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Vector spherical harmonics and their application to magnetostatics

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Abstract An alternative and somewhat systematic definition of the vector spherical harmonics, in analogy with the commonly used scalar spherical harmonics, is presented. The new set of vector spherical harmonics satisfies the properties of orthogonality and completeness, and is compared with other existing definitions of vector spherical harmonics. Some applications to problems in magnetostatics are illustrated.

Zusammenfassung Eine alternative und mehr systematische Definition vektorieller Kugelfunktionen, analog zu den gewöhnlich verwendeten skalaren Kugelfunktionen wird vorgestellt. Der neue Satz vektorieller Kugelfunktionen erfüllt die Orthogonalitäts- und Vollständigkeitsbedingungen; er wird verglichen mit vorhandenen anderen Definitionen vektorieller Kugelfunktionen und zur Veranschaulichung auf Probleme der Magnetostatik angewendet.

1. Introduction

Vector spherical harmonics (VSH) have been used in the expansion of plane waves to study the absorption and scattering of light by a sphere (see, for example, Bohren and Huffman 1983). They have also been widely used in nuclear and atomic physics (see, for example, Blatt and Weisskopf 1978).

The definitions of the various existing sets of VSH in different fields of physics are often dictated by convenience. For example, one method of defining such sets makes use of an operator which is proportional to the usual orbital angular momentum operator of quantum mechanics. When this operator acts upon the scalar spherical harmonics (SSH) function, it generates one (out of a triad of) VSH. The purpose of this note is to develop an alternate set of VSH which is particularly useful in classical electrodynamics. An alternative, simple treatment based on SSH uses the scalar Debye potentials (Gray 1978a, Gray and Nickel 1978). The layout of this paper is as follows. In §2 we present a brief review of SSH, §3 is devoted to the definition and formal properties of VSH and finally, in §4, we illustrate the usefulness of the VSH defined in §3 with several examples dealing with magnetostatic multipole moments and related electromagnetic problems.

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2. Review of scalar spherical harmonics

This review is intended primarily to define notation and underscore the parallel between the use of SSH and the VSH to be introduced in §3. Familiarity with the properties and uses of the SSH at the level of development presented in the standard electromagnetism text of Jackson (1975) will be assumed. Where possible we will follow the notation of Jackson.

A crucial property of the SSH, $Y_{lm}(\theta, \phi)$, is the completeness or closure relation, i.e. any arbitrary function g of θ, ϕ can be expanded as

$$g(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} Y_{lm}(\theta, \phi). \tag{2.1}$$

Further, if g is a function of other variables, e.g. s and r, then the expansion coefficients A_{lm} are functions of these additional variables. The computation of the expansion coefficients is made relatively simple by the orthogonality condition

$$\oint d\Omega\, Y^*_{lm}(\theta, \phi) Y_{l'm'}(\theta, \phi) = \delta_{ll'}\delta_{mm'} \tag{2.2}$$

where $d\Omega \equiv \sin\theta\, d\theta\, d\phi$ and the integral $\int d\Omega$ is over the whole range of angles θ, ϕ. The coefficients are given by

$$A_{lm} = \int d\Omega\, Y^*_{lm}(\theta, \phi) g(\theta, \phi). \tag{2.3}$$

The evaluation of the coefficients is often simplified using the symmetries of the SSH, namely,

$$\begin{aligned} Y_{lm}(\theta, \phi + \pi) &= (-1)^m Y_{lm}(\theta, \phi) \\ Y_{lm}(\pi - \theta, \phi) &= (-1)^{l+m} Y_{lm}(\theta, \phi) \\ Y_{lm}(\pi - \theta, \phi + \pi) &= (-1)^l Y_{lm}(\theta, \phi). \end{aligned} \tag{2.4}$$

One of the reasons why the SSH are useful in physics is that they behave in an exemplary way when operated upon by the Laplacian ∇^2:

$$\nabla^2 Y_{lm} = -\frac{l(l+1)}{r^2} Y_{lm}. \tag{2.5}$$

The simplifications that can be achieved with spherical harmonic expansions are evident in the Poisson equation $\nabla^2\Phi_E = -4\pi\rho$. Expanding both Φ_E and ρ,

$$\Phi_E(r, \theta, \phi) = \sum_l \sum_m C_{lm}(r) Y_{lm}(\theta, \phi) \tag{2.6}$$

$$\rho(r, \theta, \phi) = \sum_l \sum_m \rho_{lm}(r) Y_{lm}(\theta, \phi) \tag{2.7}$$

the Poisson equation becomes

$$\begin{aligned} \nabla^2\Phi_E &= \sum \sum \nabla^2 C_{lm} Y_{lm} = \sum \sum (\nabla^2 C_{lm}) Y_{lm} \\ &\quad + \sum \sum C_{lm} \nabla^2 Y_{lm} \\ &= \sum \sum \left[\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr} C_{lm}\right) - \frac{l(l+1)}{r^2}\right] Y_{lm} \\ &= -4\pi\rho = -4\pi \sum \sum \rho_{lm} Y_{lm}. \end{aligned} \tag{2.8}$$

Since the coefficients of the expansion for $\nabla^2\Phi_E$ must match the coefficient for ρ we are left with

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr} C_{lm}\right) - \frac{l(l+1)}{r^2} C_{lm} = -4\pi\rho_{lm}. \tag{2.9}$$

The angular dependences have in effect cancelled out and we need deal only with an ordinary differential equation.

To achieve a general solution of the differential equation given by equation (2.9) we start by considering the very special case $\rho_{lm} = \delta(r - r')$ and the differential equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr} f(r)\right) - \frac{l(l+1)}{r^2} f(r) = -4\pi\delta(r' - r). \tag{2.10}$$

For $r > r'$ or $r < r'$ the right-hand side of equation (2.10) vanishes and the solution is simple:

$$\begin{aligned} f &= Cr^l & r &< r' \\ f &= Dr^{-(l+1)} & r &> r'. \end{aligned} \tag{2.11}$$

Notice that we have chosen only solutions that are well behaved at the origin and at infinity. Continuity of f at $r = r'$ requires that $D = C(r')^{2l+1}$. To obtain the remaining coefficient, C, the differential equation must be integrated over an infinitesimal range of r from $r = r' - \varepsilon$ to $r = r' + \varepsilon$. This yields $C = [4\pi/(2l+1)]r'^{l-1}$. Details of the procedure to obtain C are to be found in Jackson (1975, §3.9). Introducing the usual notation

$$(r_>, r_<) = \begin{cases} (r, r') & \text{for } r > r' \\ (r', r) & \text{for } r < r' \end{cases} \tag{2.12}$$

the solution, i.e. the Green function, can be written

$$f(r, r') = \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} (r')^2. \tag{2.13}$$

To obtain the solution to equation (2.9) we need only recognise that any ρ_{lm} can be written as a superposition of delta functions. As is well known

$$\rho_{lm}(r) = \int \rho_{lm}(r')\delta(r - r')\, dr'. \tag{2.14}$$

Therefore the solution to equation (2.9) must be a similar superposition of fs as given by equation (2.13):

$$C_{lm}(r) = \int \rho_{lm}(r') \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} (r')^2\, dr'. \tag{2.15}$$

If we now require that the charge distribution ρ be bounded in extent, and we ask explicitly for the value of C_{lm} at a value of r outside the source, we have $r = r_>$ in the integrand and

$$C_{lm}(r) = \frac{4\pi}{(2l+1)r^{l+1}} \int \rho_{lm}(r') r'^{l-2}\, dr'. \tag{2.16}$$

The quantities q_{lm},

$$\begin{aligned} q_{lm} &\equiv \int \rho_{lm}(r) r^{l+2}\, dr \\ &= \int \rho(r, \theta, \phi) Y^*_{lm}(\theta, \phi) r^l\, d^3x \end{aligned} \tag{2.17}$$

($d^3x \equiv r^2\, dr\, d\Omega$) which are characteristic of the charge distribution are called its *multipole moments*, and the expression that results from substituting equations (2.16) and (2.17) back in equation (2.6),

$$\Phi_E = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{q_{lm}}{r^{l+1}} Y_{lm}(\theta, \phi) \tag{2.18}$$

is called the multipole expansion for the potential. Equations (2.17) and (2.18) provide an explicit straightforward recipe for obtaining the potential outside any given charge distribution. The entities described as scalar spherical harmonics, reviewed in this section, are also described as irreducible tensors, a notation which describes their transformations under rotation of axes.

3. Definition and properties of vector spherical harmonics

We begin this section by attempting to construct vector functions having the same desirable properties, namely, orthogonality and completeness as the SSH. Perhaps the most obvious way to do this would be to treat each of the three components of a vector as a separate scalar field:

$$\begin{aligned}\boldsymbol{E}(r,\theta,\phi) &= \hat{\boldsymbol{e}}_r E^r + \hat{\boldsymbol{e}}_\theta E^\theta + \hat{\boldsymbol{e}}_\phi E^\phi \\ &= \hat{\boldsymbol{e}}_r \sum\sum E^r_{lm}(r) Y_{lm} + \hat{\boldsymbol{e}}_\theta \sum\sum E^\theta_{lm}(r) Y_{lm} \\ &\quad + \hat{\boldsymbol{e}}_\phi \sum\sum E^\phi_{lm}(r) Y_{lm}. \end{aligned} \tag{3.1}$$

Since the SSH form a complete set such an expansion is permitted. Whether the expansion is useful is a separate question. Consider, for example, an equation like $\boldsymbol{\nabla}\cdot\boldsymbol{E}=f$. We might hope that equation (3.1) and the usual scalar spherical harmonic decomposition of the scalar function f:

$$f=\sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{lm}(r) Y_{lm}(\theta,\phi)$$

would give us a decomposition of $\boldsymbol{\nabla}\cdot\boldsymbol{E}=f$ from which the 'coefficients' E^r_{lm}, E^θ_{lm}, E^ϕ_{lm}, f_{lm} could be picked off and related. All angular variables would be swept aside and the problem would be transformed from a partial differential equation to an ordinary differential equation in r. This, however, is not the case. To see this, take the divergence of $\boldsymbol{E}$ in equation (3.1); in spherical polar coordinates we have

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{E} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2E^r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}E^\phi \\ &\quad + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E^\theta). \end{aligned} \tag{3.2}$$

If we use equation (3.1) for the second term on the right, we obtain a term proportional to $Y_{lm}/\sin\theta$. The angular dependence for the lm term is then not simply Y_{lm} and it is thus not possible to cancel all the Y_{lm} in an equation like $\boldsymbol{\nabla}\cdot\boldsymbol{E}=f$. Equation (3.1) is thus not useful. Further, an attempt to expand the cartesian components E^x, E^y, E^z in terms of Y_{lm} would be at least as frustrating and no more rewarding. This is due to the fact that the components of a vector field do not behave as a scalar field.

The above exercise seems to indicate what the correct approach should be to require from the outset that operations involving ∇ give simple expressions. In quest of this we now take a scalar field

$$f(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{lm}(r)Y_{lm}(\theta,\phi), \tag{3.3}$$

and obtain a vector field by forming its gradient:

$$\begin{aligned}\nabla f &= \sum\sum (Y_{lm}\nabla f_{lm} + f_{lm}\nabla Y_{lm}) \\ &= \sum\sum\left(\frac{\mathrm{d}}{\mathrm{d}r}f_{lm}(r)Y_{lm}\hat{\boldsymbol{e}}_r + f_{lm}\nabla Y_{lm}\right). \end{aligned} \tag{3.4}$$

Seeking simplicity we require that the gradient of a spherical harmonic expansion be itself a spherical harmonic expansion. Equation (3.4) then is the first glimpse of a vector spherical harmonic expansion. Notice that while the radial part of the vector ∇f is expanded simply with Y_{lm} (at least the first term in equation (3.1) was right!) the $\hat{\boldsymbol{e}}_\theta$, $\hat{\boldsymbol{e}}_\phi$ parts are expanded in terms of a new mathematical object, ∇Y_{lm}. This motivates the following notation

$$\boldsymbol{\Psi}_{lm}(\theta,\phi)\equiv r\nabla Y_{lm}(\theta,\phi). \tag{3.5}$$

The r factor helps making $\boldsymbol{\Psi}_{lm}$, like Y_{lm}, dimensionless.

Is this all there is? Consider another vector equation we might encounter

$$\mathbf{A}=\hat{\boldsymbol{e}}_r\times\boldsymbol{B}. \tag{3.6}$$

Now if $\boldsymbol{B}=\boldsymbol{\Psi}_{lm}$ (a one-term expansion) we have

$$\mathbf{A}=\hat{\boldsymbol{e}}_r\times\boldsymbol{\Psi}_{lm}. \tag{3.7}$$

The vector $\mathbf{A}$ cannot be expanded in terms such as $Y_{lm}\hat{\boldsymbol{e}}_r$, the reason being that $\mathbf{A}$ is orthogonal to $\hat{\boldsymbol{e}}_r$; likewise, $\mathbf{A}$ cannot be expanded in terms of $\boldsymbol{\Psi}_{lm}$, since $\mathbf{A}$ is orthogonal to $\boldsymbol{\Psi}_{lm}$. There is no alternative but to conclude that $\hat{\boldsymbol{e}}_r\times\boldsymbol{\Psi}_{lm}$ is a new type of vector which will be needed in the expansion. We thus define

$$\boldsymbol{\Phi}_{lm}\equiv\hat{\boldsymbol{e}}_r\times\boldsymbol{\Psi}_{lm}=\boldsymbol{r}\times\nabla Y_{lm}. \tag{3.8}$$

If we also allow ourselves to define a special symbol for the radial part (e.g. in equation (3.4)):

$$\mathbf{Y}_{lm}\equiv\hat{\boldsymbol{e}}_rY_{lm}. \tag{3.9}$$

We now have three *vector spherical harmonics*.

The question whether our set of VSH ($\mathbf{Y}_{lm}$, $\boldsymbol{\Psi}_{lm}$, $\boldsymbol{\Phi}_{lm}$) is complete is not a simple one and will be deferred. There is a triad of VSH for each l and m. The assumption is now made that all three varieties of the VSH constitute a complete set, i.e. any vector field can be expanded as follows:

$$\begin{aligned}\mathbf{V}(r,\theta,\phi) &= \sum\sum[V^r_{lm}\mathbf{Y}_{lm} \\ &\quad + V^{(1)}_{lm}\boldsymbol{\Psi}_{lm}+V^{(2)}_{lm}\boldsymbol{\Phi}_{lm}] \end{aligned} \tag{3.10}$$

where V^r_{lm}, $V^{(1)}_{lm}$ and $V^{(2)}_{lm}$ are the expansion coefficients analogous to f_{lm} in equation (3.3).

In physics the overwhelmingly important property of the VSH is the way they are related to the ∇ operator. Specifically, if we have any equation involving the ∇ operator (as gradient, Laplacian, curl, etc) and if all functions are expanded in spherical harmonics (scalars in scalar spherical harmonics, vectors in vector spherical harmonics), then the angular dependence will 'cancel out'. Towards this end one can readily confirm the relations among scalar and vector spherical harmonics given in the following compendium:

$$\nabla \cdot (F(r)\mathbf{Y}_{lm}) = \left(\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r} r^2 F(r)\right) Y_{lm} \tag{3.11a}$$

$$\nabla \cdot (F(r)\boldsymbol{\Psi}_{lm}) = -\frac{l(l+1)}{r} F(r) Y_{lm} \tag{3.11b}$$

$$\nabla \cdot (F(r)\boldsymbol{\Phi}_{lm}) = 0 \tag{3.11c}$$

$$\nabla \times (F(r)\mathbf{Y}_{lm}) = -\frac{F(r)}{r}\boldsymbol{\Phi}_{lm} \tag{3.12a}$$

$$\nabla \times (F(r)\boldsymbol{\Psi}_{lm}) = \left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} rF(r)\right)\boldsymbol{\Phi}_{lm} \tag{3.12b}$$

$$\nabla \times (F(r)\boldsymbol{\Phi}_{lm}) = -\left(\frac{l(l+1)}{r} F(r)\right)\mathbf{Y}_{lm} - \left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} rF(r)\right)\boldsymbol{\Psi}_{lm} \tag{3.12c}$$

$$\nabla(F(r)Y_{lm}) = \left(\frac{\mathrm{d}}{\mathrm{d}r}F(r)\right)\mathbf{Y}_{lm} + \frac{F(r)}{r}\boldsymbol{\Psi}_{lm} \tag{3.13}$$

$$\nabla^2(F(r)Y_{lm}) = \left(\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2} rF(r) - \frac{l(l+1)}{r^2}F(r)\right)Y_{lm}. \tag{3.14}$$

A few explicit values of the vector spherical harmonics $\boldsymbol{\Psi}_{lm}$ are presented in table 1. From the definitions it is an easy matter to verify that the following relations are satisfied:

$$\begin{aligned} \mathbf{Y}_{l,-m} &= (-1)^m \mathbf{Y}^*_{l,m} \\ \boldsymbol{\Psi}_{l,-m} &= (-1)^m \boldsymbol{\Psi}^*_{l,m} \\ \boldsymbol{\Phi}_{l,-m} &= (-1)^m \boldsymbol{\Phi}^*_{l,m}. \end{aligned} \tag{3.15}$$

Further, from the defining equations (3.5) and (3.8) we have

$$\begin{aligned} \frac{\partial}{\partial\theta} Y_{lm} &= (\boldsymbol{\Psi}_{lm})^\theta = (\boldsymbol{\Phi}_{lm})^\phi \\ \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} Y_{lm} &= (\boldsymbol{\Psi}_{lm})^\phi = -(\boldsymbol{\Phi}_{lm})^\theta \end{aligned} \tag{3.16}$$

which greatly simplify the tabulation of the explicit values.

To gain confidence that our mathematical arsenal does fill our needs we turn back to the problem at the beginning of this section. Utilising equations (3.3), (3.10) and (3.11) we have

$$\begin{aligned} \nabla \cdot \mathbf{V} &= \nabla \cdot \left(\sum\sum (V^r_{lm}\mathbf{Y}_{lm} + V^{(1)}_{lm}\boldsymbol{\Psi}_{lm} + V^{(2)}_{lm}\boldsymbol{\Phi}_{lm})\right) \\ &= \sum\sum\left(\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2 V^r_{lm}) - \frac{l(l+1)}{r}V^{(1)}_{lm}\right)Y_{lm} \\ &= f = \sum\sum f_{lm}(r) Y_{lm} \end{aligned} \tag{3.17}$$

therefore

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2 V^r_{lm}) - \frac{l(l+1)}{r}V^{(1)}_{lm} = f_{lm}(r). \tag{3.18}$$

Notice that the angular dependence has cancelled out as expected; one is thus left with an ordinary differential equation.

Table 1 The normalised vector spherical harmonics $\boldsymbol{\Psi}_{l,m}(\theta, \phi)$ to quadrupole order.

	$\boldsymbol{\Psi}_{lm}(\theta,\phi)$			
l	$m=0$	$m=1$	$m=2$	$m=3$
0	0			
1	$-(3/4\pi)^{1/2}\sin\theta\,\hat{\boldsymbol{e}}_\theta$	$-(3/8\pi)\exp(\mathrm{i}\phi)$ $\times(\cos\theta\,\hat{\boldsymbol{e}}_\theta + \mathrm{i}\hat{\boldsymbol{e}}_\phi)$		
2	$-3(5/4\pi)^{1/2}\sin\theta\cos\theta\,\hat{\boldsymbol{e}}_\theta$	$(15/8\pi)^{1/2}\exp(\mathrm{i}\phi)$ $\times[(1-2\cos^2\theta)\hat{\boldsymbol{e}}_\theta - \mathrm{i}\cos\theta\,\hat{\boldsymbol{e}}_\phi]$	$(15/8\pi)^{1/2}\sin\theta\exp(2\mathrm{i}\phi)$ $\times(\cos\theta\,\hat{\boldsymbol{e}}_\theta + \mathrm{i}\hat{\boldsymbol{e}}_\phi)$	
3	$-\frac{3}{2}(7/4\pi)^{1/2}$ $\times(5\sin\theta\cos^2\theta - \sin\theta)\hat{\boldsymbol{e}}_\theta$	$(21/64\pi)^{1/2}\exp(\mathrm{i}\phi)$ $\times[\cos\theta(5\cos^2\theta - 9)\hat{\boldsymbol{e}}_\theta$ $-\mathrm{i}(5\cos^2\theta - 1)\hat{\boldsymbol{e}}_\phi]$	$(105/8\pi)^{1/2}\sin\theta\exp(2\mathrm{i}\phi)$ $\times[\frac{1}{2}(3\cos^2\theta + 1)\hat{\boldsymbol{e}}_\theta$ $+\mathrm{i}\cos\theta\,\hat{\boldsymbol{e}}_\phi]$	$-3(35/64\pi)^{1/2}\sin^2\theta$ $\times\exp(3\mathrm{i}\phi)(\cos\theta\hat{\boldsymbol{e}}_\theta + \mathrm{i}\hat{\boldsymbol{e}}_\phi)$

Other definitions of VSH have been given, for example, by Hill (1954) who defines a set $[\boldsymbol{V}_{lm}, \boldsymbol{W}_{lm}, \boldsymbol{X}_{lm}]$ and by Blatt and Weisskopf (1978) who define a set $[\mathbf{Y}^{m}_{l,\alpha,1}]$ where $\alpha - l = 0, \pm 1$. The connection between these and the present vector harmonics is as follows:

$$\begin{aligned} \boldsymbol{V}_{lm} &= \mathbf{Y}^{m}_{l,l+1,1} = \left(\frac{l+1}{2l+1}\right)^{1/2} \mathbf{Y}_{lm} \\ &\quad + [(l+1)(2l+1)]^{-1/2} \boldsymbol{\Psi}_{lm} \\ \boldsymbol{W}_{lm} &= \mathbf{Y}^{m}_{l,l-1,1} = \left(\frac{l}{2l+1}\right)^{1/2} \mathbf{Y}_{lm} \\ &\quad + [l(2l+1)]^{-1/2} \boldsymbol{\Psi}_{lm} \\ \boldsymbol{X}_{lm} &= \mathbf{Y}^{m}_{l,l,1} = -\mathrm{i}[l(l+1)]^{-1/2} \boldsymbol{\Phi}_{lm}. \end{aligned} \tag{3.19}$$

Konopinski (1981) and Morse and Feshbach (1953) present in their popular books two more definitions of VSH.

We remark that with the use of group theory, the discussion by Blatt and Weisskopf and the explicit equivalences (3.19) between their spherical harmonics and the present ones should convince the reader that the vector spherical harmonics do indeed form a complete set. A simple proof of completeness based on the Debye potentials has been recently reported (Gray and Nickel 1978).

We will now show that the VSH are orthogonal in the same sense as the SSH, and that the VSH have useful symmetry properties.

First of all we note that at a point, for the same values of l, m, there is a trivial orthogonality which follows from the definition of $\mathbf{Y}_{lm}$, $\boldsymbol{\Psi}_{lm}$ and $\boldsymbol{\Phi}_{lm}$.

$$\begin{aligned} \mathbf{Y}_{lm} \cdot \boldsymbol{\Psi}_{lm} &= 0 \\ \mathbf{Y}_{lm} \cdot \boldsymbol{\Phi}_{lm} &= 0 \\ \boldsymbol{\Psi}_{lm} \cdot \boldsymbol{\Phi}_{lm} &= 0. \end{aligned} \tag{3.20}$$

More relevant are the orthogonality properties analogous to equation (2.2), and valid for all l, l', m, m'.

$$\begin{aligned} &\int \mathrm{d}\Omega\, \mathbf{Y}_{lm} \cdot \mathbf{Y}^*_{l'm'} = \delta_{ll'}\delta_{mm'} \\ &\int \mathrm{d}\Omega\, \boldsymbol{\Psi}_{lm} \cdot \boldsymbol{\Psi}^*_{l'm'} = l(l+1)\delta_{ll'}\delta_{mm'} \\ &\int \mathrm{d}\Omega\, \boldsymbol{\Phi}_{lm} \cdot \boldsymbol{\Phi}^*_{l'm'} = l(l+1)\delta_{ll'}\delta_{mm'} \\ &\int \mathrm{d}\Omega\, \mathbf{Y}_{lm} \cdot \boldsymbol{\Psi}^*_{l'm'} = \int \mathrm{d}\Omega\, \mathbf{Y}_{lm} \cdot \boldsymbol{\Phi}^*_{l'm'} \\ &\qquad = \int \mathrm{d}\Omega\, \boldsymbol{\Psi}_{lm} \cdot \boldsymbol{\Phi}^*_{l'm'} = 0. \end{aligned} \tag{3.21}$$

Although equations (3.21) are not obvious, they are easily verifiable from equation (2.2) and the definitions of the SSH. We mention in passing that the factor $l(l+1)$ in equations (3.21) could have been incorporated in the definitions of $\boldsymbol{\Psi}_{lm}$ and $\boldsymbol{\Phi}_{lm}$, to make the vector spherical harmonics orthonormal.

Using the above, the expansion coefficients in a vector spherical harmonic expansion are relatively straightforward, namely,

$$\mathbf{V} = \sum \sum (V^r_{lm} \mathbf{Y}_{lm} + V^{(1)}_{lm} \boldsymbol{\Psi}_{lm} + V^{(2)}_{lm} \boldsymbol{\Phi}_{lm}) \tag{3.22}$$

with the coefficients V^r_{lm}, $V^{(1)}_{lm}$, $V^{(2)}_{lm}$ being

$$\begin{aligned} V^r_{lm} &= \int \mathrm{d}\Omega\, \mathbf{V} \cdot \mathbf{Y}^*_{lm} \\ V^{(1)}_{lm} &= \frac{1}{l(l+1)} \int \mathrm{d}\Omega\, \mathbf{V} \cdot \boldsymbol{\Psi}_{lm} \\ V^{(2)}_{lm} &= \frac{1}{l(l+1)} \int \mathrm{d}\Omega\, \mathbf{V} \cdot \boldsymbol{\Phi}^*_{lm}. \end{aligned} \tag{3.23}$$

4. Magnetostatic multipole moments

Our goal here is to develop a formalism for multipoles of the magnetic induction field, $\boldsymbol{B}$, similar to that (see §2) for those of the $\boldsymbol{E}$ field. This will be possible because for no current ($\boldsymbol{J}$ = current density) at the field point

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c} \boldsymbol{J} = 0 \tag{4.1}$$

and therefore there exists a function Φ_M, the magnetic scalar potential, from which we can find the magnetic induction field as

$$\boldsymbol{B} = -\nabla \Phi_\mathrm{M}. \tag{4.2}$$

This scalar function can be expanded in a manner precisely analogous to the expansion in equation (2.18)

$$\Phi_\mathrm{M} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} M_{lm} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}} \tag{4.3}$$

since this is the general well behaved solution of $\nabla \cdot \boldsymbol{B} = -\nabla^2 \Phi_\mathrm{M} = 0$; the quantities M_{lm} are the magnetostatic multipole moments for the field.

Notice that in general we cannot use Φ_M directly to relate M_{lm} to the current sources $\boldsymbol{J}$, the reason being that Φ_M has meaning only where $\boldsymbol{J} = 0$.

The vector potential is related to the current sources through the equation

$$\nabla \times \boldsymbol{B} = \nabla \times (\nabla \times \mathbf{A}) = \frac{4\pi}{c} \boldsymbol{J}. \tag{4.4}$$

From equation (4.4) the differential equation for $A^{(2)}_{lm}$ is seen to be:

$$\begin{aligned} &\nabla \times \nabla \times A^{(2)}_{lm} \boldsymbol{\Phi}_{lm} = \frac{4\pi}{c} J^{(2)}_{lm} \boldsymbol{\Phi}_{lm} \\ &\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}}{\mathrm{d}r} A^{(2)}_{lm}\right) - \frac{l(l+1)}{r^2} A^{(2)}_{lm} = -\frac{4\pi}{c} J^{(2)}_{lm} \end{aligned} \tag{4.5}$$

where equations (3.12) have been used. Equation (4.5) is identical in form to equation (2.9) so that the solution is found similarly. For the delta function source, $J^{(2)}_{lm}=\delta(r-r')$, we obtain the Green function:

$$A^{(2)}_{lm}=\frac{4\pi}{(2l+1)c}\frac{r^l_<}{r^{l+1}_>}(r')^2. \tag{4.6}$$

For a field point r outside the source (r larger than any r') the solution of equation (4.5) is then

$$A^{(2)}_{lm}=\frac{4\pi}{(2l+1)cr^{l+1}}\int (r')^{l-2}J^{(2)}_{lm}(r')\,\mathrm{d}r' \tag{4.7}$$

which for now is written as follows

$$A^{(2)}_{lm}(r)=\frac{\alpha_{lm}}{r^{l+1}}. \tag{4.8}$$

The last equation above can be formally used to calculate the magnetic induction field:

$$\begin{aligned}\boldsymbol{B}&=\nabla\times\mathbf{A}=\alpha_{lm}\nabla\times\frac{\mathbf{\Phi}_{lm}}{r^{l+1}}\\ &=\alpha_{lm}\left(-\frac{\hat{\boldsymbol{e}}_r}{r^{l+2}}l(l+1)Y_{lm}+\frac{l}{r^{l+2}}\mathbf{\Psi}_{lm}\right)\\ &=l\alpha_{lm}\left[\nabla\left(\frac{Y_{lm}}{r^{l+1}}\right)\right]\end{aligned} \tag{4.9}$$

a result which is in agreement with equation (4.2). A comparison of equations (4.2), (4.3) and (4.7)–(4.9) yields

$$\Phi_{\mathrm{M}}=-\sum\sum l\alpha_{lm}\frac{Y_{lm}}{r^{l+1}} \tag{4.10}$$

therefore

$$\begin{aligned}M_{lm}&=-\frac{(2l+1)}{4\pi}l\alpha_{lm}\\ &=-\frac{l}{c}\int (r')^{l+2}J^{(2)}_{lm}(r')\,\mathrm{d}r'.\end{aligned} \tag{4.11}$$

We point out in passing that the multipole expansion of $\boldsymbol{B}$ can be derived directly, without using $\mathbf{A}$, utilising either the simple Debye potential treatment (Gray 1978a), or the simple radial component $\boldsymbol{r}\cdot\boldsymbol{B}$ (Gray 1978b). A magnetostatic multipole expansion using the scalar potential approach and employing Cartesian tensors has been recently reported (Gray 1979).

Using the definition and properties of $\mathbf{\Phi}_{lm}$, it can be readily shown that equation (4.11) can also be written in the alternative form

$$M_{lm}=\frac{-1}{(l+1)c}\int \mathrm{d}^3x Y^*_{lm}\nabla\cdot(r^l\boldsymbol{r}\times\boldsymbol{J}) \tag{4.12}$$

a result which can be used without any reference to VSH. It is worth noting that the expression for M_{lm} given in equation (4.12) has been derived for the static problem by Bronzan (1971, see also Bronzan 1982), using the magnetic scalar potential. Bronzan's derivation has been greatly simplified by Gray (1978b), who proceeds directly from the Maxwell equations.

Consider as a first example the familiar case of a circular wire loop of radius R carrying current I. Far from the loop the magnetic induction field will be predominantly a dipole field ($|\boldsymbol{B}|\sim r^{-3}$). We place the loop in the xy plane with its centre at the origin, and compute the dipole ($l=1$) coefficients from equation (4.11) and from the vectorial current density

$$\boldsymbol{J}=I\delta(\cos\theta)\frac{\delta(r-R)}{R}\hat{\boldsymbol{e}}_\phi. \tag{4.13}$$

Notice that the 'radial' delta function has the dimension of an inverse distance. The 'R' in the denominator of equation (4.13) serves then to ensure that the current density has the right dimensions.

The multipole coefficients of $\boldsymbol{J}$ are

$$\begin{aligned}J^{(2)}_{1m}&=\tfrac{1}{2}\int \mathrm{d}\Omega\boldsymbol{J}\cdot\mathbf{\Phi}^*_{1m}\\ &=\frac{I}{2R}\delta(r-R)\\ &\quad\times\int \mathrm{d}\Omega\delta(\cos\theta)\hat{\boldsymbol{e}}_\phi\cdot\{\hat{\boldsymbol{e}}_r\times r\nabla Y^*_{1m}\}\\ &=\frac{I}{2R}\delta(r-R)\int \mathrm{d}\Omega\hat{\boldsymbol{e}}_\phi\,\cdot\\ &\quad\left\{\hat{\boldsymbol{e}}_\phi\frac{\partial}{\partial\theta}Y^*_{1m}-\hat{\boldsymbol{e}}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}Y^*_{1m}\right\}\delta(\cos\theta)\\ &=\frac{I}{2R}\delta(r-R)\int_0^{2\pi}\frac{\partial}{\partial\theta}Y^*_{1m}|_{\theta=\pi/2}\,\mathrm{d}\phi.\end{aligned} \tag{4.14}$$

The dependence of Y^*_{lm} is $\exp(-im\phi)$ and $\int_0^{2\pi}\exp(-im\phi)\,\mathrm{d}\phi=0$ so the integral in equation (4.14) vanishes unless $m=0$, i.e.,

$$\begin{aligned}J^{(2)}_{1,\pm1}&=0\\ J^{(2)}_{1,0}&=\frac{I}{2R}\delta(r-R)2\pi\frac{\partial}{\partial\theta}Y^*_{10}|_{\theta=\pi/2}\\ &=-\frac{\pi I}{R}\delta(r-R)(3/4\pi)^{1/2}.\end{aligned} \tag{4.15}$$

It can be readily verified from equation (4.11) that $M_{1,\pm1}=0$. Also,

$$\begin{aligned}M_{10}&=\frac{\pi I}{Rc}(3/4\pi)^{1/2}\int r^3\delta(r-R)\,\mathrm{d}r\\ &=\frac{\pi IR^2}{c}(3/4\pi)^{1/2}.\end{aligned} \tag{4.16}$$

The magnetostatic potential, to dipole order is, therefore,

$$\Phi_{\mathrm{M}}(r)=\frac{4\pi}{3}M_{10}\frac{Y_{10}}{r^2}$$
$$=\frac{\pi IR^2\cos\theta}{cr^2}\qquad (r \text{ large}). \qquad (4.17)$$

Equation (4.17) can be written in the same form as that of an electrostatic dipole

$$\Phi_{\mathrm{M}}=\frac{\boldsymbol{m}\cdot\boldsymbol{r}}{r^3} \qquad (4.18)$$

where $\boldsymbol{m}$ denotes the magnetic dipole moment which characterises the field. The correspondence of $\boldsymbol{m}$ and M_{10} could have been written down intuitively. An electrostatic dipole coefficient is associated with a dipole component (see, e.g. Jackson 1975 p 137)

$$p^z=(\tfrac{4}{3}\pi)^{1/2}q_{10}. \qquad (4.19)$$

It follows therefore that

$$m^z=(\tfrac{4}{3}\pi)^{1/2}M_{10}=\frac{\pi IR^2}{c}. \qquad (4.20)$$

Consider as a second example a current path consisting of two D-shaped rings each of radius a, as shown in figure 1. It is 'intuitively obvious' that there will be no net dipole moment (the dipole moment of the two Ds cancel), so the distant magnetic induction field will be at most a quadrupole field.

We follow the same procedure to find M_{2m} as we did to find the dipole coefficients of the ring. The current distribution is

$$\boldsymbol{J}=\begin{cases}\dfrac{I}{a}\,\delta(\cos\theta)\delta(r-a)\hat{\boldsymbol{e}}_\phi & 0<\phi<\pi\\[2ex] -\dfrac{I}{a}\,\delta(\cos\theta)\delta(r-a)\hat{\boldsymbol{e}}_\phi & \pi<\phi<2\pi\\[2ex] 2I\delta(z)\delta(y)\hat{\boldsymbol{e}}_x & \text{on } x \text{ axis}, |x|<a.\end{cases} \qquad (4.21)$$

Since $\hat{\boldsymbol{e}}_r\cdot\boldsymbol{\Phi}^*_{lm}=0$, the third part of the current gives no contribution to $J^{(2)}_{lm}$. Consider the integration for the other contributions. The integration $\int\boldsymbol{J}\cdot\boldsymbol{\Phi}^*_{lm}\,\mathrm{d}\phi$ will have the form $\int_0^\pi\exp(-\mathrm{i}m\phi)\,\mathrm{d}\phi-\int_\pi^{2\pi}\exp(-\mathrm{i}m\phi)\,\mathrm{d}\phi$. We conclude that m must be odd for a nonvanishing $J^{(2)}_{lm}$ and that, for odd m, $\int\exp(-\mathrm{i}m\phi)\,\mathrm{d}\phi$ can be replaced by $-4\mathrm{i}/m$.

We now take precisely the same steps as we did to arrive at equation (4.14), finding now that

$$J^{(2)}_{2m}=-\frac{2\mathrm{i}I}{3am}\,\delta(r-a)\frac{\partial}{\partial\theta}\,Y^*_{2m}\big|_{\theta=\pi/2}. \qquad (4.22)$$

Substitution of this result into equation (4.11) with

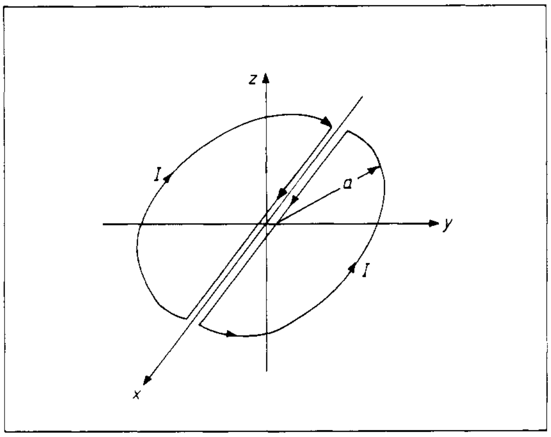

Figure 1

$l=2$ yields at once

$$M_{2,0}=M_{2,\pm2}=0$$
$$M_{2,\pm1}=-\frac{4\mathrm{i}a^3I}{3c}\left(\frac{15}{8\pi}\right)^{1/2}. \qquad (4.23)$$

The leading magnetic induction field at large distances then can be found from the potential

$$\Phi_{\mathrm{M}}=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}M_{lm}\frac{Y_{lm}}{r^{l+1}}$$
$$\simeq\frac{4\pi}{5r^3}[M_{21}Y_{21}+M_{2,-1}Y_{2,-1}] \qquad (4.24)$$
$$\Phi_{\mathrm{M}}=-\frac{4a^3I}{cr^3}\sin\theta\cos\theta\sin\phi \qquad (r \text{ large}).$$

It is possible to describe this field with a symmetric quadrupole tensor Q_{ij} from the relations (see, e.g. Jackson 1975 p 137).

$$\left.\begin{aligned}M_{22}&=0=\frac{1}{12}\left(\frac{15}{2\pi}\right)^{1/2}(Q_{xx}-2\mathrm{i}Q_{xy}-Q_{yy})\\ M_{21}&=-\frac{4\mathrm{i}a^3I}{3c}\left(\frac{15}{8\pi}\right)^{1/2}=-\frac{1}{3}\left(\frac{15}{8\pi}\right)^{1/2}(Q_{xy}-\mathrm{i}Q_{yz})\\ M_{20}&=0=\frac{1}{2}\left(\frac{5}{4\pi}\right)^{1/2}Q_{zz}.\end{aligned}\right\} \qquad (4.25)$$

For this field, then, the only nonvanishing quadrupole component is

$$Q_{yz}=-\frac{4a^3I}{c}. \qquad (4.26)$$

The magnetic induction field, of course, will be precisely the same as an $\boldsymbol{E}$ field described by equation (4.24). A simpler example where the leading multipole is the quadrupole, is two loops of equal and opposite dipoles (Gray 1979). Introductory accounts of multipole expansions, using both spherical and Cartesian tensors and the relations between them are found in the literature (see, for example, Gray and Gubbins 1983).

5. Concluding remarks

The new results of this report are the definitions of vector spherical harmonics through equations (3.5), (3.8) and (3.9). We emphasise that their development assumes nothing more than a familiarity with the properties and usage of the scalar spherical harmonics. Indeed, the vector spherical harmonics are formulated in analogy to the scalar spherical harmonics. We remark that provided one is not going to do transformations of axes, one can have scalars and vectors as they are in the present paper, but as soon as transformations are introduced the description given in this paper is not adequate.

Several convenient approaches for treating radiation fields using vector spherical harmonics have been recently reported (Gray 1978a, b, 1979, Gray and Nickel 1978, Lambert 1978). We have seen by way of illustrative examples that certain problems in magnetostatics offer straightforward solutions in terms of vector spherical harmonics. Furthermore, combining the approach by which the vector spherical harmonics are introduced in this article with the usual expressions for the multipole expansion of a plane wave, standard electromagnetic scattering problems can be easily worked out. These features should render this article useful to senior undergraduate and graduate students.

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