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A Numerical Solution of a Model for a Superconductor Field Problem

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A model of a magnetic field problem occurring in connection with Josephson junction devices is derived, and numerical solutions are obtained. The model is of mathematical interest, because the magnetic vector potential satisfies inhomogeneous Helmholtz equations in part of the region, i.e., the superconductors, and the Laplace equation elsewhere. Moreover, the inhomogeneities are the gauge constants for the potential, which are different for each superconductor, and their magnitudes are proportional to the currents flowing in the superconductors. These constants are directly related to the self and mutual inductances of the superconducting elements in the device. The numerical solution is obtained by the iterative use of a fast Poisson solver. Chebyshev acceleration is used to reduce the number of iterations required to obtain a solution. A typical problem involves solving 100,000 simultaneous equations, which the algorithm used with this model does in 20 iterations, requiring three minutes of *CPU* time on an *IBM VM*/370/168. Excellent agreement is obtained between calculated and observed values for the inductances.

1. INTRODUCTION

Josephson junction circuits are being built at the IBM Thomas J. Watson Research Center for use in high speed computers. An illustration of a prototype device [1] is shown in Fig. 1. The shaded areas are insulators, in general *SiO*, while the clear regions represent superconductors [3]. The control lines, with current I_c, are cut away in the drawing, but actually go over the top of the device. The depressed areas are where the junctions are located. In the design of such a device, one is interested in the distribution of the magnetic fields arising from the currents flowing in the superconductors, and in particular, the inductances, both self and mutual, associated with the elements of the device.

This paper describes a mathematical model of such a device, and the numerical solution of the model. The currents which determine the fields and inductances flow primarily through those superconducting portions of the device which do not comprise the actual junction, since the lengths of the junctions are small compared to the overall lengths of the superconductors, (approximately 1:15). Therefore as a first approximation, it is reasonable to take a cross section perpendicular to the current flow in the device, and to model the device as being two dimensional.

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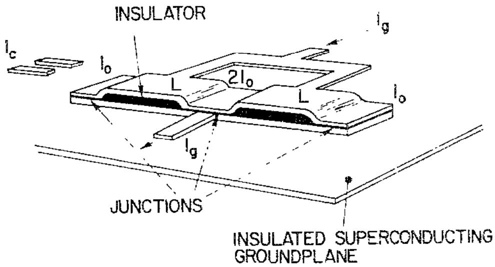

FIG. 1. An example of an interferometric device formed from three Josephson junctions. Clear areas marked L are superconductors. Shaded areas are insulators.

The permeability of the insulating materials is very close to that of vacuum. Therefore we take as our model a discrete number of superconducting rectangles in a vacuum bounded on the bottom by a superconducting ground plane. (However the method described in this paper is not limited to rectangular shapes.)

In the next section, we derive the model consisting of the basic equations which apply in the superconductors and the vacuum, coupled with various constraints. In the third section, the method of numerical solution is presented. Comparisons of the results with measured values and approximate theories are made in the last section. A more detailed report [2] contains an existence and uniqueness proof for the solution and an estimate of the numerical error. An appendix to this report gives the details of the scaling used, which results in a convenient form for the equations and yields the output in the desired units. Another appendix to this report considers a related problem, concentric superconducting annuli, which can be solved analytically.

2. Theory

A. *Basic Equations*

As discussed in the introduction, we seek to solve a two dimensional problem in which the upper half plane is a vacuum with the exception of several rectangular superconductors, and where the lower half plane is a superconducting material. Fig. 2

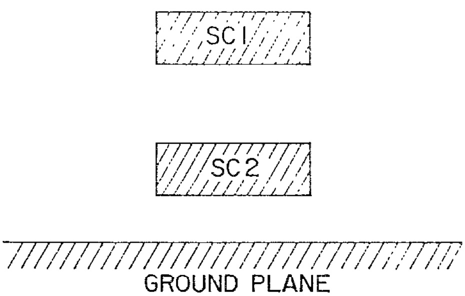

FIG. 2. A two-dimensional model of Fig. 1 in the region away from the junctions. Two rectangular superconductors surrounded by vacuum are underlain by a superconducting ground plane.

is an example of a possible configuration. The y coordinate is chosen to be perpendicular to the ground plane and directed upward, the x coordinate is directed to the right, and the origin lies on the boundary between the two half planes, as shown in Fig. 2. For a right handed coordinate system, the z coordinate is directed positively out of the page. Electric currents are assumed to flow through the rectangular superconductors, and, to have a return flow through the ground plane. The current density, $\mathbf{j}_i$, in the ith superconductor will be directed in the z direction and will be a function of x and y. From the law of Biot-Savart, the magnetic field will have x and y components only, which will be functions of x and y only.

Maxwell's equations apply both in the vacuum and in the superconductors. In c.g.s. units, the equations of interest are:

$$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{j} \tag{2.1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.2}$$

where $\mathbf{H}$ is the magnetic field, $\mathbf{B}$, the magnetic induction, $\mathbf{j}$, the current density, and c, the velocity of light. In both the superconductors and vacuum

$$\mathbf{B} = \mathbf{H},$$

and (2.2) may be replaced by

$$\nabla \cdot \mathbf{H} = 0. \tag{2.3}$$

Therefore, from Helmholtz's theorem, $\mathbf{H}$ may be obtained from a vector potential $\mathbf{A}$.

$$\mathbf{H} = \nabla \times \mathbf{A}. \tag{2.4}$$

Since $\mathbf{A}$ has only a z component and is independent of z, it follows that

$$\nabla \cdot \mathbf{A} = 0. \tag{2.5}$$

In the superconductors, London's equation

$$\nabla \times \mathbf{j} = \frac{-c}{4\pi\lambda^2}\mathbf{H} \tag{2.6}$$

also applies, where λ is the London penetration depth, possibly different for different superconductors.

If we introduce dimensionless space and time variables $(\bar{x}, \bar{y}, \bar{t})$:

$$\bar{x} = x/\lambda;\ \bar{y} = y/\lambda;\ \bar{t} = t/(4\pi\ \lambda/c), \tag{2.7}$$

equations (2.1) and (2.6) reduce to

$$\nabla \times \mathbf{H} = \mathbf{j} \tag{2.8}$$

and

$$\nabla \times \mathbf{j} = -\mathbf{H}, \tag{2.9}$$

as shown in Ref. 2. This particularly simple form applies only if λ is the same for all superconductors. Equations (2.3) and (2.4) still apply with $\mathbf{A}$ defined in terms of the new units. Taking the curl of (2.4) and using the vector relation:

$$\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2\mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$$

together with (2.5) and (2.8), yields

$$\nabla^2\mathbf{A} = -\mathbf{j}, \tag{2.10}$$

where both $\mathbf{A}$ and $\mathbf{j}$ have z components only. A comparison of (2.4) and (2.9) indicates that in a superconductor

$$\mathbf{j} = -\mathbf{A} + \nabla\phi, \tag{2.11}$$

where $\nabla\phi$ is the gradient of an arbitrary scalar field. Since both $\mathbf{j}$ and $\mathbf{A}$ have only a z component and are functions only of x and y, $\nabla\phi$ must have the same properties. Applying the curl operation to $\nabla\phi$, we obtain

$$\frac{\partial \nabla\phi}{\partial y} = -\frac{\partial \nabla\phi}{\partial x} = 0.$$

This implies that $\nabla\phi$ is a constant which we denote by X. Equation (2.11) is the gauge relation between $\mathbf{j}$ and $\mathbf{A}$. Substituting (2.11) in (2.10) yields

$$\nabla^2 A - A = X, \tag{2.12}$$

where we now consider the z component of A as a scalar. Equation (2.12) describes the behavior of A in a superconductor. In the vacuum where $\mathbf{j} = 0$, (2.10) becomes

$$\nabla^2 A = 0. \tag{2.13}$$

In problems which arise in practice, different superconductors have different penetration depths. Since they are of the order of 0.1 micrometer, we have chosen this value for λ in (2.7). It is therefore necessary to modify (2.11) and (2.12). In the ith superconductor, one obtains, as shown in the appendix of Ref. [2],

$$j_i = -\frac{1}{(\lambda_i)^2} A + X_i \tag{2.14}$$

and

$$\nabla^2 A - \frac{1}{(\lambda_i)^2} A = X_i , \tag{2.15}$$

where λ_i is the penetration depth of the ith superconductor expressed in tenths of a micrometer.

B. *Boundary Conditions*

We will first consider the appropriate boundary conditions in the absence of the ground plane. Assume that we have a single superconductor in which a known current I flows. Then from (2.8) and Stoke's formula

$$\oint \mathbf{H} \cdot \mathbf{d}l = \iint \mathbf{j} \cdot \mathbf{dS} \tag{2.16}$$

where the line integral is taken about a circle with radius R and centered at the central point of the superconductor. S is the planar surface contained in the circle. Let I be the total current flowing through S. If we assume that R is sufficiently large so that $\mathbf{H}$ may be assumed independent of azimuth, this leads to

$$I = 2\pi RH + O(R^{-3}). \tag{2.17}$$

From (2.4) the vector potential, A, must be

$$A = -\frac{I}{2\pi} \ln R + O(R^{-2}). \tag{2.18}$$

(Here the remainder is $O(R^{-2})$ because of the symmetry of the rectangle and the fact that the origin is chosen at the center of the rectangle. In the more general case, as described in the estimate of numerical error in Ref. [2], $O(R^{-2})$ would be replaced by $O(R^{-1})$.) If we have n superconductors, the effect on the far field is

$$A = -\sum_{i=1}^{n} \frac{I_i}{2\pi} \ln R_i + O(R^{-2}), \tag{2.19}$$

where R_i is calculated from the center of the ith superconductor. In doing a numerical calculation, a trade-off must be made in that the far field boundary must be far enough away so that (2.19) is a good approximation, without having the problem (i.e. domain) become too large.

The magnetic field is continuous across the boundary between a superconductor and the vacuum. This is equivalent to saying that A and its normal and tangential derivatives are continuous across the boundary.

Finally, we consider the effect of the ground plane. As an approximation, we assume that the London penetration depth for the ground plane is zero. This means that no field penetrates into the ground plane; there can be no normal component of **H** anywhere along the ground plane. In this case the surface of the ground plane may be treated as a mirror. Images of the superconductors in the upper half plane are mapped into the lower half plane (see Fig. 3), and currents of opposite sign are imagined to flow in them. This treatment of the ground plane was adopted to limit the grid size. A more recent version of the program permits the ground plane to be treated as another superconductor with a non-zero penetration depth. The image supercon-

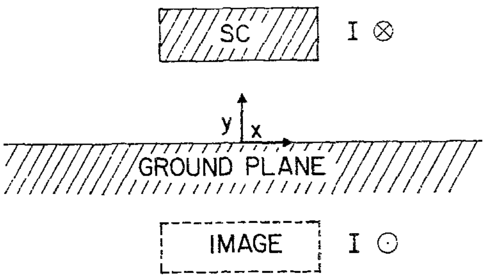

FIG. 3. The effect of the ground plane is approximated by placing a mirror image of the superconductor in the ground plane. In the image the current, I, flows in the opposite direction.

ductors will make additional contributions to the potential A in the upper half plane, which are equivalent to that caused by the actual current distribution in the ground plane. Equation (2.19) is now replaced by:

$$A = -\sum_{i=1}^{n} \frac{I_i}{2\pi} \ln R_i + \sum_{i=1}^{n} \frac{I_i}{2\pi} \ln R_i' , \tag{2.20}$$

where the R_i' are measured from the center of the image of the ith superconductor.

C. *Significance of the Constant in Equation* (2.11) - *Fluxoid*

London[3] has demonstrated that the quantity

$$\Phi_c = \oint \left(\mathbf{A} + \frac{4\pi}{c} \lambda^2 \mathbf{j}\right) \cdot \mathbf{d}l, \tag{2.21}$$

which he calls a fluxoid is constant in time. Furthermore, in a simply connected superconductor, it is identically equal to zero. Only in a multiply connected region [3], in which the line integral can be taken around a hole in the interior of the superconductor is it non-zero. The first part of the integral is equivalent to the flux flowing through the surface bounded by the loop, and often, when j is small, the flux and the fluxoid are equivalent.

Equation (2.21) is in c.g.s. units. It is shown in Ref. 2 that in our units (2.21) becomes

$$\Phi_c = \oint (\mathbf{A} + \lambda_i^2 \mathbf{j}) \cdot \mathbf{d}l, \tag{2.22}$$

where λ_i is defined at the end of Section 2A. Comparison with (2.14) shows that the constant $\lambda_i^2 X_i$ is the integrand of the fluxoid. X is directed in the z direction. If a closed loop is chosen within a long cylindrical superconducting bar, the integral of $\lambda_i^2 X_i$ will equal zero, since $\lambda_i^2 X_i$ is a constant, in agreement with London's prediction for a simply connected region. Consider, however, Fig. 4, which shows a long superconducting bar carrying current in the positive direction, with a return flow in the image bar. The image constant, X', will be the negative of X. Therefore the net fluxoid about the hole will be non-zero. $\lambda^2 X$ can be thought of then as the fluxoid per unit length.

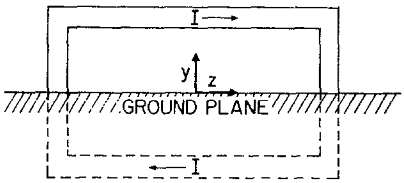

FIG. 4. Extension of Fig. 3 to a three-dimensional case. Current is assumed to flow through a long length of the superconductor in the z direction, and then to return through the image in the opposite direction. The effect of the current flow in the end regions is ignored in calculating the fluxoid.

In practical units, the energy required to establish a flux through a circuit, in which a current I flows, is:

$$\tfrac{1}{2}\Phi_c I = \tfrac{1}{2} L I^2$$

where L, the coefficient of self inductance, is equal to Φ/I, by definition. London [3] shows that a similar energy term equal to $1/2c\ \Phi_c I$ arises in superconductors. c appears here as a consequence of the units. By direct analogy to the flux case, we may define a coefficient of self inductance equal to the fluxoid divided by the current. The self inductance per unit length is therefore proportional to the constant X.

In cases involving more than one superconductor, a problem can be solved with current flowing in only one superconductor. In this manner, mutual inductances can be obtained from the constants in the non-net current carrying superconductors, since they are proportional to the fluxoids generated in these superconductors by the current flowing in the remaining superconductor.

D. *Statement of the Problem*

The problem to be solved is the following: Find A, the vector potential of the magnetic field, where A satisfies a Helmholtz equation, expression (2.15), in the superconductors and a Laplace equation, expression (2.13), in the vacuum, given Dirichlet conditions, expression (2.20), at the far boundaries, and continuity of A and its derivatives across a boundary between superconductor and vacuum. Application of expression (2.20) at the ground plane yields $A = 0$. We derive some auxiliary conditions involving X which will be useful. These conditions are a result of the continuity of A at the superconductor boundaries. In order to obtain a unique solution for A, we must know the total current flowing in each superconductor. If I_k is the total current flowing in the kth superconductor, then

$$I_k = \iint j_k \, dS_k \,, \tag{2.23}$$

the integral of the current density over the area of the kth superconductor. Substituting (2.14) in (2.24) yields

$$X_k = \left(I_k + \frac{1}{(\lambda_i)^2} \iint A \, dS_k\right) \Big/ S_k \,, \tag{2.24}$$

where S_k is the area of the kth superconductor. If we have n superconductors, we then have n auxiliary conditions of the type given in (2.24).

3. Numerical Solution

We now approximate (2.13) and (2.15) by difference equations using the standard five point difference formula for the Laplacian operator. Letting i be the index in the x direction, j, in the y direction, Δx and Δy, the mesh increments in the x and y directions respectively, we obtain

$$\frac{A_{i+1,j} - 2A_{i,j} + A_{i-1,j}}{(\Delta x)^2} + \frac{A_{i,j+1} - 2A_{i,j} + A_{i,j-1}}{(\Delta y)^2} = K\left(\frac{1}{(\lambda_k)^2} A_{i,j} + X_k\right), \tag{3.1}$$

where $K = 0$ in vacuum, $K = 1$ in the interior of the kth superconductor, $K = 1/4$ at corner points of a superconductor boundary, and $K = 1/2$ at all other points on the boundary of a superconductor. Essentially we are taking an average of the Laplace and Helmholtz equations with weights determined by the relative amounts of vacuum and superconductor contained in the four grid squares having the point (i, j) as a common corner point. This averaging procedure causes the numerical method to be consistent with the requirement of continuity of A and of its derivatives across boundaries.

The domain of the problem is chosen to be a large rectangle containing the superconducting regions. Along the boundary of this rectangle the Dirichlet conditions (2.19) or (2.20) are used with A replaced by the appropriate $A_{i,j}$ at each mesh point on the boundary.

Finally the side conditions (2.24) are discretized by replacing the integral which appears there by a numerical approximation of second order accuracy.

To solve these equations we utilize a so called fast solver. If K were everywhere zero, a fast Poisson solver of the type described by Hockney [4] could be used, or if K were constant everywhere, a fast Helmholtz solver could be used. The fact that the X's which occur in the equations for the A's, depend on the A's, (cf. (2.24)) suggests that an iterative procedure be used. Concus and Golub [5], among others, have used fast solvers in iterative procedures to solve problems involving Helmholtz equations with continuously varying coefficients. In our model we have piecewise constant coefficients. This may explain why contrary to their experience, we found that solving equation (3.1) with a fast Poisson solver worked better than using a fast Helmholtz solver.

The iterative procedure is outlined in the flow chart in Fig. 5. (N.B. The program actually solves for $u = -A$.) Initial values are chosen everywhere for A, and the X's are set equal to zero. Since the solution is unique, convergence does not depend on the

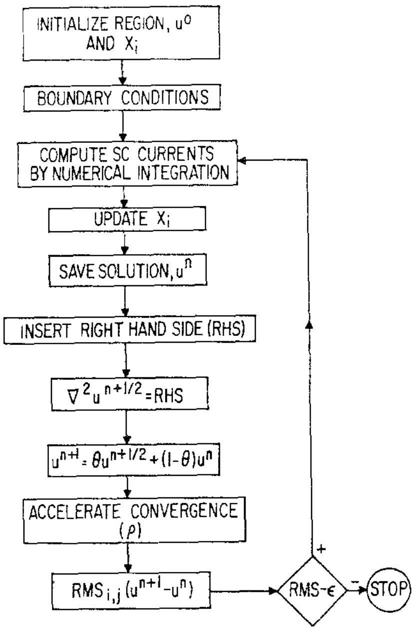

FIG. 5. Flowchart of the iterative procedure programmed to solve for $u(x, y)$ and the gauge constants X.

initial values of A. The number of iterations turned out to be relatively insensitive to the choice of initial values of A. We use expression (2.20) everywhere, except near the center of the current carrying superconductors, where (2.20) has a logarithmic singularlity. Values for the X's are then calculated from (2.24) using the initial values of A. The numerical integration scheme is quite simple. The average of the values of A at the four corner points of a grid square is taken. A summation of the averages over all grid squares is made, and then multiplied by $\Delta x\ \Delta y$.

We are now into the loop. Equations (3.1) are solved with Hockney's *POT4* program [6], which he developed at the IBM Thomas J. Watson Research Center. The right hand side of (2.24) is formed from the current X values and the previous solution for the A's. In the first iteration, the initial values occupy the storage locations of the previous solution. Designating the previous solution by A and the resultant solution of (2.24) by A^*, we form

$$A^{n+1} = \sigma A^* + (1 - \sigma) A^n \tag{3.2}$$

our new solution. The root mean square error (for all grid points) between the new and previous solution is obtained. When it falls below a chosen tolerance ϵ, an exit is made from the loop. Also values of the currents are obtained from the integral of j, as given in (2.14), for comparison with the true values.

If no exit is made from the loop, the new solution A^{n+1} is placed in the storage locations of the previous solution. New values of the X's are calculated from (2.24). Equations (3.1) are solved with new right hand sides. A new solution A^{n+2} is obtained, and so on, until the exit criterion is met. To reduce the number of iterations, Chebyshev acceleration (as described in Concus and Golub [5]) is used. Essentially this acceleration procedure makes use of the information contained in the solution for the next to last iteration to speed up convergence. The formula is

$$\tilde{A}^{n+1} = \omega_{n+1}(A^{n+1} - \tilde{A}^{n-1}) + \tilde{A}^{n-1} \tag{3.3}$$

where

$$\begin{aligned} \omega_1 &= 1 \\ \omega_2 &= 2/(2 - \rho^2) \\ \omega_{n+1} &= (1 - \rho^2\omega_n/4)^{-1} \qquad \text{for} \quad n = 2, 3, \ldots \end{aligned}$$

and $\tilde{A}^{n-1}$ is the improved value of the $(n + 1)$th iteration. ρ is an upper bound for the absolute values of the eigenvalues of the associated matrix. Once the loop is excited, the magnetic fields are calculated from A by numerical differentiation, and the current densities in the superconductors, from (2.14).

Considerable underrelaxation is required to render the iterative procedure stable. Typical values of σ in (3.2) range between 0.1 and 0.2. Nevertheless only approximately 20 iterations are required to obtain a solution of .01 % accuracy.

The program returns magnetic induction values in milleteslas and current densities in milliamperes per square micrometer.

Solutions were obtained for configurations where the inductance values had been measured in the laboratory. The numerical solutions were in agreement with the measured values to within the experimental error. Also as a convergence test, a run was made for an ϵ of .0001 %, which required 34 iterations. The solution obtained was in agreement with that obtained for an ϵ of .01 %.

4. Details for Sample Problems

A main program, unique to each configuration of superconductors, iteratively calls the Poisson solver subroutine, POT4, until a solution for the vector potential A is obtained (the programs actually work with $u = -A$). Contour plots are prepared for the vector potential A, the current densities j in each superconductor, and/or the components (and/or magnitude) of the magnetic field H.

The algorithm implemented is shown in Fig. 5. The iterative procedure terminates (converges) when the root mean square (RMS) of the pointwise difference between $\tilde{u}^{n+1}$ and $\tilde{u}^n$ is less than some prescribed tolerance ϵ.

The sample problems run used a grid of 512 points in the x direction and 256 points

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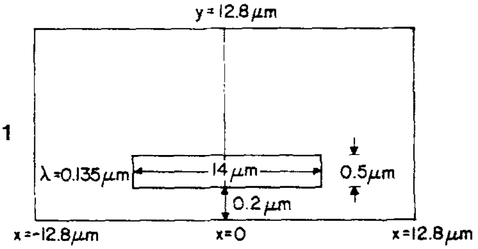


FIG. 6.1. Geometry and penetration depth for sample problem a, model of one superconductor over ground plane.

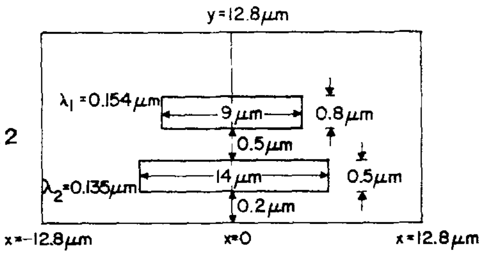
6.2


FIG. 6.2. Geometry and penetration depths for sample problem b, model of two superconductors over ground plane.

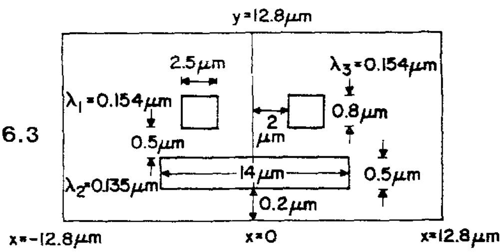
6.3


FIG. 6.3. Geometry and penetration depths for sample problem c, model of three superconductors over ground plane.

in the y direction. This required 4 megabytes of main memory. The mesh widths were $\Delta x = \Delta y = .5$. Severe under-relaxation was used with $\sigma = .1$. Minimum acceleration was attempted: $\rho = .86$. The RMS was tested against an $\epsilon = 10^{-4}$.

For the 512×256 grid each iteration took about 11 seconds. For the superconductor geometries about to be described, convergence required 16 to 18 iterations. The contour plots made averaged about 15 seconds per plot. Thus a typical run required 3 minutes of computation and two minutes of plotter preparation.

Three sample problems are considered; their geometries and penetration depths are given in Figures 6.1, 6.2, and 6.3. The bottom boundary in each case is a ground plane for which $u = -A = 0$. The problems are all run with a current of 1 milliamp applied to just one of the superconductors. All of the vector potential plots (lines of constant A) show only the lower half of the computed solution.

a) *One Superconductor Over a Ground Plane* (*Fig.* 6.1)

Fig. 7 is a contour plot of lines of constant vector potential, A. These lines are equivalent to magnetic field lines. Since the levels of the contours are uniformly chosen, the concentration of the lines of constant A are indicative of the strength or weakness of the field in a given area of the plot.

Because of symmetry only the right half of the region is plotted in Fig. 7. The wiggles on the left hand side at the center of the superconductors are an artifact of the plotting program. The field lines encircle the current source in the superconductor as expected.

W. Cheng (personal communication) has derived approximate formulas with which we may compare some of our results for the case of one superconductor over a ground plane. The formulas are for a superconductor whose x dimension (width) is greater than its distance above a second superconducting ground plane of thickness t. The normal component of the magnetic field along the vacuum-superconductor boundary is assumed to be zero. If we let $t \to \infty$ and the penetration depth in the ground plane equal zero, the inductance for the model of Fig. (6.1) is .02802 picohenries/micrometer. Our result of .02762 picohenries/micrometer differs from this by -1.44 %. The communication also contains an expression for the x component of the magnetic field at the midpoint of the bottom boundary line of the superconductor. This expression yields .08361 milliteslas. Our model gives .076 milliteslas (we only printed three decimal digits) which differes by -9.10 %. The reason for such a large discrepancy is probably that our model has only four mesh intervals between the superconductor and the ground plane.

TABLE 1

	Formula	Model	% Difference
L	.026496	.026566	.264 %
H	.08407	.084	$-.083$ %

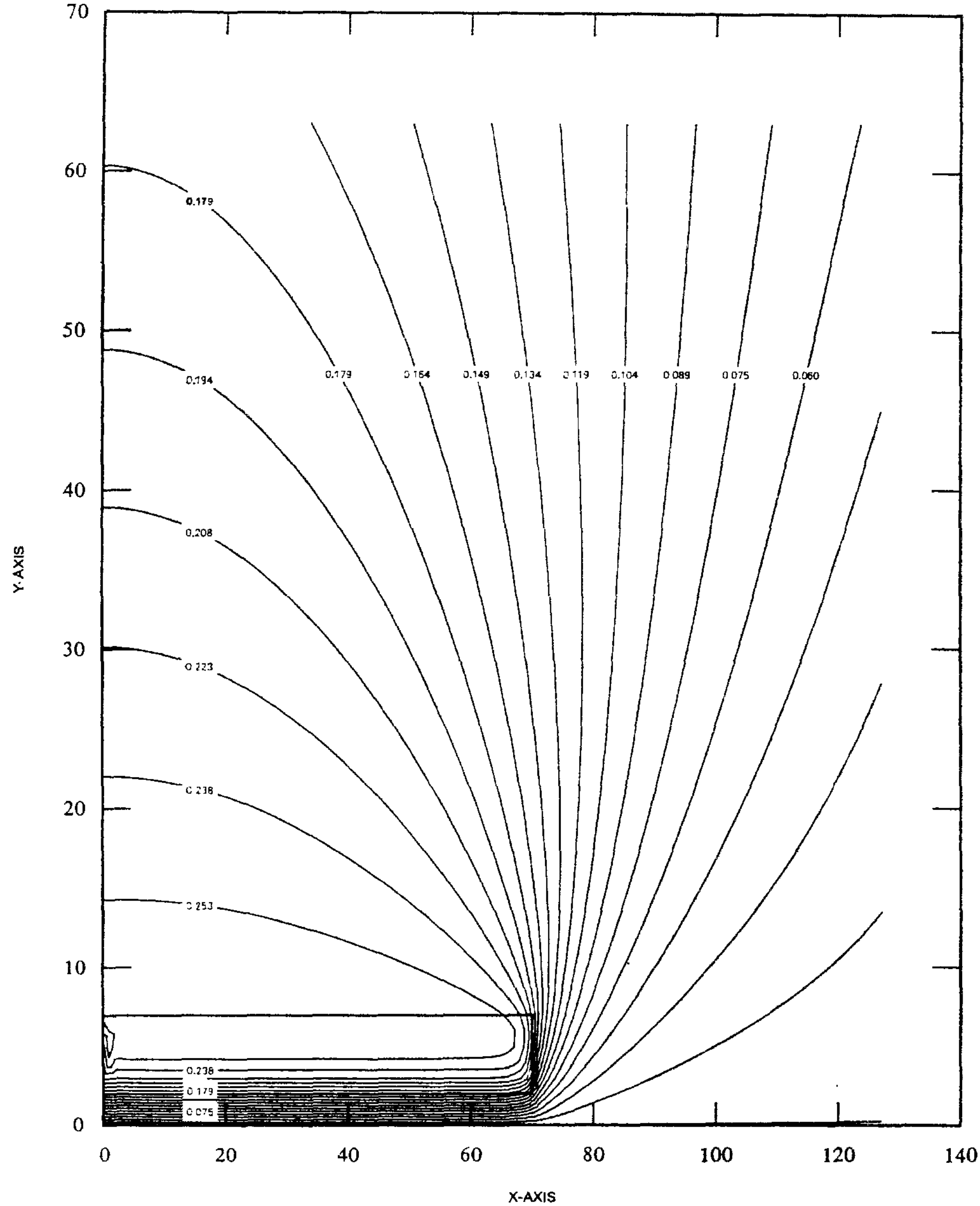


FIG. 7. Contour lines of the vector potential A for sample problem a, one superconductor over a ground plane. These lines are equivalent to magnetic field lines whose concentration is indicative of field strength.

In a comparable model in which the distance of the superconductor above the ground plane was .18 micrometers and the vertical mesh width was .02, the results from the approximate formulas and our model are given in Table 1. The agreement here for the H is excellent, and we note that there are nine mesh intervals between the superconductor and the ground plane.

b) *Two Superconductors Over a Ground Plane* (*Fig.* 6.2)

Fig. 8.1 is the field plot when current is applied to the top superconductor (SC1); Fig. 8.2 is the corresponding plot for current in the bottom superconductor (SC2). Because of symmetry only the right half of the region is shown. As in the one superconductor case, the field lines encircle the current source.

The array of constants X computed is

$$X = \begin{bmatrix} -.4577 & -.1049 \\ -.1366 & -.1513 \end{bmatrix},$$

where X_{ij} is the X computed in superconductor i with current applied to superconductor j. The inductance array which results is

$$L = \begin{bmatrix} .1136 & .025376 \\ .024914 & .02655 \end{bmatrix}.$$

The mutual inductances between the two superconductors, L_{12} and L_{21}, should be equal. Our calculated values differ by .068 %, which is within the expected numerical error.

It is interesting to observe the dynamics of the calculation. For the case with current in the top superconductor, 18 iterations were required for convergence. Table 2 shows the computed currents, I, and corresponding constants, X, as the iterative procedure was just before convergence. A current of $4\pi = 12.56637061$, which corresponds to 1 milliamp, for the scaling used, is applied to SC1 and a current of 0

TABLE II

$(I_1 = 12.56637061)$ $(I_2 = 0)$

Iteration	Current computed in *SC1*	X_1	Current computed in *SC2*	X_2	*RMS*
13	12.25196870	−.45756323	−.02618778	−.13660887	$.91985 \times 10^{-3}$
14	12.72144828	−.45799989	.02305350	−.13664628	$.57091 \times 10^{-3}$
15	12.50505022	−.45778451	.01266092	−.13661334	$.36561 \times 10^{-3}$
16	12.65669819	−.45786968	.01985031	−.13659526	$.23335 \times 10^{-3}$
17	12.53769467	−.45774422	.0028463	−.13656690	$.14749 \times 10^{-3}$
18					$.94750 \times 10^{-4}$

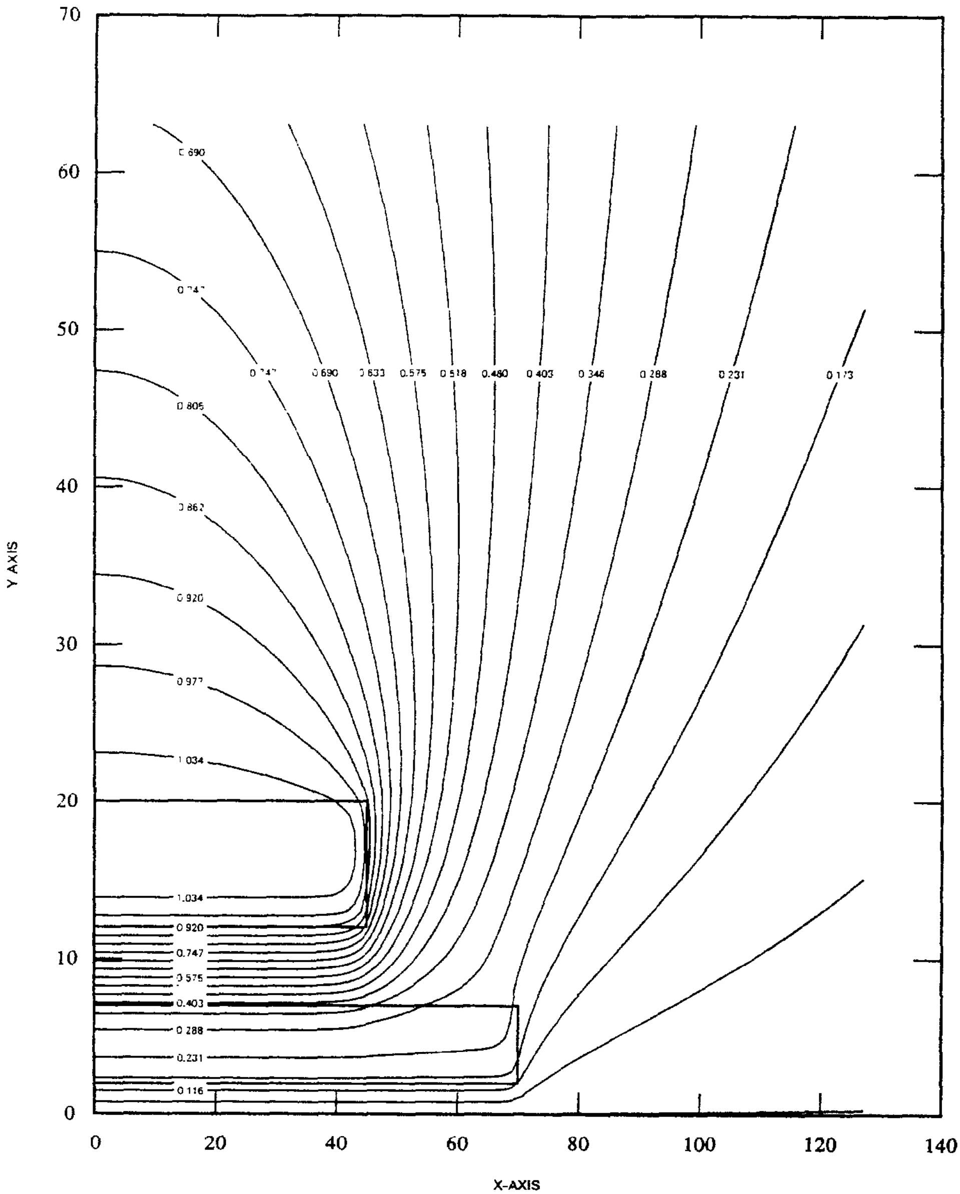


FIG. 8.1. Contour lines of the vector potential A for sample problem *b*, two superconductors over a ground plane with *current applied in the top superconductor*. These lines are equivalent to magnetic field lines whose concentration is indicative of field strength.

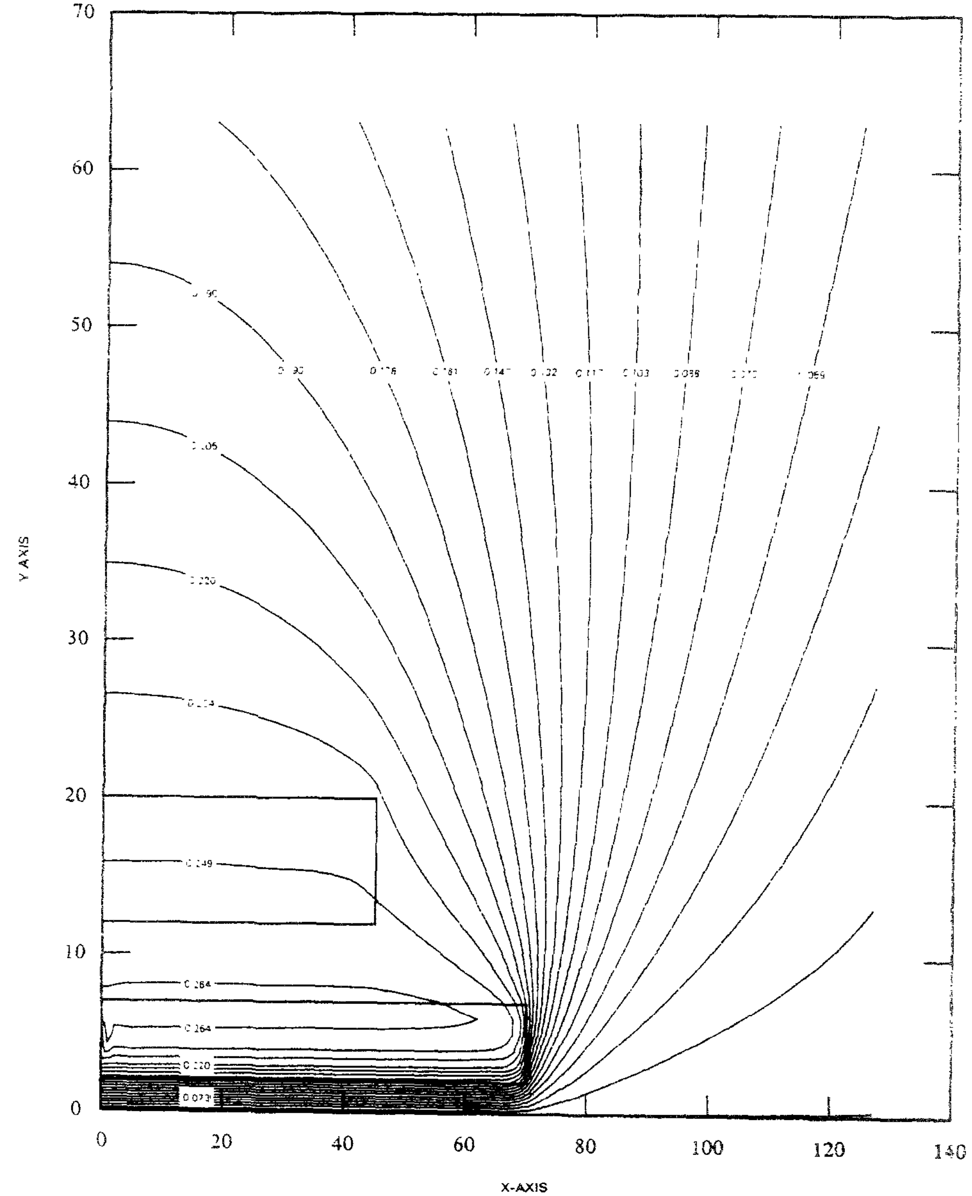


FIG. 8.2. Contour lines of the vector potential A for sample problem b, two superconductors over a ground plane with *current applied in the bottom superconductor*. These lines are equivalent to magnetic field lines whose concentration is indicative of field strength.

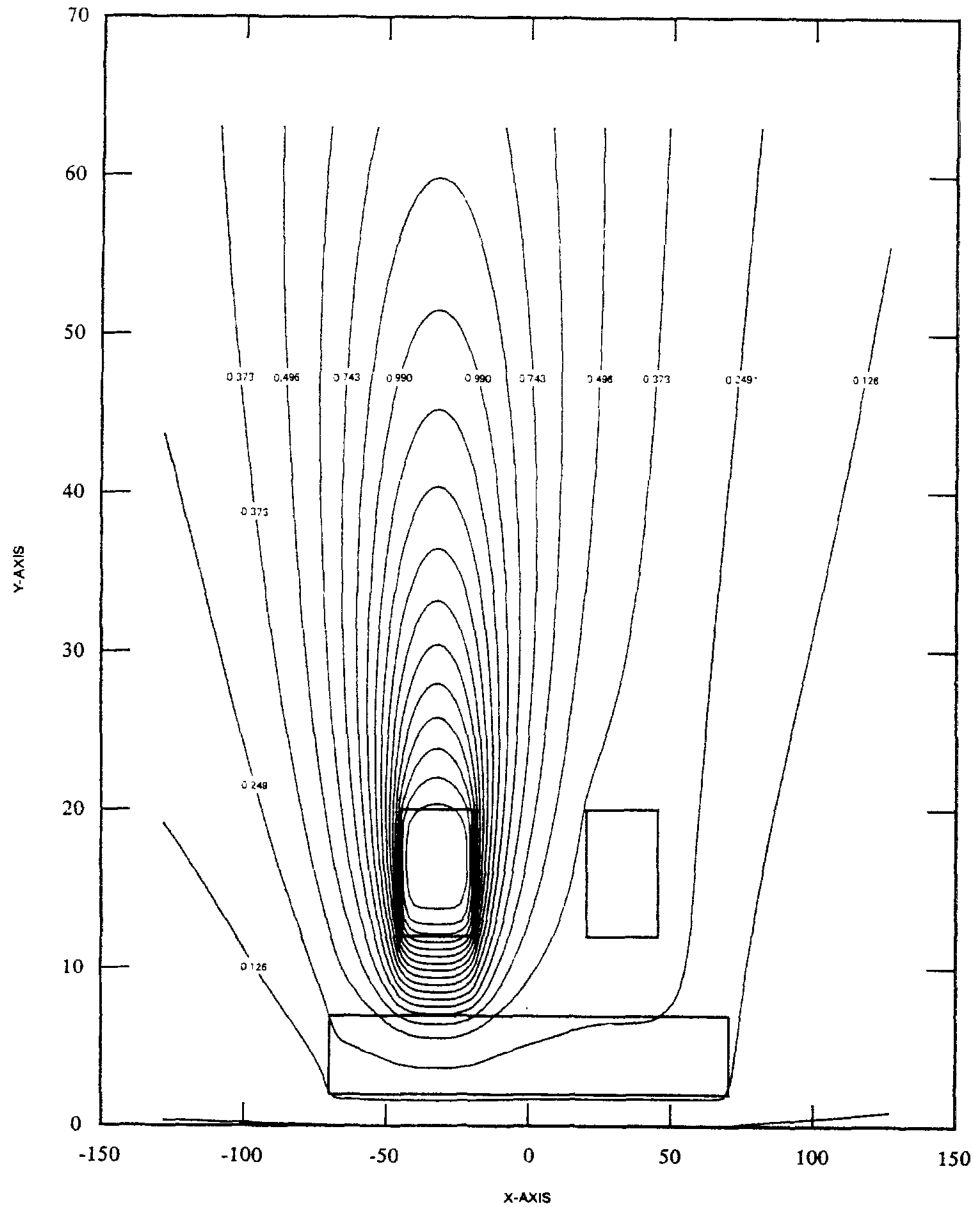


FIG. 9.1. Contour lines of the vector potential A for sample problem c, three superconductors over a ground plane with *current applied in the top left superconductor*. These lines are equivalent to magnetic field lines whose concentration is indicative of field strength.

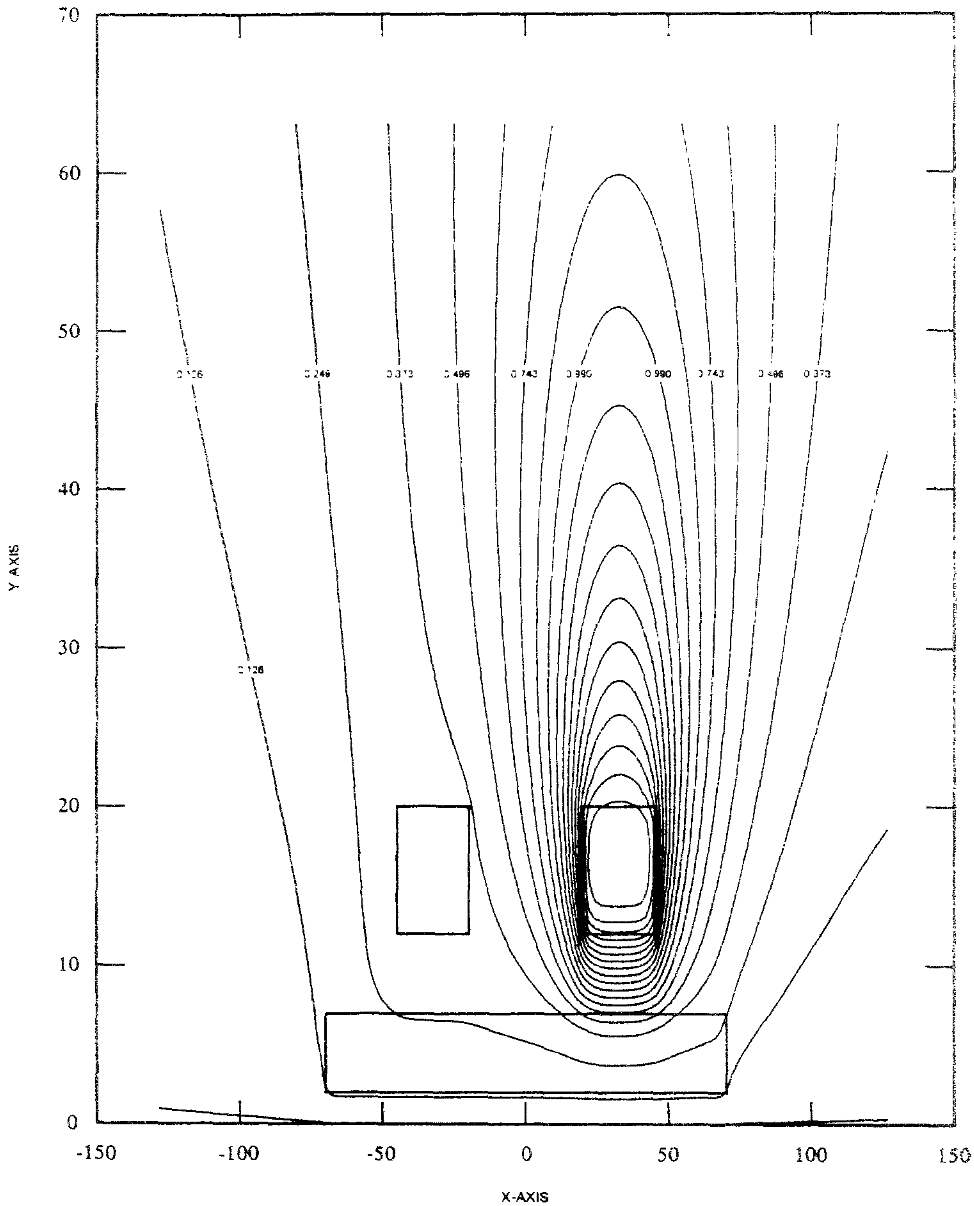


FIG. 9.2. Contour lines of the vector potential A for sample problem c, three superconductors over a ground plane with *current applied in the top right superconductor*. These lines are equivalent to magnetic field lines whose concentration is indicative of field strength.

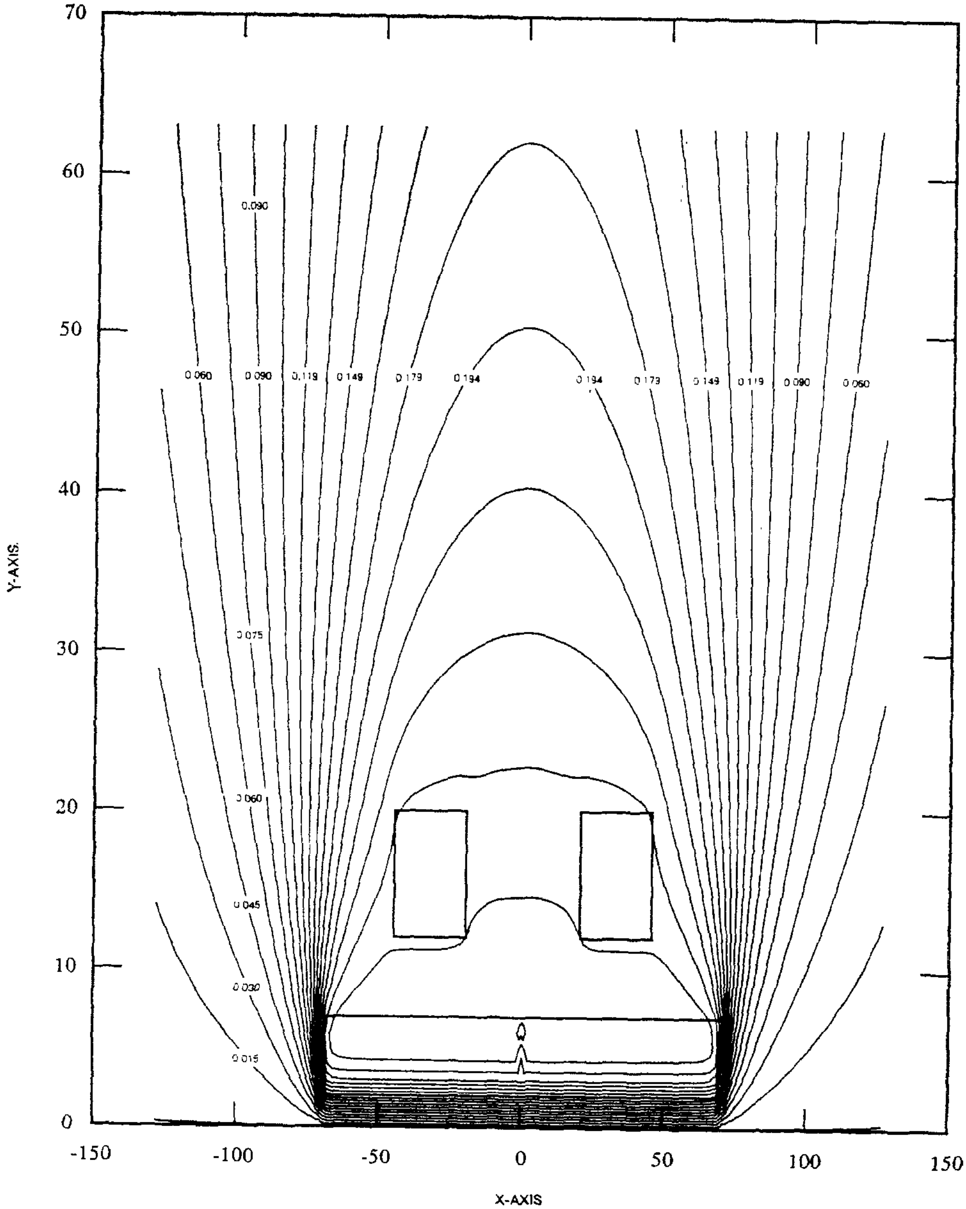


FIG. 9.3. Contour lines of the vector potential A for sample problem c, three superconductors over a ground plane with *current applied in the bottom superconductor. These lines are* equivalent to magnetic field lines whose concentration is indicative of field strength.

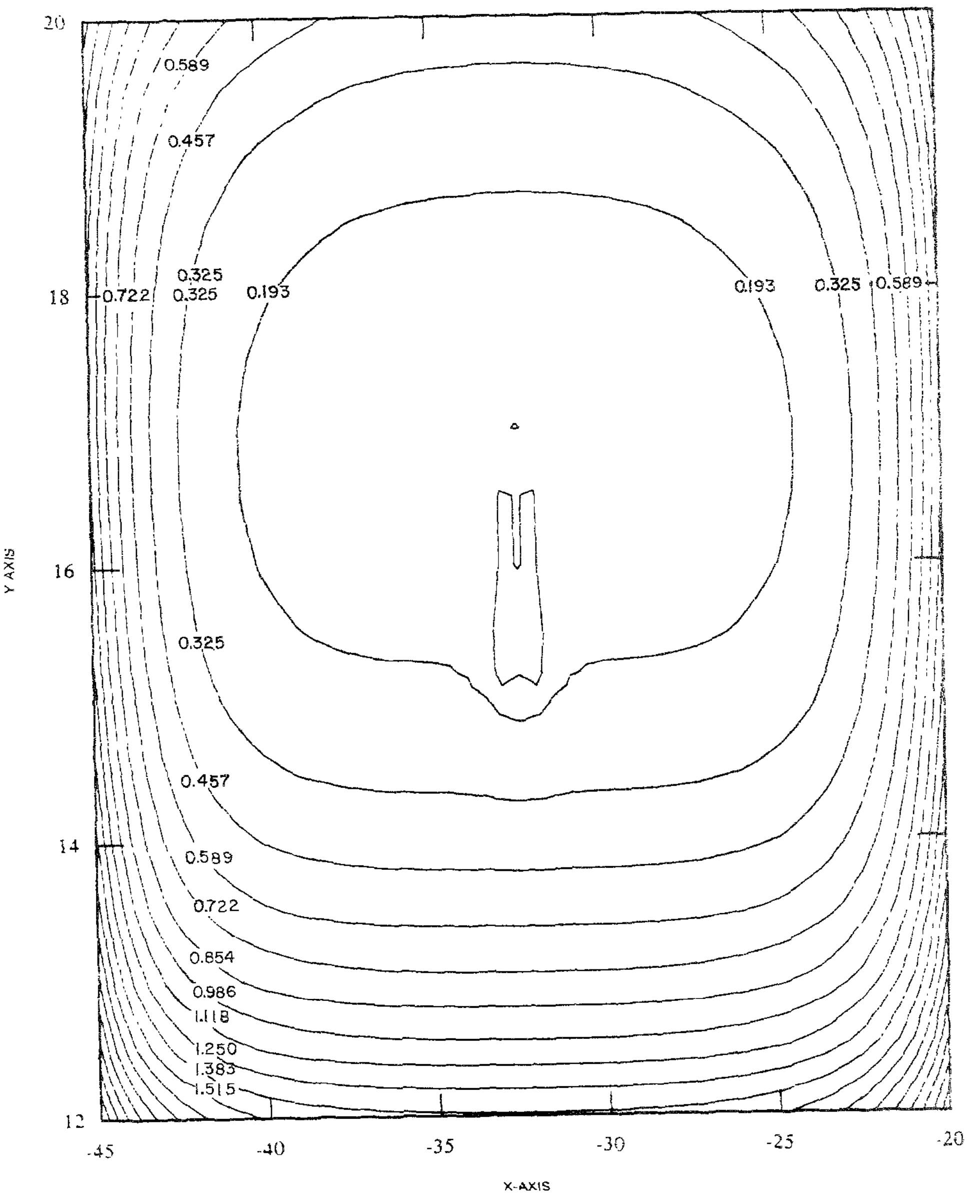


FIG. 10.1 Contour lines of the *current density in the top left superconductor* for sample problem *c*, three superconductors over a ground plane with *current applied in the top left superconductor*.

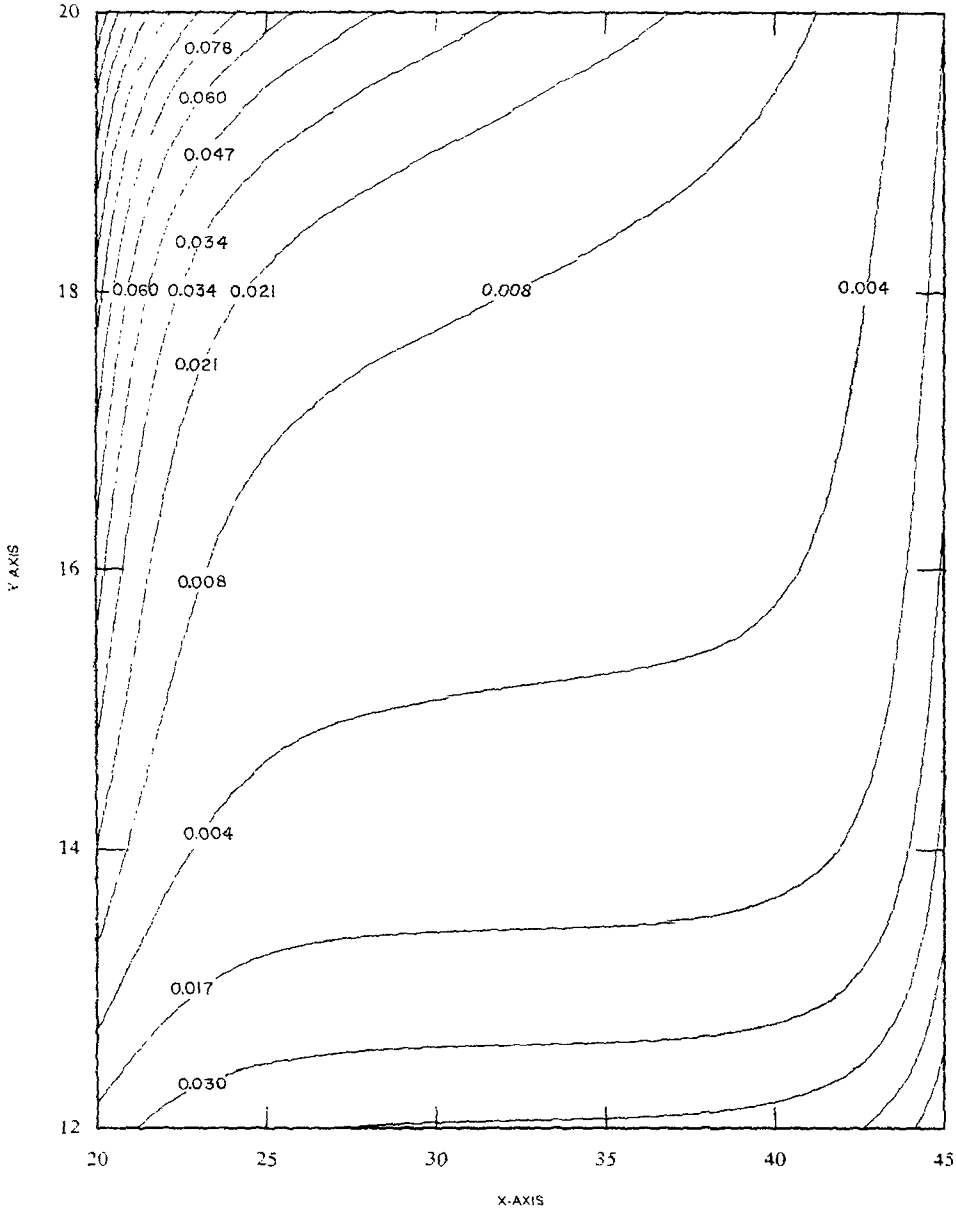


FIG. 10.2. Contour lines of the *current density in the top right superconductor* for sample problem c, three superconductors over a ground plane with *current applied in the top left superconductor*.

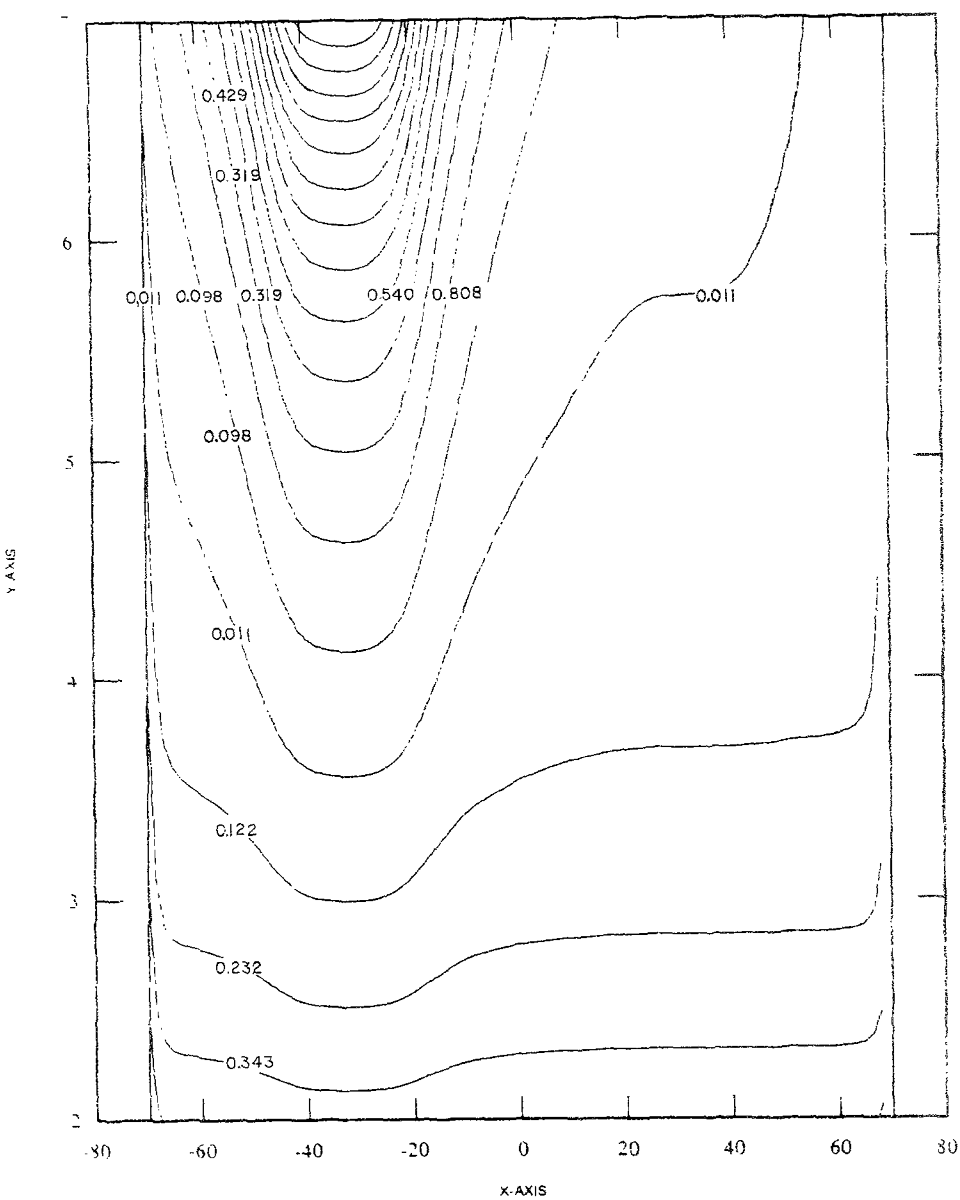


FIG. 10.3. Contour lines of the *current density in the bottom superconductor* for sample problem c, three superconductors over a ground plane with *current applied in the top left superconductor*.

to SC2. The last values were computed after the 17th iteration, and were not recomputed after convergence occurred in the 18th iteration. Both currents are clearly converging to the applied values, and could be made closer to the applied values if more iterations were made (by using a smaller ϵ).

c) *Three Superconductors Over a Group Plane* (*Fig.* 6.3)

Figures 9.1, 9.2, and 9.3 are the field plots when current is applied to the upper left superconductor (SC1), the upper right superconductor (SC3), and the bottom superconductor (SC2), respectively. The entire solutions are plotted (lower halves), since for current in either SC1 or SC3, the solution is not symmetric about the line $x = 0$. (It is symmetric with current in SC2, but we plot the whole region anyway.)

The solutions for currents in SC1 and SC3 are mirror images of each other, as we would expect. Note that here again the field lines encircle the current source.

The inductance array computed is:

$$L = \begin{bmatrix} .2368 & .024617 & .03104 \\ .024567 & .02759 & .024567 \\ .03104 & .024617 & .2368 \end{bmatrix}.$$

L_{12} and L_{21}, which should be equal, differ by .203 % as do L_{23} and L_{32}. $L_{13} = L_{31}$ and $L_{11} = L_{33}$, as would be expected from symmetry.

For current in SC1, Figures 10.1, 10.0, and 10.3 give contour plots of the current densities in SC1, SC3, and SC2. The current density contours are identical in shape to the field lines (lines of constant A) in the superconductors, since they are related by the linear transformation given in equation (2.11). Note that the current density in SC1 is everywhere positive, while in SC2 and SC3, it is both positive and negative (so that the total currents equal zero).

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