ResearchGate

See discussions, stats, and author profiles for this publication at: https://www.researchgate.net/publication/3359692

Whitney forms: A class of finite elements for three-dimensional computations in electromagnetism

Article *in* Science, Measurement and Technology, IEE Proceedings A · December 1988

DOI: 10.1049/ip-a-1:19880077 · Source: IEEE Xplore

CITATIONS
467

READS
2,278

1 author:

Alain Bossavit
École Supérieure d'Electricité

225 PUBLICATIONS **5,648** CITATIONS

SEE PROFILE

All content following this page was uploaded by Alain Bossavit on 02 March 2013.

The user has requested enhancement of the downloaded file.

Whitney forms: a class of finite elements for three-dimensional computations in electromagnetism

Alain Bossavit

Indexing terms: Electromagnetic theory, Eddy currents, Mathematical techniques

Abstract: It has been recognised that numerical computations of magnetic fields by the finite-element method may require new types of elements, whose degrees of freedom are not field values at mesh nodes, but other field-related quantities like e.g. circulations along edges of the mesh. A rationale for the use of these special 'mixed' elements can be obtained if one expresses basic equations in terms of *differential forms*, instead of vector fields. The paper gives an elementary introduction to this point of view, presents Whitney forms (the mixed finite elements alluded to), and sketches two numerical methods (dual, in some sense), for eddy-current studies, based on these elements.

1 Introduction

For those familiar with differential forms, Maxwell's equations are best expressed in the language they provide: h and e are 1-forms, i.e. forms of degree one, b and j are 2-forms. This, as we shall show in Section 2 of this paper, which consists in an elementary introduction to differential forms, means that the *circulations* of h and e along paths make sense from the physical point of view, while the *fluxes* of b and j may be understood through surfaces. Differential forms are, therefore, a useful tool, and some have argued that they should be used in electrodynamics, at least at the research level [2, 10, 12, 14], and even for teaching [13]. But it is often felt that, at least in the case of eddy-current studies, which never venture out of the nonrelativistic realm, the full power of this tool is not called for. Indeed, it can be observed that most numerical analysts and engineers engaged in the study of electromagnetism prefer to think in terms of vector fields. Moreover, representations in co-ordinates are often preferred to vector expressions. It is therefore understandable that the bulk of the effort towards finite-element modelling has consisted, up to now, in adapting to *vector*-field methods which worked well for *scalar* fields, like e.g. those used for solving the heat equation.

Paper 6282A (S8), first received 4th January and in revised form 7th June 1988

The author is with Electricité de France, 1 avenue du Général de Gaulle, 92141 Clamart, France

If such a trend were to continue, it would contrast with the present tendency to geometrisation which can be seen to pervade all physics. According to this tendency, attention should focus on geometrical entities, and not on their representations as multiples of values in some co-ordinate system. If we accept this stand, it will not be difficult to argue further than differential forms, and not vector fields, are the appropriate geometrical entities. But this will fail to convince the practising programmer who has to deal with such objects as finite-element shape functions: these seem to require a co-ordinate system for their manipulation. The situation will be different if we can define geometrical objects that are to differential forms what finite-element interpolating functions are to scalar fields.

This is precisely the definition of Whitney forms. Briefly stated (Section 3 of the paper gives a more comprehensive description), they are a family of differential forms on a simplicial mesh (i.e. a network of tetrahedra, as used in finite-element studies), defined in such a way that p-forms are determined by their integrals on p-simplices. One-forms (such as, in electricity, h and e) can then be approximated by a suitable linear combination of Whitney forms of degree 1, the coefficients being the circulations of the field along the edges of the mesh. In other words, Whitney 1-forms play the role of finite elements for h or e, but the so-called degrees of freedom are associated with edges of the mesh, and are not the values of the components of the field at mesh nodes. This justifies the nickname of 'edge elements' for Whitney 1-forms. Similarly, there are 'facet elements' which accommodate 2-forms, the degrees of freedom being fluxes across facets. 'Node elements' are just piecewise-linear functions (they are commonly called P^1), and 'volume elements' are piecewise-constant functions (similarly known as P^0).

Section 4 of the paper shows how these concepts can be used to devise approximation methods for the eddy-current equations. The two methods we propose look, in some sense, symmetrical. This symmetry, or rather this *duality*, is rooted in the duality properties of the mathematical structure that differential forms constitute.

All these arguments, we think, converge to suggest that differential forms should be used as a working tool in numerical modelling of electromagnetic problems.

This does not imply such a radical change of thinking habits as we might fear. We have tried to present the topic to simplify the transition from vector fields to differential forms, albeit at the risk of criticism from the side of mathematical tradition. (In particular, our *ad-hoc* treatment of the Hodge star operator certainly incurs such a risk. We also ignore 'twisted forms', whose relevance to Maxwell equations is stressed in the excellent book by Burke [8].)

Now, even if this plea for differential forms is not successful, Whitney forms as introduced here are interesting in their own right. Their value as finite elements for field computation has begun to be well documented [1].

Whitney forms were described in 1957, long before the use of finite elements [20]. They were rediscovered in the finite elements community since 1974, under the name of 'mixed elements', which explains why our two numerical schemes fall in the category of 'mixed methods'. The relevance of Whitney elements to mixed methods, in general, have been exposed in Reference 6.

The rest of this Introduction will be devoted to notations and to basic equations.

The three-dimensional (3D) eddy-current problem has the following mathematical description (Fig. 1). A region

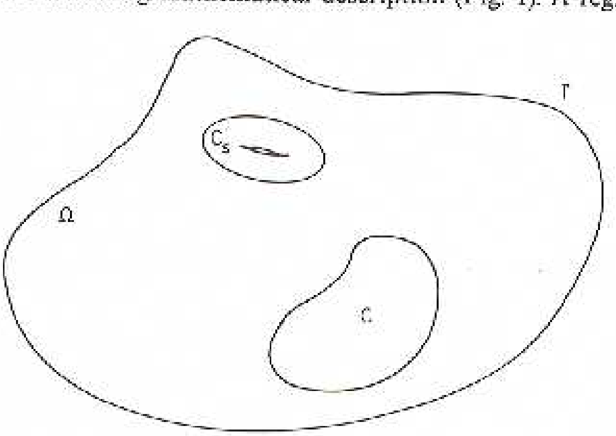


Fig. 1 *The situation: Ω is a bounded region, Γ its boundary, C a conductor and C_s an inducting coil*

Ω of space, with smooth boundary Γ, is divided up into three regions: the conductor C, the inductor C_s (s for source) and the air region. The permeability μ and the conductivity σ ($\sigma = 0$ and $\mu = \mu_0$ in air and in C_s) depend only on position, not on the values of the fields.

It will be assumed in Section 4 that domains Ω, C and $\Omega - \bar{C}$ are connected and simply-connected (no holes, no loops). Otherwise, the exposition would become too involved, without immediate benefit. But it should be stressed that the approach outlined here works as well in the general case. Indeed, the possibility of dealing with general geometries is one of the advantages of this approach, although this will not be developed here.

A current density j_s, divergence-free, null outside of C_s, is given as a function of space x and time t. We wish to compute the magnetic field h and the current density j which develop, starting from a given state of the electromagnetic field at $t = 0$. The relevant equations are well known:

$$\partial_t b + \text{curl } e = 0 \tag{1}$$

$$b = \mu h + b_s \tag{2}$$

$$\text{curl } h = j \tag{3}$$

$$j = \sigma e + j_s \tag{4}$$

where b_s is a given divergence-free field. b_s, which is also a source of the electromagnetic field, corresponds to magnets. It could be omitted (and will be dropped without further notice in Section 3), because magnets can be replaced by equivalent current distributions, but we keep it for the sake of symmetry. The following expected equation:

$$\text{div } b = 0 \tag{5}$$

is not necessary if $b = 0$ at $t = 0$, for it stems from eqn. 1. Similarly, div $j = 0$ is a consequence of eqn. 3.

We shall assume that one of the following sets of boundary conditions holds:

$$n \times h = 0 \quad \text{and} \quad n \cdot e = 0 \quad \text{on } \Gamma \tag{6}$$

(the 'normal field' condition), or

$$n \cdot b = 0 \quad \text{and} \quad n \times e = 0 \quad \text{on } \Gamma \tag{7}$$

(the 'tangential induction' condition), where n is a field of outward unit normals on Γ. As is well known, condition 6 makes sense if the permeability μ is infinite outside Γ ('in the wall', in other words), and condition 7 corresponds to the case of a superconductive wall (or, more frequently, to a symmetry plane). In the general case (μ and σ finite), formulations 6 and 7 are approximations with respect to physical reality, that are generally deemed acceptable if the boundary Γ is far enough from the region of interest. The presence of such a wall at finite distance is a simplifying assumption which we could do without. It allows us to consider a tetrahedral paving of Ω with a *finite* number of tetrahedra.

2 Eddy-current equations in the language of differential forms

2.1 Differential forms

Differential forms are fields of alternating multilinear mappings from $\mathbb{R}^p$ to $\mathbb{R}$. If ω is such a form, its value at point x is the mapping which, starting from p ordinary three-dimensional vectors $\xi_1, \ldots, \xi_p$ (which can be seen as originating from x), yields a real number $\omega(\xi_1, \ldots, \xi_p)$. 'Alternating' means that permuting two vectors inverts the sign of the result (which, thus, is 0 if $p > 3$, so only forms of degree $p = 0$, 1, 2, or 3 will be considered). We call these 'p-forms', or 'forms of degree p'. In three dimensions, differential forms can be conceived as a disguise for vector-valued fields (if $p = 1$ or 2) or scalar-valued fields ($p = 0$ or 3). For, if u is a vector field in Ω, the maps

$$\xi \to u(x) \cdot \xi \tag{8}$$

$$\{\xi, \eta\} \to u(x) \cdot (\xi \times \eta) \equiv (u(x), \xi, \eta) \tag{9}$$

where (, ,) denotes the mixed product, are alternating ones. So rule 8 yields a 1-form, which we shall note ${}^1u$, and rule 9 yields a 2-form ${}^2u$. (Conversely, given a 1- or 2-form ω, there is a vector field u such that $\omega = {}^pu$, for $p = 1$ or 2.) Similarly, to a function ϕ on Ω corresponds the 3-form ${}^3\phi$ whose value at point x is the alternating map

$$\{\xi, \eta, \zeta\} \to \phi(x)(\xi, \eta, \zeta). \tag{10}$$

The same function ϕ generates a 0-form, which, by the general definition, should be a field on Ω whose value at x is a mapping from the empty set into $\mathbb{R}$. Such a mapping is a real-valued constant. This constant we take as $\phi(x)$, hence the 0-form ${}^0\phi$. Now define the following operator d from the space of p-forms $F^p(\Omega)$ to $F^{p+1}(\Omega)$ as follows. If $p = 0$, $d\,{}^0\phi$ (the 0 is attached to ϕ, not to d) is the 1-form associated with the vector-field grad ϕ, that is

$$d\,{}^0\phi = {}^1(\text{grad } \phi) \tag{11}$$

d is similarly defined for $p = 1$ or 2:

$$d\,{}^1u = {}^2(\text{curl } u) \tag{12}$$

$$d\,{}^2u = {}^3(\text{div } u) \tag{13}$$

and, for $p = 3$, we set $d\,{}^3\phi = 0$.

In a regular course on differential geometry (see e.g. Reference 18), these definitions would work the other way, as we would, first, define d in general, then introduce grad, curl and div as particular realisations of d. The following formula, which is derived from eqns. 11–13, would then be a consequence of the definition:

$$d \circ d = 0\,. \tag{14}$$

The converse of eqn. 14, that is 'if a p-form ω is such that $d\omega = 0$, then there is a $(p-1)$-form α such that $\omega = d\alpha$', which is true if Ω is the whole space, is known as the Poincaré lemma. When the space is simply-connected, this lemma is valid for $p = 1$. Spaces for which the lemma is valid for all p are said to be contractible.

2.2 The de Rham complex

In the general case, the situation would be depicted by the following diagram (Fig. 2). (Such an algebraic structure, where there is a family of sets of the same kind and an operator d between them which has the property of eqn. 14, is known as a homological complex, and the study of such things is homological algebra. The present structure is de Rham's complex.)

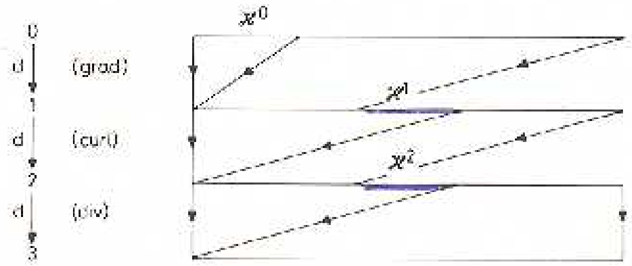


Fig. 2 *De Rham's complex*
See text for graphic conventions

Horizontal segments in Fig. 2 represent vector subspaces of $F^p(\Omega)$ for $0 \leqslant p \leqslant 3$. (The 2 or 3 subspaces displayed for a given p are, in fact, mutually orthogonal with respect to the natural scalar product

$$({}^p\phi, {}^p\phi') = \int_\Omega \phi\phi' \tag{15}$$

for $p = 0$ or 3, and

$$({}^pu, {}^pu') = \int_\Omega u \cdot u' \tag{16}$$

for $p = 1$ or 2.) Now take two slanted arrows at some level of the diagram. Between the tails of the arrows is a subspace, between their heads is its image by d. So the diagram displays the kernels of d (i.e. the constants if $p = 0$, curl-free fields if $p = 1$, div-free fields if $p = 2$), the images by d (i.e. the constants if $p = 0$, curl-free fields if $p = 1$, div-free fields if $p = 2$), the images by d (i.e. the spaces of gradients, of curls etc.) and the orthogonal complements of images with respect to kernels. These complements, here called $\mathscr{H}^0$, $\mathscr{H}^1$, $\mathscr{H}^2$, are known as 'cohomology spaces', and their dimensions (which are finite in the case of non-pathological domains) measure how short from being contractible Ω falls. More precisely, we have the following characterisations:

$$\mathscr{H}^0 = \{\phi\colon \phi \text{ is constant on connected components of } \Omega\} \tag{17}$$

$$\mathscr{H}^1 = \{{}^1u\colon \text{div } u = 0,\ \text{curl } u = 0,\ n \cdot u = 0 \quad \text{on } \Gamma\} \tag{18}$$

$$\mathscr{H}^2 = \{{}^2u\colon \text{div } u = 0,\ \text{curl } u = 0,\ n \times u = 0 \quad \text{on } \Gamma\} \tag{19}$$

$\mathscr{H}^3 = \{0\}$ and

$$\dim(\mathscr{H}^0) = \text{number of connected components of } \Omega \tag{20}$$

$$\dim(\mathscr{H}^1) = \text{number of 'loops' in } \Omega \tag{21}$$

$$\dim(\mathscr{H}^2) = \text{number of 'cavities' in } \Omega \tag{22}$$

The purpose of the de Rham complex is precisely to give a handle (through an algebraic structure which can be investigated by algebraic techniques) on such global topological properties. In particular, a domain is simply-connected if $\mathscr{H}^1 = \{0\}$, contractible if $\mathscr{H}^p = \{0\}$ for all p. In the domain, we are not interested in topology for its own sake, rather in the relations between de Rham's and Whitney's complexes, which will appear in Section 3. So, in order not to mix difficulties of independent origins, we shall assume, from now on, that $\mathscr{H}^1$ and $\mathscr{H}^2$ reduce to $\{0\}$ (Ω contractible).

Another general concept of differential geometry which can be readily introduced at this elementary level is the Hodge operator $*$ from F^p to F^{n-p} ($n = 3$ here). By definition,

$$*({}^0\phi) = {}^3\phi \qquad *({}^3\phi) = {}^0\phi \tag{23}$$

$$*({}^1u) = {}^2u \qquad *({}^2u) = {}^1u \tag{24}$$

2.3 Eddy-current equations

A first application of these considerations is Fig. 3, which displays the eddy-current equations in the language of differential forms. The diagram (whose resemblance to the one given in Reference 16 is not accidental) should be self-explanatory.

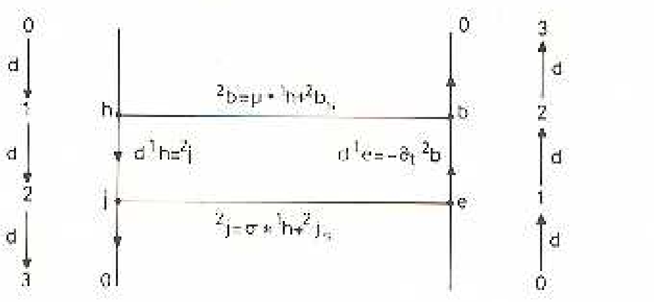


Fig. 3 *The structure of eddy-current equations*

For instance, the relation $b = \mu h + b_s$ can be rewritten as ${}^2b = \mu * ({}^1h) + {}^2b_s$, which shows the relatively complex character of such a constitutive law. This relates a 1-form (the field h) with a 2-form (the induction b) by first transforming ${}^1h$ into its Hodge dual ${}^2h$, now a 2-form, multiplying by μ, then adding the (optional) source magnetisation b_s. Clearly, the change of notation does not bring anything of interest by itself. What we gain, which is much more important, is an insight into the structure of the equations, which is exposed by Fig. 3 by the trick of locating b, h etc. at the right levels.

3 Whitney forms

3.1 Orientation

Now, as an attempt to provide a rationale for what follows, we shall outline the treatment of a particular simple case of the stationary heat-flow problem (which is structurally similar to the magnetostatics problem, as we shall see).

Let us suppose we want to find a vector field u in Ω, satisfying

$$\text{curl } u = 0 \tag{25}$$

$$\text{div } (\kappa u) = f \tag{26}$$

for a given f and a constant κ, and the boundary condition $n \cdot u = 0$ on Γ. We look for an approximation of u as

$$u = \sum_i u_i \text{ grad } w_i \tag{27}$$

where w_i are P^1 interpolants with respect to a tetrahedral paving. So, if x_i (a point in 3-space) is the position of node i, w_i is continuous, piecewise linear, with $w_i(x_j) = 0$ if $j \neq i$, and 1 if $i = j$. This way, eqn. 25 is satisfied. But eqn. 26 cannot hold, because, owing to normal discontinuities of grad w_i across facets, u in eqn. 27 and thus κu have no divergence in the usual sense (their divergence is a distribution, not a function). As condition 26 cannot be enforced in this strong form, we express it in 'weak form', i.e. we look for some u of the form given in eqn. 27, which in addition satisfies

$$-\int_\Omega \kappa u \cdot \text{grad } \phi' = \int_\Omega f\phi' \tag{28}$$

for all ϕ' which are linear combinations of the basis functions w_i.

Note that eqns. 25 and 26 may be rewritten as

$$d\,^1u = 0 \tag{29}$$

$$d\,^2(\kappa u) = {}^3f \tag{30}$$

two equations of similar form. Both cannot be exactly satisfied simultaneously in discrete form. So, in the familiar finite-element procedure described above, we choose to take one of the equations at face value (here, eqn. 25), and to deal with the other one by a transposition of the differential operator d, which shifts from the left-hand side (div in eqn. 26) to the right-hand side (grad in eqn. 28) of a scalar product of forms.

It has been known for a while, among finite-element specialists, that the other choice is feasible: treat eqn. 26 as is given and eqn. 25 in transposed form. This is known as the 'mixed' approach [7, 11]. We purport to show that this idea works for eddy-current equations as well (and even, although this will not be developed here, for a large class of equations in mathematical physics), thanks to the existence, on a given mesh of tetrahedra, of a family of piecewise polynomial differential forms (Whitney's complex) which can be described as *finite-element bases for differential forms.*

3.2 The Whitney complex

Let us consider a tetrahedral paving of Ω, with the usual conditions that two tetrahedra may have in common, a facet, an edge, a vertex, or they may be disjoint, to the exclusion of all other possibilities. Let V, E, F and T be the sets of vertices, edges, facets and tetrahedra, respectively. They will be identified by the indices of their vertices: for instance edge $\{i, j\}$, facet $\{i, j, k\}$ etc., and called by their generic name of 'p-simplices' ($p = 0$ for vertices, 1 for edges etc.).

In Reference 20, Whitney describes a family of forms with the following properties:

(i) they are polynomials of the first degree (at most) on tetrahedra

(ii) they 'match' on the facets, in a sense to be made precise below

(iii) they are uniquely determined from their integrals on p-simplices.

Properties (ii) and (iii) need some explanation.

We say that two p-forms match, or 'conform', on a surface if they assign the same values to any given set of p-vectors tangential to the surface at some point. According to the definition of forms, 3-forms match unconditionally (because 3 vectors in the same tangent plane are dependent), 2-forms ^2u match if the *normal* components of u agree on both sides, and 1-forms ^1u if *tangential* components of u agree. Zero-forms $^0\phi$ conform if ϕ is continuous.

As for property (iii), it is natural, as p-forms act (at a given point) on sets of p-vectors, to integrate the result over manifolds of dimension p. So 1-forms have integrals over edges (that of ^1u is the circulation of the vector field u along the edge). The integral of ^2u across a facet is the flux of u through this facet (the orientation of the normal is determined by the ordering of the vertices). The integral of $^3\phi$ on a tetrahedron is that of ϕ, and the integral of $^0\phi$ on 0-simplex i is simply $\phi(x_i)$.

Whitney forms are constructed as follows: consider vertex i and a point x belonging to one of the tetrahedra which share vertex i, let $\lambda_i(x)$ be the barycentric weight of x in its tetrahedron with respect to vertex i; with the convention that $\lambda_i(x) = 0$ in other cases, we obtain a continuous, piecewise-linear function λ_i (which is w_i above). Note that

$$x = \sum_{i \in N} \lambda_i(x) x_i$$

where x_i is the position of node i, and that

$$\sum_i \lambda_i(x) \equiv 1$$

Remark: It may happen that the paving contains 'curved' tetrahedra, i.e. continuous images of straight tetrahedra. In this case, $\lambda_i(x)$ is defined as the barycentric co-ordinate of the preimage of x.

Now Whitney associates with any p-simplex $i_0, i_1, \ldots i_p$ the differential form:

$$^pw_{i_0, \ldots, i_p} = p! \sum_{j=0}^{p} (-1)^j\, {}^0\lambda_{i_j}\, d\,^0\lambda_{i_0} \times \cdots \times d\,^0\lambda_{i_{j-1}} \times d\,^0\lambda_{i_{j+1}} \times \cdots \times d\,^0\lambda_{i_p} \tag{31}$$

This rather awesome expression (which our notation does not make easier to read) results in much more manageable formulas for low values of p, as we shall see.

3.3 Whitney elements

Let us begin with $p = 0$. We obtain $^0w_i = {}^0\lambda_i$, that is

$$w_i = \lambda_i \tag{32}$$

if we consider functions instead of the associated 0-forms. So Whitney elements (as we shall call the fields which correspond to Whitney forms) are just the so-called P^1 finite-element interpolants for $p = 0$.

If $p = 1$, and if i and j are the vertices of some edge, expr. 31 similarly gives a 1-form, attached to this edge

The corresponding vector field is

$$w_{ij} = \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i \,. \tag{33}$$

(We denote grad by ∇ for shortness.) Fig. 4 may help the reader to visualise it. As $\nabla \lambda_j$ is orthogonal to facet $\{i, k, l\}$

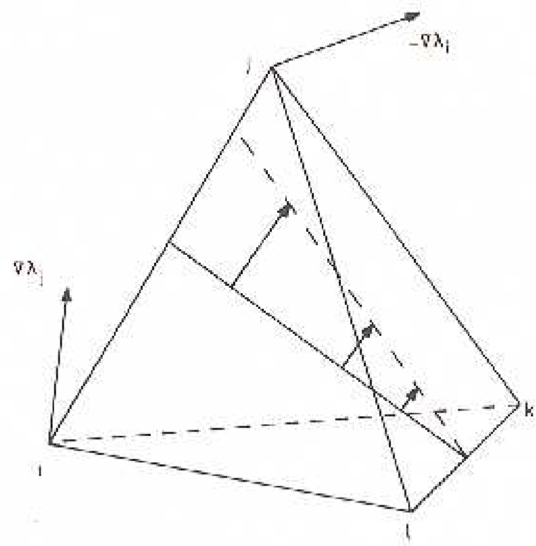


Fig. 4 *Whitney edge element* w_{ij}

and $\nabla \lambda_i$ to facet $\{j, k, l\}$, the field turns around the axis k-l (its 'central axis'), is normal to planes containing k and l, and has (at points of such a plane) a magnitude proportional to the distance to the axis. The field is nonzero over all tetrahedra which have $\{i, j\}$ as one of their edges.

As the curl of w_{ij}, which is $2\nabla\lambda_i \times \nabla\lambda_j$, exists as a function (and not only as a distribution), the tangential part of w_{ij} is continuous across facets like $\{i, j, k\}$. It is easy to check that its circulation is 1 along edge $\{i, j\}$ and 0 along all other edges. So properties (i), (ii) and (iii) are satisfied. If

$$u = \sum_{\{i,\, j\} \in E} u^{ij} w_{ij} \tag{34}$$

is a linear combination of Whitney elements of degree 1, the degrees of freedom u^{ij} are the circulations of u along the edges, hence the nickname 'edge elements'.

If $p = 2$, we obtain Whitney elements of degree 2, or 'facet elements':

$$w_{ijk} = 2(\lambda_i \nabla\lambda_j \times \nabla\lambda_k + \lambda_j \nabla\lambda_k \times \nabla\lambda_i + \lambda_k \nabla\lambda_i \times \nabla\lambda_j) \,. \tag{35}$$

Now, instead of an axial field, we have a central field (the centre is the fourth vertex) on each of the two tetrahedra which have facet $\{i, j, k\}$ in common. We imagine the field as coming from the 'source' l, growing, crossing the facet and vanishing into the 'well' l', fourth vertex of the other tetrahedron. This field has normal continuity, its flux across facet $\{ijk\}$ is 1, so such fluxes are the degrees of freedom of the element. See Fig. 5.

(For reasons which cannot be exposed here, the degrees of freedom should be localised at the centres of circles circumscribed to the facets. For edge elements, they are at mid edges.)

Whitney elements have been rediscovered in the finite-element literature since around 1974. Facet elements are described in Reference 18, although in space-dimension 2, edge elements are described in Reference 15. There, they are given in co-ordinate-dependent form:

$$w_{ij}(x) = \alpha \times x + \beta \tag{36}$$

where α and β are three-dimensional vectors, constant over tetrahedra, for edge elements, and

$$w_{ijk}(x) = \gamma x + \delta \,, \tag{37}$$

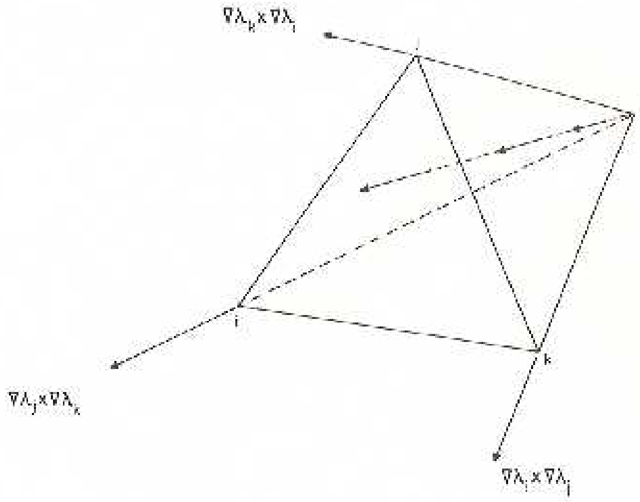


Fig. 5 *Whitney facet element* w_{ijk}

where γ is scalar and δ a vector, for facet elements. Eqns. 33 and 35, derived from Whitney's general formula, are much more convenient, in particular when finite-element assembly has to be performed.

Finally, for $p = 3$, we obtain functions w_{ijkl}, equal to a constant on tetrahedron $\{i, j, k, l\}$, to zero elsewhere. (The degrees of freedom should be localised at the centres of the circumscribed spheres.)

We shall note by W^p the vector space generated by Whitney elements of order p. The similar notation ${}^pW$ will serve, if necessary, to denote the corresponding space of differential forms. The most remarkable property of Whitney forms is

$$d^p W \subset {}^{(p+1)}W \quad (p = 0, 1, 2) \,. \tag{38}$$

Thanks to this, the complex of Whitney forms may serve in place of de Rham's complex to deal with the same kind of topological problems in an economical way (as spaces of Whitney forms are *finite*-dimensional). But their interest to us is that, as we have seen, they are ready-made finite elements for differential forms.

4 Two mixed methods

We shall now derive approximations for eqns. 1–4, by using Whitney forms. To deal with boundary conditions, we define W_0^p as the subspace of W^p of forms which vanish on Γ (those whose degrees of freedom vanish on simplices which lie entirely in Γ).

4.1 First mixed method

We look for an approximation of h in W^1, therefore

$$h = \sum_{e \in E} \bar{h}_e w_e \tag{39}$$

(recall that E is the set of edges, w_e is Whitney's element of edge e). We note $\bar{h}$ the vector of degrees of freedom:

$$\bar{h} = \{\bar{h}_e : e \in E\} \,. \tag{40}$$

If we set $j = \text{curl}\, h$, eqn. 2 is exactly satisfied, and $j \in W^2$.

From eqns. 2 and 4, b and e will be in W^1 and W^2, respectively, so eqn. 1 cannot be satisfied exactly; this was to be expected. We deal with this mismatch by shifting to the *transposed* weak form. Hence, h and e will have to satisfy

$$\frac{d}{dt}\int_\Omega \mu h \cdot h' + \int_\Omega e \cdot \text{curl } h' = 0 \quad \forall h' \in W^1 \tag{41}$$

in addition to the weak form of eqn. 3, which is

$$\int_\Omega \text{curl } h \cdot e' - \int_\Omega \sigma e \cdot e' = \int_\Omega j_s \cdot e' \quad \forall e' \in W^2 \tag{42}$$

where

$$e = \sum_{f \in F} \bar{e}_f w_f \tag{43}$$

(F is the set of facets).

This is a 'mixed', or 'two-fields' formulation [11]. It results in a differential equation in $\bar{h}$, $\bar{e}$. To see this, let us introduce the matrices A, B and C whose entries are

$$A_{e,e'} = \int_\Omega w_e \cdot w_{e'} \quad e \in E, e' \in E \tag{44}$$

$$B_{f,f'} = \int_\Omega w_f \cdot w_{f'} \quad f \in F, f' \in F \tag{45}$$

$$C_{e,f'} = \int_\Omega \text{curl } w_e \cdot w_{f'} \quad e \in E, f' \in F \tag{46}$$

and call $\bar{j}_s$ the vector of facet values such that $\bar{j}_s(f) = \int j \cdot w_f$. Matrices A and B are square-symmetric and C is rectangular. Assuming σ and μ constant for simplicity, we find the following matrix form of eqns. 41 and 42:

$$\begin{vmatrix} \frac{d}{dt}\mu A & C^t \\ C & -\sigma B \end{vmatrix} \begin{vmatrix} \bar{h} \\ \bar{e} \end{vmatrix} = \begin{vmatrix} 0 \\ \bar{j}_s \end{vmatrix} \tag{47}$$

Discretisation in time will lead to a series of linear systems with a common rigidity matrix. Owing to the peculiar form of this matrix (which is symmetric but not positive definite), finding efficient resolution methods is still a largely open problem.

In this method, the approximation of h is 'right', from a physical point of view: for h, being approximated as a 1-form, will have tangential continuity. But approximating e as a 2-form is 'wrong', because e also should have tangential continuity. Let us again stress that this is unavoidable with finite-element methods: already with the simple case detailed in Section 3.1, we could not obtain tangential and normal continuity of u simultaneously. But, if we cannot have the best of both worlds, we can minimise the effect of the mismatch on e by eliminating e from the equations (obtain e from eqn. 4 and substitute it into eqn. 41). This leads to the following variational formulation 'in h':

$$\left.\begin{aligned} &\frac{d}{dt}\int_\Omega \mu h \cdot h' + \int_C \sigma^{-1} \text{curl } h \cdot \text{curl } h' = 0 \quad \forall h' \in W^1 \\ &h \in W^1 \qquad \text{curl } h = \sum_{f \in F} \bar{j}_s(f) w_f \ \ (\sim j_s) \ \text{ where } \sigma = 0 \end{aligned}\right\} \tag{48}$$

We proposed this method in Reference 3. It has been implemented by J.C. Vérité [4, 9], then developed into an industrial code which is now in current use at Électricité de France. An example of application is given in Section 4.3.

Remark: Boundary conditions 7 are implicit in eqns. 41 and 42, in weak form. To enforce boundary conditions 6 (now in exact form), we should replace W^1 and W^2 by W^1_0 and W^2_0, respectively.

4.2 Second mixed method

We now look for an approximation of e in W^1 and
in W^2:

$$e = \sum_{e \in E} \bar{e}_e w_e \quad b = \sum_{f \in F} \bar{b}_f w_f$$

Faraday's law (eqn. 1) can then be satisfied exactly. variational form is

$$\frac{d}{dt}\int_\Omega b \cdot h' + \int_\Omega \text{curl } e \cdot h' = 0 \quad \forall h' \in W^2$$

But there is a mismatch in Ampère's theorem (eqn. 2), must now lie in W^1 and h in W^2. We cure this by tra position:

$$\int_\Omega h \,.\, \text{curl } e' - \int_\Omega \sigma e \cdot e' = \int_\Omega j_s \cdot e' \quad \forall e' \in W^1$$

is imposed on h.

The duality between the two mixed methods shou now be obvious.

Eqns. 50 and 51 are a system of differential equation which can be recast, in the same notation as above, as

$$\begin{vmatrix} \frac{d}{dt}\mu B & C \\ C^t & -\sigma A \end{vmatrix} \begin{vmatrix} \bar{h} \\ \bar{e} \end{vmatrix} = \begin{vmatrix} 0 \\ \bar{j}_s \end{vmatrix} \quad (5$$

where $\bar{j}_s$ is now the vector of *edge* components:

$$\bar{j}_s(e) = \int_\Omega j_s \cdot w_e \quad (5$$

Again, elimination of h (the wrong one in the pair h- suggests itself. Therefore,

$$\frac{d}{dt}\int_{C \cup C_s} (\sigma e + j_s) \cdot e' + \int_\Omega \mu^{-1} \text{curl } e \cdot \text{curl } e' = 0$$

$$\forall e' \in W^1 \quad (54$$

This method was proposed in Reference 5. In that paper the duality was shown, but not clearly understood. The similarity with the proposal by Pillsbury [17] of a 'modified vector potential method', based on the gauge

$$\text{div } (\sigma a) = 0 \quad (55$$

was also mentioned there. The difference is in the choice of Whitney elements to derive the discrete form of the equations.

No implementation of this method seems to have been done. As the data structures (matrices A, B etc.) are the same in both methods, it would probably be worthwhile to have them available in a single software system. For, in this way, we may hope to obtain bilateral bounds on some global quantities like magnetic energy, Joule losses etc., by solving the same problem with both methods, in a similar manner as in Reference 12.

4.3 An example

For the time being, however, the software system we have incorporates only the first method. The following example of application may help to show its possibilities.

This work has been done by Vérité [9] to assess the sensitivity of a new design of eddy-current probe to the presence of a 'transversal' crack (i.e. one which lies in a plane perpendicular to the axis) in a metallic tube (see Fig. 6). Such a probe would consist of two magnetic cores, discoid, with six poles on each core and a coil on each probe. In the actual nondestructive testing operation, it would move along the axis of the tube, and flaws in the metal would reveal themselves by a difference of impedance (as measured in a Wheatstone bridge) between the two coils.

For such applications, it is absolutely necessary to obtain a good insight into the magnetic field generated, and this requires a three-dimensional computation. By symmetry, this may be performed in a sector of 30° (Fig. 7 shows the results). We can see half of one pole in the rear and the corresponding part of the tube in the front. Shading indicates the density of the normal component (parallel to the axis) of the induced eddy currents in three selected horizontal planes. Each run has been done using about 4000 tetrahedra; of course, with edge elements for h. The computation time was 4 min on a Cray 1S, but we believe it can be much reduced by improving on the linear system solver (a biconjugate gradient solver, with rather crude preconditioning by the diagonal part of the matrix).

See also Reference 1 for information on the applicability of Whitney elements.

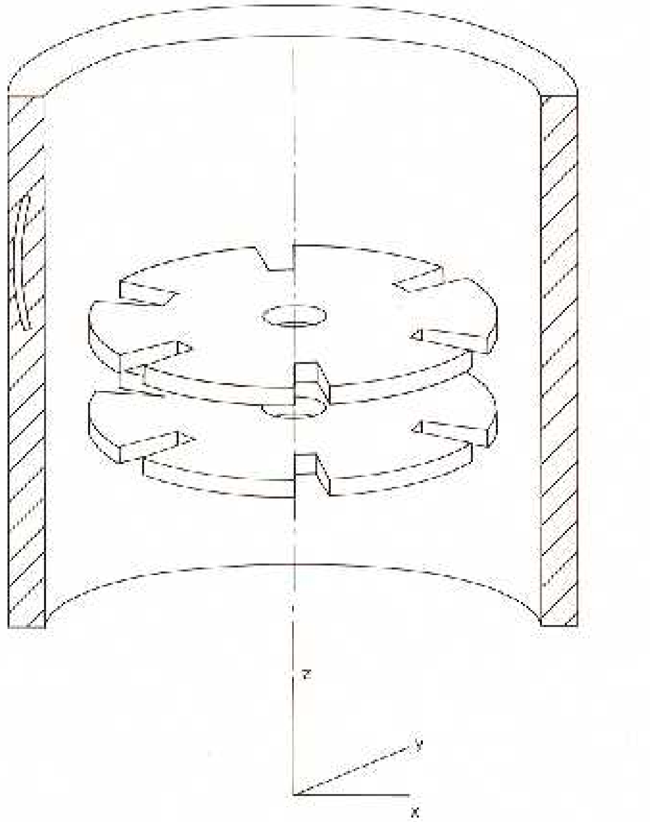
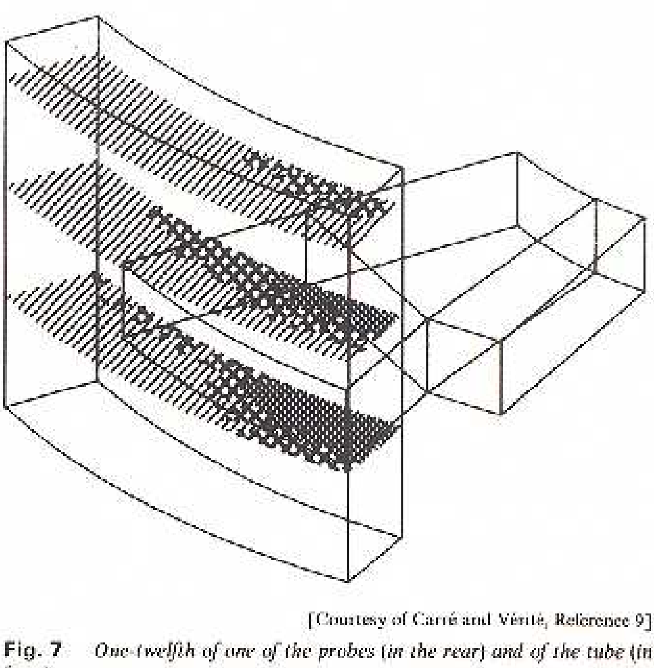


Fig. 6 *Principle of eddy-current testing: two parallel six-pole probes inside a (flawed) tube*

Coils around the poles are not shown



[Courtesy of Carré and Vérité, Reference 9]

Fig. 7 *One-twelfth of one of the probes (in the rear) and of the tube (in front)*

Eddy-current density

5 Conclusions

We have shown that, by expressing equations in the language of differential forms and by thinking of Whitney forms as finite elements, we are led almost automatically to mixed methods. In the case of eddy currents, there are two main possibilities, which result in two 'dual' mixed methods, one with emphasis on the magnetic field, the other one with emphasis on the electric field. This demonstrates the usefulness of differential forms, even at the level of finite-element discretisation.

6 Acknowledgments

I thank R. Kotiuga, of Harvard, USA, and L. Breen, of École Polytechnique, France, for their help.

7 References

1 BARTON, M.L., and CENDES, Z.J.: 'New vector finite elements for three dimensional magnetic field computation', *J. Appl. Phys.*, 1987, **61**, (8), pp. 3919–3921

2 BALDOMIR, D.: 'Differential forms and electromagnetism in 3-dimensional Euclidean space R^3', *IEE Proc. A*, 1986, **133**, (3), pp. 139–143

3 BOSSAVIT, A.: 'On finite element for the electricity equation', *in* WHITEMAN, J.R. (Ed.): 'The mathematics of finite elements and applications IV' (Academic Press, London, 1982), pp. 85–92

4 BOSSAVIT, A., and VÉRITÉ, J.C.: 'A mixed FEM-BIEM method to solve 3-D eddy current problem', *IEEE Trans.*, 1982, **MAG-18**, (2), pp. 431–435

5 BOSSAVIT, A.: 'Two dual formations of the 3-D eddy currents problem', *COMPEL*, 1985, **4**, (2), pp. 103–116

6 BOSSAVIT, A.: 'Mixed finite elements and the complex of Whitney forms', *in* WHITEMAN, J.R. (Ed.): 'The mathematics of finite elements and applications VI' (Academic Press, London, 1988), pp. 137–144

7 BREZZI, F.: 'On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers', *RAIRO Anal. Numer.*, 1974, **R2**, pp. 129–151

8 BURKE, W.: 'Applied differential geometry' (Cambridge University Press, 1985)

9 CARRE, M., and VÉRITÉ, J.C.: 'Use of 3D eddy-current code for optimizing an NDT probe', *in* WITTMANN, F.H. (Ed.): 'Structural mechanics in reactor technology' (A.A. Balkema, Rotterdam, 1987), pp. 253–257

10 DESCHAMPS, G.A.: 'Electromagnetics and differential forms', *Proc. IEEE*, 1981, **69**, (6), pp. 676–696

11 GIRAULT, V., and RAVIART, P.A.: 'Finite element approximation of the Navier Stokes equations' (Lecture Notes in Mathematics 749, Springer Verlag, Berlin, 1979)

12 HAMMOND, P., and BALDOMIR, D.: 'Dual energy methods in electromagnetism using tubes and slices', *IEE Proc. A*, 1988, **135**, (3), pp. 167–172

13 INGARDEN, R.S., and JAMIOLKOWSKI, A.: 'Classical electrodynamics' (Elsevier, Amsterdam, and PWN, Warsaw, 1985)
14 KOTIUGA, P.R.: 'Hodge decompositions and computational electromagnetics'. PhD Thesis, McGill University, Montreal, 1984
15 NÉDELEC, J.C.: 'Mixed finite elements in R^3', *Numer. Math.*, 1980, **35**, pp. 315–341
16 PENMAN, J., and FRASER, J.R.: 'Unified approach to problems in electromagnetism', *IEE Proc. A*, 1984, **131**, (1), pp. 55–61
17 PILLSBURY, R.D.: 'A three-dimensional eddy current formulation using two potentials: the magnetic vector potential and total magnetic scalar potential', *IEEE Trans.*, 1983, **MAG-19**, (6), pp. 2284–2287
18 RAVIART, P.A., and THOMAS, J.M.: 'A mixed finite element method for second order elliptic problems' (Lecture Notes in Mathematics 606, Springer Verlag, New York, 1977)
19 VON WESTENHOLZ: 'Differential forms in mathematical physics' (North-Holland, Amsterdam, 1981)
20 WHITNEY, H.: 'Geometric integration theory' (Princeton University Press, 1957)

View publication stats