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Author for correspondence:

Roger Penrose

e-mail: penroad@nexus.ox.ac.uk

Palatial twistor theory and the twistor googly problem

Roger Penrose

Mathematical Institute, Andrew Wiles Building, Woodstock Road, Oxford OX2 6GG, UK

A key obstruction to the twistor programme has been its so-called 'googly problem', unresolved for nearly 40 years, which asks for a twistor description of *right-handed* interacting massless fields (positive helicity), using the same twistor conventions that give rise to *left-handed* fields (negative helicity) in the standard 'nonlinear graviton' and Ward constructions. An explicit proposal for resolving this obstruction—*palatial twistor theory*—is put forward (illustrated in the case of gravitation). This incorporates the concept of a non-commutative holomorphic *quantized* twistor 'Heisenberg algebra', extending the sheaves of holomorphic functions of conventional twistor theory to include the operators of twistor differentiation.

1. Introduction: basic twistor algebra

The central aim of twistor theory is to provide a distinctive formalism, specific to the description of basic physics. Space–time points are taken as *secondary* constructs in the twistor approach. An individual twistor can be regarded as representing the entire history of a free classical massless particle—usually with non-zero spin, of a definite helicity s . If $s=0$, such a history is a *null geodesic*—henceforth called a *ray*—which provides the most immediate picture of what is referred to as a *null twistor*. An individual space–time point is identified in terms of the family of rays through that point. In relativity theory, such a family has the structure of a *conformal* sphere, interpreted in twistor theory as a *Riemann sphere*, i.e. a complex projective line. Twistor theory gains much of its strength from complex geometry and analysis, bringing together many features of relativity and quantum mechanics.

So long as space–time curvature can be ignored, the basic twistor space \mathbb{T} is a complex four-dimensional vector space with pseudo-Hermitian scalar product of split signature $(++--)$. Geometrical notions are often best expressed in terms of the *projective* twistor space \mathbb{PT}

(a complex projective 3-space \mathbb{CP}^3). I use abstract indices, so a twistor \mathbf{Z} (an element of \mathbb{T}) can be written Z^α . Its complex conjugate \bar{Z}_α (i.e. $\bar{\mathbf{Z}}$) is a corresponding element of the *dual* twistor space \mathbb{T}^* , the scalar product between twistors \mathbf{Z} and \mathbf{Y} being written alternatively as

$$\bar{\mathbf{Z}} \bullet \mathbf{Y} = \bar{Z}_\alpha Y^\alpha,$$

and the *norm* of \mathbf{Z} is

$$\|\mathbf{Z}\| = \bar{\mathbf{Z}} \bullet \mathbf{Z}.$$

We say that a twistor is *right-handed*, *left-handed* or *null* according as $\|\mathbf{Z}\| > 0$, $\|\mathbf{Z}\| < 0$ or $\|\mathbf{Z}\| = 0$, the respective portions of twistor space being \mathbb{T}^+ , \mathbb{T}^- and \mathbb{N} , with projective versions \mathbb{PT}^+ , \mathbb{PT}^- and \mathbb{PN} .

Any projective *null* twistor \mathbf{Z} (element of \mathbb{PN}) represents a *ray* (null geodesic) in Minkowski 4-space \mathbb{M} , where for the full correspondence we must include rays at infinity—generators of the ‘light cone at infinity’ in the *compactified* Minkowski space $\mathbb{M}^\#$, a smooth compact conformal-Lorentzian 4-space, topologically $S^1 \times S^3$, which can be identified as the space of lines through the origin in the zero-locus of a quadratic form of signature $(++--)$. The physical interpretation for a projective null twistor $\mathbb{P}\mathbf{Z}$ is thus the world-line of a classical (non-spinning) free massless particle, where idealized particles at infinity are also included. The *orthogonality* condition $\bar{\mathbf{Y}} \bullet \mathbf{Z} = 0$ (or, equivalently, its complex conjugate $\bar{\mathbf{Z}} \bullet \mathbf{Y} = 0$) between null twistors \mathbf{Y} and \mathbf{Z} has the direct space-time interpretation that the corresponding rays \mathbf{y} and \mathbf{z} in $\mathbb{M}^\#$ *intersect* (possibly at infinity). The twistor \mathbf{Z} *itself*, up to the phase freedom $\mathbf{Z} \mapsto e^{i\theta} \mathbf{Z}$ (θ real)—so that a positive real-number measure of scaling is incorporated—provides a *4-momentum* for the massless particle. We shall be seeing later that the *phase* also has a key geometrical role to play in the *palatial* twistor theory introduced in §5.

If the particle possesses a *non-zero spin*, which must be directed parallel or anti-parallel to its velocity—positive or negative *helicity* s , respectively—then its space-time trajectory is not precisely (relativistically invariantly) defined as a world-line, but can be specified in terms of its 4-momentum p_a and 6-angular momentum M^{ab} about some chosen space-time origin point \mathbf{O} . These must be subject to

$$p_a p^a = 0, \quad p_0 > 0, \quad M^{(ab)} = 0, \quad \frac{1}{2} \varepsilon_{abcd} p^b M^{cd} = s p_a$$

(curved or square brackets around indices, respectively, denoting symmetric or antisymmetric parts), where $\varepsilon_{abcd} = \varepsilon_{[abcd]}$ is the Levi-Civita tensor fixed by its component value $\varepsilon_{0123} = 1$ in a right-handed orthonormal Minkowskian frame (with time-axis basis vector δ_0^a , so p_0 is the particle’s energy, in units where the speed of light $c = 1$). The quantity s is the *helicity*, positive for right-handed spin and negative for left-handed.

These quantities are explicitly represented algebraically in terms of a twistor \mathbf{Z} (uniquely defined up to phase) and its complex conjugate dual twistor $\bar{\mathbf{Z}}$. To see this, we need the expression of a twistor \mathbf{Z} in terms of its 2-spinor parts [1,2], where we write

$$\mathbf{Z} = (\omega, \pi) \text{ or, more explicitly, in index form: } Z^\alpha = (\omega^A, \pi_{A'}).$$

The complex conjugate of \mathbf{Z} is the *dual* twistor

$$\bar{\mathbf{Z}} = (\bar{\pi}, \bar{\omega}) \text{ which, in explicit index form is: } \bar{Z}_\alpha = (\bar{\pi}_{A'}, \bar{\omega}^A).$$

A general dual twistor $\mathbf{W} = (\lambda, \mu)$ has the explicit index form $W_\alpha = (\lambda_{A'}, \mu^{A'})$.

A massless particle’s 4-momentum p_a and 6-angular momentum M^{ab} (in 2-spinor abstract-index notation [1]) can be constructed from a twistor’s spinor parts by

$$p_{AA'} = \pi_{A'} \bar{\pi}_A, \quad M^{AA'BB'} = i\omega^{(A} \bar{\pi}^{B)} \varepsilon^{A'B'} - i\bar{\omega}^{(A'} \pi^{B')} \varepsilon^{AB},$$

and all the above conditions are satisfied, provided that $\pi_{A'} \neq 0$. Conversely, the twistor \mathbf{Z} (with $\pi_{A'} \neq 0$) is determined, uniquely up to a phase multiplier $e^{i\theta}$, by p_a and M^{ab} , subject to these

conditions. The helicity s finds the very simple (and fundamental) expression

$$\begin{aligned} 2s &= \omega^A \bar{\pi}_A + \pi_{A'} \bar{\omega}^{A'} \\ &= Z^\alpha \bar{Z}_\alpha \\ &= \|\mathbf{Z}\|. \end{aligned}$$

Under a change of origin $\mathbf{O} \mapsto \mathbf{q}$, the spinor parts of \mathbf{Z} undergo

$$\omega^A \mapsto \omega^A - iq^{AA'} \pi_{A'}, \quad \pi_{A'} \mapsto \pi_{A'},$$

where q^a is the position vector $\overrightarrow{\mathbf{Oq}}$. For a dual twistor \mathbf{W} , with $W_\alpha = (\lambda_A, \mu^{A'})$, we correspondingly have

$$\lambda_A \mapsto \lambda_A, \quad \mu^{A'} \mapsto \mu^{A'} + iq^{AA'} \lambda_A.$$

This is consistent with the standard transformation of M^{ab} (and p_a) under origin change, where the position vector x^a of a space-time point \mathbf{x} correspondingly undergoes

$$x^a \mapsto x^a - q^a.$$

This is consistent also with the *incidence relation*

$$\omega = i\mathbf{x} \bullet \boldsymbol{\pi}, \quad \text{i.e. } \omega^A = i\mathbf{x}^{AA'} \pi_{A'},$$

between a twistor $\mathbf{Z} = (\omega, \boldsymbol{\pi})$ and a space-time point \mathbf{x} , which is the condition for the complex projective line \mathbf{X} (Riemann sphere) represented by \mathbf{x} in \mathbb{PT} , should pass through the point \mathbf{Z} in \mathbb{PT} . In matrix terms, this incidence relation is

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix},$$

where (t, x, y, z) are standard Minkowski space-time coordinates (with $c = 1$) for \mathbf{x} . When \mathbf{Z} is a *null* twistor, the real points \mathbf{x} which are incident with \mathbf{Z} are simply the points that constitute the ray \mathbf{z} in \mathbb{M} (at infinity, in $\mathbb{M}^\#$, if $\boldsymbol{\pi} = 0$). When \mathbf{Z} is non-null, then there are no real points incident with \mathbf{Z} . However, there will be *complex* points—points of the complexified space-time \mathbb{CM} incident with \mathbf{Z} (but if $\boldsymbol{\pi} = 0$, only at infinity, in $\mathbb{CM}^\#$). This will have importance for us in §3.

2. Wave functions for massless particles

Up to this point, we have been concerned with classical theory. *Quantum* twistor theory involves *commutation laws*

$$Z^\alpha \bar{Z}_\beta - \bar{Z}_\beta Z^\alpha = \hbar \delta_\beta^\alpha$$

and

$$Z^\alpha Z^\beta - Z^\beta Z^\alpha = 0, \quad \bar{Z}_\alpha \bar{Z}_\beta - \bar{Z}_\beta \bar{Z}_\alpha = 0$$

[2,3], where now the twistors are taken to be *linear operators* generating a non-commutative algebra \mathbb{A} (with an important role in §§5 and 6), acting on some appropriate space. We shall think of this space as a quantum '*ket-space*' [...] of some kind [4], but it is best not to be specific about this, just now. We could alternatively think of our operators as *dual* twistors, subject to the commutation laws

$$W_\alpha \bar{W}^\beta - \bar{W}^\beta W_\alpha = \hbar \delta_\beta^\alpha$$

and

$$W_\alpha W_\beta - W_\beta W_\alpha = 0, \quad \bar{W}^\alpha \bar{W}^\beta - \bar{W}^\beta \bar{W}^\alpha = 0,$$

which is the same thing as before, but with \bar{Z}_α re-labelled as W_α .

These commutation laws are *almost* implied by the standard quantum commutators for 4-position and 4-momentum

$$p_a x^b - x^b p_a = i\hbar \delta_a^b,$$

but there appears to be an additional input related to the issue of helicity. By direct calculation, we may verify that the twistor commutation laws reproduce exactly the (more complicated-looking) commutation laws for p_a and M^{ab} that arise from their roles as translation and Lorentz-rotation generators of the Poincaré group. In this calculation, there is no factor-ordering ambiguity in the expressions for p_a and M^{ab} in terms of the spinor parts of Z^α and \bar{Z}_α (owing to the symmetry brackets). Yet, the calculation for the helicity s (writing the operator as \mathbf{s}) yields

$$\mathbf{s} = \frac{1}{4}(Z^\alpha \bar{Z}_\alpha + \bar{Z}_\alpha Z^\alpha).$$

In accordance with standard quantum-mechanical procedures, in order to express wave function for massless particles in twistor terms, we need functions of Z^α that are ‘independent of \bar{Z}_β ’. This means ‘annihilated by $\partial/\partial \bar{Z}_\beta$ ’, i.e. *holomorphic* in Z^α (Cauchy–Riemann equations). Thus, a twistor wave function (in the \mathbf{Z} -description) is holomorphic in \mathbf{Z} and the operators representing Z^α and \bar{Z}_α act

$$Z^\alpha \rightsquigarrow Z^\alpha \times \quad \text{and} \quad \bar{Z}_\alpha \rightsquigarrow -\hbar \frac{\partial}{\partial Z^\alpha}.$$

Alternatively, we could be thinking of functions of \bar{Z}_α that are ‘independent of Z^β ’, i.e. *anti-holomorphic* in Z^α . Here, it would be better to re-name \bar{Z}_α as W_α and consider functions *holomorphic* in W_α . Accordingly, in the *dual* twistor \mathbf{W} -description, a wave function must be holomorphic in \mathbf{W} and we have the operators representing \bar{W}^α and W_α , again satisfying the required commutation relations, but now with

$$\bar{W}^\alpha \rightsquigarrow \hbar \frac{\partial}{\partial W_\alpha} \quad \text{and} \quad W_\alpha \rightsquigarrow W_\alpha \times.$$

If we are asking that our wave function describe a (massless) particle of *definite helicity*, then we need to put it into an eigenstate of the *helicity operator* \mathbf{s} , which, by the above, is

$$\mathbf{s} = -\frac{1}{2}\hbar \left(Z^\alpha \frac{\partial}{\partial Z^\alpha} + 2 \right),$$

in the \mathbf{Z} -description, and

$$\mathbf{s} = \frac{1}{2}\hbar \left(W_\alpha \frac{\partial}{\partial W_\alpha} + 2 \right),$$

in the \mathbf{W} -description. These are simply displaced *Euler homogeneity operators*

$$\mathbf{r} = Z^\alpha \frac{\partial}{\partial Z^\alpha} \quad \text{or} \quad \tilde{\mathbf{r}} = W_\alpha \frac{\partial}{\partial W_\alpha},$$

so a helicity eigenstate, with eigenvalue s , in the \mathbf{Z} -description requires a holomorphic twistor wave function $f(\mathbf{Z})$ that is *homogeneous* of degree

$$n = -2s - 2,$$

where I henceforth adopt $\hbar = 1$. Then $2s$ is an integer (odd for a fermion and even for a boson). In the \mathbf{W} -description, the dual twistor wave function $\tilde{f}(\mathbf{W})$ is homogeneous of degree \tilde{n} , where

$$\tilde{n} = 2s - 2.$$

In ordinary space–time terms, the position-space wave function of a massless particle of helicity $2s$ [1] satisfies a *field equation*, expressible in the 2-spinor form

$$\nabla^{AA'} \psi_{AB\dots E} = 0, \quad \square \psi = 0 \quad \text{or} \quad \nabla^{AA'} \tilde{\psi}_{A'B'\dots E'} = 0,$$

for the integer $2s$ satisfying $s < 0$, $s = 0$, or $s > 0$, respectively, where we have *total symmetry* for each of the $|2s|$ -index quantities

$$\psi_{AB\dots E} = \psi_{(AB\dots E)} \quad \text{and} \quad \tilde{\psi}_{A'B'\dots E'} = \tilde{\psi}_{(A'B'\dots E')}.$$

Some of these field equations are more familiar in *tensor* form. In the case $|s| = 1$, we can write (in abstract indices, a pair of capital spinor indices, unprimed and primed, standing for a single tensor index)

$$F_{ab} = \psi_{AB}\varepsilon_{A'B'} + \varepsilon_{AB}\tilde{\psi}_{A'B'},$$

the antisymmetric quantities ε_{AB} and $\varepsilon_{A'B'}$ giving the symplectic structure of the two-dimensional spin spaces, related to the space–time metric by

$$g_{ab} = \varepsilon_{AB} \varepsilon_{A'B'}.$$

Then our field equations become Maxwell’s free-field equations on F_{ab} ($\nabla_{[a}F_{bc]} = 0$, $\nabla^a F_{ab} = 0$). When $|s| = 2$ we write

$$K_{abcd} = \psi_{ABCD} \varepsilon_{A'B'} \varepsilon_{C'D'} + \varepsilon_{AB} \varepsilon_{CD} \tilde{\psi}_{A'B'C'D'},$$

finding that K_{abcd} has the symmetries of a trace-free Riemann (or Weyl) tensor $K_{abcd} = K_{[cd][ab]}$, $K_{[abc]d} = 0$, $K_{abc}{}^a = 0$, satisfying the vacuum Bianchi identities: $\nabla_{[a}K_{bc]de} = 0$. This gives us the weak-field (linearized) Einstein vacuum equations. Moreover, the case $|s| = 1/2$ gives the Dirac–Weyl massless neutrino (or anti-neutrino) equation.

When $|s| = 1$ or $|s| = 2$ (or $|s| = 0$) we can read our equations either classically or quantum mechanically. With quantum wave functions, we would normally demand a positive-frequency condition on the ψ and $\tilde{\psi}$ quantities; then the ψ -part would describe the left-handed (i.e. negative helicity) component of the particle’s wave function, the $\tilde{\psi}$ -part describing the right-handed (positive-helicity) component. On the other hand, we can obtain the *classical* solutions of these field equations if, instead of demanding positive frequency, we demand that F_{ab} or K_{abcd} be *real*, i.e. that the ψ -part and $\tilde{\psi}$ -part be complex conjugates

$$\tilde{\psi}_{A'B'\dots E'} = \bar{\psi}_{A'B'\dots E'}.$$

The part of F_{ab} or of K_{abcd} involving ψ is called the *anti-self-dual* part and that involving $\tilde{\psi}$ the *self-dual* part.

What is the connection between the twistor wave function $f(\mathbf{Z})$, or dual twistor wave function $\tilde{f}(\mathbf{W})$, with these space–time equations? In most direct terms, it is given (for constants k, k') by contour integrals [2]:

$$\psi_{AB\dots E}(\mathbf{x}) = k \oint_{\omega=\mathbf{i}\mathbf{x}\bullet\boldsymbol{\pi}} \frac{\partial}{\partial\omega^A} \frac{\partial}{\partial\omega^B} \cdots \frac{\partial}{\partial\omega^E} f(\boldsymbol{\omega}, \boldsymbol{\pi}) \delta\boldsymbol{\pi}, \quad \text{if } s \leq 0$$

and

$$\psi_{A'B'\dots E'}(\mathbf{x}) = k' \oint_{\omega=\mathbf{i}\mathbf{x}\bullet\boldsymbol{\pi}} \pi_{A'}\pi_{B'} \dots \pi_{E'} f(\boldsymbol{\omega}, \boldsymbol{\pi}) \delta\boldsymbol{\pi}, \quad \text{if } s \geq 0.$$

Here $AB\dots E$ or $A'B'\dots E'$ are $|2s|$ in number, and the 1-form $\delta\boldsymbol{\pi}$ is

$$\delta\boldsymbol{\pi} = \varepsilon^{FG'} \pi_F d\pi_{G'}.$$

The contour lies within the Riemann sphere, in \mathbb{PT} , of twistors $\mathbf{Z} = (\boldsymbol{\omega}, \boldsymbol{\pi})$ satisfying the incidence relation $\boldsymbol{\omega} = \mathbf{i}\mathbf{x}\bullet\boldsymbol{\pi}$ (which removes the $\boldsymbol{\omega}$ -dependence, introducing \mathbf{x} -dependence, and then the integration removes the $\boldsymbol{\pi}$ -dependence leaving us with just \mathbf{x} -dependence). Satisfaction of the field equations is an immediate consequence of these expressions. The 2-form ‘ $d^2\boldsymbol{\pi}' = (1/2) d\delta\boldsymbol{\pi} = d\pi_{0'} \wedge d\pi_{1'}$ ’ is sometimes more appropriate to use, rather than $\delta\boldsymbol{\pi}$, the contour then being two-dimensional, lying in \mathbb{T} rather than \mathbb{PT} . In the *dual* twistor description, there are corresponding expressions.

3. Twistor cohomology

For the above contour integral expressions to give non-zero answers, it is necessary that the holomorphic function f possess *singularities* in appropriate regions. The exact nature of the singularity and contour locations in relation to the domains of non-singularity of the resulting ψ and $\tilde{\psi}$ fields had initially been somewhat puzzling—as had the conformal/Poincaré non-invariance of such domains—until it later became clear [2,5] that twistor wave functions are really to be thought of as elements of *sheaf cohomology*. We shall need to understand just the basics of this. It is easiest in terms of a Čech description and this is the route that I follow here.

The issue is most clear-cut in the case of *wave functions*, and for these we require a condition of *positive frequency*. A convenient description of this is simply that our space–time functions (ψ or $\tilde{\psi}$) extend holomorphically into the *forward-tube domain* \mathbb{CM}^+ , which is the set of *complex* space–time points $x + iy$ (points of \mathbb{CM}) for which the imaginary part y is timelike and *past-pointing*. In twistor terms, complex space–time points are represented by complex projective lines in \mathbb{PT} ; those in \mathbb{CM}^+ by lines entirely within \mathbb{PT}^+ .

Accordingly, for a twistor wave function, we are interested, specifically, in holomorphic functions f whose ‘domain’ is *in some sense* the region \mathbb{PT}^+ . But what sense can this be? In order for the contour integration in the above expressions to work, we require that it have singularities in two disconnected closed subsets of \mathbb{PT}^+ which intersect all the complex lines—these being Riemann spheres—in \mathbb{PT}^+ . The function f is then holomorphic throughout the region \mathbb{PR} that is complementary, within \mathbb{PT}^+ , to these two singular regions. Thus, any complex line \mathbf{R} that lies in \mathbb{PT}^+ (\mathbf{R} representing a complex point \mathbf{r} in \mathbb{CM}^+) intersects \mathbb{PR} , and it does so in an annular way, separating the two regions where f is singular. For any such line \mathbf{R} in \mathbb{PT}^+ , the contour integral is then a loop within this annulus on the Riemann sphere, separating the two regions of singularity.

A more appropriate way to think about this geometry, however, is in terms of a *covering* of \mathbb{PT}^+ by two open sets \mathcal{U}_1 and \mathcal{U}_2 , the function f being taken holomorphic throughout their *intersection* \mathbb{PR} , within which the contour is located:

$$\mathbb{PT}^+ = \mathcal{U}_1 \cup \mathcal{U}_2 \quad \text{and} \quad \mathbb{PR} = \mathcal{U}_1 \cap \mathcal{U}_2.$$

If any function holomorphic throughout \mathcal{U}_1 or else holomorphic throughout \mathcal{U}_2 is added to f , this makes no difference to the result of the contour integral, each such additional contribution adding nothing to the answer, as the contour can be deformed away on one side or the other of the Riemann sphere. We can generalize this to coverings by many open sets $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_m$ with functions f_{ij} ($= -f_{ji}$) defined on intersections $\mathcal{U}_i \cap \mathcal{U}_j$, taken *modulo* functions h_k defined on entire sets \mathcal{U}_k in the manner of Čech (sheaf) cohomology, where refinements of such coverings and direct limits are taken when required. It is not necessary to go into the details of this here, as we shall find that 2-set coverings will be sufficient for our needs. The upshot is that twistor 1-particle wave functions should really be thought of, mathematically, as elements of holomorphic first (sheaf) cohomology on \mathbb{PT}^+ . For more details, see [2,5].

The matter that will concern us particularly, however, is not to do with detailed issues concerning this standard *linear* cohomology, but with the way in which such twistor cohomology ideas can sometimes extend to *nonlinear* generalizations that express basic physical interactions. Most specifically, I shall be concerned with the way that the linear massless fields for spin 2 ($|2s|=4$) can be generalized to nonlinear (vacuum) general relativity, and also to gauge-field interactions of electromagnetism and Yang–Mills theory ($|2s|=2$). The basic constructions, namely those of the nonlinear graviton and the Ward gauge-field construction, were found almost 40 years ago [6–8], but they dealt *only* with the *left-handed* helicity interactions ($s = -2$ and $s = -1$) in standard twistor conventions. I indicate both of these only very briefly in the next paragraph, but we shall come to the gravitational construction in more detail in §4. The *googly problem* has been the issue of extending such procedures to include, also, the *right-handed* helicity in a way that allows both to be taken into consideration simultaneously as part of the same procedure. (It may be noted that ‘googly’ is a term used in the game of cricket for a ball bowled with

right-handed helicity using the apparent action that would normally give rise to left-handed helicity.) This issue will be addressed in §§5 and 6.

The procedures of Čech first cohomology, just referred to, involve taking a covering, by open sets, of the region of twistor space under consideration and regarding a twistor function (or family of functions) as something ‘passive’ that simply resides on the pairwise intersections of the open sets of the cover, the cohomology element provided by this family of functions having no *active* geometrical influence on the twistor space itself. However, under appropriate circumstances such functions can indeed be considered to be playing an active role, such as with the *transition functions* between ‘coordinate patches’ in the piecing together of a curved manifold, or by constructing a non-trivial bundle over a given manifold by gluing together pieces of trivial bundles. These two procedures are indeed what are involved in the nonlinear graviton and Ward constructions, respectively [6–8]. In the next section, I describe (in outline) how this works in the gravitational case.

4. Nonlinear graviton construction

The nonlinear graviton construction provides a twistor representation of a curved complex-Riemannian 4-space \mathcal{M} , where we find a very simple way of ensuring that \mathcal{M} is an Einstein manifold, i.e. $R_{ab} = \Lambda g_{ab}$ (for given cosmological constant Λ). This works only in the case where \mathcal{M} has anti-self-dual Weyl curvature but is otherwise completely general. In standard flat-space twistor theory (see §1), a complex space-time point \mathbf{r} in \mathbb{CM} is represented by a complex *line* \mathbf{R} (a Riemann sphere) in \mathbb{PT} . A very similar situation holds for \mathcal{M} , applying to any small-enough open neighbourhood $\mathcal{V} \subset \mathcal{M}$ of a point $\mathbf{r}_0 \in \mathcal{M}$, and we examine how an arbitrary point \mathbf{r} of \mathcal{V} arises as a corresponding *line* \mathbf{R} (Riemann sphere in an appropriate topological family) in a certain complex 3-manifold \mathbb{PT} , a deformed version of a tubular neighbourhood \mathbb{PP} of a line \mathbf{P} in \mathbb{PT} .

A striking feature of this construction is that there is *no local* curvature information in the ‘curved’ twistor space \mathcal{T} (where \mathbb{PT} is its projective version), despite the fact that there *is* actual local curvature in the space \mathcal{V} that is constructed from \mathbb{PT} . This curvature arises entirely from *global* features of \mathcal{T} . *Locally*, \mathcal{T} would be *identical*, when $\Lambda = 0$, to the twistor space \mathbb{T} of §1. When $\Lambda \neq 0$, it would be locally identical, instead, to the twistor space arising from *de Sitter* 4-space \mathbb{D} with (positive) cosmological constant Λ (or *anti-de Sitter* space if $\Lambda < 0$). In order to clarify this distinction, I need first to be more explicit about the structure that \mathbb{T} actually acquires in order that the space-time *metric* can be encoded into \mathbb{T} ’s structure, in these two cases.

Both in the case of Minkowski space \mathbb{M} and de Sitter space \mathbb{D} (and also in anti-de Sitter space) there are particular antisymmetric 2-valent twistors, fixing the *metric* structure of the space-time, which are referred to as *infinity twistors* [2]. These are $I^{\alpha\beta}$ and $I_{\alpha\beta}$, taken to be both complex conjugates and *duals* of one another:

$$I_{\alpha\beta} = \overline{I^{\alpha\beta}}, \quad I^{\alpha\beta} = \overline{I_{\alpha\beta}},$$

and

$$I_{\alpha\beta} = \frac{1}{2} \varepsilon_{\alpha\beta\rho\sigma} I^{\rho\sigma}, \quad I^{\alpha\beta} = \frac{1}{2} \varepsilon^{\alpha\beta\rho\sigma} I_{\rho\sigma},$$

where $\varepsilon_{\alpha\beta\rho\sigma}$ and $\varepsilon^{\alpha\beta\rho\sigma}$ are Levi-Civita twistors, fixed by their antisymmetry and $\varepsilon_{0123} = 1 = \varepsilon^{0123}$ in standard twistor coordinates. For the infinity twistors, we have, in standard twistor descriptions

$$I_{\alpha\beta} = \begin{pmatrix} \frac{\Lambda}{6} \varepsilon_{AB} & 0 \\ 0 & \varepsilon_{A'B'} \end{pmatrix} \quad \text{and} \quad I^{\alpha\beta} = \begin{pmatrix} \varepsilon^{AB} & 0 \\ 0 & \frac{\Lambda}{6} \varepsilon^{A'B'} \end{pmatrix}.$$

For de Sitter space \mathbb{D} , the infinity twistors provide a *complex symplectic structure* defined by the 2-form:

$$\mathcal{I} = I_{\alpha\beta} dZ^\alpha \wedge dZ^\beta; \quad d\mathcal{I} = 0.$$

Also there is a *symplectic potential* 1-form

$$\mathcal{J} = I_{\alpha\beta} Z^\alpha dZ^\beta, \quad \text{where } \mathcal{I} = d\mathcal{J}.$$

When $\Lambda = 0$, this symplectic structure becomes degenerate, the matrices for $I_{\alpha\beta}$ and $I^{\alpha\beta}$ becoming singular. When $\Lambda \neq 0$, they are essentially inverses of one another:

$$I_{\alpha\beta} I^{\beta\gamma} = -\frac{\Lambda}{6} \delta_\alpha^\gamma,$$

but annihilate each other if $\Lambda = 0$.

To construct \mathcal{T} , we consider our above tubular neighbourhood $\mathbb{P}\mathcal{P}$, of a line \mathbf{P} in $\mathbb{P}\mathbb{T}$. Let us regard the Riemann sphere \mathbf{P} as made up of two slightly extended (open) hemispherical regions whose union is \mathbf{P} and whose intersection is an annular region, as is appropriate for the contour integrals of §2. Each hemispherical region is to be locally thickened out into the ambient complex 3-space to open sets \mathcal{U}_1 and \mathcal{U}_2 , each of Euclidean topology \mathbb{R}^6 , so that

$$\mathbb{P}\mathcal{P} = \mathcal{U}_1 \cup \mathcal{U}_2,$$

the annular intersection $\mathbb{P}\mathcal{R} = \mathcal{U}_1 \cap \mathcal{U}_2$ having topology $S^1 \times \mathbb{R}^5$, a thickened up version of the overlap between the extended hemispheres, as for the contour integrals earlier. The idea is to glue \mathcal{U}_1 to \mathcal{U}_2 in a seamless way, so that the twistor-space complex-manifold structure matches on the overlap and so also does the structure—that I call the *I-structure*—defined by $I^{\alpha\beta}$ and $I_{\alpha\beta}$, for a given value of Λ .

However, as things stand, this *I-structure* is not adequately defined, the infinity twistors being given in relation to the *non*-projective space \mathbb{T} , rather than $\mathbb{P}\mathbb{T}$. We need to phrase things in terms of ‘non-projective’ complex 4-spaces $\mathbb{P}^{-1}\mathcal{X}$, where \mathcal{X} is some subregion of $\mathbb{P}\mathbb{T}$. Here, $\mathbb{P}^{-1}\mathcal{X}$ will be a \mathbb{C}^* -bundle over \mathcal{X} . (I am excluding the origin of \mathbb{T} in these considerations.) The \mathbb{C}^* -fibres of $\mathbb{P}^{-1}\mathcal{X}$ are integral curves of the Euler operator Υ . Thus, the ‘*I-structure*’ of $\mathbb{P}\mathbb{T}$ really refers to \mathcal{T} , the Euler operator Υ being assumed given on $\mathbb{P}^{-1}\mathcal{X}$ for any \mathcal{X} in $\mathbb{P}\mathbb{T}$. We may note, also, that when Υ is applied to (i.e. contracted with) the 2-form \mathcal{I} , we get the 1-form \mathcal{J} , so that, in the presence of Υ , the *I-structure* of any such \mathcal{X} is simply equivalent to the complex symplectic structure \mathcal{I} (degenerate when $\Lambda = 0$) restricted to $\mathbb{P}^{-1}\mathcal{X}$.

We shall first see how to deform $\mathbb{P}\mathcal{P}$ merely *infinitesimally*, where we start with the two regions \mathcal{U}_1 and \mathcal{U}_2 overlapping along $\mathbb{P}\mathcal{R}$ to make $\mathbb{P}\mathcal{T}$, as in §3. We can then obtain a general *finite* small deformation, preserving the *I-structure*, by ‘exponentiating’ the procedure considered at the end of §2 for a *linearized* anti-self-dual gravitational field. There we took a ‘passive’ twistor function $f(\mathbf{Z})$, homogeneous of degree +2, defined throughout the intersection region \mathcal{R} , where $\mathbb{P}^{-1}\mathcal{U}_1$ is glued to $\mathbb{P}^{-1}\mathcal{U}_2$, but we are now to think of $f(\mathbf{Z})$ in an *active* way, telling us how to ‘slide’ the patches $\mathbb{P}^{-1}\mathcal{U}_1$ and $\mathbb{P}^{-1}\mathcal{U}_2$ across each other, preserving the matching of their *I-structures*. Infinitesimally, we can achieve this by performing an infinitesimal shift along the zero homogeneity vector field:

$$I^{\alpha\beta} \frac{\partial f}{\partial Z^\alpha} \frac{\partial}{\partial Z^\beta}.$$

We see, immediately, noting vanishing of the Lie derivative of $I^{\rho\sigma}$ with respect to this vector field (and its zero homogeneity, whereby Υ is also preserved) that this infinitesimal shift preserves the *I-structure*. When exponentiated, this provides us with a space $\mathbb{P}\mathcal{T}$ which encodes a genuinely curved complex 4-space \mathcal{V} .

It turns out that *any* complex-Riemann coformally anti-self-dual complex 4-space solution of the Einstein (Λ)-vacuum equations can be locally described from such a deformed twistor space with a globally defined *I-structure*. How does this work? First, we need the ‘lines’ within a $\mathbb{P}\mathcal{T}$ (obtained by such means). I cannot go into full details here, but theorems of Kodaira and others can be invoked to establish the fact that spaces $\mathbb{P}\mathcal{T}$, constructed in this way (and not deviating too far from the initial $\mathbb{P}\mathcal{P} \subset \mathbb{P}\mathbb{T}$) contain a complex 4-parameter family of ‘lines’ defined solely by the fact that they are imbedded Riemann spheres, with complex structure inherited from the ambient $\mathbb{P}\mathcal{T}$ and belonging to the appropriate topological family, these lines being represented as the *points*

of the space \mathcal{V} . An explicit geometrical procedure involving \mathcal{T} 's I-structure then naturally assigns a complex-Riemannian metric \mathbf{g} to \mathcal{V} [6,8], where \mathbf{g} turns out to be anti-self-dual and Einstein. (The complex-conformal anti-self-dual metric of \mathcal{V} is defined by $\mathbb{P}\mathcal{T}$'s complex-manifold structure, even without \mathcal{T} 's I-structure, simply from the fact that intersecting lines in $\mathbb{P}\mathcal{T}$ correspond to null-separated points in \mathcal{V} [6,8].)

5. Palatial patching and twistor quantization

We have seen that not only can *linearized* vacuum general relativity be encompassed in a natural (albeit somewhat surprising) way by twistor theory, but so also can the *nonlinear* theory—albeit restricted, as yet, to the left-helicity case. This provides some encouragement for the twistor programme, as a means towards describing Nature. Yet, these positive features were profoundly offset by the apparent inability of the theory to resolve the so-called ‘googly problem’, whose resolution ought to allow *both* left- and right-helicities to be twistorially represented together, according to full nonlinear Einstein theory. It is therefore fortunate that a new outlook is to hand, offering genuine hope for a complete solution to this long-standing conundrum.

The key idea behind this altered outlook is to regard the underlying twistor structure as being modelled, in effect, not on the *space* \mathbb{T} but on the non-commutative (holomorphic) twistor *quantum algebra* \mathbb{A} referred to at the beginning of §2. For a (conformally) *curved* space-time, we would have a *deformed* such algebra \mathcal{A} that is, in an appropriate *local* sense, the same as \mathbb{A} , but whose global structure would encode the entire (conformal) geometry of a given curved space-time \mathcal{M} . The algebra \mathbb{A} itself is to be thought of as, in effect, the algebra of linear operators acting on (germs of?) holomorphic entities of some twistorial kind, constituting an imagined ‘ket-space’ [...] envisaged at the right of all these operators. An indication of this is given at the end of this section, but a full understanding of what is required has not yet become altogether clear. (The choice of ket-space is basically the notion, in standard quantum mechanics, of choosing a ‘complete set of commuting variables’. It is closely related to the issue of choosing a ‘polarization’ in geometric quantization [9].) In basic terms, \mathbb{A} is to be taken as the (Heisenberg) algebra generated by Z^α and $\partial/\partial Z^\alpha$, but where *infinite series* in these (non-commuting) operators would also need to be considered as belonging to \mathbb{A} . This raises issues of *convergence* and *locality* that need to be sorted out in due course, but for present purposes I shall ignore these more subtle issues and merely explain the general idea.

As with the left-handed construction outlined in §4, we attempt to build our entire structure from pairs of local pieces, analogous to $\mathbb{P}^{-1}\mathcal{U}_1$ and $\mathbb{P}^{-1}\mathcal{U}_2$, each containing no local information specific to \mathcal{M} , whose intersection seamlessly continues the twistor structure from one to the other. Yet, the *union* $\mathbb{P}^{-1}\mathcal{U}_1 \cup \mathbb{P}^{-1}\mathcal{U}_2$ encodes the (Lorentzian) conformal structure of a local region of \mathcal{M} . When this ‘twistor structure’ appropriately incorporates an I-structure, as in §4, then \mathcal{M} automatically satisfies the Einstein Λ -vacuum equations—completely generally! At least that is the idea.

We are now concerned with patching together two relevant ‘regions’ of the algebra \mathbb{A} , corresponding to two ‘pieces’ like $\mathbb{P}^{-1}\mathcal{U}_1$ and $\mathbb{P}^{-1}\mathcal{U}_2$, rather than simply patching together two pieces of a manifold, as in §4. As yet, it is not altogether clear how this is to be interpreted, as we do not actually have a twistor *space*, like \mathbb{T} or \mathbb{PT} , leading to \mathcal{T} or \mathcal{PT} , above. Instead, we appear to have a form of *non-commutative geometry* [10] arising from some kind of deformation \mathcal{A} of the algebra \mathbb{A} .

Fortunately, for a *real* space-time \mathcal{M} , we can explicitly construct a global (real) 5-manifold \mathbb{PN} analogous to the 5-space \mathbb{PN} and, moreover, a non-projective 7-dimensional version \mathcal{N} analogous to the 7-space \mathbb{N} . The idea is to use \mathcal{N} to provide us with a key part of \mathcal{A} 's structure. The curved \mathbb{PN} is the space whose *points* represent the individual *null geodesics*—or *rays*—in \mathcal{M} . It is best to assume that \mathcal{M} is globally hyperbolic, as this ensures that \mathbb{PN} is a Hausdorff (5-real-dimensional) space [11]. If we include a (null-vector) *momentum* \mathbf{p} (index form p_a), pointing along each ray and parallel-propagated along it—to give what I call a *scaled ray*—then we get a 6-real-dimensional space \mathbb{PN} , whose points represent these scaled rays. In fact, \mathbb{PN} is naturally a *real symplectic*

6-manifold, with symplectic potential 1-form Φ and closed symplectic 2-form Σ , given [2] by

$$\Phi = p_a dx^a, \quad \Sigma = d\Phi = dp_a \wedge dx^a$$

(where, in coordinate notation, ‘ dx^a ’ stands for the coordinate 1-form basis, and in abstract indices dx^a is just a ‘Kronecker delta’ translating the abstract index on ‘ p_a ’ to conventional 1-form notation). When $\mathbb{P}\mathcal{N} = \mathbb{P}\mathbb{N}$, so our symplectic 6-manifold is the canonically given $\mathbb{P}\mathbb{N}(=\mathbb{P}\mathcal{N})$, we can apply twistor notation, finding

$$\Phi = iZ^\alpha d\bar{Z}_\alpha = -i\bar{Z}_\alpha dZ^\alpha, \quad \Sigma = d\Phi = i dZ^\alpha \wedge d\bar{Z}_\alpha.$$

Equality of the two expressions for Φ follows from the nullity of Z^α , since $0 = d(Z^\alpha \bar{Z}_\alpha) = Z^\alpha d\bar{Z} + \bar{Z}_\alpha dZ^\alpha$.

At this point, the idea is to appeal to the procedures of *geometric quantization* [9], according to which a *bundle connection* is provided, the bundle being a *circle bundle* over the symplectic manifold (here $\mathbb{P}\mathcal{N}$) under consideration, the curvature of this connection being the given 2-form Φ . In the present situation, we are fortunate in the fact that the appropriate circle bundle, namely \mathcal{N} , is already to hand, the circle being the *phase freedom* in $\pi_{A'}$, where the momentum null (co-)vector p_a , taken to scale the rays, is spinorially split:

$$p_a = \bar{\pi}_A \pi_{A'}; \quad \text{with phase freedom } \pi_{A'} \mapsto e^{i\theta} \pi_{A'} \quad (\theta \text{ real}).$$

The 2-spinor $\pi_{A'}$ is also taken parallel-propagated along each ray γ —providing a π -scaled ray Γ —the phase freedom applying to each ray as a whole. This defines our required bundle 7-space \mathcal{N} . In fact (up to a twofold sign ambiguity) the phase is geometrically realized as a null 2-plane (‘flag plane’ [1]) through the null-vector direction determined by p^a . In the *flat-space* case, this phase freedom is simply the $Z \mapsto e^{i\theta} Z$ noted in §1, our circle bundle \mathcal{N} over $\mathbb{P}\mathbb{N}$ being \mathbb{N} .

The geometric (pre-)quantization connection [9] is defined by the 1-form $\Phi (= -i\bar{Z}_\alpha dZ^\alpha$, in the canonical flat case). The connection applies to *helicity-charged* fields on $\mathbb{P}\mathcal{N}$. This ‘helicity charge’ refers to a dependence on the phase $e^{i\theta}$ above, and would be determined by the eigenvalue of the homogeneity operator Υ . This connection is consistent with the flat-space replacements

$$\bar{Z}_\alpha \rightsquigarrow -\frac{\partial}{\partial Z^\alpha} \quad \text{and} \quad \frac{\partial}{\partial \bar{Z}_\alpha} \rightsquigarrow Z^\alpha \times,$$

and gives them geometrical meaning for operations *within* \mathcal{N} also for a curved \mathcal{M} .

This does not, however, extend unambiguously to curved analogues of $\mathbb{P}\mathbb{T}^+$ and $\mathbb{P}\mathbb{T}^-$. The idea would be that, although locally possible, such extensions would be essentially *non-unique*, this freedom playing an important role for the palatial patching procedures. The key issue is the *holomorphicity* that the above replacements achieve, eliminating the anti-holomorphic quantity \bar{Z}_α in favour of the holomorphic Z^α throughout. The intention is that by such means, the algebra \mathcal{A} would be maintained as entirely holomorphic, as required.

In the case of flat (or de Sitter) space-time, the above replacements are not only well defined but also fully consistent with the twistor formalism with regard to a notion of Hermitian *inner product* $\langle g|f \rangle$. This applies to twistor functions $f(Z^\alpha)$ and $g(Z^\alpha)$ (actually representatives of first cohomology), being bilinear in $\bar{g}(W_\alpha)$ and $f(Z^\alpha)$, and is, in an appropriate sense, positive definite. Merely for notational reasons in what follows, it will be convenient to express this inner product in terms of a *holomorphic* bilinear scalar product $\{h|f\}$ between functions $h(W_\alpha) = \bar{g}(W_\alpha)$ and $f(Z^\alpha)$:

$$\begin{aligned} \{h|f\} = \langle g|f \rangle &= \oint_{\partial \rightarrow \mathbf{W} \bullet \mathbf{Z} = 0} h(\mathbf{W}) e^{\mathbf{W} \bullet \mathbf{Z}} f(\mathbf{Z}) d\mathbf{W} \wedge d\mathbf{Z} \\ &+ \int_0^\infty e^{-k} \left\{ \oint h(\mathbf{W}) (\mathbf{W} \bullet \mathbf{Z} - k)^{-1} f(\mathbf{Z}) d\mathbf{W} \wedge d\mathbf{Z} \right\} dk, \end{aligned}$$

due to Andrew Hodges (2014, unpublished data; see also [3]), involving two eight-dimensional contour integrations lying in $\mathbb{T}^* \times \mathbb{T}$, the first having a seven-dimensional (S^7) *boundary* in $\mathbf{W} \bullet \mathbf{Z} = 0$, the second being *closed*, where $d\mathbf{W}$ and $d\mathbf{Z}$ are the natural complex 4-volume forms defined on \mathbb{T}^* (by $\varepsilon^{\alpha\beta\rho\sigma}$) and on \mathbb{T} (by $\varepsilon_{\alpha\beta\rho\sigma}$), respectively.

Writing ∂_α for $\partial/\partial Z^\alpha$, we find, from various integrations by parts, that

$$\{h|\partial_\alpha f\} = -\{h|\tilde{\partial}_\alpha f\} = -\{hW_\alpha|f\}.$$

Accordingly, $\partial/\partial Z^\alpha$ and $-W_\alpha$ have identical effects, when inserted between any ‘bra’ $\{\dots|$ and ‘ket’ $|\dots\}$. This justifies the correspondence $\partial/\partial Z^\alpha \leftrightarrow -\tilde{Z}_\alpha$ of §2, since in the Hermitian inner product, the variable W_α is playing the role of \tilde{Z}_α (though independent of Z^α for the purposes of the contour integration).

A similar calculation gives us the relation $\{h|Z^\alpha|f\} = -\{\tilde{\partial}^\alpha h|f\}$ with $\tilde{\partial}^\alpha = \partial/\partial W_\alpha$ (playing the role of $\partial/\partial \tilde{Z}_\alpha$), which at first sight appears to give the correspondence $Z^\alpha \leftrightarrow -\partial/\partial \tilde{Z}_\alpha$, with the *wrong sign*. However, this is a notational confusion, because if we think of the operator $\tilde{\partial}^\alpha$ as being inserted *between* the $\{\dots|$ and the $|\dots\}$ in the expression $\{\tilde{\partial}^\alpha h|f\}$, it would really acting to the *left*, as $\tilde{\partial}^\alpha$; i.e. $\{\tilde{\partial}^\alpha h|f\} = \{h|\tilde{\partial}^\alpha|f\}$. From integration by parts, this is equivalent to $-\tilde{\partial}^\alpha$ inserted between the $\{\dots|$ and the $|\dots\}$, acting towards the *right*.

This is an important point of the notation. In §2, we were thinking of all the quantities Z^α , $\partial/\partial Z^\alpha$, \tilde{Z}_α , $\partial/\partial \tilde{Z}_\alpha$, as acting on what follows to the *right* of them (ultimately on some unspecified *ket*-space $|\dots\rangle$), and the sign in the commutation relations in §2 is dependent on this. Here, we are taking of the *ket*-space as being represented by holomorphic first cohomology in the Z^α variables. Accordingly, we indeed get consistency also with the correspondence $Z^\alpha \leftrightarrow +\partial/\partial \tilde{Z}_\alpha$ of §2. The bra space $\{\dots|$ would be represented by *anti*-holomorphic cohomology, in term of the conjugate \tilde{Z}_α variables, which would be consistent with the reverse operator ordering throughout.

6. Local twistors, palatial algebra and Einstein’s equations

When it comes to the *deformed* algebra \mathcal{A} needed for a curved space–time, we have to allow that no *specific* *ket*-space $|\dots\rangle$ be singled out globally to represent the algebra, as in §5, but that we might have different such representations, each defined only ‘locally’, in some sense, but where the ‘Heisenberg algebra’ would be independent of the particular $|\dots\rangle$ representation, and isomorphic, locally, to that provided at the end of §5, namely \mathbb{A} . A central question arises here, concerning the appropriate meaning of ‘local’ in this context. Although these issues are not yet fully resolved, we *do* have the (Hausdorff) space \mathcal{N} defined *globally*, and we have a well-defined notion of local neighbourhoods within \mathcal{N} .

In fact, assigned to each *single point* of $\mathbb{P}\mathcal{N}$, i.e. to each ray γ in \mathcal{M} (and therefore also to each point Γ of \mathcal{N} , providing a π -scaling for γ), there is a canonically and conformally invariantly defined flat twistor space \mathbb{T}_γ , which plays a role of a kind of ‘complex cotangent space’ to \mathcal{N} at Γ . This is obtained from the notions of a *local twistor*, and *local twistor transport* [2]. A local twistor is a quantity $Z^\alpha = (\omega^A, \pi_{A'})$, defined at a point \mathbf{q} of the space–time \mathcal{M} , which transforms as

$$\omega^A \mapsto \omega^A, \quad \pi_{A'} \mapsto \pi_{A'} + i\omega^A \nabla_{AA'} \log \Omega,$$

under a rescaling of \mathcal{M} ’s metric, according to $g_{ab} \mapsto \Omega^2 g_{ab}$. Since a local twistor is a quantity that can be assigned separately to each point of \mathcal{M} , we need to be careful when relating this concept to the notion of twistor used earlier, in §§1–3, where the 2-spinor pair $(\omega^A, \pi_{A'})$ referred *globally* to the (flat) space–time \mathbb{M} , assigning coordinates to the complex 4-space \mathbb{T} that is associated with \mathbb{M} . The notation in this section does correspond exactly to that of §§1–3, but only if we think of \mathbf{q} as at the particular space–time origin point \mathbf{O} .

Recall from §1 that, when the origin is displaced to a *general* point \mathbf{q} (whose position vector from \mathbf{O} is q^a), the twistor $(\omega^A, \pi_{A'})$ defined with respect to \mathbf{O} becomes $(\omega^A - iq^{AA'} \pi_{A'}, \pi_{A'})$ when defined with respect to \mathbf{q} . The local twistor perspective used here is that $(\omega^A, \pi_{A'})$, defined at the point \mathbf{O} , when carried to \mathbf{q} by *local twistor transport*, becomes $(\omega^A - iq^{AA'} \pi_{A'}, \pi_{A'})$, defined at \mathbf{q} . When $\mathcal{M} = \mathbb{M}$ (or, indeed, when \mathcal{M} is any simply connected conformally flat space–time) the notion of local twistor transport is *path-independent* but this is not true in general.

The definition of local twistor transport along a smooth curve γ in \mathcal{M} with tangent vector t^a is

$$t^a \nabla_a \omega^B = -i t^{BB'} \pi_{B'}, \quad t^a \nabla_a \pi_{B'} = -i t^{AA'} P_{AA'BB'} \omega^B,$$

where

$$P_{ab} = \frac{1}{12} R g_{ab} - \frac{1}{2} R_{ab}, \quad \text{with } R_{ac} = R_{abc}{}^b.$$

Taking γ to be a ray—which is simply connected, with topology \mathbb{R} (by \mathcal{M} 's global hyperbolicity)—we use local twistor transport to propagate (ω, π) uniquely all along γ , thereby providing us with our canonical twistor space \mathbb{T}_γ , assigned to γ . Correspondingly, we shall have spaces \mathbb{PT}_γ , \mathbb{N}_γ and \mathbb{PN}_γ , just as in §1.

We ask what the relation between each \mathbb{PN}_γ and the global space \mathbb{PN} of rays in \mathcal{M} might be. Within each \mathbb{T}_γ , the ray γ , when π -scaled to Γ , can itself be unambiguously represented by $(0, \pi_{A'})$ all along γ , this being unchanged by local twistor transport along γ (since $t^{AA'} \propto \bar{\pi}^A \pi^{A'}$ and $\pi^{A'} \pi_{A'} = 0$). When \mathcal{M} is *conformally flat* (and simply connected), the integrability of local twistor transport allows us to achieve this globally for the whole of \mathbb{PN} , where a π -scaled ray η in \mathcal{M} that meets γ in a point \mathbf{q} would be represented at \mathbf{q} by the local twistor $(0, \eta^{A'})$, in both \mathbb{T}_γ and \mathbb{T}_η , where $\eta^{A'}$ provides the direction and π -scaling for η . In fact the spaces $\mathbb{N}_\gamma - \{0\}$ are all canonically isomorphic with each other, and (locally) with \mathcal{N} itself.

On the other hand, in the general case, the *non*-integrability of local twistor transport prevents such a global twistorial representation of \mathcal{N} in this way. Instead, the idea of palatial twistor theory is that the holomorphic twistor *Heisenberg* algebras associated with each \mathbb{T}_η , though not necessarily canonically identified with one another locally, would constitute a kind of \mathbb{A} -bundle \mathcal{B} over \mathcal{N} that is *holomorphic* in an appropriate sense. Such holomorphicity would *not* arise simply from the normal geometrical structures in \mathcal{M} , in the presence of *conformal curvature*, because the complex structure of \mathbb{PT} is interpreted, in \mathbb{M} , in terms of the vanishing of *shear* in ray congruences [2], whereas the Weyl tensor provides the measure of *change* of shear along rays. Accordingly, when \mathcal{M} is conformally curved, we do not get a natural complex structure for \mathcal{N} (or, rather, natural CR-structure, like the one that \mathbb{N} inherits from \mathbb{T}). In palatial twistor theory, we obtain a holomorphic structure by removing all places where complex conjugates arise in the normal geometrical procedures and translate them into holomorphic procedures in accordance with the twistor quantization rules of §2. Most particularly, in accordance with the above relations for local twistor transport, and the twistor quantization rules, we have (like the local twistor transport equations for $(\bar{\pi}_A, \bar{\omega}^{A'})$, as the complex conjugate of $(\omega^A, \pi_{A'})$)

$$t^a \nabla_a \left\{ \frac{\partial}{\partial \pi_{B'}} \right\} = i t^{BB'} \frac{\partial}{\partial \pi_{B'}} \quad \text{and} \quad t^a \nabla_a \left\{ \frac{\partial}{\partial \omega^B} \right\} = i t^{AA'} P_{AA'BB'} \frac{\partial}{\partial \pi_{B'}}.$$

Each of the twistor spaces \mathbb{T}_γ will have its corresponding quantum algebra \mathbb{A}_γ , but we have no reason to expect a canonical isomorphism between these algebras for different rays γ , when \mathcal{M} is conformally curved. However, I would expect that we *can*, for a topologically and holomorphically trivial open region \mathcal{U} of \mathcal{N} —that I refer to as a *simple* region—*deform*, continuously and holomorphically, the various \mathbb{A}_γ s so as to obtain a holomorphic ‘trivialization’ of the portion of \mathcal{B} lying above \mathcal{U} , thereby obtaining (by no means uniquely) a single algebra $\mathcal{A}_\mathcal{U}$, continuously/holomorphically isomorphic to each \mathbb{A}_γ for $\gamma \in \mathbb{PU}$ (and therefore isomorphic to \mathbb{A}) appropriately consistently over the whole of \mathcal{U} . This notion of ‘consistency’ should demand that there be a consistent ‘ket-space’ $|\dots\rangle$ for the entire region \mathcal{U} . The point is that ‘locally’—in the sense of over *any* simple region \mathcal{U} of \mathcal{N} —we can obtain $\mathcal{A}_\mathcal{U}$, isomorphic to \mathbb{A} , with a consistent ket-space, but this consistency would not, in general, be possible globally.

Each such ‘trivial’ $\mathcal{A}_\mathcal{U}$ is to be thought of in the spirit of a ‘coordinate patch’. Over the intersection $\mathcal{U}_i \cap \mathcal{U}_j$ of two simple regions \mathcal{U}_i and \mathcal{U}_j we require consistency of the algebras $\mathcal{A}_{\mathcal{U}_i}$ and $\mathcal{A}_{\mathcal{U}_j}$ in the sense of having a continuous/holomorphic deformation of one to the other, but we do not require a common ket-space to be present. Our global notion of the quantum twistor algebra \mathcal{A} for \mathcal{N} , and assigned therefore to the whole of \mathcal{M} , is obtained by taking a covering $\{\mathcal{U}_i\}$ of \mathcal{N} by simple regions, with consistency over all (multiple non-trivial simple) intersections of the

whole of \mathcal{N} of π -scaled rays in \mathcal{M} . Our requirement is that \mathcal{A} restricts to *simple* subsets \mathcal{U} , of \mathcal{N} in the way just described above. This algebra \mathcal{A} is, of course, to be taken in entirely abstract sense, the specific $\mathcal{A}_{\mathcal{U}}$ representations obtained from *particular* coverings having no special relevance to \mathcal{A} 's structure.

In order to identify the points of \mathcal{M} in terms of the algebra \mathcal{A} , we cannot do this by means of intersections of rays, because this condition, given in §1 as $\tilde{\mathbf{Y}} \bullet \mathbf{Z} = 0$, involves a complex conjugation and does not directly survive within the local structure of \mathcal{A} . Instead, the points of \mathcal{M} have to arise by *non-local* considerations (as was the case with the nonlinear graviton construction of §4). Corresponding to any particular point \mathbf{r} of \mathcal{M} there would be a locus \mathbf{R} in $\mathbb{P}\mathcal{N}$ representing \mathbf{r} , namely the family of all rays through \mathbf{r} , topologically S^2 ($\mathbb{P}^{-1}\mathbf{R}$ acquiring a structure $\mathbb{C}^2 - \{0\}$ in \mathcal{N}), the idea being that the consistency (i.e. trivialization, in the above strong sense of having a consistent ket-space) of the \mathbb{A} -bundle over \mathbf{R} is what determines such an S^2 locus as representing a point of \mathcal{M} . For this to have a chance of working, as a sufficiently restrictive proposal for locating \mathcal{M} 's points in terms of such a twistorial construction, we must ensure that the bundle \mathcal{B} is, indeed, in an appropriate sense *holomorphic*, so that the rigidity of holomorphicity can be appealed to.

It does not seem unreasonable to me that \mathcal{M} , together with its conformal structure, would be obtainable in this kind of way. Although the nonlinear graviton construction, as outlined in §4, applies to complex space-times, so that the role of the space \mathcal{N} is not directly applicable, one can nevertheless see that in a complexified form the current construction reduces to it, in a degenerate case. The same would apply to the dual construction, where we must now employ a ket-space of *dual* twistor variables, this providing what has been referred to as a *googly* nonlinear graviton.

Yet, there is considerable mathematical speculation and conjecture in the descriptions above, aimed at the non-local coding of the structure of a general (presumably analytic) Lorentzian globally hyperbolic real 4-manifold \mathcal{M} . There are many mathematical issues that need to be sorted out, not the least being a need for some suitable generalization of the Kodaira theorem [6] that is central to the construction of §4. Moreover, none of this encodes the formulation of Einstein's equations.

It is perhaps remarkable, therefore, to find that Einstein's Λ -vacuum equations are themselves very simply encoded into this structure. For these equations provide *precisely* the necessary and sufficient condition that the local twistor spaces \mathbb{T}_{γ} possess an **I**-structure (for given Λ), as defined in §4 (being constant under local twistor transport) and we now require that all the needed continuous/holomorphic deformations of the $\mathcal{A}_{\mathcal{U}}$ algebras preserve their nature as algebras on \mathbb{T}_{γ} with this **I**-structure. If all these procedures (or something like them) indeed work as intended, with generalizations to the Yang–Mills equations and other aspects of physics, then there would appear to be significant openings for twistor theory in basic physics, not envisaged before.

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