Teorija Izračunljivosti - Homework 2

Ante Sosa, Nita Hasani

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1 Part I

1.1 Addone function

Algorithm proceeds as follows:

As we read digit by digit from input sequence, we can distinguish two general cases:

1. First non-zero digit in sequence is 1 (number isn't negative because if there are all -1s after leading 1 we have 0): Change all digits in sequence to 1.
2. First non-zero digit in sequence is -1 (number is not positive because if there are all 1s after leading -1 we have 0):
   (a) First digit is -1: change it to 1 and copy rest of sequence.
   (b) Sequence starts with 0s: change all 0s to 1s and leading -1 to 1 and copy rest of sequence.

Here is the algorithm written in recursive style:

```
add x = addone x 0
addone (1: x) 0 = 1: (addone x 1)
addone (0: x) 0 = 1: (addone x 0)
addone (-1: x) 0 = 1: (addone x -1)
addone (1: x) -1 = 1: (addone x -1)
addone (-1: x) -1 = -1: (addone x -1)
addone (0: x) -1 = 0: (addone x -1)
addone (1: x) 1 = 1: (addone x 1)
addone (-1: x) 1 = 1: (addone x 1)
addone (0: x) 1 = 1: (addone x 1)
```

1.2 Subtractone function

Algorithm proceeds as follows:

As we read digit by digit from input sequence, we can distinguish two general cases:

1. First non-zero digit in sequence is -1: Change all digits in sequence to -1.
2. First non-zero digit in sequence is 1:
   (a) First digit is 1: change it to -1 and copy rest of sequence.
   (b) Sequence starts with 0s: change all 0s to -1s and first 1 to -1 and copy rest of sequence.

Here is the algorithm written in recursive style:

```
subtract x = subtractone x 0
subtractone (1: x) 0 = -1: (subtractone x 1)
subtractone (0: x) 0 = -1: (subtractone x 0)
subtractone (-1: x) 0 = -1: (subtractone x -1)
subtractone (1: x) -1 = -1: (subtractone x -1)
subtractone (-1: x) -1 = -1: (subtractone x -1)
subtractone (0: x) -1 = -1: (subtractone x -1)
subtractone (1: x) 1 = 1: (subtractone x 1)
subtractone (-1: x) 1 = -1: (subtractone x 1)
subtractone (0: x) 1 = 0: (subtractone x 1)
```

1.3 Double function

Let $x \in [-1, 1]$ be number that input sequence is representing.

1. If sequence starts with 0 $\ldots \Rightarrow$ x $\in [-\frac{1}{2}, \frac{1}{2}]$
2. If sequence starts with 1 ... we distinguish three subcases:
   (a) First non-zero digit after leading 1 is 1 $\Rightarrow$ x $\geq \frac{1}{2}$
   (b) First non-zero digit after leading 1 is -1 $\Rightarrow$ x $\in [-\frac{1}{2}, \frac{1}{2}]$
   (c) If there are all 0s after leading 1 $\Rightarrow$ x $\in [-\frac{1}{2}, \frac{1}{2}]$
3. If sequence starts with -1 ... we distinguish three subcases:
   (a) First non-zero digit after leading -1 is -1 $\Rightarrow$ x $\leq \frac{1}{2}$
   (b) First non-zero digit after leading -1 is 1 $\Rightarrow$ x $\in [-\frac{1}{2}, \frac{1}{2}]$
   (c) If there are all 0s after leading -1 $\Rightarrow$ x $\in [-\frac{1}{2}, \frac{1}{2}]$

Here is algorithm for computing double function written in recursive style:

double (0: x) = doubleIt (x 0) [1]
double (1: x) = 1: decide (x 1) [2]
double (-1: x) = -1: decide (x -1) [3]

doubleIt (0: x) 0 = 0: doubleIt (x 0) [1]
doubleIt (-1: x) 0 = -1: doubleIt (x 0) [1]
doubleIt (1: x) 0 = 1: doubleIt (x 0) [1]

decide (-1: x) -1 = -1: negative(x) [4]
decide (1: x) -1 = doubleIt(x -1) [5]
decide (0: x) -1 = -1: decide(x -1) [6]

doubleIt (0: x) -1 = 0: doubleIt (x -1) [5]
doubleIt (-1: x) -1 = -1: doubleIt (x -1) [5]
doubleIt (1: x) -1 = 1: doubleIt (x -1) [5]

negative (1: x) = -1: (negative x) [4]
negative (-1: x) = -1: (negative x) [4]
negative (0: x) = -1: (negative x) [4]

decide (1: x) 1 = 1: positive (x 1) [7]
decide (-1: x) 1 = doubleIt (x 1) [8]
decide (0: x) 1 = 1: decide (x 1) [9]

doubleIt (0: x) 1 = 0: doubleIt (x -1) [8]
doubleIt (-1: x) 1 = -1: doubleIt (x -1) [8]
doubleIt (1: x) 1 = 1: doubleIt (x -1) [8]

positive (1: x) = 1: (positive x) [7]
positive (-1: x) = 1: (positive x) [7]
positive (0: x) = 1: (positive x) [7]

[1] Case 1)
[2] Case 2)
[3] Case 3)
[4] Case 3.1)
[5] Case 3.2)
[6] Case 3.3)
[7] Case 2.1)
[8] Case 2.2)
[9] Case 2.3)

2 Part II

There are unique functions:

$$\text{addone: } \{-1,0,1\}^{\omega} \rightarrow (\{\text{-1,0,1}\} \rightarrow \{-1,0,1\}^{\omega})$$
$$\text{add: } \{-1,0,1\}^{\omega} \rightarrow \{\text{-1,0,1}\}^{\omega}$$

satisfying the equations in Part I (1.1 Addone function).

Moreover, for all $p \in \{-1,0,1\}^{\omega}$ and $c \in \{-1,0,1\}$,

$\gamma_s(\text{add p}) = \begin{cases} \gamma_s(p)+1, & if \;\; \gamma_s(p) \leq 0 \\ 1, & if \;\; \gamma_s(p) \geq 0 \end{cases}$ $\gamma_s(\text{addone p c}) = \begin{cases} \gamma_s(p)+1, & if \;\; \gamma_s(p) \leq 0 \\ 1, & if \;\; \gamma_s(p) \geq 0 \end{cases}$ [10]

Proof.
We first show the uniqueness of function addone. We do it by proving by induction on $n \geq 0$, for all $p \in \{1,0,1\}^{\omega}$ and $e \in \{1,0,1\}$, the equations in Part I (1.1 Addone function) determine (addone p e)⌈n uniquely.
The base case $n = 0$ is trivial as (addone p e)⌈0 $= \epsilon$.
For $n > 0$, we consider the 9 different possibilities for p_0 and e.

Case 1.1) $p_0 = e = 0$
(addone p e)⌈n = (addone (0:p') 0)⌈n where $p' := p_1p_2p_3\dots$
(addone p e)⌈n = (1: (addone p' 0))⌈n
(addone p e)⌈n = 1: ((addone p' 0)⌈n-1)
The final term is uniquely determined by induction hypothesis.

Case 1.2) $p_0 = 1$, $e = 0$
(addone p e)⌈n = (addone (1:p') 0)⌈n where $p' := p_1p_2p_3\dots$
(addone p e)⌈n = (1: (addone p' 1))⌈n
(addone p e)⌈n = 1: ((addone p' 1)⌈n-1)
The final term is uniquely determined by induction hypothesis.

Case 1.3) $p_0 = -1$, $e = 0$
(addone p e)⌈n = (addone (-1:p') 0)⌈n where $p' := p_1p_2p_3\dots$
(addone p e)⌈n = (1: (addone p' -1))⌈n
(addone p e)⌈n = 1: ((addone p' -1)⌈n-1)
The final term is uniquely determined by induction hypothesis.

Case 2.1) $p_0 = 0$, $e = 1$
(addone p e)⌈n = (addone (0:p') 1)⌈n where $p' := p_1p_2p_3\dots$
(addone p e)⌈n = (1: (addone p' 1))⌈n
(addone p e)⌈n = 1: ((addone p' 1)⌈n-1)
The final term is uniquely determined by induction hypothesis.

Case 2.2) $p_0 = 1$, $e = 1$
(addone p e)⌈n = (addone (1:p') 1)⌈n where $p' := p_1p_2p_3\dots$
(addone p e)⌈n = (1: (addone p' 1))⌈n
(addone p e)⌈n = 1: ((addone p' 1)⌈n-1)
The final term is uniquely determined by induction hypothesis.

Case 2.3) $p_0 = -1$, $e = 1$
(addone p e)⌈n = (addone (-1:p') 1)⌈n where $p' := p_1p_2p_3\dots$
(addone p e)⌈n = (1: (addone p' 1))⌈n
(addone p e)⌈n = 1: ((addone p' 1)⌈n-1)
The final term is uniquely determined by induction hypothesis.

Case 3.1) $p_0 = 0$, $e = -1$
(addone p e)⌈n = (addone (0:p') -1)⌈n where $p' := p_1p_2p_3\dots$
(addone p e)⌈n = (0: (addone p' -1))⌈n
(addone p e)⌈n = 0: ((addone p' -1)⌈n-1)
The final term is uniquely determined by induction hypothesis.

Case 3.2) $p_0 = 1$, $e = -1$
(addone p e)⌈n = (addone (1:p') -1)⌈n where $p' := p_1p_2p_3\dots$

[10] Case 1)

(addone p e)$\lceil$n = (1: (addone p' -1))$\lceil$n
(addone p e)$\lceil$n = 1: ((addone p' -1)$\lceil$n-1)
The final term is uniquely determined by induction hypothesis.

Case 3.3) $p_0 = -1$, $e = -1$
(addone p e)$\lceil$n = (addone (-1:p') -1)$\lceil$n where $p' := p_1p_2p_3\ldots$
(addone p e)$\lceil$n = (-1: (addone p' -1))$\lceil$n
(addone p e)$\lceil$n = -1: ((addone p' -1)$\lceil$n-1)
The final term is uniquely determined by induction hypothesis.
In each of 9 cases, second equality it true due to defined algorithm in Part I(1.1. Addone function).
To show that addone satisfies equation [10], we prove by induction on $n \geq 0$ that, for all $p \in \{1,0,1\}^\omega$ and $c \in \{1,0,1\}$,

$$\begin{array}{ll} |\gamma_s(addone\ p\ e) - (\gamma_s(p)+1)| \leq 2^{1-n}, & \text{if } \gamma_s(p) \leq 0 \\ |\gamma_s(addone\ p\ e) - 1| \leq 2^{1-n}, & \text{if } \gamma_s(p) \geq 0 \end{array}$$

In base case $\gamma_s(addone\ p\ e)$, $\gamma_s(p)+1$ and 1 lie in $[-1,1]$, so distance between $\gamma_s(addone\ p\ e)$ and $\gamma_s(p)+1$ and distance between $\gamma_s(addone\ p\ e)$ and 1 are less then 2.
Again for $n \geq 0$, we consider the 9 different possibilities for p_0 and e.

Case 1.1) $p_0 = e = 0$
if $\gamma_s(p) \leq 0$, we have
$|\gamma_s(addone\ p\ e) - (\gamma_s(p)+1)| = |\gamma_s(addone\ 0:p'\ 0) - (\gamma_s(0:p')+1)| =$
$|\gamma_s(1:(addone\ p'\ 0)) - (\gamma_s(0:p')+1)| = \Big|\frac{1+\gamma_s(addone\ p'\ 0)}{2} - \frac{\gamma_s(p')}{2}-1| =$
$\frac{1}{2}|\gamma_s(addone\ p'\ 0)$ -$(\gamma_s(p')+1)| \leq \frac{1}{2}2^{1-(n-1)} = 2^{1-n}$
The last inequality is valid because $\gamma_s(addone\ p'\ 0)$ is negative number so it is in interval $[-1,0]$ and $\gamma_s(p')+1$ is positive such that the difference in between them is at most 1.(For example if first one is -1 second is 0).

if $\gamma_s(p) \geq 0$, we have
$|\gamma_s(addone\ p\ e) - 1| = |\gamma_s(addone\ 0:p'\ 0) - 1| =$
$|\gamma_s(1:(addone\ p'\ 0)) - 1| = \Big|\frac{1+\gamma_s(addone\ p'\ 0)}{2} - 1\,| =$
$\frac{1}{2}|\gamma_s(addone\ p'\ 0)$ -1$| \leq \frac{1}{2}2^{1-(n-1)} = 2^{1-n}$

Case 1.2) $p_0 = 1, e = 0$, we have
$|\gamma_s(addone\ p\ e) - 1| = |\gamma_s(addone\ 1:p'\ 0) - 1| =$
$|\gamma_s(1:(addone\ p'\ 1)) - 1| = \Big|\frac{1+\gamma_s(addone\ p'\ 1)}{2} - 1\,| =$
$\frac{1}{2}|\gamma_s(addone\ p'\ 1)$ -1$| \leq \frac{1}{2}2^{1-(n-1)} = 2^{1-n}$

Case 1.3) $p_0 = -1, e = 0$, we have
$|\gamma_s(addone\ p\ e) - (\gamma_s(p)+1)| = |\gamma_s(addone\ -1:p'\ 0) - (\gamma_s(-1:p')+1)| =$
$|\gamma_s(1:(addone\ p'\ -1)) - (\gamma_s(-1:p')+1)| = \Big|\frac{1+\gamma_s(addone\ p'\ -1)}{2} - \frac{1+\gamma_s(p')}{2}-1| =$
$\frac{1}{2}|\gamma_s(addone\ p'\ -1)$ -$(\gamma_s(p')+1) - 1\,| \leq \frac{1}{2}2^{1-(n-1)} = 2^{1-n}$
The last inequality is valid because $\gamma_s(addone\ p'\ -1)$ is negative number so it is in interval $[-1,0]$ and $\gamma_s(p')+1$ is positive such that the difference in between them is at most 1.(For example if first one is -1 second is 0).

Case 2.1) $p_0 = 0, e = 1$
$|\gamma_s(addone\ p\ e) - 1| = |\gamma_s(addone\ 0:p'\ 1) - 1| =$
$|\gamma_s(1:(addone\ p'\ 1)) - 1| = \Big|\frac{1+\gamma_s(addone\ p'\ 1)}{2} - 1\,| =$
$\frac{1}{2}|\gamma_s(addone\ p'\ 1)$ -1$| \leq \frac{1}{2}2^{1-(n-1)} = 2^{1-n}$

Case 2.2) $p_0 = 1, e = 1$, we have
$|\gamma_s(addone\ p\ e) - 1| = |\gamma_s(addone\ 1:p'\ 1) - 1| =$
$|\gamma_s(1:(addone\ p'\ 1)) - 1| = \Big|\frac{1+\gamma_s(addone\ p'\ 1)}{2} - 1\,| =$
$\frac{1}{2}|\gamma_s(addone\ p'\ 1)$ -1$| \leq \frac{1}{2}2^{1-(n-1)} = 2^{1-n}$

Case 2.3) $p_0 = -1, e = 1$, we have

$|\gamma_s(addone\ p\ e) - 1| = |\gamma_s(addone\ -1 : p'\ 1) - 1| =$
$|\gamma_s(1 : (addone\ p'\ 1)) - 1| = \left|\frac{1+\gamma_s(addone\ p'\ 1)}{2} - 1\right| =$
$\frac{1}{2}|\gamma_s(addone\ p'\ 1)$ -1$| \le \frac{1}{2}2^{1-(n-1)} = 2^{1-n}$

Case 3.1) $p_0 = 0, e = -1$
$|\gamma_s(addone\ p\ e) - (\gamma_s(p)+1)| = |\gamma_s(addone\ 0 : p'\ -1) - (\gamma_s(0 : p') - 1)| =$
$|\gamma_s(0 : (addone\ p'\ -1)) - (\gamma_s(0 : p') + 1)| = \left|\frac{\gamma_s(addone\ p'\ -1)}{2} - \frac{\gamma_s(p')}{2}\right.-1| =$
$\frac{1}{2}|\gamma_s(addone\ p'\ -1)$ -$(\gamma_s(p')+1) - 1| \le \frac{1}{2}2^{1-(n-1)} = 2^{1-n}$
The last inequality is valid because $\gamma_s(addone\ p'\ -1)$ is negative number so it is in interval $[-1,0]$ and $\gamma_s(p')+1$ is positive such that the difference in between them is at most 1.(For example if first one is -1 second is 0).

Case 3.2) $p_0 = 1, e = -1$, we have
$|\gamma_s(addone\ p\ e) - (\gamma_s(p)+1)| = |\gamma_s(addone\ 1 : p'\ -1) - (\gamma_s(1 : p') - 1)| =$
$|\gamma_s(1 : (addone\ p'\ -1)) - (\gamma_s(1 : p') + 1)| = \left|\frac{1+\gamma_s(addone\ p'\ -1)}{2} - \frac{1+\gamma_s(p')}{2}\right.-1| =$
$\frac{1}{2}|\gamma_s(addone\ p'\ -1)$ -$(\gamma_s(p')+1) - 1| \le \frac{1}{2}2^{1-(n-1)} = 2^{1-n}$
The last inequality is valid because $\gamma_s(addone\ p'\ -1)$ is negative number so it is in interval $[-1,0]$ and $\gamma_s(p')+1$ is positive such that the difference in between them is at most 1.(For example if first one is -1 second is 0).

Case 3.3) $p_0 = -1, e = -1$, we have
$|\gamma_s(addone\ p\ e) - (\gamma_s(p)+1)| = |\gamma_s(addone\ -1 : p'\ -1) - (\gamma_s(-1 : p') - 1)| =$
$|\gamma_s(-1 : (addone\ p'\ -1)) - (\gamma_s(-1 : p') + 1)| = \left|\frac{-1+\gamma_s(addone\ p'\ -1)}{2} - \frac{-1+\gamma_s(p')}{2}\right.-1| =$
$\frac{1}{2}|\gamma_s(addone\ p'\ -1)$ -$(\gamma_s(p')+1) - 1| \le \frac{1}{2}2^{1-(n-1)} = 2^{1-n}$
The last inequality is valid because $\gamma_s(addone\ p'\ -1)$ is negative number so it is in interval $[-1,0]$ and $\gamma_s(p')+1$ is positive such that the difference in between them is at most 1.(For example if first one is -1 second is 0).

It is proven that for each case addone satisfies furmula [10].
Finally, the uniqueness of add and equation [10] follow immediately from the results for addone.