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An Introduction to Frobenius Algebras and 2D TQFTs

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There are many reasons to study
topological quantum field theories,
but one reason is that they exhibit a
beautiful relationship between
algebra and geometry

Christopher John Schommer-Pries,
*The Classification of
Two-Dimensional Extended
Topological Field Theories*

Introduction

First we reassure the reader: this is a thesis in pure mathematics. Its goal is to develop the generator-and-relations presentation of a category that (the author has heard) plays an important role in modern theoretical physics research, and to use it to establish a link between geometry (i.e., the representations of the category of 2-dimensional oriented bordisms) and algebra (i.e., some less known kind of algebraic structures of increasing popularity among logicians and computer scientists that go under the name of “Frobenius algebras”). After reviewing some basic vocabulary from category theory in Chapter 1, we introduce oriented bordisms and their ambient category $\mathbf{Bord}(n)$ in Chapter 2, along with the *linear representations* of this category, i.e., the symmetric monoidal functors from $\mathbf{Bord}(n)$ to the (symmetric monoidal) category of (finite-dimensional) vector spaces over an arbitrary field. Topological field theories arose in the eighties as an attempt to formalize some constructions appearing in quantum field theory. A TQFT assigns to each $(n - 1)$ -dimensional “geometric object” a vector space of “states”, and to each “worldvolume” traced by a geometric object an “evolution operator” relating its initial and final states. The “geometric objects” we have in mind are $(n - 1)$ -dimensional manifolds and the “worldvolumes” they trace occur here as n -dimensional bordisms between them. While this intuitive picture alone legitimates a genuine interest in studying monoidal functors from categories of bordisms to categories of vector spaces, we will see that the rich geometric structure of the bordism category endows the category of its linear representation (i.e., a category of TQFTs) with nice algebraic properties. In Chapter 3 we introduce the notion of generators and relations for a monoidal category, and, relying on the classification theorem for 2-dimensional surfaces, we make the geometric structure alluded above explicit, presenting the 2-dimensional bordism category concretely in terms of generators and relations. From the appropriate perspective, the relations presenting this bordism category look exactly like the axioms that define a Frobenius algebra. Chapter 4 develops a powerful graphical calculus for dealing with Frobenius algebras (and more generally, with algebraic structures defined within monoidal categories in general). We employ this calculus to analyze the multiple equivalent definitions of a Frobenius algebra, which can be described in so many words just as a monoid object in the category of vector spaces which is also a comonoid object in a compatible way. With these premises in place we finally reach the main theorem of this thesis in Chapter 5, which basically states that defining a 2-dimensional TQFT is the same as choosing a commutative Frobenius

algebra. We also see how this equivalence result is just an instance of a broader theorem: $\mathbf{Bord}(2)$ is the free symmetric monoidal category over an internal Frobenius object.

The idea of writing a thesis on this topic arose from a wish to approach monoidal categories and their internal objects through a concrete example. The narrow time constraints determined the scope of this exposition, which offers nothing more than a quick informal route towards the equivalence theorem between 2-dimensional TQFTs and Frobenius algebras. The serious reader is encouraged to turn to [references]. The majority of this work clearly relies on [Koc03] and can largely be seen as a retelling of its tale of «the commutative Frobenius and the princess $\mathbf{Bord}(2)$ », even if by a less skillful bard.

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