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POTENTIAL FORMULATIONS

IN MAGNETICS

APPLYING THE

FINITE ELEMENT METHOD

Lecture notes

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Contents

1 Introduction

In electrical engineering practice, the simulation of devices, measuring arrangements and various electrical equipments is based on Maxwell's equations coupled with the constitutive relations. It is very important to take into account the hysteretic behavior as well as the vector property of the magnetic field quantities. However, in some cases it is adequate to use constant permeability or single valued nonlinearity. Hysteresis characteristics must be taken into consideration when, for example, hysteresis losses in electrical machines or effects related to the remanent magnetization are to be calculated. The second chapter is a summary of Maxwell's equations, their form in the case of nonlinear static magnetic field problems and of nonlinear eddy current field problems. Here, we give the correct description of all the possible potential formulations used in solving static magnetic field problems and eddy current field problems.

Chapter 3 presents the weighted residual method, which can be used to solve the nonlinear partial differential equations obtained from Maxwell's equations by applying potentials. Here the weighted residual method is applied to the weak formulation. The finite element method is a possible technique to solve numerically the partial differential equations formulated by using the weak form. It is noted here that the finite element method is one of the most widely used methods to solve electromagnetic field problems.

In chapter 4, we summarize the finite element method from scratch, the nodal finite elements as well as the newer vector elements. This chapter is very useful for students studying applied electromagnetics in faculties of electrical engineering, too.

The fifth chapter presents the polarization method as well as the fixed point technique to solve nonlinear electromagnetic field problems. The magnetic field intensity or the magnetic flux density is split into a linear term and a nonlinear term, as defined by the polarization method. Nonlinear equations can be formulated as a fixed point equation, which is solved iteratively by using the fixed point technique. This method results in a convergent numerical tool to solve nonlinear equations, or a system of nonlinear equations, however, the parameter of the linear term must be selected in a special way.

The last chapter is the collection of seven problems solved by the finite element method. The problems have been solved by using the user friendly graphical user interface and functions of COMSOL Multiphysics, which is a commercial finite element software.

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2 Potential formulations in electromagnetic field

The numerical analysis or the computer aided design (CAD) of an arrangement, which require electromagnetic field calculation can be characterized by the electric and magnetic field intensities and flux densities.

For determination of these quantities in the electromagnetic field, one method is to find the solution of the partial differential equations of the field quantities under prescribed boundary conditions. The mathematical description of electric and magnetic fields can be formulated by the differential form of *Maxwell's equations*, which are the collection of partial differential equations of the electric field intensity E, the magnetic field intensity H, the electric flux density D and the magnetic flux density B. The source of the electromagnetic field can be the electric current density J, the electric charge density ρ, the currents, voltages, etc. Constitutive relations between the above quantities are defined to take into account the macroscopic properties of the medium where the electromagnetic field has been studied.

It is very important to note that the numerical analysis of electromagnetic fields results in an approximate solution of the partial differential equations expressed from Maxwell's equations. The partial differential equations to be solved can be formulated for field quantities or for potentials and the resulting solution fulfilles the prescribed boundary conditions as well.

There are several studies on the basic equations of the electromagnetic fields and on their solution based on different potential formulation [3, 11, 28, 41–43, 52, 60, 61, 81–83]. In the first part of this chapter a short summary of the basic equations, i.e. Maxwell's equations, boundary and interface conditions can be found based on the above cited widely read books.

Here, we only focus on linear and nonlinear static magnetic field problems, moreover on linear and nonlinear eddy current field problems. The set of Maxwell's equations can be transformed into one or two partial differential equations by using potentials. Boundary and interface conditions can also be represented by potentials. The potential formulations are presented in the second part of this chapter.

2.1 Maxwell's equations and interface conditions

2.1.1 Maxwell's equations

A. The differential form of Maxwell's equations

Electromagnetic field simulation problems can be characterized by the field intensities and the flux densities described by the partial differential equations derived from *Maxwell's equations* under prescribed boundary conditions.

Maxwell's equations define relations for the electromagnetic field quantities and the source elements, i.e. the electric and the magnetic field intensities, the electric and the magnetic flux densities, the charge and current distribution. Expletively, the so-called constitutive relations are appended to Maxwell's equations to describe the properties of the media. The differential form of Maxwell's equations is as follows:

$$\nabla \times \boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{J}(\boldsymbol{r},t) + \frac{\partial \boldsymbol{D}(\boldsymbol{r},t)}{\partial t}, \tag{2.1}$$

$$\nabla \times \boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t}, \tag{2.2}$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r},t) = 0, \tag{2.3}$$

$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r},t) = \rho(\boldsymbol{r},t), \tag{2.4}$$

$$\begin{aligned} \boldsymbol{B}(\boldsymbol{r},t) &= \mu_0[\boldsymbol{H}(\boldsymbol{r},t) + \boldsymbol{M}(\boldsymbol{r},t)], \\ \boldsymbol{J}(\boldsymbol{r},t) &= \sigma[\boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{E}_i(\boldsymbol{r},t)], \\ \boldsymbol{D}(\boldsymbol{r},t) &= \varepsilon_0 \boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{P}(\boldsymbol{r},t). \end{aligned} \tag{2.5}$$

The presented symbols and variables with their name and unit in SI system can be found in *Table* 2.1 ($\mu_0 = 4\pi \cdot 10^{-7}$ H/m, $\varepsilon_0 = 8.854 \cdot 10^{-12}$ F/m). This is the set of partial differential equations, which can be used in the frame of any numerical field analysis tool, e.g. in the finite element method.

The field quantities are depending on space $\boldsymbol{r}$ and on time t, therefore in the following a shorter notation will be used, that is $\boldsymbol{H} = \boldsymbol{H}(\boldsymbol{r},t)$, $\boldsymbol{J} = \boldsymbol{J}(\boldsymbol{r},t)$, $\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{r},t)$, $\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{r},t)$, $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{r},t)$, $\rho = \rho(\boldsymbol{r},t)$, $\boldsymbol{M} = \boldsymbol{M}(\boldsymbol{r},t)$, $\boldsymbol{E}_i = \boldsymbol{E}_i(\boldsymbol{r},t)$, $\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{r},t)$.

The sources of the electromagnetic field are the electric current density $\boldsymbol{J}$ and the electric charge density ρ.

The first Maxwell's equation (2.1) covers the *Ampere's law* in differential form. The first term of the right-hand side is the current density of coils (generated by electrons moving inside coils) and *eddy currents* inside conducting material. The second term is the so called displacement current in dielectric media, which is generated by a time-varying electric field. This equation represents that currents in conductors or eddy currents flowing in conducting materials and time-varying electric fields generate magnetic fields as it is presented in *Fig.* 2.1.

The second Maxwell's equation (2.2) is the differential form of *Faraday's law*. This states that the time-varying magnetic field induces electric field as it is illustrated in *Fig.* 2.2.

Table 2.1. Name and SI units of symbols in Maxwell's equations

Notation	Quantity	SI unit
$\boldsymbol{H}(\boldsymbol{r},t)$	magnetic field intensity	$\frac{\mathrm{A}}{\mathrm{m}}$
$\boldsymbol{E}(\boldsymbol{r},t)$	electric field intensity	$\frac{\mathrm{V}}{\mathrm{m}}$
$\boldsymbol{B}(\boldsymbol{r},t)$	magnetic flux density	T
$\boldsymbol{D}(\boldsymbol{r},t)$	electric flux density	$\frac{\mathrm{C}}{\mathrm{m}^2}$
$\boldsymbol{J}(\boldsymbol{r},t)$	electric current density	$\frac{\mathrm{A}}{\mathrm{m}^2}$
$\rho(\boldsymbol{r},t)$	electric charge density	$\frac{\mathrm{C}}{\mathrm{m}^3}$
$\boldsymbol{M}(\boldsymbol{r},t)$	magnetization	$\frac{\mathrm{A}}{\mathrm{m}}$
$\boldsymbol{E}_i(\boldsymbol{r},t)$	impressed electric field	$\frac{\mathrm{V}}{\mathrm{m}}$
$\boldsymbol{P}(\boldsymbol{r},t)$	polarization	$\frac{\mathrm{C}}{\mathrm{m}^2}$
μ_0	permeability of vacuum	$\frac{\mathrm{H}}{\mathrm{m}}$
σ	conductivity	$\frac{\mathrm{S}}{\mathrm{m}}$
ε_0	permittivity of vacuum	$\frac{\mathrm{F}}{\mathrm{m}}$

(a) The electric current density $\boldsymbol{J}$ generates magnetic field $\boldsymbol{H}$

(b) The electric current density $\boldsymbol{J}$ and the displacement current density $\partial\boldsymbol{D}/\partial t$ generate magnetic field $\boldsymbol{H}$
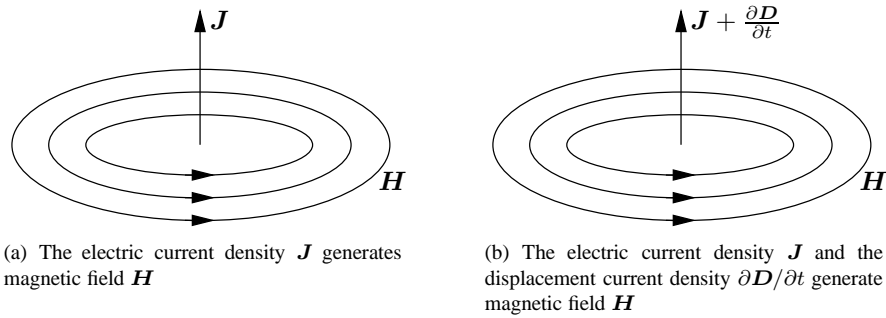
Fig. 2.1. Explanation of the first Maxwell's equation

In time-varying situation the first and the second Maxwell's equations are coupled, i.e. the time-varying magnetic field generates electric field and this electric field generates eddy currents inside the conducting materials, which currents modify the source magnetic fields.

Equation (2.3) states that the magnetic field is divergence-free, i.e. free magnetic charges do not exist physically and magnetic flux lines close upon themselves. This is the magnetic *Gauss' law*.

Equation (2.4) is the electric *Gauss' law*. This means that the source of electric field is the electric charge and electric flux lines start and close upon the charge.
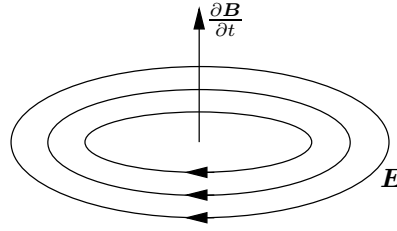
Fig. 2.2. The time variation of the magnetic flux density $\partial \boldsymbol{B}/\partial t$
generates electric field intensity $\boldsymbol{E}$

There is an important relationship between the variation of the current density and the charge density, the so-called *current continuity equation* (also called *charge conservation law*). This law is coming from the first Maxwell's equation. Taking the divergence of (2.1), the following equation can be obtained:

$$\nabla \cdot (\nabla \times \boldsymbol{H}) = \nabla \cdot \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \right) = \nabla \cdot \boldsymbol{J} + \frac{\partial}{\partial t} \nabla \cdot \boldsymbol{D}. \tag{2.6}$$

The left-hand side is equal to zero according to the identity

$$\nabla \cdot (\nabla \times \boldsymbol{v}) \equiv 0, \tag{2.7}$$

for any vector $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r}, t)$ and the second term on the right-hand side can be rewritten by applying equation (2.4). Finally the current continuity equation has the form

$$\nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0. \tag{2.8}$$

This equation means that the variation of currents and charge distribution according to space and time is depending on each other.

B. The integral form of Maxwell's equations

There is an other form of *Maxwell's equations*, the so-called integral form,

$$\oint_l \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \int_\Gamma \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \right) \cdot \mathrm{d}\Gamma, \tag{2.9}$$

$$\oint_l \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\int_\Gamma \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\Gamma, \tag{2.10}$$

$$\oint_\Gamma \boldsymbol{B} \cdot \mathrm{d}\Gamma = 0, \tag{2.11}$$

$$\oint_\Gamma \boldsymbol{D} \cdot \mathrm{d}\Gamma = \int_\Omega \rho \, \mathrm{d}\Omega. \tag{2.12}$$

The constitutive relations have the same form as in (2.5).

The relationship between the integral forms (2.9) and (2.10) and the differential forms (2.1) and (2.2) can be noticed by applying Stokes' theorem,

$$\oint_l \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l} = \int_\Gamma \nabla \times \boldsymbol{v} \cdot \mathrm{d}\Gamma, \tag{2.13}$$

where $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r}, t)$ is a space vector, which may depend on time, the vectors $\mathrm{d}\boldsymbol{l}$ and $\mathrm{d}\Gamma$ are illustrated in *Fig.* 2.3. In this figure, the relation between the circulation about the loop l and the positive (outer) normal unit vector $\boldsymbol{n}$ can be seen as well.

The first Maxwell's equation (see *Fig.* 2.3) says that the line integral of the magnetic field intensity vector along any closed loop l is equal to the sum of currents or to the surface integral of current densities flowing across the area Γ bounded by the path l. The displacement current has the same effect as the conducting current. This is the classical form of *Ampere's law*. In other words it can be stated that if a current is flowing inside a conductor, then a magnetic field is generated in the vicinity of this conductor.
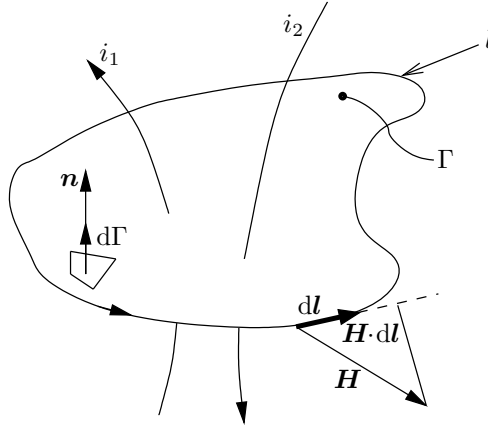
Fig. 2.3. The Ampere's law

The surface integral of source current density $\boldsymbol{J}_0 = \sigma\boldsymbol{E}_i$ is equal to the current i flowing into the surface Γ, i.e.

$$\int_\Gamma \boldsymbol{J}_0 \cdot \mathrm{d}\Gamma = i, \tag{2.14}$$

where $i = i(t)$ or $i = I$.

The second Maxwell's equation says that the line integral of the electric field intensity vector along any closed loop l is equal to the surface integral of the time variation of magnetic flux density vector across the area Γ bounded by the path l. This is the classical form of *Faraday's law*. In other words, if the flux $\Phi = \Phi(t)$ is varying with time, then a corresponding induced voltage $u_i = u_i(t)$ is generated between the two poles of a coil placed inside this time varying flux (see *Fig.* 2.4),

$$u_i(t) = -\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t}, \tag{2.15}$$
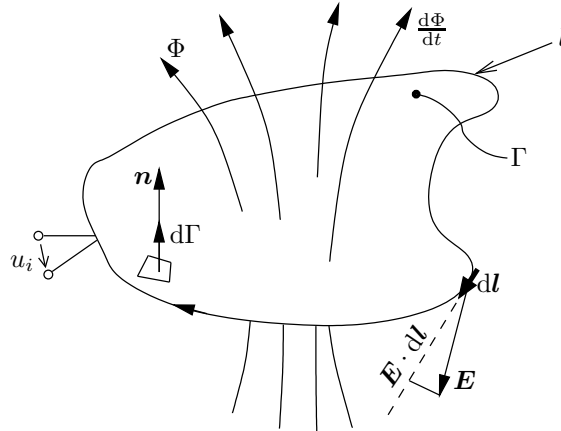
Fig. 2.4. The Faraday's law

because of definitions

$$u_i(t) = \oint_l \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}, \quad \text{and} \quad \Phi = \int_\Gamma \boldsymbol{B} \cdot \mathrm{d}\Gamma. \tag{2.16}$$

The relationship between the integral form (2.11) and (2.12) and the differential form (2.3) and (2.4) can be expressed by applying *Gauss' theorem*,

$$\oint_\Gamma \boldsymbol{v} \cdot \mathrm{d}\Gamma = \int_\Omega \nabla \cdot \boldsymbol{v} \, \mathrm{d}\Omega, \tag{2.17}$$

where $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r}, t)$ is a space vector, which may depend on time, the vector $\mathrm{d}\Gamma$ and volume Ω are represented in *Fig.* 2.5.

The third Maxwell's equation says that the surface integral of the magnetic flux density on a closed surface Γ is equal to zero, i.e. the flux lines are closed.
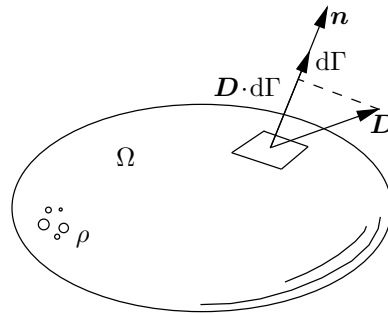
Fig. 2.5. The Gauss' law

The fourth Maxwell's equation declares that the surface integral of the electric flux density on a closed surface Γ is equal to the volume integral of the charge density enclosed by this surface representing the volume Ω as it is illustrated in *Fig.* 2.5. The flux lines are starting and closing at electric charges. The outer normal unit vector of the volume Ω is denoted by $\boldsymbol{n}$. The volume integral of charge density is equal to the charge Q,

$$\int_{\Omega} \rho \, \mathrm{d}\Omega = Q. \tag{2.18}$$

C. The constitutive relations

Equations in (2.5) collect the so-called *constitutive relations*, which – depending on the properties of the examined material – describe the relationship between field quantities.

In the simplest case these relations are linear, i.e.

$$\boldsymbol{M} = \chi \boldsymbol{H}, \quad \boldsymbol{E}_i = \boldsymbol{0}, \quad \boldsymbol{P} = \varepsilon_0 \chi_d \boldsymbol{E}, \tag{2.19}$$

where χ and χ_d are the magnetic and the dielectric susceptibility and

$$\boldsymbol{B} = \mu \boldsymbol{H}, \quad \boldsymbol{J} = \sigma \boldsymbol{E}, \quad \boldsymbol{D} = \varepsilon \boldsymbol{E}, \tag{2.20}$$

where

$$\mu = \mu_0(1 + \chi) = \mu_0 \mu_r, \tag{2.21}$$

and

$$\varepsilon = \varepsilon_0(1 + \chi_d) = \varepsilon_0 \varepsilon_r. \tag{2.22}$$

Here $\mu_r = 1 + \chi$ is the relative permeability, $\varepsilon_r = 1 + \chi_d$ is the relative permittivity of the material and the conductivity σ is constant. The second equation in (2.20) is the differential form of Ohm's law.

Constitutive relations are generally nonlinear, that is the permeability, the conductivity and the permittivity depend on the appropriate field quantities,

$$\mu = \mu(\boldsymbol{H}, \boldsymbol{B}), \quad \sigma = \sigma(\boldsymbol{E}, \boldsymbol{J}), \quad \varepsilon = \varepsilon(\boldsymbol{E}, \boldsymbol{D}). \tag{2.23}$$

This can be written in another way,

$$\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{H}), \quad \boldsymbol{J} = \boldsymbol{J}(\boldsymbol{E}), \quad \boldsymbol{D} = \boldsymbol{D}(\boldsymbol{E}), \tag{2.24}$$

where $\boldsymbol{B}(\cdot)$, $\boldsymbol{J}(\cdot)$ and $\boldsymbol{D}(\cdot)$ are operators. The first one of the above equations will be in focus in the following, which can be expressed by the hysteresis operator, represented by

$$\boldsymbol{B} = \mathscr{B}\{\boldsymbol{H}\}, \quad \text{or} \quad \boldsymbol{H} = \mathscr{B}^{-1}\{\boldsymbol{B}\}. \tag{2.25}$$

If the material properties are independent of space $\boldsymbol{r}$, they are homogeneous, otherwise they are inhomogeneous,

$$\mu = \mu(\boldsymbol{r}), \quad \sigma = \sigma(\boldsymbol{r}), \quad \varepsilon = \varepsilon(\boldsymbol{r}). \tag{2.26}$$

Constitutive relations may depend on the frequency of excitation as well,

$$\mu = \mu(f), \quad \sigma = \sigma(f), \quad \varepsilon = \varepsilon(f). \tag{2.27}$$

When the constitutive parameters depend on the direction of the applied field, the materials are anisotropic, otherwise they are isotropic (the linear equations in (2.20) states for isotropic materials). In the anisotropic case the permeability, the conductivity and the permittivity are tensors,

$$\boldsymbol{B} = [\mu]\boldsymbol{H}, \quad \boldsymbol{J} = [\sigma]\boldsymbol{E}, \quad \boldsymbol{D} = [\varepsilon]\boldsymbol{E}, \tag{2.28}$$

for example

$$[\mu] = \left[\begin{array}{ccc} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{array} \right]. \tag{2.29}$$

In the most general situation, the constitutive relations depend on all of the above variables, e.g.

$$\boldsymbol{B} = \mathscr{B}\{\boldsymbol{H}, \boldsymbol{r}, f\}. \tag{2.30}$$

D. Energy of electromagnetic fields

The *energy balance equation* of electromagnetic fields can be expressed from Maxwell's equations. After multiplying the equation (2.1) by $-\boldsymbol{E}$ and the equation (2.2) by $\boldsymbol{H}$, then summing these equations, the following expression can be obtained:

$$\boldsymbol{H} \cdot \nabla \times \boldsymbol{E} - \boldsymbol{E} \cdot \nabla \times \boldsymbol{H} = -\boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{E} \cdot \boldsymbol{J} - \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t}. \tag{2.31}$$

After applying the identity

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) = \boldsymbol{H} \cdot \nabla \times \boldsymbol{E} - \boldsymbol{E} \cdot \nabla \times \boldsymbol{H} \tag{2.32}$$

on the left-hand side and integrating the resulting equation over a volume Ω bounded by a surface Γ and using Gauss' theorem, it results in the energy balance equation of the electromagnetic fields,

$$\int_\Omega \left(\boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} \right) \mathrm{d}\Omega + \int_\Omega \boldsymbol{E} \cdot \boldsymbol{J} \, \mathrm{d}\Omega + \oint_\Gamma (\boldsymbol{E} \times \boldsymbol{H}) \cdot \mathrm{d}\Gamma = 0. \tag{2.33}$$

The first volume integral is the rate of change of energy density of electromagnetic fields $w = w(\boldsymbol{r}, t)$ in the volume Ω, i.e.

$$\frac{\partial w}{\partial t} = \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t}. \tag{2.34}$$

The second integral can be reformulated by using the second equation in (2.5),

$$\boldsymbol{E} \cdot \boldsymbol{J} = \left(\frac{\boldsymbol{J}}{\sigma} - \boldsymbol{E}_i \right) \cdot \boldsymbol{J} = \frac{|\boldsymbol{J}|^2}{\sigma} - \boldsymbol{E}_i \cdot \boldsymbol{J}. \tag{2.35}$$

This results in

$$
\int_\Omega \left(\boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} \right) \mathrm{d}\Omega + \int_\Omega \frac{|\boldsymbol{J}|^2}{\sigma} \, \mathrm{d}\Omega
$$
$$
- \int_\Omega \boldsymbol{E}_i \cdot \boldsymbol{J} \, \mathrm{d}\Omega + \oint_\Gamma (\boldsymbol{E} \times \boldsymbol{H}) \cdot \mathrm{d}\Gamma = 0. \tag{2.36}
$$

Here the second integral is the *dissipated energy* term, i.e. the *Joule heat*, the third integral contains the energy supplied by the sources and finally the last surface integral is the intensity of energy flow, i.e. the rate of change of energy crossing a unit area whose normal is oriented in the direction of the vector $\boldsymbol{E} \times \boldsymbol{H}$ (see *Fig.* 2.6(a) and *Fig.* 2.6(b)). This is the so-called *Poynting vector*,

$$
\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}. \tag{2.37}
$$

(a) The direction of Poynting vector $\boldsymbol{S}$ (b) The energy crossing the area $\mathrm{d}\Gamma$
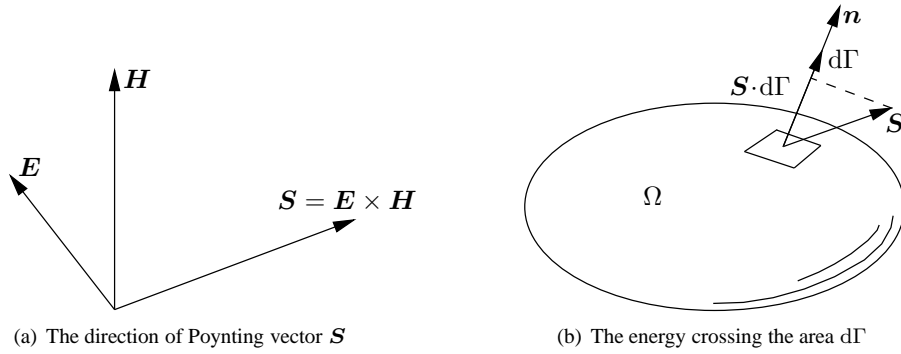
Fig. 2.6. The Poynting vector

The *energy density* $w = w(\boldsymbol{r}, t)$ in linear and isotropic media has the form

$$
w(\boldsymbol{r}, t) = \frac{1}{2} \left(\varepsilon E^2 + \mu H^2 \right). \tag{2.38}
$$

In nonlinear and anisotropic media the following expressions can be used to determine the electric and the magnetic *energy density*:

$$
w_e = \int_0^D \boldsymbol{E}(\boldsymbol{D}) \cdot \mathrm{d}\boldsymbol{D}, \quad \text{and} \quad w_m = \int_0^B \boldsymbol{H}(\boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{B}, \tag{2.39}
$$

which depend on the shape of characteristics given by the operators in (2.24).

2.1.2 Interface and boundary conditions

Field equations are valid for points in whose neighborhood the physical properties of the medium vary continuously, or in special case they are constants. Let us imagine that there is a surface bounding one medium with material parameters μ_1, σ_1 and ε_1 from

another one body with material parameters μ_2, σ_2 and ε_2, i.e. there occur sharp changes in parameters. In this situation *interface conditions* on the electromagnetic field quantities must be fulfilled.

In the case of static magnetic field problems and in eddy current field problems the open boundary is usually modeled by a sphere with a radius $r \to \infty$. The energy crossing the area of this boundary is equal to zero, because the variation of energy of electric and magnetic field is taking place inside the bounding sphere. This condition can only be fulfilled if

$$\lim_{r \to \infty} r^2 (\boldsymbol{E} \times \boldsymbol{H}) \cdot \boldsymbol{n} = 0, \tag{2.40}$$

i.e.

$$\lim_{r \to \infty} r|\boldsymbol{E}| = 0, \quad \text{and} \quad \lim_{r \to \infty} r|\boldsymbol{H}| = 0. \tag{2.41}$$

This means that the electric and the magnetic fields must vanish at infinity.

In other words, we can say that during the solution of Maxwell's equations, the interface and the boundary conditions have to be taken into account along the interface of the materials and at the boundary surfaces of the problem region. Here interface and boundary conditions are collected without deduction.

The used notations are presented in *Fig.* 2.7. The interface Γ (boundary surface) is placed between two regions denoted by Ω_1 and Ω_2 with material parameters μ_1, σ_1, ε_1 and μ_2, σ_2, ε_2, respectively. The normal unit vector $\boldsymbol{n}$ is conventionally equal to the outer normal unit vector of the medium filling the region Ω_1 (here $\boldsymbol{n} = \boldsymbol{n}_1$ and $\boldsymbol{n} = -\boldsymbol{n}_2$).
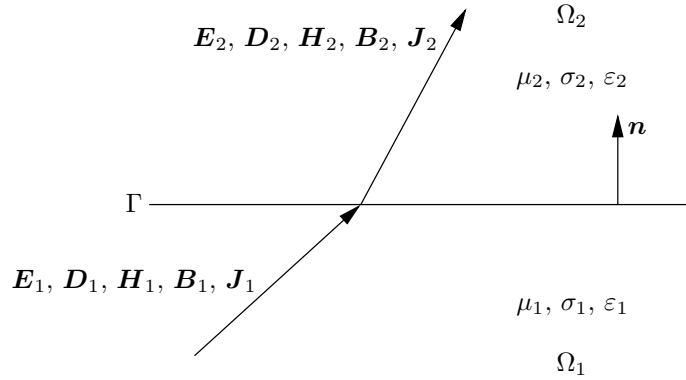
Fig. 2.7. For the behavior of electric and magnetic field quantities along the interface Γ

A. Electric and magnetic field intensity

The interface conditions prescribe continuity for the tangential component of the electric field intensity,

$$\boldsymbol{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = \boldsymbol{0}. \tag{2.42}$$

The surface current density K relates to the tangential component of the magnetic field intensity vector,

$$n \times (H_2 - H_1) = K. \tag{2.43}$$

The surface current density K is flowing on the surface tangentially to the normal vector unit n. If there is no surface current density on the interface, the tangential component of the magnetic field intensity is continuous,

$$n \times (H_2 - H_1) = 0. \tag{2.44}$$

If Γ denotes the bounding sphere of domain Ω_1, i.e. $E_2 = 0$ and $H_2 = 0$, moreover $E = E_1$ and $H = H_1$, then the *boundary conditions* can be formulated as

$$-n \times E = 0, \quad \text{or} \quad E \times n = 0, \tag{2.45}$$

and

$$-n \times H = K, \quad \text{or} \quad H \times n = K, \tag{2.46}$$

or (if $K = 0$)

$$-n \times H = 0, \quad \text{or} \quad H \times n = 0. \tag{2.47}$$

B. Electric and magnetic flux density, current density

On the interface of different dielectric materials the normal component of the electric flux density is continuous only if there is no surface charge, $\rho_s = 0$,

$$n \cdot (D_2 - D_1) = 0, \tag{2.48}$$

otherwise the normal component of the electric flux density has a jump on the interface,

$$n \cdot (D_2 - D_1) = \rho_s. \tag{2.49}$$

On the interface of different magnetic materials, the normal component of the magnetic flux density must be continuous,

$$n \cdot (B_2 - B_1) = 0. \tag{2.50}$$

The charge conservation law yields the continuity of the normal component of the conducting current in the case of eddy current field,

$$n \cdot (J_2 - J_1) = 0, \tag{2.51}$$

or generally

$$n \cdot (J_2 - J_1) + n \cdot \left(\frac{\partial D_2}{\partial t} - \frac{\partial D_1}{\partial t} \right) = 0 \tag{2.52}$$

is valid on the interface.

If Γ denotes the bounding sphere of domain Ω_1, i.e. $\boldsymbol{D}_2 = \boldsymbol{0}$, $\boldsymbol{B}_2 = \boldsymbol{0}$, $\boldsymbol{J}_2 = \boldsymbol{0}$ and $\partial \boldsymbol{D}_2/\partial t = \boldsymbol{0}$, moreover $\boldsymbol{D} = \boldsymbol{D}_1$, $\boldsymbol{B} = \boldsymbol{B}_1$ and $\boldsymbol{J} = \boldsymbol{J}_1$ then the *boundary conditions* can be formulated as

$$-\boldsymbol{n} \cdot \boldsymbol{D} = \rho_s, \quad \text{or} \quad \boldsymbol{D} \cdot \boldsymbol{n} = -\rho_s, \tag{2.53}$$

or

$$\boldsymbol{n} \cdot \boldsymbol{D} = 0, \quad \text{or} \quad \boldsymbol{D} \cdot \boldsymbol{n} = 0, \tag{2.54}$$

if $\rho_s = 0$ and

$$-\boldsymbol{n} \cdot \boldsymbol{B} = 0, \quad \text{or} \quad \boldsymbol{B} \cdot \boldsymbol{n} = 0, \tag{2.55}$$

and

$$-\boldsymbol{n} \cdot \boldsymbol{J} = 0, \quad \text{or} \quad \boldsymbol{J} \cdot \boldsymbol{n} = 0, \tag{2.56}$$

or generally

$$-\boldsymbol{n} \cdot \boldsymbol{J} - \boldsymbol{n} \cdot \frac{\partial \boldsymbol{D}}{\partial t} = 0, \quad \text{or} \quad \boldsymbol{J} \cdot \boldsymbol{n} + \frac{\partial \boldsymbol{D}}{\partial t} \cdot \boldsymbol{n} = 0. \tag{2.57}$$

It can be noted that equation (2.55) can be generalized as

$$-\boldsymbol{n} \cdot \boldsymbol{B} = b, \quad \text{or} \quad \boldsymbol{B} \cdot \boldsymbol{n} = -b, \tag{2.58}$$

where $b = b(\boldsymbol{r}, t)$ is the charge density of fictitious magnetic surface charges.

These general conditions will be rewritten in problem specific form in section 2.2.

2.1.3   Classification of Maxwell's equations

The usual classification of *Maxwell's equations* is presented here. In the simplest case the time variation of the field quantities can be neglected, i.e. $\partial/\partial t = 0$. This kind of fields is denoted as static field, when the magnetic, the electric and the current fields can be regarded independently, because there are no interactions between them. In time varying case when $\partial/\partial t \neq 0$, the magnetic and electric fields are coupled resulting the section of eddy current fields and wave propagation of electrodynamics.

(*i*) *Static magnetic field*. The time independent current density $\boldsymbol{J}_0 = \boldsymbol{J}_0(\boldsymbol{r})$ generates a time independent magnetic field intensity $\boldsymbol{H} = \boldsymbol{H}(\boldsymbol{r})$ and the time independent magnetic flux density $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{r})$. The equations are the following:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_0, \tag{2.59}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{2.60}$$

$$\boldsymbol{B} = \begin{cases} \mu_0 \boldsymbol{H}, & \text{in air,} \\ \mu_0 \mu_r \boldsymbol{H}, & \text{in magnetically linear material,} \\ \mu_0 (\boldsymbol{H} + \boldsymbol{M}), & \text{in magnetically nonlinear medium.} \end{cases} \tag{2.61}$$

In nonlinear medium, the *magnetization vector* $\boldsymbol{M} = \boldsymbol{M}(\boldsymbol{r})$ is depending on the magnetic field intensity vector, i.e. $\boldsymbol{M} = \mathscr{H}\{\boldsymbol{H}\}$. This operator can be described by hysteresis models denoted by $\boldsymbol{B} = \mathscr{B}\{\boldsymbol{H}\}$.

This *constitutive relationship* has an inverse form as well,

$$\boldsymbol{H} = \begin{cases} \nu_0 \boldsymbol{B}, & \text{in air,} \\ \nu_0 \nu_r \boldsymbol{B}, & \text{in magnetically linear material,} \\ \mathscr{B}^{-1}\{\boldsymbol{B}\}, & \text{in magnetically nonlinear medium.} \end{cases} \tag{2.62}$$

Here $\nu_0 = 1/\mu_0$, $\nu_r = 1/\mu_r$ are the reluctivity of vacuum and the relative reluctivity. In magnetically nonlinear medium it can be represented by an inverse *hysteresis operator*, $\boldsymbol{H} = \mathscr{B}^{-1}\{\boldsymbol{B}\}$.

The source current distribution is *solenoidal*, which is evident from the *current continuity equation* (2.8),

$$\nabla \cdot \boldsymbol{J}_0 = 0. \tag{2.63}$$

This means that all current lines either close upon themselves, or start and terminate at infinity.

This is the situation when magnetic field generated by current carrying coils is simulated, or the static behavior of electrical machines is the question. If $\boldsymbol{J}_0 = \boldsymbol{0}$, then a boundary value problem should be solved, which can be used to simulate e.g. the field of magnetic poles.

(*ii*) *Static electric field*. The source of electrostatic field described by the electric field intensity $\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{r})$ and by the flux density $\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{r})$ is the charge $\rho = \rho(\boldsymbol{r})$. The field equations are as follows:

$$\nabla \times \boldsymbol{E} = \boldsymbol{0}, \tag{2.64}$$

$$\nabla \cdot \boldsymbol{D} = \rho, \tag{2.65}$$

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}, \quad \text{or} \quad \boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}, \tag{2.66}$$

where $\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{r})$ is the polarization vector. For example, this is the situation when simulating the electrostatic field inside capacitors. Supposing static electric field is useful in high voltage applications, e.g. in transmission line problems.

(*iii*) *Currents in conducting materials*. The motion of charges results in currents flowing in the conducting material, which can be expressed by time independent equations

$$\nabla \times \boldsymbol{E} = \boldsymbol{0}, \tag{2.67}$$

$$\nabla \cdot \boldsymbol{J} = 0, \tag{2.68}$$

$$\boldsymbol{J} = \sigma \boldsymbol{E}, \quad \text{or} \quad \boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{E}_i). \tag{2.69}$$

Here $\boldsymbol{E}_i = \boldsymbol{E}_i(\boldsymbol{r})$ is the impressed electric field intensity. The term $\sigma \boldsymbol{E}_i$ can be represented by the source current density $\boldsymbol{J}_0 = \sigma \boldsymbol{E}_i$, i.e.

$$\boldsymbol{J} = \boldsymbol{J}_0 + \sigma \boldsymbol{E}. \tag{2.70}$$

These equations can be used efficiently when the current distribution of a coil cannot be supposed to be uniform, but can be computed as a current flow problem.

(*iv*) *Eddy current field*. In time varying case ($\partial/\partial t \neq 0$), the magnetic and electric fields are coupled, but the displacement current can be neglected in the usual frequency range, because $|\boldsymbol{J}| \gg |\partial \boldsymbol{D}/\partial t|$. The equations of 'quasi-static' field problems are as follows:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}, \tag{2.71}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{2.72}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{2.73}$$

$$\boldsymbol{B} = \begin{cases} \mu_0 \boldsymbol{H}, & \text{in air,} \\ \mu_0 \mu_r \boldsymbol{H}, & \text{in magnetically linear material,} \\ \mu_0(\boldsymbol{H} + \boldsymbol{M}), & \text{in magnetically nonlinear medium,} \end{cases} \tag{2.74}$$

$$\boldsymbol{J} = \sigma \boldsymbol{E}. \tag{2.75}$$

The constitutive relation of the magnetic material may have the inverse form (2.62).

This approximation is applicable in all power frequency applications involving metallic structures. The analysis of power losses in electrical machines or some nondestructive testing problems and the prediction of transient problems can be simulated by this kind of equations.

(*v*) *Wave propagation*. In this case all terms of Maxwell's equations are considered without any modification. This group of equations is useful in the analysis of waveguides, cavities or resonators at high frequencies.

In the following sections, we turn our attention mainly to the linear and nonlinear static magnetic field and the eddy current field problems.

2.2 Magnetostatics and eddy current field problems

In the case of magnetic materials, there are two groups of problem have to be formulated, the *static magnetic field* and the *eddy current field*. The typical structure – as they can be found in electrical engineering applications – with equations, interface and boundary conditions of these problems are formulated in this section.

2.2.1 Static magnetic fields

The typical structure of a static magnetic field problem can be seen in *Fig*. 2.8. In the case of static magnetic field, the currents flowing in coils i (represented by the source current density $\boldsymbol{J}_0$) generate electric field as well as magnetic field, but the magnetic field has no interaction with the electric field. In this case *Maxwell's equations* can be written as

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_0, \quad \text{in} \quad \Omega_0 \cup \Omega_m, \tag{2.76}$$

$$\nabla \cdot \boldsymbol{B} = 0, \quad \text{in} \quad \Omega_0 \cup \Omega_m, \tag{2.77}$$

$$\boldsymbol{B} = \begin{cases} \mu_0 \boldsymbol{H}, & \text{in air}, \Omega_0, \\ \mu_0 \mu_r \boldsymbol{H}, & \text{in magnetically linear material}, \Omega_m, \\ \mathscr{B}\{\boldsymbol{H}\} = \mu_o \boldsymbol{H} + \boldsymbol{R}, & \text{in magnetically nonlinear medium}, \Omega_m. \end{cases} \tag{2.78}$$

The last relation is defined in section 5.5.2. Taking the divergence of equation (2.76), the solenoidal property of the source current density can be obtained,

$$\nabla \cdot \boldsymbol{J}_0 = 0. \tag{2.79}$$

The constitutive relationship (2.78) can be used in its inverse form as well,

$$\boldsymbol{H} = \begin{cases} \nu_0 \boldsymbol{B}, & \text{in air}, \Omega_0, \\ \nu_0 \nu_r \boldsymbol{B}, & \text{in magnetically linear material}, \Omega_m, \\ \mathscr{B}^{-1}\{\boldsymbol{B}\} = \nu_o \boldsymbol{B} + \boldsymbol{I}, & \text{in magnetically nonlinear medium}, \Omega_m. \end{cases} \tag{2.80}$$

The last relation is defined in section 5.5.1.

A current excitation i (or a current density $\boldsymbol{J}_0$) with given amplitude is placed into air Ω_0 and symbol Ω_m represents the magnetic material (which can be represented by linear or nonlinear, isotropic or anisotropic, single valued or hysteretic model). The current density can be calculated from known current as

$$|\boldsymbol{J}_0| = \frac{Ni}{S_i}, \tag{2.81}$$

where N is the number of turns of the coil and S_i is the cross section area of the coil. The direction of $\boldsymbol{J}_0$ is generated by the geometry of the coil. The excitation current generates magnetic field $\boldsymbol{H}$ around the coil.

Maxwell's equations are valid in the problem region $\Omega = \Omega_0 \cup \Omega_m$. The problem region Ω is bounded by $\partial\Omega$, which has two disjunct parts Γ_H and Γ_B, i.e. $\partial\Omega = \Gamma_H \cup \Gamma_B$. Along the dashed line in *Fig.* 2.8 denoted by Γ_H, the tangential component of the
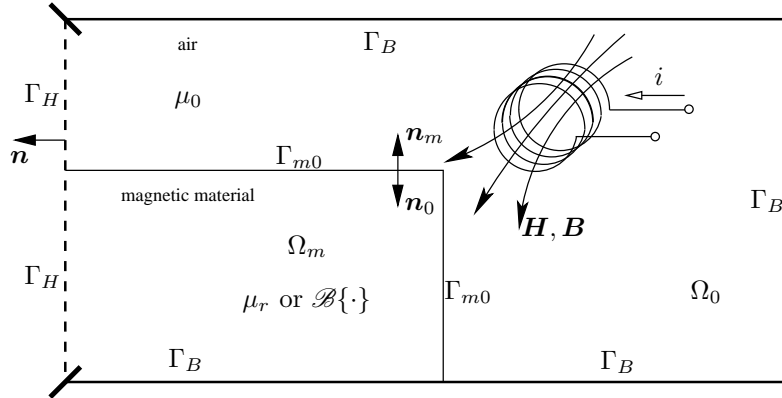
Fig. 2.8. Structure of a static magnetic field problem

magnetic field intensity is given by a known surface current density $\boldsymbol{K}$. If $\boldsymbol{K} = \boldsymbol{0}$ then Γ_H represents a symmetry plane. The boundary Γ_B is usually the closing boundary or a symmetry plane of the problem region, where the normal component of the magnetic flux density is vanishing. It is assumed to be known in a term b. Along the interface between the magnetic material and the air region Γ_{m0}, the tangential component of magnetic field intensity and the normal component of the magnetic flux density are continuous. These conditions can be formulated as (see section 2.1.2)

$$\boldsymbol{H} \times \boldsymbol{n} = \boldsymbol{K}, \quad \text{or} \quad \boldsymbol{H} \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_H, \tag{2.82}$$

and

$$\boldsymbol{B} \cdot \boldsymbol{n} = -b, \quad \text{or} \quad \boldsymbol{B} \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_B, \tag{2.83}$$

where $\boldsymbol{n}$ is the outer normal unit vector of the region, moreover

$$\boldsymbol{H}_0 \times \boldsymbol{n}_0 + \boldsymbol{H}_m \times \boldsymbol{n}_m = \boldsymbol{0}, \quad \text{and} \quad \boldsymbol{B}_0 \cdot \boldsymbol{n}_0 + \boldsymbol{B}_m \cdot \boldsymbol{n}_m = 0, \quad \text{on} \quad \Gamma_{m0}, \tag{2.84}$$

where $\boldsymbol{n}_0$, $\boldsymbol{n}_m$, $\boldsymbol{H}_0$, $\boldsymbol{H}_m$, $\boldsymbol{B}_0$ and $\boldsymbol{B}_m$ are the outer normal unit vectors of the region filled with air and with magnetic material, moreover the magnetic field intensity and the magnetic flux density in the appropriate region along the interface, respectively, and it is evident that $\boldsymbol{n}_0 = -\boldsymbol{n}_m$ along Γ_{m0}.

2.2.2 Eddy current fields

In time varying case ($\partial/\partial t \neq 0$), the electric and the magnetic fields are coupled. Currents flowing in coils generate magnetic field in the vicinity of the coil. The time variation of this magnetic field induces electric field that causes eddy currents flowing in conducting materials. The magnetic field generated by the eddy currents modifies the effect of sources. A typical structure of an eddy current field problem can be seen in *Fig.* 2.9.

In the eddy current free region denoted by Ω_n, there is a magnetic field with equations

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_0, \quad \text{in} \quad \Omega_n, \tag{2.85}$$

$$\nabla \cdot \boldsymbol{B} = 0, \quad \text{in} \quad \Omega_n, \tag{2.86}$$

$$\boldsymbol{B} = \mu \boldsymbol{H}, \quad \text{or} \quad \boldsymbol{H} = \nu \boldsymbol{B}, \quad \text{in} \quad \Omega_n. \tag{2.87}$$

Sometimes it is filled with air ($\mu = \mu_0$, or $\nu = \nu_0$).

At low frequencies and with normal conducting materials, the displacement currents are small compared with the conducting currents and they can be neglected, i.e.

$$|\partial \boldsymbol{D}/\partial t| \ll |\boldsymbol{J}|. \tag{2.88}$$

Consequently, the studied electromagnetic fields can be described by the 'quasi-static' *Maxwell's equations* written in differential form as

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}, \quad \text{in} \quad \Omega_c, \tag{2.89}$$
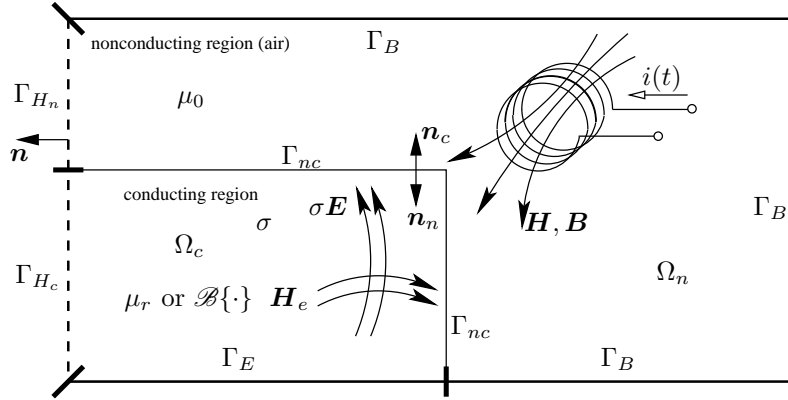
Fig. 2.9. Structure of an eddy current field problem

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \text{in} \quad \Omega_c, \tag{2.90}$$

$$\nabla \cdot \boldsymbol{B} = 0, \quad \text{in} \quad \Omega_c, \tag{2.91}$$

$$\boldsymbol{B} = \begin{cases} \mu_0\mu_r\boldsymbol{H}, & \text{in magnetically linear material,}\, \Omega_c, \\ \mathscr{B}\{\boldsymbol{H}\} = \mu_o\boldsymbol{H} + \boldsymbol{R}, & \text{in magnetically nonlinear medium,}\, \Omega_c, \end{cases} \tag{2.92}$$

$$\boldsymbol{J} = \sigma\boldsymbol{E}, \quad \text{in} \quad \Omega_c. \tag{2.93}$$

The last relation in (2.92) is defined in section 5.5.2. Of course, the *constitutive relation* (2.92) can be used in its inverse form as well,

$$\boldsymbol{H} = \begin{cases} \nu_0\nu_r\boldsymbol{B}, & \text{in magnetically linear material,}\, \Omega_c, \\ \mathscr{B}^{-1}\{\boldsymbol{B}\} = \nu_o\boldsymbol{B} + \boldsymbol{I}, & \text{in magnetically nonlinear medium,}\, \Omega_c. \end{cases} \tag{2.94}$$

The last relation is defined in section 5.5.1.

Taking the divergence of equation (2.89), the solenoidal property of the induced current density can be obtained,

$$\nabla \cdot \boldsymbol{J} = 0, \tag{2.95}$$

meaning that eddy currents close upon themselves.

A current excitation $i(t)$ (or a source current density $\boldsymbol{J}_0(t)$) with given signal shape is placed into the nonconducting region Ω_n (which is usually the air region) and the eddy current distribution $\sigma\boldsymbol{E}$ is unknown in the conducting material Ω_c (the corresponding magnetic field generated by eddy currents is denoted by $\boldsymbol{H}_e$ in the illustration in *Fig.* 2.9). Region Ω_c is sometimes filled with magnetic material (which can be represented by linear or nonlinear, isotropic or anisotropic, single valued or hysteretic model).

Static and 'quasi-static' Maxwell's equations are valid in the eddy current free region Ω_n and in the eddy current region Ω_c, respectively. The two regions are coupled through the interface Γ_{nc}. The eddy current free region Ω_n is bounded by two disjunct parts Γ_{H_n}

and Γ_B, the eddy current region is bounded by two disjunct parts Γ_{H_c} and Γ_E. Generally, the problem region $\Omega = \Omega_n \cup \Omega_c$ is bounded by $\partial\Omega = \Gamma_{H_n} \cup \Gamma_B \cup \Gamma_{H_c} \cup \Gamma_E$.

Along the dashed line placed in the eddy current free region in *Fig.* 2.9 denoted by Γ_{H_n}, the tangential component of the magnetic field intensity is given by a known surface current density $\boldsymbol{K}$. If $\boldsymbol{K} = \boldsymbol{0}$ then Γ_{H_n} represents a symmetry plane, where the tangential component of the magnetic field intensity is vanishing. The boundary Γ_B is usually the closing boundary of the problem region, where the normal component of the magnetic flux density is vanishing. It is assumed to be known by a term b. The dashed line signed in the eddy current region Γ_{H_c} usually denotes a symmetry plane, where the tangential component of the magnetic field intensity is zero and Γ_E is the boundary where the tangential component of the electric field intensity is vanishing. The surface Γ_E may also model electrodes where a voltage source is connected to the conductor. Along the interface between the two disjunct region Γ_{nc}, the tangential component of magnetic field intensity and the normal component of the magnetic flux density are continuous, moreover the normal component of the induced eddy currents are equal to zero. These conditions can be formulated as (see section 2.1.2)

$$\boldsymbol{H} \times \boldsymbol{n} = \boldsymbol{K}, \quad \text{or} \quad \boldsymbol{H} \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{H_n}, \tag{2.96}$$

and

$$\boldsymbol{B} \cdot \boldsymbol{n} = -b, \quad \text{or} \quad \boldsymbol{B} \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_B, \tag{2.97}$$

and

$$\boldsymbol{H} \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.98}$$

and

$$\boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_E, \tag{2.99}$$

where $\boldsymbol{n}$ is the outer normal unit vector of the region, moreover

$$\boldsymbol{H}_c \times \boldsymbol{n}_c + \boldsymbol{H}_n \times \boldsymbol{n}_n = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc}, \tag{2.100}$$

and

$$\boldsymbol{B}_c \cdot \boldsymbol{n}_c + \boldsymbol{B}_n \cdot \boldsymbol{n}_n = 0, \quad \text{on} \quad \Gamma_{nc}, \tag{2.101}$$

and

$$\boldsymbol{J} \cdot \boldsymbol{n}_c = 0, \quad \text{on} \quad \Gamma_{nc}, \tag{2.102}$$

where $\boldsymbol{n}_n$, $\boldsymbol{n}_c$, $\boldsymbol{H}_n$, $\boldsymbol{H}_c$, $\boldsymbol{B}_n$ and $\boldsymbol{B}_c$ are the outer normal unit vector of the region filled with air and with conducting material, moreover the magnetic field intensity and the magnetic flux density vectors in the appropriate region on the boundary, respectively and it is evident that $\boldsymbol{n}_n = -\boldsymbol{n}_c$ along Γ_{nc}.

The magnetic field must satisfy the initial conditions, as well

$$\boldsymbol{B}_n(t = 0) = \boldsymbol{B}_{n,0}, \quad \text{in} \quad \Omega_n, \quad \text{and} \quad \boldsymbol{B}_c(t = 0) = \boldsymbol{B}_{c,0}, \quad \text{in} \quad \Omega_c. \tag{2.103}$$

As a conclusion of the boundary conditions on Γ_{H_c} and on Γ_E, the normal component of the eddy current density is vanishing on the boundary Γ_{H_c} because of equations (2.98) and (2.89), while on boundary Γ_E, the normal component of magnetic flux density is equal to zero coming from equations (2.99) and (2.90), i.e.

$$\boldsymbol{J} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_{H_c}, \tag{2.104}$$

and

$$\boldsymbol{B} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_E. \tag{2.105}$$

As a proof, let us take the normal component of the first Maxwell's equation (2.89). It results in the expression $(\nabla \times \boldsymbol{H}) \cdot \boldsymbol{n} = \boldsymbol{J} \cdot \boldsymbol{n}$, i.e. $\nabla \cdot (\boldsymbol{H} \times \boldsymbol{n}) = \boldsymbol{J} \cdot \boldsymbol{n}$. The left-hand side of this equation is equal to zero on Γ_{H_c} according to (2.98), i.e. $\boldsymbol{J} \cdot \boldsymbol{n} = 0$ on Γ_{H_c}. The equation $\boldsymbol{B} \cdot \boldsymbol{n} = 0$ can be obtained similarly.

2.3   Potential formulations in static magnetic and eddy current field problems

There are several potential formulations applicable to calculate the electromagnetic field quantities, fundamentally scalar and vector potentials can be used [3–11, 13, 20–23, 25, 28, 41–43, 47, 49, 52, 65, 66, 72–75, 79–83, 87, 93]. The aim of potential formulations is to reduce the solution of Maxwell's equations to the solution of different type of partial differential equations at prescribed boundary conditions.

This section deals with the potential formulations of static magnetic field and eddy current field problems.

2.3.1   Static magnetic fields

The general definition of *static magnetic field* problems can be found in section 2.2.1. The static magnetic field is defined by Maxwell's equations (2.76), (2.77), constitutive relations in (2.78) or in (2.80), moreover the boundary and interface conditions (2.82), (2.83) and (2.84).

The static magnetic field can be described by the *reduced magnetic scalar potential* Φ, by the *total magnetic scalar potential* Ψ, by the combination of these scalar potentials ($\Phi - \Psi$-formulation), or by the *magnetic vector potential* $\boldsymbol{A}$. The combination of these formulations is also valid, resulting the $\boldsymbol{A} - \Phi$-formulation or the $\boldsymbol{A} - \boldsymbol{A} - \Phi$-formulation.

It is noted that the most economical way is using the scalar potential, however, it results in some problems, especially when iron parts with high permeability are present. In this situation the combination of the two scalar potentials can be used, but this introduces some extra conditions on the iron/air interfaces. The known rotational part of the magnetic field intensity according to coil's current must be calculated beforehand. This potential is called impressed current vector potential $\boldsymbol{T}_0$. The use of magnetic vector potential is more general, however, the number of unknowns is increasing when simulating 3D problems.

A. The reduced magnetic scalar potential, the Φ-formulation

The magnetic field intensity vector can be split in two parts as

$$\boldsymbol{H} = \boldsymbol{T}_0 + \boldsymbol{H}_m. \tag{2.106}$$

The curl of the so-called *impressed current vector potential* $\boldsymbol{T}_0$ is equal to the source current density $\boldsymbol{J}_0$ and $\boldsymbol{H}_m$ is nonrotational,

$$\nabla \times \boldsymbol{T}_0 = \boldsymbol{J}_0, \quad \text{and} \quad \nabla \times \boldsymbol{H}_m = \boldsymbol{0}. \tag{2.107}$$

The first Maxwell's equation (2.76) can be fulfilled by this decomposition,

$$\nabla \times \boldsymbol{H} = \nabla \times (\boldsymbol{T}_0 + \boldsymbol{H}_m) = \nabla \times \boldsymbol{T}_0 + \nabla \times \boldsymbol{H}_m = \boldsymbol{J}_0. \tag{2.108}$$

The divergence of $\boldsymbol{T}_0$ can be selected according to *Coulomb gauge*, i.e.

$$\nabla \cdot \boldsymbol{T}_0 = 0, \tag{2.109}$$

which selection can be useful when creating the function $\boldsymbol{T}_0$.

It is important to note that the two terms $\boldsymbol{T}_0$ and $\boldsymbol{H}_m$ can be calculated separately. The first step is the generation of the function $\boldsymbol{T}_0$ according to the source current density $\boldsymbol{J}_0$ satisfying the first equation in (2.107), then $\boldsymbol{H}_m$ can be represented by a scalar potential.

Generating the function $\boldsymbol{T}_0$. The first step is to determine the *impressed current vector potential $\boldsymbol{T}_0$*. There are many possibilities for the construction of the source term $\boldsymbol{T}_0$ from the known source current density $\boldsymbol{J}_0$. Here the four most attractive and the most widely used solutions are formulated. The impressed field can be calculated by using the Biot–Savart's law, $\boldsymbol{T}_0$ can be built up by simple analytical expressions if the geometry of coils is simple, or by the solution of a partial differential equation with appropriate boundary conditions. It must be noted that $\boldsymbol{T}_0$ is calculated in free space, i.e. $\mu = \mu_0$ must be set everywhere in the problem region.

(*i*) *Applying the Biot–Savart's law.* The impressed current vector potential $\boldsymbol{T}_0$ can be selected as the magnetic field of coils in free space $\boldsymbol{H}_s$ calculated by *Biot–Savart's law* (*Fig.* 2.10),

$$\boldsymbol{T}_0(\boldsymbol{r}_f) \equiv \boldsymbol{H}_s(\boldsymbol{r}_f) = \frac{1}{4\pi} \int_{\Omega_J} \frac{\boldsymbol{J}_0(\boldsymbol{r}_J) \times (\boldsymbol{r}_f - \boldsymbol{r}_J)}{|\boldsymbol{r}_f - \boldsymbol{r}_J|^3} \mathrm{d}\Omega_J, \tag{2.110}$$

or (*Fig.* 2.11)

$$\boldsymbol{T}_0(\boldsymbol{r}_f) \equiv \boldsymbol{H}_s(\boldsymbol{r}_f) = \frac{I_0}{4\pi} \oint_{l_I} \frac{\mathrm{d}\boldsymbol{l} \times (\boldsymbol{r}_f - \boldsymbol{r}_I)}{|\boldsymbol{r}_f - \boldsymbol{r}_I|^3}. \tag{2.111}$$

Here $\boldsymbol{r}_f$ is the field point where $\boldsymbol{H}_s(\boldsymbol{r}_f)$ is calculated, $\boldsymbol{r}_J$ and $\boldsymbol{r}_I$ are the source points where $\boldsymbol{J}_0(\boldsymbol{r}_J)$ and $I_0(\boldsymbol{r}_I)$ is given. Symbol 'O' denotes the origin of the coordinate system. It results in a divergence-free impressed current vector potential.
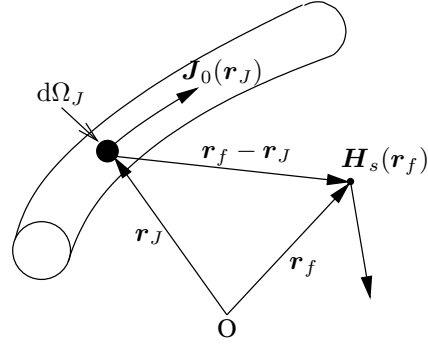
Fig. 2.10. Magnetic field intensity $\boldsymbol{H}_s$ generated by source current density
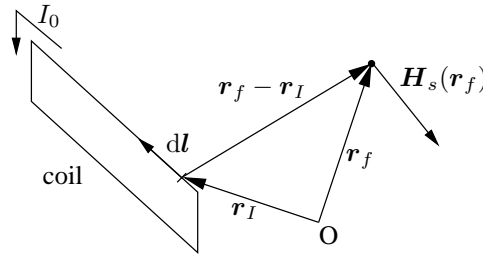
Fig. 2.11. Magnetic field intensity $\boldsymbol{H}_s$ generated by a filamentary conductor

The source term should be calculated with a given value of current ($I_0 = 1\text{A}$ is a good choice), or a given value of current density ($|\boldsymbol{J}_0| = 1\text{A/m}^2$). Then the calculated source term $\boldsymbol{T}_0$ can be multiplied by the actual value of current during simulation, e.g. when $i(t)$ is sinusoidal or $I = 0.2\text{A}$, etc.

Simple analytical formulation can be used when the geometry of the coil is simple (for example the formula $H = I_0/2r\pi$ can be used in case of one filamentary conductor in 2D simulations). However, this step can be a very time consuming task, especially in 3D situations.

(*ii*) *Creating a simple function satisfying* $\nabla \times \boldsymbol{T}_0 = \boldsymbol{J}_0$. Extremely simple analytical expressions can be formulated in case of simple coil shapes as it is illustrated by a cylindrical coil in *Fig*. 2.12. In this case the integration can be avoided and a simple, tangentially continuous analytical expression can be built up.

If $\boldsymbol{T}_0$ has only one component in the z direction, i.e. $\boldsymbol{T}_0 = T_{0,z}\mathbf{e}_z$ moreover $T_{0,z} = T_{0,z}(r)$ or $T_{0,z} = T_{0,z}(x,y)$, then source current density in the $r - \varphi$ or in the $x - y$ plane can be represented easily.

In the illustration shown in *Fig*. 2.12, the cylindrical shaped coil is placed around the z-axis with center at the origin of a cylindrical coordinate system. The inner and outer radius of the coil is R and R', respectively, the height of the coil is h and the current density is $\boldsymbol{J}_0 = |\boldsymbol{J}_0|\mathbf{e}_\varphi$. The vector potential $\boldsymbol{T}_0$ has only one component
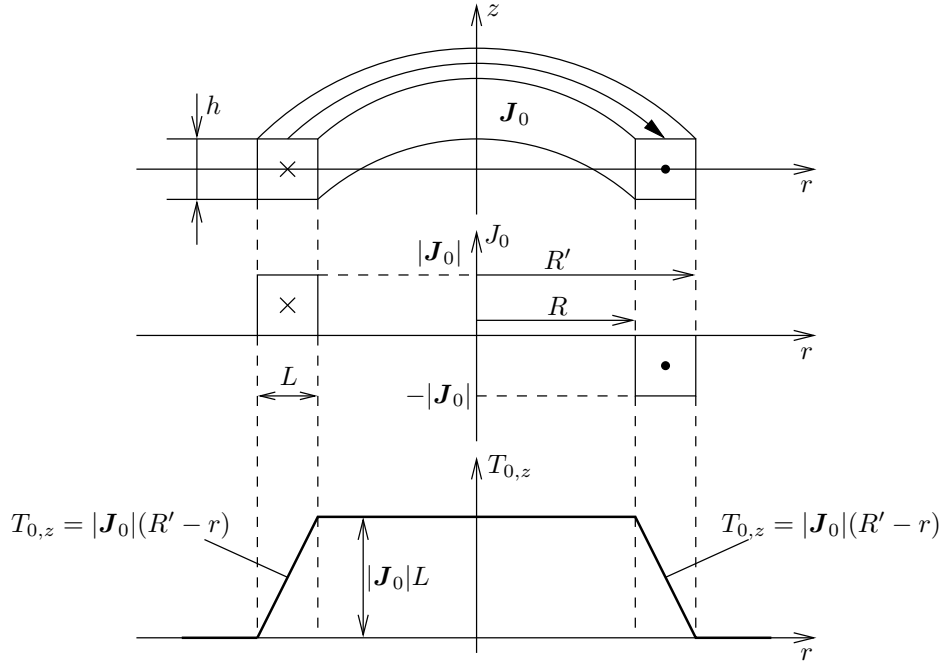
Fig. 2.12. Creating $\boldsymbol{T}_0$ for cylindrical shaped coil

in the z direction, i.e. $\boldsymbol{T}_0 = T_{0,z}\mathbf{e}_z$, where

$$T_{0,z} = \begin{cases} 0, & \text{if } |z| \geq h/2 \text{ or } r > R', \\ |\boldsymbol{J}_0|L, & \text{if } |z| < h/2 \text{ and } r \leq R, \\ |\boldsymbol{J}_0|(R'-r), & \text{if } |z| < h/2 \text{ and } r > R \text{ and } r \leq R'. \end{cases} \qquad (2.112)$$

In this case $\boldsymbol{T}_0$ is equal to zero outside the coil, it is constant inside the coil where the current density is equal to zero and it is varying linearly inside the coil where the current density is not zero. The divergence of $\boldsymbol{T}_0$ is not equal to zero.

(*iii*) *Minimizing an appropriate functional.* The following functional can be built up to find out the source term $\boldsymbol{T}_0$:

$$\mathscr{F}\{\boldsymbol{T}_0\} = \int_\Omega \left[|\nabla \times \boldsymbol{T}_0 - \boldsymbol{J}_0|^2 + (\nabla \cdot \boldsymbol{T}_0)^2 \right] \mathrm{d}\Omega, \qquad (2.113)$$

which has to be minimized. This is equivalent to the solution of the following partial differential equations defined in free space:

$$\nabla \times \boldsymbol{T}_0 = \boldsymbol{J}_0, \quad \text{in} \quad \Omega, \qquad (2.114)$$

$$\nabla \cdot \boldsymbol{T}_0 = 0, \quad \text{in} \quad \Omega. \qquad (2.115)$$

The boundary conditions

$$\boldsymbol{T}_0 \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_H, \qquad (2.116)$$

$$\boldsymbol{T}_0 \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_B \tag{2.117}$$

must also be satisfied ($\partial\Omega = \Gamma = \Gamma_H \cup \Gamma_B$). A partial differential equation with appropriate boundary conditions can be formulated for the mentioned functional (2.113) what is an advantage of this formulation.

According to the principle of minimum energy, the *functional* (2.113) has minimum value only if the vector field $\boldsymbol{T}_0$ satisfies the partial differential equations (2.114) and (2.115). This minimum is a stable configuration of the vector potential $\boldsymbol{T}_0$.

Let us first extract the *functional* (2.113),

$$\mathscr{F}\{\boldsymbol{T}_0\} = \int_\Omega \Big[|\nabla \times \boldsymbol{T}_0|^2 - 2(\nabla \times \boldsymbol{T}_0) \cdot \boldsymbol{J}_0 + |\boldsymbol{J}_0|^2 + (\nabla \cdot \boldsymbol{T}_0)^2 \Big] d\Omega, \tag{2.118}$$

and let us introduce a small *variation* of the vector potential $\boldsymbol{T}_0$ as $\boldsymbol{T}_0 + \alpha\boldsymbol{\delta T}_0$ (α is a small positive number), which results in a variation of the field energy around the stable configuration of the potential $\boldsymbol{T}_0$,

$$\begin{aligned}
\mathscr{F}\{\boldsymbol{T}_0 + \alpha\boldsymbol{\delta T}_0\} = \int_\Omega \Big[& |\nabla \times (\boldsymbol{T}_0 + \alpha\boldsymbol{\delta T}_0)|^2 \\
& -2[\nabla \times (\boldsymbol{T}_0 + \alpha\boldsymbol{\delta T}_0)] \cdot \boldsymbol{J}_0 + |\boldsymbol{J}_0|^2 \\
& +[\nabla \cdot (\boldsymbol{T}_0 + \alpha\boldsymbol{\delta T}_0)]^2 \Big] d\Omega,
\end{aligned} \tag{2.119}$$

i.e.

$$\begin{aligned}
\mathscr{F}\{\boldsymbol{T}_0 + \alpha\boldsymbol{\delta T}_0\} = \int_\Omega \Big[& |\nabla \times \boldsymbol{T}_0|^2 + 2\alpha\,(\nabla \times \boldsymbol{T}_0) \cdot (\nabla \times \boldsymbol{\delta T}_0) \\
& +\alpha^2\,|\nabla \times \boldsymbol{\delta T}_0|^2 - 2(\nabla \times \boldsymbol{T}_0) \cdot \boldsymbol{J}_0 - 2\alpha\,(\nabla \times \boldsymbol{\delta T}_0) \cdot \boldsymbol{J}_0 \\
& +|\boldsymbol{J}_0|^2 + (\nabla \cdot \boldsymbol{T}_0)^2 + 2\alpha(\nabla \cdot \boldsymbol{T}_0)(\nabla \cdot \boldsymbol{\delta T}_0) \\
& +\alpha^2\,(\nabla \cdot \boldsymbol{\delta T}_0)^2 \Big] d\Omega.
\end{aligned} \tag{2.120}$$

This variation can be represented in *Taylor series* approximation as

$$\mathscr{F}\{\boldsymbol{T}_0 + \alpha\boldsymbol{\delta T}_0\} = \mathscr{F}\{\boldsymbol{T}_0\} + \alpha\delta\mathscr{F}\{\boldsymbol{T}_0, \boldsymbol{\delta T}_0\} + \alpha^2\delta^2\mathscr{F}\{\boldsymbol{\delta T}_0\}. \tag{2.121}$$

Here

$$\begin{aligned}
\delta\mathscr{F}\{\boldsymbol{T}_0, \boldsymbol{\delta T}_0\} = 2\int_\Omega \Big[& (\nabla \times \boldsymbol{T}_0) \cdot (\nabla \times \boldsymbol{\delta T}_0) - (\nabla \times \boldsymbol{\delta T}_0) \cdot \boldsymbol{J}_0 \\
& +(\nabla \cdot \boldsymbol{T}_0)(\nabla \cdot \boldsymbol{\delta T}_0) \Big] d\Omega,
\end{aligned} \tag{2.122}$$

and

$$\delta^2 \mathscr{F}\{\boldsymbol{\delta T}_0\} = \int_\Omega \left[|\nabla \times \boldsymbol{\delta T}_0|^2 + (\nabla \cdot \boldsymbol{\delta T}_0)^2 \right] \mathrm{d}\Omega. \tag{2.123}$$

In concern with the stationary concept, the *first variation* $\delta\mathscr{F}\{\boldsymbol{T}_0, \boldsymbol{\delta T}_0\}$ has to be disappear and the *second variation* $\delta^2\mathscr{F}\{\boldsymbol{\delta T}_0\}$ has to be positive. In this case $\mathscr{F}\{\boldsymbol{T}_0\}$ leads indeed to its minimum value. The definition of the first variation is (which must be equal to zero)

$$\delta\mathscr{F}\{\boldsymbol{T}_0, \boldsymbol{\delta T}_0\} = \left. \frac{\partial\mathscr{F}\{\boldsymbol{T}_0 + \alpha\boldsymbol{\delta T}_0\}}{\partial\alpha} \right|_{\alpha=0} = 0, \tag{2.124}$$

from which

$$\int_\Omega \Big[(\nabla \times \boldsymbol{T}_0) \cdot (\nabla \times \boldsymbol{\delta T}_0) - (\nabla \times \boldsymbol{\delta T}_0) \cdot \boldsymbol{J}_0 \\ + (\nabla \cdot \boldsymbol{T}_0)(\nabla \cdot \boldsymbol{\delta T}_0) \Big] \mathrm{d}\Omega = 0, \tag{2.125}$$

and it is easy to see that the second variation (2.123) is usually positive. Using the identity

$$\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \times \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \times \boldsymbol{v}, \tag{2.126}$$

with $\boldsymbol{u} = \nabla \times \boldsymbol{T}_0$, $\boldsymbol{v} = \boldsymbol{\delta T}_0$ in the first integral and with $\boldsymbol{u} = \boldsymbol{J}_0$, $\boldsymbol{v} = \boldsymbol{\delta T}_0$ in the second integral and the identity

$$\nabla \cdot (\varphi \, \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla\varphi + \varphi\nabla \cdot \boldsymbol{v}, \tag{2.127}$$

with $\varphi = \nabla \cdot \boldsymbol{T}_0$, $\boldsymbol{v} = \boldsymbol{\delta T}_0$ in the third term results in

$$\int_\Omega \boldsymbol{\delta T}_0 \cdot (\nabla \times \nabla \times \boldsymbol{T}_0 - \nabla \times \boldsymbol{J}_0) \, \mathrm{d}\Omega \\ + \oint_\Gamma \boldsymbol{\delta T}_0 \cdot [(\nabla \times \boldsymbol{T}_0 - \boldsymbol{J}_0) \times \boldsymbol{n}] \, \mathrm{d}\Gamma \\ - \int_\Omega \boldsymbol{\delta T}_0 \cdot (\nabla\nabla \cdot \boldsymbol{T}_0) \, \mathrm{d}\Omega + \oint_\Gamma (\boldsymbol{\delta T}_0 \cdot \boldsymbol{n}) \, \nabla \cdot \boldsymbol{T}_0 \, \mathrm{d}\Gamma = 0. \tag{2.128}$$

The first boundary integral can be rewritten according to the two parts Γ_H and Γ_B of the boundary Γ,

$$\oint_\Gamma \boldsymbol{\delta T}_0 \cdot [(\nabla \times \boldsymbol{T}_0 - \boldsymbol{J}_0) \times \boldsymbol{n}] \, \mathrm{d}\Gamma \\ = \int_{\Gamma_H \cup \Gamma_B} \boldsymbol{\delta T}_0 \cdot [(\nabla \times \boldsymbol{T}_0 - \boldsymbol{J}_0) \times \boldsymbol{n}] \, \mathrm{d}\Gamma. \tag{2.129}$$

On Γ_H the Dirichlet type boundary condition $\boldsymbol{T}_0 \times \boldsymbol{n} = \boldsymbol{0}$ must be satisfied, i.e. the tangential component of the variation is equal to zero, $\boldsymbol{\delta T}_0 \times \boldsymbol{n} = \boldsymbol{0}$. According to the identity

$$\boldsymbol{\delta T}_0 \cdot [(\nabla \times \boldsymbol{T}_0 - \boldsymbol{J}_0) \times \boldsymbol{n}] = (\nabla \times \boldsymbol{T}_0 - \boldsymbol{J}_0) \cdot (\boldsymbol{n} \times \boldsymbol{\delta T}_0) \\ = -(\nabla \times \boldsymbol{T}_0 - \boldsymbol{J}_0) \cdot (\boldsymbol{\delta T}_0 \times \boldsymbol{n}), \tag{2.130}$$

the boundary integral on Γ_H is vanishing. On the rest part Γ_B the Neumann type boundary condition $(\nabla \times \boldsymbol{T}_0 - \boldsymbol{J}_0) \times \boldsymbol{n} = \boldsymbol{0}$ must be prescribed to vanish the surface integral term.

The second boundary integral can also be rewritten in the form

$$\oint_\Gamma (\boldsymbol{\delta T}_0 \cdot \boldsymbol{n}) \, \nabla \cdot \boldsymbol{T}_0 \, \mathrm{d}\Gamma = \int_{\Gamma_H \cup \Gamma_B} (\boldsymbol{\delta T}_0 \cdot \boldsymbol{n}) \, \nabla \cdot \boldsymbol{T}_0 \, \mathrm{d}\Gamma. \tag{2.131}$$

On the part Γ_B the Dirichlet type boundary condition $\boldsymbol{T}_0 \cdot \boldsymbol{n} = 0$ must be specified, consequently $\boldsymbol{\delta T}_0 \cdot \boldsymbol{n} = 0$. On the part Γ_H the Neumann type boundary condition $\nabla \cdot \boldsymbol{T}_0 = 0$ should be applied to vanish the boundary integral term. This means that an extra boundary condition must be specified on Γ_H and on Γ_B.

Finally, the integral equation (2.128) results in the following partial differential equation and the boundary conditions:

$$\nabla \times \nabla \times \boldsymbol{T}_0 - \nabla\nabla \cdot \boldsymbol{T}_0 = \nabla \times \boldsymbol{J}_0, \quad \text{in} \quad \Omega, \tag{2.132}$$

$$\boldsymbol{T}_0 \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_H, \tag{2.133}$$

$$\nabla \cdot \boldsymbol{T}_0 = 0, \quad \text{on} \quad \Gamma_H, \tag{2.134}$$

$$\boldsymbol{T}_0 \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_B, \tag{2.135}$$

$$(\nabla \times \boldsymbol{T}_0) \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_B. \tag{2.136}$$

This means that the integral equation (2.128) can be fulfilled for any variation $\boldsymbol{\delta T}_0$ if the partial differential equation (2.132) and the boundary conditions defined in (2.133)–(2.136) are solved. The solution of this boundary value problem is a divergence free impressed current vector potential and it can be solved by any numerical field calculation procedure (in this book we are focusing on the Finite Element Method).

This formulation results in *gauged impressed current vector potential*.

(*iv*) *Minimizing a functional combined with an appropriate numerical technique.* The following functional can be built up too, to find out the source term $\boldsymbol{T}_0$:

$$\mathscr{F}\{\boldsymbol{T}_0\} = \int_\Omega |\nabla \times \boldsymbol{T}_0 - \boldsymbol{J}_0|^2 \, \mathrm{d}\Omega, \tag{2.137}$$

which has to be minimized. This is equivalent to the following partial differential equation defined in free space:

$$\nabla \times \boldsymbol{T}_0 = \boldsymbol{J}_0, \quad \text{in} \quad \Omega, \tag{2.138}$$

and the boundary conditions are

$$\boldsymbol{T}_0 \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_H, \tag{2.139}$$

$$\boldsymbol{T}_0 \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_B. \tag{2.140}$$

A partial differential equation with boundary conditions can be formulated for the mentioned functional (2.137), which can be obtained in a similar way presented in the last item, only the terms according to $\nabla \cdot \boldsymbol{T}_0 = 0$ are missing. The result can be formulated as

$$\nabla \times \nabla \times \boldsymbol{T}_0 = \nabla \times \boldsymbol{J}_0, \quad \text{in} \quad \Omega, \tag{2.141}$$

$$\boldsymbol{T}_0 \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_H, \tag{2.142}$$

$$\boldsymbol{T}_0 \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_B. \tag{2.143}$$

The solution of this boundary value problem is a possible choice of the impressed current vector potential. It can be solved by a numerical field calculation procedure, which is not sensitive to Coulomb gauge.

This formulation results in an *ungauged impressed current vector potential*.

Finally $\boldsymbol{T}_0$ can be regarded as known, because this quantity is calculated before the numerical simulation.

Calculating the nonrotational part $\boldsymbol{H}_m$. The second step of Φ-formulation is the determination of the nonrotational part of the magnetic field intensity $\boldsymbol{H}_m$ in (2.106). It can be derived from the negative gradient of a *magnetic scalar potential* Φ,

$$\boldsymbol{H}_m = -\nabla \Phi, \tag{2.144}$$

because of the identity $\nabla \times (\nabla \varphi) \equiv \boldsymbol{0}$ for any scalar function $\varphi = \varphi(\boldsymbol{r})$ (or $\varphi = \varphi(\boldsymbol{r}, t)$). By this formulation the magnetic field intensity can be written as

$$\boldsymbol{H} = \boldsymbol{T}_0 - \nabla \Phi, \tag{2.145}$$

which satisfies equation (2.76) exactly. The magnetic scalar potential Φ is usually called the *reduced magnetic scalar potential*, because the source term is fundamentally hidden in $\boldsymbol{T}_0$.

Applying the linearized form of the nonlinear constitutive relation in (2.78) results in the magnetic flux density

$$\boldsymbol{B} = \mu_o \left(\boldsymbol{T}_0 - \nabla \Phi \right) + \boldsymbol{R}. \tag{2.146}$$

The divergence of magnetic flux density is equal to zero according to (2.77). Finally the linear partial differential equation of the problem has the form

$$\nabla \cdot (\mu_o \nabla \Phi) = \nabla \cdot (\mu_o \boldsymbol{T}_0) + \nabla \cdot \boldsymbol{R}, \quad \text{in} \quad \Omega, \tag{2.147}$$

which is a generalized *Laplace–Poisson equation*.

Partial differential equation (2.147) can be obtained in another way as well. Starting from the solenoidal property of the source current density defined by the equation (2.79), $\boldsymbol{J}_0$ can be written by the curl of the impressed current vector potential according to the identity $\nabla \cdot (\nabla \times \boldsymbol{v}) \equiv 0$ for any vector function $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r})$ (or $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r}, t)$),

$$\boldsymbol{J}_0 = \nabla \times \boldsymbol{T}_0. \tag{2.148}$$

Substituting the relation (2.148) into Maxwell's equation (2.76) results in

$$\nabla \times \boldsymbol{H} = \nabla \times \boldsymbol{T}_0 \quad \Rightarrow \quad \nabla \times (\boldsymbol{H} - \boldsymbol{T}_0) = \boldsymbol{0}. \tag{2.149}$$

The curl-less vector field $\boldsymbol{H} - \boldsymbol{T}_0$ can be determined by the negative gradient of the magnetic scalar potential Φ, i.e.

$$\boldsymbol{H} - \boldsymbol{T}_0 = -\nabla\Phi \quad \Rightarrow \quad \boldsymbol{H} = \boldsymbol{T}_0 - \nabla\Phi. \tag{2.150}$$

Substituting the relation of magnetic field intensity in (2.150) into (2.77) and using the linearized constitutive relation in (2.78) also leads to the partial differential equation (2.147).

Boundary conditions. Let us now define the boundary and interface conditions of the Φ-formulation.

The impressed current vector potential $\boldsymbol{T}_0$ must satisfy the symmetry conditions on symmetry planes, if any. The tangential component of the impressed current vector potential must vanish on Γ_H and its normal component must be equal to zero on Γ_B as it was mentioned in (2.133) and in (2.135).

The tangential component of the magnetic field intensity can be set to the given value as follows on Γ_H:

$$\boldsymbol{H} \times \boldsymbol{n} = (\boldsymbol{T}_0 - \nabla\Phi) \times \boldsymbol{n} = \boldsymbol{K}, \quad \text{on} \quad \Gamma_H. \tag{2.151}$$

A vector $\boldsymbol{H} = \boldsymbol{H}(\boldsymbol{r})$ can be written by the two orthogonal components as (see *Fig.* 2.13)

$$\boldsymbol{H} = (\boldsymbol{H} \cdot \boldsymbol{n}) \cdot \boldsymbol{n} + \boldsymbol{n} \times (\boldsymbol{H} \times \boldsymbol{n}). \tag{2.152}$$

The second term is the tangential component, i.e.

$$\boldsymbol{n} \times (\boldsymbol{H} \times \boldsymbol{n}) = \boldsymbol{n} \times [(\boldsymbol{T}_0 - \nabla\Phi) \times \boldsymbol{n}] = \boldsymbol{n} \times \boldsymbol{K}, \tag{2.153}$$

which can be reformulated as

$$\boldsymbol{n} \times (-\nabla\Phi \times \boldsymbol{n}) = \boldsymbol{n} \times (\boldsymbol{K} - \boldsymbol{T}_0 \times \boldsymbol{n}), \tag{2.154}$$

from which $-\nabla\Phi$ can be selected as
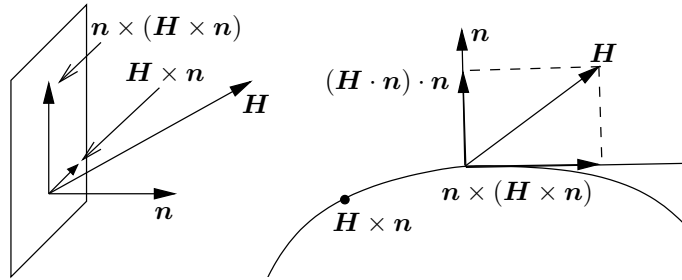
Fig. 2.13. Definition of the normal and tangential component of a vector $\boldsymbol{H}$

$$-\nabla\Phi = \boldsymbol{n} \times (\boldsymbol{K} - \boldsymbol{T}_0 \times \boldsymbol{n}), \tag{2.155}$$

because

$$\boldsymbol{n} \times (\boldsymbol{K} - \boldsymbol{T}_0 \times \boldsymbol{n}) = \boldsymbol{n} \times \{[\boldsymbol{n} \times (\boldsymbol{K} - \boldsymbol{T}_0 \times \boldsymbol{n})] \times \boldsymbol{n}\}. \tag{2.156}$$

Using the definition of the gradient of a vector field, the equation (2.155) results in the following Dirichlet boundary condition:

$$\Phi = \Phi_0, \quad \text{and} \quad \Phi_0 = \int_l \boldsymbol{n} \times (\boldsymbol{K} - \boldsymbol{T}_0 \times \boldsymbol{n}) \cdot \mathrm{d}\boldsymbol{l}, \quad \text{on} \quad \Gamma_H. \tag{2.157}$$

If one applies a known source magnetic field intensity vector $\boldsymbol{H}_0$ on the boundary Γ_H, which can be written as a surface current term as $\boldsymbol{H}_0 \times \boldsymbol{n} = \boldsymbol{K}$, or $\boldsymbol{n} \times \boldsymbol{K} = \boldsymbol{H}_0$, then

$$\Phi = \Phi_0, \quad \text{and} \quad \Phi_0 = \int_l (\boldsymbol{H}_0 - \boldsymbol{n} \times [\boldsymbol{T}_0 \times \boldsymbol{n}]) \cdot \mathrm{d}\boldsymbol{l}, \quad \text{on} \quad \Gamma_H \tag{2.158}$$

is valid. Here path l is lying on Γ_H and it can be arbitrary.

It is noted that symmetry planes are usually equi-potential surfaces.

On the rest part Γ_B, setting the normal component of the magnetic flux density results in a Neumann type boundary condition,

$$\boldsymbol{B} \cdot \boldsymbol{n} = -b \quad \Rightarrow \quad (\mu_o \boldsymbol{T}_0 - \mu_o \nabla\Phi + \boldsymbol{R}) \cdot \boldsymbol{n} = -b, \quad \text{on} \quad \Gamma_B, \tag{2.159}$$

since $\boldsymbol{B} = \mu_o (\boldsymbol{T}_0 - \nabla\Phi) + \boldsymbol{R}$.

If no Dirichlet boundary conditions are prescribed, then $\Phi = 0$ must be specified at some points. In this case the reduced magnetic scalar potential is unique.

The Φ-formulation. The partial differential equation and the boundary conditions of a static magnetic field problem solved by the reduced magnetic scalar potential can be written as

$$\nabla \cdot (\mu_o \nabla\Phi) = \nabla \cdot (\mu_o \boldsymbol{T}_0) + \nabla \cdot \boldsymbol{R}, \quad \text{in} \quad \Omega, \tag{2.160}$$

$$\Phi = \Phi_0, \quad \text{on} \quad \Gamma_H, \tag{2.161}$$

$$(\mu_o \boldsymbol{T}_0 - \mu_o \nabla\Phi + \boldsymbol{R}) \cdot \boldsymbol{n} = -b, \quad \text{on} \quad \Gamma_B, \tag{2.162}$$

where Φ_0 can be determined by the integral (2.157) or (2.158).

B. The total magnetic scalar potential, the Ψ-formulation

The Φ-formulation is satisfactory if the permeability of the medium is *not very high*. In this case the simulated magnetic field $\boldsymbol{H} = \boldsymbol{T}_0 - \nabla\Phi$ does not much differ from the source magnetic field $\boldsymbol{T}_0$. By increasing the permeability of the medium with linear constitutive relation by increasing the slope of the nonlinear characteristics (e.g. when soft magnetic materials are modeled), the resultant magnetic field $\boldsymbol{H}$ is decreasing inside the magnetic material, i.e. $\boldsymbol{T}_0 \approx \nabla\Phi$, which is a strong disadvantage in numerically point of view.

This is the so-called *cancellation error*, when two almost equal quantities are subtracted from each other.

Fortunately, the source current density $\boldsymbol{J}_0$ is usually zero in ferromagnetic parts with high value of permeability. In this case, the magnetic field intensity can be derived from the so-called *total magnetic scalar potential* Ψ as

$$\boldsymbol{H} = -\nabla\Psi, \tag{2.163}$$

since $\nabla \times \boldsymbol{H} = \boldsymbol{0}$ ($\boldsymbol{J}_0 = \boldsymbol{0}$) inside highly permeable regions.

This can be the situation of magnetic poles. However, in most practical cases, the magnetic field intensity is generated by currents flowing in coils. The source coils are usually placed somewhere around the ferromagnetic parts, but source currents are not flowing inside ferromagnetic materials. In this situation, it is possible to use the reduced magnetic scalar potential Φ in the air region and the total magnetic scalar potential Ψ inside the highly permeable region and the two formulations must be coupled through the interface between the two regions.

C. The combination of the magnetic scalar potentials, the $\Phi - \Psi$-formulation

This formulation is outdated, the method presented in the next item is more modern.

In this situation the problem region is subdivided into two regions Ω_Φ and Ω_Ψ as it is illustrated in *Fig.* 2.14 and $\Omega = \Omega_\Phi \cup \Omega_\Psi$. The domain Ω_Φ contains all coils and the permeability is usually equal to μ_0, this is the air region. Here, the magnetic field intensity is expressed as

$$\boldsymbol{H} = \boldsymbol{T}_0 - \nabla\Phi, \quad \text{in} \quad \Omega_\Phi. \tag{2.164}$$

Because of the same value of permeability in Ω_Φ, no *cancellation errors* occur there.

The rest region contains all the ferromagnetic bodies, but the source current density is equal to zero, that is why the magnetic field intensity can be derived by using the total magnetic scalar potential,

$$\boldsymbol{H} = -\nabla\Psi, \quad \text{in} \quad \Omega_\Psi. \tag{2.165}$$

The partial differential equations are as follows in the two subregions:

$$\nabla \cdot (\mu_0 \nabla\Phi) = \nabla \cdot (\mu_0 \boldsymbol{T}_0), \quad \text{in} \quad \Omega_\Phi, \tag{2.166}$$

$$\nabla \cdot (\mu_o \nabla\Psi) = \nabla \cdot \boldsymbol{R}, \quad \text{in} \quad \Omega_\Psi. \tag{2.167}$$

The boundary parts Γ_H and Γ_B are subdivided into $\Gamma_{H_\Phi}, \Gamma_{H_\Psi}, \Gamma_{B_\Phi}$ and Γ_{B_Ψ}. On the $\Gamma_{H_\Phi} \cup \Gamma_{H_\Psi}$ boundaries, Dirichlet type boundary conditions are specified according to the tangential component of the magnetic field intensity, $\boldsymbol{H} \times \boldsymbol{n} = \boldsymbol{K}$, i.e.

$$\Phi = \Phi_0, \quad \text{and} \quad \Phi_0 = \int_l \boldsymbol{n} \times (\boldsymbol{K} - \boldsymbol{T}_0 \times \boldsymbol{n}) \cdot \mathrm{d}\boldsymbol{l}, \quad \text{on} \quad \Gamma_{H_\Phi}, \tag{2.168}$$

$$\Psi = \Psi_0, \quad \text{and} \quad \Psi_0 = \int_l (\boldsymbol{n} \times \boldsymbol{K}) \cdot \mathrm{d}\boldsymbol{l}, \quad \text{on} \quad \Gamma_{H_\Psi}, \tag{2.169}$$

or
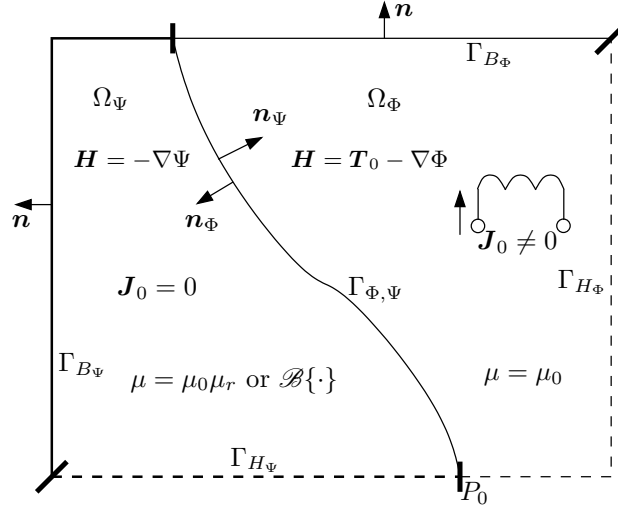
Fig. 2.14. The scheme of a static magnetic field problem with reduced
and total magnetic scalar potentials

$$\Phi = \Phi_0, \quad \text{and} \quad \Phi_0 = \int_l (\boldsymbol{H}_0 - \boldsymbol{n} \times [\boldsymbol{T}_0 \times \boldsymbol{n}]) \cdot \mathrm{d}\boldsymbol{l}, \quad \text{on} \quad \Gamma_{H_\Phi}, \tag{2.170}$$

$$\Psi = \Psi_0, \quad \text{and} \quad \Psi_0 = \int_l \boldsymbol{H}_0 \cdot \mathrm{d}\boldsymbol{l}, \quad \text{on} \quad \Gamma_{H_\Psi}. \tag{2.171}$$

On the rest part $\Gamma_{B_\Phi} \cup \Gamma_{B_\Psi}$, Neumann type boundary conditions can be specified according to the condition $\boldsymbol{B} \cdot \boldsymbol{n} = -b$,

$$(\mu_0 \boldsymbol{T}_0 - \mu_0 \nabla \Phi) \cdot \boldsymbol{n} = -b, \quad \text{on} \quad \Gamma_{B_\Phi}, \tag{2.172}$$

$$(-\mu_o \nabla \Psi + \boldsymbol{R}) \cdot \boldsymbol{n} = -b, \quad \text{on} \quad \Gamma_{B_\Psi}. \tag{2.173}$$

Interface boundary conditions must be specified along the interface between the two subregions Ω_Φ and Ω_Ψ. This is denoted by $\Gamma_{\Phi,\Psi}$, where the tangential component of the magnetic field intensity vector and the normal component of the magnetic flux density vector have to be continuous.

The condition for the tangential component of the magnetic field intensity can be written as

$$(\boldsymbol{T}_0 - \nabla \Phi) \times \boldsymbol{n}_\Phi + (-\nabla \Psi) \times \boldsymbol{n}_\Psi = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{\Phi,\Psi}, \tag{2.174}$$

which can be defined as a Dirichlet boundary condition similarly to the boundary condition (2.157)

$$\Psi = \Phi + \int_l \boldsymbol{n}_\Phi \times (\boldsymbol{T}_0 \times \boldsymbol{n}_\Phi) \cdot \mathrm{d}\boldsymbol{l}, \quad \text{on} \quad \Gamma_{\Phi,\Psi}. \tag{2.175}$$

Here $\boldsymbol{n}_\Phi$ and $\boldsymbol{n}_\Psi$ are the outer normal unit vectors of the appropriate subregion, and it is noted that $\boldsymbol{n}_\Psi = -\boldsymbol{n}_\Phi$. Moreover at a point P_0, $\Phi = \Psi$ must be specified to make the solution unique.

The condition for the normal component of the magnetic flux density can be written as

$$(\mu_0 \boldsymbol{T}_0 - \mu_0 \nabla \Phi) \cdot \boldsymbol{n}_\Phi + (-\mu_o \nabla \Psi + \boldsymbol{R}) \cdot \boldsymbol{n}_\Psi = 0, \quad \text{on} \quad \Gamma_{\Phi,\Psi}, \tag{2.176}$$

which is a Neumann type boundary condition.

The collection of partial differential equations, boundary and interface conditions of the $\Phi - \Psi$-*formulation* is as follows:

$$\nabla \cdot (\mu_0 \nabla \Phi) = \nabla \cdot (\mu_0 \boldsymbol{T}_0), \quad \text{in} \quad \Omega_\Phi, \tag{2.177}$$

$$\nabla \cdot (\mu_o \nabla \Psi) = \nabla \cdot \boldsymbol{R}, \quad \text{in} \quad \Omega_\Psi, \tag{2.178}$$

$$\Phi = \Phi_0, \quad \text{on} \quad \Gamma_{H_\Phi}, \tag{2.179}$$

$$\Psi = \Psi_0, \quad \text{on} \quad \Gamma_{H_\Psi}, \tag{2.180}$$

$$(\mu_0 \boldsymbol{T}_0 - \mu \nabla \Phi) \cdot \boldsymbol{n} = -b, \quad \text{on} \quad \Gamma_{B_\Phi}, \tag{2.181}$$

$$(-\mu_o \nabla \Psi + \boldsymbol{R}) \cdot \boldsymbol{n} = -b, \quad \text{on} \quad \Gamma_{B_\Psi}. \tag{2.182}$$

$$\Psi = \Phi + \int_l \boldsymbol{n}_\Phi \times (\boldsymbol{T}_0 \times \boldsymbol{n}_\Phi) \cdot \mathrm{d}\boldsymbol{l}, \quad \text{on} \quad \Gamma_{\Phi,\Psi}, \tag{2.183}$$

$$(\mu_0 \boldsymbol{T}_0 - \mu_0 \nabla \Phi) \cdot \boldsymbol{n}_\Phi + (-\mu_o \nabla \Psi + \boldsymbol{R}) \cdot \boldsymbol{n}_\Psi = 0, \quad \text{on} \quad \Gamma_{\Phi,\Psi}, \tag{2.184}$$

where Φ_0 and Ψ_0 can be determined by the integrals (2.168), (2.170), or by (2.169), (2.171).

D. Applying the reduced magnetic scalar potential with appropriate representation of $\boldsymbol{T}_0$

The *cancellation error* inside ferromagnetic materials with high permeability can be eliminated by the appropriate representation of the impressed current vector potential in numerical field analysis procedures.

The rotational part of the magnetic field intensity is originally a smooth function with an infinite number of continuous derivatives, whereas $\nabla \Phi$ is usually interpolated by simple functions depending on the used method, e.g. by polynomials having finite number of continuous derivatives.

A so-called *compatible approximation* should be used to interpolate the impressed field and the unknown nonrotational part. In a finite element procedure this means that the vector field $\boldsymbol{T}_0$ should be represented by edge elements and the scalar field Φ should be interpolated by nodal elements and the order of them must be the same. In this case, the Φ-formulation has been defined by (2.160)–(2.162) is satisfactory. It is advantageous, because the interface condition (2.183) can be difficult to implement, moreover *edge element* representation results in a *tangentially continuous vector field*.

E. The magnetic vector potential, the A-formulation

The *magnetic vector potential* is defined by

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \tag{2.185}$$

which satisfies (2.77) exactly, because of the identity $\nabla \cdot \nabla \times \boldsymbol{v} \equiv 0$ for any vector function $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r})$.

Substituting the definition (2.185) into the first Maxwell's equation (2.76) and using the linearized constitutive relation from (2.80), it leads to the partial differential equation

$$\nabla \times (\nu_o \nabla \times \boldsymbol{A}) = \boldsymbol{J}_0 - \nabla \times \boldsymbol{I}, \quad \text{in} \quad \Omega, \tag{2.186}$$

when using the source current density $\boldsymbol{J}_0$, or

$$\nabla \times (\nu_o \nabla \times \boldsymbol{A}) = \nabla \times \boldsymbol{T}_0 - \nabla \times \boldsymbol{I}, \quad \text{in} \quad \Omega, \tag{2.187}$$

when using the *impressed current vector potential* $\boldsymbol{T}_0$ to represent coils as it is introduced in (2.148).

To ensure the uniqueness of the magnetic vector potential, the divergence of it can be selected according to *Coulomb gauge*,

$$\nabla \cdot \boldsymbol{A} = 0. \tag{2.188}$$

This gauging is useful, because the vector potential $\boldsymbol{A}' = \boldsymbol{A} + \nabla \varphi$ also satisfies (2.77) as well as (2.186) or (2.187), because of the identity $\nabla \times (\nabla \varphi) \equiv \boldsymbol{0}$, where $\varphi = \varphi(\boldsymbol{r})$ is a scalar function. This is the reason why the magnetic vector potential is not unique.

Let us first study a two-dimensional problem, then a three-dimensional one. Gauging is satisfied automatically in 2D, but unfortunately it is not true in 3D. The origin of numerical problems is the lack of uniqueness of the magnetic vector potential.

2D problems. In 2D problems Coulomb gauge $\nabla \cdot \boldsymbol{A} = 0$ is satisfied automatically, if the source current density has only z component, the magnetic field intensity vector and the magnetic flux density vector have x and y components, i.e.

$$\begin{aligned}
\boldsymbol{J}_0 &= J_{0,z}(x, y)\, \mathbf{e}_z, \\
\boldsymbol{H} &= H_x(x, y)\, \mathbf{e}_x + H_y(x, y)\, \mathbf{e}_y, \quad \boldsymbol{B} = B_x(x, y)\, \mathbf{e}_x + B_y(x, y)\, \mathbf{e}_y.
\end{aligned} \tag{2.189}$$

In 2D, $\boldsymbol{J}_0$ is used to represent source current density. The magnetic vector potential has only z component,

$$\boldsymbol{A} = A_z(x, y)\, \mathbf{e}_z, \tag{2.190}$$

because ($A_x = 0$, $A_y = 0$ and $A_z = A_z(x, y)$)

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & A_z \end{vmatrix} = \mathbf{e}_x \frac{\partial A_z}{\partial y} - \mathbf{e}_y \frac{\partial A_z}{\partial x}, \tag{2.191}$$

i.e. $B_x(x,y) = \partial A_z/\partial y$ and $B_y(x,y) = -\partial A_z/\partial x$. The divergence of this one component vector potential is equal to zero,

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_z(x,y)}{\partial z} = 0. \tag{2.192}$$

Now, let us define the boundary conditions of the $\boldsymbol{A}$-formulation.

On the part Γ_H of the boundary, the tangential component of the magnetic field intensity vector can be set to the prescribed value by the relation

$$\boldsymbol{H} \times \boldsymbol{n} = \boldsymbol{K} \quad \Rightarrow \quad (\nu_o \nabla \times \boldsymbol{A} + \boldsymbol{I}) \times \boldsymbol{n} = \boldsymbol{K}, \quad \text{on} \quad \Gamma_H, \tag{2.193}$$

which is a Neumann type boundary condition.

The normal component of the magnetic flux density can be set as

$$\boldsymbol{B} \cdot \boldsymbol{n} = -b \quad \Rightarrow \quad (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{n} = -b, \quad \text{on} \quad \Gamma_B. \tag{2.194}$$

The left-hand side of the last formulation can be rewritten as

$$(\nabla \times \boldsymbol{A}) \cdot \boldsymbol{n} = \nabla \cdot (\boldsymbol{A} \times \boldsymbol{n}) = -b, \tag{2.195}$$

finally

$$\nabla \cdot (\boldsymbol{n} \times \boldsymbol{A}) = b, \tag{2.196}$$

i.e.

$$\boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{\alpha}, \quad \text{on} \quad \Gamma_B, \tag{2.197}$$

where $\nabla \cdot \boldsymbol{\alpha} = b$. This is a Dirichlet type boundary condition. The selection of $\boldsymbol{\alpha}$ is not evident [11], but in many practical cases $b = 0$, so

$$\boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_B \tag{2.198}$$

can be selected.

Finally, the nonlinear partial differential equation and the boundary conditions of a two-dimensional static magnetic field problem, which solution satisfies *Coulomb gauge* can be formulated as

$$\nabla \times (\nu_o \nabla \times \boldsymbol{A}) = \boldsymbol{J}_0 - \nabla \times \boldsymbol{I}, \quad \text{in} \quad \Omega, \tag{2.199}$$

$$(\nu_o \nabla \times \boldsymbol{A} + \boldsymbol{I}) \times \boldsymbol{n} = \boldsymbol{K}, \quad \text{on} \quad \Gamma_H, \tag{2.200}$$

$$\boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{\alpha}, \quad \text{on} \quad \Gamma_B. \tag{2.201}$$

3D problems. In three-dimensional problems, the uniqueness of the vector potential is not so evident and it can be prescribed by

- (*i*) implicit enforcement of Coulomb gauge (e.g. when using *nodal elements* in the finite element approximation),

- (*ii*) applying a numerical technique, which is not sensitive to Coulomb gauge (e.g. when using *vector elements* in the finite element approximation and taking care about the representation of source current density).

The first method results in the gauged, the second one in the ungauged version of the $\boldsymbol{A}$-formulation.

(*i*) *Implicit enforcement of Coulomb gauge.* When applying the partial differential equation (2.199) and boundary conditions (2.200) and (2.201) to solve a three-dimensional static magnetic field problem, the uniqueness of the vector potential is not prescribed, which results in numerical difficulties.

The Coulomb gauge can be prescribed implicitly by the following formulations. The result of this formulation is a modification of the partial differential equation (2.199), moreover additional boundary conditions are appended to the original boundary value problem.

The first step is to define the *functional* of the partial differential equation (2.199). Linear case is presented for simplicity, when $\nu_o = \nu$ and $\boldsymbol{I} = \boldsymbol{0}$,

$$\mathscr{F}\{\boldsymbol{A}\} = \frac{1}{2}\int_{\Omega}\nu|\nabla\times\boldsymbol{A}|^2\mathrm{d}\Omega - \int_{\Omega}\boldsymbol{J}_0\cdot\boldsymbol{A}\,\mathrm{d}\Omega - \int_{\Gamma_H}\boldsymbol{K}\cdot\boldsymbol{A}\,\mathrm{d}\Gamma. \qquad (2.202)$$

According to the *principle of minimum energy*, this *functional* has minimum value only if the magnetic vector potential satisfies the partial differential equation (2.199) and the boundary conditions (2.200) and (2.201). This minimum is a stable configuration of the magnetic vector potential.

Let us now modify this functional by introducing *Coulomb gauge* as

$$\begin{aligned}\mathscr{F}\{\boldsymbol{A}\} =&\frac{1}{2}\int_{\Omega}\nu|\nabla\times\boldsymbol{A}|^2\mathrm{d}\Omega - \int_{\Omega}\boldsymbol{J}_0\cdot\boldsymbol{A}\,\mathrm{d}\Omega - \int_{\Gamma_H}\boldsymbol{K}\cdot\boldsymbol{A}\,\mathrm{d}\Gamma\\ &+\frac{1}{2}\int_{\Omega}\nu(\nabla\cdot\boldsymbol{A})^2\mathrm{d}\Omega.\end{aligned} \qquad (2.203)$$

This functional has minimum value only if the magnetic vector potential satisfies the partial differential equation (2.199), the boundary conditions (2.200) and (2.201), and Coulomb gauge, however, extra boundary conditions must be appended to the boundary value problem defined by the equations (2.199), (2.200) and (2.201).

This procedure must be done, because we have two partial differential equations to be solved, (2.199) and $\nabla\cdot\boldsymbol{A} = 0$ with only one unknown $\boldsymbol{A}$. The two partial differential equations can be formulated in one partial differential equation by the help of (2.203). The partial differential equation and boundary conditions, which minimize the functional (2.203) can be derived as follows.

Let us introduce a small *variation* of the magnetic vector potential of $\boldsymbol{A}$ as $\boldsymbol{A}+\alpha\boldsymbol{\delta A}$, which results in a variation of the field energy around the stable configuration of $\boldsymbol{A}$,

$$\begin{aligned}\mathscr{F}\{\boldsymbol{A}+\alpha\boldsymbol{\delta A}\} = &\frac{1}{2}\int_{\Omega}\nu|\nabla\times(\boldsymbol{A}+\alpha\boldsymbol{\delta A})|^2\mathrm{d}\Omega\\ &-\int_{\Omega}\boldsymbol{J}_0\cdot(\boldsymbol{A}+\alpha\boldsymbol{\delta A})\,\mathrm{d}\Omega - \int_{\Gamma_H}\boldsymbol{K}\cdot(\boldsymbol{A}+\alpha\boldsymbol{\delta A})\,\mathrm{d}\Gamma\\ &+\frac{1}{2}\int_{\Omega}\nu[\nabla\cdot(\boldsymbol{A}+\alpha\boldsymbol{\delta A})]^2\mathrm{d}\Omega.\end{aligned} \qquad (2.204)$$

It is important to note that the small variation should not modify the prescribed Dirichlet boundary conditions, i.e. the vector field representing the small variation $\boldsymbol{\delta A}$ must satisfy homogeneous Dirichlet boundary condition.

The variation of the functional can be reformulated as

$$
\begin{aligned}
\mathscr{F}\{\boldsymbol{A} + \alpha \boldsymbol{\delta A}\} =& \frac{1}{2} \int_{\Omega} \nu |\nabla \times \boldsymbol{A}|^2 \mathrm{d}\Omega + \alpha \int_{\Omega} \nu (\nabla \times \boldsymbol{A}) \cdot (\nabla \times \boldsymbol{\delta A}) \, \mathrm{d}\Omega \\
&+ \frac{\alpha^2}{2} \int_{\Omega} \nu |\nabla \times \boldsymbol{\delta A}|^2 \mathrm{d}\Omega \\
&- \int_{\Omega} \boldsymbol{J}_0 \cdot \boldsymbol{A} \, \mathrm{d}\Omega - \alpha \int_{\Omega} \boldsymbol{J}_0 \cdot \boldsymbol{\delta A} \, \mathrm{d}\Omega \\
&- \int_{\Gamma_H} \boldsymbol{K} \cdot \boldsymbol{A} \, \mathrm{d}\Gamma - \alpha \int_{\Gamma_H} \boldsymbol{K} \cdot \boldsymbol{\delta A} \, \mathrm{d}\Gamma \\
&+ \frac{1}{2} \int_{\Omega} \nu (\nabla \cdot \boldsymbol{A})^2 \mathrm{d}\Omega + \alpha \int_{\Omega} \nu (\nabla \cdot \boldsymbol{A})(\nabla \cdot \boldsymbol{\delta A}) \, \mathrm{d}\Omega \\
&+ \frac{\alpha^2}{2} \int_{\Omega} \nu (\nabla \cdot \boldsymbol{\delta A})^2 \mathrm{d}\Omega.
\end{aligned}
\tag{2.205}
$$

This variation can be represented in *Taylor series* approximation as

$$
\mathscr{F}\{\boldsymbol{A} + \alpha \boldsymbol{\delta A}\} = \mathscr{F}\{\boldsymbol{A}\} + \alpha \delta \mathscr{F}\{\boldsymbol{A}, \boldsymbol{\delta A}\} + \alpha^2 \delta^2 \mathscr{F}\{\boldsymbol{\delta A}\}.
\tag{2.206}
$$

In concern with the stationary concept, the *first variation* $\delta \mathscr{F}\{\boldsymbol{A}, \boldsymbol{\delta A}\}$ has to be disappear and the *second variation* $\delta^2 \mathscr{F}\{\boldsymbol{\delta A}\}$ has to be positive. In this case $\mathscr{F}\{\boldsymbol{A}\}$ leads indeed to its minimum value. The definition of the first variation is as follows (it must be equal to zero):

$$
\delta \mathscr{F}\{\boldsymbol{A}, \boldsymbol{\delta A}\} = \left. \frac{\partial \mathscr{F}\{\boldsymbol{A} + \alpha \boldsymbol{\delta A}\}}{\partial \alpha} \right|_{\alpha = 0} = 0,
\tag{2.207}
$$

which is

$$
\begin{aligned}
\delta \mathscr{F}\{\boldsymbol{A}, \boldsymbol{\delta A}\} =& \int_{\Omega} \nu (\nabla \times \boldsymbol{A}) \cdot (\nabla \times \boldsymbol{\delta A}) \, \mathrm{d}\Omega - \int_{\Omega} \boldsymbol{J}_0 \cdot \boldsymbol{\delta A} \, \mathrm{d}\Omega \\
&- \int_{\Gamma_H} \boldsymbol{K} \cdot \boldsymbol{\delta A} \, \mathrm{d}\Gamma + \int_{\Omega} \nu (\nabla \cdot \boldsymbol{A})(\nabla \cdot \boldsymbol{\delta A}) \, \mathrm{d}\Omega = 0.
\end{aligned}
\tag{2.208}
$$

It is easy to see that the second variation

$$
\delta^2 \mathscr{F}\{\boldsymbol{\delta A}\} = \frac{1}{2} \int_{\Omega} \nu |\nabla \times \boldsymbol{\delta A}|^2 \mathrm{d}\Omega + \frac{1}{2} \int_{\Omega} \nu (\nabla \cdot \boldsymbol{\delta A})^2 \mathrm{d}\Omega
\tag{2.209}
$$

is usually positive, so (2.208) really states the minimum value of the functional (2.203).

Let us now obtain the according partial differential equation and boundary conditions for (2.208). After using the identities

$$
\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \times \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \times \boldsymbol{v},
\tag{2.210}
$$

and

$$
\nabla \cdot (\varphi \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \varphi + \varphi \, \nabla \cdot \boldsymbol{v},
\tag{2.211}
$$

with $\boldsymbol{u} = \boldsymbol{\delta A}$, $\boldsymbol{v} = \nu\nabla \times \boldsymbol{A}$, $\varphi = \nu\nabla \cdot \boldsymbol{A}$ and $\boldsymbol{v} = \boldsymbol{\delta A}$, the following relation can be obtained:

$$\begin{aligned}
\delta\mathscr{F}\{\boldsymbol{A}, \boldsymbol{\delta A}\} = &\int_\Omega \boldsymbol{\delta A} \cdot [\nabla \times (\nu\nabla \times \boldsymbol{A})]\,\mathrm{d}\Omega \\
&+ \oint_\Gamma [\boldsymbol{\delta A} \times (\nu\nabla \times \boldsymbol{A})] \cdot \boldsymbol{n}\,\mathrm{d}\Gamma - \int_\Omega \boldsymbol{J}_0 \cdot \boldsymbol{\delta A}\,\mathrm{d}\Omega \\
&- \int_{\Gamma_H} \boldsymbol{K} \cdot \boldsymbol{\delta A}\,\mathrm{d}\Gamma - \int_\Omega \boldsymbol{\delta A} \cdot [\nabla(\nu\nabla \cdot \boldsymbol{A})]\,\mathrm{d}\Omega \\
&+ \oint_\Gamma \nu\nabla \cdot \boldsymbol{A}(\boldsymbol{\delta A} \cdot \boldsymbol{n})\,\mathrm{d}\Gamma = 0.
\end{aligned} \tag{2.212}$$

This equation can be rewritten in the form

$$\begin{aligned}
\delta\mathscr{F}\{\boldsymbol{A}, \boldsymbol{\delta A}\} = &\int_\Omega \boldsymbol{\delta A} \cdot [\nabla \times (\nu\nabla \times \boldsymbol{A}) - \boldsymbol{J}_0 - \nabla(\nu\nabla \cdot \boldsymbol{A})]\,\mathrm{d}\Omega \\
&+ \int_{\Gamma_H \cup \Gamma_B} [(\nu\nabla \times \boldsymbol{A}) \times \boldsymbol{n}] \cdot \boldsymbol{\delta A}\,\mathrm{d}\Gamma - \int_{\Gamma_H} \boldsymbol{K} \cdot \boldsymbol{\delta A}\,\mathrm{d}\Gamma \\
&+ \int_{\Gamma_H \cup \Gamma_B} \nu\nabla \cdot \boldsymbol{A}(\boldsymbol{\delta A} \cdot \boldsymbol{n})\,\mathrm{d}\Gamma = 0.
\end{aligned} \tag{2.213}$$

The sum of the first and the second surface integrals on the part Γ_H is equal to zero, because of the boundary condition $(\nu\nabla \times \boldsymbol{A}) \times \boldsymbol{n} = \boldsymbol{K}$. The first boundary integral is equal to zero on the rest part Γ_B, because $\boldsymbol{\delta A}$ must satisfy homogeneous Dirichlet boundary condition, i.e. $\boldsymbol{n} \times \boldsymbol{\delta A} = \boldsymbol{0}$ on Γ_B and it can be acknowledged according to the identity $[(\nu\nabla \times \boldsymbol{A}) \times \boldsymbol{n}] \cdot \boldsymbol{\delta A} = [\boldsymbol{n} \times \boldsymbol{\delta A}] \cdot (\nu\nabla \times \boldsymbol{A})$. The last boundary integral in (2.213) can be eliminated too by some extra boundary conditions, $\nu\nabla \cdot \boldsymbol{A} = 0$ or $\boldsymbol{A} \cdot \boldsymbol{n} = 0$ should be specified somewhere, but it must be formulated after some words about the first integral.

Now, it can be seen that the partial differential equation to be solved is

$$\nabla \times (\nu\nabla \times \boldsymbol{A}) - \nabla(\nu\nabla \cdot \boldsymbol{A}) = \boldsymbol{J}_0, \quad \text{in} \quad \Omega, \tag{2.214}$$

because in this case the first integral in (2.213) is equal to zero for any value of the variation $\boldsymbol{\delta A}$.

Let us take the divergence of equation (2.214),

$$\nabla \cdot [\nabla \times (\nu\nabla \times \boldsymbol{A})] - \nabla \cdot [\nabla(\nu\nabla \cdot \boldsymbol{A})] = \nabla \cdot \boldsymbol{J}_0. \tag{2.215}$$

The first term on the left-hand side and the right-hand side are obviously equal to zero, i.e. the *Laplace equation*

$$-\nabla \cdot \nabla(\nu\nabla \cdot \boldsymbol{A}) = 0, \quad \text{in} \quad \Omega \tag{2.216}$$

for the scalar variable $\nu\nabla \cdot \boldsymbol{A}$ is left. Here the magnetic vector potential $\boldsymbol{A}$ is the only one independent variable, so only one partial differential equation should be used. Fortunately, the solution of a Laplace equation $-\nabla \cdot \nabla\varphi = 0$ ($\varphi = \varphi(\boldsymbol{r})$) can be identically equal to

zero if the potential φ is prescribed on the whole boundary as a homogeneous Dirichlet type boundary condition, or a homogeneous Dirichlet type boundary condition on one part and a homogeneous Neumann type boundary condition on the rest part of the boundary. The solution of Laplace equation

$$-\nabla \cdot \nabla\varphi = 0, \quad \text{in} \quad \Omega \tag{2.217}$$

is identically zero in Ω if

(α) $\varphi = 0$ on $\partial\Omega$, or

(β) $\varphi = 0$ on Γ_1 and $\partial\varphi/\partial n = 0$ on Γ_2 and $\partial\Omega = \Gamma_1 \cup \Gamma_2$, moreover $\boldsymbol{n}$ is the outer normal unit vector of the region Ω.

Let us follow the second way.

First, let us take the normal component of the partial differential equation (2.214) only on the boundary Γ_H,

$$\boldsymbol{n} \cdot [\nabla \times (\nu\nabla \times \boldsymbol{A})] - \boldsymbol{n} \cdot \nabla(\nu\nabla \cdot \boldsymbol{A}) = \boldsymbol{n} \cdot \boldsymbol{J}_0, \quad \text{on} \quad \Gamma_H. \tag{2.218}$$

The first term on the left-hand side is equal to zero according to

$$\boldsymbol{n} \cdot [\nabla \times (\nu\nabla \times \boldsymbol{A})] = \nabla \cdot [(\nu\nabla \times \boldsymbol{A}) \times \boldsymbol{n}] = \nabla \cdot \boldsymbol{K} = 0, \tag{2.219}$$

because of the current conversation law. The right-hand side is obviously zero, because the normal component of the source current density is equal to zero on the boundary Γ_H. Finally, a homogeneous Neumann boundary condition

$$-\boldsymbol{n} \cdot \nabla(\nu\nabla \cdot \boldsymbol{A}) = -\frac{\partial}{\partial n}(\nu\nabla \cdot \boldsymbol{A}) = 0, \quad \text{on} \quad \Gamma_H \tag{2.220}$$

is specified automatically for the scalar quantity $\nu\nabla \cdot \boldsymbol{A}$. On the rest part of the boundary Γ_B, homogeneous Dirichlet boundary condition must be specified,

$$\nu\nabla \cdot \boldsymbol{A} = 0, \quad \text{on} \quad \Gamma_B. \tag{2.221}$$

In this way $\nu\nabla \cdot \boldsymbol{A} \equiv 0$ in the whole problem region Ω and on the boundary $\Gamma_H \cup \Gamma_B$, i.e. the last surface integral in (2.213) can be eliminated.

As a last check on the uniqueness of the vector potential, let us append a $\nabla\varphi$ term to the magnetic vector potential $\boldsymbol{A}$ as $\boldsymbol{A} + \nabla\varphi$ and take the divergence of this term,

$$\nabla \cdot (\boldsymbol{A} + \nabla\varphi) = \nabla \cdot \boldsymbol{A} + \nabla \cdot \nabla\varphi = 0. \tag{2.222}$$

The first term of this equation is equal to zero according to Coulomb gauge, that is why only $\nabla \cdot \nabla\varphi = 0$ has to be analyzed. On the boundary part Γ_B, the boundary condition $\boldsymbol{n} \times (\boldsymbol{A} + \nabla\varphi) = \boldsymbol{\alpha}$ results in $\boldsymbol{n} \times \nabla\varphi = \boldsymbol{0}$ for φ, i.e. φ is constant on Γ_B. On the rest part Γ_H, the normal component of $\nabla\varphi$ can be specified as $\nabla\varphi \cdot \boldsymbol{n} = 0$. This can be obtained by the condition $\boldsymbol{A} \cdot \boldsymbol{n} = 0$ on Γ_H. Finally, it is known that if $\nabla \cdot \nabla\varphi = 0$ in Ω and $\nabla\varphi \cdot \boldsymbol{n} = \partial\varphi/\partial n = 0$ on Γ_H, then the constant value of φ on Γ_B can only be zero.

This means that $\varphi = 0$ in the problem region Ω and on the boundary $\Gamma_H \cup \Gamma_B$, i.e. the magnetic vector potential is unique.

Finally, the partial differential equation and the boundary conditions of the static magnetic field problem, which solution is unique according to *Coulomb gauge* can be written as follows:

$$\nabla \times (\nu_o \nabla \times \boldsymbol{A}) - \nabla(\nu_o \nabla \cdot \boldsymbol{A}) = \boldsymbol{J}_0 - \nabla \times \boldsymbol{I}, \quad \text{in} \quad \Omega, \tag{2.223}$$

$$(\nu_o \nabla \times \boldsymbol{A} + \boldsymbol{I}) \times \boldsymbol{n} = \boldsymbol{K}, \quad \text{on} \quad \Gamma_H, \tag{2.224}$$

$$\boldsymbol{A} \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_H, \tag{2.225}$$

$$\boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{\alpha}, \quad \text{on} \quad \Gamma_B, \tag{2.226}$$

$$\nu_o \nabla \cdot \boldsymbol{A} = 0, \quad \text{on} \quad \Gamma_B. \tag{2.227}$$

This formulation is the so-called gauged $\boldsymbol{A}$-formulation.

(*ii*) *Applying a numerical technique, which is not sensitive to Coulomb gauge.* This formulation is newer than the first one. The only one rule is that the impressed current vector potential and the unknown magnetic vector potential must be approximated by the so-called *vector elements* in the finite element procedures.

The partial differential equation and the boundary conditions of the static magnetic field problem, which solution is not sensitive to Coulomb gauge can be written as

$$\nabla \times (\nu_o \nabla \times \boldsymbol{A}) = \nabla \times \boldsymbol{T}_0 - \nabla \times \boldsymbol{I}, \quad \text{in} \quad \Omega, \tag{2.228}$$

$$(\nu_o \nabla \times \boldsymbol{A} + \boldsymbol{I}) \times \boldsymbol{n} = \boldsymbol{K}, \quad \text{on} \quad \Gamma_H, \tag{2.229}$$

$$\boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{\alpha}, \quad \text{on} \quad \Gamma_B. \tag{2.230}$$

First, $\boldsymbol{T}_0$ must be calculated by one of the methods presented on page 21. This is called ungauged $\boldsymbol{A}$-formulation.

F. Combination of the magnetic vector potential and the magnetic scalar potential, the $A - \Phi$-formulation

Applying the *magnetic vector potential $\boldsymbol{A}$* in the whole domain of a static magnetic field problem is the general solution, however, applying the total or the *reduced magnetic scalar potential* Ψ or Φ (or their combination) is a more economic way.

The combination of these methods results in the $\boldsymbol{A} - \Phi$-formulation as it is illustrated in *Fig.* 2.15. It is possible to use the magnetic vector potential only in the region filled with iron, $\Omega_{\boldsymbol{A}}$ and the reduced magnetic scalar potential in the air region Ω_Φ where μ is equal to μ_0. The two potential formulations are coupled through interface conditions along the interface $\Gamma_{\boldsymbol{A},\Phi}$. The main advantage of this formulation is that the number of unknowns in the air domain can be decreased significantly, however, extra interface conditions have to be formulated.

In the subregion $\Omega_{\boldsymbol{A}}$, the partial differential equation of the magnetic vector potential is the same as in (2.223), but $\boldsymbol{J}_0 = \boldsymbol{0}$ is true in iron. The boundary conditions on $\Gamma_{H_{\boldsymbol{A}}}$ and on $\Gamma_{B_{\boldsymbol{A}}}$ are the same as in (2.224), (2.225) and in (2.226), (2.227), respectively. The partial differential equation (2.160) and the boundary conditions (2.161), (2.162) are valid
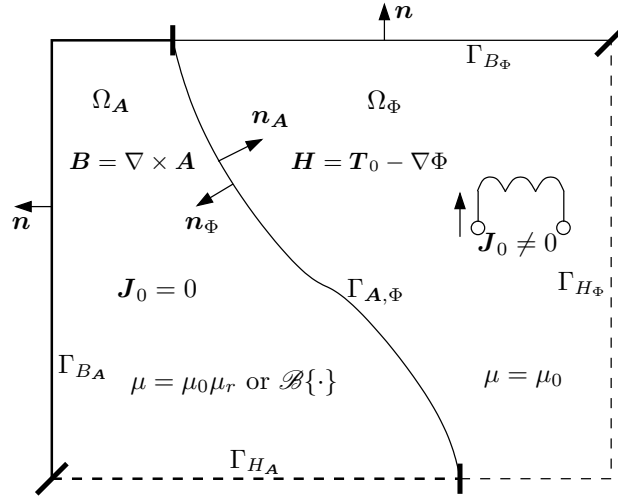
Fig. 2.15. The scheme of a static magnetic field problem with magnetic vector potential
and reduced magnetic scalar potential

in the subregion Ω_Φ partially bounded by Γ_{H_Φ} and Γ_{B_Φ}, but $\boldsymbol{R} = \boldsymbol{0}$ and $\mu = \mu_0$. The tangential component of the magnetic field intensity and the normal component of the magnetic flux density must be continuous on the interface between the two subregions, $\Gamma_{\boldsymbol{A},\Phi}$.

Finally, the partial differential equations, the boundary and interface conditions of the static magnetic field problem, which solution is unique according to *Coulomb gauge* can be written as

$$\nabla \times (\nu_o \nabla \times \boldsymbol{A}) - \nabla(\nu_0 \nabla \cdot \boldsymbol{A}) = -\nabla \times \boldsymbol{I}, \quad \text{in} \quad \Omega_{\boldsymbol{A}}, \tag{2.231}$$

$$\nabla \cdot (\mu_0 \nabla \Phi) = \nabla \cdot (\mu_0 \boldsymbol{T}_0), \quad \text{in} \quad \Omega_\Phi, \tag{2.232}$$

$$(\nu_o \nabla \times \boldsymbol{A} + \boldsymbol{I}) \times \boldsymbol{n} = \boldsymbol{K}, \quad \text{on} \quad \Gamma_{H_{\boldsymbol{A}}}, \tag{2.233}$$

$$\boldsymbol{A} \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_{H_{\boldsymbol{A}}}, \tag{2.234}$$

$$\boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{\alpha}, \quad \text{on} \quad \Gamma_{B_{\boldsymbol{A}}}, \tag{2.235}$$

$$\nu_o \nabla \cdot \boldsymbol{A} = 0, \quad \text{on} \quad \Gamma_{B_{\boldsymbol{A}}}, \tag{2.236}$$

$$\Phi = \Phi_0, \quad \text{on} \quad \Gamma_{H_\Phi}, \tag{2.237}$$

$$(\mu_0 \boldsymbol{T}_0 - \mu_0 \nabla \Phi) \cdot \boldsymbol{n} = -b, \quad \text{on} \quad \Gamma_{B_\Phi}, \tag{2.238}$$

$$(\nu_o \nabla \times \boldsymbol{A} + \boldsymbol{I}) \times \boldsymbol{n_A} + (\boldsymbol{T}_0 - \nabla \Phi) \times \boldsymbol{n}_\Phi = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{\boldsymbol{A},\Phi}, \tag{2.239}$$

$$(\nabla \times \boldsymbol{A}) \cdot \boldsymbol{n_A} + \mu_0(\boldsymbol{T}_0 - \nabla \Phi) \cdot \boldsymbol{n}_\Phi = 0, \quad \text{on} \quad \Gamma_{\boldsymbol{A},\Phi}, \tag{2.240}$$

$$\boldsymbol{A} \cdot \boldsymbol{n_A} = 0, \quad \text{on} \quad \Gamma_{\boldsymbol{A},\Phi}. \tag{2.241}$$

On the $\Gamma_{A,\Phi}$ part of the boundary of the subdomain Ω_A ($\partial\Omega_A = \Gamma_{H_A}\cup\Gamma_{B_A}\cup\Gamma_{A,\Phi}$), the normal component of the magnetic vector potential must be set to zero (see equation (2.241)). Coulomb gauge $\nabla \cdot A = 0$ is satisfied in the domain Ω_A partially bounded by $\Gamma_{H_A} \cup \Gamma_{B_A}$, but it is not true on the new boundary part $\Gamma_{A,\Phi}$. Here either the normal or the tangential component of the magnetic vector potential must be specified. According to the condition (2.239), the tangential component of the magnetic field intensity is specified by the unknown scalar potential, i.e. $(\nu_o\nabla \times A + I)\times n_A = -(T_0 - \nabla\Phi)\times n_\Phi$, which is similar to the condition (2.233). The boundary part $\Gamma_{A,\Phi}$ is similar to the part Γ_{H_A}, where the normal component of the magnetic vector potential has been specified.

Unfortunately, the behavior of the magnetic field in the vicinity of the iron/air interface may be very strange if the finite element method is used, because of the weak coupling between the two regions represented by the equations (2.239) and (2.240). The reason of it is that the normal component of the magnetic flux density in iron and the tangential component of the magnetic field intensity in air may be very low as it is illustrated in *Fig.* 2.16 (here, an iron cube is placed into a homogeneous magnetic field H_0). It is easy to see that $B \cdot n \approx 0$ on the left side of the iron cube, where $H \times n \approx 0$ as well and this strange behavior can not be eliminated by decreasing the size of mesh in the vicinity of iron/air interface. A possible solution of this numerical difficulty is moving the A/Φ interface from the iron/air interface into the air region, which results in the so-called $A - A - \Phi$-formulation (*Fig.* 2.16).
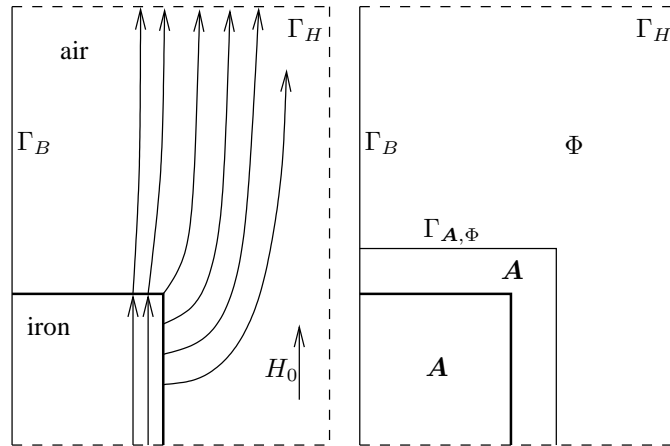
Fig. 2.16. Illustration for the $A - A - \Phi$-formulation

At the end of this section, the partial differential equations, the boundary and interface conditions of the static magnetic field problem in the case of ungauged version are also presented,

$$\nabla \times (\nu_o\nabla \times A) = -\nabla \times I, \quad \text{in} \quad \Omega_A, \tag{2.242}$$

$$\nabla \cdot (\mu_0\nabla\Phi) = \nabla \cdot (\mu_0 T_0), \quad \text{in} \quad \Omega_\Phi, \tag{2.243}$$

$$(\nu_o\nabla \times A + I)\times n = K, \quad \text{on} \quad \Gamma_{H_A}, \tag{2.244}$$

$$\boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{\alpha}, \quad \text{on} \quad \Gamma_{B_{\boldsymbol{A}}}, \tag{2.245}$$

$$\Phi = \Phi_0, \quad \text{on} \quad \Gamma_{H_\Phi}, \tag{2.246}$$

$$(\mu_0 \boldsymbol{T}_0 - \mu_0 \nabla \Phi) \cdot \boldsymbol{n} = -b, \quad \text{on} \quad \Gamma_{B_\Phi}, \tag{2.247}$$

$$(\nu_o \nabla \times \boldsymbol{A} + \boldsymbol{I}) \times \boldsymbol{n}_{\boldsymbol{A}} + (\boldsymbol{T}_0 - \nabla \Phi) \times \boldsymbol{n}_\Phi = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{\boldsymbol{A}, \Phi}, \tag{2.248}$$

$$(\nabla \times \boldsymbol{A}) \cdot \boldsymbol{n}_{\boldsymbol{A}} + \mu_0 (\boldsymbol{T}_0 - \nabla \Phi) \cdot \boldsymbol{n}_\Phi = 0, \quad \text{on} \quad \Gamma_{\boldsymbol{A}, \Phi}. \tag{2.249}$$

2.3.2 Eddy current fields

Eddy current field is defined by the equations (2.89), (2.90), (2.91), (2.92) or (2.94) and (2.93) and the boundary conditions introduced in section 2.2.2.

Two potential functions can be used in the eddy current region, either a *current vector potential* $\boldsymbol{T}$ or a *magnetic vector potential* $\boldsymbol{A}$. The current vector potential $\boldsymbol{T}$ can be coupled with a *magnetic scalar potential*, here the reduced magnetic scalar potential Φ is used. The magnetic vector potential $\boldsymbol{A}$ can be coupled with an *electric scalar potential*, denoted by V.

A. The current vector potential and magnetic scalar potential, the $\boldsymbol{T}$, Φ-formulation

The *solenoidal property* of the induced eddy current density (2.95) results in the possibility of applying the current vector potential $\boldsymbol{T}$ to represent the eddy current field in conducting materials,

$$\nabla \cdot \boldsymbol{J} = 0 \quad \Rightarrow \quad \boldsymbol{J} = \nabla \times \boldsymbol{T}, \tag{2.250}$$

because of the identity $\nabla \cdot \nabla \times \boldsymbol{v} \equiv 0$ for any vector function $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r})$, or $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r}, t)$. Substituting this relation to the first Maxwell's equation (2.89), i.e.

$$\nabla \times \boldsymbol{H} = \nabla \times \boldsymbol{T} \quad \Rightarrow \quad \nabla \times (\boldsymbol{H} - \boldsymbol{T}) = \boldsymbol{0} \tag{2.251}$$

results in the reduced magnetic scalar potential Φ as

$$\boldsymbol{H} - \boldsymbol{T} = -\nabla \Phi \quad \Rightarrow \quad \boldsymbol{H} = \boldsymbol{T} - \nabla \Phi, \tag{2.252}$$

because $\nabla \times \nabla \varphi \equiv 0$ for any scalar function $\varphi = \varphi(\boldsymbol{r})$ or $\varphi = \varphi(\boldsymbol{r}, t)$. The first Maxwell's equation (2.89) has been satisfied exactly by this formulation.

Applying the *impressed current vector potential* $\boldsymbol{T}_0$ to represent the known source current density $\boldsymbol{J}_0$ placed in the eddy current free region takes easier the coupling of the present formulation with the reduced magnetic scalar potential in the eddy current free region, i.e. appending $\boldsymbol{T}_0$ to (2.252) is advantageous,

$$\boldsymbol{H} = \boldsymbol{T}_0 + \boldsymbol{T} - \nabla \Phi. \tag{2.253}$$

This can be done, since $\nabla \times \boldsymbol{T}_0 = \boldsymbol{0}$ in the eddy current region Ω_c. It is noted that the reduced magnetic scalar potential is used in this formulation and we have to take care about the representation of $\boldsymbol{T}_0$ (see point D on page 32).

The expression $J = \sigma E = \nabla \times T$ can be written according to the constitutive relation (2.93), from which the electric field intensity E can be expressed by the current vector potential as

$$E = \frac{1}{\sigma} \nabla \times T. \tag{2.254}$$

Substituting this expression and the linearized form of the constitutive relation in (2.92) into *Faraday's law* (2.90) results in the partial differential equation

$$\nabla \times \left(\frac{1}{\sigma} \nabla \times T \right) + \mu_o \frac{\partial T}{\partial t} - \mu_o \nabla \frac{\partial \Phi}{\partial t} = -\mu_o \frac{\partial T_0}{\partial t} - \frac{\partial R}{\partial t}, \quad \text{in} \quad \Omega_c. \tag{2.255}$$

The magnetic *Gauss' law* (2.91) can be rewritten in the form

$$\nabla \cdot (\mu_o T - \mu_o \nabla \Phi) = -\nabla \cdot (\mu_o T_0) - \nabla \cdot R, \quad \text{in} \quad \Omega_c. \tag{2.256}$$

The solution of these partial differential equations with the below described boundary conditions results in two unknowns (T and Φ) of the T, Φ-formulation.

Let us now define the boundary conditions.

On the Γ_{H_c} part of the boundary, the tangential component of the magnetic field intensity vector must vanish,

$$H \times n = 0 \quad \Rightarrow \quad (T_0 + T - \nabla \Phi) \times n = 0, \quad \text{on} \quad \Gamma_{H_c}. \tag{2.257}$$

The boundary part Γ_{H_c} always represents a symmetry plane, where $T_0 \times n \equiv 0$, i.e. the Dirichlet boundary conditions

$$T \times n = 0, \quad \text{and} \quad \Phi = \Phi_0, \quad \text{on} \quad \Gamma_{H_c} \tag{2.258}$$

have to be specified and Φ_0 is constant. On Γ_{H_c}, the equation $J \cdot n = 0$ satisfies automatically, because of

$$J \cdot n = (\nabla \times T) \cdot n = \nabla \cdot (T \times n), \tag{2.259}$$

and $T \times n = 0$ is prescribed here.

On the rest of boundary Γ_E, the tangential component of the electric field intensity must be equal to zero,

$$E \times n = 0 \quad \Rightarrow \quad \left(\frac{1}{\sigma} \nabla \times T \right) \times n = 0, \quad \text{on} \quad \Gamma_E, \tag{2.260}$$

and the normal component of the magnetic flux density must vanish,

$$B \cdot n = 0 \quad \Rightarrow \quad (\mu_o T_0 + \mu_o T - \mu_o \nabla \Phi + R) \cdot n = 0, \quad \text{on} \quad \Gamma_E. \tag{2.261}$$

These are Neumann type boundary conditions.

Finally, here is the collection of partial differential equations and boundary conditions of the T, Φ-formulation, which is however, not gauged,

$$\nabla \times \left(\frac{1}{\sigma} \nabla \times T \right) + \mu_o \frac{\partial T}{\partial t} - \mu_o \nabla \frac{\partial \Phi}{\partial t} = -\mu_o \frac{\partial T_0}{\partial t} - \frac{\partial R}{\partial t}, \quad \text{in} \quad \Omega_c, \tag{2.262}$$

$$\nabla \cdot (\mu_o \boldsymbol{T} - \mu_o \nabla \Phi) = -\nabla \cdot (\mu_o \boldsymbol{T}_0) - \nabla \cdot \boldsymbol{R}, \quad \text{in} \quad \Omega_c, \tag{2.263}$$

$$\boldsymbol{T} \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.264}$$

$$\Phi = \Phi_0, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.265}$$

$$\left(\frac{1}{\sigma} \nabla \times \boldsymbol{T} \right) \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_E, \tag{2.266}$$

$$(\mu_o \boldsymbol{T}_0 + \mu_o \boldsymbol{T} - \mu_o \nabla \Phi + \boldsymbol{R}) \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_E. \tag{2.267}$$

The solution of the problem defined by the above equations and boundary conditions is not unique, because the divergence of the current vector potential has not specified yet. The *Coulomb gauge* should be used in this formulation, similarly to the gauge fixing method applied in the $\boldsymbol{A}$-formulation (see on page 34), i.e.

$$\nabla \cdot \boldsymbol{T} = 0, \quad \text{in} \quad \Omega_c \tag{2.268}$$

must be specified.

Let us first append a $-\nabla \left(1/\sigma \nabla \cdot \boldsymbol{T} \right)$ term to the left-hand side of the equation (2.262),

$$\nabla \times \left(\frac{1}{\sigma} \nabla \times \boldsymbol{T} \right) - \nabla \left(\frac{1}{\sigma} \nabla \cdot \boldsymbol{T} \right) + \mu_o \frac{\partial \boldsymbol{T}}{\partial t} - \mu_o \nabla \frac{\partial \Phi}{\partial t}$$
$$= -\mu_o \frac{\partial \boldsymbol{T}_0}{\partial t} - \frac{\partial \boldsymbol{R}}{\partial t}, \quad \text{in} \quad \Omega_c \tag{2.269}$$

Taking the divergence of this equation and taking equation (2.263) into account results in a *Laplace equation* for the scalar variable $1/\sigma \nabla \cdot \boldsymbol{T}$,

$$-\nabla \cdot \nabla \left(\frac{1}{\sigma} \nabla \cdot \boldsymbol{T} \right) = 0. \tag{2.270}$$

The solution of a Laplace equation $-\nabla \cdot \nabla \varphi = 0$ can be equal to zero if the potential φ is prescribed on the whole boundary by a homogeneous Dirichlet boundary condition, or a homogeneous Dirichlet boundary condition on one part and a homogeneous Neumann boundary condition on the rest of the boundary.

After taking the normal component of the equation (2.269) on the boundary segment Γ_E, a homogeneous Neumann boundary condition can be prescribed automatically

$$-\frac{\partial}{\partial n} \left(\frac{1}{\sigma} \nabla \cdot \boldsymbol{T} \right) = 0, \quad \text{on} \quad \Gamma_E, \tag{2.271}$$

because the normal component of the first term in (2.269),

$$\left[\nabla \times \left(\frac{1}{\sigma} \nabla \times \boldsymbol{T} \right) \right] \cdot \boldsymbol{n} = \nabla \cdot \left[\left(\frac{1}{\sigma} \nabla \times \boldsymbol{T} \right) \times \boldsymbol{n} \right] \tag{2.272}$$

is equal to zero according to the boundary condition (2.266) and the normal component of the sum of the last four terms in (2.269) is equal to zero, too, because of the condition (2.267).

Consequently, on the rest of the boundary Γ_{H_c}, the following Neumann boundary condition:

$$\frac{1}{\sigma}\nabla \cdot \boldsymbol{T} = 0, \quad \text{on} \quad \Gamma_{H_c} \tag{2.273}$$

must be specified.

In this way $1/\sigma\nabla \cdot \boldsymbol{T} \equiv 0$ in the whole problem region Ω_c and on the boundary $\Gamma_{H_c} \cup \Gamma_E$.

Finally, the partial differential equations and the boundary conditions of an eddy current field problem, which solution is unique according to *Coulomb gauge* can be written as

$$\nabla \times \left(\frac{1}{\sigma}\nabla \times \boldsymbol{T}\right) - \nabla \left(\frac{1}{\sigma}\nabla \cdot \boldsymbol{T}\right) + \mu_o\frac{\partial \boldsymbol{T}}{\partial t} - \mu_o\nabla\frac{\partial \Phi}{\partial t}$$
$$= -\mu_o\frac{\partial \boldsymbol{T}_0}{\partial t} - \frac{\partial \boldsymbol{R}}{\partial t}, \quad \text{in} \quad \Omega_c, \tag{2.274}$$

$$\nabla \cdot (\mu_o\boldsymbol{T} - \mu_o\nabla\Phi) = -\nabla \cdot (\mu_o\boldsymbol{T}_0) - \nabla \cdot \boldsymbol{R}, \quad \text{in} \quad \Omega_c, \tag{2.275}$$

$$\boldsymbol{T} \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.276}$$

$$\Phi = \Phi_0, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.277}$$

$$\frac{1}{\sigma}\nabla \cdot \boldsymbol{T} = 0, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.278}$$

$$\left(\frac{1}{\sigma}\nabla \times \boldsymbol{T}\right) \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_E, \tag{2.279}$$

$$(\mu_o\boldsymbol{T}_0 + \mu_o\boldsymbol{T} - \mu_o\nabla\Phi + \boldsymbol{R}) \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_E, \tag{2.280}$$

$$\boldsymbol{T} \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_E. \tag{2.281}$$

The last boundary condition (2.281) is introduced analogously to the boundary condition (2.225) in the $\boldsymbol{A}$-formulation. As a last check of the uniqueness of the current vector potential, let us append a $\nabla\varphi$ term to the vector potential $\boldsymbol{T}$ as $\boldsymbol{T} + \nabla\varphi$ and take the divergence of this term,

$$\nabla \cdot (\boldsymbol{T} + \nabla\varphi) = \nabla \cdot \boldsymbol{T} + \nabla \cdot \nabla\varphi = 0. \tag{2.282}$$

The first term of this equation is equal to zero according to Coulomb gauge, i.e. only $\nabla \cdot \nabla\varphi = 0$ has to be analyzed. On the boundary part Γ_{H_c}, the boundary condition $(\boldsymbol{T} + \nabla\varphi) \times \boldsymbol{n} = \boldsymbol{0}$ results in $(\nabla\varphi) \times \boldsymbol{n} = \boldsymbol{0}$, i.e. φ is constant on Γ_{H_c}. On the rest part Γ_E, the normal component of $\nabla\varphi$ can be specified as $\nabla\varphi \cdot \boldsymbol{n} = 0$. This can be obtained by introducing $\boldsymbol{T} \cdot \boldsymbol{n} = 0$ on Γ_E. Finally, it is known that $\nabla \cdot \nabla\varphi = 0$ in Ω and $\nabla\varphi \cdot \boldsymbol{n} = \partial\varphi/\partial n = 0$ on Γ_E, the constant value of φ on Γ_{H_c} can only be zero. This means that $\varphi = 0$ in the problem region Ω and on the boundary $\Gamma_{H_c} \cup \Gamma_E$, i.e. the current vector potential is unique.

B. The magnetic vector potential and electric scalar potential, the A, V-formulation

The divergence-free magnetic flux density vector can be described by the curl of the *magnetic vector potential A*, since $\nabla \cdot \nabla \times \boldsymbol{u} \equiv 0$, for any vector function $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{r})$, or $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{r},t)$, i.e.

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}. \tag{2.283}$$

This automatically enforces the satisfaction of magnetic *Gauss' law* (2.91). Substituting expression (2.283) into *Faraday's law* (2.90) results in

$$\nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t}\nabla \times \boldsymbol{A} = -\nabla \times \frac{\partial \boldsymbol{A}}{\partial t} \quad \Rightarrow \quad \nabla \times \left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} \right) = \boldsymbol{0}, \tag{2.284}$$

because rotation (i.e. derivation by space) and derivation by time can be replaced. The curl-less vector field $\boldsymbol{E} + \partial \boldsymbol{A}/\partial t$ can be derived from the so-called electric scalar potential V ($\nabla \times \nabla \varphi \equiv 0$, for any scalar functions $\varphi = \varphi(\boldsymbol{r})$, or $\varphi = \varphi(\boldsymbol{r},t)$),

$$\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} = -\nabla V, \tag{2.285}$$

and the electric field intensity vector can be described by two potentials as

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla V. \tag{2.286}$$

Substituting the relations (2.283) and (2.286) into (2.89) and using the linearized constitutive relation in (2.94) leads to the partial differential equation

$$\nabla \times (\nu_o \nabla \times \boldsymbol{A}) + \sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla V = -\nabla \times \boldsymbol{I}, \quad \text{in} \quad \Omega_c. \tag{2.287}$$

The charge conservation law (2.95) with the constitutive relation (2.93) and with the formulation (2.286) results in the second partial differential equation of this formulation,

$$-\nabla \cdot \left(\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla V \right) = 0, \quad \text{in} \quad \Omega_c. \tag{2.288}$$

There are two unknown functions (A and V), that is why two equations must be formulated, however, the second one is coming from taking the divergence of Ampere's law (2.89).

Now, let us define the boundary conditions of the problem.

On the Γ_{H_c} part of the boundary, the tangential component of the magnetic field intensity vector must vanish,

$$\boldsymbol{H} \times \boldsymbol{n} = \boldsymbol{0} \quad \Rightarrow \quad (\nu_o \nabla \times \boldsymbol{A} + \boldsymbol{I}) \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{H_c}. \tag{2.289}$$

This is a Neumann boundary condition for A. The normal component of eddy currents must be equal to zero on Γ_{H_c}, which can be formulated by the Neumann boundary condition

$$\boldsymbol{J} \cdot \boldsymbol{n} \quad \Rightarrow \quad -\sigma \frac{\partial \boldsymbol{A}}{\partial t} \cdot \boldsymbol{n} - \sigma \nabla V \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_{H_c}. \tag{2.290}$$

On the rest part of the boundary Γ_E, the tangential component of the electric field intensity must be equal to zero,

$$\boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{0} \quad \Rightarrow \quad \left(-\frac{\partial \boldsymbol{A}}{\partial t} - \nabla V\right) \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_E. \tag{2.291}$$

This boundary condition can be specified by two Dirichlet boundary conditions,

$$\boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{0}, \quad \text{and} \quad V = V_0, \quad \text{on} \quad \Gamma_E, \tag{2.292}$$

because $-\boldsymbol{A} \times \boldsymbol{n} = \boldsymbol{n} \times \boldsymbol{A}$ and V_0 is constant. Here $\boldsymbol{B} \cdot \boldsymbol{n} = 0$ satisfies explicitly, because

$$\boldsymbol{B} \cdot \boldsymbol{n} = (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{n} = \nabla \cdot (\boldsymbol{A} \times \boldsymbol{n}), \tag{2.293}$$

and $\boldsymbol{A} \times \boldsymbol{n} = \boldsymbol{0}$ has been prescribed yet.

Finally, here is the collection of equations and boundary conditions of the ungauged $\boldsymbol{A}$, V-formulation,

$$\nabla \times (\nu_o \nabla \times \boldsymbol{A}) + \sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla V = -\nabla \times \boldsymbol{I}, \quad \text{in} \quad \Omega_c, \tag{2.294}$$

$$-\nabla \cdot \left(\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla V\right) = 0, \quad \text{in} \quad \Omega_c, \tag{2.295}$$

$$(\nu_o \nabla \times \boldsymbol{A} + \boldsymbol{I}) \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.296}$$

$$-\sigma \frac{\partial \boldsymbol{A}}{\partial t} \cdot \boldsymbol{n} - \sigma \nabla V \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.297}$$

$$\boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_E, \tag{2.298}$$

$$V = V_0, \quad \text{on} \quad \Gamma_E. \tag{2.299}$$

The solution of the problem defined by the above equations and boundary conditions is not unique, because the divergence of the magnetic vector potential has not specified yet. The *Coulomb gauge* should be used in this formulation similarly to the gauge fixing method applied in $\boldsymbol{A}$-formulation of static magnetic field problems (see on page 34), i.e. $\nabla \cdot \boldsymbol{A} = 0$ must be specified.

First, let us append the left-hand side of the partial differential equation (2.294) by the term $-\nabla (\nu_o \nabla \cdot \boldsymbol{A})$,

$$\nabla \times (\nu_o \nabla \times \boldsymbol{A}) - \nabla (\nu_o \nabla \cdot \boldsymbol{A}) + \sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla V = -\nabla \times \boldsymbol{I}, \quad \text{in} \quad \Omega_c. \tag{2.300}$$

Taking the divergence of this equation and taking the equation (2.295) into account, it results in the *Laplace equation* for the scalar variable $\nu_o \nabla \cdot \boldsymbol{A}$,

$$-\nabla \cdot \nabla (\nu_o \nabla \cdot \boldsymbol{A}) = 0. \tag{2.301}$$

After taking the normal component of the equation (2.300) on the boundary segment Γ_{H_c}, a homogeneous Neumann boundary condition can be set up automatically

$$-\frac{\partial}{\partial n} (\nu_o \nabla \cdot \boldsymbol{A}) = 0, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.302}$$

because the normal component of the first and the last terms in (2.300),

$$[\nabla \times (\nu_o \nabla \times \boldsymbol{A} + \boldsymbol{I})] \cdot \boldsymbol{n} = \nabla \cdot [(\nu_o \nabla \times \boldsymbol{A} + \boldsymbol{I}) \times \boldsymbol{n}] \tag{2.303}$$

is equal to zero according to the boundary condition (2.296) and the normal component of the sum of the last two terms in (2.300) is equal to zero, too, because of the condition (2.297).

Consequently, on the rest part of the boundary Γ_E, the following Neumann type boundary condition:

$$\nu_o \nabla \cdot \boldsymbol{A} = 0, \quad \text{on} \quad \Gamma_E \tag{2.304}$$

must be specified.

In this way $\nu_o \nabla \cdot \boldsymbol{A} \equiv 0$ can be satisfied in the whole problem region Ω_c and on the boundary $\Gamma_{H_c} \cup \Gamma_E$.

Finally, the partial differential equations and the boundary conditions of an eddy current field problem, which solution is unique according to *Coulomb gauge* can be written as

$$\nabla \times (\nu_o \nabla \times \boldsymbol{A}) - \nabla (\nu_o \nabla \cdot \boldsymbol{A}) + \sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla V = -\nabla \times \boldsymbol{I}, \quad \text{in} \quad \Omega_c, \tag{2.305}$$

$$-\nabla \cdot \left(\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla V \right) = 0, \quad \text{in} \quad \Omega_c, \tag{2.306}$$

$$(\nu_o \nabla \times \boldsymbol{A} + \boldsymbol{I}) \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.307}$$

$$-\left(\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla V \right) \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.308}$$

$$\boldsymbol{A} \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.309}$$

$$\boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_E, \tag{2.310}$$

$$V = V_0, \quad \text{on} \quad \Gamma_E, \tag{2.311}$$

$$\nu_o \nabla \cdot \boldsymbol{A} = 0, \quad \text{on} \quad \Gamma_E. \tag{2.312}$$

Here, equation (2.309) is introduced according to the proof presented on page 38.

C. The modified magnetic vector potential, the $\boldsymbol{A}^{\star}$-formulation

This formulation can only be applied when the ungauged version of the $\boldsymbol{A}$, V-formulation is used.

In this situation, the electric scalar potential V can be supposed to be equal to zero, if and only if, the conductivity σ is constant. The set of partial differential equations and boundary conditions are coming from the equations (2.294)–(2.299), but the partial differential equations and boundary conditions containing V have been eliminated, i.e.

$$\nabla \times (\nu_o \nabla \times \boldsymbol{A}^{\star}) + \sigma \frac{\partial \boldsymbol{A}^{\star}}{\partial t} = -\nabla \times \boldsymbol{I}, \quad \text{in} \quad \Omega_c, \tag{2.313}$$

$$(\nu_o \nabla \times \boldsymbol{A}^{\star} + \boldsymbol{I}) \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.314}$$

$$\boldsymbol{n} \times \boldsymbol{A}^{\star} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_E. \tag{2.315}$$

2.3.3 Coupling static magnetic and eddy current fields

In most *eddy current* field problems, the conductors carrying the eddy currents are at least partially surrounded by a nonconducting medium free of eddy currents (e.g. air) where a static magnetic field is present (*Fig.* 2.17). The static magnetic field is induced both by the eddy currents and by the source current of coils. That is why the potential formulations of the static magnetic field and of the eddy current field must be coupled.
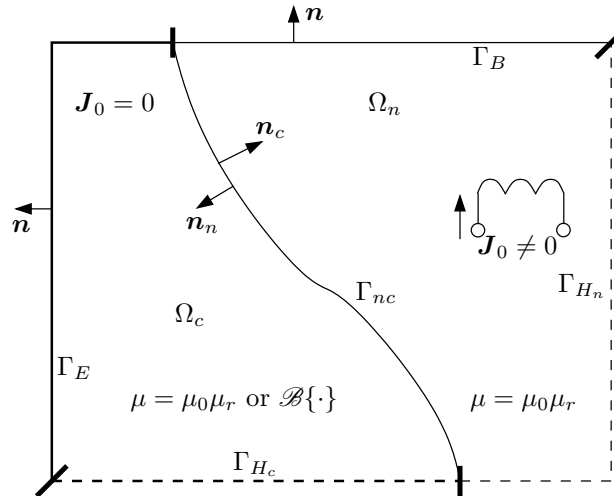
Fig. 2.17. The eddy current region is surrounded by nonconducting region

The static magnetic field in Ω_n can be described by a *magnetic scalar potential* (reduced, total or the combination of them), or by a *magnetic vector potential*. In the first case, applying the reduced magnetic scalar potential Φ is simpler to use, however, currents of coils must be represented by an impressed current vector potential T_0, which must be realized in a special way. Application of the magnetic vector potential is a more general way. Combination of the two formulations can also be supposed, e.g. $A - \Phi$, or $A - A - \Phi$ method.

The eddy current field in Ω_c can be represented by a vector potential coupled with a scalar potential. The *current vector potential* T can be coupled with the *reduced magnetic scalar potential* Φ and the *magnetic vector potential* A can be coupled with the *electric scalar potential* V.

It is important to note that the vector potentials can be either ungauged or gauged. The ungauged version has infinite number of solutions for the potentials, but electromagnetic field quantities calculated from the potentials are unique, of course, since the solution of Maxwell's equations is unique. The gauged versions result in only one unique solution for the potentials, too.

Here, the possible potential formulations are shown in the gauged and ungauged situations. The partial differential equations and the boundary conditions are based on

the last section, but the interface conditions between the regions with and without eddy currents are presented below. First, the basic formulations are shown, the T, Φ formulation in the eddy current region is coupled to the Φ formulation in the eddy current free region resulting the $T, \Phi - \Phi$ formulation, then the A, V formulation in the eddy current region is coupled to the A formulation in the eddy current free region resulting the $A, V - A$ formulation. The $T, \Phi - \Phi$ formulation can not be used when the eddy current region is *multiply connected*. This problem can be solved by the $T, \Phi - A$ formulation and by the $T, \Phi - A - \Phi$ formulation. The $A, V - A$ formulation is not an economic procedure, there are three unknown functions to represent A in air, that is why the $A, V - \Phi$ and the $A, V - A - \Phi$ formulations have been developed.

A. The gauged $T, \Phi - \Phi$ formulation

The *reduced magnetic scalar potential* Φ is used in the eddy current free region Ω_n as well as in the eddy current region Ω_c. It is a continuous scalar variable in the entire region $\Omega_c \cup \Omega_n$ and on the interface Γ_{nc}, too. The equations to be used are (2.160)–(2.162) and (2.274)–(2.281), however, some continuity equations on Γ_{nc} must be appended to these equations.

The magnetic field intensity vector is derived as $H = T_0 - \nabla\Phi$ in Ω_n and it is written as $H = T_0 + T - \nabla\Phi$ in Ω_c. The tangential component of the magnetic field intensity can be set to be continuous on Γ_{nc} by a continuous magnetic scalar potential and by setting the tangential component of the *current vector potential* equal to zero by the boundary condition $T \times n = 0$ on Γ_{nc}. It must be noted that $T_0 \times n$ is continuous. Vanishing the normal component of eddy current density on Γ_{nc} satisfies automatically, because $J = \nabla \times T$ and $J \cdot n = (\nabla \times T) \cdot n = (T \times n) \cdot \nabla$. The term in the last brackets has been set to zero, i.e. $J \cdot n = 0$ on Γ_{nc}.

The continuity of the normal component of magnetic flux density results in a Neumann type boundary condition (see (2.329)). The divergence free property of T has been defined only in the domain Ω_c and on the boundary $\Gamma_{H_c} \cup \Gamma_E$. On the rest segment $\Gamma_{nc}, 1/\sigma\nabla \cdot T = 0$ must be specified as a new Neumann type interface condition. It can be introduced similarly to (2.273).

The summarized equations of the gauged version are as follows (see *Fig.* 2.17):

$$
\nabla \times \left(\frac{1}{\sigma}\nabla \times T\right) - \nabla \left(\frac{1}{\sigma}\nabla \cdot T\right) + \mu_o \frac{\partial T}{\partial t} - \mu_o \nabla \frac{\partial \Phi}{\partial t}
$$
$$
= -\mu_o \frac{\partial T_0}{\partial t} - \frac{\partial R}{\partial t}, \quad \text{in} \quad \Omega_c, \tag{2.316}
$$

$$
\nabla \cdot (\mu_o T - \mu_o \nabla\Phi) = -\nabla \cdot (\mu_o T_0) - \nabla \cdot R, \quad \text{in} \quad \Omega_c, \tag{2.317}
$$

$$
-\nabla \cdot (\mu\nabla\Phi) = -\nabla \cdot (\mu T_0), \quad \text{in} \quad \Omega_n, \tag{2.318}
$$

$$
T \times n = 0, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.319}
$$

$$
\Phi = \Phi_0, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.320}
$$

$$
\frac{1}{\sigma}\nabla \cdot T = 0, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.321}
$$

$$\Phi = \Phi_0, \quad \text{on} \quad \Gamma_{H_n}, \tag{2.322}$$

$$\left(\frac{1}{\sigma}\nabla \times \boldsymbol{T}\right) \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_E, \tag{2.323}$$

$$(\mu_o \boldsymbol{T}_0 + \mu_o \boldsymbol{T} - \mu_o \nabla\Phi + \boldsymbol{R}) \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_E, \tag{2.324}$$

$$\boldsymbol{T} \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_E, \tag{2.325}$$

$$(\mu \boldsymbol{T}_0 - \mu \nabla\Phi) \cdot \boldsymbol{n} = -b, \quad \text{on} \quad \Gamma_B, \tag{2.326}$$

$$\Phi \text{ is continuous on } \Gamma_{nc}, \tag{2.327}$$

$$\boldsymbol{T} \times \boldsymbol{n}_c = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc}, \tag{2.328}$$

$$(\mu \boldsymbol{T}_0 - \mu \nabla\Phi) \cdot \boldsymbol{n}_n + (\mu_o \boldsymbol{T}_0 + \mu_o \boldsymbol{T} - \mu_o \nabla\Phi + \boldsymbol{R}) \cdot \boldsymbol{n}_c = 0, \quad \text{on } \Gamma_{nc}, \tag{2.329}$$

$$\frac{1}{\sigma}\nabla \cdot \boldsymbol{T} = 0, \quad \text{on} \quad \Gamma_{nc}. \tag{2.330}$$

B. The ungauged $\boldsymbol{T}, \Phi - \Phi$ formulation

The *reduced magnetic scalar potential* Φ is used in the eddy current free region Ω_n as well as in the eddy current region Ω_c and it is a scalar continuous variable in the entire region $\Omega_c \cup \Omega_n$ and on the interface Γ_{nc}, too. The equations of ungauged $\boldsymbol{T}, \Phi - \Phi$ formulation can be built up by using (2.160)–(2.162) and (2.262)–(2.267), which must be appended by some continuity equations defined on Γ_{nc}.

The interface conditions can be defined as in the gauged version, however, (2.330) is not used.

The summarized equations are as follows (see *Fig.* 2.17):

$$\nabla \times \left(\frac{1}{\sigma}\nabla \times \boldsymbol{T}\right) + \mu_o \frac{\partial \boldsymbol{T}}{\partial t} - \mu_o \nabla \frac{\partial \Phi}{\partial t} = -\mu_o \frac{\partial \boldsymbol{T}_0}{\partial t} - \frac{\partial \boldsymbol{R}}{\partial t}, \quad \text{in} \quad \Omega_c, \tag{2.331}$$

$$\nabla \cdot (\mu_o \boldsymbol{T} - \mu_o \nabla\Phi) = -\nabla \cdot (\mu_o \boldsymbol{T}_0) - \nabla \cdot \boldsymbol{R}, \quad \text{in} \quad \Omega_c, \tag{2.332}$$

$$-\nabla \cdot (\mu \nabla\Phi) = -\nabla \cdot (\mu \boldsymbol{T}_0), \quad \text{in} \quad \Omega_n, \tag{2.333}$$

$$\boldsymbol{T} \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.334}$$

$$\Phi = \Phi_0, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.335}$$

$$\Phi = \Phi_0, \quad \text{on} \quad \Gamma_{H_n}, \tag{2.336}$$

$$\left(\frac{1}{\sigma}\nabla \times \boldsymbol{T}\right) \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_E, \tag{2.337}$$

$$(\mu_o \boldsymbol{T}_0 + \mu_o \boldsymbol{T} - \mu_o \nabla\Phi + \boldsymbol{R}) \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_E, \tag{2.338}$$

$$(\mu \boldsymbol{T}_0 - \mu \nabla\Phi) \cdot \boldsymbol{n} = -b, \quad \text{on} \quad \Gamma_B, \tag{2.339}$$

$$\Phi \text{ is continuous on } \Gamma_{nc}, \tag{2.340}$$

$$\boldsymbol{T} \times \boldsymbol{n}_c = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc}, \tag{2.341}$$

$$(\mu \boldsymbol{T}_0 - \mu \nabla\Phi) \cdot \boldsymbol{n}_n + (\mu_o \boldsymbol{T}_0 + \mu_o \boldsymbol{T} - \mu_o \nabla\Phi + \boldsymbol{R}) \cdot \boldsymbol{n}_c = 0, \quad \text{on } \Gamma_{nc}. \tag{2.342}$$

Note. The $\boldsymbol{T}, \Phi - \Phi$ formulation can not be used in case of *multiply connected region*. In this situation the formulations $\boldsymbol{T}, \Phi - \boldsymbol{A}$ or $\boldsymbol{T}, \Phi - \boldsymbol{A} - \Phi$ can be used.

Note. The $\boldsymbol{T}, \Phi - \Phi$ formulation can not be useful in case of *multiply connected region*, however, a simple modification can be introduced in this case. The holes filled with nonconducting materials ($\sigma = 0$) can be replaced by conducting materials with very small conductivity ($\sigma \approx 0$ comparing with the conductivity of the conducting material, e.g. $\sigma = \sigma_c/a$, where a is constant).

C. The gauged $\boldsymbol{A}, V - \boldsymbol{A}$ formulation

The *magnetic vector potential* $\boldsymbol{A}$ is used in this formulation throughout the region $\Omega_c \cup \Omega_n$ and the *electric scalar potential* V only in Ω_c. Here, the equations (2.223)–(2.227) and (2.305)–(2.312) have to be used to prepare the formulation, but the set of these equations have to be appended by continuity equations defined on Γ_{nc}.

The magnetic vector potential is continuous, meaning that the tangential and the normal component of the magnetic vector potential are continuous on Γ_{nc}. The continuity of the tangential component of the magnetic vector potential immediately enforces the continuity of the normal component of the magnetic flux density from (2.101) and (2.357),

$$
\begin{aligned}
(\nabla \times \boldsymbol{A}) \cdot \boldsymbol{n}_c + (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{n}_n &= \nabla \cdot (\boldsymbol{A} \times \boldsymbol{n}_c) + \nabla \cdot (\boldsymbol{A} \times \boldsymbol{n}_n) \\
&= \nabla \cdot (\boldsymbol{A} \times \boldsymbol{n}_c + \boldsymbol{A} \times \boldsymbol{n}_n) = 0.
\end{aligned}
\tag{2.343}
$$

The continuity of the tangential component of the magnetic field intensity vector must be prescribed by an additional interface condition on Γ_{nc} (see the Neumann type condition (2.359)). It is obvious that the normal component of the eddy current density must vanish on Γ_{nc}. The divergence free property of $\boldsymbol{A}$ has been defined only in the domain $\Omega_c \cup \Omega_n$ bounded by $\Gamma_{H_c} \cup \Gamma_E \cup \Gamma_{H_n} \cup \Gamma_B$, but not on Γ_{nc}. On the part Γ_{nc} $\nu \nabla \cdot \boldsymbol{A}$ must be continuous, which can be prescribed by a Neumann type condition (see (2.361)).

The summarized equations are as follows (see *Fig.* 2.17):

$$
\nabla \times (\nu_o \nabla \times \boldsymbol{A}) - \nabla (\nu_o \nabla \cdot \boldsymbol{A}) + \sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla V = -\nabla \times \boldsymbol{I}, \quad \text{in} \quad \Omega_c, \tag{2.344}
$$

$$
-\nabla \cdot \left(\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla V \right) = 0, \quad \text{in} \quad \Omega_c, \tag{2.345}
$$

$$
\nabla \times (\nu \nabla \times \boldsymbol{A}) - \nabla (\nu \nabla \cdot \boldsymbol{A}) = \boldsymbol{J}_0, \quad \text{in} \quad \Omega_n, \tag{2.346}
$$

$$
(\nu_o \nabla \times \boldsymbol{A} + \boldsymbol{I}) \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.347}
$$

$$
-\left(\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla V \right) \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.348}
$$

$$
\boldsymbol{A} \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.349}
$$

$$
(\nu \nabla \times \boldsymbol{A}) \times \boldsymbol{n} = \boldsymbol{K}, \quad \text{on} \quad \Gamma_{H_n}, \tag{2.350}
$$

$$
\boldsymbol{A} \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_{H_n}, \tag{2.351}
$$

$$\boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_E, \tag{2.352}$$

$$V = V_0, \quad \text{on} \quad \Gamma_E, \tag{2.353}$$

$$\nu_o \nabla \cdot \boldsymbol{A} = 0, \quad \text{on} \quad \Gamma_E, \tag{2.354}$$

$$\boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{\alpha}, \quad \text{on} \quad \Gamma_B, \tag{2.355}$$

$$\nu \nabla \cdot \boldsymbol{A} = 0, \quad \text{on} \quad \Gamma_B, \tag{2.356}$$

$$\boldsymbol{n}_c \times \boldsymbol{A} + \boldsymbol{n}_n \times \boldsymbol{A} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc}, \tag{2.357}$$

$$\boldsymbol{A} \cdot \boldsymbol{n}_c + \boldsymbol{A} \cdot \boldsymbol{n}_n = 0, \quad \text{on} \quad \Gamma_{nc}, \tag{2.358}$$

$$(\nu_o \nabla \times \boldsymbol{A} + \boldsymbol{I}) \times \boldsymbol{n}_c + (\nu \nabla \times \boldsymbol{A}) \times \boldsymbol{n}_n = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc}, \tag{2.359}$$

$$-\left(\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla V \right) \cdot \boldsymbol{n}_c = 0, \quad \text{on} \quad \Gamma_{nc}, \tag{2.360}$$

$$(\nu_o \nabla \cdot \boldsymbol{A}) \, \boldsymbol{n}_c + (\nu \nabla \cdot \boldsymbol{A}) \, \boldsymbol{n}_n = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc}. \tag{2.361}$$

Note. It must be highlighted that the interface condition (2.358) must be eliminated, if the permeability is changing abruptly along the interface Γ_{nc}. In this case Γ_{nc} is an iron/air interface (remember *Fig.* 2.16). The condition $\nabla \cdot \boldsymbol{A} = 0$ on Γ_{nc} can be satisfied according to the discretization. The problem can be solved by the ungauged $\boldsymbol{A}, V - \boldsymbol{A}$ formulation too, presented in the next item.

D. The ungauged $A, V - A$ formulation

The *magnetic vector potential $\boldsymbol{A}$* is used in this formulation in the region $\Omega_c \cup \Omega_n$ and the *electric scalar potential V* only in Ω_c. The equations of ungauged $\boldsymbol{A}, V - \boldsymbol{A}$ formulation can be built up by using (2.228)–(2.230) and (2.294)–(2.299), which must be appended by some continuity equations defined on Γ_{nc}.

The interface conditions can be defined as in the case of gauged version, however, equations (2.349), (2.351), (2.354), (2.356), (2.358) and (2.361) are not used. Here, $\boldsymbol{T}_0$ is used to represent the source current density $\boldsymbol{J}_0$.

The summarized equations are as follows (see *Fig.* 2.17):

$$\nabla \times (\nu_o \nabla \times \boldsymbol{A}) + \sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla V = -\nabla \times \boldsymbol{I}, \quad \text{in} \quad \Omega_c, \tag{2.362}$$

$$-\nabla \cdot \left(\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla V \right) = 0, \quad \text{in} \quad \Omega_c, \tag{2.363}$$

$$\nabla \times (\nu \nabla \times \boldsymbol{A}) = \nabla \times \boldsymbol{T}_0, \quad \text{in} \quad \Omega_n, \tag{2.364}$$

$$(\nu_o \nabla \times \boldsymbol{A} + \boldsymbol{I}) \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.365}$$

$$-\left(\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla V \right) \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.366}$$

$$(\nu \nabla \times \boldsymbol{A}) \times \boldsymbol{n} = \boldsymbol{K}, \quad \text{on} \quad \Gamma_{H_n}, \tag{2.367}$$

$$\boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_E, \tag{2.368}$$

$$V = V_0, \quad \text{on} \quad \Gamma_E, \tag{2.369}$$

$$\boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{\alpha}, \quad \text{on} \quad \Gamma_B, \tag{2.370}$$

$$\boldsymbol{n}_c \times \boldsymbol{A} + \boldsymbol{n}_n \times \boldsymbol{A} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc}, \tag{2.371}$$

$$(\nu_o \nabla \times \boldsymbol{A} + \boldsymbol{I}) \times \boldsymbol{n}_c + (\nu \nabla \times \boldsymbol{A}) \times \boldsymbol{n}_n = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc}, \tag{2.372}$$

$$-\left(\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla V\right) \cdot \boldsymbol{n}_c = 0, \quad \text{on} \quad \Gamma_{nc}. \tag{2.373}$$

Note. If σ is constant in Ω_c and the ungauged $\boldsymbol{A}, V - \boldsymbol{A}$ formulation is used, then $V = 0$ can be selected. This results in the ungauged $\boldsymbol{A}^\star - \boldsymbol{A}$ formulation.

Note. A more economical way of $\boldsymbol{A}$-based formulations is using the reduced magnetic scalar potential in the air region resulting the $\boldsymbol{A}, V - \Phi$ and the $\boldsymbol{A}, V - \boldsymbol{A} - \Phi$ formulations.

E. The ungauged $A^\star - A$ formulation

The equations of this formulation are (2.228)–(2.230) and (2.313)–(2.315) appended by the continuity equations (2.100) and (2.101) (see *Fig.* 2.17),

$$\nabla \times (\nu_o \nabla \times \boldsymbol{A}^\star) + \sigma \frac{\partial \boldsymbol{A}^\star}{\partial t} = -\nabla \times \boldsymbol{I}, \quad \text{in} \quad \Omega_c, \tag{2.374}$$

$$\nabla \times (\nu \nabla \times \boldsymbol{A}) = \nabla \times \boldsymbol{T}_0, \quad \text{in} \quad \Omega_n, \tag{2.375}$$

$$(\nu_o \nabla \times \boldsymbol{A}^\star + \boldsymbol{I}) \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.376}$$

$$(\nu \nabla \times \boldsymbol{A}) \times \boldsymbol{n} = \boldsymbol{K}, \quad \text{on} \quad \Gamma_{H_n}, \tag{2.377}$$

$$\boldsymbol{n} \times \boldsymbol{A}^\star = \boldsymbol{0}, \quad \text{on} \quad \Gamma_E, \tag{2.378}$$

$$\boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{\alpha}, \quad \text{on} \quad \Gamma_B, \tag{2.379}$$

$$\boldsymbol{n}_c \times \boldsymbol{A}^\star + \boldsymbol{n}_n \times \boldsymbol{A} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc}, \tag{2.380}$$

$$(\nu_o \nabla \times \boldsymbol{A}^\star + \boldsymbol{I}) \times \boldsymbol{n}_c + (\nu \nabla \times \boldsymbol{A}) \times \boldsymbol{n}_n = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc}. \tag{2.381}$$

F. The gauged $T, \Phi - A$ formulation

The $\boldsymbol{T}, \Phi - \Phi$ formulation is not capable of treating *multiply connected conductors* when all the eddy current free region is described by the reduced magnetic scalar potential Φ. This difficulty can be overcome by applying the magnetic vector potential in the eddy current free region, Ω_n. This is called $\boldsymbol{T}, \Phi - \boldsymbol{A}$ formulation.

In this formulation, the gauged *current vector potential* $\boldsymbol{T}$ with the *reduced magnetic scalar potential* Φ is used in the eddy current region Ω_c and the gauged *magnetic vector potential* $\boldsymbol{A}$ is used in the region free of eddy currents, Ω_n. The equations to be used are (2.274)–(2.281) and (2.223)–(2.227). Some continuity equations and boundary conditions defined on Γ_{nc} must be appended.

The magnetic field intensity is derived as $\boldsymbol{H} = \boldsymbol{T}_0 + \boldsymbol{T} - \nabla\Phi$ in Ω_c and $\boldsymbol{H} = \nu\nabla \times \boldsymbol{A}$ in Ω_n, while the magnetic flux density can be written as $\boldsymbol{B} = \mu_o(\boldsymbol{T}_0 + \boldsymbol{T} - \nabla\Phi) + \boldsymbol{R}$

in Ω_c and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ in Ω_n. The continuity equations (2.100) and (2.101) can be reformulated by these equations.

The potential functions are not continuous on Γ_{nc}, therefore this surface acts as a boundary of the two subregions.

It must be noted here that the uniqueness of a vector potential can be ensured by defining either its tangential component and its divergence or its normal component and the tangential component of its curl.

The tangential component of the current vector potential can not be set to zero on the interface Γ_{nc} since this would imply that no net current can flow around the hole filled by $\boldsymbol{A}$. That is why the other conditions must be prescribed here, i.e. $\boldsymbol{T} \cdot \boldsymbol{n} = 0$ on Γ_{nc}. The curl of the current vector potential is the electric field intensity since $\boldsymbol{E} = 1/\sigma \nabla \times \boldsymbol{T}$, which tangential component is continuous on any surface, i.e. $\boldsymbol{E}_c \times \boldsymbol{n}_c + \boldsymbol{E}_n \times \boldsymbol{n}_n = \boldsymbol{0}$ must be satisfied. From the second Maxwell's equation (2.90)

$$(\nabla \times \boldsymbol{E}) \cdot \boldsymbol{n} = -\frac{\partial}{\partial t}(\nabla \times \boldsymbol{A}) \cdot \boldsymbol{n} \ \Rightarrow \ \nabla \cdot (\boldsymbol{E} \times \boldsymbol{n}) = \nabla \cdot \left[-\frac{\partial}{\partial t}(\boldsymbol{A} \times \boldsymbol{n}) \right], \quad (2.382)$$

from which

$$\boldsymbol{E} \times \boldsymbol{n} = \left(\frac{1}{\sigma} \nabla \times \boldsymbol{T} \right) \times \boldsymbol{n} = -\frac{\partial}{\partial t} \boldsymbol{A} \times \boldsymbol{n}. \quad (2.383)$$

This is a condition for the tangential component of the magnetic vector potential too, which uniqueness on Γ_{nc} can be ensured by prescribing its divergence.

The summarized equations of the gauged version are as follows (see *Fig.* 2.17):

$$\nabla \times \left(\frac{1}{\sigma} \nabla \times \boldsymbol{T} \right) - \nabla \left(\frac{1}{\sigma} \nabla \cdot \boldsymbol{T} \right) + \mu_o \frac{\partial \boldsymbol{T}}{\partial t} - \mu_o \nabla \frac{\partial \Phi}{\partial t}$$
$$= -\mu_o \partial \boldsymbol{T}_0 / \partial t - \partial \boldsymbol{R} / \partial t, \text{ in } \Omega_c, \quad (2.384)$$

$$\nabla \cdot (\mu_o \boldsymbol{T} - \mu_o \nabla \Phi) = -\nabla \cdot (\mu_o \boldsymbol{T}_0) - \nabla \cdot \boldsymbol{R}, \quad \text{in} \quad \Omega_c, \quad (2.385)$$

$$\nabla \times (\nu \nabla \times \boldsymbol{A}) - \nabla(\nu \nabla \cdot \boldsymbol{A}) = \boldsymbol{J}_0, \quad \text{in} \quad \Omega_n, \quad (2.386)$$

$$\boldsymbol{T} \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{H_c}, \quad (2.387)$$

$$\Phi = \Phi_0, \quad \text{on} \quad \Gamma_{H_c}, \quad (2.388)$$

$$\frac{1}{\sigma} \nabla \cdot \boldsymbol{T} = 0, \quad \text{on} \quad \Gamma_{H_c}, \quad (2.389)$$

$$(\nu \nabla \times \boldsymbol{A}) \times \boldsymbol{n} = \boldsymbol{K}, \quad \text{on} \quad \Gamma_{H_n}, \quad (2.390)$$

$$\boldsymbol{A} \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_{H_n}, \quad (2.391)$$

$$\left(\frac{1}{\sigma} \nabla \times \boldsymbol{T} \right) \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_E, \quad (2.392)$$

$$(\mu_o \boldsymbol{T}_0 + \mu_o \boldsymbol{T} - \mu_o \nabla \Phi + \boldsymbol{R}) \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_E, \quad (2.393)$$

$$\boldsymbol{T} \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_E, \quad (2.394)$$

$$\boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{\alpha}, \quad \text{on} \quad \Gamma_B, \quad (2.395)$$

$$\nu\nabla \cdot \boldsymbol{A} = 0, \quad \text{on} \quad \Gamma_B, \tag{2.396}$$

$$(\boldsymbol{T}_0 + \boldsymbol{T} - \nabla\Phi) \times \boldsymbol{n}_c + (\nu\nabla \times \boldsymbol{A}) \times \boldsymbol{n}_n = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc}, \tag{2.397}$$

$$(\mu_o\boldsymbol{T}_0 + \mu_o\boldsymbol{T} - \mu_o\nabla\Phi + \boldsymbol{R}) \cdot \boldsymbol{n}_c + (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{n}_n = 0, \quad \text{on} \quad \Gamma_{nc}, \tag{2.398}$$

$$\boldsymbol{T} \cdot \boldsymbol{n}_c = 0, \quad \text{on} \quad \Gamma_{nc}, \tag{2.399}$$

$$\left(\frac{1}{\sigma}\nabla \times \boldsymbol{T}\right) \times \boldsymbol{n}_c - \frac{\partial \boldsymbol{A}}{\partial t} \times \boldsymbol{n}_n = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc}, \tag{2.400}$$

$$\nu\nabla \cdot \boldsymbol{A} = 0, \quad \text{on} \quad \Gamma_{nc}. \tag{2.401}$$

G. The ungauged $T, \Phi - A$ formulation

In this formulation, the ungauged *current vector potential* $\boldsymbol{T}$ with the reduced magnetic scalar potential Φ is used in the eddy current region Ω_c and the ungauged *magnetic vector potential* $\boldsymbol{A}$ is used in the region free of eddy currents, Ω_n.

The ungauged $T, \Phi - A$ formulation can be built up by using (2.262)–(2.267) and (2.228)–(2.230), which must be appended by some continuity and boundary conditions defined on Γ_{nc}.

The interface conditions can be defined as in the gauged version, however, equations (2.389), (2.391), (2.394), (2.396), (2.399) and (2.401) are not used. The impressed current vector potential $\boldsymbol{T}_0$ is used in this formulation to represent $\boldsymbol{J}_0$.

The summarized equations of the gauged version are as follows (see *Fig.* 2.17):

$$\nabla \times \left(\frac{1}{\sigma}\nabla \times \boldsymbol{T}\right) + \mu_o\frac{\partial \boldsymbol{T}}{\partial t} - \mu_o\nabla\frac{\partial \Phi}{\partial t} = -\mu_o\frac{\partial \boldsymbol{T}_0}{\partial t} - \frac{\partial \boldsymbol{R}}{\partial t}, \quad \text{in} \quad \Omega_c, \tag{2.402}$$

$$\nabla \cdot (\mu_o\boldsymbol{T} - \mu_o\nabla\Phi) = -\nabla \cdot (\mu_o\boldsymbol{T}_0) - \nabla \cdot \boldsymbol{R}, \quad \text{in} \quad \Omega_c, \tag{2.403}$$

$$\nabla \times (\nu\nabla \times \boldsymbol{A}) = \nabla \times \boldsymbol{T}_0, \quad \text{in} \quad \Omega_n, \tag{2.404}$$

$$\boldsymbol{T} \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.405}$$

$$\Phi = \Phi_0, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.406}$$

$$(\nu\nabla \times \boldsymbol{A}) \times \boldsymbol{n} = \boldsymbol{K}, \quad \text{on} \quad \Gamma_{H_n}, \tag{2.407}$$

$$\left(\frac{1}{\sigma}\nabla \times \boldsymbol{T}\right) \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_E, \tag{2.408}$$

$$(\mu_o\boldsymbol{T}_0 + \mu_o\boldsymbol{T} - \mu_o\nabla\Phi + \boldsymbol{R}) \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_E, \tag{2.409}$$

$$\boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{\alpha}, \quad \text{on} \quad \Gamma_B, \tag{2.410}$$

$$(\boldsymbol{T}_0 + \boldsymbol{T} - \nabla\Phi) \times \boldsymbol{n}_c + (\nu\nabla \times \boldsymbol{A}) \times \boldsymbol{n}_n = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc}, \tag{2.411}$$

$$(\mu_o\boldsymbol{T}_0 + \mu_o\boldsymbol{T} - \mu_o\nabla\Phi + \boldsymbol{R}) \cdot \boldsymbol{n}_c + (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{n}_n = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc}, \tag{2.412}$$

$$\left(\frac{1}{\sigma}\nabla \times \boldsymbol{T}\right) \times \boldsymbol{n}_c - \frac{\partial \boldsymbol{A}}{\partial t} \times \boldsymbol{n}_n = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc}. \tag{2.413}$$

Note. A more economical way of the $T, \Phi - A$ formulations is using the reduced magnetic scalar potential in a subregion of Ω_n. This is the $T, \Phi - A - \Phi$ formulations.

H. The gauged $T, \Phi - A - \Phi$ formulation

This is a modification of the gauged $T, \Phi - A$ formulation. Applying the *magnetic vector potential* in the static magnetic field region is not economical, because it requires three scalar unknown functions (A_x, A_y and A_z). This can be reduced to one by using the *reduced magnetic scalar potential* in a subregion of air, which is simple connected.

The *current vector potential* T with the reduced magnetic scalar potential Φ is used in the eddy current region Ω_c, the magnetic vector potential A is used in the holes of the eddy current region, which is free of eddy currents and denoted by Ω_A. The rest region is described by the reduced magnetic scalar potential Φ and this region is denoted by Ω_Φ. The scheme of this formulation can be seen in *Fig.* 2.18. The problem region has three disjunct parts, $\Omega = \Omega_c \cup \Omega_{n_A} \cup \Omega_{n_\Phi}$. This means that the boundary of eddy current free region and the interface have two disjunct parts, $\Gamma_{H_n} = \Gamma_{H_A} \cup \Gamma_{H_\Phi}$, $\Gamma_B = \Gamma_{B_A} \cup \Gamma_{B_\Phi}$, $\Gamma_{nc} = \Gamma_{nc_A} \cup \Gamma_{nc_\Phi}$. The further interface between Ω_{n_A} and Ω_{n_Φ} is $\Gamma_{A,\Phi}$.
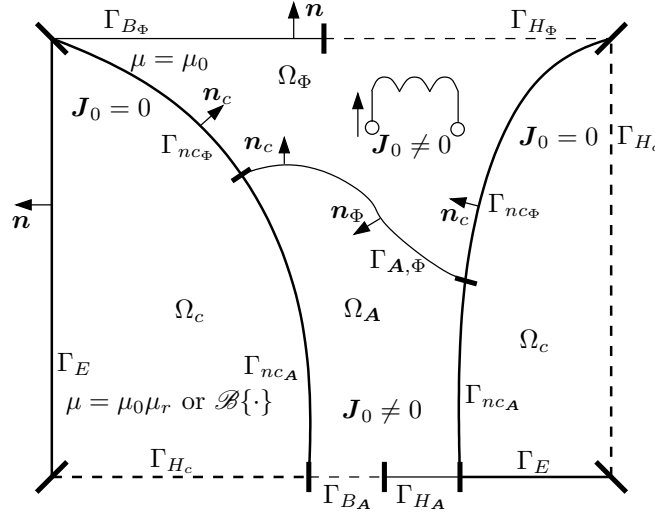
Fig. 2.18. Applying A and Φ in the air region in the case of multiply connected
eddy current region

The equations to be used are (2.274)–(2.281), (2.223)–(2.227) and (2.160)–(2.162) in the subregion Ω_c, Ω_{n_A} and Ω_{n_Φ}, respectively, which must be appended by some continuity equations and boundary conditions defined on Γ_{nc} and $\Gamma_{A,\Phi}$.

The interface conditions on Γ_{nc_Φ} are the same as on Γ_{nc} in the gauged $T, \Phi - \Phi$ formulation, see (2.327)–(2.330). The interface conditions on Γ_{nc_A} are the same as on Γ_{nc} in the gauged $T, \Phi - A$ formulation, see (2.397)–(2.401).

The tangential component of the magnetic field intensity and the normal component of the magnetic flux density must be continuous on the interface $\Gamma_{A,\Phi}$, moreover this is a boundary of the region Ω_{n_A}, where $A \cdot n = 0$ is prescribed because of the condition for the tangential component of the magnetic field intensity, which is a condition for $(\nu \nabla \times A) \times n$ (see (2.439)).

The summarized equations of the gauged version are as follows:

$$\nabla \times \left(\frac{1}{\sigma} \nabla \times \boldsymbol{T} \right) - \nabla \left(\frac{1}{\sigma} \nabla \cdot \boldsymbol{T} \right) + \mu_o \frac{\partial \boldsymbol{T}}{\partial t} - \mu_o \nabla \frac{\partial \Phi}{\partial t}$$
$$= - \mu_o \partial \boldsymbol{T}_0 / \partial t - \partial \boldsymbol{R} / \partial t, \text{ in } \Omega_c, \tag{2.414}$$

$$\nabla \cdot (\mu_o \boldsymbol{T} - \mu_o \nabla \Phi) = -\nabla \cdot (\mu_o \boldsymbol{T}_0) - \nabla \cdot \boldsymbol{R}, \quad \text{in} \quad \Omega_c, \tag{2.415}$$

$$-\nabla \cdot (\mu \nabla \Phi) = -\nabla \cdot (\mu \boldsymbol{T}_0), \quad \text{in} \quad \Omega_{n_\Phi}, \tag{2.416}$$

$$\nabla \times (\nu \nabla \times \boldsymbol{A}) - \nabla(\nu \nabla \cdot \boldsymbol{A}) = \boldsymbol{J}_0, \quad \text{in} \quad \Omega_{n_{\boldsymbol{A}}}, \tag{2.417}$$

$$\boldsymbol{T} \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.418}$$

$$\Phi = \Phi_0, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.419}$$

$$\frac{1}{\sigma} \nabla \cdot \boldsymbol{T} = 0, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.420}$$

$$(\nu \nabla \times \boldsymbol{A}) \times \boldsymbol{n} = \boldsymbol{K}, \quad \text{on} \quad \Gamma_{H_{\boldsymbol{A}}}, \tag{2.421}$$

$$\boldsymbol{A} \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_{H_{\boldsymbol{A}}}, \tag{2.422}$$

$$\Phi = \Phi_0, \quad \text{on} \quad \Gamma_{H_\Phi}, \tag{2.423}$$

$$\left(\frac{1}{\sigma} \nabla \times \boldsymbol{T} \right) \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_E, \tag{2.424}$$

$$(\mu_o \boldsymbol{T}_0 + \mu_o \boldsymbol{T} - \mu_o \nabla \Phi + \boldsymbol{R}) \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_E, \tag{2.425}$$

$$\boldsymbol{T} \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_E, \tag{2.426}$$

$$\boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{\alpha}, \quad \text{on} \quad \Gamma_{B_{\boldsymbol{A}}}, \tag{2.427}$$

$$\nu \nabla \cdot \boldsymbol{A} = 0, \quad \text{on} \quad \Gamma_{B_{\boldsymbol{A}}}, \tag{2.428}$$

$$(\mu \boldsymbol{T}_0 - \mu \nabla \Phi) \cdot \boldsymbol{n} = -b, \quad \text{on} \quad \Gamma_{B_\Phi}, \tag{2.429}$$

$$\Phi \text{ is continuous on } \Gamma_{nc_\Phi}, \tag{2.430}$$

$$\boldsymbol{T} \times \boldsymbol{n}_c = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc_\Phi}, \tag{2.431}$$

$$(\mu \boldsymbol{T}_0 - \mu \nabla \Phi) \cdot \boldsymbol{n}_\Phi + (\mu_o \boldsymbol{T}_0 + \mu_o \boldsymbol{T} - \mu_o \nabla \Phi + \boldsymbol{R}) \cdot \boldsymbol{n}_c = 0, \text{ on } \Gamma_{nc_\Phi}, \tag{2.432}$$

$$\frac{1}{\sigma} \nabla \cdot \boldsymbol{T} = 0, \quad \text{on} \quad \Gamma_{nc_\Phi}, \tag{2.433}$$

$$(\boldsymbol{T}_0 + \boldsymbol{T} - \nabla \Phi) \times \boldsymbol{n}_c + (\nu \nabla \times \boldsymbol{A}) \times \boldsymbol{n}_{\boldsymbol{A}} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc_{\boldsymbol{A}}}, \tag{2.434}$$

$$(\mu_o \boldsymbol{T}_0 + \mu_o \boldsymbol{T} - \mu_o \nabla \Phi + \boldsymbol{R}) \cdot \boldsymbol{n}_c + (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{n}_{\boldsymbol{A}} = 0, \quad \text{on} \quad \Gamma_{nc_{\boldsymbol{A}}}, \tag{2.435}$$

$$\boldsymbol{T} \cdot \boldsymbol{n}_c = 0, \quad \text{on} \quad \Gamma_{nc_{\boldsymbol{A}}}, \tag{2.436}$$

$$\left(\frac{1}{\sigma} \nabla \times \boldsymbol{T} \right) \times \boldsymbol{n}_c - \frac{\partial \boldsymbol{A}}{\partial t} \times \boldsymbol{n}_{\boldsymbol{A}} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc_{\boldsymbol{A}}}, \tag{2.437}$$

$$\nu \nabla \cdot \boldsymbol{A} = 0, \quad \text{on} \quad \Gamma_{nc_{\boldsymbol{A}}}, \tag{2.438}$$

$$(\boldsymbol{T}_0 - \nabla \Phi) \times \boldsymbol{n}_\Phi + (\nu \nabla \times \boldsymbol{A}) \times \boldsymbol{n}_{\boldsymbol{A}} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{\boldsymbol{A}, \Phi}, \tag{2.439}$$

$$(\mu \boldsymbol{T}_0 - \mu \nabla \Phi) \cdot \boldsymbol{n}_\Phi + (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{n}_{\boldsymbol{A}} = 0, \quad \text{on} \quad \Gamma_{\boldsymbol{A}, \Phi}, \tag{2.440}$$

$$\boldsymbol{A} \cdot \boldsymbol{n}_{\boldsymbol{A}} = 0, \quad \text{on} \quad \Gamma_{\boldsymbol{A}, \Phi}. \tag{2.441}$$

I. The ungauged $T, \Phi - A - \Phi$ formulation

Applying the same notations presented in the last item, the summarized equations of the gauged version can be set up, however, J_0 is represented by T_0. The scheme of this formulation can be seen in *Fig.* 2.18.

The equations to be used are (2.262)–(2.267), (2.228)–(2.230) and (2.160)–(2.162) in the subregion Ω_c, Ω_{n_A} and Ω_{n_Φ}, respectively. Some continuity equations and boundary conditions defined on Γ_{nc_Φ} (see (2.340)–(2.342)), on Γ_{nc_A} (see (2.412)–(2.413)) and on $\Gamma_{A,\Phi}$ (see (2.439) and (2.440)) must be appended.

Finally, the system of equations of the ungauged $T, \Phi - A - \Phi$ formulation are as follows:

$$\nabla \times \left(\frac{1}{\sigma}\nabla \times T\right) + \mu_o \frac{\partial T}{\partial t} - \mu_o \nabla \frac{\partial \Phi}{\partial t} = -\mu_o \frac{\partial T_0}{\partial t} - \frac{\partial R}{\partial t}, \quad \text{in} \quad \Omega_c, \quad (2.442)$$

$$\nabla \cdot (\mu_o T - \mu_o \nabla \Phi) = -\nabla \cdot (\mu_o T_0) - \nabla \cdot R, \quad \text{in} \quad \Omega_c, \quad (2.443)$$

$$-\nabla \cdot (\mu \nabla \Phi) = -\nabla \cdot (\mu T_0), \quad \text{in} \quad \Omega_{n_\Phi}, \quad (2.444)$$

$$\nabla \times (\nu \nabla \times A) = \nabla \times T_0, \quad \text{in} \quad \Omega_{n_A}, \quad (2.445)$$

$$T \times n = 0, \quad \text{on} \quad \Gamma_{H_c}, \quad (2.446)$$

$$\Phi = \Phi_0, \quad \text{on} \quad \Gamma_{H_c}, \quad (2.447)$$

$$(\nu \nabla \times A) \times n = K, \quad \text{on} \quad \Gamma_{H_A}, \quad (2.448)$$

$$\Phi = \Phi_0, \quad \text{on} \quad \Gamma_{H_\Phi}, \quad (2.449)$$

$$\left(\frac{1}{\sigma}\nabla \times T\right) \times n = 0, \quad \text{on} \quad \Gamma_E, \quad (2.450)$$

$$(\mu_o T_0 + \mu_o T - \mu_o \nabla \Phi + R) \cdot n = 0, \quad \text{on} \quad \Gamma_E, \quad (2.451)$$

$$n \times A = \alpha, \quad \text{on} \quad \Gamma_{B_A}, \quad (2.452)$$

$$(\mu T_0 - \mu \nabla \Phi) \cdot n = -b, \quad \text{on} \quad \Gamma_{B_\Phi}, \quad (2.453)$$

$$\Phi \text{ is continuous on } \Gamma_{nc_\Phi}, \quad (2.454)$$

$$T \times n = 0, \quad \text{on} \quad \Gamma_{nc_\Phi}, \quad (2.455)$$

$$(\mu T_0 - \mu \nabla \Phi) \cdot n_n + (\mu_o T_0 + \mu_o T - \mu_o \nabla \Phi + R) \cdot n_c = 0, \quad \text{on } \Gamma_{nc_\Phi}, \quad (2.456)$$

$$(T_0 + T - \nabla \Phi) \times n_c + (\nu \nabla \times A) \times n_A = 0, \quad \text{on} \quad \Gamma_{nc_A}, \quad (2.457)$$

$$(\mu_o T_0 + \mu_o T - \mu_o \nabla \Phi + R) \cdot n_c + (\nabla \times A) \cdot n_A = 0, \quad \text{on} \quad \Gamma_{nc_A}, \quad (2.458)$$

$$\left(\frac{1}{\sigma}\nabla \times T\right) \times n_c - \frac{\partial A}{\partial t} \times n_A = 0, \quad \text{on} \quad \Gamma_{nc_A}, \quad (2.459)$$

$$(T_0 - \nabla \Phi) \times n_\Phi + (\nu \nabla \times A) \times n_A = 0, \quad \text{on} \quad \Gamma_{A,\Phi}, \quad (2.460)$$

$$(\mu T_0 - \mu \nabla \Phi) \cdot n_\Phi + (\nabla \times A) \cdot n_A = 0, \quad \text{on} \quad \Gamma_{A,\Phi}. \quad (2.461)$$

J. The gauged $A, V - \Phi$ formulation

This formulation is a modification of the $A, V - A$ potential formulation. The *magnetic vector potential* in the eddy current free region Ω_n has been replaced by the *reduced magnetic scalar potential* Φ. The two potentials A and V remain in Ω_c. The number of unknown potential functions is decreased from three to one in Ω_n, but it can be used only if the eddy current region Ω_c is a simple connected domain.

The equations to be used are (2.305)–(2.312) and (2.160)–(2.162), moreover some continuity equations and boundary conditions defined on Γ_{nc} must be appended to these equations.

The tangential component of the magnetic field intensity and the normal component of the magnetic flux density must be continuous on the interface Γ_{nc}, moreover the normal component of the eddy current density must vanish there. The interface is a boundary of the A-region, too, i.e. the uniqueness of the magnetic vector potential must be satisfied on Γ_{nc} as well. It can be performed by using the condition $A \cdot n = 0$, because of the condition for the tangential component of the magnetic field intensity, which is a condition for $(\nu \nabla \times A) \times n$ (see (2.473)).

The summarized equations are as follows (see *Fig.* 2.17):

$$\nabla \times (\nu_o \nabla \times A) - \nabla (\nu_o \nabla \cdot A) + \sigma \frac{\partial A}{\partial t} + \sigma \nabla V = -\nabla \times I, \ \ \text{in} \ \Omega_c, \qquad (2.462)$$

$$-\nabla \cdot \left(\sigma \frac{\partial A}{\partial t} + \sigma \nabla V \right) = 0, \quad \text{in} \quad \Omega_c, \qquad (2.463)$$

$$-\nabla \cdot (\mu \nabla \Phi) = -\nabla \cdot (\mu T_0), \quad \text{in} \quad \Omega_n, \qquad (2.464)$$

$$(\nu_o \nabla \times A + I) \times n = 0, \quad \text{on} \quad \Gamma_{H_c}, \qquad (2.465)$$

$$-\left(\sigma \frac{\partial A}{\partial t} + \sigma \nabla V \right) \cdot n = 0, \quad \text{on} \quad \Gamma_{H_c}, \qquad (2.466)$$

$$A \cdot n = 0, \quad \text{on} \quad \Gamma_{H_c}, \qquad (2.467)$$

$$\Phi = \Phi_0, \quad \text{on} \quad \Gamma_{H_n}, \qquad (2.468)$$

$$n \times A = 0, \quad \text{on} \quad \Gamma_E, \qquad (2.469)$$

$$V = V_0, \quad \text{on} \quad \Gamma_E, \qquad (2.470)$$

$$\nu_o \nabla \cdot A = 0, \quad \text{on} \quad \Gamma_E, \qquad (2.471)$$

$$(\mu T_0 - \mu \nabla \Phi) \cdot n = -b, \quad \text{on} \quad \Gamma_B, \qquad (2.472)$$

$$(\nu_o \nabla \times A + I) \times n_c + (T_0 - \nabla \Phi) \times n_n = 0, \quad \text{on} \quad \Gamma_{nc}, \qquad (2.473)$$

$$(\nabla \times A) \cdot n_c + (\mu T_0 - \mu \nabla \Phi) \cdot n_n = 0, \quad \text{on} \quad \Gamma_{nc}, \qquad (2.474)$$

$$-\left(\sigma \frac{\partial A}{\partial t} + \sigma \nabla V \right) \cdot n_c = 0, \quad \text{on} \quad \Gamma_{nc}, \qquad (2.475)$$

$$A \cdot n_c = 0, \quad \text{on} \quad \Gamma_{nc}. \qquad (2.476)$$

Note. In the case of *multiply connected region*, the so-called gauged $A, V - A - \Phi$ formulation can be used.

K. The ungauged $A, V - \Phi$ formulation

This formulation is a modification of the $A, V - A$ potential formulation, the *magnetic vector potential* in the eddy current free region Ω_n has been replaced by the *reduced magnetic scalar potential* Φ. The two potentials A and V remain in Ω_c. The number of unknown potential functions is decreased from three to one in Ω_n, but it can be used only if the eddy current region Ω_c is a simple connected domain.

The equations of the ungauged $A, V - \Phi$ formulation can be built up by using the relations (2.294)–(2.299) and (2.160)–(2.162), which are completed by some continuity and boundary conditions defined on Γ_{nc}. The interface conditions can be defined as in the gauged version, however, equations (2.467), (2.471) and (2.476) are not used.

The summarized equations are as follows (see *Fig.* 2.17):

$$\nabla \times (\nu_o \nabla \times A) + \sigma \frac{\partial A}{\partial t} + \sigma \nabla V = -\nabla \times I, \quad \text{in} \quad \Omega_c, \tag{2.477}$$

$$-\nabla \cdot \left(\sigma \frac{\partial A}{\partial t} + \sigma \nabla V \right) = 0, \quad \text{in} \quad \Omega_c, \tag{2.478}$$

$$-\nabla \cdot (\mu \nabla \Phi) = -\nabla \cdot (\mu T_0), \quad \text{in} \quad \Omega_n, \tag{2.479}$$

$$(\nu_o \nabla \times A + I) \times n = 0, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.480}$$

$$-\left(\sigma \frac{\partial A}{\partial t} + \sigma \nabla V \right) \cdot n = 0, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.481}$$

$$\Phi = \Phi_0, \quad \text{on} \quad \Gamma_{H_n}, \tag{2.482}$$

$$n \times A = 0, \quad \text{on} \quad \Gamma_E, \tag{2.483}$$

$$V = V_0, \quad \text{on} \quad \Gamma_E, \tag{2.484}$$

$$(\mu T_0 - \mu \nabla \Phi) \cdot n = -b, \quad \text{on} \quad \Gamma_B, \tag{2.485}$$

$$(\nu_o \nabla \times A + I) \times n_c + (T_0 - \nabla \Phi) \times n_n = 0, \quad \text{on} \quad \Gamma_{nc}, \tag{2.486}$$

$$(\nabla \times A) \cdot n_c + (\mu T_0 - \mu \nabla \Phi) \cdot n_n = 0, \quad \text{on} \quad \Gamma_{nc}, \tag{2.487}$$

$$-\left(\sigma \frac{\partial A}{\partial t} + \sigma \nabla V \right) \cdot n_c = 0, \quad \text{on} \quad \Gamma_{nc}. \tag{2.488}$$

Note. If the conductivity of the media placed in Ω_c was constant, then $V = 0$ could be supposed. This results in the ungauged $A^\star - \Phi$ formulation. This is based on the equations (2.477)–(2.488) neglecting the term $\sigma \nabla V$ in (2.477) and ignoring the equations (2.478), (2.481), (2.484) and (2.488).

Note. In the case of multiply connected region, the so-called $A, V - A - \Phi$ formulation can be used.

L. The gauged $A, V - A - \Phi$ formulation

The aim of this formulation is to modify the $A, V - \Phi$ potential formulation in such a way that the resulting formulation will be able to simulate *multiply connected regions* by introducing the *magnetic vector potential* in the holes placed in the eddy current region. The region of holes filled with nonconducting material is denoted by Ω_{n_A}. The two potentials A and V remain in Ω_c and Φ in the remaining part of the eddy current free region Ω_{n_Φ}, i.e. $\Omega_n = \Omega_{n_A} \cup \Omega_{n_\Phi}$. The problem region has three disjunct parts, that is $\Omega = \Omega_c \cup \Omega_{n_A} \cup \Omega_{n_\Phi}$. This means that the boundaries of eddy current free region and the interface have two disjunct parts, $\Gamma_{H_n} = \Gamma_{H_A} \cup \Gamma_{H_\Phi}$, $\Gamma_B = \Gamma_{B_A} \cup \Gamma_{B_\Phi}$, $\Gamma_{nc} = \Gamma_{nc_A} \cup \Gamma_{nc_\Phi}$. The further interface between Ω_{n_A} and Ω_{n_Φ} is $\Gamma_{A,\Phi}$. The scheme is plotted in *Fig.* 2.18.

The equations to be used are (2.305)–(2.312), (2.223)–(2.227) and (2.160)–(2.162) in the subregion Ω_c, Ω_{n_A} and Ω_{n_Φ}, respectively, which must be completed by some continuity equations and boundary conditions defined on Γ_{nc} and $\Gamma_{A,\Phi}$.

The interface conditions on Γ_{nc_A} are the same as on Γ_{nc} in the gauged $A, V - A$ formulation, see (2.357)–(2.361). The interface conditions on Γ_{nc_Φ} are the same as on Γ_{nc} in the gauged $A, V - \Phi$ formulation, see (2.473)–(2.476). The tangential component of the magnetic field intensity and the normal component of the magnetic flux density must be continuous on the interface $\Gamma_{A,\Phi}$, moreover this is a boundary of the region Ω_{n_A}, where $A \cdot n = 0$ is prescribed because of the condition for the tangential component of the magnetic field intensity, which is a condition for $(\nu \nabla \times A) \times n$ (see (2.514)). It is noted that the last three boundary conditions are the same as in the gauged $T, \Phi - A - \Phi$ formulation.

The summarized equations are as follows:

$$\nabla \times (\nu_o \nabla \times A) - \nabla (\nu_o \nabla \cdot A) + \sigma \frac{\partial A}{\partial t} + \sigma \nabla V = -\nabla \times I, \text{ in } \Omega_c, \quad (2.489)$$

$$-\nabla \cdot \left(\sigma \frac{\partial A}{\partial t} + \sigma \nabla V \right) = 0, \quad \text{in} \quad \Omega_c, \quad (2.490)$$

$$\nabla \times (\nu \nabla \times A) - \nabla (\nu \nabla \cdot A) = J_0, \quad \text{in} \quad \Omega_{n_A}, \quad (2.491)$$

$$-\nabla \cdot (\mu \nabla \Phi) = -\nabla \cdot (\mu T_0), \quad \text{in} \quad \Omega_{n_\Phi}, \quad (2.492)$$

$$(\nu_o \nabla \times A + I) \times n = 0, \quad \text{on} \quad \Gamma_{H_c}, \quad (2.493)$$

$$-\left(\sigma \frac{\partial A}{\partial t} + \sigma \nabla V \right) \cdot n = 0, \quad \text{on} \quad \Gamma_{H_c}, \quad (2.494)$$

$$A \cdot n = 0, \quad \text{on} \quad \Gamma_{H_c}, \quad (2.495)$$

$$(\nu \nabla \times A) \times n = K, \quad \text{on} \quad \Gamma_{H_A}, \quad (2.496)$$

$$A \cdot n = 0, \quad \text{on} \quad \Gamma_{H_A}, \quad (2.497)$$

$$\Phi = \Phi_0, \quad \text{on} \quad \Gamma_{H_\Phi}, \quad (2.498)$$

$$n \times A = 0, \quad \text{on} \quad \Gamma_E, \quad (2.499)$$

$$V = V_0, \quad \text{on} \quad \Gamma_E, \quad (2.500)$$

$$\nu_o \nabla \cdot \boldsymbol{A} = 0, \quad \text{on} \quad \Gamma_E, \tag{2.501}$$

$$\boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{\alpha}, \quad \text{on} \quad \Gamma_{B_{\boldsymbol{A}}}, \tag{2.502}$$

$$\nu \nabla \cdot \boldsymbol{A} = 0, \quad \text{on} \quad \Gamma_{B_{\boldsymbol{A}}}, \tag{2.503}$$

$$(\mu \boldsymbol{T}_0 - \mu \nabla \Phi) \cdot \boldsymbol{n} = -b, \quad \text{on} \quad \Gamma_{B_{\Phi}}, \tag{2.504}$$

$$(\nu_o \nabla \times \boldsymbol{A} + \boldsymbol{I}) \times \boldsymbol{n}_c + (\nu \nabla \times \boldsymbol{A}) \times \boldsymbol{n}_{\boldsymbol{A}} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc_{\boldsymbol{A}}}, \tag{2.505}$$

$$(\nu_o \nabla \cdot \boldsymbol{A}) \boldsymbol{n}_c + (\nu \nabla \cdot \boldsymbol{A}) \boldsymbol{n}_{\boldsymbol{A}} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc_{\boldsymbol{A}}}, \tag{2.506}$$

$$-\left(\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla V\right) \cdot \boldsymbol{n}_c = 0, \quad \text{on} \quad \Gamma_{nc_{\boldsymbol{A}}}, \tag{2.507}$$

$$\boldsymbol{n}_c \times \boldsymbol{A} + \boldsymbol{n}_n \times \boldsymbol{A} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc_{\boldsymbol{A}}}, \tag{2.508}$$

$$\boldsymbol{A} \cdot \boldsymbol{n}_c + \boldsymbol{A} \cdot \boldsymbol{n}_n = 0, \quad \text{on} \quad \Gamma_{nc_{\boldsymbol{A}}}, \tag{2.509}$$

$$(\nu_o \nabla \times \boldsymbol{A} + \boldsymbol{I}) \times \boldsymbol{n}_c + (\boldsymbol{T}_0 - \nabla \Phi) \times \boldsymbol{n}_{\Phi} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc_{\Phi}}, \tag{2.510}$$

$$(\nabla \times \boldsymbol{A}) \cdot \boldsymbol{n}_{\boldsymbol{A}} + (\mu \boldsymbol{T}_0 - \mu \nabla \Phi) \cdot \boldsymbol{n}_{\Phi} = 0, \quad \text{on} \quad \Gamma_{nc_{\Phi}}, \tag{2.511}$$

$$-\left(\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla V\right) \cdot \boldsymbol{n}_c = 0, \quad \text{on} \quad \Gamma_{nc_{\Phi}}, \tag{2.512}$$

$$\boldsymbol{A} \cdot \boldsymbol{n}_c = 0, \quad \text{on} \quad \Gamma_{nc_{\Phi}}, \tag{2.513}$$

$$(\nu \nabla \times \boldsymbol{A}) \times \boldsymbol{n}_{\boldsymbol{A}} + (\boldsymbol{T}_0 - \nabla \Phi) \times \boldsymbol{n}_{\Phi} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{\boldsymbol{A},\Phi}, \tag{2.514}$$

$$(\nabla \times \boldsymbol{A}) \cdot \boldsymbol{n}_{\boldsymbol{A}} + (\mu \boldsymbol{T}_0 - \mu \nabla \Phi) \cdot \boldsymbol{n}_{\Phi} = 0, \quad \text{on} \quad \Gamma_{\boldsymbol{A},\Phi}, \tag{2.515}$$

$$\boldsymbol{A} \cdot \boldsymbol{n}_{\boldsymbol{A}} = 0, \quad \text{on} \quad \Gamma_{\boldsymbol{A},\Phi}. \tag{2.516}$$

M. The ungauged $\boldsymbol{A}, V - \boldsymbol{A} - \Phi$ formulation

Applying the same notations and theory, presented in the last item, the summarized equations of the gauged version can be set up, but the impressed current vector potential $\boldsymbol{T}_0$ is used to represent $\boldsymbol{J}_0$, see *Fig.* 2.18.

The equations to be used are (2.294)–(2.299), (2.228)–(2.230) and (2.160)–(2.162) in the subregion Ω_c, $\Omega_{n_{\boldsymbol{A}}}$ and $\Omega_{n_{\Phi}}$, respectively, which must be completed by some continuity equations and boundary conditions defined on $\Gamma_{nc_{\Phi}}$ (see (2.486)–(2.488)), on $\Gamma_{nc_{\boldsymbol{A}}}$ (see (2.371)–(2.373)) and on $\Gamma_{\boldsymbol{A},\Phi}$ (see (2.514) and (2.515)).

Finally, the system of equations of the ungauged $\boldsymbol{A}, V - \boldsymbol{A} - \Phi$ formulation are as follows:

$$\nabla \times (\nu_o \nabla \times \boldsymbol{A}) + \sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla V = -\nabla \times \boldsymbol{I}, \quad \text{in} \quad \Omega_c, \tag{2.517}$$

$$-\nabla \cdot \left(\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla V\right) = 0, \quad \text{in} \quad \Omega_c, \tag{2.518}$$

$$\nabla \times (\nu \nabla \times \boldsymbol{A}) = \nabla \times \boldsymbol{T}_0, \quad \text{in} \quad \Omega_{n_{\boldsymbol{A}}}, \tag{2.519}$$

$$-\nabla \cdot (\mu \nabla \Phi) = -\nabla \cdot (\mu \boldsymbol{T}_0), \quad \text{in} \quad \Omega_{n_{\Phi}}, \tag{2.520}$$

$$(\nu_o \nabla \times \boldsymbol{A} + \boldsymbol{I}) \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.521}$$

$$-\left(\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla V \right) \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_{H_c}, \tag{2.522}$$

$$(\nu \nabla \times \boldsymbol{A}) \times \boldsymbol{n} = \boldsymbol{K}, \quad \text{on} \quad \Gamma_{H_{\boldsymbol{A}}}, \tag{2.523}$$

$$\Phi = \Phi_0, \quad \text{on} \quad \Gamma_{H_\Phi}, \tag{2.524}$$

$$\boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_E, \tag{2.525}$$

$$V = V_0, \quad \text{on} \quad \Gamma_E, \tag{2.526}$$

$$\boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{\alpha}, \quad \text{on} \quad \Gamma_{B_{\boldsymbol{A}}}, \tag{2.527}$$

$$(\mu \boldsymbol{T}_0 - \mu \nabla \Phi) \cdot \boldsymbol{n} = -b, \quad \text{on} \quad \Gamma_{B_\Phi}, \tag{2.528}$$

$$(\nu_o \nabla \times \boldsymbol{A} + \boldsymbol{I}) \times \boldsymbol{n}_c + (\nu \nabla \times \boldsymbol{A}) \times \boldsymbol{n_A} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc_{\boldsymbol{A}}}, \tag{2.529}$$

$$-\left(\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla V \right) \cdot \boldsymbol{n}_c = 0, \quad \text{on} \quad \Gamma_{nc_{\boldsymbol{A}}}, \tag{2.530}$$

$$\boldsymbol{n}_c \times \boldsymbol{A} + \boldsymbol{n}_n \times \boldsymbol{A} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc_{\boldsymbol{A}}}, \tag{2.531}$$

$$(\nu_o \nabla \times \boldsymbol{A} + \boldsymbol{I}) \times \boldsymbol{n}_c + (\boldsymbol{T}_0 - \nabla \Phi) \times \boldsymbol{n}_\Phi = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{nc_\Phi}, \tag{2.532}$$

$$(\nabla \times \boldsymbol{A}) \cdot \boldsymbol{n_A} + (\mu \boldsymbol{T}_0 - \mu \nabla \Phi) \cdot \boldsymbol{n}_\Phi = 0, \quad \text{on} \quad \Gamma_{nc_\Phi}, \tag{2.533}$$

$$-\left(\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla V \right) \cdot \boldsymbol{n}_c = 0, \quad \text{on} \quad \Gamma_{nc_\Phi}, \tag{2.534}$$

$$(\nu \nabla \times \boldsymbol{A}) \times \boldsymbol{n_A} + (\boldsymbol{T}_0 - \nabla \Phi) \times \boldsymbol{n}_\Phi = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{\boldsymbol{A}, \Phi}, \tag{2.535}$$

$$(\nabla \times \boldsymbol{A}) \cdot \boldsymbol{n_A} + \mu (\boldsymbol{T}_0 - \nabla \Phi) \cdot \boldsymbol{n}_\Phi = 0, \quad \text{on} \quad \Gamma_{\boldsymbol{A}, \Phi}. \tag{2.536}$$

3 Weak formulation of nonlinear static and eddy current field problems

There are several methods to solve the partial differential equations of the electromagnetic fields based on the *weighted residual method* [42]. The first group is based on the direct form of the weighted residual method, as the finite difference method [29, 42, 52, 84] or the classical Galerkin technique [42]. The weak form of the weighted residual method results in the variational method [41] and the finite element method [11, 12, 19, 46, 52, 59, 68, 71, 77, 78, 94, 95], finally the boundary element method [16] and Trefftz method [42] are coming from the inverse form of the weighted residual method.

The numerical analysis of electromagnetic field problems with the aid of the Finite Element Method (FEM) has been one of the main directions of research in computational electromagnetics [11, 12, 19, 46, 52, 59, 68, 71, 77, 78, 94, 95]. This is the most widely used technique to approximate the solution of the partial differential equations. The basis of this extensively studied method is the weak formulation of partial differential equations, which also will be presented in this chapter, especially for magnetostatics and for eddy current fields as well as the potential formulations of these fields.

FEM is presented in the next chapter.

3.1   The weighted residual and Galerkin's method

This section presents the *weighted residual method*, which is a family of methods for solving partial differential equations. We are focusing only on the partial differential equations obtained from the static magnetic field problems and from the eddy current field problems. The application of *Galerkin's method* to solve partial differential equations is one possibility, however, it is the most widely used technique. The finite element method is based on the Galerkin's method of the weighted residual method [3, 5, 11–13, 19, 21, 28, 41–43, 46, 52, 59, 68, 71, 77, 78, 81–83, 94, 95].

The weighted residual method can be applied to minimizes the residual of a partial differential equation. The best approximation for the potentials can be obtained when the integral of the residual of the partial differential equation multiplied by a weighting function over the problem domain is equal to zero. The weighting function can be arbitrary, but in Galerkin's method, the weighting functions are selected to be the same as those used for expansion of the approximate solution.

3.1.1   Differential equations in electromagnetic field computation

The order of partial differential equations in electromagnetic field computation problems is two. Generally these equations have the form

$$-\nabla \cdot [c \nabla \Phi(\boldsymbol{r},t)] + d\frac{\partial \Phi(\boldsymbol{r},t)}{\partial t} + e\frac{\partial^2 \Phi(\boldsymbol{r},t)}{\partial t^2} = f(\boldsymbol{r},t), \tag{3.1}$$

or

$$\nabla \times [c \nabla \times \boldsymbol{A}(\boldsymbol{r},t)] - \nabla[c \nabla \cdot \boldsymbol{A}(\boldsymbol{r},t)] + d\frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} + e\frac{\partial^2 \boldsymbol{A}(\boldsymbol{r},t)}{\partial t^2} = \boldsymbol{F}(\boldsymbol{r},t), \tag{3.2}$$

or

$$\nabla \times [c \nabla \times \boldsymbol{A}(\boldsymbol{r},t)] + d\frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} + e\frac{\partial^2 \boldsymbol{A}(\boldsymbol{r},t)}{\partial t^2} = \boldsymbol{F}(\boldsymbol{r},t). \tag{3.3}$$

Here c, d and e are known parameters of the medium to be analyzed, which can be linear or nonlinear. Functions $f(\boldsymbol{r},t)$ and $\boldsymbol{F}(\boldsymbol{r},t)$ are known source terms. The unknown scalar function $\Phi(\boldsymbol{r},t)$ and the unknown vector function $\boldsymbol{A}(\boldsymbol{r},t)$ are depending on space $\boldsymbol{r}$ and time t as well.

According to the parameters c, d and e, the following three groups of partial differential equations can be formulated in the analyzed domain Ω: elliptic, parabolic and hyperbolic type partial differential equation.

The *elliptic type partial differential equations* have the form ($d = e = 0$)

$$-\nabla \cdot [c \nabla \Phi(\boldsymbol{r},t)] = f(\boldsymbol{r},t), \quad \text{in} \quad \Omega, \tag{3.4}$$

or

$$\nabla \times [c \nabla \times \boldsymbol{A}(\boldsymbol{r},t)] - \nabla[c \nabla \cdot \boldsymbol{A}(\boldsymbol{r},t)] = \boldsymbol{F}(\boldsymbol{r},t), \quad \text{in} \quad \Omega, \tag{3.5}$$

or

$$\nabla \times [c \nabla \times \boldsymbol{A}(\boldsymbol{r},t)] = \boldsymbol{F}(\boldsymbol{r},t), \quad \text{in} \quad \Omega. \tag{3.6}$$

The first one is the Poisson equation of static fields. If $f(\boldsymbol{r},t) = 0$, then Laplace equation can be obtained.

The *parabolic type partial differential equations* represent the diffusion equation (in this case $e = 0$),

$$-\nabla \cdot [c \nabla \Phi(\boldsymbol{r},t)] + d\frac{\partial \Phi(\boldsymbol{r},t)}{\partial t} = f(\boldsymbol{r},t), \quad \text{in} \quad \Omega, \tag{3.7}$$

or

$$\nabla \times [c\, \nabla \times \boldsymbol{A}(\boldsymbol{r},t)] - \nabla[c\, \nabla \cdot \boldsymbol{A}(\boldsymbol{r},t)] + d\frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} = \boldsymbol{F}(\boldsymbol{r},t), \quad \text{in} \quad \Omega, \quad (3.8)$$

or

$$\nabla \times [c\, \nabla \times \boldsymbol{A}(\boldsymbol{r},t)] + d\frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} = \boldsymbol{F}(\boldsymbol{r},t), \quad \text{in} \quad \Omega. \qquad (3.9)$$

Finally, wave equation is a *hyperbolic type partial differential equation* ($d = 0$),

$$-\nabla \cdot [c\, \nabla \Phi(\boldsymbol{r},t)] + e\frac{\partial^2 \Phi(\boldsymbol{r},t)}{\partial t^2} = f(\boldsymbol{r},t), \quad \text{in} \quad \Omega, \qquad (3.10)$$

or

$$\nabla \times [c\, \nabla \times \boldsymbol{A}(\boldsymbol{r},t)] - \nabla[c\, \nabla \cdot \boldsymbol{A}(\boldsymbol{r},t)] + e\frac{\partial^2 \boldsymbol{A}(\boldsymbol{r},t)}{\partial t^2} = \boldsymbol{F}(\boldsymbol{r},t), \quad \text{in} \quad \Omega, \quad (3.11)$$

or

$$\nabla \times [c\, \nabla \times \boldsymbol{A}(\boldsymbol{r},t)] + e\frac{\partial^2 \boldsymbol{A}(\boldsymbol{r},t)}{\partial t^2} = \boldsymbol{F}(\boldsymbol{r},t), \quad \text{in} \quad \Omega. \qquad (3.12)$$

Only the elliptic and the parabolic partial differential equations of the scalar function and of the vector function will be presented in this book, because they represent the static magnetic field and the eddy current field problems.

The following shorter notations will be used below: $\Phi = \Phi(\boldsymbol{r},t)$, $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{r},t)$, $f = f(\boldsymbol{r},t)$ and $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r},t)$.

The left-hand side of the above partial differential equations is usually called the differential operator of the actually used potential function.

The partial differential equations are usually given in the domain Ω, which has a boundary, $\Gamma = \partial\Omega$. As it was presented in the previous section, there are Dirichlet and Neumann type boundary conditions on the disjunct parts of the boundary. Here, Γ_D and Γ_N are the two parts of the boundary where Dirichlet and Neumann boundary conditions are prescribed, respectively and $\Gamma = \Gamma_D \cup \Gamma_N$.

Dirichlet boundary condition is specified on Γ_D, i.e.

$$\Phi = g, \quad \text{or} \quad \boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{G}, \quad \text{or} \quad \boldsymbol{A} \cdot \boldsymbol{n} = G, \quad \text{on} \quad \Gamma_D. \qquad (3.13)$$

Here, the functions $g = g(\boldsymbol{r},t)$, $\boldsymbol{G} = \boldsymbol{G}(\boldsymbol{r},t)$ and $G = G(\boldsymbol{r},t)$ are known on the part of boundary, Γ_D. Here $\boldsymbol{n}$ is the outer normal unit vector of the domain Ω.

Neumann boundary condition is given on the rest Γ_N,

$$(c\nabla\Phi) \cdot \boldsymbol{n} = c\frac{\partial \Phi}{\partial n} = h, \quad \text{or} \quad (c\nabla \times \boldsymbol{A}) \times \boldsymbol{n} = \boldsymbol{H},$$
$$\text{or} \quad c\nabla \cdot \boldsymbol{A} = H, \quad \text{on} \quad \Gamma_N. \qquad (3.14)$$

Here, the functions $h = h(\boldsymbol{r},t)$, $\boldsymbol{H} = \boldsymbol{H}(\boldsymbol{r},t)$ and $H = H(\boldsymbol{r},t)$ are known on the boundary part Γ_N.

3.1.2   The weighted residual method

A. Weighted residual of elliptic differential equations with scalar potential

The potential function Φ can be approximated by a function $\tilde{\Phi}$, i.e. $\Phi \simeq \tilde{\Phi}$, but this function does not satisfy the differential equation exactly, however, it is possible to select a function $\tilde{\Phi}$, which satisfies Dirichlet boundary conditions exactly (see next sections).

The weighted residual method is based on the *inner product* of the differential operator of the partial differential equation and a weighting function $N = N(\boldsymbol{r})$, defined as

$$< N, \mathcal{PDE} >= \int_\Omega N\left[-\nabla \cdot (c\,\nabla\Phi)\right]\,\mathrm{d}\Omega. \tag{3.15}$$

In this case N is a real function. Using the identity $\nabla \cdot (\varphi\boldsymbol{v}) = \boldsymbol{v}\cdot\nabla\varphi + \varphi\nabla\cdot\boldsymbol{v}$ and $\varphi = N$ and $\boldsymbol{v} = c\,\nabla\Phi$, the inner product has the form

$$-\int_\Omega N\nabla \cdot (c\,\nabla\Phi)\,\mathrm{d}\Omega = \int_\Omega c\,\nabla\Phi\cdot\nabla N\,\mathrm{d}\Omega - \oint_\Gamma N(c\nabla\Phi\cdot\boldsymbol{n})\,\mathrm{d}\Gamma. \tag{3.16}$$

Let us determine the following inner product in the same way:

$$-\int_\Omega \Phi\nabla \cdot (c\,\nabla N)\,\mathrm{d}\Omega = \int_\Omega c\,\nabla\Phi\cdot\nabla N\,\mathrm{d}\Omega - \oint_\Gamma \Phi(c\nabla N\cdot\boldsymbol{n})\,\mathrm{d}\Gamma, \tag{3.17}$$

i.e. Φ and N are replaced, then

$$\begin{aligned}
-\int_\Omega N\nabla \cdot (c\,\nabla\Phi)\,\mathrm{d}\Omega = &-\int_\Omega \Phi\nabla \cdot (c\,\nabla N)\,\mathrm{d}\Omega \\
&+ \oint_\Gamma \left[\Phi(c\nabla N\cdot\boldsymbol{n}) - N(c\nabla\Phi\cdot\boldsymbol{n})\right]\,\mathrm{d}\Gamma,
\end{aligned} \tag{3.18}$$

because the first integrals on the right in (3.16) and in (3.17) are the same.

This relation is valid for the approximating trial function $\tilde{\Phi}$ as well,

$$\begin{aligned}
-\int_\Omega N\nabla \cdot (c\,\nabla\tilde{\Phi})\,\mathrm{d}\Omega = &-\int_\Omega \tilde{\Phi}\nabla \cdot (c\,\nabla N)\,\mathrm{d}\Omega \\
&+ \oint_\Gamma \left[\tilde{\Phi}(c\nabla N\cdot\boldsymbol{n}) - N(c\nabla\tilde{\Phi}\cdot\boldsymbol{n})\right]\,\mathrm{d}\Gamma.
\end{aligned} \tag{3.19}$$

By subtracting equation (3.18) from (3.19) and applying the definition of elliptic partial differential equation $-\nabla \cdot (c\,\nabla\Phi) = f$, the following equation can be obtained:

$$\begin{aligned}
\int_\Omega N\left[-\nabla \cdot (c\,\nabla\tilde{\Phi}) - f\right]\,\mathrm{d}\Omega = &-\int_\Omega (\tilde{\Phi} - \Phi)\nabla \cdot (c\,\nabla N)\,\mathrm{d}\Omega \\
&+ \oint_\Gamma \left\{(\tilde{\Phi} - \Phi)(c\nabla N\cdot\boldsymbol{n}) - N[c\nabla(\tilde{\Phi} - \Phi)\cdot\boldsymbol{n}]\right\}\,\mathrm{d}\Gamma.
\end{aligned} \tag{3.20}$$

Let us denote the two disjunct segments of the boundary,

$$
\int_\Omega N \left[-\nabla \cdot (c\,\nabla\tilde{\Phi}) - f \right] d\Omega = -\int_\Omega (\tilde{\Phi} - \Phi)\nabla \cdot (c\,\nabla N)\,d\Omega
$$
$$
+ \int_{\Gamma_D \cup \Gamma_N} \left\{ (\tilde{\Phi} - \Phi)(c\nabla N \cdot \boldsymbol{n}) - N[c\nabla(\tilde{\Phi} - \Phi)\cdot\boldsymbol{n}] \right\} d\Gamma.
\tag{3.21}
$$

The potential function Φ is prescribed on Γ_D, $\Phi = g$ and the normal component of the gradient of Φ is given on Γ_N, $-\nabla\Phi \cdot \boldsymbol{n} = h$ (see equations (3.13) and (3.14)). It is possible to select the approximation function to satisfy Dirichlet condition exactly, but

$$
N = 0, \quad \text{on} \quad \Gamma_D
\tag{3.22}
$$

should be specified in this case. These result in

$$
\int_\Omega N \left[-\nabla \cdot (c\,\nabla\tilde{\Phi}) - f \right] d\Omega = -\int_\Omega (\tilde{\Phi} - \Phi)\nabla \cdot (c\,\nabla N)\,d\Omega
$$
$$
+ \int_{\Gamma_N} (\tilde{\Phi} - \Phi)(c\nabla N \cdot \boldsymbol{n})d\Gamma - \int_{\Gamma_N} N\left[(c\nabla\tilde{\Phi}) \cdot \boldsymbol{n} - h \right]d\Gamma.
\tag{3.23}
$$

Supposing that the difference between the unknown potential function Φ and the approximation function $\tilde{\Phi}$ is equal to zero, i.e. the first and the second integrals on the right of (3.23) disappear, it results in

$$
\int_\Omega N \left[-\nabla \cdot (c\,\nabla\tilde{\Phi}) - f \right] d\Omega + \int_{\Gamma_N} N \left[(c\nabla\tilde{\Phi}) \cdot \boldsymbol{n} - h \right] d\Gamma = 0.
\tag{3.24}
$$

This is the sum of two terms, the first one is the inner product of the partial differential equation and the weighting function, the second term is the Neumann type boundary condition multiplied by the weighting function, integrated only on Γ_N. The Dirichlet boundary condition can be specified by an appropriate selection of the approximation function on Γ_D. This is called the *weak formulation* of elliptic differential equations with scalar potential.

B. Weighted residual of elliptic differential equations with vector potential

The potential function $\boldsymbol{A}$ can be approximated by a function $\tilde{\boldsymbol{A}}$, i.e. $\boldsymbol{A} \simeq \tilde{\boldsymbol{A}}$, but this function does not satisfy the differential equation exactly, however, it is possible to select a function $\tilde{\boldsymbol{A}}$, which satisfies Dirichlet boundary conditions exactly (see next sections).

Let us first determine the inner product of the differential operator and a weighting function $\boldsymbol{W} = \boldsymbol{W}(\boldsymbol{r})$,

$$
< \boldsymbol{W}, \mathcal{PDE} >= \int_\Omega \boldsymbol{W} \cdot [\nabla \times (c\,\nabla \times \boldsymbol{A}) - \nabla(c\,\nabla \cdot \boldsymbol{A})]\,d\Omega.
\tag{3.25}
$$

Here, $\boldsymbol{W}$ is a real function. Using the identity

$$
\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \times \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \times \boldsymbol{v}
\tag{3.26}
$$

with $\boldsymbol{u} = c\,\nabla \times \boldsymbol{A}$ and $\boldsymbol{v} = \boldsymbol{W}$, moreover the relation

$$\nabla \cdot (\varphi \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \varphi + \varphi \nabla \cdot \boldsymbol{v}, \tag{3.27}$$

with $\varphi = c\,\nabla \cdot \boldsymbol{A}$ and $\boldsymbol{v} = \boldsymbol{W}$, the inner product has the form

$$
\begin{aligned}
&\int_\Omega \boldsymbol{W} \cdot [\nabla \times (c\,\nabla \times \boldsymbol{A}) - \nabla(c\,\nabla \cdot \boldsymbol{A})]\,\mathrm{d}\Omega \\
&= \int_\Omega c\,\nabla \times \boldsymbol{A} \cdot \nabla \times \boldsymbol{W}\,\mathrm{d}\Omega + \int_\Omega c\,\nabla \cdot \boldsymbol{A}\,\nabla \cdot \boldsymbol{W}\,\mathrm{d}\Omega \\
&\quad + \oint_\Gamma [(c\,\nabla \times \boldsymbol{A}) \times \boldsymbol{W}] \cdot \boldsymbol{n}\,\mathrm{d}\Gamma - \oint_\Gamma c\,\nabla \cdot \boldsymbol{A}\,(\boldsymbol{W} \cdot \boldsymbol{n})\,\mathrm{d}\Gamma.
\end{aligned}
\tag{3.28}
$$

Let us next determine the following inner product in the same way:

$$
\begin{aligned}
&\int_\Omega \boldsymbol{A} \cdot [\nabla \times (c\,\nabla \times \boldsymbol{W}) - \nabla(c\,\nabla \cdot \boldsymbol{W})]\,\mathrm{d}\Omega \\
&= \int_\Omega c\,\nabla \times \boldsymbol{A} \cdot \nabla \times \boldsymbol{W}\,\mathrm{d}\Omega + \int_\Omega c\,\nabla \cdot \boldsymbol{A}\,\nabla \cdot \boldsymbol{W}\,\mathrm{d}\Omega \\
&\quad + \oint_\Gamma [(c\,\nabla \times \boldsymbol{W}) \times \boldsymbol{A}] \cdot \boldsymbol{n}\,\mathrm{d}\Gamma - \oint_\Gamma c\,\nabla \cdot \boldsymbol{W}\,(\boldsymbol{A} \cdot \boldsymbol{n})\,\mathrm{d}\Gamma,
\end{aligned}
\tag{3.29}
$$

then

$$
\begin{aligned}
&\int_\Omega \boldsymbol{W} \cdot [\nabla \times (c\,\nabla \times \boldsymbol{A}) - \nabla(c\,\nabla \cdot \boldsymbol{A})]\,\mathrm{d}\Omega \\
&= \int_\Omega \boldsymbol{A} \cdot [\nabla \times (c\,\nabla \times \boldsymbol{W}) - \nabla(c\,\nabla \cdot \boldsymbol{W})]\,\mathrm{d}\Omega \\
&\quad + \oint_\Gamma [(c\,\nabla \times \boldsymbol{A}) \times \boldsymbol{W}] \cdot \boldsymbol{n}\,\mathrm{d}\Gamma - \oint_\Gamma c\,\nabla \cdot \boldsymbol{A}\,(\boldsymbol{W} \cdot \boldsymbol{n})\,\mathrm{d}\Gamma \\
&\quad - \oint_\Gamma [(c\,\nabla \times \boldsymbol{W}) \times \boldsymbol{A}] \cdot \boldsymbol{n}\,\mathrm{d}\Gamma + \oint_\Gamma c\,\nabla \cdot \boldsymbol{W}\,(\boldsymbol{A} \cdot \boldsymbol{n})\,\mathrm{d}\Gamma.
\end{aligned}
\tag{3.30}
$$

Applying the identity $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{n} = (\boldsymbol{n} \times \boldsymbol{a}) \cdot \boldsymbol{b} = -(\boldsymbol{a} \times \boldsymbol{n}) \cdot \boldsymbol{b}$ in the first ($\boldsymbol{a} = c\,\nabla \times \boldsymbol{A}$, $\boldsymbol{b} = \boldsymbol{W}$) and in the third ($\boldsymbol{a} = c\,\nabla \times \boldsymbol{W}$, $\boldsymbol{b} = \boldsymbol{A}$) boundary integral terms, it results in

$$
\begin{aligned}
&\int_\Omega \boldsymbol{W} \cdot [\nabla \times (c\,\nabla \times \boldsymbol{A}) - \nabla(c\,\nabla \cdot \boldsymbol{A})]\,\mathrm{d}\Omega \\
&= \int_\Omega \boldsymbol{A} \cdot [\nabla \times (c\,\nabla \times \boldsymbol{W}) - \nabla(c\,\nabla \cdot \boldsymbol{W})]\,\mathrm{d}\Omega \\
&\quad - \oint_\Gamma [(c\,\nabla \times \boldsymbol{A}) \times \boldsymbol{n}] \cdot \boldsymbol{W}\,\mathrm{d}\Gamma - \oint_\Gamma c\,\nabla \cdot \boldsymbol{A}\,(\boldsymbol{W} \cdot \boldsymbol{n})\,\mathrm{d}\Gamma \\
&\quad + \oint_\Gamma [(c\,\nabla \times \boldsymbol{W}) \times \boldsymbol{n}] \cdot \boldsymbol{A}\,\mathrm{d}\Gamma + \oint_\Gamma c\,\nabla \cdot \boldsymbol{W}\,(\boldsymbol{A} \cdot \boldsymbol{n})\,\mathrm{d}\Gamma.
\end{aligned}
\tag{3.31}
$$

This relation is valid for the approximating trial function $\tilde{\boldsymbol{A}}$ as well,

$$
\begin{aligned}
&\int_{\Omega} \boldsymbol{W} \cdot [\nabla \times (c\,\nabla \times \tilde{\boldsymbol{A}}) - \nabla(c\,\nabla \cdot \tilde{\boldsymbol{A}})]\,\mathrm{d}\Omega \\
&= \int_{\Omega} \tilde{\boldsymbol{A}} \cdot [\nabla \times (c\,\nabla \times \boldsymbol{W}) - \nabla(c\,\nabla \cdot \boldsymbol{W})]\,\mathrm{d}\Omega \\
&\quad - \oint_{\Gamma} [(c\,\nabla \times \tilde{\boldsymbol{A}}) \times \boldsymbol{n}] \cdot \boldsymbol{W}\,\mathrm{d}\Gamma - \oint_{\Gamma} c\,\nabla \cdot \tilde{\boldsymbol{A}}\,(\boldsymbol{W} \cdot \boldsymbol{n})\,\mathrm{d}\Gamma \\
&\quad + \oint_{\Gamma} [(c\,\nabla \times \boldsymbol{W}) \times \boldsymbol{n}] \cdot \tilde{\boldsymbol{A}}\,\mathrm{d}\Gamma + \oint_{\Gamma} c\,\nabla \cdot \boldsymbol{W}\,(\tilde{\boldsymbol{A}} \cdot \boldsymbol{n})\,\mathrm{d}\Gamma .
\end{aligned}
\tag{3.32}
$$

By subtracting equation (3.31) from equation (3.32) and applying the definition of elliptic partial differential equation $\nabla \times (c\,\nabla \times \boldsymbol{A}) - \nabla(c\,\nabla \cdot \boldsymbol{A}) = \boldsymbol{F}$, the following equation can be obtained:

$$
\begin{aligned}
&\int_{\Omega} \boldsymbol{W} \cdot [\nabla \times (c\,\nabla \times \tilde{\boldsymbol{A}}) - \nabla(c\,\nabla \cdot \tilde{\boldsymbol{A}}) - \boldsymbol{F}]\,\mathrm{d}\Omega \\
&= \int_{\Omega} (\tilde{\boldsymbol{A}} - \boldsymbol{A}) \cdot [\nabla \times (c\,\nabla \times \boldsymbol{W}) - \nabla(c\,\nabla \cdot \boldsymbol{W})]\,\mathrm{d}\Omega \\
&\quad - \oint_{\Gamma} \left\{ [c\,\nabla \times (\tilde{\boldsymbol{A}} - \boldsymbol{A})] \times \boldsymbol{n} \right\} \cdot \boldsymbol{W}\,\mathrm{d}\Gamma - \oint_{\Gamma} c\,\nabla \cdot (\tilde{\boldsymbol{A}} - \boldsymbol{A})\,(\boldsymbol{W} \cdot \boldsymbol{n})\,\mathrm{d}\Gamma \\
&\quad + \oint_{\Gamma} [(c\,\nabla \times \boldsymbol{W}) \times \boldsymbol{n}] \cdot (\tilde{\boldsymbol{A}} - \boldsymbol{A})\,\mathrm{d}\Gamma + \oint_{\Gamma} c\,\nabla \cdot \boldsymbol{W}\,[(\tilde{\boldsymbol{A}} - \boldsymbol{A}) \cdot \boldsymbol{n}]\,\mathrm{d}\Gamma .
\end{aligned}
\tag{3.33}
$$

The boundary can be split in two disjunct segments, $\Gamma = \Gamma_D \cup \Gamma_N$. Dirichlet boundary condition is prescribed only on the segment of the boundary surface Γ_D and Neumann boundary condition is prescribed only on the segment Γ_N.

The first and the second boundary integral terms contain Neumann type boundary conditions, $(c\nabla \times \boldsymbol{A}) \times \boldsymbol{n} = \boldsymbol{H}$, or $c\,\nabla \cdot \boldsymbol{A} = H$, respectively. The last boundary integral terms contain Dirichlet type boundary conditions, $\boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{G}$, or $\boldsymbol{A} \cdot \boldsymbol{n} = G$. These are coming from (3.13) and (3.14).

Dirichlet boundary conditions can be specified exactly by an appropriate selection of the approximation function. In this case

$$
\boldsymbol{n} \times \boldsymbol{W} = \boldsymbol{0}, \quad \text{or} \quad \boldsymbol{W} \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_D
\tag{3.34}
$$

must be specified.

Supposing that the difference between the unknown potential function $\boldsymbol{A}$ and the approximation function $\tilde{\boldsymbol{A}}$ is zero, i.e.

$$
R(\tilde{\boldsymbol{A}}, \boldsymbol{W}) = \int_{\Omega} (\tilde{\boldsymbol{A}} - \boldsymbol{A}) \cdot [\nabla \times (c\,\nabla \times \boldsymbol{W}) - \nabla(c\,\nabla \cdot \boldsymbol{W})]\,\mathrm{d}\Omega = 0,
\tag{3.35}
$$

it results in the weak formulation of the elliptic differential equations with vector potential,

$$
\begin{aligned}
&\int_{\Omega} \boldsymbol{W} \cdot [\nabla \times (c\,\nabla \times \tilde{\boldsymbol{A}}) - \nabla(c\,\nabla \cdot \tilde{\boldsymbol{A}}) - \boldsymbol{F}]\,\mathrm{d}\Omega \\
&+ \int_{\Gamma_N} [(c\,\nabla \times \tilde{\boldsymbol{A}}) \times \boldsymbol{n} - \boldsymbol{H}] \cdot \boldsymbol{W}\,\mathrm{d}\Gamma + \int_{\Gamma_N} (c\,\nabla \cdot \tilde{\boldsymbol{A}} - H)\,(\boldsymbol{W} \cdot \boldsymbol{n})\,\mathrm{d}\Gamma = 0.
\end{aligned}
\tag{3.36}
$$

The first term is the inner product of the partial differential equation and the weighting function, the second and third terms are the Neumann type boundary conditions multiplied by the weighting function, integrated only on Γ_N. Dirichlet boundary conditions can be approximated exactly on Γ_D.

The above formulation is valid if the gauged vector potential is used. The formulation of ungauged vector potential is as follows:

$$
\begin{aligned}
&\int_\Omega \boldsymbol{W} \cdot [\nabla \times (c\,\nabla \times \tilde{\boldsymbol{A}}) - \boldsymbol{F}]\,\mathrm{d}\Omega \\
&+ \int_{\Gamma_N} [(c\,\nabla \times \tilde{\boldsymbol{A}}) \times \boldsymbol{n} - \boldsymbol{H}] \cdot \boldsymbol{W}\,\mathrm{d}\Gamma = 0,
\end{aligned}
\tag{3.37}
$$

because the form of partial differential equation is the following: $\nabla \times (c\,\nabla \times \boldsymbol{A}) = \boldsymbol{F}$.

C. Weighted residual of parabolic differential equations

As it is presented in the last section, the partial differential equations of a transient eddy current field problem has some terms containing time derivative of the potentials. These terms are appended to the integrand of the weighted residual, however, additional initial conditions must be specified as

$$
\Phi(t = 0) = \Phi_0, \quad \text{and} \quad \boldsymbol{A}(t = 0) = \boldsymbol{A}_0.
\tag{3.38}
$$

3.2   Approximation of unknown functions and weighting functions

In the previous section, we have defined the weighted residual of the static magnetic field problem and the eddy current field problem, but we have not introduced the approximating functions and the weighting functions, yet [5,11–13,19,42,46,52,59,68,71,77,78,94,95].

The approximating scalar and vector potential functions and the scalar and vector weighting functions are selected as

$$
\Phi(\boldsymbol{r}, t) \simeq \widetilde{\Phi}(\boldsymbol{r}, t) = \sum_{i=1}^I Q_i(\boldsymbol{r})\,\varphi_i(t), \;\; \boldsymbol{A}(\boldsymbol{r}, t) \simeq \widetilde{\boldsymbol{A}}(\boldsymbol{r}, t) \sum_{j=1}^J \boldsymbol{Q}_j(\boldsymbol{r})\,a_j(t),
\tag{3.39}
$$

and

$$
N(\boldsymbol{r}) = \sum_{k=1}^K P_k(\boldsymbol{r})\,b_k, \quad \boldsymbol{W}(\boldsymbol{r}) = \sum_{l=1}^L \boldsymbol{P}_l(\boldsymbol{r})\,c_l,
\tag{3.40}
$$

where $Q_i(\boldsymbol{r})$, $\boldsymbol{Q}_j(\boldsymbol{r})$, $P_k(\boldsymbol{r})$ and $\boldsymbol{P}_l(\boldsymbol{r})$ are the basis functions of the approximation, $\varphi_i(t)$, $a_j(t)$, b_k and c_l are the unknown coefficients, which must be determined.

This is the most general form of approximation and weighting function. According to the selection of the basis functions, the weighted residual method results in several technique.

The weighted residual method has three forms as it is shown in *Fig.* 3.1,

(*i*) the direct form,

(*ii*) the weak form,

(*iii*) the inverse form.

Applying the *direct form* of the weighted residual method, i.e. using directly e.g. (3.24), (3.36) or (3.37) and the formulations in (3.39) and (3.40) results in a system of linear equations, which solution gives the unknown coefficients $\varphi_i(t)$ or $a_j(t)$ of the approximating potential functions. The finite difference method and the general

$$Q = P, \boldsymbol{Q} = \boldsymbol{P} \qquad\qquad Q \neq P, \boldsymbol{Q} \neq \boldsymbol{P}$$

Direct form

Bubnov–Galerkin methods

Moment and finite difference methods

Weak form

Finite element method

General weak form

Inverse form

Trefftz method

Boundary element method
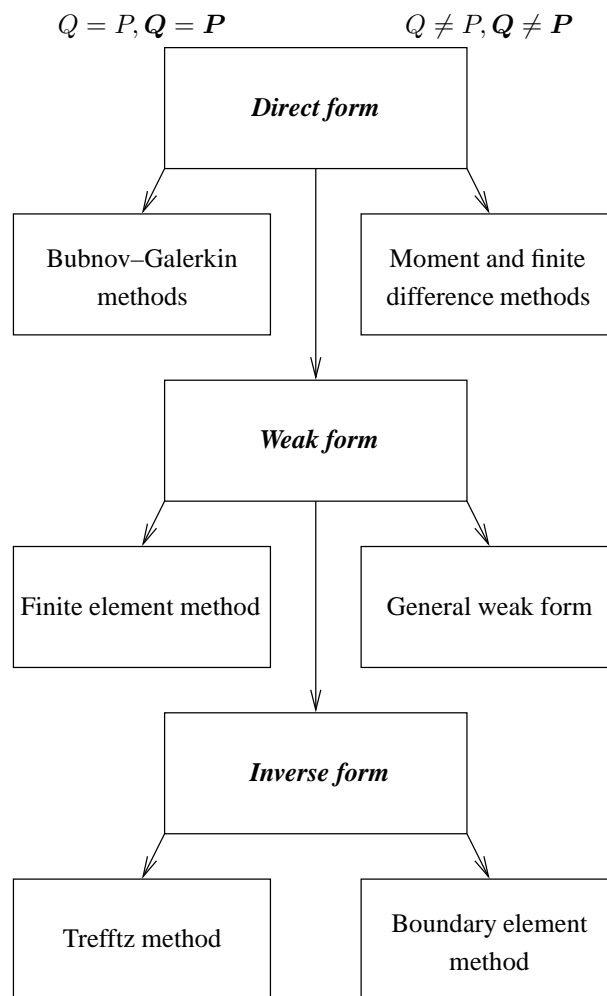
Fig. 3.1. The numerical field analysis methods

Bubnov–Galerkin method belong to this group. The finite difference method is a popular and widely used technique. The disadvantage of this form is that the basis functions must be differentiated twice, because the differential operator is a second order one.

The *weak formulation* of the weighted residual method can be obtained when applying the rule of integration by parts to decrease the order of the differential operator in the inner product. This means that the order of approximating function can be just one. The finite element method or the general weak form can be derived from this group of the weighted residual method, moreover the weak formulation of static magnetic field problems and eddy current field problems is presented in the next section. In the case of finite element method, the weighting function and the basis function of the approximating function are the same.

The boundary element method and the Trefftz method are based on the *inverse form* of partial differential equations. In this situation, the weighting function is a solution of the homogeneous partial differential equation.

Further analysis of the direct form and the inverse form is out of scope of this book, since we are focusing on the weak formulation, which is the way to the Finite Element Method.

3.3   The weak formulation with Galerkin's method

The finite element method use the the weak formulation with *Galerkin's method* when the basis functions of the approximating function and the weighting function are the same. Here, the weak formulations of the potential formulations according to Galerkin's method are presented, which are appropriate in the finite element method. In the following $N = N(\boldsymbol{r})$ denotes the scalar weighting function as well as the basis functions of approximating function and $\boldsymbol{W} = \boldsymbol{W}(\boldsymbol{r})$ denotes the vector weighting function as well as the basis functions of approximating function [5, 11–13, 19, 42, 46, 52, 59, 68, 71, 77, 78, 94, 95].

Scalar potentials $\Phi = \Phi(\boldsymbol{r})$, or $\Phi = \Phi(\boldsymbol{r}, t)$ are approximated by an expansion in terms of I elements of an entire function set N_i,

$$\Phi \simeq \widetilde{\Phi} = \Phi_D + \sum_{i=1}^{I} N_i \Phi_i, \tag{3.41}$$

where Φ_D satisfies the prescribed non homogeneous *Dirichlet type boundary conditions*,

$$\Phi_D = g, \quad \text{on} \quad \Gamma_D, \tag{3.42}$$

that is why the elements of the function set must prescribe homogeneous Dirichlet type boundary condition,

$$N_i = 0, \quad \text{on} \quad \Gamma_D. \tag{3.43}$$

Finally, non homogeneous Dirichlet boundary conditions can be prescribed directly by the help of the function Φ_D.

Vector potentials $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{r})$, or $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{r}, t)$ are approximated by an expansion in terms of J elements of an entire function set $\boldsymbol{W}_j$,

$$\boldsymbol{A} \simeq \widetilde{\boldsymbol{A}} = \boldsymbol{A}_D + \sum_{j=1}^{J} \boldsymbol{W}_j A_j, \tag{3.44}$$

where $\boldsymbol{A}_D$ satisfies the prescribed non homogeneous Dirichlet boundary conditions,

$$\boldsymbol{n} \times \boldsymbol{A}_D = \boldsymbol{G}, \quad \text{or} \quad \boldsymbol{A}_D \cdot \boldsymbol{n} = G, \quad \text{on} \quad \Gamma_D, \tag{3.45}$$

that is why the elements of the function set must prescribe homogeneous Dirichlet type boundary condition,

$$\boldsymbol{n} \times \boldsymbol{W}_j = \boldsymbol{0}, \quad \text{or} \quad \boldsymbol{W}_j \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_D. \tag{3.46}$$

Finally, non homogeneous Dirichlet type boundary conditions can be prescribed directly by the help of $\boldsymbol{A}_D$.

Shape functions N_i and $\boldsymbol{W}_j$ are the elements of an entire function set, which can be defined in many ways. The definition of these elements are presented in chapter 3.3.18.

In the following $\widetilde{\Phi}$, $\widetilde{\Psi}$, $\widetilde{\boldsymbol{A}}$, $\widetilde{V}$ $(\widetilde{v})$ and $\widetilde{\boldsymbol{T}}$ will denote the approximated unknown potential functions.

3.3.1 The reduced magnetic scalar potential, the Φ-formulation

The weak formulation of the partial differential equation (2.160) and the Neumann type boundary condition (2.162) can be written as

$$\int_{\Omega} N_k \nabla \cdot (\mu_o \nabla \widetilde{\Phi}) \, \mathrm{d}\Omega + \int_{\Gamma_B} N_k \left(b + \mu_o \widetilde{\boldsymbol{T}}_0 \cdot \boldsymbol{n} - \mu_o \nabla \widetilde{\Phi} \cdot \boldsymbol{n} + \widetilde{\boldsymbol{R}} \cdot \boldsymbol{n} \right) \mathrm{d}\Gamma$$
$$= \int_{\Omega} N_k \nabla \cdot (\mu_o \widetilde{\boldsymbol{T}}_0) \, \mathrm{d}\Omega + \int_{\Omega} N_k \nabla \cdot \widetilde{\boldsymbol{R}} \, \mathrm{d}\Omega, \quad k = 1, \cdots, I, \tag{3.47}$$

where

$$N_k = 0, \quad \text{on} \quad \Gamma_H. \tag{3.48}$$

The second order derivative in the first integral can be reduced to first order by using the identity

$$\nabla \cdot (\varphi \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \varphi + \varphi \nabla \cdot \boldsymbol{v} \tag{3.49}$$

with the notations $\varphi = N_k$ and $\boldsymbol{v} = \mu_o \nabla \widetilde{\Phi}$ and the Gauss' theorem

$$-\int_{\Omega} \mu_o \nabla N_k \cdot \nabla \widetilde{\Phi} \, \mathrm{d}\Omega + \oint_{\Gamma} \mu_o N_k \nabla \widetilde{\Phi} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma$$
$$+ \int_{\Gamma_B} N_k \left(b + \mu_o \widetilde{\boldsymbol{T}}_0 \cdot \boldsymbol{n} - \mu_o \nabla \widetilde{\Phi} \cdot \boldsymbol{n} + \widetilde{\boldsymbol{R}} \cdot \boldsymbol{n} \right) \mathrm{d}\Gamma =$$
$$-\int_{\Omega} \mu_o \nabla N_k \cdot \widetilde{\boldsymbol{T}}_0 \, \mathrm{d}\Omega + \oint_{\Gamma} \mu_o N_k \widetilde{\boldsymbol{T}}_0 \cdot \boldsymbol{n} \, \mathrm{d}\Gamma$$
$$-\int_{\Omega} \nabla N_k \cdot \widetilde{\boldsymbol{R}} \, \mathrm{d}\Omega + \oint_{\Gamma} N_k \widetilde{\boldsymbol{R}} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma, \tag{3.50}$$

where the right-hand side has been modified with notations $\varphi = N_k$ and $\boldsymbol{v} = \mu_o \widetilde{\boldsymbol{T}}_0$ and $\boldsymbol{v} = \widetilde{\boldsymbol{R}}$, too. The boundary integrals on the right-hand side are valid over the boundary of the whole problem region $\Gamma = \Gamma_H \cup \Gamma_B$, i.e.

$$
\begin{aligned}
& -\int_\Omega \mu_o \nabla N_k \cdot \nabla \widetilde{\Phi} \, \mathrm{d}\Omega + \int_{\Gamma_H \cup \Gamma_B} \mu_o N_k \nabla \widetilde{\Phi} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma \\
& + \int_{\Gamma_B} N_k \left(b + \mu_o \widetilde{\boldsymbol{T}}_0 \cdot \boldsymbol{n} - \mu_o \nabla \widetilde{\Phi} \cdot \boldsymbol{n} + \widetilde{\boldsymbol{R}} \cdot \boldsymbol{n} \right) \mathrm{d}\Gamma = \\
& -\int_\Omega \mu_o \nabla N_k \cdot \widetilde{\boldsymbol{T}}_0 \, \mathrm{d}\Omega + \int_{\Gamma_H \cup \Gamma_B} \mu_o N_k \widetilde{\boldsymbol{T}}_0 \cdot \boldsymbol{n} \, \mathrm{d}\Gamma \\
& -\int_\Omega \nabla N_k \cdot \widetilde{\boldsymbol{R}} \, \mathrm{d}\Omega + \int_{\Gamma_H \cup \Gamma_B} N_k \widetilde{\boldsymbol{R}} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma.
\end{aligned}
\tag{3.51}
$$

It is easy to see that the terms $\mu_o \widetilde{\boldsymbol{T}}_0 \cdot \boldsymbol{n}$, $\mu_o \nabla \widetilde{\Phi} \cdot \boldsymbol{n}$ and $\widetilde{\boldsymbol{R}} \cdot \boldsymbol{n}$ on the part Γ_B are disappearing,

$$
\begin{aligned}
& -\int_\Omega \mu_o \nabla N_k \cdot \nabla \widetilde{\Phi} \, \mathrm{d}\Omega + \int_{\Gamma_H} \mu_o N_k \nabla \widetilde{\Phi} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma + \int_{\Gamma_B} N_k b \, \mathrm{d}\Gamma = \\
& -\int_\Omega \mu_o \nabla N_k \cdot \widetilde{\boldsymbol{T}}_0 \, \mathrm{d}\Omega + \int_{\Gamma_H} \mu_o N_k \widetilde{\boldsymbol{T}}_0 \cdot \boldsymbol{n} \, \mathrm{d}\Gamma \\
& -\int_\Omega \nabla N_k \cdot \widetilde{\boldsymbol{R}} \, \mathrm{d}\Omega + \int_{\Gamma_H} N_k \widetilde{\boldsymbol{R}} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma.
\end{aligned}
\tag{3.52}
$$

On the rest part of the boundary Γ_H, the shape function N_k is equal to zero because of Dirichlet type boundary condition (2.161).

Finally, the following weak form can be obtained for the Φ-formulation ($k = 1, \cdots, I$):

$$
\int_\Omega \mu_o \nabla N_k \cdot \nabla \widetilde{\Phi} \, \mathrm{d}\Omega = \int_\Omega \mu_o \nabla N_k \cdot \widetilde{\boldsymbol{T}}_0 \, \mathrm{d}\Omega + \int_\Omega \nabla N_k \cdot \widetilde{\boldsymbol{R}} \, \mathrm{d}\Omega + \int_{\Gamma_B} N_k b \, \mathrm{d}\Gamma.
\tag{3.53}
$$

3.3.2   Combination of magnetic scalar potentials, $\Phi - \Psi$-formulation

The partial differential equation (2.177) in Ω_Φ, the partial differential equation (2.178) in Ω_Ψ, the Neumann type boundary conditions (2.181) and (2.182) on $\Gamma_{B_\Phi} \cup \Gamma_{B_\Psi}$ and the interface condition (2.184) on $\Gamma_{\Phi,\Psi}$ can be summarized in the following formula:

$$
\begin{aligned}
& \int_{\Omega_\Phi} N_k \nabla \cdot (\mu \nabla \widetilde{\Phi}) \, \mathrm{d}\Omega + \int_{\Omega_\Psi} N_k \nabla \cdot (\mu_o \nabla \widetilde{\Psi}) \, \mathrm{d}\Omega \\
& + \int_{\Gamma_{B_\Phi}} N_k \left(b + \mu \widetilde{\boldsymbol{T}}_0 \cdot \boldsymbol{n} - \mu \nabla \widetilde{\Phi} \cdot \boldsymbol{n} \right) \mathrm{d}\Gamma \\
& + \int_{\Gamma_{B_\Psi}} N_k \left(b - \mu_o \nabla \widetilde{\Psi} \cdot \boldsymbol{n} + \widetilde{\boldsymbol{R}} \cdot \boldsymbol{n} \right) \mathrm{d}\Gamma \\
& + \int_{\Gamma_{\Phi,\Psi}} N_k \left[(\mu \widetilde{\boldsymbol{T}}_0 - \mu \nabla \widetilde{\Phi}) \cdot \boldsymbol{n}_\Phi + (-\mu_o \nabla \widetilde{\Psi} + \widetilde{\boldsymbol{R}}) \cdot \boldsymbol{n}_\Psi \right] \mathrm{d}\Gamma \\
& = \int_{\Omega_\Phi} N_k \nabla \cdot (\mu \widetilde{\boldsymbol{T}}_0) \, \mathrm{d}\Omega + \int_{\Omega_\Psi} N_k \nabla \cdot \widetilde{\boldsymbol{R}} \, \mathrm{d}\Omega, \quad k = 1, \cdots, I.
\end{aligned}
\tag{3.54}
$$

Here

$$N_k = 0, \quad \text{on} \quad \Gamma_{H_\Phi} \cup \Gamma_{H_\Psi}. \tag{3.55}$$

The second order derivatives in the first and in the second integrals can be reduced to first order and the right-hand side can also be manipulated by using the identity

$$\nabla \cdot (\varphi \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \varphi + \varphi \nabla \cdot \boldsymbol{v} \tag{3.56}$$

with the notations $\varphi = N_k$ and $\boldsymbol{v} = \mu \nabla \widetilde{\Phi}$, then $\varphi = N_k$ and $\boldsymbol{v} = \mu_o \nabla \widetilde{\Psi}$, moreover $\varphi = N_k$ and $\boldsymbol{v} = \mu \widetilde{\boldsymbol{T}}_0$, finally $\varphi = N_k$ and $\boldsymbol{v} = \widetilde{\boldsymbol{R}}$ and the Gauss' theorem

$$
\begin{aligned}
&- \int_{\Omega_\Phi} \mu \nabla N_k \cdot \nabla \widetilde{\Phi} \, \mathrm{d}\Omega + \int_{\Gamma_{H_\Phi} \cup \Gamma_{B_\Phi} \cup \Gamma_{\Phi,\Psi}} \mu N_k \nabla \widetilde{\Phi} \cdot \boldsymbol{n}_\Phi \, \mathrm{d}\Gamma \\
&- \int_{\Omega_\Psi} \mu_o \nabla N_k \cdot \nabla \widetilde{\Psi} \, \mathrm{d}\Omega + \int_{\Gamma_{H_\Psi} \cup \Gamma_{B_\Psi} \cup \Gamma_{\Phi,\Psi}} \mu_o N_k \nabla \widetilde{\Psi} \cdot \boldsymbol{n}_\Psi \, \mathrm{d}\Gamma \\
&+ \int_{\Gamma_{B_\Phi}} N_k \left(b + \mu \widetilde{\boldsymbol{T}}_0 \cdot \boldsymbol{n}_\Phi - \mu \nabla \widetilde{\Phi} \cdot \boldsymbol{n}_\Phi \right) \mathrm{d}\Gamma \\
&+ \int_{\Gamma_{B_\Psi}} N_k \left(b - \mu_o \nabla \widetilde{\Psi} \cdot \boldsymbol{n}_\Psi + \widetilde{\boldsymbol{R}} \cdot \boldsymbol{n}_\Psi \right) \mathrm{d}\Gamma \\
&+ \int_{\Gamma_{\Phi,\Psi}} N_k \left[(\mu \widetilde{\boldsymbol{T}}_0 - \mu \nabla \widetilde{\Phi}) \cdot \boldsymbol{n}_\Phi + (-\mu_o \nabla \widetilde{\Psi} + \widetilde{\boldsymbol{R}}) \cdot \boldsymbol{n}_\Psi \right] \mathrm{d}\Gamma = \\
&- \int_{\Omega_\Phi} \mu \nabla N_k \cdot \widetilde{\boldsymbol{T}}_0 \, \mathrm{d}\Omega + \int_{\Gamma_{H_\Phi} \cup \Gamma_{B_\Phi} \cup \Gamma_{\Phi,\Psi}} \mu N_k \widetilde{\boldsymbol{T}}_0 \cdot \boldsymbol{n}_\Phi \, \mathrm{d}\Gamma \\
&- \int_{\Omega_\Psi} \nabla N_k \cdot \widetilde{\boldsymbol{R}} \, \mathrm{d}\Omega + \int_{\Gamma_{H_\Psi} \cup \Gamma_{B_\Psi} \cup \Gamma_{\Phi,\Psi}} N_k \widetilde{\boldsymbol{R}} \cdot \boldsymbol{n}_\Psi \, \mathrm{d}\Gamma.
\end{aligned}
\tag{3.57}
$$

Here the orientation of normal unit vectors is also indicated, i.e. $\boldsymbol{n}_\Phi$ is pointing outward the subregion Ω_Φ and $\boldsymbol{n}_\Psi$ is pointing outward the subregion Ω_Ψ (see *Fig.* 2.14).

The terms $\mu \widetilde{\boldsymbol{T}}_0 \cdot \boldsymbol{n}_\Phi$, $\mu \nabla \widetilde{\Phi} \cdot \boldsymbol{n}_\Phi$, $\mu_o \nabla \widetilde{\Psi} \cdot \boldsymbol{n}_\Psi$ and $\widetilde{\boldsymbol{R}} \cdot \boldsymbol{n}_\Psi$ on Γ_{B_Φ} and on Γ_{B_Ψ} are disappearing in the boundary integral terms. All the terms on $\Gamma_{\Phi,\Psi}$ have also pairs in the first, second and in the last two surface integrals. The rest parts are Γ_H type boundaries where Dirichlet type boundary condition is prescribed, that is why $N_k = 0$ can be supposed there.

Finally the weak formulation of the $\Phi - \Psi$-formulation can be written as

$$
\begin{aligned}
&\int_{\Omega_\Phi} \mu \nabla N_k \cdot \nabla \widetilde{\Phi} \, \mathrm{d}\Omega + \int_{\Omega_\Psi} \mu_o \nabla N_k \cdot \nabla \widetilde{\Psi} \, \mathrm{d}\Omega \\
&= \int_{\Omega_\Phi} \mu \nabla N_k \cdot \widetilde{\boldsymbol{T}}_0 \, \mathrm{d}\Omega + \int_{\Omega_\Psi} \nabla N_k \cdot \widetilde{\boldsymbol{R}} \, \mathrm{d}\Omega + \int_{\Gamma_{B_\Phi} \cup \Gamma_{B_\Psi}} N_k b \, \mathrm{d}\Gamma,
\end{aligned}
\tag{3.58}
$$

where $k = 1, \cdots, I$.

3.3.3   The magnetic vector potential, the A-formulation with implicit enforcement of Coulomb gauge

The weak formulation is built up by using the partial differential equation (2.223) and the Neumann type boundary conditions (2.224) and (2.227). The weak formulation is,

$$
\begin{aligned}
&\int_\Omega \boldsymbol{W}_k \cdot \left[\nabla \times (\nu_o \nabla \times \widetilde{\boldsymbol{A}}) - \nabla(\nu_o \nabla \cdot \widetilde{\boldsymbol{A}}) \right] \mathrm{d}\Omega \\
&+ \int_{\Gamma_H} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}} \right) \times \boldsymbol{n} - \boldsymbol{K} \right] \mathrm{d}\Gamma \\
&+ \int_{\Gamma_B} \nu_0 \nabla \cdot \widetilde{\boldsymbol{A}} \left(\boldsymbol{W}_k \cdot \boldsymbol{n} \right) \mathrm{d}\Gamma = \int_\Omega \boldsymbol{W}_k \cdot \boldsymbol{J}_0 \, \mathrm{d}\Omega - \int_\Omega \boldsymbol{W}_k \cdot (\nabla \times \widetilde{\boldsymbol{I}}) \, \mathrm{d}\Omega,
\end{aligned}
\tag{3.59}
$$

where

$$
\boldsymbol{n} \times \boldsymbol{W}_k = \boldsymbol{0}, \quad \text{on} \quad \Gamma_B,
\tag{3.60}
$$

and

$$
\boldsymbol{W}_k \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_H,
\tag{3.61}
$$

where $k = 1, \cdots, J$. The second order derivatives in the first integral can be reduced to first order one and the last integral can be reformulated by using the following identities:

$$
\nabla \cdot (\varphi \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \varphi + \varphi \nabla \cdot \boldsymbol{v},
\tag{3.62}
$$

and

$$
\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \times \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \times \boldsymbol{v},
\tag{3.63}
$$

with the notations $\varphi = \nu_o \nabla \widetilde{\boldsymbol{A}}$, $\boldsymbol{v} = \boldsymbol{W}_k$, $\boldsymbol{u} = \nu_o \nabla \times \widetilde{\boldsymbol{A}}$ and $\boldsymbol{u} = \widetilde{\boldsymbol{I}}$ and the Gauss' theorem, i.e.

$$
\begin{aligned}
&\int_\Omega \left[\nu_o \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}} \right) + \nu_o \nabla \cdot \boldsymbol{W}_k \, \nabla \cdot \widetilde{\boldsymbol{A}} \right] \mathrm{d}\Omega \\
&+ \int_{\Gamma_H \cup \Gamma_B} \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{W}_k \right] \cdot \boldsymbol{n} \, \mathrm{d}\Gamma - \int_{\Gamma_H \cup \Gamma_B} \nu_o \nabla \cdot \widetilde{\boldsymbol{A}} \left(\boldsymbol{W}_k \cdot \boldsymbol{n} \right) \mathrm{d}\Gamma \\
&+ \int_{\Gamma_H} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}} \right) \times \boldsymbol{n} - \boldsymbol{K} \right] \mathrm{d}\Gamma + \int_{\Gamma_B} \nu_o \nabla \cdot \widetilde{\boldsymbol{A}} \left(\boldsymbol{W}_k \cdot \boldsymbol{n} \right) \mathrm{d}\Gamma \\
&= \int_\Omega \boldsymbol{W}_k \cdot \boldsymbol{J}_0 \, \mathrm{d}\Omega - \int_\Omega (\nabla \times \boldsymbol{W}_k) \cdot \widetilde{\boldsymbol{I}} \, \mathrm{d}\Omega - \int_{\Gamma_H \cup \Gamma_B} (\widetilde{\boldsymbol{I}} \times \boldsymbol{W}_k) \cdot \boldsymbol{n} \, \mathrm{d}\Gamma.
\end{aligned}
\tag{3.64}
$$

The first surface integral can be reformulated according to

$$
\begin{aligned}
\left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{W}_k \right] \cdot \boldsymbol{n} &= \left[\boldsymbol{n} \times \left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} \right) \right] \cdot \boldsymbol{W}_k \\
&= -\boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{n} \right].
\end{aligned}
\tag{3.65}
$$

The last integral can be rearranged by using the identity

$$-(\widetilde{\boldsymbol{I}} \times \boldsymbol{W}_k) \cdot \boldsymbol{n} = -(\boldsymbol{n} \times \widetilde{\boldsymbol{I}}) \cdot \boldsymbol{W}_k = \boldsymbol{W}_k \cdot (\widetilde{\boldsymbol{I}} \times \boldsymbol{n}), \tag{3.66}$$

i.e.

$$\begin{aligned}
&\int_{\Omega} \left[\nu_o \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}} \right) + \nu_o \nabla \cdot \boldsymbol{W}_k \, \nabla \cdot \widetilde{\boldsymbol{A}} \right] \mathrm{d}\Omega \\
&- \int_{\Gamma_H \cup \Gamma_B} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{n} \right] \mathrm{d}\Gamma - \int_{\Gamma_H \cup \Gamma_B} \nu_o \nabla \cdot \widetilde{\boldsymbol{A}} \left(\boldsymbol{W}_k \cdot \boldsymbol{n} \right) \mathrm{d}\Gamma \\
&+ \int_{\Gamma_H} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}} \right) \times \boldsymbol{n} - \boldsymbol{K} \right] \mathrm{d}\Gamma + \int_{\Gamma_B} \nu_o \nabla \cdot \widetilde{\boldsymbol{A}} \left(\boldsymbol{W}_k \cdot \boldsymbol{n} \right) \mathrm{d}\Gamma \\
&= \int_{\Omega} \boldsymbol{W}_k \cdot \boldsymbol{J}_0 \, \mathrm{d}\Omega - \int_{\Omega} (\nabla \times \boldsymbol{W}_k) \cdot \widetilde{\boldsymbol{I}} \, \mathrm{d}\Omega + \int_{\Gamma_H \cup \Gamma_B} \boldsymbol{W}_k \cdot (\widetilde{\boldsymbol{I}} \times \boldsymbol{n}) \, \mathrm{d}\Gamma.
\end{aligned} \tag{3.67}$$

Now, it is easy to see that the first and the last surface integrals are vanishing on Γ_H. On the rest Γ_B, the Dirichlet boundary condition (2.226) must be satisfied, i.e. $\boldsymbol{n} \times \boldsymbol{W}_k = \boldsymbol{0}$ and

$$\begin{aligned}
\left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{W}_k \right] \cdot \boldsymbol{n} &= [\boldsymbol{W}_k \times \boldsymbol{n}] \cdot \left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} \right) \\
&= -[\boldsymbol{n} \times \boldsymbol{W}_k] \cdot \left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} \right).
\end{aligned} \tag{3.68}$$

This is the reason why the first and the last boundary integral terms can be eliminated. It is evident that the second boundary integral is vanishing on Γ_B (because of the fourth boundary integral term) and it is equal to zero on the rest part of the boundary because of the Dirichlet boundary condition (2.225), so $\boldsymbol{W}_k \cdot \boldsymbol{n} = 0$ on Γ_H.

Finally, the following weak formulation can be given to the $\boldsymbol{A}$-formulation appended by the implicit Coulomb gauge enforcement:

$$\begin{aligned}
&\int_{\Omega} \left[\nu_o \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}} \right) + \nu_o \nabla \cdot \boldsymbol{W}_k \, \nabla \cdot \widetilde{\boldsymbol{A}} \right] \mathrm{d}\Omega \\
&= \int_{\Omega} \boldsymbol{W}_k \cdot \boldsymbol{J}_0 \, \mathrm{d}\Omega + \int_{\Gamma_H} \boldsymbol{W}_k \cdot \boldsymbol{K} \, \mathrm{d}\Gamma - \int_{\Omega} (\nabla \times \boldsymbol{W}_k) \cdot \widetilde{\boldsymbol{I}} \, \mathrm{d}\Omega,
\end{aligned} \tag{3.69}$$

where $k = 1, \cdots, J$.

3.3.4 The magnetic vector potential, the A-formulation with a numerical technique, which is not sensitive to Coulomb gauge

The partial differential equation (2.228) and the Neumann boundary condition (2.229) can be summarized in

$$\begin{aligned}
&\int_{\Omega} \boldsymbol{W}_k \cdot \left[\nabla \times (\nu_o \nabla \times \widetilde{\boldsymbol{A}}) \right] \mathrm{d}\Omega \\
&+ \int_{\Gamma_H} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}} \right) \times \boldsymbol{n} - \boldsymbol{K} \right] \mathrm{d}\Gamma \\
&= \int_{\Omega} \boldsymbol{W}_k \cdot \left(\nabla \times \widetilde{\boldsymbol{T}}_0 \right) \mathrm{d}\Omega - \int_{\Omega} \boldsymbol{W}_k \cdot \left(\nabla \times \widetilde{\boldsymbol{I}} \right) \mathrm{d}\Omega, \quad k = 1, \cdots, J.
\end{aligned} \tag{3.70}$$

Here

$$n \times W_k = 0, \quad \text{on} \quad \Gamma_B. \tag{3.71}$$

The second order derivative in the first integral can be reduced to first order one by using the identity

$$\nabla \cdot (u \times v) = v \cdot \nabla \times u - u \cdot \nabla \times v, \tag{3.72}$$

with the notations $v = W_k$ and $u = \nu \nabla \times \widetilde{A}$. The right-hand side can also be simplified by applying the same identity with the notations $v = W_k$, $u = \widetilde{T}_0$ and $u = \widetilde{I}$. After using Gauss' theorem the following equation can be obtained:

$$\int_\Omega \nu_o (\nabla \times W_k) \cdot \left(\nabla \times \widetilde{A}\right) d\Omega + \int_{\Gamma_H \cup \Gamma_B} \left[\left(\nu_o \nabla \times \widetilde{A}\right) \times W_k\right] \cdot n \, d\Gamma$$
$$+ \int_{\Gamma_H} W_k \cdot \left[\left(\nu_o \nabla \times \widetilde{A} + \widetilde{I}\right) \times n - K\right] d\Gamma$$
$$= \int_\Omega (\nabla \times W_k) \cdot \widetilde{T}_0 \, d\Omega + \int_{\Gamma_H \cup \Gamma_B} \left(\widetilde{T}_0 \times W_k\right) \cdot n \, d\Gamma \tag{3.73}$$
$$- \int_\Omega (\nabla \times W_k) \cdot \widetilde{I} \, d\Omega$$
$$- \int_{\Gamma_H \cup \Gamma_B} \left(\widetilde{I} \times W_k\right) \cdot n \, d\Gamma.$$

The first boundary integral can be eliminated by the procedure applied in equations above (3.65)–(3.68). The integrand of the boundary integral term remained on the right-hand side can be rewritten in the following forms:

$$\left(\widetilde{T}_0 \times W_k\right) \cdot n = \left(n \times \widetilde{T}_0\right) \cdot W_k = (W_k \times n) \cdot \widetilde{T}_0 = -(n \times W_k) \cdot \widetilde{T}_0. \tag{3.74}$$

The second one is equal to zero on Γ_H, because the tangential component of the impressed current vector potential is vanishing there, the last one is useful on the rest Γ_B of the boundary, because of the Dirichlet type boundary condition (2.230). As a consequence of these manipulations, the boundary integral terms can be eliminated.

Finally, the weak formulation of the A-formulation, which must be realized by a numerical technique, which is not sensitive to Coulomb gauge is as follows:

$$\int_\Omega \nu_o (\nabla \times W_k) \cdot \left(\nabla \times \widetilde{A}\right) d\Omega = \int_\Omega (\nabla \times W_k) \cdot \widetilde{T}_0 \, d\Omega$$
$$+ \int_{\Gamma_H} W_k \cdot K \, d\Gamma - \int_\Omega (\nabla \times W_k) \cdot \widetilde{I} \, d\Omega, \quad k = 1, \cdots, J. \tag{3.75}$$

It is possible to select the impressed current vector potential $\widetilde{T}_0$ such that it satisfies the Dirichlet boundary condition

$$\widetilde{T}_0 \times n = K, \quad \text{on} \quad \Gamma_H. \tag{3.76}$$

In this case the according boundary integral term on the right-hand side on the boundary part Γ_H can be reformulated as

$$\int_{\Gamma_H} (\widetilde{\boldsymbol{T}}_0 \times \boldsymbol{W}_k) \cdot \boldsymbol{n} \, \mathrm{d}\Gamma = \int_{\Gamma_H} (\boldsymbol{n} \times \widetilde{\boldsymbol{T}}_0) \cdot \boldsymbol{W}_k \, \mathrm{d}\Gamma =$$
$$-\int_{\Gamma_H} (\widetilde{\boldsymbol{T}}_0 \times \boldsymbol{n}) \cdot \boldsymbol{W}_k \, \mathrm{d}\Gamma = -\int_{\Gamma_H} \boldsymbol{K} \cdot \boldsymbol{W}_k \, \mathrm{d}\Gamma, \tag{3.77}$$

i.e. the weak formulation of the ungauged $\boldsymbol{A}$-formulation can be written as,

$$\int_\Omega \nu_o (\nabla \times \boldsymbol{W}_k) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}}\right) \mathrm{d}\Omega = \int_\Omega (\nabla \times \boldsymbol{W}_k) \cdot \widetilde{\boldsymbol{T}}_0 \, \mathrm{d}\Omega$$
$$-\int_\Omega (\nabla \times \boldsymbol{W}_k) \cdot \widetilde{\boldsymbol{I}} \, \mathrm{d}\Omega, \tag{3.78}$$

where $k = 1, \cdots, J$.

3.3.5 Combination of the magnetic vector potential and the magnetic scalar potential, the $A - \Phi$-formulation

There are two unknown potentials in this formulation, the *magnetic vector potential $\boldsymbol{A}$* in the iron region and the *reduced magnetic scalar potential* Φ in the air region, that is why two equations must be realized. The first is coming from the region $\Omega_{\boldsymbol{A}}$, the second is from the region Ω_Φ, but the two equations are coupled through the interface $\Gamma_{\boldsymbol{A},\Phi}$. It is important to note that the resulting system of equations should be symmetric.

The first weak formulation is based on the partial differential equation (2.231), on the Neumann boundary conditions (2.233), (2.236) and on the vector type interface condition (2.239). The integral equation is very similar to (3.59),

$$\int_{\Omega_{\boldsymbol{A}}} \boldsymbol{W}_k \cdot \left[\nabla \times (\nu_o \nabla \times \widetilde{\boldsymbol{A}}) - \nabla(\nu_o \nabla \cdot \widetilde{\boldsymbol{A}}) \right] \mathrm{d}\Omega$$
$$+ \int_{\Gamma_{H_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}}\right) \times \boldsymbol{n} - \boldsymbol{K} \right] \mathrm{d}\Gamma + \int_{\Gamma_{B_{\boldsymbol{A}}}} \nu_o \nabla \cdot \widetilde{\boldsymbol{A}} \, (\boldsymbol{W}_k \cdot \boldsymbol{n}) \, \mathrm{d}\Gamma$$
$$+ \int_{\Gamma_{\boldsymbol{A},\Phi}} \boldsymbol{W}_k \cdot \left[(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}}) \times \boldsymbol{n}_{\boldsymbol{A}} + \left(\widetilde{\boldsymbol{T}}_0 - \nabla\widetilde{\Phi}\right) \times \boldsymbol{n}_\Phi \right] \mathrm{d}\Gamma = \tag{3.79}$$
$$- \int_{\Omega_{\boldsymbol{A}}} \boldsymbol{W}_k \cdot \left(\nabla \times \widetilde{\boldsymbol{I}}\right) \mathrm{d}\Omega, \quad k = 1, \cdots, J,$$

where

$$\boldsymbol{n} \times \boldsymbol{W}_k = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{B_{\boldsymbol{A}}}, \tag{3.80}$$

and

$$\boldsymbol{W}_k \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_{H_{\boldsymbol{A}}} \cup \Gamma_{\boldsymbol{A},\Phi}. \tag{3.81}$$

Obtaining the weak form of this weak formulation is very similar to the manipulations starting on page 78. After applying the identities (3.62), (3.63), (3.65) and (3.66), the following equation can be obtained:

$$
\begin{aligned}
&\int_{\Omega_{\boldsymbol{A}}} \left[\nu_o \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}} \right) + \nu_o \nabla \cdot \boldsymbol{W}_k \nabla \cdot \widetilde{\boldsymbol{A}} \right] \mathrm{d}\Omega \\
&- \int_{\Gamma_{H_{\boldsymbol{A}}} \cup \Gamma_{B_{\boldsymbol{A}}} \cup \Gamma_{\boldsymbol{A},\Phi}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{n} \right] \mathrm{d}\Gamma \\
&- \int_{\Gamma_{H_{\boldsymbol{A}}} \cup \Gamma_{B_{\boldsymbol{A}}} \cup \Gamma_{\boldsymbol{A},\Phi}} \nu_o \nabla \cdot \widetilde{\boldsymbol{A}} \left(\boldsymbol{W}_k \cdot \boldsymbol{n} \right) \mathrm{d}\Gamma \\
&+ \int_{\Gamma_{H_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}} \right) \times \boldsymbol{n} - \boldsymbol{K} \right] \mathrm{d}\Gamma \\
&+ \int_{\Gamma_{B_{\boldsymbol{A}}}} \nu_o \nabla \cdot \widetilde{\boldsymbol{A}} \left(\boldsymbol{W}_k \cdot \boldsymbol{n} \right) \mathrm{d}\Gamma \\
&+ \int_{\Gamma_{\boldsymbol{A},\Phi}} \boldsymbol{W}_k \cdot \left[(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}}) \times \boldsymbol{n}_{\boldsymbol{A}} + \left(\widetilde{\boldsymbol{T}}_0 - \nabla \widetilde{\Phi} \right) \times \boldsymbol{n}_{\Phi} \right] \mathrm{d}\Gamma = \\
&- \int_{\Omega_{\boldsymbol{A}}} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \widetilde{\boldsymbol{I}} \mathrm{d}\Omega + \int_{\Gamma_{H_{\boldsymbol{A}}} \cup \Gamma_{B_{\boldsymbol{A}}} \cup \Gamma_{\boldsymbol{A},\Phi}} \boldsymbol{W}_k \cdot \left(\widetilde{\boldsymbol{I}} \times \boldsymbol{n} \right) \mathrm{d}\Gamma.
\end{aligned}
\tag{3.82}
$$

The boundary of the region $\Omega_{\boldsymbol{A}}$ is $\partial \Omega_{\boldsymbol{A}} = \Gamma_{H_{\boldsymbol{A}}} \cup \Gamma_{B_{\boldsymbol{A}}} \cup \Gamma_{\boldsymbol{A},\Phi}$ (see *Fig.* 2.15). It should be noted that the normal unit vector $\boldsymbol{n}$ in the first and in the second boundary integral terms must be replaced by $\boldsymbol{n}_{\boldsymbol{A}}$ on the interface $\Gamma_{\boldsymbol{A},\Phi}$. The first boundary integral is vanishing on the part $\Gamma_{H_{\boldsymbol{A}}}$ and on $\Gamma_{\boldsymbol{A},\Phi}$ according to the third and to the last boundary integral terms on the left-hand side, moreover it is equal to zero according to the Dirichlet boundary condition (2.235) on $\Gamma_{B_{\boldsymbol{A}}}$. The second boundary integral term is vanishing on $\Gamma_{B_{\boldsymbol{A}}}$, because of the fourth boundary integral term and it is equal to zero on the rest part $\Gamma_{H_{\boldsymbol{A}}} \cup \Gamma_{\boldsymbol{A},\Phi}$ because the Dirichlet boundary conditions (2.234) and (2.241) must be satisfied there. The term in the third and in the fifth boundary integrals according to the nonlinear residual $\widetilde{\boldsymbol{I}}$ are vanishing because of the same terms in the last boundary integral. The last integral is equal to zero on $\Gamma_{B_{\boldsymbol{A}}}$ since (2.235) is prescribed there.

Finally, the following equation can be written:

$$
\begin{aligned}
&\int_{\Omega_{\boldsymbol{A}}} \left[\nu_o \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}} \right) + \nu_o \nabla \cdot \boldsymbol{W}_k \nabla \cdot \widetilde{\boldsymbol{A}} \right] \mathrm{d}\Omega \\
&- \int_{\Gamma_{\boldsymbol{A},\Phi}} \boldsymbol{W}_k \cdot \left(\nabla \widetilde{\Phi} \times \boldsymbol{n}_{\Phi} \right) \mathrm{d}\Gamma \\
&= \int_{\Gamma_{H_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \boldsymbol{K} \, \mathrm{d}\Gamma - \int_{\Gamma_{\boldsymbol{A},\Phi}} \boldsymbol{W}_k \cdot \left(\widetilde{\boldsymbol{T}}_0 \times \boldsymbol{n}_{\Phi} \right) \mathrm{d}\Gamma \\
&- \int_{\Omega_{\boldsymbol{A}}} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \widetilde{\boldsymbol{I}} \mathrm{d}\Omega.
\end{aligned}
\tag{3.83}
$$

Unfortunately, this is not the final weak form. One more modification is necessary because of symmetry, but this will be presented after the second weak form.

The partial differential equation in (2.232), the Neumann type boundary condition (2.238) and the interface condition with scalar type (2.240) build the second equation, which is similar to (3.47),

$$
\int_{\Omega_\Phi} N_k \nabla \cdot (\mu \nabla \widetilde{\Phi}) \, \mathrm{d}\Omega + \int_{\Gamma_{B_\Phi}} N_k \left(b + \mu \widetilde{\boldsymbol{T}}_0 \cdot \boldsymbol{n} - \mu \nabla \widetilde{\Phi} \cdot \boldsymbol{n} \right) \mathrm{d}\Gamma
$$
$$
+ \int_{\Gamma_{\boldsymbol{A},\Phi}} N_k \left[(\nabla \times \widetilde{\boldsymbol{A}}) \cdot \boldsymbol{n_A} + \mu \left(\widetilde{\boldsymbol{T}}_0 - \nabla \widetilde{\Phi} \right) \cdot \boldsymbol{n_\Phi} \right] \mathrm{d}\Gamma \qquad (3.84)
$$
$$
= \int_{\Omega_\Phi} N_k \nabla \cdot (\mu \widetilde{\boldsymbol{T}}_0) \, \mathrm{d}\Omega,
$$

where

$$
N_k = 0, \quad \text{on} \quad \Gamma_{H_\Phi}, \qquad (3.85)
$$

and $k = 1, \cdots, I$. Obtaining the weak form is very similar to that on page 75. After using the manipulations (3.49), (3.50) and (3.51), but supposing linear media in Ω_Φ, the following equation can be obtained:

$$
- \int_{\Omega_\Phi} \mu \nabla N_k \cdot \nabla \widetilde{\Phi} \, \mathrm{d}\Omega + \int_{\Gamma_{H_\Phi} \cup \Gamma_{B_\Phi} \cup \Gamma_{\boldsymbol{A},\Phi}} \mu N_k \nabla \widetilde{\Phi} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma
$$
$$
+ \int_{\Gamma_{B_\Phi}} N_k \left(b + \mu \widetilde{\boldsymbol{T}}_0 \cdot \boldsymbol{n} - \mu \nabla \widetilde{\Phi} \cdot \boldsymbol{n} \right) \mathrm{d}\Gamma
$$
$$
+ \int_{\Gamma_{\boldsymbol{A},\Phi}} N_k \left[(\nabla \times \widetilde{\boldsymbol{A}}) \cdot \boldsymbol{n_A} + \mu \left(\widetilde{\boldsymbol{T}}_0 - \nabla \widetilde{\Phi} \right) \cdot \boldsymbol{n_\Phi} \right] \mathrm{d}\Gamma = \qquad (3.86)
$$
$$
- \int_{\Omega_\Phi} \mu \nabla N_k \cdot \widetilde{\boldsymbol{T}}_0 \, \mathrm{d}\Omega + \int_{\Gamma_{H_\Phi} \cup \Gamma_{B_\Phi} \cup \Gamma_{\boldsymbol{A},\Phi}} \mu N_k \widetilde{\boldsymbol{T}}_0 \cdot \boldsymbol{n} \, \mathrm{d}\Gamma.
$$

The first surface integral is vanishing on $\Gamma_{B_\Phi} \cup \Gamma_{\boldsymbol{A},\Phi}$, because of the same term with opposite sign in the second and in the third surface integrals (of course $\boldsymbol{n} = \boldsymbol{n_\Phi}$ on $\Gamma_{\boldsymbol{A},\Phi}$). The last surface integral is vanishing on the same part of the boundary according to the second and the third boundary integrals. The first as well as the last surface integrals are equal to zero on the rest part Γ_{H_Φ}, because $N_k = 0$ there according to the Dirichlet boundary condition (2.237). The result is as follows:

$$
- \int_{\Omega_\Phi} \mu \nabla N_k \cdot \nabla \widetilde{\Phi} \, \mathrm{d}\Omega + \int_{\Gamma_{\boldsymbol{A},\Phi}} N_k (\nabla \times \widetilde{\boldsymbol{A}}) \cdot \boldsymbol{n_A} \mathrm{d}\Gamma =
$$
$$
- \int_{\Omega_\Phi} \mu \nabla N_k \cdot \widetilde{\boldsymbol{T}}_0 \, \mathrm{d}\Omega - \int_{\Gamma_{B_\Phi}} N_k b \, \mathrm{d}\Gamma. \qquad (3.87)
$$

It can be seen that the surface integral terms on the left-hand side of equations (3.83) and (3.87) have no got the same format, however, – from the finite element point of view – they should have. The surface integral term of (3.83) can be rewritten as

$$
- \int_{\Gamma_{\boldsymbol{A},\Phi}} \boldsymbol{W}_k \cdot \left(\nabla \widetilde{\Phi} \times \boldsymbol{n_\Phi} \right) \mathrm{d}\Gamma
$$
$$
= \int_{\Gamma_{\boldsymbol{A},\Phi}} \boldsymbol{W}_k \cdot \left(\nabla \widetilde{\Phi} \times \boldsymbol{n_A} \right) \mathrm{d}\Gamma = \int_{\Gamma_{\boldsymbol{A},\Phi}} \boldsymbol{n_A} \cdot \left(\boldsymbol{W}_k \times \nabla \widetilde{\Phi} \right) \mathrm{d}\Gamma. \qquad (3.88)
$$

The identity

$$\nabla \times (\varphi \boldsymbol{v}) = \varphi \nabla \times \boldsymbol{v} - \boldsymbol{v} \times \nabla \varphi \tag{3.89}$$

should be applied as

$$\boldsymbol{n} \cdot [\nabla \times (\varphi \boldsymbol{v})] = \varphi [\boldsymbol{n} \cdot (\nabla \times \boldsymbol{v})] - \boldsymbol{n} \cdot (\boldsymbol{v} \times \nabla \varphi) \tag{3.90}$$

with the notations $\varphi = \widetilde{\Phi}$ and $\boldsymbol{v} = \boldsymbol{W}_k$, i.e.

$$\int_{\Gamma_{\boldsymbol{A},\Phi}} \boldsymbol{n_A} \cdot \left(\boldsymbol{W}_k \times \nabla \widetilde{\Phi} \right) \mathrm{d}\Gamma = \int_{\Gamma_{\boldsymbol{A},\Phi}} \widetilde{\Phi} [\boldsymbol{n_A} \cdot (\nabla \times \boldsymbol{W}_k)] \, \mathrm{d}\Gamma$$
$$- \int_C \widetilde{\Phi} \boldsymbol{W}_k \cdot \mathrm{d}\boldsymbol{l}, \tag{3.91}$$

where curve C is bounding the interface between the $\boldsymbol{A}$-region and the Φ-region. If this surface is closed (e.g. when symmetry planes are not taken into account), the curve integral can be eliminated. Otherwise, the curve C meets with boundary type of Γ_B and/or Γ_H (*Fig.* 2.15). In the first case, $\boldsymbol{n} \times \boldsymbol{W}_k = 0$, i.e. $\boldsymbol{W}_k \cdot \mathrm{d}\boldsymbol{l} = 0$. In the second case Φ is given by (2.237), i.e. $\Phi = \Phi_0$. Finally,

$$\int_C \widetilde{\Phi} \boldsymbol{W}_k \cdot \mathrm{d}\boldsymbol{l} = \int_{C_H} \Phi_0 \boldsymbol{W}_k \cdot \mathrm{d}\boldsymbol{l}, \tag{3.92}$$

where C_H denotes the path lying on $\Gamma_{\boldsymbol{A},\Phi}$ and meeting Γ_H.

The weak form the gauged $\boldsymbol{A} - \Phi$-formulation can be collected as follows:

$$\int_{\Omega_{\boldsymbol{A}}} \left[\nu_o (\nabla \times \boldsymbol{W}_k) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}} \right) + \nu_o \nabla \cdot \boldsymbol{W}_k \nabla \cdot \widetilde{\boldsymbol{A}} \right] \mathrm{d}\Omega$$
$$+ \int_{\Gamma_{\boldsymbol{A},\Phi}} \widetilde{\Phi} [(\nabla \times \boldsymbol{W}_k) \cdot \boldsymbol{n_A}] \, \mathrm{d}\Gamma$$
$$= \int_{\Gamma_{H_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \boldsymbol{K} \, \mathrm{d}\Gamma + \int_{\Gamma_{\boldsymbol{A},\Phi}} \boldsymbol{W}_k \cdot \left(\widetilde{\boldsymbol{T}}_0 \times \boldsymbol{n_A} \right) \mathrm{d}\Gamma \tag{3.93}$$
$$+ \int_{C_H} \Phi_0 \boldsymbol{W}_k \cdot \mathrm{d}\boldsymbol{l}$$
$$- \int_{\Omega_{\boldsymbol{A}}} (\nabla \times \boldsymbol{W}_k) \cdot \widetilde{\boldsymbol{I}} \mathrm{d}\Omega,$$

where $k = 1, \cdots, J$ and

$$- \int_{\Omega_\Phi} \mu \nabla N_k \cdot \nabla \widetilde{\Phi} \, \mathrm{d}\Omega + \int_{\Gamma_{\boldsymbol{A},\Phi}} N_k (\nabla \times \widetilde{\boldsymbol{A}}) \cdot \boldsymbol{n_A} \mathrm{d}\Gamma =$$
$$- \int_{\Omega_\Phi} \mu \nabla N_k \cdot \widetilde{\boldsymbol{T}}_0 \, \mathrm{d}\Omega - \int_{\Gamma_{B_\Phi}} N_k b \, \mathrm{d}\Gamma, \tag{3.94}$$

with $k = 1, \cdots, I$.

3.3.6 The gauged $T, \Phi - \Phi$ formulation

There are two unknown potentials in this formulation, the current vector potential T in the eddy current region Ω_c and the reduced magnetic scalar potential Φ in the whole region $\Omega_c \cup \Omega_n$, that is two equations must be worked out. These are coming from the partial differential equations (2.316)–(2.318) and the boundary and interface conditions (2.319)–(2.330). It is important to note that the resulting system of equations should be symmetric.

The first weak formulation is based on the partial differential equation (2.316), on the Neumann type boundary conditions (2.321), (2.323) and on the interface condition (2.330),

$$
\int_{\Omega_c} W_k \cdot \left[\nabla \times \left(\frac{1}{\sigma} \nabla \times \widetilde{T} \right) - \nabla \left(\frac{1}{\sigma} \nabla \cdot \widetilde{T} \right) + \mu_o \frac{\partial \widetilde{T}}{\partial t} - \mu_o \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right] d\Omega
$$
$$
+ \int_{\Gamma_{H_c} \cup \Gamma_{nc}} \frac{1}{\sigma} \nabla \cdot \widetilde{T} (W_k \cdot n) \, d\Gamma + \int_{\Gamma_E} W_k \cdot \left[\left(\frac{1}{\sigma} \nabla \times \widetilde{T} \right) \times n \right] d\Gamma = \quad (3.95)
$$
$$
- \int_{\Omega_c} \mu_o W_k \cdot \frac{\partial \widetilde{T}_0}{\partial t} \, d\Omega - \int_{\Omega_c} W_k \cdot \frac{\partial \widetilde{R}}{\partial t} \, d\Omega, \quad k = 1, \cdots, J,
$$

where

$$
W_k \cdot n = 0, \quad \text{on} \quad \Gamma_E, \tag{3.96}
$$

and

$$
W_k \times n = 0, \quad \text{on} \quad \Gamma_{H_c} \cup \Gamma_{nc}. \tag{3.97}
$$

The second order derivatives in the first integral can be reduced to first order one by using the identity

$$
\nabla \cdot (u \times v) = v \cdot \nabla \times u - u \cdot \nabla \times v, \tag{3.98}
$$

with the notations $v = W_k$ and $u = 1/\sigma \nabla \times \widetilde{T}$ and the identity

$$
\nabla \cdot (\varphi v) = v \cdot \nabla \varphi + \varphi \nabla \cdot v \tag{3.99}
$$

with the notations $\varphi = 1/\sigma \nabla \cdot \widetilde{T}$ and $v = W_k$ also must be used. This results in

$$
\int_{\Omega_c} \left[\frac{1}{\sigma} (\nabla \times W_k) \cdot \left(\nabla \times \widetilde{T} \right) + \frac{1}{\sigma} \nabla \cdot W_k \nabla \cdot \widetilde{T} \right] d\Omega
$$
$$
+ \int_{\Omega_c} \left[\mu_o W_k \cdot \frac{\partial \widetilde{T}}{\partial t} - \mu_o W_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right] d\Omega
$$
$$
+ \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc}} \left\{ \left[\left(\frac{1}{\sigma} \nabla \times \widetilde{T} \right) \times W_k \right] \cdot n - \frac{1}{\sigma} \nabla \cdot \widetilde{T} (W_k \cdot n) \right\} d\Gamma \quad (3.100)
$$
$$
+ \int_{\Gamma_{H_c} \cup \Gamma_{nc}} \frac{1}{\sigma} \nabla \cdot \widetilde{T} (W_k \cdot n) \, d\Gamma + \int_{\Gamma_E} W_k \cdot \left[\left(\frac{1}{\sigma} \nabla \times \widetilde{T} \right) \times n \right] d\Gamma =
$$
$$
- \int_{\Omega_c} \mu_o W_k \cdot \frac{\partial \widetilde{T}_0}{\partial t} \, d\Omega - \int_{\Omega_c} W_k \cdot \frac{\partial \widetilde{R}}{\partial t} \, d\Omega.
$$

The first and the last boundary integral terms on the left are vanishing on the boundary part Γ_E. The first integrand of the first boundary integral term can be reformulated as

$$
\begin{aligned}
&\left[\left(\frac{1}{\sigma}\nabla \times \widetilde{\boldsymbol{T}}\right) \times \boldsymbol{W}_k\right] \cdot \boldsymbol{n} \\
&= \left[\boldsymbol{n} \times \left(\frac{1}{\sigma}\nabla \times \widetilde{\boldsymbol{T}}\right)\right] \cdot \boldsymbol{W}_k \\
&= -\boldsymbol{W}_k \cdot \left[\left(\frac{1}{\sigma}\nabla \times \widetilde{\boldsymbol{T}}\right) \times \boldsymbol{n}\right],
\end{aligned}
\tag{3.101}
$$

which is the same as the last one with opposite sign. The first term of the first boundary integral term is equal to zero on the rest part $\Gamma_{H_c} \cup \Gamma_{nc}$, because of the Dirichlet type boundary and interface conditions (2.319) and (2.328), i.e.

$$
\boldsymbol{W}_k \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{H_c} \cup \Gamma_{nc}.
\tag{3.102}
$$

The second term of the first and the second boundary integral terms are vanishing on $\Gamma_{H_c} \cup \Gamma_{nc}$ and the first one is equal to zero on the rest part Γ_E because of the Dirichlet type boundary condition (2.325) and

$$
\boldsymbol{W}_k \cdot \boldsymbol{n} = 0
\tag{3.103}
$$

there.

Finally, the first equation of the weak form is the following:

$$
\begin{aligned}
&\int_{\Omega_c} \left[\frac{1}{\sigma}\left(\nabla \times \boldsymbol{W}_k\right) \cdot \left(\nabla \times \widetilde{\boldsymbol{T}}\right) + \frac{1}{\sigma}\nabla \cdot \boldsymbol{W}_k \nabla \cdot \widetilde{\boldsymbol{T}}\right] \, \mathrm{d}\Omega \\
&+ \int_{\Omega_c} \left[\mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \boldsymbol{W}_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t}\right] \, \mathrm{d}\Omega = \\
&- \int_{\Omega_c} \mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \, \mathrm{d}\Omega \\
&- \int_{\Omega_c} \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \, \mathrm{d}\Omega,
\end{aligned}
\tag{3.104}
$$

where $k = 1, \cdots, J$.

The partial differential equations (2.317) and (2.318), moreover the Neumann type boundary conditions (2.324), (2.326), (2.329) can be summarized in the weak formulation formulation presented next.

The time derivative of these partial differential equations and the according Neumann type boundary conditions must be performed, anyway the resulting system of equations will not be symmetric. It is noted that it is useful to multiply the partial differential equations (2.317) and (2.318) by -1.

After taking the time derivative, the following form can be obtained:

$$-\int_{\Omega_c} N_k \nabla \cdot \left(\mu_o \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right) d\Omega$$

$$+ \int_{\Omega_n} N_k \nabla \cdot \left(\mu \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right) d\Omega$$

$$+ \int_{\Gamma_E} N_k \left(\mu_o \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} + \mu_o \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \nabla \frac{\partial \widetilde{\Phi}}{\partial t} + \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \right) \cdot \boldsymbol{n} \, d\Gamma$$

$$+ \int_{\Gamma_B} N_k \left(\frac{\partial b}{\partial t} + \mu \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \cdot \boldsymbol{n} - \mu \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \cdot \boldsymbol{n} \right) d\Gamma$$

$$+ \int_{\Gamma_{nc}} N_k \left(\mu \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} - \mu \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right) \cdot \boldsymbol{n}_n d\Gamma \qquad (3.105)$$

$$+ \int_{\Gamma_{nc}} N_k \left(\mu_o \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} + \mu_o \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \nabla \frac{\partial \widetilde{\Phi}}{\partial t} + \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \right) \cdot \boldsymbol{n}_c d\Gamma$$

$$= \int_{\Omega_c} N_k \nabla \cdot \left(\mu_o \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \right) d\Omega$$

$$+ \int_{\Omega_n} N_k \nabla \cdot \left(\mu \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \right) d\Omega$$

$$+ \int_{\Omega_c} N_k \nabla \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \, d\Omega,$$

where

$$N_k = 0, \quad \text{on} \quad \Gamma_{H_c} \cup \Gamma_{H_n}, \qquad (3.106)$$

and

$$k = 1, \cdots, I. \qquad (3.107)$$

The first, the second and the last three integral terms can be reformulated by the use of the identity

$$\nabla \cdot (\varphi \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \varphi + \varphi \nabla \cdot \boldsymbol{v} \qquad (3.108)$$

with the notations $\varphi = N_k$ and $\boldsymbol{v} = \mu_o \partial \widetilde{\boldsymbol{T}}/\partial t$, or $\boldsymbol{v} = \mu \partial \widetilde{\boldsymbol{T}}/\partial t$, $\boldsymbol{v} = \mu_o \nabla \partial \widetilde{\Phi}/\partial t$, or $\boldsymbol{v} = \mu \nabla \partial \widetilde{\Phi}/\partial t$ and $\boldsymbol{v} = \mu_o \partial \widetilde{\boldsymbol{T}}_0/\partial t$, or $\boldsymbol{v} = \mu \partial \widetilde{\boldsymbol{T}}_0/\partial t$ and $\boldsymbol{v} = \partial \widetilde{\boldsymbol{R}}/\partial t$.

The above manipulations result in the following equation:

$$
\int_{\Omega_c} \mu_o \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} \mathrm{d}\Omega - \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc}} N_k \, \mu_o \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma
$$

$$
- \int_{\Omega_c} \mu_o \nabla N_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \mathrm{d}\Omega - \int_{\Omega_n} \mu \nabla N_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \mathrm{d}\Omega
$$

$$
+ \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc}} N_k \, \mu_o \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_B \cup \Gamma_{H_n} \cup \Gamma_{nc}} N_k \, \mu \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_E} N_k \left(\mu_o \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} + \mu_o \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \nabla \frac{\partial \widetilde{\Phi}}{\partial t} + \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \right) \cdot \boldsymbol{n} \, \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_B} N_k \left(\frac{\partial b}{\partial t} + \mu \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \cdot \boldsymbol{n} - \mu \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \cdot \boldsymbol{n} \right) \mathrm{d}\Gamma \qquad (3.109)
$$

$$
+ \int_{\Gamma_{nc}} N_k \left(\mu \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} - \mu \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right) \cdot \boldsymbol{n}_n \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_{nc}} N_k \left(\mu_o \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} + \mu_o \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \nabla \frac{\partial \widetilde{\Phi}}{\partial t} + \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \right) \cdot \boldsymbol{n}_c \, \mathrm{d}\Gamma =
$$

$$
- \int_{\Omega_c} \mu_o \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \mathrm{d}\Omega - \int_{\Omega_n} \mu \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \mathrm{d}\Omega
$$

$$
+ \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc}} N_k \, \mu_o \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma + \int_{\Gamma_B \cup \Gamma_{H_n} \cup \Gamma_{nc}} N_k \, \mu \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma
$$

$$
- \int_{\Omega_c} \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \mathrm{d}\Omega + \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc}} N_k \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma.
$$

It must be noted that the boundary of the region Ω_c is $\partial \Omega_c = \Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc}$ and the boundary of the region Ω_n is $\partial \Omega_n = \Gamma_B \cup \Gamma_{H_n} \cup \Gamma_{nc}$.

The fourth boundary integral term defined on Γ_E is vanishing, because of the same terms with opposite sign in the eighth, first, second and the last boundary integral terms. The sixth and seventh boundary terms defined on Γ_{nc} are vanishing, too, because of the same terms with opposite sign in the ninth, the third, moreover the eighth, the first, the second and the last boundary integral terms. The first and the second boundary integral terms are vanishing on the rest Γ_{H_c}, because of the Dirichlet type boundary conditions (2.320), i.e. $N_k = 0$ on Γ_{H_c}. The last two terms in the fifth boundary integral, defined on Γ_B, are vanishing according to the terms in the third and in the ninth boundary integrals. The third one is equal to zero on Γ_{H_n} because of (2.320). The last three integrals are equal to zero on the rest part Γ_{H_c} and on Γ_{H_n} as well.

Finally, the following weak formulation can be obtained:

$$
\int_{\Omega_c} \mu_o \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} d\Omega - \int_{\Omega_c} \mu_o \nabla N_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} d\Omega - \int_{\Omega_n} \mu \nabla N_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} d\Omega =
$$

$$
- \int_{\Omega_c} \mu_o \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} d\Omega - \int_{\Omega_n} \mu \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} d\Omega \tag{3.110}
$$

$$
- \int_{\Gamma_B} N_k \frac{\partial b}{\partial t} d\Gamma - \int_{\Omega_c} \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} d\Omega.
$$

Finally, the weak formulation of the $\boldsymbol{T}, \Phi - \Phi$ formulation satisfying Coulomb gauge is the following:

$$
\int_{\Omega_c} \left[\frac{1}{\sigma} (\nabla \times \boldsymbol{W}_k) \cdot \left(\nabla \times \widetilde{\boldsymbol{T}} \right) + \frac{1}{\sigma} \nabla \cdot \boldsymbol{W}_k \nabla \cdot \widetilde{\boldsymbol{T}} \right] d\Omega
$$

$$
+ \int_{\Omega_c} \left[\mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \boldsymbol{W}_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right] d\Omega = \tag{3.111}
$$

$$
- \int_{\Omega_c} \mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} d\Omega - \int_{\Omega_c} \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} d\Omega, \quad k = 1, \cdots, J,
$$

$$
- \int_{\Omega_c} \mu_o \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} d\Omega + \int_{\Omega_c} \mu_o \nabla N_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} d\Omega + \int_{\Omega_n} \mu \nabla N_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} d\Omega
$$

$$
= \int_{\Omega_c} \mu_o \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} d\Omega + \int_{\Omega_n} \mu \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} d\Omega \tag{3.112}
$$

$$
+ \int_{\Gamma_B} N_k \frac{\partial b}{\partial t} d\Gamma + \int_{\Omega_c} \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} d\Omega, \quad k = 1, \cdots, I.
$$

3.3.7 The ungauged $\boldsymbol{T}, \Phi - \Phi$ formulation

There are two unknown potentials in this formulation, too, the current vector potential $\boldsymbol{T}$ in the eddy current region Ω_c and the reduced magnetic scalar potential Φ in the whole region $\Omega_c \cup \Omega_n$, that is two equations must be realized, coming from the partial differential equations (2.331)–(2.333) and the boundary and interface conditions (2.334)–(2.342).

The first weak formulation is based on the partial differential equation (2.331) and on the Neumann boundary condition (2.337),

$$
\int_{\Omega_c} \boldsymbol{W}_k \cdot \left[\nabla \times \left(\frac{1}{\sigma} \nabla \times \widetilde{\boldsymbol{T}} \right) + \mu_o \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right] d\Omega
$$

$$
+ \int_{\Gamma_E} \boldsymbol{W}_k \cdot \left[\left(\frac{1}{\sigma} \nabla \times \widetilde{\boldsymbol{T}} \right) \times \boldsymbol{n} \right] d\Gamma = - \int_{\Omega_c} \mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} d\Omega \tag{3.113}
$$

$$
- \int_{\Omega_c} \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} d\Omega.
$$

Here

$$\boldsymbol{W}_k \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{H_c} \cup \Gamma_{nc}, \tag{3.114}$$

and $k = 1, \cdots, J$. The second order derivatives in the first integral can be reduced to first order one by using the identity

$$\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \times \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \times \boldsymbol{v}, \tag{3.115}$$

with the notations $\boldsymbol{v} = \boldsymbol{W}_k$ and $\boldsymbol{u} = 1/\sigma \nabla \times \widetilde{\boldsymbol{T}}$,

$$
\begin{aligned}
&\int_{\Omega_c} \left[\frac{1}{\sigma} (\nabla \times \boldsymbol{W}_k) \cdot \left(\nabla \times \widetilde{\boldsymbol{T}} \right) + \mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \boldsymbol{W}_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right] \mathrm{d}\Omega \\
&+ \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc}} \left[\left(\frac{1}{\sigma} \nabla \times \widetilde{\boldsymbol{T}} \right) \times \boldsymbol{W}_k \right] \cdot \boldsymbol{n} \, \mathrm{d}\Gamma \\
&+ \int_{\Gamma_E} \boldsymbol{W}_k \cdot \left[\left(\frac{1}{\sigma} \nabla \times \widetilde{\boldsymbol{T}} \right) \times \boldsymbol{n} \right] \mathrm{d}\Gamma = \\
&- \int_{\Omega_c} \mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \, \mathrm{d}\Omega - \int_{\Omega_c} \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \, \mathrm{d}\Omega.
\end{aligned}
\tag{3.116}
$$

The first and the second boundary integral terms are vanishing on the boundary part Γ_E, the integrand of the first boundary integral term can be reformulated as

$$
\begin{aligned}
&\left[\left(\frac{1}{\sigma} \nabla \times \widetilde{\boldsymbol{T}} \right) \times \boldsymbol{W}_k \right] \cdot \boldsymbol{n} \\
&= \left[\boldsymbol{n} \times \left(\frac{1}{\sigma} \nabla \times \widetilde{\boldsymbol{T}} \right) \right] \cdot \boldsymbol{W}_k \\
&= - \boldsymbol{W}_k \cdot \left[\left(\frac{1}{\sigma} \nabla \times \widetilde{\boldsymbol{T}} \right) \times \boldsymbol{n} \right],
\end{aligned}
\tag{3.117}
$$

which is the same as the second one with opposite sign. The first boundary integral term is equal to zero on the rest part $\Gamma_{H_c} \cup \Gamma_{nc}$ because of the Dirichlet type boundary and interface conditions (2.334) and (2.341), i.e. $\boldsymbol{W}_k \times \boldsymbol{n} = \boldsymbol{0}$ on $\Gamma_{H_c} \cup \Gamma_{nc}$.

Finally, the first equation of the weak form is the following:

$$
\begin{aligned}
&\int_{\Omega_c} \left[\frac{1}{\sigma} (\nabla \times \boldsymbol{W}_k) \cdot \left(\nabla \times \widetilde{\boldsymbol{T}} \right) + \mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \boldsymbol{W}_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right] \mathrm{d}\Omega = \\
&- \int_{\Omega_c} \mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \, \mathrm{d}\Omega - \int_{\Omega_c} \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \, \mathrm{d}\Omega, \quad k = 1, \cdots, J.
\end{aligned}
\tag{3.118}
$$

Fortunately, the weak formulation of the partial differential equations (2.332) and (2.333) and the Neumann type boundary conditions (2.338), (2.339), (2.342) is the same as presented in the last point H.

The weak formulation of the ungauged $\boldsymbol{T}, \Phi - \Phi$ formulation is the following:

$$\int_{\Omega_c} \left[\frac{1}{\sigma} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{T}} \right) + \mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \boldsymbol{W}_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right] \mathrm{d}\Omega =$$

$$- \int_{\Omega_c} \mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \, \mathrm{d}\Omega \qquad\qquad (3.119)$$

$$- \int_{\Omega_c} \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \, \mathrm{d}\Omega,$$

and

$$- \int_{\Omega_c} \mu_o \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} \mathrm{d}\Omega + \int_{\Omega_c} \mu_o \nabla N_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \mathrm{d}\Omega$$

$$+ \int_{\Omega_n} \mu \nabla N_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \mathrm{d}\Omega$$

$$= \int_{\Omega_c} \mu_o \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \mathrm{d}\Omega + \int_{\Omega_n} \mu \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \mathrm{d}\Omega \qquad (3.120)$$

$$+ \int_{\Gamma_B} N_k \frac{\partial b}{\partial t} \, \mathrm{d}\Gamma + \int_{\Omega_c} \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \, \mathrm{d}\Omega.$$

This weak formulation must be realized by a numerical technique, which is not sensitive to Coulomb gauge. Here $k = 1, \cdots, J$ and $k = 1, \cdots, I$, respectively.

3.3.8 The gauged $A, V - A$ formulation

There are two unknown potentials in this formulation, the magnetic vector potential $\boldsymbol{A}$ in the entire problem region $\Omega_c \cup \Omega_n$ and the electric scalar potential V defined only in the eddy current region Ω_c, consequently two equations are needed. These are coming from the partial differential equations, the boundary and interface conditions given by (2.344)–(2.361). It is important to note that the resulting system of equations should be symmetric, we are going to take care about it.

First of all, *electric scalar potential V* is replaced by the function $v = v(\boldsymbol{r}, t)$ defined by

$$v = \int_{-\infty}^{t} V(\tau) \, \mathrm{d}\tau, \qquad\qquad (3.121)$$

from which

$$V = \frac{\partial v}{\partial t}. \qquad\qquad (3.122)$$

The resulting system of equations will be symmetric by this modification, and this function is approximated by $\widetilde{v}$.

The first weak formulation is based on the partial differential equations (2.344) and (2.346), on the Neumann type boundary conditions (2.347), (2.350), (2.354), (2.356) and on the interface conditions (2.359), (2.361),

$$
\begin{aligned}
&\int_{\Omega_c} \boldsymbol{W}_k \cdot \left[\nabla \times \left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} \right) - \nabla \left(\nu_o \nabla \cdot \widetilde{\boldsymbol{A}} \right) \right] \mathrm{d}\Omega \\
&+ \int_{\Omega_c} \boldsymbol{W}_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \mathrm{d}\Omega \\
&+ \int_{\Omega_n} \boldsymbol{W}_k \cdot \left[\nabla \times \left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) - \nabla \left(\nu \nabla \cdot \widetilde{\boldsymbol{A}} \right) \right] \mathrm{d}\Omega \\
&+ \int_{\Gamma_E} \nu_o \nabla \cdot \widetilde{\boldsymbol{A}} (\boldsymbol{W}_k \cdot \boldsymbol{n}) \, \mathrm{d}\Gamma + \int_{\Gamma_B} \nu \nabla \cdot \widetilde{\boldsymbol{A}} (\boldsymbol{W}_k \cdot \boldsymbol{n}) \, \mathrm{d}\Gamma \\
&+ \int_{\Gamma_{H_c}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}} \right) \times \boldsymbol{n} \right] \mathrm{d}\Gamma \qquad (3.123) \\
&+ \int_{\Gamma_{H_n}} \boldsymbol{W}_k \cdot \left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{n} - \boldsymbol{K} \right] \mathrm{d}\Gamma \\
&+ \int_{\Gamma_{nc}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}} \right) \times \boldsymbol{n}_c + \left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{n}_n \right] \mathrm{d}\Gamma \\
&+ \int_{\Gamma_{nc}} \left[\nu \nabla \cdot \widetilde{\boldsymbol{A}} (\boldsymbol{W}_k \cdot \boldsymbol{n}_c) + \nu \nabla \cdot \widetilde{\boldsymbol{A}} (\boldsymbol{W}_k \cdot \boldsymbol{n}_n) \right] \mathrm{d}\Gamma = \int_{\Omega_n} \boldsymbol{W}_k \cdot \boldsymbol{J}_0 \, \mathrm{d}\Omega \\
&- \int_{\Omega_c} \boldsymbol{W}_k \cdot (\nabla \times \widetilde{\boldsymbol{I}}) \, \mathrm{d}\Omega,
\end{aligned}
$$

where

$$
\boldsymbol{n} \times \boldsymbol{W}_k = \boldsymbol{0}, \quad \text{on} \quad \Gamma_E \cup \Gamma_B, \tag{3.124}
$$

and

$$
\boldsymbol{W}_k \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_{H_c} \cup \Gamma_{H_n}, \tag{3.125}
$$

and here $k = 1, \cdots, J$. The second order derivatives in the first and in the third integrals can be reduced to first order one by using the identity

$$
\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \times \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \times \boldsymbol{v}, \tag{3.126}
$$

with the notations $\boldsymbol{v} = \boldsymbol{W}_k$ and $\boldsymbol{u} = \nu_o \nabla \times \widetilde{\boldsymbol{A}}$, or $\boldsymbol{u} = \nu \nabla \times \widetilde{\boldsymbol{A}}$ and the identity

$$
\nabla \cdot (\varphi \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \varphi + \varphi \nabla \cdot \boldsymbol{v} \tag{3.127}
$$

with the notations $\varphi = \nu_o \nabla \cdot \widetilde{\boldsymbol{A}}$, or $\varphi = \nu \nabla \cdot \widetilde{\boldsymbol{A}}$ and $\boldsymbol{v} = \boldsymbol{W}_k$ also must be used. The last integral on the right-hand side can also be rewritten by $\boldsymbol{v} = \boldsymbol{W}_k$ and $\boldsymbol{u} = \widetilde{\boldsymbol{I}}$. This results in

$$\int_{\Omega_c} \left[\nu_o \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}} \right) + \nu_o \nabla \cdot \boldsymbol{W}_k \nabla \cdot \widetilde{\boldsymbol{A}} \right] d\Omega$$

$$+ \int_{\Omega_c} \boldsymbol{W}_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) d\Omega$$

$$+ \int_{\Omega_n} \left[\nu \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}} \right) + \nu \nabla \cdot \boldsymbol{W}_k \nabla \cdot \widetilde{\boldsymbol{A}} \right] d\Omega$$

$$+ \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc}} \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{W}_k \right] \cdot \boldsymbol{n} \, d\Gamma$$

$$+ \int_{\Gamma_B \cup \Gamma_{H_n} \cup \Gamma_{nc}} \left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{W}_k \right] \cdot \boldsymbol{n} \, d\Gamma$$

$$- \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc}} \nu_o \nabla \cdot \widetilde{\boldsymbol{A}} (\boldsymbol{W}_k \cdot \boldsymbol{n}) \, d\Gamma$$

$$- \int_{\Gamma_B \cup \Gamma_{H_n} \cup \Gamma_{nc}} \nu \nabla \cdot \widetilde{\boldsymbol{A}} (\boldsymbol{W}_k \cdot \boldsymbol{n}) \, d\Gamma \qquad (3.128)$$

$$+ \int_{\Gamma_E} \nu_o \nabla \cdot \widetilde{\boldsymbol{A}} (\boldsymbol{W}_k \cdot \boldsymbol{n}) \, d\Gamma + \int_{\Gamma_B} \nu \nabla \cdot \widetilde{\boldsymbol{A}} (\boldsymbol{W}_k \cdot \boldsymbol{n}) \, d\Gamma$$

$$+ \int_{\Gamma_{H_c}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}} \right) \times \boldsymbol{n} \right] d\Gamma$$

$$+ \int_{\Gamma_{H_n}} \boldsymbol{W}_k \cdot \left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{n} - \boldsymbol{K} \right] d\Gamma$$

$$+ \int_{\Gamma_{nc}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}} \right) \times \boldsymbol{n}_c + \left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{n}_n \right] d\Gamma$$

$$+ \int_{\Gamma_{nc}} \left[\nu_o \nabla \cdot \widetilde{\boldsymbol{A}} (\boldsymbol{W}_k \cdot \boldsymbol{n}_c) + \nu \nabla \cdot \widetilde{\boldsymbol{A}} (\boldsymbol{W}_k \cdot \boldsymbol{n}_n) \right] d\Gamma = \int_{\Omega_n} \boldsymbol{W}_k \cdot \boldsymbol{J}_0 \, d\Omega$$

$$- \int_{\Omega_c} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \widetilde{\boldsymbol{I}} \, d\Omega - \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc}} \left(\widetilde{\boldsymbol{I}} \times \boldsymbol{W}_k \right) \cdot \boldsymbol{n} \, d\Gamma.$$

The first and the second boundary integrals are vanishing on the boundary part $\Gamma_{H_c} \cup \Gamma_{nc}$ and on $\Gamma_{H_n} \cup \Gamma_{nc}$, respectively, because of the seventh, eighth, ninth and the last boundary integral terms after using the identity

$$\left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{W}_k \right] \cdot \boldsymbol{n} = \left[\boldsymbol{n} \times \left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \right] \cdot \boldsymbol{W}_k$$

$$= - \boldsymbol{W}_k \cdot \left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{n} \right]. \qquad (3.129)$$

The first and the second as well as the last boundary integral terms are equal to zero on the rest parts Γ_E and Γ_B as well, because of the Dirichlet type boundary condition (2.352) and (2.355), i.e. $\boldsymbol{W}_k \times \boldsymbol{n} = \boldsymbol{0}$ on these boundaries. The third and the fourth boundary integral terms are vanishing on Γ_E, on Γ_B and on Γ_{nc}, because of the fifth, the sixth and the last boundary integral terms on the left-hand side. On the part Γ_{H_c} and on Γ_{H_n} these integral terms are equal to zero, too, because of Dirichlet type boundary conditions (2.349) and (2.351), i.e. $\boldsymbol{W}_k \cdot \boldsymbol{n} = 0$ there. The last boundary integral term containing

the nonlinear residual term $\widetilde{\boldsymbol{I}}$ is vanishing, because of the same terms on the left-hand side on $\Gamma_{H_c} \cup \Gamma_{nc}$ and the Dirichlet type boundary condition (2.352). Here the identity

$$-(\widetilde{\boldsymbol{I}} \times \boldsymbol{W}_k) \cdot \boldsymbol{n} = -(\boldsymbol{n} \times \widetilde{\boldsymbol{I}}) \cdot \boldsymbol{W}_k = (\widetilde{\boldsymbol{I}} \times \boldsymbol{n}) \cdot \boldsymbol{W}_k \tag{3.130}$$

must be used on the right-hand side.

Finally, the first equation of the weak form is the following:

$$\int_{\Omega_c} \left[\nu_o \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}} \right) + \nu_o \nabla \cdot \boldsymbol{W}_k \nabla \cdot \widetilde{\boldsymbol{A}} \right] \mathrm{d}\Omega$$

$$+ \int_{\Omega_c} \boldsymbol{W}_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \mathrm{d}\Omega$$

$$+ \int_{\Omega_n} \left[\nu \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}} \right) + \nu \nabla \cdot \boldsymbol{W}_k \nabla \cdot \widetilde{\boldsymbol{A}} \right] \mathrm{d}\Omega \tag{3.131}$$

$$= \int_{\Omega_n} \boldsymbol{W}_k \cdot \boldsymbol{J}_0 \, \mathrm{d}\Omega + \int_{\Gamma_{H_n}} \boldsymbol{W}_k \cdot \boldsymbol{K} \, \mathrm{d}\Gamma - \int_{\Omega_c} (\nabla \times \boldsymbol{W}_k) \cdot \widetilde{\boldsymbol{I}} \, \mathrm{d}\Omega,$$

and $k = 1, \cdots, J$.

The partial differential equation (2.345), with the Neumann type boundary condition (2.348) and the interface condition (2.360) can be summarized in the formulation presented next. Here, the conditions are multiplied by -1,

$$- \int_{\Omega_c} N_k \nabla \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \mathrm{d}\Omega$$

$$+ \int_{\Gamma_{H_c} \cup \Gamma_{nc}} N_k \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \cdot \boldsymbol{n} \, \mathrm{d}\Gamma = 0, \tag{3.132}$$

where

$$N_k = 0, \quad \text{on} \quad \Gamma_E, \tag{3.133}$$

and $k = 1, \cdots, I$. The first volume integral can be reformulated by the use of the identity

$$\nabla \cdot (\varphi \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \varphi + \varphi \nabla \cdot \boldsymbol{v} \tag{3.134}$$

with the notations $\varphi = N_k$ and $\boldsymbol{v} = \sigma \partial \widetilde{\boldsymbol{A}} / \partial t + \sigma \nabla \partial \widetilde{v} / \partial t$,

$$\int_{\Omega_c} \nabla N_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \mathrm{d}\Omega$$

$$- \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc}} N_k \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \cdot \boldsymbol{n} \, \mathrm{d}\Gamma \tag{3.135}$$

$$+ \int_{\Gamma_{H_c} \cup \Gamma_{nc}} N_k \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \cdot \boldsymbol{n} \, \mathrm{d}\Gamma = 0.$$

The boundary integral terms are vanishing on $\Gamma_{H_c} \cup \Gamma_{nc}$ and the first one is equal to zero because $N_k = 0$ on Γ_E according to Dirichlet type boundary condition (2.353).

Finally, the following weak formulation can be written:

$$\int_{\Omega_c} \nabla N_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \mathrm{d}\Omega = 0, \quad k = 1, \cdots, I. \tag{3.136}$$

Finally, the weak formulation of the $\boldsymbol{A}, V - \boldsymbol{A}$ formulation satisfying Coulomb gauge is the following:

$$\int_{\Omega_c} \left[\nu_o \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}} \right) + \nu_o \nabla \cdot \boldsymbol{W}_k \, \nabla \cdot \widetilde{\boldsymbol{A}} \right] \mathrm{d}\Omega$$

$$+ \int_{\Omega_c} \boldsymbol{W}_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \mathrm{d}\Omega$$

$$+ \int_{\Omega_n} \left[\nu \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}} \right) + \nu \nabla \cdot \boldsymbol{W}_k \, \nabla \cdot \widetilde{\boldsymbol{A}} \right] \mathrm{d}\Omega \tag{3.137}$$

$$= \int_{\Omega_n} \boldsymbol{W}_k \cdot \boldsymbol{J}_0 \, \mathrm{d}\Omega + \int_{\Gamma_{H_n}} \boldsymbol{W}_k \cdot \boldsymbol{K} \, \mathrm{d}\Gamma$$

$$- \int_{\Omega_c} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \widetilde{\boldsymbol{I}} \, \mathrm{d}\Omega,$$

$$\int_{\Omega_c} \nabla N_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \mathrm{d}\Omega = 0. \tag{3.138}$$

Here $k = 1, \cdots, J$ and $k = 1, \cdots, I$, respectively.

3.3.9   The ungauged $A, V - A$ formulation

There are two unknown potentials in this formulation, the magnetic vector potential $\boldsymbol{A}$ in the entire problem region $\Omega_c \cup \Omega_n$ and the electric scalar potential V defined only in the eddy current region Ω_c, consequently two equations are needed. These are coming from the partial differential equations, the boundary and interface conditions given by (2.362)–(2.373). It is important to note that the resulting system of equations should be symmetric, we are going to take care about it. The electric scalar potential V is replaced and approximated by $v = v(\boldsymbol{r}, t)$ defined by (3.122) and $\widetilde{v}$, respectively.

The impressed current vector potential $\boldsymbol{T}_0$ has the property that

$$\nabla \times \boldsymbol{T}_0 = \begin{cases} \boldsymbol{J}_0, & \text{in} \quad \Omega_n, \\ \boldsymbol{0}, & \text{in} \quad \Omega_c, \end{cases} \tag{3.139}$$

that is why the term $\nabla \times \boldsymbol{T}_0$ can be appended to the right-hand side of (2.362). In this case it is easier to obtain the weak formulation.

The first weak formulation is based on the partial differential equations (2.362) and (2.364), on the Neumann type boundary conditions (2.365), (2.367) and on the interface condition (2.372),

$$\int_{\Omega_c} \boldsymbol{W}_k \cdot \left[\nabla \times \left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} \right) \right] \mathrm{d}\Omega + \int_{\Omega_c} \boldsymbol{W}_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \mathrm{d}\Omega$$

$$+ \int_{\Omega_n} \boldsymbol{W}_k \cdot \left[\nabla \times \left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \right] \mathrm{d}\Omega$$

$$+ \int_{\Gamma_{H_c}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}} \right) \times \boldsymbol{n} \right] \mathrm{d}\Gamma$$

$$+ \int_{\Gamma_{H_n}} \boldsymbol{W}_k \cdot \left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{n} - \boldsymbol{K} \right] \mathrm{d}\Gamma \qquad (3.140)$$

$$+ \int_{\Gamma_{nc}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}} \right) \times \boldsymbol{n}_c + \left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{n}_n \right] \mathrm{d}\Gamma$$

$$= \int_{\Omega_c \cup \Omega_n} \boldsymbol{W}_k \cdot \left(\nabla \times \widetilde{\boldsymbol{T}}_0 \right) \mathrm{d}\Omega - \int_{\Omega_c} \boldsymbol{W}_k \cdot \left(\nabla \times \widetilde{\boldsymbol{I}} \right) \mathrm{d}\Omega,$$

where $\boldsymbol{n} \times \boldsymbol{W}_k = \boldsymbol{0}$ on $\Gamma_E \cup \Gamma_B$, and $k = 1, \cdots, J$. The second order derivative in the first and in the third integrals can be reduced to first order one by using the identity

$$\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \times \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \times \boldsymbol{v}, \qquad (3.141)$$

with the notation $\boldsymbol{v} = \boldsymbol{W}_k$ and $\boldsymbol{u} = \nu_o \nabla \times \widetilde{\boldsymbol{A}}$, or $\boldsymbol{u} = \nu \nabla \times \widetilde{\boldsymbol{A}}$, moreover the integrals on the right-hand side can also be reformulated by the notation $\boldsymbol{v} = \boldsymbol{W}_k$, $\boldsymbol{u} = \widetilde{\boldsymbol{T}}_0$ and $\boldsymbol{u} = \widetilde{\boldsymbol{I}}$, finally

$$\int_{\Omega_c} \nu_o \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}} \right) \mathrm{d}\Omega + \int_{\Omega_c} \boldsymbol{W}_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \mathrm{d}\Omega$$

$$+ \int_{\Omega_n} \nu \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}} \right) \mathrm{d}\Omega$$

$$+ \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc}} \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{W}_k \right] \cdot \boldsymbol{n} \, \mathrm{d}\Gamma$$

$$+ \int_{\Gamma_B \cup \Gamma_{H_n} \cup \Gamma_{nc}} \left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{W}_k \right] \cdot \boldsymbol{n} \, \mathrm{d}\Gamma$$

$$+ \int_{\Gamma_{H_c}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}} \right) \times \boldsymbol{n} \right] \mathrm{d}\Gamma$$

$$+ \int_{\Gamma_{H_n}} \boldsymbol{W}_k \cdot \left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{n} - \boldsymbol{K} \right] \mathrm{d}\Gamma \qquad (3.142)$$

$$+ \int_{\Gamma_{nc}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}} \right) \times \boldsymbol{n}_c + \left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{n}_n \right] \mathrm{d}\Gamma$$

$$= \int_{\Omega_c \cup \Omega_n} (\nabla \times \boldsymbol{W}_k) \cdot \widetilde{\boldsymbol{T}}_0 \mathrm{d}\Omega + \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc}} (\widetilde{\boldsymbol{T}}_0 \times \boldsymbol{W}_k) \cdot \boldsymbol{n} \mathrm{d}\Gamma$$

$$+ \int_{\Gamma_B \cup \Gamma_{H_n} \cup \Gamma_{nc}} (\widetilde{\boldsymbol{T}}_0 \times \boldsymbol{W}_k) \cdot \boldsymbol{n} \mathrm{d}\Gamma$$

$$- \int_{\Omega_c} (\nabla \times \boldsymbol{W}_k) \cdot \widetilde{\boldsymbol{I}} \mathrm{d}\Omega - \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc}} (\widetilde{\boldsymbol{I}} \times \boldsymbol{W}_k) \cdot \boldsymbol{n} \mathrm{d}\Gamma$$

can be obtained. The first and the second boundary integral terms are vanishing on the boundary part $\Gamma_{H_c} \cup \Gamma_{nc}$ and on $\Gamma_{H_n} \cup \Gamma_{nc}$, respectively, because of the third, fourth and fifth boundary integral terms after using the identity

$$\left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}}\right) \times \boldsymbol{W}_k\right] \cdot \boldsymbol{n} = -\boldsymbol{W}_k \cdot \left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}}\right) \times \boldsymbol{n}\right]. \tag{3.143}$$

The first and the second boundary integral terms are equal to zero on the rest parts Γ_E and Γ_B, because of the Dirichlet type boundary condition (2.368) and (2.370), $\boldsymbol{W}_k \times \boldsymbol{n} = \boldsymbol{0}$ on these boundaries. The first and the second boundary integral terms on the right-hand side vanish on Γ_{nc}, because $\boldsymbol{n} = \boldsymbol{n}_c$ in the first and $\boldsymbol{n} = \boldsymbol{n}_n$ in the second boundary integral on the right-hand side, moreover $\boldsymbol{n}_c = -\boldsymbol{n}_n$. On the rest parts Γ_{H_c} and Γ_{H_n}, $\widetilde{\boldsymbol{T}}_0 \times \boldsymbol{n} \equiv \boldsymbol{0}$, because these are symmetry planes where the tangential component of the magnetic field intensity is equal to zero and the impressed current vector potential must satisfy this homogeneous Dirichlet type condition, moreover, on Γ_E and Γ_B, the Dirichlet type boundary condition (2.368) and (2.370) must satisfy, i.e. $\boldsymbol{W}_k \times \boldsymbol{n} = \boldsymbol{0}$ on these boundaries. The last boundary integral term on the right-hand side is vanishing, too, because of the same terms on the left-hand side and $\boldsymbol{W}_k \times \boldsymbol{n} = \boldsymbol{0}$ on Γ_E and

$$-(\widetilde{\boldsymbol{I}} \times \boldsymbol{W}_k) \cdot \boldsymbol{n} = -(\boldsymbol{n} \times \widetilde{\boldsymbol{I}}) \cdot \boldsymbol{W}_k = (\widetilde{\boldsymbol{I}} \times \boldsymbol{n}) \cdot \boldsymbol{W}_k. \tag{3.144}$$

Finally, the first equation of the weak form is the following:

$$\int_{\Omega_c} \nu_o \left(\nabla \times \boldsymbol{W}_k\right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}}\right) \mathrm{d}\Omega + \int_{\Omega_c} \boldsymbol{W}_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t}\right) \mathrm{d}\Omega$$

$$+ \int_{\Omega_n} \nu \left(\nabla \times \boldsymbol{W}_k\right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}}\right) \mathrm{d}\Omega = \int_{\Omega_c \cup \Omega_n} \left(\nabla \times \boldsymbol{W}_k\right) \cdot \widetilde{\boldsymbol{T}}_0 \, \mathrm{d}\Omega \tag{3.145}$$

$$+ \int_{\Gamma_{H_n}} \boldsymbol{W}_k \cdot \boldsymbol{K} \, \mathrm{d}\Gamma - \int_{\Omega_c} \left(\nabla \times \boldsymbol{W}_k\right) \cdot \widetilde{\boldsymbol{I}} \, \mathrm{d}\Gamma.$$

It is possible to select the impressed current vector potential $\boldsymbol{T}_0$ such that it satisfies the Dirichlet type boundary condition

$$\boldsymbol{T}_0 \times \boldsymbol{n} = \boldsymbol{K}, \quad \text{on} \quad \Gamma_{H_n}. \tag{3.146}$$

In this case the first boundary integral term on the right-hand side of (3.142) on the boundary part Γ_{H_n} can be reformulated as

$$\int_{\Gamma_{H_n}} (\widetilde{\boldsymbol{T}}_0 \times \boldsymbol{W}_k) \cdot \boldsymbol{n} \, \mathrm{d}\Gamma = \int_{\Gamma_{H_n}} (\boldsymbol{n} \times \widetilde{\boldsymbol{T}}_0) \cdot \boldsymbol{W}_k \, \mathrm{d}\Gamma =$$

$$-\int_{\Gamma_{H_n}} (\widetilde{\boldsymbol{T}}_0 \times \boldsymbol{n}) \cdot \boldsymbol{W}_k \, \mathrm{d}\Gamma = -\int_{\Gamma_{H_n}} \boldsymbol{K} \cdot \boldsymbol{W}_k \, \mathrm{d}\Gamma, \tag{3.147}$$

i.e. the boundary integral term on the right-hand side of (3.145) can be neglected.

The weak formulation of the partial differential equation (2.363), the Neumann type boundary condition (2.366) and the interface condition (2.373) can be obtained in the same way presented in the last item, see the equations (3.132)–(3.136).

Finally, the weak formulation of the ungauged $A, V - A$ formulation is the following:

$$
\int_{\Omega_c} \nu_o \left(\nabla \times \boldsymbol{W}_k\right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}}\right) \mathrm{d}\Omega + \int_{\Omega_c} \boldsymbol{W}_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t}\right) \mathrm{d}\Omega
$$
$$
+ \int_{\Omega_n} \nu \left(\nabla \times \boldsymbol{W}_k\right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}}\right) \mathrm{d}\Omega \tag{3.148}
$$
$$
= \int_{\Omega_c \cup \Omega_n} \left(\nabla \times \boldsymbol{W}_k\right) \cdot \widetilde{\boldsymbol{T}}_0 \, \mathrm{d}\Omega - \int_{\Omega_c} \left(\nabla \times \boldsymbol{W}_k\right) \cdot \widetilde{\boldsymbol{I}} \, \mathrm{d}\Gamma,
$$

$$
\int_{\Omega_c} \nabla N_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t}\right) \mathrm{d}\Omega = 0. \tag{3.149}
$$

This weak formulation must be realized by a numerical technique, which is not sensitive to Coulomb gauge, moreover $k = 1, \cdots, J$ and $k = 1, \cdots, I$, respectively.

3.3.10   The ungauged $A^\star - A$ formulation

The only one unknown potential in this formulation is the magnetic vector potential A in the entire problem region $\Omega_c \cup \Omega_n$, but it is usually denoted by $A^\star$ in the eddy current region Ω_c. The weak formulation can be obtained from the partial differential equations, the boundary and interface conditions given by (2.374)–(2.380).

The impressed current vector potential $\boldsymbol{T}_0$ has the property shown in (3.139), that is why the term $\nabla \times \boldsymbol{T}_0$ can be appended to the right-hand side of (2.374). In this case it is easier to obtain the weak formulation.

The weak formulation is based on the partial differential equations (2.374) and (2.375), on the Neumann type boundary conditions (2.376), (2.377) and on the interface condition (2.381),

$$
\int_{\Omega_c} \boldsymbol{W}_k \cdot \left[\nabla \times \left(\nu_o \nabla \times \widetilde{\boldsymbol{A}}^\star\right)\right] \mathrm{d}\Omega + \int_{\Omega_c} \sigma \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{A}}^\star}{\partial t} \mathrm{d}\Omega
$$
$$
+ \int_{\Omega_n} \boldsymbol{W}_k \cdot \left[\nabla \times \left(\nu \nabla \times \widetilde{\boldsymbol{A}}\right)\right] \mathrm{d}\Omega
$$
$$
+ \int_{\Gamma_{H_c}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}}^\star + \widetilde{\boldsymbol{I}}\right) \times \boldsymbol{n}\right] \mathrm{d}\Gamma
$$
$$
+ \int_{\Gamma_{H_n}} \boldsymbol{W}_k \cdot \left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}}\right) \times \boldsymbol{n} - \boldsymbol{K}\right] \mathrm{d}\Gamma \tag{3.150}
$$
$$
+ \int_{\Gamma_{nc}} \boldsymbol{W}_k \cdot \left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}}^\star + \widetilde{\boldsymbol{I}}\right) \times \boldsymbol{n}_c + \left(\nu \nabla \times \widetilde{\boldsymbol{A}}\right) \times \boldsymbol{n}_n\right] \mathrm{d}\Gamma
$$
$$
= \int_{\Omega_c \cup \Omega_n} \boldsymbol{W}_k \cdot \left(\nabla \times \widetilde{\boldsymbol{T}}_0\right) \mathrm{d}\Omega - \int_{\Omega_c} \boldsymbol{W}_k \cdot \left(\nabla \times \widetilde{\boldsymbol{I}}\right) \mathrm{d}\Omega.
$$

Here

$$
\boldsymbol{n} \times \boldsymbol{W}_k = \boldsymbol{0}, \quad \text{on} \quad \Gamma_E \cup \Gamma_B, \tag{3.151}
$$

and $k = 1, \cdots, J$. The first, the third and the last two integrals can be reformulated as it was presented in the last item.

Finally, the weak formulation of the ungauged $\boldsymbol{A}^\star - \boldsymbol{A}$ formulation is the following:

$$\int_{\Omega_c} \nu_o \left(\nabla \times \boldsymbol{W}_k\right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}}^\star\right) \mathrm{d}\Omega + \int_{\Omega_c} \sigma \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{A}}^\star}{\partial t} \mathrm{d}\Omega$$

$$+ \int_{\Omega_n} \nu \left(\nabla \times \boldsymbol{W}_k\right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}}\right) \mathrm{d}\Omega = \int_{\Omega_c \cup \Omega_n} \left(\nabla \times \boldsymbol{W}_k\right) \cdot \widetilde{\boldsymbol{T}}_0 \, \mathrm{d}\Omega \qquad (3.152)$$

$$- \int_{\Omega_c} \left(\nabla \times \boldsymbol{W}_k\right) \cdot \widetilde{\boldsymbol{I}} \, \mathrm{d}\Gamma,$$

and $k = 1, \cdots, J$. This weak formulation must be realized by a numerical technique, which is not sensitive to Coulomb gauge.

3.3.11   The gauged $\boldsymbol{T}, \Phi - \boldsymbol{A}$ formulation

The current vector potential $\boldsymbol{T}$ with the reduced magnetic scalar potential Φ is used in the eddy current region Ω_c and the magnetic vector potential $\boldsymbol{A}$ is used in the region free of eddy currents, Ω_n. There are three unknown potentials, $\boldsymbol{T}$, Φ and $\boldsymbol{A}$, which can be calculated by the following weak formulation obtained from the gauged version of the $\boldsymbol{T}, \Phi - \boldsymbol{A} - \Phi$ formulation (see on page 101):

$$\int_{\Omega_c} \left[\frac{1}{\sigma} \left(\nabla \times \boldsymbol{W}_k\right) \cdot \left(\nabla \times \widetilde{\boldsymbol{T}}\right) + \frac{1}{\sigma} \nabla \cdot \boldsymbol{W}_k \nabla \cdot \widetilde{\boldsymbol{T}}\right] \mathrm{d}\Omega$$

$$+ \int_{\Omega_c} \left[\mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \boldsymbol{W}_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t}\right] \mathrm{d}\Omega$$

$$- \int_{\Gamma_{nc}} \boldsymbol{W}_k \cdot \left(\frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \times \boldsymbol{n}_{\boldsymbol{A}}\right) \mathrm{d}\Gamma = \qquad\qquad (3.153)$$

$$- \int_{\Omega_c} \mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \, \mathrm{d}\Omega$$

$$- \int_{\Omega_c} \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \, \mathrm{d}\Omega, \quad k = 1, \cdots, J,$$

$$- \int_{\Omega_c} \mu_o \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} \, \mathrm{d}\Omega + \int_{\Omega_c} \mu_o \nabla N_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \, \mathrm{d}\Omega$$

$$- \int_{\Gamma_{nc}} N_k \left(\nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t}\right) \cdot \boldsymbol{n}_{\boldsymbol{A}} \mathrm{d}\Gamma$$

$$= \int_{\Omega_c} \mu_o \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \, \mathrm{d}\Omega \qquad\qquad (3.154)$$

$$+ \int_{\Omega_c} \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \, \mathrm{d}\Omega, \quad k = 1, \cdots, I,$$

$$- \int_{\Omega_n} \left[\nu \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) + \nu \nabla \cdot \boldsymbol{W}_k \, \nabla \cdot \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right] \mathrm{d}\Omega$$

$$- \int_{\Gamma_{nc}} \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} \cdot \left(\boldsymbol{W}_k \times \boldsymbol{n}_{\boldsymbol{A}} \right) \mathrm{d}\Gamma - \int_{\Gamma_{nc}} \frac{\partial \widetilde{\Phi}}{\partial t} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \boldsymbol{n}_{\boldsymbol{A}} \mathrm{d}\Gamma =$$

$$- \int_{\Omega_n} \boldsymbol{W}_k \cdot \frac{\partial \boldsymbol{J}_0}{\partial t} \, \mathrm{d}\Omega - \int_{\Gamma_{H_n}} \boldsymbol{W}_k \cdot \frac{\partial \boldsymbol{K}}{\partial t} \, \mathrm{d}\Gamma \tag{3.155}$$

$$- \int_{\Gamma_{nc}} \boldsymbol{W}_k \cdot \left(\frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \times \boldsymbol{n}_{\boldsymbol{A}} \right) \mathrm{d}\Gamma + \int_C \frac{\partial \widetilde{\Phi}}{\partial t} \boldsymbol{W}_k \cdot \mathrm{d}\boldsymbol{l}, \quad k = 1, \cdots, J.$$

3.3.12   The ungauged $T, \Phi - A$ formulation

The current vector potential $\boldsymbol{T}$ with the reduced magnetic scalar potential Φ is used in the eddy current region Ω_c and the magnetic vector potential $\boldsymbol{A}$ is used in the region free of eddy currents, Ω_n. There are three unknown potentials, $\boldsymbol{T}$, Φ and $\boldsymbol{A}$, which can be calculated by the following weak formulation coming from the ungauged version of the $T, \Phi - A - \Phi$ formulation (see on page 109):

$$\int_{\Omega_c} \left[\frac{1}{\sigma} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{T}} \right) + \mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \boldsymbol{W}_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right] \mathrm{d}\Omega$$

$$- \int_{\Gamma_{nc}} \boldsymbol{W}_k \cdot \left(\frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \times \boldsymbol{n}_{\boldsymbol{A}} \right) \mathrm{d}\Gamma = - \int_{\Omega_c} \mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \, \mathrm{d}\Omega \tag{3.156}$$

$$- \int_{\Omega_c} \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \, \mathrm{d}\Omega,$$

$$- \int_{\Omega_c} \mu_o \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} \, \mathrm{d}\Omega + \int_{\Omega_c} \mu_o \nabla N_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \, \mathrm{d}\Omega$$

$$- \int_{\Gamma_{nc}} N_k \left(\nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) \cdot \boldsymbol{n}_{\boldsymbol{A}} \, \mathrm{d}\Gamma \tag{3.157}$$

$$= \int_{\Omega_c} \mu_o \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \, \mathrm{d}\Omega + \int_{\Omega_c} \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \, \mathrm{d}\Gamma,$$

$$- \int_{\Omega_n} \nu \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) \mathrm{d}\Omega - \int_{\Gamma_{nc}} \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} \cdot \left(\boldsymbol{W}_k \times \boldsymbol{n}_{\boldsymbol{A}} \right) \mathrm{d}\Gamma$$

$$- \int_{\Gamma_{nc}} \frac{\partial \widetilde{\Phi}}{\partial t} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \boldsymbol{n}_{\boldsymbol{A}} \mathrm{d}\Gamma = \tag{3.158}$$

$$- \int_{\Omega_n} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \mathrm{d}\Omega - \int_C \frac{\partial \widetilde{\Phi}}{\partial t} \boldsymbol{W}_k \cdot \mathrm{d}\boldsymbol{l},$$

where $k = 1, \cdots, J, \, k = 1, \cdots, I$ and $k = 1, \cdots, J$, respectively.

3.3.13 The gauged $T, \Phi - A - \Phi$ formulation

The current vector potential T with the reduced magnetic scalar potential Φ is used in the eddy current region Ω_c, the magnetic vector potential A is used in the holes, Ω_{n_A} and the reduced magnetic scalar potential Φ is used in the rest region free of eddy currents, Ω_{n_Φ}. There are three unknown potentials, T, Φ and A, so three equations must be set up.

It is important to note that the resulting system of equations should be symmetric. That is the reason why the time derivative of partial differential equations (2.415), (2.416) and (2.417) and the according Neumann type boundary and interface conditions must be taken. This will be highlighted while obtaining the formulation.

The first weak formulation can be obtained from the partial differential equations (2.414) and the Neumann type boundary and interface conditions (2.420), (2.424), (2.433) and (2.437),

$$
\begin{aligned}
&\int_{\Omega_c} \boldsymbol{W}_k \cdot \left[\nabla \times \left(\frac{1}{\sigma} \nabla \times \widetilde{\boldsymbol{T}} \right) - \nabla \left(\frac{1}{\sigma} \nabla \cdot \widetilde{\boldsymbol{T}} \right) \right] \mathrm{d}\Omega \\
&+ \int_{\Omega_c} \boldsymbol{W}_k \cdot \left[\mu_o \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right] \mathrm{d}\Omega \\
&+ \int_{\Gamma_{H_c} \cup \Gamma_{nc_\Phi}} \frac{1}{\sigma} \nabla \cdot \widetilde{\boldsymbol{T}} (\boldsymbol{W}_k \cdot \boldsymbol{n}) \, \mathrm{d}\Gamma \\
&+ \int_{\Gamma_E} \boldsymbol{W}_k \cdot \left[\left(\frac{1}{\sigma} \nabla \times \widetilde{\boldsymbol{T}} \right) \times \boldsymbol{n} \right] \mathrm{d}\Gamma \\
&+ \int_{\Gamma_{nc_A}} \boldsymbol{W}_k \cdot \left[\left(\frac{1}{\sigma} \nabla \times \widetilde{\boldsymbol{T}} \right) \times \boldsymbol{n}_c - \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \times \boldsymbol{n}_A \right] \mathrm{d}\Gamma = \\
&- \int_{\Omega_c} \mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \, \mathrm{d}\Omega - \int_{\Omega_c} \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \, \mathrm{d}\Omega, \quad k = 1, \cdots, J,
\end{aligned}
\tag{3.159}
$$

where

$$
\boldsymbol{W}_k \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{H_c} \cup \Gamma_{nc_\Phi},
\tag{3.160}
$$

and

$$
\boldsymbol{W}_k \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_E \cup \Gamma_{nc_A}.
\tag{3.161}
$$

The second order derivatives in the first two integrals can be reduced to first order one by using the identity

$$
\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \times \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \times \boldsymbol{v},
\tag{3.162}
$$

with the notations $\boldsymbol{v} = \boldsymbol{W}_k$ and $\boldsymbol{u} = 1/\sigma \nabla \times \widetilde{\boldsymbol{T}}$ and the identity

$$
\nabla \cdot (\varphi \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \varphi + \varphi \nabla \cdot \boldsymbol{v}
\tag{3.163}
$$

with the notations $\varphi = 1/\sigma \nabla \cdot \widetilde{\boldsymbol{T}}$ and $\boldsymbol{v} = \boldsymbol{W}_k$ also must be used. This results in

$$\int_{\Omega_c} \left[\frac{1}{\sigma} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{T}} \right) + \frac{1}{\sigma} \nabla \cdot \boldsymbol{W}_k \, \nabla \cdot \widetilde{\boldsymbol{T}} \right] \mathrm{d}\Omega$$

$$+ \int_{\Omega_c} \left[\mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \boldsymbol{W}_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right] \mathrm{d}\Omega$$

$$+ \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{nc_{\Phi}}} \left[\left(\frac{1}{\sigma} \nabla \times \widetilde{\boldsymbol{T}} \right) \times \boldsymbol{W}_k \right] \cdot \boldsymbol{n} \, \mathrm{d}\Gamma$$

$$- \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{nc_{\Phi}}} \frac{1}{\sigma} \nabla \cdot \widetilde{\boldsymbol{T}} (\boldsymbol{W}_k \cdot \boldsymbol{n}) \, \mathrm{d}\Gamma \qquad (3.164)$$

$$+ \int_{\Gamma_{H_c} \cup \Gamma_{nc_{\Phi}}} \frac{1}{\sigma} \nabla \cdot \widetilde{\boldsymbol{T}} (\boldsymbol{W}_k \cdot \boldsymbol{n}) \, \mathrm{d}\Gamma + \int_{\Gamma_E} \boldsymbol{W}_k \cdot \left[\left(\frac{1}{\sigma} \nabla \times \widetilde{\boldsymbol{T}} \right) \times \boldsymbol{n} \right] \mathrm{d}\Gamma$$

$$+ \int_{\Gamma_{nc_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \left[\left(\frac{1}{\sigma} \nabla \times \widetilde{\boldsymbol{T}} \right) \times \boldsymbol{n}_c - \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \times \boldsymbol{n}_{\boldsymbol{A}} \right] \mathrm{d}\Gamma =$$

$$- \int_{\Omega_c} \mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \, \mathrm{d}\Omega - \int_{\Omega_c} \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \, \mathrm{d}\Omega.$$

The boundary integral terms on the boundary part Γ_E are vanishing as well as the same term on the boundary term $\Gamma_{nc_{\boldsymbol{A}}}$, because the first boundary integral can be reformulated as

$$\left[\left(\frac{1}{\sigma} \nabla \times \widetilde{\boldsymbol{T}} \right) \times \boldsymbol{W}_k \right] \cdot \boldsymbol{n} = -\boldsymbol{W}_k \cdot \left[\left(\frac{1}{\sigma} \nabla \times \widetilde{\boldsymbol{T}} \right) \times \boldsymbol{n} \right]. \qquad (3.165)$$

The first first boundary integral is equal to zero on the rest part $\Gamma_{H_c} \cup \Gamma_{nc_{\Phi}}$ because of the Dirichlet type boundary and interface conditions (2.418) and (2.431), i.e. $\boldsymbol{W}_k \times \boldsymbol{n} = \boldsymbol{0}$ on $\Gamma_{H_c} \cup \Gamma_{nc_{\Phi}}$. The second boundary integral term is vanishing on $\Gamma_{H_c} \cup \Gamma_{nc_{\Phi}}$ because of the third boundary integral term, as well as on the rest part $\Gamma_E \cup \Gamma_{nc_{\boldsymbol{A}}}$ because of the Dirichlet type boundary and interface conditions (2.426) and (2.436), $\boldsymbol{W}_k \cdot \boldsymbol{n} = 0$ there.

Finally, the first equation of the weak form is the following:

$$\int_{\Omega_c} \left[\frac{1}{\sigma} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{T}} \right) + \frac{1}{\sigma} \nabla \cdot \boldsymbol{W}_k \, \nabla \cdot \widetilde{\boldsymbol{T}} \right] \mathrm{d}\Omega$$

$$+ \int_{\Omega_c} \left[\mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \boldsymbol{W}_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right] \mathrm{d}\Omega$$

$$- \int_{\Gamma_{nc_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \left(\frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \times \boldsymbol{n}_{\boldsymbol{A}} \right) \mathrm{d}\Gamma = - \int_{\Omega_c} \mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \, \mathrm{d}\Omega \qquad (3.166)$$

$$- \int_{\Omega_c} \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \, \mathrm{d}\Omega,$$

where $k = 1, \cdots, J$.

The second weak equation of this potential formulation is coming from the time derivative of the partial differential equations (2.415) and (2.416) and the Neumann type

boundary and interface conditions (2.425), (2.429), (2.432), (2.435) and (2.440). It is noted that it is useful to multiply the partial differential equations (2.415) and (2.416) by -1. After taking the time derivative, the following formula can be obtained:

$$
\begin{aligned}
&-\int_{\Omega_c} N_k \nabla \cdot \left(\mu_o \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} \right) d\Omega + \int_{\Omega_c} N_k \nabla \cdot \left(\mu_o \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right) d\Omega \\
&+ \int_{\Omega_{n_\Phi}} N_k \nabla \cdot \left(\mu \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right) d\Omega \\
&+ \int_{\Gamma_E} N_k \left(\mu_o \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} + \mu_o \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \nabla \frac{\partial \widetilde{\Phi}}{\partial t} + \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \right) \cdot \boldsymbol{n}\, d\Gamma \\
&+ \int_{\Gamma_{B_\Phi}} N_k \left(\frac{\partial b}{\partial t} + \mu \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \cdot \boldsymbol{n} - \mu \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \cdot \boldsymbol{n} \right) d\Gamma \\
&+ \int_{\Gamma_{nc_\Phi}} N_k \left(\mu \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} - \mu \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right) \cdot \boldsymbol{n}_\Phi\, d\Gamma \\
&+ \int_{\Gamma_{nc_\Phi}} N_k \left(\mu_o \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} + \mu_o \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \nabla \frac{\partial \widetilde{\Phi}}{\partial t} + \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \right) \cdot \boldsymbol{n}_c\, d\Gamma \\
&+ \int_{\Gamma_{nc_{\boldsymbol{A}}}} N_k \left(\nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) \cdot \boldsymbol{n}_{\boldsymbol{A}}\, d\Gamma \\
&+ \int_{\Gamma_{nc_{\boldsymbol{A}}}} N_k \left(\mu_o \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} + \mu_o \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \nabla \frac{\partial \widetilde{\Phi}}{\partial t} + \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \right) \cdot \boldsymbol{n}_c\, d\Gamma \\
&+ \int_{\Gamma_{\boldsymbol{A},\Phi}} N_k \left[\left(\nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) \cdot \boldsymbol{n}_{\boldsymbol{A}} + \left(\mu \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} - \mu \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right) \cdot \boldsymbol{n}_\Phi \right] d\Gamma \\
&= \int_{\Omega_c} N_k \nabla \cdot \left(\mu_o \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \right) d\Omega \\
&+ \int_{\Omega_{n_\Phi}} N_k \nabla \cdot \left(\mu \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \right) d\Omega + \int_{\Omega_c} N_k \nabla \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t}\, d\Omega,
\end{aligned}
\tag{3.167}
$$

where

$$
N_k = 0, \quad \text{on} \quad \Gamma_{H_c} \cup \Gamma_{H_\Phi},
\tag{3.168}
$$

and $k = 1, \cdots, I$. The first, the second, the third on the left and the three integral terms on the right can be reformulated by the use of the identity

$$
\nabla \cdot (\varphi \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \varphi + \varphi \nabla \cdot \boldsymbol{v}
\tag{3.169}
$$

with the notations $\varphi = N_k$ and $\boldsymbol{v} = \mu_o \partial \widetilde{\boldsymbol{T}} / \partial t$, or $\boldsymbol{v} = \mu_o \nabla \partial \widetilde{\Phi} / \partial t$, or $\boldsymbol{v} = \mu \nabla \partial \widetilde{\Phi} / \partial t$ and $\boldsymbol{v} = \mu_o \partial \widetilde{\boldsymbol{T}}_0 / \partial t$, or $\boldsymbol{v} = \mu \partial \widetilde{\boldsymbol{T}}_0 / \partial t$, or $\boldsymbol{v} = \partial \widetilde{\boldsymbol{R}} / \partial t$.

Finally, the following equation can be obtained:

$$
\int_{\Omega_c} \mu_o \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t}\, \mathrm{d}\Omega - \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{nc_\Phi}} N_k\, \mu_o \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} \cdot \boldsymbol{n}\, \mathrm{d}\Gamma
$$

$$
- \int_{\Omega_c} \mu_o \nabla N_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t}\, \mathrm{d}\Omega - \int_{\Omega_{n_\Phi}} \mu \nabla N_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t}\, \mathrm{d}\Omega
$$

$$
+ \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{nc_\Phi}} N_k\, \mu_o \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \cdot \boldsymbol{n}\, \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_{B_\Phi} \cup \Gamma_{H_\Phi} \cup \Gamma_{nc_\Phi} \cup \Gamma_{\boldsymbol{A},\Phi}} N_k\, \mu \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \cdot \boldsymbol{n}\, \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_E} N_k \left(\mu_o \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} + \mu_o \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \nabla \frac{\partial \widetilde{\Phi}}{\partial t} + \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \right) \cdot \boldsymbol{n}\, \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_{B_\Phi}} N_k \left(\frac{\partial b}{\partial t} + \mu \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \cdot \boldsymbol{n} - \mu \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \cdot \boldsymbol{n} \right)\, \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_{nc_\Phi}} N_k \left(\mu \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} - \mu \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right) \cdot \boldsymbol{n}_\Phi\, \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_{nc_\Phi}} N_k \left(\mu_o \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} + \mu_o \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \nabla \frac{\partial \widetilde{\Phi}}{\partial t} + \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \right) \cdot \boldsymbol{n}_c\, \mathrm{d}\Gamma \qquad (3.170)
$$

$$
+ \int_{\Gamma_{nc_{\boldsymbol{A}}}} N_k \left(\nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) \cdot \boldsymbol{n}_{\boldsymbol{A}}\, \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_{nc_{\boldsymbol{A}}}} N_k \left(\mu_o \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} + \mu_o \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \nabla \frac{\partial \widetilde{\Phi}}{\partial t} + \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \right) \cdot \boldsymbol{n}_c\, \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_{\boldsymbol{A},\Phi}} N_k \left[\left(\nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) \cdot \boldsymbol{n}_{\boldsymbol{A}} + \left(\mu \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} - \mu \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right) \cdot \boldsymbol{n}_\Phi \right]\, \mathrm{d}\Gamma =
$$

$$
- \int_{\Omega_c} (\nabla N_k) \cdot \left(\mu_o \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \right)\, \mathrm{d}\Omega - \int_{\Omega_{n_\Phi}} (\nabla N_k) \cdot \left(\mu \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \right)\, \mathrm{d}\Omega
$$

$$
+ \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{nc_\Phi}} N_k \mu_o \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \cdot \boldsymbol{n}\, \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_{B_\Phi} \cup \Gamma_{H_\Phi} \cup \Gamma_{nc_\Phi} \cup \Gamma_{\boldsymbol{A},\Phi}} N_k \mu \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \cdot \boldsymbol{n}\, \mathrm{d}\Gamma
$$

$$
- \int_{\Omega_c} (\nabla N_k) \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t}\, \mathrm{d}\Gamma + \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{nc_\Phi}} N_k \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \cdot \boldsymbol{n}\, \mathrm{d}\Gamma.
$$

It must be noted that the boundary of the region Ω_c is $\partial \Omega_c = \Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc_\Phi} \cup \Gamma_{nc_{\boldsymbol{A}}}$ and the boundary of the region Ω_n is $\partial \Omega_n = \Gamma_B \cup \Gamma_{H_n} \cup \Gamma_{nc_\Phi} \cup \Gamma_{nc_{\boldsymbol{A}}}$.

The fourth boundary integral term is vanishing on Γ_E because of the same terms with opposite sign in the first on the right and on the left, second on the left and the last on the right boundary integrals. The second and the third fractions in the fifth boundary integrals are vanishing according to the second one on the right and the third one on the left on Γ_{B_Φ}. The interface integral term on Γ_{nc_Φ} is vanishing according to the same terms with opposite sign in the first two on the right and the first, second and third boundary integral terms on the left. The ninth boundary integral and the second term in the tenth integral are vanishing, because of the terms in the first, second, third and the first two on the right. The $\partial \widetilde{R}/\partial t$ terms are vanishing, too, on $\Gamma_{nc_\Phi} \cup \Gamma_{nc_A}$, because of the same terms on the left and on the right. Dirichlet type boundary condition (see (2.419) and (2.423)) is prescribed on Γ_{H_c} and on Γ_{H_Φ}, where $N_k = 0$, i.e. the according boundary terms are vanishing on $\Gamma_{H_c} \cup \Gamma_{H_\Phi}$. Finally, the following weak formulation can be obtained:

$$
\begin{aligned}
&-\int_{\Omega_c} \mu_o \nabla N_k \cdot \frac{\partial \widetilde{T}}{\partial t}\, \mathrm{d}\Omega + \int_{\Omega_c} \mu_o \nabla N_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t}\, \mathrm{d}\Omega + \int_{\Omega_{n_\Phi}} \mu \nabla N_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t}\, \mathrm{d}\Omega \\
&- \int_{\Gamma_{nc_A} \cup \Gamma_{A,\Phi}} N_k \left(\nabla \times \frac{\partial \widetilde{A}}{\partial t} \right) \cdot \boldsymbol{n_A}\, \mathrm{d}\Gamma \\
&= \int_{\Omega_c} \mu_o \nabla N_k \cdot \frac{\partial \widetilde{T}_0}{\partial t} \mathrm{d}\Omega + \int_{\Omega_{n_\Phi}} \mu \nabla N_k \cdot \frac{\partial \widetilde{T}_0}{\partial t}\, \mathrm{d}\Omega + \int_{\Gamma_{B_\Phi}} N_k \frac{\partial b}{\partial t}\, \mathrm{d}\Gamma \\
&+ \int_{\Omega_c} \nabla N_k \cdot \frac{\partial \widetilde{R}}{\partial t}\, \mathrm{d}\Omega, \quad k = 1, \cdots, I.
\end{aligned}
\tag{3.171}
$$

This equation has been multiplied by -1 again, because of symmetry conditions.

The last weak equation of this potential formulation is coming from the time derivative of the partial differential equation (2.417), the Neumann boundary and interface conditions (2.421), (2.428), (2.434), (2.438) and (2.439),

$$
\begin{aligned}
&\int_{\Omega_{n_A}} \boldsymbol{W}_k \cdot \left[\nabla \times \left(\nu \nabla \times \frac{\partial \widetilde{A}}{\partial t} \right) - \nabla \left(\nu \nabla \cdot \frac{\partial \widetilde{A}}{\partial t} \right) \right] \mathrm{d}\Omega \\
&+ \int_{\Gamma_{H_A}} \boldsymbol{W}_k \cdot \left[\left(\nu \nabla \times \frac{\partial \widetilde{A}}{\partial t} \right) \times \boldsymbol{n} - \frac{\partial \boldsymbol{K}}{\partial t} \right] \mathrm{d}\Gamma + \int_{\Gamma_{B_A} \cup \Gamma_{nc_A}} \nu \nabla \cdot \frac{\partial \widetilde{A}}{\partial t}\, (\boldsymbol{W}_k \cdot \boldsymbol{n})\, \mathrm{d}\Gamma \\
&+ \int_{\Gamma_{nc_A}} \boldsymbol{W}_k \cdot \left[\left(\frac{\partial \widetilde{T}_0}{\partial t} + \frac{\partial \widetilde{T}}{\partial t} - \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right) \times \boldsymbol{n}_c + \left(\nu \nabla \times \frac{\partial \widetilde{A}}{\partial t} \right) \times \boldsymbol{n_A} \right] \mathrm{d}\Gamma \\
&+ \int_{\Gamma_{A,\Phi}} \boldsymbol{W}_k \cdot \left[\left(\frac{\partial \widetilde{T}_0}{\partial t} - \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right) \times \boldsymbol{n}_\Phi + \left(\nu \nabla \times \frac{\partial \widetilde{A}}{\partial t} \right) \times \boldsymbol{n_A} \right] \mathrm{d}\Gamma \\
&= \int_{\Omega_{n_A}} \boldsymbol{W}_k \cdot \frac{\partial \boldsymbol{J}_0}{\partial t}\, \mathrm{d}\Omega.
\end{aligned}
\tag{3.172}
$$

Here

$$
\boldsymbol{W}_k \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{B_A},
\tag{3.173}
$$

and

$$\boldsymbol{W}_k \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_{H_{\boldsymbol{A}}} \cup \Gamma_{\boldsymbol{A},\Phi}, \tag{3.174}$$

moreover $k = 1, \cdots, J$. The second order derivatives in the first integral can be reduced to first order one by using the following identities:

$$\nabla \cdot (\varphi \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \varphi + \varphi \nabla \cdot \boldsymbol{v}, \tag{3.175}$$

and

$$\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \times \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \times \boldsymbol{v}, \tag{3.176}$$

with the notations $\varphi = \nu \nabla \widetilde{\boldsymbol{A}}$, $\boldsymbol{v} = \boldsymbol{W}_k$ and $\boldsymbol{u} = \nu \nabla \times \widetilde{\boldsymbol{A}}$, i.e.

$$
\begin{aligned}
&\int_{\Omega_{n_{\boldsymbol{A}}}} \left[\nu \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) + \nu \nabla \cdot \boldsymbol{W}_k \, \nabla \cdot \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right] \mathrm{d}\Omega \\
&+ \int_{\Gamma_{H_{\boldsymbol{A}}} \cup \Gamma_{B_{\boldsymbol{A}}} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{\boldsymbol{A},\Phi}} \left[\left(\nu \nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) \times \boldsymbol{W}_k \right] \cdot \boldsymbol{n} \, \mathrm{d}\Gamma \\
&- \int_{\Gamma_{H_{\boldsymbol{A}}} \cup \Gamma_{B_{\boldsymbol{A}}} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{\boldsymbol{A},\Phi}} \nu \nabla \cdot \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \left(\boldsymbol{W}_k \cdot \boldsymbol{n} \right) \mathrm{d}\Gamma \\
&+ \int_{\Gamma_{H_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \left[\left(\nu \nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) \times \boldsymbol{n} - \frac{\partial \boldsymbol{K}}{\partial t} \right] \mathrm{d}\Gamma \\
&+ \int_{\Gamma_{B_{\boldsymbol{A}}} \cup \Gamma_{nc_{\boldsymbol{A}}}} \nu \nabla \cdot \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \left(\boldsymbol{W}_k \cdot \boldsymbol{n} \right) \mathrm{d}\Gamma \\
&+ \int_{\Gamma_{nc_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \left[\left(\frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} + \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right) \times \boldsymbol{n}_c + \left(\nu \nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) \times \boldsymbol{n}_{\boldsymbol{A}} \right] \mathrm{d}\Gamma \\
&+ \int_{\Gamma_{\boldsymbol{A},\Phi}} \boldsymbol{W}_k \cdot \left[\left(\frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} - \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right) \times \boldsymbol{n}_\Phi + \left(\nu \nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) \times \boldsymbol{n}_{\boldsymbol{A}} \right] \mathrm{d}\Gamma \\
&= \int_{\Omega_{n_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \frac{\partial \boldsymbol{J}_0}{\partial t} \, \mathrm{d}\Omega.
\end{aligned} \tag{3.177}
$$

The first surface integral can be reformulated according to

$$\left[\left(\nu \nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) \times \boldsymbol{W}_k \right] \cdot \boldsymbol{n} = -\boldsymbol{W}_k \cdot \left[\left(\nu \nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) \times \boldsymbol{n} \right], \tag{3.178}$$

and it is vanishing on $\Gamma_{H_{\boldsymbol{A}}} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{\boldsymbol{A},\Phi}$ according to the third, fifth and sixth integrals. On the rest part, $\Gamma_{B_{\boldsymbol{A}}}$ it is equal to zero because of the Dirichlet type boundary condition (2.427) and $\boldsymbol{W}_k \times \boldsymbol{n} = \boldsymbol{0}$ here. The second boundary integral term is vanishing on $\Gamma_{B_{\boldsymbol{A}}} \cup \Gamma_{nc_{\boldsymbol{A}}}$ and it is equal to zero on the rest part $\Gamma_{H_{\boldsymbol{A}}} \cup \Gamma_{\boldsymbol{A},\Phi}$, because of the fourth boundary integral term and the Dirichlet type conditions (2.422), (2.441), respectively.

The following weak formulation can be obtained after the above manipulations:

$$
\begin{aligned}
-\int_{\Omega_{n_A}} &\left[\nu \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) + \nu \nabla \cdot \boldsymbol{W}_k \, \nabla \cdot \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right] \mathrm{d}\Omega \\
-\int_{\Gamma_{nc_A}} &\frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} \cdot \left(\boldsymbol{W}_k \times \boldsymbol{n}_A \right) \mathrm{d}\Gamma - \int_{\Gamma_{nc_A} \cup \Gamma_{A,\Phi}} \frac{\partial \widetilde{\Phi}}{\partial t} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \boldsymbol{n}_A \mathrm{d}\Gamma = \\
-\int_{\Omega_{n_A}} &\boldsymbol{W}_k \cdot \frac{\partial \boldsymbol{J}_0}{\partial t} \mathrm{d}\Omega - \int_{\Gamma_{H_A}} \boldsymbol{W}_k \cdot \frac{\partial \boldsymbol{K}}{\partial t} \mathrm{d}\Gamma \\
-\int_{\Gamma_{nc_A} \cup \Gamma_{A,\Phi}} &\boldsymbol{W}_k \cdot \left(\frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \times \boldsymbol{n}_A \right) \mathrm{d}\Gamma + \int_C \frac{\partial \widetilde{\Phi}}{\partial t} \boldsymbol{W}_k \cdot \mathrm{d}\boldsymbol{l},
\end{aligned}
\tag{3.179}
$$

where $k = 1, \cdots, J$. Some notes must be appended here about the boundary integral terms. First of all, the integral equation has been multiplied by -1, because of symmetry conditions. The terms $\boldsymbol{W}_k \cdot (\partial \widetilde{\boldsymbol{T}}_0/\partial t) \times \boldsymbol{n}_c$ on Γ_{nc_A} and $\boldsymbol{W}_k \cdot (\partial \widetilde{\boldsymbol{T}}_0/\partial t) \times \boldsymbol{n}_\Phi$ on $\Gamma_{A,\Phi}$ have been put to the right-hand side and notations $\boldsymbol{n}_A = -\boldsymbol{n}_c$ and $\boldsymbol{n}_A = -\boldsymbol{n}_\Phi$ have been used. Because of symmetry, the term $\boldsymbol{W}_k \cdot (\partial \widetilde{\boldsymbol{T}}/\partial t) \times \boldsymbol{n}_c$ on Γ_{nc_A} is manipulated as

$$
\boldsymbol{W}_k \cdot \left(\frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} \times \boldsymbol{n}_c \right) = -\boldsymbol{W}_k \cdot \left(\frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} \times \boldsymbol{n}_A \right) = \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} \cdot \left(\boldsymbol{W}_k \times \boldsymbol{n}_A \right).
\tag{3.180}
$$

The third term $\boldsymbol{W}_k \cdot (\nabla \partial \widetilde{\Phi}/\partial t \times \boldsymbol{n}_c)$, or $\boldsymbol{W}_k \cdot (\nabla \partial \widetilde{\Phi}/\partial t \times \boldsymbol{n}_\Phi)$ on $\Gamma_{nc_A} \cup \Gamma_{A,\Phi}$ can be reformulated as

$$
\int_{\Gamma_{nc_A} \cup \Gamma_{A,\Phi}} \boldsymbol{W}_k \cdot \left(\nabla \frac{\partial \widetilde{\Phi}}{\partial t} \times \boldsymbol{n}_A \right) \mathrm{d}\Gamma = \int_{\Gamma_{nc_A} \cup \Gamma_{A,\Phi}} \boldsymbol{n}_A \cdot \left(\boldsymbol{W}_k \times \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right) \mathrm{d}\Gamma, \tag{3.181}
$$

after using $\boldsymbol{n}_A = -\boldsymbol{n}_c$ and $\boldsymbol{n}_A = -\boldsymbol{n}_\Phi$. The identity

$$
\nabla \times (\varphi \boldsymbol{v}) = \varphi \nabla \times \boldsymbol{v} - \boldsymbol{v} \times \nabla \varphi \tag{3.182}
$$

must be applied as

$$
\boldsymbol{n} \cdot [\nabla \times (\varphi \boldsymbol{v})] = \varphi [\boldsymbol{n} \cdot (\nabla \times \boldsymbol{v})] - \boldsymbol{n} \cdot (\boldsymbol{v} \times \nabla \varphi) \tag{3.183}
$$

with the notations $\varphi = \partial \widetilde{\Phi}/\partial t$ and $\boldsymbol{v} = \boldsymbol{W}_k$, i.e.

$$
\begin{aligned}
&\int_{\Gamma_{nc_A} \cup \Gamma_{A,\Phi}} \boldsymbol{n}_A \cdot \left(\boldsymbol{W}_k \times \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right) \mathrm{d}\Gamma \\
&= \int_{\Gamma_{nc_A} \cup \Gamma_{A,\Phi}} \frac{\partial \widetilde{\Phi}}{\partial t} [\boldsymbol{n}_A \cdot (\nabla \times \boldsymbol{W}_k)] \mathrm{d}\Gamma - \int_C \frac{\partial \widetilde{\Phi}}{\partial t} \boldsymbol{W}_k \cdot \mathrm{d}\boldsymbol{l},
\end{aligned}
\tag{3.184}
$$

where curve C is bounding the interface between the $\boldsymbol{A}$-region and the region where Φ can be found, i.e. $\Omega_c \cup \Omega_{n_\Phi}$. If this surface is closed (e.g. when symmetry planes are not taken into account), the curve integral can be eliminated. Otherwise, the curve C meets

with boundary type of $\Gamma_{B_A} \cup \Gamma_E$ and/or $\Gamma_{H_c} \cup \Gamma_{H_A}$. In the first case, $\boldsymbol{n} \times \boldsymbol{W}_k = 0$ (see (2.427)), i.e. $\boldsymbol{W}_k \cdot \mathrm{d}\boldsymbol{l} = 0$. In the second case Φ is given by (2.419), i.e. $\Phi = \Phi_0$. Finally,

$$\int_C \widetilde{\Phi} \boldsymbol{W}_k \cdot \mathrm{d}\boldsymbol{l} = \int_{C_H} \Phi_0 \boldsymbol{W}_k \cdot \mathrm{d}\boldsymbol{l}, \tag{3.185}$$

where C_H denotes the path lying on $\Gamma_{A,\Phi}$ and meeting Γ_H.

The weak equations of the gauged $\boldsymbol{T}, \Phi - \boldsymbol{A} - \Phi$ formulation are:

$$\int_{\Omega_c} \left[\frac{1}{\sigma} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{T}} \right) + \frac{1}{\sigma} \nabla \cdot \boldsymbol{W}_k \nabla \cdot \widetilde{\boldsymbol{T}} \right] \mathrm{d}\Omega$$

$$+ \int_{\Omega_c} \left[\mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \boldsymbol{W}_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right] \mathrm{d}\Omega$$

$$- \int_{\Gamma_{nc_A}} \boldsymbol{W}_k \cdot \left(\frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \times \boldsymbol{n_A} \right) \mathrm{d}\Gamma = \tag{3.186}$$

$$- \int_{\Omega_c} \mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \mathrm{d}\Omega - \int_{\Omega_c} \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \mathrm{d}\Omega,$$

$$- \int_{\Omega_c} \mu_o \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} \mathrm{d}\Omega + \int_{\Omega_c} \mu_o \nabla N_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \mathrm{d}\Omega$$

$$+ \int_{\Omega_{n_\Phi}} \mu \nabla N_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \mathrm{d}\Omega - \int_{\Gamma_{nc_A} \cup \Gamma_{A,\Phi}} N_k \left(\nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) \cdot \boldsymbol{n_A} \, \mathrm{d}\Gamma$$

$$= \int_{\Omega_c} \mu_o \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \mathrm{d}\Omega + \int_{\Omega_{n_\Phi}} \mu \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \mathrm{d}\Omega \tag{3.187}$$

$$+ \int_{\Gamma_{B_\Phi}} N_k \frac{\partial b}{\partial t} \, \mathrm{d}\Gamma + \int_{\Omega_c} \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \mathrm{d}\Omega,$$

$$- \int_{\Omega_{n_A}} \left[\nu \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) + \nu \nabla \cdot \boldsymbol{W}_k \nabla \cdot \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right] \mathrm{d}\Omega$$

$$- \int_{\Gamma_{nc_A}} \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} \cdot \left(\boldsymbol{W}_k \times \boldsymbol{n_A} \right) \mathrm{d}\Gamma - \int_{\Gamma_{nc_A} \cup \Gamma_{A,\Phi}} \frac{\partial \widetilde{\Phi}}{\partial t} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \boldsymbol{n_A} \mathrm{d}\Gamma =$$

$$- \int_{\Omega_{n_A}} \boldsymbol{W}_k \cdot \frac{\partial \boldsymbol{J}_0}{\partial t} \mathrm{d}\Omega - \int_{\Gamma_{H_A}} \boldsymbol{W}_k \cdot \frac{\partial \boldsymbol{K}}{\partial t} \mathrm{d}\Gamma \tag{3.188}$$

$$- \int_{\Gamma_{nc_A} \cup \Gamma_{A,\Phi}} \boldsymbol{W}_k \cdot \left(\frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \times \boldsymbol{n_A} \right) \mathrm{d}\Gamma + \int_C \frac{\partial \widetilde{\Phi}}{\partial t} \boldsymbol{W}_k \cdot \mathrm{d}\boldsymbol{l}.$$

Here $k = 1, \cdots, J$, $k = 1, \cdots, I$ and $k = 1, \cdots, J$, respectively.

3.3.14 The ungauged $T, \Phi - A - \Phi$ formulation

The current vector potential T with the reduced magnetic scalar potential Φ is used in the eddy current region Ω_c, the magnetic vector potential A is used in the holes, Ω_{n_A} and the reduced magnetic scalar potential Φ is used in the rest region free of eddy currents, Ω_{n_Φ}. There are three unknown potentials, T, Φ and A, so three equations must be set up.

It is important to note that the resulting system of equations should be symmetric. That is the reason why the time derivative of the partial differential equations (2.443), (2.444) and (2.445) and the according Neumann type boundary and interface conditions are taken. This will be highlighted while obtaining the formulation.

The first weak formulation is coming from the partial differential equations (2.442) and the Neumann type boundary and interface conditions (2.450) and (2.459),

$$
\begin{aligned}
&\int_{\Omega_c} \boldsymbol{W}_k \cdot \left[\nabla \times \left(\frac{1}{\sigma} \nabla \times \widetilde{\boldsymbol{T}} \right) + \mu_o \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right] \mathrm{d}\Omega \\
&+ \int_{\Gamma_E} \boldsymbol{W}_k \cdot \left[\left(\frac{1}{\sigma} \nabla \times \widetilde{\boldsymbol{T}} \right) \times \boldsymbol{n} \right] \mathrm{d}\Gamma \\
&+ \int_{\Gamma_{nc_A}} \boldsymbol{W}_k \cdot \left[\left(\frac{1}{\sigma} \nabla \times \widetilde{\boldsymbol{T}} \right) \times \boldsymbol{n}_c - \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \times \boldsymbol{n}_A \right] \mathrm{d}\Gamma = \\
&- \int_{\Omega_c} \mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \, \mathrm{d}\Omega - \int_{\Omega_c} \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \, \mathrm{d}\Omega,
\end{aligned}
\tag{3.189}
$$

where

$$
\boldsymbol{W}_k \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_{H_c} \cup \Gamma_{nc_\Phi},
\tag{3.190}
$$

and $k = 1, \cdots, J$. The second order derivative in the first integral can be reduced to first order one by using the identity

$$
\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \times \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \times \boldsymbol{v},
\tag{3.191}
$$

with the notations $\boldsymbol{v} = \boldsymbol{W}_k$ and $\boldsymbol{u} = 1/\sigma \nabla \times \widetilde{\boldsymbol{T}}$, i.e.

$$
\begin{aligned}
&\int_{\Omega_c} \left[\frac{1}{\sigma} (\nabla \times \boldsymbol{W}_k) \cdot \left(\nabla \times \widetilde{\boldsymbol{T}} \right) + \mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \boldsymbol{W}_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right] \mathrm{d}\Omega \\
&+ \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc_A} \cup \Gamma_{nc_\Phi}} \left[\left(\frac{1}{\sigma} \nabla \times \widetilde{\boldsymbol{T}} \right) \times \boldsymbol{W}_k \right] \cdot \boldsymbol{n} \, \mathrm{d}\Gamma \\
&+ \int_{\Gamma_E} \boldsymbol{W}_k \cdot \left[\left(\frac{1}{\sigma} \nabla \times \widetilde{\boldsymbol{T}} \right) \times \boldsymbol{n} \right] \mathrm{d}\Gamma \\
&+ \int_{\Gamma_{nc_A}} \boldsymbol{W}_k \cdot \left[\left(\frac{1}{\sigma} \nabla \times \widetilde{\boldsymbol{T}} \right) \times \boldsymbol{n}_c - \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \times \boldsymbol{n}_A \right] \mathrm{d}\Gamma = \\
&- \int_{\Omega_c} \mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \, \mathrm{d}\Omega - \int_{\Omega_c} \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t} \, \mathrm{d}\Omega.
\end{aligned}
\tag{3.192}
$$

The boundary integral terms on the boundary part Γ_E are vanishing as well as the same term on the boundary term Γ_{nc_A}, because the first term of the first boundary integral can be reformulated as

$$
\begin{aligned}
&\left[\left(\frac{1}{\sigma}\nabla \times \widetilde{\boldsymbol{T}}\right) \times \boldsymbol{W}_k\right] \cdot \boldsymbol{n} \\
&= \left[\boldsymbol{n} \times \left(\frac{1}{\sigma}\nabla \times \widetilde{\boldsymbol{T}}\right)\right] \cdot \boldsymbol{W}_k \\
&= -\boldsymbol{W}_k \cdot \left[\left(\frac{1}{\sigma}\nabla \times \widetilde{\boldsymbol{T}}\right) \times \boldsymbol{n}\right].
\end{aligned}
\tag{3.193}
$$

The first boundary integral is equal to zero on the rest part $\Gamma_{H_c} \cup \Gamma_{nc_\Phi}$ because of the Dirichlet type boundary and interface conditions (2.446) and (2.455), i.e. $\boldsymbol{W}_k \times \boldsymbol{n} = \boldsymbol{0}$ on $\Gamma_{H_c} \cup \Gamma_{nc_\Phi}$. Finally, the first equation of the weak form is the following:

$$
\begin{aligned}
&\int_{\Omega_c}\left[\frac{1}{\sigma}\left(\nabla \times \boldsymbol{W}_k\right) \cdot \left(\nabla \times \widetilde{\boldsymbol{T}}\right) + \mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \mu_o \boldsymbol{W}_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t}\right] \mathrm{d}\Omega \\
&- \int_{\Gamma_{nc_A}} \boldsymbol{W}_k \cdot \left(\frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \times \boldsymbol{n}_A\right) \mathrm{d}\Gamma = -\int_{\Omega_c} \mu_o \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t}\, \mathrm{d}\Omega \\
&- \int_{\Omega_c} \boldsymbol{W}_k \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t}\, \mathrm{d}\Omega,
\end{aligned}
\tag{3.194}
$$

where $k = 1, \cdots, J$.

The second weak equation is coming from the time derivative of the partial differential equations (2.443) and (2.444) and the Neumann type boundary and interface conditions (2.451), (2.453), (2.456), (2.458) and (2.461). It is noted that it is useful to multiply the partial differential equations (2.443) and (2.444) by -1.

The same form can be obtained as in the gauged $\boldsymbol{T}, \Phi - \boldsymbol{A} - \Phi$ formulation, see (3.167), so the weak formulation is the following:

$$
\begin{aligned}
&- \int_{\Omega_c} \mu_o \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t}\, \mathrm{d}\Omega + \int_{\Omega_c} \mu_o \nabla N_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t}\, \mathrm{d}\Omega + \int_{\Omega_{n_\Phi}} \mu \nabla N_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t}\, \mathrm{d}\Omega \\
&- \int_{\Gamma_{nc_A} \cup \Gamma_{A,\Phi}} N_k \left(\nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t}\right) \cdot \boldsymbol{n}_A\, \mathrm{d}\Gamma \\
&= \int_{\Omega_c} \mu_o \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t}\, \mathrm{d}\Omega + \int_{\Omega_{n_\Phi}} \mu \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t}\, \mathrm{d}\Omega + \int_{\Gamma_{B_\Phi}} N_k \frac{\partial b}{\partial t}\, \mathrm{d}\Gamma \\
&+ \int_{\Omega_c} \nabla N_k \cdot \frac{\partial \widetilde{\boldsymbol{R}}}{\partial t}\, \mathrm{d}\Omega,
\end{aligned}
\tag{3.195}
$$

where $k = 1, \cdots, I$.

The last weak equation of this potential formulation is coming from the time derivative of the partial differential equations (2.445) and the Neumann type boundary and interface conditions (2.448), (2.457) and (2.460),

$$
\int_{\Omega_{n_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \left[\nabla \times \left(\nu \nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) \right] \mathrm{d}\Omega
$$

$$
+ \int_{\Gamma_{H_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \left[\left(\nu \nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) \times \boldsymbol{n} - \frac{\partial \boldsymbol{K}}{\partial t} \right] \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_{nc_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \left[\left(\frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} + \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right) \times \boldsymbol{n}_c + \left(\nu \nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) \times \boldsymbol{n}_{\boldsymbol{A}} \right] \mathrm{d}\Gamma \qquad (3.196)
$$

$$
+ \int_{\Gamma_{\boldsymbol{A},\Phi}} \boldsymbol{W}_k \cdot \left[\left(\frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} - \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right) \times \boldsymbol{n}_\Phi + \left(\nu \nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) \times \boldsymbol{n}_{\boldsymbol{A}} \right] \mathrm{d}\Gamma
$$

$$
= \int_{\Omega_{n_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \left(\nabla \times \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \right) \mathrm{d}\Omega, \quad k = 1, \cdots, J.
$$

Here $\boldsymbol{W}_k \times \boldsymbol{n} = \boldsymbol{0}$ on $\Gamma_{B_{\boldsymbol{A}}}$ is used. The second order derivative in the first integral can be reduced to first order one and the integral on the right-hand side can be reformulated by using the following identity:

$$
\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \times \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \times \boldsymbol{v}, \qquad (3.197)
$$

with the notations $\boldsymbol{v} = \boldsymbol{W}_k$, $\boldsymbol{u} = \nu \nabla \times (\partial \widetilde{\boldsymbol{A}}/\partial t)$ and $\boldsymbol{u} = \partial \widetilde{\boldsymbol{T}}_0/\partial t$ i.e.

$$
\int_{\Omega_{n_{\boldsymbol{A}}}} \nu \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) \mathrm{d}\Omega
$$

$$
+ \int_{\Gamma_{H_{\boldsymbol{A}}} \cup \Gamma_{B_{\boldsymbol{A}}} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{\boldsymbol{A},\Phi}} \left[\left(\nu \nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) \times \boldsymbol{W}_k \right] \cdot \boldsymbol{n} \, \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_{H_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \left[\left(\nu \nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) \times \boldsymbol{n} - \frac{\partial \boldsymbol{K}}{\partial t} \right] \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_{nc_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \left[\left(\frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} + \frac{\partial \widetilde{\boldsymbol{T}}}{\partial t} - \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right) \times \boldsymbol{n}_c + \left(\nu \nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) \times \boldsymbol{n}_{\boldsymbol{A}} \right] \mathrm{d}\Gamma \qquad (3.198)
$$

$$
+ \int_{\Gamma_{\boldsymbol{A},\Phi}} \boldsymbol{W}_k \cdot \left[\left(\frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} - \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \right) \times \boldsymbol{n}_\Phi + \left(\nu \nabla \times \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} \right) \times \boldsymbol{n}_{\boldsymbol{A}} \right] \mathrm{d}\Gamma
$$

$$
= \int_{\Omega_{n_{\boldsymbol{A}}}} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \, \mathrm{d}\Omega
$$

$$
+ \int_{\Gamma_{H_{\boldsymbol{A}}} \cup \Gamma_{B_{\boldsymbol{A}}} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{\boldsymbol{A},\Phi}} \left(\frac{\partial \widetilde{\boldsymbol{T}}_0}{\partial t} \times \boldsymbol{W}_k \right) \cdot \boldsymbol{n} \, \mathrm{d}\Gamma.
$$

The first surface integral can be reformulated according to

$$
\left[\left(\nu\nabla\times\frac{\partial\widetilde{\boldsymbol{A}}}{\partial t}\right)\times\boldsymbol{W}_k\right]\cdot\boldsymbol{n} = -\boldsymbol{W}_k\cdot\left[\left(\nu\nabla\times\frac{\partial\widetilde{\boldsymbol{A}}}{\partial t}\right)\times\boldsymbol{n}\right], \tag{3.199}
$$

and it is vanishing on $\Gamma_{H_{\boldsymbol{A}}}\cup\Gamma_{nc_{\boldsymbol{A}}}\cup\Gamma_{\boldsymbol{A},\Phi}$ according to the second, third and fourth integral terms. On the rest part, $\Gamma_{B_{\boldsymbol{A}}}$, it is equal to zero because of the Dirichlet type boundary condition (2.452) and $\boldsymbol{W}_k\times\boldsymbol{n}=\boldsymbol{0}$ here. The last boundary integral term can be reformulated as

$$
\left(\frac{\partial\widetilde{\boldsymbol{T}}_0}{\partial t}\times\boldsymbol{W}_k\right)\cdot\boldsymbol{n}_{\boldsymbol{A}} = -\boldsymbol{W}_k\cdot\left(\frac{\partial\widetilde{\boldsymbol{T}}_0}{\partial t}\times\boldsymbol{n}_{\boldsymbol{A}}\right). \tag{3.200}
$$

This is the reason why the last boundary integral term is vanishing on $\Gamma_{nc_{\boldsymbol{A}}}\cup\Gamma_{\boldsymbol{A},\Phi}$, because of the same terms in the third and fourth boundary integral terms and $\boldsymbol{n}=-\boldsymbol{n}_c$ and $\boldsymbol{n}=-\boldsymbol{n}_\Phi$. The last boundary integral term is equal to zero on $\Gamma_{B_{\boldsymbol{A}}}$ as well, since (2.452) is a Dirichlet type boundary condition, i.e. $\boldsymbol{W}_k\times\boldsymbol{n}=\boldsymbol{0}$ here. According to the time derivative of (3.146) and (3.147), the surface current density $\boldsymbol{K}$ can be eliminated from the formulation and the last boundary integral term is vanishing on the rest part $\Gamma_{H_{\boldsymbol{A}}}$. The following weak formulation can be obtained after the above manipulations:

$$
-\int_{\Omega_{n_{\boldsymbol{A}}}}\nu\left(\nabla\times\boldsymbol{W}_k\right)\cdot\left(\nabla\times\frac{\partial\widetilde{\boldsymbol{A}}}{\partial t}\right)\,\mathrm{d}\Omega
$$

$$
-\int_{\Gamma_{nc_{\boldsymbol{A}}}}\frac{\partial\widetilde{\boldsymbol{T}}}{\partial t}\cdot\left(\boldsymbol{W}_k\times\boldsymbol{n}_{\boldsymbol{A}}\right)\mathrm{d}\Gamma-\int_{\Gamma_{nc_{\boldsymbol{A}}}\cup\Gamma_{\boldsymbol{A},\Phi}}\frac{\partial\widetilde{\Phi}}{\partial t}\left(\nabla\times\boldsymbol{W}_k\right)\cdot\boldsymbol{n}_{\boldsymbol{A}}\mathrm{d}\Gamma = \tag{3.201}
$$

$$
-\int_{\Omega_{n_{\boldsymbol{A}}}}\left(\nabla\times\boldsymbol{W}_k\right)\cdot\frac{\partial\widetilde{\boldsymbol{T}}_0}{\partial t}\mathrm{d}\Omega+\int_{C_H}\frac{\partial\widetilde{\Phi}}{\partial t}\boldsymbol{W}_k\cdot\mathrm{d}\boldsymbol{l}.
$$

Some notes must be appended here about the boundary integral terms. First of all, the equation has been multiplied by -1, because of symmetry conditions, moreover notations $\boldsymbol{n}_{\boldsymbol{A}}=-\boldsymbol{n}_c$ and $\boldsymbol{n}_{\boldsymbol{A}}=-\boldsymbol{n}_\Phi$ have been used. The term $\boldsymbol{W}_k\cdot(\partial\widetilde{\boldsymbol{T}}/\partial t)\times\boldsymbol{n}_c$ on $\Gamma_{nc_{\boldsymbol{A}}}$ as well as the third term $\boldsymbol{W}_k\cdot(\nabla\partial\widetilde{\Phi}/\partial t\times\boldsymbol{n}_c)$, or $\boldsymbol{W}_k\cdot(\nabla\partial\widetilde{\Phi}/\partial t\times\boldsymbol{n}_\Phi)$ on $\Gamma_{nc_{\boldsymbol{A}}}\cup\Gamma_{\boldsymbol{A},\Phi}$ have been reformulated as it is presented in the last item, see the manipulations (3.180)–(3.185).

Finally, the weak formulation of the ungauged $\boldsymbol{T},\Phi-\boldsymbol{A}-\Phi$ formulation can be summarized as

$$
\int_{\Omega_c}\left[\frac{1}{\sigma}\left(\nabla\times\boldsymbol{W}_k\right)\cdot\left(\nabla\times\widetilde{\boldsymbol{T}}\right)+\mu_o\boldsymbol{W}_k\cdot\frac{\partial\widetilde{\boldsymbol{T}}}{\partial t}-\mu_o\boldsymbol{W}_k\cdot\nabla\frac{\partial\widetilde{\Phi}}{\partial t}\right]\mathrm{d}\Omega
$$

$$
-\int_{\Gamma_{nc_{\boldsymbol{A}}}}\boldsymbol{W}_k\cdot\left(\frac{\partial\widetilde{\boldsymbol{A}}}{\partial t}\times\boldsymbol{n}_{\boldsymbol{A}}\right)\mathrm{d}\Gamma = -\int_{\Omega_c}\mu_o\boldsymbol{W}_k\cdot\frac{\partial\widetilde{\boldsymbol{T}}_0}{\partial t}\,\mathrm{d}\Omega \tag{3.202}
$$

$$
-\int_{\Omega_c}\boldsymbol{W}_k\cdot\frac{\partial\widetilde{\boldsymbol{R}}}{\partial t}\,\mathrm{d}\Omega,
$$

$$- \int_{\Omega_c} \mu_o \nabla N_k \cdot \frac{\partial \widetilde{T}}{\partial t} \, \mathrm{d}\Omega + \int_{\Omega_c} \mu_o \nabla N_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \, \mathrm{d}\Omega + \int_{\Omega_{n_\Phi}} \mu \nabla N_k \cdot \nabla \frac{\partial \widetilde{\Phi}}{\partial t} \, \mathrm{d}\Omega$$

$$- \int_{\Gamma_{nc_A} \cup \Gamma_{A,\Phi}} N_k \left(\nabla \times \frac{\partial \widetilde{A}}{\partial t} \right) \cdot n_A \, \mathrm{d}\Gamma$$

$$= \int_{\Omega_c} \mu_o \nabla N_k \cdot \frac{\partial \widetilde{T}_0}{\partial t} \, \mathrm{d}\Omega + \int_{\Omega_{n_\Phi}} \mu \nabla N_k \cdot \frac{\partial \widetilde{T}_0}{\partial t} \, \mathrm{d}\Omega + \int_{\Gamma_{B_\Phi}} N_k \frac{\partial b}{\partial t} \, \mathrm{d}\Gamma \tag{3.203}$$

$$+ \int_{\Omega_c} \nabla N_k \cdot \frac{\partial \widetilde{R}}{\partial t} \, \mathrm{d}\Omega,$$

$$- \int_{\Omega_{n_A}} \nu \left(\nabla \times W_k \right) \cdot \left(\nabla \times \frac{\partial \widetilde{A}}{\partial t} \right) \mathrm{d}\Omega$$

$$- \int_{\Gamma_{nc_A}} \frac{\partial \widetilde{T}}{\partial t} \cdot \left(W_k \times n_A \right) \mathrm{d}\Gamma - \int_{\Gamma_{nc_A} \cup \Gamma_{A,\Phi}} \frac{\partial \widetilde{\Phi}}{\partial t} \left(\nabla \times W_k \right) \cdot n_A \mathrm{d}\Gamma = \tag{3.204}$$

$$- \int_{\Omega_{n_A}} \left(\nabla \times W_k \right) \cdot \frac{\partial \widetilde{T}_0}{\partial t} \mathrm{d}\Omega + \int_{C_H} \frac{\partial \widetilde{\Phi}}{\partial t} W_k \cdot \mathrm{d}l.$$

Here $k = 1, \cdots, J$, $k = 1, \cdots, I$ and $k = 1, \cdots, J$, respectively.

3.3.15 The gauged $A, V - \Phi$ formulation

The magnetic vector potential A and the electric scalar potential V are used in the eddy current region Ω_c and the reduced magnetic scalar potential Φ is used in the region free of eddy currents, Ω_n. The three unknown potentials, A, V and Φ can be obtained from the weak formulation of the more general gauged $A, V - A - \Phi$ method (see on page 114),

$$\int_{\Omega_c} \left[\nu_o \left(\nabla \times W_k \right) \cdot \left(\nabla \times \widetilde{A} \right) + \nu_o \nabla \cdot W_k \nabla \cdot \widetilde{A} \right] \mathrm{d}\Omega$$

$$+ \int_{\Omega_c} W_k \cdot \left(\sigma \frac{\partial \widetilde{A}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \mathrm{d}\Omega + \int_{\Gamma_{nc}} \widetilde{\Phi} \left(\nabla \times W_k \right) \cdot n_A \mathrm{d}\Gamma \tag{3.205}$$

$$= \int_{\Gamma_{nc}} W_k \cdot (\widetilde{T}_0 \times n_A) \mathrm{d}\Gamma + \int_C \Phi_0 W_k \cdot \mathrm{d}l - \int_{\Omega_c} \left(\nabla \times W_k \right) \cdot \widetilde{I} \, \mathrm{d}\Omega,$$

$$\int_{\Omega_c} \nabla N_k \cdot \left(\sigma \frac{\partial \widetilde{A}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \mathrm{d}\Omega = 0, \tag{3.206}$$

$$- \int_{\Omega_n} \mu \nabla N_k \cdot \nabla \widetilde{\Phi} \, \mathrm{d}\Omega + \int_{\Gamma_{nc}} N_k (\nabla \times \widetilde{A}) \cdot n_A \mathrm{d}\Gamma =$$

$$- \int_{\Omega_n} \mu \nabla N_k \cdot \widetilde{T}_0 \, \mathrm{d}\Omega - \int_{\Gamma_B} N_k b \, \mathrm{d}\Gamma. \tag{3.207}$$

Here $k = 1, \cdots, J$, $k = 1, \cdots, I$ and $k = 1, \cdots, I$, respectively.

3.3.16   The ungauged $A, V - \Phi$ formulation

The magnetic vector potential A with the electric scalar potential V is used in the eddy current region Ω_c and the reduced magnetic scalar potential Φ is used in the region free of eddy currents, Ω_n. The three unknown potentials, A, V and Φ, can be obtained by reducing the weak formulation of the more general ungauged $A, V - A - \Phi$ method (see on page 119),

$$
\begin{aligned}
&\int_{\Omega_c} \nu_o \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}} \right) \mathrm{d}\Omega \\
&+ \int_{\Omega_c} \boldsymbol{W}_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \mathrm{d}\Omega \\
&+ \int_{\Gamma_{nc_\Phi}} \widetilde{\Phi} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \boldsymbol{n}_{\boldsymbol{A}} \mathrm{d}\Gamma \\
&= \int_C \Phi_0 \boldsymbol{W}_k \cdot \mathrm{d}\boldsymbol{l} \\
&- \int_{\Omega_c} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \widetilde{\boldsymbol{I}} \, \mathrm{d}\Omega,
\end{aligned}
\tag{3.208}
$$

$$
\int_{\Omega_c} \nabla N_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \mathrm{d}\Omega = 0,
\tag{3.209}
$$

$$
\begin{aligned}
&- \int_{\Omega_n} \mu \nabla N_k \cdot \nabla \widetilde{\Phi} \, \mathrm{d}\Omega \\
&+ \int_{\Gamma_{nc}} N_k (\nabla \times \widetilde{\boldsymbol{A}}) \cdot \boldsymbol{n}_{\boldsymbol{A}} \mathrm{d}\Gamma = \\
&- \int_{\Omega_n} \mu \nabla N_k \cdot \widetilde{\boldsymbol{T}}_0 \, \mathrm{d}\Omega \\
&- \int_{\Gamma_B} N_k b \, \mathrm{d}\Gamma.
\end{aligned}
\tag{3.210}
$$

Here $k = 1, \cdots, J$, $k = 1, \cdots, I$ and $k = 1, \cdots, I$, respectively.

3.3.17   The gauged $A, V - A - \Phi$ formulation

The magnetic vector potential A with the electric scalar potential V is used in the eddy current region Ω_c, the magnetic vector potential A is used in the holes, Ω_{n_A} and the reduced magnetic scalar potential Φ is used in the rest region free of eddy currents, Ω_{n_Φ}.

There are three unknown potentials, A, V and Φ, that is three equations must be set up.

It is important to note that the resulting system of equations should be symmetric, we are going to take care about it.

The first weak formulation is based on the partial differential equations (2.489) and (2.491), on the Neumann type boundary conditions (2.493), (2.496), (2.501), (2.503) and on the interface conditions (2.505), (2.506), (2.510), (2.514)

$$
\int_{\Omega_c} \boldsymbol{W}_k \cdot \left[\nabla \times \left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} \right) - \nabla \left(\nu_o \nabla \cdot \widetilde{\boldsymbol{A}} \right) \right] \mathrm{d}\Omega
$$

$$
+ \int_{\Omega_c} \boldsymbol{W}_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \mathrm{d}\Omega
$$

$$
+ \int_{\Omega_{n_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \left[\nabla \times \left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) - \nabla \left(\nu \nabla \cdot \widetilde{\boldsymbol{A}} \right) \right] \mathrm{d}\Omega
$$

$$
+ \int_{\Gamma_E} \nu_o \nabla \cdot \widetilde{\boldsymbol{A}} (\boldsymbol{W}_k \cdot \boldsymbol{n}) \, \mathrm{d}\Gamma + \int_{\Gamma_{B_{\boldsymbol{A}}}} \nu \nabla \cdot \widetilde{\boldsymbol{A}} (\boldsymbol{W}_k \cdot \boldsymbol{n}) \, \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_{H_c}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}} \right) \times \boldsymbol{n} \right] \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_{H_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{n} - \boldsymbol{K} \right] \mathrm{d}\Gamma \tag{3.211}
$$

$$
+ \int_{\Gamma_{nc_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}} \right) \times \boldsymbol{n}_c + \left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{n}_{\boldsymbol{A}} \right] \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_{nc_{\boldsymbol{A}}}} \left[\nu_o \nabla \cdot \widetilde{\boldsymbol{A}} (\boldsymbol{W}_k \cdot \boldsymbol{n}_c) + \nu \nabla \cdot \widetilde{\boldsymbol{A}} (\boldsymbol{W}_k \cdot \boldsymbol{n}_{\boldsymbol{A}}) \right] \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_{nc_\Phi}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}} \right) \times \boldsymbol{n}_c + \left(\widetilde{\boldsymbol{T}}_0 - \nabla \widetilde{\Phi} \right) \times \boldsymbol{n}_\Phi \right] \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_{\boldsymbol{A},\Phi}} \boldsymbol{W}_k \cdot \left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{n}_{\boldsymbol{A}} + \left(\widetilde{\boldsymbol{T}}_0 - \nabla \widetilde{\Phi} \right) \times \boldsymbol{n}_\Phi \right] \mathrm{d}\Gamma
$$

$$
= \int_{\Omega_n} \boldsymbol{W}_k \cdot \boldsymbol{J}_0 \, \mathrm{d}\Omega - \int_{\Omega_c} \boldsymbol{W}_k \cdot (\nabla \times \widetilde{\boldsymbol{I}}) \, \mathrm{d}\Omega,
$$

where

$$
\boldsymbol{W}_k \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_E \cup \Gamma_{B_{\boldsymbol{A}}}, \tag{3.212}
$$

and

$$
\boldsymbol{W}_k \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_{H_c} \cup \Gamma_{nc_\Phi} \cup \Gamma_{H_{\boldsymbol{A}}} \cup \Gamma_{\boldsymbol{A},\Phi}, \tag{3.213}
$$

moreover $k = 1, \cdots, J$. The second order derivatives in the first and in the third integrals can be reduced to first order one by using the identity

$$
\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \times \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \times \boldsymbol{v}, \tag{3.214}
$$

with the notations $\boldsymbol{v} = \boldsymbol{W}_k$ and $\boldsymbol{u} = \nu_o \nabla \times \widetilde{\boldsymbol{A}}$, or $\boldsymbol{u} = \nu \nabla \times \widetilde{\boldsymbol{A}}$ and the identity

$$
\nabla \cdot (\varphi \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \varphi + \varphi \nabla \cdot \boldsymbol{v} \tag{3.215}
$$

with the notations $\varphi = \nu_o \nabla \cdot \widetilde{\boldsymbol{A}}$, or $\varphi = \nu \nabla \cdot \widetilde{\boldsymbol{A}}$ and $\boldsymbol{v} = \boldsymbol{W}_k$ also must be used. The last integral on the right-hand side can also be modified by using the notations $\boldsymbol{v} = \boldsymbol{W}_k$ and $\boldsymbol{u} = \widetilde{\boldsymbol{I}}$. This results in

$$
\int_{\Omega_c} \left[\nu_o \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}} \right) + \nu_o \nabla \cdot \boldsymbol{W}_k \nabla \cdot \widetilde{\boldsymbol{A}} \right] \mathrm{d}\Omega
$$

$$
+ \int_{\Omega_c} \boldsymbol{W}_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \mathrm{d}\Omega
$$

$$
+ \int_{\Omega_{n_{\boldsymbol{A}}}} \left[\nu \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}} \right) + \nu \nabla \cdot \boldsymbol{W}_k \nabla \cdot \widetilde{\boldsymbol{A}} \right] \mathrm{d}\Omega
$$

$$
+ \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{nc_\Phi}} \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{W}_k \right] \cdot \boldsymbol{n} \, \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_{B_{\boldsymbol{A}}} \cup \Gamma_{H_{\boldsymbol{A}}} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{\boldsymbol{A},\Phi}} \left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{W}_k \right] \cdot \boldsymbol{n} \, \mathrm{d}\Gamma
$$

$$
- \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{nc_\Phi}} \nu_o \nabla \cdot \widetilde{\boldsymbol{A}} (\boldsymbol{W}_k \cdot \boldsymbol{n}) \, \mathrm{d}\Gamma
$$

$$
- \int_{\Gamma_{B_{\boldsymbol{A}}} \cup \Gamma_{H_{\boldsymbol{A}}} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{\boldsymbol{A},\Phi}} \nu \nabla \cdot \widetilde{\boldsymbol{A}} (\boldsymbol{W}_k \cdot \boldsymbol{n}) \, \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_E} \nu_o \nabla \cdot \widetilde{\boldsymbol{A}} (\boldsymbol{W}_k \cdot \boldsymbol{n}) \, \mathrm{d}\Gamma + \int_{\Gamma_{B_{\boldsymbol{A}}}} \nu \nabla \cdot \widetilde{\boldsymbol{A}} (\boldsymbol{W}_k \cdot \boldsymbol{n}) \, \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_{H_c}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}} \right) \times \boldsymbol{n} \right] \mathrm{d}\Gamma \tag{3.216}
$$

$$
+ \int_{\Gamma_{H_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{n} - \boldsymbol{K} \right] \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_{nc_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}} \right) \times \boldsymbol{n}_c + \left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{n}_{\boldsymbol{A}} \right] \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_{nc_{\boldsymbol{A}}}} \left[\nu_o \nabla \cdot \widetilde{\boldsymbol{A}} (\boldsymbol{W}_k \cdot \boldsymbol{n}_c) + \nu \nabla \cdot \widetilde{\boldsymbol{A}} (\boldsymbol{W}_k \cdot \boldsymbol{n}_{\boldsymbol{A}}) \right] \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_{nc_\Phi}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}} \right) \times \boldsymbol{n}_c + \left(\widetilde{\boldsymbol{T}}_0 - \nabla \widetilde{\Phi} \right) \times \boldsymbol{n}_\Phi \right] \mathrm{d}\Gamma
$$

$$
+ \int_{\Gamma_{\boldsymbol{A},\Phi}} \boldsymbol{W}_k \cdot \left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{n}_{\boldsymbol{A}} + \left(\widetilde{\boldsymbol{T}}_0 - \nabla \widetilde{\Phi} \right) \times \boldsymbol{n}_\Phi \right] \mathrm{d}\Gamma
$$

$$
= \int_{\Omega_n} \boldsymbol{W}_k \cdot \boldsymbol{J}_0 \, \mathrm{d}\Omega - \int_{\Omega_c} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \widetilde{\boldsymbol{I}} \, \mathrm{d}\Gamma
$$

$$
- \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{nc_\Phi}} \left(\widetilde{\boldsymbol{I}} \times \boldsymbol{W}_k \right) \cdot \boldsymbol{n} \, \mathrm{d}\Gamma.
$$

The first and the second boundary integrals are vanishing on $\Gamma_{H_c} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{nc_\Phi}$ and on $\Gamma_{H_{\boldsymbol{A}}} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{\boldsymbol{A},\Phi}$, respectively, because of the seventh, the first term of the eighth, the ninth, the first term of eleventh and the twelfth boundary integrals after using

$$\left[\left(\nu\nabla\times\widetilde{\boldsymbol{A}}\right)\times\boldsymbol{W}_k\right]\cdot\boldsymbol{n} = -\boldsymbol{W}_k\cdot\left[\left(\nu\nabla\times\widetilde{\boldsymbol{A}}\right)\times\boldsymbol{n}\right]. \tag{3.217}$$

The first and the second boundary integral terms are equal to zero on the rest parts Γ_E and $\Gamma_{B_{\boldsymbol{A}}}$ as well, because of the Dirichlet type boundary conditions (2.499) and (2.502), i.e. $\boldsymbol{W}_k\times\boldsymbol{n} = \boldsymbol{0}$ on these boundaries. The third and the fourth boundary integral terms are vanishing on $\Gamma_E\cup\Gamma_{nc\boldsymbol{A}}$, on $\Gamma_{B_{\boldsymbol{A}}}\cup\Gamma_{nc\boldsymbol{A}}$, because of the fifth, the sixth and the tenth boundary integral terms. On the rest part $\Gamma_{H_c}\cup\Gamma_{nc\Phi}$ and on $\Gamma_{H_{\boldsymbol{A}}}\cup\Gamma_{\boldsymbol{A},\Phi}$ these integral terms are equal to zero, too, because of Dirichlet type boundary conditions (2.495), (2.513), (2.497) and (2.516), i.e. $\boldsymbol{W}_k\cdot\boldsymbol{n} = 0$ there. The last boundary integral term on the right is vanishing on $\Gamma_{H_c}\cup\Gamma_{nc\boldsymbol{A}}\cup\Gamma_{nc\Phi}$, because of the same terms on the left-hand side and the identity $-(\widetilde{\boldsymbol{I}}\times\boldsymbol{W}_k)\cdot\boldsymbol{n} = (\widetilde{\boldsymbol{I}}\times\boldsymbol{n})\cdot\boldsymbol{W}_k$ must be used. It is equal to zero on the rest part Γ_E, because of the Dirichlet type boundary condition (2.499).

The last two interface integral terms on the left-hand side containing $\nabla\widetilde{\Phi}$ can be reformulated in the same way as in the $\boldsymbol{A}-\Phi$ formulation on page 83, from (3.88) to (3.92).

Finally, the first equation of the weak form is the following:

$$\int_{\Omega_c}\left[\nu_o\left(\nabla\times\boldsymbol{W}_k\right)\cdot\left(\nabla\times\widetilde{\boldsymbol{A}}\right) + \nu_o\nabla\cdot\boldsymbol{W}_k\nabla\cdot\widetilde{\boldsymbol{A}}\right]d\Omega$$

$$+\int_{\Omega_c}\boldsymbol{W}_k\cdot\left(\sigma\frac{\partial\widetilde{\boldsymbol{A}}}{\partial t} + \sigma\nabla\frac{\partial\widetilde{v}}{\partial t}\right)d\Omega$$

$$+\int_{\Omega_{n_{\boldsymbol{A}}}}\left[\nu\left(\nabla\times\boldsymbol{W}_k\right)\cdot\left(\nabla\times\widetilde{\boldsymbol{A}}\right) + \nu\nabla\cdot\boldsymbol{W}_k\nabla\cdot\widetilde{\boldsymbol{A}}\right]d\Omega$$

$$+\int_{\Gamma_{nc\Phi}\cup\Gamma_{\boldsymbol{A},\Phi}}\widetilde{\Phi}\left(\nabla\times\boldsymbol{W}_k\right)\cdot\boldsymbol{n}_{\boldsymbol{A}}d\Gamma = \int_{\Omega_{n_{\boldsymbol{A}}}}\boldsymbol{W}_k\cdot\boldsymbol{J}_0\,d\Omega \tag{3.218}$$

$$+\int_{\Gamma_{H_{\boldsymbol{A}}}}\boldsymbol{W}_k\cdot\boldsymbol{K}d\Gamma + \int_{\Gamma_{nc\Phi}\cup\Gamma_{\boldsymbol{A},\Phi}}\boldsymbol{W}_k\cdot(\widetilde{\boldsymbol{T}}_0\times\boldsymbol{n}_{\boldsymbol{A}})d\Gamma$$

$$+\int_C\Phi_0\boldsymbol{W}_k\cdot d\boldsymbol{l} - \int_{\Omega_c}\left(\nabla\times\boldsymbol{W}_k\right)\cdot\widetilde{\boldsymbol{I}}\,d\Omega.$$

Here, all the normal unit vectors are changed to $\boldsymbol{n}_{\boldsymbol{A}}$, pointing out from the regions where $\boldsymbol{A}$ can be found ($\Omega_c\cup\Omega_{n_{\boldsymbol{A}}}$) and $k = 1,\cdots,J$.

The partial differential equation (2.490), the Neumann type boundary and interface conditions (2.494), (2.507) and (2.512) can be summarized in the following formulation. Here, the conditions are multiplied by -1,

$$-\int_{\Omega_c}N_k\nabla\cdot\left(\sigma\frac{\partial\widetilde{\boldsymbol{A}}}{\partial t} + \sigma\nabla\frac{\partial\widetilde{v}}{\partial t}\right)d\Omega$$

$$+\int_{\Gamma_{H_c}\cup\Gamma_{nc\boldsymbol{A}}\Gamma_{nc\Phi}}N_k\left(\sigma\frac{\partial\widetilde{\boldsymbol{A}}}{\partial t} + \sigma\nabla\frac{\partial\widetilde{v}}{\partial t}\right)\cdot\boldsymbol{n}\,d\Gamma = 0, \tag{3.219}$$

where $k = 1,\cdots,I$. The first integral terms can be reformulated by the use of the identity

$$\nabla\cdot(\varphi\boldsymbol{v}) = \boldsymbol{v}\cdot\nabla\varphi + \varphi\nabla\cdot\boldsymbol{v} \tag{3.220}$$

with the notations $\varphi = N_k$ and $\boldsymbol{v} = \sigma\partial\widetilde{\boldsymbol{A}}/\partial t + \sigma\nabla\partial\widetilde{v}/\partial t$,

$$\int_{\Omega_c} \nabla N_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \mathrm{d}\Omega$$

$$- \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{nc_\Phi}} N_k \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \cdot \boldsymbol{n} \, \mathrm{d}\Gamma \qquad (3.221)$$

$$+ \int_{\Gamma_{H_c} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{nc_\Phi}} N_k \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \cdot \boldsymbol{n} \, \mathrm{d}\Gamma = 0.$$

The boundary integral terms are vanishing on $\Gamma_{H_c} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{nc_\Phi}$ and the first one is equal to zero because $N_k = 0$ on Γ_E according to Dirichlet type boundary condition (2.500). Finally, the following weak formulation can be obtained:

$$\int_{\Omega_c} \nabla N_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \mathrm{d}\Omega = 0, \quad k = 1, \cdots, I. \qquad (3.222)$$

The partial differential equation (2.492), the Neumann type boundary condition (2.504) and the interface conditions (2.511) and (2.515) build the third weak formulation,

$$\int_{\Omega_{n_\Phi}} N_k \nabla \cdot (\mu \nabla \widetilde{\Phi}) \, \mathrm{d}\Omega + \int_{\Gamma_{B_\Phi}} N_k \left(b + \mu \widetilde{\boldsymbol{T}}_0 \cdot \boldsymbol{n} - \mu \nabla \widetilde{\Phi} \cdot \boldsymbol{n} \right) \mathrm{d}\Gamma$$

$$+ \int_{\Gamma_{nc_\Phi} \cup \Gamma_{\boldsymbol{A},\Phi}} N_k \left[(\nabla \times \widetilde{\boldsymbol{A}}) \cdot \boldsymbol{n}_+ \boldsymbol{A} \mu \left(\widetilde{\boldsymbol{T}}_0 - \nabla \widetilde{\Phi} \right) \cdot \boldsymbol{n}_\Phi \right] \mathrm{d}\Gamma \qquad (3.223)$$

$$= \int_{\Omega_{n_\Phi}} N_k \nabla \cdot (\mu \widetilde{\boldsymbol{T}}_0) \, \mathrm{d}\Omega,$$

where $k = 1, \cdots, I$. The first integral terms can be reformulated by the use of the identity

$$\nabla \cdot (\varphi \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \varphi + \varphi \nabla \cdot \boldsymbol{v} \qquad (3.224)$$

with the notations $\varphi = N_k$ and $\boldsymbol{v} = \mu \nabla \widetilde{\Phi}$,

$$- \int_{\Omega_{n_\Phi}} \mu \nabla N_k \cdot \nabla \widetilde{\Phi} \, \mathrm{d}\Omega + \int_{\Gamma_{H_\Phi} \cup \Gamma_{B_\Phi} \cup \Gamma_{nc_\Phi} \cup \Gamma_{\boldsymbol{A},\Phi}} \mu N_k \nabla \widetilde{\Phi} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma$$

$$+ \int_{\Gamma_{B_\Phi}} N_k \left(b + \mu \widetilde{\boldsymbol{T}}_0 \cdot \boldsymbol{n} - \mu \nabla \widetilde{\Phi} \cdot \boldsymbol{n} \right) \mathrm{d}\Gamma$$

$$+ \int_{\Gamma_{nc_\Phi} \cup \Gamma_{\boldsymbol{A},\Phi}} N_k \left[(\nabla \times \widetilde{\boldsymbol{A}}) \cdot \boldsymbol{n}_{\boldsymbol{A}} + \mu \left(\widetilde{\boldsymbol{T}}_0 - \nabla \widetilde{\Phi} \right) \cdot \boldsymbol{n}_\Phi \right] \mathrm{d}\Gamma = \qquad (3.225)$$

$$- \int_{\Omega_{n_\Phi}} \mu \nabla N_k \cdot \widetilde{\boldsymbol{T}}_0 \, \mathrm{d}\Omega + \int_{\Gamma_{H_\Phi} \cup \Gamma_{B_\Phi} \cup \Gamma_{nc_\Phi} \cup \Gamma_{\boldsymbol{A},\Phi}} \mu N_k \widetilde{\boldsymbol{T}}_0 \cdot \boldsymbol{n} \, \mathrm{d}\Gamma.$$

The first surface integral is vanishing on the part $\Gamma_{B_\Phi} \cup \Gamma_{nc_\Phi} \cup \Gamma_{\boldsymbol{A},\Phi}$ of the boundary because of the same term with opposite sign in the second and in the third surface integrals (of course $\boldsymbol{n} = \boldsymbol{n}_\Phi$ on $\Gamma_{nc_\Phi} \cup \Gamma_{\boldsymbol{A},\Phi}$). The last surface integral is vanishing on the same part of the boundary according to the second and the third boundary integrals. The first as

well as the last surface integrals are equal to zero on the rest part Γ_{H_Φ}, because $N_k = 0$ there according to the Dirichlet type boundary condition (2.498). The result of this math is as follows:

$$-\int_{\Omega_{n_\Phi}} \mu \nabla N_k \cdot \nabla \widetilde{\Phi}\, \mathrm{d}\Omega + \int_{\Gamma_{nc_\Phi} \cup \Gamma_{\boldsymbol{A},\Phi}} N_k (\nabla \times \widetilde{\boldsymbol{A}}) \cdot \boldsymbol{n_A} \mathrm{d}\Gamma =$$
$$-\int_{\Omega_{n_\Phi}} \mu \nabla N_k \cdot \widetilde{\boldsymbol{T}}_0\, \mathrm{d}\Omega - \int_{\Gamma_{B_\Phi}} N_k b\, \mathrm{d}\Gamma, \quad k = 1, \cdots, I. \tag{3.226}$$

Finally, the three equations of the weak formulation of the gauged $\boldsymbol{A}, V - \boldsymbol{A} - \Phi$ method is the following:

$$\int_{\Omega_c} \left[\nu_o \left(\nabla \times \boldsymbol{W}_k\right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}}\right) + \nu_o \nabla \cdot \boldsymbol{W}_k \nabla \cdot \widetilde{\boldsymbol{A}} \right] \mathrm{d}\Omega$$
$$+ \int_{\Omega_c} \boldsymbol{W}_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \mathrm{d}\Omega$$
$$+ \int_{\Omega_{n_{\boldsymbol{A}}}} \left[\nu \left(\nabla \times \boldsymbol{W}_k\right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}}\right) + \nu \nabla \cdot \boldsymbol{W}_k \nabla \cdot \widetilde{\boldsymbol{A}} \right] \mathrm{d}\Omega$$
$$+ \int_{\Gamma_{nc_\Phi} \cup \Gamma_{\boldsymbol{A},\Phi}} \widetilde{\Phi} \left(\nabla \times \boldsymbol{W}_k\right) \cdot \boldsymbol{n_A} \mathrm{d}\Gamma \tag{3.227}$$
$$= \int_{\Omega_{n_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \boldsymbol{J}_0\, \mathrm{d}\Omega + \int_{\Gamma_{H_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \boldsymbol{K} \mathrm{d}\Gamma$$
$$+ \int_{\Gamma_{nc_\Phi} \cup \Gamma_{\boldsymbol{A},\Phi}} \boldsymbol{W}_k \cdot (\widetilde{\boldsymbol{T}}_0 \times \boldsymbol{n_A}) \mathrm{d}\Gamma + \int_C \Phi_0 \boldsymbol{W}_k \cdot \mathrm{d}\boldsymbol{l}$$
$$- \int_{\Omega_c} (\nabla \times \boldsymbol{W}_k) \cdot \widetilde{\boldsymbol{I}}\, \mathrm{d}\Omega,$$

$$\int_{\Omega_c} \nabla N_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \mathrm{d}\Omega = 0, \tag{3.228}$$

$$-\int_{\Omega_{n_\Phi}} \mu \nabla N_k \cdot \nabla \widetilde{\Phi}\, \mathrm{d}\Omega + \int_{\Gamma_{nc_\Phi} \cup \Gamma_{\boldsymbol{A},\Phi}} N_k (\nabla \times \widetilde{\boldsymbol{A}}) \cdot \boldsymbol{n_A} \mathrm{d}\Gamma =$$
$$-\int_{\Omega_{n_\Phi}} \mu \nabla N_k \cdot \widetilde{\boldsymbol{T}}_0\, \mathrm{d}\Omega - \int_{\Gamma_{B_\Phi}} N_k b\, \mathrm{d}\Gamma. \tag{3.229}$$

Here $k = 1, \cdots, J, k = 1, \cdots, I, k = 1, \cdots, I$, respectively.

3.3.18 The ungauged $A, V - A - \Phi$ formulation

The magnetic vector potential $\boldsymbol{A}$ with the electric scalar potential V is used in the eddy current region Ω_c, the magnetic vector potential $\boldsymbol{A}$ is used in the holes, $\Omega_{n_{\boldsymbol{A}}}$ and the reduced magnetic scalar potential Φ is used in the rest region free of eddy currents, Ω_{n_Φ}.

There are three unknown potentials, $\boldsymbol{A}$, V and Φ, so three equations must be set up. It is important to note that the resulting system of equations should be symmetric.

The source current density $\boldsymbol{J}_0$ is represented by the impressed current vector potential $\boldsymbol{T}_0$.

The first weak formulation is based on the partial differential equations (2.517) and (2.519), on the Neumann type boundary conditions (2.521), (2.523) and on the interface conditions (2.529), (2.532), (2.535). The term $\nabla \times \boldsymbol{T}_0$ has been appended to the right-hand side of the partial differential equation (2.517), similarly to (2.519), according to the definition (3.139). The weak formulation is as follows:

$$
\begin{aligned}
&\int_{\Omega_c} \boldsymbol{W}_k \cdot \left[\nabla \times \left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} \right) \right] \mathrm{d}\Omega \\
&+ \int_{\Omega_c} \boldsymbol{W}_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \mathrm{d}\Omega \\
&+ \int_{\Omega_{n_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \left[\nabla \times \left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \right] \mathrm{d}\Omega \\
&+ \int_{\Gamma_{H_c}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}} \right) \times \boldsymbol{n} \right] \mathrm{d}\Gamma \\
&+ \int_{\Gamma_{H_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{n} - \boldsymbol{K} \right] \mathrm{d}\Gamma \\
&+ \int_{\Gamma_{nc_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}} \right) \times \boldsymbol{n}_c + \left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{n}_{\boldsymbol{A}} \right] \mathrm{d}\Gamma \\
&+ \int_{\Gamma_{nc_\Phi}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}} \right) \times \boldsymbol{n}_c + \left(\widetilde{\boldsymbol{T}}_0 - \nabla \widetilde{\Phi} \right) \times \boldsymbol{n}_\Phi \right] \mathrm{d}\Gamma \\
&+ \int_{\Gamma_{\boldsymbol{A},\Phi}} \boldsymbol{W}_k \cdot \left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}} \right) \times \boldsymbol{n}_{\boldsymbol{A}} + \left(\widetilde{\boldsymbol{T}}_0 - \nabla \widetilde{\Phi} \right) \times \boldsymbol{n}_\Phi \right] \mathrm{d}\Gamma \\
&= \int_{\Omega_c \cup \Omega_{n_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot (\nabla \times \widetilde{\boldsymbol{T}}_0) \mathrm{d}\Omega \\
&\quad - \int_{\Omega_c} \boldsymbol{W}_k \cdot (\nabla \times \widetilde{\boldsymbol{I}}) \mathrm{d}\Omega,
\end{aligned}
\tag{3.230}
$$

where

$$
\boldsymbol{W}_k \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{on} \quad \Gamma_E \cup \Gamma_{B_{\boldsymbol{A}}}, \tag{3.231}
$$

and $k = 1, \cdots, J$. The second order derivatives in the first and in the third integrals can be reduced to first order one by using the identity

$$
\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \times \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \times \boldsymbol{v}, \tag{3.232}
$$

with the notations $\boldsymbol{v} = \boldsymbol{W}_k$ and $\boldsymbol{u} = \nu_o \nabla \times \widetilde{\boldsymbol{A}}$, or $\boldsymbol{u} = \nu \nabla \times \widetilde{\boldsymbol{A}}$. The integrals on the right-hand side can also be reformulated by $\boldsymbol{v} = \boldsymbol{W}_k$, $\boldsymbol{u} = \widetilde{\boldsymbol{T}}_0$ and $\boldsymbol{u} = \widetilde{\boldsymbol{I}}$. This results in

$$\int_{\Omega_c} \nu_o \left(\nabla \times \boldsymbol{W}_k\right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}}\right) \mathrm{d}\Omega + \int_{\Omega_c} \boldsymbol{W}_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t}\right) \mathrm{d}\Omega$$

$$+ \int_{\Omega_{n_{\boldsymbol{A}}}} \nu \left(\nabla \times \boldsymbol{W}_k\right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}}\right) \mathrm{d}\Omega$$

$$+ \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{nc_\Phi}} \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}}\right) \times \boldsymbol{W}_k\right] \cdot \boldsymbol{n} \, \mathrm{d}\Gamma$$

$$+ \int_{\Gamma_{B_{\boldsymbol{A}}} \cup \Gamma_{H_{\boldsymbol{A}}} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{\boldsymbol{A},\Phi}} \left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}}\right) \times \boldsymbol{W}_k\right] \cdot \boldsymbol{n} \, \mathrm{d}\Gamma$$

$$+ \int_{\Gamma_{H_c}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}}\right) \times \boldsymbol{n}\right] \mathrm{d}\Gamma$$

$$+ \int_{\Gamma_{H_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}}\right) \times \boldsymbol{n} - \boldsymbol{K}\right] \mathrm{d}\Gamma$$

$$+ \int_{\Gamma_{nc_{\boldsymbol{A}}}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}}\right) \times \boldsymbol{n}_c + \left(\nu \nabla \times \widetilde{\boldsymbol{A}}\right) \times \boldsymbol{n}_{\boldsymbol{A}}\right] \mathrm{d}\Gamma \qquad (3.233)$$

$$+ \int_{\Gamma_{nc_\Phi}} \boldsymbol{W}_k \cdot \left[\left(\nu_o \nabla \times \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{I}}\right) \times \boldsymbol{n}_c + \left(\widetilde{\boldsymbol{T}}_0 - \nabla \widetilde{\Phi}\right) \times \boldsymbol{n}_\Phi\right] \mathrm{d}\Gamma$$

$$+ \int_{\Gamma_{\boldsymbol{A},\Phi}} \boldsymbol{W}_k \cdot \left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}}\right) \times \boldsymbol{n}_{\boldsymbol{A}} + \left(\widetilde{\boldsymbol{T}}_0 - \nabla \widetilde{\Phi}\right) \times \boldsymbol{n}_\Phi\right] \mathrm{d}\Gamma$$

$$= \int_{\Omega_c \cup \Omega_{n_{\boldsymbol{A}}}} \left(\nabla \times \boldsymbol{W}_k\right) \cdot \widetilde{\boldsymbol{T}}_0 \, \mathrm{d}\Omega$$

$$+ \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{nc_\Phi}} \left(\widetilde{\boldsymbol{T}}_0 \times \boldsymbol{W}_k\right) \cdot \boldsymbol{n} \, \mathrm{d}\Gamma$$

$$+ \int_{\Gamma_{B_{\boldsymbol{A}}} \cup \Gamma_{H_{\boldsymbol{A}}} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{\boldsymbol{A},\Phi}} \left(\widetilde{\boldsymbol{T}}_0 \times \boldsymbol{W}_k\right) \cdot \boldsymbol{n} \, \mathrm{d}\Gamma$$

$$- \int_{\Omega_c} \left(\nabla \times \boldsymbol{W}_k\right) \cdot \widetilde{\boldsymbol{I}} \, \mathrm{d}\Gamma - \int_{\Gamma_E \cup \Gamma_{H_c} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{nc_\Phi}} \left(\widetilde{\boldsymbol{I}} \times \boldsymbol{W}_k\right) \cdot \boldsymbol{n} \, \mathrm{d}\Gamma.$$

The first and the second boundary integral terms are vanishing on the boundary part $\Gamma_{H_c} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{nc_\Phi}$ and on $\Gamma_{H_{\boldsymbol{A}}} \cup \Gamma_{nc_{\boldsymbol{A}}} \cup \Gamma_{\boldsymbol{A},\Phi}$, respectively, because of the third, fourth, fifth and the first term of the sixth and of the seventh boundary integral terms after using the identity

$$\left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}}\right) \times \boldsymbol{W}_k\right] \cdot \boldsymbol{n} = -\boldsymbol{W}_k \cdot \left[\left(\nu \nabla \times \widetilde{\boldsymbol{A}}\right) \times \boldsymbol{n}\right]. \qquad (3.234)$$

The first and the second boundary integral terms are equal to zero on the rest parts Γ_E and $\Gamma_{B_{\boldsymbol{A}}}$ as well, because of the Dirichlet type boundary condition (2.525) and (2.527), i.e. $\boldsymbol{W}_k \times \boldsymbol{n} = \boldsymbol{0}$ on these boundaries.

The first and the second boundary integral terms on the right-hand side are vanishing on Γ_{nc_Φ} and on $\Gamma_{\boldsymbol{A},\Phi}$ according to the sixth and the seventh terms and that

$$\boldsymbol{W}_k \cdot \left(\widetilde{\boldsymbol{T}}_0 \times \boldsymbol{n}_\Phi\right) = \boldsymbol{n}_\Phi \cdot \left(\boldsymbol{W}_k \times \widetilde{\boldsymbol{T}}_0\right) = -\boldsymbol{n}_\Phi \cdot \left(\widetilde{\boldsymbol{T}}_0 \times \boldsymbol{W}_k\right), \qquad (3.235)$$

and $-\boldsymbol{n}_\Phi = \boldsymbol{n}_c$ and $-\boldsymbol{n}_\Phi = \boldsymbol{n}_A$ in the first and in the second case, respectively. On Γ_{nc_A}, they compensate each others, because the normal unit vectors have opposite direction. Dirichlet type boundary conditions (2.525) and (2.527) are satisfied on Γ_E and on Γ_B, that is why these two surface integrals are equal to zero on these parts of the boundary. On the rest segments, the tangential component of the impressed current vector potential must be equal to zero, $\boldsymbol{n} \times \widetilde{\widetilde{\boldsymbol{T}}}_0 = \boldsymbol{0}$, however, it is useful to apply (3.146), because the surface current term $\boldsymbol{K}$ can be eliminated on the left-hand side.

The surface integral terms containing the nonlinear residual term $\widetilde{\boldsymbol{I}}$ are vanishing on $\Gamma_{H_c} \cup \Gamma_{nc_A} \cup \Gamma_{nc_\Phi}$, because of the same terms on the left-hand side. On Γ_E, $\boldsymbol{W}_k \times \boldsymbol{n} = \boldsymbol{0}$ according to the Dirichlet type boundary condition (2.525).

The interface integral terms containing $\nabla\widetilde{\Phi}$ can be reformulated in the same way as in the $\boldsymbol{A} - \Phi$ formulation on page 83, from (3.88) to (3.92) and see *Fig.* 2.18.

Finally, the first equation of the weak form is the following:

$$
\begin{aligned}
&\int_{\Omega_c} \nu_o \left(\nabla \times \boldsymbol{W}_k\right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}}\right) \mathrm{d}\Omega \\
&+ \int_{\Omega_c} \boldsymbol{W}_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t}\right) \mathrm{d}\Omega \\
&+ \int_{\Omega_{n_A}} \nu \left(\nabla \times \boldsymbol{W}_k\right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}}\right) \mathrm{d}\Omega \\
&+ \int_{\Gamma_{nc_\Phi} \cup \Gamma_{A,\Phi}} \widetilde{\Phi} \left(\nabla \times \boldsymbol{W}_k\right) \cdot \boldsymbol{n}_A \mathrm{d}\Gamma \\
&= \int_{\Omega_c \cup \Omega_{n_A}} \left(\nabla \times \boldsymbol{W}_k\right) \cdot \widetilde{\boldsymbol{T}}_0 \,\mathrm{d}\Omega \\
&+ \int_C \Phi_0 \boldsymbol{W}_k \cdot \mathrm{d}\boldsymbol{l} - \int_{\Omega_c} \left(\nabla \times \boldsymbol{W}_k\right) \cdot \widetilde{\boldsymbol{I}} \,\mathrm{d}\Omega,
\end{aligned}
\tag{3.236}
$$

and $k = 1, \cdots, J$. Here, all the normal unit vectors are changed to $\boldsymbol{n}_A$, pointing out from the regions where $\boldsymbol{A}$ can be found ($\Omega_c \cup \Omega_{n_A}$).

The partial differential equation (2.518), the Neumann type boundary and interface conditions (2.522), (2.530) and (2.534) can be summarized in the following formulation. Here, the conditions are multiplied by -1,

$$
\begin{aligned}
&- \int_{\Omega_c} N_k \nabla \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t}\right) \mathrm{d}\Omega \\
&+ \int_{\Gamma_{H_c} \cup \Gamma_{nc_A} \Gamma_{nc_\Phi}} N_k \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t}\right) \cdot \boldsymbol{n} \,\mathrm{d}\Gamma = 0,
\end{aligned}
\tag{3.237}
$$

which is the same as in the gauged $\boldsymbol{A}, V - \boldsymbol{A} - \Phi$ formulation presented in the last item. The weak formulation is repeated here,

$$
\int_{\Omega_c} \nabla N_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t}\right) \mathrm{d}\Omega = 0, \quad k = 1, \cdots, I.
\tag{3.238}
$$

The partial differential equation (2.520), the Neumann boundary condition (2.528) and the interface conditions (2.533) and (2.536) build the third weak formulation,

$$\int_{\Omega_{n_\Phi}} N_k \nabla \cdot (\mu \nabla \widetilde{\Phi}) \, \mathrm{d}\Omega + \int_{\Gamma_{B_\Phi}} N_k \left(b + \mu \widetilde{\boldsymbol{T}}_0 \cdot \boldsymbol{n} - \mu \nabla \widetilde{\Phi} \cdot \boldsymbol{n} \right) \mathrm{d}\Gamma$$

$$+ \int_{\Gamma_{nc_\Phi} \cup \Gamma_{\boldsymbol{A},\Phi}} N_k \left[(\nabla \times \widetilde{\boldsymbol{A}}) \cdot \boldsymbol{n}_{\boldsymbol{A}} + \mu \left(\widetilde{\boldsymbol{T}}_0 - \nabla \widetilde{\Phi} \right) \cdot \boldsymbol{n}_\Phi \right] \mathrm{d}\Gamma \tag{3.239}$$

$$= \int_{\Omega_{n_\Phi}} N_k \nabla \cdot (\mu \widetilde{\boldsymbol{T}}_0) \, \mathrm{d}\Omega.$$

Obtaining the weak formulation can be realized in the same way as it was presented in the last item, in the gauged $\boldsymbol{A}, V - \boldsymbol{A} - \Phi$ formulation,

$$- \int_{\Omega_{n_\Phi}} \mu \nabla N_k \cdot \nabla \widetilde{\Phi} \, \mathrm{d}\Omega + \int_{\Gamma_{nc_\Phi} \cup \Gamma_{\boldsymbol{A},\Phi}} N_k (\nabla \times \widetilde{\boldsymbol{A}}) \cdot \boldsymbol{n}_{\boldsymbol{A}} \mathrm{d}\Gamma =$$

$$- \int_{\Omega_{n_\Phi}} \mu \nabla N_k \cdot \widetilde{\boldsymbol{T}}_0 \, \mathrm{d}\Omega - \int_{\Gamma_{B_\Phi}} N_k b \, \mathrm{d}\Gamma, \quad k = 1, \cdots, I. \tag{3.240}$$

Finally, the three equations of the weak formulation of the ungauged $\boldsymbol{A}, V - \boldsymbol{A} - \Phi$ method is the following:

$$\int_{\Omega_c} \nu_o \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}} \right) \mathrm{d}\Omega + \int_{\Omega_c} \boldsymbol{W}_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \mathrm{d}\Omega$$

$$+ \int_{\Omega_{n_{\boldsymbol{A}}}} \nu \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}} \right) \mathrm{d}\Omega$$

$$+ \int_{\Gamma_{nc_\Phi} \cup \Gamma_{\boldsymbol{A},\Phi}} \widetilde{\Phi} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \boldsymbol{n}_{\boldsymbol{A}} \mathrm{d}\Gamma = \int_{\Omega_{n_{\boldsymbol{A}}}} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \widetilde{\boldsymbol{T}}_0 \, \mathrm{d}\Omega \tag{3.241}$$

$$+ \int_C \Phi_0 \boldsymbol{W}_k \cdot \mathrm{d}\boldsymbol{l} - \int_{\Omega_c} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \widetilde{\boldsymbol{I}} \, \mathrm{d}\Omega, \quad k = 1, \cdots, J,$$

$$\int_{\Omega_c} \nabla N_k \cdot \left(\sigma \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t} + \sigma \nabla \frac{\partial \widetilde{v}}{\partial t} \right) \mathrm{d}\Omega = 0, \quad k = 1, \cdots, I, \tag{3.242}$$

$$- \int_{\Omega_{n_\Phi}} \mu \nabla N_k \cdot \nabla \widetilde{\Phi} \, \mathrm{d}\Omega + \int_{\Gamma_{nc_\Phi} \cup \Gamma_{\boldsymbol{A},\Phi}} N_k (\nabla \times \widetilde{\boldsymbol{A}}) \cdot \boldsymbol{n}_{\boldsymbol{A}} \mathrm{d}\Gamma =$$

$$- \int_{\Omega_{n_\Phi}} \mu \nabla N_k \cdot \widetilde{\boldsymbol{T}}_0 \, \mathrm{d}\Omega - \int_{\Gamma_{B_\Phi}} N_k b \, \mathrm{d}\Gamma, \quad k = 1, \cdots, I. \tag{3.243}$$

4 The finite element method

By using scalar and vector potentials, Maxwell's equations can be transformed into partial differential equations as it is introduced in section 2.3. Generally, the partial differential equations can be solved by numerical methods [3,5,11,12,19,28,41–43,46,52,59,68,71, 77,78,81–83,94,95]. One of these numerical methods is the finite element method, which is based on the weak formulation of the partial differential equations, as it is presented in the previous chapter.

The basis of numerical techniques is to reduce the partial differential equations to algebraic ones whose solution gives an approximation of the unknown potentials and electromagnetic field quantities. This reduction can be done by discretizing the partial differential equations in time if necessary and in space. The potential functions, the approximation method and the generated mesh distinguish the numerical field solvers.

This section summarize the finite element method as a CAD technique in electrical engineering to obtain the electromagnetic field quantities in the case of static magnetic field and eddy current field problems. Here, we show how to discretize the analyzed domain with finite elements, how to approximate potential functions with nodal and vector shape functions and how to build up the system of equations, which solution obtain the unknown potentials.

4.1   Fundamentals of FEM

The *Finite Element Method* (FEM) is the most popular and the most flexible numerical technique to determine the approximate solution of the partial differential equations in engineering [3,5,11,12,19,28,41–43,46,52,59,68,71,77,78,81–83,94,95]. For example, commercially available FEM software package is *COMSOL Multiphysics*, which is able to solve one, two and three-dimensional problems [19,88] (see last chapter). A free mesh generator with a built-in CAD engine and post-processor is Gmsh [89].

The main steps of simulation with FEM are illustrated in *Fig.* 4.1. Firstly, in the model specification phase, the model of the real life problem, which simulation require electromagnetic field calculations must be set up, i.e. we have to find out the partial differential equations, which must be solved with prescribed boundary and continuity

conditions. We have to find out, whether it is a linear or a nonlinear problem and how the characteristics look like. After selecting potentials, the weak formulation of these partial differential equations must be worked out as well. It is depending on the problem, of course, but the chosen mathematical model of the arrangement should be adequate to calculate electromagnetic field quantities in the given accuracy. The geometry of the problem must be defined by a CAD software tool, e.g. by using a user friendly interface, see e.g. *Fig.* 4.2.

The next step is the *preprocessing* task. Here we have to give the values of different parameters, such as the material properties, i.e. the constitutive relations, the excitation signal and the others. The geometry can be simplified according to symmetries or axial symmetries.

The geometry of the problem must be discretized by a *FEM mesh*. The fundamental idea of FEM is to divide the problem region to be analyzed into smaller finite elements with given shape. A finite element can be e.g. *triangle* (*Fig.* 4.3(a)) or *quadrangle* (*Fig.* 4.3(b)) in 2D, e.g. *tetrahedron* (*Fig.* 4.4(a)) or *hexahedra* (*Fig.* 4.4(b)) in 3D. A triangle has three vertices 1, 2 and 3, here numbered counter-clockwise and has 3 edges. The quadrangle element has 4 nodes and 4 edges. A tetrahedral element has 4 vertices
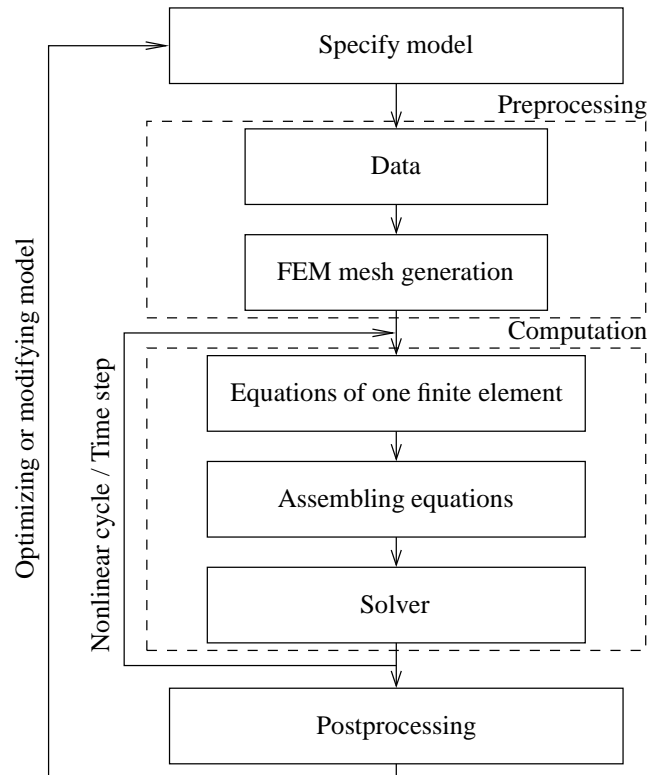
Fig. 4.1. Steps of simulation by FEM
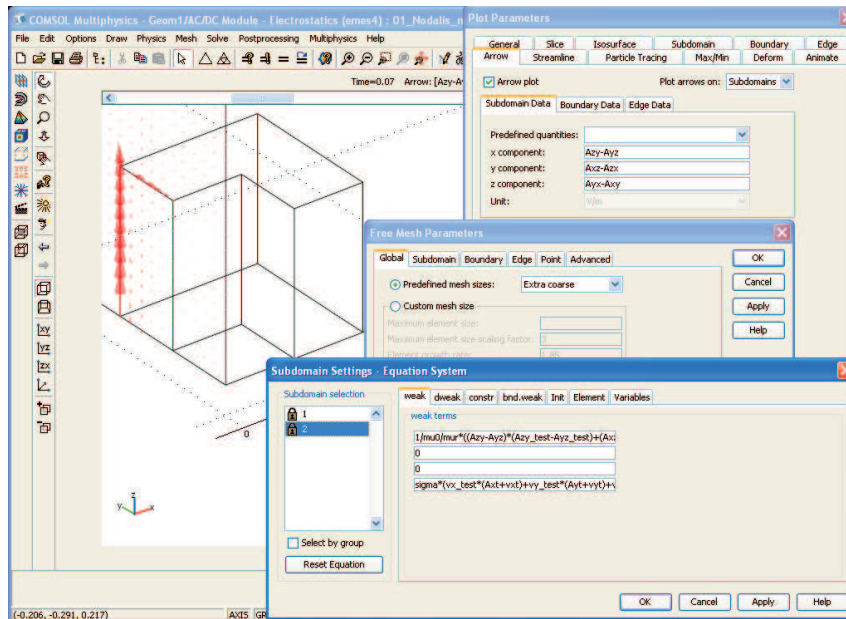
Fig. 4.2. COMSOL Multiphysics, a CAD environment to solve electromagnetic
field problems

and 6 edges and a hexahedral element has 8 nodes and 12 edges (the usual numbering of
nodes is shown in the illustrations).

There are some simple rules, how to generate a mesh. Neither overlapping nor holes
are allowed in the generated finite element mesh. If material interface are included in
the problem region, the configuration of mesh must be adapted to these boundaries, i.e.
interfaces coincide with finite element interfaces.

FEM mesh, as two illustrations, generated by COMSOL Multiphysics [88] can be

(a) Triangular element   (b) Quadrangle element
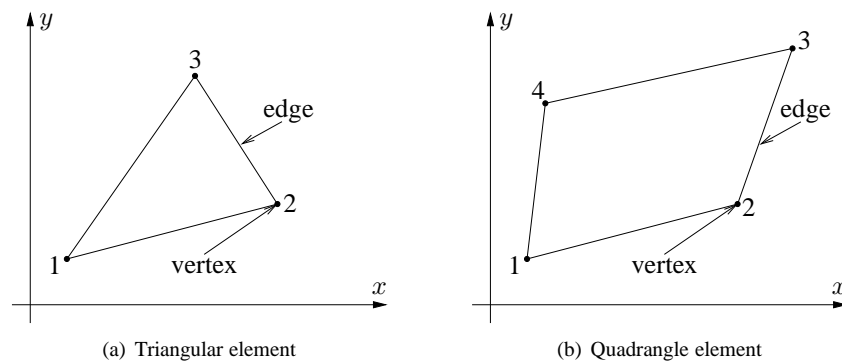
Fig. 4.3. Typical finite elements in the two-dimensional $x - y$ plane

(a) Tetrahedral element   (b) Hexahedral element
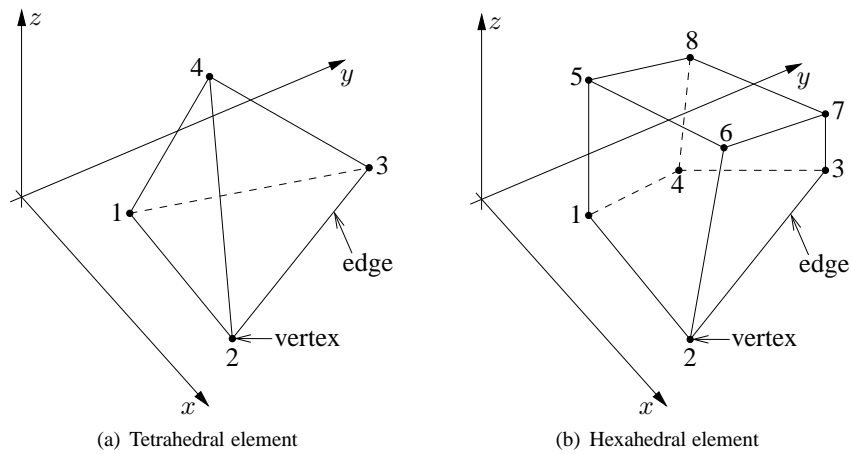
Fig. 4.4. Typical finite elements in 3D

seen in *Fig.* 4.5 and in *Fig.* 4.6. The first 2D illustration (*Fig.* 4.5) shows the mesh of a horseshoe-shaped permanent magnet. The two ends are pre-magnetized in different directions. The second illustration (*Fig.* 4.6) presents a model of a micro-scale square inductor, used for LC bandpass filters in microelectromechanical systems. The model geometry consists of the spiral-shaped inductor and the air surrounding it (the mesh in air is not shown). The outer dimensions of the model geometry are around 0.3 mm. These illustrations are cited from the Model Documentation of COMSOL Multiphysics.
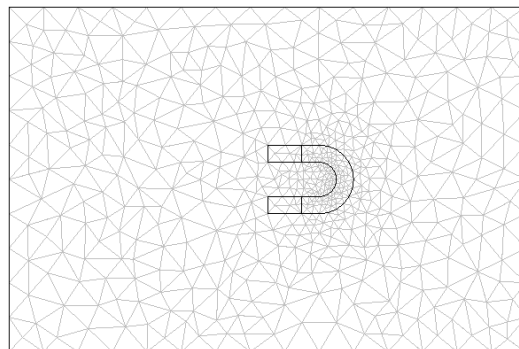
Fig. 4.5. COMSOL model of a permanent magnet, geometry is meshed by triangles

The next step in FEM simulations is solving the problem. The FEM equations, based on the weak formulations, must be set up in the level of one finite element, then these equations must be assembled through the FEM mesh. *Assembling* means that the global system of equations is built up, which solution is the approximation of the introduced potential. The obtained global system of algebraic equations is linear, or nonlinear but
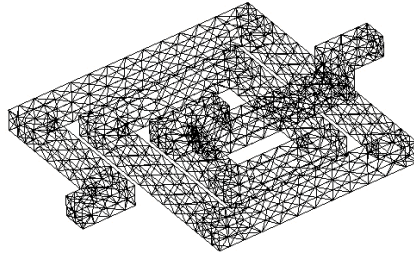
Fig. 4.6. COMSOL model of a micro-scale square inductor, geometry is discretized by
tetrahedral shape finite elements

linearized, depending on the medium to be analyzed. Then this global system of equations
must be solved by a solver. The computation may contain iteration if the constitutive
equations are nonlinear. This is the situation when simulating ferromagnetic materials
with nonlinear characteristics. Iteration means that the system of equations must be set
up and must be solved step by step until convergence is reached. If the problem is time
dependent, then the solution must be worked out at every discrete time instant.

The result of computations is the approximated potential value in the FEM mesh. Any
electromagnetic field quantity (e.g. magnetic field intensity, or magnetic flux density,
etc.) can be calculated by using the potentials at the *postprocessing* stage. Capacitance,
inductance, energy, force and other quantities can also be calculated. The postprocessing
give a chance to modify the geometry, the material parameters or the FEM mesh to
get more accurate result. The COMSOL Multiphysics [88] has been used to show two
examples about postprocessing. The pattern of the magnetic field around the permanent
magnet is well known through experiments (see *Fig.* 4.7). Figure 4.8 shows the electric
potential in the inductor and the magnetic flux lines. The thickness of the flow lines
represents the magnitude of the magnetic flux.
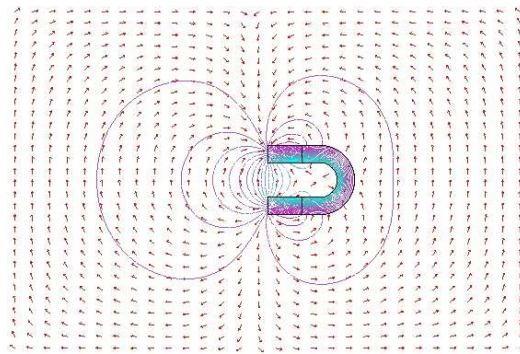
Fig. 4.7. COMSOL solution of the static magnetic field generated
by a permanent magnet
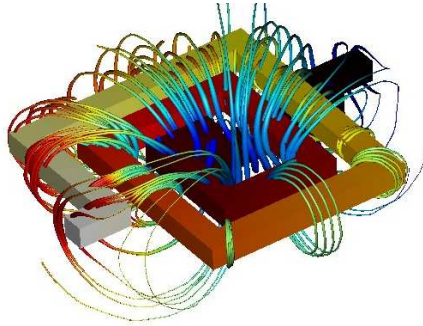
Fig. 4.8. Electric potential in the device and magnetic flux lines around the device,
the problem has been solved by COMSOL

4.2 Approximating potentials with shape functions

It has been shown in section 2.3 that the introduced potential function can be scalar valued
(e.g. the magnetic scalar potential Φ, or the electric scalar potential V), or vector valued
(e.g. the current vector potential T, or the magnetic vector potential A).

The scalar potential functions can be approximated by so-called *nodal shape functions*
and the vector potential functions can be approximated by either nodal or so-called *vector
shape functions*, also called *edge shape functions*. Generally, a shape function is a simple
continuous polynomial function defined in a finite element and it is depending on the type
of the used finite element.

Shape functions have the following general properties [59]:

(*i*) Each shape function is defined in the entire problem region;

(*ii*) Each scalar shape function corresponds to just one nodal point and each vector
shape function corresponds to just one edge;

(*iii*) Each scalar shape function is nonzero over just those finite elements that contain its
nodal point and equals to zero over all other elements. Each vector shape function
in nonzero over just those finite elements that contain its edge and equals to zero
over all other elements;

(*iv*) The scalar shape function has a value unity at its nodal point and zero at all other
nodal points. The line integral of a vector shape function is equal to one along its
edge and the line integral of it is equal to zero along the other edges;

(*v*) The shape functions are linearly independent, i.e. no shape function equals a linear
combination of the other shape functions.

The accuracy of solution obtained by FEM can be increased in three ways. The
first one is increasing the number of finite elements, i.e. decreasing the element size.
It is called *h-FEM*. The second way is to increase the degree of polynomials building
up a shape function (e.g. using Lagrange or Legendre interpolation functions). This

is the so-called *p-FEM*. The mixture of these methods results in *hp-FEM*. Potentials approximated by *h*-version or *p*-version are assigned in the indices of the potentials.

4.2.1   Nodal finite elements

Scalar potential functions can be represented by a linear combination of shape functions associated with nodes of the finite element mesh. Within a finite element, a scalar potential function $\Phi = \Phi(\boldsymbol{r}, t)$ is approximated by

$$\Phi \simeq \sum_{i=1}^{m} N_i \Phi_i, \tag{4.1}$$

where $N_i = N_i(\boldsymbol{r})$ and $\Phi_i = \Phi_i(t)$ are the *nodal shape functions* and the value of potential function corresponding to the ith node, respectively. The number of degrees of freedom is $m = 2$ in 1D problems, $m = 3$ in a 2D problem using triangular FEM mesh and $m = 4$ in a 3D arrangement meshed by tetrahedral elements and the shape functions are linear. The nodal shape functions can be defined by the relation

$$N_i = \left\{ \begin{array}{ll} 1, & \text{at the node } i, \\ 0, & \text{at other nodes.} \end{array} \right. \tag{4.2}$$

(*i*) *In 1D*, the linear shape functions can be build up by

$$N_1 = \frac{x_2 - x}{x_2 - x_1}, \quad \text{and} \quad N_2 = \frac{x - x_1}{x_2 - x_1}, \tag{4.3}$$

where x_1 and x_2 are the coordinates of the boundaries of one finite element. The linear shape functions are plotted in *Fig.* 4.9. It is easy to control the equation (4.2).
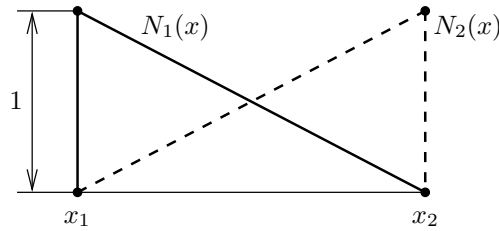
Fig. 4.9. The 1D linear shape functions $N_1(x)$ and $N_2(x)$

If the values of the potential are known in the two boundary points x_1 and x_2, then the potential can be determined easily inside the finite element $x_1 \le x \le x_2$ as (see *Fig.* 4.10)

$$\Phi = N_1\, \Phi_1 + N_2\, \Phi_2 = \frac{x_2 - x}{x_2 - x_1}\Phi_1 + \frac{x - x_1}{x_2 - x_1}\Phi_2. \tag{4.4}$$

Of course, it is valid in the other finite elements as well, e.g. if $x_2 \le x \le x_3$, then

$$\Phi = N_1\, \Phi_2 + N_2\, \Phi_3 = \frac{x_3 - x}{x_3 - x_2}\Phi_2 + \frac{x - x_2}{x_3 - x_2}\Phi_3, \tag{4.5}$$

and N_1, N_2 are shifted to the second finite element.

The scalar potential is continuous in the whole 1D region. It is noted here that the accuracy of approximation can be increased by decreasing the length of the elements, especially where the rate of change of the solution is large, e.g. between x_3 and x_4 in *Fig.* 4.10. Here, the mesh can be very fine and higher order approximation can results in better solution.
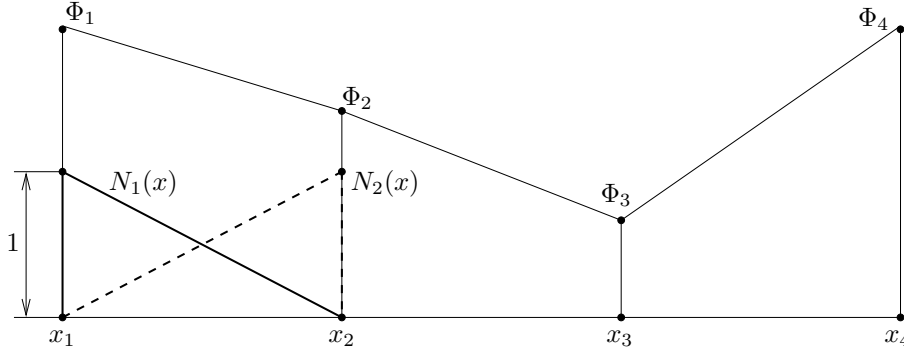
Fig. 4.10. Known potential values are approximated by linear functions

One way to build up higher order shape functions is using *Lagrange interpolation functions*, defined by the formula

$$N_i(x) = \prod_{j=1,\, j\neq i}^{m} \frac{x - x_j}{x_i - x_j}. \tag{4.6}$$

The order is $m-1$ and $N_i(x)$ is equal to one in the node i and equal to zero in all the other nodes. Here, second and third order approximations are shown.

The second order approximation can be defined by 3 quadratic shape functions (i.e. $m = 3$ in (4.1), see *Fig.* 4.11),

$$N_1 = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}, \tag{4.7}$$
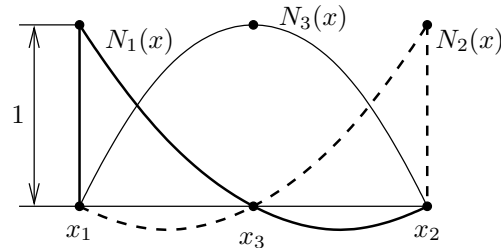
Fig. 4.11. The 1D quadratic shape functions $N_1(x)$, $N_2(x)$ and $N_3(x)$

$$N_2 = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}, \tag{4.8}$$

$$N_3 = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}, \tag{4.9}$$

and the new point x_3 is placed in the center of the element,

$$x_3 = \frac{x_1 + x_2}{2}. \tag{4.10}$$

The third order approximation can be defined by 4 cubic shape functions ($m = 4$ in (4.1), see *Fig.* 4.12),

$$N_1 = \frac{(x - x_2)(x - x_3)(x - x_4)}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)}, \tag{4.11}$$

$$N_2 = \frac{(x - x_1)(x - x_3)(x - x_4)}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)}, \tag{4.12}$$

$$N_3 = \frac{(x - x_1)(x - x_2)(x - x_4)}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)}, \tag{4.13}$$

$$N_4 = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)}, \tag{4.14}$$

and the new points x_3 and x_4 are placed inside the element as

$$x_3 = \frac{1(x_1 + x_2)}{3}, \quad x_4 = \frac{2(x_1 + x_2)}{3}. \tag{4.15}$$
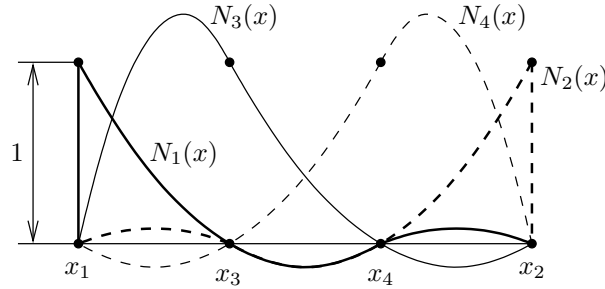
Fig. 4.12. The 1D cubic shape functions $N_1(x)$, $N_2(x)$, $N_3(x)$ and $N_4(x)$

With this technique, the interpolation functions of any order can be defined and the equation (4.2) can be controlled.

Figure 4.13 shows the higher order approximation of the potential plotted in *Fig.* 4.10. This illustration shows the applicability of higher order functions.

(*ii*) *2D* linear shape functions can be built up as follows when using a finite element mesh with *triangular finite elements*. Linear basis functions can be introduced by using
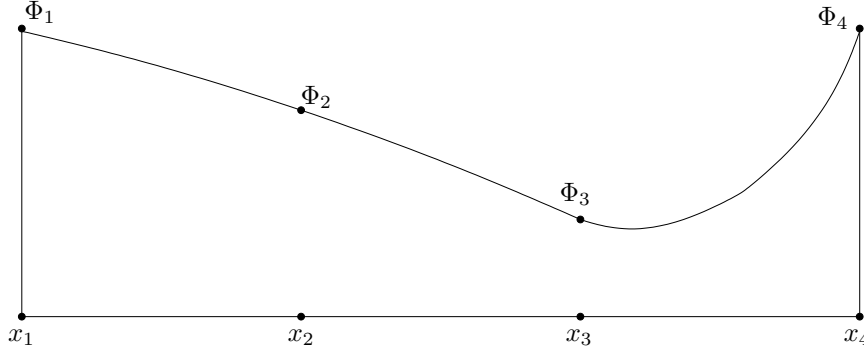
Fig. 4.13. Known potential values are approximated by quadratic functions

the so-called *barycentric coordinate system* in a triangle as follows. The area of a triangle is denoted by Δ and it can be calculated as

$$\Delta = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}, \tag{4.16}$$

where (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the coordinates of the three nodes of the triangle in the global coordinate system building an anticlockwise sequence. The *area functions* (see *Fig.* 4.14) of a given point inside the triangle with coordinates (x, y) can be calculated as

$$\Delta_1 = \frac{1}{2} \begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}, \ \Delta_2 = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x & y \\ 1 & x_3 & y_3 \end{vmatrix}, \ \Delta_3 = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{vmatrix}, \tag{4.17}$$

i.e. $\Delta_1 = \Delta_1(x, y)$, $\Delta_2 = \Delta_2(x, y)$ and $\Delta_3 = \Delta_3(x, y)$ are depending on the coordinates x and y.

The barycentric coordinates $L_i = L_i(x, y)$ can be defined by the above area functions as

$$L_i = \frac{\Delta_i}{\Delta}, \qquad i = 1, 2, 3. \tag{4.18}$$

Three linear shape functions $N_i = N_i(x, y)$ can be described as

$$N_i = L_i, \qquad i = 1, 2, 3. \tag{4.19}$$

The shape function N_i is equal to 1 at the ith node of the triangle and it is equal to zero at the other two nodes, because Δ_i is equal to Δ at node i and it is equal to zero at the other two nodes. That is why the relation (4.2) is satisfied. It is obvious that the three shape functions are linearly independent.

The linear shape functions N_i $(i = 1, 2, 3)$ vary linearly over the triangle, because the fraction Δ_i/Δ measures the perpendicular distance of the point (x, y) toward the vertex opposite to node i as it is illustrated in *Fig.* 4.15 and the linear shape function is constant along such a line. The three linear shape functions are shown in *Fig.* 4.16.
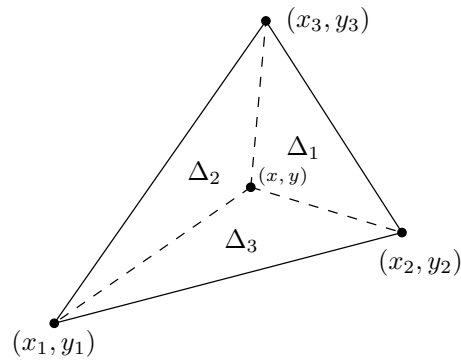
Fig. 4.14. The area function of a triangle
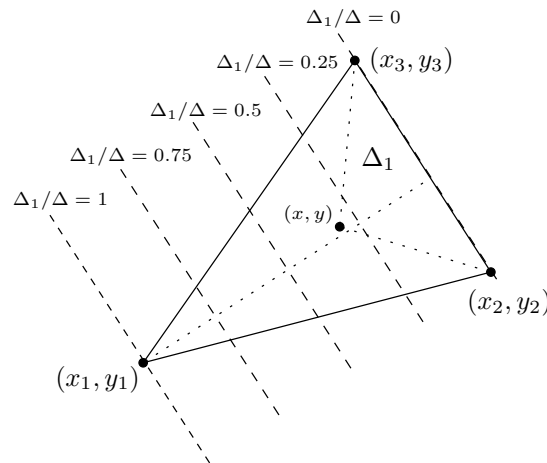
Fig. 4.15. Fraction Δ_i/Δ measures the perpendicular distance of the point (x, y) toward the vertex opposite to node i (here $i = 1$)

If the potential at the nodes is known, then a linear approximation of the potential function can be represented by (4.1). The derivative of a first order approximation is zeroth order, i.e. constant. The magnetic field intensity $\boldsymbol{H}$, or the magnetic flux density $\boldsymbol{B}$ are constant within a triangle, if these are obtained from a first order approximation by $\boldsymbol{H} = -\nabla\Phi$, or $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. This may results in inaccurate solution. This is the reason why higher order approximations are studied. Here, only the second and the third order approximations are shown.

Higher order shape functions can also be built up by using the barycentric coordinates L_1, L_2 and L_3 introduced above in (4.18) [46].

(a) $N_1(x,y)$

(b) $N_2(x,y)$

(c) $N_3(x,y)$
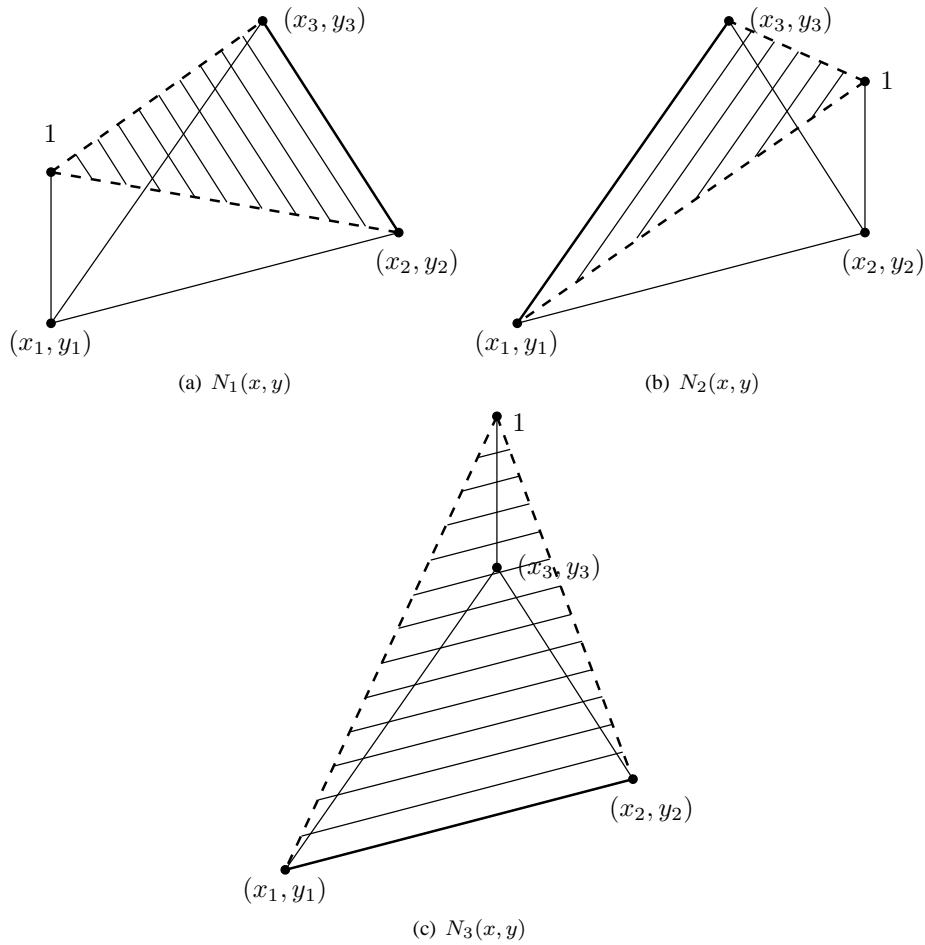
Fig. 4.16. The 2D linear shape functions $N_1(x,y)$, $N_2(x,y)$ and $N_3(x,y)$

A polynomial of order n must contain all possible terms $x^p y^q$, $0 \leq p+q \leq n$, as it is presented by *Pascal's triangle*,

$$1$$

$$x \quad y$$

$$x^2 \quad xy \quad y^2$$

$$x^3 \quad x^2y \quad xy^2 \quad y^3 \qquad \cdots$$

The first row contains the only one term of the zeroth order polynomials, the second, third and fourth rows contain the terms of the first, second and third order polynomials. Pascal's triangle can be used to generate the elements of a polynomial with given order.

Such a polynomial contains

$$m = \frac{(n+1)(n+2)}{2} \tag{4.20}$$

elements altogether, i.e. $m = 1$, $m = 3$, $m = 6$ and $m = 10$ in the case of zeroth, first, second and third order polynomials. It means that m coefficients must be expressed, finally m points must be placed within a triangle. Pascal's triangle can be continued, of course.
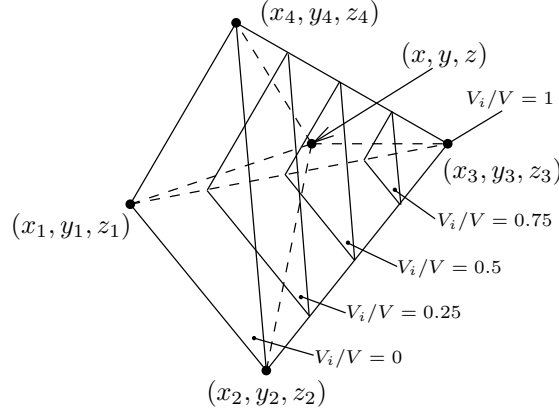
The interpolation function of order n can be constructed as

$$N_i = P_I^n(L_1)\, P_J^n(L_2)\, P_K^n(L_3), \quad \text{where} \quad I + J + K = n, \tag{4.21}$$

and the integers I, J and K label the nodes within the triangle, resulting in a numbering scheme. Figure 4.17, *Fig.* 4.18 and *Fig.* 4.19 illustrate the numbering scheme of the first, the second and the third order approximations. It is noted that points must be inserted not only the edges, but inside the triangle, if $n > 2$.

The polynomials $P_I^n(L_1)$, $P_J^n(L_2)$ and $P_K^n(L_3)$ are defined as

$$P_I^n(L_1) = \prod_{p=0}^{I-1} \frac{n\,L_1 - p}{I - p} = \frac{1}{I!} \prod_{p=0}^{I-1} (n\,L_1 - p), \quad \text{if} \quad I > 0, \tag{4.22}$$

$$P_J^n(L_2) = \prod_{p=0}^{J-1} \frac{n\,L_2 - p}{J - p} = \frac{1}{J!} \prod_{p=0}^{J-1} (n\,L_2 - p), \quad \text{if} \quad J > 0, \tag{4.23}$$

$$P_K^n(L_3) = \prod_{p=0}^{K-1} \frac{n\,L_3 - p}{K - p} = \frac{1}{K!} \prod_{p=0}^{K-1} (n\,L_3 - p), \quad \text{if} \quad K > 0, \tag{4.24}$$

and as a definition

$$P_0^n = 1. \tag{4.25}$$

If $n = 1$, then $m = 3$, i.e. (see *Fig.* 4.17)

$$N_1 = P_1^1(L_1)\, P_0^1(L_2)\, P_0^1(L_3) = L_1, \tag{4.26}$$

$$N_2 = P_0^1(L_1)\, P_1^1(L_2)\, P_0^1(L_3) = L_2, \tag{4.27}$$

$$N_3 = P_0^1(L_1)\, P_0^1(L_2)\, P_1^1(L_3) = L_3, \tag{4.28}$$

since

$$P_1^1(L_i) = \prod_{p=0}^{1-1} \frac{1\,L_i - p}{1 - p} = L_i, \tag{4.29}$$

as it was mentioned in (4.19).

If $n = 2$, then $m = 6$, i.e. (see *Fig.* 4.18)

$$N_1 = P_2^2(L_1)\, P_0^2(L_2)\, P_0^2(L_3) = L_1(2\,L_1 - 1), \tag{4.30}$$
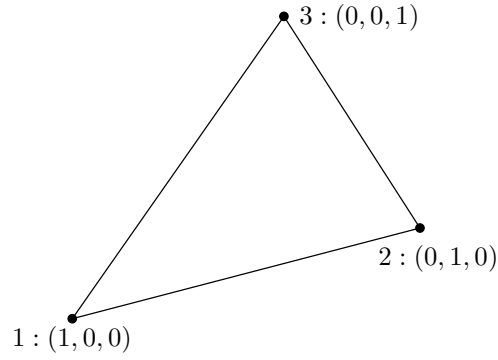
Fig. 4.17. Numbering scheme for linear element, $n = 1$
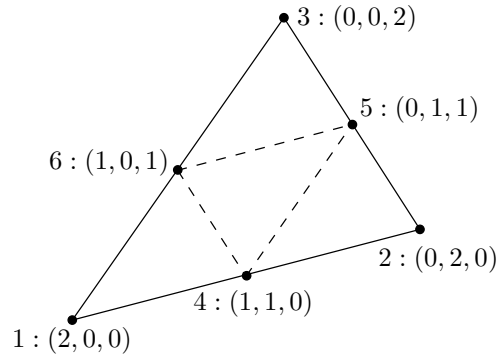
Fig. 4.18. Numbering scheme for quadratic element, $n = 2$
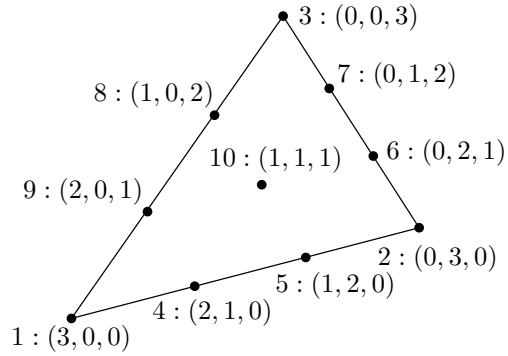
Fig. 4.19. Numbering scheme for cubic element, $n = 3$

$$N_2 = P_0^2(L_1)\, P_2^2(L_2)\, P_0^2(L_3) = L_2(2\, L_2 - 1), \tag{4.31}$$

$$N_3 = P_0^2(L_1)\, P_0^2(L_2)\, P_2^2(L_3) = L_3(2\, L_3 - 1), \tag{4.32}$$

$$N_4 = P_1^2(L_1)\, P_1^2(L_2)\, P_0^2(L_3) = 4\, L_1\, L_2, \tag{4.33}$$

$$N_5 = P_0^2(L_1)\, P_1^2(L_2)\, P_1^2(L_3) = 4\, L_2\, L_3, \tag{4.34}$$

$$N_6 = P_1^2(L_1)\, P_0^2(L_2)\, P_1^2(L_3) = 4\, L_1\, L_3, \tag{4.35}$$

because

$$P_1^2(L_i) = \prod_{p=0}^{1-1} \frac{2\, L_i - p}{1 - p} = 2\, L_i, \tag{4.36}$$

and

$$P_2^2(L_i) = \prod_{p=0}^{2-1} \frac{2\, L_i - p}{2 - p} = \frac{2\, L_i}{2}\, \frac{2\, L_i - 1}{1} = L_i(2\, L_i - 1). \tag{4.37}$$

Figure 4.20 shows the shape functions N_1 and N_4. The other shape functions look like these, N_2 and N_3 are the same as N_1, moreover N_5 and N_6 are the same as N_4, but they must be rotated to the corresponding nodes.

(a) $N_1(x, y)$ (b) $N_4(x, y)$
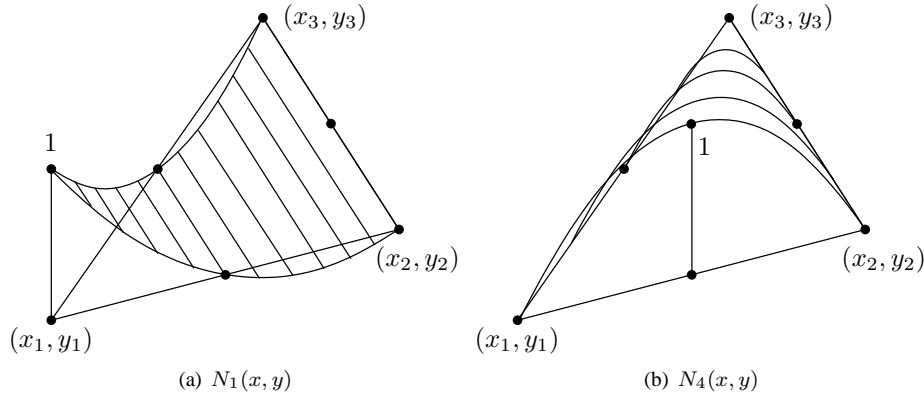
Fig. 4.20. The 2D quadratic shape functions $N_1(x, y)$ and $N_4(x, y)$

Finally, if $n = 3$, $m = 10$, shape functions can be constructed as (see *Fig.* 4.21)

$$N_1 = P_3^3(L_1)\, P_0^3(L_2)\, P_0^3(L_3) = \frac{1}{2} L_1(3\, L_1 - 1)(3\, L_1 - 2), \tag{4.38}$$

$$N_2 = P_0^3(L_1)\, P_3^3(L_2)\, P_0^3(L_3) = \frac{1}{2} L_2(3\, L_2 - 1)(3\, L_2 - 2), \tag{4.39}$$

$$N_3 = P_0^3(L_1)\, P_0^3(L_2)\, P_3^3(L_3) = \frac{1}{2} L_3(3\, L_3 - 1)(3\, L_3 - 2), \tag{4.40}$$

$$N_4 = P_2^3(L_1)\, P_1^3(L_2)\, P_0^3(L_3) = \frac{9}{2} L_1(3\, L_1 - 1)L_2, \tag{4.41}$$

$$N_5 = P_1^3(L_1)\, P_2^3(L_2)\, P_0^3(L_3) = \frac{9}{2} L_2(3\, L_2 - 1)L_1, \tag{4.42}$$

$$N_6 = P_0^3(L_1)\, P_2^3(L_2)\, P_1^3(L_3) = \frac{9}{2} L_2(3\,L_2 - 1)L_3, \tag{4.43}$$

$$N_7 = P_0^3(L_1)\, P_1^3(L_2)\, P_2^3(L_3) = \frac{9}{2} L_3(3\,L_3 - 1)L_2, \tag{4.44}$$

$$N_8 = P_1^3(L_1)\, P_0^3(L_2)\, P_2^3(L_3) = \frac{9}{2} L_3(3\,L_3 - 1)L_1, \tag{4.45}$$

$$N_9 = P_2^3(L_1)\, P_0^3(L_2)\, P_1^3(L_3) = \frac{9}{2} L_1(3\,L_1 - 1)L_3, \tag{4.46}$$

$$N_{10} = P_1^3(L_1)\, P_1^3(L_2)\, P_1^3(L_3) = 27\, L_1\, L_2\, L_3, \tag{4.47}$$

because

$$P_1^3(L_i) = \prod_{p=0}^{1-1} \frac{3\,L_i - p}{1 - p} = 3\,L_i, \tag{4.48}$$

$$P_2^3(L_i) = \prod_{p=0}^{2-1} \frac{3\,L_i - p}{2 - p} = \frac{3\,L_i}{2}\,\frac{3\,L_i - 1}{1} = \frac{3}{2} L_i(3\,L_i - 1), \tag{4.49}$$

$$P_3^3(L_i) = \prod_{p=0}^{3-1} \frac{3\,L_i - p}{3 - p} = \frac{3\,L_i}{3}\,\frac{3\,L_i - 1}{2}\,\frac{3\,L_i - 2}{1}$$
$$= \frac{1}{2} L_i(3\,L_i - 1)(3\,L_i - 2). \tag{4.50}$$

These functions satisfy the condition (4.2). Figure 4.21 shows the shape functions N_1 and N_5, as examples. The other shape functions look like these, N_2 and N_3 are the same as N_1, N_4, N_6, N_7, N_8 and N_9 look like N_5, but they must be imagined at the corresponding nodes. The shape function N_{10} is equal to one at the center of mass of the triangle and equal to zero on the other nine nodes.

(a) $N_1(x, y)$ (b) $N_5(x, y)$
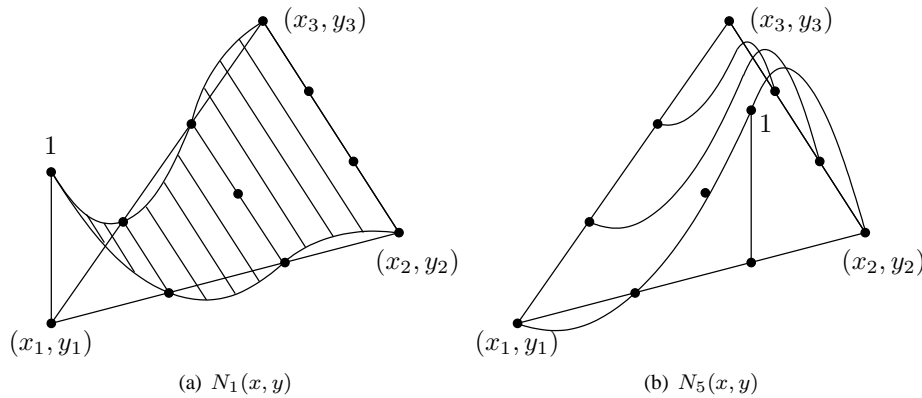
Fig. 4.21. The 2D cubic shape functions $N_1(x, y)$ and $N_5(x, y)$

The scalar potential along any edge of a triangle is the linear combination of the values defined in the points of this edge (see *Fig.* 4.17, *Fig.* 4.18, *Fig.* 4.19), so that if two triangles share the same vertice, the potential will be continuous across the interface element boundary. This means that the approximate solution is continuous everywhere, however, its normal derivate is not.

It is easy to see that the 1D shape functions are the same as the functions along the edges of a triangle.

(***iii***) *3D* linear shape functions can be worked out as follows when using tetrahedral finite elements. Linear basis functions can be introduced again by using the *barycentric coordinate system*. The volume of a tetrahedron is denoted by V and it can be expressed as

$$V = \frac{1}{6} \begin{vmatrix} x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \\ x_4 - x_2 & y_4 - y_2 & z_4 - z_2 \\ x_4 - x_3 & y_4 - y_3 & z_4 - z_3 \end{vmatrix}, \tag{4.51}$$

where (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) are the coordinates of the four nodes of the tetrahedron as shown in *Fig.* 4.22. The volume functions according to a given point inside the tetrahedron with coordinates (x, y, z) can be calculated as

$$V_1 = \frac{1}{6} \begin{vmatrix} x_4 - x & y_4 - y & z_4 - z \\ x_4 - x_2 & y_4 - y_2 & z_4 - z_2 \\ x_4 - x_3 & y_4 - y_3 & z_4 - z_3 \end{vmatrix}, \tag{4.52}$$

$$V_2 = \frac{1}{6} \begin{vmatrix} x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \\ x_4 - x & y_4 - y & z_4 - z \\ x_4 - x_3 & y_4 - y_3 & z_4 - z_3 \end{vmatrix}, \tag{4.53}$$

$$V_3 = \frac{1}{6} \begin{vmatrix} x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \\ x_4 - x_2 & y_4 - y_2 & z_4 - z_2 \\ x_4 - x & y_4 - y & z_4 - z \end{vmatrix}, \tag{4.54}$$

$$V_4 = \frac{1}{6} \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x - x_2 & y - y_2 & z - z_2 \\ x - x_3 & y - y_3 & z - z_3 \end{vmatrix}. \tag{4.55}$$

The *barycentric coordinates* $L_i = L_i(x, y, z)$ of a tetrahedron can be formulated as

$$L_i = \frac{V_i}{V}, \qquad i = 1, 2, 3, 4. \tag{4.56}$$

Four linear shape functions $N_i = N_i(x, y, z)$ correspondingly to the four nodes are

$$N_i = L_i, \qquad i = 1, 2, 3, 4. \tag{4.57}$$

A shape function N_i is equal to 1 at the ith node of the tetrahedron, moreover it is equal to zero at the other three nodes and it is varying linearly within the tetrahedron, because the fraction V_i/V measures the perpendicular distance of the point (x, y, z) toward the facet opposite to node i as it is illustrated in *Fig.* 4.23 and the linear shape function is constant along such a surface. That is why the relation (4.2) is satisfied. It is obvious that the four shape functions are linearly independent.
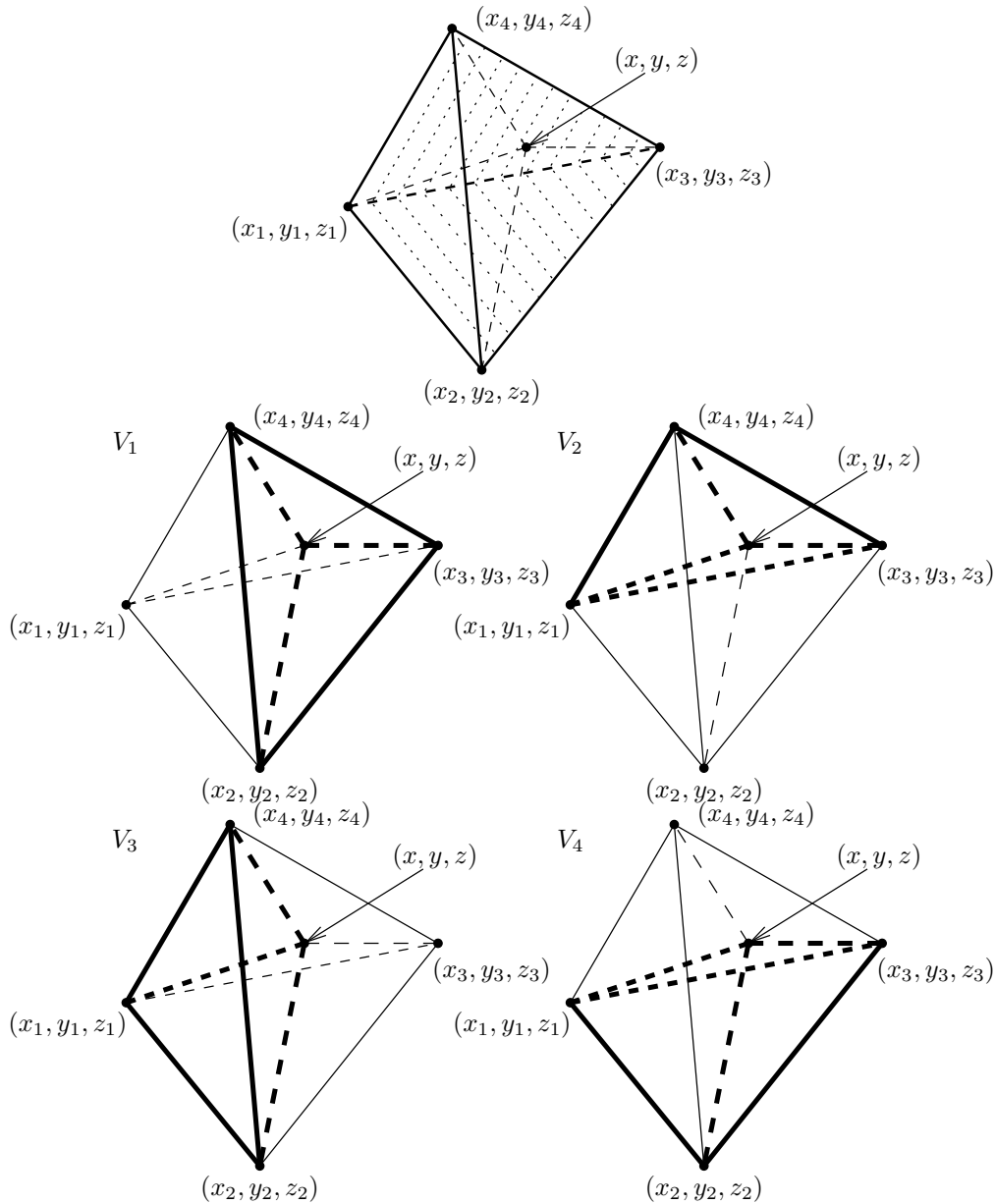
Fig. 4.22. The volume functions in a tetrahedron

The higher order shape functions can be worked out similarly as it was mentioned in the case of triangular elements [46]. The barycentric coordinates L_1, L_2, L_3 and L_4 can be used. A polynomial of order n must contain all possible terms $x^p y^q z^r$, $0 \leq p+q+r \leq n$

and a polynomial contains

$$m = \frac{(n+1)(n+2)(n+3)}{6} \tag{4.58}$$

elements altogether, i.e. $m = 1$, $m = 4$, $m = 10$ and $m = 20$ in the case of zeroth, first, second and third order polynomials. It means that m coefficients must be expressed and m points must be placed within a tetrahedron.

Fig. 4.23. Fraction V_i/V measures the perpendicular distance of the point (x, y, z)
toward the facet opposite to node i (here $i = 3$)

The interpolation function of order n can be constructed as

$$N_i = P_I^n(L_1) P_J^n(L_2) P_K^n(L_3) P_L^n(L_4), \quad \text{where} \quad I + J + K + L = n, \tag{4.59}$$

where the integers I, J, K and L label the nodes within the tetrahedra, resulting in a numbering scheme. Figure 4.24, *Fig.* 4.25 and *Fig.* 4.26 illustrate the numbering scheme of the first, the second and the third order approximations.

The polynomials $P_I^n(L_1)$, $P_J^n(L_2)$, $P_K^n(L_3)$ and $P_L^n(L_4)$ are defined in the same way as it was presented in the 2D situation, see definitions (4.22)–(4.25).

If $n = 1$, then $m = 4$, i.e. (see *Fig.* 4.24)

$$N_1 = P_1^1(L_1) P_0^1(L_2) P_0^1(L_3) P_0^1(L_4) = L_1, \tag{4.60}$$

$$N_2 = P_0^1(L_1) P_1^1(L_2) P_0^1(L_3) P_0^1(L_4) = L_2, \tag{4.61}$$

$$N_3 = P_0^1(L_1) P_0^1(L_2) P_1^1(L_3) P_0^1(L_4) = L_3, \tag{4.62}$$

$$N_4 = P_0^1(L_1) P_0^1(L_2) P_0^1(L_3) P_1^1(L_4) = L_4, \tag{4.63}$$

since (4.29) as it was mentioned in (4.57).

If $n = 2$, then $m = 10$, i.e. (see *Fig.* 4.25)

$$N_1 = P_2^2(L_1) P_0^2(L_2) P_0^2(L_3) P_0^2(L_4) = L_1(2 L_1 - 1), \tag{4.64}$$

$$N_2 = P_0^2(L_1)\,P_2^2(L_2)\,P_0^2(L_3)\,P_0^2(L_4) = L_2(2\,L_2 - 1), \tag{4.65}$$

$$N_3 = P_0^2(L_1)\,P_0^2(L_2)\,P_2^2(L_3)\,P_0^2(L_4) = L_3(2\,L_3 - 1), \tag{4.66}$$

$$N_4 = P_0^2(L_1)\,P_0^2(L_2)\,P_0^2(L_3)\,P_2^2(L_4) = L_4(2\,L_4 - 1), \tag{4.67}$$

$$N_5 = P_1^2(L_1)\,P_1^2(L_2)\,P_0^2(L_3)\,P_0^2(L_4) = 4\,L_1\,L_2, \tag{4.68}$$

$$N_6 = P_0^2(L_1)\,P_1^2(L_2)\,P_1^2(L_3)\,P_0^2(L_4) = 4\,L_2\,L_3, \tag{4.69}$$

$$N_7 = P_1^2(L_1)\,P_0^2(L_2)\,P_1^2(L_3)\,P_0^2(L_4) = 4\,L_1\,L_3, \tag{4.70}$$

$$N_8 = P_1^2(L_1)\,P_0^2(L_2)\,P_0^2(L_3)\,P_1^2(L_4) = 4\,L_1\,L_4, \tag{4.71}$$

$$N_9 = P_0^2(L_1)\,P_0^2(L_2)\,P_1^2(L_3)\,P_1^2(L_4) = 4\,L_3\,L_4, \tag{4.72}$$

$$N_{10} = P_0^2(L_1)\,P_1^2(L_2)\,P_0^2(L_3)\,P_1^2(L_4) = 4\,L_2\,L_4, \tag{4.73}$$

because (4.36) and (4.37).

Finally, if $n = 3$, $m = 20$ shape functions can be constructed as (see *Fig.* 4.26)

$$N_1 = P_3^3(L_1)\,P_0^3(L_2)\,P_0^3(L_3)\,P_0^3(L_4) = \frac{1}{2}L_1(3\,L_1 - 1)(3\,L_1 - 2), \tag{4.74}$$

$$N_2 = P_0^3(L_1)\,P_3^3(L_2)\,P_0^3(L_3)\,P_0^3(L_4) = \frac{1}{2}L_2(3\,L_2 - 1)(3\,L_2 - 2), \tag{4.75}$$

$$N_3 = P_0^3(L_1)\,P_0^3(L_2)\,P_3^3(L_3)\,P_0^3(L_4) = \frac{1}{2}L_3(3\,L_3 - 1)(3\,L_3 - 2), \tag{4.76}$$

$$N_4 = P_0^3(L_1)\,P_0^3(L_2)\,P_0^3(L_3)\,P_3^3(L_4) = \frac{1}{2}L_4(3\,L_4 - 1)(3\,L_4 - 2), \tag{4.77}$$

$$N_5 = P_2^3(L_1)\,P_1^3(L_2)\,P_0^3(L_3)\,P_0^3(L_4) = \frac{9}{2}L_1(3\,L_1 - 1)L_2, \tag{4.78}$$

$$N_6 = P_1^3(L_1)\,P_2^3(L_2)\,P_0^3(L_3)\,P_0^3(L_4) = \frac{9}{2}L_2(3\,L_2 - 1)L_1, \tag{4.79}$$

$$N_7 = P_0^3(L_1)\,P_2^3(L_2)\,P_1^3(L_3)\,P_0^3(L_4) = \frac{9}{2}L_2(3\,L_2 - 1)L_3, \tag{4.80}$$

$$N_8 = P_0^3(L_1)\,P_1^3(L_2)\,P_2^3(L_3)\,P_0^3(L_4) = \frac{9}{2}L_3(3\,L_3 - 1)L_2, \tag{4.81}$$

$$N_9 = P_1^3(L_1)\,P_0^3(L_2)\,P_2^3(L_3)\,P_0^3(L_4) = \frac{9}{2}L_3(3\,L_3 - 1)L_1, \tag{4.82}$$

$$N_{10} = P_2^3(L_1)\,P_0^3(L_2)\,P_1^3(L_3)\,P_0^3(L_4) = \frac{9}{2}L_1(3\,L_1 - 1)L_3, \tag{4.83}$$

$$N_{11} = P_1^3(L_1)\,P_0^3(L_2)\,P_0^3(L_3)\,P_2^3(L_4) = \frac{9}{2}L_4(3\,L_4 - 1)L_1, \tag{4.84}$$

$$N_{12} = P_2^3(L_1)\,P_0^3(L_2)\,P_0^3(L_3)\,P_1^3(L_4) = \frac{9}{2}L_1(3\,L_1 - 1)L_4, \tag{4.85}$$

$$N_{13} = P_0^3(L_1)\,P_1^3(L_2)\,P_0^3(L_3)\,P_2^3(L_4) = \frac{9}{2}L_4(3\,L_4 - 1)L_2, \tag{4.86}$$

$$N_{14} = P_0^3(L_1)\,P_2^3(L_2)\,P_0^3(L_3)\,P_1^3(L_4) = \frac{9}{2}L_2(3\,L_2 - 1)L_4, \tag{4.87}$$

$$N_{15} = P_0^3(L_1)\,P_0^3(L_2)\,P_1^3(L_3)\,P_2^3(L_4) = \frac{9}{2}L_4(3\,L_4 - 1)L_3, \tag{4.88}$$

$$N_{16} = P_0^3(L_1)\,P_0^3(L_2)\,P_2^3(L_3)\,P_1^3(L_4) = \frac{9}{2}L_3(3\,L_3 - 1)L_4, \tag{4.89}$$

$$N_{17} = P_1^3(L_1)\,P_1^3(L_2)\,P_1^3(L_3)\,P_0^3(L_4) = 27\,L_1\,L_2\,L_3, \tag{4.90}$$

$$N_{18} = P_1^3(L_1)\,P_1^3(L_2)\,P_0^3(L_3)\,P_1^3(L_4) = 27\,L_1\,L_2\,L_4, \tag{4.91}$$

$$N_{19} = P_1^3(L_1)\,P_0^3(L_2)\,P_1^3(L_3)\,P_1^3(L_4) = 27\,L_1\,L_3\,L_4, \tag{4.92}$$

$$N_{20} = P_0^3(L_1)\,P_1^3(L_2)\,P_1^3(L_3)\,P_1^3(L_4) = 27\,L_2\,L_3\,L_4, \tag{4.93}$$

because of the equations (4.48), (4.49) and (4.50).

The scalar potential along any edge of a tetrahedron is the linear combination of the values defined on the points of the given edge, so that if two tetrahedra share the same facet, the potential will be continuous across this interface. This means that the approximate solution is continuous everywhere, however, its normal derivate is not.

If potentials at the nodes are known, then a linear approximation of the potential function can be represented by (4.1).

The sum of all nodal shape functions is equal to 1, hence the sum of their gradient is zero,

$$\sum_{i=1}^{m} N_i = 1, \quad \text{and} \quad \sum_{i=1}^{m} \nabla N_i = 0. \tag{4.94}$$

This means that the maximal number of linearly independent nodal basis functions is m and the maximal number of linearly independent gradients of the nodal basis functions is $m - 1$, i.e. shape functions are linearly independent but their gradients are not.
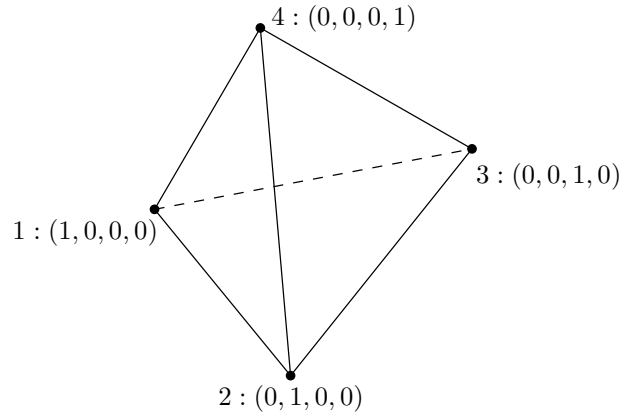
Fig. 4.24. Numbering scheme for linear element, $n = 1$
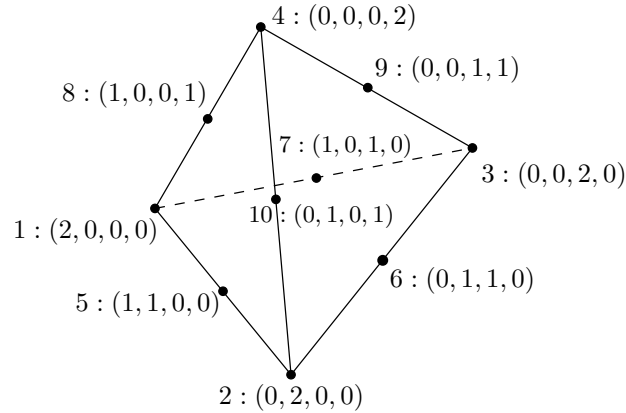
Fig. 4.25. Numbering scheme for quadratic element, $n = 2$
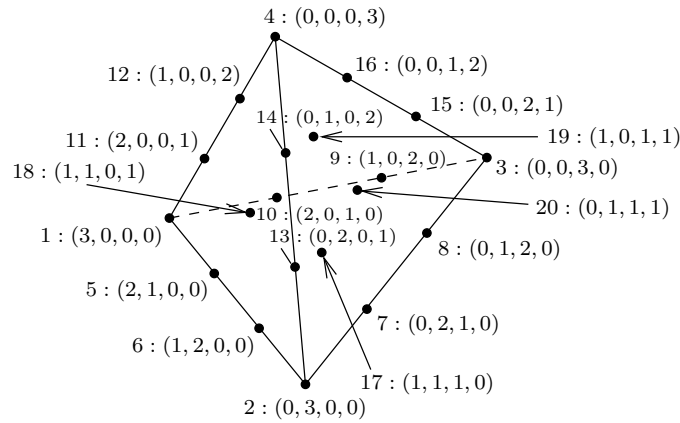
Fig. 4.26. Numbering scheme for cubic element, $n = 3$

4.2.2   Edge finite elements

Vector potentials can be represented either by nodal shape functions or by so-called *edge shape functions* [33, 46, 58, 86, 90–92]. Edge shape functions are also called *vector shape functions*.

The natural approach is to treat the vector field $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{r}, t)$ as two or three coupled scalar fields $T_x = T_x(\boldsymbol{r}, t)$, $T_y = T_y(\boldsymbol{r}, t)$ and $T_z = T_z(\boldsymbol{r}, t)$, i.e.

$$\boldsymbol{T} = T_x \boldsymbol{e}_x + T_y \boldsymbol{e}_y, \tag{4.95}$$

and

$$\boldsymbol{T} = T_x \boldsymbol{e}_x + T_y \boldsymbol{e}_y + T_z \boldsymbol{e}_z \tag{4.96}$$

in 2D and in 3D situations, respectively, $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$ are the orthogonal unit vectors in the $x - y$ and in the $x - y - z$ plane.

Nodal shape functions can be used in this case as well, as it was presented for scalar potentials in the previous section, however, each node has two or three unknowns. Nodal shape functions can be applied to approximate the scalar components of the vector field $\boldsymbol{T}$. For example in 3D, $\boldsymbol{T}$ can be approximated as

$$
\begin{aligned}
\boldsymbol{T} &\simeq \sum_{i=1}^{m} N_i \left(T_{x,i}\, \boldsymbol{e}_x + T_{y,i}\, \boldsymbol{e}_y + T_{z,i}\, \boldsymbol{e}_z \right) \\
&= \sum_{i=1}^{m} N_i T_{x,i}\, \boldsymbol{e}_x + \sum_{i=1}^{m} N_i T_{y,i}\, \boldsymbol{e}_y + \sum_{i=1}^{m} N_i T_{z,i}\, \boldsymbol{e}_z.
\end{aligned}
\tag{4.97}
$$

Here $N_i = N_i(\boldsymbol{r})$ are the usual nodal shape functions defined by (4.2) and $T_{x,i} = T_{x,i}(t)$, $T_{y,i} = T_{y,i}(t)$, $T_{z,i} = T_{z,i}(t)$ are the values of components of the approximated vector potential at node i. The number of degrees of freedom is $2m$ in a 2D problem using triangular mesh and $3m$ in a 3D arrangement meshed by tetrahedral elements.

Nodal shape functions are used to approximate gauged vector potentials, which was the first in the history of finite element method in electromagnetics. Unfortunately, there are some problems when the usual nodal based finite elements are used to interpolate vector potentials. The lack of enforcement of the divergence condition (lack of gauging) results in a system of algebraic equations, which has infinite number of solution and the application of iterative solvers sometimes fails. We have to take care about the Coulomb gauge. There are problems on the iron/air interface when using the magnetic vector potential approximated by nodal elements and extra interface conditions must be set up to solve this problem.

Fortunately, vector shape functions have been developed in the last decades, which application in static and eddy current field problems is more and more popular, because of their advantages. The use of *edge shape functions* solves the problems described above. Some illustrations and examples are shown in chapter 6, which aims to compare not only the different potential functions, but the performance of the nodal and edge representation of vector potentials. It will be shown that the divergence of vector shape functions is equal to zero, that is why, gauging is satisfied automatically. The ungauged potential functions are approximated by vector elements. Vector shape functions are usually called edge shape functions, because they are associated to the edges of the FEM mesh. Vector shape functions are more and more popular in wave problems, too.

Instead of scalar shape functions, *vector shape functions* (or *edge shape functions*) $\boldsymbol{W}_i = \boldsymbol{W}_i(\boldsymbol{r})$ can be applied to approximate a vector potential $\boldsymbol{T}$,

$$
\boldsymbol{T} \simeq \sum_{i=1}^{k} \boldsymbol{W}_i\, T_i,
\tag{4.98}
$$

where $T_i = T_i(t)$ is the line integral of the vector potential $\boldsymbol{T}$ along the edge i. First order vector shape functions are defined by the line integral

$$
\int_l \boldsymbol{W}_i \cdot \mathrm{d}\boldsymbol{l} = \left\{ \begin{array}{ll} 1, & \text{along edge } i, \\ 0, & \text{along other edges,} \end{array} \right.
\tag{4.99}
$$

i.e. the line integral of the vector shape function $\boldsymbol{W}_i$ along the ith edge is equal to one. In other words, the vector shape function $\boldsymbol{W}_i$ has tangential component only along the ith edge and it has only normal component along the other edges, because $\boldsymbol{W}_i \cdot \mathrm{d}\boldsymbol{l}$ is equal to zero only if the vectors $\boldsymbol{W}_i$ and $\mathrm{d}\boldsymbol{l}$ are perpendicular to each others and $|\boldsymbol{W}_i||\mathrm{d}\boldsymbol{l}| > 0$. Moreover, in 3D case, the vector shape function $\boldsymbol{W}_i$ has zero tangential component along every facet of the 3D finite element, which not share the edge i.

If two triangles share the same vertices, the tangential component of the approximated vector potential will be continuous across the interface element boundary. This is true in 3D case as well, moreover, if two tetrahedra share the same facet, the tangential component of the vector potential will be continuous across this interface. This means that the tangential component of the approximate solution is continuous everywhere, however, its normal component is not. In the words of equations, according to the definition (4.99), the line integral of the vector potential along the mth edge is equal to T_m, i.e.

$$
\int_{l_m} \boldsymbol{T} \cdot \mathrm{d}\boldsymbol{l} = \int_{l_m} \left(\sum_{i=1}^{k} \boldsymbol{W}_i T_i \right) \cdot \mathrm{d}\boldsymbol{l} = \sum_{i=1}^{k} \int_{l_m} (\boldsymbol{W}_i T_i) \cdot \mathrm{d}\boldsymbol{l}
$$
$$
= T_m \int_{l_m} \boldsymbol{W}_m \cdot \mathrm{d}\boldsymbol{l} = T_m.
$$

(4.100)

That is why, edge shape functions are also called *tangentially continuous shape functions*.

The gradients of the nodal shape functions are in the function space spanned by the edge basis functions, that is

$$
\nabla N_j = \sum_{i=1}^{k} c_{ji} \boldsymbol{W}_i, \quad j = 1, \cdots, m-1,
$$

(4.101)

where $\sum_{i=1}^{k} c_{ji}^2 > 0$. Taking the curl of each equation in (4.101) results in

$$
\sum_{i=1}^{k} c_{ji} \nabla \times \boldsymbol{W}_i = \boldsymbol{0}, \quad j = 1, \cdots, m-1,
$$

(4.102)

because $\nabla \times (\nabla \varphi) \equiv 0$. This shows that the maximal number of linearly independent curls of the edge basis functions is $k - (m-1)$. The interdependence of the curls of the edge basis functions means that an ungauged formulation leads to a singular, positive semidefinit finite element curl-curl matrix. Singular systems can be solved by iterative methods, if the right-hand side of the system of equations is consistent. We took care about it when obtaining the weak formulations of the ungauged version of potentials, because excitation current density has been taken into account by the use of impressed current vector potential, $\boldsymbol{T}_0$.

The vector function

$$
\boldsymbol{w}_{ij} = L_i \nabla L_j - L_j \nabla L_i
$$

(4.103)

will be applied to construct the edge shape functions, because it can be used in functions, which satisfies (4.99) and (4.100). In 2D, L_i ($i = 1, 2, 3$) are the *barycentric coordinates*

of the triangle defined by (4.18). In 3D, L_i ($i = 1, 2, 3, 4$) are the barycentric coordinates of the tetrahedron defined by (4.56). According to the notations in (4.103), the edges of a finite element are pointing from node i to node j, as it can be seen in *Fig.* 4.27 and in *Fig.* 4.28.

The vector field $\boldsymbol{w}_{ij}$ has the following important properties, which proofs the use of vector function $\boldsymbol{w}_{ij}$ as vector shape function.

(*i*) Let $\boldsymbol{e}_{ij}$ is a unit vector pointing from node i to node j, then

$$\boldsymbol{e}_{ij} \cdot \boldsymbol{w}_{ij} = \frac{1}{l_{ij}}, \tag{4.104}$$

where l_{ij} is the length of edge $\{i, j\}$. This means that $\boldsymbol{w}_{ij}$ has constant tangential component along the edge $\{i, j\}$.

Since L_i and L_j are linear functions that vary from node i to node j from 1 to 0 and from 0 to 1, respectively, we have $\boldsymbol{e}_{ij} \cdot \nabla L_i = -1/l_{ij}$ and $\boldsymbol{e}_{ij} \cdot \nabla L_j = 1/l_{ij}$, finally

$$\boldsymbol{e}_{ij} \cdot \boldsymbol{w}_{ij} = L_i \frac{1}{l_{ij}} + L_j \frac{1}{l_{ij}} = \frac{L_i + L_j}{l_{ij}} = \frac{1}{l_{ij}}, \tag{4.105}$$

because $L_i + L_j = 1$ along the edge $\{i, j\}$. See, for example *Fig.* 4.16 and let $i = 1$, $j = 2$, so $N_1 = L_1$ is decreasing along edge $\{1, 2\}$ and $N_2 = L_2$ is increasing along the same edge. See also *Fig.* 4.9, from which it is easy to see the gradients $\boldsymbol{e}_{ij} \cdot \nabla L_i = -1/l_{ij}$ and $\boldsymbol{e}_{ij} \cdot \nabla L_j = 1/l_{ij}$.

(*ii*) In 2D, the function L_i varies linearly from node i to the opposite edge $\{j, k\}$ (see e.g. N_1 in *Fig.* 4.16, $i = 1$, $j = 2$, $k = 3$), i.e. the vector field ∇L_i is perpendicular to this edge, but L_i is zero there, that is why $\boldsymbol{w}_{ij}$ is perpendicular to the edge $\{j, k\}$,

$$\boldsymbol{w}_{ij} = -L_j \nabla L_i, \quad \text{on the edge} \quad \{j, k\}, \tag{4.106}$$

and the length of this vector is decreasing from node j to k according to L_j. On the other hand, the function L_j varies linearly from node j to the opposite edge $\{k, i\}$
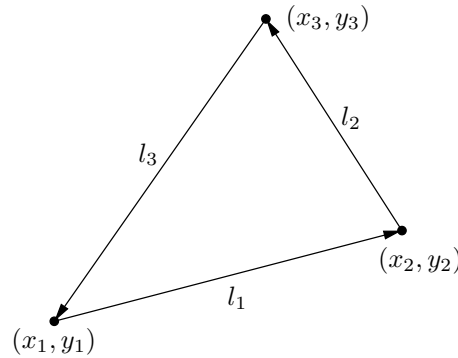
Fig. 4.27. The definition of edges with local directions of the triangular finite element
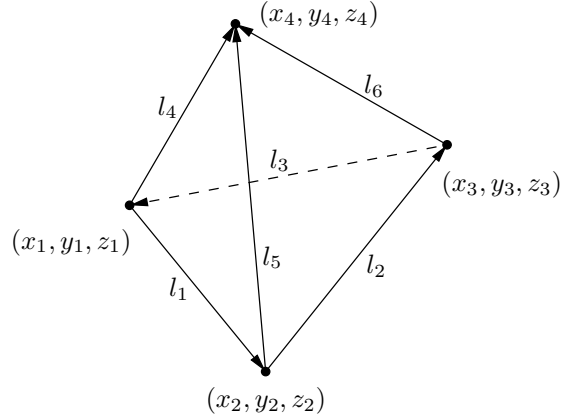
Fig. 4.28. The definition of edges with local directions of the tetrahedral finite element

(see N_2 in *Fig.* 4.16), i.e. ∇L_j is perpendicular to this edge, but L_j is zero there and $\boldsymbol{w}_{ij}$ is perpendicular to the edge $\{k, i\}$,

$$\boldsymbol{w}_{ij} = L_i \nabla L_j, \quad \text{on the edge} \quad \{k, i\}, \tag{4.107}$$

and the length of this vector is decreasing from node i to k according to L_i.

This with item (i) means that the vector function $\boldsymbol{w}_{ij}$ has tangential component only on the edge $\{i, j\}$ and it is perpendicular to the other edges.

In 3D, this is valid to the whole triangular facet with the bounding edges opposite to a node, see e.g. *Fig.* 4.23.

(*iii*) The vector field $\boldsymbol{w}_{ij}$ is divergence-free,

$$\begin{aligned} \nabla \cdot \boldsymbol{w}_{ij} &= \nabla \cdot (L_i \nabla L_j - L_j \nabla L_i) = \nabla \cdot (L_i \nabla L_j) - \nabla \cdot (L_j \nabla L_i) \\ &= \nabla L_i \cdot \nabla L_j + L_i \nabla \cdot \nabla L_j - \nabla L_j \cdot \nabla L_i - L_j \nabla \cdot \nabla L_i = 0, \end{aligned} \tag{4.108}$$

by using the identity

$$\nabla \cdot (\varphi \, \boldsymbol{v}) = \nabla \varphi \cdot \boldsymbol{v} + \varphi \nabla \cdot \boldsymbol{v} \tag{4.109}$$

with the notations $\varphi = L_i$, $\boldsymbol{v} = \nabla L_j$ in the second and $\varphi = L_j$, $\boldsymbol{v} = \nabla L_i$ in the last term. The barycentric coordinates are linear functions of the coordinates and their gradient is constant, which divergence is equal to zero, i.e. the second and fourth terms are vanishing. The first and the third terms are equal, finally, $\nabla \cdot \boldsymbol{w}_{ij} = 0$.

(*iv*) The vector field $\boldsymbol{w}_{ij}$ has constant curl,

$$\begin{aligned} \nabla \times \boldsymbol{w}_{ij} &= \nabla \times (L_i \nabla L_j - L_j \nabla L_i) = \nabla \cdot (L_i \nabla L_j) - \nabla \cdot (L_j \nabla L_i) \\ &= L_i \nabla \times \nabla L_j - \nabla L_j \times \nabla L_i - L_j \nabla \times \nabla L_i + \nabla L_i \times \nabla L_j \\ &= 2 \nabla L_i \times \nabla L_j, \end{aligned} \tag{4.110}$$

by using the identity

$$\nabla \times (\varphi \, \boldsymbol{v}) = \varphi \nabla \times \boldsymbol{v} - \boldsymbol{v} \times \nabla \varphi \qquad (4.111)$$

with the notations $\varphi = L_i$, $\boldsymbol{v} = \nabla L_j$ in the first and $\varphi = L_j$, $\boldsymbol{v} = \nabla L_i$ in the second term. The first and the third terms are equal to zero because of the identity $\nabla \times \nabla \varphi \equiv 0$ for any function φ. The second term can be reformulated by $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$, finally, the result is constant, because the gradients of the barycentric coordinates are constant.

First, the edge shape functions defined on triangles based on (4.103) are collected. The basic 2D vector shape functions $\boldsymbol{W}_i$ can be constructed by using the first order nodal shape functions,

$$\boldsymbol{W}_1 = l_1(N_1 \nabla N_2 - N_2 \nabla N_1)\delta_1, \qquad (4.112)$$

$$\boldsymbol{W}_2 = l_2(N_2 \nabla N_3 - N_3 \nabla N_2)\delta_2, \qquad (4.113)$$

$$\boldsymbol{W}_3 = l_3(N_3 \nabla N_1 - N_1 \nabla N_3)\delta_3. \qquad (4.114)$$

Here l_i (*Fig.* 4.27) denotes the length of the ith edges of the triangle and it is used to normalize the edge shape function according to (4.104). The edge basis function $\boldsymbol{W}_i$ ($i = 1, 2, 3$) has tangential component only along the ith edge and it is perpendicular to the other two edges as represented in *Fig.* 4.29(a)-4.29(c). It is easy to see that an edge shape function has magnitude and direction. The value of δ_i is equal to ± 1, depending on whether the local direction of the edge is the same as the global direction or opposite (see *Fig.* 4.27 for local direction). This set of vector functions is called zeroth order vector shape functions.

If the approximation of the vector function $\boldsymbol{T}$ is known along the edges of the mesh, then (4.98) can be used to interpolate the function anywhere and in linear case $K = 3$.

Higher order vector shape functions can be constructed by using the vector function $\boldsymbol{w}_{ij}$ defined by (4.103), too. This vector function must be multiplied by a complete interpolatory polynomial, which results in the higher order vector shape functions. First and second order polynomials will be used to build up first and second order vector shape functions. Here, we follow [46], the method is as follows.

First of all, an indexing sequence must be set up, which is similar to the method used to build up the scalar shape functions, because the higher order vector shape functions are based on the Lagrange polynomials and (4.103). In the case of first order approximation, the numbering scheme of the third order scalar interpolation can be used and the points are shown in *Fig.* 4.30, *Fig.* 4.31 and *Fig.* 4.32 must be used to represent first order vector shape functions associated to the edge $\{1, 2\}$, $\{2, 3\}$ and $\{3, 1\}$, respectively. In the case of second order approximation, the numbering scheme of the fourth order scalar interpolation can be used and the interpolation points shown in *Fig.* 4.33, *Fig.* 4.34 and *Fig.* 4.35 must be used to represent second order vector shape functions associated to the edge $\{1, 2\}$, $\{2, 3\}$ and $\{3, 1\}$, respectively. The interpolation points have been selected in this special way, because the interpolation of field vectors along vertices has been avoided, i.e. the points have been shifted inside the triangle and the indexing scheme of order $n + 2$ is used to represent the vector interpolation of order n. This is called global numbering and denoted by (I, J, K) on the 'big' triangle, local numbering means the numbering scheme with the real order (i, j, k) defined over the 'small' triangle.
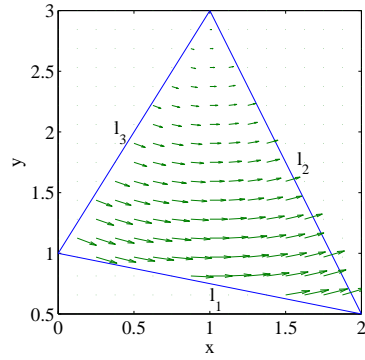
(a) The edge shape function $\boldsymbol{W}_1$
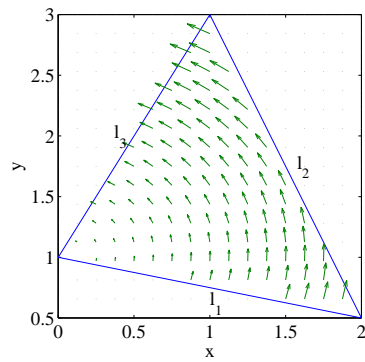
(b) The edge shape function $\boldsymbol{W}_2$
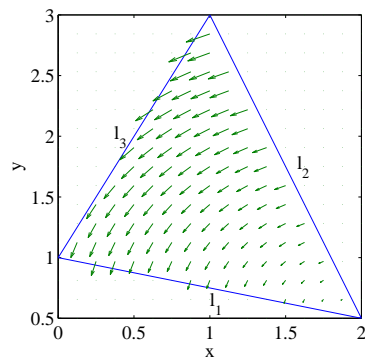
(c) The edge shape function $\boldsymbol{W}_3$

Fig. 4.29. The 2D edge shape functions
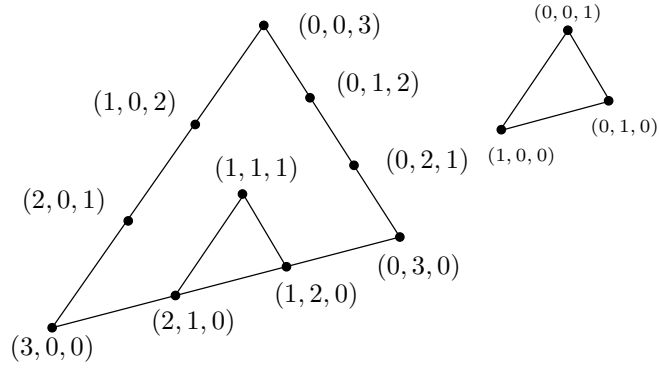
Fig. 4.30. Numbering scheme for the first order vector element associated with $\boldsymbol{w}_{12}$
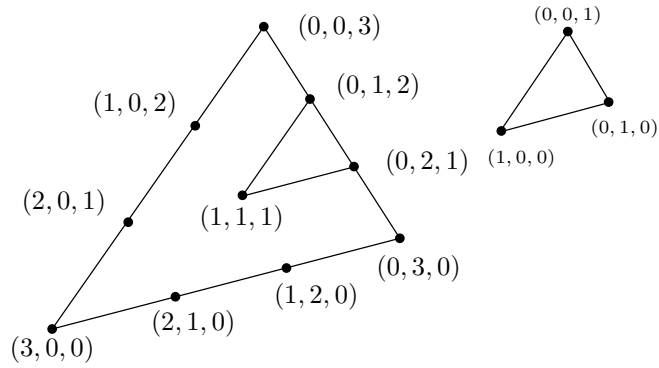
Fig. 4.31. Numbering scheme for the first order vector element associated with $\boldsymbol{w}_{23}$
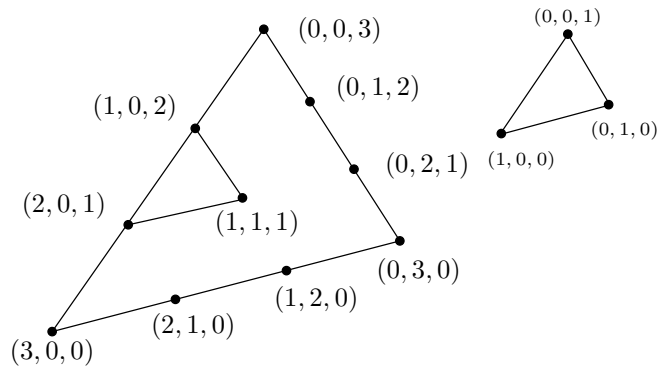
Fig. 4.32. Numbering scheme for the first order vector element associated with $\boldsymbol{w}_{31}$
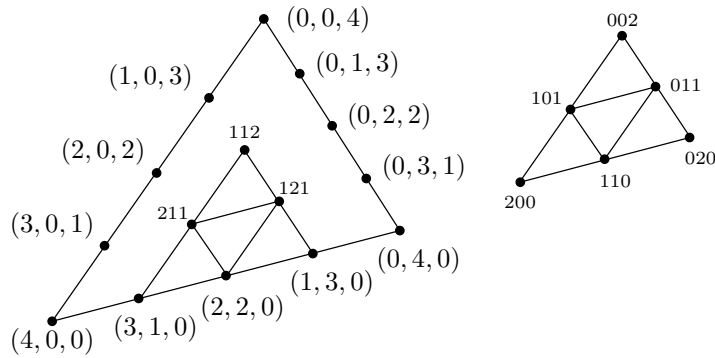
Fig. 4.33. Numbering scheme for the second order vector element associated with w_{12}
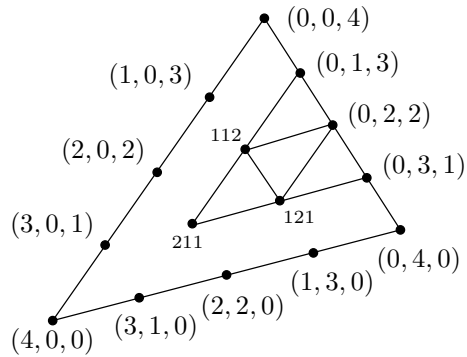
Fig. 4.34. Numbering scheme for the second order vector element associated with w_{23}
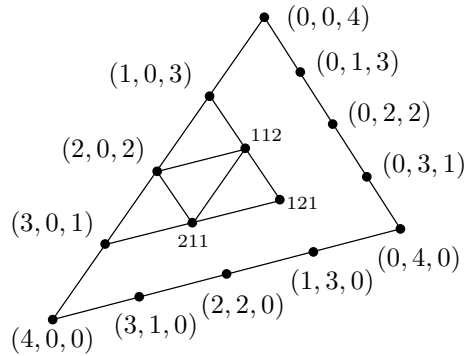
Fig. 4.35. Numbering scheme for the second order vector element associated with w_{31}

It is noted here that the *COMSOL Multiphysics* software uses this kind of vector shape functions, however, $n = 0$, $n = 1$ and $n = 2$ are named as linear, quadratic and cubic vector shape functions.

The vector function $\boldsymbol{w}_{ab}$ (associated to the edge pointing from node a to node b) can be multiplied by the Lagrange polynomials as

$$\boldsymbol{W}_{ab}^{IJK} = \alpha_{ab}^{IJK}\, P_i^n(l_1)\, P_j^n(l_2)\, P_k^n(l_3)\, \boldsymbol{w}_{ab}, \tag{4.115}$$

where n is the order of approximation and the integers i, j and k satisfy $i + j + k = n$ (see the small triangles in *Fig.* 4.30-*Fig.* 4.35).

If $n = 0$, the basic vector shape functions can be obtained, because $P_m^0(\cdot) = 1$ and α can be selected as the length of the appropriate edge, l_{ab}, since α_{ab}^{IJK} is a normalizing factor. The barycentric coordinates l_1, l_2 and l_3 are imagined in the small triangles. The transformation between local and global numbering is as follows:

$$i = I - 1, \quad j = J - 1, \quad k = K, \quad \text{on the edge}\quad \{1, 2\}, \tag{4.116}$$

$$i = I, \quad j = J - 1, \quad k = K - 1, \quad \text{on the edge}\quad \{2, 3\}, \tag{4.117}$$

$$i = I - 1, \quad j = J, \quad k = K - 1, \quad \text{on the edge}\quad \{3, 1\}. \tag{4.118}$$

The relation between the barycentric coordinates of the small and the big triangles is as follows:

$$l_1 = L_1 \frac{n+2}{n}, \quad l_2 = L_2 \frac{n+2}{n} - \frac{1}{n}, \quad l_3 = L_3 \frac{n+2}{n} - \frac{1}{n}. \tag{4.119}$$

Using these relations, (4.115) can be written as (let here $\{ab\} = \{23\}$ for simplicity)

$$\boldsymbol{W}_{23}^{IJK} = \alpha_{23}^{IJK}\, P_I^n\left(L_1 \frac{n+2}{n}\right) P_{J-1}^n\left(L_2 \frac{n+2}{n} - \frac{1}{n}\right)$$
$$P_{K-1}^n\left(L_3 \frac{n+2}{n} - \frac{1}{n}\right) \boldsymbol{w}_{23}. \tag{4.120}$$

According to (4.22), *Lagrange polynomials* can be reformulated as

$$P_I^n\left(L_1 \frac{n+2}{n}\right) = \frac{1}{I!} \prod_{p=0}^{I-1} \left(n\, L_1 \frac{n+2}{n} - p\right)$$
$$= \frac{1}{I!} \prod_{p=0}^{I-1} [(n+2)L_1 - p] = P_I^{n+2}(L_1), \quad \text{if}\quad I > 0, \tag{4.121}$$

and

$$P_{J-1}^n\left(L_2 \frac{n+2}{n} - \frac{1}{n}\right) = \frac{1}{(J-1)!} \prod_{p=0}^{J-2} \left[n\left(L_2 \frac{n+2}{n} - \frac{1}{n}\right) - p\right]$$
$$= \frac{1}{(J-1)!} \prod_{p=0}^{J-2} [(n+2)L_2 - 1 - p]$$
$$= \frac{1}{(J-1)!} \prod_{p=0}^{J-2} \left[(n+2)\left(L_2 - \frac{1}{n+2}\right) - p\right]$$
$$= P_{J-1}^{n+2}\left(L_2 - \frac{1}{n+2}\right), \quad \text{if}\quad J > 1. \tag{4.122}$$

The so-called shifted *Silvester polynomials* can be used to simplify the relations above [46],

$$S_J^{n+2}(L_2) = P_{J-1}^{n+2}\left(L_2 - \frac{1}{n+2}\right) = \frac{1}{(J-1)!}\prod_{p=0}^{J-1}\left[(n+2)L_2 - p\right]. \quad (4.123)$$

Finally, the higher order vector shape functions can be formulated as follows by using the Lagrange and Silvester polynomials:

$$\boldsymbol{W}_{12}^{IJK} = \alpha_{12}^{IJK}\, S_I^{n+2}(L_1)\, S_J^{n+2}(L_2)\, P_K^{n+2}(L_3)\, \boldsymbol{w}_{12}, \quad (4.124)$$

$$\boldsymbol{W}_{23}^{IJK} = \alpha_{23}^{IJK}\, P_I^{n+2}(L_1)\, S_J^{n+2}(L_2)\, S_K^{n+2}(L_3)\, \boldsymbol{w}_{23}, \quad (4.125)$$

$$\boldsymbol{W}_{31}^{IJK} = \alpha_{31}^{IJK}\, S_I^{n+2}(L_1)\, P_J^{n+2}(L_2)\, S_K^{n+2}(L_3)\, \boldsymbol{w}_{31}. \quad (4.126)$$

The parameter denoted by α is a normalization factor, which must have the value such that the line integral of vector shape function $\boldsymbol{W}_{ab}^{IJKL}$ is equal to 1 on the edge pointing from node a to node b. Here,

$$P_0^n(\cdot) = 1, \quad \text{and} \quad S_1^n(\cdot) = 1. \quad (4.127)$$

The number of vector basis functions is

$$k = (n+1)(n+3). \quad (4.128)$$

There is one shape function associated with the introduced interpolation nodes on the edges. It means $3(n+1)$ basis functions. There are three basis functions for an interior interpolation point, because every interpolation point inside the triangle is used to build all the vector shape functions in the three edges. Since a surface vector has only two degrees of freedom, these three basis functions are not independent and one of them must be discarded. This results in $n(n+1)$ interior basis functions. In total, the number of shape functions is $3(n+1) + n(n+1) = (n+3)(n+1)$.

In the case of first order approximation $n = 0$ and $k = 3$. In the case of second order approximation $n = 1$ and $K = 8$. In the case of third order approximation $n = 2$ and $k = 15$ and so on.

The first order vector shape functions are as follows (presented in *Fig.* 4.36) from (4.124)–(4.127):

$$\boldsymbol{W}_1 = \boldsymbol{W}_{12}^{120} = \alpha_{12}^{120}\, S_1^3(L_1)\, S_2^3(L_2)\, P_0^3(L_3)\, \boldsymbol{w}_{12} = \alpha_{12}^{120}(3\,L_2 - 1)\, \boldsymbol{w}_{12}, \quad (4.129)$$

$$\boldsymbol{W}_2 = \boldsymbol{W}_{12}^{210} = \alpha_{12}^{210}\, S_2^3(L_1)\, S_1^3(L_2)\, P_0^3(L_3)\, \boldsymbol{w}_{12} = \alpha_{12}^{210}(3\,L_1 - 1)\, \boldsymbol{w}_{12}, \quad (4.130)$$

$$\boldsymbol{W}_3 = \boldsymbol{W}_{12}^{111} = \alpha_{12}^{111}\, S_1^3(L_1)\, S_1^3(L_2)\, P_1^3(L_3)\, \boldsymbol{w}_{12} = \alpha_{12}^{111}\, 3\,L_3\, \boldsymbol{w}_{12}, \quad (4.131)$$

$$\boldsymbol{W}_4 = \boldsymbol{W}_{23}^{012} = \alpha_{23}^{012}\, P_0^3(L_1)\, S_1^3(L_2)\, S_2^3(L_3)\, \boldsymbol{w}_{23} = \alpha_{23}^{012}(3\,L_3 - 1)\, \boldsymbol{w}_{23}, \quad (4.132)$$

$$\boldsymbol{W}_5 = \boldsymbol{W}_{23}^{021} = \alpha_{23}^{021}\, P_0^3(L_1)\, S_2^3(L_2)\, S_1^3(L_3)\, \boldsymbol{w}_{23} = \alpha_{23}^{021}(3\,L_2 - 1)\, \boldsymbol{w}_{23}, \quad (4.133)$$

$$\boldsymbol{W}_6 = \boldsymbol{W}_{31}^{102} = \alpha_{31}^{102}\, S_1^3(L_1)\, P_0^3(L_2)\, S_2^3(L_3)\, \boldsymbol{w}_{31} = \alpha_{31}^{102}(3\,L_3 - 1)\, \boldsymbol{w}_{31}, \quad (4.134)$$

$$\boldsymbol{W}_7 = \boldsymbol{W}_{31}^{201} = \alpha_{31}^{201}\, S_2^3(L_1)\, P_0^3(L_2)\, S_1^3(L_3)\, \boldsymbol{w}_{31} = \alpha_{31}^{201}(3\,L_1 - 1)\, \boldsymbol{w}_{31}, \quad (4.135)$$

$$\boldsymbol{W}_8 = \boldsymbol{W}_{31}^{111} = \alpha_{31}^{111}\, S_1^3(L_1)\, P_1^3(L_2)\, S_1^3(L_3)\, \boldsymbol{w}_{31} = \alpha_{31}^{111}\, 3\,L_2\, \boldsymbol{w}_{31}. \quad (4.136)$$

(a) $\boldsymbol{W}_1 = \boldsymbol{W}_{12}^{120}$

(b) $\boldsymbol{W}_2 = \boldsymbol{W}_{12}^{210}$

(c) $\boldsymbol{W}_4 = \boldsymbol{W}_{23}^{012}$

(d) $\boldsymbol{W}_5 = \boldsymbol{W}_{23}^{021}$

(e) $\boldsymbol{W}_6 = \boldsymbol{W}_{31}^{102}$

(f) $\boldsymbol{W}_7 = \boldsymbol{W}_{31}^{201}$

(g) $\boldsymbol{W}_3 = \boldsymbol{W}_{12}^{111}$

(h) $\boldsymbol{W}_8 = \boldsymbol{W}_{31}^{111}$
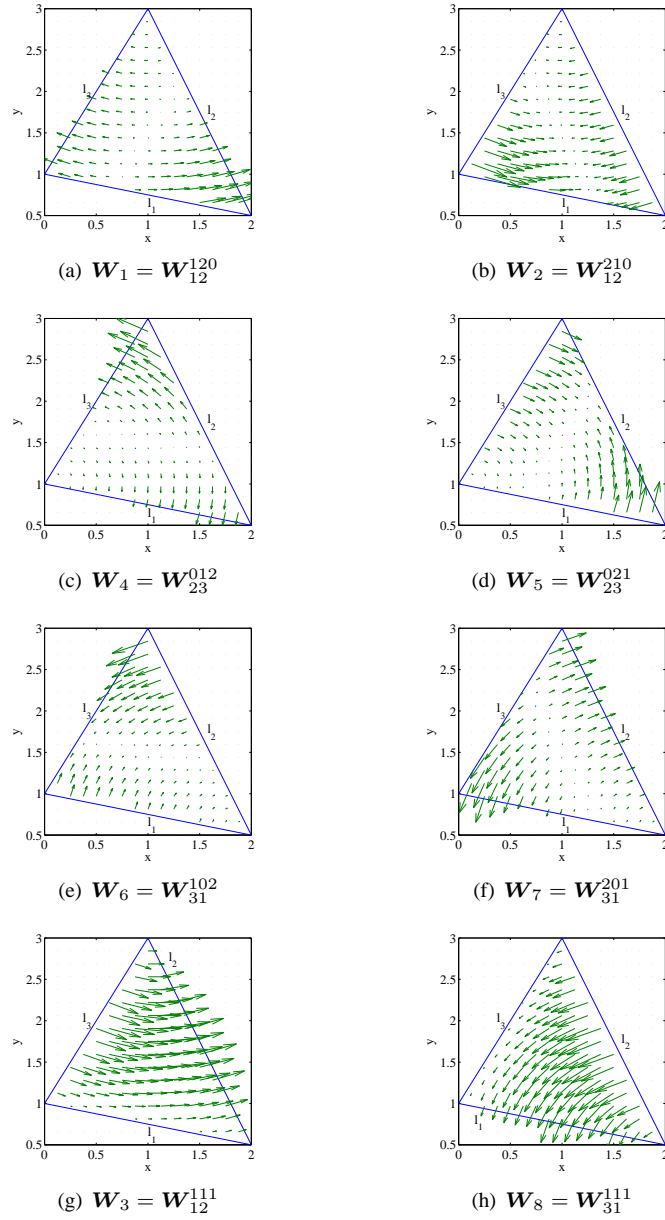
Fig. 4.36. First order vector shape functions, $n = 1$, $k = 8$

The second order vector shape functions are as follows using (4.124)–(4.127):

$$
\begin{aligned}
\boldsymbol{W}_1 = \boldsymbol{W}_{12}^{310} &= \alpha_{12}^{310}\, S_3^4(L_1)\, S_1^4(L_2)\, P_0^4(L_3)\, \boldsymbol{w}_{12} \\
&= \alpha_{12}^{310}\frac{1}{2}(4\,L_1 - 1)(4\,L_1 - 2)\, \boldsymbol{w}_{12},
\end{aligned}
\tag{4.137}
$$

$$\boldsymbol{W}_2 = \boldsymbol{W}_{12}^{220} = \alpha_{12}^{220} \, S_2^4(L_1) \, S_2^4(L_2) \, P_0^4(L_3) \, \boldsymbol{w}_{12}$$
$$= \alpha_{12}^{220}(4\,L_1 - 1)(4\,L_2 - 1)\,\boldsymbol{w}_{12}, \tag{4.138}$$

$$\boldsymbol{W}_3 = \boldsymbol{W}_{12}^{130} = \alpha_{12}^{130} \, S_1^4(L_1) \, S_3^4(L_2) \, P_0^4(L_3) \, \boldsymbol{w}_{12}$$
$$= \alpha_{12}^{130}\frac{1}{2}(4\,L_2 - 1)(4\,L_2 - 2)\,\boldsymbol{w}_{12}, \tag{4.139}$$

$$\boldsymbol{W}_4 = \boldsymbol{W}_{12}^{211} = \alpha_{12}^{211} \, S_2^4(L_1) \, S_1^4(L_2) \, P_1^4(L_3) \, \boldsymbol{w}_{12}$$
$$= \alpha_{12}^{211}(4\,L_1 - 1)4\,L_3\,\boldsymbol{w}_{12}, \tag{4.140}$$

$$\boldsymbol{W}_5 = \boldsymbol{W}_{12}^{121} = \alpha_{12}^{121} \, S_1^4(L_1) \, S_2^4(L_2) \, P_1^4(L_3) \, \boldsymbol{w}_{12}$$
$$= \alpha_{12}^{121}(4\,L_2 - 1)4\,L_3\,\boldsymbol{w}_{12}, \tag{4.141}$$

$$\boldsymbol{W}_6 = \boldsymbol{W}_{23}^{031} = \alpha_{23}^{031} \, P_0^4(L_1) \, S_3^4(L_2) \, S_1^4(L_3) \, \boldsymbol{w}_{23}$$
$$= \alpha_{23}^{031}\frac{1}{2}(4\,L_2 - 1)(4\,L_2 - 2)\,\boldsymbol{w}_{23}, \tag{4.142}$$

$$\boldsymbol{W}_7 = \boldsymbol{W}_{23}^{022} = \alpha_{23}^{022} \, P_0^4(L_1) \, S_2^4(L_2) \, S_2^4(L_3) \, \boldsymbol{w}_{23}$$
$$= \alpha_{23}^{022}(4\,L_2 - 1)(4\,L_3 - 1)\,\boldsymbol{w}_{23}, \tag{4.143}$$

$$\boldsymbol{W}_8 = \boldsymbol{W}_{23}^{013} = \alpha_{23}^{013} \, P_0^4(L_1) \, S_1^4(L_2) \, S_3^4(L_3) \, \boldsymbol{w}_{23}$$
$$= \alpha_{23}^{013}\frac{1}{2}(4\,L_3 - 1)(4\,L_3 - 2)\,\boldsymbol{w}_{23}, \tag{4.144}$$

$$\boldsymbol{W}_9 = \boldsymbol{W}_{23}^{121} = \alpha_{23}^{121} \, P_1^4(L_1) \, S_2^4(L_2) \, S_1^4(L_3) \, \boldsymbol{w}_{23}$$
$$= \alpha_{23}^{121}(4\,L_1)(4\,L_2 - 1)\,\boldsymbol{w}_{23}, \tag{4.145}$$

$$\boldsymbol{W}_{10} = \boldsymbol{W}_{23}^{112} = \alpha_{23}^{112} \, P_1^4(L_1) \, S_1^4(L_2) \, S_2^4(L_3) \, \boldsymbol{w}_{23}$$
$$= \alpha_{23}^{112}(4\,L_1)(4\,L_3 - 1)\,\boldsymbol{w}_{23}, \tag{4.146}$$

$$\boldsymbol{W}_{11} = \boldsymbol{W}_{31}^{103} = \alpha_{31}^{103} \, S_1^4(L_1) \, P_0^4(L_2) \, S_3^4(L_3) \, \boldsymbol{w}_{31}$$
$$= \alpha_{31}^{103}\frac{1}{2}(4\,L_3 - 1)(4\,L_3 - 2)\,\boldsymbol{w}_{31}, \tag{4.147}$$

$$\boldsymbol{W}_{12} = \boldsymbol{W}_{31}^{202} = \alpha_{31}^{202} \, S_2^4(L_1) \, P_0^4(L_2) \, S_2^4(L_3) \, \boldsymbol{w}_{31}$$
$$= \alpha_{31}^{202}(4\,L_1 - 1)(4\,L_3 - 1)\,\boldsymbol{w}_{31}, \tag{4.148}$$

$$\boldsymbol{W}_{13} = \boldsymbol{W}_{31}^{301} = \alpha_{31}^{301} \, S_3^4(L_1) \, P_0^4(L_2) \, S_1^4(L_3) \, \boldsymbol{w}_{31}$$
$$= \alpha_{31}^{301}\frac{1}{2}(4\,L_1 - 1)(4\,L_1 - 2)\,\boldsymbol{w}_{31}, \tag{4.149}$$

$$\boldsymbol{W}_{14} = \boldsymbol{W}_{31}^{112} = \alpha_{31}^{112} \, S_1^4(L_1) \, P_1^4(L_2) \, S_2^4(L_3) \, \boldsymbol{w}_{31}$$
$$= \alpha_{31}^{112}(4\,L_2)(4\,L_3 - 1)\,\boldsymbol{w}_{31}, \tag{4.150}$$

$$\boldsymbol{W}_{15} = \boldsymbol{W}_{31}^{211} = \alpha_{31}^{211} \, S_2^4(L_1) \, P_1^4(L_2) \, S_1^4(L_3) \, \boldsymbol{w}_{31}$$
$$= \alpha_{31}^{211}(4\,L_2)(4\,L_1 - 1)\,\boldsymbol{w}_{31}. \tag{4.151}$$

The second order shape functions associated to the edge $\{1, 2\}$ can be seen in *Fig.* 4.37. The edge shape functions associated to the other two edges can be imagined as the rotation

(a) $\boldsymbol{W}_1 = \boldsymbol{W}_{12}^{310}$

(b) $\boldsymbol{W}_2 = \boldsymbol{W}_{12}^{220}$

(c) $\boldsymbol{W}_3 = \boldsymbol{W}_{12}^{130}$

(d) $\boldsymbol{W}_4 = \boldsymbol{W}_{12}^{211}$

(e) $\boldsymbol{W}_5 = \boldsymbol{W}_{12}^{121}$
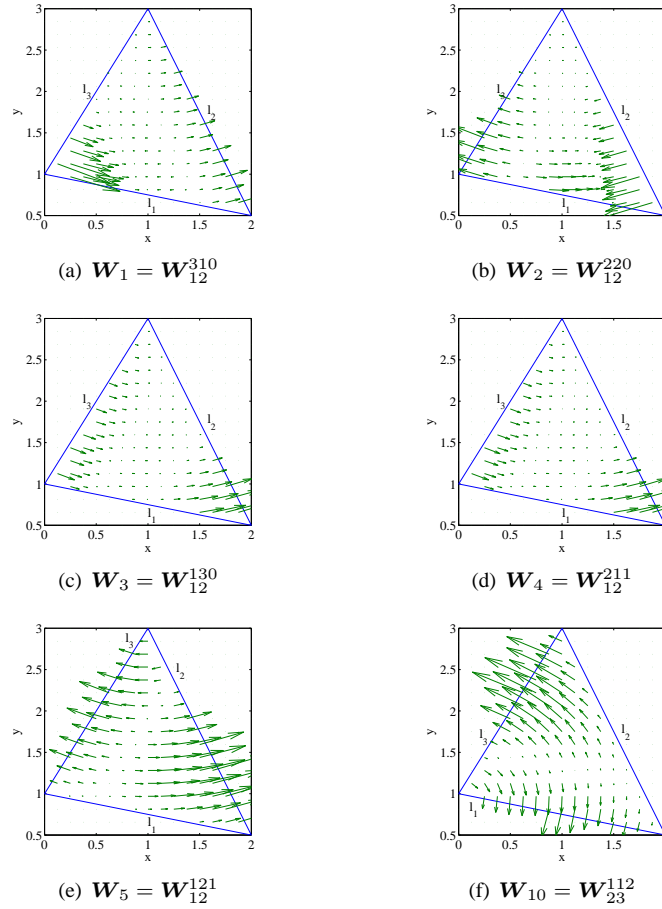
(f) $\boldsymbol{W}_{10} = \boldsymbol{W}_{23}^{112}$

Fig. 4.37. Second order vector shape functions, $n = 2$, $K = 15$

of the vector functions shown in *Fig.* 4.37. For example, $\boldsymbol{W}_{10} = \boldsymbol{W}_{23}^{112}$ is also plotted, which is the same as $\boldsymbol{W}_5 = \boldsymbol{W}_{12}^{121}$, but along the edge $\{2, 3\}$.

The *vector shape functions* in 3D can be constructed as the extension of the above presented 2D realization. Three-dimensional zeroth order edge shape functions can be constructed as [46],

$$\boldsymbol{W}_1 = l_1(N_1\nabla N_2 - N_2\nabla N_1)\delta_1, \tag{4.152}$$

$$\boldsymbol{W}_2 = l_2(N_2\nabla N_3 - N_3\nabla N_2)\delta_2, \tag{4.153}$$

$$\boldsymbol{W}_3 = l_3(N_3\nabla N_1 - N_1\nabla N_3)\delta_3, \tag{4.154}$$

$$\boldsymbol{W}_4 = l_4(N_1\nabla N_4 - N_4\nabla N_1)\delta_4, \tag{4.155}$$

$$\boldsymbol{W}_5 = l_5(N_2\nabla N_4 - N_4\nabla N_2)\delta_5, \tag{4.156}$$

$$\boldsymbol{W}_6 = l_6(N_3 \nabla N_4 - N_4 \nabla N_3)\delta_6. \tag{4.157}$$

Here l_i (*Fig.* 4.28) is the length of the edges and it is used to normalize the edge shape function according to (4.104). The value of δ_i is also equal to ± 1 depending on whether the local direction of the edge is the same as the global direction or opposite. The edge definition employed in my analysis can be seen in *Fig.* 4.28.

If the approximation of the vector function $\boldsymbol{T}$ is known along the edges of the mesh, then (4.98) can be used to interpolate the function anywhere and in linear case $k = 6$.

To construct higher order vector basis functions, the points of interpolation polynomials are arranged in a pyramid format to build an applicable numbering scheme (I, J, K, L) and $I, J, K, L = 0, 1, \cdots, n + 2$, where n is the order of the element. The illustration of numbering scheme in 3D is not easy, but it can be construct as follows. Let us imagine the same numbering scheme on the triangular facets of the tetrahedron as in *Fig.* 4.33-*Fig.* 4.35 and the integers I, J, K and L can be set up according to the facets.

The vector shape functions of order n are given as [46],

$$\boldsymbol{W}_{12}^{IJKL} = \alpha_{12}^{IJKL} \, S_I^{n+2}(L_1) \, S_J^{n+2}(L_2) \, P_K^{n+2}(L_3) \, P_L^{n+2}(L_4) \, \boldsymbol{w}_{12}, \tag{4.158}$$

$$\boldsymbol{W}_{23}^{IJKL} = \alpha_{23}^{IJKL} \, P_I^{n+2}(L_1) \, S_J^{n+2}(L_2) \, S_K^{n+2}(L_3) \, P_L^{n+2}(L_4) \, \boldsymbol{w}_{23}, \tag{4.159}$$

$$\boldsymbol{W}_{31}^{IJKL} = \alpha_{31}^{IJKL} \, S_I^{n+2}(L_1) \, P_J^{n+2}(L_2) \, S_K^{n+2}(L_3) \, P_L^{n+2}(L_4) \, \boldsymbol{w}_{31}, \tag{4.160}$$

$$\boldsymbol{W}_{14}^{IJKL} = \alpha_{14}^{IJKL} \, S_I^{n+2}(L_1) \, P_J^{n+2}(L_2) \, P_K^{n+2}(L_3) \, S_L^{n+2}(L_4) \, \boldsymbol{w}_{14}, \tag{4.161}$$

$$\boldsymbol{W}_{24}^{IJKL} = \alpha_{24}^{IJKL} \, P_I^{n+2}(L_1) \, S_J^{n+2}(L_2) \, P_K^{n+2}(L_3) \, S_L^{n+2}(L_4) \, \boldsymbol{w}_{24}, \tag{4.162}$$

$$\boldsymbol{W}_{34}^{IJKL} = \alpha_{34}^{IJKL} \, P_I^{n+2}(L_1) \, P_J^{n+2}(L_2) \, S_K^{n+2}(L_3) \, S_L^{n+2}(L_4) \, \boldsymbol{w}_{34}. \tag{4.163}$$

The parameters denoted by α are normalization factors, which must have the value such that the line integral of the vector shape function $\boldsymbol{W}_{ab}^{IJKL}$ is equal to 1 on the edge pointing from node a to node b.

The number of edge shape functions when defining the nth order family is

$$k = \frac{(n+1)(n+3)(n+4)}{2}. \tag{4.164}$$

For each interpolation point on the edge, there is one corresponding vector shape function, which means $6(n+1)$ functions. For each interpolation point on the face of a tetrahedron there are three vector functions, but one of them is depending on the other two and it must be discarded, finally there are $4n(n+1)$ vector shape functions defined on the four facets. For each interpolation points inside the element there are six basis functions. A 3D vector has only three degree of freedom, that is why three vector basis functions must be discarded resulting in $n(n-1)(n+1)/2$ vector basis functions. Totally, there are $6(n+1) + 4n(n+1) + n(n-1)(n+1)/2 = (n+1)(n+3)(n+4)/2$ vector shape functions. There are $k = 6$, $k = 20$ and $k = 45$ shape functions for $n = 0$, $n = 1$ and $n = 2$, respectively.

As an example, the following vector shape functions can be set up when $n = 1$, $\boldsymbol{W}_{12}^{2100}$, $\boldsymbol{W}_{12}^{1200}$, $\boldsymbol{W}_{23}^{0210}$, $\boldsymbol{W}_{23}^{0120}$, $\boldsymbol{W}_{31}^{1020}$, $\boldsymbol{W}_{31}^{2010}$, $\boldsymbol{W}_{14}^{2001}$, $\boldsymbol{W}_{14}^{1002}$, $\boldsymbol{W}_{24}^{0201}$, $\boldsymbol{W}_{24}^{0102}$, $\boldsymbol{W}_{34}^{0021}$, $\boldsymbol{W}_{34}^{0012}$, $\boldsymbol{W}_{12}^{1110}$, $\boldsymbol{W}_{31}^{1110}$, $\boldsymbol{W}_{14}^{1011}$, $\boldsymbol{W}_{34}^{1011}$, $\boldsymbol{W}_{23}^{0111}$, $\boldsymbol{W}_{24}^{0111}$, $\boldsymbol{W}_{12}^{1101}$, $\boldsymbol{W}_{24}^{1101}$.

5 The polarization method and the fixed point technique

This short chapter, based on the publications [14,17,24,34–39,50,51,53,69,70] presents a method to handle nonlinear characteristics of ferromagnetic materials in electromagnetic field computation and a technique to solve the resulting system of *nonlinear equations*. According to the so-called *polarization method*, the output of the nonlinear model is split in two parts, a linear part and a nonlinear part. The system of nonlinear equations can be solved by the *fixed point iteration* technique.

First, the general form of nonlinear equations is shortly presented, then a simple illustrative example is solved in the first part of this chapter. In the second part, the most important definitions and formulations are shown, which are important to understand the introduction of the polarization technique and the application of the fixed point based iteration technique.

5.1   Solution of nonlinear equations

A *nonlinear equation*

$$F(x) = 0, \tag{5.1}$$

or a system of nonlinear equations $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ (here $\mathbf{F}$ is a vector valued function) generally can be solved by iterative methods. Iteration means a repeated sequence of operations, which must lead to the solution of the nonlinear equation. There are two main groups of methods, the *successive approximation based methods* and the techniques based on *Newton's method*.

Here, only the first family will be analyzed, the so-called *fixed point iteration* scheme is introduced to solve nonlinear electromagnetic field problems.

The *successive approximation method* is based on the following form of nonlinear equations:

$$x = f(x), \tag{5.2}$$

which is usually called the fixed point form of nonlinear equations (the system of nonlinear equations can be represented by the vector valued function $\mathbf{x} = \mathbf{f}(\mathbf{x})$).

Let us follow the iteration sequence illustrated in *Fig.* 5.1. Here, the fixed point equation $x = f(x)$ is solved. A simple function $y = f(x)$, as well as the function $y = x$ are presented here. The point denoted by c, where $c = f(c)$, is called fixed point, which is the solution of the nonlinear equation and it is the common point of the functions $y = f(x)$ and $y = x$. The iteration can be started by any arbitrary point x_0, $y_0 = f(x_0)$ can be calculated by using the function $f(x)$. Value y_0 is the input of the second step of iteration, i.e. $x_1 = y_0 = f(x_0)$ as it is illustrated by the linear function $y = x$. This results in $y_1 = f(x_1)$ and $x_2 = y_1 = f(x_1)$. This iteration scheme can be generalized as

$$x_{n+1} = f(x_n), \quad n = 0, 1, \cdots . \tag{5.3}$$

The sequence shown in *Fig.* 5.1 leads to the solution of the nonlinear equation $x = f(x)$, which is called the fixed point, because its coordinates are fixed, they do not change if $n \to \infty$. This iteration is convergent, meaning that the distance $|x_{n+1} - x_n|$ is decreasing by increasing the index n. The condition

$$|x_{n+1} - x_n| < \varepsilon \tag{5.4}$$

can be used to stop the iteration, where ε is a small positive real number.

The example shown in *Fig.* 5.2 shows an iteration, which is not convergent. This nonlinear equation can not be solved by the above presented iteration.

The following can be summarized after this illustration. Let us denote the set of the input (argument) and of the output (value) of the function f by $\mathbb{I}$ and $\mathbb{O}$, respectively. Convergent iteration can be realized only if $\mathbb{O} \subset \mathbb{I}$ and function f only has solution in $\mathbb{I}$,
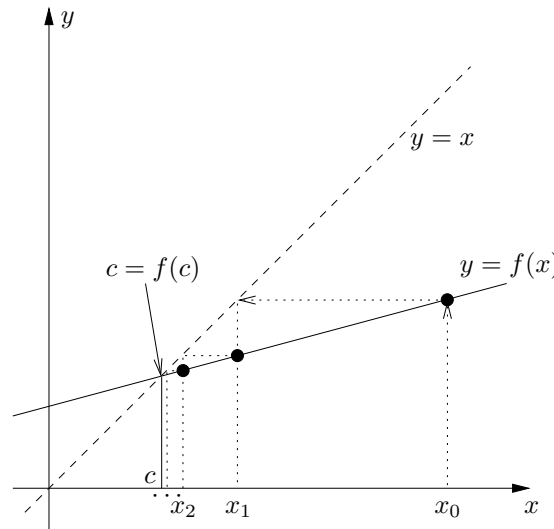
Fig. 5.1. Convergent iteration to a fixed point
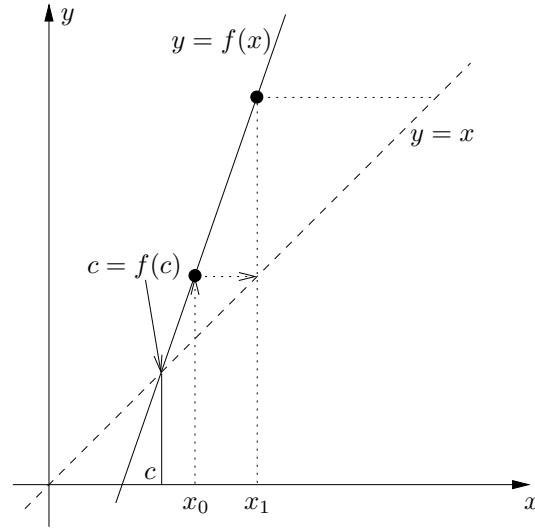
Fig. 5.2. Iteration does not converge to the fixed point

if it is continuous in $\mathbb{I}$. Figure 5.1 illustrates that the rate of change of function $y = f(x)$ should not exceed the rate of change of the function $y = x$, i.e.

$$\left| \frac{\mathrm{d}f(x)}{\mathrm{d}x} \right| < \frac{\mathrm{d}x}{\mathrm{d}x} \quad \Rightarrow \quad \left| \frac{\mathrm{d}f(x)}{\mathrm{d}x} \right| < 1. \tag{5.5}$$

The iteration scheme (5.3) is convergent in this case. Moreover, it can be seen in *Fig.* 5.1, that the smaller the rate of change of function f, the smaller the number of iterations needed to reach the fixed point.

In practical situations, the derivative of function f can not be calculated analytically, or it is a time consuming task. In this case the formulation

$$\frac{|f(x) - f(y)|}{|x - y|} < 1 \tag{5.6}$$

can be applied. The mapping f is said to be *contraction*, if

$$|f(x) - f(y)| \leq q|x - y|, \quad \text{for all } x, y \in \mathbb{I}, \tag{5.7}$$

and $q < 1$, because it decreases the distance between the points x and y.

Equation (5.1) can be reformulated as $x = x \pm F(x)$. Positive or negative sign must be used if function $F(x)$ is decreasing or increasing, respectively. Let us suppose that $F(x)$ is increasing, i.e. $x = x - F(x)$ can be used and $f(x) = x - F(x)$, from which the following condition can be obtained after (5.5):

$$\left| \frac{\mathrm{d}f(x)}{\mathrm{d}x} \right| = \left| 1 - \frac{\mathrm{d}F(x)}{\mathrm{d}x} \right| < 1, \tag{5.8}$$

i.e. $|\mathrm{d}F(x)/\mathrm{d}x| < 1$ must be satisfied. This can not be said generally, that is why the following modification must be used:

$$F(x) = 0 \quad \Rightarrow \quad f(x) = x - \lambda F(x), \tag{5.9}$$

where λ is a free parameter. In this case

$$\left| \frac{\mathrm{d}f(x)}{\mathrm{d}x} \right| = \left| 1 - \lambda \frac{\mathrm{d}F(x)}{\mathrm{d}x} \right| < 1, \tag{5.10}$$

and $|\lambda\, \mathrm{d}F(x)/\mathrm{d}x| < 1$ can be set easily by selecting λ. The same can be obtained if $F(x)$ is decreasing.

If the solution, i.e. the fixed point of $f(x)$ is located in the interval $x \in [a, \cdots, b]$ and $m = \min \mathrm{d}F(x)/\mathrm{d}x$, $M = \max \mathrm{d}F(x)/\mathrm{d}x$ are the minimum and maximum slope of function $F(x)$ in the interval $[a, \cdots, b]$, then the following inequality can be written:

$$0 \leq 1 - \lambda M \leq 1 - \lambda \frac{\mathrm{d}F(x)}{\mathrm{d}x} \leq 1 - \lambda m \leq 1. \tag{5.11}$$

From the first inequality, $0 = 1 - \lambda M$ can be selected, i.e.

$$\lambda = \frac{1}{M} \tag{5.12}$$

can be used as a general rule to satisfy $|\lambda\, \mathrm{d}F(x)/\mathrm{d}x| < 1$, i.e. the condition (5.5). The following simple example shows the application of this procedure.

Illustration.   Let us try to find the solution of the nonlinear equation $x = \cos x$. It can be performed by the iterative sequence

$$x_{n+1} = \cos x_n, \quad n = 0, 1, \cdots, \tag{5.13}$$
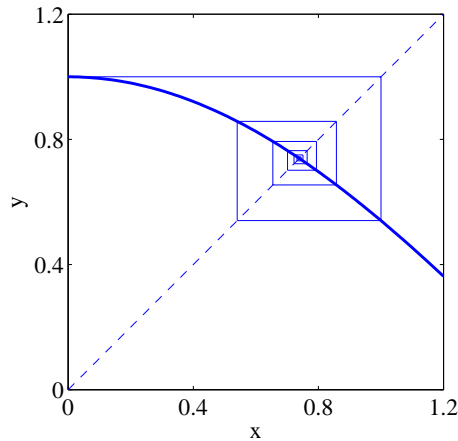
Fig. 5.3. Solution of the nonlinear equation $x = \cos x$ by the fixed point method
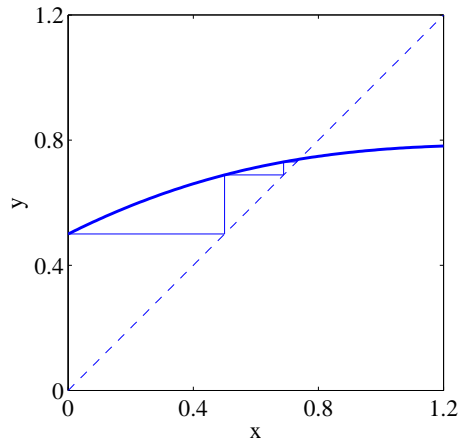
Fig. 5.4. Solution of the nonlinear equation $x = \cos x$ by the fixed point method, when $\lambda = 1/2$

and let $x_0 = 0$ and let the interval of search be $x \in [0, \cdots, \pi/2]$. This iteration sequence can be seen in *Fig.* 5.3. Here, the solution $c = 0.7390851332$ can be reached after 58 iteration steps and the stopping criterion is $|x_{n+1} - x_n| < 10^{-10}$.

The problem can be formulated as $F(x) = x - \cos x = 0$, too and it can be written as $x = x - \lambda(x - \cos x)$, i.e. $f(x) = x - \lambda(x - \cos x)$. The maximum value of the derivative of the function $F(x)$ in the interval $x \in [0, \cdots, \pi/2]$ is $\max(1 + \sin x) = 2$. In the case of $\lambda = 1/2$, 13 steps are enough to reach the error criterion $|x_{n+1} - x_n| < 10^{-10}$. The
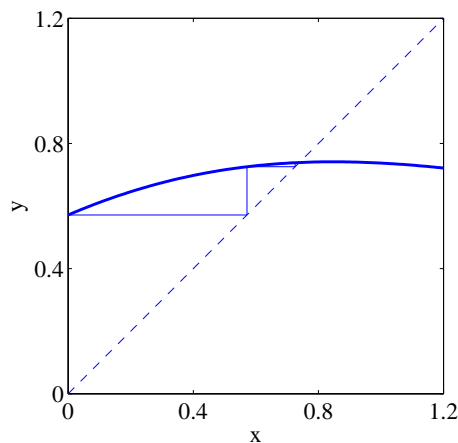
Fig. 5.5. Solution of the nonlinear equation $x = \cos x$ by the fixed point method, if $\lambda = 1/1.75$

iteration sequence can be seen in *Fig.* 5.4. By decreasing the interval, $\lambda = 1/1.75$ can be selected and the fixed point can be reached after 8 steps (see *Fig.* 5.5), however, 58 steps are needed without λ.

It is easy to see that $\mathrm{d}f(x)/\mathrm{d}x$ can be set smaller and smaller by the selection of the appropriate value of λ, but the fixed point does not change.

The following scheme is a very important conclusion of this illustrative introduction. The *fixed point scheme*

$$x_{n+1} = x_n - \lambda F(x_n) \equiv f(x_n), \quad n = 0, 1, \cdots \tag{5.14}$$

with appropriate value of λ is convergent, i.e. the mapping defined by f is a *contraction mapping*. The convergence speed can be increased by the appropriate selection of λ. This formulation will be used in the next sections.

5.2   Nonlinearity in electromagnetic field simulation

Design and simulation of electrical engineering applications containing ferromagnetic parts require the accurate modeling of hysteresis characteristics and the implementation of the models into a procedure to solve the nonlinear partial differential equations derived from Maxwell's equations with appropriate boundary conditions. The partial differential equations are generally nonlinear because of the nonlinear characteristics of ferromagnetic materials.

The numerical analysis of electromagnetic fields can be characterized by the electric and magnetic field intensities and flux densities formulated by Maxwell's equations, which are the collection of partial differential equations of the electric field intensity $\boldsymbol{E}$, the magnetic field intensity $\boldsymbol{H}$, the electric flux density $\boldsymbol{D}$ and the magnetic flux density $\boldsymbol{B}$.

Constitutive relations between the above quantities are defined to take into account the macroscopic properties of the medium where the electromagnetic field has been studied. The constitutive relation between the magnetic field intensity and the magnetic flux density is nonlinear in this work, given by the operator $\boldsymbol{B} = \mathscr{B}\{\boldsymbol{H}\}$ or $\boldsymbol{H} = \mathscr{B}^{-1}\{\boldsymbol{B}\}$. These nonlinear characteristics can be reformulated by introducing the polarization method and the resulting system of nonlinear equations can be solved by the fixed point technique. The constitutive relations can be rewritten in a form similar to (5.14).

According to the *polarization method*, the magnetic flux density can be split in two parts as

$$\boldsymbol{B} = \mu \boldsymbol{H} + \boldsymbol{\beta}, \tag{5.15}$$

where $\mu \boldsymbol{H}$ is a linear term, because μ is supposed to be constant and nonlinearity is hidden in the second term $\boldsymbol{\beta}$. It is a magnetic flux density like quantity. The question is the appropriate value of the parameter μ. This representation can be reformulated as

$$\boldsymbol{\beta} = \boldsymbol{B} - \mu \boldsymbol{H} = \boldsymbol{B} - \mu \mathscr{B}^{-1}\{\boldsymbol{B}\}. \tag{5.16}$$

Here, the inverse type hysteresis characteristics are used and the right-hand side of this equation is similar to the formula (5.14), which we are going to denote by

$$f_{\boldsymbol{B}}(\boldsymbol{B}) = \boldsymbol{B} - \mu \mathscr{B}^{-1}\{\boldsymbol{B}\}. \tag{5.17}$$

This equation can be used to obtain the optimal value of μ, which results in a convergent fixed point iteration sequence and it is as fast as possible.

There is another *formula of polarization*,

$$\boldsymbol{H} = \nu\boldsymbol{B} + \boldsymbol{\eta}, \tag{5.18}$$

where $\nu\boldsymbol{B}$ is a linear term, because ν is supposed to be constant. Nonlinear behavior is hidden in the second term $\boldsymbol{\eta}$, which is a magnetic field intensity like quantity. The question is the appropriate value of the parameter ν. By using the direct model, this representation can be reformulated as

$$\boldsymbol{\eta} = \boldsymbol{H} - \nu\boldsymbol{B} = \boldsymbol{H} - \nu\mathscr{B}\{\boldsymbol{H}\}. \tag{5.19}$$

The right-hand side of this equation is similar to the formula (5.14), which we are going to denote by

$$f_{\boldsymbol{H}}(\boldsymbol{H}) = \boldsymbol{H} - \nu\mathscr{B}\{\boldsymbol{H}\}. \tag{5.20}$$

This equation can be used to obtain the optimal value of ν, which results in a convergent fixed point iteration scheme and it is as fast as possible.

It is noted that the functions $f_{\boldsymbol{B}}(\boldsymbol{B})$ and $f_{\boldsymbol{H}}(\boldsymbol{H})$ have the same form. Here, the parameters μ and ν are supposed to be constant for simplicity.

It is important to note that the nonlinear equations are solved by iterative methods, the nth step of iteration is denoted by the superscript (n) in the following.

By using the formula (5.15) or (5.18) in the constitutive relations defined in Maxwell's equations, the solution of the *nonlinear partial differential equations* with appropriate boundary conditions can be formulated as

$$\boldsymbol{B}^{(n)} = \mathscr{M}\{\boldsymbol{\beta}^{(n-1)}\}, \tag{5.21}$$

or

$$\boldsymbol{H}^{(n)} = \mathscr{M}\{\boldsymbol{\eta}^{(n-1)}\}, \tag{5.22}$$

where the operator $\mathscr{M}\{\cdot\}$ represents the set of Maxwell's equations and the boundary conditions. The starting $\boldsymbol{\beta}^{(0)}$ and $\boldsymbol{\eta}^{(0)}$ are arbitrary. The value of electromagnetic field quantities in the nth step are depending on the value of $\boldsymbol{\beta}$ or $\boldsymbol{\eta}$ in the $(n-1)^{\text{th}}$ step, these represent source like quantities. The source of electromagnetic fields (e.g. the electric current density $\boldsymbol{J}$) is not changing during fixed point iteration, i.e. fixed point iteration has no got physical meaning.

By using the magnetic flux density or the magnetic field intensity satisfying Maxwell's equations and boundary conditions, the nonlinear terms can be updated by the relation (5.16) or (5.19),

$$\boldsymbol{\beta}^{(n)} = f_{\boldsymbol{B}}\{\boldsymbol{B}^{(n)}\}, \tag{5.23}$$

or

$$\boldsymbol{\eta}^{(n)} = f_{\boldsymbol{H}}\{\boldsymbol{H}^{(n)}\}. \tag{5.24}$$

The above sequence can be rewritten in the form of the fixed point equation (5.3),

$$\boldsymbol{\beta}^{(n)} = f_{\boldsymbol{B}}\{\mathscr{M}\{\boldsymbol{\beta}^{(n-1)}\}\} = \phi_{\boldsymbol{\beta}}\{\boldsymbol{\beta}^{(n-1)}\}, \tag{5.25}$$

or

$$\boldsymbol{\eta}^{(n)} = f_{\boldsymbol{H}}\{\mathscr{M}\{\boldsymbol{\eta}^{(n-1)}\}\} = \phi_{\boldsymbol{\eta}}\{\boldsymbol{\eta}^{(n-1)}\}. \tag{5.26}$$

This kind of nonlinear equation can be formulated by using the polarization method and the convergence and speed of convergence can be set by the appropriate value of μ or ν. It is noted that these parameters are constant.

A fixed point technique was proposed for the first time in electrical engineering for solving nonlinear electric circuits. The method based on *Picard–Banach theorem* requires a *contractive mapping* defined in *metric spaces*. A great benefit of contractive mapping is that uniqueness and existence of a solution are guaranteed by *Banach's theorem*. The main disadvantage of this method is its slow convergence.

The *polarization method* combined with the *fixed point technique* is one of the most popular technique in electromagnetic field analysis due to its advantages,

(*i*) it is robust since it is proved to be convergent with any kind of monotonic, Lipschitz continuous, nonlinear relationship, also in the presence of inflection points,

(*ii*) preliminary operations, in order to define a suitable starting value of residual term, are not required because the process is proved to be convergent for any trial value,

(*iii*) at each iteration the updates are performed only on the right-hand side of the linearized system of equations, without modifications of the stiffness matrix, which can be computed once and for all,

(*iv*) it requires no constraint on the smoothness of the magnetic characteristics so a simple piecewise linear representation can be assumed.

First, metric spaces and some definitions are given in the following sections, then Banach fixed point theorem is introduced and used.

5.3 Metric spaces

5.3.1 Definition of metric spaces

Let $\mathbb{X}$ be a set. The function

$$d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_+^1 \tag{5.27}$$

is a function of two variables $x, y \in \mathbb{X}$, which results in a positive real number, i.e. $d(x, y) \in \mathbb{R}_+^1$. This function called metric on the set $\mathbb{X}$, such that

(*i*) the metric d is nonnegative,

$$d(x, y) \geq 0, \tag{5.28}$$

and $d(x, y) = 0$ if and only if $x = y$,

(*ii*) the metric d is symmetric

$$d(x, y) = d(y, x), \tag{5.29}$$

and

(*iii*) the triangle inequality is true, i.e.

$$d(x, y) \leq d(x, z) + d(z, y). \tag{5.30}$$

These are the axioms of the metric space and x, y, $z \in \mathbb{X}$. The function d is also called distance defined on the set $\mathbb{X}$. The metric space is a set together with a metric on this set.

A simple example of a metric space is the set of real numbers $\mathbb{R}$ with the usual metric

$$d(x, y) = |x - y|, \quad x, y \in \mathbb{R}. \tag{5.31}$$

Another usual metric space can be defined by using the Eucleidan metric, i.e.

$$d(x, y) = \sqrt{\sum_{k=1}^{n} |x_k - y_k|^2}, \tag{5.32}$$

where $x = [x_1, \cdots, x_n]$ and $y = [y_1, \cdots, y_n]$ are two points, or by the equation

$$d(x, y) = \sqrt{\int_{\Omega} |x(\boldsymbol{r}) - y(\boldsymbol{r})|^2 \mathrm{d}\Omega}, \tag{5.33}$$

where $\Omega \in \mathbb{R}^3$. There are many other metrices, e.g. the Hamming distance between two digital words, the maximum absolute value of the difference between two vectors and so on.

In some cases, metric $d(x, y)$ is denoted by $||x - y||$.

5.3.2   Definitions

Convergent sequence.   Let $\{x_1, x_2, \cdots, x_n\}$ be a sequence in the metric space $\mathbb{X}$. This sequence converges to $x \in \mathbb{X}$, i.e.

$$\lim_{n \to \infty} x_n = x, \tag{5.34}$$

if for each $\varepsilon > 0$ there exists N such that

$$d(x, x_n) < \varepsilon, \quad \text{if} \quad n \geq N. \tag{5.35}$$

Cauchy sequence.   Let $\mathbb{X}$ be a metric space with metric d. A sequence given as $\{x_1, x_2, \cdots, x_n\}$ in $\mathbb{X}$ is said to be a Cauchy sequence if for each $\varepsilon > 0$ there exists N such that $n \geq N$ and $m \geq N$ implies

$$d(x_n, n_m) < \varepsilon. \tag{5.36}$$

It is evident that a convergent sequence is *Cauchy sequence*.

Complete metric space. A metric space $\mathbb{X}$ is said to be complete if each Cauchy sequence in $\mathbb{X}$ is convergent to a point in $\mathbb{X}$. A subset of $\mathbb{X}$ (denoted by $\mathbb{A} \subset \mathbb{X}$) is said to be complete if each Cauchy sequence in $\mathbb{A}$ is convergent to a point in $\mathbb{A}$.

The definition of complete metric spaces is very important in Banach fixed point theorem. Before defining this theorem, some useful definitions are repeated here.

Let f be a mapping from a metric space $\mathbb{X}$ to itself, i.e. $f : \mathbb{X} \to \mathbb{X}$ (it can be defined on a subset of the metric space $\mathbb{X}$, of course). The mapping f is usually an operator or a function defined on the metric space $\mathbb{X}$.

Nonexpansive mapping. The mapping f is said to be *nonexpansive*, if

$$||f(x) - f(y)|| \leq ||x - y||, \quad \text{for all } x, y \in \mathbb{X}. \tag{5.37}$$

Lipschitzian mapping. The mapping f is said to be L-Lipschitzian, if

$$||f(x) - f(y)|| \leq L||x - y||, \quad \text{for all } x, y \in \mathbb{X}. \tag{5.38}$$

where $L > 0$ is a positive constant, called *Lipschitz constant*.

Contraction mapping. The mapping f is said to be Banach contraction, if f is L-Lipschitzian, moreover Lipschitz constant is lower than 1 (denoted by q),

$$||f(x) - f(y)|| \leq q||x - y||, \quad \text{for all } x, y \in \mathbb{X}. \tag{5.39}$$

5.3.3 Banach fixed point theorem

The *Banach fixed point theorem* (*contraction mapping principle*) was formulated by Stefan Banach (1892–1945) in 1922 [51].

The fixed point of a function $f : \mathbb{X} \to \mathbb{X}$ is a point $x \in \mathbb{X}$ such that

$$f(x) = x. \tag{5.40}$$

Banach fixed point theorem is as follows. Let $\mathbb{X}$ be a complete metric space. Let $f : \mathbb{X} \to \mathbb{X}$ be a contraction mapping from metric space $\mathbb{X}$ to itself with contractivity coefficient q. Let $x_0 \in \mathbb{X}$ be an arbitrary starting point of the iteration

$$x_{n+1} = f(x_n), \quad n \geq 0. \tag{5.41}$$

Mapping f has a unique fixed point c and the sequence x_n converges to c, i.e.

$$f(c) = c, \tag{5.42}$$

and

$$d(c, x_n) \leq q^n d(c, x_0), \tag{5.43}$$

or equivalently

$$d(c, x_{n+1}) \leq \frac{q}{1-q} d(x_{n+1}, x_n), \quad \text{or} \quad d(c, x_n) \leq \frac{q^n}{1-q} d(x_1, x_0). \tag{5.44}$$

The last inequality is an applicable estimate of the error in x_n if q is known.

The speed of convergence is depending on the value of the contractivity coefficient q. The smaller the value of q the faster the convergence, i.e. the aim is to find as small contractivity coefficient as possible.

5.4   The optimal value of parameters μ and ν

5.4.1   Using the inverse characteristics

The fixed point iteration (5.25) is convergent if the mapping $\phi_{\boldsymbol{\beta}}\{\cdot\}$ is contraction, i.e. (5.39) must be satisfied in every point of the domain under investigation,

$$||\phi_{\boldsymbol{\beta}}\{\boldsymbol{\beta}_1\} - \phi_{\boldsymbol{\beta}}\{\boldsymbol{\beta}_2\}|| \leq q||\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2||, \quad \text{for all } \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \tag{5.45}$$

and $0 < q < 1$. Here $|| \cdot ||$ means the square of the inner product,

$$||\boldsymbol{x}|| = \sqrt{<\boldsymbol{x}, \boldsymbol{x}>} = \sqrt{\int_{\Omega} \boldsymbol{x} \cdot \boldsymbol{x} \, \mathrm{d}\Omega} = \sqrt{\int_{\Omega} |\boldsymbol{x}|^2 \mathrm{d}\Omega}. \tag{5.46}$$

The relation (5.45) is difficult to use, a simpler relation can be formulated, as it is proved in the following.

Fortunately, the set of Maxwell's equations is nonexpansive, i.e.

$$||\mathscr{M}\{\boldsymbol{\beta}_1\} - \mathscr{M}\{\boldsymbol{\beta}_2\}|| \leq ||\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2||, \quad \text{for all } \boldsymbol{\beta}_1, \boldsymbol{\beta}_2. \tag{5.47}$$

Moreover, the parameter μ can be selected such a way that the operator (5.17) is a contraction mapping,

$$||f_{\boldsymbol{B}}\{\boldsymbol{B}_1\} - f_{\boldsymbol{B}}\{\boldsymbol{B}_2\}|| \leq q||\boldsymbol{B}_1 - \boldsymbol{B}_2||, \quad \text{for all } \boldsymbol{B}_1, \boldsymbol{B}_2, \tag{5.48}$$

where $0 < q < 1$ must be satisfied. Relation (5.48) together with (5.47) is equivalent with (5.45) and this results in the condition for the parameter μ.

Here, $\boldsymbol{B}_1 = \mathscr{M}\{\boldsymbol{\beta}_1\}$ and $\boldsymbol{B}_2 = \mathscr{M}\{\boldsymbol{\beta}_2\}$ and the right-hand side of relation (5.48) can be reformulated by using (5.47),

$$\begin{aligned} ||f_{\boldsymbol{B}}\{\boldsymbol{B}_1\} - f_{\boldsymbol{B}}\{\boldsymbol{B}_2\}|| \leq &q||\boldsymbol{B}_1 - \boldsymbol{B}_2|| = q||\mathscr{M}\{\boldsymbol{\beta}_1\} - \mathscr{M}\{\boldsymbol{\beta}_2\}|| \\ \leq &q||\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2||. \end{aligned} \tag{5.49}$$

The left-hand side of (5.48) can also be rewritten as

$$||f_{\boldsymbol{B}}\{\mathscr{M}\{\boldsymbol{\beta}_1\}\} - f_{\boldsymbol{B}}\{\mathscr{M}\{\boldsymbol{\beta}_2\}\}|| \leq q||\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2||, \tag{5.50}$$

which is the same as the contraction mapping (5.45) by using (5.25).

This means that the mapping $\phi_{\boldsymbol{\beta}}\{\cdot\}$ is a contraction if the mapping $f_{\boldsymbol{B}}\{\cdot\}$ defined by (5.17) is a contraction. Finally, the parameter μ can be selected so that $f_{\boldsymbol{B}}\{\cdot\}$ is a contraction. A possible solution is shown in the following.

The relation (5.48) can be rewritten as

$$\frac{||f_{\boldsymbol{B}}\{\boldsymbol{B}_1\} - f_{\boldsymbol{B}}\{\boldsymbol{B}_2\}||}{||\boldsymbol{B}_1 - \boldsymbol{B}_2||} \leq q, \quad \text{for all } \boldsymbol{B}_1, \boldsymbol{B}_2, \tag{5.51}$$

i.e.

$$\frac{||\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2||}{||\boldsymbol{B}_1 - \boldsymbol{B}_2||} = \frac{||\Delta\boldsymbol{\beta}||}{||\Delta\boldsymbol{B}||} \le q, \quad \text{for all } \boldsymbol{B}_1, \boldsymbol{B}_2, \tag{5.52}$$

by using the notations $\Delta\boldsymbol{\beta} = \boldsymbol{\beta}_1 - \boldsymbol{\beta}_2$ and $\Delta\boldsymbol{B} = \boldsymbol{B}_1 - \boldsymbol{B}_2$. The integral in (5.46) can be omitted, if inequality (5.52) is true in every point of the domain Ω. These quantities can be decomposed into their three orthogonal components,

$$\Delta\boldsymbol{\beta} = \Delta\beta_x \mathbf{e}_x + \Delta\beta_y \mathbf{e}_y + \Delta\beta_z \mathbf{e}_z, \tag{5.53}$$

and

$$\Delta\boldsymbol{B} = \Delta B_x \mathbf{e}_x + \Delta B_y \mathbf{e}_y + \Delta B_z \mathbf{e}_z, \tag{5.54}$$

where $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ are the three orthogonal unit vectors.

The inequality (5.52) can be rewritten as

$$\frac{\Delta\boldsymbol{\beta} \cdot \Delta\boldsymbol{\beta}}{\Delta\boldsymbol{B} \cdot \Delta\boldsymbol{B}} = \frac{\Delta\beta_x^2 + \Delta\beta_y^2 + \Delta\beta_z^2}{\Delta B_x^2 + \Delta B_y^2 + \Delta B_z^2} \le q^2. \tag{5.55}$$

The upper bound of this fraction is $\Delta\beta_x^2/\Delta B_x^2$, or $\Delta\beta_y^2/\Delta B_y^2$, or $\Delta\beta_z^2/\Delta B_z^2$.

The upper bound $\Delta\beta_x^2/\Delta B_x^2$ can be rewritten as

$$\left(\frac{\Delta\beta_x}{\Delta B_x}\right)^2 = \left(\frac{\Delta B_x - \mu\Delta H_x}{\Delta B_x}\right)^2 = \left(1 - \frac{\mu}{\mu_{\Delta_x}}\right)^2, \tag{5.56}$$

where μ_{Δ_x} is the differential permeability,

$$\mu_{\Delta_x} = \frac{\Delta B_x}{\Delta H_x} = \frac{B_{x,1} - B_{x,2}}{H_{x,1} - H_{x,2}} \tag{5.57}$$

which has two extreme values along the hysteresis curve

$$\mu_{\min} = \min_{H_{x,1}, H_{x,2}} \frac{B_{x,1} - B_{x,2}}{H_{x,1} - H_{x,2}}, \quad \text{and} \quad \mu_{\max} = \max_{H_{x,1}, H_{x,2}} \frac{B_{x,1} - B_{x,2}}{H_{x,1} - H_{x,2}}. \tag{5.58}$$

Here $\mu_{\min}$ and $\mu_{\max}$ are the minimum and maximum slope of the hysteresis characteristics. The inequality

$$\mu_{\min} \le \mu \le \mu_{\max} \tag{5.59}$$

is obvious, from which the inequalities

$$1 - \frac{\mu}{\mu_{\min}} < 0, \quad \text{and} \quad 1 - \frac{\mu}{\mu_{\max}} > 0, \tag{5.60}$$

moreover

$$\left(1 - \frac{\mu}{\mu_{\Delta_x}}\right)^2 \le \left(1 - \frac{\mu}{\mu_{\min}}\right)^2, \quad \text{and} \quad \left(1 - \frac{\mu}{\mu_{\Delta_x}}\right)^2 \le \left(1 - \frac{\mu}{\mu_{\max}}\right)^2 \tag{5.61}$$

are valid.

These expressions result in

$$\left(1 - \frac{\mu}{\mu_{\Delta_x}}\right)^2 \leq \left(1 - \frac{\mu}{\mu_{\min}}\right)^2 \leq q^2, \quad \text{and} \quad \left(1 - \frac{\mu}{\mu_{\Delta_x}}\right)^2 \leq \left(1 - \frac{\mu}{\mu_{\max}}\right)^2 \leq q^2,$$

(5.62)

i.e.

$$-\left(1 - \frac{\mu}{\mu_{\min}}\right) \leq q, \quad \text{and} \quad 1 - \frac{\mu}{\mu_{\max}} \leq q.$$

(5.63)

The minus sign on the left of the first inequality is coming from (5.60).

From the last expressions, the inequalities

$$\mu_{\min}(1 + q) \geq \mu, \quad \text{and} \quad \mu_{\max}(1 - q) \leq \mu$$

(5.64)

are coming, which can be written as

$$0 < \mu_{\max}(1 - q) \leq \mu \leq \mu_{\min}(1 + q) < 2\mu_{\min},$$

(5.65)

because $0 < q < 1$. This inequality occurs surely, if

$$\mu_{\max}(1 - q) \leq \mu_{\min}(1 + q),$$

(5.66)

hence

$$\frac{\mu_{\max} - \mu_{\min}}{\mu_{\max} + \mu_{\min}} \leq q < 1.$$

(5.67)

The least value of q is

$$q = \frac{\mu_{\max} - \mu_{\min}}{\mu_{\max} + \mu_{\min}},$$

(5.68)

and the corresponding optimal value of permeability is obtained from one of the equations in (5.64) by substituting the least value of q in (5.68),

$$\mu_o = \frac{2\,\mu_{\max}\,\mu_{\min}}{\mu_{\max} + \mu_{\min}}.$$

(5.69)

The subscript o refers to the *optimal value of permeability*.

The same formulations can be done to obtain the optimal value of permeability used in the y and z directions, too. If the three characteristics are the same, then μ is constant. This is the situation, when isotropic hysteresis model is used. Generally, the permeability is a tensor with three constants in the diagonal.

Finally, in this case the inverse type hysteresis model must be applied whose input and output are the magnetic flux density and the magnetic field intensity, respectively, $\boldsymbol{H} = \mathscr{B}^{-1}\{\boldsymbol{B}\}$. Equations (5.15) and (5.16) can be rewritten as

$$\boldsymbol{B} = \mu_o \boldsymbol{H} + \boldsymbol{\beta}, \quad \text{where} \quad \mu_o = \frac{2\,\mu_{\max}\,\mu_{\min}}{\mu_{\max} + \mu_{\min}},$$

(5.70)

and

$$\boldsymbol{\beta} = \boldsymbol{B} - \mu_o \mathscr{B}^{-1}\{\boldsymbol{B}\}. \tag{5.71}$$

The value of $\mu_{\min}$ and $\mu_{\max}$ is defined in (5.58). In this case, the sequence (5.25) can be used in the nonlinear electromagnetic field computation. However, we use its inverse, summarized in section 5.5.1.

5.4.2 Using the direct characteristics

The fixed point iteration (5.26) is convergent if the mapping $\phi_{\boldsymbol{\eta}}\{\cdot\}$ is contraction mapping, i.e. (5.39) must be satisfied,

$$||\phi_{\boldsymbol{\eta}}\{\boldsymbol{\eta}_1\} - \phi_{\boldsymbol{\eta}}\{\boldsymbol{\eta}_2\}|| \leq q||\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2||, \quad \text{for all } \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \tag{5.72}$$

and $0 < q < 1$. It is easy to see that this relation is very similar to (5.45). That is why, the optimal value of reluctivity in (5.20) can be obtained in the same way as it was presented in the previous section, however, $\boldsymbol{\beta}$, $\boldsymbol{B}$ and μ must be changed to $\boldsymbol{\eta}$, $\boldsymbol{H}$ and ν, respectively.

Finally, in this case the direct type hysteresis model must be applied, which input and output are the magnetic field intensity and the magnetic flux density, respectively, $\boldsymbol{B} = \mathscr{B}\{\boldsymbol{H}\}$. Equations (5.18) and (5.19) can be rewritten as

$$\boldsymbol{H} = \nu_o \boldsymbol{B} + \boldsymbol{\eta}, \quad \text{where} \quad \nu_o = \frac{2\,\nu_{\max}\,\nu_{\min}}{\nu_{\max} + \nu_{\min}}, \tag{5.73}$$

and

$$\boldsymbol{\eta} = \boldsymbol{H} - \nu_o \mathscr{B}\{\boldsymbol{H}\}. \tag{5.74}$$

The value of $\nu_{\min}$ and $\nu_{\max}$ are the minimum and the maximum slope of the inverse hysteresis characteristics. In this case, the sequence (5.26) can be used in the nonlinear electromagnetic field computation. The subscript o refers to the *optimal value of reluctivity*.

5.5 The applied formulation, summary

5.5.1 Using the inverse characteristics

It may be better to use the reluctivity ν when applying the *inverse hysteresis model*, because $\nu(B) = \partial H / \partial B$.

Let us now multiply the relation (5.70) by $\nu_o = 1/\mu_o$,

$$\nu_o \boldsymbol{B} = \boldsymbol{H} + \nu_o \boldsymbol{\beta}, \tag{5.75}$$

i.e.

$$\boldsymbol{H} = \nu_o \boldsymbol{B} - \nu_o \boldsymbol{\beta}. \tag{5.76}$$

Here, we are going to denote the second term $-\nu_o \boldsymbol{\beta}$ by $\boldsymbol{I}$,

$$\boldsymbol{H} = \nu_o \boldsymbol{B} + \boldsymbol{I}, \tag{5.77}$$

and from (5.70),

$$\nu_o = \frac{\nu_{\max} + \nu_{\min}}{2}, \tag{5.78}$$

where $\nu_{\max}$ and $\nu_{\min}$ are the maximum and minimum slope of the inverse hysteresis characteristics. It is evident that $\nu_{\max} = 1/\mu_{\min}$ and $\nu_{\min} = 1/\mu_{\max}$. This formulation is more convenient when obtaining the partial differential equations of electromagnetic field problems, i.e. the magnetic field intensity vector is defined by the sum of two terms, $\boldsymbol{H} = \nu_o \boldsymbol{B} + \boldsymbol{I}$. There is a condition obtained from (5.65), which must be satisfied,

$$\nu_o > \frac{\nu_{\max}}{2}. \tag{5.79}$$

The nonlinear iteration can be summarized as follows. The iteration can be started by an arbitrary value of $\boldsymbol{I}^{(0)}$, then in the nth iteration step ($n > 0$),

(*i*) the magnetic flux density $\boldsymbol{B}^{(n)}$ can be calculated by solving the partial differential equations obtained from Maxwell's equations and using $\boldsymbol{I}^{(n-1)}$, in other words, $\boldsymbol{B}^{(n)} = \mathscr{M}\{\boldsymbol{I}^{(n-1)}\}$,

(*ii*) the magnetic field intensity $\boldsymbol{H}^{(n)}$ can be calculated by applying the inverse type hysteresis model, $\boldsymbol{H}^{(n)} = \mathscr{B}^{-1}\{\boldsymbol{B}^{(n)}\}$,

(*iii*) the nonlinear residual term can be updated by using the magnetic flux density and magnetic field intensity,

$$\boldsymbol{I}^{(n)} = \boldsymbol{H}^{(n)} - \nu_o \boldsymbol{B}^{(n)} = \mathscr{B}^{-1}\{\boldsymbol{B}^{(n)}\} - \nu_o \boldsymbol{B}^{(n)}, \tag{5.80}$$

(*iv*) and the sequence defined by steps (i)–(iii) must be repeated until convergence. Convergence criterion can be

$$||\boldsymbol{I}^{(n)} - \boldsymbol{I}^{(n-1)}|| < \varepsilon, \tag{5.81}$$

where ε is a small positive real number. Convergence criterion can be defined by using the magnetic field intensity or the magnetic flux density, as well.

5.5.2   Using the direct characteristics

It may be better to use the permeability μ when applying the *direct hysteresis model*, because $\mu(H) = \partial B / \partial H$.

Let us now multiply the relation (5.73) by $\mu_o = 1/\nu_o$,

$$\mu_o \boldsymbol{H} = \boldsymbol{B} + \mu_o \boldsymbol{\eta}, \tag{5.82}$$

i.e.

$$\boldsymbol{B} = \mu_o \boldsymbol{B} - \mu_o \boldsymbol{\eta}. \tag{5.83}$$

Here, we are going to denote the second term $-\mu_o \boldsymbol{\eta}$ by $\boldsymbol{R}$,

$$\boldsymbol{B} = \mu_o \boldsymbol{H} + \boldsymbol{R}, \tag{5.84}$$

and from (5.73),

$$\mu_o = \frac{\mu_{\max} + \mu_{\min}}{2}, \tag{5.85}$$

where $\mu_{\max}$ and $\mu_{\min}$ are the maximum and minimum slope of the direct hysteresis characteristics. It is evident that $\mu_{\max} = 1/\nu_{\min}$ and $\mu_{\min} = 1/\nu_{\max}$. This formulation is more convenient when obtaining the partial differential equations of electromagnetic field problems, i.e. the magnetic flux density vector is defined by the sum of two terms, $\boldsymbol{B} = \mu_o \boldsymbol{H} + \boldsymbol{R}$. There is a condition obtained from the ν-version of (5.65), which must be satisfied,

$$\mu_o > \frac{\mu_{\max}}{2}. \tag{5.86}$$

The nonlinear iteration can be summarized as follows. The iteration can be started by an arbitrary value of $\boldsymbol{R}^{(0)}$, then in the nth iteration step ($n > 0$),

(i) the magnetic field intensity $\boldsymbol{H}^{(n)}$ can be calculated by solving the partial differential equations obtained from Maxwell's equations and using $\boldsymbol{R}^{(n-1)}$, in other words, $\boldsymbol{H}^{(n)} = \mathscr{M}\{\boldsymbol{R}^{(n-1)}\}$,

(ii) the magnetic flux density $\boldsymbol{B}^{(n)}$ can be calculated by applying the inverse hysteresis model, $\boldsymbol{B}^{(n)} = \mathscr{B}\{\boldsymbol{H}^{(n)}\}$,

(iii) the nonlinear residual term can be updated by using the magnetic field intensity and magnetic flux density,

$$\boldsymbol{R}^{(n)} = \boldsymbol{B}^{(n)} - \mu_o \boldsymbol{H}^{(n)} = \mathscr{B}\{\boldsymbol{H}^{(n)}\} - \mu_o \boldsymbol{H}^{(n)}, \tag{5.87}$$

(iv) and the sequence defined by steps (i)–(iii) must be repeated until convergence. Convergence criterion can be

$$||\boldsymbol{R}^{(n)} - \boldsymbol{R}^{(n-1)}|| < \varepsilon, \tag{5.88}$$

where ε is a small positive real number. Convergence criterion can be defined by using the magnetic field intensity or the magnetic flux density, as well.

5.5.3 Proof of the nonexpansive property of Maxwell's equations

Theorem (5.47) is reformulated by using the notations in the formulation $\boldsymbol{B} = \mu \boldsymbol{H} + \boldsymbol{R}$,

$$||\mathscr{M}\{\boldsymbol{R}_1\} - \mathscr{M}\{\boldsymbol{R}_2\}|| \leq ||\boldsymbol{R}_1 - \boldsymbol{R}_2||, \quad \text{for all } \boldsymbol{R}_1, \boldsymbol{R}_2. \tag{5.89}$$

Let $(\boldsymbol{H}_1, \boldsymbol{B}_1, \boldsymbol{R}_1)$, $(\boldsymbol{H}_2, \boldsymbol{B}_2, \boldsymbol{R}_2)$ be two fields and polarizations having the same boundary conditions and same current densities $\boldsymbol{J}$. The solution of Maxwell's equations is unique, that is why the difference of these fields $\boldsymbol{H}_d = \boldsymbol{H}_1 - \boldsymbol{H}_2$, $\boldsymbol{B}_d = \boldsymbol{B}_1 - \boldsymbol{B}_2$ verifies

$$< \boldsymbol{B}_d, \boldsymbol{H}_d > = \int_\Omega \boldsymbol{B}_d \cdot \boldsymbol{H}_d \, \mathrm{d}\Omega = 0. \tag{5.90}$$

This inner product can be reformulated as

$$< \boldsymbol{B}_d, \nu(\boldsymbol{B}_d - \boldsymbol{R}_d) >=< \boldsymbol{B}_d, \nu \boldsymbol{B}_d > - < \boldsymbol{B}_d, \nu \boldsymbol{R}_d >= 0, \tag{5.91}$$

where $\nu = 1/\mu$ and $\boldsymbol{R}_d = \boldsymbol{R}_1 - \boldsymbol{R}_2$. It can be rewritten as

$$< \boldsymbol{B}_d, \nu \boldsymbol{B}_d >=< \boldsymbol{B}_d, \nu \boldsymbol{R}_d > . \tag{5.92}$$

According to Schwartz inequality

$$|< \boldsymbol{x}, \boldsymbol{y} >| \leq ||\boldsymbol{x}|| \, ||\boldsymbol{y}|| \tag{5.93}$$

the upper bound of the right-hand side of (5.92) can be predicted,

$$< \boldsymbol{B}_d, \nu \boldsymbol{R}_d > \leq ||\boldsymbol{B}_d|| \, ||\nu \boldsymbol{R}_d||. \tag{5.94}$$

It is obvious from (5.91) that the value of inner product $< \boldsymbol{B}_d, \nu \boldsymbol{R}_d >$ is always positive because $< \boldsymbol{B}_d, \nu \boldsymbol{B}_d >$ is always positive. Finally

$$< \boldsymbol{B}_d, \nu \boldsymbol{B}_d >\equiv ||\boldsymbol{B}_d|| \, ||\nu \boldsymbol{B}_d|| \leq ||\boldsymbol{B}_d|| \, ||\nu \boldsymbol{R}_d||, \tag{5.95}$$

i.e. the inequality

$$||\nu \boldsymbol{B}_d|| \leq ||\nu \boldsymbol{R}_d|| \tag{5.96}$$

is true. It can be written as

$$||\nu(\boldsymbol{B}_1 - \boldsymbol{B}_2)|| \leq ||\nu(\boldsymbol{R}_1 - \boldsymbol{R}_2)||. \tag{5.97}$$

Parameter ν can be omitted if it is constant. Generally ν is tensor.

This theorem can be proved by applying the formula $\boldsymbol{H} = \nu \boldsymbol{B} + \boldsymbol{I}$, too, in a similar way.

6 Application of the finite element method

6.1 Introduction

This chapter contains some illustrative examples, which have been solved on the basis of the previous methods by the help of *COMSOL Multiphysics* [19, 88]. This commercial finite element software has the advantage that the weak formulation of a problem can be inserted easily and it is not necessary to implement mesh generators, solvers and postprocessors. The user can focus on the problem and on the weak formulation, can design the geometry of the problem under test, can insert the weak form, the boundary and interface conditions according to the applied potential formulation. It is available to select nodal or edge finite elements. Using higher order approximation is also possible, COMSOL Multiphysics uses the scalar shape functions with the order from one to five and vector shape functions with the order from one to three. The mesh generation is performed automatically. Many solvers are implemented in COMSOL Multiphysics, so all of them can be tested and the best one, the most applicable to the problem can be selected. The postprocessor is perfect, simulated data and electromagnetic field quantities obtained from the potentials can also be simulated and plotted easily. These properties are very useful from the point of view of research, because the time consuming programming task can be eliminated.

Seven problems are shown in the following.

The first 2D magnetostatic problem illustrates how to select the impressed current vector potential, T_0 when applying magnetic scalar potential formulations. The problem contains a C-shaped magnet and the magnetic field is generated by a coil. Here, we show the false solution according to the cancellation error, the correct solution by the help of the two scalar potentials (total and reduced), finally the edge element representation of the impressed current vector potential will be proposed. The reference solution is obtained by the magnetic vector potential. Edge element representation of T_0 is very advantageous and useful in the other formulations based on the ungauged magnetic vector potential, too.

The second 3D problem containing an iron cube placed in a homogeneous magnetic field illustrates the effect of gauging on the magnetic vector potential $\mathbf{A}$. The problem has been solved without and with Coulomb gauge and $\mathbf{A}$ is approximated by nodal finite elements. Unfortunately, the solution of the gauged version is not correct on the iron/air interfaces, that is why two potentials are used, one in the air region and an other one in the iron part and only the tangential component is prescribed to be continuous on the iron/air interfaces. This modification can be implemented easily by COMSOL Multiphysics. Edge element representation of ungauged $\mathbf{A}$ is much better, because it results in less computation time. We will show the $\mathbf{A} - \Phi$ and $\mathbf{A} - \mathbf{A} - \Phi$ formulations as well. The aim of these methods is to reduce the degree of freedom. The problem is solved by the reduced magnetic scalar potential, too, as a reference solution, because no currents are present.

The third problem is a modification of the second one. It contains an iron cube made of conducting material placed in a homogeneous magnetic field, too. The problem illustrates the effect of mesh generation on skin effect problems. The problem has been solved by using the Coulomb gauged $\mathbf{A}, V - \mathbf{A}$ and $\mathbf{T}, \Phi - \Phi$ formulations and by the ungauged version of them. The gauged vector potentials are approximated by nodal finite elements, the ungauged vector potentials are approximated by vector finite elements.

The fourth problem is a Benchmark problem containing an excitation coil and steel plates around it. First, the problem is supposed to be linear, then nonlinear with saturating characteristics, but hysteresis is neglected. The problem has been solved by nodal and edge element representation of the magnetic vector potential and by reduced magnetic scalar potential combined with the edge element representation of the impressed current vector potential. The magnetic flux density inside the steel plates simulated by the reduced magnetic scalar potential is a little larger than the result calculated by the magnetic vector potential formulations. However, simulated results are close to measured ones.

After linear simulations, the nonlinear problem has been solved by the fixed point technique.

The next one is a Benchmark problem again. It consists of a plate made of aluminum containing a hole in it and an excitation coil generates time varying magnetic field. The effect of eddy currents are taken into account by applying different potential formulations. The problem contains a multiply connected region and the behavior of formulations are studied. The time varying problem has been solved in frequency domain and simulated results are compared with measured ones.

The last problem is the analysis of a vector hysteresis measurement system, which is under construction in the Laboratory of Electromagnetic Fields, at the Department of Telecommunications, Széchenyi István University, Győr. The simulations have been implemented in the time domain because of the hysteresis characteristics and we show a time stepping scheme with the fixed point technique. Hysteresis is taken into account inside the specimen by a 2D isotropic vector hysteresis model.

6.2   Illustration how to select T_0, the C-magnet

The first problem focuses on how to represent the impressed current vector potential T_0 to eliminate *cancellation error* on the interface between iron and air when applying the *magnetic scalar potential formulations* (the Φ-formulation, the Ψ-formulation and the $\Phi - \Psi$-formulation) [8, 11, 72, 79, 80, 87].

A simple 2D example has been chosen for illustration: a C-shaped yoke is magnetized by a coil as it can be seen in *Fig.* 6.1. The yoke is made of iron and a linear characteristics have been supposed with permeability $\mu = 1000\mu_0$. The coil has only two turns placed symmetrically to the x-axis and the value of DC current is $I = 1\,\text{A}$ ($x_{+I} = x_{-I} = 0$, $y_{+I} = \pm 105\,\text{mm}$, $y_{-I} = \pm 35\,\text{mm}$).
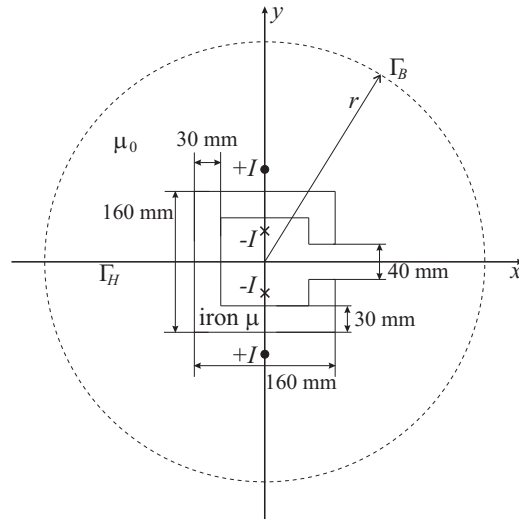
Fig. 6.1. C-shaped yoke magnetized by a coil

Only the half of geometry has been analyzed because of symmetry along the line $y = 0$. The tangential component of magnetic field intensity is vanishing here, i.e. the line $y = 0$ is a Γ_H type boundary. *Artificial far boundary* is supposed at the distance $r = 800\,\text{mm}$, where $\boldsymbol{B} \cdot \boldsymbol{n} = 0$. It is denoted by Γ_B. The value of r, i.e. the place of the artificial far boundary can be determined after some trials, the boundary condition along Γ_B should not modify the magnetic field in the problem region.

First of all, the problem has been solved by using the magnetic vector potential $\boldsymbol{A}$ as a reference solution, because the problem is 2D and the partial differential equation (2.199) with boundary conditions (2.200) and (2.201) can be used. Here $\boldsymbol{K} = \boldsymbol{0}$ and $\boldsymbol{\alpha} = \boldsymbol{0}$. The magnetic flux density $\boldsymbol{B}$ inside and around the yoke can be seen in *Fig.* 6.2(a) and in *Fig.* 6.2(b). The system of linear equations has been solved by the direct solver *UMFPACK* [19, 88].

It is easy to see that the magnetic field intensity is perpendicular to the line $y = 0$, i.e. to the boundary Γ_H and the magnetic flux lines follow the shape of the yoke.

(a) The magnetic flux density inside and around the yoke

(b) A magnified plot
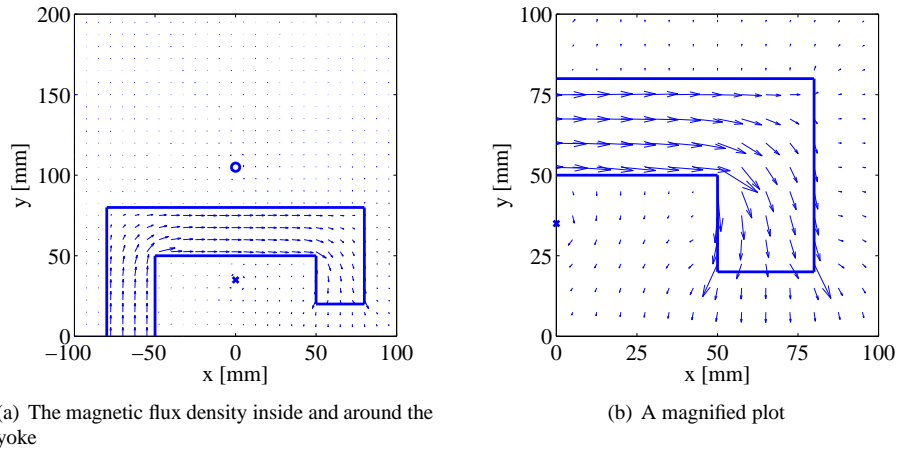
Fig. 6.2. The magnetic flux density simulated by using the magnetic vector potential $\boldsymbol{A}$ as a reference solution

The absolute value of magnetic flux density along the line $x = -65, \cdots, 65 \, \text{mm}$, $y = 65 \, \text{mm}$ (along the center of upper leg of the yoke) can be seen in *Fig.* 6.3. This result is used as a reference solution in the following.

The *impressed current vector potential* has two components and it can be determined easily by using the well known relation of the magnetic field intensity according to one *infinite long filamentary conductor* $H = I/2R\pi$ [28, 81], i.e.

$$\boldsymbol{T}_0 = T_{0,x}\boldsymbol{e}_x + T_{0,y}\boldsymbol{e}_y, \tag{6.1}$$

where
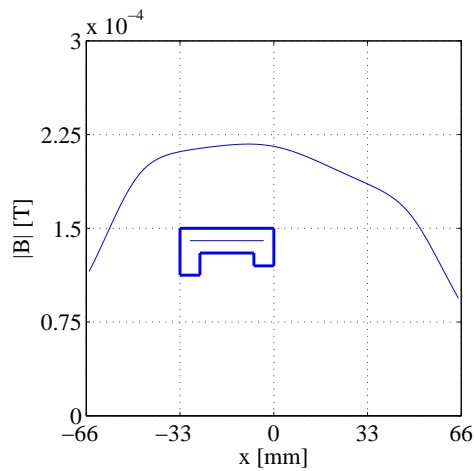
Fig. 6.3. The absolute value of magnetic flux density along the line $x = -65, \cdots, 65 \, \text{mm}$, $y = 65 \, \text{mm}$ calculated by the magnetic vector potential

$$T_{0,x} = \frac{I}{2\pi} \cdot \left[\frac{y - 0.105}{x^2 + (y - 0.105)^2} + \frac{y + 0.105}{x^2 + (y + 0.105)^2} \right]$$
$$- \frac{I}{2\pi} \cdot \left[\frac{y - 0.035}{x^2 + (y - 0.035)^2} + \frac{y + 0.035}{x^2 + (y + 0.035)^2} \right], \tag{6.2}$$

$$T_{0,y} = - \frac{I}{2\pi} \cdot \left[\frac{x}{x^2 + (y - 0.105)^2} + \frac{x}{x^2 + (y + 0.105)^2} \right]$$
$$- \frac{I}{2\pi} \cdot \left[\frac{x}{x^2 + (y - 0.035)^2} + \frac{x}{x^2 + (y + 0.035)^2} \right]. \tag{6.3}$$

The impressed current vector potential can be calculated by analytical expression in nodes of the finite element mesh and the reduced scalar potential can be used in the simulations. It is noted that there is no node where the filamentary conductors are placed. The partial differential equation (2.160) with the boundary conditions (2.161) and (2.162) can be applied to solve the problem. Here $\Phi_0 = 0$ since $K = 0$ and the tangential component of T_0 is equal to zero along Γ_H, $T_0 \times n = 0$, moreover $b = 0$ on Γ_B.

The formulation is very simple, however, the solution is catastrophic in the iron region as it can be seen in *Fig.* 6.4(a) (here first order elements are used). The absolute value of magnetic flux along the line $x = -65, \cdots, 65$ mm, $y = 65$ mm calculated by the reduced magnetic scalar potential can be seen in *Fig.* 6.4(b).

(a) The simulated magnetic flux density inside and around the yoke

(b) The absolute value of magnetic flux along the line $x = -65, \cdots, 65$ mm, $y = 65$ mm
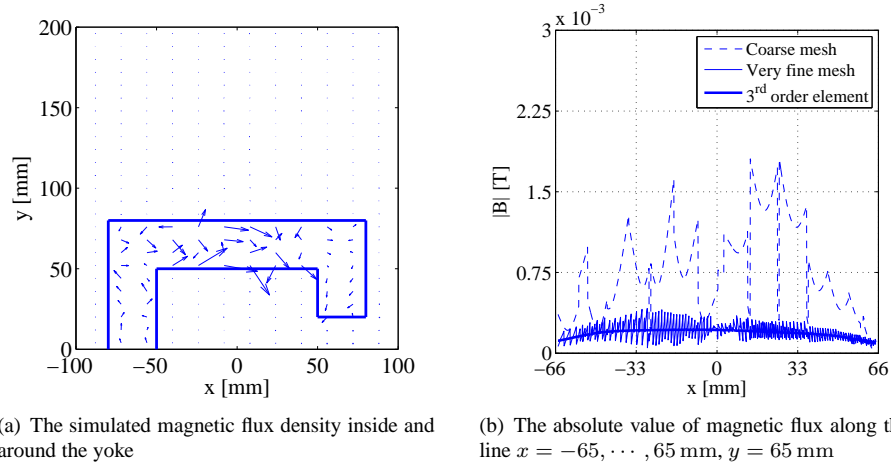
Fig. 6.4. The distribution of magnetic flux density vector is catastrophic when using the reduced magnetic scalar potential Φ

The accuracy of solution can not be increased very well by increasing the number of finite elements, however, applying higher order approximation results in better distribution of the magnetic flux density. Here, the coarse mesh consists of 782 triangles with 419 unknowns, the very fine mesh consists of 50048 triangles with 25241 unknowns and the

approximation is linear. The geometry discretized by 3128 triangles when third order approximation is applied and the number of unknowns is 14239.

Let us now compare the value of impressed current vector potential T_0 and the gradient of the reduced magnetic scalar potential $\nabla\Phi$. It can be seen in *Fig.* 6.5 for different value of the relative permeability μ_r of the iron yoke. The difference between the two quantities is decreasing by increasing the relative permeability, i.e. $|H| = |T_0 - \nabla\Phi|$ is decreasing inside the C-shaped yoke. The impressed current vector potential is expressed by analytical formulation, while the magnetic scalar potential is approximated by first order polynomial. The difference field can be very inaccurate if they are almost the same, which results in numerical difficulties. This problem is usually referred as *cancellation error*. This is the reason why the accuracy can be increased by increasing the order of approximating polynomials, however, this is not a general rule (the impressed current vector potential is very simple in this situation).
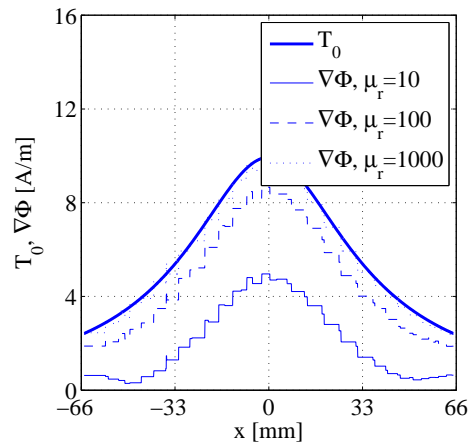
Fig. 6.5. Comparison of the impressed current vector potential and the gradient of the reduced magnetic scalar potential

There are two widely used solutions for this problem. The first is a combination of the reduced and the total scalar magnetic potential, the second is applying edge elements for the representation of impressed field.

The origin of cancellation error is the presence of impressed current vector potential inside regions with high permeability. In iron regions the total scalar potential Ψ without T_0 should be used and the reduced scalar potential Φ with T_0 should represent the magnetic field in the air region. In this case the partial differential equations (2.177) and (2.178) must be solved satisfying the boundary conditions (2.179), (2.180), (2.181) and (2.182), moreover the interface conditions (2.183) and (2.184). In this example, $\Phi_0 = \Psi_0 = 0$, because $K = 0$ and $T_0 \times n = 0$, furthermore $b = 0$. The line integral in (2.183) can now be expressed by analytical formula, however, this is not general.

Here a coarse mesh with 782 elements and 419 unknowns has been used again. Figure 6.6 shows the comparison between results simulated by the reduced scalar potential

formulation with third order approximation and the two potential approach with linear and second order approximation. The first order approximation of $\Phi - \Psi$-formulation is much better than the linear one in *Fig.* 6.4(b), however, the mesh is the same. The result calculated by the second order elements ($\Phi - \Psi$-formulation) is almost the same as the third order one (Φ-formulation).
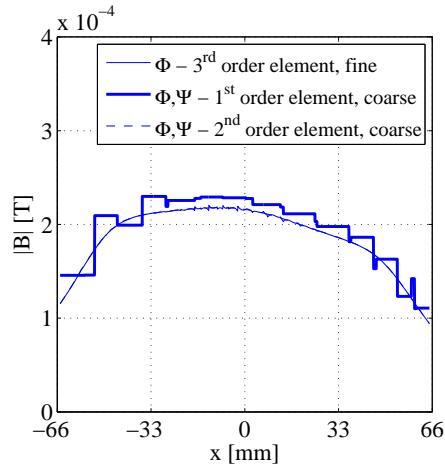
Fig. 6.6. Cancellation error can be eliminated by the $\Phi - \Psi$-formulation
(the first and the last curves are practically the same)
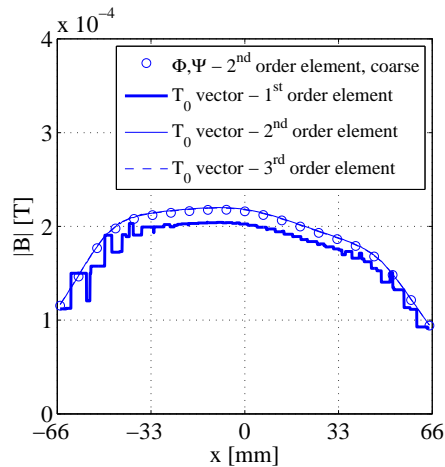
Fig. 6.7. Cancellation error can be eliminated by applying edge elements in the
approximation of impressed field (the last two curves are practically the same)

If the impressed current vector potential and the gradient of the reduced scalar potential are in the same function space, then the *cancellation error* can be eliminated as it is presented in *Fig.* 6.7. Here the result of second order approximation simulated by the $\Phi - \Psi$-formulation has been compared by the first, the second and the third order vector approximation of impressed field. The impressed current vector potential $\boldsymbol{T}_0$ has been represented by edge elements and the reduced scalar potential with the same order has been applied in the simulations. The second and the third order approximations are almost the same, however, the first order one is also accurate.

The difference between the magnetic flux density simulated by the magnetic vector potential and by the last method is maximum $3 \cdot 10^{-6}$, i.e. the relative error is $1.8\,\%$.

6.3 Iron cube in homogeneous magnetic field

The problem is a very simple structure, an iron cube is situated in a homogeneous magnetic field [72]. The linear model can be seen in *Fig.* 6.8 ($\boldsymbol{B}_0 = (1\,\mathrm{T})\boldsymbol{e}_z$, $\mu = 1000\mu_0$).
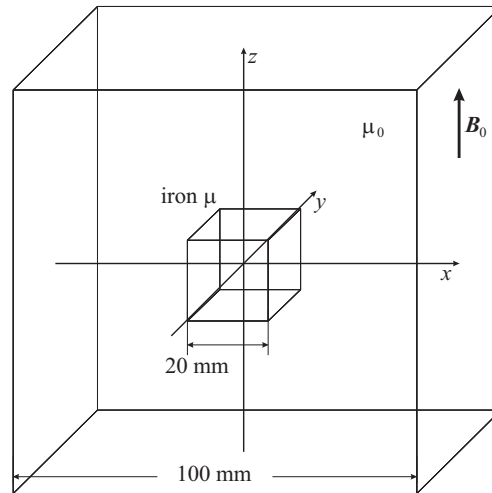
Fig. 6.8. Iron cube in a homogeneous magnetic field

The three planes $x = 0$, $y = 0$ and $z = 0$ are symmetry planes. On the planes $x = 0$ and $y = 0$ the normal component of the magnetic flux density is vanishing, $\boldsymbol{B} \cdot \boldsymbol{n} = 0$, i.e. they are Γ_B type boundaries, the plane $z = 0$ is a Γ_H boundary, where the tangential component of the magnetic field intensity is equal to zero, $\boldsymbol{H} \times \boldsymbol{n} = \boldsymbol{0}$. *Artificial far boundaries* at $x = \pm 50\,\mathrm{mm}$, $y = \pm 50\,\mathrm{mm}$ and $z = \pm 50\,\mathrm{mm}$ are assumed, where the cube does not affect the magnetic field. The first and the second boundaries are Γ_B type where the source magnetic flux is given by boundary conditions defined by potentials, the artificial far boundaries placed at $z = \pm 50\,\mathrm{mm}$ are Γ_H type, where $\boldsymbol{H} \times \boldsymbol{n} = \boldsymbol{0}$. These simplifications result in the analysis of the eighth of the whole problem, $x \geq 0$, $y \geq 0$ and $z \geq 0$ (see *Fig.* 6.9).
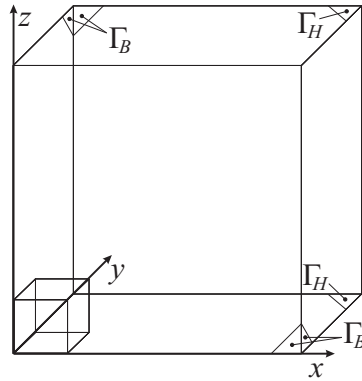
Fig. 6.9. The analyzed region, the type of boundaries is also indicated

The aim of simulations is to compare the different potential formulations, the gauged and the ungauged magnetic vector potential approximated by nodal elements, the magnetic scalar potential, the combination of the gauged magnetic vector potential and the magnetic scalar potential, finally the magnetic vector potential approximated by vector elements.

First, the effect of gauging on the magnetic vector potential A has been studied. The vector field of magnetic vector potential is not unique if *Coulomb gauge* is not used, but the magnetic field calculated from A is unique and of course, it is correct. The behavior of the ungauged magnetic vector potential is very strange as it is presented in *Fig.* 6.10(a). It is a very important experiment that the number of iterations of iterative solvers needed for the solution can be very high, because of the ill-conditioned mass matrix. The solvers sometimes do not converge in the case of ungauged vector potential (the *conjugate gradient method* with algebraic multigrid preconditioner has been applied in the simulations). Coulomb gauge $\nabla \cdot A = 0$ can be easily satisfied by the modified

(a) The ungauged magnetic vector potential

(b) The gauged magnetic vector potential
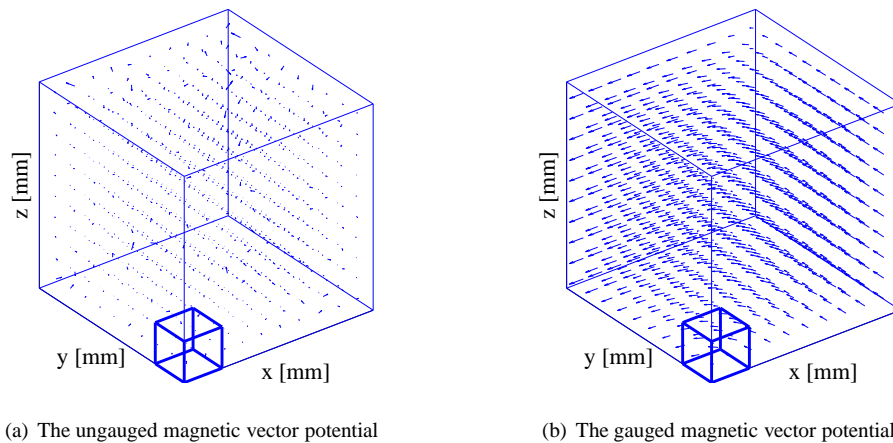
Fig. 6.10. The effect of gauging on the magnetic vector potential

partial differential equation (2.223). The gauged magnetic vector potential as well as
the boundary conditions according to (2.224)–(2.227) can be seen in *Fig.* 6.10(b). The
number of iterations of the solver is decreasing by introducing Coulomb gauge (*Fig.* 6.11).

The resulting magnetic flux density and the gauged magnetic vector potential can be
seen in *Fig.* 6.12. The results seem to be accurate, moreover the number of iterations
is decreasing. However, let us take a closer look at the solution along the line $y = 0$,
$z = 0$ (*Fig.* 6.13(a)) and $y = 10$mm, $z = 0$ (*Fig.* 6.13(b)). The gauged solution is not the
same as the ungauged one, especially in the second case. The problem is coming from
the abrupt change of the permeability along the iron/air interface.
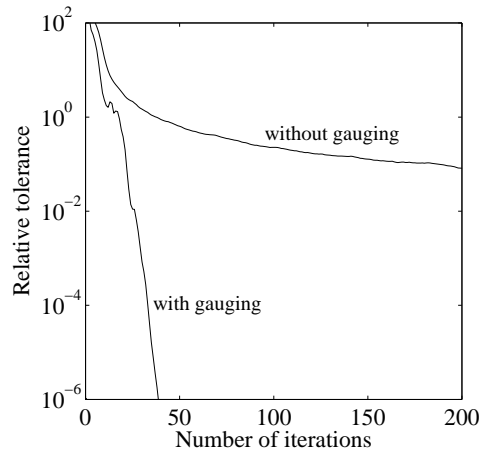
Fig. 6.11. The number of iterations is decreasing by introducing gauging
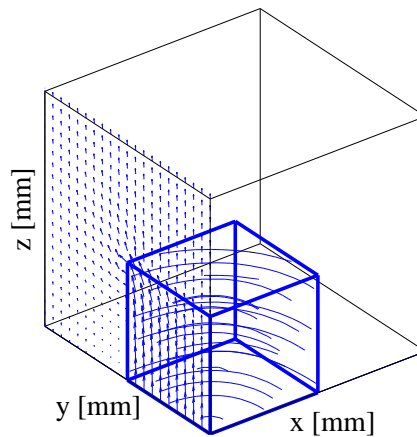
Fig. 6.12. Magnetic flux density as vectors and the gauged magnetic vector potential in
iron as streamlines

The problem can be solved by the following modifications:

(*i*) The solution is correct if gauging is not used, however, solving the system of the resulting equations can be very difficult, so this is not an advantageous way;

(*ii*) Applying nodal finite elements to approximate the magnetic vector potential results in continuous normal and tangential components of the magnetic vector potential. Allowing the jump of the normal component of the magnetic vector potential can solve efficiently the problem. This means that a magnetic vector potential A_1 has to be used inside iron and a magnetic vector potential A_2 has to be used in the air
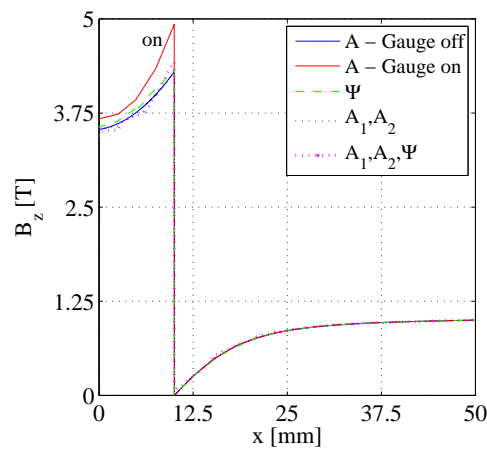
(a) Along the line $y = 0$, $z = 0$
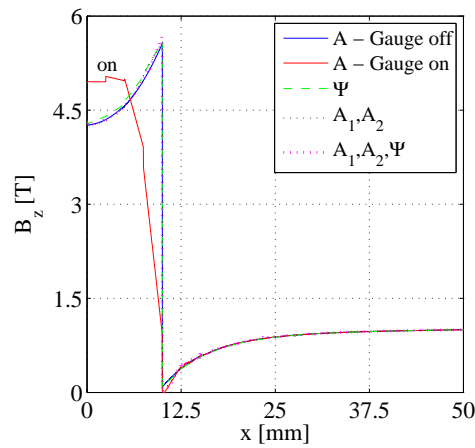
(b) Along the line $y = 10$mm, $z = 0$

Fig. 6.13. Distribution of the magnetic flux density

region. There are two unknown vector functions along the iron/air interface, but $\boldsymbol{A} \times \boldsymbol{n}$ must be continuous;

(*iii*) Using *edge elements* to represent unknown vector potentials.

The problem can be solved easily by the *reduced magnetic scalar potential* Ψ as well. The main advantage of using magnetic scalar potential is the few number of unknowns, because it can be decreased from three per node to only one per node.

A very advantageous formulation can be built up by applying the magnetic vector potential inside iron and the magnetic scalar potential in the air region ($\boldsymbol{A} - \Psi$ formulation). Unfortunately, the behavior of the magnetic field is very strange on the iron/air interface, which is caused by the weak coupling of the normal component of the magnetic flux density and of the tangential component of the magnetic field intensity along the iron/air interface. If the $\boldsymbol{A}/\Psi$ interface is moved from the iron/air interface into the air region, the problem disappears resulting the $\boldsymbol{A} - \boldsymbol{A} - \Psi$ formulation. Comparison between the results of the two potential formulations can be seen in *Fig*. 6.14. The disadvantage of this formulation is the high number of iterations and the strange behavior of error during the iterations, but this can be eliminated by an appropriate solver, as it is presented in *Fig*. 6.15.

(a) By $\boldsymbol{A} - \Psi$-formulation   (b) By $\boldsymbol{A} - \boldsymbol{A} - \Psi$-formulation
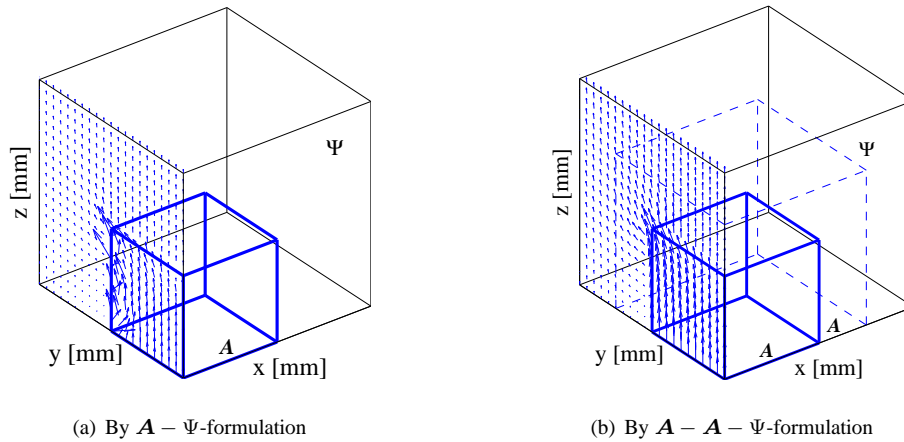
Fig. 6.14. Magnetic flux density calculated by using Φ in the air region

The value of flux inside the cube over the surface $z = 0$, $\int_{\Gamma} B_z \mathrm{d}\Gamma$ has been computed and compared. The reference result of paper [72] is $0.417 \, \mathrm{mVs}$. The number of elements in the tetrahedral mesh, the number of equations, the number of iterations (conjugate gradient method with algebraic multigrid preconditioner) and the resulting flux are shown in *Table*. 6.1. The magnetic flux calculated by the magnetic vector formulation is a little lower, the magnetic flux calculated by using the magnetic scalar potential is a little higher when using the same mesh.

Comparison between the performance and results of different formulations is collected in *Table*. 6.1.
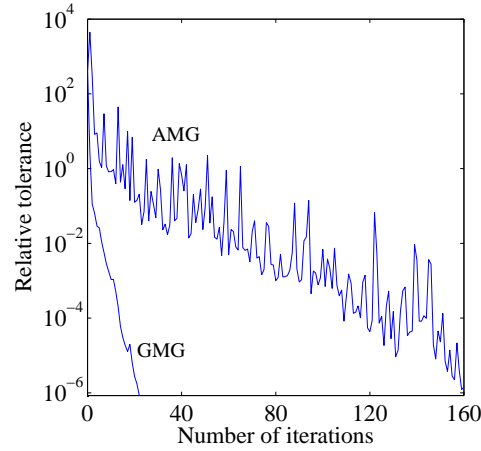
Fig. 6.15. The performance of iteration is sensitive for the preconditioner when
applying magnetic scalar potentail in the air region
(AMG-Algebraic Multigrid, GMG-Geometric Multigrid)

Table 6.1. Comparison between the performance and results of different formulations

No. of	$\boldsymbol{A}$-formulation			$\boldsymbol{A}_1 - \boldsymbol{A}_2$-formulation			Ψ-formulation		
No. of	No. of	No. of	Flux	No. of	No. of	Flux	No. of	No. of	Flux
FEs	eqs.	iter.	[mVs]	eqs.	iter.	[mVs]	eqs.	iter.	[mVs]
146	915	11	0.339	972	11	0.339	306	5	0.346
2552	12216	20	0.399	12309	21	0.399	3919	10	0.438
16838	73812	39	0.411	74139	40	0.411	24990	12	0.423
103034	434256	88	0.414	435087	96	0.414	145989	16	0.419

6.4   Conducting cube in homogeneous magnetic field

The problem is a modification of the previous one, i.e. a conducting cube made of iron
is situated in a homogeneous magnetic field resulting a linear eddy current field problem.
The linear model can be seen in *Fig.* 6.8, too ($\boldsymbol{B}_0 = (1\,\text{T})\boldsymbol{e}_z$, $\mu = 1000\mu_0$, but the
conductivity of the cube is $\sigma = 2 \cdot 10^6\,\text{S/m}$).

   The problem has been solved in the time domain and the simulated frequencies are
$f = 0.1\,\text{Hz}$, $f = 1\,\text{Hz}$, $f = 10\,\text{Hz}$ and $f = 100\,\text{Hz}$, and the *skin depth* is $\delta = 35.59\,\text{mm}$,
$\delta = 11.25\,\text{mm}$, $\delta = 3.559\,\text{mm}$ and $\delta = 1.125\,\text{mm}$, respectively, calculated by the
relation [28, 43, 81]

$$\delta = \sqrt{\frac{1}{\pi f \mu \sigma}}. \tag{6.4}$$

The number of calculated periods is 3 and every periods are divided into 72 time instants,
i.e. $\omega \Delta t = 5°$, where ω is the angular frequency of the external magnetic field and Δt is
the time step.

The problem has been solved by the $\boldsymbol{A}, V - \boldsymbol{A}$ and by the $\boldsymbol{T}, \Phi - \Phi$ formulations and the vector potentials have been approximated by nodal and vector shape functions of second order. In case of nodal approximation, the $\boldsymbol{A}_1, V - \boldsymbol{A}_2$ formulation is used, i.e. the magnetic vector potential $\boldsymbol{A}_1$ is used in the cube and $\boldsymbol{A}_2$ in the air region, similarly to that presented in the last example. The aim was to compare the results obtained by these four formulations.

The finite element mesh of the problem can be seen in *Fig.* 6.16 in case of $f = 0.1\,\mathrm{Hz}$ and $f = 1\,\mathrm{Hz}$. Here, only the discretized cube is shown. The variation of the z component of magnetic flux density inside the cube is almost linear, because the frequency is small enough. The four methods result in practically the same magnetic flux density as it is plotted in *Fig.* 6.17.
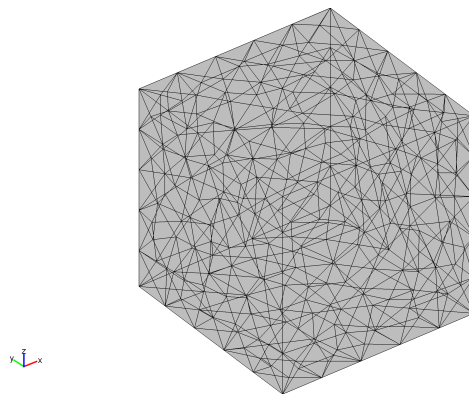
Fig. 6.16. The FEM mesh of the eighth of the problem when $f = 0.1\,\mathrm{Hz}$ and $f = 1\,\mathrm{Hz}$ (the mesh in air is not shown)

(a) Magnetic flux inside the cube, $f = 0.1\,\mathrm{Hz}$

(b) Magnetic flux inside the cube, $f = 1\,\mathrm{Hz}$
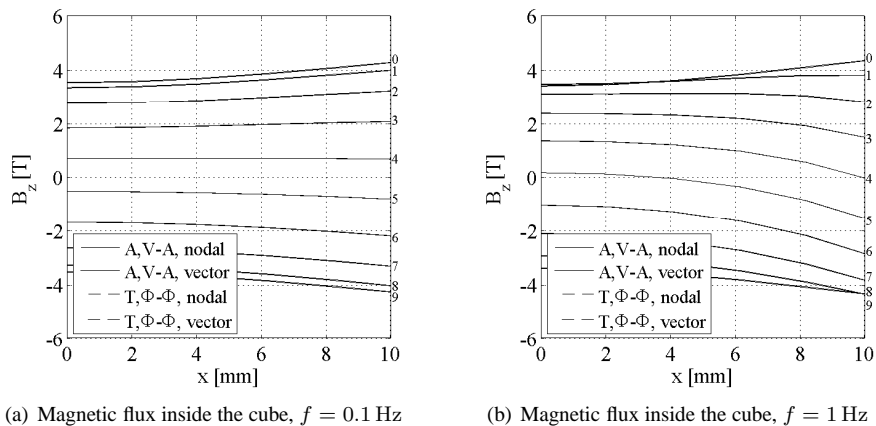
Fig. 6.17. Magnetic flux density calculated by using the four potential formulations along the line $x \in [0, \cdots, 10]\mathrm{mm}$, $y = 10\,\mathrm{mm}$, $z = 0\,\mathrm{mm}$

Here, the z component of the magnetic flux density is shown in the third period, when the external flux density is decreasing from $+1\,\mathrm{T}$ to $-1\,\mathrm{T}$ (curves are denoted by the integer k and curves are plotted at the time instants $t_k = (2.25 + k/18)/f$). In small frequency, the eddy currents have very small effect.

It is our experience that worse results can be obtained at higher frequency when using the same mesh. The effect of eddy currents is higher and higher when increasing the frequency of the excitation and it is observable as the so-called skin effect [28, 43, 81], i.e. the magnetic flux density and the magnetic field intensity are crowded out from the interior parts of the cube. When the frequency is high enough, the magnetic field can almost be equal to zero in the central part of the cube.

The changing (i.e. the gradient) of the magnetic field can be very small inside the cube and it can be very fast close to the surface when the frequency is higher and higher. The information, i.e. the change of magnetic field is crowded out to the surface, that is why finer and finer mesh close to the surface and as coarse mesh as possible in the interior space should be generated. Increasing the degree of the approximating functions is not the only way to solve this problem, the mesh must also be modified.

After some trials, the finite element mesh plotted in *Fig.* 6.18 and in *Fig.* 6.19 are generated to reach accurate results at $f = 10\,\mathrm{Hz}$ and $f = 100\,\mathrm{Hz}$, respectively. The mesh is finer close to the surface and it is coarser inside the cube. This modification has solved the problem of worse results as it can be seen in *Fig.* 6.20 and in *Fig.* 6.21 when the frequency is $f = 10\,\mathrm{Hz}$ and $f = 100\,\mathrm{Hz}$, respectively. The figures show the difference between the solutions calculated by using the $\boldsymbol{A}, V - \boldsymbol{A}$ and the $\boldsymbol{T}, \Phi - \Phi$ formulations. The magnetic flux density simulated by the magnetic vector potential is first order, because the magnetic vector potential is second order. However the magnetic flux density calculated by the second order current vector potential and the magnetic scalar potential is smoother than the one calculated by the magnetic vector potential. Even so, the results are comparable.
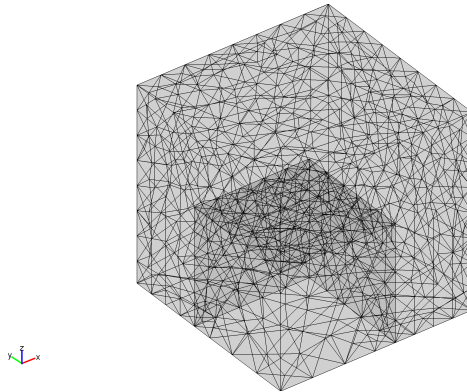
Fig. 6.18. The FEM mesh of the eighth of the problem when $f = 10\,\mathrm{Hz}$
(only the cube is shown)
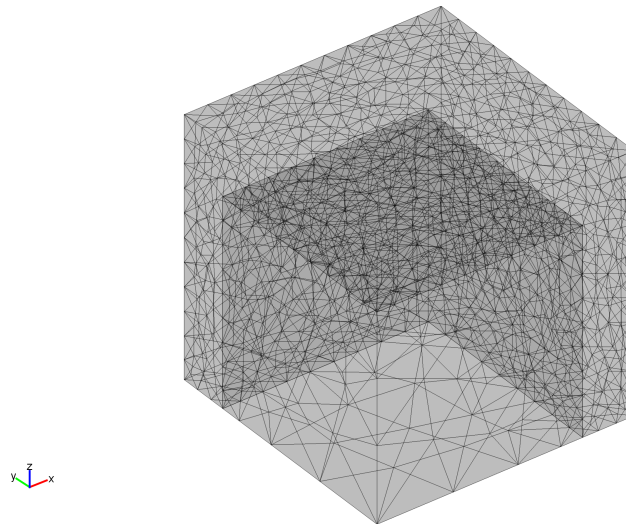
Fig. 6.19. The FEM mesh of the eighth of the problem when $f = 100\,\mathrm{Hz}$
(only the cube is shown)

(a) Magnetic flux inside cube, calculated by the $\boldsymbol{A}, V - \boldsymbol{A}$ formulation

(b) Magnetic flux inside cube, calculated by the $\boldsymbol{T}, \Phi - \Phi$ formulation
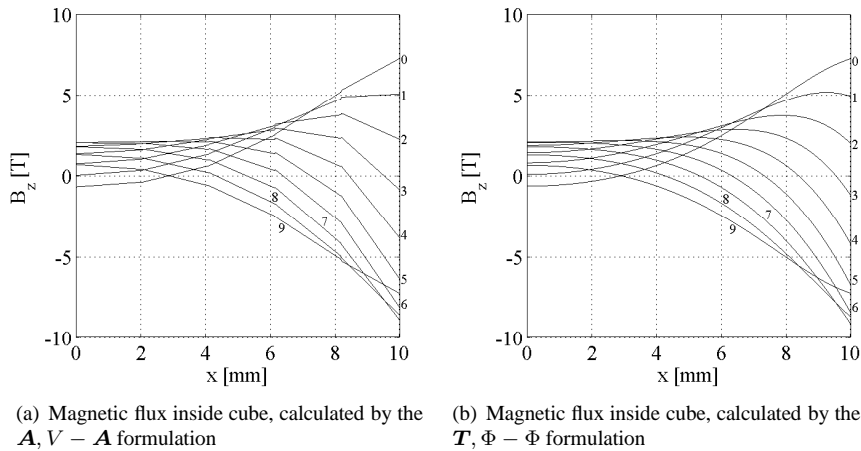
Fig. 6.20. The magnetic flux density at $f = 10\,\mathrm{Hz}$ along the line $x \in [0, \cdots, 10]\mathrm{mm}$,
$y = 10\,\mathrm{mm}$, $z = 0\,\mathrm{mm}$

The magnetic field is almost equal to zero when the distance measured from the surface is higher than 5δ [28, 43, 81]. It can be seen easily in *Fig.* 6.21.

As a consequence of the above simulations, we can say that the magnetic flux density simulated by nodal or vector representation of the unknown potentials leads to the same results, however, this is not true for simulation time. The computation time is smaller in

(a) Magnetic flux inside cube, calculated by the $A, V - A$ formulation

(b) Magnetic flux inside cube, calculated by the $T, \Phi - \Phi$ formulation
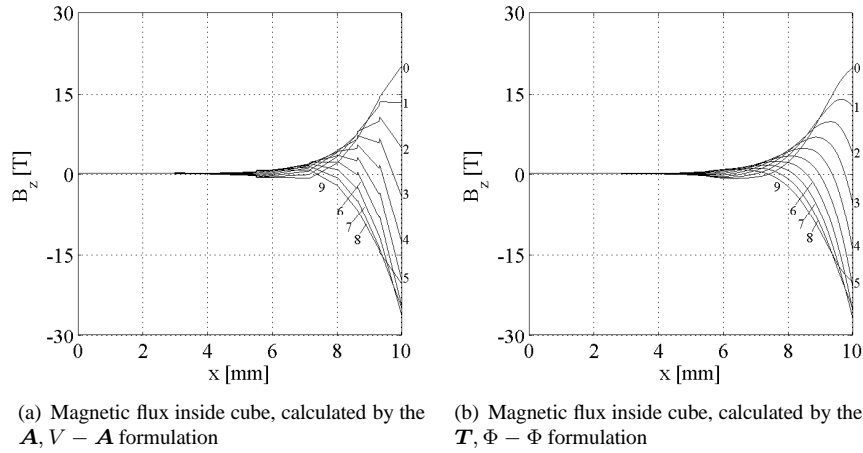
Fig. 6.21. The magnetic flux density at $f = 100\,\text{Hz}$ along the line $x \in [0, \cdots, 10]\text{mm}$, $y = 10\,\text{mm}$, $z = 0\,\text{mm}$

the case of vector representation of the magnetic vector potential and in the case of nodal approximation of the current vector potential at low frequency, but in the case of vector representation of the current vector potential in higher frequency. Results are presented in *Table* 6.2 and in *Table* 6.3. Here, DOF means the number of degree of freedom, i.e. the number of unknowns. The number of steps means the average number of steps of the whole time stepping scheme.

It is important to note that the ungauged $A, V - A$ formulation can be reformulated by using $V = 0$ as $A^\star - A$ formulation. The performance of this formulation is very bad, the number of step is 1022, 1014 and 928 in the case of $f = 0.1\,\text{Hz}$, $f = 1\,\text{Hz}$ and $f = 10\,\text{Hz}$, moreover the convergence was very weak when $f = 100\,\text{Hz}$. The mesh was the same as it is presented in *Table* 6.2.

Table 6.2. Computational costs of using the magnetic vector potential

Formulation	Frequency [Hz]	DOF	No. of steps
$A, V - A$, nodal	0.1	20180	105
$A, V - A$, vector	0.1	25324	50
$A, V - A$, nodal	1	20180	112
$A, V - A$, vector	1	25324	50
$A, V - A$, nodal	10	59579	219
$A, V - A$, vector	10	76356	72
$A, V - A$, nodal	100	105008	495
$A, V - A$, vector	100	136843	90

Table 6.3. Computational costs of using the current vector potential

Formulation	Frequency [Hz]	DOF	No. of steps
$T, \Phi - \Phi$, nodal	0.1	15622	17
$T, \Phi - \Phi$, vector	0.1	20456	32
$T, \Phi - \Phi$, nodal	1	15622	19
$T, \Phi - \Phi$, vector	1	20456	34
$T, \Phi - \Phi$, nodal	10	49458	53
$T, \Phi - \Phi$, vector	10	66773	49
$T, \Phi - \Phi$, nodal	100	86400	383
$T, \Phi - \Phi$, vector	100	117297	98

6.5   Steel plates around a coil, linear problem

This problem is a modification of Problem No. 10 of the TEAM Workshops [25, 63, 72]. The original task is to compute the transient eddy currents in saturated steel plates around racetrack-shaped coil excited by exponentially rised current. Here, the problem is simplified, the current as well as the permeability of steel are constant and static magnetic field has been simulated.

The problem has been solved by the ungauged A-formulation represented by edge finite elements first, then by the gauged A-formulation approximated by nodal FEM to study the effect of gauging. Here the *generalized minimum residual method* (*GMRES*) with *algebraic multigrid preconditioner* has been applied to solve the linear system of equations. The FEM mesh of the eighth of the problem can be seen in *Fig.* 6.22 (it is enough to simulate the eighth of the problem because of symmetry). The convergence of the ungauged version is very bad, but gauging can speed up the solution of the system
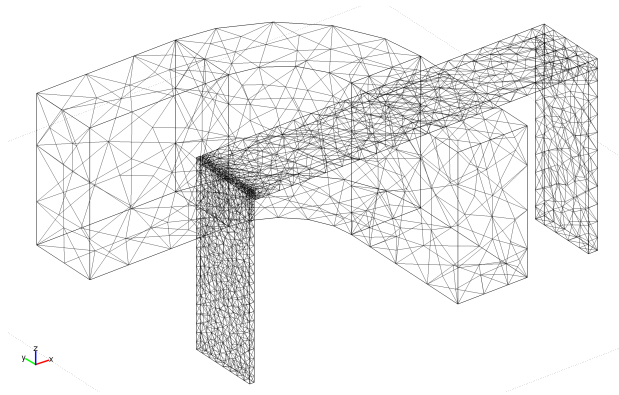
Fig. 6.22. The FEM mesh of the eighth of the problem

of linear equations (e.g. a mesh with 76662 tetrahedral elements and second order FEM approximation results in 328116 equations, which can not be solved without gauging – normalized residual was only 0.13 after 500 steps and it did not decrease –, however, it is 274 steps with gauging when the error limit is 10^{-6}). Unfortunately, the magnetic flux density calculated from the gauged magnetic vector potential is not correct, especially in the vicinity of iron/air interface, because of the continuous normal component of the magnetic vector potential. Generating finer and finer mesh does not solve this problem. If the jump of the normal component of magnetic vector potential is allowed, the problem can be eliminated, however, the number of unknowns is increasing a bit as well as the number of iterations (in this case, the number of unknowns is 341109 and the number of iterations is 780). The magnetic flux density vectors inside the plate and inside the central plate can be seen in *Fig.* 6.23 and in *Fig.* 6.24.
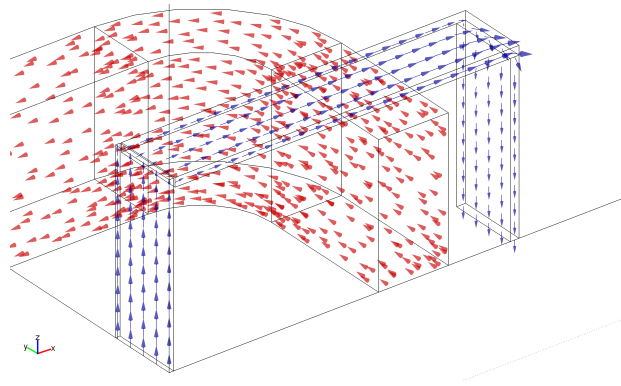
Fig. 6.23. The magnetic flux is driven by the plates
(current density inside coil can also be seen)

(a) Continuous magnetic vector potential

(b) Normal component of A is not continuous
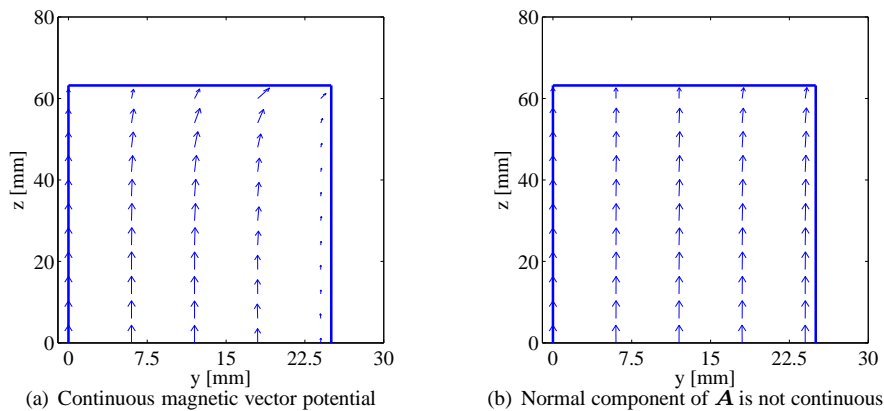
Fig. 6.24. Magnetic flux density vector calculated by using the gauged magnetic vector potential approximated by nodal finite elements

The magnetic flux should be almost constant inside the specimen, *Fig.* 6.25 shows the results simulated by different A-formulations using the same mesh. It must be noted that the solution without gauge fix has been solved by the direct solver *SPOOLES*.
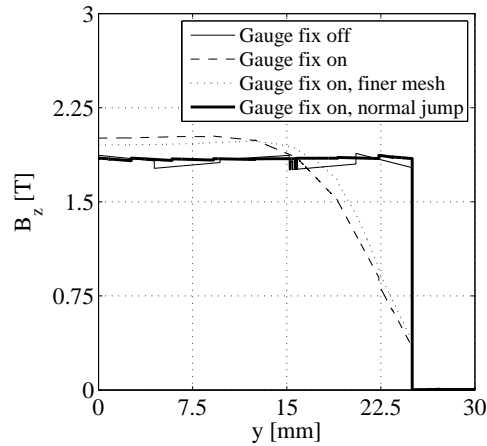
Fig. 6.25. The z component of the magnetic flux density, B_z, along the line
$x = 0, y = 0, \cdots, 30\,\mathrm{mm}, z = 0$
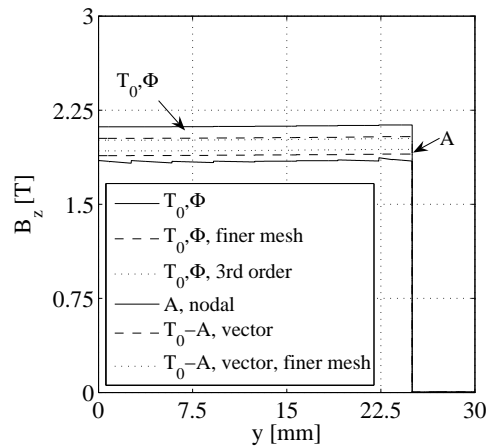
Fig. 6.26. The z component of the magnetic flux density, B_z, along the line
$x = 0, y = 0, \cdots, 30\,\mathrm{mm}, z = 0$ simulated by different formulations

The problem has been solved by the ungauged version of the magnetic vector potential formulation, too. In this case, the coil's current must be represented by the impressed current vector potential T_0 approximated by edge elements. Edge elements are used to represent the magnetic vector potential, too, which results in larger value of the magnetic

flux density (see *Fig.* 6.26). The number of unknowns is 95397 and 290123 for a coarse and a fine mesh. In the first situation 55 and 112 steps needed to calculate the ungauged impressed field and the magnetic vector potential, respectively. In the case of fine mesh 93 and 349 steps needed. The ungauged version results in a faster solution than the gauged one.

Next, the problem has been solved by the reduced magnetic scalar potential Φ. The coil's current has been represented by T_0 approximated by edge elements, the magnetic scalar potential has been approximated by nodal elements. Figure 6.26 shows the results simulated by different formulations. It can be seen that the magnetic flux density inside the magnetic material is larger than the result obtained by the magnetic vector potential represented by nodal elements. The result calculated by the reduced scalar potential formulation can be smaller and smaller by increasing the number of unknowns or the order of approximation (the number of unknowns here is 14710 and 173163 for a coarse and a fine mesh). It must be noted that the order of T_0 and of Φ must be the same.

6.6 Steel plates around a coil, nonlinear problem

The modified version of Problem No. 10 of the TEAM Workshops have been solved when the nonlinearity of the steel plates is taken into account. Measured data is known from the paper [25, 63, 72].

The measured magnetization curve of steel plates can be seen in *Fig.* 6.27. This curve can be modeled by simple functions, like the inverse tangent function, however, here a neural network has been trained to represent the measured data.
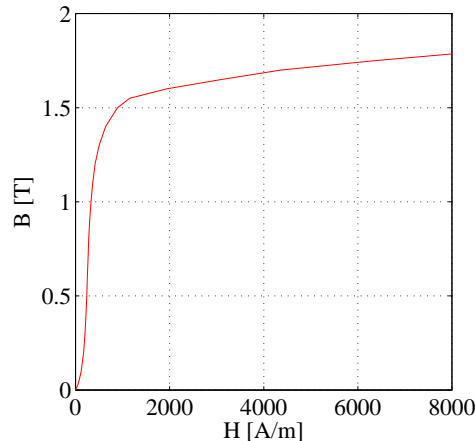
Fig. 6.27. The measured magnetization curve of steel plates

The problem has been solved by first order reduced magnetic scalar potential Φ and by first order ungauged magnetic vector potential. The problem region is discretized by a mesh used in the last example, too, it consists of 76662 elements. The number of unknowns is 95397 for T_0, 14710 for Φ and 95397 for A.

The magnetic field intensity vector $\boldsymbol{H}$ can be calculated directly when the reduced magnetic scalar potential Φ is used, obtained from the finite element method,

$$\boldsymbol{H} = \boldsymbol{T}_0 - \nabla\Phi. \tag{6.5}$$

In this case, the direct model, $\boldsymbol{B} = \mathscr{B}\{\boldsymbol{H}\}$ can be used to calculate the magnetic flux density, then

$$\boldsymbol{B} = \mu_o \boldsymbol{H} + \boldsymbol{R} \quad \Rightarrow \quad \boldsymbol{R} = \boldsymbol{B} - \mu_o \boldsymbol{H} \tag{6.6}$$

is used to update the residual term $\boldsymbol{R}$. After it, the updated finite element equations must be solved again and the procedure described here has to be repeated until convergence.

The magnetic flux density vector $\boldsymbol{B}$ can be calculated when the magnetic vector potential $\boldsymbol{A}$ is used, obtained from the finite element method,

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}. \tag{6.7}$$

In this case, the inverse model, $\boldsymbol{H} = \mathscr{B}^{-1}\{\boldsymbol{B}\}$ can be used to calculate the actual magnetic field intensity, then the formula

$$\boldsymbol{H} = \nu_o \boldsymbol{B} + \boldsymbol{I} \quad \Rightarrow \quad \boldsymbol{I} = \boldsymbol{H} - \nu_o \boldsymbol{B} \tag{6.8}$$

is used to update the residual term $\boldsymbol{I}$. After it, the updated finite element equations must be solved again and the procedure described here has to be repeated until convergence.

The magnetic field quantities as well as the residual terms are defined on the points of the mesh, where the scalar potentials are.
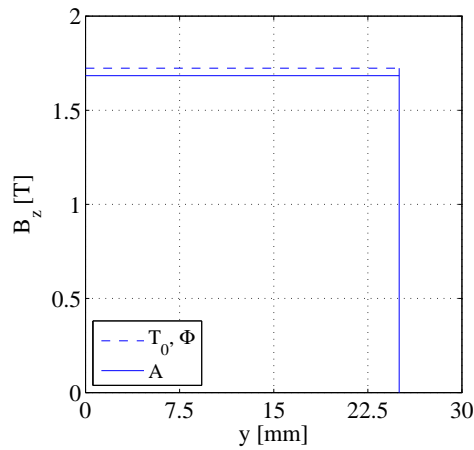
Fig. 6.28. The z component of the magnetic flux density, B_z, along the line $x = 0$, $y = 0, \cdots, 30\,\mathrm{mm}$, $z = 0$ simulated by different formulations

Updating the nonlinear residual terms $\boldsymbol{R}$, or $\boldsymbol{I}$ means that the right-hand side of the linearized, assembled system of equations is changing from iteration to iteration. The difference between two iteration steps (i.e. $\Delta\boldsymbol{R}$, $\Delta\boldsymbol{I}$, $\Delta\boldsymbol{H}$, or $\Delta\boldsymbol{B}$) is decreasing during the convergence algorithm. The iteration can be stopped if a predefined error limit is reached.

The fixed point iteration scheme with 154 and 143 steps found the solution by using the Φ and the $\boldsymbol{A}$-formulation, respectively. The z component of the resulting magnetic flux density along the line $x = 0$, $y = 0, \cdots, 30\,\mathrm{mm}$, $z = 0$ can be seen in *Fig.* 6.28. The averaged value of measured data is $B_z = 1.67\,\mathrm{T}$ [72]. Simulated results are $B_z = 1.72\,\mathrm{T}$, moreover $B_z = 1.68\,\mathrm{T}$ by the Φ-formulation and by the $\boldsymbol{A}$-formulation, respectively.

6.7 Asymmetrical conductor with a hole

Benchmark problem No. 7 of the TEAM workshop consists of an asymmetrical conductor with a hole [30,40,64,85], see *Fig.* 6.29. The conductor is made of aluminum. The source of magnetic field is the sinusoidal current flowing in the racetrack-shaped coil placed above the plate eccentrically. The conductivity of the plate is $\sigma = 3.526 \cdot 10^7\,\mathrm{S/m}$, the maximum *ampere-turn* is 2742 AT (the ampere-turn (AT) is the unit of magnetomotive force, represented by a direct current of one ampere flowing in a single-turn loop in vacuum), the frequencies are $f = 50\,\mathrm{Hz}$ and $f = 200\,\mathrm{Hz}$.

The amplitude of the source current density of the coil can be calculated by the formula $|\boldsymbol{J}| = 2742/(0.1 \cdot 0.025)\,\mathrm{A/m^2}$, where the denominator is the cross-section area of the coil. The impressed current vector potential $\boldsymbol{T}_0$ must be used to represent the current of coil in the case of ungauged $\boldsymbol{A}$-formulation and Φ-formulation.

The problem is a linear quasi-steady state eddy current field problem with multiply connected eddy current region, because there is a hole inside the conducting material.

Paper [30] is a summary of the measurement system and the comparison between 25 solutions with different methods and formulations and the measurements [1, 2, 7, 15, 26, 27, 32, 48, 62, 67, 76]. Here, the problem has been solved by the formulations introduced previously. The problem has been discretized by tetrahedral finite elements. The air region has been closed by an artificial far boundary as it can be seen in *Fig.* 6.30. The radius of the bounding sphere has been determined after some trials. Figure 6.31 shows the discretization of the plate and the coil. After some trials, the third order approximation on a mesh with 8604 elements has been selected.

The weak formulations have been transformed to the *frequency domain* by replacing the operator $\partial/\partial t$ by the multiplier $\mathrm{j}\omega$, i.e. the resulting system of equations is complex. The weak form of the gauged $\boldsymbol{A}, V - \boldsymbol{A}$ formulation (3.137)–(3.138) in the frequency domain can be summarized as

$$
\int_{\Omega_c \cup \Omega_n} \left[\nu \left(\nabla \times \boldsymbol{W} \right) \cdot \left(\nabla \times \widetilde{\boldsymbol{A}} \right) + \nu \nabla \cdot \boldsymbol{W} \, \nabla \cdot \widetilde{\boldsymbol{A}} \right] \mathrm{d}\Omega
$$
$$
+ \mathrm{j}\omega \int_{\Omega_c} \boldsymbol{W} \cdot \left(\sigma \widetilde{\boldsymbol{A}} + \sigma \widetilde{v} \right) \mathrm{d}\Omega = \int_{\Omega_n} \boldsymbol{W} \cdot \boldsymbol{J}_0 \, \mathrm{d}\Omega + \int_{\Gamma_{H_n}} \boldsymbol{W} \cdot \boldsymbol{K} \mathrm{d}\Gamma,
\tag{6.9}
$$

$$
\mathrm{j}\omega \int_{\Omega_c} \nabla N \cdot \left(\sigma \widetilde{\boldsymbol{A}} + \sigma \widetilde{v} \right) \mathrm{d}\Omega = 0.
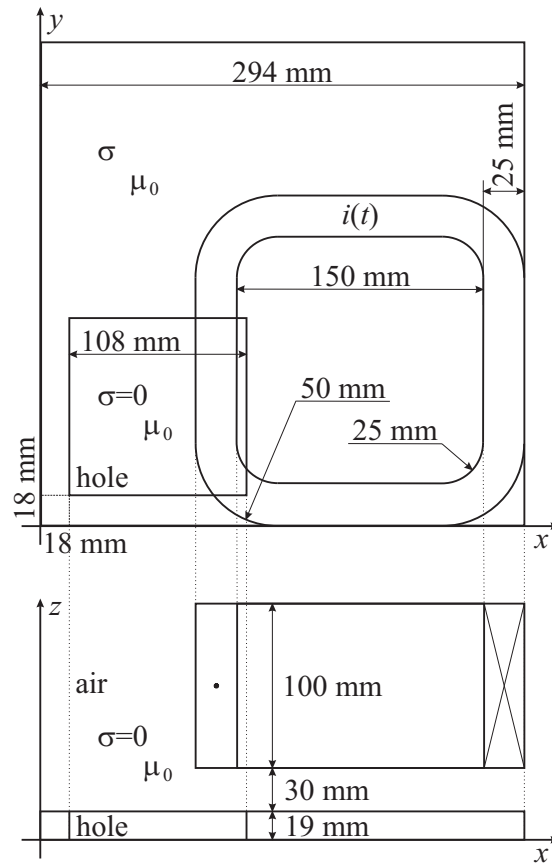\tag{6.10}
$$

Fig. 6.29. A plate beneath a coil

The other formulations in the frequency domain can be obtained similarly. It is noted that the problem based on nodal finite elements has been solved by the *GMRES* (*Generalized Minimum Residual Method*) solver with the *Incomplete LU preconditioner* . If the vector functions have been approximated by vector elements, the GMRES solver with *SSOR* (*Symmetric Successive Over-Relaxation*) preconditioner has been applied. Our experiment is that the other solvers of COMSOL Multiphysics are very slow in this case [18].

The reference solution is obtained by the $A, V - A$-formulation. The eddy currents have a path around the hole filled with air, as it can be seen in *Fig.* 6.32 and in *Fig.* 6.33, for $f = 50$ Hz and $f = 200$ Hz, respectively. The use of the reduced magnetic scalar potential Φ in the eddy current free region results in false solution, as it is illustrated in *Fig.* 6.34. This is the situation when applying the $A, V - \Phi$-formulation or the $T, \Phi - \Phi$-formulation aiming the reduction of the degree of freedom.
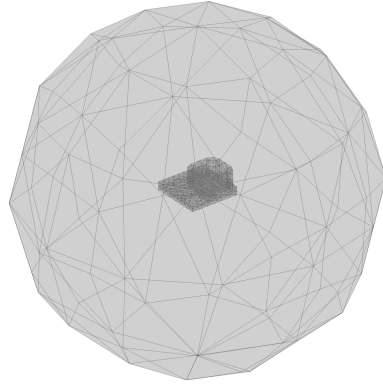
Fig. 6.30. Finite element mesh of the problem
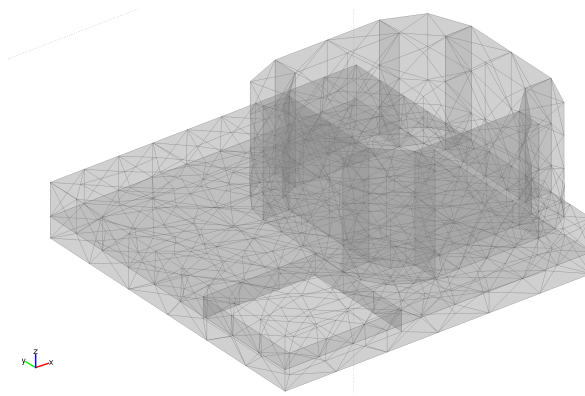
Fig. 6.31. Finite element mesh of the plate and its surround

The problem of the $\boldsymbol{A}, V - \Phi$-formulation can be solved by applying the $\boldsymbol{A}, V - \boldsymbol{A} - \Phi$-formulation, when the magnetic vector potential $\boldsymbol{A}$ is introduced in the hole, too. The false solution of the $\boldsymbol{T}, \Phi - \Phi$-formulation can be corrected by employing the $\boldsymbol{T}, \Phi - \boldsymbol{A}$-formulation or the $\boldsymbol{T}, \Phi - \boldsymbol{A} - \Phi$-formulation, when the magnetic vector potential $\boldsymbol{A}$ is introduced in the whole air region, or only in the hole, respectively. The other possibility is filling the hole with a conducting material with very low conductivity, i.e. the $\boldsymbol{T}, \Phi - \Phi$-formulation can be used without any modification. The conductivity of air region in the hole can be determined by trial and error method, i.e. $\sigma_{\mathrm{air}} = \sigma_{\mathrm{aluminum}}/c$, where c is a constant. Increasing c results in a model, which is closer and closer to the original one, however, the system of linear equations may become ill-conditioned and the number of iterations may increase.

The nodal $\boldsymbol{A}, V - \boldsymbol{A}$-formulation as well as the nodal $\boldsymbol{A}, V - \boldsymbol{A} - \Phi$-formulation can be sped up by using edge element representation of the unknown magnetic vector potential.
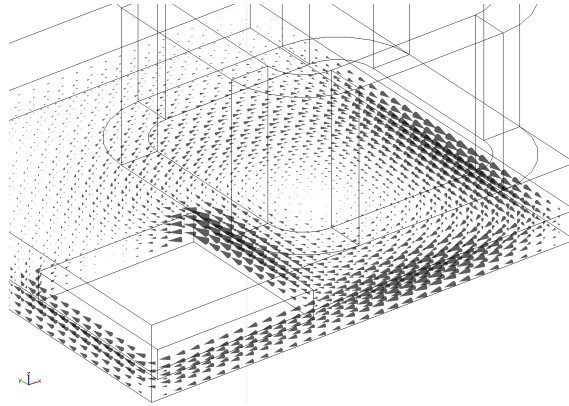
Fig. 6.32. Real part of eddy current field inside the aluminum plate at $f = 50\,\mathrm{Hz}$
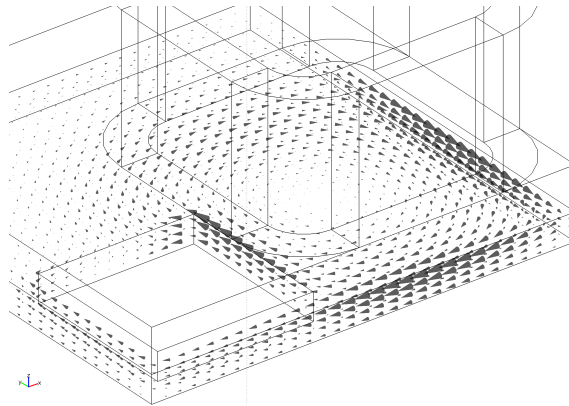
Fig. 6.33. Real part of eddy current field inside the aluminum plate at $f = 200\,\mathrm{Hz}$

The convergence of the gauged $\boldsymbol{A}, V - \boldsymbol{A} - \Phi$-formulation is extremely slow. According to [11], the reason of this inferior performance is the only Neumann type boundary conditions (2.507) and (2.512) on $\Gamma_{nc_{\boldsymbol{A}}}$ and on $\Gamma_{nc_{\Phi}}$, respectively, of the electric scalar potential V, since, there is no plane of symmetry. However, the performance of the ungauged $\boldsymbol{A}, V - \boldsymbol{A} - \Phi$-formulation is much better. The fastest solver can be obtained by using the nodal $\boldsymbol{T}, \Phi - \boldsymbol{A} - \Phi$-formulation, however, the performance of the nodal $\boldsymbol{T}, \Phi - \Phi$-formulation (the hole is filled with conducting media) is also satisfactory. It is our experience that the use of edge elements in the $\boldsymbol{T}, \Phi$-based methods slow down the solvers. In the case of edge element representation of the magnetic vector potential, $V = 0$ can be supposed resulting the $\boldsymbol{A}^{\star} - \boldsymbol{A}$-formulation or the $\boldsymbol{A}^{\star} - \boldsymbol{A} - \Phi$-formulation. These methods have the advantage that the number of unknowns is decreasing, however, the solution time is increasing.

Comparison of measured and simulated results in *Fig.* 6.35 and in *Fig.* 6.36 show
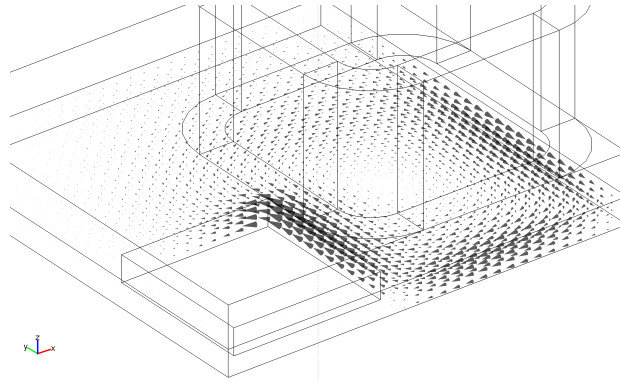
Fig. 6.34. False eddy current distribution inside the aluminum plate at $f = 50\,\mathrm{Hz}$ obtained by the $\boldsymbol{A}, V - \Phi$-formulation or by the $\boldsymbol{T}, \Phi - \Phi$-formulation (real part)

that the solution of different potential formulations is practically identical and close to measurements. The solution is complex, following [30], the quantity

$$C = \mathrm{sign}(C_r)\sqrt{C_r^2 + C_i^2} \tag{6.11}$$

has been determined, where C_r and C_i are the real and the imaginary parts of the complex quantity, respectively (e.g. of the magnetic flux density) and $\mathrm{sign}(C_r)$ is the sign of the real part. Finally, C can be plotted and analyzed (*Table*. 6.4, where CPU time means the results of an AMD Athlon 64 X2 Dual Core Processor 4600+ 2.41GHz computer with 4GByte RAM).

6.8 Rotational single sheet tester

The example closing this chapter contains an ongoing research, which aims to build up a vector hysteresis measurement apparatus. The CAD design of the arrangement is based on the finite element method.

The *rotational single sheet tester* is one of the possible measurement arrangements of the two-dimensional vector hysteresis properties. In this case, the specimen has round shape and this is called *RRSST system* [31, 44, 45, 54–57].

The RRSST system is an induction motor, which rotor has been removed and the round-shaped specimen has been installed in this place. The magnetic field inside the specimen can be generated by a special two phase winding excited by two independent current generators. The two orthogonal components of the magnetic field intensity vector and of the magnetic flux density vector inside the specimen can be measured by a sensor system. The tangential component of the magnetic field intensity can be measured by two coils placed onto the surface of the specimen (H-coils), the magnetic flux density inside the specimen can be measured by two coils slipped into holes of the specimen (B-coils). The block diagram can be seen in *Fig.* 6.37.

Table 6.4. Comparison between the performance and results of different formulations

Method	Frequency [Hz]	DOF	Iteration	CPU time [sec]
$\boldsymbol{A}, V - \boldsymbol{A}$, nodal	50	122020	532	1924
$\boldsymbol{A}, V - \boldsymbol{A}$, nodal	200	122020	857	2454
$\boldsymbol{A}, V - \boldsymbol{A}$, vector	50	132344	132	616
$\boldsymbol{A}, V - \boldsymbol{A}$, vector	200	132344	132	616
$\boldsymbol{A}, V - \boldsymbol{A} - \Phi$, nodal	50	55554	980	1450
$\boldsymbol{A}, V - \boldsymbol{A} - \Phi$, nodal	200	55554	9995	9919
$\boldsymbol{A}, V - \boldsymbol{A} - \Phi$, vector	50	58377	185	259
$\boldsymbol{A}, V - \boldsymbol{A} - \Phi$, vector	200	58377	180	255
$\boldsymbol{T}, \Phi - \Phi$, nodal	50	53204	50	319
$\boldsymbol{T}, \Phi - \Phi$, nodal	200	53204	60	345
$\boldsymbol{T}, \Phi - \Phi$, vector	50	55949	216	278
$\boldsymbol{T}, \Phi - \Phi$, vector	200	55949	277	355
$\boldsymbol{T}, \Phi - \boldsymbol{A} - \Phi$, nodal	50	53422	47	340
$\boldsymbol{T}, \Phi - \boldsymbol{A} - \Phi$, nodal	200	53422	44	325
$\boldsymbol{T}, \Phi - \boldsymbol{A} - \Phi$, vector	50	55999	2353	2409
$\boldsymbol{T}, \Phi - \boldsymbol{A} - \Phi$, vector	200	55999	1092	1171
$\boldsymbol{A}^{\star} - \boldsymbol{A}$, vector	50	128952	405	1598
$\boldsymbol{A}^{\star} - \boldsymbol{A}$, vector	200	128952	301	1213
$\boldsymbol{A}^{\star} - \boldsymbol{A} - \Phi$, vector	50	54147	436	456
$\boldsymbol{A}^{\star} - \boldsymbol{A} - \Phi$, vector	200	54147	328	384

The stator core is made of laminated iron, that is eddy currents have been neglected there. Eddy currents have been taken into account only inside the specimen. According to the preliminary studies of the problem [54–57], the magnetic field intensity is much smaller inside the stator core than inside the specimen, so a linear characteristics have been supposed inside the stator core ($\mu_r = 4000$) and a nonlinear hysteretic one inside the specimen. The hysteresis characteristics of the material have been simulated by the isotropic neural network based vector model. The x component of the applied hysteresis characteristics can be seen in *Fig.* 6.38. Here, the terms $\mu_o H_x$ and R_x are also presented.

The arrangement has been simulated by the gauged $\boldsymbol{T}, \Phi - \Phi$-formulation. The $x - y$ plane of the arrangement has been discretized by triangular mesh (*Fig.* 6.39), which has been extruded in the z direction resulting in prism elements. The mesh consists of 94512 prism elements. The impressed current vector potential $\boldsymbol{T}_0$ has been represented by edge elements (see *Fig.* 6.40), the unknown current vector potential $\boldsymbol{T}$ and the unknown reduced magnetic scalar potential Φ have been approximated by nodal elements. The number of unknowns is 68893 and 5800 vector hysteresis models with 20 scalar models per one vector model, i.e. the full number of scalar models is 116000. There are 3 layers inside the specimen.
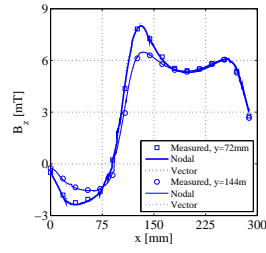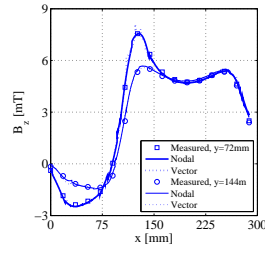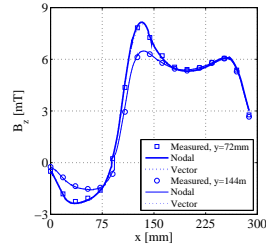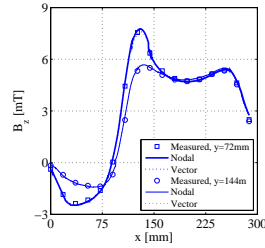
(a) $\boldsymbol{A}, V - \boldsymbol{A}$-formulation, $f = 50\,\text{Hz}$

(b) $\boldsymbol{A}, V - \boldsymbol{A}$-formulation, $f = 200\,\text{Hz}$

(c) $\boldsymbol{A}, V - \boldsymbol{A} - \Phi$-formulation, $f = 50\,\text{Hz}$

(d) $\boldsymbol{A}, V - \boldsymbol{A} - \Phi$-formulation, $f = 200\,\text{Hz}$
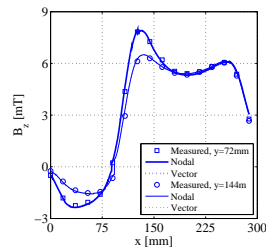
(e) $\boldsymbol{T}, \Phi - \Phi$-formulation, $f = 50\,\text{Hz}$
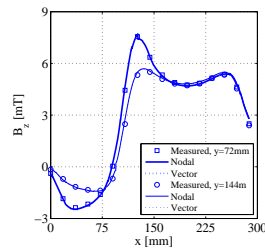
(f) $\boldsymbol{T}, \Phi - \Phi$-formulation, $f = 200\,\text{Hz}$
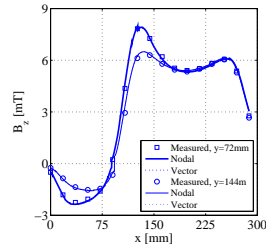
(g) $\boldsymbol{T}, \Phi - \boldsymbol{A} - \Phi$-formulation, $f = 50\,\text{Hz}$
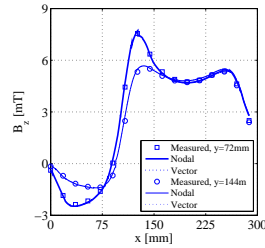
(h) $\boldsymbol{T}, \Phi - \boldsymbol{A} - \Phi$-formulation, $f = 200\,\text{Hz}$

Fig. 6.35. The z component of the magnetic flux density along $x = 0, \cdots, 288\,\text{mm}$, $y = 72\,\text{mm}$ and $y = 144\,\text{mm}$, $z = 34\,\text{mm}$, simulated by different formulations

(a) $\boldsymbol{A}, V - \boldsymbol{A}$-formulation, $f = 50\,\mathrm{Hz}$

(b) $\boldsymbol{A}, V - \boldsymbol{A}$-formulation, $f = 200\,\mathrm{Hz}$

(c) $\boldsymbol{A}, V - \boldsymbol{A} - \Phi$-formulation, $f = 50\,\mathrm{Hz}$

(d) $\boldsymbol{A}, V - \boldsymbol{A} - \Phi$-formulation, $f = 200\,\mathrm{Hz}$

(e) $\boldsymbol{T}, \Phi - \Phi$-formulation, $f = 50\,\mathrm{Hz}$

(f) $\boldsymbol{T}, \Phi - \Phi$-formulation, $f = 200\,\mathrm{Hz}$

(g) $\boldsymbol{T}, \Phi - \boldsymbol{A} - \Phi$-formulation, $f = 50\,\mathrm{Hz}$

(h) $\boldsymbol{T}, \Phi - \boldsymbol{A} - \Phi$-formulation, $f = 200\,\mathrm{Hz}$
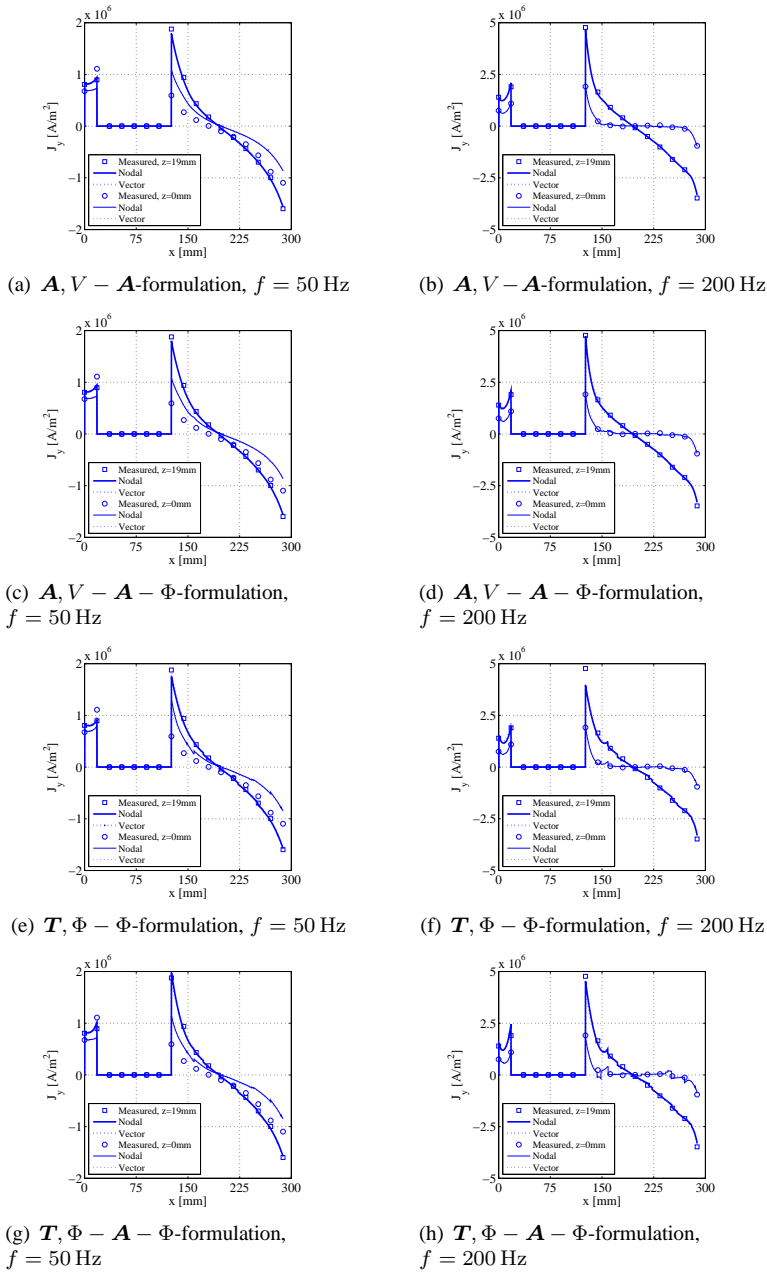
Fig. 6.36. The y component of the eddy current density, J_y, along the two lines $x = 0, \cdots, 288\,\mathrm{mm}$, $y = 72\,\mathrm{mm}$, $z = 0\,\mathrm{mm}$ and $z = 19\,\mathrm{mm}$, simulated by different formulations
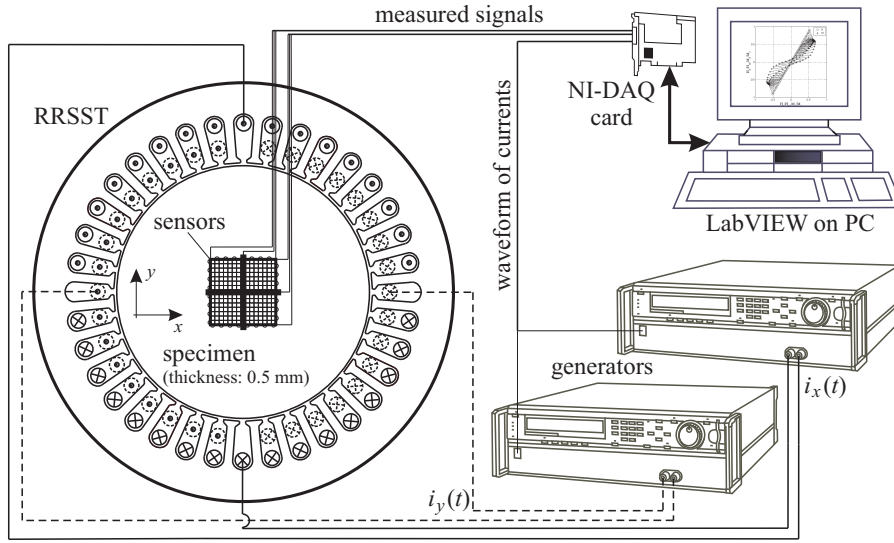
Fig. 6.37. The block diagram of the measurement system
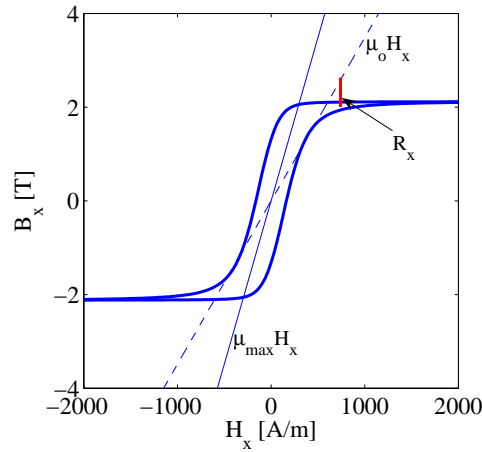
Fig. 6.38. The hysteresis characteristics and the terms $\mu_o H_x$ and R_x

The system of the nonlinear ordinary differential equations (3.111) and (3.112) must be solved in the time domain, because of the hysteresis characteristics and it has been solved in the $(n+1)^{\text{th}}$ time step by the following scheme for the general variable $\boldsymbol{a}$:

$$\frac{\boldsymbol{a}^{(n+1)} - \boldsymbol{a}^{(n)}}{\Delta t} = \theta \frac{\partial \boldsymbol{a}^{(n+1)}}{\partial t} + (1-\theta)\frac{\partial \boldsymbol{a}^{(n)}}{\partial t}, \tag{6.12}$$

where Δt is the time step of the time discretization and $\theta \in [0, \cdots, 1]$ is a parameter [41].
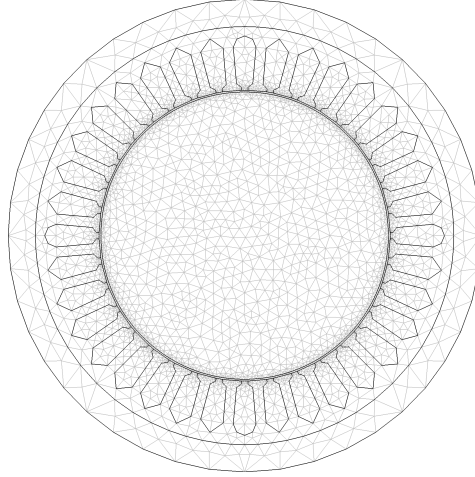
Fig. 6.39. The 2D mesh of the arrangement, which is extruded in z direction

First, the equations (3.111) and (3.112) must be multiplied by $\theta \Delta t$ and it is supposed in the time step $(n+1)$,

$$
\begin{aligned}
& \int_{\Omega_c} \left[\frac{\theta \Delta t}{\sigma} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \boldsymbol{T}^{(n+1)} \right) + \frac{\theta \Delta t}{\sigma} \nabla \cdot \boldsymbol{W}_k \nabla \cdot \boldsymbol{T}^{(n+1)} \right] \mathrm{d}\Omega \\
& + \int_{\Omega_c} \left[\mu_o \theta \Delta t \boldsymbol{W}_k \cdot \frac{\partial \boldsymbol{T}^{(n+1)}}{\partial t} - \mu_o \theta \Delta t \boldsymbol{W}_k \cdot \nabla \frac{\partial \Phi^{(n+1)}}{\partial t} \right] \mathrm{d}\Omega = \\
& - \int_{\Omega_c} \mu_o \theta \Delta t \boldsymbol{W}_k \cdot \frac{\partial \boldsymbol{T}_0^{(n+1)}}{\partial t} \, \mathrm{d}\Omega - \int_{\Omega_c} \theta \Delta t \boldsymbol{W}_k \cdot \frac{\partial \boldsymbol{R}^{(n+1)}}{\partial t} \, \mathrm{d}\Omega,
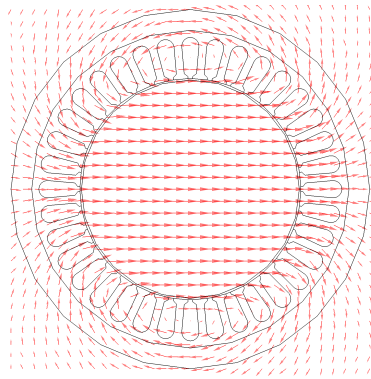\end{aligned}
\tag{6.13}
$$

Fig. 6.40. The x component of the impressed current vector potential $\boldsymbol{T}_0$

$$- \int_{\Omega_c} (\nabla N_k) \cdot \left(\mu_o \theta \Delta t \frac{\partial \boldsymbol{T}^{(n+1)}}{\partial t} \right) \mathrm{d}\Omega$$

$$+ \int_{\Omega_c \cup \Omega_n} (\nabla N_k) \cdot \left(\mu_o \theta \Delta t \nabla \frac{\partial \Phi^{(n+1)}}{\partial t} \right) \mathrm{d}\Omega$$

$$= \int_{\Omega_c \cup \Omega_n} (\nabla N_k) \cdot \left(\mu_o \theta \Delta t \frac{\partial \boldsymbol{T}_0^{(n+1)}}{\partial t} \right) \mathrm{d}\Omega \qquad (6.14)$$

$$+ \int_{\Omega_c} (\nabla N_k) \cdot \left(\theta \Delta t \frac{\partial \boldsymbol{R}^{(n+1)}}{\partial t} \right) \mathrm{d}\Omega$$

$$+ \int_{\Gamma_B} N_k \theta \Delta t \frac{\partial b^{(n+1)}}{\partial t} \, \mathrm{d}\Gamma.$$

Next, equations (3.111)–(3.112) are multiplied by $(1 - \theta)\Delta t$ in the time step (n),

$$\int_{\Omega_c} \frac{(1-\theta)\Delta t}{\sigma} (\nabla \times \boldsymbol{W}_k) \cdot \left(\nabla \times \boldsymbol{T}^{(n)} \right) \mathrm{d}\Omega$$

$$+ \int_{\Omega_c} \frac{(1-\theta)\Delta t}{\sigma} \nabla \cdot \boldsymbol{W}_k \, \nabla \cdot \boldsymbol{T}^{(n)} \mathrm{d}\Omega$$

$$+ \int_{\Omega_c} \mu_o (1-\theta)\Delta t \boldsymbol{W}_k \cdot \frac{\partial \boldsymbol{T}^{(n)}}{\partial t} \mathrm{d}\Omega$$

$$- \int_{\Omega_c} \mu_o (1-\theta)\Delta t \boldsymbol{W}_k \cdot \nabla \frac{\partial \Phi^{(n)}}{\partial t} \mathrm{d}\Omega = \qquad (6.15)$$

$$- \int_{\Omega_c} \mu_o (1-\theta)\Delta t \boldsymbol{W}_k \cdot \frac{\partial \boldsymbol{T}_0^{(n)}}{\partial t} \mathrm{d}\Omega$$

$$- \int_{\Omega_c} (1-\theta)\Delta t \boldsymbol{W}_k \cdot \frac{\partial \boldsymbol{R}^{(n)}}{\partial t} \mathrm{d}\Omega,$$

and

$$- \int_{\Omega_c} (\nabla N_k) \cdot \left(\mu_o (1-\theta)\Delta t \frac{\partial \boldsymbol{T}^{(n)}}{\partial t} \right) \mathrm{d}\Omega$$

$$+ \int_{\Omega_c \cup \Omega_n} (\nabla N_k) \cdot \left(\mu_o (1-\theta)\Delta t \nabla \frac{\partial \Phi^{(n)}}{\partial t} \right) \mathrm{d}\Omega$$

$$= \int_{\Omega_c \cup \Omega_n} (\nabla N_k) \cdot \left(\mu_o (1-\theta)\Delta t \frac{\partial \boldsymbol{T}_0^{(n)}}{\partial t} \right) \mathrm{d}\Omega \qquad (6.16)$$

$$+ \int_{\Omega_c} (\nabla N_k) \cdot \left((1-\theta)\Delta t \frac{\partial \boldsymbol{R}^{(n)}}{\partial t} \right) \mathrm{d}\Omega$$

$$+ \int_{\Gamma_B} N_k (1-\theta)\Delta t \frac{\partial b^{(n)}}{\partial t} \, \mathrm{d}\Gamma.$$

Finally, the equations (6.13), (6.15) and (6.14), (6.16) must be added using (6.12),

$$
\int_{\Omega_c} \frac{\theta \Delta t}{\sigma} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \boldsymbol{T}^{(n+1)} \right) d\Omega
$$

$$
+ \int_{\Omega_c} \frac{\theta \Delta t}{\sigma} \nabla \cdot \boldsymbol{W}_k \, \nabla \cdot \boldsymbol{T}^{(n+1)} d\Omega
$$

$$
+ \int_{\Omega_c} \left[\mu_o \boldsymbol{W}_k \cdot \boldsymbol{T}^{(n+1)} - \mu_o \boldsymbol{W}_k \cdot \nabla \Phi^{(n+1)} \right] d\Omega =
$$

$$
- \int_{\Omega_c} \mu_o \boldsymbol{W}_k \cdot \left(\boldsymbol{T}_0^{(n+1)} - \boldsymbol{T}_0^{(n)} \right) d\Omega
$$

$$
- \int_{\Omega_c} \boldsymbol{W}_k \cdot \left(\boldsymbol{R}^{(n+1)} - \boldsymbol{R}^{(n)} \right) d\Omega \tag{6.17}
$$

$$
- \int_{\Omega_c} \frac{(1-\theta) \Delta t}{\sigma} \left(\nabla \times \boldsymbol{W}_k \right) \cdot \left(\nabla \times \boldsymbol{T}^{(n)} \right) d\Omega
$$

$$
- \int_{\Omega_c} \frac{(1-\theta) \Delta t}{\sigma} \nabla \cdot \boldsymbol{W}_k \, \nabla \cdot \boldsymbol{T}^{(n)} d\Omega
$$

$$
+ \int_{\Omega_c} \left[\mu_o \boldsymbol{W}_k \cdot \boldsymbol{T}^{(n)} - \mu_o \boldsymbol{W}_k \cdot \nabla \Phi^{(n)} \right] d\Omega,
$$

and

$$
- \int_{\Omega_c} \left(\nabla N_k \right) \cdot \left(\mu_o \boldsymbol{T}^{(n+1)} \right) d\Omega
$$

$$
+ \int_{\Omega_c \cup \Omega_n} \left(\nabla N_k \right) \cdot \left(\mu_o \nabla \Phi^{(n+1)} \right) d\Omega
$$

$$
= \int_{\Omega_c \cup \Omega_n} \left(\nabla N_k \right) \cdot \mu_o \left(\boldsymbol{T}_0^{(n+1)} - \boldsymbol{T}_0^{(n)} \right) d\Omega
$$

$$
+ \int_{\Omega_c} \left(\nabla N_k \right) \cdot \left(\boldsymbol{R}^{(n+1)} - \boldsymbol{R}^{(n)} \right) d\Omega \tag{6.18}
$$

$$
+ \int_{\Gamma_B} N_k \left(b^{(n+1)} - b^{(n)} \right) d\Gamma
$$

$$
- \int_{\Omega_c} \left(\nabla N_k \right) \cdot \left(\mu_o \boldsymbol{T}^{(n)} \right) d\Omega
$$

$$
+ \int_{\Omega_c \cup \Omega_n} \left(\nabla N_k \right) \cdot \left(\mu_o \nabla \Phi^{(n)} \right) d\Omega.
$$

Different schemes can be realized by selecting θ, if $\theta = 1$, $\theta = 0$, $\theta = 1/2$ or $\theta = 2/3$, the scheme is the *backward Euler scheme*, the *forward Euler scheme*, the *Crank–Nicolson scheme* or the *Galerkin scheme*, respectively. Here, the Galerkin scheme has been used. The other formulations can be formulated in a similar way.

The problem is nonlinear, which must be solved iteratively. The applied *fixed point based algorithm* is the following.

1. Starting from demagnetized state, $\boldsymbol{H} = \boldsymbol{0}$, $\boldsymbol{B} = \boldsymbol{0}$. Initialize variables, $\boldsymbol{T}^{(0)} = \boldsymbol{0}$, $\Phi^{(0)} = 0$, $\boldsymbol{R}^{(0)} = \boldsymbol{0}$, $n = 0$, $k = 1$ (k is the index of fixed point iteration, for short, it is denoted only if necessary). Calculate μ_o using the x component of the isotropic vector hysteresis characteristics;

2. Solve the equations (6.17) and (6.18) in the time step $(n + 1)$;

3. Calculate the magnetic field intensity inside the specimen in the time step $(n + 1)$ by $\boldsymbol{H}^{(n+1)} = \boldsymbol{T}_0^{(n+1)} + \boldsymbol{T}^{(n+1)} - \nabla\Phi^{(n+1)}$;

4. Calculate the magnetic flux density by the direct isotropic vector hysteresis model, $\boldsymbol{B}^{(n+1)} = \mathscr{B}\{\boldsymbol{H}^{(n+1)}\}$;

5. Update the residual term $\boldsymbol{R}_{k+1}^{(n+1)} = \boldsymbol{B}^{(n+1)} - \mu_o \boldsymbol{H}^{(n+1)}$, which is the new value of $\boldsymbol{R}^{(n+1)}$ in the equations (6.17) and (6.18);

6. Repeat from step 2 until the procedure is not convergent and $k \leftarrow k + 1$. The criteria of stopping the sequence 2–5 is

$$\left\| \boldsymbol{R}_{k+1}^{(n+1)} - \boldsymbol{R}_k^{(n+1)} \right\| < \varepsilon_R, \tag{6.19}$$

or

$$\left\| \boldsymbol{H}_{k+1}^{(n+1)} - \boldsymbol{H}_k^{(n+1)} \right\| < \varepsilon_H, \tag{6.20}$$

or

$$\left\| \boldsymbol{B}_{k+1}^{(n+1)} - \boldsymbol{B}_k^{(n+1)} \right\| < \varepsilon_B, \tag{6.21}$$

where ε_R, ε_H and ε_B are predefined error limits, e.g. $\varepsilon_R = 10^{-6}$ and $\|\cdot\|$ is a norm. Here, the following norm has been used:

$$\sqrt{\frac{1}{N_h} \sum_{i=1}^{N_h} \left| \boldsymbol{H}_{k+1,i}^{(n+1)} - \boldsymbol{H}_{k,i}^{(n+1)} \right|}, \tag{6.22}$$

where N_h is the length of vector storing the values of $\boldsymbol{H}$, moreover k is the index of fixed point iteration. In other words, the iteration is stopping if two subsequent solution are close enough to each others. If time step $(n + 1)$ is convergent, then the next time step must be iterated.

Every data is modified by the iterated $\boldsymbol{R}^{(n+1)}$ in time step $(n + 1)$.

The independent excitation currents have been prescribed by the functions

$$i_x(t) = I_x \sin(\omega t + \alpha), \tag{6.23}$$

and

$$i_y(t) = I_y \sin(\omega t + \beta). \tag{6.24}$$

The amplitude and the phase of currents define the polar angle and the amplitude of the magnetic field intensity vector or of the magnetic flux density vector. Controlling the flux or the magnetic field can be worked out by an iterative feedback algorithm. Here the relationship between the currents and the magnetic field intensity is studied at $f = 5\,\mathrm{Hz}$, $f = 50\,\mathrm{Hz}$ and $f = 500\,\mathrm{Hz}$. The conductivity of the specimen is $\sigma = 2 \cdot 10^6\mathrm{S/m}$, the value of μ_o is about $4000\mu_0$ and the peak value of currents is $1\,\mathrm{A}$. The thickness of the sample is $0.5\,\mathrm{mm}$.

In *Fig.* 6.41, the magnetic field intensity has been increased in the x direction then it has been rotated in the counter clockwise direction and only the stationary state of the two orthogonal components of the magnetic field intensity at point $x = y = z = 0$ is plotted.
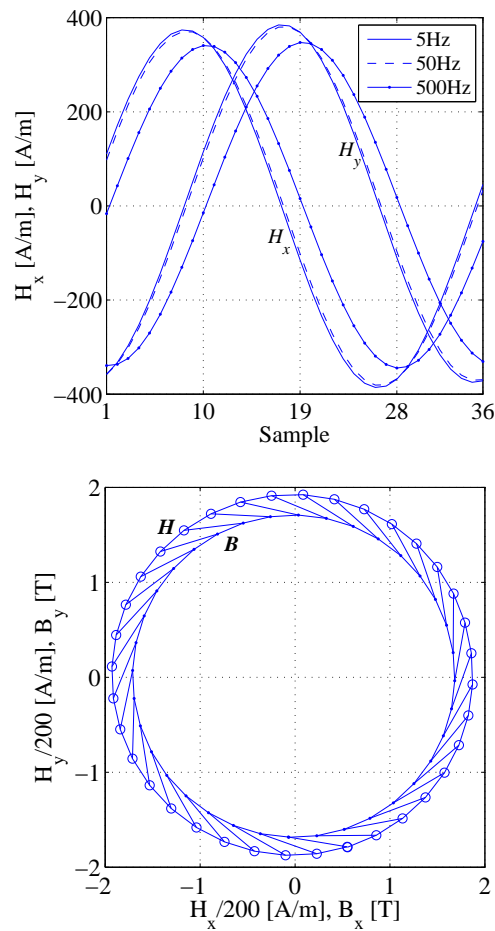
Fig. 6.41. Variation of x and y components of the magnetic field intensity vector and the loci of the magnetic field intensity and the magnetic flux density at the point
$$x = y = z = 0$$

The effect of eddy currents can be sensed at $f = 500\,\text{Hz}$, because the magnetic field has some phase shift and its amplitude becomes smaller, however, the result according to $f = 50\,\text{Hz}$ seems to be almost the same as the results of $f = 5\,\text{Hz}$ calculations. Figure 6.41 shows the rotating magnetic field intensity vector and the magnetic flux density vector when $f = 5\,\text{Hz}$. The effect of hysteresis can be seen in the figure, because the magnetic flux density has some delay, which moreover means that the material is not in the saturation state.

Figure 6.42 shows the time variation of the x and y components of the magnetic field intensity at some points inside the specimen ($f = 50\,\text{Hz}$). The coordinates are given in the legend of the figure in mm and $z = 0$.
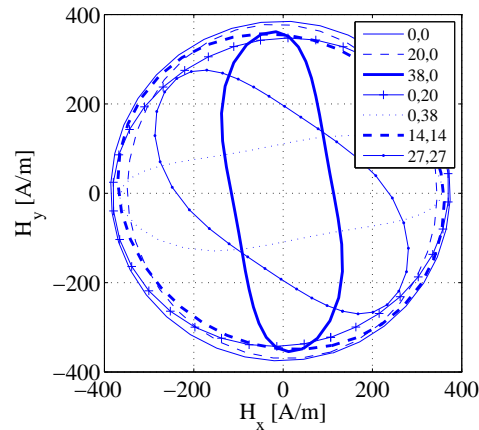
Fig. 6.42. Loci of magnetic field inside the specimen, rotating field

Figure 6.43 shows the loci of the magnetic field intensity vector in linear excitation when $i_x(t) = i_y(t)$. The coordinates are given again in the legend of the figure in mm and $z = 0$.
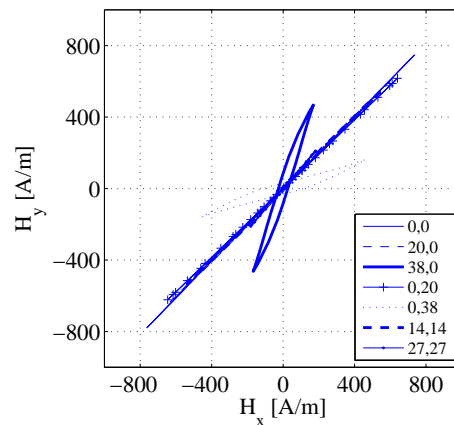

Fig. 6.43. Loci of magnetic field inside the specimen, linear field

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