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LETTERS AND COMMENTS

Comment on 'Elementary analysis of the special relativistic combination of velocities, Wigner rotation and Thomas precession'

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Abstract
Significant difficulties in understanding the meaning of calculations presented by O'Donnell and Visser (2011 *Eur. J. Phys.* **32** 1033), as well as an incorrect interpretation of the formula for the Thomas precession, are pointed out. A straightforward method to attribute the formal calculations performed by the authors to the relativistic effect of Wigner or Thomas rotation is proposed.

(Some figures may appear in colour only in the online journal)

1. Introduction

Recently O'Donnell and Visser [1] presented an analysis of the special relativity relativistic combination of velocities that results in the Wigner rotation or Thomas precession. To demonstrate the effects the authors considered two different situations in which Alice and Bob, each travelling in a spaceship, and a mission control placed on Earth took part.

(a) Alice's spacecraft has a velocity $\vec{v}_1$ with respect to a mission control frame and Bob's spacecraft moves with a velocity $\vec{v}_2$ with respect to Alice's spacecraft. Then Bob's spacecraft's velocity as seen in the mission control frame is $\vec{v}_{21}$.

(b) Alice's spacecraft has a velocity $\vec{v}_2$ with respect to a mission control frame and Bob's spacecraft moves with a velocity $\vec{v}_1$ with respect to Alice's spacecraft. In this case Bob's spacecraft's velocity measured in the mission control frame is $\vec{v}_{12}$.

While in Galilean spacetime $\vec{v}_{21} = \vec{v}_{12}$, this is not the case in special relativity. Although the magnitudes of $\vec{v}_{21}$ and $\vec{v}_{12}$ are the same, the two velocities do not point in the same direction—there is an angle $\vec{\Omega}$ between them that is identified in [1] with the Wigner rotation angle. If the velocity $\vec{v}_2$ is replaced by an infinitesimal velocity $\mathrm{d}\vec{v}_2$, the angle of rotation becomes infinitesimal $\mathrm{d}\vec{\Omega}$ and $\mathrm{d}\vec{\Omega}/\mathrm{d}t$ is called the Thomas precession rate.

The authors of [1] provide concise methods to derive explicit expressions for $\vec{\Omega}$ and $\mathrm{d}\vec{\Omega}/\mathrm{d}t$ that are technically clear and elegant. Unfortunately the reader of [1] encounters serious difficulties in interpretation of the applied calculations and the results. What is more, there are some flaws in [1] concerning the explanation of the findings for the Thomas precession rate.

2. Correct expression for the Thomas precession

A particularly significant mistake that contributes to the existing confusion regarding the formula for the Thomas precession was made by O'Donnell and Visser in interpreting their own result concerning the Thomas precession. They obtained an expression for the Thomas precession rate in the form (see equations (8) and (26) in [1]):

$$\frac{\mathrm{d}\vec{\Omega}}{\mathrm{d}t} = \vec{v}_1 \times \vec{a}\left(\frac{\gamma_1}{1+\gamma_1}\right), \tag{1}$$

where the authors set $c = 1$. It is explained that $\vec{a} = \mathrm{d}\vec{v}_2/\mathrm{d}t$ is the 'acceleration experienced by Bob'. The authors note that their result (1) differs by the factor γ_1 from the more well known one achieved by Møller [2] (so also by Thomas [3], Jackson [4], and many others authors, including the author of this comment [5, 6]). In fact, the result (1) also differs from the standard one by the minus sign, which is overlooked in [1]. O'Donnell and Visser maintain that their expression for the Thomas precession is correct in the mission control's frame and the standard one given by Thomas and others is related to Alice's reference frame (see footnote 5 and the comments at equations (71) and (72) in [1]). This view is supported by referring to papers by Malykin and Ritus (see in [1]). At this point, however, O'Donnell and Visser make a twofold mistake: they incorrectly interpret both their own result (1) and the one obtained by Thomas, Møller, Jackson and others.

First, one can easily verify that the Thomas precession rate calculated by Thomas [3], Møller [2] and Jackson [4] refers to a laboratory frame, i.e. just to the mission control frame and not the Alice one, as O'Donnell and Visser suggest. What is more, the idea that the Thomas precession is observed in Alice's frame is completely misleading. Alice cannot see any precession of Bob's spacecraft because Alice's frame and Bob's frame are connected by a *single* Lorentz boost and the Thomas precession is a result of at least two non-collinear Lorentz boosts.

As concerns the result (1) presented in [1] as equations (8), (26) and (71), the authors made the interpretational mistake because they overlooked that the 'acceleration' $\vec{a} = \mathrm{d}\vec{v}_2/\mathrm{d}t$ they use is combined from quantities coming from different reference frames: $\mathrm{d}\vec{v}_2$ is Bob's spacecraft's velocity change in the *Alice frame*, while $\mathrm{d}t$ is a time interval in the *mission control frame*. It follows that $\vec{a} = \mathrm{d}\vec{v}_2/\mathrm{d}t$ is *not* the acceleration measured in the mission control frame (the laboratory frame). To get the acceleration in the mission control frame first we find Bob's spacecraft's velocity in this frame. According to the Lorentz transformation it is given as:

$$(\mathrm{d}\vec{v}_2)_{\mathrm{Lab}} = \mathrm{d}\vec{v}_{2_\parallel} + \frac{\mathrm{d}\vec{v}_{2_\perp}}{\gamma_1}, \tag{2}$$

where the indices $\parallel$ and $\perp$ refer to the direction of the velocity $\vec{v}_1$. Due to the cross product of $\mathrm{d}\vec{v}_2$ with the velocity $\vec{v}_1$ occurring in the expression for the Thomas precession (see equation (24) in [1]), only the component perpendicular to $\vec{v}_1$ is meaningful. From (2) we get:

$$\frac{\mathrm{d}\vec{v}_{2_\perp}}{\mathrm{d}t} = \gamma_1 \frac{(\mathrm{d}\vec{v}_{2_\perp})_{\mathrm{Lab}}}{\mathrm{d}t} \tag{3}$$

or

$$\vec{a}_\perp = \gamma_1 (\vec{a}_\perp)_{\mathrm{Lab}}. \tag{4}$$

It follows that if we introduce in equation (1) the laboratory acceleration $\vec{a}_{\mathrm{Lab}}$ instead of the hybrid acceleration $\vec{a} = \mathrm{d}\vec{v}_2/\mathrm{d}t$ used unconsciously by O'Donnell and Visser, we get the Thomas precession rate in the laboratory frame (the mission control frame) in the form:

$$\frac{\mathrm{d}\vec{\Omega}}{\mathrm{d}t} = \vec{v}_1 \times \vec{a}_{\mathrm{Lab}} \left(\frac{\gamma_1^2}{1+\gamma_1} \right), \tag{5}$$

which is precisely the result obtained by Thomas, Møller and Jackson (but with the minus sign, as has already been pointed out above). The alleged discrepancy due to the factor γ_1 between the result obtained by O'Donnell and Visser and the one of Thomas then vanishes. Taking into account this correction, the paper [1] confirms Thomas's result and proves indirectly that the Malykin and Ritus claims on the Thomas precession rate are wrong.

As concerns Ritus's paper (reference 7 in [1]) promoting the incorrect expression for the Thomas precession, the author made a mistake when he introduced his equations (47) and (48) and stated that the spin rotates by an angle ω while the velocity of particle changes from $\vec{v}$ to $\vec{v}_2 \equiv -\vec{w}''$. Simply, no process in which the particle velocity changes from $\vec{v}$ to $\vec{v}_2 \equiv -\vec{w}''$ is considered earlier in his work and therefore none of his equations could be applied to this situation. The equations (17) and (28) that Ritus uses extensively in his reasoning refer to a motion when the velocity $\vec{v}_2 \equiv -\vec{w}''$ is achieved from a velocity $\vec{v}_1 = \vec{u}'$, not $\vec{v}$. (That Ritus confuses the order of boosts is clearly seen when he writes equation (18) with the wrong order of velocities in the vector product or derives equation (55) from (17).) It follows then that the description above equation (32) in Ritus's paper and especially the discussion preceding equations (47) and (48) is wrong; when the spin rotates by the angle ω the particle velocity changes from $\vec{v}$ to $\vec{w}$, i.e. rotates by an angle δ, not θ. Therefore, to get the rotation rate, the angle of spin rotation should be divided by $\mathrm{d}t$, as Møller did, and not by the enlarged time interval $\Delta t = \gamma\,\mathrm{d}t$, as Ritus wants.

In turn, in Malykin's paper (reference 6 in [1]) the reader is informed that the results for the Thomas precession obtained by L H Thomas himself, and many acknowledged experts on special relativity such as Møller, Jackson and Rohrlich [7], and Misner, Thorne and Wheeler [8], are wrong. Closer inspection of Malykin's text shows, however, that the author misunderstood the work by Thomas. His first equation for $\mathrm{d}\boldsymbol{\phi}$, which he attributes to Thomas is incorrectly interpreted as pertaining to a laboratory frame (cf the original work by Thomas [3], where one finds the unnumbered equation below equation (3.3) with a description). What is more, the second of Malykin's equations for $\boldsymbol{\Omega}_T$, which the author ascribes to Møller, is treated as inconsistent (due to the sign) with the first one. Meanwhile, both these results belong to Thomas's theory (the second is incorporated into Thomas's equation (3.5), first line, in the term $+\mathbf{w}_0 \times \mathrm{d}\boldsymbol{\phi}/\mathrm{d}s_0$, which is equivalent to $+\boldsymbol{\Omega}_T \times \mathbf{w}_0$, i.e. the Thomas precession rate registered in the laboratory frame). The difference in the sign follows from the fact that the first equation, for $\mathrm{d}\boldsymbol{\phi}$, shows a rotation between two instantaneous rest frames, each of which is connected with the laboratory frame by a pure boost (these instantaneous rest frames are denoted in [3] as (R_0, T_0) and (R_1, T_1) and as x' and x'' in Jackson's textbook [4]—probably the clearest presentation of the Thomas approach). In turn, the second Malykin's equation, for $\boldsymbol{\Omega}_T$, is the rotation rate between a rest frame fixed to the spin (the frame (ρ, τ) in [3] or the frame x''' in [4]) and the laboratory frame. The difference in sign is clearly explained in [4], where the author obtains a result for $\Delta\boldsymbol{\Omega}$ (a counterpart of $\mathrm{d}\boldsymbol{\phi}$ in Malykin's paper), but the Thomas precession is correctly calculated using $\Delta\boldsymbol{\Omega}$ with the *minus sign* (cf equation (11.119)), which represents a rotation of the frame x''' with respect to the laboratory frame. Although the first and second equations in Malykin's work are consistent and correct, the author states that they are wrong. To defend the interpretation of his equation (3) as the correct expression for the Thomas precession, Malykin quotes uncritically Ritus's work, which contains the mistakes indicated in this comment. Malykin also claims that the same expression as his (3) was achieved by Baylis ([104] in his paper) and Strandberg ([127] in his paper). Meanwhile, Baylis's result agrees with Thomas's, but it is simply misinterpreted by Malykin; the spatial part of four-velocity $\vec{u} = \gamma\vec{v}$ is confused with the ordinary velocity $\vec{v}$. The same mistake is made by Malykin in his equation (15), where factors $\gamma_1\gamma_2$ are missing. One should also notice that in the Baylis work the velocity differentiation is performed with respect to the proper time and not the laboratory one. In turn, Strandberg provides an incorrect result for the Thomas precession; in his reasoning the quantity $\mathrm{d}\vec{v}$ belongs to the laboratory frame and not to the commoving one, as the author claims. Similar mistakes one can find in other cited papers used by Malykin to justify the erroneous expression for the Thomas precession rate. The common mistake made by many authors is that they do not distinguish carefully between the frames x', x'' and x''', which influence the sign of the Thomas rotation. Finally, let us mention that the example of a clock in circular motion given by Malykin in section 4.6 is completely misleading, as it has nothing in common with the Thomas precession. Simply, the same behaviour of the hour hand would be observed if the clock moved *rectilinearly*.

Summarizing, the flaws indicated above entail that Ritus's and Malykin's conclusions are unreliable and are not suitable to be used as a support or additional confirmation of results achieved by other authors.

3. Spacecraft rotation and the angle $\vec{\Omega}$ between the velocities $\vec{v}_{21}$ and $\vec{v}_{12}$

It is well known that two non-collinear boosts are equivalent to a boost and a rotation called the Wigner rotation. Bob's spacecraft undergoes the Wigner rotation in both situations (a) and (b). O'Donnell and Visser prove some nontrivial fact that the angle of the Wigner rotation in each of these cases is equal to the angle between the velocities $\vec{v}_{21}$ and $\vec{v}_{12}$ achieved by Bob in the respective situations. Unfortunately, the reason why such the equality holds in special relativity remains obscure in [1]. Below we propose a simple method to explain this property. The method allows also to verify directly the sign of the Wigner rotation (incorrectly described in [1]) and prove that the angle of the Wigner rotation is wrongly indicated in the O'Donnell and Visser figures.

Let us refer to case (a) and, to make our considerations more transparent, assume that the second boost is made perpendicularly to the velocity $\vec{v}_1$. The resulting velocity $\vec{v}_{21}$ of Bob's spacecraft must then be written carefully as:

$$\vec{v}_{21}^{M} = v_1\hat{e}_1^{M} + \frac{v_2}{\gamma_1}\hat{e}_2^{M}. \tag{6}$$

(Note that the notation used by O'Donnell and Visser in their equations (3) and (4), as well as (13) and (14) is misleading because it suggests that the authors add vectors from different frames of reference.) In our notation the index 'M' indicates that the labelled quantities are established in the mission control frame. The versor $\hat{e}_1^{M}$ points in the direction of $\vec{v}_1$ and the versor $\hat{e}_2^{M}$ is oriented along the direction of the second boost, i.e. $\hat{e}_1^{M} \perp \hat{e}_2^{M}$ (figure 1). Certainly, the perpendicularity of the second boost is established in Alice's frame but it is equivalent to the perpendicularity in the mission control frame.

Now, continuing with situation (a), one may ask: What is the velocity of the mission control frame as seen by Bob in this case? To obtain the velocity of the mission control with respect to Bob's spacecraft we have to perform a boost with the velocity $-\vec{v}_2$ from Bob's frame to Alice's and next the second boost with the velocity $-\vec{v}_1$, perpendicular to $-\vec{v}_2$, from Alice's spacecraft to the mission control frame. As we can see, in this case we deal with relativistic

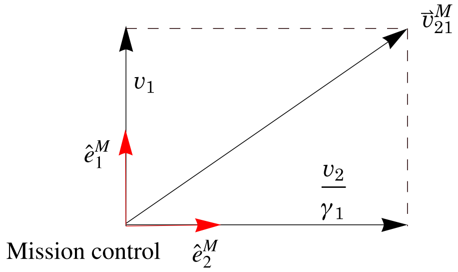

Figure 1. The velocity $\vec{v}_{21}^{M}$ of Bob's spacecraft as seen in the mission control frame.

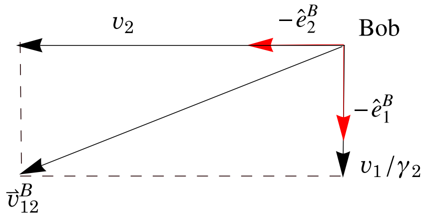

Figure 2. The velocity $\vec{v}_{12}^{B}$ of the mission control as seen in Bob's frame.

combination of velocities $-\vec{v}_2$ and $-\vec{v}_1$ and the resulting velocity can be written, by imitating the notation of equation (4) in [1], as:

$$\vec{v}_{12}^{B} = v_2\big(-\hat{e}_2^B\big) + \frac{v_1}{\gamma_2}\big(-\hat{e}_1^B\big). \tag{7}$$

The index 'B' indicates that the mission control velocity $\vec{v}_{12}^{B}$ and the perpendicular unit vectors $\big(-\hat{e}_2^B\big)$, $\big(-\hat{e}_1^B\big)$, pointing along the velocity $-\vec{v}_2$ and along the second boost, respectively, are defined in Bob's frame (see figure 2).

It appears that the components of the velocity $\vec{v}_{12}^{B}$ are numerically the same as the components of the velocity $\vec{v}_{12}$ defined by equation (4) in [1]. Therefore the velocity $\vec{v}_{12}$ considered by O'Donnell and Visser as pertaining to the completely different situation (b) appears to also be meaningful in the case (a). Both $\vec{v}_{21}$ and $\vec{v}_{12}$ can be referred to one and the same situation (a), which is not pointed out by O'Donnell and Visser. This fact, however, is crucial for providing an interpretation of the results achieved in [1].

To demonstrate the Wigner rotation performed by Bob's spacecraft and its connection with the velocities $\vec{v}_{21}$ and $\vec{v}_{12}$ one can merge figures 1 and 2 and get a full description of case (a). In effect, we obtain figure 3, where it is taken into account that the $\vec{v}_{21}^{M}$ and $\vec{v}_{12}^{B}$ have the same direction because they represent one and the same relative velocity between Bob and the mission control. Nevertheless, from equations (6) and (7) it follows that the component of $\vec{v}_{21}^{M}$ along the versor $\hat{e}_1^M$ does not agree with the component of $\vec{v}_{12}^{B}$ along the versor $\big(-\hat{e}_1^B\big)$. The same refers to the components of these vectors along the versors $\hat{e}_2^M$ and $\big(-\hat{e}_2^B\big)$. This means that Bob's frame connected with the versors $\big(-\hat{e}_1^B\big)$ and $\big(-\hat{e}_2^B\big)$ is rotated with respect to the mission control frame, defined by $\hat{e}_1^M$ and $\hat{e}_2^M$, by an angle Ω. Because the vector $\vec{v}_{12}^{B}$ has the same components as the O'Donnell and Visser vector $\vec{v}_{12}$, the magnitude of rotation of the Bob frame axes is precisely the same as the angle between the vectors $\vec{v}_{12}$ and $\vec{v}_{21}$. In this way one can justify the study [1] on the orientation of the vectors $\vec{v}_{21}$ and $\vec{v}_{12}$ as being applicable to the Wigner or Thomas rotation performed by Bob's spacecraft itself when it acquires the velocity $\vec{v}_{21}$ or the velocity $\vec{v}_{12}$.

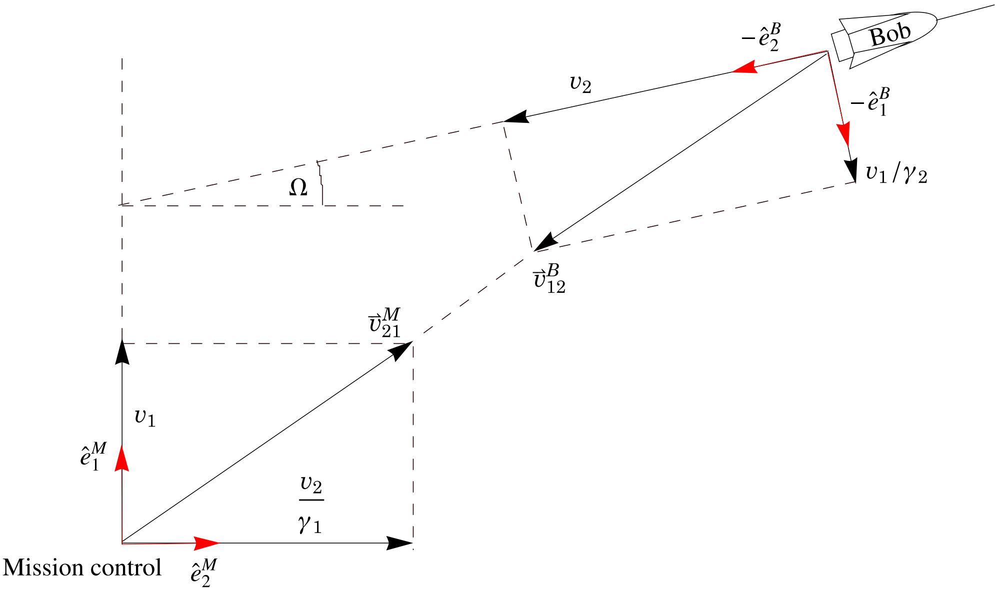

Figure 3. The direction of the velocities $\vec{v}_{21}^M$ and $\vec{v}_{12}^B$ is regarded as the same both in the mission control frame and Bob's frame because both of them represent the relative velocity between the two frames. Also, $||\vec{v}_{21}^M|| = ||\vec{v}_{12}^B||$. However, the components of these velocities are different in the respective frames. This signifies that the frames are rotated one with respect to the other by an angle Ω.

The authors of [1] do not explain in the text how Bob's spaceship is oriented during travel. To make the argumentation reasonable one has to assume that Bob's spacecraft orientation is fixed with respect to the frame of reference defined by the versors $\left(-\hat{e}_1^B\right)$ and $\left(-\hat{e}_2^B\right)$. Figures 1, 3, 4, 5 in [1] suggest that Bob's spacecraft axis is oriented along the velocity $\vec{v}_2$. If so, the angle of the Wigner rotation Ω is indicated incorrectly. It is *not* the angle between Bob's spaceship's axis and the velocity $\vec{v}_{21}$, but rather between the versors $\hat{e}_1^M$ and $\hat{e}_1^B$(or $\hat{e}_2^M$ and $\hat{e}_2^B$), which is clearly seen in our figure 3.

As mentioned above, the expression (26) in [1], recalled here as equation (1), is incorrect not only due to the fact that γ_1 is missing, but also concerning the sign. Actually, it can be verified on the basis of figure 3 that if the vector $\vec{\Omega}$ is to represent Bob's frame axes rotation with respect to the mission control frame, it points in the direction of $\hat{e}_2^M \times \hat{e}_1^M$, or equivalently $\vec{a} \times \vec{v}_1$ in the case of the Thomas precession (when $\vec{a} = \mathrm{d}\vec{v}_2/\mathrm{d}t$), which does not agree with equation (26) in [1].

In conclusion, we believe that this comment contributes to elucidation of the problem of the Thomas precession and hope that the explanations given above provide clear interpretational fundamentals that enable the calculations presented by O'Donnell and Visser to be well understood and appreciated.

Acknowledgments

I wish to thank Vladimir Hnizdo for pointing out the problem with the results obtained in [1] and for an inspiring discussion on the Thomas precession. I am grateful to the anonymous referee for suggestions that improved the form of this comment.

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