# Notes on Prozorov-Giannetta
# Meissner-London state in superconductors of rectangular cross section in a perpendicular magnetic field

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## 1 Weak formulation

The solution consists of two stages:

(A) Solve for the magnetic scalar potential $\Psi$, imposing the desired fluxoid quantization condition through the inhomogeneous boundary term $B_{QS}$

(B) Solve for the shifted vector potential $\mathbf{A}^*$ and the phase $\theta$.

Because both vector- and scalar-valued fields are involved, the finite-element function space will be a direct product of $Q_p(\Omega_{\text{in}} \cup \Omega_{\text{out}})$ (for the scalar fields) and $Q_p^3(\Omega_{\text{in}} \cup \Omega_{\text{out}})$ (for the vector fields).

**TODO.** Investigate possible issues with convergence related to choice of finite-element spaces and the LBB condition for saddle-point problems. I believe the above (naive) choices should be OK, but there may be better choices / I may have misunderstood the convergence criteria.

**Definition 1** (Basis functions)**.** Let

- $\varphi_i$ denote the scalar basis elements
- $\mathbf{\Phi}_i$ denote the vector basis elements

**Definition 2** (System matrices)**.** The following matrices will be used to assemble the linear systems for (A) and (B):

$$M_{ij}^{(\mathrm{int})} := \int_{\Omega_{(\mathrm{int})}} \boldsymbol{\nabla}\varphi_i \cdot \boldsymbol{\nabla}\varphi_j \,\mathrm{d}\Omega \tag{0.1}$$

$$M_{ij}^{(\mathrm{ext})} := \int_{\Omega_{(\mathrm{ext})}} \boldsymbol{\nabla}\varphi_i \cdot \boldsymbol{\nabla}\varphi_j \,\mathrm{d}\Omega \tag{0.2}$$

$$C_{ij}^{(\mathrm{int/ext})} := \int_{\Omega_{(\mathrm{int/ext})}} (\boldsymbol{\nabla}\times\boldsymbol{\Phi}_i) \cdot (\boldsymbol{\nabla}\times\boldsymbol{\Phi}_j) \,\mathrm{d}\Omega \tag{0.3}$$

$$D_{ij}^{(\mathrm{int/ext})} := \int_{\Omega_{(\mathrm{int/ext})}} (\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}_i)(\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}_j) \,\mathrm{d}\Omega \tag{0.4}$$

$$S_{ij}^{(\mathrm{int/ext})} := \int_{\Omega_{(\mathrm{int/ext})}} \boldsymbol{\Phi}_i \cdot \boldsymbol{\Phi}_j \,\mathrm{d}\Omega \tag{0.5}$$

$$T_{ij}^{(\mathrm{int/ext})} := \int_{\Omega_{(\mathrm{int/ext})}} \boldsymbol{\Phi}_i \cdot \boldsymbol{\nabla}\varphi_j \,\mathrm{d}\Omega \tag{0.6}$$

$$E_{ij}^{(\mathrm{ext})} := \int_{\Omega_{(\mathrm{ext})}} (\boldsymbol{\nabla}\times\boldsymbol{\Phi}_i) \cdot \boldsymbol{\nabla}\varphi_j \,\mathrm{d}\Omega \tag{0.7}$$

$$H_{i}^{(\mathrm{ext})} := \int_{\Gamma_H} \boldsymbol{\Phi}_i \cdot \mathbf{H}_{(\mathrm{ext})} \,\mathrm{d}\Omega \tag{0.8}$$

$$B_{i}^{(\mathrm{QS})} := \int_{S_a} \varphi_i B_{QS} \,\mathrm{d}S \tag{0.9}$$

The weak formulation of the problem is then

$$M_{ij}^{(\mathrm{ext})}\boldsymbol{\Psi}_j = B_i^{(\mathrm{QS})} \tag{A}$$

$$\begin{cases} \left[C_{ij}^{(\mathrm{int})} + pD_{ij}^{(\mathrm{int})} + \frac{1}{\lambda^2}S_{ij}^{(\mathrm{int})}\right] A_j^* = \frac{\Phi_0}{2\pi\lambda^2}T_{ij}^{(\mathrm{int})}\theta_i \\ \frac{1}{\mu_0\lambda^2}\widetilde{T}_{ij}^{(\mathrm{int})}A_j^* = \frac{\Phi_0}{2\pi\mu_0\lambda^2}M_{ij}^{(\mathrm{int})}\theta_i \\ \left[C_{ij}^{(\mathrm{ext})} + pD_{ij}^{(\mathrm{ext})}\right] A_i^* = -E_{ij}^{(\mathrm{ext})}\Psi_j + \mathbf{H}_i^{(\mathrm{ext})} \end{cases} \tag{B}$$