

AN INTERPRETATION OF CLASSICAL YANG-MILLS THEORY

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We present a new formulation, in twistor space, of the classical second-order Yang-Mills field equations.

Recently Ward [1], following earlier constructions by Penrose [2] and Sparling [3], has shown that the self-dual Yang-Mills equations [4] have a simple interpretation when transformed into an auxiliary space, twistor space, of three complex dimensions. The construction is valid in Minkowski or euclidean space; it has yielded striking results in euclidean space [5].

It is natural to wonder whether an analogous construction exists for the second order Yang-Mills equations, $D_\mu F_{\mu\nu} = 0$. In this letter, such a construction will be proposed. It is applicable for any gauge group and in Minkowski or euclidean space; it is likely to be of interest mainly in Minkowski space. Similar ideas have been developed independently by P. Yasskin, J. Isenberg and P. Green, and will be described by them in a separate publication.

The basic difference between the self-dual equation $F_{\mu\nu} = \tilde{F}_{\mu\nu}$ and the second-order field equation $D_\mu F_{\mu\nu} = 0$ is that the former is an algebraic equation and the latter a differential equation for the field strength tensor. Ward [1], and also Yang [6], and Zakharov and Belavin [7], were able to interpret the self-dual equation as an integrability condition, stating the vanishing of the field strength on certain two-dimensional planes. To find a construction analogous to Ward's, one must be able to interpret Yang-Mills theory as an integrability condition. Since the second-order equation in Minkowski space has no such interpretation, we will imbed Minkowski space in a larger space in which such an interpretation will be possible. The second-order equation in Minkowski space will be equivalent to cer-

tain first-order equations in the larger space.

We will give two formalisms of this type, one for ordinary Yang-Mills theory and one for supersymmetric Yang-Mills theory. The first formalism can be derived from the second.

For constructions of the Penrose-Ward type, we must regard the space-time coordinates x^0, \dots, x^3 as complex variables. Physical Minkowski space consists of the points x^0 imaginary, x^1, x^2 , and x^3 real. All considerations will be local; whenever we refer to a space, we mean really a suitable open subset of that space.

Now, consider an eight-dimensional space with coordinates y^μ and z^ν , $\mu, \nu = 1, \dots, 4$. On this space, consider a gauge field that satisfies the first order equations

$$\begin{aligned} [D/Dy^\mu, D/Dy^\nu] &= \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} [D/Dy^\alpha, D/Dy^\beta], \\ [D/Dz^\mu, D/Dz^\nu] &= -\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} [D/Dz^\alpha, D/Dz^\beta], \\ [D/Dy^\mu, D/Dz^\nu] &= 0. \end{aligned} \quad (1)$$

The first two equations say that the field is self-dual as a function of the y^μ and anti-self-dual as a function of the z^ν .

In this eight-dimensional world, we interpret the diagonal subspace $y^\mu = z^\mu$ as physical Minkowski space. The first main claim is that the desired second-order equations on the physical subspace are consequences of eqs. (1). In fact, letting $x^\mu = \frac{1}{2}(y^\mu + z^\mu)$, $w^\mu = \frac{1}{2}(y^\mu - z^\mu)$, the second-order equation on the diagonal can be written as a double commutator,

$$[D/Dx^\mu, [D/Dx^\mu, D/Dx^\nu]] = 0. \quad (2)$$

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Since $D/Dx^\mu = D/Dy^\mu + D/Dz^\mu$, eq. (2) can be written

$$\left[\frac{D}{Dy^\mu} + \frac{D}{Dz^\mu}, \left[\frac{D}{Dy^\mu} + \frac{D}{Dz^\mu}, \frac{D}{Dy^\nu} + \frac{D}{Dz^\nu} \right] \right] = 0. \quad (3)$$

Eq. (3) is a sum of eight terms. By using eq. (1) together with Jacobi identities, one can see that each of these is separately zero.

Now we may ask the converse. Given a gauge field $A_\mu(x)$ defined at $w = 0$, under what conditions and to what extent can it be defined for nonzero w consistently with eq. (1)? To answer this, we expand A_μ in powers of w around $w = 0$ and attempt to satisfy eq. (1) order by order. Whether or not $A_\mu(x)$ satisfies the Yang–Mills field equations at $w = 0$, it can be defined up to and including terms of order w consistently with eq. (1). But the term quadratic in w can be defined in accord with eq. (1) if and only if the field equations are satisfied at $w = 0$. Specifically, the appropriate formula is

$$A_\mu^w = -\frac{1}{2} F_{\mu\nu} w_\nu - \frac{1}{3} (w^\alpha D/Dx^\alpha) \tilde{F}_{\mu\nu} w_\nu, \quad (4)$$

$$A_\mu^x = A_\mu(w = 0) - \tilde{F}_{\mu\nu} w_\nu - \frac{1}{2} (w^\alpha D/Dx^\alpha) F_{\mu\nu} w_\nu,$$

where $F_{\mu\nu}$ and D/Dx^α are evaluated at $w = 0$. One can check that eq. (4) satisfies eq. (1) to this order if and only if $D_\mu F_{\mu\nu} = 0$ at $w = 0$, while otherwise there is no satisfactory way to modify eq. (4).

Even if $D_\mu F_{\mu\nu} = 0$ at $w = 0$, it is, in the non-abelian case, generally not possible to define A_μ to cubic order in w consistently with eq. (1). The field equations merely correspond to the ability to define the quadratic term in w in a way that agrees with eq. (1).

All of this has a simple interpretation in the language of twistor space and complex geometry, as follows. Consider, in complex Minkowski space M , the space Q of all lightlike lines. Q has complex dimension five; it can be regarded as the set of all (U^α, V_β) in $CP^3 \times CP^3$ with $U^\alpha V_\alpha = 0$.

By analogy with Ward's construction, an arbitrary gauge field in M , not necessarily satisfying any equation, is equivalent to a vector bundle on Q that is trivial when restricted to any subspace $CP^1 \times CP^1$ in Q corresponding to the lines through a fixed point in $M^{\neq 1}$.

^{†1} Here and elsewhere, mathematically minded readers must remember that all constructions are local and, for instance, "every" $CP^1 \times CP^1$ means every one in a suitable open set.

A basic ingredient in this equivalence is the fact that any gauge field is integrable when restricted to a single line.

How can we interpret solutions of eq. (1)? Eq. (1) is equivalent to the integrability of the gauge field on certain four-dimensional surfaces in (y, z) -space, namely the cartesian product of a left-handed null two plane in y with a right-handed null two plane in z . The space of all such surfaces is $CP^3 \times CP^3$, and the space of all such surfaces through a fixed point (y, z) is a subspace $CP^1 \times CP^1$ of $CP^3 \times CP^3$ (with one CP^1 in each CP^3). By exact analogy with Ward, a solution of eq. (1) is equivalent to a vector bundle on $CP^3 \times CP^3$ that is trivial on each $CP^1 \times CP^1$.

Thinking now of M as the diagonal in (y, z) space and combining these observations, we learn that a gauge field on M can be extended to a solution of eq. (1) if and only if the corresponding vector bundle on Q can be extended to a bundle on $CP^3 \times CP^3$.

However, the Yang–Mills equations correspond not to a genuine extension from M into (y, z) space but merely to the ability to define the quadratic term in such an extension. But under Ward's construction, the expansion in powers of w from M into (y, z) space corresponds to a power series expansion from Q into $CP^3 \times CP^3$. And the quadratic term in eq. (4) corresponds to the cubic term in an expansion from Q into $CP^3 \times CP^3$. Therefore, we conclude: a solution of the Yang–Mills field equations is equivalent to a vector bundle on Q which is trivial on each $CP^1 \times CP^1$ and can be extended up to cubic order in a formal power series expansion to a bundle on $CP^3 \times CP^3$.

There is another way to describe this result. It involves a consideration of supersymmetric Yang–Mills theory [8] in superspace.

Superspace [9] is the enlargement of Minkowski space with anticommuting coordinates, θ^{ai} and $\tilde{\theta}_i^{\alpha'}$, $i = 1, \dots, N$, of Lorentz type $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, as well as the ordinary coordinates x^μ . (We will temporarily think in terms of real superspace, so x^μ is real and θ the complex conjugate of $\tilde{\theta}$.)

Now we ask: what object in superspace corresponds to the lightlike lines of Minkowski space [10]?

In the two-component notation [11], a lightlike vector in Minkowski space is of the form $v^{\alpha\alpha'} = \lambda^\alpha \lambda'^{\alpha'}$, λ^α being an arbitrary pair of complex numbers. Translations in the direction $v^{\alpha\alpha'}$ are generated by $D = i\lambda^\alpha \lambda'^{\alpha'} \times \partial/\partial x^{\alpha\alpha'}$. But in superspace, of course, there are square

roots of D . In fact, letting $Q_\alpha = \partial/\partial\theta^\alpha + i\tilde{\theta}^{\alpha'} \partial/\partial x^{\alpha\alpha'}$, $\tilde{Q}_{\alpha'} = \partial/\partial\tilde{\theta}^{\alpha'} + i\theta^\alpha \partial/\partial x^{\alpha\alpha'}$ (we are suppressing the internal symmetry index carried by θ), we see that $T = 1/\sqrt{2} (\lambda^\alpha Q_\alpha + \lambda^{*\alpha'} \tilde{Q}_{\alpha'})$ is the square root of D : $T^2 = D$.

Just as we regard D as generating ordinary translations in the lightlike direction $\lambda^\alpha \lambda^{*\alpha'}$, we may regard T , the square root of D , as generating fermionic translations in this direction.

Just as in Minkowski space, a lightlike line is the orbit of a point under the action of D , in superspace we may define a lightlike line as the orbit of a point under the joint action of D and T .

In other words, in Minkowski space a lightlike line is the set of all $x^{\alpha\alpha'} = c^{\alpha\alpha'} + t\lambda^\alpha \lambda^{*\alpha'}$ with c and λ fixed but t arbitrary. We define a lightlike line in superspace as the set of all $x^{\alpha\alpha'} = c^{\alpha\alpha'} + t\lambda^\alpha \lambda^{*\alpha'}$, $\theta^\alpha = \eta^\alpha + \epsilon\lambda^\alpha$, $\tilde{\theta}^{\alpha'} = \eta^{*\alpha'} + \epsilon^*\lambda^{*\alpha'}$, with c , λ , and η fixed but t and ϵ arbitrary. (η and ϵ are anticommuting.) For extended supersymmetry, $N > 1$, we introduce arbitrary ϵ^i , $i = 1, \dots, N$, and write $\theta^{\alpha i} = \eta^{\alpha i} + \epsilon^i \lambda^\alpha$, $\tilde{\theta}_i^{\alpha'} = \eta_i^{*\alpha'} + \epsilon_i^* \lambda^{*\alpha'}$, where $\eta^{\alpha i}$ are constants.

Additional motivations for this viewpoint will be described later.

These "lightlike lines" are not truly one-dimensional. They have one ordinary dimension plus N fermionic dimensions.

In Minkowski space, because lightlike lines are one-dimensional, integrability on lines is a trivial condition. We used this in saying that an arbitrary Minkowski space gauge field, not necessarily satisfying any equations, is equivalent to a vector bundle on Q , the space of lines.

In the supersymmetric case, integrability on lines is not a trivial condition, because the lines are not truly one-dimensional.

As N increases, the lightlike lines have more and more anticommuting dimensions, and integrability on them becomes a more and more stringent condition.

Let us recall that supersymmetric Yang–Mills theory exists for $N = 1, 2, 3$, and 4 . The $N = 3$ and $N = 4$ theories are actually equivalent.

For $N = 1$ and $N = 2$, integrability on lines eliminates some degrees of freedom but does not imply real equations of motion.

For $N = 3$ integrability on lines turns out to be precisely equivalent to the equations of motion of the supersymmetric gauge theory. For $N = 4$ there is a simi-

lar statement, with some qualifications. We will mainly discuss the case $N = 3$.

We will explain these statements more fully below; let us first describe their consequences.

For a construction of the Penrose–Ward type, one must regard the space–time coordinates as complex variables, so henceforth the x^μ are complex and the $(0, \frac{1}{2})$ spinors $\tilde{\theta}$ are no longer complex conjugates of the $(\frac{1}{2}, 0)$ spinors θ . We will, following Ferber [10], describe the supersymmetric space of (complex) lightlike lines.

We recall that twistor space is \mathbb{CP}^3 , the projective space of four complex variables Z^α . The supersymmetric analogue is $\mathbb{CP}^{3,N}$, the projective space of four complex variables Z^α and N anticommuting variables ζ^k . Just as the conformal algebra $SU(2, 2)$ acts naturally on \mathbb{CP}^3 , so the superconformal algebra $SU(2, 2|N)$ acts naturally on $\mathbb{CP}^{3,N}$.

The space of all lightlike lines in Minkowski space is, as we have said, the space Q of all (Z^α, W_β) in $\mathbb{CP}^3 \times \mathbb{CP}^3$ such that $Z^\alpha W_\alpha = 0$. Likewise, using Ferber's formulas, the space of all supersymmetric lightlike lines is the space $\bar{Q}(N)$ of all $(Z^\alpha, \zeta^k; W_\beta, \psi_l)$ in $\mathbb{CP}^{3,N} \times \mathbb{CP}^{3,N}$ with $Z^\alpha W_\alpha - \zeta^k \psi_k = 0$. Let us emphasize that, using Ferber's formulas, one can see that points in \bar{Q} do correspond to supersymmetric lightlike lines as we have defined them before. The space of all lines through a fixed superspace point is a subspace of $\bar{Q}(N)$ of type $\mathbb{CP}^1 \times \mathbb{CP}^1$; in this respect supersymmetry introduces no modification.

In ordinary Minkowski space, integrability on lines is a trivial condition. Hence vector bundles on Q , trivial on each $\mathbb{CP}^1 \times \mathbb{CP}^1$, correspond to gauge fields on Minkowski space not necessarily satisfying any equation. But in the supersymmetric case, for $N = 3$, the equations of motion are simply the statement of integrability on lines, and are therefore precisely the condition that enables one to introduce a vector bundle on the space of lines. So we can conclude: solutions of the supersymmetric equations for $N = 3$ are equivalent to vector bundles on $\bar{Q}(3)$ trivial on each $\mathbb{CP}^1 \times \mathbb{CP}^{1 \neq 1}$.

Now let us make contact with the first half of this paper. Solutions of the supersymmetric Yang–Mills equations include, as special cases, solutions of the ordinary Yang–Mills equations, with the scalar and spinor fields set equal to zero. For such solutions, we will see that the supersymmetric construction reduces to the previous one.

Solutions with the scalar and spinor fields equal to zero correspond to vector bundles in which the transition functions can be chosen as functions only of Z^α and W_β , not of ξ^k and ψ_I . These transition functions are not defined for arbitrary Z and W . Rather, because of the defining relation $Z^\alpha W_\alpha = \xi^k \psi_k$ of \bar{Q} , and the fact that for $N = 3$, $(\xi^k \psi_k)^4 = 0$, they are defined only subject to a condition $(Z^\alpha W_\alpha)^4 = 0$. So a vector bundle on $\bar{Q}(3)$ with transition functions independent of ξ^k and ψ_I is equivalent to a bundle on the third infinitesimal neighborhood of Q in $\mathbb{CP}^3 \times \mathbb{CP}^3$. But we have already seen in the first half of the paper that such bundles are equivalent to Yang–Mills solutions.

It remains now to show that for $N = 3$, integrability on lines constitutes the equations of motion.

Integrability on a line means that the translation operators along that line satisfy an algebra unmodified by the gauge field. For a line in the direction $\lambda^\alpha \tilde{\lambda}^{\alpha'}$, these operators are $D = \lambda^\alpha \tilde{\lambda}^{\alpha'} iD/DX^{\alpha\alpha'}$, $T_i = \lambda^\alpha Q_{\alpha i}$, $\tilde{T}^j = \tilde{\lambda}^{\alpha'} \tilde{Q}_{\alpha'}^j$. Integrability means that

$$\{T_i, T_j\} = \{\tilde{T}^i, \tilde{T}^j\} = 0, \quad \{T_i, \tilde{T}^j\} = 2\delta_i^j D, \quad (5)$$

exactly as in the absence of a gauge field. Eq. (5) is true for arbitrary λ and $\tilde{\lambda}$ if and only if

$$\begin{aligned} \{Q_{\alpha i}, Q_{\beta j}\} + \{Q_{\beta i}, Q_{\alpha j}\} &= 0, \\ \{\tilde{Q}_{\alpha'}^i, \tilde{Q}_{\beta'}^j\} + \{\tilde{Q}_{\beta'}^i, \tilde{Q}_{\alpha'}^j\} &= 0, \\ \{Q_{\alpha i}, Q^{\alpha' j}\} &= 2\delta_i^j D/DX^{\alpha\alpha'}. \end{aligned} \quad (6)$$

For $N = 1$ and $N = 2$ one may recognize eqs. (6) as the constraint equations of these theories [8]. But for $N = 3$, eqs. (6) are equivalent to the supersymmetric equations of motion [12].

We will only illustrate the computations with an example, in the abelian case for simplicity. Let $\phi_{ij} = \epsilon^{\alpha\beta} \{Q_{\alpha i}, Q_{\beta j}\}$. Eqs. (6) plus Bianchi identities imply that $Q_{\alpha k} \phi_{ij} = \epsilon_{ijk} \lambda_\alpha$ for some spinor λ_α . We claim that by virtue of eqs. (6), λ_α satisfies the Dirac equation. In fact,

$$\begin{aligned} D_{\alpha\alpha'} \lambda_\beta &= Q_{\alpha 1} \tilde{Q}_{\alpha'}^1 \lambda_\beta = Q_{\alpha 1} \tilde{Q}_{\alpha'}^1 Q_{\beta 2} \phi_{31} \\ &= -Q_{\alpha 1} Q_{\beta 2} \tilde{Q}_{\alpha'}^1 \phi_{31} = -Q_{\alpha 1} Q_{\beta 2} \tilde{Q}_{\alpha'}^2 \phi_{32} \\ &= -Q_{\beta 2} \tilde{Q}_{\alpha'}^2 Q_{\alpha 1} \phi_{32} = Q_{\beta 2} \tilde{Q}_{\alpha'}^2 \lambda_\alpha = D_{\beta\alpha'} \lambda_\alpha. \end{aligned}$$

Each step in this calculation follows from eqs. (6) plus Bianchi identities. But in a two-component language, the Dirac equation is $D_{\alpha\alpha'} \lambda_\beta - D_{\beta\alpha'} \lambda_\alpha = 0$, so we have shown that λ satisfies the Dirac equation.

This is one of the equations of motion; a detailed analysis shows that they all follow from eqs. (6) and that eqs. (6) imply no additional conditions.

Many questions remain. To what extent can this construction yield explicit solutions? Does an analogue exist for Einstein's equations? Is there a similar supersymmetric construction in the case $N = 4$?^{‡2} If it exists, it is probably more powerful than the construction we have given for $N = 3$. And can the *quantum* gauge field theory be transformed into twistor space?

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^{‡2} A word of explanation should be added here concerning the situation for $N = 4$. For $N = 4$, integrability on lines is equivalent to a certain supersymmetric system of equations, and, by the reasoning in the text, solutions of those equations are equivalent to bundles on $\bar{Q}(4)$. The equations obtained in this way, however, are not the desired equations of Gliozzi, Scherk, and Olive and of Brink, Schwarz, and Scherk but an enlarged system with various unphysical properties. To obtain the desired equations, one must impose in the notation of the text, an extra condition $\phi_{ij} = \epsilon_{ijkl} \phi^{kl} \mp 1$. It is this condition whose translation into twistor space is not clear. For $N = 3$, instead, no such extra condition is needed.

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