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Regularized Maxwell Equations and Nodal Finite Elements for Electromagnetic Field Computations

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Abstract *This article presents an alternative approach to the usual finite element formulation based on edge elements and double-curl Maxwell equations. This alternative approach is based on nodal elements and regularized Maxwell equations. The advantage is that, without adding extra unknowns (such as Lagrange multipliers), it provides spurious-free solutions and well-conditioned matrices. The drawback is that a globally wrong solution is obtained when the electromagnetic field has a singularity in the problem domain. The main objective of this work is to obtain accurate solutions with nodal elements and the regularized formulation, even in the presence of electromagnetic field singularities.*

Keywords finite element method, regularized Maxwell equations, weighted regularization, singularities, intersection of dielectrics and nodal elements

Introduction

The typical approach when solving a general electromagnetic problem with the finite element method (FEM) is to use edge elements and double-curl Maxwell equations. These edge elements, proposed by Nedelec (1980), seem to be the answer to most of the drawbacks exhibited by FEM when applied to electromagnetism (Salazar-Palma et al., 1998; Jin, 2002). With edge elements, spurious-free solutions are obtained, boundary conditions are easier to implement, and the normal discontinuity and tangential continuity between different media are automatically satisfied. In addition, they present a better behavior in nonconvex domains than Lagrangian elements (Webb, 1993). However, using edge elements with the double-curl formulation has also disadvantages (Mur, 1994, 1998). The most important flaw is the matrices produced, which are ill-conditioned and, in problems with a high number of unknowns, can even be singular (Lager, 1996). The use of potentials or Lagrange multipliers can improve the conditioning of the matrix (Ciarlet, 2005; Hiptmair, 2002), but the number of unknowns is increased by the presence of these scalar functions. Therefore, although edge elements present several advantageous features, there can be trouble when trying to solve big problems where the use of direct methods, or good preconditioning, can be limited by computer hardware. Because of this, it would be desirable to explore alternative FEM formulations that provide matrices that are easy to solve by means of iterative solvers.

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The alternative approach proposed in this work is based on nodal elements and regularized Maxwell equations (Hazard & Lenoir, 1996; Costabel & Dauge, 2002). The good point of this proposal is that it provides spurious-free solutions with well-conditioned matrices and, moreover, only the three components of **E** or **H** are the unknowns; that is, there is no need of extra functions such as Lagrange multipliers or scalar potentials. On the other hand, new difficulties arise that were not present in the classical formulation. The main drawback is that if the electromagnetic field has a singularity in the problem domain, a *globally wrong* solution is obtained. Also, boundary conditions and field discontinuities are more laborious to implement. This article explains how to overcome these difficulties. The main objective is to demonstrate that accurate solutions can be obtained using nodal elements and the regularized formulation. The comparative performance of the classical *edge-double-curl* formulation versus the *nodal-regularized* formulation presented in this article will be the topic of future work.

In the first two sections, the classical double-curl formulation and the regularized Maxwell formulation are exposed, both adapted to electromagnetic problems in frequency domain. The formulations are written to emphasize their differences in the differential and weak form and also in the boundary conditions. This method can use **E** or **H** as the primary unknown but, in this work, for simplicity, only the **E** field is used. The next section explains how to deal with field discontinuities and nodal elements, not only how to solve problems with a discontinuity surface present, but also how to work with the intersection of three or more different materials. Following that, the problem of the singularities is described: why it is produced and how to overcome this critical question. Finally, in the last sections, several examples in two and three dimensions are shown to check the accuracy of the method.

Double-Curl Maxwell Equations and Edge Elements

The generic problem to be solved is to find the electric field **E** in a domain Ω with boundary $\partial\Omega$ produced by a divergence-free source **J** driven at a frequency ω. The equations describing this situation, the so-called *double-curl Maxwell equations*, are

$$\begin{gathered}\nabla \times \left(\frac{1}{\mu}\nabla \times \mathbf{E}\right) - \omega^2\varepsilon\mathbf{E} = -j\omega\mathbf{J} \quad \text{in } \Omega \\ \hat{\mathbf{n}} \times \mathbf{E} = 0 \quad \text{in } \Gamma,\end{gathered} \tag{1}$$

where Γ is the surface of a perfect electric conductor (PEC). In an open domain, the Silver-Müller radiation boundary condition must be added at infinity:

$$\lim_{r\to\infty}\oint_{\partial\Omega_r} \|\hat{\mathbf{n}} \times \nabla \times \mathbf{E} - (j\omega\sqrt{\varepsilon\mu})\mathbf{E}\|^2 = 0. \tag{2}$$

If $\partial\Omega$ is a waveguide port, the boundary conditions will be adapted to take into account the specific modes of the waveguide. For instance, in a rectangular waveguide port with only the fundamental mode TE_{10} propagating,

$$\hat{\mathbf{n}} \times (\nabla \times \mathbf{E}) - \gamma_{10}(\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{E}) = \mathbf{U} \tag{3}$$

holds, where γ_{10} is the propagation constant of the fundamental mode $\mathbf{E}_{10}$ and

$$\begin{aligned} \mathbf{U} &= -2\gamma_{10}(\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{E}_{10}), \\ \mathbf{U} &= 0 \end{aligned} \tag{4}$$

for the input and the output port, respectively (Jin, 2002).

The above set of equations can be solved using an equivalent weak formulation, that is, if

$$\mathbf{H}_0(curl;\Omega) := \{\mathbf{F} \in \mathbf{L}^2(\Omega) \mid \nabla \times \mathbf{F} \in \mathbf{L}^2(\Omega), \hat{\mathbf{n}} \times \mathbf{F} = 0 \text{ in } \Gamma\} \tag{5}$$

is defined, solving Eq. (1) is equivalent to finding $\mathbf{E} \in \mathbf{H}_0(curl;\Omega)$ such that $\forall \mathbf{F} \in \mathbf{H}_0(curl;\Omega)$ holds:

$$\int_\Omega \frac{1}{\mu}(\nabla \times \mathbf{E}) \cdot (\nabla \times \bar{\mathbf{F}}) - \omega^2 \int_\Omega \varepsilon \mathbf{E} \cdot \bar{\mathbf{F}} + \mathbf{B.C.}|_{\partial\Omega} = -j\omega \int_\Omega \mathbf{J} \cdot \bar{\mathbf{F}}, \tag{6}$$

where the expression $\mathbf{B.C.}|_{\partial\Omega}$ is the term that takes into account the radiation boundary conditions or the modes in a waveguide port. This weak formulation, discretized with edge elements, is the classical way to use the FEM when applied to electromagnetic field problems. The advantages and disadvantages of this approach were cited in the introduction of this article.

Regularized Maxwell Equations and Nodal Elements

An alternative for solving Eq. (1) is to use an equivalent system of second-order differential equations, the so-called *regularized Maxwell equations*:

$$\begin{aligned} \nabla \times \left(\frac{1}{\mu}\nabla \times \mathbf{E}\right) - \bar{\varepsilon}\nabla\left(\frac{1}{\bar{\varepsilon}\varepsilon\mu}\nabla \cdot (\varepsilon\mathbf{E})\right) - \omega^2\varepsilon\mathbf{E} &= -j\omega\mathbf{J}, \quad \text{in } \Omega, \\ \nabla \cdot (\varepsilon\mathbf{E}) &= 0 \quad \text{in } \Gamma, \\ \hat{\mathbf{n}} \times \mathbf{E} &= 0 \quad \text{in } \Gamma, \end{aligned} \tag{7}$$

where Γ is, again, the surface of a PEC, and $\bar{\varepsilon}$ is the complex conjugate of ε. In an open domain, the Silver-Müller radiation boundary condition of Eq. (2) must be adapted to the regularization

$$\begin{aligned} \lim_{r\to\infty} \oint_{\partial\Omega_r} \|\hat{\mathbf{n}} \times \nabla \times \mathbf{E} + j\omega\sqrt{\varepsilon\mu}(\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{E})\|^2 &= 0, \\ \lim_{r\to\infty} \oint_{\partial\Omega_r} |\nabla \cdot \mathbf{E} + j\omega\sqrt{\varepsilon\mu}(\hat{\mathbf{n}} \cdot \mathbf{E})|^2 &= 0. \end{aligned} \tag{8}$$

If $\partial\Omega$ is a waveguide port, the boundary conditions must also be adapted to the regularization; for instance, in a rectangular waveguide port with only the fundamental mode TE_{10} propagating, the condition

$$\hat{\mathbf{n}} \cdot \mathbf{E} = 0 \tag{9}$$

must be added to Eq. (3).

The above differential equation can be solved using an equivalent weak formulation; that is, if

$$\mathbf{H}_0(curl, \mathrm{div}; \Omega) := \{\mathbf{F} \in \mathbf{L}^2(\Omega) \mid \nabla \times \mathbf{F} \in \mathbf{L}^2(\Omega), \nabla \cdot (\varepsilon \mathbf{F}) \in L^2(\Omega),$$
$$\hat{\mathbf{n}} \times \mathbf{F} = 0 \text{ in } \Gamma\} \tag{10}$$

is defined, solving Eq. (7) is equivalent to finding $\mathbf{E} \in \mathbf{H}_0(curl, \mathrm{div}; \Omega)$ such that $\forall \mathbf{F} \in \mathbf{H}_0(curl, \mathrm{div}; \Omega)$

$$\int_\Omega \frac{1}{\mu}(\nabla \times \mathbf{E}) \cdot (\nabla \times \bar{\mathbf{F}}) + \int_\Omega \frac{1}{\bar{\varepsilon}\varepsilon\mu}(\nabla \cdot (\varepsilon \mathbf{E})) \cdot (\nabla \cdot (\bar{\varepsilon}\bar{\mathbf{F}}))$$
$$- \omega^2 \int_\Omega \varepsilon \mathbf{E} \cdot \bar{\mathbf{F}} + \mathbf{R.B.C.}|_{\partial\Omega} = -j\omega \int_\Omega \mathbf{J} \cdot \bar{\mathbf{F}} \tag{11}$$

holds. The expression $\mathbf{R.B.C.}|_{\partial\Omega}$ is the term, properly adapted to the regularization, that takes into account the radiation boundary conditions or the modes in a waveguide port.

In Hazard and Lenoir (1996), the equivalence between the *classical* problem in Eqs. (1)-(6) and the *regularized* problem of Eqs. (7)-(11) is demonstrated. The regularized formulation has the characteristics required for an efficient FEM simulation; that is, it calculates spurious-free solutions with $\mathbf{E}$ as the only unknown, and it produces well-conditioned matrices. Nodal elements can be used as the finite element base, being careful to explicitly consider the discontinuities between different media, as explained in the next section.

Although Eq. (11) looks like a penalized method, the regularized formulation has no undetermined constants, and with the help of the extra boundary conditions, the problem is well-posed. It is worth emphasizing the importance of the extension of the boundary conditions in the regularized formulation: if these extra boundary conditions are omitted, the number of iterations needed for an iterative solver to achieve convergence or even obtain spurious solutions can be severely increased.

Nodal Elements and Field Discontinuities

As mentioned in the previous section, nodal elements are used along with the regularized formulation. Due to the fact that these elements impose normal and tangential continuity, the discontinuities between different media must explicitly be considered. To do so, the technique explained in Paulsen et al. (1988) is employed. This technique consists of defining two different nodes—one on each side of the discontinuity—and during the assembly procedure, relate the two nodes as follows:

$$\begin{pmatrix} E_x^+ \\ E_y^+ \\ E_z^+ \end{pmatrix} = \begin{pmatrix} n_x^2\xi + 1 & n_x^2 n_y^2 \xi & n_x^2 n_z^2 \xi \\ n_x^2 n_y^2 \xi & n_y^2 \xi + 1 & n_y^2 n_z^2 \xi \\ n_x^2 n_z^2 \xi & n_y^2 n_z^2 \xi & n_z^2 \xi + 1 \end{pmatrix} \begin{pmatrix} E_x^- \\ E_y^- \\ E_z^- \end{pmatrix} \tag{12}$$

being

$$\xi = \frac{\epsilon^- - j(\sigma^-/\omega)}{\epsilon^+ - j(\sigma^+/\omega)} - 1;$$

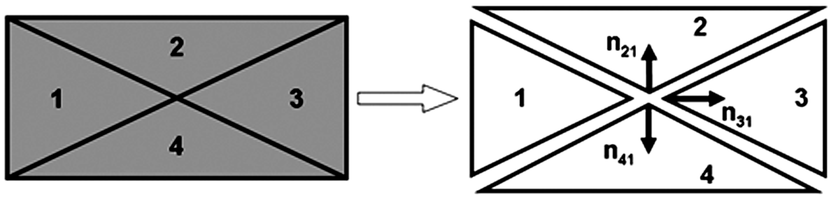

Figure 1. Intersection of four different materials. The interface between different materials is considered as two separated surfaces related by Eq. (12). The vectors $\mathbf{n}_{21}$, $\mathbf{n}_{31}$, and $\mathbf{n}_{41}$ are the unit normal to the surfaces limiting the volume of materials 2, 3, and 4, respectively. In the intersection, Eq. (12) is applied to the pairs 2-1, 3-1, and 4-1.

$\mathbf{n} = (n_x, n_y, n_z)$ is the unit normal at the surface, and superscripts "+" and "−" denote each side of the discontinuity surface. In this procedure, the node on side "+" is removed from the total linear system with the help of Eq. (12), and only the unknowns on the side "−" are solved. No extra unknowns are added, and the symmetry of the total FEM matrix is retained if the side "+" is removed using Eq. (12) and its transpose (Boyse et al., 1992).

Although this technique works well for discontinuity surfaces, a procedure is needed to deal with the intersection of three or more different materials. An attempt was made in Paulsen et al. (1987), but the finite element bases appearing there do not belong to $\mathbf{H}_0(curl, \mathrm{div}; \Omega)$ and cannot be used to discretize Eq. (11). The approach here to overcome this setback consists of a simple extension of the method presented at the beginning of this section. For instance, in a situation like the one shown in Figure 1, four different nodes at the center are defined. One of the nodes defined at the center plays the role of "−" in Eq. (12). In this case, the selected node is in material 1. Then, during the assembly procedure, Eq. (12) is applied to the pairs 2-1, 3-1, and 4-1 with unit normals $\mathbf{n}_{21}$, $\mathbf{n}_{31}$, and $\mathbf{n}_{41}$, respectively. Nodes 2, 3, and 4 are removed from the total linear system, and only the unknowns of node 1 are solved. The best choice for the role of "−" is the node that is in the material with the smallest $|\epsilon - j(\sigma/\omega)|$. It was observed that this option always gives the lowest number of iterations when solving the total FEM matrix with iterative solvers.

Regularized Formulation and Field Singularities

In Hazard and Lenoir (1996), it is shown that solving Eq. (11) analytically is equivalent to solving Eq. (6). However, care must be taken when solving Eq. (11) numerically with nodal finite elements in a nonconvex polyhedral domain. If $\mathbf{V}_h$ is the vectorial space spanned by K nodal basis functions $N_i(\mathbf{r})$,

$$\mathbf{V}_h := \left\{ \mathbf{u}_h | \mathbf{u}_h = \hat{\mathbf{x}} \sum_{i=1}^{K} c_{x_i} N_i(\mathbf{r}) + \hat{\mathbf{y}} \sum_{i=1}^{K} c_{y_i} N_i(\mathbf{r}) + \hat{\mathbf{z}} \sum_{i=1}^{K} c_{z_i} N_i(\mathbf{r}), c_i \in \mathbb{C} \right\}$$

and

$$\mathbf{H}_0^1(\Omega) := \left\{ \mathbf{F} \in L_2(\Omega) \mid \frac{\partial \mathbf{F}}{\partial x}, \frac{\partial \mathbf{F}}{\partial y}, \frac{\partial \mathbf{F}}{\partial z} \in L_2(\Omega), \hat{\mathbf{n}} \times \mathbf{F} = 0 \text{ in } \Gamma \right\},$$

and it is clear that $\mathbf{V}_h$ is included in $\mathbf{H}_0^1(\Omega)$. On the other hand, for nonconvex polyhedral domains, $\mathbf{H}_0^1(\Omega)$ is strictly included in $\mathbf{H}_0(curl, \mathrm{div}; \Omega)$, and moreover, $\mathbf{H}_0^1(\Omega)$ is closed in $\mathbf{H}_0(curl, \mathrm{div}; \Omega)$ (Costabel, 1991; Costabel & Dauge, 2002). As a consequence of this theorem, if $\mathbf{E}$, the analytical solution of Eq. (11) belongs to $\mathbf{H}_0(curl, \mathrm{div}; \Omega)$ but not to $\mathbf{H}_0^1(\Omega)$; then, it is impossible to approximate $\mathbf{E}$ using nodal elements. In fact, such an approximation is impossible using any $\mathbf{H}^1$-conforming finite element discretization (Costabel & Dauge, 1999). This situation happens, for instance, when the electric field is singular in the corners or edges of a PEC. In other words, for nonconvex polyhedral domains,

$$\mathbf{V}_h \subseteq \mathbf{H}_0^1(\Omega) \subset \mathbf{H}_0(curl, \mathrm{div}; \Omega).$$

If the electric field is singular at some point of the domain, the analytical solution $\mathbf{E}$ belongs to $\mathbf{H}_0(curl, \mathrm{div}; \Omega)$ but not to $\mathbf{H}_0^1(\Omega)$. This means that $\mathbf{E}$ cannot be approximated with nodal elements and Eq. (11), because any approximation of $\mathbf{E}$, regardless of the element size or the polynomial order used in the discretization, will belong to $\mathbf{H}_0^1(\Omega)$. In fact, what is approximated using Eq. (11) and nodal elements is the Garlekin projection of $\mathbf{E}$ onto $\mathbf{H}_0^1(\Omega)$, which is, in general, globally different to $\mathbf{E}$. An example of this behavior is shown in Figure 2.

A good option to overcome the problem with the singularities in the regularized formulation is to follow the *weighted regularized Maxwell equations* (WRME) *method* explained in Costabel and Dauge (2002). There are other possibilities, such as Dhia et al. (1999) or Boyse and Seidl (1994), but the WRME method is more general and robust. In the WRME method, the divergence term of Eq. (11) is multiplied by a geometry-dependent weight. This weight tends to zero when approaching to a field singularity. To be more specific, the weight τ is

$$\tau := \left(\prod_{corners} r^{\gamma_c} \right) \cdot \left(\prod_{edges} \rho^{\gamma_e} \right), \tag{13}$$

where r and ρ are the distances from a point in the domain to the corner or edge where the field is singular. The coefficients γ_c and γ_e only depend on the geometry and can be calculated theoretically (Costabel & Dauge, 2002). The WRME method defines the space

$$\mathbf{X} := \{\mathbf{u} \in \mathbf{H}_0(curl; \Omega) \mid \tau(\nabla \cdot \mathbf{u}) \in L^2_{loc}(\Omega)\}$$

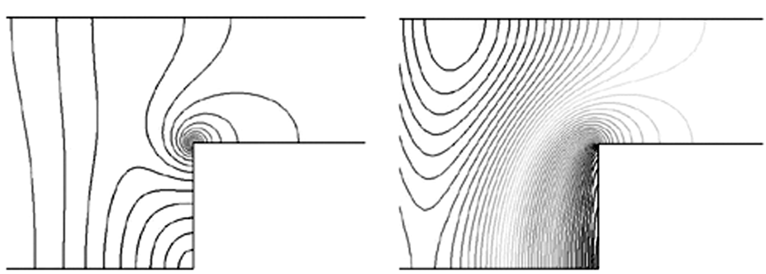

Figure 2. $\|\mathbf{E}\|$ in a waveguide step discontinuity: left, reference solution and right, same problem solved with the regularized formulation.

and solves the problem of finding $\mathbf{E} \in \mathbf{X}$ such that $\forall \mathbf{F} \in \mathbf{X}$ holds

$$\int_\Omega \frac{1}{\mu}(\nabla \times \mathbf{E}) \cdot (\nabla \times \bar{\mathbf{F}}) + \int_\Omega \frac{\tau}{\bar{\varepsilon}\varepsilon\mu}(\nabla \cdot (\varepsilon \mathbf{E})) \cdot (\nabla \cdot (\bar{\varepsilon}\bar{\mathbf{F}}))$$

$$- \omega^2 \int_\Omega \varepsilon \mathbf{E} \cdot \bar{\mathbf{F}} + \mathbf{W.R.B.C.}|_{\partial\Omega} = -j\omega \int_\Omega \mathbf{J} \cdot \bar{\mathbf{F}}, \tag{14}$$

where the expression $\mathbf{W.R.B.C.}|_{\partial\Omega}$ is the term, properly adapted to the weighted regularization, that takes into account the radiation boundary conditions or the modes in a waveguide port. In Costabel and Dauge (2002), it is demonstrated that solving Eq. (14) is equivalent to solving the Maxwell equations and also that $\mathbf{H}_0^1(\Omega)$ is dense in $\mathbf{X}$. These results imply that nodal elements converge to the right electromagnetic field solution even in the presence of singularities.

The method proposed in this work is a simplification of the WRME method. Instead of Eq. (13), the divergence term of Eq. (11) is multiplied by a weight that is equal to zero in the elements near a singularity and equal to one in the rest. The same idea appears in Preis et al. (2000) for eddy current problems with the $\mathbf{T} - \Omega$ formulation and also in Kaltenbacher and Reitzinger (2002) for magnetostatic problems with the potential $\mathbf{A}$ formulation. It is worth mentioning that the WRME method always needs finer meshes and more iterations of the iterative solvers to achieve the same accuracy as the simplified formulation. This was observed for the 2D problems in the next section. No testing of the WRME method for 3D problems was performed.

Henceforth, when a problem is said to be solved with one ungauged layer (UL), it means that only the elements with a node in contact with a singularity have a weight equal to zero. If a problem is said to be solved with 2 ULs, it means that the weight is set to zero in the elements with a node in contact with a singularity and also in the elements that have a node in contact with the elements of the first layer. An equivalent definition is applied for 3 ULs, 4 ULs, and so on. Only the elements in the UL have a weight equal to zero; for the rest, the weight is equal to one. These ULs must be applied in all places where the field is singular. If a singularity point is kept without this treatment, the simulation can give a globally wrong solution. To know the places where the field is singular, analyze the geometry and look for (Bladel, 1991):

- reentrant corners and edges of PECs,
- corners and edges of dielectrics, and
- intersection of several dielectrics.

Once these points are located, the number of ULs depends on the size and order of the elements, as shown in the next sections. If, for some reason, very small elements are required around the singularity, the number of ULs must be increased. Not doing so is equivalent to use Eq. (11) instead of Eq. (14). To distinguish the simplified formulation from the WRME method, it is called HORUS (high-order elements ungauged near the singularity).

Validation Examples in Two Dimensions

This section shows some 2D configurations used to check the accuracy of HORUS (Figures 3, 4, 5, and 6). The geometries employed are four different discontinuities in a

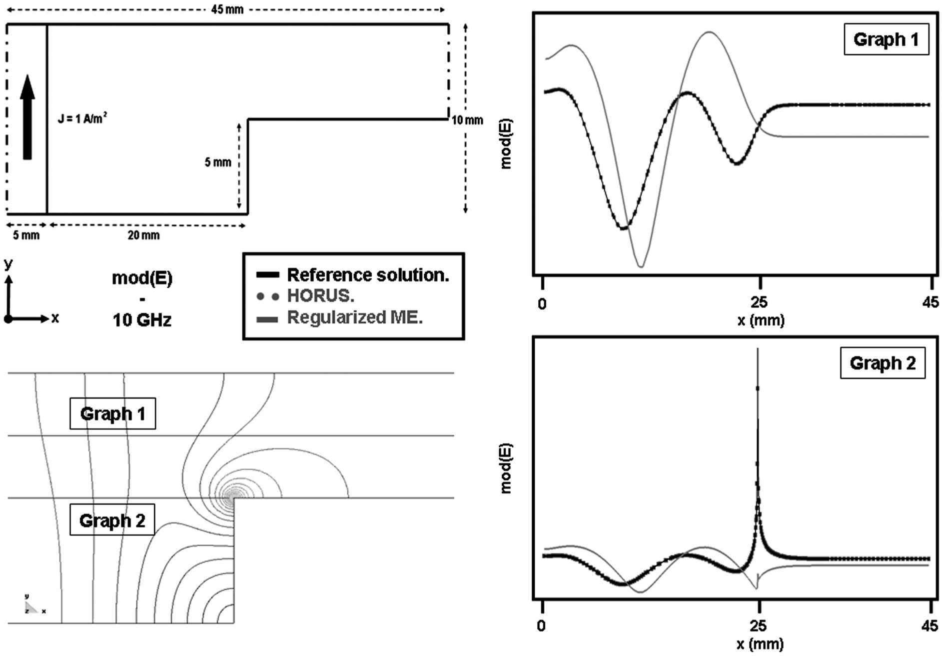

Figure 3. Step discontinuity in a parallel-plate waveguide.

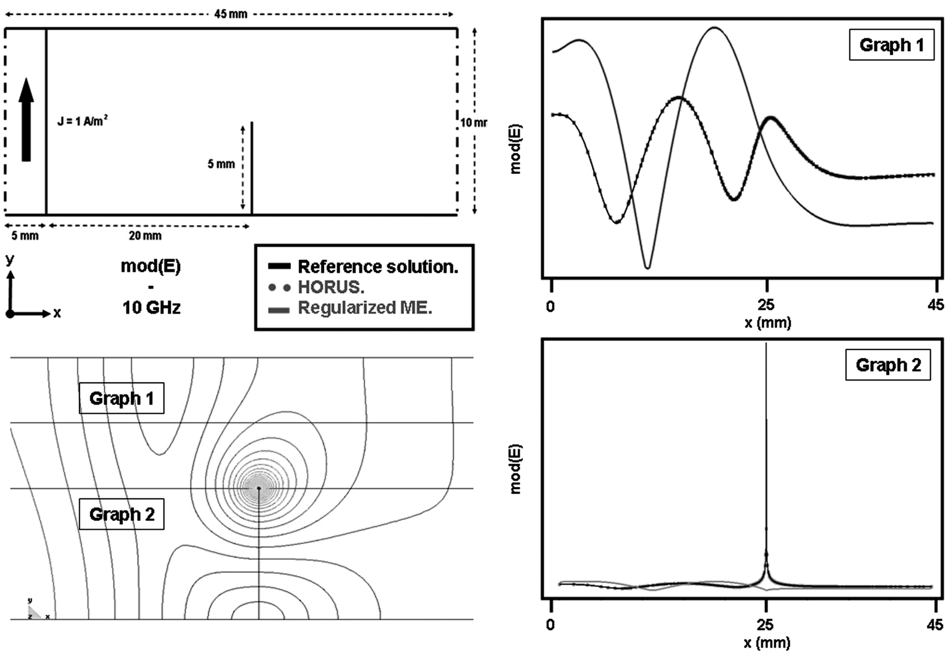

Figure 4. Sheet singularity in a parallel-plate waveguide.

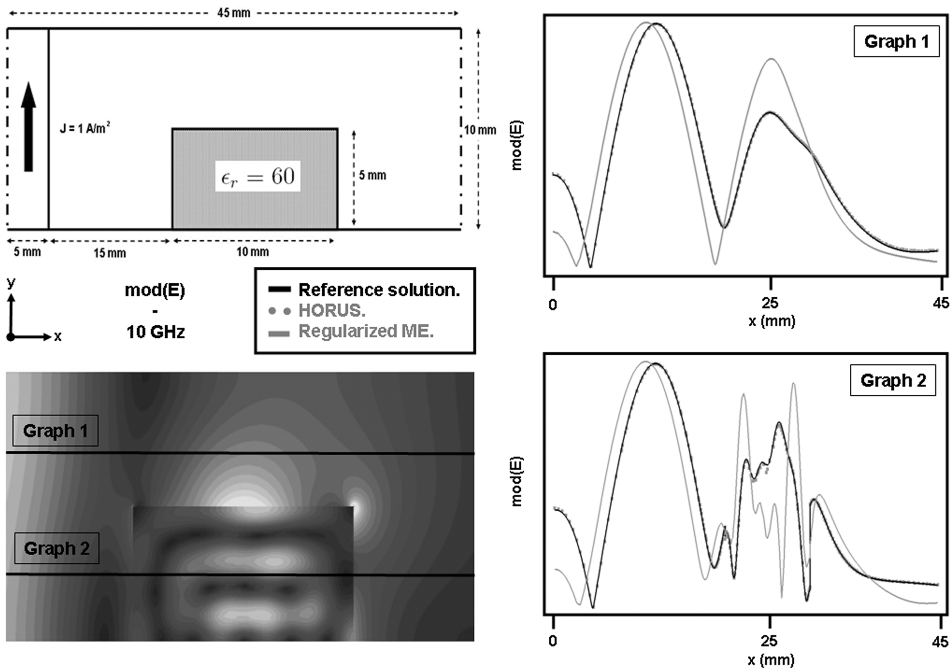

Figure 5. Dielectric in a parallel-plate waveguide.

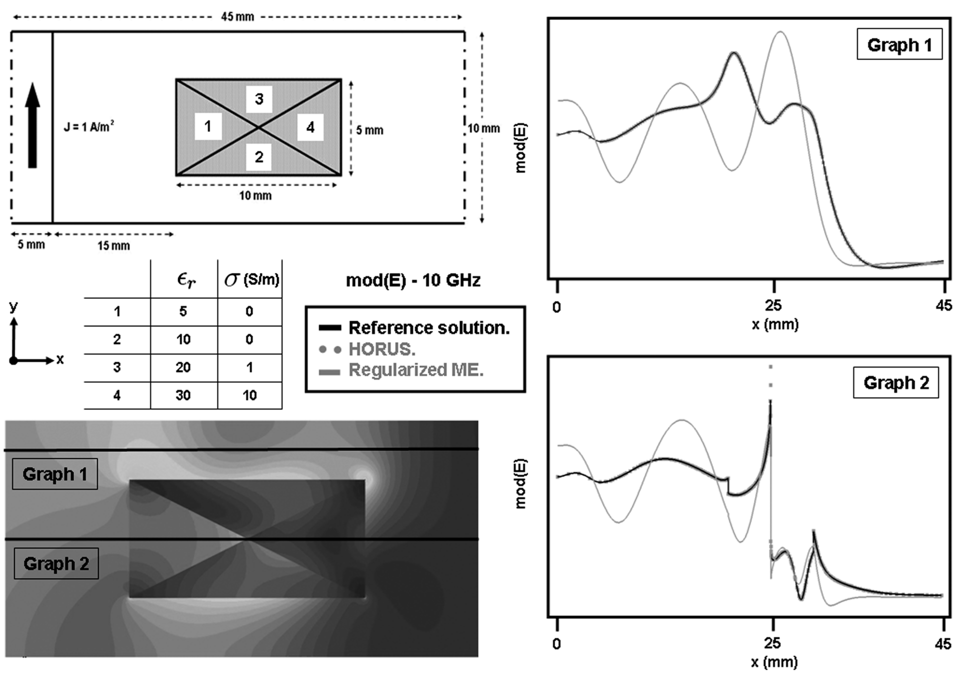

Figure 6. Intersection of dielectrics in parallel-plate waveguide.

parallel-plate waveguide. The problems are driven by a current density $J_y = 1$ A/m^2 at a frequency of 10 GHz. The walls of the waveguide are PECs. The discontinuous lines at the sides symbolize the application of the first-order absorbing boundary conditions (first ABCs). The reference solution $\mathbf{E} = (E_x, E_y)$ is obtained by the differentiation of H_z with the method explained in Jin (2002). On the other hand, HORUS calculates $\mathbf{E} = (E_x, E_y)$ directly from Eq. (14), with the weight τ as it is explained in the previous section. The unit normals at the nodes placed in singular corners are calculated as a geometric average, and they are used in Eq. (12), or in the PEC condition, like in any other node (Boyse et al., 1992). HORUS was tested with triangular nodal elements from the first to sixth order. From the third- to sixth-order elements and 1 UL, accurate solutions were obtained even near the singularity. With second-order elements and 3 ULs, correct solutions were also obtained. With first-order elements, it was not possible to obtain accurate solutions even with 6 ULs. In the graphs of Figures 3, 4, 5, and 6, it can be seen that the reference solution and HORUS overlap perfectly, even in the neighborhood of the singularity. In the same graphs, the result of using Eq. (11) without taking into account the effect of the singularities is also represented (graph *Regularized ME*).

Validation Examples in Three Dimensions

This section presents some 3D configurations solved with HORUS. Figures 7, 8, and 9 display three microwave filters taken from Montejo-Garai and Zapata (1995), Mansour et al. (1988), and Katzier (1984). HORUS calculates the scattering parameters of this microwave filter applying Eqs. (3), (4), and (9) in the waveguide ports. The elements used were second- and third-order tetrahedral nodal elements. With third-order elements and 1 UL, excellent results were obtained, as can be seen in Figures 7, 8, and 9. With second-order elements and 3 ULs, good results can also be obtained, but less accurate than with third-order elements. Nonetheless, although second-order elements are less accurate than third-order elements, they are computationally more efficient and very useful for some applications.

The last validation example (Figures 10, 11, 12, and 13) is a PROCOM MU9-XP4 antenna (PROCOM, Gørløse, Denmark) installed in a motorbike. This problem was solved in the frame of the project PROFIT SANTTRA (Sistema de Antenas para Transceptores de Radio, Ref: FIT-330210-2006-44) based on the European project PIDEA-EUREKA SMART (Smart Antennas System for Radio Transceivers). EADS Secure Networks (2008) performed the simulation with the method of moments (MoM) implemented in the commercial code FEKO (FEKO, 2008), and the results were compared with HORUS. The objective was to calculate the specific absorption rate (SAR) distribution produced by the antenna in a body placed at a distance 0.6 m. The antenna is fed by a coaxial source with 10 W (40 dBm) at a frequency of 406 MHz. The body has a density $\rho = 1{,}000$ Kg/m^3 with electric properties: $\epsilon_r = 41.3$, $\sigma = 0.58$ S/m, and $\mu_r = 1$. A PEC ground is placed 1.22 m below the feeding point of the antenna. A PEC topcase is placed 0.15 m below the same feeding point (Figures 10 and 11). HORUS employed tetrahedral nodal elements of the second order with 3 ULs in the edges of the topcase and also 3 ULs in the edges of the base and tip of the antenna. Extended first-order ABCs (Eq. (8)) were used as boundary conditions. The results of the simulations are shown in Figures 12 and 13. The maximum SAR value obtained with HORUS was 0.06 W/Kg, located in the neck. The maximum SAR value obtained with FEKO was 0.07 W/Kg, also located in the neck. The SAR averaged over the 10-g cube was 0.04 W/Kg (HORUS) and 0.06 W/Kg (FEKO).

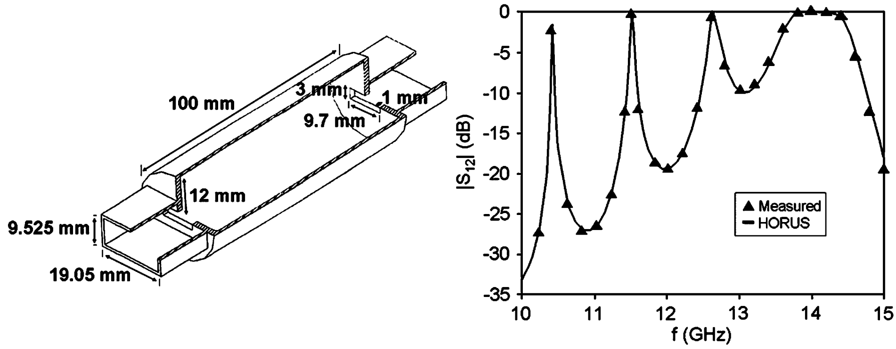

Figure 7. Cylindrical cavity filter measured in Montejo-Garai and Zapata (1995). HORUS used tetrahedral third-order nodal elements, and the weight in Eq. (14) was set to zero only in the elements with a node in contact with the edges of the coupling slots (1 UL).

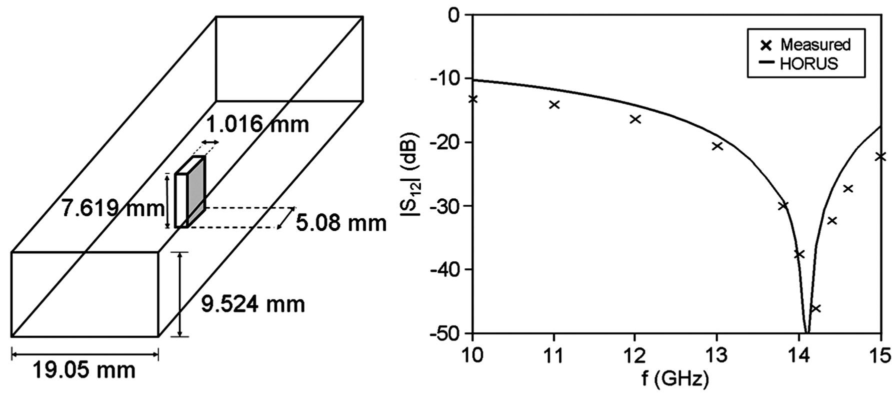

Figure 8. Ridge waveguide measured in Mansour et al. (1988). HORUS used tetrahedral third-order nodal elements, and the weight in Eq. (14) was set to zero only in the elements with a node in contact with the edges of the ridge (1 UL).

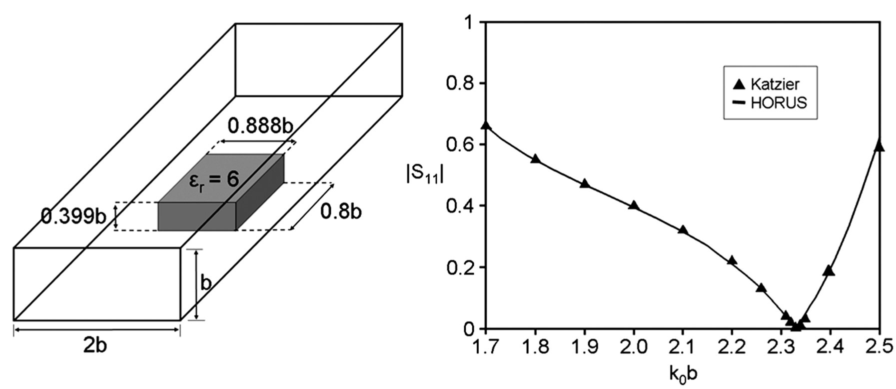

Figure 9. Dielectric in a rectangular waveguide. Reference scattering parameters values obtained from Katzier (1984). HORUS used tetrahedral third-order nodal elements, and the weight in Eq. (14) was set to zero only in the elements with a node in contact with the edges of the dielectric (1 UL).

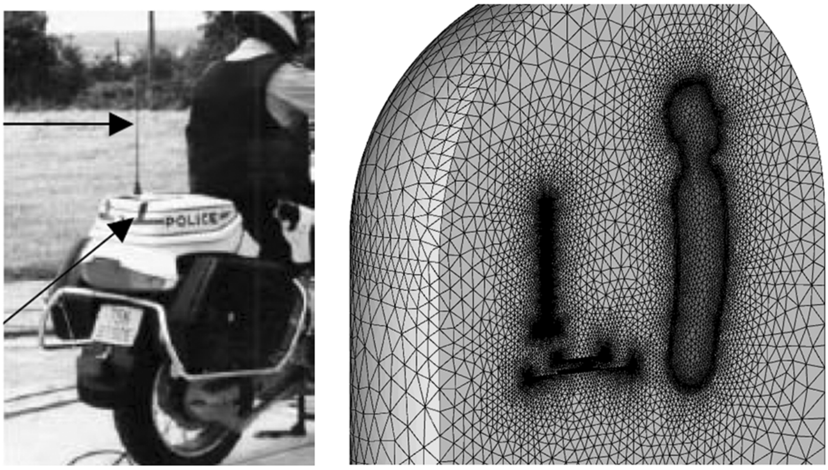

Figure 10. PROCOM MU9-XP4 antenna installed in a motorbike: left, real geometry; and right, FEM mesh used for HORUS simulations. The mesh is composed of tetrahedral second-order nodal elements.

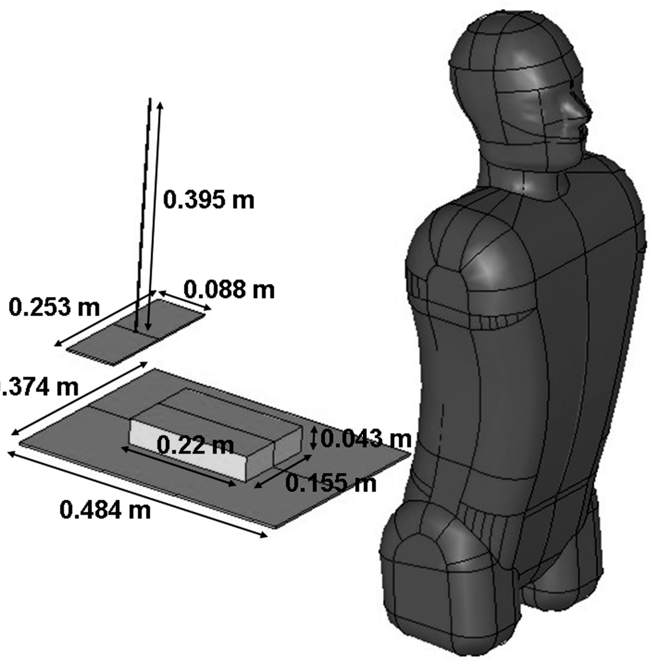

Figure 11. PROCOM MU9-XP4 antenna installed in a motorbike. Detailed geometry.

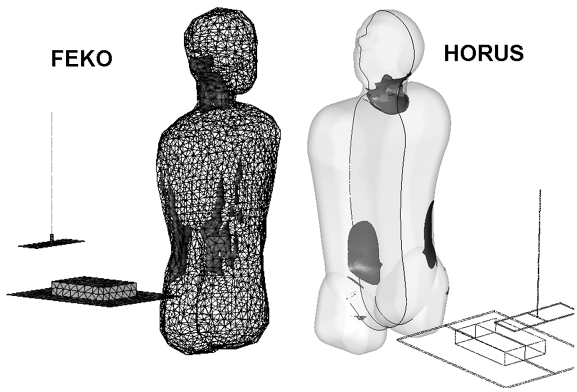

Figure 12. Iso-surfaces for SAR = 0.03 W/Kg: left, FEKO results and right, HORUS results.

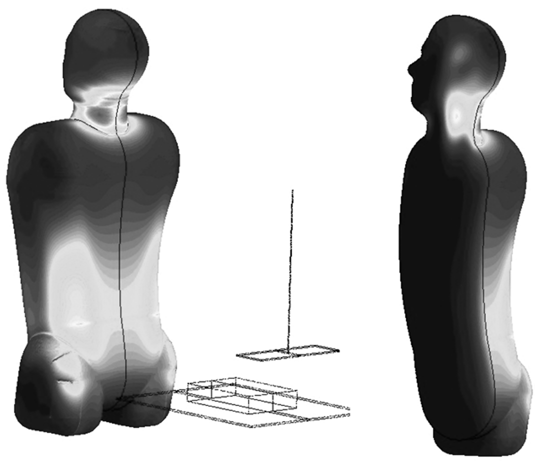

Figure 13. SAR distribution calculated with HORUS. Maximum value is 0.06 W/Kg, located in the neck.

Conclusion

It is demonstrated in this article that an accurate solution can be obtained with nodal elements and the regularized formulation even in the presence of electromagnetic field singularities. To do so, a simplified version of the WRME method was used. This simplification consists of a weight equal to zero applied in the elements near a singularity and equal to one in the rest. The number of layers of elements whose weight is equal to zero depends on their size and order. For a given element order, this number is fixed, but if for some reason, it is meshed with very small elements near a singularity, this number must be increased to obtain a correct solution. It is necessary to further study the relationship between the number of elements with a weight equal to zero and the element size, element order, and singularity order. Also, a deeper theoretical knowledge of how this simplification affects the well-conditioning and convergence of the regularized formulation is needed. In future work, a comparative performance of HORUS with other methods will be given.

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References

Bladel, J. V. 1991. *Singular electromagnetic fields and sources*. New York: IEEE Press.

Boyse, W. E., D. R. Lynch, K. D. Paulsen, & G. N. Minerbo. 1992. Nodal-based finite-element modeling of Maxwell's equations. *IEEE Trans. Antennas Propagat.* 40:642–651.

Boyse, W. E., & A. A. Seidl. 1994. A hybrid finite element method for 3-D scattering using nodal and edge elements. *IEEE Trans. Antennas Propagat.* 42:1436–1442.

Ciarlet, P. 2005. Augmented formulations for solving Maxwell equations. *Comput. Methods Appl. Mechan. Eng.* 194:559–586.

Costabel, M. 1991. A coercive bilinear form for Maxwell's equations. *J. Math. Anal. Appl.* 157:527–541.

Costabel, M., & M. Dauge. 1999. Maxwell and Lamé eigenvalues on polyhedral. *Math. Methods Appl. Sci.* 22:243–258.

Costabel, M., & M. Dauge. 2002. Weighted regularization of Maxwell equations in polyhedral domains. *Numer. Math.* 93:239–277.

Dhia, A. B.-B., C. Hazard, & S. Lohrengel. 1999. A singular field method for the solution of Maxwell's equations in polyhedral domains. *SIAM J. Appl. Math.* 59:2028–2044.

EADS Secure Networks. 2008. *EADS-SN*. Available on-line at http://www.eads.net (accessed 19 January 2010). Elancourt, France: Author.

FEKO. 2008. *EM software & systems*. Available on-line at http://www.feko.info (accessed 19 January 2010).

Hazard, C., & M. Lenoir. 1996. On the solution of the time-harmonic scattering problems for Maxwell's equations. *SIAM J. Math. Anal.* 27:1597–1630.

Hiptmair, R. 2002. Finite elements in computational electromagnetism. *Acta Numer.* 1:237–339.

Jin, J. 2002. *The finite element method in electromagnetics*, 2nd ed. New York: John Wiley & Sons.

Kaltenbacher, M., & S. Reitzinger. 2002. Appropiate finite-element formulation for 3-D electromagnetic-field problems. *IEEE Trans. Magnet.* 38:513–516.

Katzier, H. 1984. Streuverhalten elektromagnetischer Wellen bei sprunghaften Übergängen geschirmter dielektrischer leitungen. *Arch. Elek. Übertragung* 38:290–296.

Lager, I. E. 1996. *Finite element modelling of static and stationary electric and magnetic fields*. Ph.D. Thesis, Delft University, Delft, The Netherlands.

Mansour, R. R., R. S. K. Tong, & R. H. Machpie. 1988. Simplified description of the field distribution in finlines and ridge waveguides and its application to the analysis of E-plane discontinuities. *IEEE Trans. Microw. Theory Techniq.* 36:1825–1832.

Montejo-Garai, J. R., & J. Zapata. 1995. Full-wave design and realization of multicoupled dual-mode circular waveguide filters. *IEEE Trans. Microw. Theory Techniq.* 43:1290–1297.

Mur, G. 1994. Edge elements, their advantages and their disadvantages. *IEEE Trans. Magnet.* 30:3552–3557.

Mur, G. 1998. The fallacy of edge elements. *IEEE Trans. Magnet.* 34:3244–3247.

Nedelec, J. C. 1980. Mixed finite elements in R^3. *Numer. Math.* 35:315–341.

Paulsen, K. D., D. R. Lynch, & J. W. Strohbehn. 1987. Numerical treatment of boundary conditions at points connecting more than two electrically distinct regions. *Commun. Appl. Numer. Methods* 3:53–62.

Paulsen, K. D., D. R. Lynch, & J. W. Strohbehn. 1988. Three-dimensional finite, boundary, and hybrid element solutions of the Maxwell equations for lossy dielectric media. *IEEE Trans. Microw. Theory Techniq.* 36:682–693.

Preis, K., O. Bíró, & I. Ticar. 2000. Gauged current vector potential and reetrant corners in the FEM analysis of 3D eddy currents. *IEEE Trans. Magnet.* 36:840–843.

Salazar-Palma, M., T. K. Sarkar, L.-E. García Castillo, T. Roy, & A. Djordjevic. 1998. *Iterative and self-adaptive finite-elements in electromagnetic modeling*. Boston–London: Artech House Publishers.

Webb, J. P. 1993. Edge elements and what they can do for you. *IEEE Trans. Magnet.* 29:1460–1465.