Contents

Chapter 1

Categorical preliminaries

We begin by recalling some useful notions from category theory, which we will encounter throughout this thesis, assuming the reader is already familiar with its fundamentals.

1.1 Monoidal Categories

Perché servono le categorie monoidali per questo lavoro? Valutare se scriverlo qui o in introduzione.

Definition 1.1 Monoidal categories A (weak) *monoidal category* is a sextuple $(\mathcal{C}, \square, \eta, \alpha, l, r)$ consisting of:

- a category $\mathcal{C}$,
- a bifuctor $\square \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
- a functor $\eta \colon \mathbf{1} \to \mathcal{C}$ identifying an object $\eta(1) = I \in \mathrm{Obj}(\mathcal{C})$,
- a natural isomorphism called *associator* α,

$$\begin{tikzcd} & \mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[dl, "\square \times \mathrm{id}_{\mathcal{C}}"'] \arrow[dr, "\mathrm{id}_{\mathcal{C}} \times \square"] & \\ \mathcal{C} \times \mathcal{C} \arrow[dr, "\square"'] \arrow[rr, phantom, "\xRightarrow{\alpha}"] & & \mathcal{C} \times \mathcal{C} \arrow[dl, "\square"] \\ & \mathcal{C} & \end{tikzcd}$$

so that $\alpha_{A,B,C} \colon (A \square B) \square C \xrightarrow{\cong} A \square (B \square C)$ for every $A, B, C \in \mathrm{Obj}(\mathcal{C})$

$$\begin{tikzcd}
& (A \square B) \square (C \square D) \arrow[dr, "\alpha_{A,B,C\square D}"] & \\
((A \square B) \square C) \square D \arrow[ur, "\alpha_{A\square B,C,D}"] \arrow[d, "\alpha_{A,B,C}\square\mathrm{id}_D"'] & & A \square (B \square (C \square D)) \\
(A \square (B \square C)) \square D \arrow[rr, "\alpha_{A,B\square C,D}"] & & A \square ((B \square C) \square D) \arrow[u, "\mathrm{id}_A\square\alpha_{B,C,D}"']
\end{tikzcd}$$

Figure 1.1: Associativity coherence

$$\begin{tikzcd}
(A \square I) \square B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "r_A\square\mathrm{id}_B"'] & & A \square (I \square B) \arrow[dl, "\mathrm{id}_A\square l_B"] \\
& A \square B &
\end{tikzcd}$$

Figure 1.2: Unit coherence

- and two natural isomorphisms called *left* and *right unitors* l and r.

$$\begin{tikzcd}
& \mathcal{C}\times\mathcal{C} \arrow[dr, "\square"] \arrow[d, phantom, "\Downarrow l"] & & \mathcal{C}\times\mathcal{C} \arrow[dl, "\square"'] \arrow[d, phantom, "\Downarrow r"] & \\
\mathbf{1}\times\mathcal{C} \arrow[ur, "\eta\times\mathrm{id}_{\mathcal{C}}"] \arrow[rr, "\pi"'] & & \mathcal{C} & & \mathcal{C}\times\mathbf{1} \arrow[ul, "\mathrm{id}_{\mathcal{C}}\times\eta"'] \arrow[ll, "\pi"]
\end{tikzcd}$$

  so that $l_A \colon I \square A \overset{\cong}{\to} A$ and $r_A \colon A \square I \overset{\cong}{\to} A$ for every $A \in \mathrm{Obj}(\mathcal{C})$.

This maps must satisfy the so called coherence requirements, depicted in 1.1 and 1.2, for every $A, B, C, D \in \mathrm{Obj}(\mathcal{C})$.

It is important to notice how, by definition, the naturality of such transformations means that all these isomorphisms are compatible with the arrows in $\mathcal{C}$.

Definition 1.2 Strict monoidal categories A monoidal category is *strict* if all the natural transformations α, l, r are identities.

We can say that a strict monoidal category is a category $\mathcal{C}$ equipped with an associative bifunctor $\square\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and an object $I\in \mathrm{Obj}(\mathcal{C})$ which is a left and a right unit for $\square$.

We'd now like to define morphisms between monoidal categories as functors that preserve the monoidal structure. However we need to precise to which "degree" such structure needs to be preserved. As for monoidal categories, the weakest definition will also be the most intricate.

Definition 1.3 Monoidal functors Let $(\mathcal{C},\square,I,\alpha^{\mathcal{C}},l^{\mathcal{C}},r^{\mathcal{C}})$ and $(\mathcal{D},\boxtimes,J,\alpha^{\mathcal{D}},l^{\mathcal{D}},r^{\mathcal{D}})$ be two monoidal categories. A monoidal functor is a triple (F,φ,ε) consisting of

- a functor
$$F\colon \mathcal{C}\to\mathcal{D}$$
- a natural transformation
$$\varphi_{A,B}\colon F(A)\boxtimes F(B)\to F(A\,\square\, B)$$
for every $A,B\in\mathrm{Obj}(\mathcal{C})$
- a unique morphism
$$\varepsilon\colon J\to F(I)$$

making the following diagrams commute

1. *(Associativity)*

$$\begin{tikzcd}
(F(A)\boxtimes F(B))\boxtimes F(C) \arrow[r, "\alpha^{\mathcal{D}}_{F(A),F(B),F(C)}"] \arrow[d, "\varphi_{A,B}\boxtimes \mathrm{id}_{F(C)}"] & F(A)\boxtimes (F(B)\boxtimes F(C)) \arrow[d, "\mathrm{id}_{F(A)}\boxtimes\varphi_{B,C}"] \\
F(A\,\square\, B)\boxtimes F(C) \arrow[d, "\varphi_{A\square B,C}"] & F(A)\boxtimes F(B\,\square\, C) \arrow[d, "\varphi_{A,B\square C}"] \\
F((A\,\square\, B)\,\square\, C) \arrow[r, "\alpha^{\mathcal{C}}_{A,B,C}"] & F(A\,\square\,(B\,\square\, C))
\end{tikzcd}$$

2. *(Unitality)*

$$\begin{tikzcd}
F(A)\boxtimes J \arrow[r, "r^{\mathcal{D}}_{F(A)}"] \arrow[d, "\mathrm{id}_{F(A)}\boxtimes\varepsilon"'] & F(A) \\
F(A)\boxtimes F(I) \arrow[r, "\varphi_{A,I}"] & F(A\,\square\, I) \arrow[u, "F\circ r^{\mathcal{C}}_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
J\boxtimes F(A) \arrow[r, "l^{\mathcal{D}}_{F(A)}"] \arrow[d, "\varepsilon\boxtimes\mathrm{id}_{F(A)}"'] & F(A) \\
F(I)\boxtimes F(A) \arrow[r, "\varphi_{I,A}"] & F(I\,\square\, A) \arrow[u, "F\circ l^{\mathcal{C}}_A"']
\end{tikzcd}$$

for every $A,B,C\in\mathrm{Obj}(\mathcal{C})$

Definition 1.4 Strong monoidal functors Let $(\mathcal{C}, \square, I)$ and $(\mathcal{D}, \boxtimes, J)$ be two monoidal categories. A monoidal functor $(F, \varphi, \varepsilon)$ from $\mathcal{C}$ to $\mathcal{D}$ is *strong monoidal* if φ and ε are isomorphism.

We hence have $F(A) \boxtimes F(B) \cong F(A \square B)$ and $J \cong F(I)$.

Definition 1.5 Strict monoidal functors Let $(\mathcal{C}, \square, I)$ and $(\mathcal{D}, \boxtimes, J)$ be two monoidal categories. A monoidal functor $(F, \varphi, \varepsilon)$ from $\mathcal{C}$ to $\mathcal{D}$ is *strict monoidal* if φ and ε are identities.

In terms of objects and arrows we are asking that

$$F(A \square B) = F(A) \boxtimes F(B), \qquad F(I) = J$$

$$F(f \square g) = F(f) \boxtimes F(g)$$

$$F(\alpha^{\mathcal{C}}_{A,B,C}) = \alpha^{\mathcal{D}}_{F(A),F(B),F(C)} \qquad F(l^{\mathcal{C}}_A) = l^{\mathcal{D}}_{F(A)} \qquad F(r^{\mathcal{C}}_A) = r^{\mathcal{D}}_{F(A)}$$

for every $A, B, C \in \mathrm{Obj}(\mathcal{C})$ and $f, g \in \mathrm{Arr}(\mathcal{C})$.

The composition of two monoidal functors is again a monoidal functor, and identity functors are monoidal. We hence have a category **MonCat** where objects are monoidal categories and arrows are monoidal functors. There is also a full subcategory where objects are strict monoidal categories.

ripensare a come dire questa cosa. (come devono essere i funtori? strong?)

Remark 1.6. As the reader can imagine, working with weak monoidal categories can become a really tedious process. The following result mitigates this difficulty, by justifying the common practice of treating all monoidal categories as strictly monoidal ones. The proof and a more in depth explanation can be found in [Mac78] and [Tru20].

Theorem 1.7 Strictification (See [Mac78], Chapter XI, Section 3) *Every monoidal category is categorically equivalent, via strong monoidal functors, to a strict monoidal category.*

Essentially, what follows is that from now on we can assume, without loss of generalities, that monoidal categories are strict. This is possible because the equivalence provided by the above theorem ensures that all coherence diagram commute up to unique canonical isomorphism, as long as we don't require functors between such categories to be strict.

Definition 1.8 Monoidal natural transformations Let $(\mathcal{C}, \square, I)$ and $(\mathcal{D}, \boxtimes, J)$ be two monoidal categories and let $(F, \varphi, \varepsilon)$ and (G, ϕ, e) be two monoidal functors from $\mathcal{C}$ to $\mathcal{D}$. A *monoidal natural transformation* is a natural transformation $u \colon F \to G$

such that the following diagrams commute

$$\begin{tikzcd}
F(A) \boxtimes F(B) \arrow[r, "u_A \boxtimes u_B"] \arrow[d, "\varphi_{A,B}"] & G(A) \boxtimes G(B) \arrow[d, "\phi_{A,B}"] \\
F(A \square B) \arrow[r, "u_{A \square B}"] & G(A \square B)
\end{tikzcd}
\qquad
\begin{tikzcd}
J \arrow[r, "\varepsilon"] \arrow[dr, "e"'] & F(I) \arrow[d, "u_I"] \\
 & G(I)
\end{tikzcd}$$

for every $A, B \in \mathrm{Obj}(\mathcal{C})$.

Given two monoidal categories $(\mathcal{C}, \square, I)$, $(\mathcal{D}, \boxtimes, J)$ we can then consider the category $\mathbf{MonCat}(\mathcal{C}, \mathcal{D})$ where objects are monoidal functors from $\mathcal{C}$ to $\mathcal{D}$ and arrows are monoidal natural transformations between such functors.

1.2 Adding a twist

Definition 1.9 Braided monoidal categories A *braided monoidal category* is a monoidal category $(\mathcal{C}, \square, I)$ equipped with a natural isomorphisms

$$\beta_{A,B} \colon A \square B \to B \square A$$

for every $A, B \in \mathrm{Obj}(\mathcal{C})$, called the *braiding*, such that the following diagrams commute

$$\begin{tikzcd}
 & A \square (B \square C) \arrow[r, "\beta_{A,B \square C}"] & (B \square C) \square A \arrow[dr, "\alpha_{B,C,A}"] & \\
(A \square B) \square C \arrow[ur, "\alpha_{A,B,C}"] \arrow[dr, "\beta_{A,B} \square \mathrm{id}_C"'] & & & B \square (C \square A) \\
 & (B \square A) \square C \arrow[r, "\alpha_{B,A,C}"] & B \square (A \square C) \arrow[ur, "\mathrm{id}_B \square \beta_{A,C}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (A \square B) \square C \arrow[r, "\beta_{A \square B,C}"] & C \square (A \square B) \arrow[dr, "\alpha^{-1}_{C,A,B}"] & \\
A \square (B \square C) \arrow[ur, "\alpha^{-1}_{A,B,C}"] \arrow[dr, "\mathrm{id}_A \square \beta_{B,C}"'] & & & (C \square A) \square B \\
 & A \square (C \square B) \arrow[r, "\alpha^{-1}_{A,C,B}"] & (A \square C) \square B \arrow[ur, "\beta_{A,C} \square \mathrm{id}_A"'] &
\end{tikzcd}$$

The definition implies compatibility with the unital structure:

Lemma 1.10 *Let $(\mathcal{C}, sq, I, \beta)$ be a braided monoidal category. Then the following* *diagrams commute for every* $A \in \mathrm{Obj}(\mathcal{C})$.

$$
\begin{tikzcd}
A \,\square\, I \arrow[rr, "\beta_{A,I}"] \arrow[dr, "r_A"'] & & I \,\square\, A \arrow[dl, "l_A"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
I \,\square\, A \arrow[rr, "\beta_{I,A}"] \arrow[dr, "l_A"'] & & A \,\square\, I \arrow[dl, "r_A"] \\
& A &
\end{tikzcd}
$$

is it useful? do i need to prove it?

Again, by 1.6, when considering a braided monoidal categories we can restrict to the following definition.

Definition 1.11 (Semi)strict braided monoidal categories A (semi)strict *braided monoidal category* is a (strict) monoidal category $(\mathcal{C}, \square, I)$ equipped with a natural isomorphism

$$\beta_{A,B} \colon A \,\square\, B \to B \,\square\, A$$

for every $A, B \in \mathrm{Obj}(\mathcal{C})$, called the *braiding*, such that the following diagrams commute

$$
\begin{tikzcd}
A \,\square\, B \,\square\, C \arrow[rr, "\beta_{A,B\square C}"] \arrow[dr, "\beta_{A,B}\square\mathrm{id}_C"'] & & B \,\square\, C \,\square\, A \\
& B \,\square\, A \,\square\, C \arrow[ur, "\mathrm{id}_B\square\beta_{A,C}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \,\square\, B \,\square\, C \arrow[rr, "\beta_{A\square B,C}"] \arrow[dr, "\mathrm{id}_A\square\beta_{B,C}"'] & & C \,\square\, A \,\square\, B \\
& A \,\square\, C \,\square\, B \arrow[ur, "\beta_{A,C}\square\mathrm{id}_A"'] &
\end{tikzcd}
$$

Remark 1.12. It would be misleading to name these categories "strict braided monoidal". While the underlying monoidal structure is strict, we are not asking for the braiding itself to be strict.

Definition 1.13 Symmetric monoidal categories A *symmetric monoidal category* is a braided monoidal category $(\mathcal{C}, \square, I, \beta)$ where the braiding β is symmetric, that is if

$$\beta_{B,A} \circ \beta_{A,B} = \mathrm{id}_{A\square B}$$

for every $A, B \in \mathrm{Obj}(\mathcal{C})$.

Definition 1.14 Braided monoidal functors Let $(\mathcal{C}, \square, I, \beta)$ and $(\mathcal{D}, \boxtimes, J, \gamma)$ be two braided monoidal categories. A monoidal functor $(F, \varphi, \varepsilon)$ from $\mathcal{C}$ to $\mathcal{D}$ is a *braided monoidal functor* if the diagram

$$
\begin{tikzcd}
F(A) \boxtimes F(B) \arrow[r, "\gamma_{F(A),F(B)}"] \arrow[d, "\varphi_{A,B}"] & F(B) \boxtimes F(A) \arrow[d, "\varphi_{B,A}"] \\
F(A \,\square\, B) \arrow[r, "F\circ\beta_{A,B}"] & F(B \,\square\, A)
\end{tikzcd}
$$

commutes for every $A, B \in \mathrm{Obj}(\mathcal{C})$.

When the braiding β is symmetric we say the functor is *symmetric monoidal.*

As we noticed for monoidal categories, we have that the composition of braided (resp. symmetric) monoidal functors is a braided (resp. symmetric) monoidal functor and the identity monoidal functor is braided (resp.symmetric). We can hence define the categories **BrMonCat** and **SymMonCat**. Again, similarly to the plain monoidal case, given two braided (resp. symmetric) monoidal categories $(\mathcal{C}, \square, I, \beta)$, $(\mathcal{D}, \boxtimes, J, \gamma)$ we can consider the category $\mathbf{BrMonCat}(\mathcal{C}, \mathcal{D})$ (resp. $\mathbf{SymMonCat}(\mathcal{C}, \mathcal{D})$) where objects are braided (resp. symmetric) monoidal functors form $\mathcal{C}$ to $\mathcal{D}$ and arrows are monoidal natural transformations between such functors.

Chapter 2

Bordisms and TQFTs

We denote with $\mathbb{R}^n$ the n-dimensional Euclidian space composed of all ordered n-tuples $(x_1, ..., x_n)$ of real numbers. We denote with $\mathbb{H}^n$ the set $\{x \in \mathbb{R}^n : x_n \geq 0\}$ equipped with the topology induced by $\mathbb{R}^n$. Recall that an n-dimensional *topological manifold* is a second-countable Hausdorff topological space locally homeomorphic to $\mathbb{R}^n$. An n-dimensional *topological manifold with boundary* is a second-countable Hausdorff topological space locally homeomorphic to $\mathbb{H}^n$. When saying we have a *local chart* on a topological manifold M, we are referring to a pair (U, u) where $U \subseteq X$ is an open subset of X and $u: U \to \underline{U} \subseteq \mathbb{R}^n$ is an homeomorphism. Two intersecting charts (U, u), (V, v) on M are *(C^∞) compatible* if the transition maps $u \circ v^{-1}$ and $v \circ u^{-1}$ are infinitely differentiable. An *atlas* $\mathcal{A}$ on M is a collection $\{(U_i, u_i)\}_{i \in I}$ of compatible charts such that $\{U_i\}_{i \in I}$ is an open covering of M. Two atlases $\mathcal{A}$, $\mathcal{B}$ are *compatible* if $\mathcal{A} \cup \mathcal{B}$ is an atlas. A *smooth (C^∞) manifold* is a second-countable Hausdorff topological space M equipped with a *smooth structure*, i.e. an equivalence class of compatible atlases. In the same vein we can define a *smooth manifold with boundary*. A point $x \in M$ is a *boundary point* if it is mapped, through some local chart, into the boundary $\{x \in \mathbb{R}^n : x_n = 0\}$ of $\mathbb{H}^n$. The set of all boundary points of an n-dimensional mandifold M is an $(n-1)$-dimensional manifold we'll denote with ∂M, the *boundary* of M. The boundary of a manifold can also be empty. In such way every manifold can be seen as a manifold with boundary. Finally, we define a *closed manifold* as a compact manifold with no boundary.

2.1 Bordisms

Having recalled the basic framework, we continue with some more precise notions from differential geometry. We'll assume all manifolds to be smooth ones.

Definition 2.1 Orientation of a vector space Let V be a finite dimensional real vector space. We say that two ordered basis $\mathcal{B}_1$, $\mathcal{B}_2$ have the *same orientation* (resp. *opposite orientation* if the linear transformation carrying one to the other has positive (resp. negative) determinant. An *orientation* on V is given by associating a sign (either $+$ or $-$) to each ordered basis, following the rule just stated.

Definition 2.2 Linear maps and orientation Let V, W be two ordered finite dimensional real vector spaces. A linear map $f: V \to W$ is *orientation preserving* if it sends positive basis into positive basis and *orientation reversing* if positive bases are sent into negative ones.

Definition 2.3 Oriented manifolds An *orientation* of a (smooth) manifold M is a choice of orientation of its tangent bundle TM. In making such choice we require that the differentials of the transition functions preserve orientations. We say a smooth manifold M is *orientable* when it admits an orientation.

add the notion of tangent bundle in the "recall"

Each orientable connected smooth manifold admits two possible orientations. An orientable manifold with $k \in [0, \infty)$ connected components admits 2^k possible orientations. The empty manifold has exactly one orientation.

Definition 2.4 Orientation of a product Let M, N be two oriented manifolds, where at least one of them has no boundary. The product $M \times N$ acquires an orientation such that for each point $(x, y) \in M \times N$, if $\{v_1, \dots, v_m\}$ is a positive basis for $T_x(M)$ and $\{w_1, \dots, w_n\}$ is a positive basis for $T_y(N)$, then $\{v_1, \dots, v_m, w_1, \dots, w_n\}$ is a positive basis for $T_{(x,y)}(M \times N)$.

Example 2.5. Take a circle $\mathbb{S}^1$ with the usual counterclockwise orientation and the interval $I := [0, 1]$ with its standard orientation. The products $\mathbb{S} \times I$ and $I \times \mathbb{S}$ then have opposite orientations. We hence need to be careful when choosing the order of such factors.

Definition 2.6 Let M be a manifold with boundary and $p \in \partial M$. A vector $v \in T_pM \setminus T_p(\partial M)$ is *inward-pointing* if for some $\varepsilon > 0$ there exists a smooth curve $\gamma: [0, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$. A vector $v \in T_pM \setminus T_p(\partial M)$ is *outward-pointing* if there exists such a curve whose domain is $(-\varepsilon, 0]$.

By considering some smooth chart $(U \ni p, \varphi = (x^i))$, we then have that the inward-pointing vectors in T_pM are exactly the ones with $x^n > 0$ and the outward-pointing ones are those for which $x^n < 0$.

Definition 2.7 The induced orientation on a boundary Let M be an oriented manifold with boundary. Its boundary ∂M inherits a canonical orientation, defined as follows. Consider an outward pointing vector $n \in T_xM \setminus T_x(\partial M)$; for any basis $\{t_1, \dots, t_n\}$ of $T_x(\partial M)$ we say it to be a *positive* (resp. *negative*) if $\{n, t_1, \dots, t_n\}$ is a positive (resp. negative) basis for T_xM.

Following such convention, the oriented cylinders above will induce opposite orientations on the two components of the boundary.

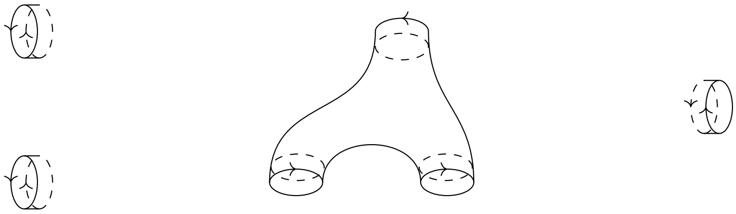
We are finally ready to give the following definition.

Definition 2.8 Oriented bordisms Let Σ_0, Σ_1 be two oriented closed manifolds of dimension $n-1$. An *n-dimensional oriented bordism* from Σ_0 to Σ_1 is a triple (M, i_0, i_1) consisting of

- an n-dimensional oriented manifold M with boundary,
- two orientation preserving embeddings

$$i_0 \colon \Sigma_0 \times [0, \varepsilon) \to M \qquad i_1 \colon \Sigma_1 \times (1-\varepsilon, 1] \to M$$

  definining an *in-boundary* $(\partial M)_0 := i_0(\Sigma_0, 0)$ and an *out-boundary* $(\partial M)_0 := i_1(\Sigma_1, 1)$, such that $\partial M = (\partial M)_0 \sqcup (\partial M)_1$.



Example 2.9.

add "incoming/outcoming" and embeddings

The role of the embeddings i_0, i_1 in the above definition is to identify a collar of the in-boundary and the out-boundary. To make this precise, we again go back to differential geometry [Lee03].

Definition 2.10 Collars Let M be a manifold with boundary. A neighbourhood of ∂M is called a *collar neighbourhood* if it is the image of a smooth embedding $\partial M \times [0, \varepsilon) \to M$, that restircts to the identification $\partial M \times \{0\} \to \partial M$.

Theorem 2.11 Collar neighbourhood theorem *Let M be a smooth manifold with nonempty boundary. Then ∂M has a collar neighbourhood.*

This guarantees that the notion of an oriented bordism is well defined.

Theorem 2.12 Gluing smooth manifolds along their boundaries *Let M, N be two manifolds with non empty boundaries ∂M, ∂N. Suppose we have a diffeomorphism between the two boundaries and consider the topological pushout $M \sqcup_{\partial M \cong \partial N} N$. Such topological manifold then has a smooth structure, compatible* *with the smooth structures on M and N. If M, N are both compact, $M \sqcup_{\partial M \cong \partial N} N$ is compact. If M, N are both connected, $M \sqcup_{\partial M \cong \partial N} N$ is connected.*

add a graphical example. For reference Freed (page 10)

Remark 2.13. While we do not give a formal proof of such result, we observe that the smooth structure on the topological pushout is not unique and relies on the choice of collars. Our definition of bordisms with fixed collars is given precisely to specify this choice, making the gluing to be exactly the pushout in the proper category (which we can imagine as the category of "manifolds with defined collars"). However, different choices of collars give rise to diffeomorphic smooth structures.

Theorem 2.14 *Let Σ be an out-boundary of a bordism M_0 and an in-boundary of a bordism M_1 and consider $M_0 \sqcup_\Sigma M_1$ the pushout through Σ of the two topological manifolds. Let α, β be two smooth structures on $M_0 \sqcup_\Sigma M_1$, which both induce the original smooth structures on M_0 and M_1 (via pullback along the inclusion maps). Then there is a diffeomorphism $\phi\colon (M_0 \sqcup_\Sigma M_1, \alpha) \to (M_0 \sqcup_\Sigma M_1, \beta)$ such that its restricition on Σ is the identity* id_Σ.

2.2 A category of oriented boridsms

Our goal is now to construct a category of oriented n-dimensional bordisms. The intuitive idea behind it is to take closed oriented $(n-1)$-dimensional manifolds as objects and oriented bordisms between them as morphisms. The proper definition, however, requires some more refinement. We begin by adressing some techinical issues. While equipping each bordism with an explicit choice of a collar allows us to properly define a gluing, this approach fails when trying to define a strict identity morphism. Given a closed oriented $(n-1)$-dimensional manifold Σ, a candidate for id_Σ is given by the cylinder $\Sigma \times [0,1]$ with some choice of collars. However, gluing it to a bordism M with one of the boundaries equal to Σ, only gives us a manifold *diffeomorphic* to M. By considering the underlying topological spaces we easily understand that the only way to define a strict identity in this setting is to consider Σ itself as a bordism, but this would not satisfy our definition which requires it to be an n-dimensional manifold. Finally we recall how, when gluing manifolds, different choices of collars give rise to different but diffeomorphic smooth structures. This motivates the following framework: we define the bordism category by taking as morphisms bordisms "up to diffeomorphism".

Definition 2.15 Equivalent bordisms Let (M, i_0, i_1), (M', i'_0, i'_1) be two oriented bordisms, both from Σ_0 to Σ_1. We say M and M' are *equivalent bordisms* if there exists an orientation preserving diffeomorphism $\psi\colon M \to M'$ making the following diagram commute.

$$\begin{tikzcd}
 & M \arrow[dd, dashed, "\psi"] & \\
\Sigma_0 \times \{0\} \cong \Sigma_0 \arrow[ur, "i_0"] \arrow[dr, "i'_0"'] & & \Sigma_1 \cong \Sigma_1 \times \{1\} \arrow[ul, "i_1"'] \arrow[dl, "i'_1"] \\
 & M' &
\end{tikzcd}$$

This clearly defines an equivalence relation between bordisms. We can then consider the equivalence classes $[(M, i_0, i_1)]$.

Remark 2.16. In defining the equivalence class we are, in a way, forgetting the collar choice we gave in the definition of a bordism. Indeed, when considering two bordisms (M, i_0, i_1), (M, j_0, j_1), we are asking for

$$i_0(\Sigma_0, 0) = (\partial M)_0 = j_0(\Sigma_0, 0) \qquad j_0(\Sigma_1, 1) = (\partial M)_1 = j_1(\Sigma_1, 1)$$

hence making the diagram commute by just choosing the identity map on M. From now on, when referring to an equivalence class of bordisms, we can forget the collars data and only consider the diffeomorphisms $\Sigma_0 \cong (\partial M)_0$ and $\Sigma_1 \cong (\partial M)_1$. We'll then just say that M is a bordism from Σ_0 to Σ_1, without specifying further informations.

Lemma 2.17 Composition of cobordism classes *Given a bordism M from Σ_0 to Σ_1 and a bordism N from Σ_1 to Σ_2, we define their composition MN from Σ_0 to Σ_2 as follows: we take any representative from each class, glue them and consider the equivalence class of the resulting gluing. This composition is well defined.*

Proof. Take the bordisms (M, i_0, i_1), (M', i'_0, i'_1) from the first equivalence class and (N, i_0, i_1), (N', i'_0, i'_1) from the second. This means we have two diffeomorphisms ψ_M, ψ_N such that:

$$\begin{tikzcd}
 & M \arrow[dd, "\psi_0"] & & N \arrow[dd, "\psi_1"] & \\
\Sigma_0 \arrow[ur] \arrow[dr] & & \Sigma_1 \arrow[ul] \arrow[dl] \arrow[ur] \arrow[dr] & & \Sigma_2 \arrow[ul] \arrow[dl] \\
 & M' & & N' &
\end{tikzcd}$$

We can then consider the gluings (MN, i_0, j_1) and $(M'N', i'_0, j'_1)$. By taking the pushout of the two diffeomorphisms in the category of continuous maps we get an homeomorphism $\psi \colon MN \to M'N'$.

come si comporta? preserva la struttura liscia? Se lo fa sistema il finale

Through such homeomorphism we can define a new smooth structure on $M'N'$, which by the previous theorem 2.1 is diffeomorphic to the one induced by the gluing. $\square$

Definition 2.18 The category of oriented n-dimensional bordisms $\mathbf{Bord}(n)$ is defined as follows.

- The objects are closed oriented $(n-1)$-dimensional manifolds
- For any $\Sigma_0, \Sigma_1 \in \mathrm{Obj}(\mathbf{Bord}(n))$, morphisms are the equivalence classes of bordisms $M\colon \Sigma_0 \to \Sigma_1$
- Composition of morphisms is obtained by gluing
- For each object Σ, the identity map id_Σ is given by the bordism $\Sigma \times [0,1]$

2.19 (Disjoint union of cobordisms). Given two bordisms $M\colon \Sigma_0 \to \Sigma_1$ and $N\colon \Sigma_0' \to \Sigma_1'$, their disjoint union $M \amalg N$ naturally defines a bordism from $\Sigma_0 \amalg \Sigma_0'$ to $\Sigma_1 \amalg \Sigma_1'$. This operation is precisely the coproduct in the category of smooth manifold, equipped with the unique orientation agreeing with the ones on M and N.

Lemma 2.20 *Let $\mathcal{C}$ be a category that admits coproducts. Then $(\mathcal{C}, \amalg, 0)$, where $\amalg$ denotes the coproduct and 0 the initial object, is a monoidal category.*

Proposition 2.21 *The category $\boldsymbol{Bord}(n)$ has a monoidal structure given by the disjoint union of manifolds (and the initial object, being the empty manifold $\varnothing$).*

Proposition 2.22 *Any diffeomorphism of $(n-1)$-dimensional manifolds Σ_0, Σ_1 define an equivalence class of bordisms $M\colon \Sigma_0 \to \Sigma_1$.*

2.23 (The twist bordism). The twist diffeomorphism of manifolds $\sigma\colon \Sigma \amalg \Sigma' \to \Sigma' \amalg \Sigma$ defines a cobordism in $\mathbf{Bord}(n)$ which we'll denote as $T_{\Sigma,\Sigma'}\colon \Sigma \amalg \Sigma' \to \Sigma' \amalg \Sigma$.

Proposition 2.24 *The category $\boldsymbol{Bord}(n)$ has a symmetric monoidal structure $(\boldsymbol{Bord}(n), \amalg, \varnothing, T)$*

2.3 Topological Quantum Field Theories

Definition 2.25 An *n-dimensional topological quantum field theory* is a symmetric monoidal functor from $(\mathbf{Bord}(n), \amalg, \varnothing, T)$ to $(\mathbf{Vect}, \otimes, \Bbbk, \sigma)$

Historically, the first axiomatization of TQFTs appeared in the late 1980s. The first explicit set of axioms was given by Atiyah in 1988. Let us see how they compare with the functorial definition we just stated.

Definition 2.26 Atiyah's axioms of a TQFT An n-dimensional topological quantm field theory consists of the following data:

- a $\Bbbk$-vector space $Z(\Sigma)$ associated to each closed oriented $(n-1)$-dimensional manifold Σ
- a $\Bbbk$-linear map $Z(M)\colon Z(\Sigma_0) \to Z(\Sigma_1)$ associated to each n-dimensional

bordism M from Σ_0 to Σ_1

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