Because this surface divergence would leave the momentum budget unbalanced, it is necessary, in order to satisfy the principle of conservation of momentum, to assume that *surface forces* of the magnitude $S_{\nu x}$, $S_{\nu y}$, $S_{\nu z}$ are present at the surface of the superconductor; there these forces act on the metal and transform the excess momentum of the (**h**, **j**)-field into mechanical momentum. This momentum may then be propagated by an elastic reaction of the material and be translated into strains, or it may serve to move the whole body to another place. While the magnetic field continuously penetrates the superconductor and while the currents are also continuously distributed over its interior, the seat of the mechanical forces is located strictly at its surface. It is the kinetic tensor $\Lambda(j_{s_i}j_{s_k} - \frac{1}{2}\mathbf{j}_s^2\delta_{ik})$ which assures the transmission of the volume forces to the surface of the superconductor, where they are transformed into surface forces. This result is quite plausible if we recall that no interaction has been assumed between the supercurrent and the lattice except the kind of coupling which is realized at the surface of the superconductor and which is due to the fact that both superelectrons and lattice extend over the same volume.

It is noteworthy that the forces of inertia, $-\Lambda[\mathbf{j}_s \times \mathbf{curl}\,\mathbf{j}_s]$, are capable of balancing the Lorentz force, $(1/c)[\mathbf{j}_s \times \mathbf{h}]$. This possibility was not recognized at first [2] and it seemed necessary to assume that the Lorentz force is balanced by an electrostatic field, that is, by a kind of Hall effect. On the other hand, it seemed difficult to accept the possibility of electrostatic fields in a superconductor; actually they have never been found. We now can see why a Hall effect is neither necessary nor possible in superconductors.

§ 10. THE SUPERPOTENTIAL. GAUGE INVARIANCE

(a) In eq. (23) of § 8 we introduced the momentum field $\mathbf{p}_s$ of the supercurrent. In non-relativistic approximation it is defined by

$$\begin{aligned} \mathbf{p}_s &= m\mathbf{v}_s + \frac{e}{c}\mathbf{A} \\ &= \frac{e}{c}(\Lambda c\mathbf{j}_s + \mathbf{A}) \end{aligned} \tag{1}$$

where **A** is the vector potential which, together with the scalar potential ϕ, describes the electromagnetic field by

$$\begin{aligned} \mathbf{h} &= \mathbf{curl}\,\mathbf{A} \\ \mathbf{e} &= -\mathbf{grad}\,\phi - \frac{\dot{\mathbf{A}}}{c} \end{aligned} \tag{2}$$

[2] H. A. Lorentz, *Leiden Comm.*, Suppl. No. 50 (1924).

Equation (I), § 3, simply says

$$\mathbf{curl}\,\mathbf{p}_s = 0 \tag{I}$$

and eq. (II):

$$\frac{\partial \mathbf{p}_s}{\partial t} = -e\,\mathbf{grad}\,\phi \tag{II}$$

From these equations the conservation of the fluxoids through any hole is very simply derived since

$$\frac{e}{c}\dot{\Phi} = \frac{d}{dt}\oint \mathbf{p}_s\cdot d\mathbf{s} = -e\oint \mathbf{grad}\,\phi\cdot d\mathbf{s} = 0$$

because ϕ is a single-valued function.

Because of (I) we may express $\mathbf{p}_s$ as the gradient of a scalar, the "superpotential" χ (compare eq. 24 of § 8):

$$\mathbf{p}_s = \mathbf{grad}\,\chi \tag{3}$$

The function χ is defined only within the superconductor and, hence, *need not be single-valued* in multiply connected superconductors. Consider a multiply connected superconductor containing the fluxoids $\Phi_1, \Phi_2, \cdots \Phi_k, \cdots$:

$$\frac{e}{c}\Phi_k = \oint_{(k)} \mathbf{p}_s\cdot d\mathbf{s} = \oint_{(k)} \mathbf{grad}\,\chi\cdot d\mathbf{s} = <\chi>_k \tag{4}$$

Here $<\chi>_k$ denotes the *modulus* of χ with respect to the hole k, that is, the increase of χ if we follow it continuously on a path embracing the hole k just once. Thus χ cannot generally be a single-valued function. It behaves like a helical staircase. Equation (4) states that the moduli of χ are identical with the fluxoids multiplied by e/c.

The vector potential is not uniquely defined by (2). We may still prescribe the sources of **A** as we like. For quasistationary conditions, to which we will confine ourselves in this section, it is customary to impose on **A** the supplementary condition

$$\mathbf{div}\,\mathbf{A} = 0 \tag{5}$$

Because of the equation of continuity and (VI′, § 3) this means, for $\mathbf{p}_s$,

$$\mathbf{div}\,\mathbf{p}_s = 0$$

(b) Even with this supplement (5), the vector potential is not uniquely defined. It can be replaced by an "equivalent" **A′** connected with **A** by a so-called "gauge transformation":

$$\mathbf{A}' = \mathbf{A} + \mathbf{grad}\,k \tag{6}$$

where k is an arbitrary continuous function of the space coordinates with continuous derivatives. If $\mathbf{A}'$ is to have a vanishing divergence as well as $\mathbf{A}$ (eq. 5), then k has to fulfill the equation

$$\nabla^2 k = 0 \tag{7}$$

Evidently $\mathbf{A}$ and $\mathbf{A}'$ define the same field (2). Hence we shall call them "equivalent." The function k, in contrast to the superpotential χ, is defined in the whole space, not only in the superconductor. This means that k, in contrast to χ, has to be *single-valued* in order to be regular everywhere.

From (6) it follows that $\mathbf{p}_s$ is not uniquely determined by the definition (1). However, the difference $\mathbf{p}_s - (e/c)\mathbf{A}$ has a well-defined meaning. Hence, if $\mathbf{A}$ is replaced by an equivalent $\mathbf{A}'$ according to (6), then $\mathbf{p}_s$ has to be replaced at the same time by

$$\mathbf{p}_s' = \mathbf{p}_s + \frac{e}{c}\,\mathbf{grad}\,k \tag{8}$$

χ is not completely determined either. It is subject to the transformation

$$\chi' = \chi + \frac{e}{c}k \tag{9}$$

whenever $\mathbf{A}$ is transformed by the gauge transformation (6). Since k is always single-valued, it follows that the moduli of χ are left unchanged by the transformation (9); they are "gauge invariants."

(c) It is possible and convenient to choose a *standard value* for the vector potential $\mathbf{A}$ and therewith for the momentum $\mathbf{p}_s$ within the superconductor. We may, for instance, choose a value of $\mathbf{A}$ such that at the surface of the superconductor its normal component disappears:

$$A_\nu = 0 \tag{10}$$

Indeed, if we originally had another vector potential $\mathbf{A}'$ which did not fulfill (10), it must be connected with $\mathbf{A}$ by a transformation of the type (6):

$$\mathbf{A}' = \mathbf{A} + \mathbf{grad}\,k$$

where k, according to (7), fulfills the equation

$$\nabla^2 k = 0$$

and, according to (10), the boundary condition at the surface:

$$\frac{\partial k}{\partial \nu} = A_\nu' \tag{10'}$$

It follows from well-known theorems of potential theory that, except for a still arbitrary, meaningless, additive constant, there is just *one* single-valued solution k within the superconductor which satisfies (7) and (10′). Accordingly for a given magnetic field $\mathbf{h}$ there is now *just one standard potential* $\mathbf{A}$, which, besides, is so defined as to disappear in the absence of any field.

Correspondingly the standard momentum field of the superelectrons, $\mathbf{p}_s$, is now uniquely determined. It fulfills the following equations:

$$\mathbf{curl}\,\mathbf{p}_s = 0$$

$$\frac{\partial \mathbf{p}_s}{\partial t} = -e\,\mathbf{grad}\,\phi$$

$$\mathbf{div}\,\mathbf{p}_s = 0$$

and at its boundary:

$$p_{s_\nu} = e\Lambda j_\nu \tag{11}$$

For a *simply connected superconductor* these equations have a single solution, if j_ν is given on the entire surface. Especially for an *isolated* superconductor, where $j_\nu = 0$, this solution is well known, from irrotational hydrodynamics, to be

$$\mathbf{p}_s = 0$$

Hence the standard vector potential is defined in such a way that for an isolated simply connected superconductor we have

$$c\Lambda\mathbf{j}_s = -\mathbf{A} \tag{12}$$

Of course, this relation cannot be generally valid, as it is not gauge-invariant.

§ 11. UNIQUENESS THEOREMS

The uniqueness properties of the theory of superconductivity touch upon a very essential point. The "perfect-conductor" theory implied an infinity of states belonging to the same external conditions, frozen-in non-equilibria, which offended the good taste of most physicists. Only after Meissner's discovery did it become clear that in actual fact not as many states are possible within a superconductor as would be expected for a perfect conductor. Our electrodynamics was constructed in such a way as to take cognizance of this situation. Indeed, we have seen that this theory, in principle, excludes a whole ensemble of electronic motions which would be possible in a perfect conductor. However, we have not yet rigorously shown that this reduction has actually led us