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BESSEL FUNCTIONS OF PURELY IMAGINARY ORDER, WITH AN APPLICATION TO SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS HAVING A LARGE PARAMETER*

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Abstract. Bessel functions of purely imaginary order are examined. Solutions of both the modified and unmodified Bessel equations are defined which, when their order is purely imaginary and their argument is real and positive, are pairs of real numerically satisfactory functions. Recurrence relations, analytic continuation formulas, power series representations, Wronskian relations, integral representations, behavior at singularities, and asymptotic forms of the zeros are derived for these numerically satisfactory functions. Also, asymptotic expansions in terms of elementary and Airy functions are derived for the Bessel functions when their order is purely imaginary and of large absolute value.

Second-order linear ordinary differential equations having a large parameter and a simple pole are then examined, for the case where the exponent of the pole is complex. Asymptotic expansions are derived for the solutions in terms of the numerically satisfactory Bessel functions of purely imaginary order.

Key words. asymptotic analysis, Bessel functions, ordinary differential equations, zeros

AMS(MOS) subject classifications. primary 33A40; secondary 34E20

1. Introduction and summary. The purpose of this paper is to investigate solutions of both the unmodified and the modified Bessel equations (see (2.1) and (3.1)). We consider the case where the parameter μ in the equations is purely imaginary, so that the solutions are of purely imaginary order.

Consider first the asymptotic behavior of Bessel functions. This is an area that has been extensively studied, reflecting the importance of Bessel functions in many areas of mathematics and physics. Uniform asymptotic expansions of modified and unmodified Bessel functions of complex argument and large positive order are available in terms of both elementary and Airy functions, see § 7 of Chap. 10 and § 10 of Chap. 11 in Olver (1974). (We will refer to Olver's book frequently, and therefore here and throughout we use the abbreviation "Chap." to refer to a chapter of that text.) Expansions for complex orders with positive (nonzero) real part are also available: see § 8 of Chap. 10.

When the parameter $\mu \equiv i\nu$ is purely imaginary, however, the picture is less complete; only the modified Bessel function of the third kind $K_{i\nu}(z)$ seems to have been extensively studied. Uniform asymptotic expansions in terms of Airy functions have been derived for $K_{i\nu}(\nu z)$, ν real and large, by Balogh (1967). The positive zeros of $K_{i\nu}(z)$ have also been investigated by a number of authors (see, e.g., Ferreira and Sesma (1970), Laforgia (1986)). For other asymptotic results concerning Bessel functions of purely imaginary order, see Jeffreys (1962, pp. 90–91) and Falcão (1973).

Regarded as a Bessel function of purely imaginary order, the function $K_{i\nu}(x)$ is unique in two respects. First, it alone of the standard Bessel functions is real when the argument x is positive: the Bessel functions $J_{i\nu}(x)$, $Y_{i\nu}(x)$, $H_{i\nu}^{(1)}(x)$, $H_{i\nu}^{(2)}(x)$, and the modified Bessel function $I_{i\nu}(x)$ are all complex when ν and x are real and nonzero.

Second, $K_{i\nu}(x)$ is recessive as $x \to +\infty$, a property that makes the function useful in certain physical problems, such as hydrodynamical, quantum mechanical, and

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diffraction theories. Also, this property allows us to readily identify the function with asymptotic solutions of (2.1) that it satisfies.

An example in which Bessel functions of purely imaginary order play an important role in quantum mechanics is the problem of s-wave scattering by a reduced exponential potential; see Kogan and Galitsky (1963, pp. 334–341) and Joachain (1975, p. 317). For other applications see Hemker (1974).

On account of the two above-mentioned properties, $K_{i\nu}(x)$ remains one of a pair of solutions of (3.1) on the x interval $(0, \infty)$ that are numerically satisfactory; see Miller (1950). $K_{i\nu}(x)$ is oscillatory in a neighborhood of the $x=0$, and is exponential in a neighborhood of $x=\infty$. An appropriate numerically satisfactory companion would be a real solution which is of equal amplitude and $\pi/2$ out of phase in the oscillatory region. We introduce a function, denoted by $L_{i\nu}(x)$, which fulfills these criteria (see (2.2)).

Solutions of the unmodified Bessel equation (3.1) that are real when μ is purely imaginary, and $z \equiv x$ is positive, are oscillatory throughout $0<x<\infty$. We introduce two real solutions, denoted by $F_{i\nu}(x)$ and $G_{i\nu}(x)$, that are $\pi/2$ out of phase on $0<x<\infty$, have equal amplitudes of oscillation at $x=\infty$ for all $\nu>0$, and have asymptotically equal amplitudes of oscillation throughout $0<x<\infty$ as $\nu\to\infty$ (see (3.2) and (3.3)).

The plan of this paper is as follows. In §§ 2 and 3 we derive a number of results concerning $K_\mu(z)$, $L_\mu(z)$, $F_\mu(z)$, and $G_\mu(z)$, most of which pertain to μ being purely imaginary. We record recurrence relations, analytic continuation formulas, power series representations, Wronskian relations, connection formulas, integral representations, and asymptotic behavior at $z=0$ and ∞. These results follow from standard results concerning Bessel functions, the latter being found, for example, by perusing Olver (1974), and in most instances details of their derivations have been omitted.

In §§ 2 and 3 the zeros of the four functions are also examined; asymptotic formulas are derived for the zeros including those of $K_{i\nu}(x)$, which are of importance in certain physical problems, such as in quantum mechanics (see Gray, Mathews, and MacRobert (1952)).

In §§ 4 and 5 we examine the asymptotic behavior of the four functions as $\nu\to\infty$. As has already been noted, the modified Bessel function $K_{i\nu}(\nu z)$ has been studied by Balogh (1967). The corresponding asymptotic expansion for $L_{i\nu}(\nu z)$ (as well as that for $I_{i\nu}(\nu z)$) is derived, using the theory of Chap. 11. The theory of Chap. 10 is applied to deriving asymptotic expansions, involving elementary functions, for $F_{i\nu}(\nu z)$ and $G_{i\nu}(\nu z)$ (as well as for the Hankel functions $H^{(1)}_{i\nu}(\nu z)$ and $H^{(2)}_{i\nu}(\nu z)$).

One example of a useful application for these asymptotic results is to problems of high-frequency wave propagation in inhomogeneous media with linear velocity profiles (see, for example, Gupta (1965)). For detailed discussions of this class of problem see Brekhovskikh (1960). In his paper Gupta uses expansions (29) for $\arg \nu = \pi/2$, and although this is not justified, we shall see that the first of these represents the (dominant) real part when $y=-i\sigma$, $1<\sigma<\infty$.

In § 7 asymptotic solutions are constructed for equations of the form

$$\frac{d^2w}{dx^2}=\{u^2f(x)+g(x)\}w, \tag{1.1}$$

in which u is a large parameter, the independent variable x lies in an open (finite or infinite) interval, and at some point $x=x_0$, $f(x)$ has a simple pole and $(x-x_0)^2g(x)$ is analytic. It is supposed that there are no other transition points (zeros of $f(x)$ or singularities) in the x interval under consideration.

We consider the case where

$$\lim_{x\to x_0}(x-x_0)^2 g(x) \equiv -\frac{\nu^2+1}{4} < -\frac{1}{4} \qquad (\nu>0), \tag{1.2}$$

which corresponds to the exponents of the pole $x=x_0$ being complex (see, e.g., § 4 of Chap. 5).

The complementary problem, where the exponents are real, has been tackled by Olver (see §§ 1-4 of Chap. 12). We proceed in a manner similar to Olver's. By means of the Liouville transformation, formulas (2.02) and (2.03) of Chap. 12, our equation (1.1) is transformed to the form (7.1), where ν is positive (compare (2.05) of Chap. 12). Equation (7.1) is the focus of our attention; asymptotic solutions are constructed in terms of the Bessel functions of purely imaginary order $K_{i\nu}(x)$, $L_{i\nu}(x)$, $F_{i\nu}(x)$, and $G_{i\nu}(x)$. Using auxiliary functions for these four functions (given in § 6), we derive error bounds for the asymptotic expansions.

It will be assumed that the reader is familiar with the results in Chaps. 10 and 11 and §§ 1-5 of Chap. 12.

2. Modified Bessel functions of purely imaginary order. The modified Bessel functions $I_\mu(z)$ and $K_\mu(z)$ compose a numerically satisfactory pair of solutions of the modified Bessel equation

$$\frac{d^2w}{dz^2}+\frac{1}{z}\frac{dw}{dz}-\left(1+\frac{\mu^2}{z^2}\right)w=0 \tag{2.1}$$

in the half-plane $|\arg z| \leqq \pi/2$, for all complex values of μ such that $\operatorname{Re}\mu \geqq 0$. By "numerically satisfactory" we mean a pair of linearly independent solutions that satisfy the criteria of Miller (1950). When μ is purely imaginary, however, the function $I_\mu(z)$ has the undesirable property of being complex on the positive real z axis. Therefore we now introduce the following function:

$$L_\mu(z)=\frac{\pi i}{2\sin(\mu\pi)}\{I_\mu(z)+I_{-\mu}(z)\} \qquad (\mu\neq 0), \tag{2.2}$$

which will be seen to be an appropriate numerically satisfactory companion to $K_{i\nu}(x)$, where ν is real and nonzero, and x is real and positive. Note that $L_\mu(z)$ is not defined when $\mu=0$. It is not possible to define a numerically satisfactory companion to $K_{i\nu}(x)$ which remains finite as $\nu\to 0$.

The definition of $L_\mu(z)$ should be compared with the definition of $K_\mu(z)$:

$$K_\mu(z)=\frac{\pi}{2\sin(\mu\pi)}\{I_{-\mu}(z)-I_\mu(z)\}. \tag{2.3}$$

The purpose of this section is to record some relevant properties of $L_\mu(z)$ and $K_\mu(z)$, with emphasis on the case where μ is purely imaginary. Throughout, μ denotes a complex parameter, and ν denotes *a positive (nonzero) parameter*. When the independent variable z is real and positive we denote it by x.

Recurrence relations. The functions $e^{\mu\pi i}K_\mu(z)$ and $e^{\mu\pi i}L_\mu(z)$ satisfy the same recurrence relations as $I_\mu(z)$, viz.

$$\begin{aligned} &I_{\mu-1}(z)-I_{\mu+1}(z)=(2\mu/z)I_\mu(z), \qquad && I_{\mu-1}(z)+I_{\mu+1}(z)=2I'_\mu(z),\\ &I_{\mu+1}(z)=-(\mu/z)I_\mu(z)+I'_\mu(z), && I_{\mu-1}(z)=(\mu/z)I_\mu(z)+I'_\mu(z). \end{aligned} \tag{2.4}$$

Analytic continuation. For any integer m

$$K_\mu(z\,e^{m\pi i})=\cos(m\mu\pi)K_\mu(z)-\sin(m\mu\pi)L_\mu(z), \tag{2.5a}$$

$$L_\mu(z\,e^{m\pi i})=\cos(m\mu\pi)L_\mu(z)+\sin(m\mu\pi)K_\mu(z). \tag{2.5b}$$

Power series representation. A power series expansion for $L_\mu(z)$ in ascending powers of z is readily derived from (2.2) together with the well-known power series for $I_{\pm\mu}(z)$ (see formula (10.01) of Chap. 2). When $\mu=i\nu$ and $z=x$ this power series can be expressed as

$$L_{i\nu}(x)=\left(\frac{\nu\pi}{\sinh(\nu\pi)}\right)^{1/2}\sum_{s=0}^{\infty}\frac{(x^2/4)^s\cos(\nu\ln(x/2)-\phi_{\nu,s})}{s![(\nu^2)(1^2+\nu^2)\cdots(s^2+\nu^2)]^{1/2}}, \tag{2.6}$$

where

$$\phi_{\nu,s}=\arg\{\Gamma(1+s+i\nu)\}. \tag{2.7}$$

(For each s we define the branch of (2.7) so that $\phi_{\nu,s}$ is continuous for $0<\nu<\infty$, with $\lim_{\nu\to0}\phi_{\nu,s}=0$.)

From the definition (2.3) of $K_\mu(z)$ we derive in a similar manner

$$K_{i\nu}(x)=-\left(\frac{\nu\pi}{\sinh(\nu\pi)}\right)^{1/2}\sum_{s=0}^{\infty}\frac{(x^2/4)^2\sin(\nu\ln(x/2)-\phi_{\nu,s})}{s![(\nu^2)(1^2+\nu^2)\cdots(s^2+\nu^2)]^{1/2}}. \tag{2.8}$$

Connection formulas.

$$L_{-\mu}(z)=-L_\mu(z),\qquad K_{-\mu}(z)=K_\mu(z). \tag{2.9}$$

Wronskian.

$$\mathscr{W}\{K_\mu(z),L_\mu(z)\}=\frac{\pi i}{\sin(\mu\pi)z}. \tag{2.10}$$

Integral representations. To derive an integral representation for $L_{i\nu}(z)$ we first express the function as a linear combination of Hankel functions. From the definition of $L_{i\nu}(z)$ and § 8.1 of Chap. 7 we obtain

$$L_{i\nu}(z)=\frac{\pi}{2\sinh(\nu\pi)}\{e^{-(\nu\pi/2)}\cosh(\nu\pi)H_{i\nu}^{(1)}(z\,e^{i\pi/2})+e^{(\nu\pi/2)}H_{i\nu}^{(2)}(z\,e^{i\pi/2})\}. \tag{2.11}$$

Next, the Hankel functions in (2.11) are expressed by their Sommerfeld integral representations (equation (4.19) of Chap. 7, with $\alpha=\pi/2$). The resulting integral representation for $L_{i\nu}(z)$ can be re-expressed, via a splitting into three integrals followed by appropriate changes of integration variables, in the following form:

$$\begin{aligned}L_{i\nu}(z)&=[\sinh(\nu\pi)]^{-1}\int_0^{\pi}e^{z\cos\theta}\cosh(\nu\theta)\,d\theta\\&\quad-\int_0^{\infty}e^{-z\cosh t}\sin(\nu t)\,dt,\qquad |\arg z|<\pi/2.\end{aligned} \tag{2.12}$$

It is immediately seen from (2.12) that $L_{i\nu}(x)$ is real for $0<x<\infty$. The modified Bessel function $K_{i\nu}(z)$ has the known integral representation

$$K_{i\nu}(z)=\int_0^{\infty}e^{-z\cosh t}\cos(\nu t)\,dt,\qquad |\arg z|<\pi/2, \tag{2.13}$$

and from this it is seen that $K_{i\nu}(x)$, too, is real for $0<x<\infty$.

Behavior at the singularities $z=0, \infty$. If $\nu(>0)$ is fixed and $x\to 0^+$, then

$$K_{i\nu}(x) = -\left(\frac{\pi}{\nu \sinh(\nu\pi)}\right)^{1/2} \{\sin(\nu \ln(x/2) - \phi_{\nu,0}) + O(x^2)\}, \tag{2.14}$$

$$L_{i\nu}(x) = \left(\frac{\pi}{\nu \sinh(\nu\pi)}\right)^{1/2} \{\cos(\nu \ln(x/2) - \phi_{\nu,0}) + O(x^2)\}. \tag{2.15}$$

(Note that the amplitudes of oscillation of $L_{i\nu}(x)$ and $K_{i\nu}(x)$ in a neighborhood of the origin become unbounded as $\nu \to 0$.) As $z\to\infty$

$$K_{i\nu}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\left\{1+O\left(\frac{1}{z}\right)\right\}, \qquad |\arg z| \leqq 3\pi/2-\delta, \tag{2.16}$$

$$L_{i\nu}(z) = \frac{1}{\sinh(\nu\pi)}\left(\frac{\pi}{2z}\right)^{1/2} e^{z}\left\{1+O\left(\frac{1}{z}\right)\right\}, \qquad |\arg z| \leqq \pi/2-\delta, \tag{2.17}$$

where δ is *an arbitrary small positive constant* (a convention used throughout). It should be emphasized that in (2.17) we have neglected exponentially small contributions (in Poincaré's sense), and as such the error term in this asymptotic formula can be large near the boundary of the region of validity. Inclusion of the exponentially small terms will result both in an extension of the region of validity, and increased accuracy (cf. Exercise 13.2 of Chap. 7).

Zeros. When $\nu>0$ it is known that $K_{i\nu}(x)$ has an infinite number of simple positive zeros in $0<x<\nu$, and no zeros in $\nu \leqq x < \infty$. We denote the zeros by $\{k_{\nu,s}\}_{s=1}^{\infty}$, such that

$$\nu > k_{\nu,1} > k_{\nu,2} > k_{\nu,3} > \cdots > 0, \tag{2.18}$$

$$\lim_{s\to\infty} k_{\nu,s} = 0. \tag{2.19}$$

LEMMA 1. *When $\nu>0$, $L_{i\nu}(x)$ has an infinite number of simple positive zeros, denoted by $\{l_{\nu,s}\}_{s=1}^{\infty}$, say, such that*

$$l_{\nu,1} > k_{\nu,1} > l_{\nu,2} > k_{\nu,2} > l_{\nu,3} > \cdots > 0, \tag{2.20}$$

$$\lim_{s\to\infty} l_{\nu,s} = 0. \tag{2.21}$$

Proof. The asymptotic behavior of $L_{i\nu}(x)$ near $x=0$ (see (2.15)) shows that the function has an infinite number of positive zeros. Using the Wronskian relation (2.10) and arguing along the lines of the proof of Theorem 7.1 of Chap. 7 we see that the zeros $k_{\nu,s}$, $l_{\nu,s}$ $(s=0,1,2,\cdots)$ are interlaced. It remains then to show that $l_{\nu,1}>k_{\nu,1}$. Suppose that $k_{\nu,1}>l_{\nu,1}(>k_{\nu,2})$. From (2.16) it is seen that $K_{i\nu}(x)$ is positive for $x>k_{\nu,1}$, and therefore negative in $(k_{\nu,2}, k_{\nu,1})$; in particular $K_{i\nu}(l_{\nu,1})<0$. From (2.17) it is seen that $L_{i\nu}(x)$ is positive in $(l_{\nu,1},\infty)$, and therefore $L'_{i\nu}(l_{\nu,1})>0$. Thus the assumption $k_{\nu,1}>l_{\nu,1}$ implies

$$K_{i\nu}(l_{\nu,1})L'_{i\nu}(l_{\nu,1})<0,$$

which contradicts the fact that the Wronskian (2.10) is positive for $0<x<\infty$. Thus $l_{\nu,1}>k_{\nu,1}$ as asserted. □

Asymptotic approximations for the zeros $\{k_{\nu,s}\}_{s=1}^{\infty}$ of $K_{i\nu}(x)$ can be derived from (2.14), and also from the asymptotic expansions given in § 4 (see (4.7) and (4.8)). We now record asymptotic approximations for the zeros which can be derived from these results. First, consider the asymptotic behavior of the zeros as $\nu\to\infty$: from (4.3), (4.7), and (4.8), together with the theory of § 6 of Chap. 11, we can show that

$$k_{\nu,s} = \nu X(-\nu^{-2/3}a_s) + s^{-1/3}O(\nu^{-2/3}) + O(\nu^{-1}), \tag{2.22}$$

as $\nu\to\infty$, uniformly for all positive integers s. Here $X(\zeta)$ is defined implicitly by the equation

$$\frac{2}{3}\zeta^{3/2}=\ln\left\{\frac{1+(1-X^2)^{1/2}}{X}\right\}-(1-X^2)^{1/2}, \tag{2.23}$$

and $\{a_s\}_{s=1}^{\infty}$ denote the (negative) zeros of the Airy function $\mathrm{Ai}(x)$, with the usual convention

$$0>a_1>a_2>\cdots. \tag{2.24}$$

For *fixed* s the sth zero of $K_{i\nu}(x)$ takes the simplified form

$$k_{\nu,s}=\nu+a_s(\nu/2)^{1/3}+\tfrac{3}{20}a_s^2(\nu/2)^{-1/3}+O(\nu^{-2/3}), \tag{2.25}$$

as $\nu\to\infty$.

Next, we consider the form of the zeros for fixed ν, as $s\to\infty$. We know that $k_{\nu,s}\to 0$ in this case; from the first two terms in the power series (2.8) we find that as $s\to\infty$ $(x\to 0)$

$$k_{\nu,s}=2\,e^{-(1/\nu)(s\pi-\phi_{\nu,0})}\left\{1+\frac{e^{-(2/\nu)(s\pi-\phi_{\nu,0})}}{(1+\nu^2)}+O(e^{-(4s\pi/\nu)})\right\}, \tag{2.26}$$

for fixed ν. We remark that it is not obvious that the right-hand side (RHS) of (2.26) is the sth zero of $K_{i\nu}(x)$, as opposed to, say, the $(s+1)$th zero. We now show that the RHS of (2.26) indeed represents $k_{\nu,s}$. To do so consider (2.22): this is uniformly valid for all integers s, and therefore we can consider the limiting form of this expression as $s\to\infty$, with ν large but now assumed fixed. On employing the approximation

$$a_s=-(3\pi(4s-1)/8)^{2/3}+O(s^{-4/3}) \tag{2.27}$$

(see (5.05) of Chap. 11) together with (2.23) we find that

$$\nu X(-\nu^{-2/3}a_s)\sim 2\,e^{-(1/\nu)(s\pi-\nu\ln\nu+\nu-\pi/4)}\quad\text{as } s\to\infty. \tag{2.28}$$

From the definition (2.7) of $\phi_{\nu,s}$, and the asymptotic form (see Abramowitz and Stegun (1965, p. 257))

$$\arg\{\Gamma(iy)\}\sim y\ln(y)-y-\frac{\pi}{4}\qquad(y\to+\infty)$$

we find that for large ν

$$\phi_{\nu,0}\sim\nu\ln(\nu)-\nu+\frac{\pi}{4}. \tag{2.29}$$

Thus on comparing (2.22), (2.28), and (2.29) with the RHS of (2.26) we deduce that the latter represents $k_{\nu,s}$ for large fixed ν and $s\to\infty$. By a continuity argument this is true for nonlarge values of ν as well.

Finally, we record corresponding asymptotic forms for $\{l_{\nu,s}\}_{s=1}^{\infty}$. As $\nu\to\infty$

$$l_{\nu,s}=\nu X(-\nu^{-2/3}b_s)+s^{-1/3}O(\nu^{-2/3})+O(\nu^{-1}), \tag{2.30}$$

uniformly for all positive integers s. Here $X(\zeta)$ is again given by (2.23), and $\{b_s\}_{s=1}^{\infty}$ denote the (negative) zeros of the Airy function $\mathrm{Bi}(x)$, in ascending order of absolute magnitude.

For fixed s and $\nu\to\infty$

$$l_{\nu,s}=\nu+b_s(\nu/2)^{1/3}+\tfrac{3}{20}b_s^2(\nu/2)^{-1/3}+O(\nu^{-2/3}). \tag{2.31}$$

For fixed ν and $s\to\infty$

$$l_{\nu,s}=2\,e^{-(1/\nu)((s-1/2)\pi-\phi_{\nu,0})}\left\{1+\frac{e^{-(2/\nu)((s-1/2)\pi-\phi_{\nu,0})}}{(1+\nu^2)}+O(e^{-(4\pi/\nu)(s-1/2)})\right\}. \tag{2.32}$$

3. Unmodified Bessel functions of purely imaginary order. Standard solutions of the unmodified Bessel equation

$$\frac{d^2w}{dz^2}+\frac{1}{z}\frac{dw}{dz}+\left(1-\frac{\mu^2}{z^2}\right)w=0, \tag{3.1}$$

are the Bessel functions of the first and second kinds $J_\mu(z)$, $Y_\mu(z)$, and the Bessel functions of the third kind (Hankel functions) $H_\mu^{(1)}(z)$, $H_\mu^{(2)}(z)$. The characterizing properties of these functions are the following:

(i) $J_\mu(z)$ is recessive at the regular singularity $z=0$ when $\operatorname{Re}\mu>0$ or $\mu=0$, and moreover is real on the positive real z axis when μ is real.

(ii) $Y_\mu(z)$ is real for positive z and real μ, and for large positive z has the same amplitude of oscillation as $J_\mu(z)$ and is out of phase by $\pi/2$.

(iii) For all μ, $H_\mu^{(1)}(z)$ is recessive at infinity in the sector $\delta \leqq \arg z \leqq \pi-\delta$, and $H_\mu^{(2)}(z)$ is recessive in the conjugate sector.

Thus, when μ is real and nonnegative, $J_\mu(z)$ and $Y_\mu(z)$ are a numerically satisfactory pair on $0<x<\infty$, and when $\operatorname{Re}\mu>0$ or $\mu=0$, $J_\mu(z)$ and $H_\mu^{(1)}(z)$ are a numerically satisfactory pair throughout the sector $0\leqq \arg z\leqq \pi$, $J_\mu(z)$, $H_\mu^{(2)}(z)$ being the numerically satisfactory pair in the conjugate sector.

When $\arg \mu=\pm\pi/2$ no solution is recessive at the origin, and the Hankel functions $H_\mu^{(1)}(z)$ and $H_\mu^{(2)}(z)$ compose a numerically satisfactory pair throughout $|\arg z|\leqq \pi$. However, these functions, as well as $J_\mu(z)$ and $Y_\mu(z)$, are not real on the real z axis when their orders are purely imaginary. We therefore now introduce two new Bessel functions that *are* real when $z\equiv x$ is positive and $\arg\mu=\pm\pi/2$, and moreover are numerically satisfactory when x and $|\mu|$ are not both small. We define

$$F_\mu(z)=\tfrac{1}{2}\{e^{\mu\pi i/2}H_\mu^{(1)}(z)+e^{-\mu\pi i/2}H_\mu^{(2)}(z)\}, \tag{3.2}$$

$$G_\mu(z)=\frac{1}{2i}\{e^{\mu\pi i/2}H_\mu^{(1)}(z)-e^{-\mu\pi i/2}H_\mu^{(2)}(z)\}. \tag{3.3}$$

From the relations $H_\mu^{(1)}(z)=\overline{H_{\bar\mu}^{(2)}(\bar z)}$, $H_{-\mu}^{(2)}(z)=e^{-\mu\pi i}H_\mu^{(2)}(z)$, where bars denote complex conjugates, it is readily verified that $F_{i\nu}(x)$ and $G_{i\nu}(x)$ are real for $x>0$. (Note that $F_0(z)=J_0(z)$ and $G_0(z)=Y_0(z)$.) Also, from the following alternative representations, which can be derived from standard results (see, e.g., (4.12), of Chap. 7)

$$F_\mu(z)=\tfrac{1}{2}\sec(\mu\pi/2)\{J_\mu(z)+J_{-\mu}(z)\}, \tag{3.4a}$$

$$G_\mu(z)=\tfrac{1}{2}\operatorname{cosec}(\mu\pi/2)\{J_\mu(z)-J_{-\mu}(z)\}, \tag{3.4b}$$

it is seen that $\cos(\mu\pi/2)F_\mu(z)$ and $\sin(\mu\pi/2)G_\mu(z)$ satisfy the same recurrence relations as $J_\mu(z)$, namely (2.3) with $I_{\mu+1}(z)$, $I_{\mu-1}(z)$, $I_\mu(z)$, $I'_\mu(z)$ replaced by $-J_{\mu+1}(z)$, $J_{\mu-1}(z)$, $J_\mu(z)$, $J'_\mu(z)$, respectively.

We now record other properties of $F_\mu(z)$ and $G_\mu(z)$.

Analytic continuation. For any integer m

$$F_\mu(z\,e^{m\pi i})=\cos(m\mu\pi)F_\mu(z)+i\sin(m\mu\pi)\tan(\mu\pi/2)G_\mu(z), \tag{3.5}$$

$$G_\mu(z\,e^{m\pi i})=i\sin(m\mu\pi)\cot(\mu\pi/2)F_\mu(z)+\cos(m\mu\pi)G_\mu(z). \tag{3.6}$$

Connection formulas.

$$F_{-\mu}(z)=F_\mu(z),\qquad G_{-\mu}(z)=G_\mu(z). \tag{3.7}$$

Power series representations. For purely imaginary order and positive argument we have

$$F_{i\nu}(x)=\left(\frac{2\nu\tanh(\nu\pi/2)}{\pi}\right)^{1/2} \cdot \sum_{s=0}^{\infty}\frac{(-1)^s(x^2/4)^s\cos(\nu\ln(x/2)-\phi_{\nu,s})}{s![(\nu^2)(1+\nu^2)\cdots(s^2+\nu^2)]^{1/2}}, \tag{3.8}$$

$$G_{i\nu}(x)=\left(\frac{2\nu\coth(\nu\pi/2)}{\pi}\right)^{1/2} \cdot \sum_{s=0}^{\infty}\frac{(-1)^s(x^2/4)^s\sin(\nu\ln(x/2)-\phi_{\nu,s})}{s![(\nu^2)(1+\nu^2)\cdots(s^2+\nu^2)]^{1/2}}, \tag{3.9}$$

where $\phi_{\nu,s}$ is defined by (2.7).

Wronskian.

$$\mathscr{W}\{F_\mu(z), G_\mu(z)\}=2/(\pi z). \tag{3.10}$$

Integral representations. The Schläfli-type representations are readily shown to be

$$F_\mu(z)=\frac{1}{2\pi i\cos(\mu\pi/2)}\int_{\infty-\pi i}^{\infty+\pi i}e^{z\sinh t}\cosh(\mu t)\,dt, \qquad |\arg z|<\pi/2, \tag{3.11}$$

$$G_\mu(z)=\frac{-1}{2\pi i\sin(\mu\pi/2)}\int_{\infty-\pi i}^{\infty+\pi i}e^{z\sinh t}\sinh(\mu t)\,dt, \qquad |\arg z|<\pi/2, \tag{3.12}$$

where the path of integration is indicated in Fig. 1.

For purely imaginary order these integrals can be re-expressed as

$$F_{i\nu}(z)=\frac{1}{\pi}\operatorname{sech}(\nu\pi/2)\int_0^{\pi}\cos(z\sin\theta)\cosh(\nu\theta)\,d\theta -\frac{2}{\pi}\sinh(\nu\pi/2)\int_0^{\infty}e^{-z\sinh t}\sin(\nu t)\,dt, \qquad |\arg z|<\pi/2, \tag{3.13}$$

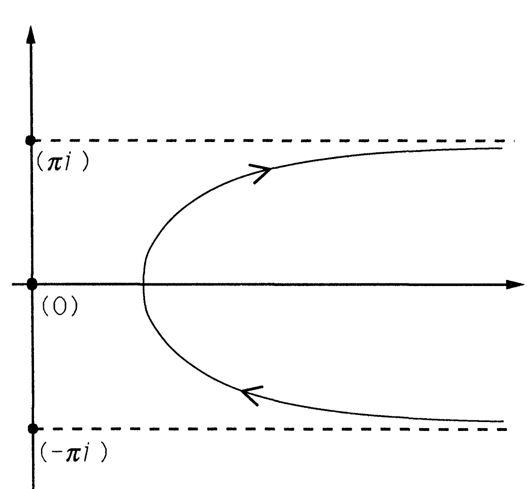

FIG. 1. *t plane.*

$$G_{i\nu}(z)=\frac{1}{\pi}\operatorname{cosech}(\nu\pi/2)\int_0^{\pi}\sin(z\sin\theta)\sinh(\nu\theta)\,d\theta$$
$$-\frac{2}{\pi}\cosh(\nu\pi/2)\int_0^{\infty}e^{-z\sinh t}\cos(\nu t)\,dt,\qquad |\arg z|<\pi/2. \tag{3.14}$$

Behavior at the singularities $z=0,\infty$. If ν (greater than zero) is fixed and $x\to 0^+$, then

$$F_{i\nu}(x)=\left(\frac{2\tanh(\nu\pi/2)}{\nu\pi}\right)^{1/2}\{\cos(\nu\ln(x/2)-\phi_{\nu,0})+O(x^2)\}, \tag{3.15}$$

$$G_{i\nu}(x)=\left(\frac{2\coth(\nu\pi/2)}{\nu\pi}\right)^{1/2}\{\sin(\nu\ln(x/2)-\phi_{\nu,0})+O(x^2)\}. \tag{3.16}$$

Note that in a neighborhood of $x=0$ the amplitude of oscillation of $F_{i\nu}(x)$ and $G_{i\nu}(x)$ tends to 1 and ∞, respectively, as $\nu\to 0$.

As $z\to\infty$ in $|\arg z|\leqq \pi-\delta$

$$F_{i\nu}(z)\sim\left(\frac{2}{\pi z}\right)^{1/2}\left\{\cos(z-\pi/4)\sum_{s=0}^{\infty}(-1)^s\frac{A_{2s}(i\nu)}{z^{2s}}\right.$$
$$\left.-\sin(z-\pi/4)\sum_{s=0}^{\infty}(-1)^s\frac{A_{2s+1}(i\nu)}{z^{2s+1}}\right\}, \tag{3.17}$$

$$G_{i\nu}(z)\sim\left(\frac{2}{\pi z}\right)^{1/2}\left\{\sin(z-\pi/4)\sum_{s=0}^{\infty}(-1)^s\frac{A_{2s}(i\nu)}{z^{2s}}\right.$$
$$\left.+\cos(z-\pi/4)\sum_{s=0}^{\infty}(-1)^s\frac{A_{2s+1}(i\nu)}{z^{2s+1}}\right\}, \tag{3.18}$$

where the A_s are given by (4.02) of Chap. 7.

Zeros. From the foregoing results it is evident that $F_{i\nu}(x)$ and $G_{i\nu}(x)$ have an infinite number of zeros in both the x intervals $(0,\delta)$ and $[\delta,\infty)(\delta>0)$. A convenient notation for the zeros of $F_{i\nu}(x)$ is $\{f_{\nu,s}^{(<)}\}_{s=1}^{\infty}$ and $\{f_{\nu,s}^{(>)}\}_{s=1}^{\infty}$, where

$$\nu\tau>f_{\nu,1}^{(<)}>f_{\nu,2}^{(<)}>f_{\nu,3}^{(<)}>\cdots>0, \tag{3.19}$$

$$\nu\tau\leqq f_{\nu,1}^{(>)}<f_{\nu,2}^{(>)}<f_{\nu,3}^{(>)}<\cdots<\infty, \tag{3.20}$$

with τ being the positive constant defined by (5.6) below. Using the same convention we can denote the zeros of $G_{i\nu}(x)$ by $\{g_{\nu,s}^{(<)}\}_{s=1}^{\infty}$ and $\{g_{\nu,s}^{(>)}\}_{s=1}^{\infty}$. The zeros of $F_{i\nu}(x)$ and $G_{i\nu}(x)$ are simple and interlaced (cf. Lemma 1).

Asymptotic approximations for the zeros can be established in a similar manner to those derived in § 2 for the zeros of $K_{i\nu}(x)$ and $L_{i\nu}(x)$. For large ν the theory of § 8 of Chap. 6 (in particular § 8.5) can be applied to the uniform asymptotic expansions (5.15) and (5.16) (with $n=0$) which are given in § 5 below. We obtain the following asymptotic forms:

$$f_{\nu,s}^{(>)}=\nu Z\left\{\frac{(4s-1)\pi}{4\nu}\right\}+O\left(\frac{1}{\nu}\right)Z'\left\{\frac{(4s-1)}{4\nu}+O\left(\frac{1}{\nu^2}\right)\right\}, \tag{3.21}$$

$$g_{\nu,s}^{(>)}=\nu Z\left\{\frac{(4s-3)\pi}{4\nu}\right\}+O\left(\frac{1}{\nu}\right)Z'\left\{\frac{(4s-3)}{4\nu}+O\left(\frac{1}{\nu^2}\right)\right\}, \tag{3.22}$$

as $\nu\to\infty$, uniformly for all s. Here it is assumed that ν is sufficiently large to ensure that

$$|\varepsilon_{1,1}(\nu,0)|+|\varepsilon_{1,2}(\nu,0)|<\sqrt{2}. \tag{3.23}$$

This ensures that none of the zeros can take the value $\nu\tau$. In the Appendix we give a sufficient condition for (3.23) to hold, and it is seen that ν does not have to be very large for the inequality to hold (see (A6)). The function $Z(\xi)$ is defined implicitly by the equation

$$\xi=(1+Z^2)^{1/2}+\ln\left\{\frac{Z}{1+(1+Z^2)^{1/2}}\right\}. \tag{3.24}$$

The corresponding approximations for $f_{\nu,s}^{(<)}$ and $g_{\nu,s}^{(<)}$ are given by (3.21) and (3.22), respectively, with s replaced by $-s+1$.

For fixed s, (3.21) and (3.22) can be simplified by Taylor's theorem to give

$$f_{\nu,s}^{(>)}=\nu\tau+\frac{\tau(4s-1)\pi}{4(1+\tau^2)^{1/2}}+O\left(\frac{1}{\nu}\right), \tag{3.25}$$

$$g_{\nu,s}^{(>)}=\nu\tau+\frac{\tau(4s-3)\pi}{4(1+\tau^2)^{1/2}}+O\left(\frac{1}{\nu}\right), \tag{3.26}$$

as $\nu\to\infty$. Again, on replacing s by $-s+1$ in (3.25) and (3.26) we obtain the corresponding formulas for $f_{\nu,s}^{(<)}$ and $g_{\nu,s}^{(<)}$.

When ν is fixed, but still satisfying (3.23), and $x\to 0^+$ we employ the first two terms of (3.8) and (3.9) to obtain the approximations

$$f_{\nu,s}^{(>)}=2\,e^{-(1/\nu)((s-1/2)\pi-\phi_{\nu,0})}\left\{1-\frac{e^{-(2/\nu)((s-1/2)\pi-\phi_{\nu,0})}}{(1+\nu^2)}+O(e^{-(4\pi/\nu)(s-1/2)})\right\}, \tag{3.27}$$

$$g_{\nu,s}^{(>)}=2\,e^{-(1/\nu)(s\pi-\phi_{\nu,0})}\left\{1-\frac{e^{-(2/\nu)(s\pi-\phi_{\nu,0})}}{(1+\nu^2)}+O(e^{-(4s\pi/\nu)})\right\}, \tag{3.28}$$

as $s\to\infty$. Justification that these approximations represent the sth zero to the left of the point $x=\nu\tau$ follows in a similar manner to that of (2.26) and (2.32).

Finally, for fixed ν (satisfying (3.23)) and $s\to\infty$ we find from (3.17) that

$$f_{\nu,s}^{(>)}=\left(s-\frac{1}{4}\right)\pi+\frac{4\nu^2+1}{2(4s-1)\pi}-\frac{(4\nu^2+1)(28\nu^2+31)}{6(4s-1)^3\pi^3}+O(s^{-5}); \tag{3.29}$$

compare (6.03) of Chap. 7. Likewise, from (3.18) we find that

$$g_{\nu,s}^{(>)}=\left(s-\frac{3}{4}\right)\pi+\frac{4\nu^2+1}{2(4s-3)\pi}-\frac{(4\nu^2+1)(28\nu^2+31)}{6(4s-3)^3\pi^3}+O(s^{-5}). \tag{3.30}$$

4. Asymptotic expansions for modified Bessel functions of purely imaginary order. The modified Bessel functions $z^{1/2}K_{i\nu}(\nu z)$, $z^{1/2}L_{i\nu}(\nu z)$, as well as the analytic continuations $z^{1/2}K_{i\nu}(\nu z\,e^{\pi i})$, $z^{1/2}K_{i\nu}(\nu z\,e^{-\pi i})$, satisfy

$$\frac{d^2w}{dz^2}=\left\{\nu^2\frac{z^2-1}{z^2}-\frac{1}{4z^2}\right\}w, \tag{4.1}$$

which is characterized by having a regular singularity at $z=0$, an irregular singularity at $z=\infty$, and turning points at $z=\pm 1$. We apply the theory of a turning point in the complex plane (Theorem 9.1 of Chap. 11) to obtain asymptotic approximations, for large ν, in terms of Airy functions.

The first step is to transform (4.1) to the form

$$d^2W/d\zeta^2=\{-\nu^2\zeta+\phi(\zeta)\}W, \tag{4.2}$$

which is achieved by the following Liouville transformation:

$$\frac{2}{3}\zeta^{3/2}(z)=\ln\left\{\frac{1+(1-z^2)^{1/2}}{z}\right\}-(1-z^2)^{1/2}, \tag{4.3}$$

$$W(\zeta)=\left(\frac{1-z^2}{z^2\zeta}\right)^{1/4}w(z). \tag{4.4}$$

This is precisely the Liouville transformation of § 10 of Chap. 11 and the reader is referred to this section for full details. It is seen from (10.04) in Chap. 11 that

$$\phi(\zeta)=\psi(-\zeta)=\frac{5}{16\zeta^2}-\frac{\zeta z^2(z^2+4)}{4(z^2-1)^3}. \tag{4.5}$$

In the notation of § 10 of Chap. 11 solutions of (4.2) are $W_{2n+1,0}(\nu,-\zeta)$, $W_{2n+1,1}(\nu,-\zeta)$, and $W_{2n+1,1}(\nu,-\zeta)$; see (9.02), (10.06), (10.07), (10.14), and (10.23) of Chap. 11. It can be shown by induction from (10.06) and (10.07) that

$$A_s(-\zeta)=(-1)^sA_s(\zeta), \qquad B_s(-\zeta)=(-1)^sB_s(\zeta), \tag{4.6}$$

and therefore for $j=0,\pm1$, $n=0,1,2,\cdots$, and $\nu>0$

$$\begin{aligned}W_{2n+1,j}(\nu,-\zeta)&=\mathrm{Ai}\,(-\nu^{2/3}\zeta\,e^{-2\pi ij/3})\sum_{s=0}^{n}(-1)^s\frac{A_s(\zeta)}{\nu^{2s}}\\&\quad+\frac{\mathrm{Ai}'\,(-\nu^{2/3}\zeta\,e^{-2\pi ij/3})}{\nu^{4/3}}\sum_{s=0}^{n}(-1)^s\frac{B_s(\zeta)}{\nu^{2s}}+\varepsilon_{2n+1,j}(\nu,-\zeta).\end{aligned} \tag{4.7}$$

Bounds on the error terms $\varepsilon_{2n+1,j}$ are furnished by (9.03) of Chap. 11. The solutions above are to be identified with solutions of (4.1). First, since $z^{1/2}K_{i\nu}(\nu z)$ and $z^{1/2}(\zeta/(1-z^2))^{1/4}W_{2n+1,0}(\nu,-\zeta)$ are solutions of (4.1) that share the same recessive property at $z=+\infty$ ($\zeta=-\infty$), it follows that they are proportional to one another. The constant of proportionality can be determined by comparing the behavior of both functions at $z=\infty$, $\zeta=-\infty$; from (2.16), and from Chap. 11, (1.07), (10.08), (10.14), (10.23), we find

$$K_{i\nu}(\nu z)=\frac{\pi\,e^{-\nu\pi/2}}{\nu^{1/3}}\left(\frac{4\zeta}{1-z^2}\right)^{1/4}W_{2n+1,0}(\nu,-\zeta), \tag{4.8}$$

a result first given by Balogh (1967). (See also Exercise 10.6 of Chap. 11.)

The identification of $W_{2n+1,1}(\nu,-\zeta)$ is similar. Both this function, regarded as a function of z, and the modified Bessel function $K_{i\nu}(\nu z\,e^{-\pi i})$ are recessive at $z=\infty$ when $\pi/2<\arg z<\pi$. (We are restricting our attention to $|\arg z|<\pi$; $K_{i\nu}(\nu z\,e^{-\pi i})$ is of course also recessive at $z=\infty$ in $\pi\leqq\arg z<3\pi/2$.) It follows that there exists a constant c such that

$$K_{i\nu}(\nu z\,e^{-\pi i})=c\left(\frac{4\zeta}{1-z^2}\right)^{1/4}W_{2n+1,1}(\nu,-\zeta). \tag{4.9}$$

By comparing both sides as $\zeta\to\infty\,e^{-\pi i/3}$ we find that

$$c=\frac{\pi\,e^{\pi i/3}\,e^{\nu\pi/2}}{\nu^{1/3}} \tag{4.10}$$

Likewise, it can be shown that

$$K_{i\nu}(\nu z\,e^{\pi i})=\frac{\pi\,e^{-\pi i/3}\,e^{\nu\pi/2}}{\nu^{1/3}}\left(\frac{4\zeta}{1-z^2}\right)^{1/4}W_{2n+1,-1}(\nu,-\zeta). \tag{4.11}$$

This completes the identification of the asymptotic solutions (4.7). It remains to derive an asymptotic expansion for $L_{i\nu}(\nu z)$, and to do so we employ the analytic continuation formula (2.5a). On setting $m=\pm 1$ in this equation, and then eliminating $K_\mu(z)$ from the resulting two equations, we derive the relation

$$L_\mu(z)=\frac{1}{2\sin(\mu\pi)}\{K_\mu(z\,e^{-\pi i})-K_\mu(z\,e^{\pi i})\}. \tag{4.12}$$

We now replace z by νz in (4.12), set $\mu=i\nu$, and employ (4.9)-(4.11) to obtain the following identification:

$$\begin{aligned} L_{i\nu}(\nu z)&=\frac{\pi\, e^{\nu\pi/2}}{2i\nu^{1/3}\sinh(\nu\pi)}\left(\frac{4\zeta}{1-z^2}\right)^{1/4} \\ &\quad\cdot\{e^{\pi i/3}W_{2n+1,1}(\nu,-\zeta)-e^{-\pi i/3}W_{2n+1,-1}(\nu,-\zeta)\}. \end{aligned} \tag{4.13}$$

On employing (4.7), together with (8.04) of Chap. 11, we can re-express this as

$$\begin{aligned} L_{i\nu}(\nu z)&=\frac{\pi\, e^{\nu\pi/2}}{2\nu^{1/3}\sinh(\nu\pi)}\left(\frac{4\zeta}{1-z^2}\right)^{1/4} \\ &\quad\cdot\left[\mathrm{Bi}(-\nu^{2/3}\zeta)\sum_{s=0}^{n}(-1)^s\frac{A_s(\zeta)}{\nu^{2s}}+\frac{\mathrm{Bi}'(-\nu^{2/3}\zeta)}{\nu^{4/3}}\sum_{s=0}^{n-1}(-1)^s\frac{B_s(\zeta)}{\nu^{2s}}\right. \\ &\qquad\left.+\{e^{-\pi i/6}\varepsilon_{2n+1,1}(\nu,-\zeta)+e^{\pi i/6}\varepsilon_{2n+1,-1}(\nu,-\zeta)\}\right], \end{aligned} \tag{4.14}$$

an expansion that is uniformly valid for $\nu>0$ and $|\arg z|\leqq\pi-\delta$. We emphasize that both the expansions (4.8) and (4.14) are uniformly valid in a neighborhood of the singularity $z=0$, provided $|\arg z|\leqq\pi-\delta$. Use of (4.8) and (4.14) can be restricted to the half-plane $|\arg z|\leqq\pi/2$, extensions to other ranges of arg z being achieved via the analytic continuation formulas (2.5a, b).

An asymptotic expansion for $I_{i\nu}(\nu z)$ is also readily derived from the foregoing results; from the identity

$$I_{i\nu}(\nu z)=\frac{\mathrm{sech}(\nu\pi)}{2\pi i}\{e^{\nu\pi}K_{i\nu}(\nu z\,e^{-\pi i})-e^{-\nu\pi}K_{i\nu}(\nu z\,e^{\pi i})\},$$

and (4.9)-(4.11) we obtain

$$\begin{aligned} I_{i\nu}(\nu z)&=\frac{e^{\nu\pi/2}}{2\nu^{1/3}}\left(\frac{4\zeta}{1-z^2}\right)^{1/4}\left[\mathrm{Bi}_\nu(-\nu^{2/3})\sum_{s=0}^{n}(-1)^s\frac{A_s(\zeta)}{\nu^{2s}}\right. \\ &\qquad+\frac{\mathrm{Bi}_\nu'(-\nu^{2/3})}{\nu^{4/3}}\sum_{s=0}^{n-1}(-1)^s\frac{B_s(\zeta)}{\nu^{2s}} \\ &\qquad+\mathrm{sech}(\nu\pi)\{e^{\pi(\nu-i/6)}\varepsilon_{2n+1,1}(\nu,-\zeta) \\ &\qquad\left.+e^{-\pi(\nu-i/6)}\varepsilon_{2n+1,-1}(\nu,-\zeta)\}\right], \end{aligned} \tag{4.15}$$

where

$$\mathrm{Bi}_\nu(z)=\mathrm{Bi}(z)-i\tanh(\nu\pi)\,\mathrm{Ai}(z). \tag{4.16}$$

Again, this expansion is uniformly valid for $\nu>0$, $|\arg z|\leqq\pi-\delta$.

Finally, in this section we derive Debye-type expansions for $L_{i\nu}(\nu x)$, i.e., asymptotic expansions involving elementary functions that are uniformly valid for positive x. The corresponding Debye-type expansions for $K_{i\nu}(\nu x)$ are well known (see, e.g., Magnus, Oberhettinger, and Soni (1966, p. 141)). The following expansions are not valid near the turning point $x = 1$, and therefore we must consider the x intervals $(0, 1-\delta]$, $[1+\delta, \infty)$ separately.

First, consider the case $1+\delta \leqq x < \infty$. On applying the Liouville transformation to (4.1) (with z replaced by x), we obtain the transformed equation

$$d^2\Theta/d\eta^2 = \{\nu^2 + \chi(\eta)\}\Theta, \tag{4.17}$$

where

$$\eta(x) = \int_1^x \frac{(t^2-1)^{1/2}}{t}\,dt = (x^2-1)^{1/2} - \sec^{-1}(x), \tag{4.18}$$

$$\Theta(\eta) = (d\eta/dx)^{1/2} w(x) = x^{-1/2}(x^2-1)^{1/4} w(x), \tag{4.19}$$

$$\chi(\eta) = -\frac{x^2(4+x^2)}{4(x^2-1)}; \tag{4.20}$$

see § 2.1 of Chap. 10. On applying Theorem 3.1 of Chap. 10 we obtain the following solution of (4.17):

$$\Theta_n(\nu, \eta) = e^{-\nu\eta} \sum_{s=0}^{n-1} (-1)^s \frac{V_s(q)}{\nu^s} + \varepsilon_n(\nu, \eta), \tag{4.21}$$

where

$$q = (x^2-1)^{-1/2}, \tag{4.22}$$

$$V_0(q) = 1, \tag{4.23}$$

$$V_{s+1}(q) = \tfrac{1}{2}q^2(q^2+1)V_s'(q) + \tfrac{1}{8}\int_0^q V_s(t)(5t^2+1)\,dt \qquad (s \geqq 1). \tag{4.24}$$

A bound for $\varepsilon_n(\nu, \eta)$ is furnished by (3.04) of Chap. 10; for our purposes it suffices to observe that

$$\varepsilon_n(\nu, \eta) = e^{-\nu\eta} O(\nu^{-n})$$

as $\nu \to \infty$, uniformly for $1+\delta \leqq x < \infty$. It is possible to carry error bounds throughout the following analysis, but we will not pursue this.

Since both $K_{i\nu}(\nu x)$ and $\Theta_n(\nu, \eta)$ are recessive as $x \to \infty$ it follows that they are multiples of one another. By comparing both functions as $x \to \infty$ we find that

$$K_{i\nu}(\nu x) = (\pi/(2\nu))^{1/2} e^{-\nu\pi/2}(x^2-1)^{-1/4}\Theta_n(\nu, \eta). \tag{4.25}$$

Next, on identifying the left-hand side (LHS) of (4.25) with (4.8), employing asymptotic expansions for $\mathrm{Ai}(x)$, $\mathrm{Ai}'(x)$ of large positive argument (see Chap. 11, (1.07)), and equating coefficients of ν^{-s}, we arrive at the following relations for each $s \geqq 0$:

$$\sum_{j=0}^{s} (-1)^j \eta^{-2(s-j)} u_{2(s-j)} A_j(\zeta) + (-\zeta)^{1/2} \cdot \sum_{j=0}^{s-1} (-1)^j \eta^{-2(s-j)+1} v_{2(s-j)-1} B_j(\zeta) = V_{2s}(q), \tag{4.26}$$

$$\sum_{j=0}^{s} (-1)^j \eta^{-2(s-j)-1} u_{2(s-j)+1} A_j(\zeta) + (-\zeta)^{1/2} \cdot \sum_{j=0}^{s} (-1)^j \eta^{-2(s-j)} v_{2(s-j)} B_j(\zeta) = V_{2s+1}(q), \tag{4.27}$$

where $u_0 = v_0 = 1$ and

$$u_s = \frac{(2s+1)(2s+3)(2s+5)\cdots(6s-1)}{(216)^s s!}, \qquad v_s = -\frac{(6s+1)}{(6s-1)} u_s \qquad (s \geqq 1).$$

We now are in a position to derive a Debye-type asymptotic expansion for $L_{i\nu}(\nu x)$ for $x > 1$. From (4.14), (4.26), and (4.27), together with (1.07) and (1.16) of Chap. 11, we have for $1+\delta \leqq x < \infty$ and $\nu \to +\infty$:

$$L_{i\nu}(\nu x) \sim \left(\frac{\pi}{2\nu}\right)^{1/2} \frac{e^{\nu\pi/2}}{\sinh(\nu\pi)} (x^2-1)^{-1/4} e^{\nu\eta} \sum_{s=0}^{\infty} \frac{V_s(q)}{\nu^s}, \tag{4.28}$$

where $\eta(x)$, q, and $V_s(q)$ are given by (4.18) and (4.22)-(4.24). In a similar manner we can show that for $0 < x \leqq 1-\delta$, $\nu \to +\infty$,

$$\begin{aligned} L_{i\nu}(\nu x) \sim \left(\frac{\pi}{2\nu}\right)^{1/2} &\frac{e^{\nu\pi/2}}{\sinh(\nu\pi)} (1-x^2)^{-1/4} \\ &\cdot \left[-\sin(\nu\hat{\eta} - \pi/4) \sum_{s=0}^{n} \frac{V_{2s}(i\hat{q})}{\nu^{2s}} \right. \\ &\quad \left. + \cos(\nu\hat{\eta} - \pi/4) \sum_{s=0}^{\infty} \frac{iV_{2s+1}(i\hat{q})}{\nu^{2s+1}} \right], \end{aligned} \tag{4.29}$$

where

$$\hat{q}(x) = (1-x^2)^{-1/2}, \tag{4.30}$$

$$\hat{\eta}(x) = \int_x^1 \frac{(1-t^2)^{1/2}}{t} dt = \ln\left\{\frac{1+(1-x^2)^{1/2}}{x}\right\} - (1-x^2)^{1/2}. \tag{4.31}$$

5. Asymptotic expansions for unmodified Bessel functions of purely imaginary order. The unmodified Bessel functions $z^{1/2}F_{i\nu}(\nu z)$, $z^{1/2}G_{i\nu}(\nu z)$, $z^{1/2}H_{i\nu}^{(1)}(\nu z)$, and $z^{1/2}H_{i\nu}^{(2)}(\nu z)$ satisfy the equation

$$\frac{d^2w}{dz^2} = \left\{-\nu^2 \frac{1+z^2}{z^2} - \frac{1}{4z^2}\right\} w, \tag{5.1}$$

which is characterized by having a regular singularity at $z=0$, an irregular singularity at $z=\infty$, and turning points at $z=\pm i$ (where the results of § 4 are applicable).

We restrict our attention to the half-plane $|\arg z| < \pi/2$, with $\nu > 0$, and apply the Liouville transformation of § 7 in Chap. 10. The effect of this transformation is to throw (5.1) into the form

$$d^2W/d\xi^2 = \{-\nu^2 + \psi(\xi)\} W, \tag{5.2}$$

where

$$\xi(z) = (1+z^2)^{1/2} + \ln\left\{\frac{z}{1+(1+z^2)^{1/2}}\right\}, \tag{5.3}$$

$$W(\xi) = \left(\frac{1+z^2}{z^2}\right)^{1/4} w(z), \tag{5.4}$$

$$\psi(\xi) = \frac{z^2(4-z^2)}{4(1+z^2)^3}. \tag{5.5}$$

Before proceeding further let us introduce a constant τ, defined by

$$\tau = (\tau_0 - 1)^{1/2}, \tag{5.6}$$

where τ_0 is the positive root of the equation $\coth \tau_0 = \tau_0$; from Exercise 8.1 of Chap. 10 and (5.3) it is seen that $z = \tau$ is the point that is mapped to $\xi = 0$, i.e.,

$$\xi(\tau) = 0 \qquad (\tau = 0.6627\cdots). \tag{5.7}$$

On applying Theorem 3.1 of Chap. 10 to the transformed equation (5.2) we obtain the following solutions:

$$W_{n,1}(\nu, \xi) = e^{i\nu\xi} \sum_{s=0}^{n-1} \frac{U_s(p)}{(i\nu)^s} + \varepsilon_{n,1}(\nu, \xi), \tag{5.8}$$

$$W_{n,2}(\nu, \xi) = e^{-i\nu\xi} \sum_{s=0}^{n-1} \frac{(-1)^s U_s(p)}{(i\nu)^s} + \varepsilon_{n,2}(\nu, \xi), \tag{5.9}$$

where

$$p = (1+z^2)^{-1/2}, \tag{5.10}$$

and the coefficients $U_s(p)$ are given by (7.10) of Chap. 10, and are related to the V_s of the previous section by

$$U_s(p) = (-i)^s V_s(ip) \qquad (s = 0, 1, 2, \cdots). \tag{5.11}$$

Our choice of reference points for the solutions is $\alpha_1 = +i\infty$, $\alpha_2 = -i\infty$; with these choices the error term $\varepsilon_{n,1}$ is bounded by (3.04) of Chap. 10 for all points in $|\arg z| \leqq \pi/2$ except those on the finite interval $z = i\sigma$, $0 \leqq \sigma \leqq 1$, and at $z = -i$, with the corresponding bound for $\varepsilon_{n,2}$ being valid in the conjugate region.

We now identify the solutions (5.8) and (5.9) with Bessel functions. First, we see that for some constant c_1

$$H_{i\nu}^{(1)}(\nu z) = c_1(1+z^2)^{-1/4} W_{n,1}(\nu, \xi), \tag{5.12}$$

since both functions are solutions of Bessel's equation and share the same recessive property as $z \to +i\infty$. By comparing both sides as $z \to +i\infty$ (see (4.03) of Chap. 7) we find that

$$c_1 = (2/(\pi\nu))^{1/2} e^{\nu\pi/2} e^{-i\pi/4}. \tag{5.13}$$

Likewise we find that

$$H_{i\nu}^{(2)}(\nu z) = (2/(\pi\nu))^{1/2} e^{-\nu\pi/2} e^{i\pi/4}(1+z^2)^{-1/4} W_{n,2}(\nu, \xi). \tag{5.14}$$

Asymptotic expansions for $F_{i\nu}(\nu z)$ and $G_{i\nu}(\nu z)$ are now obtainable from the above expansions and the relations (3.2) and (3.3): for $\nu > 0$, $|\arg z| < \pi/2$ we have

$$\begin{aligned} F_{i\nu}(\nu z) = \left(\frac{2}{\pi\nu}\right)^{1/2} &(1+z^2)^{-1/4} \\ &\cdot \Bigg[\cos(\nu\xi - \pi/4) \sum_{s=0}^{n} \frac{(-1)^s U_{2s}(p)}{\nu^{2s}} \\ &\quad + \sin(\nu\xi - \pi/4) \sum_{s=0}^{n-1} \frac{(-1)^s U_{2s+1}(p)}{\nu^{2s+1}} \\ &\quad + \frac{1}{2}\{e^{-i\pi/4}\varepsilon_{2n+1,1}(\nu, \xi) + e^{i\pi/4}\varepsilon_{2n+1,2}(\nu, \xi)\}\Bigg], \end{aligned} \tag{5.15}$$

$$G_{i\nu}(\nu z)=\left(\frac{2}{\pi\nu}\right)^{1/2}(1+z^2)^{-1/4}$$

$$\cdot\left[\sin(\nu\xi-\pi/4)\sum_{s=1}^{n}\frac{(-1)^s U_{2s}(p)}{\nu^{2s}}\right.$$

(5.16)

$$-\cos(\nu\xi-\pi/4)\sum_{s=0}^{n-1}\frac{(-1)^s U_{2s+1}(p)}{\nu^{2s+1}}$$

$$\left.+\frac{1}{2i}\{e^{-i\pi/4}\varepsilon_{2n+1,1}(\nu,\xi)-e^{i\pi/4}\varepsilon_{2n+1,2}(\nu,\xi)\}\right].$$

Asymptotic expansions for $J_{i\nu}(\nu z)$ and $Y_{i\nu}(\nu z)$ can also be obtained in a similar manner. Note that the above expansions are uniformly valid in a neighborhood of $z=0$, provided $-\pi/2\leqq \arg z\leqq \pi/2-\delta$ for (5.12), $-\pi/2+\delta\leqq \arg z\leqq \pi/2$ for (5.14), and $|\arg z|\leqq \pi/2-\delta$ for both (5.15) and (5.16).

6. Auxiliary functions. For differential equations of the type (1.1), with (1.2) applying, asymptotic solutions will be obtained involving Bessel functions and modified Bessel functions of purely imaginary order. In order to construct error bounds it is necessary to define auxiliary weight, modulus, and phase functions for these functions, as Olver did for the corresponding problem of Chap. 12 (see § 1.3).

First we define auxiliary functions for $K_{i\nu}(x)$ and $L_{i\nu}(x)$. *Let* $x=\chi_\nu$ be the largest positive root of

(6.1) $$K_{i\nu}(x)-L_{i\nu}(x)=0.$$

Since the LHS of (6.1) is negative as $x\to\infty$, and positive at $x=l_{\nu,1}$ (see (2.16), (2.17), and (2.20)), it follows that

(6.2) $$l_{\nu,1}<\chi_\nu<\infty.$$

We now define a weight function $E_\nu^{(1)}(x)$ by

(6.3a) $$E_\nu^{(1)}(x)=1 \qquad (0\leqq x\leqq \chi_\nu),$$

(6.3b) $$E_\nu^{(1)}(x)=\left\{\frac{L_{i\nu}(x)}{K_{i\nu}(x)}\right\}^{1/2} \qquad (\chi_\nu\leqq x<\infty).$$

From the definition above it is seen that $E_\nu^{(1)}(x)$ is a positive continuous function of x, and moreover is nondecreasing, as can be seen from the equation

(6.4) $$\frac{d\{E_\nu^{(1)}(x)\}^2}{dx}=\frac{\pi}{\sinh(\nu\pi)xK_{i\nu}^2(x)} \qquad (\chi_\nu<x<\infty),$$

which can be derived by differentiating (6.3b) and employing (2.10).

Having defined a weight function we now introduce modulus and phase functions; we define them by the relations

(6.5a) $$K_{i\nu}(x)=-(E_\nu^{(1)}(x))^{-1}M_\nu^{(1)}(x)\sin\theta_\nu^{(1)}(x),$$

(6.5b) $$L_{i\nu}(x)=E_\nu^{(1)}(x)M_\nu^{(1)}(x)\cos\theta_\nu^{(1)}(x),$$

or explicitly

(6.6a) $$M_\nu^{(1)}(x)=\{K_{i\nu}^2(x)+L_{i\nu}^2(x)\}^{1/2},$$
$$\theta_\nu^{(1)}(x)=-\tan^{-1}\{K_{i\nu}(x)/L_{i\nu}(x)\} \qquad (0<x\leqq\chi_\nu),$$

$$M_\nu^{(1)}(x)=\{2K_{i\nu}(x)L_{i\nu}(x)\}^{1/2},$$
(6.6b)
$$\theta_\nu^{(1)}(x)=-\frac{\pi}{4}\qquad(\chi_\nu\leqq x<\infty),$$

the branch of the inverse tangent being chosen so that $\theta_\nu(x)$ is continuous for $0<x<\infty$. On differentiating and using (2.10) we find that

$$\frac{d\theta_\nu^{(1)}(x)}{dx}=\frac{\pi}{\sinh(\nu\pi)x(M_\nu^{(1)}(x))^2}>0\qquad(0<x\leqq\chi_\nu)\tag{6.7}$$

and therefore as x decreases from χ_ν to zero, $\theta_\nu^{(1)}(x)$ decreases monotonically from $-\pi/4$ to $-\infty$. This fact, together with (2.20), (6.2), and (6.5a, b), shows that

$$\theta_\nu^{(1)}(k_{\nu,s})=-s\pi,\qquad\theta_\nu^{(1)}(l_{\nu,s})=-(s-\tfrac{1}{2})\pi.\tag{6.8}$$

For fixed $\nu>0$, the following asymptotic behavior of the auxiliary functions can readily be derived from the definitions above and the results of § 2.

As $x\to0^+$

$$M_\nu^{(1)}(x)\to\left(\frac{\pi}{\nu\sinh(\nu\pi)}\right)^{1/2},\qquad\theta_\nu^{(1)}(x)\sim\nu\ln\left(\frac{x}{2}\right);\tag{6.9}$$

as $x\to\infty$

$$E_\nu^{(1)}(x)\sim\frac{e^x}{(\sinh(\nu\pi))^{1/2}},\qquad M_\nu^{(1)}(x)\sim\left(\frac{\pi}{x\sinh(\nu\pi)}\right)^{1/2}.\tag{6.10}$$

Next, we must introduce auxiliary functions for the derivatives of the modified Bessel functions. We define

$$K'_{i\nu}(x)=(E_\nu^{(1)}(x))^{-1}N_\nu^{(1)}(x)\sin\omega_\nu^{(1)}(x),\tag{6.11}$$

$$L'_{i\nu}(x)=E_\nu^{(1)}(x)N_\nu^{(1)}(x)\cos\omega_\nu^{(1)}(x),\tag{6.12}$$

giving

$$N_\nu^{(1)}(x)=\{K_{i\nu}'^2(x)+L_{i\nu}'^2(x)\}^{1/2}\tag{6.13a}$$

$$\omega_\nu^{(1)}(x)=\tan^{-1}\{K'_{i\nu}(x)/L'_{i\nu}(x)\}\qquad(0<x\leqq\chi_\nu),\tag{6.13b}$$

$$N_\nu^{(1)}(x)=\left(\frac{K_{i\nu}'^2(x)L_{i\nu}^2(x)+L_{i\nu}'^2(x)K_{i\nu}^2(x)}{K_{i\nu}(x)L_{i\nu}(x)}\right)^{1/2}\tag{6.14a}$$

$$\omega_\nu^{(1)}(x)=\tan^{-1}\left(\frac{K'_{i\nu}(x)L_{i\nu}(x)}{L'_{i\nu}(x)K_{i\nu}(x)}\right)\qquad(\chi_\nu\leqq x<\infty).\tag{6.14b}$$

The branches of the inverse tangents are chosen so that $\omega_\nu^{(1)}(x)$ is continuous for $0<x<\infty$ with $\omega_\nu^{(1)}(x)\to-\pi/4$ as $x\to\infty$. From the following equation (which can be deduced from (2.1), (2.10), and (6.11)-(6.13)):

$$\frac{d\omega_\nu^{(1)}(x)}{dx}=\frac{\pi(x^2-\nu^2)}{\sinh(\nu\pi)x^3(N_\nu^{(1)}(x))^2}\qquad(0<x\leqq\chi_\nu),$$

it is seen that $\omega_\nu^{(1)}(x)$ is monotonically decreasing for $0<x<\hat{\chi}_\nu$, where $\hat{\chi}_\nu=\min(\nu,\chi_\nu)$.

The asymptotic behavior of $N_\nu^{(1)}(x)$ is as follows. As $x\to0^+$

$$N_\nu^{(1)}(x)\sim\frac{2}{x}\left(\frac{\nu\pi}{\sinh(\nu\pi)}\right)^{1/2}.\tag{6.15}$$

As $x\to\infty$

$$N_\nu^{(1)}(x)\sim\left(\frac{\pi}{x\sinh(\nu\pi)}\right)^{1/2}.\tag{6.16}$$

Auxiliary functions for the unmodified Bessel functions $F_{i\nu}(x)$ and $G_{i\nu}(x)$ are defined in a similar manner. These functions are oscillatory, of bounded amplitude, throughout the x interval $(0, \infty)$, and as such a weight function does not strictly need to be introduced. However, although the amplitudes of both functions are equal for large ν, this is not the case when ν is small; near the origin the amplitudes of the two functions are quite disparate as $\nu \to 0$ (see (3.15) and (3.16)). Thus, to sharpen subsequent error bounds, we introduce a weight function $E_\nu^{(2)}(x)$ for $G_{i\nu}(x)$ that is continuous in x and decreases monotonically from the value $\coth(\nu\pi/2)$ at $x=0$ to unity at $x=\infty$. Our choice, one of the simplest, is

$$E_\nu^{(2)}(x) = \frac{1+x}{\tanh(\nu\pi/2)+x}. \tag{6.17}$$

We now introduce modulus and phase functions in the usual manner. We define

$$F_{i\nu}(x) = M_\nu^{(2)}(x)\cos\theta_\nu^{(2)}(x), \tag{6.18}$$

$$G_{i\nu}(x) = E_\nu^{(2)}(x)M_\nu^{(2)}(x)\sin\theta_\nu^{(2)}(x), \tag{6.19}$$

so that

$$M_\nu^{(2)}(x) = \left\{F_{i\nu}^2(x) + \left(\frac{G_{i\nu}(x)}{E_\nu^{(2)}(x)}\right)^2\right\}^{1/2}, \tag{6.20}$$

$$\theta_\nu^{(2)}(x) = \tan^{-1}\left\{\frac{G_{i\nu}(x)}{E_\nu^{(2)}(x)F_{i\nu}(x)}\right\}. \tag{6.21}$$

On differentiating (6.21) and employing (3.10) we arrive at the equation

$$E_\nu^{(2)}(x)(M_\nu^{(2)}(x))^2\frac{d\theta_\nu^{(2)}(x)}{dx} = \frac{2}{\pi x} + F_{i\nu}(x)G_{i\nu}(x)\frac{1-\tanh(\nu\pi/2)}{(\tanh(\nu\pi/2)+x)(1+x)}.$$

From (3.15)–(3.18), therefore, it is seen that $d\theta_\nu^{(2)}(x)/dx$ is positive for both sufficiently small and sufficiently large x, for each fixed positive value of ν. $\theta_\nu^{(2)}(x)$ is thus monotonically increasing for large x, and with this fact in mind we define the branch of the inverse tangent in (6.21) so that $\theta_\nu^{(2)}(x)$ is continuous for $0<x<\infty$, and also

$$\theta_\nu^{(2)}(x) = x - \frac{\pi}{4} + o(1) \quad \text{as } x\to\infty. \tag{6.22}$$

The asymptotic behavior of $M_\nu^{(2)}(x)$ is as follows. As $x \to 0^+$

$$M_\nu^{(2)}(x) \to \left(\frac{2\tanh(\nu\pi/2)}{\nu\pi}\right)^{1/2}. \tag{6.23}$$

As $x \to \infty$

$$M_\nu^{(2)}(x) \sim \left(\frac{2}{\pi x}\right)^{1/2}. \tag{6.24}$$

Finally, we define modulus and phase functions for the derivatives of $F_{i\nu}(x)$ and $G_{i\nu}(x)$ by

$$F'_{i\nu}(x) = N_\nu^{(2)}(x)\cos\omega_\nu^{(2)}(x), \tag{6.25}$$

$$G'_{i\nu}(x) = E_\nu^{(2)}(x)N_\nu^{(2)}(x)\sin\omega_\nu^{(2)}(x), \tag{6.26}$$

or explicitly

$$N_\nu^{(2)}(x) = \left\{F_{i\nu}'^2(x) + \left(\frac{G'_{i\nu}(x)}{E_\nu^{(2)}(x)}\right)^2\right\}^{1/2}, \tag{6.27}$$

$$\omega_\nu^{(2)}(x) = \tan^{-1}\left\{\frac{G'_{i\nu}(x)}{E_\nu^{(2)}(x)F'_{i\nu}(x)}\right\}. \tag{6.28}$$

The derivative of (6.28) can be shown to be

$$\begin{aligned} x^2E_\nu^{(2)}(x)(N_\nu^{(2)}(x))^2\frac{d\omega_\nu^{(2)}(x)}{dx} \\ = \frac{2(x^2+\nu^2)}{\pi x} + \frac{x^2F'_{i\nu}(x)G'_{i\nu}(x)(1-\tanh(\nu\pi))}{(\tanh(\nu\pi/2)+x)(1+x)}, \end{aligned} \tag{6.29}$$

and since the product $F'_{i\nu}(x)G'_{i\nu}(x)$ is $O(1/x)$ as $x\to\infty$ it follows that $d\omega_\nu^{(2)}(x)/dx$ is positive for sufficiently large x. The function $\omega_\nu^{(2)}(x)$ is thus monotonically increasing as $x\to\infty$, and therefore we can define the branch of the inverse tangent of (6.28) so that $\omega_\nu^{(2)}(x)$ is continuous for $0<x<\infty$, with the stipulation that

$$\omega_\nu^{(2)}(x) = x+\frac{\pi}{4}+o(1) \quad \text{as } x\to\infty. \tag{6.30}$$

The asymptotic forms of $N_\nu^{(2)}(x)$ are

$$N_\nu^{(2)}(x) \sim \frac{2}{x}\left(\frac{2\nu\tanh(\nu\pi/2)}{\pi}\right)^{1/2} \quad \text{as } x\to 0^+, \tag{6.31}$$

$$N_\nu^{(2)}(x) \sim \left(\frac{2}{\pi x}\right)^{1/2} \quad \text{as } x\to\infty. \tag{6.32}$$

7. Asymptotic expansions for solutions of a differential equation with a large parameter and a simple pole. We now turn our attention to differential equations of the form

$$\frac{d^2W}{d\zeta^2} = \left\{\frac{u^2}{4\zeta} - \frac{\nu^2+1}{4\zeta^2} + \frac{\psi(\zeta)}{\zeta}\right\}W, \tag{7.1}$$

where u and ν are positive parameters, and $\psi(\zeta)$ is analytic in some interval $[0,\beta)$, where β is positive (and possibly infinite). Our task is to construct asymptotic solutions for (7.1) for large u, analogous to those of Olver, who in Chap. 12 considered the complementary problem where the coefficient of ζ^{-2} is greater than or equal to $-\frac{1}{4}$.

We start by considering the comparison equation to (7.1):

$$\frac{d^2W}{d\zeta^2} = \left\{\frac{u^2}{4\zeta} - \frac{\nu^2+1}{4\zeta^2}\right\}W, \tag{7.2}$$

which has exact solutions $\zeta^{1/2}\mathscr{L}_{i\nu}(u\zeta^{1/2})$, where $\mathscr{L}_{i\nu}$ denotes $K_{i\nu}$ or $L_{i\nu}$, or any linear combination of the two. These solutions are in fact the first terms in asymptotic expansions of solutions of (7.1); we seek formal series solutions of the form

$$W = \zeta^{1/2}\mathscr{L}_{i\nu}(u\zeta^{1/2})\sum_{s=0}^{\infty}\frac{C_s(\zeta)}{u^{2s}} + \frac{\zeta}{u}\mathscr{L}'_{i\nu}(u\zeta^{1/2})\sum_{s=0}^{\infty}\frac{D_s(\zeta)}{u^{2s}} \qquad (\zeta>0). \tag{7.3}$$

On substituting the series above into (7.1) and comparing like powers of u we find that the series formally satisfies the equation if both

$$\zeta C''_s(\zeta) + C'_s(\zeta) - \psi(\zeta)C_s(\zeta) + D'_s(\zeta) - \nu^2 D'_{s-1}(\zeta) + \tfrac{1}{2}D_s(\zeta) = 0, \tag{7.4}$$

$$C'_{s+1}(\zeta) + \zeta D''_s(\zeta) + D'_s(\zeta) - \psi(\zeta)D_s(\zeta) = 0. \tag{7.5}$$

These two equations can be integrated to give the following two recursion relations for the coefficients:

$$D_s(\zeta) = -C'_s(\zeta) + \zeta^{-1/2}\int_0^\zeta t^{-1/2}\{\psi(t)C_s(t) - C'_s(t)/2 + \nu^2 D'_{s-1}(t)\}\,dt, \tag{7.6}$$

$$C_{s+1}(\zeta) = -\zeta D_s'(\zeta) + \int \psi(\zeta) D_s(\zeta)\, d\zeta. \tag{7.7}$$

Without loss of generality we set $C_0(\zeta) = 1$. Since, by hypotheses, $\psi(\zeta)$ is analytic, so too are the coefficients C_s, D_s in the ζ interval $[0, \beta)$ (see Chap. 11, Lemma 7.1).

Before we state our theorem on error bounds, let us define certain constants that appear:

$$\lambda_1^{(1)}(\nu) = \sup\{\sigma(\nu) x (M_\nu^{(1)}(x))^2\}, \tag{7.8}$$

$$\lambda_2^{(1)}(\nu) = \sup\{\sigma(\nu) x |K_{i\nu}(x)| E_\nu^{(1)}(x) M_\nu^{(1)}(x)\}, \tag{7.9}$$

$$\lambda_3^{(1)}(\nu) = \sup\{\sigma(\nu) x |L_{i\nu}(x)| (E_\nu^{(1)}(x))^{-1} M_\nu^{(1)}(x)\}, \tag{7.10}$$

where

$$\sigma(\nu) = 2 \sinh(\nu\pi)/\pi,$$

each supremum being taken over the finite x interval $(0, \infty)$. It is readily shown that each supremum exists and is finite for every positive value of ν.

THEOREM 1. *With the conditions stated at the beginning of this section,* (7.1) *has, for each positive value of u and ν and each integer n, the following two solutions, which are repeatedly differentiable in the ζ interval* $(0, \beta)$:

$$\begin{aligned} W_{2n+1,1}(u, \zeta) = \zeta^{1/2} K_{i\nu}(u\zeta^{1/2}) \sum_{s=0}^{n} \frac{C_s(\zeta)}{u^{2s}} + \frac{\zeta}{u} K_{i\nu}'(u\zeta^{1/2}) \\ \cdot \sum_{s=0}^{n-1} \frac{D_s(\zeta)}{u^{2s}} + \varepsilon_{2n+1,1}(u, \zeta), \end{aligned} \tag{7.11}$$

$$\begin{aligned} W_{2n+1,2}(u, \zeta) = \zeta^{1/2} L_{i\nu}(u\zeta^{1/2}) \sum_{s=0}^{n} \frac{C_s(\zeta)}{u^{2s}} + \frac{\zeta}{u} L_{i\nu}'(u\zeta^{1/2}) \\ \cdot \sum_{s=0}^{n-1} \frac{D_s(\zeta)}{u^{2s}} + \varepsilon_{2n+1,2}(u, \zeta), \end{aligned} \tag{7.12}$$

where

$$\begin{aligned} &\frac{|\varepsilon_{2n+1,1}(u, \zeta)|}{\zeta^{1/2} M_\nu^{(1)}(u\zeta^{1/2})}, \frac{|\partial \varepsilon_{2n+1,1}(u, \zeta)/\partial \zeta|}{\{\zeta^{-1/2} M_\nu^{(1)}(u\zeta^{1/2}) + u N_\nu^{(1)}(u\zeta^{1/2})\}/2} \\ &\leqq \lambda_2^{(1)}(\nu) (E_\nu^{(1)}(u\zeta^{1/2}))^{-1} \exp\left\{\frac{\lambda_1^{(1)}(\nu)}{u} \mathcal{V}_{\zeta,\beta}(\zeta^{1/2} D_0(\zeta))\right\} \\ &\cdot \frac{\mathcal{V}_{\zeta,\beta}(\zeta^{1/2} D_n(\zeta))}{u^{2n+1}}, \end{aligned} \tag{7.13}$$

$$\begin{aligned} &\frac{|\varepsilon_{2n+1,2}(u, \zeta)|}{\zeta^{1/2} M_\nu^{(1)}(u\zeta^{1/2})}, \frac{|\partial \varepsilon_{2n+1,2}(u, \zeta)/\partial \zeta|}{\{\zeta^{-1/2} M_\nu^{(1)}(u\zeta^{1/2}) + u N_\nu^{(1)}(u\zeta^{1/2})\}/2} \\ &\leqq \lambda_3^{(1)}(\nu) E_\nu^{(1)}(u\zeta^{1/2}) \exp\left\{\frac{\lambda_1^{(1)}(\nu)}{u} \mathcal{V}_{0,\zeta}(\zeta^{1/2} D_0(\zeta))\right\} \\ &\cdot \frac{\mathcal{V}_{0,\zeta}(\zeta^{1/2} D_n(\zeta))}{u^{2n+1}}. \end{aligned} \tag{7.14}$$

The derivation of these error bounds is similar to that of Theorem 4.1 of Chap. 12, and details will not be included here.

It remains to construct asymptotic solutions for (7.1) in the interval $(\alpha, 0)$, where α is negative (possibly infinite) constant. On replacing ζ by $|\zeta|e^{\pi i}$ in (7.3) it is readily verified that formal series solutions of (7.1) are given by

$$W = |\zeta|^{1/2}\mathscr{C}_{i\nu}(u|\zeta|^{1/2}) \sum_{s=0}^{\infty} \frac{C_s(\zeta)}{u^{2s}} + \frac{|\zeta|}{u} \mathscr{C}'_{i\nu}(u|\zeta|^{1/2}) \sum_{s=0}^{\infty} \frac{D_s(\zeta)}{u^{2s}} \qquad (\zeta<0), \tag{7.15}$$

where $\mathscr{C}_{i\nu}$ denotes $F_{i\nu}$ or $G_{i\nu}$, or any linear combination of the two. The coefficients $C_s(\zeta)$ and $D_s(\zeta)$ here are understood to be the analytic continuations across $\zeta = 0$ of those satisfying (7.4) and (7.5); thus (7.7) still holds, and (7.6) is replaced by

$$D_s(\zeta) = -C'_s(\zeta) + |\zeta|^{-1/2} \int_\zeta^0 |t|^{-1/2}\{\psi(t)C_s(t) - C''_s(t)/2 + \nu^2 D'_{s-1}(t)\}\, dt. \tag{7.16}$$

As before, we introduce three constants that appear in subsequent error bounds. We define

$$\lambda_1^{(2)}(\nu) = \sup\{\pi x E_\nu^{(2)}(x)(M_\nu^{(2)}(x))^2\}, \tag{7.17}$$

$$\lambda_2^{(2)}(\nu) = \sup\{\pi x |F_{i\nu}(x)| E_\nu^{(2)}(x) M_\nu^{(2)}(x)\}, \tag{7.18}$$

$$\lambda_3^{(2)}(\nu) = \sup\{\pi x |G_{i\nu}(x)| M_\nu^{(2)}(x)\}, \tag{7.19}$$

each supremum being taken over the x interval $(0, \infty)$; again, it is readily verified that each supremum exists and is finite for every positive value of ν.

We may now state the following theorem concerning error bounds.

THEOREM 2. *With the conditions stated at the beginning of this section,* (7.1) *has, for each positive value of u and ν and each integer n, the following two solutions, which are repeatedly differentiable in the ζ interval* $(\alpha, 0)$:

$$W_{2n+1,3}(u,\zeta) = |\zeta|^{1/2}F_{i\nu}(u|\zeta|^{1/2}) \sum_{s=0}^{n} \frac{C_s(\zeta)}{u^{2s}} + \frac{|\zeta|}{u} F'_{i\nu}(u|\zeta|^{1/2}) \sum_{s=0}^{n-1} \frac{D_s(\zeta)}{u^{2s}} + \varepsilon_{2n+1,3}(u,\zeta), \tag{7.20}$$

$$W_{2n+1,4}(u,\zeta) = |\zeta|^{1/2}G_{i\nu}(u|\zeta|^{1/2}) \sum_{s=0}^{n} \frac{C_s(\zeta)}{u^{2s}} + \frac{|\zeta|}{u} G'_{i\nu}(u|\zeta|^{1/2}) \sum_{s=0}^{n-1} \frac{D_s(\zeta)}{u^{2s}} + \varepsilon_{2n+1,4}(u,\zeta), \tag{7.21}$$

where

$$\frac{|\varepsilon_{2n+1,3}(u,\zeta)|}{|\zeta|^{1/2}M_\nu^{(2)}(u|\zeta|^{1/2})}, \frac{|\partial\varepsilon_{2n+1,3}(u,\zeta)/\partial\zeta|}{\{|\zeta|^{-1/2}M_\nu^{(2)}(u|\zeta|^{1/2}) + uN_\nu^{(2)}(u|\zeta|^{1/2})\}/2} \le \lambda_2^{(2)}(\nu) \exp\left\{\frac{\lambda_1^{(2)}(\nu)}{u} \mathscr{V}_{\zeta,0}(|\zeta|^{1/2}D_0(\zeta))\right\} \cdot \frac{\mathscr{V}_{\zeta,0}(|\zeta|^{1/2}D_n(\zeta))}{u^{2n+1}}, \tag{7.22}$$

$$\frac{|\varepsilon_{2n+1,4}(u,\zeta)|}{|\zeta|^{1/2}M_\nu^{(2)}(u|\zeta|^{1/2})},\frac{|\partial\varepsilon_{2n+1,4}(u,\zeta)/\partial\zeta|}{\{|\zeta|^{-1/2}M_\nu^{(2)}(u|\zeta|^{1/2})+uN_\nu^{(2)}(u|\zeta|^{1/2})\}/2}$$
$$\leqq \lambda_3^{(2)}(\nu)E_\nu^{(2)}(u|\zeta|^{1/2})\exp\left\{\frac{\lambda_1^{(2)}(\nu)}{u}\mathcal{V}_{\alpha,\zeta}(|\zeta|^{1/2}D_0(\zeta))\right\} \tag{7.23}$$
$$\cdot\frac{\mathcal{V}_{\alpha,\zeta}(|\zeta|^{1/2}D_n(\xi))}{u^{2n+1}}.$$

Final remarks. (i) We have assumed that $\psi(\zeta)$ is infinitely differentiable in (α, β). If we do not require asymptotic expansions for solutions of (7.1), but just a finite number of terms in the approximations, the requirement of analyticity of $\psi(\zeta)$ can be relaxed to that of finite differentiability.

(ii) Since we have derived explicit error bounds on the approximations, it has not been necessary to impose any restrictions on the dependence of $\psi(\zeta)$ on u, other than that it be a continuous function of u; if the dependence of ψ on u adversely affects the asymptotic validity of an approximation it will be reflected in the error bound.

(iii) To facilitate identification of solutions it is desirable that the asymptotic solutions be uniformly valid on the semi-infinite ζ intervals $(-\infty, 0)$ and $(0,\infty)$. For this it is necessary that the variations of $|\zeta|^{1/2}D_s(\zeta)$ $(s=0,1,2,\cdots,n)$ converge at $\zeta=\pm\infty$. Sufficient conditions for this to be true are given in Exercise 4.2 of Chap. 12.

(iv) The error bounds can be used to deduce the asymptotic behavior of the four solutions, both respect to the independent variable ζ and the asymptotic variable u. For instance the solution $W_{2n+1,1}(u,\zeta)$ is seen to be recessive as $\zeta\to\beta$, a property that uniquely characterizes the solution if $\beta=\infty$. Likewise, the solutions $W_{2n+1,2}(u,\zeta)$ and $W_{2n+1,3}(u,\zeta)$ can be identified by their behavior as $\zeta\to 0$ (with the aid of (2.15) and (3.15)), and $W_{2n+1,4}(u,\zeta)$ can be identified by its behavior as $\zeta\to\alpha$.

Finally, consider the asymptotic behavior of the four solutions as $u\to\infty$. If the variations in the error bounds are bounded functions of u then, in the manner of § 5.2 of Chap. 12, it can be shown that the RHS of (7.3) provides a uniform compound expansion of $W_{2n+1,1}(u,\zeta)$ and $W_{2n+1,2}(u,\zeta)$, for $\mathcal{L}=K$ and L, respectively, to $2n+1$ terms. A similar argument holds for (7.15). The existence of solutions that are independent of n and have the infinite series (7.3) or (7.15) as compound asymptotic expansions may be established by the method of § 6 of Chap. 10.

Appendix. We investigate how large ν should be to ensure that (3.23) holds. First, we observe that $\varepsilon_{1,1}(\nu, 0)$ is bounded by

$$|\varepsilon_{1,1}(\nu,0)|\leqq 2\exp\left\{\frac{2\mathcal{V}_\Lambda(U_1)}{\nu}\right\}\frac{\mathcal{V}_\Lambda(U_1)}{\nu}, \tag{A.1}$$

where

$$U_1(p)=(3p-5p^3)/24, \tag{A.2}$$

and p is given by (5.10). The bound (A.1) corresponds to (7.14) of Chap. 10, the only difference being that the reference point for $\varepsilon_{1,1}(\nu,\xi)$ is $\xi=i\infty$, and that the path of integration Λ must be an $(i\nu\xi)$-progressive path linking $\xi=i\infty$ to $\xi=0$, or correspondingly $p=0$ to $p=(1+\tau^2)^{-1/2}$. (For a definition of a progressive path, see p. 222 of Chap. 6.)

Our choice of Λ is as follows. For large positive R let Γ_R be the path linking $\xi=iR$ to $\xi=0$, consisting of the union of a circular arc from $\xi=iR$ to $\xi=R$ with a real segment from $\xi=R$ to $\xi=0$. It is seen that Γ_R is an $(i\nu\xi)$-progressive path. The

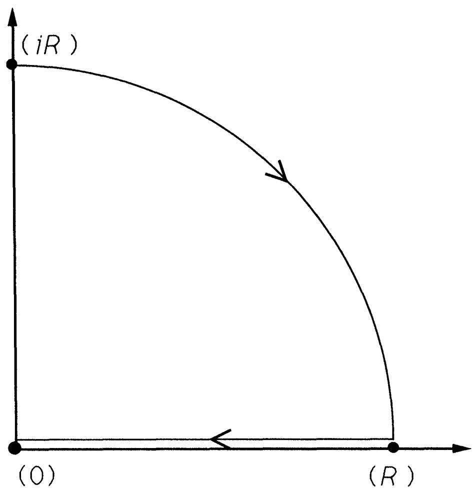

FIG. 2a. *Path Γ_R in ξ plane.*

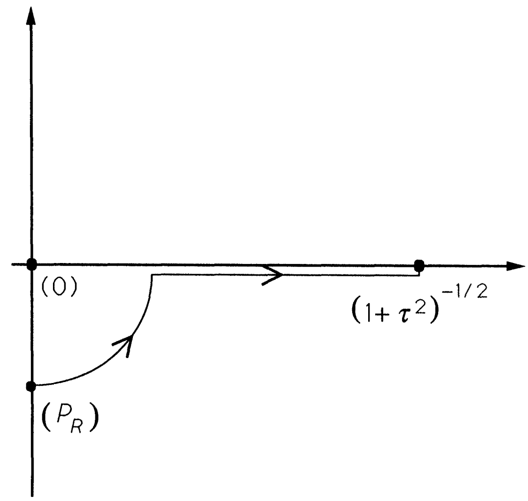

FIG. 2b. *Path γ_R in p plane.*

corresponding path in the p plane, γ_R, links $p=p_R$ to $p=(1+\tau^2)^{-1/2}$, where $p_R=-i/R+O(iR^{-2})$; see Fig. 2a, b. We take our variation path Λ to be the limit of Γ_R as $R\to\infty$. In the p plane the ξ-path Λ corresponds to the real segment $0\leqq p\leqq(1+\tau^2)^{-1/2}$; note that we can neglect the contribution to the variation from the vanishingly small arc near $p=0$.

Thus with our choice of Λ

$$\mathcal{V}_\Lambda(U_1)=\int_0^{(1+\tau^2)^{-1/2}}\frac{|1-5p^2|}{8}\,dp=\frac{1}{6\sqrt{5}}+\frac{2-3\tau}{24(1+\tau^2)^{3/2}}=0.0091\cdots. \tag{A.3}$$

Now, since the RHS of (A.1) is a monotonically decreasing function of ν, it follows that it is bounded above by $1/\sqrt{2}$ if

$$2\mathcal{V}_\Lambda(U_1)/\nu<\nu_0, \tag{A.4}$$

where ν_0 is the root of the equation

$$\nu_0\, e^{\nu_0}=1/\sqrt{2}\qquad(\nu_0=0.4506\cdots). \tag{A.5}$$

By symmetry $|\varepsilon_{1,2}(\nu,0)|$ is also bounded above by $1/\sqrt{2}$ if (A.4) holds, and so, in conclusion, we have shown that a sufficient condition for (3.23) to hold is for

$$\nu>2\mathcal{V}_\Lambda(U_1)/\nu_0=0.4039\cdots. \tag{A.6}$$

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