  $\mathcal{F}(a^k,b^l)\simeq \mathcal{F}(g,a^{k-\bar{a}},b^l)$ .

Case m=1. Since  $\mathcal{F}(g^2,a^{k-\bar{a}},b^l)$  is contractible, we conclude by (1.4.6) that  $\mathcal{F}(g,a^k,b^l)\simeq \mathcal{F}(g,a^k,b^l)/\mathcal{F}(g^2,a^{k-\bar{a}},b^l)=\mathcal{F}(Q(S,\pi\downarrow S))$ . By Lemma 1.4.8, and the properties of the resonance functors, we have

$$\mathcal{F}(g,a^k,b^l) \simeq \mathcal{F}(\bar{b},1^l) \wedge (\mathcal{F}(1^k)/\mathcal{F}(\bar{a},1^{k-\bar{a}})) \simeq \mathcal{F}(\bar{b},1^l) \wedge \mathrm{susp}\,(\mathcal{F}(\bar{a},1^{k-\bar{a}})). \tag{1.4.7}$$

By the repeated usage of (1.4.7) we obtain (1.4.4).  $\square$ 

## 1.5 SEQUENTIAL RESONANCES

1.5.1 The structure theory of strata associated to sequential resonances

**Definition 1.5.1** Let  $\lambda = (\lambda_1, \dots, \lambda_n)$ ,  $\lambda_1 \leq \dots \leq \lambda_n$ , be a number partition. We call  $\lambda$  sequential if, whenever  $\sum_{i \in I} \lambda_i = \sum_{j \in J} \lambda_j$ , and  $q \in I$ , such that  $q = \max(I \cup J)$ , then there exists  $\widetilde{J} \subseteq J$ , such that  $\lambda_q = \sum_{j \in \widetilde{J}} \lambda_j$ .

Correspondingly, we call a resonance S sequential, if it can be associated to a sequential partition.

Note that the set of sequential partitions is closed under removing blocks.

#### **Examples 1.5.2** Sequential partitions.

- (1) All partitions whose blocks are equal to powers of some number;
- (2)  $(a^k, b^l, 1^m)$ , such that a > bl; more generally  $(a_1^{k_1}, \dots, a_t^{k_t}, 1^m)$ , such that  $a_i > \sum_{j=i+1}^t a_j k_j$ , for all  $i \in [t]$ .

Through the rest of this subsection, we let  $\lambda$  be as in the Definition 1.5.1. For such  $\lambda$  we use the following additional notations:

- $mm(\lambda) = |\{i \in [n] \mid \lambda_i = \lambda_n\}|$ . In other words  $\lambda_{n-mm(\lambda)} \neq \lambda_{n-mm(\lambda)+1} = \cdots = \lambda_n$ .
- $I(\lambda) \subseteq [n]$  is the lexicographically maximal set (see below the convention that we use to order lexicographically), such that  $|I(\lambda)| \ge 2$ , and  $\lambda_n = \sum_{i \in I(\lambda)} \lambda_i$ . Note that it may happen that  $I(\lambda)$  does not exist, in which case  $\mathcal{F}(\lambda) \simeq \mathcal{F}(\lambda_1, \dots, \lambda_{n-mm(\lambda)}) \wedge \mathcal{F}(1^{mm(\lambda)})$ , and can be dealt with by induction.

Let n be a positive integer. We use the following convention for the lexicographic order on [n]. For  $A = \{a_1, \ldots, a_k\}$ ,  $B = \{b_1, \ldots, b_m\}$ ,  $A, B \subseteq [n]$ ,  $a_1 \leq \ldots \leq a_k$ ,  $b_1 \leq \ldots \leq b_m$ , we say that A is lexicographically larger than B if, either  $A \supseteq B$ , or there exists  $q < \min(k, m)$ , such that  $a_k = b_m$ ,  $a_{k-1} = b_{m-1}$ ,  $\ldots$ ,  $a_{k-q+1} = b_{m-q+1}$ , and  $a_{k-q} > b_{m-q}$ .

**Proposition 1.5.3** If  $\lambda = (\lambda_1, \dots, \lambda_n)$ ,  $\lambda_1 \leq \dots \leq \lambda_n$ , is a sequential partition, then so is  $\bar{\lambda} = (\lambda_{j_1}, \dots, \lambda_{j_t}, \sum_{i \in I(\lambda)} \lambda_i)$ , where  $t = n - |I(\lambda)|$ , and  $\{j_1, \dots, j_t\} = [n] \setminus I(\lambda)$ .

**Proof.** Let  $\bar{\lambda}_1 = \lambda_{j_1}, \dots, \bar{\lambda}_t = \lambda_{j_t}, \ \bar{\lambda}_{t+1} = \sum_{i \in I(\lambda)} \lambda_i$ . We need to check the condition of the Definition 1.5.1 for the identity

$$\sum_{i \in I} \bar{\lambda}_i = \sum_{j \in J} \bar{\lambda}_j. \tag{1.5.1}$$

If  $t+1\not\in I\cup J$ , then it follows from the assumption that  $\lambda$  is sequential. Assume  $t+1\in I$ . If  $\bar{\lambda}_j=\lambda_n$ , for some  $j\in J$ , take  $\widetilde{J}=\{j\}$ , and we are done. If  $\bar{\lambda}_i=\lambda_n$ , for some  $i\in I\setminus\{t+1\}$ , then, since  $\lambda$  is sequential, there exists  $\widetilde{J}\subseteq J$ , such that  $\sum_{j\in\widetilde{J}}\bar{\lambda}_j=\lambda_n=\bar{\lambda}_{t+1}$ , and we are done again.

Finally, assume  $\bar{\lambda}_i \neq \lambda_n$ , for  $i \in (I \cup J) \setminus \{t+1\}$ . Substituting  $\lambda_n$  instead of  $\bar{\lambda}_{t+1}$  into the identity (1.5.1) is allowed, since  $\lambda_n$  does not appear among  $\{\bar{\lambda}_i\}_{i\in(I\cup J)\setminus\{t+1\}}$ . This gives us an identity for  $\lambda$ , and again, since  $\lambda$  is sequential, we find the desired set  $\widetilde{J} \subseteq J$ , such that  $\sum_{i\in\widetilde{J}} \bar{\lambda}_i = \bar{\lambda}_{t+1}$ .  $\square$ 

Let  $S \in \mathcal{R}_n$  be the set of all elements of  $\{-1,0,1\}^n$ , which are orthogonal to the vector  $\lambda = (\lambda_1, \dots, \lambda_n)$ . Clearly,  $[S] = \lambda$ . Let  $\pi \in P(n)$  be the partition whose only nonsingleton block is given by  $I(\lambda)$ . The next lemma expresses the main combinatorial property of sequential partitions.

**Lemma 1.5.4** Let  $\tau \in P(n)$  be a partition which has only one nonsingleton block B, and assume  $\lambda_n = \sum_{i \in B} \lambda_i$ . Then  $\tau \in \pi \downarrow S$ .

**Proof.** Assume there exists partitions  $\tau$  as in the formulation of the lemma, such that  $\tau \not\in \pi \downarrow S$ . Choose one so that the block B is lexicographically largest possible. Let  $C = B \cap I(\lambda)$ . By the definition of  $I(\lambda)$ , and the choice of B, we have  $\sum_{i \in I(\lambda) \setminus C} \lambda_i = \sum_{j \in B \setminus C} \lambda_j$ , and  $q \in I(\lambda) \setminus C$ , where  $q = \max((I(\lambda) \cup B) \setminus C)$ .

Since partition  $\lambda$  is sequential, there exists  $D \subseteq B \setminus C$ , such that  $\lambda_q = \sum_{j \in D} \lambda_j$ . Let  $\gamma \in P(n)$  be the partition whose only nonsingleton block is  $G = (B \setminus D) \cup \{q\}$ . Clearly,  $\sum_{i \in G} \lambda_i = \lambda_n$ , and  $|G| \geq 2$ . By the choice of q, G is lexicographically larger than B, hence  $\gamma \in \pi \downarrow S$ .

Let furthermore  $\tilde{\gamma} \in P(n)$  be the partition having two nonsingleton blocks: D and G. By the Definition 1.3.1(2) if  $\gamma \in \pi \downarrow S$ , then  $\tilde{\gamma} \in \pi \downarrow S$ . By the Definition 1.3.1(3), if  $\tilde{\gamma} \in \pi \downarrow S$ , then  $\tau \in \pi \downarrow S$ , which yields a contradiction.  $\square$ 

Let  $T \in \mathcal{R}_{n-mm(\lambda)}$  be the set of all elements of  $\{-1,0,1\}^{n-mm(\lambda)}$ , which are orthogonal to the vector  $(\lambda_1,\ldots,\lambda_{n-mm(\lambda)})$ . Let  $\nu \in P(n-mm(\lambda))$  be the partition whose only nonsingleton block is given by  $I(\lambda)$ . We are now ready to state the combinatorial result which is crucial for our topological applications.

## Lemma 1.5.5

$$[Q(S, \pi \downarrow S)] = [Q(T, \nu \downarrow T)] \times (1^{mm(\lambda)}). \tag{1.5.2}$$

**Proof.** By definition we must verify that the sets of partitions at infinity and the surviving elements coincide on both sides of the equation (1.5.2).

Let us start with the partitions at infinity. Once filtered through the Proposition 1.4.1, the identity  $\pi \downarrow S = (\nu \downarrow T) \times P(\{n-mm(\lambda)+1,\ldots,n\})$  becomes essentially tautological. Both sides consist of the partitions  $\tau = (\tau_1,\ldots,\tau_k) \in P(n)$ , such that the number partition  $(\sum_{i\in\tau_1}\lambda_i,\ldots,\sum_{i\in\tau_k}\lambda_i)$  can be obtained from the number partition  $(\lambda_{j_1},\ldots,\lambda_{j_t},\sum_{i\in I(\lambda)}\lambda_i)$ , where  $\{j_1,\ldots,j_t\}=[n]\setminus I(\lambda)$ , by summing parts.

Let us now look at the surviving elements. Obviously,  $S \setminus (\pi \downarrow S) \supseteq (T \setminus (\nu \downarrow T)) \times U$ , where  $U \in \mathcal{R}_k$ , such that  $[U] = (1^{mm(\lambda)})$ , and we need to show the converse inclusion. Let  $x = (x_1, \dots, x_n) \in S$ , such that  $\sum_{i=n-mm(\lambda)+1}^n x_i \neq 0$  (otherwise  $x \in (T \setminus (\nu \downarrow T)) \times U$ ), we can assume  $\sum_{i=n-mm(\lambda)+1}^n x_i > 0$ . Then, since S is a sequential resonance, there exists  $y = (y_1, \dots, y_n) \in S$ , such that

- if  $y_i \neq 0$ , then  $x_i = y_i$ ;
- $|\operatorname{Plus}(y)| = 1$ , and  $\operatorname{Plus}(y) \subseteq \{n mm(\lambda) + 1, \dots, n\}$ .

This means that  $y \in S^{\tau}$ , for some  $\tau \in P(n)$ , which satisfies the conditions of the Lemma 1.5.4, which implies that  $\tau \in \pi \downarrow S$ . On the other hand,  $y \in S^{\tau}$  necessitates  $x \in S^{\tau}$ , and hence  $x \notin S \setminus (\pi \downarrow S)$ . This finishes the proof of the lemma.  $\square$ 

Just as before, this combinatorial fact about the resonances translates into a topological statement, which can be further strengthened by requiring some additional properties from  $\lambda$ .

**Definition 1.5.6** *Let*  $\lambda = (\lambda_1, ..., \lambda_n)$ ,  $\lambda_1 \leq ... \leq \lambda_n$ , *be a sequential partition, and let*  $q = \max I(\lambda)$ .  $\lambda$  *is called* **strongly sequential**, *if there exists*  $J \subseteq I(\lambda) \setminus \{q\}$ , *such that*  $\lambda_q = \sum_{i \in J} \lambda_i$  (note that we do not require  $|J| \geq 2$ ).

We are now in a position to prove the main topological structure theorem concerning the sequential resonances.

**Theorem 1.5.7** Let  $\mathcal{F}$  be as in the Theorem 1.4.2. Let  $\lambda$  be a sequential partition, such that  $I(\lambda)$  exists, then

- (1) if  $mm(\lambda) \geq 2$ , then  $\mathcal{F}(\lambda)$  is contractible;
- (2) if  $mm(\lambda) = 1$ , then  $\mathcal{F}(\lambda) \simeq \mathcal{F}(Q(T, \nu \downarrow T)) \wedge \mathcal{F}(1)$ , and we have the inclusion triple  $\mathcal{F}(\mu) \stackrel{i}{\hookrightarrow} \mathcal{F}(\lambda_1, \dots, \lambda_{n-1}) \to \mathcal{F}(Q(T, \nu \downarrow T))$ , where  $\mu = (\lambda_{j_1}, \dots, \lambda_{j_t})$ ,  $\{j_1, \dots, j_t\} = [n] \setminus I(\lambda)$ , and  $\nu \in P(n mm(\lambda))$  is the partition whose only nonsingleton block is given by  $I(\lambda)$ .

If moreover  $\lambda$  is strongly sequential, then the map i is homotopic to a trivial map (mapping everything to a point), hence the triple splits and we conclude that

$$\mathcal{F}(\lambda) \simeq (\mathcal{F}(1) \wedge \mathcal{F}(\lambda_1, \dots, \lambda_{n-1})) \vee susp(\mathcal{F}(1) \wedge \mathcal{F}(\mu)).$$
 (1.5.3)

#### Proof.

(1) We use induction on  $\sum_{i=1}^{n-mm(\lambda)} \lambda_i$ . If  $I(\lambda)$  does not exist, then  $\lambda_n$  is independent, i.e.,  $\mathcal{F}(\lambda) \simeq \mathcal{F}(\lambda_1, \dots, \lambda_{n-mm(\lambda)}) \times \mathcal{F}(1^{mm(\lambda)})$ , and hence  $\mathcal{F}(\lambda)$  is contractible. Otherwise consider the inclusion triple

$$\mathcal{F}(\bar{\lambda}) \hookrightarrow \mathcal{F}(\lambda) \to \mathcal{F}(\lambda)/\mathcal{F}(\bar{\lambda}) = \mathcal{F}(Q(S, \pi \downarrow S)),$$
 (1.5.4)

where  $\bar{\lambda} = (\lambda_{j_1}, \dots, \lambda_{j_t}, \sum_{i \in I(\lambda)} \lambda_i)$ , and  $\pi \in P(n)$  is the partition whose only nonsingleton block is given by  $I(\lambda)$ . By the induction assumption  $\mathcal{F}(\bar{\lambda})$  is contractible. On the other hand, by Lemma 1.5.5,  $\mathcal{F}(Q(S, \pi \downarrow S)) \simeq \mathcal{F}(Q(T, \nu \downarrow T)) \wedge \mathcal{F}(1^{mm(\lambda)})$ , which is also contractible if  $mm(\lambda) \geq 2$ .

(2) If  $mm(\lambda) = 1$ , then we can conclude from (1.5.4) that  $\mathcal{F}(\lambda) \simeq \mathcal{F}(1) \wedge \mathcal{F}(Q(T, \nu \downarrow T))$ . Next, consider the inclusion triple

$$\mathcal{F}(\mu) \stackrel{i}{\hookrightarrow} \mathcal{F}(\lambda_1, \dots, \lambda_{n-1}) \to \mathcal{F}(Q(T, \nu \downarrow T)).$$
 (1.5.5)

If  $\lambda$  is strongly sequential, then there exists  $J \subseteq I(\lambda) \setminus \{q\}$ , such that  $\lambda_q = \sum_{i \in J} \lambda_i$  (here  $q = \max I(\lambda)$ ). The map i factors:

$$\mathcal{F}(\mu) \stackrel{i_1}{\hookrightarrow} \mathcal{F}(\lambda_{p_1}, \dots, \lambda_{p_{n-1-|J|}}, \sum_{i \in I(\lambda)} \lambda_i) \stackrel{i_2}{\hookrightarrow} \mathcal{F}(\lambda_1, \dots, \lambda_{n-1}), \tag{1.5.6}$$

where  $\{p_1,\ldots,p_{n-1-|J|}\}=[n-1]\setminus J$ . Since  $(\lambda_{p_1},\ldots,\lambda_{p_{n-1-|J|}},\sum_{i\in I(\lambda)}\lambda_i)$  is sequential, and  $mm((\lambda_{p_1},\ldots,\lambda_{p_{n-1-|J|}},\sum_{i\in I(\lambda)}\lambda_i))\geq 2$ , we can conclude that the middle space in (1.5.6) is contractible, and hence i in (1.5.5) is homotopic to a trivial map. This yields the conclusion.  $\square$ 

# 1.5.2 Resonances $(a^k, b^l, 1^m)$

We give here the first application of the structure theory described in the previous subsection.

**Theorem 1.5.8** Let a, b, k, l, m, r be positive integers, such that b > 1,  $m \ge r$ , and a = bl + r. Then

$$\mathcal{F}(a^k, b^l, 1^m) \simeq susp\left(\mathcal{F}(1^k) \wedge \mathcal{F}(a, 1^{m-r})\right) \vee \left(\mathcal{F}(1^k) \wedge \mathcal{F}(b^l, 1^m)\right). \tag{1.5.7}$$

**Remark 1.5.9** The restriction  $m \ge r$  is unimportant. Indeed, if m < r, then a > bl + m, hence a is not involved in any resonance other than a = a. This implies that  $\mathcal{F}(a^k, b^l, 1^m) = \mathcal{F}(1^k) \times \mathcal{F}(b^l, 1^m)$ , and we have determined the homotopy type of  $\mathcal{F}(a^k, b^l, 1^m)$  by the previous computations.

#### **Proof of the Theorem 1.5.8.**

Obviously, the condition a > bl guarantees that the partition  $(a^k, b^l, 1^m)$  is sequential, hence the Theorem 1.5.7 is valid. It follows that if  $k \ge 2$ , then  $\mathcal{F}(a^k, b^l, 1^m)$  is contractible, hence (1.5.7) is true.

Furthermore, if  $l \ge 2$ , or, l = 1 and  $m \ge b$ , then  $(a, b^l, 1^m)$  is strongly sequential, hence in this case (1.5.3) is valid, which in new notations becomes precisely the equation (1.5.7).

Finally, assume l=1 and  $b>m\geq r\geq 1$ . Let a=b+d. If  $\mathcal{F}(a,1^{m-d})$  or  $\mathcal{F}(b,1^m)$  is contractible, then the map i in the inclusion triple  $\mathcal{F}(a,1^{m-d})\stackrel{i}{\hookrightarrow} \mathcal{F}(b,1^m)\to \mathcal{F}(b,1^m)/\mathcal{F}(a,1^{m-d})$  is homotopic to a trivial map, and we again conclude (1.5.7). If both of these spaces are not contractible then  $\mathcal{F}(a,1^{m-d})\simeq S^{2y+\epsilon_2+1}$  and  $\mathcal{F}(b,1^m)\simeq S^{2x+\epsilon_1+1}$ , where nonnegative integers  $x,y,\epsilon_1,\epsilon_2$  are defined by

$$m = bx + \epsilon_1, \ m - d = (b + d)y + \epsilon_2, \ \epsilon_1, \epsilon_2 \in \{0, 1\}.$$
 (1.5.8)

Let us show that  $2x+\epsilon_1>2y+\epsilon_2$ . If x>y, then  $2x+\epsilon_1\geq 2x\geq 2y+2>2y+\epsilon_2$ . From (1.5.8) we have that  $b(x-y)=d+dy+\epsilon_2-\epsilon_1$ . If  $x\leq y$ , then the left hand side is nonpositive. On the other hand, since  $d\geq 1$ , the right hand side is nonnegative. Hence, both sides are equal to 0, which implies  $x=y, d=\epsilon_1=1$ ,  $\epsilon_2=y=0$ . This yields  $2x+\epsilon_1>2y+\epsilon_2$ .

The homotopic triviality of the map i follows now from the fact that the homotopy groups of a sphere are trivial up to the dimension of that sphere, i.e.,  $\pi_k(S^n) = 0$ , for  $0 \le k \le n - 1$ .  $\square$ 

## 1.5.3 Resonances consisting of powers

Let us fix an integer  $a \ge 2$ . In this subsection we will study the topology of the strata indexed by the following class of partitions: all number partitions whose blocks are powers of a. Let  $\Lambda(a)$  denote the set of all such partitions.

It is convenient to introduce a different notation for the partitions in this class. Let  $\Lambda_a(\alpha_n, \alpha_{n-1}, \dots, \alpha_0)$ , where  $\alpha_n, \alpha_{n-1}, \dots, \alpha_0$  are nonnegative integers, denote  $\lambda \in \Lambda(a)$ , which consists of  $\alpha_n$  parts equal to  $a^n$ ,  $\alpha_{n-1}$  parts equal to  $a^{n-1}$ , ...,  $\alpha_0$  parts equal to 1. For example  $(8, 4, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1) = \Lambda_2(1, 1, 3, 6)$ .

Obviously, all partitions from  $\Lambda(a)$  are strongly sequential, hence the Theorem 1.5.7 applies, and it yields:

- (1) if  $\alpha_n \geq 2$ , then  $\mathcal{F}(\Lambda_a(\alpha_n, \alpha_{n-1}, \dots, \alpha_0))$  is contractible;
- (2) if  $I(\Lambda_a(1, \alpha_{n-1}, \dots, \alpha_0))$  exists, then

$$\mathcal{F}(\Lambda_a(1,\alpha_{n-1},\ldots,\alpha_0)) \simeq (\mathcal{F}(1) \wedge \mathcal{F}(\Lambda_a(\alpha_{n-1},\ldots,\alpha_0))) \vee (S^1 \wedge \mathcal{F}(1) \wedge \mathcal{F}(\Lambda_a(1,\beta_{n-1},\ldots,\beta_0))), \quad (1.5.9)$$

where  $\Lambda_a(1, \beta_{n-1}, \dots, \beta_0)$  is obtained from  $\Lambda_a(1, \alpha_{n-1}, \dots, \alpha_0)$  by removing the blocks indexed by  $I(\Lambda_a(1, \alpha_{n-1}, \dots, \alpha_0))$ .

(3) If  $I(\Lambda_a(1, \alpha_{n-1}, \dots, \alpha_0))$  does not exist, then

$$\mathcal{F}(\Lambda_a(1,\alpha_{n-1},\ldots,\alpha_0)) \simeq \mathcal{F}(1) \wedge \mathcal{F}(\Lambda_a(\alpha_{n-1},\ldots,\alpha_0)).$$
 (1.5.10)

It is immediate from the formulae (1.5.9) and (1.5.10) that each topological space  $\mathcal{F}(\Lambda_a(\alpha_n,\alpha_{n-1},\ldots,\alpha_0))$  is homotopy equivalent to a wedge of spaces of the form  $\mathcal{F}(1)^{\alpha} \wedge S^{\beta}$ , where  $\mathcal{F}(1)^{\alpha}$  means an  $\alpha$ -fold smash product of  $\mathcal{F}(1)$ . The natural combinatorial question which arises is how to enumerate these spaces. We shall now construct a combinatorial model: a weighted graph which yields such an enumeration.

**Definition 1.5.10** Let  $\lambda = \Lambda_a(\alpha_n, \alpha_{n-1}, \ldots, \alpha_0)$ ), and set  $\alpha_{-1} = 1$ .  $\Gamma_{\lambda}$  is a directed weighted graph on the set of vertices  $\{n, \ldots, 0, -1\}$  whose edges and weights are defined by the following rule. For  $x, x+d \in \{-1, 0, \ldots, n\}$ ,  $d \geq 1$ , there exists an edge e(x, x+d) (the edge is directed from x to x+d) if and only if  $\alpha_x > 0$ ,  $\alpha_{x+d} > 0$ , and

$$a^{d} \mid a^{d-1}\alpha_{x+d-1} + a^{d-2}\alpha_{x+d-2} + \dots + a\alpha_{x+1} + \alpha_{x} - 1.$$

In this case the weight of the edge is defined as

$$w(x, x+d) = (a^{d-1}\alpha_{x+d-1} + a^{d-2}\alpha_{x+d-2} + \dots + a\alpha_{x+1} + \alpha_x - 1) \cdot a^{-d}.$$

Note that if  $d \ge 2$  and there exists an edge e(x, x + d), then there exists an edge e(x, x + d - 1).

We call a directed path in  $\Gamma_{\lambda}$  complete if it starts in -1 and ends in n. Let  $\gamma$  be a complete path in  $\Gamma_{\lambda}$  consisting of t edges,  $\gamma = (e(x_0, x_1), \dots, e(x_{t-1}, x_t))$ , where  $x_0 = -1$ , and  $x_t = n$ . The weight of  $\gamma$  is defined to be the pair  $(l(\gamma), w(\gamma))$ , where  $l(\gamma) = t$ , and  $w(\gamma) = \sum_{i=1}^t w(x_{i-1}, x_i)$ .

**Theorem 1.5.11** Let  $\lambda \in \Lambda(a)$ ,  $\lambda = \Lambda_a(1, \alpha_{n-1}, \dots, \alpha_0)$ , then

$$\mathcal{F}(\lambda) \simeq \bigvee_{\gamma} (\mathcal{F}(1)^{l(\gamma) + w(\gamma)} \wedge S^{w(\gamma)}), \tag{1.5.11}$$

where the wedge is taken over all complete paths of  $\Gamma_{\lambda}$ .

**Proof.** We use induction on  $\sum_{i=0}^{n} \alpha_i$ . As the base of the induction we take the case  $\lambda = \Lambda_a(1, \underbrace{0, \dots, 0}_n)$ . In this case  $\Gamma_\lambda$  is a graph with only one edge e(-1, n),

w(-1,n)=0. Thus, there is only one complete path. It has weight (1,0), and  $\mathcal{F}(\lambda)\simeq\mathcal{F}(1)$ .

Next, we prove the induction step. We break up the proof in three cases. Let  $t \in [n-1]$  be the maximal index for which  $\alpha_t \neq 0$ .

**Case 1.**  $I(\lambda)$  does not exist.

By (1.5.10) we have

$$\mathcal{F}(\lambda) \simeq \mathcal{F}(1) \wedge \mathcal{F}(\Lambda_a(\alpha_{n-1}, \dots, \alpha_0)).$$
 (1.5.12)

On the other hand,  $I(\lambda)$  does not exist if and only if  $a^n > \alpha_{n-1}a^{n-1} + \cdots + \alpha_1a + \alpha_0$ . We also know that  $\lambda \neq \Lambda_a(1,0,\ldots,0)$ , i.e.,  $\alpha_{n-1}a^{n-1} + \cdots + \alpha_1a + \alpha_0 > 0$ . This implies that there is at most one edge of the type e(x,n). This edge exists if and only if  $\alpha_x = 1$ , and  $\alpha_{n-1} = \cdots = \alpha_{x+1} = 0$ , in which case w(x,n) = 0.

If this edge does not exist then there are no complete paths in  $\Gamma_{\lambda}$  and, at the same time  $\mathcal{F}(\Lambda_a(\alpha_{n-1},\ldots,\alpha_0))$  is contractible by the previous observations. This agrees with (1.5.11).

If, on the other hand, this edge does exist, then all complete paths  $\gamma$  must be of the type  $\gamma = (\tilde{\gamma}, e(x, n))$ , where  $\tilde{\gamma}$  is a complete path from -1 to x. Also in this case (1.5.12) agrees with (1.5.11).

Case 2.  $I(\lambda)$  exists and  $\alpha_t \geq 2$ .

In this case  $\mathcal{F}(\Lambda_a(\alpha_{n-1},\ldots,\alpha_0))$  is contractible, and

$$\mathcal{F}(\lambda) \simeq S^1 \wedge \mathcal{F}(1) \wedge \mathcal{F}(\Lambda_a(1, \beta_{n-1}, \dots, \beta_0)),$$
 (1.5.13)

where  $\beta_{n-1}, \ldots, \beta_0$  are as in (1.5.9).

Let  $q \in \{n-1, \ldots, 0, -1\}$  be the maximal index for which  $\beta_q \neq 0$  (we assume  $\beta_{-1} = 1$ ). Let  $\tilde{\lambda} = (1, \beta_{n-1}, \ldots, \beta_0)$ . We can describe the graph  $\Gamma_{\tilde{\lambda}}$ : it is obtained from  $\Gamma_{\lambda}$  by

- (1) removing the edges which have one of the endpoints in the set  $\{n-1,\ldots,q+1\}$ ;
- (2) decreasing the weight of every existing edge e(x, n) by 1;
- (3) keeping all existing edges with the old weights on the set  $\{q, q-1, \ldots, -1\}$ .

This operation on  $\Gamma_{\lambda}$  is well-defined, since there can be no edges in  $\Gamma_{\Lambda}$  of the type e(x,n), for  $x\in\{n-1,\ldots,q+1\}$ , and since the weight of edges e(x,n), for  $x\in\{q,\ldots,0,-1\}$  must be at least 1, as  $b_q\neq 0$ . Furthermore, it is clear from the

above combinatorial description of  $\Gamma_{\tilde{\lambda}}$ , that the set of the complete paths of  $\Gamma_{\tilde{\lambda}}$  is the same as that of  $\Gamma_{\lambda}$ , and that the weights of the edges in these paths are also the same except for the edge with the endpoint n, whose weight has been decreased by 1. Thus, (1.5.13) agrees with (1.5.11) in this case.

## **Case 3.** $I(\lambda)$ exists and $\alpha_t = 1$ .

This case is rather similar to the case 2, except that there is an edge e(t, n) of weight 0. Thus,  $\Gamma_{\tilde{\lambda}}$  bookkeeps all the complete paths of  $\Gamma_{\lambda}$ , except for the ones which have this edge e(t, n).

However, the first term of the right hand side of (1.5.9) bookkeeps the paths  $(\tilde{\gamma}, e(t, n))$ , just like in the case 1. Since the set of all complete paths of  $\Gamma_{\lambda}$  is the disjoint union of the sets of those paths which contain e(t, n), and those which do not, we again get that (1.5.9) provides the inductive step for (1.5.11).  $\square$ 

## **Examples 1.5.12**

- (1) Let  $\lambda = (a, 1^l)$ , for  $a \geq 2$ . Then  $\Gamma_{\lambda}$  is a graph on the vertex set  $\{1, 0, -1\}$  having either one or two edges:
  - 1. it has in any case the edge e(-1, 0), w(-1, 0) = 0;
  - 2. if a divides l, then it has the edge e(-1,1), in which case w(-1,1) = l/a;
  - 3. if a divides l-1, then it has the edge e(0,1), in which case w(0,1)=(l-1)/a.

Clearly the Theorem 1.5.11 agrees with the Theorem 1.4.2. Indeed, if  $\epsilon \notin \{0,1\}$  (where  $\epsilon$  is taken from the formulation of the Theorem 1.4.2), then there are no complete paths in  $\Gamma_{\lambda}$ . If  $\epsilon = 0$ , then there is one path (-1,1) of weight (1,l/a); and if  $\epsilon = 1$ , then there is one path ((-1,0),(0,1)) of weight (2,(l-1)/a). Thus, (1.5.11) and (1.4.1) are equivalent in this case.

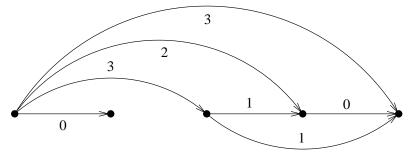


Figure 1.1.  $\Gamma_{(8,4,2,2,2,1,1,1,1,1,1)}$ .

(2) Let  $\lambda = (8, 4, 2, 2, 2, 1, 1, 1, 1, 1, 1) = \Lambda_2(1, 1, 3, 6)$ . The graph  $\Gamma_{\lambda}$  is shown on the Figure 1.1. It has 4 directed paths from -1 to 3 and, by the Theorem 1.5.11,

we have

$$\mathcal{F}(\lambda) \simeq (\mathcal{F}(1)^3 \wedge S^2) \vee (\mathcal{F}(1)^5 \wedge S^3) \vee (\mathcal{F}(1)^6 \wedge S^4) \vee (\mathcal{F}(1)^7 \wedge S^4),$$
 in particular  $\Sigma_{\lambda}^{\mathbb{R}} \simeq S^5 \vee S^8 \vee S^{10} \vee S^{11}$ .

#### 1.6 Complexity of resonances

The main idea of all our previous computations was to find, for a given n-cut S, a partition  $\pi \in P(n)$ , such that span  $(S \setminus (\pi \downarrow S)) \neq S$ . Intuitively speaking, shrinking the substratum corresponding to  $\tilde{\pi}S$ , where un  $(\tilde{\pi}) = \pi$ , essentially reduces the set of linear identities in S. It is easy to construct examples when such  $\pi$  does not exist, e.g., Example 1.2.2(4).

These observations lead us to introduce a formal notion of complexity of a resonance.

#### **Definition 1.6.1**

1) For  $S \in \mathcal{R}_n$ , the **complexity** of S is denoted c(S) and is defined by:

$$c(S) = \min\{|\Pi| \mid \Pi \subseteq P(n), span(S \setminus (\Pi \downarrow S)) \neq S\}. \tag{1.6.1}$$

2) We define the complexity of an n-resonance to be the complexity of one of its representing cuts. Clearly, it does not depend on the choice of the representative.

**Remark 1.6.2** The number c(S) would not change if we required the partitions in  $\Pi$  to have one block of size 2, and all other blocks of size 1.

The higher is the complexity of a resonance [S], the less it is likely that one can succeed with analyzing its topological structure using the method decribed in this chapter. This is because one would need to take a quotient by a union of c([S]) strata and it might be difficult to get a hold on the topology of that union.

We finish by constructing for an arbitrary  $n \in \mathbb{N}$ , a resonance of complexity n. Let  $\lambda_n = (a_1, \ldots, a_n, b_1, \ldots, b_n)$ , such that  $a_i, b_i \in \mathbb{N}$ , for  $i, j \in [n]$ , and all other linear identities among  $a_i$ 's and  $b_i$ 's with coefficients  $\pm 1, 0$  are generated by such identities. In other words, the cut S associated to  $\lambda$  is equal to the set

$$\left\{ (x_1, \dots, x_n, y_1, \dots, y_n) \in \{-1, 0, 1\}^{2n} \, \Big| \, \sum_{i=1}^n y_i = 0, \, x_i + y_i = 0, \, \forall i \in [n] \right\}.$$
(1.6.2)

It is not difficult to construct such  $\lambda_n$  directly:

1) Choose  $a_1, \ldots, a_n$ , such that the only linear identities with coefficients  $\pm 1, 0$  on the set  $a_1, a_1, a_2, a_2, \ldots, a_n, a_n$  are of the form  $a_i = a_i$ ; in other words, there are

no linear identities with coefficients  $\pm 2, \pm 1, 0$  on the set  $a_1, \ldots, a_n$ . One example is provided by the choice  $a_1 = 1, a_2 = 3, \ldots, a_n = 3^{n-1}$ .

2) Let  $b_i = N + a_i$ , for  $i \in [n]$ , where N is sufficiently large. As the proof of the Proposition 1.6.3 will show, it is enough to choose  $N > 2\sum_{i=1}^{n} \lambda_i$ . This bound is far from sharp, but it is sufficient for our purposes.

**Proposition 1.6.3** Let  $S_n$  be the n-cut associated to the ordered sequence of natural numbers  $\lambda_n$  described above, then  $c(S_n) = n$ .

**Proof.** First, let us verify that the cut  $S_n$  associated to  $\lambda_n$  is equal to the one described in (1.6.2). Take  $(x_1, \ldots, x_n, y_1, \ldots, y_n) \in S_n$ .

Assume first that  $\sum_{i=1}^n y_i \neq 0$ . Then,  $(x_1, \dots, x_n, y_1, \dots, y_n)$  stands for the identity

$$\sum_{i \in I_1} a_i + \sum_{j \in J_1} b_j = \sum_{i \in I_2} a_i + \sum_{j \in J_2} b_j, \tag{1.6.3}$$

such that  $|J_1| \ge |J_2| + 1$ . This implies that N is equal to some linear combination of  $a_i$ 's with coefficients  $\pm 2, \pm 1, 0$ . This leads to contradiction, since  $N > 2 \sum_{i=1}^{n} \lambda_i$ .

Thus, we know that  $\sum_{i=1}^{n} y_i = 0$ . Cancelling  $N \cdot |J_1|$  out of (1.6.3) we get an identity with coefficients  $\pm 2, \pm 1, 0$  on the set  $a_1, \ldots, a_n$ . By the choice of  $a_i$ 's, this identity must be trivial, which amounts exactly to saying that  $x_i + y_i = 0$ , for  $i \in [n]$ .

Second, it is a trivial observation that  $c(S_n) \leq n$ . Indeed, let  $\pi_i \in P(n)$  be a partition with only one nonsingleton block (1, n+i), for  $i \in [n]$ . Then span  $(S_n \setminus (\{\pi_1, \dots, \pi_n\} \downarrow S_n)) \neq S_n$ , since for any  $(x_1, \dots, x_n, y_1, \dots, y_n) \in S_n \setminus (\{\pi_1, \dots, \pi_n\} \downarrow S_n)$ , we have  $x_1 = 0$ .

Finally, let us see that  $c(S_n) > n - 1$ . As we have remarked after the Definition refdf6.1, it is enough to consider the case when the partitions of  $\Pi$  have one block of size 2, and the rest are singletons. Let us call the identity  $a_i + b_j = a_j + b_i$  the elementary identity indexed (i, j).

From the definition of the closure operation  $\downarrow$  it is clear that an elementary identity indexed (i,j) is not in  $S_n \setminus (\Pi \downarrow S_n)$  if and only if the partition whose only nonsingleton block is (i,n+j) belongs to  $\Pi$ , or the partition whose only nonsingleton block is (j,n+i) belongs to  $\Pi$ . That is because the only reason this identity would not be in  $S_n \setminus (\Pi \downarrow S_n)$  would be that one of these two partitions is in  $\Pi \downarrow S_n$ . But, if such a partition is in  $\Pi \downarrow S_n$ , then it must be in  $\Pi$ : moves (2) of the Definition 1.3.1 can never produce a partition whose only nonsingleton block has size 2, while the moves (3) of the Definition 1.3.1 may only interchange between partitions (i, n+j) and (j, n+i) in our specific situation. Thus, we can conclude that if  $|\Pi| \leq n-1$ , then at most n-1 elementary identities are not in  $S_n \setminus (\Pi \downarrow S_n)$ .

Next, we note that for any distinct  $i,j,k\in[n]$ , the elementary identities (i,j) and (j,k) imply the elementary identity (i,k). Let us now think of elementary identities as edges in a complete graph on n vertices,  $K_n$ . Then, any set M of elementary identities corresponds to a graph G on n vertices, and the collection of the elementary identities which lie in the span M is encoded by the transitive closure of G. It is a well known combinatorial fact that  $K_n$  is (n-1)-connected, which means that removal of at most n-1 edges from it leaves a connected graph. Hence, if we remove at most n-1 edges from  $K_n$  and then take the transitive closure, we get  $K_n$  again. Thus, if  $|\Pi| \leq n-1$ , all elementary identities lie in span  $(S_n \setminus (\Pi \downarrow S_n))$ . Since the elementary identities generate the whole  $S_n$ , we conclude that  $S_n = \operatorname{span}(S_n \setminus (\Pi \downarrow S_n))$ , hence  $c(S_n) > n-1$ .  $\square$ 

#### CHAPTER 2

# INCIDENCE COMBINATORICS OF RESOLUTIONS

#### 2.1 The motivation for the abstract framework

In this chapter we introduce notions of *combinatorial blowups*, *building sets*, and *nested sets*, for an arbitrary meet-semilattice. The definitions are given on a purely order-theoretic level without any reference to geometry. This provides a common abstract framework for the incidence combinatorics occurring in at least two different situations in algebraic geometry: the construction of De Concini-Procesi models of subspace arrangements [DP95], and the resolution of singularities in toric varieties.

The various parts of this abstract framework have received different emphasis within different situations: while the notion of combinatorial blowups clearly specializes to stellar subdivisions of defining fans in the context of toric varieties, building sets and nested sets were introduced in the context of model constructions by De Concini & Procesi [DP95] (earlier and in a more special setting by Fulton & MacPherson [FM94]), from where we adopt our terminology. This correspondence however is not complete: the building sets in [DP95, FM94] are not canonical, they depend on the geometry, while ours do not. See Section 2.4 for further details.

It was proved in [DP95] that a sequence of blowups within an arrangement of complex linear subspaces leads from the intersection stratification of complex space given by the maximal subspaces of the arrangement to an arrangement model stratified by divisors with normal crossings. In the context of toric varieties, there exist many different procedures for stellar subdivisions of a defining fan that result in a simplicial fan, so-called simplicial resolutions.

The purpose of our Main Theorem 2.3.4 is to unify these two situations on the combinatorial level: a sequence of combinatorial blowups, performed on a (combinatorial) building set in linear extension compatible order, transforms the initial semilattice to a semilattice where all intervals are boolean algebras, more precisely to the face poset of the corresponding simplicial complex of nested sets. In particular, the structure of the resulting semilattice can be fully described by the initial data of nested sets. Both the formulation and the proof of our main theorem are purely combinatorial.

We sketch the content of this chapter:

**Section 2.2.** After providing some basic poset terminology, we define building sets and nested sets for meet-semilattices in purely order-theoretic terms and develop general structure theory for these notions.

**Section 2.3.** We define combinatorial blowups of meet-semilattices, and study their effect on building sets and nested sets. The section contains our Main Theorem 2.3.4 which describes the result of blowing up the elements of a building set in terms of the initial nested set complex.

The next two sections are devoted to relating our abstract framework to two different contexts in algebraic geometry.

**Section 2.4.** We briefly review the construction of De Concini-Procesi models for subspace arrangements. After that, we show that the change of the incidence combinatorics of the stratification in a single construction step is described by a combinatorial blowup of the semilattice of strata, and trace their resolution procedure step-by-step.

**Section 2.5.** We draw the connection to simplicial resolutions of toric varieties: we recognize stellar subdivisions as combinatorial blowups of the face posets of defining fans and discuss the notions of building and nested sets in this context.

#### 2.2 Building sets and nested sets of meet-semilattices

## 2.2.1 Irreducible elements in posets

We assume known the parts of the terminology of posets described in Appendix B. A poset is called *irreducible* if it is not a direct product of two other posets, both consisting of at least two elements. For a poset P with a unique minimal element  $\hat{0}$ , we call  $I(P) = \{x \in P \mid [\hat{0}, x] \text{ is irreducible } \}$  the *set of irreducible elements* in P. In particular, the minimal element  $\hat{0}$  and all atoms of P are irreducible elements in P. For  $x \in P$ , we call  $D(x) = \max(I(P)_{\leq x})$  the *set of elementary divisors* of x – a term which is explained by the following proposition:

**Proposition 2.2.1** Let P be a poset with a unique minimal element  $\hat{0}$ . For  $x \in P$  there exists a unique finest decomposition of the interval  $[\hat{0},x]$  in P as a direct product, which is given by an isomorphism  $\varphi_x^{\text{el}}:\prod_{j=1}^l [\hat{0},y_j] \xrightarrow{\cong} [\hat{0},x]$ , with  $\varphi_x^{\text{el}}(\hat{0},\ldots,y_j,\ldots,\hat{0})=y_j$  for  $j=1,\ldots,l$ . The factors of this decomposition are the intervals below the elementary divisors of  $x:\{y_1,\ldots,y_l\}=D(x)$ .

**Proof.** Whenever a poset with a minimal element  $\hat{0}$  is represented as a direct product, all elements which have more than one coordinate different from  $\hat{0}$  are

reducible. Hence, if  $\prod_{i=1}^{l} [\hat{0}, y_j] \cong [\hat{0}, x]$ , and the  $y_j$  are irreducible for  $j = 1, \dots, l$ , then  $\{y_1,\ldots,y_l\} = D(x)$ .  $\square$ 

#### 2.2.2 Building sets

In this subsection we define the notion of building sets of a semilattice and develop their structure theory.

**Definition 2.2.2** Let  $\mathcal{L}$  be a semilattice. A subset  $\mathcal{G}$  in  $\mathcal{L}$  is called a **building set** of  $\mathcal{L}$  if for any  $x \in \mathcal{L}$  and  $\max \mathcal{G}_{\leq x} = \{x_1, \dots, x_k\}$  there is an isomorphism of posets

$$\varphi_x: \prod_{j=1}^k [\hat{0}, x_j] \stackrel{\cong}{\longrightarrow} [\hat{0}, x]$$
 (2.2.1)

with  $\varphi_x(\hat{0},\ldots,x_j,\ldots,\hat{0}) = x_j$  for  $j=1,\ldots,k$ . We call  $F(x) = \max \mathcal{G}_{\leq x}$  the set of factors of x in G.

The next proposition provides several equivalent conditions for a subset of  $\mathcal{L}$ to be a building set.

**Proposition 2.2.3** For a semilattice  $\mathcal{L}$  and a subset  $\mathcal{G}$  of  $\mathcal{L}$  the following are equivalent:

- (1)  $\mathcal{G}$  is a building set of  $\mathcal{L}$ ;
- (2)  $\mathcal{G}\supseteq I(\mathcal{L})$ , and for every  $x\in\mathcal{L}$  with  $D(x)=\{y_1,\ldots,y_l\}$  the elementary divisors of x, there exists a partition  $\pi_x = \pi_1 | \dots | \pi_k$ , of the set [l], with blocks  $\pi_t = \{i_1, \dots, i_{|\pi_t|}\}$ , for  $t \in [k]$ , such that the elements in the set  $\max \mathcal{G}_{\leq x} = 1$  $\{x_1, \ldots, x_k\}$  are of the form  $x_t = \varphi_x^{el}(\hat{0}, \ldots, \hat{0}, y_{i_1}, \hat{0}, \ldots, \hat{0}, y_{i_2}, \hat{0}, \ldots, \hat{0}, y_{i_3}, \hat{0}, \ldots, \hat{0}, y_{i_2}, \hat{0}, \ldots, \hat{0}, y_{i_3}, \dots, \hat{0}, \dots, \hat{0}, y_{i_3}, \dots, \hat{0},$  $y_{i_{|\pi_{*}|}}, \hat{0}).$

Informally speaking, the factors of x in G are products of disjoint sets of elementary divisors of x.

- (3)  $\mathcal{G}$  generates  $\mathcal{L}$  by  $\vee$ , and for any  $x \in \mathcal{L}$ , any  $\{y, y_1, \dots, y_t\} \subseteq \max \mathcal{G}_{\leq x}$ , and  $z \in \mathcal{L}$  with z < y, we have  $\mathcal{G}_{\leq y} \cap \mathcal{G}_{\leq z \vee y_1 \vee ... \vee y_t} = \mathcal{G}_{\leq z}$ .
- (4)  $\mathcal{G}$  generates  $\mathcal{L}$  by  $\vee$ , and for any  $x \in \mathcal{L}$ , any  $\{y, y_1, \dots, y_t\} \subseteq \max \mathcal{G}_{\leq x}$ , and  $z \in \mathcal{L}$  with z < y, the following two conditions are satisfied:
  - $i) \quad \mathcal{G}_{\leq u} \cap \mathcal{G}_{\leq u_1 \vee \dots \vee u_t} = \{\hat{0}\}$  $\begin{array}{ll} i) & \mathcal{G}_{\leq y} \cap \mathcal{G}_{\leq y_1 \vee \ldots \vee y_t} = \{0\} & \text{``disjointness,''} \\ \ddot{u}) & z \vee y_1 \vee \ldots \vee y_t < y \vee y_1 \vee \ldots \vee y_t & \text{``necessity.''} \end{array}$

#### Proof.

 $\underline{(2)}\Rightarrow(1)$ : The decomposition of the interval  $[\hat{0},x]$  into intervals below the elements in  $\max \mathcal{G}_{\leq x}$  follows from Proposition 2.2.1 by assembling factors  $[\hat{0},y_j]$  with maximal elements indexed by elements from the same block of the partition  $\pi_x$  into one factor.

 $\underline{(1)}\Rightarrow(3)$ : (3) is a direct consequence of  $[\hat{0},x]$  decomposing into a direct product of the form described in the definition of building sets.

 $\underline{(3)}\Rightarrow\underline{(4)}$ : i) follows by setting  $z=\hat{0}$  in (3). Equality in  $\ddot{u}$ ) implies with (3) that  $\overline{\mathcal{G}}_{\leq y}=\overline{\mathcal{G}}_{\leq z}$ , in particular,  $y\in\mathcal{G}_{\leq z}$  – a contradiction to z< y.

(4) $\Rightarrow$ (1): For  $x \in \mathcal{L}$  and max  $\mathcal{G}_{\leq x} = \{x_1, \dots, x_k\}$  consider the poset map

$$\phi: \prod_{j=1}^k [\hat{0}, x_j] \longrightarrow [\hat{0}, x], \quad (\alpha_1, \dots, \alpha_k) \longmapsto \alpha_1 \vee \dots \vee \alpha_k.$$

i)  $\phi$  is surjective: For  $\hat{0} \neq y \leq x$ , let  $\max \mathcal{G}_{\leq y} = \{y_1, \dots, y_t\}$ . First, we have  $\bigvee_{i=1}^t y_i = y$ , since  $\mathcal{G}$  generates  $\mathcal{L}$  by  $\vee$ . Second, define  $\gamma_j = \bigvee_{y_i \in S_j} y_i$ , with  $S_j = (\max \mathcal{G}_{\leq y}) \cap \mathcal{G}_{\leq x_j}$ , for  $j = 1, \dots, k$ . Clearly,  $\gamma_j \in [\hat{0}, x_j]$ , and  $\bigcup_{j=1}^k S_j = \max \mathcal{G}_{\leq y}$ , since  $\mathcal{G}_{\leq y} \subseteq \mathcal{G}_{\leq x}$ . Hence,  $\phi(\gamma_1, \dots, \gamma_k) = \bigvee_{i=1}^t y_i = y$ .

ii)  $\phi$  is injective: a) Assume  $\phi(\alpha_1,\ldots,\alpha_k)=\phi(\beta_1,\ldots,\beta_k)=y\neq x$ , and let  $\max\mathcal{G}_{\leq y}=\{y_1,\ldots,y_t\}$ . By induction on the number of elements in  $[\hat{0},x]$  we can assume that  $[\hat{0},y]$  decomposes as a direct product  $[\hat{0},y]\cong\prod_{i=1}^t[\hat{0},y_i]$ . Moreover, the subsets  $S_j$  of  $\max\mathcal{G}_{\leq y}$  defined in i) actually partition  $\max\mathcal{G}_{\leq y}$  as follows from the disjointness property applied to pairwise intersections of the  $\mathcal{G}_{\leq x_j}$ . Thus,  $[\hat{0},y]\cong\prod_{j=1}^k[\hat{0},\gamma_j]$ , with elements  $\gamma_j\in[\hat{0},x_j]$  as above, and it follows that  $\alpha_j=\beta_j=\gamma_j$  for  $j=1,\ldots,k$ .

b) Assume that  $\phi(\alpha_1, \ldots, \alpha_k) = \phi(\beta_1, \ldots, \beta_k) = x$ . By the necessity property it follows that  $\alpha_j = \beta_j = x_j$  for  $j = 1, \ldots, k$ .  $\square$ 

**Remark 2.2.4** The definition of building sets and of irreducible elements, as well as the characterization of building sets in Proposition 2.2.3 (2), are independent of the existence of a join operation and can be formulated for any poset with a unique minimal element.

We gather a few important properties of building sets.

**Proposition 2.2.5** *For a building set* G *of* L*, the following holds:* 

- (1) Let  $x \in \mathcal{L}$ ,  $F(x) = \{x_1, \dots, x_k\}$  the set of factors of x in  $\mathcal{G}$ , and  $\hat{0} \neq y \in \mathcal{G}$  with  $y \leq x$ . Then there exists a unique  $j \in \{1, \dots, k\}$  such that  $y \leq x_j$ ; i.e.,  $F(x) = \max \mathcal{G}_{\leq x}$  induces a partition of  $\mathcal{G}_{\leq x} \setminus \{\hat{0}\}$ .
- (2) For  $x \in \mathcal{L}$  and  $x_0 \in F(x)$ ,

$$\bigvee (F(x) \setminus \{x_0\}) < \bigvee F(x) = x,$$

i.e., each factor of x in G is needed to generate x.

(3) If  $h_1, \ldots, h_k$  in  $\mathcal{G}$  are such that  $(h_i, \bigvee_{j=1}^k h_j] \cap \mathcal{G} = \emptyset$  for  $i = 1, \ldots, k$ , then  $F(\bigvee_{j=1}^k h_j) = \{h_1, \ldots, h_k\}.$ 

**Proof.** (1) is a consequence of Proposition 2.2.3 (4)i), as was noted already in the proof of (4) $\Rightarrow$ (1), part ii) a), in the previous proposition. Taking the full set of factors and setting  $z = \hat{0}$  in Proposition 2.2.3 (4)i, yields (2). For (3) note that  $\{h_1, \ldots, h_k\} \subseteq F(\bigvee_{j=1}^k h_j)$  by assumption. If  $\{h_1, \ldots, h_k\}$  were not the complete set of factors, we would obtain a contradiction to (2).  $\square$ 

#### 2.2.3 Nested sets

In this subsection we define the notion of nested subsets of a building set of a semilattice and prove some of their properties.

**Definition 2.2.6** Let  $\mathcal{L}$  be a semilattice and  $\mathcal{G}$  a building set of  $\mathcal{L}$ . A subset N in  $\mathcal{G}$  is called **nested** if, for any set of incomparable elements  $x_1, \ldots, x_t$  in N of cardinality at least two, the join  $x_1 \vee \ldots \vee x_t$  exists and does not belong to  $\mathcal{G}$ . The nested sets in  $\mathcal{G}$  form an abstract simplicial complex, denoted  $\mathcal{N}(\mathcal{G})$ .

Note that the elements of  $\mathcal{G}$  are the vertices of the complex of nested sets  $\mathcal{N}(\mathcal{G})$ . Moreover, the order complex of  $\mathcal{G}$  is a subcomplex of  $\mathcal{N}(\mathcal{G})$ , since linearly ordered subsets of  $\mathcal{G}$  are nested.

**Proposition 2.2.7** For a given semilattice  $\mathcal{L}$  and a subset N of a building set  $\mathcal{G}$  of  $\mathcal{L}$ , the following are equivalent:

- (1) N is nested.
- (2) Whenever  $x_1, \ldots, x_t$  are noncomparable elements in N, the join  $x_1 \vee \ldots \vee x_t$  exists, and  $F(x_1 \vee \ldots \vee x_t) = \{x_1, \ldots, x_t\}$ .

- (3) There exists a chain  $C \subseteq \mathcal{L}$ , such that  $N = \bigcup_{x \in C} F(x)$ .
- (4)  $N \in \Lambda$ , where  $\Lambda$  is the maximal subset of  $2^{\mathcal{G}}$ , for which the following three conditions are satisfied:
  - (o)  $\emptyset \in \Lambda$ , and  $\{g\} \in \Lambda$ , for  $g \in \mathcal{G}$ ;
  - (i) if  $N \in \Lambda$  and  $x \in \max N$ , then  $N_{\leq x} \in \Lambda$ ;
  - (ii) if  $N \in \Lambda$ , then  $\max N = F(\bigvee \max N)$ .

#### Proof.

(1) $\Rightarrow$ (2): Let N be a nested set, and let  $M = \{x_1, \ldots, x_t\} \subseteq N$  be a set of incomparable elements with  $\bigvee_{i=1}^t x_i \not\in \mathcal{G}$ . We can assume that for some  $x_j$  we have  $(x_j, \bigvee_{i=1}^t x_i] \cap \mathcal{G} \neq \emptyset$ , otherwise the claim follows by Proposition 2.2.5 (3). Without loss of generality, we may assume that there exists an element y, such that  $y \in (x_1, \bigvee_{i=1}^t x_i] \cap \mathcal{G}$ , and  $y \in \max \mathcal{G}_{\leq \bigvee M}$ . Define  $M' = \{x_1, \ldots, x_t\} \cap \mathcal{G}_{\leq y} = \{x_1 = x_{j_0}, x_{j_1}, \ldots, x_{j_k}\}$ , and  $z = \bigvee_{l=0}^k x_{j_l}$ . Since  $M' = \{x_{j_0}, x_{j_1}, \ldots, x_{j_k}\}$  is nested (it is a subset of N), we have the strict inequality z < y. Furthermore,

$$\bigvee_{i=1}^{t} x_i = z \vee \bigvee (M \setminus M') \le z \vee \bigvee (\max G_{\le \bigvee M} \setminus \{y\}) < \bigvee_{i=1}^{t} x_i,$$

where the first inequality follows from Proposition 2.2.5 (1) and the second inequality from Proposition 2.2.5 (2). We thus arrive to a contradiction, which finishes the proof.

- $(2)\Rightarrow(1)$ : Obvious.
- $(2)\Rightarrow(3)$ : Let N be a set satisfying condition (2). Fix a particular linear extension  $\{x_1,\ldots,x_k\}$  on the partial order of N, and define  $\alpha_j=x_1\vee\ldots\vee x_j$ , for  $j=1,\ldots,k$ . By (2) we have  $F(\alpha_j)=\max\{x_1,\ldots,x_j\}$ , and therefore  $x_j\in F(\alpha_j)$  and  $x_{j+1}\not\in F(\alpha_j)$  for  $j=1,\ldots,k$ . Hence, the  $\alpha_j$ 's are different and form a chain  $C=\alpha_1<\alpha_2<\cdots<\alpha_k$ . By construction,  $N=\bigcup_{x\in C}F(x)$ .
- $(1),(2)\Rightarrow(4)$ : Let N be a nested set, we shall prove that  $N\in\Lambda$  by induction on the size of N:
  - 1. if |N| = 0, then  $N \in \Lambda$  by condition (o);
  - 2. if  $|N| \ge 1$ , then  $\max N = F(\bigvee \max N)$  by condition (2). Furthermore, since  $|N_{< x}| < |N|$ , and  $N_{< x}$  is nested (it is a subset of N),  $N_{< x} \in \Lambda$  by induction. Hence  $N \in \Lambda$ .

Let  $y \in N' \cap F(\alpha_s)$ . If  $|N' \cap F(\alpha_s)| > 1$ ,

$$y < \bigvee (N' \cap F(\alpha_s)) \le \bigvee N' \le \alpha_s$$

where the strict inequality is a consequence of the necessity property for building sets. Thus,  $\bigvee N' \notin \mathcal{G}$ . If  $|N' \cap F(\alpha_s)| = 1$ , we have  $y < \bigvee N' \leq \alpha_s$ , due to N' being an antichain with |N'| > 1, and again  $\bigvee N' \notin \mathcal{G}$ .

 $(4)\Rightarrow(3)$ : We need the following fact:

**Fact.** If there exist elements  $x_1, ..., x_t$  and  $y_1, ..., y_k$  in  $\mathcal{L}$ , such that  $x_t > y_j$ , for j = 1, ..., k, and  $F(\bigvee_{i=1}^t x_i) = \{x_1, ..., x_t\}$ , and  $F(\bigvee_{j=1}^k y_j) = \{y_1, ..., y_k\}$ , then  $F(x_1 \lor ... \lor x_{t-1} \lor y_1 \lor ... \lor y_k) = \{x_1, ..., x_{t-1}, y_1, ..., y_k\}$ .

Once the fact above is proved, one can derive (3) as follows. For  $N \in \Lambda$  we shall form a chain  $C = (\alpha_1 < \ldots < \alpha_{|N|})$  such that  $N = \bigcup_{i=1}^{|N|} F(\alpha_i)$ . First, choose a linear extension  $\{x_1,\ldots,x_t\}$  of N. Then, set  $\alpha_t = \bigvee \max N$ ,  $\alpha_{t-1} = \bigvee \max(N\setminus\{x_t\})$ ,  $\alpha_{t-2} = \bigvee \max(N\setminus\{x_t,x_{t-1}\})$ , and so on. By (4)(ii), we have  $F(\alpha_t) = \max N$ . Applying (4)(i) to  $x_t \in \max N$ , and (4)(ii) to  $N_{< x_t}$ , we obtain  $F(\bigvee \max N_{< x_t}) = \max N_{< x_t}$ . Taking into account the fact above, we conclude that  $F(\alpha_{t-1}) = \max(N\setminus\{x_t\})$ , and, using the same argument iteratively, we arrive to  $N = \bigcup_{i=1}^t F(\alpha_i)$ .

**Proof of the fact.** Set  $\alpha = x_1 \vee \ldots \vee x_{t-1} \vee y_1 \vee \ldots \vee y_k$ . Since  $\alpha \leq \bigvee_{i=1}^t x_i$ , the factors of  $\alpha$  can be partitioned into groups of elements below the  $x_i$  for  $i = 1, \ldots, t$ , by Proposition 2.2.5 (1). Since  $x_i \leq \alpha$  for  $i = 1, \ldots, t-1$ , we obtain  $F(\alpha) = \{x_1, \ldots, x_{t-1}, \gamma_1, \ldots, \gamma_m\}$  with  $\gamma_j \leq x_t$  for  $j = 1, \ldots, m$ .

Again using Proposition 2.2.5 (1), the  $y_1, \ldots, y_k$  can be partitioned into groups below the factors  $\gamma_j$  for  $j=1,\ldots,m$ . The occurrence of one strict inequality  $\bigvee \{y_l \mid y_l \leq \gamma_j\} < \gamma_j$ , for some  $j \in [m]$ , yields a contradiction to

$$\alpha = \bigvee_{i=1}^{t-1} x_i \vee \bigvee_{j=1}^k y_j = \bigvee_{i=1}^{t-1} x_i \vee \bigvee_{j=1}^m \gamma_j,$$

due to the necessity property of building sets. Moreover, since the  $y_i$  are factors themselves, joins of more than two of the  $y_i$ 's are not elements of  $\mathcal{G}$ . Thus,  $y_i = \gamma_i$ , for  $i = 1, \ldots, k = m$ , as claimed.  $\square$ 

### 2.3 SEQUENCES OF COMBINATORIAL BLOWUPS

We introduce the notion of a combinatorial blowup of an element in a semilattice and prove that the set of semilattices is closed under this operation.

#### 2.3.1 Combinatorial blowups

**Definition 2.3.1** For a semilattice  $\mathcal{L}$  and an element  $\alpha \in \mathcal{L}$  we define a poset  $\mathrm{Bl}_{\alpha}\mathcal{L}$ , the **combinatorial blowup of**  $\mathcal{L}$  **at**  $\alpha$ , as follows:

- $\circ$  elements of  $\mathrm{Bl}_{\alpha}\mathcal{L}$ :
  - (1)  $y \in \mathcal{L}$ , such that  $y \ngeq \alpha$ ;
  - (2)  $[\alpha, y]$ , for  $y \in \mathcal{L}$ , such that  $y \not\geq \alpha$  and  $(y \vee \alpha)_{\mathcal{L}}$  exists (in particular,  $[\alpha, \hat{0}]$  can be thought of as the result of blowing up  $\alpha$ );
- $\circ$  order relations in  $\mathrm{Bl}_{\alpha}\mathcal{L}$ :
  - (1) y > z in  $\operatorname{Bl}_{\alpha} \mathcal{L}$  if y > z in  $\mathcal{L}$ ;
  - (2)  $[\alpha, y] > [\alpha, z]$  in  $\operatorname{Bl}_{\alpha} \mathcal{L}$  if y > z in  $\mathcal{L}$ ;
  - (3)  $[\alpha, y] > z$  in  $\operatorname{Bl}_{\alpha} \mathcal{L}$  if  $y \geq z$  in  $\mathcal{L}$ ;

where in all three cases  $y, z \not> \alpha$ .

Note that the atoms in  $\mathrm{Bl}_{\alpha}\mathcal{L}$  are the atoms of  $\mathcal{L}$  together with the element  $[\alpha,\hat{0}]$ . It is easy, albeit tedious, to check that the class of (meet-)semilattices is closed under combinatorial blowups.

**Lemma 2.3.2** *Let*  $\mathcal{L}$  *be a semilattice and*  $\alpha \in \mathcal{L}$ *, then*  $\mathrm{Bl}_{\alpha}\mathcal{L}$  *is a semilattice.* 

**Proof.** The joins in Bl  $_{\alpha}\mathcal{L}$  are defined by the rule

$$([\alpha, y] \vee [\alpha, z])_{\mathrm{Bl}_{\alpha}\mathcal{L}} = [\alpha, (y \vee z)_{\mathcal{L}}],$$
$$([\alpha, y] \vee z)_{\mathrm{Bl}_{\alpha}\mathcal{L}} = [\alpha, (y \vee z)_{\mathcal{L}}],$$
$$(y \vee z)_{\mathrm{Bl}_{\alpha}\mathcal{L}} = (y \vee z)_{\mathcal{L}},$$

which is applicable only if  $(y \vee z)_{\mathcal{L}}$  exists, otherwise the corresponding joins in  $\operatorname{Bl}_{\alpha}\mathcal{L}$  do not exist. Also, the first and second formulae are applicable only in the case  $(y \vee z)_{\mathcal{L}} \not\geq \alpha$ , otherwise the corresponding joins do not exist.

We check this by considering four possible cases separately:

$$\begin{cases} [\alpha, x] \ge [\alpha, y] \\ [\alpha, x] \ge [\alpha, z] \end{cases} \Leftrightarrow \begin{cases} x \ge y \\ x \ge z \end{cases} \Leftrightarrow x \ge (y \lor z)_{\mathcal{L}} \Leftrightarrow [\alpha, x] \ge [\alpha, (y \lor z)_{\mathcal{L}}].$$

$$\begin{cases} [\alpha, x] \geq [\alpha, y] \\ [\alpha, x] \geq z \end{cases} \Leftrightarrow \begin{cases} x \geq y \\ x \geq z \end{cases} \Leftrightarrow x \geq (y \vee z)_{\mathcal{L}} \Leftrightarrow [\alpha, x] \geq [\alpha, (y \vee z)_{\mathcal{L}}].$$

$$\begin{cases} x \geq_{\operatorname{Bl}_{\alpha} \mathcal{L}} y \\ x \geq_{\operatorname{Bl}_{\alpha} \mathcal{L}} z \\ x \not\geq \alpha \end{cases} \Leftrightarrow \begin{cases} x \geq_{\mathcal{L}} y \\ x \geq_{\mathcal{L}} z \\ x \not\geq \alpha \end{cases} \Leftrightarrow \begin{cases} x \geq (y \vee z)_{\mathcal{L}} \\ x \not\geq \alpha \end{cases} \Rightarrow \begin{cases} y \vee z \not\geq \alpha \\ x \geq (y \vee z)_{\mathcal{L}}. \end{cases}$$

$$\begin{cases} [\alpha, x] \geq y \\ [\alpha, x] \geq z \end{cases} \Leftrightarrow \begin{cases} x \geq_{\mathcal{L}} y \\ x \geq_{\mathcal{L}} z \\ x, y, z \not\geq \alpha \end{cases} \Leftrightarrow \begin{cases} x \geq (y \vee z)_{\mathcal{L}} \\ x, (y \vee z)_{\mathcal{L}} \not\geq \alpha \end{cases} \Rightarrow [\alpha, x] \geq (y \vee z)_{\operatorname{Bl}_{\alpha} \mathcal{L}}.$$

Observe that it is possible that  $(x \vee y)_{\mathcal{L}}$  exists, while  $(x \vee y)_{\mathrm{Bl}_{\alpha}\mathcal{L}}$  does not.  $\square$ 

## 2.3.2 Blowing up building sets

In this subsection we prove that if one combinatorially blows up a building set of a semilattice in any chosen linear extension order, then one ends up with the face poset of the simplicial complex of nested sets of this building set. The following proposition provides the essential step for the proof.

**Proposition 2.3.3** *Let*  $\mathcal{L}$  *be a semilattice,*  $\mathcal{G}$  *a building set of*  $\mathcal{L}$ *, and*  $\alpha \in \max \mathcal{G}$ . *Then,*  $\widetilde{\mathcal{G}} = (\mathcal{G} \setminus \{\alpha\}) \cup \{[\alpha, \hat{0}]\}$  *is a building set of*  $\mathrm{Bl}_{\alpha}\mathcal{L}$ . *Furthermore, the nested subsets of*  $\widetilde{\mathcal{G}}$  *are precisely the nested subsets of*  $\mathcal{G}$  *with*  $\alpha$  *replaced by*  $[\alpha, \hat{0}]$ .

**Proof.** It is easy to see that  $\widetilde{\mathcal{G}}$  is a building set of  $\mathrm{Bl}_{\alpha}\mathcal{L}$ . Indeed, given  $x \in \mathcal{L} \setminus \mathcal{L}_{\geq \alpha}$ , (2.2.1) is obvious for  $x \in \mathrm{Bl}_{\alpha}\mathcal{L}$ , and, if  $(x \vee \alpha)_{\mathcal{L}}$  exists, it follows for  $[\alpha, x] \in \mathrm{Bl}_{\alpha}\mathcal{L}$  from the identity

$$[\hat{0}, [\alpha, x]]_{\mathrm{Bl}_{\alpha}\mathcal{L}} = [\hat{0}, x]_{\mathrm{Bl}_{\alpha}\mathcal{L}} \times B_1,$$

where  $B_1$  is the subposet consisting of the two comparable elements  $\hat{0} < [\alpha, \hat{0}]$ .

Let us now see that the sets of nested subsets of  $\mathcal{G}$  and  $\widetilde{\mathcal{G}}$  are the same when replacing  $\alpha$  by  $[\alpha, \hat{0}]$ :

Let N be a nested set in  $\mathcal{G}$ , not containing  $\alpha$ . For incomparable elements  $x_1, \ldots, x_t$  in  $N, \bigvee_{i=1}^t x_i \not\geq \alpha$ , since otherwise we had

$$\alpha \in \max \mathcal{G}_{\leq \bigvee x_i} = F(\bigvee_{i=1}^t x_i) = \{x_1, \dots, x_t\}$$

by Proposition 2.2.7(2). Thus,  $\bigvee_{i=1}^t x_i$  exists in  $\operatorname{Bl}_{\alpha} \mathcal{L}$  and  $\bigvee_{i=1}^t x_i \notin \widetilde{\mathcal{G}}$ . Hence, N is nested in  $\widetilde{\mathcal{G}}$ . A nested subset in  $\widetilde{\mathcal{G}}$  not containing  $[\alpha, \hat{0}]$  is obviously nested in  $\mathcal{G}$ .

Let now N be nested in  $\mathcal G$  containing  $\alpha$ , and set  $\widetilde N=(N\setminus\{\alpha\})\cup\{[\alpha,\hat 0]\}$ . Subsets of incomparable elements in  $\widetilde N$  not containing  $[\alpha,\hat 0]$  can be dealt with as above. Thus assume that  $[\alpha,\hat 0],x_1,\ldots,x_t$  are incomparable in  $\widetilde N$ . Then,  $x_1,\ldots,x_t$  are incomparable in the nested set N, and, as above, we conclude that  $\bigvee_{i=1}^t x_i$  exists and  $\bigvee_{i=1}^t x_i \not\geq \alpha$ . Moreover,  $\alpha\vee\bigvee_{i=1}^t x_i$  exists in  $\mathcal L$  (joins of nested sets always exist!), thus,  $[\alpha,\bigvee_{i=1}^t x_i]=[\alpha,\hat 0]\vee\bigvee_{i=1}^t x_i$  exists in  $\mathrm{Bl}_{\alpha}\mathcal L$  and is obviously not contained in  $\widetilde \mathcal G$ . We conclude that  $\widetilde N$  is nested in  $\widetilde \mathcal G$ .

Vice versa, let  $\widetilde{N}$  be nested in  $\widetilde{\mathcal{G}}$  containing  $[\alpha,\hat{0}]$ , and set  $N=(\widetilde{N}\setminus\{[\alpha,\hat{0}]\})\cup\{\alpha\}$ . Again it suffices to consider subsets of incomparable elements  $\alpha,x_1,\ldots,x_t$  in N. With  $[\alpha,\hat{0}],x_1,\ldots,x_t$  incomparable in  $\widetilde{N}$ ,  $[\alpha,\hat{0}]\vee\bigvee_{i=1}^t x_i=[\alpha,\bigvee_{i=1}^t x_i]$  exists in  $\mathrm{Bl}_{\alpha}\mathcal{L}$ , thus  $\alpha\vee\bigvee_{i=1}^t x_i$  exists in  $\mathcal{L}$ . Incomparability implies that  $\alpha\vee\bigvee_{i=1}^t x_i>\alpha$ , and thus  $\alpha\vee\bigvee_{i=1}^t x_i\not\in\mathcal{G}$ . We conclude that N is nested in  $\mathcal{G}$ .  $\square$ 

By iterating the combinatorial blowup described in Proposition 2.3.3 through all of  $\mathcal{G}$ , we obtain the following theorem, which serves as a motivation for the entire development.

**Theorem 2.3.4** Let  $\mathcal{L}$  be a semilattice and  $\mathcal{G}$  a building set of  $\mathcal{L}$  with some chosen linear extension:  $\mathcal{G} = \{G_1, \ldots, G_t\}$ , with  $G_i > G_j$  implying i < j. Let  $\mathrm{Bl}_k \mathcal{L}$  denote the result of subsequent blowups  $\mathrm{Bl}_{G_k}(\mathrm{Bl}_{G_{k-1}}(\ldots \mathrm{Bl}_{G_1}\mathcal{L}))$ . Then the final semilattice  $\mathrm{Bl}_t \mathcal{L}$  is equal to the face poset of the simplicial complex  $\mathcal{N}(\mathcal{G})$ .

**Proof.** The building set  $\mathcal{G}_t$  of  $\mathrm{Bl}_t\mathcal{L}$  that results from iterated application of Proposition 2.3.3 obviously is the set of atoms  $\mathfrak{A}$  in  $\mathrm{Bl}_t\mathcal{L}$ . Every element  $x \in \mathrm{Bl}_t\mathcal{L}$  is the join of atoms below it:  $x = \bigvee \mathfrak{A}_{\leq x}$ . The subset  $\mathfrak{A}_{\leq x}$  of  $\mathcal{G}_t$  is nested, in particular, it is the set of factors of x in  $\mathrm{Bl}_t\mathcal{L}$  with respect to  $\mathcal{G}_t$  (Proposition 2.2.7(2)). Proposition 2.2.5(2) implies that the interval  $[\hat{0},x]$  in  $\mathrm{Bl}_t\mathcal{L}$  is boolean. We conclude that  $\mathrm{Bl}_t\mathcal{L}$  is the face poset of a simplicial complex with faces in one-to-one correspondence with the nested sets in  $\mathcal{G}_t$ , which in turn correspond to the nested sets in  $\mathcal{G}$  by Proposition 2.3.3.  $\square$ 

#### 2.4 DE CONCINI-PROCESI MODELS OF SUBSPACE ARRANGEMENTS

#### 2.4.1 Building sets for subspace arrangements

Let  $\mathcal{A} = \{A_1, \dots, A_n\}$  be an arrangement of linear subspaces in  $\mathbb{C}^d$ . Much effort has been spent on describing the cohomology of the complement  $\mathcal{M}(\mathcal{A}) = \mathbb{C}^d \setminus \bigcup \mathcal{A}$  of such an arrangement and, in particular, on answering the question whether the cohomology algebra is completely determined by the combinatorial data of the arrangement. Here, combinatorial data is understood as the lattice  $\mathcal{L}(\mathcal{A})$  of intersections of subspaces of  $\mathcal{A}$  ordered by reverse inclusion together

with the complex codimensions of the intersections. A major step towards the solution of this problem (for a complete answer see [DGM00, dLS01]) was the construction of smooth models for the complement  $\mathcal{M}(\mathcal{A})$  by De Concini & Procesi [DP95] that allowed for an explicit description of rational models for  $\mathcal{M}(\mathcal{A})$  following [Mo78]. The De Concini-Procesi models for arrangements in turn are one instance in a sequence of model constructions reaching from compactifications of symmetric spaces [DP83, DP85], over the Fulton-MacPherson compactifications of configuration spaces [FM94] to the general framework of wonderful conical compactifications proposed by MacPherson & Procesi [MP98].

Given a complex subspace arrangement  $\mathcal{A}$  in  $\mathbb{C}^d$ , De Concini & Procesi describe a smooth irreducible variety Y together with a proper map  $\pi:Y\longrightarrow\mathbb{C}^d$  such that  $\pi$  is isomorphism over  $\mathcal{M}(\mathcal{A})$ , and the complement of the preimage of  $\mathcal{M}(\mathcal{A})$  is a union of irreducible divisors with normal crossings in Y. The model Y can be constructed by a sequence of blowups of smooth subvarieties that is prescribed by the stratification of complex space induced by the arrangement.

In order to enumerate the strata in the intersection stratification of Y given by the irreducible divisors, De Concini & Procesi introduced the notions of building sets, nested sets and irreducible elements as follows:

**Definition 2.4.1** ([DP95, §2]) Let  $\mathcal{L}(A)$  be the intersection lattice of an arrangement A of linear subspaces in a finite dimensional complex vector space. Consider the lattice  $\mathcal{L}(A)^*$  formed by the orthogonal complements of intersections ordered by inclusion.

- (1) For  $U \in \mathcal{L}(\mathcal{A})^*$ ,  $U = \bigoplus_{i=1}^k U_i$  with  $U_i \in \mathcal{L}(\mathcal{A})^*$ , is called a **decomposition** of U if for any  $V \subseteq U$ ,  $V \in \mathcal{L}(\mathcal{A})^*$ ,  $V = \bigoplus_{i=1}^k (U_i \cap V)$  and  $U_i \cap V \in \mathcal{L}(\mathcal{A})^*$  for i = 1, ..., k.
- (2) Call  $U \in \mathcal{L}(A)^*$  irreducible if it does not admit a non-trivial decomposition.
- (3) A subset  $\mathcal{G} \subseteq \mathcal{L}(\mathcal{A})^*$  is called a **building set** for  $\mathcal{A}$  if for any  $U \in \mathcal{L}(\mathcal{A})^*$  and  $G_1, \ldots, G_k$  maximal in  $\mathcal{G}$  below  $U, U = \bigoplus_{i=1}^k G_i$  is a decomposition (the  $\mathcal{G}$ -decomposition) of U.
- (4) A subset  $S \subseteq G$  is called **nested** if for any set of non-comparable elements  $U_1, \ldots, U_k$  in S,  $U = \bigoplus_{i=1}^k U_i$  is the G-decomposition of U.

Note that  $\mathcal{L}(\mathcal{A})^*$  coincides with  $\mathcal{L}(\mathcal{A})$  as abstract lattices. We will therefore talk about irreducible elements, building sets and nested sets in  $\mathcal{L}(\mathcal{A})$  without explicitly referring to the dual setting of the preceding definition.

The notions of Definition 2.4.1 are in part based on the earlier notions introduced by Fulton & MacPherson in [FM94] to study compactifications of configuration spaces. Our terminology is naturally adopted from [FM94, DP95]. Building sets and nested sets in the sense of De Concini & Procesi are building and nested sets for the intersection lattices of subspace arrangements in our combinatorial sense (see Proposition 2.4.5 (1) below). However, there are differences. The opposite is not true: A combinatorial building set for the intersection lattice of a subspace arrangement is not necessarily a building set for this arrangement in the sense of De Concini & Procesi, neither are irreducible elements in the sense of De Concini & Procesi irreducible in our sense.

**Example 2.4.2** Consider the following arrangement of 3 subspaces in  $\mathbb{C}^4$ :

$$A_1: z_4 = 0, \quad A_2: z_1 = z_2 = 0, \quad A_3: z_1 = z_3 = 0.$$

The intersection lattice is a boolean algebra on 3 elements.  $\{A_1, A_2, A_3\} \subseteq \mathcal{L}(\mathcal{A})$  is a combinatorial building set, in fact, it is the set of irreducibles in the abstract lattice. However, the minimal building set in the sense of De Concini & Procesi is  $\{A_1, A_2, A_3, A_2 \cap A_3\}$ .

The main difference can be formulated in the following way: our constructions are order-theoretically canonical for a given semilattice. The set of combinatorial building sets, in particular the set of irreducible elements, depends only on the semilattice itself and not on the geometry of the subspace arrangement which it encodes. See Proposition 2.4.5 for the complete explanation.

#### 2.4.2 Local subspace arrangements

In order to trace the De Concini-Procesi construction step by step we need the more general notion of a local subspace arrangement.

**Definition 2.4.3** Let M be a smooth complex d-dimensional manifold, and let A be a union of finitely many smooth complex submanifolds of M such that all non-empty intersections of submanifolds in A are connected smooth complex submanifolds. A is called a **local subspace arrangement** if for any  $x \in A$  there exists an open set N in M with  $x \in N$ , a subspace arrangement  $\widetilde{A}$  in  $\mathbb{C}^d$ , and a biholomorphic map  $\phi: N \to \mathbb{C}^d$ , such that  $\phi(N \cap A) = \widetilde{A}$ .

Given a subspace arrangement  $\mathcal{A}$ , the initial ambient space  $\mathbb{C}^d$  of  $\mathcal{M}(\mathcal{A})$  carries a natural stratification by the subspaces of  $\mathcal{A}$  and their intersections, the poset of strata being the intersection lattice  $\mathcal{L}(\mathcal{A})$  of the arrangement. For a local subspace arrangement  $\mathcal{A} = \{A_1, \dots, A_n\}$  in M we again consider the stratification

of M by all possible intersections of the  $A_i$ 's, just like in the global case. The poset of strata is also denoted by  $\mathcal{L}(\mathcal{A})$  and is called the intersection semilattice (it is a lattice if the intersection of all maximal strata is nonempty).

**Definition 2.4.4** Let A be a local subspace arrangement and  $\mathcal{L}(A)$  its intersection semilattice. For  $U \in \mathcal{L}(A)$ ,  $U_1, \ldots, U_k \in \mathcal{L}(A)$  are said to form a **decomposition** of U if for any  $x \in U$  there exists an open set N with  $x \in N$  and a biholomorphic map  $\phi: N \to \mathbb{C}^d$ , such that  $\phi(N \cap U_1), \ldots, \phi(N \cap U_k)$  form a decomposition of  $\phi(N \cap U)$  in the sense of Definition 2.4.1(1).

As in the global case,  $\mathcal{G} \subseteq \mathcal{L}(\mathcal{A})$  is a **building set** for  $\mathcal{A}$  if for any  $U \in \mathcal{L}(\mathcal{A})$ , the set of strata  $\max \mathcal{G}_{\leq U}$  gives a decomposition of U.

We shall refer to these building sets as *geometric* building sets. The difference between combinatorial building sets and geometric ones is contained in the dimension function as is explained in the following proposition.

**Proposition 2.4.5** *Let* A *be a local subspace arrangement with intersection semilattice*  $\mathcal{L}(A)$ .

- (1) If  $G \subseteq \mathcal{L}(A)$  is a geometric building set of A, then it is a combinatorial building set.
- (2) If  $G \subseteq \mathcal{L}(A)$  is a combinatorial building set of  $\mathcal{L}(A)$ , and for any  $x \in \mathcal{L}(A)$  the sum of codimensions of its factors is equal to the codimension of x, then G is a geometric building set.

**Proof.** In both cases it is enough to consider the case when A is a subspace arrangement.

(1) Consider  $\mathcal{G}$  as a subset of  $\mathcal{L}(\mathcal{A})^*$ , then, for  $U \in \mathcal{G}$ , the isomorphism  $\varphi_U$  requested in Definition 2.2.2 is given by taking direct sums:

$$\varphi_U: \prod_{j=1}^k \left[\hat{0}, G_j\right] \stackrel{\oplus_{j=1}^k}{\longrightarrow} \left[\hat{0}, U\right],$$

where  $G_1, \ldots, G_k$  are maximal in  $\mathcal{G}$  below U.

- (2) For  $U \in \mathcal{L}(\mathcal{A})^*$ , the set  $\{U_1, \dots, U_k\} = \max \mathcal{G}_{\leq U}$  gives a decomposition of U because:
  - a) By the definition of  $\mathcal{L}(\mathcal{A})^*$  and the definition of combinatorial building sets, we have  $U = \operatorname{span}(U_1, \dots, U_k)$ , and, since  $\sum_{i=1}^k \dim U_i = \dim U$ , we have  $U = \bigoplus_{i=1}^k U_i$ ;
  - b) for any  $V \subseteq U$ ,  $\bigoplus_{i=1}^k (U_i \cap V) \subseteq V = \operatorname{span}(U_1 \wedge V, \dots, U_k \wedge V) \subseteq \bigoplus_{i=1}^k (U_i \cap V)$ , where " $\wedge$ " denotes the meet operation in  $\mathcal{L}(\mathcal{A})^*$ , hence  $V = \bigoplus_{i=1}^k (U_i \cap V)$ .  $\square$

## 2.4.3 Intersection stratification of local arrangements after blowup

Let a space X be given with an intersection stratification induced by a local subspace arrangement, and let G be a stratum in X. In the blowup of X at G,  $\operatorname{Bl}_G X$ , we find the following maximal strata:

- $\circ$  maximal strata in X that do not intersect with G,
- o blowups of maximal strata V at  $G \cap V$ ,  $\operatorname{Bl}_{G \cap V} V$ , where V is maximal in X and intersects G,
- $\circ$  the exceptional divisor  $\widetilde{G}$  replacing G.

We consider the intersection stratification of  $\mathrm{Bl}_G X$  induced by these maximal strata. We will later see (proof of Proposition 2.4.7) that in case G is maximal in a building set for the local arrangement in X, then the union of maximal strata in  $\mathrm{Bl}_G X$  is again a local arrangement with induced intersection stratification. In general, this is not the case, see Remark 2.4.6.

For ease of notation, let us agree here that formally blowing up an empty (non-existing) stratum has no effect on the space. We think about a stratum Y in X, intersection of all maximal strata  $V_1, \ldots, V_t$  that contain Y, as being replaced by the intersection of corresponding maximal strata in  $\mathrm{Bl}_G X$ :

$$\operatorname{Bl}_{G \cap V_1} V_1 \cap \ldots \cap \operatorname{Bl}_{G \cap V_t} V_t,$$
 (2.4.1)

(recall that  $\operatorname{Bl}_{G\cap V_j}V_j=V_j$  for  $G\cap V_j=\emptyset$ ). The intersection (2.4.1) being empty means that the stratum Y vanishes under blowup of G. For notational convenience, we most often retain names of strata under blowups, thereby referring to the replacement of strata described above.

**Remark 2.4.6** Let us mention here that blowing up a stratum in a local subspace arrangement does not necessarily result in a local subspace arrangement again. Consider the following arrangement of 2 planes and 1 line in  $\mathbb{C}^3$ :

$$A_1: y-z=0, A_2: y+z=0, L: x=y=0.$$

After blowing up L, the planes  $A_1$  and  $A_2$  are replaced by complex line bundles over  $\mathbb{C}\mathrm{P}^1$ , which have in common their zero section Z and a complex line Y; L is replaced by a direct product of  $\mathbb{C}$  and  $\mathbb{C}\mathrm{P}^1$ , which intersects both line bundles in Z. The new maximal strata fail to form a local subspace arrangement in the point  $Z \cap Y$ .

## 2.4.4 Tracing incidence structure during arrangement model construction

We now give a more detailed description of the model construction by De Concini & Procesi via successive blowups, and then proceed with linking our notion of combinatorial blowups to the context of arrangement models.

Let  $\mathcal{A}$  be a complex subspace arrangement,  $\mathcal{G} \subseteq \mathcal{L}(\mathcal{A})$  a geometric building set for  $\mathcal{A}$ , and  $\{G_1,\ldots,G_t\}$  some linear extension of the partial containment order on associated strata in  $\mathbb{C}^d$  such that  $G_k \supset G_l$  implies l < k. The De Concini-Procesi model  $Y = Y_{\mathcal{G}}$  of  $\mathcal{M}(\mathcal{A})$  is the result of blowing up the strata indexed by elements of  $\mathcal{G}$  in the given order. Note that the linear order was chosen so that at each step the stratum which is to be blown up does *not* contain any other stratum indexed by an element of  $\mathcal{G}$ . At each step we consider intersection stratifications as described above, and we denote the poset of strata after blowup of  $G_i$  with  $\mathcal{L}_i^{\mathcal{G}}(\mathcal{A})$ . For the case of a stratum  $G_i$  being empty after previous blowups remember our agreement of considering blowups of  $\emptyset$  as having no effect on a space. The later Proposition 2.4.7 however shows that strata indexed by elements in  $\mathcal{G}$  do not disappear during the sequence of blowups.

Let us remark that the combinatorial data of the initial stratification, i.e., of the arrangement, prescribes much of the geometry of  $Y_{\mathcal{G}}$ : the complement  $Y_{\mathcal{G}} \setminus \mathcal{M}(\mathcal{A})$  is a union of smooth irreducible divisors indexed by elements of  $\mathcal{G}$ , and these divisors intersect if and only if the set of indices is nested in  $\mathcal{G}$  [DP95, Thm 3.2].

**Proposition 2.4.7** Let A be an arrangement of complex subspaces, G a building set for A in the sense of De Concini & Procesi, and  $\{G_1, \ldots, G_t\}$  some linear extension of the partial containment order on associated strata as described above. Let  $\mathrm{Bl}_i^G(A)$  denote the geometric result of successively blowing up strata  $G_1, \ldots, G_i$ , for  $1 \leq i \leq t$ . Then,

(1) The poset of strata  $\mathcal{L}_i^{\mathcal{G}}(\mathcal{A})$  of  $\mathrm{Bl}_i^{\mathcal{G}}(\mathcal{A})$  can be described as the result of a sequence of combinatorial blowups of the intersection lattice  $\mathcal{L} = \mathcal{L}(\mathcal{A})$ :

$$\mathcal{L}_{i}^{\mathcal{G}}(\mathcal{A}) = \operatorname{Bl}_{i}(\mathcal{L}), \quad \text{for } 1 \leq i \leq t.$$

(Recall that 
$$\operatorname{Bl}_{i}(\mathcal{L}) = \operatorname{Bl}_{G_{i}}(\operatorname{Bl}_{G_{i-1}}(\ldots \operatorname{Bl}_{G_{1}}\mathcal{L}))$$
 for  $1 \leq i \leq t$ .)

(2) The union of maximal strata  $\mathcal{A}_i^{\mathcal{G}}$  in  $\mathrm{Bl}_i^{\mathcal{G}}(\mathcal{A})$  is a local subspace arrangement, with  $\mathcal{G}$  in  $\mathcal{L}_i^{\mathcal{G}}(\mathcal{A})$  being a building set for  $\mathcal{A}_i^{\mathcal{G}}$  in the sense of Definition 2.4.4. (Recall that  $\mathcal{G}$  here refers to the preimages of the original strata in  $\mathcal{G} \subseteq \mathcal{L}(\mathcal{A})$  under the sequence of blowups.)

**Proof.** We proceed by induction on the number of blowups. The induction start is obvious, since the lattice of strata  $\mathcal{L}_0^{\mathcal{G}}(\mathcal{A})$  of the initial stratification of  $\mathbb{C}^d$  coincides

with the intersection lattice  $\mathcal{L}(\mathcal{A}) = \mathrm{Bl}_0(\mathcal{L})$  of the arrangement  $\mathcal{A}$ . The union of maximal strata is the arrangement  $\mathcal{A}$  itself with its given building set  $\mathcal{G}$ .

Assume that  $\mathcal{L}_{i-1}^{\mathcal{G}}(\mathcal{A}) = \operatorname{Bl}_{i-1}(\mathcal{L})$  for some  $1 \leq i \leq t$ , the union of maximal strata  $\mathcal{A}_{i-1}^{\mathcal{G}}$  in  $\operatorname{Bl}_{i-1}^{\mathcal{G}}(\mathcal{A})$  being a local arrangement, and  $\mathcal{G}$  a building set for  $\mathcal{L}_{i-1}^{\mathcal{G}}(\mathcal{A})$ . Let  $G = G_i$  be the next stratum to be blown up. First, we proceed in 4 steps to show that  $\mathcal{L}_i^{\mathcal{G}}(\mathcal{A}) = \operatorname{Bl}_i(\mathcal{L})$ . In 2 further steps we then verify the claims in (2).

**Step 1:** Assign strata of  $\mathrm{Bl}_{i}^{\mathcal{G}}(\mathcal{A})$  to elements in  $\mathrm{Bl}_{i}(\mathcal{L})$ .

We distinguish two types of elements in  $Bl_i(\mathcal{L})$ :

$$\begin{array}{ll} \text{Type I}: & Y \quad \text{with } Y \in \operatorname{Bl}_{i-1}(\mathcal{L}) \text{ and } Y \not \geq G \,, \\ \text{Type II}: & [G,Y] \quad \text{with } Y \in \operatorname{Bl}_{i-1}(\mathcal{L}) \,, \ Y \not \geq G \,, \\ & \text{and } Y \vee G \text{ exists in } \operatorname{Bl}_{i-1}(\mathcal{L}) \,. \end{array}$$

To  $Y \in \mathrm{Bl}_i(\mathcal{L})$  of type I, assign  $\mathrm{Bl}_{G \cap Y} Y$  (recall that blowing up an empty stratum does not change the space). Note that  $\dim \mathrm{Bl}_{G \cap Y} Y = \dim Y$ .

To  $[G,Y] \in \operatorname{Bl}_i(\mathcal{L})$  of type  $\mathbb{I}$ , assign  $(\operatorname{Bl}_{G\cap Y}Y) \cap \widetilde{G}$ , where  $\widetilde{G}$  denotes the exceptional divisor that replaces G in  $\operatorname{Bl}_i^{\mathcal{G}}(\mathcal{A})$ . This description comprises  $\widetilde{G}$  being assigned to  $[G,\hat{0}]$ . Note that  $\dim(\operatorname{Bl}_{G\cap Y}Y) \cap \widetilde{G} = \dim Y - 1$ .

**Step 2:** Reverse inclusion order on the assigned spaces coincides with the partial order on  $\mathrm{Bl}_i(\mathcal{L})$ .

(1)  $X, Y \in \text{Bl}_{i}(\mathcal{L})$ , both of type I:

$$X \, \leq_{\operatorname{Bl}_{i}(\mathcal{L})} Y \, \Leftrightarrow \, X \, \leq_{\operatorname{Bl}_{i-1}(\mathcal{L})} Y \, \Leftrightarrow \, X \supseteq_{\operatorname{Bl}_{i-1}^{\mathcal{G}}(\mathcal{A})} Y \, \Leftrightarrow \, \operatorname{Bl}_{G \cap X} X \supseteq \operatorname{Bl}_{G \cap Y} Y,$$

where " $\Leftarrow$ " in the last equivalence can be seen by first noting that  $Y \setminus (G \cap Y) \subseteq X \setminus (G \cap X)$ , and then comparing points in the exceptional divisors.

(2)  $X, [G, Y] \in \mathrm{Bl}_{i}(\mathcal{L}), X$  of type I, [G, Y] of type II: As above we conclude

$$\begin{array}{cccc} X \leq_{\operatorname{Bl}_{i}(\mathcal{L})} [G,Y] & \Leftrightarrow & X \leq_{\operatorname{Bl}_{i-1}(\mathcal{L})} Y \\ & \Leftrightarrow & X \supseteq_{\operatorname{Bl}_{i-1}^{\mathcal{G}}(\mathcal{A})} Y & \Rightarrow & \operatorname{Bl}_{G \cap X} X \supseteq \operatorname{Bl}_{G \cap Y} Y \cap \widetilde{G} \,. \end{array}$$

To prove the converse is rather subtle. Note first that  $G \cap Y \subseteq G \cap X$ . Assume that G strictly contains  $G \cap X$ , then both  $G \cap X$  and  $G \cap Y$  are not in the building set due to the linear order chosen on G, and G is a factor of both  $G \cap X$  and  $G \cap Y$ . Let  $F(G \cap X) = \{G, G_1, \ldots, G_k\}$ ,  $F(G \cap Y) = \{G, H_1, \ldots, H_t\}$ . X written as a join of elements in  $Bl_{i-1}(\mathcal{L})$  below the factors of  $G \cap X$  reads

$$X = g_X \vee Z_1 \vee \ldots \vee Z_k$$

for some  $g_X \in [\hat{0}, G]$ ,  $Z_i \in [\hat{0}, G_i]$ , for i = 1, ..., k. If  $Z_i < G_i$ , for some  $i \in \{1, ..., k\}$ , we have

$$G \vee X = G \vee (g_X \vee Z_1 \vee \ldots \vee Z_i \vee \ldots \vee Z_k)$$

$$\leq G \vee (g_X \vee G_1 \vee \ldots \vee Z_i \vee \ldots \vee G_k)$$

$$< G \vee G_1 \vee \ldots \vee G_k = G \vee X,$$

by the "necessity" property of the Proposition 2.2.3(4), yielding a contradiction. Hence,

$$X = q_X \vee G_1 \vee \ldots \vee G_k$$

and similarly,  $Y = g_Y \vee H_1 \vee \ldots \vee H_t$  for some  $g_Y \in [\hat{0}, G]$ .

For each  $j \in \{1, ..., k\}$  there exists a unique  $i_j \in \{1, ..., t\}$  such that  $G_j \le H_{i_j}$  by Proposition 2.2.5(1). Thus,  $\bigvee G_i < \bigvee H_j$ , and, for showing that  $X \le Y$ , it is enough to see that  $g_X \le g_Y$ .

We show that in an open neighborhood of any point  $y \in G \cap Y$ ,  $g_Y \subseteq g_X$ . This yields our claim since strata in  $\mathrm{Bl}_{i-1}^{\mathcal{G}}(\mathcal{A})$  have pairwise transversal intersections: if they coincide locally, they must coincide globally. With  $\mathcal{A}_{i-1}^{\mathcal{G}}$  being a local arrangement, there exists an open neighborhood of  $y \in G \cap Y$  where the stratification is biholomorphic to a stratification induced by a subspace arrangement. We tacitly work in the arrangement setting, using that  $(\mathrm{Bl}_{i-1}(\mathcal{L}))_{\leq G \vee Y}$  is the intersection lattice of a product arrangement. The  $\mathcal{G}$ -decomposition of  $(G \vee Y)^{\perp}$  described in Definition 2.4.4 yields (when transferred to the primal setting):

$$g_Y = \operatorname{span}(G, Y)$$
.

Analogously,  $g_X = \operatorname{span}(G, X)$ .

In the linear setting we are concerned with, we interpret points in the exceptional divisor of a blowup as follows:

$$\operatorname{Bl}_{G\cap Y}Y\cap\widetilde{G}\ =\ \{(a,\operatorname{span}(p,G\cap Y))\mid a\in G\cap Y,\ p\in Y\setminus (G\cap Y)\}\ .\ \ (2.4.2)$$

In terms of this description, the inclusion map  $\mathrm{Bl}_{G\cap Y}Y\cap\widetilde{G}\hookrightarrow\mathrm{Bl}_G(\mathrm{Bl}_{i-1}^{\mathcal{G}}(\mathcal{A}))$  reads

$$(a,\operatorname{span}(p,G\cap Y))\longmapsto (a,\operatorname{span}(p,G))\,.$$

Therefore,  $\operatorname{Bl}_{G\cap Y}Y\cap\widetilde{G}$  being contained in  $\operatorname{Bl}_{G\cap X}X\subseteq\operatorname{Bl}_G(\operatorname{Bl}_{i-1}^{\mathcal{G}}(\mathcal{A}))$  means that for  $(a,\operatorname{span}(p,G\cap Y))\in\operatorname{Bl}_{G\cap Y}Y\cap\widetilde{G}$  there exists  $q\in X\setminus(G\cap X)$  such that  $\operatorname{span}(p,G)=\operatorname{span}(q,G)$ . In particular,  $\operatorname{span}(Y,G)\subseteq\operatorname{span}(X,G)$ , which by our previous arguments implies that  $Y\subseteq X$ .

We assumed above that  $G \supset G \cap X$ . If  $G \cap X$  coincides with G, i.e., X contains G, then  $g_X = X$  and a similar reasoning applies to see that  $Y \subseteq X$ . Similarly for  $G \cap X = G \cap Y = G$ .

(3) [G, X],  $[G, Y] \in \operatorname{Bl}_{i}(\mathcal{L})$ , both of type **I**:

$$\begin{split} [G,X] \leq_{\operatorname{Bl}_i(\mathcal{L})} [G,Y] &\; \Leftrightarrow \;\; X \leq_{\operatorname{Bl}_{i-1}(\mathcal{L})} Y \\ &\; \Leftrightarrow \;\; X \supseteq_{\operatorname{Bl}_{i-1}^{\mathcal{G}}(\mathcal{A})} Y \; \Leftrightarrow \; \operatorname{Bl}_{G\cap X} X \cap \widetilde{G} \supseteq \operatorname{Bl}_{G\cap Y} Y \cap \widetilde{G} \,, \end{split}$$

where " $\Leftarrow$ " follows from (2) and  $\operatorname{Bl}_{G\cap X}X\supseteq\operatorname{Bl}_{G\cap X}X\cap\widetilde{G}\supseteq\operatorname{Bl}_{G\cap Y}Y\cap\widetilde{G}$ .

**Step 3:** Each of the assigned spaces is the intersection of maximal strata in  $\mathrm{Bl}_{i}^{\mathcal{L}}(\mathcal{A})$ .

It is enough to show that spaces assigned to elements of type I in  $\mathrm{Bl}_i(\mathcal{L})$  are intersections of new maximal strata. Those associated to elements of type II then are intersections as well by definition.

Let  $Y \in Bl_i(\mathcal{L})$ ,  $Y \not\geq G$ , and  $Y = \bigcap_{i=1}^t V_i$  with  $V_1, \ldots, V_t$  the maximal strata in  $Bl_{i-1}^{\mathcal{G}}(\mathcal{A})$  containing Y. We claim that

$$\operatorname{Bl}_{G \cap Y} Y = \bigcap_{i=1}^{t} \operatorname{Bl}_{G \cap V_{i}} V_{i}. \tag{2.4.3}$$

For the inclusion " $\subseteq$ " note that  $\operatorname{Bl}_{G\cap Y}Y\subseteq\operatorname{Bl}_{G\cap V_i}V_i$  is a direct consequence of  $Y\subseteq V_i$  as discussed in Step 2 (1).

For the reverse inclusion we need the following identity:

$$\bigvee_{i=1}^{t} (G \wedge V_i) = G \wedge Y. \tag{2.4.4}$$

This identity holds in any semilattice without referring to G being an element of the building set.

Let  $\alpha \in \cap_{i=1}^t \operatorname{Bl}_{G \cap V_i} V_i$ . In case  $\alpha \in \cap_{i=1}^t V_i \setminus (G \cap V_i)$ , we conclude that  $\alpha \in Y \setminus (G \cap Y)$ . We thus assume that  $\alpha$  is contained in the intersection of exceptional divisors  $\widetilde{G \cap V_i}$ ,  $i=1,\ldots,t$ . We again switch to local considerations in the neighborhood of a point  $y \in G \cap Y$ , using that it carries a stratification biholomorphic to an arrangement stratification.

Using the description (2.4.2) of points in exceptional divisors that are created by blowups in the arrangement setting,  $\alpha \in \bigcap_{i=1}^t \widetilde{G \cap V_i} \subseteq \bigcap_{i=1}^t \operatorname{Bl}_{G \cap V_i} V_i$  means that there exist  $a \in \bigcap_{i=1}^t (G \cap V_i)$ , and  $p_i \in V_i \setminus (G \cap V_i)$  for  $i = 1, \ldots, t$ , with

$$\alpha = (a, \operatorname{span}(p_i, G \cap V_i)) \in \operatorname{Bl}_{G \cap V_i} V_i.$$

In particular,  $\operatorname{span}(p_i,G)=\operatorname{span}(p_j,G)$  for  $1\leq i,j\leq t.$  Thus,

$$\operatorname{span}(p_j, G) \subseteq \bigcap_{i=1}^t \operatorname{span}(V_i, G) = \operatorname{span}(Y, G)$$

using the identity (2.4.4). We conclude that there exists  $y \in Y \setminus (G \cap Y)$  such that  $\operatorname{span}(y, G) = \operatorname{span}(p_j, G)$  for all  $j \in \{1, \dots, k\}$ , hence

$$\alpha = (a, \operatorname{span}(y, G \cap Y)) \in \operatorname{Bl}_{G \cap Y} Y.$$

Though we are for the moment not concerned with the case of  $Y \subseteq G$ , we note for later reference that (2.4.3) remains true, with  $\operatorname{Bl}_Y Y = \emptyset$  meaning that the intersection on the right-hand side is empty. Following the proof of the inclusion " $\supseteq$ " in (2.4.3) for  $G \cap Y = Y$ , we first find that the intersection of blowups can only contain points in the exceptional divisors. Assuming  $\alpha \in \bigcap_{i=1}^t G \cap V_i$  we arrive to a contradiction when concluding that  $\operatorname{span}(p_j, G) \subseteq \bigcap_{i=1}^t \operatorname{span}(V_i, G) = \operatorname{span}(Y, G) = G$  for  $j = 1, \ldots, t$ .

**Step 4:** Any intersection of maximal strata in  $\mathrm{Bl}_i^{\mathcal{G}}(\mathcal{A})$  occurs as an assigned space.

Every intersection involving the exceptional divisor  $\widetilde{G}$  occurs if we can show that all other intersections occur (intersections that additionally involve  $\widetilde{G}$  then are assigned to corresponding elements of type  $\mathbb{I}$ ).

Consider  $W = \bigcap_{i=1}^t \operatorname{Bl}_{G \cap V_i} V_i$ , where the  $V_i$  are maximal strata in  $\operatorname{Bl}_{i-1}^{\mathcal{G}}(\mathcal{A})$ ; recall here that a blowup in an empty stratum does not alter the space. We can assume that  $\bigcap_{i=1}^t V_i \neq \emptyset$ , otherwise the intersection W were empty. With the identity (2.4.3) in Step 3 we conclude that either  $W = \emptyset$  (in case  $\bigcap_{i=1}^t V_i \subseteq G$ ) or  $W = \operatorname{Bl}_{G \cap \bigcap_{i=1}^t V_i} \bigcap_{i=1}^t V_i$ , in which case it is assigned to the element  $\bigcap_{i=1}^t V_i$  in  $\operatorname{Bl}_i(\mathcal{L})$ .

**Step 5:**  $A_i^{\mathcal{G}}$  *is a local subspace arrangement in*  $\mathrm{Bl}_i^{\mathcal{G}}(\mathcal{A})$ .

It follows from the description (2.4.3) of strata in  $\mathrm{Bl}_i^{\mathcal{G}}(\mathcal{A})$  that all intersections of maximal strata are connected and smooth. It remains to show that  $\mathcal{A}_i^{\mathcal{G}}$  locally looks like a subspace arrangement. Let  $y \in \mathcal{A}_i^{\mathcal{G}}$ . We can assume that y lies in the exceptional divisor  $\widetilde{G}$ . Let  $x \in G \subseteq \mathcal{A}_{i-1}^{\mathcal{G}}$  be the image of y under the blowdown map.

We first give a local description around x in  $\mathcal{A}_{i-1}^{\mathcal{G}}$ . By induction hypothesis, there exists a neighborhood N of x, and an arrangement of linear subspaces  $\mathcal{B}$  in  $\mathbb{C}^n$  such that the pair  $(N, \mathcal{A}_{i-1}^{\mathcal{G}} \cap N)$  is biholomorphic to the pair  $(\mathbb{C}^n, \mathcal{B})$ . We can assume that under this biholomorphic map, x is mapped to the origin. Let  $T = \bigcap_{B \in \mathcal{B}} B$  and note that  $G \cap N$  is mapped to some subspace  $\Gamma$  in  $\mathcal{B}$ .

With G being maximal in the building set for  $\mathcal{A}_{i-1}^{\mathcal{G}}$ ,  $\mathcal{B}/T$  is a product arrangement with one of the factors being an arrangement in  $\Gamma/T$ . More precisely, there exists a subspace  $\Gamma' \subseteq \mathbb{C}^n$ , and two subspace arrangements,  $\mathcal{C}$  in  $\Gamma/T$  and  $\mathcal{C}'$  in  $\Gamma/T$ , such that

(1) 
$$\Gamma/T \oplus \Gamma'/T \oplus T = \mathbb{C}^n$$
,

$$(2) \mathcal{B} = \{ A \oplus \Gamma'/T \oplus T \mid A \in \mathcal{C} \} \cup \{ \Gamma/T \oplus A' \oplus T \mid A' \in \mathcal{C}' \}.$$

Blowing up G in  $\mathrm{Bl}_{i-1}^{\mathcal{G}}(\mathcal{A})$  locally corresponds to blowing up  $\Gamma$  in  $\mathbb{C}^n$ . Let t be the point on the special divisor  $\widetilde{\Gamma}$  corresponding to  $y \in \widetilde{G}$ , thus t maps to the origin in  $\mathbb{C}^n$  under the blowdown map. A neighborhood of t in  $\mathrm{Bl}_{\Gamma}\mathbb{C}^n$  is an n-dimensional open ball which can be parameterized as a direct sum

$$M \oplus M' \oplus I \oplus T$$
.

Here, M is an open ball around 0 in  $\Gamma/T$ , M' is an open ball on the unit sphere in  $\Gamma'/T$  around the point of intersection with the line  $\langle p \rangle$  in  $\Gamma'/T$  that defines t as a point in the exceptional divisor,  $t=(0,\operatorname{span}(p,\Gamma))\in\widetilde{\Gamma}$  (compare (2.4.2)), and I an open unit ball in  $\mathbb{C}$ .

The maximal strata in this neighborhood are the following:

- $\circ$  the hyperplane  $M \oplus M' \oplus \{0\} \oplus T$ , as the exceptional divisor,
- $\circ (M \cap A) \oplus M' \oplus I \oplus T$ , replacing  $A \oplus \Gamma'/T \oplus T$  after blowup,
- o  $M \oplus (M' \cap A') \oplus I \oplus T$ , replacing  $\Gamma/T \oplus A' \oplus T$  after blowup for  $A' \neq 0$ .

This proves that around t in  $\mathrm{Bl}_{\Gamma}\mathbb{C}^n$  we have the structure of a local subspace arrangement, which in turn shows the local arrangement property around y in  $\mathcal{A}_i^{\mathcal{G}}$ . **Step 6:**  $\mathcal{G}$  is a building set for  $\mathcal{A}_i^{\mathcal{G}}$  in the sense of Definition 2.4.4.

 $\mathcal G$  is a combinatorial building set by Proposition 2.3.3. Complementing this with the dimension information about the strata, we conclude, by Proposition 2.4.5(2), that  $\mathcal G$  is a geometric building set.  $\square$ 

#### 2.5 SIMPLICIAL RESOLUTIONS OF TORIC VARIETIES

The study of toric varieties has proved to be a field of fruitful interplay between algebraic and convex geometry: toric varieties are determined by rational polyhedral fans, and many of their algebraic geometric properties are reflected by combinatorial properties of their defining fans.

We recall one such correspondence – between subdivisions of fans and special toric morphisms – and show that so-called stellar subdivisions are instances of combinatorial blowups. This allows us to apply our Main Theorem in the present context: Given a polyhedral fan, we specify a class of *simplicial* subdivisions, and, interpreting our notions of building sets and nested sets, we describe the incidence combinatorics of the subdivisions in terms of the combinatorics of the

initial fan. For background material on toric varieties we refer to the standard sources [Da78, Od88, Ful93, Ew96].

Let  $X_{\Sigma}$  be a toric variety defined by a rational polyhedral fan  $\Sigma$ . Any subdivision of  $\Sigma$  gives rise to a proper, birational toric morphism between the associated toric varieties (cf [Da78, 5.5.1]). In particular, simplicial subdivisions yield toric morphisms from quasi-smooth toric varieties to the initial variety – so-called *simplicial resolutions*. Quasi-smooth toric varieties being rational homology manifolds, such morphisms can replace smooth resolutions for (co)homological considerations.

We define a particular, elementary, type of subdivisions:

**Definition 2.5.1** Let  $\Sigma = {\sigma}_{\sigma \in \Sigma} \subseteq \mathbb{R}^d$  be a polyhedral fan, i.e., a collection of closed polyhedral cones  $\sigma$  in  $\mathbb{R}^d$  such that  $\sigma \cap \tau$  is a cone in  $\Sigma$  for any  $\sigma, \tau \in \Sigma$ . Let  $\operatorname{cone}(x)$  be a ray in  $\mathbb{R}^d$  generated by  $x \in \operatorname{relint} \tau$  for some  $\tau \in \Sigma$ . The stellar subdivision  $\operatorname{sd}(\Sigma, x)$  of  $\Sigma$  in x is given by the collection of cones

$$(\Sigma \setminus \operatorname{star}(\tau, \Sigma)) \cup \{\operatorname{cone}(x, \rho) | \rho \subseteq \sigma \text{ for some } \sigma \in \operatorname{star}(\tau, \Sigma)\},$$

where  $\operatorname{star}(\tau, \Sigma) = \{ \sigma \in \Sigma \mid \tau \subseteq \sigma \}$ , and  $\operatorname{cone}(x, \rho)$  the closed polyhedral cone spanned by  $\rho$  and x. If only concerned with the combinatorics of the subdivided fan, we also talk about stellar subdivision of  $\Sigma$  in  $\tau$ ,  $\operatorname{sd}(\Sigma, \tau)$ , meaning any stellar subdivision in x for  $x \in \operatorname{relint} \tau$ .

**Proposition 2.5.2** *Let*  $\mathcal{F}(\Sigma)$  *be the face poset of a polyhedral fan*  $\Sigma$ , *i.e., the set of closed cones in*  $\Sigma$  *ordered by inclusion, together with the zero cone*  $\{0\}$  *as a minimal element. For*  $\tau \in \Sigma$ , *the face poset of the stellar subdivision of*  $\Sigma$  *in*  $\tau$  *can be described as the combinatorial blowup of*  $\mathcal{F}(\Sigma)$  *at*  $\tau$ :

$$\mathcal{F}(\mathrm{sd}(\Sigma, \tau)) = \mathrm{Bl}_{\tau} \mathcal{F}(\Sigma).$$

**Proof.** We observe that removing  $\operatorname{star}(\tau, \Sigma)$  from  $\Sigma$  corresponds to removing  $\mathcal{F}(\Sigma)_{\geq \tau}$  from  $\mathcal{F}(\Sigma)$ , while adding cones, as described in Definition 2.5.1, corresponds to extending  $\mathcal{F}(\Sigma) \setminus \mathcal{F}(\Sigma)_{\geq \tau}$  by elements  $[\tau, \rho]$  for  $\rho \in \mathcal{F}(\Sigma)$ ,  $\rho \subseteq \sigma$  for some  $\sigma \in \operatorname{star}(\tau, \Sigma)$ . The comparison of order relations is straightforward.  $\square$ 

We apply our Main Theorem to the present context.

**Theorem 2.5.3** Let  $\Sigma$  be a polyhedral fan in  $\mathbb{R}^d$  with face poset  $\mathcal{F}(\Sigma)$ . Let  $\mathcal{G} \subseteq \mathcal{F}(\Sigma)$  be a building set of  $\mathcal{F}(\Sigma)$  in the sense of Definition 2.2.2,  $\mathcal{N}(\mathcal{G})$  the complex of nested sets in  $\mathcal{G}$  (cf. Definition 2.2.6). Then, the consecutive application of stellar subdivisions in every cone  $G \in \mathcal{G}$  in a non-increasing order yields a simplicial subdivision of  $\Sigma$  with face poset equal to the face poset of  $\mathcal{N}(\mathcal{G})$ .

As examples of building sets for face lattices of polyhedral fans let us mention:

- (1) the full set of faces, with the corresponding complex of nested sets being the order complex of  $\mathcal{F}(\Sigma)$  (stellar subdivision in all cones results in the barycentric subdivision of the fan);
- (2) the set of rays together with the non-simplicial faces of  $\Sigma$ ;
- (3) the set of irreducible elements in  $\mathcal{F}(\Sigma)$ : the set of rays together with all faces of  $\Sigma$  that are not products of some of their proper faces.

Remark 2.5.4 For a smooth toric variety  $X_{\Sigma}$ , the union of closed codimension 1 torus orbits is a local subspace arrangement, in particular, the codimension 1 orbits form a divisor with normal crossings, [Ful93, p. 100]. The intersection stratification of this local arrangement coincides with the torus orbit stratification of the toric variety. For any face  $\tau$  in the defining fan  $\Sigma$ , the torus orbit  $\mathcal{O}_{\tau}$  together with all orbits corresponding to rays in  $\Sigma$  form a geometric building set. Our proof in the subsection 2.4.4 applies in this context with  $\mathcal{O}_{\tau}$  playing the role of G. We conclude that under blowup of  $X_{\Sigma}$  in the closed torus orbit  $\mathcal{O}_{\tau}$ , the incidence combinatorics of torus orbits changes exactly in the way described by a stellar subdivision of  $\Sigma$  in  $\tau$ . This is the combinatorial part of the well-known fact that in the smooth case, the blowup of  $X_{\Sigma}$  in a torus orbit  $\mathcal{O}_{\tau}$  corresponds to a regular stellar subdivision of the fan  $\Sigma$  in  $\tau$  [MO73].

### PART II

# COMPLEXES OF TREES AND QUOTIENT CONSTRUCTIONS

In the reproof of chance Lies the true proof of men

-William Shakespeare, Troilus and Cressida

#### CHAPTER 3

# RATIONAL HOMOLOGY OF SPACES OF COMPLEX MONIC POLYNOMIALS WITH MULTIPLE ROOTS AND COMPLEXES OF MARKED FORESTS

## 3.1 THE STRATIFICATION BY ROOT MULTIPLICITIES OF THE SPACE OF COMPLEX MONIC POLYNOMIALS

Let n be an integer,  $n \geq 2$ . We view n-dimensional complex space  $\mathbb{C}^n$  as the space of all monic polynomials with complex coefficients of degree n by identifying  $a = (a_0, \ldots, a_{n-1}) \in \mathbb{C}^n$  with  $f_a(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ . To each  $\lambda = (\lambda_1, \ldots, \lambda_t) \vdash n$  one can associate a topological space as follows (we refer the reader to the Appendix A for a description of our conventions on the terminology of number and set partitions).

**Definition 3.1.1**  $\widetilde{\Sigma}_{\lambda}$  is the set of all  $a \in \mathbb{C}^n$ , for which the roots of  $f_a(z)$  can be partitioned into sets of sizes  $\lambda_1, \ldots, \lambda_t$ , so that within each set the roots are equal. Clearly,  $\widetilde{\Sigma}_{\lambda}$  is a closed subset of  $\mathbb{C}^n$ . Let  $\Sigma_{\lambda}^{S^2}$  be the one-point compactification of  $\widetilde{\Sigma}_{\lambda}$ .

It is easy to see that the Definition 3.1.1 is equivalent to the description of  $\Sigma_{\lambda}^{S^2}$  given in the Chapter 1, so our notation is consistent in that point. In this chapter we shall focus on the (reduced) rational Betti numbers of the spaces  $\Sigma_{\lambda}^{S^2}$ . In [Ar70a], V.I. Arnol'd has computed  $\tilde{\beta}_*(\Sigma_{\lambda}^{S^2},\mathbb{Q})$  for  $\lambda=(k^m,1^{n-km})$ .

**Theorem 3.1.2** (Arnol'd, [Ar70a]). Let  $\lambda = (k^m, 1^{n-km})$  for some natural numbers  $k \geq 2$ , m, and  $n \geq km$ . Then

$$\tilde{\beta}_i(\Sigma_{\lambda}^{S^2}, \mathbb{Q}) = \begin{cases} 1, & \text{for } i = 2l(\lambda); \\ 0, & \text{otherwise.} \end{cases}$$
 (3.1.1)

In [SuW97] Sundaram and Welker conjectured that

**Conjecture 3.1.3** For any number partition  $\lambda$ ,  $\tilde{\beta}_i(\Sigma_{\lambda}^{S^2}, \mathbb{Q}) = 0$  unless  $i = 2l(\lambda)$ .

In this chapter we shall give a new, combinatorial proof of the Theorem 3.1.2 and disprove Conjecture 3.1.3. To do that, we shall introduce a family of topological spaces  $X_{\lambda,\mu}$ , indexed by pairs of number partitions  $(\lambda,\mu)$ , satisfying  $\lambda \vdash \mu$ .  $X_{\lambda,\mu}$  will be defined so that the following equality is satisfied

$$\tilde{\beta}_i(\Sigma_{\lambda}^{S^2}, \mathbb{Q}) = \sum_{\lambda \vdash \mu \vdash n} \tilde{\beta}_{i-2l(\mu)-1}(X_{\lambda,\mu}, \mathbb{Q}). \tag{3.1.2}$$

Here is the summary of the chapter.

**Section 3.2.** We define the topological spaces  $X_{\lambda,\mu}$  and derive (3.1.2).

Section 3.3. We give a combinatorial description of the cell structure of the triangulated spaces  $X_{\lambda,\mu}$  in terms of marked forests, see Theorem 3.3.4. This description is the backbone of the chapter, it serves as both language and intuition for the material in the subsequent sections. One consequence of Theorem 3.3.4 is that homology groups of  $X_{\lambda,\mu}$  may be computed from a chain complex, whose components are freely generated by marked forests, and the boundary operator is described in terms of a combinatorial operation on such forests (deletion of level sets).

Section 3.4. We prove a general theorem about collapsibility of certain triangulated spaces. The direct combinatorial argument is heavily relying on the combinatorial cell description, from Section 3.3, of  $X_{\lambda,\mu}$ . We would like to mention that for the special cases  $\lambda=(k,1^{n-k})$  and  $\lambda=(k^m)$  the new proof of the Theorem 3.1.2 was also found in [SuW97], the argument there also made use of the Theorem 3.2.2 (as Proposition 3.5.7 shows, the case  $\lambda=(k^m)$  is especially simple). However, the Theorem 3.4.2 is the first combinatorial (modulo Theorem 3.2.2) proof of the result of Arnol'd in the general case.

**Section 3.5.** We disprove the conjecture of Sundaram and Welker. Besides giving a counterexample we prove this conjecture for a class of number partitions, which we call generic partitions.

#### 3.2 Orbit arrangements and spaces $X_{\lambda,\mu}$

#### 3.2.1 Reformulation in the language of orbit arrangements

Following [SuW97], we give a different interpretation of the numbers  $\tilde{\beta}_i(\Sigma_{\lambda}^{S^2}, \mathbb{Q})$ , for general  $\lambda$ . First, let us observe that the symmetric group  $\mathcal{S}_n$  acts on  $\mathbb{C}^n$  by permuting the coordinates, so we can consider the space  $\mathbb{C}^n/\mathcal{S}_n$  endowed with the quotient topology. It is a classical fact that the map  $\phi: \mathbb{C}^n \to \mathbb{C}^n/\mathcal{S}_n$ , mapping a polynomial to the (unordered) set of its roots, is a homeomorphism, which extends to the one-point compactifications. Therefore  $\mathbb{C}^n \cong \mathbb{C}^n/\mathcal{S}_n$  and

$$\Sigma_{\lambda}^{S^2} \cong \phi(\Sigma_{\lambda}^{S^2}) = \phi(\widetilde{\Sigma}_{\lambda}) \cup \{\infty\}.$$

 $\phi(\widetilde{\Sigma}_{\lambda})$  can be viewed as the configuration space of n unmarked points on  $\mathbb C$  such that the number partition given by the coincidences among the points is refined by  $\lambda$ . For example,  $\phi(\widetilde{\Sigma}_{(2,1^{n-2})})$  is the configuration space of n unmarked points on  $\mathbb C$  such that at least 2 points coincide. Using this point of view,  $\phi(\Sigma_{\lambda}^{S^2})$  can be described in the language of orbit subspace arrangements.

**Definition 3.2.1** For  $\pi \vdash [n]$ ,  $\pi = (S_1, \ldots, S_t)$ ,  $S_j = \{i_1^j, \ldots, i_{|S_j|}^j\}$ ,  $1 \le j \le t$ ,  $K_{\pi}$  is the subspace given by the equations  $x_{i_1^1} = \cdots = x_{i_{|S_1|}^1}, \ldots, x_{i_1^t} = \cdots = x_{i_{|S_t|}^t}$ . For  $\lambda \vdash n$ , set  $I_{\lambda} = \{\pi \vdash [n] \mid \text{type } (\pi) = \lambda\}$  and define  $\mathcal{A}_{\lambda} = \{K_{\pi} \mid \pi \in I_{\lambda}\}$ .  $\mathcal{A}_{\lambda}$ 's are called **orbit arrangements**.

The orbit arrangements were introduced in [Bj94] and studied in further detail in [Ko97]. They provide the appropriate language to describe  $\phi(\Sigma_{\lambda}^{S^2})$ , indeed

$$\phi(\Sigma_{\lambda}^{S^2}) = \Gamma_{\mathcal{A}_{\lambda}}^{\mathcal{S}_n}.$$
 (3.2.1)

An important special case is that of the braid arrangement  $\mathcal{A}_{n-1} = \mathcal{A}_{(2,1^{n-2})}$ , which corresponds under  $\phi$  to  $\widetilde{\Sigma}_{(2,1^{n-2})}$ , the space of all monic complex polynomials of degree n with at least one multiple root. The name "braid arrangement" stems from the fact that  $\mathbb{C}^n \setminus V_{\mathcal{A}_{n-1}}$  is a classifying space of the colored braid group, see [Ar69]. The intersection lattice  $\mathcal{L}_{\mathcal{A}_{n-1}}$  is usually denoted  $\Pi_n$ . It is the poset consisting of all set partitions of [n], where the partial order relation is refinement. Furthermore, for  $\lambda \vdash n$ , the intersection lattice of  $\mathcal{A}_{\lambda}$  is denoted  $\Pi_{\lambda}$ . It is the subposet of  $\Pi_n$  consisting of all elements which are joins of elements of type  $\lambda$ , with the minimal element  $\hat{0}$  attached.

#### 3.2.2 Applying Sundaram-Welker formula

The following formula of S. Sundaram and V. Welker, [SuW97], is vital for our approach.

**Theorem 3.2.2** ([SuW97, Theorem 2.4(ii) and Lemma 2.7(ii)]).

Let A be an arbitrary subspace arrangement in  $\mathbb{C}^n$  with an action of a finite group  $G \subset \mathbf{U}_n(\mathbb{C})$ . Let  $\mathcal{D}_A$  be the intersection of  $V_A$  with the (2n-1)-sphere (often called the link of A). Then there is the following isomorphism of G-modules.

$$\widetilde{H}_{i}(\mathcal{D}_{\mathcal{A}}, \mathbb{Q}) \cong_{G} \bigoplus_{x \in \mathcal{L}_{\mathcal{A}}^{>\hat{0}}/G} \operatorname{Ind}_{Stab(x)}^{G}(\widetilde{H}_{i-\dim x}(\Delta(\mathcal{L}_{\mathcal{A}}(\hat{0}, x)), \mathbb{Q})),$$
 (3.2.2)

where the sum is taken over representatives of the orbits of G in  $\mathcal{L}_A \setminus \{\hat{0}\}$ , under the action of G, one representative for each orbit.

Clearly  $\Gamma_A^G \cong \operatorname{susp}(\mathcal{D}_A/G)$ . Recall that if a finite group G acts on a finite cell complex K then  $\tilde{\beta}_i(K/G,\mathbb{Q})$  is equal to the multiplicity of the trivial representation in the induced representation of G on the  $\mathbb{Q}$ -vector space  $\tilde{H}_i(K,\mathbb{Q})$ , see for example [Co56, Theorem 1], [Br72]. Hence, it follows from (3.2.2), and the Frobenius reciprocity law, that

$$\tilde{\beta}_{i}(\Gamma_{\mathcal{A}}^{G}, \mathbb{Q}) = \sum_{x \in \mathcal{L}_{\mathcal{A}}^{>\hat{0}}/G} \tilde{\beta}_{i-\dim x - 1}(\Delta(\mathcal{L}_{\mathcal{A}}(\hat{0}, x))/\operatorname{Stab}(x), \mathbb{Q}), \tag{3.2.3}$$

where Stab (x) denotes the stabilizer of x.

#### 3.2.3 Spaces $X_{\lambda,\mu}$ and their properties

Let us now restate this identity in the special case of orbit arrangements. As mentioned above, the intersection lattice of  $\mathcal{A}_{\lambda}$  is  $\Pi_{\lambda}$ . It has an action of the symmetric group  $\mathcal{S}_n$ , which, for any  $\pi \in \Pi_{\lambda}$  induces an action of  $\operatorname{Stab}(\pi)$  on  $\Delta(\Pi_{\lambda}(\hat{0},\pi))$ .

**Notation.** Let  $X_{\lambda,\mu}$  denote the topological space  $\Delta(\Pi_{\lambda}(\hat{0},\pi))/Stab(\pi)$ , where the set partition  $\pi$  has type  $\mu$ . If there is no set partition  $\pi \in \Pi_{\lambda}$  of type  $\mu$ , i.e., if  $\mu$  cannot be obtained as a join of  $\lambda$ 's, then let  $X_{\lambda,\mu}$  be a point.

For fixed  $\mu$ , the space  $X_{\lambda,\mu}$  does not depend on the choice of  $\pi$ . Observe that  $X_{\lambda,\mu}$  is in general not a simplicial complex, however it is a triangulated space, (a regular CW complex with each cell being a simplex, see [GeM96, Chapter I, Section 1]), with its cell structure inherited from the simplicial complex  $\Delta(\Pi_{\lambda}(\hat{0},\pi))$ . In general, whenever G is a finite group which acts on a poset P in an order-preserving way,  $\Delta(P)/G$  is a triangulated space whose cells are orbits of simplices of  $\Delta(P)$  under the action of G; this is obviously not true in general for an action of a finite group on a finite simplicial complex.

Clearly, (3.2.1) together with (3.2.3), and the fact that  $\phi$  is a homeomorphism, implies (3.1.2). Let us quickly analyze (3.1.2).  $X_{\lambda,\lambda}=\emptyset$  makes a contribution 1 in dimension  $2l(\lambda)$ . Assume  $\mu \neq \lambda$ , then  $1 \leq l(\mu) \leq l(\lambda)-1$  and  $X_{\lambda,\mu} \neq \emptyset$ . dim  $X_{\lambda,\mu}=l(\lambda)-l(\mu)-1$ , hence  $\tilde{\beta}_{i-2l(\mu)-1}(X_{\lambda,\mu},\mathbb{Q})=0$  unless  $0 \leq i-2l(\mu)-1 \leq l(\lambda)-l(\mu)-1$ , that is  $2l(\mu)+1 \leq i \leq l(\lambda)+l(\mu)$ . It follows from (3.1.2) that

$$\tilde{\beta}_{2l(\lambda)}(\Sigma_{\lambda}^{S^2},\mathbb{Q})=1, \text{ and } \tilde{\beta}_i(\Sigma_{\lambda}^{S^2},\mathbb{Q})=0 \text{ unless } 3\leq i \leq 2l(\lambda).$$

The purpose of this chapter is to investigate the values  $\tilde{\beta}_i(\Sigma_{\lambda}^{S^2},\mathbb{Q})$  for  $3 \leq i \leq 2l(\lambda)-1$ , by studying  $\tilde{\beta}_i(X_{\lambda,\mu},\mathbb{Q})$ . We shall prove that the latter are equal to 0 for a certain set of pairs  $(\lambda,\mu)$ ,  $\lambda \vdash \mu$ , of partitions, including the case in Theorem 3.1.2,  $(\lambda=(k^m,1^{n-km}),\mu$  is arbitrary such that  $\lambda \vdash \mu$ ), and we shall give an example that this is not the case in general.

#### 3.3 The cell structure of $X_{\lambda,\mu}$ and marked forests

#### 3.3.1 The terminology of marked forests

In order to index the simplices of  $X_{\lambda,\mu}$  we need to introduce some terminology for certain types of trees with additional data. For an arbitrary forest of rooted trees T (we only consider finite graphs), let V(T) denote the set of the vertices of T,  $R(T) \subseteq V(T)$  denote the set of the roots of T and  $L(T) \subseteq V(T)$  denote the set of the leaves of T. For any integer  $i \geq 0$ , let  $l_i(T)$  be the number of  $v \in V(T)$  such that, v has distance i to the root in its connected component.

**Definition 3.3.1** A forest of rooted trees T is called a **graded forest of rank** r if  $l_{r+2}(T) = 0$ ,  $l_{r+1}(T) = |L(T)|$ , and the sequence  $l_0(T), \ldots, l_{r+1}(T)$  is strictly increasing.

For  $v, w \in V(T)$ , w is called a *child* of v if there is an edge between w and v and the unique path from w to the corresponding root passes through v. For  $v \in V(T)$ , we call the distance from v to the closest leaf the *height* of v. For example, in a graded forest of rank r, leaves have height 0 and roots have height r+1.

**Definition 3.3.2** A marked forest of rank r is a pair  $(T, \eta)$ , where T is a graded forest of rank r and  $\eta$  is a function from V(T) to the set of natural numbers such that for any vertex  $v \in V(T) \setminus L(T)$  we have

$$\eta(v) = \sum_{w \in \text{children}(v)} \eta(w). \tag{3.3.1}$$

We remark that the set of the marked forests of rank r, such that not all leaves have label 1, is equal to the set of graded forests of rank r+1. Indeed, instead of labeling the vertices with natural numbers so that (3.3.1) is satisfied, one can as well attach a new level of leaves so that each "old leaf" v has  $\eta(v)$  children. Then the old labels will correspond to the numbers of the new leaves below each vertex. For our context it is more convenient to use labels rather than auxiliary leaves, i.e., it is more handy to label all vertices rather than just the leaves, so we stick to the terminology of Definition 3.3.2.

For a marked forest  $(T, \eta)$  of rank r and  $0 \le i \le r + 1$ , we have a number partition  $\lambda_i(T, \eta) = \{\eta(v) \mid v \text{ has height } i\}$ . Clearly  $\lambda_0(T, \eta) \vdash \ldots \vdash \lambda_r(T, \eta) \vdash \lambda_{r+1}(T, \eta)$ .

**Definition 3.3.3** *Let*  $\lambda \vdash \mu \vdash n$ ,  $\lambda \neq \mu$ .  $A(\lambda, \mu)$ -forest of rank r is a marked forest of rank r,  $(T, \eta)$  such that  $\mu = \lambda_{r+1}(T, \eta)$  and  $\lambda \vdash \lambda_0(T, \eta)$ .

To simplify the language, we call  $((2, 1^{n-2}), \mu)$ -forests simply  $\mu$ -forests, and  $((2, 1^{n-2}), (n))$ -forests simply n-trees.

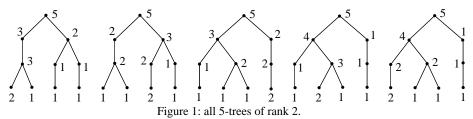


Figure 3.1. All 5-trees of rank 2.

Whenever  $(T,\eta)$  is a  $(\lambda,\mu)$ -forest of rank r and  $0 \le i \le r$ , we can obtain a  $(\lambda,\mu)$ -forest  $(T^i,\eta^i)$  of rank r-1 by deleting from T all the vertices of height i and connecting the vertices of height i+1 to their grandchildren (unless i=0);  $\eta^i$  is the restriction of  $\eta$  to  $V(T^i)$ . In other words,  $(T^i,\eta^i)$  is obtained from  $(T,\eta)$  by removing the entire ith level, counting from the leaves, and filling in the gap in an obvious way. This allows us to define a boundary operator by

$$\partial(T,\eta) = \sum_{i=0}^{r} (-1)^{i} (T^{i}, \eta^{i}). \tag{3.3.2}$$

This paves the way to explicit combinatorial computations by means of marked forests.

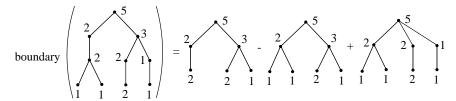


Figure 3.2. An example of a boundary computation.

For a given set partition  $\pi$  one can define the notion of a  $\pi$ -forest  $(T,\zeta)$  of rank r almost identically to the case of number partitions described above. The difference is that  $\zeta$  maps V(T) to the set of finite sets, rather than the set of natural numbers. The condition (3.3.1) is replaced by

$$\zeta(v) = \bigcup_{w \in \text{children}(v)} \zeta(w), \tag{3.3.3}$$

and  $\pi = \{\zeta(v) \mid v \in R(T)\}$ . For  $0 \le i \le r+1$ , analogously to  $\lambda_i(T, \eta)$ , we define  $\pi_i(T, \zeta)$  to be the set partition which is read off from the vertices of T having height i.

Let  $\mu$  be the type of  $\pi$ , then there exists a canonical  $\mu$ -forest  $(T, |\zeta|)$  associated to each  $\pi$ -forest  $(T, \zeta)$ , where  $|\zeta|$  is obtained as the composition of  $\zeta$  with the map which maps finite sets to their sizes.

#### 3.3.2 The main theorem

Let us describe how to associate a  $(\lambda, \mu)$ -forest,  $\psi(\sigma)$ , of rank r to an r-simplex  $\sigma$  of  $X_{\lambda,\mu}$ . The simplex  $\sigma$  is an Stab  $(\pi)$ -orbit of r-simplices of  $\Delta(\Pi_{\lambda}(\hat{0},\pi))$ , where  $\pi$  is a set partition of type  $\mu$ . Take a representative of this orbit, a chain  $c=(x_r>\cdots>x_0)$ . Now we define  $\psi(\sigma)=(T,\eta)$ . Each element  $x_i$  corresponds to the ith level in T, counting from the leaves. Each block b of  $x_i$  corresponds to a node in the tree; on this node we define the value of  $\eta$  to be |b|. We define the edges of the tree T by connecting each node corresponding to a block b of  $x_i$  to all nodes corresponding to the blocks of  $x_{i-1}$  contained in b, we do that for all b and a. The top a0 the edges from the top level to the a1 th level connect each block of a2 to the blocks of a3 and the edges from the top level to the a4 th level connect each block of a5 to the blocks of a5 contained in it. For example, the value of a6 on the a6 orbit of the chain a7 contained in it. For example, the value of a7 on the Figure 3.1.

We are now ready to state and prove the main result of this section.

**Theorem 3.3.4** Assume  $\lambda \vdash \mu \vdash n$ ,  $\lambda \neq \mu$ . The correspondence  $\psi$  of the r-simplices of  $X_{\lambda,\mu}$  and  $(\lambda,\mu)$ -forests  $(T,\eta)$  of rank r is a bijection. Under this bijection, the boundary operator of the triangulated space  $X_{\lambda,\mu}$  corresponds to the boundary operator described in (3.3.2).

In particular, the simplices of  $\Delta(\Pi_n)/\mathcal{S}_n$  along with the cell inclusion structure are described by the *n*-trees. Indeed,  $\Delta(\Pi_5)/\mathcal{S}_5$  is shown in the Figure 3.3.

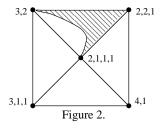


Figure 3.3

The five triangles may be labeled by the five 5-trees of rank 2 in Figure 3.1.

**Proof of the Theorem 3.3.4.** By the definitions of  $\Pi_n$  and of  $\Delta$ , the r-simplices of  $\Delta(\Pi_n)$  can be indexed by ([n])-trees of rank r (we write ([n]) to emphasize that the set [n] is viewed here as a set partition consisting of only one set). Furthermore, the cell inclusions in  $\Delta(\Pi_n)$  correspond to level deletion in ([n])-trees as is described above for the case of number partitions, because the levels in the ([n])-trees correspond to the elements of  $\Pi_n$ , and the edges in the ([n])-trees correspond to block inclusions of two consecutive elements in the chain.

More generally, the r-simplices  $\sigma$  of  $\Delta(\Pi_{\lambda}(\hat{0}, \pi))$  can be indexed by  $\pi$ -forests  $(T(\sigma), \zeta(\sigma))$  such that  $\lambda$  refines the type of  $\pi_0(T(\sigma), \zeta(\sigma))$ . The definition of  $\psi$ 

can now be rephrased as associating to  $\sigma$  the  $(\lambda, \mu)$ -forest  $(T(\sigma), |\zeta(\sigma)|)$ , where  $\mu = \text{type } \pi$ .

The group action of  $\operatorname{Stab}(\pi)$  on  $\Delta(\Pi_{\lambda}(\hat{0},\pi))$  corresponds to relabeling elements within the sets of  $\zeta(\sigma)$ . This shows that for  $g \in \operatorname{Stab}(\pi)$  we have  $T(g\sigma) = T(\sigma)$  and  $|\zeta(g\sigma)| = |\zeta(\sigma)|$ . Therefore  $\psi(\sigma)$  is well-defined, it does not depend on the choice of the representative of the, corresponding to  $\sigma$ ,  $\operatorname{Stab}(\pi)$ -orbit of chains.

 $\psi$  is surjective, we shall now show that it is also injective. If  $\sigma_1, \sigma_2$  are two different r-simplices of  $\Delta(\Pi_\lambda(\hat{0},\pi))$  such that  $T(\sigma_1)=T(\sigma_2)$  and  $|\zeta(\sigma_1)|=|\zeta(\sigma_2)|$ , then there exists  $g\in \operatorname{Stab}(\pi)$  such that  $\zeta(g\sigma_2)=\zeta(\sigma_1)$ . Indeed, let  $T=T(\sigma_1)=T(\sigma_2)$  and let  $\alpha_1$ , resp.  $\alpha_2$ , be the string concatenated from the values of  $\zeta(\sigma_1)$ , resp.  $\zeta(\sigma_2)$ , on the leaves of T; the order of leaves of T is arbitrary, but the same for  $T(\sigma_1)$  and  $T(\sigma_2)$ , the order of elements within each  $\zeta(\sigma_1)(v)$ , resp.  $\zeta(\sigma_2)(v)$ , for  $v\in L(T)$  is also chosen arbitrarily. Then  $g\in \mathcal{S}_n$  which maps  $\alpha_2$  to  $\alpha_1$  satisfies the necessary conditions:  $\zeta(g\sigma_2)=\zeta(\sigma_1)$  on the leaves of T, and hence by (3.3.3) on all vertices of T. Furthermore, since  $g\pi=g\pi_{r+1}(T,\zeta(\sigma_2))=\pi_{r+1}(T,\zeta(g\sigma_2))=\pi_{r+1}(T,\zeta(\sigma_1))=\pi$ , we have  $g\in\operatorname{Stab}(\pi)$ .

This shows that  $\psi$  is a bijection. Since the levels of the  $(\lambda, \mu)$ -forests correspond to the Stab  $(\pi)$ -orbits of the vertices of  $\Delta(\Pi_{\lambda}(\hat{0}, \pi))$  (hence to the vertices of  $X_{\lambda,\mu}$ ), the boundary operator of  $X_{\lambda,\mu}$  corresponds under  $\psi$  to the level deletion in  $(\lambda, \mu)$ -forests, i.e., the boundary operator described in (3.3.2).  $\square$ 

#### 3.3.3 Remarks

- 1. While the presence of the root in an ([n])-tree is just a formality (two marked ([n])-trees are equal iff the deletion of the root gives equal marked forests), the presence of the roots in a  $\pi$ -forest is vital. In fact, if roots were not taken into account (as seems natural, since the partition read off from the roots does not correspond to any vertex in  $\Delta(\Pi_{\lambda}(\hat{0},\pi))$ ) the argument above would be false already for vertices: if  $\tau_1,\tau_2\in\Pi_{\lambda}(\hat{0},\pi)$ , such that type  $(\tau_1)=$  type  $(\tau_2)$  (i.e., the corresponding marked forests of rank 0 are equal once the roots are removed), there may not exist  $g\in$ Stab  $(\pi)$ , such that  $g\tau_2=\tau_1$  (although such  $g\in \mathcal{S}_n$  certainly exists).
- 2. Marked forests equipped with an order on the children of each vertex were used by Vassiliev, [Vas94], to label cells in a certain CW-complex structure on the space  $\widetilde{\mathbb{R}}^n(m)$ , the one-point compactification of the configuration space of m unmarked distinct points in  $\mathbb{R}^n$ . Vassiliev's cell decomposition of  $\widetilde{\mathbb{R}}^n(m)$  is a generalization of the earlier Fuchs' cell decomposition of  $\widetilde{\mathbb{R}}^2(m) = \widetilde{\mathbb{C}}(m)$ , [Fuc70], which allowed Fuchs to compute the ring  $H^*(Br(m), \mathbb{Z}_2)$ , where Br(m) is Artin's braid group on m strings, see also [Vai78]. Beyond a certain similarity of the combi-

natorial objects used for labeling the cells, cf. [Vas94, Lemma 3.3.1] and Theorem 3.3.4, the connection between the results of this chapter and the results of Vassiliev and Fuchs seems unclear.

As yet another instance of a similar situation, we would like to mention the labeling of the components in the stratification of  $\overline{M}_{0,n}$  (the Deligne-Knudsen-Mumford compactification of the moduli space of stable projective complex curves of genus 0 with n punctures) with trees with n labeled leaves, see, e.g., [FM94, Kn83].

#### 3.4 A NEW PROOF OF A THEOREM OF ARNOL'D

#### 3.4.1 Formulation of the main theorem and its corollaries

In this section we take a look at a rather general question of which  $\mathbb{Q}$ -acyclicity of the spaces  $X_{\lambda,\mu}$  is a special case:

Let  $\pi \in \Pi_n$  and let Q be an Stab  $(\pi)$ -invariant subposet of  $\Pi_n(\hat{0}, \pi)$ . When is the multiplicity of the trivial representation in the induced representation of Stab  $(\pi)$  on  $\widetilde{H}_i(\Delta(Q), \mathbb{Q})$  equal to 0 for all i, in other words, when is  $\Delta(Q)/\text{Stab}(\pi)$   $\mathbb{Q}$ -acyclic?

**Definition 3.4.1** Let  $\Lambda$  be a subset of the set of all number partitions of n such that  $(1^n), (n) \notin \Lambda$ . Define  $\Pi_{\Lambda}$  to be the subposet of  $\Pi_n$  consisting of all set partitions  $\pi$  such that  $(\text{type } \pi) \in \Lambda$ .

Clearly,  $\Pi_{\Lambda}$  is  $\mathcal{S}_n$ -invariant and, more generally,  $\Pi_{\Lambda}(\hat{0}, \pi)$  is Stab  $(\pi)$ -invariant. Vice versa, any  $\mathcal{S}_n$ -invariant subposet of  $\Pi_n \setminus \{(\{1\}, \dots, \{n\}), ([n])\}$  is of the form  $\Pi_{\Lambda}$  for some  $\Lambda$ .

The following theorem is the main result of this section. The proof is a combination of the language of marked forests from Section 3.3 and the ideas used in the proof of [Ko00a, Theorem 4.1].

**Theorem 3.4.2** Let  $2 \le k < n$ . Assume  $\Lambda$  is a subset of the set of all number partitions of n such that  $(1^n), (n) \notin \Lambda$  and  $\Lambda$  satisfies the following condition:

Condition 
$$C_k$$
. If  $\mu \in \Lambda$ , such that  $\mu = (\mu_1, \dots, \mu_t)$ , where  $\mu_i = kq_i + r_i$ ,  $0 \le r_i < k$  for  $i \in [t]$ , then  $\gamma_k(\mu) = (k^{q_1 + \dots + q_t}, 1^{r_1 + \dots + r_t}) \in \Lambda$ .

Then for any  $\mu \in \Lambda \cup \{(n)\}$  the triangulated space  $X_{\Lambda,\mu} = \Delta(\Pi_{\Lambda}(\hat{0},\pi))/Stab(\pi)$ , where  $\mu = \text{type } \pi$ , is collapsible, in particular the multiplicity of the trivial representation in the induced  $Stab(\pi)$ -representation on  $\widetilde{H}_i(\Delta(\Pi_{\Lambda}(\hat{0},\pi)),\mathbb{Q})$  is equal to 0 for all i.

**Remark 3.4.3** Consider the following special case: k = 2 and  $\Lambda$  is equal to the set of all number partitions of n except for  $(1^n)$  and (n). Then the Condition  $C_k$  is obviously satisfied. Since in this case  $X_{\Lambda,\mu} = \Delta(\Pi_n)/\mathcal{S}_n$ , we conclude that the complex  $\Delta(\Pi_n)/\mathcal{S}_n$  is collapsible.

Slightly more generally, we have the following result.

**Corollary 3.4.4** Assume  $\lambda = (k^m, 1^{n-km})$ ,  $\lambda \vdash \mu$ , then  $X_{\lambda,\mu}$  is collapsible. In particular,  $X_{\lambda,\mu}$  is  $\mathbb{Q}$ -acyclic, therefore Theorem 3.1.2 follows.

**Proof.** Clearly  $X_{\lambda,\mu} = X_{\Lambda,\mu}$  for  $\Lambda = \{\tau \mid \lambda \vdash \tau \vdash \mu, \tau \neq (n)\}$ . It is easy to check that Condition  $C_k$  is satisfied for the case  $\lambda = (k^m, 1^{n-km})$ , therefore Theorem 3.1.2 follows from Theorem 3.4.2 via (3.1.2).  $\square$ 

Another consequence of Theorem 3.4.2 concerns the multiplicity of the trivial character in certain representations of the symmetric group.

Let k be a field such that either char  $\mathbf{k}=0$  or char  $\mathbf{k}>n$ . Following a conjecture of R. Stanley, [St82, page 151], P. Hanlon has proved in [Ha83, Theorem 3.1] that if  $\Pi_n^t$  denotes the  $\{1,\ldots,t\}$  rank selection of the partition lattice, then the multiplicity of the trivial character in the natural representation  $\mathcal{S}_n \to GL(H_i(\Delta(\Pi_n^t),\mathbf{k}))$  induced by the standard permutation  $\mathcal{S}_n$ -representation on the set [n], is equal to 0 for all i and t. The following corollary generalizes his result.

**Corollary 3.4.5** Assume  $\Lambda$  is as in the Theorem 3.4.2, then the multiplicity of the trivial character in the representation  $S_n \to GL(H_i(\Delta(\Pi_{\Lambda}), \mathbf{k}))$  is 0 for all i.

**Proof.** We know that the complex  $X_{\Lambda} = \Delta(\Pi_{\Lambda})/\mathcal{S}_n$  is collapsible. By [Co56, Theorem 1],  $\beta_i(X_{\Lambda}, \mathbf{k})$  is equal to the multiplicity of the trivial character in the representation  $\mathcal{S}_n \to GL(H_i(\Delta(\Pi_{\Lambda}), \mathbf{k}))$ . Thus the statement of the corollary is equivalent to saying that  $X_{\Lambda}$  is  $\mathbf{k}$ -acyclic, which in turn is immediate from Theorem 3.4.2.  $\square$ 

Theorem 3.4.2 can be viewed as an attempt to provide a common framework for these results in the spirit of the question stated in the beginning of this section.

#### 3.4.2 Auxiliary propositions

First we need some terminology. For an arbitrary cell complex  $\Delta$  we denote by  $V(\Delta)$  the set of vertices of  $\Delta$ . Assume  $\Delta$  is a regular CW complex and  $\Delta'$  is its subcomplex. We denote the set of the simplices of  $\Delta$  which are not simplices of  $\Delta'$  by  $\Delta \setminus \Delta'$ . We use the sign  $\succ$  to denote the cover relation in the cell structure of  $\Delta$ .

Assume that, in addition,  $\Delta$  is a triangulated space with some linear order  $\ll$  on the set of vertices. For  $\sigma \in \Delta \setminus \Delta'$  we may write  $\sigma = (x_1, \dots, x_t)$ , this notation is slightly inaccurate since the set of vertices does not determine the simplex uniquely, all we mean is that  $\sigma$  has vertices  $x_1 \ll \dots \ll x_t$ . In that case, we let  $\xi(\sigma) = i$  if  $x_1, \dots, x_{i-1} \in V(\Delta')$  and  $x_i \notin V(\Delta')$ .

**Proposition 3.4.6** Let  $\Delta$  be a regular CW complex and  $\Delta'$  a subcomplex of  $\Delta$ , then the following are equivalent:

- a) there is a sequence of collapses leading from  $\Delta$  to  $\Delta'$ ;
- b) there is a matching of cells of  $\Delta \setminus \Delta'$ :  $\sigma \leftrightarrow \phi(\sigma)$ , such that  $\phi(\sigma) \succ \sigma$  and there is no sequence  $\sigma_1, \ldots, \sigma_t \in \Delta \setminus \Delta'$  such that  $\phi(\sigma_1) \succ \sigma_2, \phi(\sigma_2) \succ \sigma_3, \ldots, \phi(\sigma_t) \succ \sigma_1$  (such matching is called acyclic).

#### Proof.

<u>a)</u>  $\Rightarrow$  <u>b)</u>: Let the elementary collapses define the matching  $\phi$ . Assume there is a sequence  $\sigma_1, \ldots, \sigma_t \in \Delta \setminus \Delta'$  such that  $\phi(\sigma_1) \succ \sigma_2, \phi(\sigma_2) \succ \sigma_3, \ldots, \phi(\sigma_t) \succ \sigma_1$ . Without loss of generality we can assume that the collapse  $(\sigma_1, \phi(\sigma_1))$  precedes collapses  $(\sigma_i, \phi(\sigma_i))$  for  $2 \le i \le t$ . Then  $\phi(\sigma_t) \succ \sigma_1$  yields a contradiction.

b) ⇒ a): The proof is again very easy, various versions of it were given in [Fo98, Corollary 3.5], [BBLSW, Proposition 3.7], and [Ko00a, Theorem 3.2].  $\Box$ 

**Proposition 3.4.7** Let  $\Delta$  be a triangulated space with some linear order  $\ll$  on its set of vertices  $V(\Delta)$ . Let  $V' \subseteq V(\Delta)$  and  $\Delta'$  be the subcomplex of  $\Delta$  induced by V'. Assume we have a partition  $V(\Delta) = \bigcup_{z \in V'} V_z$  such that  $z = \min_{\ll} V_z$ . For  $\sigma \in \Delta \setminus \Delta'$ , let  $\chi(\sigma) \in V'$  be defined by  $x_{\xi(\sigma)} \in V_{\chi(\sigma)}$ . Finally assume that the following condition is satisfied:

Condition  $\aleph$ . If  $\sigma \in \Delta \setminus \Delta'$ ,  $\sigma = (x_1, \dots, x_t)$ , is such that either  $\xi(\sigma) = 1$  or  $x_{\xi(\sigma)-1} \neq \chi(\sigma)$ , then there exists a unique simplex  $\sigma' = (x_1, \dots, x_{\xi(\sigma)-1}, \chi(\sigma), x_{\xi(\sigma)}, \dots, x_t)$  such that  $\sigma' \setminus \chi(\sigma) = \sigma$ .

Then there is a sequence of collapses leading from  $\Delta$  to  $\Delta'$ .

**Proof.** Let U denote the set of all  $\sigma \in \Delta \setminus \Delta'$ ,  $\sigma = (x_1, \ldots, x_t)$ , such that  $x_{\xi(\sigma)-1} \neq \chi(\sigma)$  or  $\xi(\sigma) = 1$ . The matching  $\phi$  is defined by Condition  $\aleph$ : for  $\sigma \in U$  we set  $\phi(\sigma) = \sigma'$ . By Proposition 3.4.6 it is enough to check that this matching is acyclic.

For  $\sigma \in U$  we have  $\xi(\phi(\sigma)) = \xi(\sigma) + 1$ . Moreover, if  $\phi(\sigma) \succ \sigma'$  and  $\sigma' \in U$ , then  $\sigma' = \phi(\sigma) \setminus x_{\xi(\sigma)}$ , hence  $\xi(\sigma') \geq \xi(\phi(\sigma))$ . Therefore, if there is a sequence  $\sigma_1, \ldots, \sigma_t \in \Delta \setminus \Delta'$  such that  $\phi(\sigma_1) \succ \sigma_2, \phi(\sigma_2) \succ \sigma_3, \ldots, \phi(\sigma_t) \succ \sigma_1$ , then we have  $\xi(\sigma_1) < \xi(\phi(\sigma_1)) \leq \xi(\sigma_2) < \xi(\phi(\sigma_2)) \leq \cdots < \xi(\phi(\sigma_t)) \leq \xi(\sigma_1)$  which yields a contradiction.  $\square$ 

#### *3.4.3 Proof of Theorem 3.4.2*

We define a  $(\Lambda, \mu)$ -forest of rank r to be a marked forest  $(T, \eta)$  of rank r such that  $\lambda_{r+1}(T, \eta) = \mu$  and  $\lambda_i(T, \eta) \in \Lambda$ , for  $0 \le i \le r$ . It follows from the discussion in Section 3.3 and in particular from Theorem 3.3.4 that the r-simplices of  $X_{\Lambda,\mu}$  can be indexed by  $(\Lambda, \mu)$ -forests of rank r so that the boundary relation of  $X_{\Lambda,\mu}$  corresponds to level deletion in the marked forests.

We call number partitions of the form  $(k^m, 1^{n-km})$ , for some m, special. Let K be the subcomplex of  $X_{\Lambda,\mu}$  induced by the set of all special partitions. We adopt the notations  $\xi(\sigma)$  and  $\chi(\sigma)$  used in Proposition 3.4.7 to the context of  $X_{\Lambda,\mu}$  and its subcomplex K. The linear order on  $V(X_{\Lambda,\mu})$  can be taken to be any linear extension of the partial order on  $V(X_{\Lambda,\mu})$  given by the negative of the length function. The partition of  $V(X_{\Lambda,\mu})$  is given by: for  $v \in V(X_{\Lambda,\mu}) \setminus V(K)$ ,  $z \in V(K)$ , we have  $v \in V_z$  iff  $z = \gamma_k(v)$ .

Let us show that the subcomplex K satisfies Condition  $\aleph$ . Let  $\sigma \in X_{\Lambda,\mu} \setminus K$ ,  $\sigma = (x_1, \ldots, x_t)$ , and assume  $\xi(\sigma) = 1$  or  $\chi(\sigma) \neq x_{\xi(\sigma)-1}$ . In the language of marked forests this can be reformulated as:  $\sigma$  is a  $(\Lambda, \mu)$ -forest  $(T, \eta)$  of rank t such that  $\lambda_{\xi(\sigma)-1}(T,\eta)$  is not special and if  $\xi(\sigma) > 1$  then  $\lambda_0(T,\eta),\ldots,$   $\lambda_{\xi(\sigma)-2}(T,\eta)$  are special, and  $\lambda_{\xi(\sigma)-2}(T,\eta) \neq \gamma_k(\lambda_{\xi(\sigma)-1}(T,\eta))$ . In other words, on all vertices of height 0 to  $\xi(\sigma)-2$  the function  $\eta$  takes only values 1 or k and for the vertices of height  $\xi(\sigma)-1$  it is no longer true. Moreover, there exists a vertex of height  $\xi(\sigma)-1$  which has at least k children on which  $\eta$  is equal to 1. It is now clear that there exists a unique  $(\Lambda,\mu)$ -forest  $(\widetilde{T},\widetilde{\eta})$  of rank r+1 such that

- $\bullet \ (\widetilde{T}^{\xi(\sigma)-1},\widetilde{\eta}^{\xi(\sigma)-1})=(T,\eta);$
- $\lambda_{\xi(\sigma)-1}(\widetilde{T},\widetilde{\eta}) = \gamma_k(\lambda_{\xi(\sigma)}(\widetilde{T},\widetilde{\eta}))$ , i.e.,  $\widetilde{\eta}$  takes only values 1 or k on the vertices of height  $\xi(\sigma)-1$  and each vertex of height  $\xi(\sigma)$  in  $(\widetilde{T},\widetilde{\eta})$  has no more than k-1 children labeled 1.

To construct  $(\widetilde{T},\widetilde{\eta})$ , extend  $(T,\eta)$  by splitting each vertex of height  $\xi(\sigma)-1$  into vertices marked k and 1 so that the number of k's is maximized. The uniqueness of  $(\widetilde{T},\widetilde{\eta})$  follows from the definition of the notion of isomorphism of marked forests.

We have precisely checked Condition  $\aleph$  and therefore by Proposition 3.4.6 we conclude that there is a sequence of collapses leading from  $X_{\Lambda,\mu}$  to K.

It remains to see that K is collapsible. If  $\mu=(n)$ , then K is a simplex, so we can assume that  $\mu\in\Lambda$ . If  $\mu=\gamma_k(\mu)$ , then K is again a simplex. Otherwise it is easy to see that there is a unique vertex in  $X_{\Lambda,\mu}$  labeled  $\gamma_k(\mu)$  and that K is a cone with an apex in this vertex.  $\square$ 

#### 3.5 On a conjecture of Sundaram and Welker

#### 3.5.1 A counterexample to the general conjecture

The original formulation of Conjecture 3.1.3 in [SuW97] was

**Conjecture 3.5.1** [SuW97, Conjectures 4.12 and 4.13]. Let  $\lambda$  and  $\mu$  be different set partitions, such that  $\lambda \vdash \mu$ . Let  $\pi \in \Pi_{\lambda}$  be a partition of type  $\mu$ . Then the multiplicity of the trivial representation in the Stab  $(\pi)$ -module  $H_*(\Delta(\Pi_{\lambda}(\hat{0}, \pi)), \mathbb{Q})$  is 0.

In our terms Conjecture 3.5.1 is equivalent to

**Conjecture 3.5.2** For  $\lambda \vdash \mu$ ,  $\lambda \neq \mu$ , the space  $X_{\lambda,\mu}$  is  $\mathbb{Q}$ -acyclic.

We shall give an example when  $X_{\lambda,\mu}$  is not even connected. It turns out that if one is only interested in counting the number of connected components of  $X_{\lambda,\mu}$ , then there is a simpler poset model which we now proceed to describe.

**Definition 3.5.3** Assume  $\lambda \vdash \mu \vdash n$ ,  $\lambda \neq \mu$ . The  $(\lambda, \mu)$ -forests of rank 0 can be partially ordered as follows:  $(T_1, \eta_1) \prec (T_2, \eta_2)$  if there exists a  $(\lambda, \mu)$ -forest  $(T, \eta)$  of rank 1 such that  $(T_1, \eta_1) = (T^1, \eta^1)$  and  $(T_2, \eta_2) = (T^0, \eta^0)$ . We call the obtained poset  $P_{\lambda,\mu}$ .

In other words, elements of  $P_{\lambda,\mu}$  are number partitions  $\tau \neq \mu$  such that  $\lambda \vdash \tau \vdash \mu$ , together with a bracketing which shows how to form  $\mu$  out of  $\tau$ , the order of the brackets and of the terms within the brackets is neglected. For example (1,1,1)(3,1)(2,2) and (3)(2,1,1)(2,1,1) are two different elements of  $P_{(2,1^9),(4^2,3)}$ , while (1,1,1)(2,2)(3,1) is equal to the first mentioned element. These bracketed partitions are ordered by refinement, preserving the bracket structure.

**Proposition 3.5.4**  $X_{\lambda,\mu}$  and  $\Delta(P_{\lambda,\mu})$  have the same number of connected components, i.e.,  $\beta_0(X_{\lambda,\mu}) = \beta_0(\Delta(P_{\lambda,\mu}))$ .

**Proof.** We know that  $\Delta(P_{\lambda,\mu})$  and  $X_{\lambda,\mu}$  have the same set of vertices and that there is an edge between two vertices a and b of  $\Delta(P_{\lambda,\mu})$  iff  $a \prec b$  or  $b \prec a$ , which is, by the Definition 3.5.3, the case iff there is an edge between the corresponding vertices of  $X_{\lambda,\mu}$ . This shows that  $\Delta(P_{\lambda,\mu})$  and  $X_{\lambda,\mu}$  have the same number of connected components.  $\square$ 

Note that 1-skeleta of  $X_{\lambda,\mu}$  and  $\Delta(P_{\lambda,\mu})$  need not be equal.  $\Delta(P_{\lambda,\mu})$  can intuitively be thought of as a simplicial complex obtained by forgetting the multiplicities of simplices in the triangulated space  $X_{\lambda,\mu}$ .

**Counterexample.** For n=23,  $\lambda=(7,6,4,3,2,1)$ ,  $\mu=(10,8,5)$ ,  $X_{\lambda,\mu}$  is disconnected.  $P_{\lambda,\mu}$  is shown on the Figure 3.4. Clearly  $\Delta(P_{\lambda,\mu})$  is not connected, hence, by the Proposition 3.5.4, neither is  $X_{\lambda,\mu}$ , which disproves Conjecture 3.5.1.

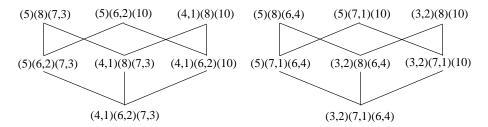


Figure 3.4

**Remark 3.5.5** In the counterexample above, one can actually verify that  $X_{\lambda,\mu} = \Delta(P_{\lambda,\mu})$ . However, we choose to use posets  $P_{\lambda,\mu}$  for two reasons:

- 1. it is easier to produce series of counterexamples to Sundaram-Welker conjecture using  $\Delta(P_{\lambda,\mu})$  rather than  $X_{\lambda,\mu}$ ;
- 2. we feel that posets  $P_{\lambda,\mu}$  are of independent interest, since they are in a certain sense the "naive" quotient of  $\Pi_{\lambda}(\hat{0},\pi)$  by  $Stab(\pi)$ .

We believe that, in general, connected components of  $X_{\lambda,\mu}$  may be not acyclic.

#### 3.5.2 *Verification of the conjecture in a special case*

**Definition 3.5.6** We say that a number partition  $\lambda = (\lambda_1, \dots, \lambda_t)$  is generic (also called free of resonances in [ShW98], having no equal sub-sums in [Ko97]) if whenever  $\sum_{i \in I} \lambda_i = \sum_{j \in J} \lambda_j$ , for some  $I, J \subseteq [t]$ , we have  $\{\lambda_i\}_{i \in I} = \{\lambda_j\}_{j \in J}$  as multisets.

For example  $\lambda = (k^m)$  is generic.

**Proposition 3.5.7** If  $\lambda$  is generic, then the stratum  $\Sigma_{\lambda}^{S^2}$  is homeomorphic to the  $2l(\lambda)$ -dimensional sphere.

**Proof.** For a generic partition 
$$\lambda = (\underbrace{\lambda_1, \dots, \lambda_1}_{k_1}, \underbrace{\lambda_2, \dots, \lambda_2}_{k_2}, \dots, \underbrace{\lambda_t, \dots, \lambda_t}_{k_t})$$
 we have  $\Sigma_{\lambda}^{S^2} \cong (S^2)^{(k_1)} \wedge (S^2)^{(k_2)} \wedge \dots \wedge (S^2)^{(k_t)} \cong S^{2k_1 + \dots + 2k_t} = S^{2l(\lambda)}$ .  $\square$ 

#### CHAPTER 4

## COMPLEXES OF DIRECTED TREES AND THEIR QUOTIENTS

#### 4.1 The objects of study and the main questions

To any directed graph one can associate an abstract simplicial complex in the following way.

**Definition 4.1.1** Let G be an arbitrary directed graph.  $\Delta(G)$  is the simplicial complex  $\Delta(G)$  constructed as follows: the vertices of  $\Delta(G)$  are given by the edges of G and faces are all directed forests which are subgraphs of G.

In [St97] R. Stanley asked the following two questions.

**Question 1.** Let  $G_n$  be the complete directed graph on n vertices, i.e., a graph having exactly one edge in each direction between any pair of vertices, all together n(n-1) edges. The complex  $\Delta(G_n)$  is obviously pure, but is it shellable?

There has been a recent upsurge of activity in studying the homotopy type of simplicial complexes constructed from monotone properties of graphs: the vertices of such a complex are all possible edges of the graph and the simplices are given by graphs which satisfy given monotone property, see [BBLSW, BWe98]. The question above can be reformulated in this language: what is the homotopy type of the simplicial complex corresponding to the monotone property of a graph being a directed forest?

**Question 2.** *Is the complex*  $\Delta(G)$  *shellable in general?* 

In general, one may ask what are the homology groups  $H_*(\Delta(G))$  and whether they can be linked to the combinatorial invariants of the graph G in a simple way.

In this chapter we answer affirmatively the Question 1 in Theorem 4.3.1 and negatively the Question 2 in Example 4.2.2, Section 4.2. Furthermore, we start the general investigation by computing the homology groups of  $\Delta(G)$  for the cases when G is essentially a tree (see Definition 4.4.1) and when G is a double directed cycle.

There is a natural action of  $S_n$  on  $\Delta(G_n)$  induced by the standard permutation action of  $S_n$  on [n], thus one can form the topological quotient  $X_n = \Delta(G_n)/S_n$ ;

for example, the case n=3 is shown on the Figure 4.3. The quotient complexes of combinatorially defined spaces still tend to be rather complicated in general, however sometimes (as is the case for  $\Delta(\Pi_n)/\mathcal{S}_n$ ) their homology groups are torsion free.

**Question 3.** Are the homology groups  $H_i(X_n, \mathbb{Z})$  torsion free in general?

In Section 4.5 we show that the groups  $H_*(X_n, \mathbb{Z})$  are, in general, not free, and also give a formula for  $\beta_{n-2}(X_n, \mathbb{Q})$ .

#### 4.2 First examples and properties

#### 4.2.1 Conventions

For brevity, we write  $H_*(G)$  instead of  $H_*(\Delta(G), \mathbb{Z})$ . We will also use the following standard fact: if  $\Delta$  is a simplicial complex and  $F_1, \ldots, F_t$  is the set of maximal simplices such that  $\Delta \setminus \bigcup_{i=1}^t F_i$  is contractible, then  $\Delta \simeq \bigvee_{i=1}^t S^{\dim F_i}$ . In this case, we call  $\{F_1, \ldots, F_t\}$  a *generating set* for  $\Delta$ . Clearly, for a fixed  $\Delta$ , there may be several generating sets, but the multiset  $\{\dim F_1, \ldots, \dim F_t\}$  is defined uniquely by  $\Delta$ .

Let  $\Delta$  be a simplicial complex, F be one of its maximal simplices and  $\tilde{F}$  be a subsimplex of F, such that  $\dim F = \dim \tilde{F} + 1$  and F is the only maximal simplex which contains  $\tilde{F}$ . Then, removing F and  $\tilde{F}$  from  $\Delta$  is called an *elementary collapse*. Obviously,  $\Delta \setminus \{\tilde{F}, F\}$  is a strong deformation retract of  $\Delta$ . For further references on the topological concepts used here we refer the reader to the Appendix D, the textbook by J. Munkres, [Mu84], and the thorough survey article by A. Björner, [Bj95].

#### 4.2.2 First examples

Let us give a couple of examples to illustrate how irregular  $\Delta(G)$  can be.

**Example 4.2.1** A graph G for which  $\Delta(G)$  is not pure.

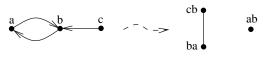


Figure 4.1

**Example 4.2.2** The complex  $\Delta(G)$  does not have to be shellable. Let  $G = C_5$ , a double directed cycle with 5 vertices, see the subsection 4.3 for the definition. It is easy to see that  $\Delta(C_5)$  is a pure simplicial complex of dimension 3. On the other hand, by Proposition 4.4.8,  $\Delta(C_5) \simeq S^2 \vee S^3 \vee S^3$ . This implies that  $H_2(\Delta(C_5), \mathbb{Z}) = \mathbb{Z}$ , in particular  $\Delta(C_5)$  is not shellable.

#### 4.2.3 Elementary properties

It is not difficult to derive the simplest properties of our construction. For example  $\Delta(G_1)*\Delta(G_2)=\Delta(G_1\uplus G_2)$ . Also it is easy to characterize those graphs G for which  $\Delta(G)$  is pure of full dimension. Namely, for  $x\in V(G)$ , let  $S(x)=\{y\,|\,(y\to x)\in E(G)\}$ . Then we have

**Proposition 4.2.3** Assume V(G) = n. The following are equivalent

- a)  $\Delta(G)$  has a maximal simplex of dimension less than n-1;
- b) there exist two disjoint subtrees  $T_1$  and  $T_2$  of G with roots  $x_1$ , resp.  $x_2$  such that  $S(x_1) \subseteq V(T_1)$ ,  $S(x_2) \subseteq V(T_2)$ . In particular, the sets  $S(x_1)$  and  $S(x_2)$  are disjoint.

In other words,  $\Delta(G)$  is pure of dimension n-1 iff b) is not true.

#### Proof.

 $\underline{a)\Rightarrow b}$ : Let F be a maximal simplex with fewer than n-1 edges. Then F defines a forest consisting of two or more trees. Let  $T_1$  and  $T_2$  be two different maximal subtrees of this forest with roots  $x_1$ , resp.  $x_2$ . If there exists  $y\in S(x_1)\setminus V(T_1)$ , then  $(y\to x_1)\cup F$  is a simplex of  $\Delta(G)$ . This contradicts to the fact that F is a maximal simplex. Thus  $S(x_1)\subseteq V(T_1)$  and, symmetrically,  $S(x_2)\subseteq V(T_2)$ .

 $\underline{b}) \Rightarrow a$ ): Let F be an arbitrary maximal simplex of  $\Delta(G)$  such that  $E(T_1) \cup \overline{E(T_2)} \subseteq F$ . Since  $S(x_1) \subseteq V(T_1)$  and  $S(x_2) \subseteq V(T_2)$ , there are no edges in F which point to either  $x_1$  or  $x_2$ . This proves that  $|F| \leq n - 2$ .  $\square$ 

#### 4.3 Graphs with complete source

#### 4.3.1 Shellability of complexes of directed trees

In the next theorem we answer the first question of R. Stanley.

**Theorem 4.3.1** If the graph G has a complete source, then the complex  $\Delta(G)$  is shellable. In particular  $\Delta(G_n)$  is shellable for any  $n \geq 1$ . More precisely,  $\Delta(G_n)$  is homotopy equivalent to a wedge of  $(n-1)^{n-1}$  spheres of dimension (n-2) and the representatives of the cohomology classes are labeled by the spanning directed trees having vertex 1 as a leaf.

**Proof.** Let G be a directed graph on n vertices labeled by the set [n] and 1 be the complete source of G. Clearly every partial subforest of G can be completed to a tree: just add edges pointing from the complete source to the roots of the trees

of this subforest (except for the one which contains the complete source). Thus  $\Delta(G)$  is pure. We shall now describe a labeling of the maximal faces of G.

For an edge  $(x \to y)$  we define  $\lambda(x \to y) = x$ . For a graph H we define  $\lambda(H) = (\lambda_1, \ldots, \lambda_m)$ , where m = |E(H)|,  $\{\lambda_1, \ldots, \lambda_m\} = \{\lambda(e) \mid e \in E(H)\}$  (as multisets) and  $\lambda_1 \leq \ldots \leq \lambda_m$ . The function  $\lambda$  describes a labeling of maximal faces of  $\Delta(G)$  by sequences of n-1 numbers. Let us order these sequences (and hence the maximal faces) in a lexicographic order. We shall next verify that this ordering is a shelling order.

Let A and B be directed trees on n vertices, such that

$$\lambda(A) = (\alpha_1, \dots, \alpha_{n-1}) \preceq (\beta_1, \dots, \beta_{n-1}) = \lambda(B).$$

Let C be a graph defined by V(C) = [n],  $E(C) = E(B) \cap E(A)$ . Clearly,  $\lambda(C)$  is a substring of  $\lambda(A)$  and  $\lambda(B)$ . C is a forest, denote its trees by  $T_1, \ldots, T_s$  such that  $1 \in V(T_1)$  and let  $r_i$  be the root of  $T_i$ . It is clear that edges from  $E(A) \setminus E(B)$  as well as from  $E(B) \setminus E(A)$  have the form  $(a \to r_i)$  for some  $a \in [n]$ ,  $i \in [s]$ .

Choose  $(x \to y) \in E(B) \setminus E(A)$  such that  $x \neq 1$ . It must exist, since otherwise we have  $\alpha_i = \beta_i$ ,  $i \in [n-1]$ , and all of the edges from  $E(A) \setminus E(B)$  and  $E(B) \setminus E(A)$  are  $(1 \to r_i)$ , for  $i = 2, \ldots, s$ , which would imply A = B.

Let B' be defined by  $E(B') = E(B) \setminus \{(x \to y)\}$ . B' consists of two trees T' and T'' with roots r' and r''. Assume that  $1 \in V(T')$ . Define a new tree  $\tilde{B}$  by

$$E(\tilde{B}) = E(B') \uplus \{(1 \rightarrow r'')\}.$$

It is clear that  $\lambda(\tilde{B}) = (\lambda(B) \setminus \{x\}) \uplus \{1\}$  (as multisets), hence  $\lambda(\tilde{B}) \preceq \lambda(B)$ . Furthermore, by construction, C is a subgraph of  $\tilde{B}$ . Hence we have verified Condition (S), thus  $\Delta(G)$  is shellable.

It is now easy to choose representatives of cohomology classes of  $\Delta(G)$ . They are labeled by maximal faces A, such that  $1 \notin \lambda(A)$  (the representatives themselves are functions which evaluate to 1 on such A and 0 on all other maximal faces). Furthermore, when  $G = G_n$ , it is easy to enumerate all such maximal faces. Denote this number by f(n). The condition  $1 \notin \lambda(A)$  simply means that 1 is a leaf in A, so we can instead consider trees on n-1 vertices with some marked vertex. The number of such trees is clearly (n-1)g(n-1), where g(n) is the number of all rooted labeled trees on n vertices. Since it is well known that  $g(n) = n^{n-1}$  we conclude that  $f(n) = (n-1)g(n-1) = (n-1)^{n-1}$ .  $\square$ 

In the last part of the argument above we have have found a new proof of the combinatorial formula  $\tilde{\chi}(\Delta(G_n)) = (n-1)^{n-1}$ , cf. [Pi96, St99].

#### 4.3.2 Algebraic consequences of the shelling

The result of Theorem 4.3.1 can be reformulated in the algebraic language as follows.

**Corollary 4.3.2** *Pick an index set*  $\Gamma \subseteq [n] \times [n]$  *such that*  $(1, i) \in \Gamma$  *for*  $2 \le i \le n$ ,  $(i, i) \notin \Gamma$  *for*  $1 \le i \le n$ . *For an arbitrary field* k, *let*  $k[\Gamma] = k[x_{ij}]_{(i,j)\in\Gamma}$ . *Let* I *be the ideal of*  $k[\Gamma]$  *generated by the following monomials:* 

- $x_{ij}x_{kj}$ , for  $(i,j),(k,j)\in\Gamma$ ;
- $x_{i_1i_2}x_{i_2i_3}\dots x_{i_ti_1}$ , for  $t \geq 2$ ,  $(i_t, i_1), (i_j, i_{j+1}) \in \Gamma$ , for  $j \in [t-1]$ .

Then the ring  $k[\Gamma]/I$  is Cohen-Macaulay (i.e., the dimension of  $k[\Gamma]/I$  is equal to its depth, see [BH93] for further details).

**Proof.** The ring  $k[\Gamma]/I$  is the Stanley-Reisner ring of  $\Delta(G)$ , where G is a directed graph with a complete source such that  $E(G) = \Gamma$ . See [Re76, St96] for the definition and properties of Stanley-Reisner rings.  $\square$ 

#### 4.3.3 A few words on the related $S_n$ -representations

The natural action of the symmetric group  $S_n$  on  $V(G_n)$  induces an action on  $\Delta(G_n)$  in an obvious way. This action determines a linear representation of  $S_n$  on  $H_{n-2}(G_n)$ . It would be interesting to understand the structure of this representation better, but it seems to be hard. However, R. Stanley was able to compute its character, [St97]. We study the topological quotient  $\Delta(\Pi_n)/S_n$  in Section 4.5.

Consider instead a slightly different, but better behaving action. Let us act with  $S_{n-1}$  on  $\Delta(G_n)$  by permuting the vertices  $\{2,\ldots,n\}$ . It follows from the description of the cohomology classes of  $\Delta(G_n)$  that  $S_{n-1}$  permutes them and thus the representation of  $S_{n-1}$  is a permutation representation:  $S_{n-1}$  permutes double-rooted trees on n-1 vertices. Again, the structure of this representation, such as decomposition into irreducibles, is unclear.

#### 4.4 Computations for other classes of graphs

#### 4.4.1 Graphs which are essentially trees

**Definition 4.4.1** A graph G is called **essentially a tree** if it turns into an undirected tree when one replaces all directed edges/pairs of directed edges going in opposite direction by an edge.

The following 3 propositions will provide us with a procedure to compute homology groups of  $\Delta(G)$  when G is essentially a tree.

**Proposition 4.4.2** Let G be a directed graph and let  $x \in V(G)$  with  $S(x) = \{y\}$ , for some  $y \in V(G)$ . If no edge of G has x as a source, then  $\Delta(G)$  is contractible.

**Proof.**  $\Delta(G)$  is a cone with apex  $(y \to x)$ .  $\square$ 

**Proposition 4.4.3** Let G be a graph,  $x \in V(G)$ , such that  $(x \to y), (y \to x) \in E(G)$  and there are no other edges where x is a source or a sink. Then  $\Delta(G) \simeq \sup \Delta(\tilde{G})$ , where  $\tilde{G}$  is defined by  $V(\tilde{G}) = V(G) \setminus \{x\}$ ,  $E(\tilde{G}) = E(G) \setminus (\{(x \leftarrow y)\} \uplus \{(y \leftarrow z) \mid z \in V(G)\})$ .

**Proof.** Let  $A = \operatorname{st}(x \to y)$ ,  $B = \operatorname{st}(y \to x)$ . Then  $\Delta(G) = A \cup B$  since every forest not containing edge  $(x \to y)$  can be extended with the edge  $(y \to x)$ .  $A \cap B$  contains those forests which can be extended with both  $(x \to y)$  and  $(y \to x)$ . This means we have to delete all edges having x or y as a sink, which gives  $A \cap B = \Delta(\tilde{G})$ . Both A and B are contractible, hence, by [Bj95, Lemma 10.4(ii)],  $\Delta(G) \simeq \operatorname{susp} \Delta(\tilde{G})$ .  $\square$ 

**Proposition 4.4.4** Let G be a graph and  $x_1, \ldots, x_k, y \in V(G)$  such that  $(x_1 \to y), \ldots, (x_k \to y) \in E(G)$  and there are no further edges which have  $x_i$  as a source or a sink for  $i \in [k]$ .

Assume furthermore, that for some  $z \in V(G) \setminus \{x_1, \ldots, x_k, y\}$  one of the following is true:

- a)  $(y \to z) \in E(G)$ ;
- b)  $(z \to y) \in E(G)$ ;
- c)  $(y \to z), (z \to y) \in E(G);$

and there are no other edges having y as a sink or a source.

Then, corresponding to these cases, we have

a) 
$$\Delta(G) \simeq \operatorname{susp}_k \Delta(\tilde{G})$$
, where  $V(\tilde{G}) = V(G) \setminus \{x_1, \dots, x_k\}$ ,  $E(\tilde{G}) = E(G) \setminus \{(x_1 \to y), \dots, (x_k \to y), (z \to y)\}$ ;

- b)  $\Delta(G) \simeq \sup_{k+1} \Delta(G')$ , where G' is the subgraph of G induced by  $V(G) \setminus \{x_1, \ldots, x_k, y\}$ ;
- c) If k = 1, then  $\Delta(G) \simeq susp \ \Delta(G')$ ; if  $k \geq 2$ , then, for all  $t \in \mathbb{Z}$ , we have  $H_t(G) = H_{t-1}(G') \oplus (H_{t-1}(\tilde{G}))^{\oplus (k-1)}$ .

**Proof.** Set  $A_i = \operatorname{st}(x_i \to y)$  for  $i \in [k]$ . Clearly, all  $A_i$  are contractible and the intersection of two or more of them is equal to  $\Delta(\tilde{G})$ . Furthermore, in case a) we have  $\Delta(G) = \bigcup_{i=1}^k A_i$  and hence the conclusion of a) follows by [Bj95, Lemma 10.4(ii)].

Assume now that b) or c) holds. In both cases  $(z \to y) \in E(G)$ . Set  $B = \operatorname{st}(z \to y)$ , then  $\Delta(G) = (\bigcup_{i=1}^k A_i) \cup B$ . In the case b) the intersection of any two or more of the complexes  $A_1, \ldots, A_k, B$  is equal to  $\Delta(G')$ , hence the conclusion of b) again follows by [Bj95, Lemma 10.4(ii)].

Let us show c) by induction on k. Assume k=1, then  $\Delta(G)=A_1\cup B$ . Both  $A_1$  and B are contractible, hence by [Bj95, Lemma 10.4(ii)] we get  $\Delta(G)\simeq \sup (A_1\cap B)=\sup \Delta(G')$ .

Assume now that  $k \geq 2$ . Let  $A = (\bigcup_{i=1}^{k-1} A_i) \cup B$  and  $A' = A_{k-1} \cup A_k$ . Then  $A \cup A' = \Delta(G)$  and  $A \cap A' = A_{k-1}$ , which is a cone. Thus

$$H_t(G) = H_t(A) \oplus H_t(A') = H_{t-1}(G') \oplus (H_{t-1}(\tilde{G}))^{\oplus (k-2)} \oplus H_{t-1}(\tilde{G}),$$

where the first equality follows by a Mayer-Vietoris argument and the second equality follows from the induction assumption.  $\Box$ 

So, given any graph G, which is essentially a tree, we have recursive procedure to compute homology groups  $H_*(G)$ . If some class of trees which behaves well under recursion is specified, then closed formulae can be derived if so desired. Observe, for example, that if G is a double directed tree with two leaves or more attached to the same vertex, then  $\Delta(G)$  is contractible (just apply Proposition 4.4.3 and then Proposition 4.4.2).

#### 4.4.2 Double directed strings

Another interesting specific example is a double directed string on n+1 vertices  $L_n$ , which is defined by  $V(L_n)=[n+1]$ ,  $E(L_n)=\{(i\to i+1),(i+1\to i)\,|\,i\in[n]\}$ . Let  $\tilde{L}_n$  be the directed graph defined by  $V(\tilde{L}_n)=[n+2]$ ,  $E(\tilde{L}_n)=E(L_n)\uplus\{(n+1\leftarrow n+2)\}$ . The complexes  $\Delta(L_n)$  and  $\Delta(\tilde{L}_n)$  have an alternative description.

**Definition 4.4.5** Complex  $\mathcal{L}_n$  has n vertices indexed by the set [n] and  $F \in 2^{[n]}$  is a face of  $\mathcal{L}_n$  iff it does not contain  $\{i, i+1\}$  for  $i \in [n-1]$ .

It is easy to see that

$$\mathcal{L}_{2n} \simeq \Delta(L_n), \quad \mathcal{L}_{2n+1} \simeq \Delta(\tilde{L}_n).$$
 (4.4.1)

#### **Proposition 4.4.6**

- (1)  $\mathcal{L}_{n+3} \simeq susp \mathcal{L}_n$ .
- (2) The generating simplices for  $\mathcal{L}_n$  are  $(2, 5, \dots, 3k + 2)$ , if n = 3k + 2, and  $(2, 5, \dots, 3k 1)$ , if n = 3k.
- (3)  $\mathcal{L}_1$  is contractible,  $\mathcal{L}_2 = S^0$ , and  $\mathcal{L}_3 \simeq S^0$ . Hence

$$\mathcal{L}_n \simeq \begin{cases} S^{k-1}, & \text{if } n = 3k; \\ a \text{ point}, & \text{if } n = 3k+1; \\ S^k, & \text{if } n = 3k+2. \end{cases}$$

$$(4.4.2)$$

**Proof.** Let  $C = \mathcal{L}_{n+3} \setminus \{3\}$ . Since every maximal simplex of C contains exactly one of the vertices of 1 and 2, we have  $C \simeq \text{susp } (\mathcal{L}_{n+3} \setminus \{1,2,3\}) \simeq \text{susp } (\mathcal{L}_n)$ . Let us show that C is a deformation retract of  $\mathcal{L}_{n+3}$ . Order the simplices of  $\mathcal{L}_{n+3}$  which have vertex 3, but do not have vertex 1 in any order  $S_1, \ldots, S_k$  respecting inclusions, i.e., if  $S_j$  is a subcomplex of  $S_i$ , then i < j. Remove pairs of simplices  $(S_i, S_i \uplus \{1\})$  in the increasing order of i. These removals are elementary collapses, they correspond to deformation retracts. Since  $\mathcal{L}_{n+3} = C \uplus \{S_i \mid i \in [k]\} \uplus \{S_i \uplus \{1\} \mid i \in [k]\}$ , we conclude that C is a deformation retract of  $\mathcal{L}_{n+3}$ . This proves (1).

(2) follows from (1) and the fact that if  $\Delta$  is a simplicial complex which is homotopy equivalent to a sphere and F is its generating simplex, then  $F \cup \{a\}$  is a generating simplex of  $\Delta * \{a,b\} = \sup \Delta$ . To see this, it is enough to show that  $\Delta * \{a,b\} \setminus \{F \cup \{a\}\}$  is contractible. Indeed, removing F and  $F \cup \{b\}$  from  $\Delta * \{a,b\} \setminus \{F \cup \{a\}\}$  is an elementary collapse, and  $\Delta * \{a,b\} \setminus \{F,F \cup \{a\},F \cup \{b\}\} = (\Delta \setminus F) * \{a,b\}$ , which is contractible. The verification of (3) is left to the reader.  $\Box$ 

Using (4.4.1) and (4.4.2) we immediately derive that

$$\Delta(L_n) \simeq \begin{cases} S^{2k-1}, & \text{if } n = 3k; \\ S^{2k}, & \text{if } n = 3k+1; \\ \text{a point,} & \text{if } n = 3k+2. \end{cases}$$

#### 4.4.3 *Cycles*

Let  $n \geq 3$  and denote by  $C_n$  a double directed cycle, i.e., a directed graph defined by  $V(C_n) = \mathbb{Z}_n$ ,  $E(C_n) = \{(i \to i+1), (i+1 \to i) \mid i \in \mathbb{Z}_n\}$ . In this section we determine the homotopy type of  $\Delta(C_n)$ .

Again we would like to formulate our complexes in a slightly different language. **Definition 4.4.7** Complex  $C_n$  has n vertices indexed by the set  $\mathbb{Z}_n$  and  $F \in 2^{\mathbb{Z}_n}$  is a face of  $C_n$  iff it does not contain  $\{i, i+1\}$  for  $i \in \mathbb{Z}_n$ .

Similar to before, we have a relation  $\Delta(C_n) = \tilde{\mathcal{C}}_{2n}$ , where  $\tilde{\mathcal{C}}_{2n}$  is obtained from  $\mathcal{C}_{2n}$  by deleting the simplices  $(1, 3, \dots, 2n - 1)$  and  $(2, 4, \dots, 2n)$ .

**Proposition 4.4.8** *The homotopy type of*  $C_n$  *is given by* 

$$C_n \simeq \begin{cases} S^{k-1} \vee S^{k-1}, & \text{if } n = 3k; \\ S^{k-1}, & \text{if } n = 3k+1; \\ S^k, & \text{if } n = 3k+2. \end{cases}$$
(4.4.3)

**Therefore** 

$$\Delta(C_n) \simeq \begin{cases} S^{2k-1} \vee S^{2k-1} \vee S^{3k-2} \vee S^{3k-2}, & \text{if } n = 3k; \\ S^{2k} \vee S^{3k-1} \vee S^{3k-1}, & \text{if } n = 3k+1; \\ S^{2k} \vee S^{3k} \vee S^{3k}, & \text{if } n = 3k+2. \end{cases}$$
(4.4.4)

**Proof.** Let  $A = \operatorname{st}_{\mathcal{C}_n}(1)$  and  $B = \mathcal{C}_n \setminus \{1\}$ . Then  $A \cup B = \mathcal{C}_n$ ,  $A \cap B = \mathcal{C}_n \setminus \{1, 2, n\} = \mathcal{L}_{n-3}$ , A is contractible and  $B = \mathcal{L}_{n-1}$ .

If n=3k+2, then B is contractible, hence  $\mathcal{C}_n \simeq \text{susp } (A \cap B) = \text{susp } (\mathcal{L}_{n-3}) \simeq S^k$  and  $(1,4,\ldots,n-1)$  is the generating simplex.

If n = 3k + 1, then  $A \cap B$  is contractible, hence  $C_n \simeq B = L_{n-1} \simeq S^{k-1}$ .

If n=3k, let  $F=(3,6,\ldots,n)$  be a generating simplex of B. Since  $F\in B\setminus (A\cap B)$ , one can shrink  $B\setminus F$  to a point inside  $\mathcal{C}_n$  and obtain

$$C_n \simeq B \vee \text{susp } (A \cap B) \simeq \mathcal{L}_{3k-1} \vee \mathcal{L}_{3k} \simeq S^{k-1} \vee S^{k-1}.$$

Furthermore the generating simplices are  $(3, 6, \ldots, n)$  and  $(1, 4, \ldots, n-2)$ . This shows (4.4.3).

Let us now see (4.4.4). Let us single out the following simplices of  $\mathcal{C}_{2n}$ :  $F_1=(1,3,\ldots,2n-1),\ F_2=(2,4,\ldots,2n),\ \tilde{F}_1=(1,3,\ldots,2n-3),\ \tilde{F}_2=(2,4,\ldots,2n-2).$  Recall that  $\Delta(C_n)=\mathcal{C}_{2n}\setminus\{F_1,F_2\}.$   $F_1$ , resp.  $F_2$ , is the only maximal simplex which contains  $\tilde{F}_1$ , resp.  $\tilde{F}_2$ , hence to remove  $F_1,\ F_2,\ \tilde{F}_1,\ \text{ and }\tilde{F}_2,\ \text{ from }\mathcal{C}_{2n}\ \text{ means to perform two elementary collapses. Let } \hat{\mathcal{C}}_{2n}=\mathcal{C}_{2n}\setminus\{F_1,F_2,\tilde{F}_1,\tilde{F}_2\}.$  Let us show that the boundaries of  $\tilde{F}_1$  and  $\tilde{F}_2$  can be shrunk to a point within  $\hat{\mathcal{C}}_{2n}$ .

Simplices  $F_1$  and  $\tilde{F}_1$  lie in the subcomplex  $C_{2n} \setminus \{2n\} \simeq L_{2n-1}$ . We can choose  $(2, 5, \ldots, 2n-2)$ , if 2n = 3k + 1, and  $(2, 5, \ldots, 2n-1)$ , if 2n = 3k, as a generating simplex of  $(C_{2n} \setminus \{2n\}) \setminus \{F_1, \tilde{F}_1\}$ .  $\tilde{F}_1$  does not contain this generating simplex, hence its boundary can be shrunk to a point within  $(C_{2n} \setminus \{2n\}) \setminus \{F_1, \tilde{F}_1\}$ ,

and hence within  $\hat{C}_{2n}$ . Analogously the boundary of  $\tilde{F}_2$  can be shrunk to a point within  $\hat{C}_{2n}$ .

Thus we obtain 
$$\Delta(C_n) \simeq \mathcal{C}_{2n} \setminus \{F_1, F_2\} = \hat{\mathcal{C}}_{2n} \cup \{\tilde{F}_1, \tilde{F}_2\} \simeq \hat{\mathcal{C}}_{2n} \vee S^{n-2} \vee S^{n-2} \otimes \mathcal{C}_{2n} \vee S^{n-2} \otimes \mathcal{C}_{2n} \vee S^{n-2} \otimes \mathcal{C}_{2n} \otimes \mathcal{C}_{2n}$$

#### 4.5 $S_n$ -quotients of complexes of directed forests

#### 4.5.1 A combinatorial description for the cell structure of $X_n$

As mentioned in the introduction, let  $X_n$  be the topological quotient  $\Delta(G_n)/\mathcal{S}_n$ . Clearly, the action of  $\mathcal{S}_n$  on  $\Delta(G_n)$  is not free. What is worse, the elements of  $\mathcal{S}_n$  may fix the simplices of  $\Delta(G_n)$  without fixing them pointwise: for example for n=3 the permutation (23) "flips" the 1-simplex given by the directed tree  $2 \longleftarrow 1 \longrightarrow 3$ . Therefore, one does not have a bijection between the orbits of simplices of  $\Delta(G_n)$  and simplices of  $X_n$ .

To rectify the situation, let us consider the barycentric subdivision  $B_n = \operatorname{Bsd}(\Delta(G_n))$ . We have a simplicial  $\mathcal{S}_n$ -action on  $B_n$  induced from the  $\mathcal{S}_n$ -action on  $\Delta(G_n)$  and, clearly,  $B_n/\mathcal{S}_n$  is homeomorphic to  $X_n$ . Furthermore, if an element of  $\mathcal{S}_n$  fixes a simplex of  $B_n$  then it fixes it pointwise. In this situation, it is well-known, e.g., see [Br72], that the quotient projection  $B_n \to X_n$  induces a simplicial structure on  $X_n$ , in which simplices of  $X_n$  correspond to  $\mathcal{S}_n$ -orbits of the simplices of  $B_n$  with appropriate boundary relation.

Let us now give a combinatorial description of the  $\mathcal{S}_n$ -orbits of the simplices of  $B_n$ . Let  $\sigma$  be a simplex of  $B_n$ , then  $\sigma$  is a chain  $(T_1, T_2, \ldots, T_{\dim(\sigma)+1})$  of forests on n labeled vertices, such that  $T_i$  is a subgraph of  $T_{i+1}$ , for  $i=1,\ldots,\dim(\sigma)$ . One can view this data in a slightly different way: it is a forest with  $\dim(\sigma)+1$  integer labels on edges (labels on different edges may coincide). Indeed, given a chain of forests as above, take the forest  $T_{\dim(\sigma)+1}$  and put label 1 an all edges of the forest  $T_1$ , label 2 on all edges of  $T_2$ , which are not labeled yet, etc. Vice versa, given a forest T with a labeling, let  $T_1$  be the forest consisting of all edges of T with one of the two smallest label, let  $T_2$  be the forest consisting of all edges of T with one of the two smallest labels, etc. To make the described correspondence a bijection, one should identify all labeled forests on which labelings produce the same order on edges.

Formally: the p-simplices of  $B_n$  are in bijection with the set of all pairs  $(T, \phi^T)$ , where T is a directed forest on n labeled vertices and  $\phi^T : E(T) \to \mathbb{Z}$ , such that  $|\operatorname{Im} \phi^T| = p+1$ , modulo the following equivalence relation:  $(T_1, \phi^{T_1}) \sim (T_2, \phi^{T_2})$  if  $T_1 = T_2$  and there exists an order-preserving injection  $\psi : \mathbb{Z} \to \mathbb{Z}$ , such that  $\phi^{T_1} \circ \psi = \phi^{T_2}$ .

The boundary operator can be described as follows: for a p-simplex  $(T, \phi^T)$ ,

 $p \ge 1$ , we have

$$\partial(T, \phi^T) = \sum_{i=1}^{p+1} (-1)^{p+i+1} (T_i, \phi^{T_i}),$$

where, for  $i=1,\ldots,p$ , we have  $T_i=T$  and  $\phi^{T_i}$  takes the same values as  $\phi^T$  except for the edges on which  $\phi^T$  takes ith and (i+1)st largest values (say a and b), on these edges  $\phi^{T_i}$  takes value a. Furthermore,  $T_{p+1}$  is obtained from T be removing the edges with the highest value of  $\phi^T$ ,  $\phi^{T_{p+1}}$  is the restriction of  $\phi^T$ . Of course, this description of the boundary map is just a rephrasing of the deletion of the ith forest from the chain of forests in the original description. However, we will find it more convenient to work with the labeled forests rather than the chains of forests.

The orbits of the action of  $S_n$  can be obtained by forgetting the numbering of the vertices. Thus, using the fact that simplices of  $X_n$  and  $S_n$ -orbits of simplices of  $B_n$  are the same thing, we get the following description.

The p-simplices of  $X_n$  are in bijection with pairs  $(T, \phi^T)$ , where T is a directed forest on n unlabeled vertices and  $\phi^T$  is an edge labeling of T with p+1 labels, modulo a certain equivalence relation. This equivalence relation and the boundary operator are exactly as in the description of simplices of  $B_n$ .

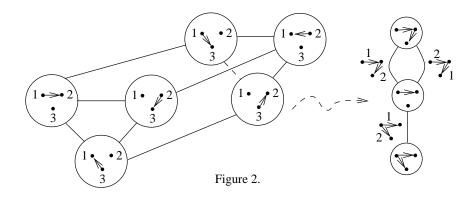


Figure 4.2

On Figure 4.2 we show the case n=3: on the left hand side we have  $\Delta(G_3)$ , on the right hand side is  $X_3=\Delta(G_3)/\mathcal{S}_3$ . The labeled forests next to the edges indicate the bijection described above, labeling on the forests corresponding to the vertices in  $X_3$  is omitted.  $\mathcal{S}_3$  acts on  $\Delta(G_3)$  as follows: 3-cycles act as rotations around the line which goes through the middles of the triangles, each transposition acts as a central symmetry on one of the quadrangles, and as a "flip" on the edge which is parallel to that quadrangle.

#### 4.5.2 Filtration and description of the $E^1$ tableau

There is a natural filtration on the chain complex associated to the simplicial structure on  $X_n$  described above. Let  $F_i$  be the union of all simplices  $(T, \phi^T)$  where T has at most i edges. Clearly,  $\emptyset = F_0 \subset F_1 \subset \ldots \subset F_{n-1} = X_n$ .

Recall that  $E_{p,k}^1 = H_p(F_k, F_{k-1})$ , here we use the indexing from the Appendix D. In other words, the homology is computed with "truncated" boundary operator: the last term, where some edges are deleted from the forest, is omitted. Clearly,

$$E_{p,k}^1 = \bigoplus_T H_p(E_T),$$
 (4.5.1)

where the sum is over all forests with k edges and  $E_T$  is a chain complex generated by the simplices  $(T, \phi^T)$ , for various labelings  $\phi^T$ , with the truncated boundary operator as above.

Let us now describe a simplicial complex whose reduced homology groups, after a shift in the index by 1, are equal to the nonreduced homology groups of  $E_T$ . The arrangement of k(k-1)/2 hyperplanes  $x_i = x_j$  in  $\mathbb{R}^k$  cuts the space  $S^{k-1} \cap H$  into simplices, where H is the hyperplane given by the equation  $x_1 + x_2 + \cdots + x_k = 0$ . Denote this simplicial complex  $A_k$ . The permutation action of  $\mathcal{S}_k$  on [k] induces an  $\mathcal{S}_k$ -action on  $A_k$ . It is easy to see that if an element of  $\mathcal{S}_k$  fixes a simplex of  $A_k$ , then it fixes it pointwise. Hence, for any subgroup  $\Gamma \subseteq \mathcal{S}_k$ , the  $\Gamma$ -orbits of the simplices of  $A_k$  are in a natural bijection with the simplices of  $A_k/\Gamma$ .

Let T be an arbitrary forest with n vertices and k edges. Assume that vertices, resp. edges, are labeled with numbers  $1, \ldots, n$ , resp.  $1, \ldots, k$ .  $\mathcal{S}_n$  acts on [n] by permutation, let  $\operatorname{Stab}(T)$  be stabilizer of T under this action, that is the maximal subgroup of  $\mathcal{S}_n$  which fixes T. Then  $\operatorname{Stab}(T)$  acts on E(T), i.e., we have a homomorphism  $\chi:\operatorname{Stab}(T)\to \mathcal{S}_k$ . Let  $\mathcal{S}(T)=\operatorname{Im}\chi$ . Clearly  $\mathcal{S}(T)$  does not depend on the choice of the labeling of vertices. However, relabeling the edges changes  $\mathcal{S}(T)$  to a conjugate subgroup. Therefore, for a forest T without labeling on vertices and edges,  $\mathcal{S}(T)$  can be defined, but only up to a conjugation.

**Proposition 4.5.1** The chain complex of  $A_k/\mathcal{S}(T)$  and  $E_T$  (with a shift by 1 in the indexing) are isomorphic. In particular,  $\widetilde{H}_p(A_k/\mathcal{S}(T)) = H_{p+1}(E_T)$ .

**Proof.** Label the k edges of T with numbers  $1, \ldots, k$ . As mentioned above, the p-simplices of  $E_T$  are in bijection with labelings of the edges of T with numbers  $1, \ldots, p+1$  (using each number at least once). Taking in account the chosen labeling of the edges, this is the same as to divide the set [k] into an ordered tuple of p+1 non-empty sets, modulo the symmetries of [k] induced by the symmetries of T. Clearly, these symmetries of [k] are precisely the elements of  $\mathcal{S}(T)$ .

The (p-1)-simplices of  $A_k$  are in bijection with dividing [k] into an ordered tuple of p+1 non-empty sets: by the values of the coordinates. Therefore we conclude that the p-simplices of  $E_T$  are in a natural bijection with the (p-1)-simplices of  $A_k/\mathcal{S}(T)$ . Here the unique 0-simplex of  $E_T$ , (T,1), (1 is the constant function taking value 1), corresponds in  $A_k/\mathcal{S}(T)$  to the empty set, which is a (-1)-simplex. One verifies immediately that the boundary operators of  $E_T$  and  $A_k/\mathcal{S}(T)$  commute with the described bijection. Therefore  $E_T$  and  $A_k/\mathcal{S}(T)$  are isomorphic as chain complexes (after a shift in the indexing). In particular,  $\widetilde{H}_p(A_k/\mathcal{S}(T)) = H_{p+1}(E_T)$ .  $\square$ 

#### 4.5.3 $\mathbb{Q}$ coefficients

Proposition 4.5.1 allows us to give a description of  $E^1_{*,*}$ -entries in the case when the homology groups are computed with rational coefficients.

Indeed, it is well known that, when a finite group  $\Gamma$  acts on a finite simplicial complex X, one has  $\widetilde{H}_i(X/\Gamma,\mathbb{Q})=\widetilde{H}_i^\Gamma(X,\mathbb{Q})$ , where  $\widetilde{H}_i^\Gamma(X,\mathbb{Q})$  is the maximal vector subspace of  $\widetilde{H}_i(X,\mathbb{Q})$  on which  $\Gamma$  acts trivially (more generally  $\mathbb{Q}$  can be replaced with a field whose characteristic does not divide  $|\Gamma|$ ). Since  $A_k$  is homeomorphic to  $S^{k-2}$  we have  $\widetilde{H}_{k-2}(A_k,\mathbb{Q})=\mathbb{Q}$  and  $\widetilde{H}_i(A_k,\mathbb{Q})=0$  for  $i\neq k-2$ .

It is easy to compute  $\widetilde{H}_{k-2}^{\mathcal{S}(T)}(A_k,\mathbb{Q})$ . In fact, for  $\pi \in \mathcal{S}_k$ ,  $\alpha \in \widetilde{H}_{k-2}(A_k,\mathbb{Q})$ , one has  $\pi(\alpha) = (-1)^{\operatorname{sgn} \pi} \alpha$ , where sgn denotes the sign homomorphism sgn:  $\mathcal{S}_k \to \{-1,1\}$ . Therefore

$$\widetilde{H}_{k-2}(A_k/\mathcal{S}(T),\mathbb{Q}) = \widetilde{H}_{k-2}^{\mathcal{S}(T)}(A_k,\mathbb{Q}) = \begin{cases} \mathbb{Q}, & \text{if } \mathcal{S}(T) \subseteq \mathcal{A}_k, \\ 0, & \text{otherwise,} \end{cases}$$

where  $A_k$  is the alternating group,  $A_k = \operatorname{sgn}^{-1}(1)$ .

Combined with the Proposition 4.5.1 this gives  $H_i(E_T,\mathbb{Q})=\mathbb{Q}$ , if i=|E(T)|-1 and  $\mathcal{S}(T)\subseteq\mathcal{A}_{|E(T)|}$ , and  $H_i(E_T,\mathbb{Q})=0$  in all other cases. Therefore it follows from (4.5.1) that  $\mathrm{rk}\,E^1_{k-1,k}=f_{k,n}$ , where  $f_{k,n}$  is equal to the number of forests T with k edges and n vertices, such that  $\mathcal{S}(T)\subseteq\mathcal{A}_k$ .  $\mathrm{rk}\,E^1_{p,k}=0$  for  $p\neq k-1$ . Note that  $\beta_i(X_n,\mathbb{Q})=0$ , for  $i\neq n-2$ , because  $\beta_i(\Delta(G_n),\mathbb{Q})=0$ , for  $i\neq n-2$  (by the Theorem 4.3.1), and  $\beta_i(X_n,\mathbb{Q})=\beta_i^{\mathcal{S}_n}(\Delta(G_n),\mathbb{Q})$ . In particular, by computing the Euler characteristic of  $X_n$  in two different ways, we obtain

**Theorem 4.5.2** For 
$$n \geq 3$$
,  $\beta_{n-2}(X_n, \mathbb{Q}) = \sum_{k=2}^{n-1} (-1)^{n+k+1} f_{k,n}$ .

The first values of  $f_{k,n}$  are given in the Table 4.3. Note that there are zeroes on and below the main diagonal and that the rows stabilize at the entry (k, 2k-1) (for  $k \ge 2$ ).

$k \backslash n$	1	2	3	4	5	6
1	0	1	1	1	1	1
2	0	0	1	1	1	1
3	0	0	0	2	3	3
4	0	0	0	0	4	7
5	0	0	0	0	0	8

Table 4.3

#### 4.5.4 $\mathbb{Z}$ coefficients

The case of integer coefficients is more complicated. In general, we do not even know the entries of the first tableau. However, we do know that it is different from the rational case, i.e., torsion may occur.

For example, let T be the forest with 8 vertices and 6 edges depicted on Figure 4.4. Clearly,  $\mathcal{S}(T) = \{\mathrm{id}, (12)(34)(56)\}$ . It is easy to see that  $A_6/\mathcal{S}(T)$  is a double suspension (by which we mean suspension of suspension) of  $\mathbb{RP}^2$ , thus the only nonzero homology group is  $\widetilde{H}_3(A_6/\mathcal{S}(T), \mathbb{Z}) = \mathbb{Z}_2$ . In particular,  $E_{4,6}^1$  is not free.

Figure 4.4

On the positive side, we can describe the values which  $d^1$  takes on the "rational" generators of  $E^1_{*,*}$ . Let us call a forest *admissible* if  $\mathcal{S}(T) \subseteq \mathcal{A}_{|E(T)|}$ . For every admissible forest T with k edges we fix some order on the edges, i.e., a bijection  $\psi_T: E(T) \to [k]$ . This determines uniquely an integer generator  $e_T$  of  $H_{k-1}(E_T, \mathbb{Z})$  by

$$e_T = \sum_{\mathcal{S}(T)g} \operatorname{sgn}(g)(T, g \circ \psi_T), \tag{4.5.2}$$

where we sum over all right cosets of  $\mathcal{S}(T)$ , (we choose one representative for each coset). Observe that the sign of g, resp. the simplex  $(T, g \circ \psi_T)$ , are the same for different representatives of the same right coset class, because  $\mathcal{S}(T) \subseteq \mathcal{A}_k$ , resp. by the definition of  $\mathcal{S}(T)$ .

**Proposition 4.5.3** For an admissible forest T, we have

$$d^{1}(e_{T}) = \sum_{\alpha} sgn\left(\tilde{\psi}_{T,\alpha} \circ \psi_{T}^{-1}\right) \lambda_{T,\alpha} e_{T \setminus \alpha}, \tag{4.5.3}$$

where the sum is over S(T)-orbits of E(T), for which there exists a representative  $\alpha$ , such that  $T \setminus \alpha$  is admissible, we choose one representative for each orbit; note that the admissibility of  $T \setminus \alpha$  depends only on the S(T)-orbit of  $\alpha$ , not on the choice of the representative. Notation in the formula:  $T \setminus \alpha$  denotes the forest obtained from T by removing the edge  $\alpha$ ;  $\tilde{\psi}_{T,\alpha}: E(T) \to [k]$  is defined by  $\tilde{\psi}_{T,\alpha}|_{T\setminus \alpha} = \psi_{T\setminus \alpha}$  and  $\tilde{\psi}_{T,\alpha}(\alpha) = k$ ;  $\lambda_{T,\alpha} = [S(T \setminus \alpha): \tilde{S}(T)]$ , where  $\tilde{S}(T)$  consists of those permutations of edges of  $T \setminus \alpha$  which can be extended to T by fixing the additional edge.

**Proof.** For an admissible forest T with k edges and a bijection  $\phi: E(T) \to [k]$ , let  $(\widetilde{T}, \widetilde{\phi})$  denote a face simplex of  $(T, \phi)$ , where  $\widetilde{T}$  is obtained from T by removing the edge with the highest label,  $\widetilde{\phi}$  is the restriction of  $\phi$  to  $\widetilde{T}$ . In our notations  $(\widetilde{T}, \widetilde{\phi}) = (T \setminus \phi^{-1}(k), \phi|_{E(T \setminus \phi^{-1}(k))})$ . However, for convenience, we use the notation "tilde" in the rest of the proof.

According to the general theory for spectral sequences,  $d^1(e_T) = \partial(e_T)$ , where  $\partial$  denotes the usual boundary operator, and we view  $\partial(e_T)$  as embedded into the relative homology group  $H_{k-2}(F_{k-1},F_{k-2})$ .  $\partial(e_T)$  is a linear combination of simplices which are obtained from the simplices  $(T,g\circ\psi_T)$  by either merging two labels, or omitting the edge with the top label.  $e_T\in H_{k-1}(F_k,F_{k-1})$  means that the application of the "truncated" boundary operator to  $e_T$  gives 0, therefore all the simplices obtained by merging two labels will cancel out. Furthermore, since  $\partial(e_T)\in H_{k-2}(F_{k-1},F_{k-2})$ ,  $\dim F_{k-1}=k-2$ , and the group  $H_{k-2}(F_{k-1},F_{k-2})$  is freely generated by  $e_U$ , where U is an admissible forest with k-1 edges, we can conclude that also the contributions  $(\widetilde{T},\widetilde{\phi})$ , where  $\widetilde{T}$  is not admissible, will cancel out. Combining these arguments with (4.5.2) we obtain:

$$d^{1}(e_{T}) = \sum_{\mathcal{S}(T)g} \operatorname{sgn}(g)(\widetilde{T}, \widetilde{g \circ \psi_{T}}), \tag{4.5.4}$$

where we have only those terms left in the sum, for which  $\widetilde{T}$  is admissible. After regrouping we get

$$\sum_{\mathcal{S}(T)g} \operatorname{sgn}(g)(\widetilde{T}, \widetilde{g \circ \psi_T}) = \sum_{\alpha} \sum_{\mathcal{S}(T)g} \operatorname{sgn}(g)(\widetilde{T}, \widetilde{g \circ \psi_T}), \tag{4.5.5}$$

where in the second term the first sum is taken over all  $\mathcal{S}(T)$ -orbits of [k], for which  $\widetilde{T}$  is admissible, while the second sum is taken over all right cosets  $\mathcal{S}(T)g$  which have a representative g such that  $g \circ \psi_T(\alpha) = k$ , we take one representative per coset. To verify (4.5.5) we just need to observe that the  $\mathcal{S}(T)$ -orbit of  $(g \circ \psi_T)^{-1}(k)$  does not depend on the choice of the representative of  $\mathcal{S}(T)g$ ; this follows from the definition of  $\mathcal{S}(T)$ .

Finally, one can see that, for  $\alpha$  being an edge of T, such that  $T \setminus \alpha$  is admissible,

$$\sum_{\mathcal{S}(T)g}\operatorname{sgn}(g)(\widetilde{T},\widetilde{g\circ\psi_T})=\operatorname{sgn}(\widetilde{\psi}_{T,\alpha}\circ\psi_T^{-1})\lambda_{T,\alpha}\sum_{\mathcal{S}(T\setminus\alpha)h}\operatorname{sgn}(h)(T\setminus\alpha,h\circ\psi_{T\setminus\alpha}),$$
(4.5.6)

where the sum in the first term is again taken over all right cosets S(T)g which have a representative g such that  $g \circ \psi_T(\alpha) = k$ , and the sum in the second term is simply over all right cosets of  $S(T \setminus \alpha)$ .

Indeed, on the left hand side we have a sum over all labelings of E(T) with numbers  $1,\ldots,k$ , such that  $\alpha$  gets a label k, and we consider these labelings up to a symmetry of T; each labeling comes in with a sign of the permutation g, which is obtained by reading off this labeling in the order prescribed by  $\psi_T$ . On the right hand side the same sum is regrouped, using the observation that to label E(T) with [k], so that  $\alpha$  gets a label k, is the same as to label  $E(T \setminus \alpha)$  with [k-1]. The only details which need attention are the multiplicity and the sign.

Every  $\mathcal{S}(T)$ -orbit of labelings of E(T) with [k] so that  $\alpha$  gets a label k corresponds to  $[\mathcal{S}(T \setminus \alpha) : \widetilde{\mathcal{S}}(T)]$  of  $\mathcal{S}(T \setminus \alpha)$ -orbits of labelings of  $E(T \setminus \alpha)$  with [k-1], since we identify labelings by the actions of different groups:  $\mathcal{S}(T \setminus \alpha) \supseteq \widetilde{\mathcal{S}}(T)$ . Each of this  $\mathcal{S}(T \setminus \alpha)$ -orbits comes with the same sign, because  $\mathcal{S}(T \setminus \alpha) \subseteq \mathcal{A}_{k-1}$ . The sign  $\operatorname{sgn}(\tilde{\psi}_{T,\alpha} \circ \psi_T^{-1})$  corresponds to the change of the order in which we read off the edges: instead of reading them off according to  $\psi_T$ , we first read off along  $\psi_{T \setminus \alpha}$  and then read off the edge  $\alpha$  last. Formally:  $g \circ \psi_T = \tilde{h} \circ \tilde{\psi}_{T,\alpha}$ , and  $\operatorname{sgn}\tilde{h} = \operatorname{sgn}h$ , hence  $\operatorname{sgn}g = \operatorname{sgn}h \operatorname{sgn}(\tilde{\psi}_{T,\alpha} \circ \psi_T^{-1})$ , where  $\tilde{h}$  is defined by  $\tilde{h}|_{[k-1]} = h$ ,  $\tilde{h}(k) = k$ .

Combining (4.5.4), (4.5.5) and (4.5.6) we obtain (4.5.3).  $\Box$ 

#### 4.5.5 Homology groups of $X_n$ for n = 2, 3, 4, 5, 6

 $X_2$  is just a point. As shown in Figure 4.3,  $X_3 \simeq S^1$ , where  $\simeq$  denotes homotopy equivalence. With a bit of labor, one can manually verify that  $X_4 \simeq S^2$ . Furthermore, one can see that  $H_3(X_5,\mathbb{Z}) = \mathbb{Z}^2$  and  $\widetilde{H}_i(X_5,\mathbb{Z}) = 0$  for  $i \neq 3$ . We leave this to the reader, while confining ourselves to the case n=6. On Figure 4.5 we have all forests on 6 vertices. We denote some of the forests by two digits. The numbers over the edges denote the order in which we read the labels, i.e., the bijection  $\psi_T$ .

It is easy to see that  $A_k/\mathcal{S}(T)$  is homeomorphic to  $S^{k-2}$  for all admissible T, and is contractible otherwise. The only nontrivial cases are 41, 47, 48, 51, 55, and 59, all of which can be verified directly. Therefore, the only nontrivial entries of  $E^1_{*,*}$  ( $\mathbb{Z}$  coefficients) will lie on the (k-1,k)-diagonal. Thus  $H_*(X_6,\mathbb{Z})$  can be computed from the chain complex  $0 \leftarrow E^1_{0,1} \xleftarrow{d^1} E^1_{1,2} \xleftarrow{d^1} E^1_{2,3} \xleftarrow{d^1} E^1_{3,4} \xleftarrow{d^1}$ 

$$E_{4.5}^1 \leftarrow 0.$$

By Proposition 4.5.3 we have the following relations:

$$\begin{array}{lll} d^1(11)=0, & d^1(21)=0, \\ d^1(31)=2\cdot 21, & d^1(32)=21, & d^1(33)=21, \\ d^1(41)=32-33, & d^1(42)=31-32-33, & d^1(43)=31-2\cdot 33, \\ d^1(44)=0, & d^1(45)=31-32-33, & d^1(46)=32-33, \\ & d^1(47)=0, & \end{array}$$

$$d^{1}(51) = 41 - 46,$$

$$d^{1}(53) = -42 + 45,$$

$$d^{1}(55) = 41 - 43 + 45,$$

$$d^{1}(56) = 42 + 43 + 44 - 46,$$

$$d^{1}(54) = 2 \cdot 41 + 42 - 43 - 46 + 2 \cdot 47,$$

$$d^{1}(56) = 42 + 44 - 45 - 2 \cdot 47,$$

$$d^{1}(57) = -43 + 44 + 45 + 46,$$

$$d^{1}(58) = 2 \cdot 44 + 2 \cdot 47,$$

here the two-digit strings denote the corresponding forests on Figure 4.5. Thus  $\widetilde{H}_3(X_6,\mathbb{Z})=\mathbb{Z}_2$ ,  $\widetilde{H}_4(X_6,\mathbb{Z})=\mathbb{Z}^3$  and  $\widetilde{H}_i(X_6,\mathbb{Z})=0$  for  $i\neq 3,4$ .

Therefore we conclude that 6 is the smallest value of n, for which the homology groups  $H_*(X_n, \mathbb{Z})$  are not free.

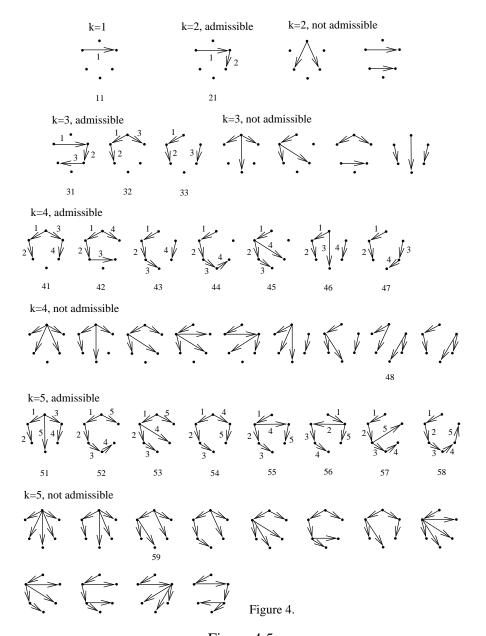


Figure 4.5

#### CHAPTER 5

#### GROUP ACTIONS ON POSETS

#### 5.1 PREAMBLE

Assume that we have a finite group G acting on a poset P in an order-preserving way. The purpose of this chapter is to compare the various constructions of the quotient, associated with this action. Our basic suggestion is to view P as a category and the group action as a functor from G to  $\mathbf{Cat}$ . Then, it is natural to define P/G to be the colimit of this functor. As a result P/G is in general a category, not a poset.

After getting a hand on the formal setting in Section 5.2 we proceed in Section 5.3 with imposing different conditions on the group action. We give conditions for each of the following properties to be satisfied:

- (1) the morphisms of P/G are exactly the orbits of the morphisms of P, we call it *regularity*;
- (2) the quotient construction commutes with Quillen's nerve functor;
- (3) P/G is again a poset.

Furthermore, we study the class of categories which can be seen as the "quotient closure" of the set of all finite posets: loopfree categories.

#### 5.2 FORMALIZATION OF GROUP ACTIONS AND THE MAIN QUESTION

#### 5.2.1 Preliminaries

For a small category K denote the set of its objects by  $\mathcal{O}(K)$  and the set of its morphisms by  $\mathcal{M}(K)$ . For every  $a \in \mathcal{O}(K)$  there is exactly one identity morphism which we denote  $\mathrm{id}_a$ , this allows us to identify  $\mathcal{O}(K)$  with a subset of  $\mathcal{M}(K)$ . If m is a morphism of K from a to b, we write  $m \in \mathcal{M}_K(a,b)$ ,  $\partial^{\bullet} m = a$  and  $\partial_{\bullet} m = b$ . The morphism m has an inverse  $m^{-1} \in \mathcal{M}_K(b,a)$ , if  $m \circ m^{-1} = \mathrm{id}_a$  and  $m^{-1} \circ m = \mathrm{id}_b$ . If only the identity morphisms have inverses in K then K is said to be a category without inverses.

We denote the category of all small categories by  $\operatorname{Cat}$ . If  $K_1, K_2 \in \mathcal{O}(\operatorname{Cat})$  we denote by  $\mathcal{F}(K_1, K_2)$  the set of functors from  $K_1$  to  $K_2$ . We need three full subcategories of  $\operatorname{Cat}$ : P the category of posets, (which are categories with at most one morphism, denoted  $(x \to y)$ , between any two objects x, y), L the category of loopfree categories (see Definition 5.3.9), and  $\operatorname{Grp}$  the category of groups, (which are categories with a single element, morphisms given by the group elements and the law of composition given by group multiplication). Finally, 1 is the terminal object of  $\operatorname{Cat}$ , that is, the category with one element, and one (identity) morphism. The other two categories we use are  $\operatorname{Top}$ , the category of topological spaces, and  $\operatorname{SS}$ , the category of simplicial sets.

We are also interested in the functors  $\Delta: \mathbf{Cat} \to \mathbf{SS}$  and  $\mathcal{R}: \mathbf{SS} \to \mathbf{Top}$ . The composition is denoted  $\tilde{\Delta}: \mathbf{Cat} \to \mathbf{Top}$ . Here,  $\Delta$  is the nerve functor, see Appendix B, or [Qu73, Qu78, Se68]. In particular, the simplices of  $\Delta(K)$  are chains of morphisms in K, with degenerate simplices corresponding to chains that include identity morphisms, see [GeM96, We94].  $\mathcal{R}$  is the topological realization functor, see [Mil57].

We recall here the definition of a colimit (see [ML98, Mit65]).

**Definition 5.2.1** Let  $K_1$  and  $K_2$  be categories and  $X \in \mathcal{F}(K_1, K_2)$ . A **sink** of X is a pair consisting of  $L \in \mathcal{O}(K_2)$ , and a collection of morphisms  $\{\lambda_s \in \mathcal{M}_{K_2}(X(s), L)\}_{s \in \mathcal{O}(K_1)}$ , such that if  $\alpha \in \mathcal{M}_{K_1}(s_1, s_2)$  then  $\lambda_{s_2} \circ X(\alpha) = \lambda_{s_1}$ . (One way to think of this collection of morphisms is as a natural transformation between the functors X and  $X' = X_1 \circ X_2$ , where  $X_2$  is the terminal functor  $X_2 : K_1 \to \mathbf{1}$  and  $X_1 : \mathbf{1} \to K_2$  takes the object of  $\mathbf{1}$  to L). When  $(L, \{\lambda_s\})$  is universal with respect to this property we call it the **colimit** of X and write L = colim X.

#### 5.2.2 Definition of the quotient and formulation of the main problem

Our main object of study is described in the following definition.

**Definition 5.2.2** We say that a group G acts on a category K if there is a functor  $A_K : G \to \mathbf{Cat}$  which takes the unique object of G to K. The colimit of  $A_K$  is called the **quotient** of K by the action of G and is denoted by K/G.

To simplify notations, we identify  $\mathcal{A}_K g$  with g itself. Furthermore, in Definition 5.2.2 the category  $\mathbf{Cat}$  can be replaced with any category C, then  $K, K/G \in \mathcal{O}(C)$ . Important special case is  $C = \mathbf{SS}$ . It arises when  $K \in \mathcal{O}(\mathbf{Cat})$  and we consider  $\mathrm{colim}\,\Delta \circ \mathcal{A}_K = \Delta(K)/G$ .

**Main Problem.** Understand the relation between the topological and the categorical quotients, that is, between  $\Delta(K/G)$  and  $\Delta(K)/G$ .

To start with, by the universal property of colimits there exists a canonical surjection  $\lambda: \Delta(K)/G \to \Delta(K/G)$ . In the next section we give combinatorial conditions under which this map is an isomorphism.

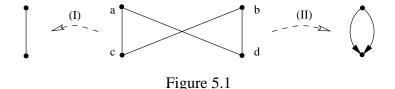
The general theory tells us that if G acts on the category K, then the colimit K/G exists, since **Cat** is cocomplete. We shall now give an explicit description.

# An explicit description of the category K/G.

When x is a morphism of K, denote by Gx the orbit of x under the action of G. We have  $\mathcal{O}(K/G) = \{Ga \mid a \in \mathcal{O}(K)\}$ . The situation with morphisms is more complicated. Define a relation  $\leftrightarrow$  on the set  $\mathcal{M}(K)$  by setting  $x \leftrightarrow y$ , iff there are decompositions  $x = x_1 \circ \ldots \circ x_t$  and  $y = y_1 \circ \ldots \circ y_t$  with  $Gy_i = Gx_i$  for all  $i \in [t]$ . The relation  $\leftrightarrow$  is reflexive and symmetric since G has identity and inverses, however it is not in general transitive. Let  $\sim$  be the transitive closure of  $\leftrightarrow$ , it is clearly an equivalence relation. Denote the  $\sim$  equivalence class of x by [x]. Note that  $\sim$  is the minimal equivalence relation on  $\mathcal{M}(K)$  closed under the G action and under composition; that is, with  $a \sim ga$  for any  $g \in G$ , and if  $x \sim x'$  and  $y \sim y'$  and  $x \circ x'$  and  $y \circ y'$  are defined then  $x \circ x' \sim y \circ y'$ . It is not difficult to check that the set  $\{[x] \mid x \in \mathcal{M}(K)\}$  with the relations  $\partial_{\bullet}[x] = [\partial_{\bullet}x]$ ,  $\partial_{\bullet}[x] = [\partial_{\bullet}x]$  and  $[x] \circ [y] = [x \circ y]$  (whenever the composition  $x \circ y$  is defined), are the morphisms of the category K/G.

Note that if P is a poset with a G action, the quotient taken in  $\mathbf{Cat}$  need not be a poset, and hence may differ from the poset quotient.

**Example 5.2.3** Let P be the center poset in the figure below. Let  $S_2$  act on P by simultaneously permuting a with b and c with d. (I) shows  $P/S_2$  in P and (II) shows  $P/S_2$  in P and the functor P (the canonical surjection P is an isomorphism), whereas the quotient in P does not.



#### 5.3 CONDITIONS ON GROUP ACTIONS

## 5.3.1 Outline of the results and surjectiveness of the canonical map

In this section we consider combinatorial conditions for a group G acting on a category K which ensure that the quotient by the group action commutes with

the nerve functor. If  $\mathcal{A}_K: G \to \mathbf{Cat}$  is a group action on a category K then  $\Delta \circ \mathcal{A}_K: G \to \mathbf{SS}$  is the associated group action on the nerve of K. It is clear that  $\Delta(K/G)$  is a sink for  $\Delta \circ \mathcal{A}_K$ , and hence, as previously mentioned, the universal property of colimits gives a canonical map  $\lambda: \Delta(K)/G \to \Delta(K/G)$ . We wish to find conditions under which  $\lambda$  is an isomorphism.

First we prove in Proposition 5.3.1 that  $\lambda$  is always surjective. Furthermore, Ga = [a] for  $a \in \mathcal{O}(K)$ , which means that, restricted to 0-skeleta,  $\lambda$  is an isomorphism. If the two simplicial spaces were simplicial complexes (only one face for any fixed vertex set), this would suffice to show isomorphism. Neither one is a simplicial complex in general, but while the quotient of a complex  $\Delta(K)/G$  can have simplices with fairly arbitrary face sets in common,  $\Delta(K/G)$  has only one face for any fixed edge set, since it is a nerve of a category. Thus for  $\lambda$  to be an isomorphism it is necessary and sufficient to find conditions under which

- 1)  $\lambda$  is an isomorphism restricted to 1-skeleta;
- 2)  $\Delta(K)/G$  has only one face with any given set of edges.

We will give conditions equivalent to  $\lambda$  being an isomorphism, and then give some stronger conditions that are often easier to check, the strongest of which is also inherited by the action of any subgroup H of G acting on K.

First note that a simplex of  $\Delta(K/G)$  is a sequence  $([m_1], \ldots, [m_t])$ ,  $m_i \in \mathcal{M}(K)$ , with  $\partial_{\bullet}[m_{i-1}] = \partial^{\bullet}[m_i]$ , which we will call a *chain*. On the other hand a simplex of  $\Delta(K)/G$  is an orbit of a sequence  $(n_1, \ldots, n_t)$ ,  $n_i \in \mathcal{M}(K)$ , with  $\partial_{\bullet}n_{i-1} = \partial^{\bullet}n_i$ , which we denote  $G(n_1, \ldots, n_t)$ . The canonical map  $\lambda$  is given by  $\lambda(G(n_1, \ldots, n_t)) = ([n_1], \ldots, [n_t])$ .

**Proposition 5.3.1** Let K be a category and G a group acting on K. The canonical map  $\lambda : \Delta(K)/G \to \Delta(K/G)$  is surjective.

**Proof.** By the above description of  $\lambda$  it suffices to fix a chain  $([m_1], \ldots, [m_t])$  and find a chain  $(n_1, \ldots, n_t)$  with  $[n_i] = [m_i]$ . The proof is by induction on t. The case t = 1 is obvious, just take  $n_1 = m_1$ .

Assume now that we have found  $n_1, \ldots, n_{t-1}$ , so that  $[n_i] = [m_i]$ , for  $i = 1, \ldots, t-1$ , and  $n_1, \ldots, n_{t-1}$  compose, i.e.,  $\partial {}^{\bullet} n_i = \partial {}_{\bullet} n_{i+1}$ , for  $i = 1, \ldots, t-2$ . Since  $[\partial {}_{\bullet} n_{t-1}] = [\partial {}_{\bullet} m_{t-1}] = [\partial {}^{\bullet} m_t]$ , we can find  $g \in G$ , such that  $g \partial {}^{\bullet} m_t = \partial {}_{\bullet} n_{t-1}$ . If we now take  $n_t = g m_t$ , we see that  $n_{t-1}$  and  $n_t$  compose, and  $[n_t] = [m_t]$ , which provides a proof for the induction step.  $\square$ 

97

# 5.3.2 Conditions for injectiveness of the canonical projection

**Definition 5.3.2** Let K be a category and G a group acting on K. We say that this action satisfies **Condition (R)** if the following is true: If  $x, y_a, y_b \in \mathcal{M}(K)$ ,  $\partial_{\bullet} x = \partial^{\bullet} y_a = \partial^{\bullet} y_b$  and  $Gy_a = Gy_b$ , then  $G(x \circ y_a) = G(x \circ y_b)$ .

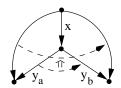


Figure 5.2

We say in such case that G acts regularly on K.

**Proposition 5.3.3** *Let* K *be a category and* G *a group acting on* K. *This action satisfies Condition* (R) *iff the canonical surjection*  $\lambda : \Delta(K)/G \to \Delta(K/G)$  *is injective on* 1-*skeleta.* 

**Proof.** The injectiveness of  $\lambda$  on 1-skeleta is equivalent to requiring that Gm = [m], for all  $m \in \mathcal{M}(K)$ , while Condition (R) is equivalent to requiring that  $G(m \circ Gn) = G(m \circ n)$ , for all  $m, n \in \mathcal{M}(K)$  with  $\partial_{\bullet} m = \partial^{\bullet} n$ ; here  $m \circ Gn$  means the set of all  $m \circ gn$  for which the composition is defined.

Assume that  $\lambda$  is injective on 1-skeleta. The we have the following computation:

$$G(m \circ Gn) = Gm \circ Gn = [m] \circ [n] = [m \circ n] = G(m \circ n),$$

hence the Condition (R) is satisfied.

Reversely, assume that the Condition (R) is satisfied, that is  $G(m \circ Gn) = G(m \circ n)$ . Since the equivalence class [m] is generated by G and composition, it suffices to show that orbits are preserved by composition, which is precisely  $G(m \circ Gn) = G(m \circ n)$ .  $\square$ 

The following theorem is the main result of this chapter. It provides us with combinatorial conditions which are equivalent to  $\lambda$  being an isomorphism.

**Theorem 5.3.4** Let K be a category and G a group acting on K. The following two assertions are equivalent for any  $t \geq 2$ :

- (1<sub>t</sub>) Condition (C<sub>t</sub>). If  $m_1, \ldots, m_{t-1}, m_a, m_b \in \mathcal{M}(K)$  with  $\partial^{\bullet} m_i = \partial_{\bullet} m_{i-1}$  for all  $2 \leq i \leq t-1$ ,  $\partial^{\bullet} m_a = \partial^{\bullet} m_b = \partial_{\bullet} m_{t-1}$ , and  $Gm_a = Gm_b$ , then there is some  $g \in G$  such that  $gm_a = m_b$  and  $gm_i = m_i$  for  $1 \leq i \leq t-1$ .
- (2<sub>t</sub>) The canonical surjection  $\lambda: \Delta(K)/G \to \Delta(K/G)$  is injective on t-skeleta.

In particular,  $\lambda$  is an isomorphism iff  $(C_t)$  is satisfied for all  $t \geq 2$ . If this is the case, we say that Condition (C) is satisfied

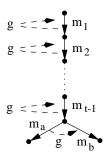


Figure 5.3

**Proof.**  $(1_t)$  is equivalent to  $G(m_1, \ldots, m_t) = G(m_1, \ldots, m_{t-1}, Gm_t)$ ; this notation is used, as before, for all sequences  $(m_1, \ldots, m_{t-1}, gm_t)$  which are chains, that is for which  $m_1 \circ \ldots \circ m_{t-1} \circ gm_t$  is defined.  $(2_t)$  implies Condition (R) above, and so can be restated as  $G(m_1, \ldots, m_t) = (Gm_1, \ldots, Gm_t)$ .

$$(2_{t}) \Rightarrow (1_{t}) : G(m_{1}, \dots, m_{t}) = (Gm_{1}, \dots, Gm_{t}) = G(Gm_{1}, \dots, Gm_{t})$$

$$\supseteq G(m_{1}, \dots, m_{t-1}, Gm_{t}) \supseteq G(m_{1}, \dots, m_{t}).$$

$$(1_{2}) \Rightarrow (2_{2}) : G(m_{1}, m_{2}) = G(m_{1}, Gm_{2})$$

$$= \{g_{1}(m_{1}, g_{2}m_{2}) \mid \partial_{\bullet}m_{1} = \partial^{\bullet}g_{2}m_{2}\}$$

$$= \{(g_{1}m_{1}, g_{2}m_{2}) \mid \partial_{\bullet}g_{1}m_{1} = \partial^{\bullet}g_{2}m_{2}\} = (Gm_{1}, Gm_{2}).$$

$$(1_{t}) \Rightarrow (2_{t}), t \ge 3 : \text{We use induction on } t.$$

$$(G(m_{1}, \dots, m_{t}) = G(m_{1}, \dots, m_{t-1}, Gm_{t})$$

$$= \{(gm_{1}, \dots, gm_{t-1}, \tilde{g}m_{t} \mid \partial_{\bullet}gm_{t-1} = \partial^{\bullet}\tilde{g}m_{t})\}$$

$$= \{(g_{1}m_{1}, \dots, g_{t}m_{t}) \mid \partial_{\bullet}g_{i}m_{i} = \partial^{\bullet}g_{i+1}m_{i+1}, i \in [t-1]\}$$

$$= (Gm_{1}, \dots, Gm_{t}). \square$$

**Example 5.3.5** A group action which satisfies Condition  $(C_t)$ , but does not satisfy Condition  $(C_{t+1})$ . Let  $P_{t+1}$  be the order sum of t+1 copies of the 2-element antichain. The automorphism group of  $P_{t+1}$  is the direct product of t+1 copies of  $\mathbb{Z}_2$ . Take G to be the index 2 subgroup consisting of elements with an even number of nonidentity terms in the product.

The following condition implies Condition (C), and is often easier to check.

**Condition** (S). There exists a set  $\{S_m\}_{m \in \mathcal{M}(K)}, S_m \subseteq \text{Stab}(m)$ , such that

- (1)  $S_m \subseteq S_{\partial \bullet_m} \subseteq S_{m'}$ , for any  $m' \in \mathcal{M}(K)$ , such that  $\partial \bullet m' = \partial \bullet m$ ;
- (2)  $S_{\partial \bullet_m}$  acts transitively on  $\{gm \mid g \in \operatorname{Stab}(\partial \bullet_m)\}$ , for any  $m \in \mathcal{M}(K)$ .

## **Proposition 5.3.6** *Condition (S) implies Condition (C).*

**Proof.** Let  $m_1, \ldots, m_{t-1}, m_a, m_b$  and g be as in Condition (C), then, since  $g \in \operatorname{Stab}(\partial^{\bullet} m_a)$ , there must exist  $\tilde{g} \in S_{\partial^{\bullet} m_a}$  such that  $\tilde{g}(m_a) = m_b$ . From (1) above one can conclude that  $\tilde{g}(m_i) = m_i$ , for  $i \in [t-1]$ .  $\square$ 

We say that the **strong** Condition (S) is satisfied if Condition (S) is satisfied with  $S_a = \text{Stab}(a)$ . Clearly, in such a case part (2) of the Condition (S) is obsolete.

**Example 5.3.7** A group action satisfying Condition (S), but not the strong Condition (S). Let  $K = \mathcal{B}_n$ , lattice of all subsets of [n] ordered by inclusion, and let  $G = \mathcal{S}_n$  act on  $\mathcal{B}_n$  by permuting the ground set [n]. Clearly, for  $A \subseteq [n]$ , we have  $Stab(A) = \mathcal{S}_A \times \mathcal{S}_{[n]\setminus A}$ , where, for  $X \subseteq [n]$ ,  $\mathcal{S}_X$  denotes the subgroup of  $\mathcal{S}_n$  which fixes elements of  $[n] \setminus X$  and acts as a permutation group on the set X. Since A > B means  $A \supset B$ , condition (1) of (S) is not satisfied for  $S_A = Stab(A)$ :  $\mathcal{S}_A \times \mathcal{S}_{[n]\setminus A} \not\supseteq \mathcal{S}_B \times \mathcal{S}_{[n]\setminus B}$ . However, we can set  $S_A = \mathcal{S}_A$ . It is easy to check that for this choice of  $\{S_A\}_{A \in \mathcal{B}_n}$  Condition (S) is satisfied.

We close the discussion of the conditions stated above by the following proposition.

## **Proposition 5.3.8**

- 1) The sets of group actions which satisfy Condition (C) or Condition (S) are closed under taking the restriction of the group action to a subcategory.
- 2) Assume a finite group G acts on a poset P, so that Condition (S) is satisfied. Let  $x \in P$  and  $S_x \subseteq H \subseteq Stab(x)$ , then Condition (S) is satisfied for the action of H on  $P_{\leq x}$ .
- 3) Assume a finite group G acts on a category K, so that Condition (S) is satisfied with  $S_a$  =Stab(a) (strong version), and H is a subgroup of G. Then the strong version of Condition (S) is again satisfied for the action of H on K.

**Proof.** 1) and 3) are obvious. To show 2) observe that for  $a \le x$  we have  $S_a \subseteq S_x \subseteq H$ , hence  $S_a \subseteq H \cap \operatorname{Stab}(a)$ . Thus condition (1) remains true. Condition (2) is true since  $\{g(b) \mid g \in \operatorname{Stab}(a)\} \supseteq \{g(b) \mid g \in \operatorname{Stab}(a) \cap H\}$ .  $\square$ 

## 5.3.3 Conditions for the categories to be closed under taking quotients

Next, we are concerned with finding out what categories one may get as a quotient of a poset by a group action. In particular, we ask: *in which cases is the quotient again a poset?* To answer that question, it is convenient to use the following class of categories.

**Definition 5.3.9** A category is called **loopfree** if it has no inverses and no non-identity automorphisms.

Intuitively, one may think of loopfree categories as those which can be drawn so that all nontrivial morphisms point down. To familiarize us with the notion of a loopfree category we make the following observations:

- K is loopfree iff for any  $x, y \in \mathcal{O}(K)$ ,  $x \neq y$ , only one of the sets  $\mathcal{M}_K(x, y)$  and  $\mathcal{M}_k(y, x)$  is non-empty and  $\mathcal{M}_K(x, x) = \{id_x\}$ ;
- a poset is a loopfree category;
- a barycentric subdivision of an arbitrary category is a loopfree category;
- a barycentric subdivision of a loopfree category is a poset;
- if K is a loopfree category, then there exists a partial order  $\geq$  on the set  $\mathcal{O}(K)$  such that  $\mathcal{M}_K(x,y) \neq \emptyset$  implies  $x \geq y$ .

**Definition 5.3.10** Suppose K is a small category, and  $T \in \mathcal{F}(K,K)$ . We say that T is **horizontal** if for any  $x \in \mathcal{O}(K)$ , if  $T(x) \neq x$ , then  $\mathcal{M}_K(x,T(x)) = \mathcal{M}_K(T(x),x) = \emptyset$ . When a group G acts on K, we say that the action is horizontal if each  $g \in G$  is a horizontal functor.

When K is a finite loopfree category, the action is always horizontal. Another example of horizontal actions is given by rank preserving action on a (not necessarily finite) poset. We have the following useful property:

**Proposition 5.3.11** Let P be a finite loopfree category and  $T \in \mathcal{F}(P, P)$  be a horizontal functor. Let  $\tilde{T} \in \mathcal{F}(\Delta(P), \Delta(P))$  be the induced functor, i.e.,  $\tilde{T} = \Delta(T)$ . Then  $\Delta(P_T) = \Delta(P)_{\tilde{T}}$ , where  $P_T$  denotes the subcategory of P fixed by T and  $\Delta(P)_{\tilde{T}}$  denotes the subcomplex of  $\Delta(P)$  fixed by  $\tilde{T}$ .

**Proof.** Obviously,  $\Delta(P_T) \subseteq \Delta(P)_{\tilde{T}}$ . On the other hand, if for some  $x \in \Delta(P)$  we have  $\tilde{T}(x) = x$ , then the minimal simplex  $\sigma$ , which contains x, is fixed as a set and, since the order of simplices is preserved by T,  $\sigma$  is fixed by T pointwise, thus  $x \in \Delta(P_T)$ .  $\square$ 

The class of loopfree categories can be seen as the closure of the class of posets under the operation of taking the quotient by a horizontal group action. More precisely, we have:

**Proposition 5.3.12** The quotient of a loopfree category by a horizontal action is again a loopfree category. In particular, the quotient of a poset by a horizontal action is a loopfree category.

101

**Proof.** Let K be a loopfree category and assume G acts on K horizontally. First observe that  $\mathcal{M}_{K/G}([x]) = \{\mathrm{id}_{[x]}\}$ . Because if  $m \in \mathcal{M}_{K/G}([x])$ , then there exist  $x_1, x_2 \in \mathcal{O}(K)$ ,  $\tilde{m} \in \mathcal{M}_K(x_1, x_2)$ , such that  $[x_1] = [x_2]$ ,  $[\tilde{m}] = m$ . Then  $gx_1 = x_2$  for some  $g \in G$ , hence, since g is a horizontal functor,  $x_1 = x_2$  and since K is loopfree we get  $\tilde{m} = \mathrm{id}_{x_1}$ .

Let us show that for  $[x] \neq [y]$  at most one of the sets  $M_{K/G}([x], [y])$  and  $M_{K/G}([y], [x])$  is nonempty. Assume the contrary and pick  $m_1 \in M_{K/G}([x], [y])$ ,  $m_2 \in M_{K/G}([y], [x])$ . Then there exist  $x_1, x_2, y_1, y_2 \in \mathcal{O}(K)$ ,  $\tilde{m}_1 \in \mathcal{M}_K(x_1, y_1)$ ,  $\tilde{m}_2 \in \mathcal{M}_K(y_2, x_2)$  such that  $[x_1] = [x_2] = [x]$ ,  $[y_1] = [y_2] = [y]$ ,  $[\tilde{m}_1] = [m_1]$ ,  $[\tilde{m}_2] = [m_2]$ . Choose  $g \in G$  such that  $gy_1 = y_2$ . Then  $[gx_1] = [x_2] = [x]$  and we have  $g\tilde{m}_1 \in \mathcal{M}_K(gx_1, y_2)$ , so  $\tilde{m}_2 \circ g\tilde{m}_1 \in \mathcal{M}_K(gx_1, x_2)$ . Since K is loopfree we conclude that  $gx_1 = x_2$ , but then both  $\mathcal{M}_K(x_2, y_2)$  and  $\mathcal{M}_K(y_2, x_2)$  are nonempty, which contradicts to the fact that K is loopfree.  $\square$ 

Next, we shall state a condition under which the quotient of a loopfree category is a poset.

**Proposition 5.3.13** Let K be a loopfree category and let G act on K. The following two assertions are equivalent:

- (1) Condition (SR). If  $x, y \in \mathcal{M}(K)$ ,  $\partial^{\bullet} x = \partial^{\bullet} y$  and  $G\partial_{\bullet} x = G\partial_{\bullet} y$ , then Gx = Gy.
- (2) G acts regularly on K and K/G is a poset.

**Proof.**  $(2) \Rightarrow (1)$ . Follows immediately from the regularity of the action of G and the fact that there must be only one morphism between  $[\partial \bullet x] (= [\partial \bullet y])$  and  $[\partial \bullet x] (= [\partial \bullet y])$ .



Figure 5.4

 $(1) \Rightarrow (2)$ . Obviously (SR)  $\Rightarrow$  (R), hence the action of G is regular. Furthermore, if  $x, y \in \mathcal{M}(K)$  and there exist  $g_1, g_2 \in G$  such that  $g_1 \partial^{\bullet} x = \partial^{\bullet} y$  and  $g_2 \partial_{\bullet} x = \partial_{\bullet} y$ , then we can replace x by  $g_1 x$  and reduce the situation to the one described in Condition (SR), namely that  $\partial^{\bullet} x = \partial^{\bullet} y$ . Applying Condition (SR) and acting with  $g_1^{-1}$  yields the result.  $\square$ 

When K is a poset, Condition (SR) can be stated in simpler terms.

**Condition (SRP).** If  $a, b, c \in K$ , such that  $a \ge b$ ,  $a \ge c$  and there exists  $g \in G$  such that g(b) = c, then there exists  $\tilde{g} \in G$  such that  $\tilde{g}(a) = a$  and  $\tilde{g}(b) = c$ .

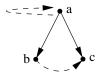


Figure 5.5

That is, for any  $a, b \in P$ , such that  $a \ge b$ , we require that the stabilizor of a acts transitively on Gb.

**Proposition 5.3.14** Let P be a poset and assume G acts on P. The action of G on P induces an action on the barycentric subdivision BdP (the poset of all chains of P ordered by inclusion). This action satisfies Condition (S), hence it is regular and  $\Delta(BdP)/G \cong \Delta((BdP)/G)$ . Moreover, if the action of G on P is horizontal, then (BdP)/G is a poset.

**Proof.** Let us choose chains b, c and  $a=(a_1>\cdots>a_t)$ , such that  $a\geq b$  and  $a\geq c$ . Then  $b=(a_{i_1}>\cdots>a_{i_l})$ ,  $c=(a_{j_1}>\cdots>a_{j_l})$ . Assume also that there exists  $g\in G$  such that  $g(a_{i_s})=a_{j_s}$  for  $s\in [l]$ . If g fixes a then it fixes every  $a_i$ ,  $i\in [t]$ , hence b=c and Condition (S) follows.

If, moreover, the action of G is horizontal, then again  $a_{is} = a_{js}$ , for  $s \in [l]$ , hence b = c and Condition (SRP) follows.  $\square$ 

#### APPENDIX A

# COMBINATORIAL TOOLS

#### A.1 NUMBER AND SET PARTITIONS

Let n be a natural number. We denote the set  $\{1, \ldots, n\}$  by [n].

**Definition A.1.1** A number partition of n is a set  $\{\lambda_1, \ldots, \lambda_t\}$  of natural numbers, such that  $\lambda_1 + \cdots + \lambda_t = n$ .

The usual convention is to write  $\lambda = (\lambda_1, \dots, \lambda_t)$ , where  $\lambda_1 \geq \dots \geq \lambda_t$ , and  $\lambda \vdash n$ . The *length* of  $\lambda$ , denoted  $l(\lambda)$ , is the number of components of  $\lambda$ , say, in the previous sentence  $l(\lambda) = t$ . We also use the power notation:  $(n^{\alpha_n}, \dots, 1^{\alpha_1}) = \underbrace{(n, \dots, n, \dots, \underbrace{1, \dots, 1}_{\alpha_1})}$ .

**Definition A.1.2** We say that  $\pi$  is an **ordered set partition** of [n] with m parts (sometimes called blocks) when  $\pi = (\pi_1, \ldots, \pi_m)$ ,  $\pi_i \neq \emptyset$ ,  $[n] = \bigcup_{i=1}^m \pi_i$ , and  $\pi_i \cap \pi_j = \emptyset$ , for  $i \neq j$ . If the order of the parts is not specified, then  $\pi$  is just called a **set partition**.

We denote the set of all set partitions, resp. ordered set partitions, of a set A by P(A), resp. OP(A). For P([n]), resp. OP([n]), we use the shorthand notations P(n), resp. OP(n). Furthermore, for every set A, we let un  $: OP(A) \to P(A)$  be the map which takes the ordered partition to the associated unordered partition.

Whenever we write  $\pi \vdash [n]$ , it implicitly implies that  $\pi$  is a set partition, as opposed to a number partition. A set partition  $\pi \vdash [n]$ ,  $\pi = (S_1, \ldots, S_t)$ , is said to have  $type \ \lambda$ , where  $\lambda \vdash n$  is the number partition  $\lambda = \{|S_1|, \ldots, |S_t|\}$ .

## **Definition A.1.3**

- (1) For two set partitions  $\pi, \tilde{\pi} \vdash S$ ,  $\pi = (S_1, \ldots, S_t)$ ,  $\tilde{\pi} = (\tilde{S}_1, \ldots, \tilde{S}_q)$  we write  $\pi \vdash \tilde{\pi}$ , and say that  $\pi$  **refines**  $\tilde{\pi}$ , if there exists  $\iota \vdash [t]$ ,  $\iota = \{I_1, \ldots, I_q\}$ , such that  $\tilde{S}_i = \bigcup_{j \in I_i} S_j$ , for  $i \in [q]$ .
- (2) Analogously, for two number partitions  $\lambda = (\lambda_1, \dots, \lambda_t)$ ,  $\mu = (\mu_1, \dots, \mu_q)$  we write  $\lambda \vdash \mu$ , and say that  $\lambda$  **refines**  $\mu$ , if there exists  $\iota \vdash [t]$ ,  $\iota = \{I_1, \dots, I_q\}$ , such that  $\mu_i = \sum_{j \in I_i} \lambda_j$ , for  $i \in [q]$ .

Clearly  $\pi \vdash [n]$  and  $\lambda \vdash n$  are special cases of these notations. Finally observe that if  $\pi, \tilde{\pi}$  are two set partitions, such that  $\pi \vdash \tilde{\pi}$ , then (type  $\tilde{\pi}$ )  $\vdash$  (type  $\tilde{\pi}$ ).

## A.2 GRAPHS

**Definition A.2.1** A directed graph G is a pair of sets (V(G), E(G)) such that  $E(G) \subseteq V(G) \times V(G) \setminus \{(x,x) \mid x \in V(G)\}$ . V(G) is called the set of vertices of G, E(G), the set of edges of G.

To support the intuition, we sometimes write  $(x \to y)$  instead of (x,y) and call this an *edge from* x *to* y. A directed graph H = (V(H), E(H)) is called a *subgraph* of G if  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ . Furthermore, H is called a *subgraph induced by* V(H) if  $E(H) = E(G) \cap (V(H) \times V(H))$ . For  $x, y \in V(G)$ , a *directed path* from x to y is an ordered tuple  $((x_1, x_2), (x_2, x_3), \dots, (x_{k-1}, x_k))$ , such that  $(x_i, x_{i+1}) \in E(G)$ , for  $i \in [k-1]$ , and  $x_1 = x, x_k = y$ .

#### **Definition A.2.2**

- a) A directed graph G is called a **directed tree** with root  $x \in V(G)$  if for every  $y \in V(G)$  there is a unique directed path from x to y.
- b) A directed graph G is called a **directed forest** if there exists a decomposition  $V(G) = \biguplus_{i \in I} A_i$ , (where  $\biguplus$  means disjoint union), such that each subgraph induced by  $A_i$ , for  $i \in I$ , is a directed tree, and there are no edges between  $A_i$  and  $A_j$  for  $i \neq j$ .

If G is a directed forest,  $x \in V(G)$ , and there are no edges  $(y \to x)$ , for  $y \in V(G)$ , we say that x is a *root*.

**Definition A.2.3** We say that  $x \in V(G)$  is a **complete source** of G if  $(x \to y) \in E(G)$  for all  $y \in V(G) \setminus \{x\}$ .

#### APPENDIX B

# POSETS AND RELATED TOPOLOGICAL CONSTRUCTIONS

#### B.1 BASIC NOTIONS

All posets discussed in this thesis are finite.

**Definition B.1.1** A poset  $\mathcal{L}$  is called a **meet-semilattice** if any two elements  $x, y \in \mathcal{L}$  have a greatest lower bound, i.e., the set  $\{z \in \mathcal{L} \mid z \leq x, z \leq y\}$  has a maximal element, called the meet,  $x \wedge y$ , of x and y.

For a subset  $A = \{a_1, \ldots, a_t\} \subseteq \mathcal{L}$  we let  $\bigwedge A = a_1 \wedge \ldots \wedge a_t$  denote the unique greatest lower bound of A, called the *meet*. In particular, meet-semilattices have a unique minimal element denoted  $\hat{0}$ . Minimal elements in  $\mathcal{L} \setminus \{\hat{0}\}$  are called the *atoms* in  $\mathcal{L}$ .

Symmetrically, meet-semilattices share the following property: for any subset  $A = \{a_1, \ldots, a_t\} \subseteq \mathcal{L}$  the set  $\{x \in \mathcal{L} \mid x \geq a \text{ for all } a \in A\}$  is either empty or it has a unique minimal element, denoted  $\bigvee A = a_1 \vee \ldots \vee a_t$ , called the *join* of A. If the meet-semilattice needs to be specified, we write  $(\bigvee A)_{\mathcal{L}} = (a_1 \vee \ldots \vee a_t)_{\mathcal{L}}$  for the join of A in  $\mathcal{L}$ . For brevity, we talk about semilattices throughout the second chapter, meaning meet-semilattices.

**Definition B.1.2** For arbitrary posets P and Q, the poset  $P \times Q$ , called the **direct product**, consists of all pairs (p,q),  $p \in P$ ,  $q \in Q$ , ordered by the rule:  $(p,q) \leq (\tilde{p},\tilde{q})$  iff  $(p \leq \tilde{p}$  and  $q \leq \tilde{q})$ .

Let P be an arbitrary poset. For  $x \in P$  set:  $P_{\leq x} = \{y \in P \mid y \leq x\}$ ;  $P_{< x}$ , and  $P_{\geq x}$ ,  $P_{> x}$  are defined analogously. For subsets  $\mathcal{G} \subseteq P$  with the induced order, and  $x \in P$ , we define  $\mathcal{G}_{\leq x} = \{y \in \mathcal{G} \mid y \leq x\}$ , and  $\mathcal{G}_{< x}$  again analogously. For intervals in P we use the following standard notations:  $[x,y] = \{z \in P \mid x \leq z \leq y\}$ ,  $[x,y) = \{z \in P \mid x \leq z < y\}$ , etc. We refer to [St86, Ch. 3] for further details.

#### B.2 Order complexes of posets

**Definition B.2.1** For a poset P, let  $\Delta(P)$  denote the **nerve** of P viewed as a category in the usual way: it is a simplicial complex with i-dimensional simplices

corresponding to chains of i+1 elements of P (chains are totally ordered sets of elements of P). In particular, vertices of  $\Delta(P)$  correspond to the elements of P. We call  $\Delta(P)$  the **order complex** of P.

The concept of the nerve of a category goes back at least to D. Quillen, [Qu73], and probably even further back to G. Segal, [Se68]. In its combinatorial guise of the order complex, it appears in the Goresky-MacPherson formula and serves as one of the main bridges between combinatorics and topology.

## **B.3** SHELLABILITY

**Definition B.3.1** A simplicial complex  $\Delta$  is called **shellable** if there exists an ordering  $F_1, \ldots, F_t$  of the maximal faces of  $\Delta$ , such that  $F_{i+1} \cap (\cup_{j=1}^i F_j)$  is a pure simplicial complex of dimension dim  $F_{i+1} - 1$  for all  $i \in [t-1]$ .

Such an ordering is said to satisfy Condition (S). Sometimes it is useful to replace Condition (S) with an equivalent Condition (S'): for  $1 \le i < k \le t$  there exist  $j \le k$  and  $x \in F_k$  such that  $F_i \cap F_k \subseteq F_j \cap F_k = F_k \setminus \{x\}$ .

If a simplicial complex is shellable, then  $\Delta$  is homotopy equivalent to a wedge of spheres, indexed by those simplices  $F_i$ , for which  $F_i \cap (\bigcup_{j=1}^{i-1} F_j)$  is equal to the full boundary of  $F_i$ , and each sphere has the dimension  $\dim F_i$ , for the corresponding i. In particular, the representatives of cohomology classes are given by the cochains dual to these simplices. See [Bj80, Bj95] for more information on shellability.

#### APPENDIX C

# SUBSPACE ARRANGEMENTS

#### C.1 DEFINITION AND RELATED CONSTRUCTIONS

**Definition C.1.1** A set  $\mathcal{A} = \{\mathcal{K}_1, \dots, \mathcal{K}_t\}$  of affine linear subspaces in a vector space  $\mathcal{V}$ , such that  $\mathcal{K}_i \not\subseteq \mathcal{K}_j$  for  $i \neq j$ , is called a **subspace arrangement** in  $\mathcal{V}$ . If all the subspaces  $\mathcal{K}_1, \dots, \mathcal{K}_t$  are also required to contain the origin, then the subspace arrangement is called **central**.

The topological spaces which one customarily associates to a subspace arrangement are:

- $V_A = \bigcup_{i=1}^t \mathcal{K}_i$ , the union of subspaces;
- $\mathcal{M}_{\mathcal{A}} = \mathcal{V} \setminus V_{\mathcal{A}}$ , the *complement* of the arrangement.

**Definition C.1.2** Let A be a subspace arrangement in  $\mathbb{C}^n$ , and let G be a subgroup of  $\mathbf{GL}_n(\mathbb{C})$ , such that  $V_A$  is invariant under the action of G. We say that G acts on A. In that case,  $\Gamma_A^G$  denotes the one-point compactification of  $V_A/G$ .

#### C.2 GORESKY-MACPHERSON THEOREM

The intersection data of a subspace arrangement may be represented by a poset.

**Definition C.2.1** To a subspace arrangement A in V one can associate a partially ordered set  $\mathcal{L}_A$ , called the **intersection semilattice** of A. The set if elements of  $\mathcal{L}_A$  is  $\{K \subseteq \mathbb{C}^n \mid \exists I \subseteq [t], \text{ such that } \bigcap_{i \in I} \mathcal{K}_i = K\} \cup \{V\} \text{ with the order given by reversing inclusions: } x \leq_{\mathcal{L}_A} y \text{ iff } x \supseteq y. \text{ That is, the minimal element of } \mathcal{L}_A \text{ is } V$ , also customarily denoted  $\hat{0}$ , and the maximal element is  $\bigcap_{K \in \mathcal{A}} K$ .

The following theorem describes the cohomology groups of the complement of a subspace arrangement in terms of the homology groups of the order complexes of the intervals in the corresponding intersection lattices. **Theorem C.2.2** (Goresky & MacPherson, [GoM88]). Let A be a central subspace arrangement in  $\mathbb{C}^n$ , or in  $\mathbb{R}^n$ , and let  $\mathcal{L}_A$  denote its intersection lattice, then

$$\widetilde{H}^{i}(\mathcal{M}_{\mathcal{A}}) \simeq \bigoplus_{x \in \mathcal{L}_{\mathcal{A}}^{\geq \hat{0}}} \widetilde{H}_{codim_{\mathbb{R}}(x) - i - 2}(\Delta(\hat{0}, x)).$$

This provided another strong motivation to study the order complex construction and to develop technical tools such as lexicographic shellability, see e.g., [Bj94, Ko97].

#### APPENDIX D

# TOPOLOGICAL TOOLS

## D.1 OPERATIONS ON TOPOLOGICAL SPACES

For a topological space X,  $\widetilde{H}_i(X)$ , resp.  $\widetilde{H}^i(X)$ , denotes the ith reduced homology, resp. cohomology, group of X; while  $\widetilde{\beta}_i(X)$  denotes the ith reduced Betti number of X.

Throughout this thesis we use the operations on topological spaces described in this subsection.

**Definition D.1.1** Let (X, x) and (Y, y) be two pointed topological spaces. (1) The **wedge** of X and Y, denoted  $X \vee Y$ , is the pointed topological space

$$((X \cup Y)/(x \sim y), z),$$

where the base point z is given by the equivalence class of x (and hence of y). (2) The **smash product** of X and Y, denoted  $X \wedge Y$ , is the pointed topological space obtained as the quotient space  $X \times Y/\sim$ , where the equivalence relation is given by:  $(\tilde{x}, y) \sim (x, \tilde{y})$ , for any  $\tilde{x} \in X$ ,  $\tilde{y} \in Y$ ; with the base point being the equivalence class of (x, y).

The wedge and the smash products enjoy a variety of properties:

- they are commutative and associative;
- $X \vee \mathsf{pt} \cong \mathsf{pt} \vee X \cong X$ ;
- $X \wedge S^0 \cong S^0 \wedge X \cong X$ :
- $S^n \wedge S^m \cong S^{n+m}$ :
- $X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z)$ .

Furthermore, there are important special cases.

**Definition D.1.2** Let X be a topological space, the **suspension** of X, denoted susp X, is the quotient topological space  $X \times [0,1]/\sim$ , where the equivalence relation is given by  $(x_1,0) \sim (x_2,0)$ , and  $(x_1,1) \sim (x_2,1)$ , for any  $x_1,x_2 \in X$ .

Clearly, susp  $X \cong X \wedge S^1$ .

**Definition D.1.3** Let X and Y be two topological spaces. The **join** of X and Y, denoted X \* Y, is the quotient topological space  $X \times [0,1] \times Y / \sim$ , where the equivalence relation  $\sim$  is given by  $(x,0,y_1) \sim (x,0,y_2)$ , and  $(x_1,1,y) \sim (x_2,1,y)$ , for any  $x,x_1,x_2 \in X$ , and  $y,y_1,y_2 \in Y$ .

For simplicial complexes one can use the alternative, more explicit definition of the join.

**Definition D.1.4** Let X and Y be two simplicial complexes, then Z = X \* Y is the simplicial complex defined by:

- the set of vertices of Z is equal to the disjoint union of the sets of vertices of X and Y;
- the subset  $\Sigma$  of the set of vertices of Z is a simplex iff  $\Sigma = A \cup B$ , where A is a simplex in X and B is a simplex in Y.

# D.2 SPECTRAL SEQUENCES

A spectral sequence associated with a chain complex C and a filtration F on C is a sequence of 2-dimensional tableaux  $(E^r_{*,*})_{r=0}^{\infty}$ , where every component  $E^r_{k,i}$  is a vector space (for simplicity we first consider only field coefficients),  $E^r_{k,i}=0$  unless  $k\geq -1$  and  $i\geq 0$ , and a sequence of differential maps  $(d^r)_{r=0}^{\infty}$  such that

- (0)  $E_{k,i}^0 = F_i C_k / F_{i-1} C_k$ ;
- (1)  $d^r: E^r_{k,i} \longrightarrow E^r_{k-1,i-r}, \forall k, i \in \mathbb{Z};$
- (2)  $E_{*,*}^{r+1} = H_*(E_{*,*}^r, d^r)$ , in other words

$$E_{k,i}^{r+1} = \operatorname{Ker}\left(E_{k,i}^r \xrightarrow{d^r} E_{k-1,i-r}^r\right) / \operatorname{Im}\left(E_{k+1,i+r}^r \xrightarrow{d^r} E_{k,i}^r\right); \quad (D.2.1)$$

(3) for all  $k \in \mathbb{Z}$ ,

$$H_k(C) = \bigoplus_{i \in \mathbb{Z}} E_{k,i}^{\infty}.$$
 (D.2.2)

## Comments.

- 0. It follows from (0) and (2) that  $E_{k,i}^{1} = H_{k}(F_{i}, F_{i-1})$ .
- 1. In the general case  $E_{k,i}^{\infty}$  is defined using the notion of convergence of the spectral sequence. We will not explain this notion in general, since for the spectral

sequence that we consider only a finite number of components in every tableau  $E^r_{*,*}$  are different from zero, so there exists  $N \in \mathbb{N}$ , such that  $d^r = 0$  for  $r \geq N$ . Then, one sets  $E^\infty_{*,*} = E^N_{*,*}$ , and so  $H_k(C) = \bigoplus_{i \in \mathbb{Z}} E^N_{k,i}$ .

- 2. For the case of integer coefficients, (D.2.2) becomes more involved: rather than just summing the entries of  $E^{\infty}_{*,*}$  one needs to solve extension problems to get  $H_*(C)$ . This difficulty will not arise in our applications, so we refer the interested reader to [McC85] for the detailed explanation of this phenomena. When considering integer coefficients,  $E^r_{*,*}$  are not vector spaces, but just abelian groups.
- 3. We would like to warn the reader that our indexing is different from the standard (but more convenient for our purposes). The standard indexing is more convenient for the spectral sequences associated to fibrations, an instance we do not discuss in this thesis.

Spectral sequences constitute a convenient tool for computing the homology groups of a simplicial complex. A few good sources for a more comprehensive further reading are [McC85, Sp66, Mas52].

# **BIBLIOGRAPHY**

- [An99] L. Anderson, *Matroid bundles*, New perspectives in algebraic combinatorics, (Berkeley, CA, 1996–97), pp. 1–21, Math. Sci. Res. Inst. Publ. **38**, Cambridge University Press, Cambridge, 1999.
- [Ar69] V.I. Arnol'd, *The cohomology ring of the group of dyed braids*, Mat. Zametki **5** (1969), 227–231, (Russian).
- [Ar70a] V.I. Arnol'd, *Topological invariants of algebraic functions*, Trans. Moscow Math. Soc. **21**, (1970), 30–52.
- [Ar70b] V.I. Arnol'd, On some topological invariants of algebraic functions, II, Func. Anal. Pril. **4** (1970), no. 2, 1–9, (Russian).
- [AGV85] V.I. Arnol'd, S.M. Gusein-Zade, A.N. Varchenko, *Singularities of Differentiable Maps, Vol. I*, Translated from the Russian by Ian Porteous and Mark Reynolds, Monographs in Math. Vol. **82**, Birkhäuser Boston, Boston, MA, 1985.
- [BBLSW] E. Babson, A. Björner, S. Linusson, J. Shareshian, V. Welker, *Complexes of not i-connected graphs*, Topology **38** (1999), no. 2, 271–299.
- [Bj80] A. Björner, *Shellable and Cohen-Macaulay partially ordered sets*, Trans. Amer. Math. Soc. **260**, (1980), 159–183.
- [Bj94] A. Björner, *Subspace arrangements*, in "First European Congress of Mathematics, (Paris 1992)" (eds. A. Joseph et al), Progr. Math. **119**, Birkhäuser Basel, 1994, pp. 321–370.
- [Bj95] A. Björner, *Topological Methods*, in "Handbook of Combinatorics" (eds. R. Graham, M. Grötschel and L. Lovász), Elsevier, Amsterdam, 1995, pp. 1819–1872.
- [BLSWZ] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, G.M. Ziegler, *Oriented Matroids*, Encyclopedia of Mathematics, Cambridge University Press, Cambridge, 1992.
- [BWa83] A. Björner, M. Wachs, *On lexicographically shellable posets*, Trans. Amer. Math. Soc. **277** (1983), 323–341.

[BWa96] A. Björner, M. Wachs, *Shellable non-pure complexes and posets I*, Trans. Amer. Math. Soc. **348** (1996), 1299–1327.

- [BWa97] A. Björner, M. Wachs, *Nonpure shellable complexes and posets II*, Trans. Amer. Math. Soc. **349** (1997), 3945–3975.
- [BWe98] A. Björner, V. Welker, *Complexes of directed graphs*, SIAM J. Discrete Math. **12** (1999), no. 4, 413–424.
- [Br72] G.E. Bredon, *Introduction to compact transformation groups*, Pure and Applied Mathematics, Vol. **46**, Academic Press, New York-London, 1972.
- [BH93] W. Bruns, J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, 1993.
- [Co56] P.E. Conner, *Concerning the action of a finite group*, Proc. Nat. Acad. Sci. U.S.A. **142**, (1956), 349–351.
- [Da78] V.I. Danilov, *The geometry of toric varieties*, Russ. Math. Surv. **33** (1978), pp. 97–154.
- [DP83] C. De Concini, C. Procesi, *Complete symmetric varieties*, Invariant theory (Montecatini, 1982), pp. 1–44, Lecture Notes in Math. **996**, Springer, Berlin-New York, 1983.
- [DP85] C. De Concini, C. Procesi, *Complete symmetric varieties, II*, Intersection theory. Algebraic groups and related topics (Kyoto/Nagoya, 1983), pp. 481–513, Adv. Stud. Pure Math. **6**, North-Holland, Amsterdam-New York, 1985.
- [DP95] C. De Concini, C. Procesi, Wonderful models of subspace arrangements, Selecta Math. (N.S.) 1 (1995), no. 3, 459–494.
- [DGM00] P. Deligne, M. Goresky, R. MacPherson, L'algèbre de cohomologie du complément, dans un espace affine, d'une famille finie de sous-espaces affines, (French) [Cohomology algebra of the complement, in an affine space, of a finite family of affine subspaces], Michigan Math. J. 48 (2000), 121–136.
- [dLS01] M. de Longueville, C. Schultz, *The cohomology rings of complements of subspace arrangements*, Math. Ann. **319** (2001), no. 4, 625–646.

[Ew96] G. Ewald, *Combinatorial Convexity and Algebraic Geometry*, Graduate Texts in Mathematics **168**, Springer-Verlag, New York, 1996.

- [FH01] E.R. Fadell, S.Y. Husseini, *Geometry and topology of configura*tion spaces, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2001.
- [FZ00] E.-M. Feichtner, G.M. Ziegler, *The integral cohomology algebras of ordered configuration spaces of spheres*, Doc. Math. **5** (2000), 115–139 (electronic).
- [Fo98] R. Forman, *Morse theory for cell complexes*, Adv. Math. **134** (1998), no. 1, 90–145.
- [Fuc70] D.B. Fuchs, *Cohomology of braid group* mod 2, Func. Anal. Pril. **4** (1970), no. 2, 62–75, (Russian).
- [Ful93] W. Fulton, *Introduction to Toric Varieties*, Annals of Mathematics Studies 131, The William H. Roever Lectures in Geometry, Princeton University Press, Princeton, 1993.
- [FM94] W. Fulton, R. MacPherson, A compactification of configuration spaces, Ann. of Math. **139** (1994), 183–225.
- [GeM92] I.M. Gelfand, R. MacPherson, *A combinatorial formula for the Pontrjagin classes*, Bull. Amer. Math. Soc. (N.S.) **26** (1992), no. 2, 304–309.
- [GeM96] S. Gelfand, Y. Manin, *Methods of homological algebra*, Translated from the 1988 Russian original, Springer-Verlag, Berlin, 1996.
- [GoM88] M. Goresky, R. MacPherson, *Stratified Morse Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. **14**, Springer-Verlag, Berlin/Heidelberg/New York, 1988.
- [Ha83] P. Hanlon, A proof of a conjecture of Stanley concerning partitions of a set, European J. Combin. **4** (1983), no. 2, 137–141.
- [Kn83] F. Knudsen, Projectivity of the moduli space of stable curves, II: the stacks  $M_{q,n}$ , Math. Scand. **52** (1983), 1225–1265.
- [Ko97] D.N. Kozlov, General lexicographic shellability and orbit arrangements, Ann. Comb. 1 (1997), no. 1, 67–90.

[Ko99a] D.N. Kozlov, *Rational homology of spaces of complex monic polynomials with multiple roots*, preprint, Institute for Advanced Study 1999, to appear in Mathematika.

- [Ko99b] D.N. Kozlov, *Complexes of directed trees*, J. Combin. Theory Ser. A **88** (1999), no. 1, 112–122.
- [Ko00a] D.N. Kozlov, Collapsibility of  $\Delta(\Pi_n)/S_n$  and some related CW complexes, Proc. Amer. Math. Soc. 128 (2000), no. 8, 2253–2259.
- [Ko00b] D.N. Kozlov, *Topology of spaces of hyperbolic polynomials and combinatorics of resonances*, preprint 2000, submitted.
- [Ko01a] D.N. Kozlov, Spectral sequences on combinatorial simplicial complexes, J. Algebraic Combin. **14** (2001), no. 1, 27–48.
- [Ko01b] D.N. Kozlov, Resonance category, preprint 2001, submitted.
- [Ko01c] D.N. Kozlov, *Relative Resonance Category*, in preparation.
- [Lo78] L. Lovász, *Kneser's conjecture, chromatic number, and homotopy*, J. Combin. Theory Ser. A **25** (1978), no. 3, 319–324.
- [ML98] S. Mac Lane, Categories for the working mathematician, Second edition, Graduate Texts in Mathematics 5, Springer-Verlag, New York, 1998.
- [Mac93] R. MacPherson, Combinatorial differential manifolds, in Topological methods in modern mathematics: A symposium in honor of John Milnor's sixtieth birthday (Stony Brook, NY, 1991), Publish or Perish, Houston, TX, 1993.
- [MP98] R. MacPherson, C. Procesi, *Making conical compactifications won-derful*, Selecta Math. (N.S.) **4** (1998), no. 1, 125–139.
- [Mas52] W.S. Massey, *Exact couples in algebraic topology I, II*, Ann. of Math. **56** (1952), 363–396.
- [McC85] J. McCleary, *User's Guide To Spectral Sequences*, Mathematics Lecture Series **12**, Publish or Perish, Inc., Wilmington, DE, 1985.
- [Mil57] J. Milnor, *The geometric realization of semi-simplicial complex*, Ann. of Math. **65** (1957), 357–362.

[Mit65] B. Mitchell, *Theory of categories*, Pure and Applied Mathematics, Vol. XVII, Academic Press, New York-London, 1965.

- [MO73] K. Miyake, T. Oda, Almost homogeneous algebraic varieties under algebraic torus action; in: Manifolds, Tokyo 1973 (A. Hattori, ed.), University of Tokyo Press, 1975, pp. 373–381.
- [Mo78] J.W. Morgan, *The algebraic topology of smooth algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. No. 48, (1978), 137–204.
- [Mu84] J.R. Munkres, *Elements of Algebraic Topology*, Addison-Wesley Publishing Company, Menlo Park, CA, 1984.
- [Od88] T. Oda, *Convex Bodies and Algebraic Geometry*, (translated from Japanese), Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Bd. 15, Springer-Verlag, Berlin, 1988.
- [OS80] P. Orlik, L. Solomon, *Combinatorics and topology of complements of hyperplanes*, Inventiones Math. **56** (1980), 167–189.
- [Pi96] J. Pitman, *Coalescent random trees*, Tech. Rep. No. 457, Dept. of Statistics, University of California at Berkeley, 1996.
- [Qu73] D. Quillen, *Higher algebraic K-theory I*, Lecture Notes in Mathematics **341**, Springer-Verlag, Berlin, 1973, pp. 85–148.
- [Qu78] D. Quillen, Homotopy properties of the poset of nontrivial p-sub-groups of a group, Adv. Math. **28** (1978), no. 2, 101–128.
- [Re76] G.A. Reisner, *Cohen-Macaulay quotients of polynomial rings*, Adv. Math. **21** (1976), no. 1, 30–49.
- [Se68] G. Segal, *Classifying spaces and spectral sequences*, Inst. Hautes Études Sci. Publ. Math. No. **34** (1968), 105–112.
- [Sh00] B. Shapiro, Private communication, 2000.
- [ShW98] B. Shapiro, V. Welker, Combinatorics and topology of stratifications of the space of monic polynomials with real coefficients, Results Math. 33 (1998), no. 3–4, 338–355.
- [Sp66] E. Spanier, *Algebraic Topology*, McGraw Hill Book Co., New York-Toronto, Ont.-London, 1966.

[St82] R.P. Stanley, *Some aspects of groups acting on finite posets*, J. Combin. Theory Ser. A **32** (1982), no. 2, 132–161.

- [St86] R.P. Stanley, *Enumerative combinatorics, Vol. I*, The Wadsworth & Brooks/Cole Mathematics Series, Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1986.
- [St96] R.P. Stanley, *Combinatorics and commutative algebra*, Progress in Mathematics **41**, Birkhäuser Boston, Inc., Boston, MA, 2nd Edition, 1996.
- [St97] R.P. Stanley, Private communication, 1997.
- [St99] R.P. Stanley, *Enumerative Combinatorics*, vol. II, Cambridge Studies in Advanced Mathematics **62**, Cambridge University Press, Cambridge, 1999.
- [Su94] S. Sundaram, The homology representations of the symmetric group on Cohen-Macaulay subposets of the partition lattice, Adv. Math. **104** (1994), no. 2, 225-296.
- [SuW97] S. Sundaram, V. Welker, Group actions on arrangements of linear subspaces and applications to configuration spaces, Trans. Amer. Math. Soc. **349** (1997), no. 4, 1389–1420.
- [Vai78] F.V. Vainshtain, *Cohomology of the braid groups*, Func. Anal. Appl. **12** (1978), 135–137.
- [Vas94] V.A. Vassiliev, Complements of Discriminants of Smooth Maps: Topology and Applications, Transl. Math. Monographs, vol. 98, Amer. Math. Soc., Providence, RI, 1994. Revised Edition.
- [Vas98] V.A. Vassiliev, Homology of spaces of homogeneous polynomials in  $\mathbb{R}^2$  without multiple zeros, in: Local and global problems of singularity theory (Russian), Tr. Mat. Inst. Steklova **221** (1998), 143–148; translation in Proc. Steklov Inst. Math. **1998**, no. 2 (221), 133–138.
- [We94] C. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics **38**, Cambridge University Press, Cambridge, 1994.
- [Yu97] S. Yuzvinsky, Cohomology bases for the De Concini-Procesi models of hyperplane arrangements and sums over trees, Invent. Math. 127 (1997), no. 2, 319–335.

[Zi92] G.M. Ziegler, *Combinatorial Models of Subspace Arrangements*, Habilitations-Schrift, TU Berlin, April 1992.

- [Zi93] G.M. Ziegler, *What is a complex matroid?*, Discrete Comput. Geom. **10** (1993), no. 3, 313–348.
- [ZZ93] G.M. Ziegler, R.T. Živaljević, *Homotopy types of subspace arrangements via diagrams of spaces*, Math. Ann. **295** (1993), 527–548.