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Extended Translation Invariance and Associated Gauge Fields

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Gauge fields together with non-linear field equations to govern them are introduced by requiring that the Lagrangian should be invariant under an extended translation in space-time, i. e. a translation in which four parameters are replaced by four arbitrary coordinate-dependent functions. A prescription is given to convert a non-invariant canonical ("pseudo") energy-momentum tensor into an invariant one.

The symmetric part of these field equations is examined for the two cases: (1) under linear and non-relativistic approximation, it reduces to the classical gravitational-field equation, (2) for static and spherically symmetric field, its solution is shown to correspond to Schwarzschild's solution. The antisymmetric part has no classical analogues, for there are no sources of skew-symmetric energy-momentum tensors in the classical experiments. A reasonable method is proposed to eliminate this redundant field.

§ 1. Introduction

Since it was suggested that the electromagnetic interaction is best understood in terms of a principle of gauge invariance, under a gauge transformation with a coordinate-dependent function, there have been a number of attempts to deduce the existence of gauge fields coupled to conserved currents, starting with the idea of extended transformations.[1]

It was shown that the invariance under the n-parameter Lie group of transformation referred to space-time and/or fields leads to the conservation of n generators. Further, invariance requirement under an extended transformation, i.e. a transformation whose n parameters are replaced by n space-time dependent functions, necessitates the introduction of n (generally) non-commuting vector fields together with field equations which they must obey.[2),3)]

The purpose of this paper is to deduce the existence of a gravitational field from the translational invariance in an extended sense just mentioned above. In order to construct the gravitational interaction, Utiyama has proposed to introduce 24 new field variables by postulating the invariance under an extended four-dimensional rotation which is specified by six skew-symmetric arbitrary functions $\omega_{ij}(x)$.[3)] However, the self-inconsistency of his scheme was pointed out by Kibble who has claimed that it is necessary to consider the extension of full 10-parameter inhomogeneous Lorentz group in place of the restricted six-parameter group.[4)] Then, our method is different from both of them and will be shown to be one of the simplest ways of discussing the gravitational interaction within the Lagrangian formalism of the unquantized fields in that we need the minimal transformation group (translation group) and its extension necessary and sufficient to deduce it.*)

In the following section, a general formulation of the extended translation is given within the classical Lagrangian framework and a prescription is presented to convert a "pseudo" energy-momentum tensor into an invariant energy-momentum tensor. In § 3, we apply it to the system consisting of the spinor field and the new fields.

We identify the symmetric part of these new fields with the classical gravitational field by means of the linear approximation to the non-linear field variables. In § 4, for the purposes of comparison we shall consider the static and spherically symmetric field in which the exact solution of Einstein's equation of gravitation has been well known and verified by the observations. In § 5, elimination of the antisymmetric part of the new fields will be attempted and the final section is devoted to a discussion of the results.

§ 2. General formulation

We start with the Lagrangian density**),***)

$$L_0 = L_0(q^A, q^A_{,k}, x_k), \qquad q^A_{,k} = \partial q^A/\partial x_k,$$

where q^A are a set of fields, $(A=1, 2, \cdots, N)$. The action integral referred to an arbitrary four-dimensional domain Σ,

$$I(\Sigma) = \int_\Sigma L_0(x)\, d^4x, \tag{2·1}$$

is invariant under the following infinitesimal transformation:

$$\left.\begin{aligned} q^A(x) &\to q'^A(x') = q^A(x) + \delta q^A(x), \\ x_k &\to x_k' = x_k + \delta x_k, \end{aligned}\right\} \tag{2·2}$$

if the following identity holds true at any world points (independent of the behavior of q^A and its derivatives):

$$L_0 + L_0 \delta x_{k',k} = \delta * L_0 + (L_0 \delta x_k)_{,k} \equiv 0, \tag{2·3}$$

where $\delta * L_0 = \delta L_0 - L_{0,k} \delta x_k$ is often called a substantial variation in the sense that a variation caused by the coordinate transformation is subtracted.****) Up to the variation of first order, it follows that

*) Einstein's theory of general relativity has been based on the general covariance under the extended translation within the classical mechanics.

**) It is assumed that the Lagrangian to be considered hereinafter contains the first derivatives of the field variable at most.

***) We use the imaginary fourth coordinate $x_4 = ict$.

****) A summation convention for dummy indices is used throughout this paper.

$$I(\Sigma) \longrightarrow I'(\Sigma) = \int_{\Sigma} L_0'(x)\,(1+\delta x_{k,k})\,d^4x\,, \tag{2·1'}$$

which is used in deriving the above identity. Consider a translation

$$\delta q^A = 0\,, \quad \delta x_k = \epsilon_k\,, \quad (\epsilon_k\text{: infinitesimal parameter}), \tag{2·4}$$

then the following identity must obtain in order to preserve the invariance under this transformation:

$$\frac{\partial L_0}{\partial x_k} = \frac{dL_0}{dx_k} - \frac{\partial L}{\partial q^A}\frac{\partial q^A}{\partial x_k} - \frac{\partial L}{\partial q^A_{,l}}\frac{\partial q^A_{,l}}{\partial x_k} \equiv 0\,, \tag{2·5}$$

which obviously implies that the invariant Lagrangian under the translation has no explicit x-dependence, hence we shall consider exclusively the Lagrangian

$$L_0 = L_0(q^A,\, q^A_{,k})\,. \tag{2·6}$$

Equation (2·3) is rewritten as

$$[L_0]_{q^A}\delta^* q^A + S_{k,k} \equiv 0\,, \tag{2·7}$$

and the equation of motion is abbreviated:

$$[L_0]_{q^A} = \frac{\partial L_0}{\partial q^A} - \left(\frac{\partial L_0}{\partial q^A_{,k}}\right)_{,k} = 0\,,^{*)} \tag{2·8}$$

where

$$\begin{aligned} S_k &= L_{0,q^A_{,k}}\delta q^A - T_{lk}\delta x_l\,, \qquad \partial L_0/\partial q^A_{,k} = L_{0,q^A_{,k}}\,,\\ T_{lk} &= L_{0,q^A_{,k}} q^A_{,l} - \delta_{lk} L_0\,. \end{aligned} \tag{2·9}$$

If the action integral is invariant under the translation (2·4), the conservation law of the energy-momentum tensor defined above follows

$$T_{kl,l} = 0 \tag{2·10}$$

on account of the field equation (2·8).

Next we consider the extended translation**)

$$\delta q^A = 0\,, \qquad \delta x^\mu = \varepsilon^\mu(x)\,, \tag{2·11}$$

$$(\varepsilon^\mu(x);\ \text{infinitesimal arbitrary function}).$$

The invariance property of L_0 under the translation (2·4) breaks down in this case; the variation of the derivative does not vanish,

$$\delta q^A_{,\mu} = (\delta q^A)_{,\mu} - q^A_{,\nu}\delta x^\nu_{,\mu} = -\varepsilon^\nu_{,\mu} q^A_{,\nu}\,. \tag{2·12}$$

We shall further require the invariance of the action integral under the extended

*) This is called the Euler equation and derived by postulating $\delta^* I = 0$ under the condition that $\delta^* q^A$ should vanish on the boundary surface of the integration domain.

**) In this case the Greek indices are used for conveniences.

translation, by defining the covariant derivative through which the new field $a_k{}^\mu(x)$ is introduced so as to satisfy our postulate:

$$D_k q^A = \{\delta_k{}^\mu + a_k{}^\mu(x)\} q^A_{,\mu} = b_k{}^\mu(x) q^A_{,\mu}, \tag{2·13}$$

$$\delta D_k q^A = 0\,. \tag{2·14}$$

In order to satisfy Eq. (2·14), it follows immediately,

$$\delta b_k{}^\mu = \varepsilon^\mu_{,\nu} b_k{}^\nu. \tag{2·15}$$

Therefore we recover the invariance of the action integral even under the extended translation (i) by simply replacing $q^A_{,k}$ in the original Lagrangian by the covariant derivative $D_k q^A$ defined above;

$$L_0(q^A, q^A_{,k}) \longrightarrow L'(q^A, q^A_{,k}, b_k{}^\mu) = L''(q^A, D_k q^A)$$
$$= L_0(q^A, q^A_{,k} \longrightarrow D_k q^A), \tag{2·16}$$

hence its variation associated with (2·11) vanishes identically

$$\delta L' = 0\,, \tag{2·16'}$$

and further (ii) by multiplying L' by a certain function $b(x)$ so as to satisfy the required identity (2·3):

$$\delta \boldsymbol{L} + \boldsymbol{L}\varepsilon^\mu_{,\mu} = 0\,,$$
$$\boldsymbol{L} = bL'. \tag{2·17}$$

Accordingly, the transformation property of b has to be

$$\delta b = -\varepsilon^\mu_{,\mu} b\,. \tag{2·18}$$

Next tasks are then to construct such a function $b(x)$ and the invariant field strength from $b_k{}^\mu$ and its first derivatives. For these purposes, it is necessary to define the field $b_{k\mu}$ inverse to $b_k{}^\mu$ from the following orthogonal relations:

$$b_{k\mu} b_l{}^\mu = \delta_{kl}\,,$$
$$b_{k\mu} b_k{}^\nu = \delta_\mu{}^\nu. \tag{2·19}$$

Consequently, it follows

$$\delta b_{k\mu} = -\varepsilon^\nu_{,\mu} b_{k\nu}\,,$$

hence we choose

$$b = \det(b_{k\mu}),$$

because it has the desired property (2·18). In other words, the invariant volume element becomes bd^4x instead of d^4x. Suppose that we obtain a free Lagrangian L^G for the new field, the action integral turns out

$$I(\Sigma) = \int_\Sigma d^4x \boldsymbol{L}\,,$$

where

$$\boldsymbol{L}=\boldsymbol{L}'+\boldsymbol{L}^{G}=b(L'+L^{G}),^{*)} \tag{2·20}$$

and L^G consists of the invariant field strength. We shall write for short

$$(Q^{\alpha})=(q^{A},\ b_{k\mu}). \tag{2·21}$$

The invariance of the action integral follows from the following identity analogous to (2·3):

$$\delta*\boldsymbol{L}+(\boldsymbol{L}\varepsilon^{\mu})_{,\mu}=0\,,$$

which is just shown above to hold by means of our prescription. The above identity is rewritten as before,

$$[\boldsymbol{L}]_{Q^{\alpha}}\delta*Q^{\alpha}+\boldsymbol{S}^{\mu}_{,\mu}=0\,, \tag{2·22}$$

where

$$\left.\begin{aligned}&\boldsymbol{S}^{\mu}=\boldsymbol{L}_{,Q^{\alpha}_{\ \mu}}\delta Q^{\alpha}-\boldsymbol{T}_{\nu}{}^{\mu}\varepsilon^{\nu},\\&\boldsymbol{T}_{\nu}{}^{\mu}=\boldsymbol{L}_{,Q^{\alpha}_{\ \mu}}Q^{\alpha}_{,\nu}-\delta_{\nu}{}^{\mu}\boldsymbol{L}={}^{m}\boldsymbol{T}_{\nu}{}^{\mu}+\tilde{\boldsymbol{t}}_{\nu}{}^{\mu},\end{aligned}\right\}^{**)} \tag{2·23}$$

$$\left.\begin{aligned}&{}^{m}\boldsymbol{T}_{\nu}{}^{\mu}=\boldsymbol{L}'_{,q^{A}_{\ \mu}}q^{A}_{,\nu}-\delta_{\nu}{}^{\mu}\boldsymbol{L}',\\&\tilde{\boldsymbol{t}}_{\nu}{}^{\mu}=\boldsymbol{L}^{G}_{,b_{k\lambda,\mu}}b_{k\lambda,\nu}-\delta_{\nu}{}^{\mu}\boldsymbol{L}^{G}.\end{aligned}\right\} \tag{2·24}$$

From (2·22), we obtain

$$\begin{aligned}&(-\varepsilon^{\mu})\{[\boldsymbol{L}']_{q^{A}}q^{A}_{,\mu}+[\boldsymbol{L}]_{b_{k\nu}}b_{k\nu,\mu}-([\boldsymbol{L}]_{b_{k\nu}}b_{k\mu})_{,\nu}\}\\&-\{[\boldsymbol{L}]_{b_{k\mu}}b_{k\nu}\varepsilon^{\nu}+\boldsymbol{L}^{G}_{,b_{k\lambda,\mu}}b_{k\nu}\varepsilon^{\nu}_{,\lambda}+({}^{m}\boldsymbol{T}_{\nu}{}^{\mu}+\tilde{\boldsymbol{t}}_{\nu}{}^{\mu})\varepsilon^{\nu}\}_{,\mu}\equiv0\,.\end{aligned} \tag{2·25}$$

As ε^{ν}, $\varepsilon^{\nu}_{,\mu}$ and $\varepsilon^{\nu}_{,\mu\lambda}$ are chosen arbitrarily inside the integration domain Σ, the second term of Eq. (2·25) resolves itself into the three identities

$$\varepsilon^{\nu}\,;\qquad([\boldsymbol{L}]_{b_{k\mu}}b_{k\nu})_{,\mu}+({}^{m}\boldsymbol{T}_{\nu}{}^{\mu}+\tilde{\boldsymbol{t}}_{\nu}{}^{\mu})_{,\mu}\equiv0\,, \tag{2·26}$$

$$\varepsilon^{\nu}_{,\mu}\,;\qquad[\boldsymbol{L}]_{b_{k\mu}}b_{k\nu}+({}^{m}\boldsymbol{T}_{\nu}{}^{\mu}+\tilde{\boldsymbol{t}}_{\nu}{}^{\mu})+(\boldsymbol{L}^{G}_{,b_{k\mu,\lambda}}b_{k\nu})_{,\lambda}\equiv0\,, \tag{2·27}$$

$$\varepsilon^{\nu}_{,\mu\lambda}\boldsymbol{L}^{G}_{,b_{k\lambda,\mu}}b_{k\nu}\equiv0\,, \tag{2·28}$$

among which there are only two independent identities, for the last identity implies

$$\boldsymbol{L}^{G}_{,b_{k\lambda,\mu}}+\boldsymbol{L}^{G}_{,b_{k\mu,\lambda}}=0 \tag{2·28'}$$

and the differentiation of the second identity yields the first one, by making use of (2·28′). Furthermore it suggests that the invariant field strength must contain an antisymmetric combination of $b_{k\lambda,\mu}$ with respect to the Greek subscripts and

*) In the standard terminology of tensor algebra, $\boldsymbol{L}$ is named the "tensor density".

**) ${}^{m}\boldsymbol{T}_{\nu}{}^{\mu}$, $\tilde{\boldsymbol{t}}_{\nu}{}^{\mu}$ are the "canonical" energy-momentum tensors.

then the contraction of these indices has to be performed; finally we have*)

$$c_{klm}=2b_{k[\mu,\nu]}b_l{}^\mu b_m{}^\nu,$$
$$=2b_{k\nu}b_{[l}{}^\mu b_{m]}{}^\nu{}_{,\mu}$$

with

$$\delta c_{klm}=0\,.$$

It should be noticed that there exist infinitely many conservation laws for generalized energy-momentum tensors in addition to the one implied directly by (2·26), ($f^{(\nu)}$: an arbitrary function),

$$\left.\begin{aligned}&\boldsymbol{T}^{(f)\mu}=f^{(\nu)}({}^m\boldsymbol{T}_\nu{}^\mu+\tilde{\boldsymbol{t}}_\nu{}^\mu)-f^{(\nu)}_{,\lambda}(\boldsymbol{L}^G_{,b_{k\mu,\lambda}}b_{k\nu}),\\&\boldsymbol{T}^{(f)\mu}_{,\mu}=0\,.^{**)}\end{aligned}\right\}\qquad(2\cdot29)$$

Under the extended translation only the Greek indices are associated with the transformation properties: Then there arises a question, "What role does the Latin index play?" To see it, we consider the four-dimensional rotation of the field variables only,

$$\left.\begin{aligned}&\delta x^\mu=0\,,\\&\delta q^A=Tq^A,\end{aligned}\right\}\qquad(2\cdot30)$$

under the assumption that L' is kept invariant under (2·30) and L_0 under the Lorentz transformation specified by

$$\delta x_k=\omega_{kl}x_l\,,\qquad(\omega_{(kl)}=0)\,,$$
$$\delta q^A=Tq^A.$$

The assumed invariance properties respectively yield the following identities:

$$L''_{,q^A}Tq^A+L''_{,D_kq^A}TD_kq^A+L'_{,b_k{}^\mu}\delta b_k{}^\mu=0\,,\qquad(2\cdot31)$$
$$L_{0,q^A}Tq^A+L_{0,q^A_{,k}}Tq^A_{,k}-L_{0,q^A_{,k}}q^A_{,l}\omega_{lk}=0\,,$$

and (2·31) passes into

$$(\delta b_k{}^\mu-\omega_{kl}b_l{}^\mu)L''_{,D_kq^A}q^A_{,\mu}=0\,,$$

by making use of the relation (2·16). Hence the transformation property is established,

$$\delta b_k{}^\mu=\omega_{kl}b_l{}^\mu,$$

that is, the Latin index is related to the four-dimensional rotation and $b_k{}^\mu$ transforms as a four vector under it (the same is true for $b_{k\mu}$ too).

The quantity $b_k{}^\mu$ is the contravariant vector as it transforms contragradiently

*) $A_{(\mu\nu)}=(1/2)(A_{\mu\nu}+A_{\nu\mu})$, $A_{[\mu\nu]}=(1/2)(A_{\mu\nu}-A_{\nu\mu})$.

**) The conserved quantities will be explicitly given later in an invariant manner.

to $\delta q^A_{,\mu}$, and $b_{k\mu}$ the covariant vector as it transforms cogradiently to (2·12). $b_k{}^\mu$ is to be referred to as a "vierbein" system because of its dual character under the extended translation and the four-dimensional rotation specified by (2·11) and (2·30), respectively. These situations are made clear and are summed up by the following statement. Under the combination of these two independent transformations,

$$\delta x^\mu = \varepsilon^\mu(x), \tag{2·30'}$$

$$\delta q^A = Tq^A, \tag{2·30''}$$

L' stays invariant if $b_k{}^\mu$ (or equivalently $b_{k\mu}$) transforms as follows:

$$\left.\begin{aligned}\delta b_k{}^\mu &= \varepsilon^\mu_{,\nu} b_k{}^\nu + \omega_{kl} b_l{}^\mu,\\ \delta b_{k\mu} &= -\varepsilon^\nu_{,\mu} b_{k\nu} + \omega_{kl} b_{l\mu}.\end{aligned}\right. \tag{2·30'''}$$

The field strength c_{klm} is reducible under the four-dimensional rotation; the irreducible parts of it are calculated by means of the standard method:

i) an irreducible tensor of rank 3,

$$c^T_{klm} = c_{(kl)m} - (1/3)\,(\delta_{kl} c_m{}^V - \delta_{m(k} c_{l)}{}^V),$$

ii) a vector,

$$c_k{}^V = c_{mmk} = (b b_k{}^\mu)_{,\mu}/b\,,$$

iii) an axial vector,

$$c_k{}^A = i\varepsilon_{klmn} c_{lmn}/6\,.$$

The tensor c_{klm} is represented in terms of these irreducible tensors,

$$c_{klm} = (4/3)\, c^T_{k[lm]} + (2/3)\,\delta_{k[l} c^V_{m]} + i\varepsilon_{klmn} c_n{}^A.$$

We shall require that L^G should be of the quadratic form in the first derivative of $b_k{}^\mu$. Thus, we choose

$$\boldsymbol{L}^G = b\,(\alpha c^T_{klm} c^T_{klm} + \beta c_k{}^V c_k{}^V + \gamma c_k{}^A c_k{}^A + \delta)\,{}^{*)} \tag{2·32}$$

with the arbitrary constants α, β, γ, δ. Inserting the above Lagrangian into (2·27), we obtain after some algebra

$$\boldsymbol{B}_{kl} = {}^m\boldsymbol{T}_{kl}\,, \tag{2·33}$$

$$\boldsymbol{B}_{kl} = -b_m{}^\mu b F_{klm,\,\mu} + c_m{}^V \boldsymbol{F}_{kml} + (1/2)\, c_{lmn} \boldsymbol{F}_{kmn} - c_{mnk} \boldsymbol{F}_{mnl} + \delta_{kl} \boldsymbol{L}^G, \tag{2·34}$$

$$\boldsymbol{F}_{klm} = 4b\,\{\alpha c^T_{k[lm]} + \beta \delta_{k[l} c^V_{m]} - (1/6)\, i\gamma \varepsilon_{klmn} c_n{}^A\}, \tag{2·35}$$

$$\left.\begin{aligned}{}^m\boldsymbol{T}_{kl} &= b_k{}^\nu b_{l\mu}\, {}^m\boldsymbol{T}_\nu{}^\mu = b\,{}^m T_{kl}\\ &= \boldsymbol{L}''_{,D_l q^A} D_k q^A - \delta_{kl} \boldsymbol{L}'',{}^{**)}\end{aligned}\right\} \tag{2·36}$$

*) We shall assume invariance under space reflection. If we do not assume it, we can add a linear sum of the following two terms: $bc_k{}^V c_k{}^A$, $i\varepsilon_{klmn} bc^T{}_{jkl} c^T{}_{jmn}$.

**) It is also rewritten as $-\boldsymbol{L}'_{,b_{k\mu}} b_{l\mu}$.

where all the tensors are converted into the local tensors in order to preserve the invariance under the extended translation.

We shall manage to write down the equation of motion (2·33) in a simpler form analogous to the divergent form (2·27),

$$-(b_l{}^\mu b_m{}^\nu \boldsymbol{F}_{klm})_{,\nu} = b_l{}^\mu ({}^m\boldsymbol{T}_{kl} + \boldsymbol{t}_{kl}), \tag{2·37}$$

where

$$\begin{aligned}\boldsymbol{t}_{kl} &= b_k{}^\nu b_{l\mu} \boldsymbol{t}_\nu{}^\mu = \boldsymbol{L}^G_{,b_l{}^\mu} b_k{}^\mu \\ &= c_{mnk} \boldsymbol{F}_{mnl} - \delta_{kl} \boldsymbol{L}^G,\end{aligned} \tag{2·38}$$

$$\begin{aligned}\tilde{\boldsymbol{t}}_{kl} &= b_k{}^\nu b_{l\mu} \tilde{\boldsymbol{t}}_\nu{}^\mu \\ &= \boldsymbol{t}_{kl} + b_k{}^\mu b_m{}^\nu b_{n\mu,\nu} \boldsymbol{F}_{nml}.\end{aligned} \tag{2·39}$$

It should be emphasized that $\boldsymbol{t}_{kl}$ remains invariant under the extended translation while $\tilde{\boldsymbol{t}}_{kl}$ is not invariant, as is easily shown (hence a tilde has been attached to the canonical energy-momentum tensor density $\tilde{\boldsymbol{t}}_\nu{}^\mu$). From (2·37), we obtain the conservation law

$$\{b_l{}^\mu ({}^m\boldsymbol{T}_{kl} + \boldsymbol{t}_{kl})\}_{,\mu} = 0, \tag{2·40}$$

which is essentially the same as (2·26), although $\boldsymbol{t}_\nu{}^\mu$ is preferred to $\tilde{\boldsymbol{t}}_\nu{}^\mu$. In a manner similar to that stated above, (2·29) turns out

$$\left.\begin{aligned}&(b_l{}^\mu \boldsymbol{T}_l{}^{(f)})_{,\mu} = 0, \\ &\boldsymbol{T}_l{}^{(f)} = f_{(k)} ({}^m\boldsymbol{T}_{kl} + \boldsymbol{t}_{kl}) - f_{(k),\nu} b_m{}^\nu \boldsymbol{F}_{klm}, \\ &(f_{(k)}(x) \text{ is an arbitrary function}).\end{aligned}\right\} \tag{2·41}$$

Hence there exist indenumerable conserved quantities

$$\boldsymbol{P}^{(f)} = \int d^3x b_l{}^0 \boldsymbol{T}_l{}^{(f)}, \tag{2·42}$$

where of course the conserved vector corresponding to (2·40) is included by a particular choice in $f_{(k)}(x)$:

$$\begin{aligned}&(f_{(k)}) = (f^l_{(k)}) = (\delta_k{}^l), \\ &\boldsymbol{P}_k = \int d^3x b_l{}^0 ({}^m\boldsymbol{T}_{kl} + \boldsymbol{t}_{kl}).\end{aligned} \tag{2·43}$$

A function $f_{(k)}$ needs not to be a vector under the four-dimensional rotation.

Before closing this section, we shall resolve (2·33) into the symmetric and skew-symmetric parts**)

*) $\tilde{\boldsymbol{t}}_{kl}$ corresponds to the famed "pseudo" energy-momentum tensor of the gravitational field which does not transform as a tensor under the general coordinate transformation.

**) Just recall that the symmetrized (not the canonical) energy-momentum tensor of the matter source is of physical significance.

$$\boldsymbol{B}_{(kl)} = {}^m\boldsymbol{T}_{(kl)}, \tag{2·44}$$

$$\boldsymbol{B}_{[kl]} = {}^m\boldsymbol{T}_{[kl]}, \tag{2·45}$$

and further attempt to convert them into the divergent forms similar to (2·37),

$$\left.\begin{aligned} &-\{bb_l{}^\mu b_m{}^\nu (3\alpha(c_{klm} - (3/2)\,i\varepsilon_{klmn}c_n{}^A) - 2(\alpha - 2\beta)\,\delta_{k[l}c^V_{m]})\}{}_{,\nu} \\ &\qquad = b_l{}^\mu ({}^m\boldsymbol{T}_{(kl)} + \boldsymbol{t}^1_{kl}) + 2(\alpha+\beta)\,bb_l{}^\mu b^\nu_{[k}c^V_{l]}{}_{,\nu}\,, \end{aligned}\right\} \tag{2·46}$$

$$\left.\begin{aligned} &-\{bb_l{}^\mu b_m{}^\nu (3/2)\,i(\alpha - (4/9)\gamma)\,\varepsilon_{klmn}c_n{}^A\}{}_{,\nu} \\ &\qquad = b_l{}^\mu ({}^m\boldsymbol{T}_{[kl]} + \boldsymbol{t}^2_{kl}) - 2(\alpha+\beta)\,bb_l{}^\mu b^\nu_{[k}c^V_{l]}{}_{,\nu}\,, \end{aligned}\right\} \tag{2·47}$$

with

$$\left.\begin{aligned} \boldsymbol{t}^2_{kl} &= \boldsymbol{t}_{kl} - \boldsymbol{t}^1_{kl} \\ &= b(3/4)\,i\{\alpha - (4/9)\gamma\}\,\varepsilon_{kmnj}c_{lmn}c_j{}^A, \end{aligned}\right\} \tag{2·48}$$

where $\boldsymbol{t}^2_{kl}$ does not contain an arbitrary constant β; it can be rewritten in terms of the irreducible tensor components,

$$\boldsymbol{t}^2_{kl} = (1/2)\,ib\{\alpha - (4/9)\gamma\}\,\{2\varepsilon_{kmnj}c^T_{lmn}c_j{}^A + \varepsilon_{klmj}c_m{}^V c_j{}^A + 3i(\delta_{kl}c_m{}^A c_m{}^A - c_k{}^A c_l{}^A)\}. \tag{2·48'}$$

In deriving (2·46) and (2·47) a special care is taken in order to eliminate the second derivate of the field variables from the definitions of $\boldsymbol{t}^1_{kl}$ and $\boldsymbol{t}^2_{kl}$, by dint of the useful identity

$$b_k{}^\mu c_{klm}{}_{,\mu} + c_k{}^V c_{klm} = 2b^\mu_{[l}c^V_{m]}{}_{,\mu}\,. \tag{2·49}$$

If we put

$$\alpha + \beta = 0\,, \tag{2·50}$$

the above two equations (2·46) and (2·47) yield after the differentiation

$$\{b_l{}^\mu ({}^m\boldsymbol{T}_{(kl)} + \boldsymbol{t}^1_{kl})\}{}_{,\mu} = 0\,, \tag{2·51}$$

$$\{b_l{}^\mu ({}^m\boldsymbol{T}_{[kl]} + \boldsymbol{t}^2_{kl})\}{}_{,\mu} = 0\,. \tag{2·52}$$

§ 3. Linear approximation in spinor-vierbein interaction

The field equations proposed in § 2 are non-linear with respect to the field variables $b_k{}^\mu$. We know that a linear theory (Newton's theory) accounts, with a considerable degree of accuracy, for the motion of bodies under the gravitational forces. We shall discuss the interaction between the spinor field and the vierbein field $b_k{}^\mu$ by assuming both the difference

$$b_{k\mu} - \delta_{k\mu} = a_{k\mu}$$

and its first derivatives are so small as compared to unity that the quadratic terms in $a_k{}^\mu$ and/or its derivatives lead only to secondary effects and are hereafter neglected. In this linear approximation, all the Greek indices are replaced by the Latin indices as there remain no differences between them. From the orthogonal relations (2·19), it follows that

$$a_k{}^l = -a_{lk}\,,$$

and the various non-linear quantities pass into the linearized ones,

$$\left.\begin{aligned} &c_{klm} = 2a_{k[l'm]}\,,\\ &c^T_{klm} = a_{(kl)'m} - a_{(km'l)} - (1/3)\,(\delta_{kl}c_m{}^V - \delta_{m(k}c^V_{l)})\,,\\ &c_k{}^V = 2a_{l[l'k]}\,,\\ &c_k{}^A = (1/3)\,i\varepsilon_{klmn}a_{lm'n}\,. \end{aligned}\right\} \tag{3·1}$$

The field equations (2·44) and (2·45) become

$$\begin{aligned} &-3\alpha\Box a_{(kl)} + (2\alpha-\beta)\,(a_{(km)'ml} + a_{(lm)'mk})\\ &\quad + (\alpha-2\beta)\,\{\delta_{kl}(\Box a - a_{(mn)'mn}) - a_{'kl}\} + (\alpha+\beta)\,(a_{[km]'ml} + a_{[lm]'mk})\\ &\quad + \delta\cdot\delta_{kl} = {}^mT_{(kl)}\,, \end{aligned} \tag{3·2}$$

$$\begin{aligned} &-\{\alpha-(4/9)\gamma\}\Box a_{[kl]} + \{\beta+(4/9)\gamma\}\,(a_{[lm]'mk} - a_{[km]'ml})\\ &\quad -(\alpha+\beta)\,(a_{(lm)'mk} - a_{(km)'ml}) = {}^mT_{[kl]} \end{aligned} \tag{3·3}$$

with

$$a = a_{mm}\,,\qquad \Box = \partial_m\partial_m\,.$$

Provided that the relation (2·50) holds, these two equations are completely decomposed into the symmetric and anti-symmetric parts,

$$-3\alpha\{\Box a_{(kl)} + a_{'kl} - a_{(km)'ml} - a_{(lm)'mk} - \delta_{kl}(\Box a - a_{(mn)'mn})\} + \delta\cdot\delta_{kl} = {}^mT_{(kl)}\,, \tag{3·4}$$

$$-\{\alpha-(4/9)\gamma\}\,(\Box a_{[kl]} + a_{[lm]'mk} - a_{[km]'ml}) = {}^mT_{[kl]}\,. \tag{3·5}$$

With the help of the convenient notations

$$\left.\begin{aligned} &\alpha = (3\kappa^2)^{-1},\qquad \alpha-(4/9)\gamma = \lambda^{-2},\\ &a_{(kl)} - (1/2)\,\delta_{kl}a = \kappa S_{kl}\,,\\ &a_{[kl]} = \lambda A_{kl}\,, \end{aligned}\right\} \tag{3·6}$$

we shall be able to simplify the form of these equations to some extent,

$$\Box S_{kl} - 2S_{(km'ml)} + \delta_{kl}(S_{mn'mn} - \kappa\delta) = -\kappa{}^mT_{(kl)}\,,^{*)} \tag{3·7}$$

$$\Box A_{kl} - 2A_{[km'ml]} = -\kappa{}^mT_{[kl]}\,. \tag{3·8}$$

Further by imposing the generalized Lorentz conditions on S_{kl} and A_{kl}

*) $\kappa\delta$ corresponds to the cosmological constant, hence it will be neglected hereinafter.

$$\left.\begin{aligned} S_{km'm} &= 0\,, \\ A_{km'm} &= 0\,, \end{aligned}\right\} \tag{3·9}$$

we obtain

$$\Box S_{kl} = -\kappa^m T_{(kl)}\,, \tag{3·10}$$

$$\Box A_{kl} = -\lambda^m T_{[kl]}\,. \tag{3·11}$$

On the other hand, we choose the Lagrangian of a spinor field, say an electron, as a matter field,

$$L_0 = (1/2)\,(\bar{\psi}\gamma_k\psi_{,k} - \bar{\psi}_{,k}\gamma_k\psi) + m\bar{\psi}\psi\,, \tag{3·12}$$

hence we obtain the invariant substitute of it by the prescription stated in § 2,

$$L' = (1/2)\,b_k{}^{\mu}(\bar{\psi}\gamma_k\psi_{,\mu} - \bar{\psi}_{,\mu}\gamma_k\psi) + m\bar{\psi}\psi\,. \tag{3·13}$$

The equation of motion derived from $\boldsymbol{L}' = bL'$ is

$$b_k{}^{\mu}\gamma_k\psi_{,\mu} + (1/2)\,c_k{}^{V}\gamma_k\psi + m\psi = 0\,, \tag{3·14}$$

which is reduced by the linear approximation (3·1) and the condition of divergence-free (3·9) to

$$\gamma_k\{\partial_k - \kappa(S_{kl} - (1/2)\delta_{kl}S)\,\partial_l + \lambda A_{kl}\partial_l - (1/4)\kappa S_{,k}\}\,\psi + m\psi = 0\,, \tag{3·14'}$$
$$(S_{mm} = S)\,.$$

On multiplying the dual operator, we obtain the differential equation of second order,

$$\{(1+\kappa S)\,\Box - 2\kappa S_{kl}\partial_k\partial_l - (1/4)\,\kappa(\Box S)\}\,\psi$$
$$+ i\,\{\kappa(S_{km'l}\partial_m - (1/2)\,S_{,l}\partial_k) + \lambda A_{lm'k}\partial_m\}\,\sigma_{kl}\psi - m^2\psi = 0\,, \tag{3·15}$$

where

$$\gamma_k\gamma_l = \delta_{kl} + i\sigma_{kl}\,.$$

Now let us proceed to the non-relativistic limit of (3·10), (3·11) and (3·15);

$$\left.\begin{aligned} &\Box S_{00} = -\kappa^m T_{00} = -\kappa\rho\,, \\ &\Box S_{a0} = 0 = \Box S_{ab}\,, \qquad (a,\ b = 1,\ 2,\ 3)\,, \\ &\Box A_{kl} = 0\,, \end{aligned}\right\} \tag{3·16}$$

$$E\psi = \{p^2/(2m) - (\kappa m/2)\,S_{00} + (\kappa^2/8m)^m T_{00}\}\,\psi, \tag{3·17}$$

where the well-known relations

$$i\partial_t\psi = (m+E)\,\psi\,, \qquad -i\partial_a\psi = p_a\psi\,, \qquad (a = 1,\ 2,\ 3)$$

*) One is permitted to set $S_{kl} = 0 = A_{kl}$ except for S_{00} only, as there are no components of sources for them.

are used and ρ denotes the density of matter sources.

Equation (3·17) turns out,

$$E\psi=\{p^2/(2m)-(\kappa m/2)S_{00}\}\psi\,, \tag{3·18}$$

where we neglected the last term containing κ^2.

Upon comparing the potential term in this Schroedinger equation with the Newtonian potential of gravitation φ, we find that

$$\varphi=-(\kappa/2)S_{00}=a_{44}\,. \tag{3·19}$$

If the field does not change quickly with time, i.e. S_{00} is almost static, it follows from (3·16) and (3·19)

$$\Delta\varphi=-(\kappa/2)\Delta S_{00}=(\kappa^2/2)\rho\,,$$

with which the Newtonian equation

$$\Delta\varphi=4\pi k\rho$$

is to be compared. In this manner we are able to determine the coupling constant of the symmetric field,

$$\left.\begin{array}{l}\kappa=\sqrt{8\pi k}\,,\\ (k=1.06\times10^{-9}g^{-2}=5.2\times10^{-67}\,\mathrm{cm}^2 \quad \text{in the natural units}).\end{array}\right\} \tag{3·20}$$

We close this section by remarking that the coupling constant of the anti-symmetric field A_{kl} cannot be determined as there seems no source of skew-symmetric energy-momentum tensors in the classical experiments.

§ 4. Comparison with Einstein's theory

In this section we shall compare the symmetric part of our equation (2·44) with Einstein's equation of gravitation, by defining the symmetric metric tensor by

$$g^{\mu\nu}=b_k{}^\mu b_k{}^\nu\,, \qquad g_{\mu\nu}=b_{k\mu}b_{k\nu}\,. \tag{4·1}$$

The Christoffel three-index symbol of the second kind is given by

$$\Gamma^\lambda_{\mu\nu}=b_l{}^\lambda\{b_{k(\mu}b_{m\nu)}c_{klm}+b_{l(\mu',\nu)}\}\,. \tag{4·2}$$

The Einstein equation takes the form

$$G_{\mu\nu}=R_{\mu\nu}-(1/2)Rg_{\mu\nu}+\kappa^2 g_{\mu\nu}\delta=\kappa^{2m}T_{(\mu\nu)}\,,$$

where

$$R_{\mu\nu}=2\{\Gamma^\lambda_{\mu[\nu',\lambda]}-\Gamma^\rho_{\mu[\lambda}\Gamma^\lambda_{\nu]\rho}\}\,,$$

$$R=g^{\mu\nu}R_{\mu\nu}\,.$$

Transition from the Greek indices into the Latin ones (as already shown in (2·36), for example) yields

$$G_{kl}=b_k{}^\mu b_l{}^\nu G_{\mu\nu}=\kappa^{2m}T_{(kl)}\,, \tag{4·3}$$

with which our symmetric equation (2·44) (devided by b)

$$\kappa^2 B_{(kl)}=\kappa^{2m}T_{(kl)} \tag{4·4}$$

should be compared. By the use of the previous notation (3·7) provided that the relation (2·50) holds, the difference of these two equations is represented by

$$\kappa^2 B_{(kl)}-G_{kl}=-(4\lambda^2)^{-1}(-8i\varepsilon_{(kmni}c_{l)mn}c_i{}^A+6c_k{}^A c_k{}^A+3\delta_{kl}c_m{}^A c_m{}^A)\,. \tag{4·5}$$

It should be noticed that each term of the difference contains an axial vector $c_k{}^A$. Now we have to check whether it vanishes or not.

Apart from the linear approximation where the above difference vanishes exactly, there has been the well-known solution of Schwarzschild in the spherically symmetric field. In particular, we consider its static case: the field variables do not depend on time and a mass point is situated at the origin under the influence of the spherically symmetric forces. In this situation, the components of $b_{k\mu}$ transform according to the laws

$$\left.\begin{aligned}&\delta b_{a\alpha}=\omega_{ab}b_{b\alpha}+\omega_{\alpha\beta}b_{a\beta}\,,\\&\delta b_{a4}=\omega_{ab}b_{b4}\,,\\&\delta b_{4\alpha}=\omega_{\alpha\beta}b_{4\beta}\,,\\&\delta b_{44}=0\,,\quad (a,b,\alpha,\beta=1,2,3)\,,\end{aligned}\right\} \tag{4·6}$$

under the three-dimensional rotation

$$\begin{aligned}&\delta x^\alpha=\omega_{\alpha\beta}x^\beta,\quad (\omega_{(\alpha\beta)}=0)\,,\\&\delta x^4=0\,.\end{aligned} \tag{4·7}$$

It is easy to construct the general form of $b_{k\mu}$ such that it has the required transformation properties mentioned above.

$$\left.\begin{aligned}&b_{a\alpha}=\delta_{a\alpha}(1+A)+BX_aX_\alpha\,,\\&b_{a4}=iCX_a\,,\\&b_{4\alpha}=iDX_\alpha\,,\\&b_{44}=1+E\,,\end{aligned}\right\} \tag{4·8}$$

where

$$X_a=x^a/r\,,\qquad r^2=x^2\,,$$

and $A, B, \cdots, E$ are functions of r only.

A simple calculation yields by making use of these general forms,

$$c_k{}^A=0\,. \tag{4·9}$$

Consequently, the equivalence between the solution of Schwarzschild and ours is established.

§ 5. Elimination of the antisymmetric field

We shall discuss various choices in the arbitray constants introduced in the free Lagrangian L^G for the $b_k{}^\mu$ field (2·32), except for a trivial case, $\alpha=\beta=\gamma=0$. In particular we lay emphasis on the possible vanishing of the skew-symmetric energy-momentum tensor generated by a matter field about which no definite statements have been given in the previous sections. The well-known procedure of symmetrizing energy-momentum tensors by means of adding the canonical spin angular-momentum to it cannot be applied to the present case.

To be specific, we shall base our arguments on the spinor Lagrangian (3·12). As it is invariant under the translation in the usual sense (see (2·4)), there are the conservation laws of the symmetric and antisymmetric energy-momentum tensors separately,

$$T_{(kl)}{}^{\cdot l}=0\,, \tag{5·1}$$

$$T_{[kl]}{}^{\cdot l}=0\,, \tag{5·2}$$

where

$$T_{lk}=(1/2)\,(\bar{\psi}\gamma_k\psi_{\cdot l}-\bar{\psi}_{\cdot l}\gamma_k\psi)\,.$$

As stated at the end of § 2, we could obtain the separate conservation laws (2·51) and (2·52) closely similar to the above ones, provided that α and β satisfy the relation given by (2·50). Now we shall assume it, although we cannot find any *a priori* reason to require such separate conservation laws in an extended sense; however, the result obtained in § 4 seems to support our assumption. Various cases are investigated in order.

Case 1: $\alpha+\beta=0$.

The field equations for the $b_k{}^\mu$ field become

$$-\{b_l{}^\mu b_m{}^\nu b3\alpha[c_{klm}-(3/2)\,i\varepsilon_{klmn}c_n{}^A-2\delta_{k[l}c^V_{m]}]\}_{\cdot\nu}=b_l{}^\mu({}^m\boldsymbol{T}_{(kl)}+\boldsymbol{t}^1_{kl})\,, \tag{5·3}$$

$$-\{b_l{}^\mu b_m{}^\nu b(3/2)\,i[\alpha-(4/9)\gamma]\varepsilon_{klmn}c_n{}^A\}_{\cdot\nu}=b_l{}^\mu({}^m\boldsymbol{T}_{[kl]}+\boldsymbol{t}^2_{kl})\,. \tag{5·4}$$

As is easily shown, it is impossible to make both sides of Eq. (5·4) vanish identically without further conditions.

Case 2: $\alpha+\beta=0$ and $\alpha-(4/9)\gamma=0$.

We find

$$\boldsymbol{t}^2_{kl}=0\,,\qquad \boldsymbol{t}^1_{kl}=\boldsymbol{t}_{kl} \tag{5·5}$$

and

$$\boldsymbol{B}_{[kl]}=0\,. \tag{5·6}$$

However, the skew-symmetric part of the energy-momentum tensor ${}^m\boldsymbol{T}_{[kl]}$ cannot vanish identically so long as the matter field exists. It is interesting to check whether its derivatives should vanish or not. By making use of the equation of motion (3·14) and the generalized Klein-Gordon equation which is derived from (3·14) by multiplying a proper dual operator,

$$\begin{aligned}(\Box^G - m^2)\psi = \{i\sigma_{mn}D_mD_n + c_m{}^V D_m + (1/2)\, b_m{}^\mu c^V_{m',\mu} \\ + (i/2)\,\sigma_{mn} b_m{}^\mu c^V_{n',\mu} + (1/4)\, c_m{}^V c_m{}^V\}\psi,^{*)} \\ \Box^G = D_m D_m\,, \qquad (5\cdot 7)\end{aligned}$$

we observe the connection between the energy-momentum tensor and its derivative as follows

$$^m\boldsymbol{T}_{lk} = (1/2)\, bb_l{}^\mu (\bar\psi\gamma_k\psi_{,\mu} - \bar\psi_{,\mu}\gamma_k\psi) = b^m T_{lk}\,, \qquad (5\cdot 8)$$

$$(b_l{}^{\mu m}\boldsymbol{T}_{kl})_{,\mu} = -c_{mkl}{}^m\boldsymbol{T}_{ml}\,, \qquad (5\cdot 8')$$

$$\begin{aligned}(b_l{}^{\mu m}\boldsymbol{T}_{lk})_{,\mu} = -b(1/4)\,\varepsilon_{klmn}c_{jmn}(\bar\psi\gamma_5\gamma_l\psi_{,j} - \bar\psi_{,j}\gamma_5\gamma_l\psi) + \\ + (1/2)\,(bb_m{}^\mu c_{mkl}\bar\psi\gamma_l\psi)_{,\mu}\,, \qquad (5\cdot 9)\end{aligned}$$

$$(b_l{}^\mu \boldsymbol{t}_{kl})_{,\mu} \doteq -c_{mkl}\boldsymbol{t}_{ml} + 2b_k{}^\nu b_{n[\mu',\nu]}(b_l{}^\mu b_m{}^\lambda \boldsymbol{F}_{nlm})_{,\lambda}\,. \qquad (5\cdot 10)$$

In fact, the sum of (5·8) and (5·10) vanishes if the field equation for the $b_k{}^\mu$ field is employed. Thus, the derivative of the skew-symmetric energy-momentum tensor cannot be made zero with any choices in the free parameters.

Case 3: $\alpha+\beta=0$, $\alpha-(4/9)\gamma=0$ and we add an axial-vector interaction to the matter field.

First, we consider the local homogeneous Lorentz transformation (compare with (2·30)),

$$\left.\begin{aligned}&\delta x^\mu = 0\,, \\ &\delta\psi = (i/4)\,\omega_{kl}(x)\,\sigma_{kl}\psi\,, \quad (\omega_{(kl)}(x)=0)\,, \\ &\delta\bar\psi = -(i/4)\,\omega_{kl}(x)\,\bar\psi\sigma_{kl}\,.\end{aligned}\right\} \qquad (5\cdot 11)^{**)}$$

Under this transformation, the modified Lagrangian L' (3·13) is no longer invariant. According to our recipe we should introduce new fields in order to make the theory invariant.***) Instead of introducing new fields, however, we shall here add some interaction terms consisting of the gravitational field strength. There are the vector, the axial-vector and the tensor couplings constructed in terms of $c_k{}^V$, $c_k{}^A$ and c^T_{klm} in the form of tri-linear interactions. On examining the respective transformation properties, we find a promising one, that is, an

*) Its linearized form is given by (3·15). It should be noticed that the covariant derivative does not commute each other, yielding the invariant field strength, $[D_k, D_l] = c_{mkl}D_m$.

**) $\omega_{kl}(x)$ and its first derivative are assumed to vanish on the boundary surface of the integration domain.

***) Detailed discussion of it will be made in a forthcoming paper.

axial-vector coupling,

$$L^A = -(3i/4)\, c_k{}^A \bar{\psi}\gamma_5\gamma_k\psi\,, \tag{5·12}$$

and the modified Lagrangian

$$L' + L^A \tag{5·13}$$

remains invariant under the extended four-dimensional rotation (and of course under the extended translation). From (5·13), the equation of motion is replaced by

$$b_k{}^\mu\gamma_k\psi_{\cdot\mu} + (1/2)\, c_k{}^V\gamma_k\psi - (3i/4)\, c_k{}^A\gamma_5\gamma_k\psi + m\psi = 0\,. \tag{5·14}$$

It should be noticed, on the other hand, that the action integral

$$I(\Sigma) = \int_\Sigma d^4x \boldsymbol{L}^G$$

is kept invariant even under (5·11), because of the particular choice in the arbitrary coefficient (see (5·6)),

$$\delta I = \int_\Sigma \boldsymbol{B}_{kl}\omega_{kl} d^4x = \int_\Sigma \boldsymbol{B}_{(kl)}\omega_{kl} d^4x = 0\,.$$

Now the total Lagrangian density becomes

$$\boldsymbol{L}' + \boldsymbol{L}^A + \boldsymbol{L}^G.$$

Let us calculate the contribution from L^A to the energy-momentum tensor ${}^mT_{kl}$ (5·8); it is denoted by T^A_{kl},

$$\begin{aligned} T^A_{kl} = (1/4)\, &[(1/2)\,\varepsilon_{kijm}c_{lij} - \varepsilon_{lijm}c_{ijk} - \varepsilon_{kljm}c_j{}^V]\bar{\psi}\gamma_5\gamma_m\psi \\ &- (1/4)\,\varepsilon_{kljm}b_j{}^\nu[\bar{\psi}_{\cdot\nu}\gamma_5\gamma_m\psi + \bar{\psi}\gamma_5\gamma_m\psi_{\cdot\nu}]. \end{aligned} \tag{5·15}$$

The sum of the anti-symmetric parts

$$\begin{aligned} {}^mT_{[kl]} + T^A_{[kl]} = {} & (1/2)\, b^\mu_{[k}(\bar{\psi}\gamma_{l]}\psi_{\cdot\mu} - \bar{\psi}_{\cdot\mu}\gamma_{l]}\psi) - (3i/4)\, c^A_{[k}\bar{\psi}\gamma_5\gamma_{l]}\psi \\ & - (1/4)\, \{c_m{}^V\bar{\psi}\gamma_k\gamma_l\gamma_m\psi + b_m{}^\mu(\bar{\psi}\gamma_k\gamma_l\gamma_m\psi_{\cdot\mu} + \bar{\psi}_{\cdot\mu}\gamma_k\gamma_l\gamma_m\psi) \\ & - \delta_{kl}[c_m{}^V\bar{\psi}\gamma_m\psi + b_m{}^\mu(\bar{\psi}\gamma_m\psi_{\cdot\mu} + \bar{\psi}_{\cdot\mu}\gamma_m\psi)]\} + (1/2)\, c^V_{[k}\bar{\psi}\gamma_{l]}\psi \\ & + (1/2)\, b^\mu_{[k}(\bar{\psi}\gamma_{l]}\psi_{\cdot\mu} + \bar{\psi}_{\cdot\mu}\gamma_{l]}\psi) \end{aligned}$$

is shown to vanish after some algebra with the aid of the equation of motion (5·14) and the similar one for $\bar{\psi}$. Consequently, we have managed to symmetrize the energy-momentum tensor of the matter field; in this case the additional interaction Lagrangian of an axial-vector type plays the similar role to that of the canonical spin angular momentum played in a conventional manner for a flat-space. The additional interaction gives rise to a spin interaction with a spinor field, which is explicitly observed in the non-relativistic limit.

§ 6. Discussion

We have discussed the extension of the translation group and introduced the new field including the gravitational field in a manner quite analogous to the electromagnetic case. It should be stressed that the geometrical interpretation in terms of a Riemannian space may be given if desired and if necessary.

What differs from the electromagnetic field lies in the fact that our field strength is decomposed into the three irreducible parts. The free Lagrangian for the new fields is constructed with the four arbitrary constants.*)

In performing the linear approximation to the complicated non-linear field equations and making comparison between the Einstein's equation of gravitation and ours, we have assumed one relation among the arbitrary constants. This assumption is equivalent to the requirement that there should exist the conservation laws of the energy-momentum tensors quite similar to those implied by the translational invariance in a narrow sense for a free matter field. The relation assumed above has enabled us 1) to decompose the linearized field equations into one for the symmetric field variable and the other for the antisymmetric field variable, respectively, and 2) to identify the symmetric solution of the non-linear field equations with Schwarzschild's solution in the spherical symmetry.

We have been incapable of making any definite and conclusive statement concerning the antisymmetric part of the field equations, except for the proposed prescription to eliminate it as a redundant field by adding the tri-linear interaction Lagrangian of the axial-vector type besides the particular choice in the free parameters. In this case the total Lagrangian keeps invariant even under the extended four-dimensional rotation.

In a forthcoming paper, the extension of the homogeneous Lorentz group and its detailed consequences will be reported, where it is emphatically aimed to introduce a massive gauge field in an invariant-theoretic way.

References

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3) R. Utiyama, Phys. Rev. **101** (1956), 1597.
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D. W. Sciama, *Festschrift for Infeld* (Pergamon Press, New York, 1961), p. 415. We are indebted to Dr. Kei Semba for informing us of this paper.

*) The field strength of the electromagnetic field is irreducible for itself and there remains only one free parameter in producing the free electromagnetic Lagrangian.