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# An Introduction to Frobenius Algebras and 2D TQFTs

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# Contents

# Introduction

> There are many reasons to study topological quantum field theories, but one reason is that they exhibit a beautiful relationship between algebra and geometry
>
> Christopher John Schommer-Pries, *The Classification of Two-Dimensional Extended Topological Field Theories*

First we reassure the reader: this is a thesis in pure mathematics. Its goal is to develop the generator-and-relations presentation of a category that (the author has heard) plays an important role in modern theoretical physics research, and to use it to establish a link between geometry (i.e., the representations of the category of 2-dimensional oriented bordisms) and algebra (i.e., some less known kind of algebraic structures of increasing popularity among logicians and computer scientists that go under the name of "Frobenius algebras"). After reviewing some basic vocabulary from category theory in Chapter 1, we introduce oriented bordisms and their ambient category **Bord**$(n)$ in Chapter 2, along with the *linear representations* of this category, i.e., the symmetric monoidal functors from **Bord**$(n)$ to the (symmetric monoidal) category of (finite-dimensional) vector spaces over an arbitrary field. Things become even more concrete and structured in Chapter 3 where we consider the particular case of dimension 2. Indeed, having an already established classification of topological surfaces allows us to give a presentation of the category **Bord**$(2)$ in terms of generators and relations and ask ourselves what a representation of such category looks like. At the end we will discover this defines exactly a structure of Frobenius algebra. In Chapter 4 we'll have a close encounter with these algebras, through the help of a graphical language. With these premises in place we finally reach the main equivalence in Chapter 5 which basically states that defining a 2-dimensional TQFT is the same as choosing a commutative Frobenius algebra (and viceversa). We also see how this is just an instance of a broader concept: defining free monoidal categories over some particular objects.

The idea of writing a thesis on this topic arose from a (desire???) to approach monoidal categories and their internal objects through a concrete example. During the work, I (??) got lost many times when trying to expand on other concepts. This is also due to the fact that the right setting as of nowadays seems to be the one of higher categories and extended bordisms. Due to the limited amount of time, I could only take a look at these more advanced topics, which could not make it in the thesis.

The majority of this work is clearly based on [Koc03] and can largely be seen as a retelling of its tale of *the commutative Frobenius and the princess* **Bord**$(2)$, even if by a less skillful bard.

Chapter 1

# Categorical preliminaries

We begin by recalling some useful notions from category theory, which we will encounter throughout this thesis, assuming the reader is already familiar with its fundamentals.

## 1.1 Monoidal Categories

**Definition 1.1 (Monoidal categories).** A (weak) *monoidal category* is a sextuple $(\mathcal{C}, \square, \eta, \alpha, l, r)$ consisting of:

- a category $\mathcal{C}$,
- a bifunctor $\square \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
- a functor $\eta \colon \mathbf{1} \to \mathcal{C}$ identifying an object $\eta(1) = I \in \mathrm{Obj}(\mathcal{C})$[1],
- a natural isomorphism called *associator* $\alpha$,

$$
\begin{tikzcd}
 & \mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[dl, "\square \times \mathrm{id}_{\mathcal{C}}"'] \arrow[dr, "\mathrm{id}_{\mathcal{C}} \times \square"] & \\
\mathcal{C} \times \mathcal{C} \arrow[dr, "\square"'] \arrow[rr, phantom, "\overset{\alpha}{\Longrightarrow}"] & & \mathcal{C} \times \mathcal{C} \arrow[dl, "\square"] \\
 & \mathcal{C} &
\end{tikzcd}
$$

so that $\alpha_{A,B,C} \colon (A \,\square\, B) \,\square\, C \xrightarrow{\cong} A \,\square\, (B \,\square\, C)$ for every $A, B, C \in \mathrm{Obj}(\mathcal{C})$

$$\begin{tikzcd}
 & & (A \,\Box\, B) \,\Box\, (C \,\Box\, D) \arrow[drr, "\alpha_{A,B,C\Box D}"] & & \\
((A \,\Box\, B) \,\Box\, C) \,\Box\, D \arrow[urr, "\alpha_{A\Box B,C,D}"] \arrow[dr, "\alpha_{A,B,C}\Box\mathrm{id}_D"'] & & & & A \,\Box\, (B \,\Box\, (C \,\Box\, D)) \\
 & (A \,\Box\, (B \,\Box\, C)) \,\Box\, D \arrow[rr, "\alpha_{A,B\Box C,D}"] & & A \,\Box\, ((B \,\Box\, C) \,\Box\, D) \arrow[ur, "\mathrm{id}_A\Box\alpha_{B,C,D}"'] &
\end{tikzcd}$$

Figure 1.1: Associativity coherence

$$\begin{tikzcd}
(A \,\Box\, I) \,\Box\, B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "r_A\Box\mathrm{id}_B"'] & & A \,\Box\, (I \,\Box\, B) \arrow[dl, "\mathrm{id}_A\Box l_B"] \\
 & A \,\Box\, B &
\end{tikzcd}$$

Figure 1.2: Unit coherence

- and two natural isomorphisms called *left* and *right unitors* $l$ and $r$

$$\begin{tikzcd}
 & \mathcal{C}\times\mathcal{C} \arrow[dr, "\Box"] \arrow[d, phantom, "\Downarrow l"] & & \mathcal{C}\times\mathcal{C} \arrow[dl, "\Box"'] \arrow[d, phantom, "\Downarrow r"] & \\
\mathbf{1}\times\mathcal{C} \arrow[ur, "\eta\times\mathrm{id}_{\mathcal{C}}"] \arrow[rr, "\pi"'] & & \mathcal{C} & & \mathcal{C}\times\mathbf{1} \arrow[ul, "\mathrm{id}_{\mathcal{C}}\times\eta"'] \arrow[ll, "\pi"]
\end{tikzcd}$$

  so that $l_A \colon I \,\Box\, A \overset{\cong}{\to} A$ and $r_A \colon A \,\Box\, I \overset{\cong}{\to} A$ for every $A \in \mathrm{Obj}(\mathcal{C})$.

These maps must satisfy the so-called coherence requirements, depicted in Figure 1.1 and Figure 1.2, for every $A, B, C, D \in \mathrm{Obj}(\mathcal{C})$.

It is important to notice that naturality of such transformations is not for free, as some proof depend on naturality to work.

**Definition 1.2 (Strict monoidal categories).** A monoidal category is *strict* if all the natural transformations $\alpha$, $l$, $r$ are identities.

We can say that a strict monoidal category is a category $\mathcal{C}$ equipped with an associative bifunctor $\Box \colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and an object $I \in \mathrm{Obj}(\mathcal{C})$ which is a left and

[1] We will hence equivalentely refer to a monoidal category as $(\mathcal{C}, \Box, I, \alpha, l, r)$

a right unit for $\square$. In fancy words, a strict monoidal category is a (strict) monoid *internal* to the category of categories and functors.

From now on, when considering a weak monoidal category $(\mathcal{C}, \square, I, \alpha^{\mathcal{C}}, l^{\mathcal{C}}, r^{\mathcal{C}})$, we will refer to it with using the notation $(\mathcal{C}, \square, I)$. This is an abuse of notation that allows for brevity.

**Proposition 1.3.** *Let $\mathcal{C}$ be a category that admits products. Then $(\mathcal{C}, \times, 1)$, where $\times$ denotes the product and $1$ the terminal object, is a (weak) monoidal category.*

*Proof.* Consider $A, B, C \in \mathrm{Obj}(\mathcal{C})$. By the universal property of the product, we can define an isomorphism $\alpha_{A,B,C}$ between $(A \times B) \times C$ and $A \times (B \times C)$, with which we can define the *associator* $\alpha$. Then, since for any $A \in \mathrm{Obj}(\mathcal{C})$ there exists a unique arrow to the terminal object 1, we have that

$$1 \xleftarrow{\;!\;} A \xrightarrow{\mathrm{id}_A} A$$

defines a product for 1 and $A$. The uniqueness of the product provides an isomorphism $1 \times A \cong A$ and, similarly, $A \times 1 \cong A$. Through such maps, we define *left* and *right unitors*. Using such isomorphisms one can also check that coherence holds. $\square$

We call this structure a *cartesian* monoidal category. The dual statement also holds.

**Proposition 1.4.** *Let $\mathcal{C}$ be a category that admits coproducts. Then $(\mathcal{C}, +, 0)$, where $+$ denotes the coproduct and $0$ the initial object, is a (weak) monoidal category.*

We call this structure a *cocartesian* monoidal category.

Example 1.5. Examples of monoidal categories, which we will encounter throughout this thesis, are:

- $(M, \mu, 1_M)$, the discrete monoidal category on a monoid $M$
- $(\mathbf{Set}, \times, \{*\})$, the cartesian monoidal category of sets and functions between them
- $(\mathbf{Set}, \amalg, \varnothing)$, the cocartesian monoidal category of sets and functions between them
- $(\mathbf{Cat}, \times, \mathbf{1})$, the cartesian monoidal category of categories and functors between them
- $(\Delta, +, [0])$, the strict monoidal simplicial category. A more in-depth description of it can be found in Definition 5.11.
- $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk)$, the monoidal category of vector spaces over the field $\Bbbk$ and linear maps between them, with the monoidal structure given by the tensor product

We'd now like to define morphisms between monoidal categories as functors that preserve the monoidal structure. However, we need to be precise about the degree to which such a structure needs to be preserved. As for monoidal categories, the weakest definition will also be the most intricate.

**Definition 1.6 (Monoidal functors).** Let $(\mathcal{C}, \Box, I)$ and $(\mathcal{D}, \boxtimes, J)$ be two (weak) monoidal categories. A monoidal functor is a triple $(F, \varphi, \varepsilon)$ consisting of

- a functor
$$F \colon \mathcal{C} \to \mathcal{D}$$
- a natural transformation
$$\varphi_{A,B} \colon F(A) \boxtimes F(B) \to F(A \Box B)$$
for every $A, B \in \mathrm{Obj}(\mathcal{C})$
- a morphism
$$\varepsilon \colon J \to F(I)$$

making the following diagrams commute

1. *(Associativity)*

$$\begin{tikzcd}
(F(A) \boxtimes F(B)) \boxtimes F(C) \arrow[rr, "\alpha^{\mathcal{D}}_{F(A),F(B),F(C)}"] \arrow[d, "\varphi_{A,B} \boxtimes \mathrm{id}_{F(C)}"] & & F(A) \boxtimes (F(B) \boxtimes F(C)) \arrow[d, "\mathrm{id}_{F(A)} \boxtimes \varphi_{B,C}"] \\
F(A \Box B) \boxtimes F(C) \arrow[d, "\varphi_{A \Box B, C}"] & & F(A) \boxtimes F(B \Box C) \arrow[d, "\varphi_{A, B \Box C}"] \\
F((A \Box B) \Box C) \arrow[rr, "F(\alpha^{\mathcal{C}}_{A,B,C})"] & & F(A \Box (B \Box C))
\end{tikzcd}$$

2. *(Unitality)*

$$\begin{tikzcd}
F(A) \boxtimes J \arrow[r, "r^{\mathcal{D}}_{F(A)}"] \arrow[d, "\mathrm{id}_{F(A)} \boxtimes \varepsilon"'] & F(A) \\
F(A) \boxtimes F(I) \arrow[r, "\varphi_{A,I}"] & F(A \Box I) \arrow[u, "F \circ r^{\mathcal{C}}_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
J \boxtimes F(A) \arrow[r, "l^{\mathcal{D}}_{F(A)}"] \arrow[d, "\varepsilon \boxtimes \mathrm{id}_{F(A)}"'] & F(A) \\
F(I) \boxtimes F(A) \arrow[r, "\varphi_{I,A}"] & F(I \Box A) \arrow[u, "F \circ l^{\mathcal{C}}_A"']
\end{tikzcd}$$

for every $A, B, C \in \mathrm{Obj}(\mathcal{C})$

*Remark* 1.7. This notion can be found in the literature as *lax monoidal functor*. We can also define an *oplax monoidal functor* between two categories $\mathcal{C}$ and $\mathcal{D}$ as a lax monoidal functor $\mathcal{C}^{op} \to \mathcal{D}^{op}$. In this case, we will have natural transformations $\varphi_{A,B} \colon F(A \Box B) \to F(A) \boxtimes F(B)$ and $\varepsilon \colon F(I) \to J$. The idea behind these two notions is that a *lax* monoidal functor is the one preserving the structure of an internal monoid, while an *oplax* monoidal functor preserves the internal comonoid structure. The definition of an internal monoid (resp. comonoid) will be properly given in 5.2 (resp. 5.3).

**Definition 1.8 (Strong monoidal functors).** Let $(\mathcal{C}, \square, I)$ and $(\mathcal{D}, \boxtimes, J)$ be two (weak) monoidal categories. A monoidal functor $(F, \varphi, \varepsilon)$ from $\mathcal{C}$ to $\mathcal{D}$ is *strong monoidal* if $\varphi$ and $\varepsilon$ are isomorphisms.

We hence have $F(A) \boxtimes F(B) \cong F(A \square B)$ and $J \cong F(I)$.

**Definition 1.9 (Strict monoidal functors).** Let $(\mathcal{C}, \square, I)$ and $(\mathcal{D}, \boxtimes, J)$ be two (weak) monoidal categories. A monoidal functor $(F, \varphi, \varepsilon)$ from $\mathcal{C}$ to $\mathcal{D}$ is *strict monoidal* if $\varphi$ and $\varepsilon$ are identities.

The conditions for a strict monoidal functor are asking that

$$F(A \square B) = F(A) \boxtimes F(B), \qquad F(I) = J$$

$$F(f \square g) = F(f) \boxtimes F(g)$$

$$F(\alpha^{\mathcal{C}}_{A,B,C}) = \alpha^{\mathcal{D}}_{F(A),F(B),F(C)} \qquad F(l^{\mathcal{C}}_A) = l^{\mathcal{D}}_{F(A)} \qquad F(r^{\mathcal{C}}_A) = r^{\mathcal{D}}_{F(A)}$$

for every $A, B, C \in \mathrm{Obj}(\mathcal{C})$ and $f, g \in \mathrm{Arr}(\mathcal{C})$.

*Remark* 1.10. Let $(\mathcal{C}, \times_{\mathcal{C}}, 1_{\mathcal{C}})$, $(\mathcal{D}, \times_{\mathcal{D}}, 1_{\mathcal{D}})$ be two cartesian monoidal categories and $F\colon \mathcal{C} \to \mathcal{D}$ a functor between them. By the universal property of the terminal object, there exists a unique arrow $\varepsilon\colon F(1_{\mathcal{C}}) \to 1_{\mathcal{D}}$, and by the universal property of the product, we have

$$\begin{tikzcd}
F(A) & F(A) \times_{\mathcal{D}} F(B) \arrow[l, "\pi_{F(A)}"'] \arrow[r, "\pi_{F(B)}"] & F(B) \\
& F(A \times_{\mathcal{C}} B) \arrow[ul, "F(\pi_A)"] \arrow[u, dashed, "\exists!"] \arrow[ur, "F(\pi_B)"'] &
\end{tikzcd}$$

Again, by relying on universal properties, we are able to prove that associativity and unitality hold. We hence have that $F\colon \mathcal{C} \to \mathcal{D}$ is *oplax* monoidal without further requirements. Similarly, every functor $F\colon \mathcal{C}^{op} \to \mathcal{D}^{op}$ is automatically *lax* monoidal.

Conversely, let $(\mathcal{C}, +_{\mathcal{C}}, 0_{\mathcal{C}})$, $(\mathcal{D}, +_{\mathcal{D}}, 0_{\mathcal{D}})$ be two cocartesian monoidal categories and $F\colon \mathcal{C} \to \mathcal{D}$ a functor between them. Then $F$ is a *lax* monoidal functor. Similarly, every functor $F\colon \mathcal{C}^{op} \to \mathcal{D}^{op}$ is automatically *oplax* monoidal.

The composition of two (lax, oplax, strong, strict) monoidal functors is again a (lax, oplax, strong, strict) monoidal functor, and identity functors are monoidal. We hence have a category $\mathbf{MonCat}_{\mathrm{lax}}$ where objects are (weak) monoidal categories and arrows are lax monoidal functors. Similarly, $\mathbf{MonCat}_{\mathrm{strong}}$ defines the category whose objects are (weak) monoidal categories and arrows are strong monoidal functors. There is also a full subcategory where objects are strict monoidal categories and arrows are strong monoidal functors.

*Remark* 1.11. As the reader can imagine, working with weak monoidal categories can become a really tedious process. The following result mitigates this difficulty, justifying working with monoidal categories *as if they were strict*, forgetting about associators and unitors without losing generality. Its proof and a more in-depth explanation can be found in [Mac78] and [Tru20].

**Theorem 1.12 (Strictification).** (See [Mac78], Chapter XI, Section 3) *Every monoidal category is categorically equivalent, via strong monoidal functors, to a strict monoidal category.*

Essentially, what follows is that from now on we can assume, without loss of generality, that monoidal categories *are* strict. This is possible because the equivalence provided by the above theorem ensures that all coherence diagrams commute up to a unique canonical isomorphism, as long as we do not require functors between such categories to be strict. To do so, from now on, when considering the functor category **MonCat**, we are referring to $\mathbf{MonCat}_{\text{strong}}$.

*Remark* 1.13. The strictification of the monoidal category of finite dimensional vectors spaces over a field $k$ is the category (cf. [Wal92, 2.§2]) $\mathbb{N}[k]$ whose objects are iterated powers $k^n$ of $k$ with itself; the monoidal structure is given by product of natural numbers, i.e. as the strict equality

$$k^n \oplus k^m = k^{nm}$$

Note that $\mathbb{N}[k]$ is a PROP in the sense of [Mac65, Ch. V, §24]; the sum of natural numbers gives $\mathbb{N}[k]$ another monoidal structure $\oplus$, in that

$$k^n \oplus k^m = k^{n+m}$$

and moreover $\mathbb{N}[k]$ is a categorified version of a *semiring*, in that 'product distributes over sum',

$$k^{n(p+q)} = k^{np} \oplus k^{nq}.$$

**Definition 1.14 (Monoidal natural transformations).** Let $(\mathcal{C}, \square, I)$ and $(\mathcal{D}, \boxtimes, J)$ be two monoidal categories and let $(F, \varphi, \varepsilon)$ and $(G, \phi, e)$ be two monoidal functors from $\mathcal{C}$ to $\mathcal{D}$. A *monoidal natural transformation* is a natural transformation $u \colon F \to G$ such that the following diagrams commute

$$\begin{tikzcd} F(A) \boxtimes F(B) \arrow[r, "u_A \boxtimes u_B"] \arrow[d, "\varphi_{A,B}"] & G(A) \boxtimes G(B) \arrow[d, "\phi_{A,B}"] \\ F(A \square B) \arrow[r, "u_{A \square B}"] & G(A \square B) \end{tikzcd} \qquad \begin{tikzcd} J \arrow[r, "\varepsilon"] \arrow[dr, "e"'] & F(I) \arrow[d, "u_I"] \\ & G(I) \end{tikzcd}$$

for every $A, B \in \mathrm{Obj}(\mathcal{C})$.

*Remark* 1.15. Let $(\mathcal{C}, \times_{\mathcal{C}}, 1_{\mathcal{C}})$, $(\mathcal{D}, \times_{\mathcal{D}}, 1_{\mathcal{D}})$ be two cartesian monoidal categories and let $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{C} \to \mathcal{D}$ be two functors between them. Consider any natural transformation $u\colon F \Rightarrow G$ and the diagrams

$$\begin{tikzcd} F(A) \times_{\mathcal{D}} F(B) \arrow[r, "u_A \times_{\mathcal{D}} u_B"] & G(A) \times_{\mathcal{D}} G(B) \\ F(A \times_{\mathcal{C}} B) \arrow[u, "\varphi_{A,B}"] \arrow[r, "u_{A \times_{\mathcal{C}} B}"] & G(A \times_{\mathcal{C}} B) \arrow[u, "\phi_{A,B}"] \end{tikzcd} \qquad \begin{tikzcd} 1_{\mathcal{C}} & F(1_{\mathcal{C}}) \arrow[l, "\varepsilon"'] \arrow[d, "u_I"] \\ & G(1_{\mathcal{C}}) \arrow[ul, "e"] \end{tikzcd}$$

By naturality of $u$, one checks the two diagrams commute without asking further conditions to hold. We hence have that every natural transformation between two functors defined on cartesian monoidal categories is monoidal. The same holds when the two categories are cocartesian monoidal.

## 1.2 Adding a twist

**Definition 1.16 (Braided monoidal categories).** A *braided monoidal category* is a monoidal category $(\mathcal{C}, \square, I)$ equipped with a natural isomorphisms

$$\beta_{A,B}\colon A \,\square\, B \to B \,\square\, A$$

for every $A, B \in \mathrm{Obj}(\mathcal{C})$, called the *braiding*, such that the following diagrams commute

$$\begin{tikzcd} & A \square (B \square C) \arrow[r, "\beta_{A,B\square C}"] & (B \square C) \square A \arrow[dr, "\alpha_{B,C,A}"] & \\ (A \square B) \square C \arrow[ur, "\alpha_{A,B,C}"] \arrow[dr, "\beta_{A,B} \square \mathrm{id}_C"'] & & & B \square (C \square A) \\ & (B \square A) \square C \arrow[r, "\alpha_{B,A,C}"] & B \square (A \square C) \arrow[ur, "\mathrm{id}_B \square \beta_{A,C}"'] & \end{tikzcd}$$

$$\begin{tikzcd} & (A \square B) \square C \arrow[r, "\beta_{A\square B,C}"] & C \square (A \square B) \arrow[dr, "\alpha^{-1}_{C,A,B}"] & \\ A \square (B \square C) \arrow[ur, "\alpha^{-1}_{A,B,C}"] \arrow[dr, "\mathrm{id}_A \square \beta_{B,C}"'] & & & (C \square A) \square B \\ & A \square (C \square B) \arrow[r, "\alpha^{-1}_{A,C,B}"] & (A \square C) \square B \arrow[ur, "\beta_{A,C} \square \mathrm{id}_A"'] & \end{tikzcd}$$

The definition implies compatibility with the unital structure. For a proof of the following result, we refer to [Koc03].

**Proposition 1.17.** *Let $(\mathcal{C}, \square, I, \beta)$ be a braided monoidal category. Then the following diagrams commute for every $A \in \mathrm{Obj}(\mathcal{C})$.*

$$
\begin{tikzcd}
A \square I \arrow[rr, "\beta_{A,I}"] \arrow[dr, "r_A"'] & & I \square A \arrow[dl, "l_A"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
I \square A \arrow[rr, "\beta_{I,A}"] \arrow[dr, "l_A"'] & & A \square I \arrow[dl, "r_A"] \\
& A &
\end{tikzcd}
$$

Again, by Theorem 1.12, when considering braided monoidal categories, we can restrict to the following definition.

**Definition 1.18 ((Semi)strict braided monoidal categories).** A semistrict *braided monoidal category* is a strict monoidal category $(\mathcal{C}, \square, I)$ equipped with a natural isomorphism

$$\beta_{A,B} \colon A \square B \to B \square A$$

for every $A, B \in \mathrm{Obj}(\mathcal{C})$, called the *braiding*, such that the following diagrams commute

$$
\begin{tikzcd}
A \square B \square C \arrow[rr, "\beta_{A,B\square C}"] \arrow[dr, "\beta_{A,B}\square \mathrm{id}_C"'] & & B \square C \square A \\
& B \square A \square C \arrow[ur, "\mathrm{id}_B \square \beta_{A,C}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \square B \square C \arrow[rr, "\beta_{A\square B,C}"] \arrow[dr, "\mathrm{id}_A\square \beta_{B,C}"'] & & C \square A \square B \\
& A \square C \square B \arrow[ur, "\beta_{A,C} \square \mathrm{id}_A"'] &
\end{tikzcd}
$$

*Remark* 1.19. It would be misleading to name these categories "strict braided monoidal". While the underlying monoidal structure is strict, we are not asking for the braiding itself to be strict.

**Definition 1.20 (Symmetric monoidal categories).** A *symmetric monoidal category* is a braided monoidal category $(\mathcal{C}, \square, I, \beta)$ where the braiding $\beta$ is symmetric, that is

$$\beta_{B,A} \circ \beta_{A,B} = \mathrm{id}_{A \square B}$$

for every $A, B \in \mathrm{Obj}(\mathcal{C})$.

Let us show some examples of symmetric monoidal categories.

Example 1.21. The monoidal category $(\mathbf{Vect}_{\mathbb{k}}, \otimes, \mathbb{k})$ is equipped with a canonical symmetry defined by

$$
\begin{aligned}
\sigma \colon V \otimes W &\to W \otimes V \\
v \otimes w &\mapsto w \otimes v
\end{aligned}
$$

Example 1.22 (Free symmetric monoidal category on a single object). We define the *symmetric monoidal category* **Sym** by considering

- the sets of natural numbers $[n] = 0, \ldots, n-1$ as objects. We also define $[0] = \varnothing$.

- the elements of the symmetric group $\mathrm{Sym}(n)$ as morphism $\mathrm{Hom}([n],[n])$. When $[n] \neq [m]$, we have $\mathrm{Hom}([m],[n]) = \varnothing$.

The monoidal operation $+$ is defined by juxtaposition[2], with the empty set $\varnothing = [0]$ as the unit object. The symmetric structure is defined by interchanging factors (we define the *twist* from $[m]+[n]$ to $[n]+[m]$ by the association $\{0_m, \dots, (m-1)_m, 0_n, \dots, (n-1)_n\} \mapsto \{0_n, \dots, (n-1)_n, 0_m, \dots, (m-1)_m\}$).

Example 1.23 (Free braided monoidal category on a single object). Similarly to the previous case, we define the *braided monoidal category* $(\mathbf{Braid}, +, [0])$ by taking the Artin braid groups $\mathrm{Braid}(n)$ (see [KT08])a s morphisms.

It is important to understand that being a symmetric monoidal category involves defining a structure, not merely possessing a property. This distinction is clearly shown by the following example.

Example 1.24 (A category with more than one symmetric structure). Let $\mathbf{grVect}_{\Bbbk}$ be the category of *graded vector spaces* (i.e. direct sums of vector spaces $V = \bigoplus_{n\in\mathbb{Z}} V_n$ together with linear maps $f = \bigoplus_{n\in\mathbb{Z}} f_n$ respecting the grading)[3]. The monoidal structure defined for $\mathbf{Vect}$ restricts to the graded setting and defines the monoidal category $(\mathbf{grVect}_{\Bbbk}, \otimes, \Bbbk)$.

In this setting, we have a canonical symmetry $v \otimes w \mapsto w \otimes v$, but we can also define a different isomorphism, known as the *Koszul's sign change*

$$\begin{aligned} \kappa\colon V \otimes W &\to W \otimes V \\ v \otimes w &\mapsto (-1)^{\deg(v)\deg(w)} w \otimes v \end{aligned}$$

Both $(\mathbf{grVect}_{\Bbbk}, \otimes, \Bbbk, \sigma)$ and $(\mathbf{grVect}_{\Bbbk}, \otimes, \Bbbk, \kappa)$ are symmetric monoidal categories, yet they are distinct and carry different internal structures.

**Definition 1.25 (Braided monoidal functors).** Let $(\mathcal{C}, \square, I, \beta)$ and $(\mathcal{D}, \boxtimes, J, \gamma)$ be two braided monoidal categories. A monoidal functor $(F, \varphi, \varepsilon)$ from $\mathcal{C}$ to $\mathcal{D}$ is a *braided monoidal functor* if the diagram

$$\begin{tikzcd} F(A) \boxtimes F(B) \arrow[r, "\gamma_{F(A),F(B)}"] \arrow[d, "\varphi_{A,B}"] & F(B) \boxtimes F(A) \arrow[d, "\varphi_{B,A}"] \\ F(A \square B) \arrow[r, "F \circ \beta_{A,B}"] & F(B \square A) \end{tikzcd}$$

commutes for every $A, B \in \mathrm{Obj}(\mathcal{C})$.

[2]We can see an explicit definition of such operation in Definition 5.12, for the case of $\Delta$

[3]What we are describing are actually $\mathbb{Z}$-graded vector spaces. In general, we can define $G$-graded vector spaces as $V = \bigoplus_{g\in G} V_g$ for any group $G$.

When the braiding $\beta$ is symmetric, we say the functor is *symmetric monoidal.*

As we noticed for monoidal categories, we have that the composition of braided (resp. symmetric) monoidal functors is a braided (resp. symmetric) monoidal functor, and the identity monoidal functor is braided (resp., symmetric). We can hence define the categories **BrMonCat** and **SymMonCat**. Again, similarly to the plain monoidal case, given two braided (resp. symmetric) monoidal categories $(\mathcal{C}, \Box, I, \beta)$, $(\mathcal{D}, \boxtimes, J, \gamma)$ we can consider the category $\mathbf{BrMonCat}(\mathcal{C}, \mathcal{D})$ (resp. $\mathbf{SymMonCat}(\mathcal{C}, \mathcal{D})$) where objects are braided (resp. symmetric) monoidal functors form $\mathcal{C}$ to $\mathcal{D}$ and arrows are monoidal natural transformations between such functors.

## 1.3 Rigid categories

We include a brief treatment of the concept of duals in a monoidal category, a notion that will recur again throughout this thesis. We refer to [Sel10].

**Definition 1.26 (Exact pairing, left and right duals).** Let $(\mathcal{C}, \Box, I)$ be a (without loss of generalities, strict) monoidal category. An *exact pairing* between two objects $A$ and $B$ is given by a pair of morphisms $\eta \colon I \to B \otimes A$ and $\varepsilon \colon A \otimes B \to I$, such that the following triangles commutes

$$
\begin{tikzcd}
A \arrow[r, "\mathrm{id}_A \otimes \eta"] \arrow[dr, "\mathrm{id}_A"'] & A \otimes B \otimes A \arrow[d, "\varepsilon \otimes \mathrm{id}_A"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\eta \otimes \mathrm{id}_B"] \arrow[dr, "\mathrm{id}_B"'] & B \otimes A \otimes B \arrow[d, "\mathrm{id}_B \otimes \varepsilon"] \\
& B
\end{tikzcd}
$$

In such exact pairing, the object $B$ is called *right dual* of $A$ and $A$ is called *left dual* of $B$.

When the monoidal category is symmetric, left and right duals coincide; we refer to either as the *dual*.We call every object that admits categorical duals *fully dualizable.*

**Definition 1.27 (Rigid categories).** A monoidal category $(\mathcal{C}, \Box, I)$ is called *rigid* if every of its object has both left and right duals.

# Chapter 2

# Bordisms and TQFTs

We denote with $\mathbb{R}^n$ the $n$-dimensional Euclidean space composed of all ordered $n$-tuples $(x^1, ....., x^n)$ of real numbers. We denote with $\mathbb{H}^n$ the set $\{x \in \mathbb{R}^n : x^n \geq 0\}$ equipped with the topology induced by $\mathbb{R}^n$. Recall that an $n$-dimensional *topological manifold* is a second-countable Hausdorff topological space locally homeomorphic to $\mathbb{R}^n$. An $n$-dimensional *topological manifold with boundary* is a second-countable Hausdorff topological space locally homeomorphic to $\mathbb{H}^n$. When we say that we have a *local chart* on a topological manifold $M$, we are referring to a pair $(U, u)$ where $U \subseteq X$ is an open subset of $X$ and $u \colon U \to \underline{U} \subseteq \mathbb{R}^n$ is an homeomorphism. Two charts $(U, u)$, $(V, v)$ on $M$ are *($C^\infty$)-compatible* if $U \cap V = \emptyset$ or if $U \cap V \neq \emptyset$ and the transition maps $u \circ v^{-1}$ and $v \circ u^{-1}$ are infinitely differentiable. An *atlas* $\mathcal{A}$ on $M$ is a collection $\{(U_i, u_i)\}_{i \in I}$ of compatible charts such that $\{U_i\}_{i \in I}$ is an open covering of $M$. Two atlases $\mathcal{A}$, $\mathcal{B}$ are *compatible* if $\mathcal{A} \cup \mathcal{B}$ is an atlas. A *smooth ($C^\infty$-)manifold* is a second-countable Hausdorff topological space $M$ equipped with a *smooth structure*, i.e. an equivalence class of compatible atlases. In the same vein, we can define a *smooth manifold with boundary*. A point $x \in M$ is a *boundary point* if it is mapped, through some local chart, into the boundary $\{x \in \mathbb{R}^n : x^n = 0\}$ of $\mathbb{H}^n$. The set of all boundary points of an $n$-dimensional manifold $M$ is an $(n-1)$-dimensional manifold we'll denote with $\partial M$, the *boundary* of $M$. The boundary of a manifold can also be empty. In this way, every manifold can be seen as a manifold with boundary. In the following, a compact manifold with no boundary is a *closed manifold*.

## 2.1 Bordisms

Having recalled the basic framework, we continue with some more precise notions from differential geometry. We'll assume all manifolds to be $C^\infty$.

**Definition 2.1 (Orientation of a vector space).** Let $V$ be a finite-dimensional real vector space. We say that two ordered bases $\mathcal{B}_1$, $\mathcal{B}_2$ have the *same orientation* (resp. *opposite orientation*) if the linear transformation carrying one into the other has positive (resp. negative) determinant. An *orientation* on $V$ is given by associating a sign (either $+$ or $-$) to each ordered basis, following the rule just stated.

**Definition 2.2 (Linear maps and orientation).** Let $V$, $W$ be two oriented, finite-dimensional real vector spaces. A linear map $f\colon V \to W$ is *orientation preserving* if it sends positive bases into positive bases and *orientation reversing* if positive bases are sent into negative ones.

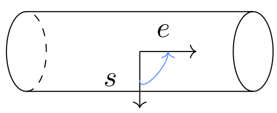
**Definition 2.3 (Oriented manifolds).** An *orientation* of a (smooth) manifold $M$ is a choice of orientation for each tangent space $T_xM$. In making such a choice, we require that the differentials of the transition functions preserve orientations. We say a smooth manifold $M$ is *orientable* when it admits an orientation.

A differentiable map $f\colon M \to M'$ between two oriented (smooth) manifolds is *orientation preserving* if, for each $x \in M$, the differential $df_x\colon T_xM \to T_{f(x)}M'$ is orientation preserving.

Each orientable connected smooth manifold admits two possible orientations. An orientable manifold with $k \geq 0$ connected components admits $2^k$ possible orientations. The empty manifold has exactly one orientation.

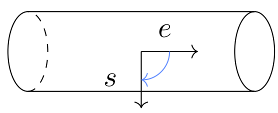
**Definition 2.4 (Orientation of a product).** Let $M$, $N$ be two oriented manifolds, where at least one of them has no boundary. The product $M \times N$ acquires an orientation such that for each point $(x,y) \in M \times N$, if $\{v_1,\dots,v_m\}$ is a positive basis for $T_x(M)$ and $\{w_1,\dots,w_n\}$ is a positive basis for $T_y(N)$, then $\{v_1,\dots,v_m,w_1,\dots,w_n\}$ is a positive basis for $T_{(x,y)}(M \times N)$.

*Example* 2.5. Take a circle $\mathbb{S}^1$ with the usual counterclockwise orientation and the interval $I \coloneqq [0,1]$ with its standard orientation. The products $\mathbb{S}^1 \times I$ and $I \times \mathbb{S}^1$ then have opposite orientations. We hence need to be careful when choosing the order of such factors.




$\mathbb{S}^1 \times I$

$\{s,e\}$ positive basis for $T_x(\mathbb{S}^1 \times I)$




$I \times \mathbb{S}^1$

$\{e,s\}$ positive basis for $T_x(I \times \mathbb{S}^1)$

**Definition 2.6.** Let $M$ be a manifold with boundary and $p \in \partial M$. A vector $v \in T_pM \setminus T_p(\partial M)$ is *inward-pointing* if for some $\varepsilon > 0$ there exists a smooth curve $\gamma\colon [0,\varepsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$. A vector $v \in T_pM \setminus T_p(\partial M)$ is *outward-pointing* if there exists such a curve whose domain is $(-\varepsilon, 0]$.

By considering some smooth chart $(U \ni p, \varphi = (x^i))$, we then have that the inward-pointing vectors in $T_pM$ are exactly the ones with $x^n > 0$ and the outward-pointing ones are those for which $x^n < 0$. To avoid confusion, we remember that this correspondence depends on the space $\mathbb{H}^n$ we chose when defining manifolds with a boundary, in this case $\mathbb{H}^n = \{x^n \geq 0\}$.

**Definition 2.7 (The induced orientation on a boundary).** Let $M$ be an oriented manifold with boundary. Its boundary $\partial M$ inherits a canonical orientation[1], defined as follows. Consider an outward pointing vector $n \in T_xM \setminus T_x(\partial M)$; for any basis $\{t_1, \dots, t_n\}$ of $T_x(\partial M)$ we say it to be a *positive* (resp. *negative*) if $\{n, t_1, \dots, t_n\}$ is a positive (resp. negative) basis for $T_xM$.

*Example* 2.8. Following such a convention, the oriented cylinders above will induce opposite orientations on the two components of the boundary.


We are finally ready to give the following definition.

**Definition 2.9 (Oriented bordisms).** Let $\Sigma_0$, $\Sigma_1$ be two oriented closed manifolds of dimension $n-1$. An *$n$-dimensional oriented bordism* from $\Sigma_0$ to $\Sigma_1$ is a triple $(M, i_0, i_1)$ consisting of

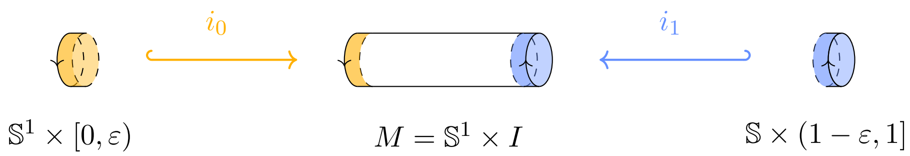
- an $n$-dimensional oriented manifold $M$ with boundary,
- two orientation-preserving embeddings

$$i_0 \colon \Sigma_0 \times [0, \varepsilon) \to M \qquad i_1 \colon \Sigma_1 \times (1-\varepsilon, 1] \to M$$

  definining an *in-boundary* $(\partial M)_0 := i_0(\Sigma_0, 0)$ and an *out-boundary* $(\partial M)_1 := i_1(\Sigma_1, 1)$, such that $\partial M = (\partial M)_0 \sqcup (\partial M)_1$.

*Example* 2.10. Let us draw an example of a 2-dimensional bordism $B \colon \mathbb{S}^1 \to \mathbb{S}^1$. We consider a 2-dimensional oriented cylinder $M = \mathbb{S}^1 \times I$ and the corresponding embeddings $i_0$, $i_1$, defining its orientation as a bordism.

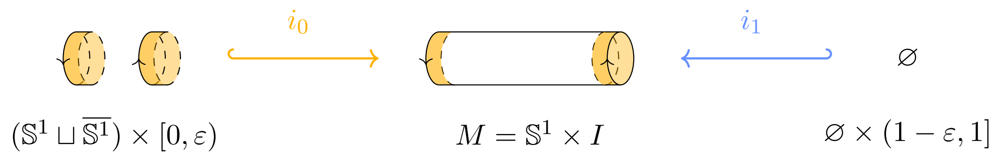
Looking at the first picture in Example 2.8, we notice how to define a bordism from $\mathbb{S}^1$ to itself, the cylinder we are considering has boundary equal to $\partial M = \mathbb{S}^1 \sqcup (-\mathbb{S}^1)$.




[1]This is just a matter of convention, which we refer to as *outward normal first*. Equivalently, one could define the induced orientation using the opposite convention, called *outward normal last*.

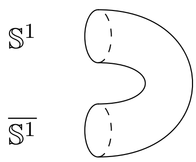
The figure shows how the choice of intervals we made in the definition assures that the inclusion maps are both orientation-preserving.

We can also see that the choice of in-boundaries and out-boundaries does not depend on the orientation of the manifold $M$, but on the ones of the copies of $\mathbb{S}^1$. For example, we could consider the same manifold $M$ and equip it with two in-boundaries and zero out-boundaries, as depicted below. In this case $i_0$ will be defined as the obvious inclusion on the component $\mathbb{S}^1 \times [0, \varepsilon)$ and as $i_0(x, t) = (x, 1 - t)$ on the component $\overline{\mathbb{S}^1} \times [0, \varepsilon)$.




*Remark* 2.11. These examples let us notice why we asked for both the $n$-dimensional manifold and the $(n-1)$-dimensional manifolds to be oriented. Indeed, by specifying orientations on all manifolds, we are, in a way, fixing an in/out boundary distinction. For example, given the manifold $M = \mathbb{S}^1 \times I$ equipped with the orientation above, it would not be possible to define a bordism from $\mathbb{S}^1$ to $\overline{\mathbb{S}^1}$, since we would need to have $\partial M = \mathbb{S}^1 \sqcup (-\overline{\mathbb{S}^1}) = \mathbb{S}^1 \sqcup \mathbb{S}^1$.

From now on, we will draw in-boundaries on the left and out-boundaries on the right, to avoid further confusion. Thus, the previous cylinder representing a bordism $M \colon \mathbb{S}^1 \sqcup \overline{\mathbb{S}^1} \to \varnothing$ will be depicted as follows:




The role of the embeddings $i_0$, $i_1$ in the above definition is to put some sort of "collar" around the in-boundary and the out-boundary. Let us state this properly.

**Definition 2.12 (Collars [Lee03]).** Let $M$ be a manifold with boundary. A neighbourhood of $\partial M$ is called a *collar neighbourhood* if it is the image of a smooth embedding $\partial M \times [0, \varepsilon) \to M$ that restricts to the identification $\partial M \times \{0\} \to \partial M$.

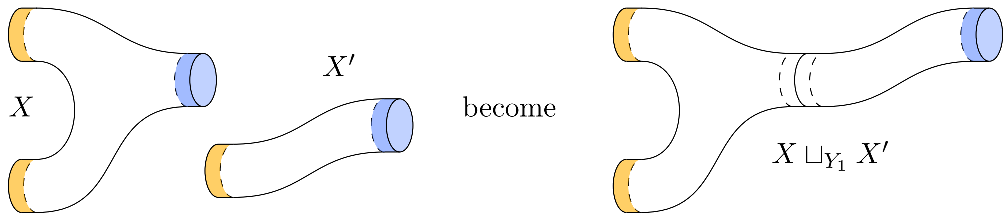
**Theorem 2.13 (Collar neighbourhood theorem).** *Let $M$ be a smooth manifold with nonempty boundary. Then $\partial M$ has a collar neighbourhood.*

This guarantees that the notion of an oriented bordism is well defined.

**Theorem 2.14 (Gluing smooth manifolds along their boundaries).** *Let $M$, $N$ be two manifolds with nonempty boundaries $\partial M$, $\partial N$. Suppose we have a diffeomorphism between the two boundaries and consider the topological pushout $M \sqcup_{\partial M \cong \partial N} N$. Such a topological manifold then has a smooth structure, compatible* *with the smooth structures on $M$ and $N$. If $M$, $N$ are both compact, $M \sqcup_{\partial M \cong \partial N} N$ is compact. If $M$, $N$ are both connected, $M \sqcup_{\partial M \cong \partial N} N$ is connected.*

*Remark* 2.15. While we do not give a formal proof of such a result, we observe that the smooth structure on the topological pushout is not unique and relies on the choice of collars. Our definition of bordisms with fixed collars, taken from [MS25] and [Fre19], is given precisely to specify this choice, making the glueing to be exactly the pushout in the proper category (which we can imagine as the category of "manifolds with defined collars"). However, different choices of collars give rise to diffeomorphic smooth structures.

*Example* 2.16. We illustrate such a procedure with the following example, considering bordisms of dimension 2. We take a bordism $(X, i_0, i_1)$ from $\Sigma_0 = \mathbb{S}^1 \sqcup \mathbb{S}^1$ to $\Sigma_1 = \mathbb{S}^1$ and a bordism $(X', i'_0, i'_1)$ from $\Sigma_1 = \mathbb{S}^1$ to $\Sigma_2 = \mathbb{S}^1$ and glue them along the common boundary $\Sigma_1$. Since our goal is to define a sort of "composition" we will always glue an out-boundary to an in-boundary.




More is actually true when considering the glueing of two oriented bordisms.

**Theorem 2.17.** *Let $\Sigma$ be an out-boundary of a bordism $M_0$ and an in-boundary of a bordism $M_1$, and consider $M_0 \sqcup_\Sigma M_1$ the pushout through $\Sigma$ of the two topological manifolds. Let $\alpha$, $\beta$ be two smooth structures on $M_0 \sqcup_\Sigma M_1$, which both induce the original smooth structures on $M_0$ and $M_1$ (via pullback along the inclusion maps). Then there is a diffeomorphism $\phi\colon (M_0 \sqcup_\Sigma M_1, \alpha) \to (M_0 \sqcup_\Sigma M_1, \beta)$ such that its restriction on $\Sigma$ is the identity* $\mathrm{id}_\Sigma$.

## 2.2 A category of oriented boridsms

Our goal is now to construct a category of oriented $n$-dimensional bordisms. The intuitive idea behind it is to take closed oriented $(n-1)$-dimensional manifolds as objects and oriented bordisms between them as morphisms. The proper definition, however, requires some more refinement. We begin by addressing some technical issues. While equipping each bordism with an explicit choice of a collar allows us to properly define a glueing, this approach fails when trying to define a strict identity morphism. Given a closed oriented $(n-1)$-dimensional manifold $\Sigma$, a candidate for $\mathrm{id}_\Sigma$ is given by the cylinder $\Sigma \times [0, 1]$ with some choice of collars. However, glueing it to a bordism $M$ with one of the boundaries equal to $\Sigma$ only gives us a manifold *diffeomorphic* to $M$. By considering the underlying topological spaces, we easily understand that the only way to define a strict identity in this setting is to consider

$\Sigma$ itself as a bordism, but this would not satisfy our definition, which requires it to be an $n$-dimensional manifold. Finally, we recall how, when glueing manifolds, different choices of collars give rise to different but diffeomorphic smooth structures. This motivates the following framework: we define the bordism category by taking as morphisms bordisms "up to diffeomorphism".

**Definition 2.18 (Equivalent bordisms).** Let $(M, i_0, i_1)$, $(M', i'_0, i'_1)$ be two oriented bordisms, both from $\Sigma_0$ to $\Sigma_1$. We say $M$ and $M'$ are *equivalent bordisms* if there exists an orientation-preserving diffeomorphism $\psi\colon M \to M'$ making the following diagram commute.

$$\begin{tikzcd}
& M \arrow[dd, dashed, "\psi"] & \\
\Sigma_0 \times \{0\} \cong \Sigma_0 \arrow[ur, "i_0"] \arrow[dr, "i'_0"'] & & \Sigma_1 \cong \Sigma_1 \times \{1\} \arrow[ul, "i_1"'] \arrow[dl, "i'_1"] \\
& M' &
\end{tikzcd}$$

This clearly defines an equivalence relation between bordisms. We can then consider the equivalence classes $[(M, i_0, i_1)]$.

*Remark* 2.19. In defining the equivalence class, we are in a way forgetting the collar choice we gave in the definition of a bordism. Indeed, when considering two bordisms $(M, i_0, i_1)$, $(M, j_0, j_1)$, we are asking for

$$i_0(\Sigma_0, 0) = (\partial M)_0 = j_0(\Sigma_0, 0) \qquad j_0(\Sigma_1, 1) = (\partial M)_1 = j_1(\Sigma_1, 1)$$

hence making the diagram commute by just choosing the identity map on $M$. From now on, when referring to an equivalence class of bordisms, we can forget the collar data and only consider the diffeomorphisms $\Sigma_0 \cong (\partial M)_0$ and $\Sigma_1 \cong (\partial M)_1$. We'll then just say that $M$ is a bordism from $\Sigma_0$ to $\Sigma_1$, without specifying further information.

**Lemma 2.20 (Composition of cobordism classes).** *Given a bordism $M$ from $\Sigma_0$ to $\Sigma_1$ and a bordism $N$ from $\Sigma_1$ to $\Sigma_2$, we define their composition $MN$ from $\Sigma_0$ to $\Sigma_2$ as follows: we take any representative from each class, glue them and consider the equivalence class of the resulting glueing. This composition is well defined.*

*Proof.* Take the bordisms $(M, i_0, i_1)$, $(M', i'_0, i'_1)$ from the first equivalence class and $(N, i_0, i_1)$, $(N', i'_0, i'_1)$ from the second. This means we have two diffeomorphisms $\psi_M$, $\psi_N$ such that:

$$\begin{tikzcd}
& M \arrow[dd, "\psi_0"] & & N \arrow[dd, "\psi_1"] & \\
\Sigma_0 \arrow[ur] \arrow[dr] & & \Sigma_1 \arrow[ul] \arrow[dl] \arrow[ur] \arrow[dr] & & \Sigma_2 \arrow[ul] \arrow[dl] \\
& M' & & N' &
\end{tikzcd}$$

We can then consider the gluings $(MN, i_0, j_1)$ and $(M'N', i'_0, j'_1)$. By taking the pushout of the two diffeomorphisms in the category of continuous maps, we get a homeomorphism $\psi\colon MN \to M'N'$. Through such a homeomorphism, we can define a new smooth structure on $M'N'$, which by Theorem 2.17 is diffeomorphic to the one induced by the glueing. □

**Definition 2.21 (A category of oriented bordisms).** The category of oriented n-dimensional bordisms $\mathbf{Bord}(n)$ is defined as follows.

- The objects are closed oriented $(n-1)$-dimensional manifolds
- For any $\Sigma_0, \Sigma_1 \in \mathrm{Obj}(\mathbf{Bord}(n))$, morphisms are the equivalence classes of bordisms $M\colon \Sigma_0 \to \Sigma_1$
- Composition of morphisms is obtained by glueing
- For each object $\Sigma$, the identity map $\mathrm{id}_\Sigma$ is given by the bordism $\Sigma \times [0,1]$

**(Disjoint union of bordisms).** Given two bordisms $M\colon \Sigma_0 \to \Sigma_1$ and $N\colon \Sigma'_0 \to \Sigma'_1$, their disjoint union $M \amalg N$ naturally defines a bordism from $\Sigma_0 \amalg \Sigma'_0$ to $\Sigma_1 \amalg \Sigma'_1$. This operation is precisely the coproduct in the category of smooth manifolds, equipped with the unique orientation agreeing with the ones on $M$ and $N$.

**Proposition 2.22.** *The category* $\boldsymbol{Bord}(n)$ *has a monoidal structure given by the disjoint union of manifolds (and the initial object, being the empty manifold* $\varnothing$*).*

*Proof.* Since the disjoint union is the coproduct in the category $\mathbf{Bord}(n)$, the result follows from Proposition 1.4. □

**Proposition 2.23 (Embedding diffeomorphisms in the bordism category).** *Any diffeomorphism of* $(n-1)$*-dimensional manifolds* $\Sigma_0$, $\Sigma_1$ *define an equivalence class of (invertible) bordisms* $M\colon \Sigma_0 \to \Sigma_1$.

*Proof.* Let $\Sigma$ be a closed (oriented) $(n-1)$-dimensional manifold and $f\colon \Sigma \to \Sigma$ a diffeomorphism. By considering $X = \Sigma \times [0,1]$ equipped with the usual orientation and the obvious inclusions $i_0\colon \Sigma \cong \Sigma \times \{0\} \hookrightarrow X$ and $i_1\colon \Sigma \cong \Sigma \times \{1\} \hookrightarrow X$ we define a bordism (or better, an equivalence class of bordisms) $X\colon \Sigma \to \Sigma$. The same procedure can be applied when considering two diffeomorphic $(n-1)$-dimensional manifolds and the diffeomorphism $f\colon \Sigma_0 \to \Sigma_1$ between them. We can indeed consider the $n$-dimensional manifold $X = \Sigma_0 \times I$ and the inclusions $i_0\colon \Sigma_0 \to X$, $i_1\colon \Sigma_1 \cong \Sigma_0 \to X$ as in the previous case. □

We also notice that given two composable diffeomorphisms $f\colon \Sigma_0 \to \Sigma_1$, $g\colon \Sigma_1 \to \Sigma_2$, we have the corresponding bordisms $X_f$, $X_g$ and their composition $X_f X_g$.

**Proposition 2.24.** *Two diffeomorphisms $f\colon \Sigma_0 \to \Sigma_1$, $g\colon \Sigma_0 \to \Sigma_1$ give rise to the same bordism class if and only if they are smoothly homotopic.*

*Proof.* Recall that $f\colon \Sigma_0 \to \Sigma_1$, $g\colon \Sigma_0 \to \Sigma_1$ are (smoothly) homotopic if there exists a smooth map $\phi\colon \Sigma_0 \times [0,1] \to \Sigma_1$ such that $\phi(x,0) = f(x)$ and $\phi(x,1) = g(x)$. Equivalently, we are asking for the following diagram to be commutative.

$$\begin{tikzcd}
\Sigma_0 \arrow[r] \arrow[dr, "f"'] & \Sigma_0 \times I \arrow[d, "\phi"] & \Sigma_0 \arrow[l] \arrow[dl, "g"] \\
& \Sigma_1 &
\end{tikzcd}$$

By composition with the inclusion $\Sigma_1 \hookrightarrow \Sigma_1 \times I$ and with $g^{-1}\colon \Sigma_1 \to \Sigma_0$ we have

$$\begin{tikzcd}
\Sigma_0 \arrow[r] \arrow[dr, "f"'] & \Sigma_0 \times I \arrow[d, "\phi"] & \Sigma_0 \arrow[l] \arrow[dl, "g"] & \Sigma_1 \arrow[l, "g^{-1}"] \\
& \Sigma_1 \arrow[d] & & \\
& \Sigma_1 \times I & &
\end{tikzcd}$$

which gives the following commutative diagram.

$$\begin{tikzcd}
& \Sigma_0 \times I \arrow[dd, "\phi"] & \\
\Sigma_0 \arrow[ur, "\mathrm{id}"] \arrow[dr, "f"'] & & \Sigma_1 \arrow[ul, "g^{-1}"'] \arrow[dl, "\mathrm{id}"] \\
& \Sigma_1 \times I &
\end{tikzcd}$$

Even if the naming of the arrows is a bit sloppy, this defines the equivalence between the two bordisms. To prove the converse we take two equivalent bordisms $\Sigma_0 \times I$ and $\Sigma_1 \times I$ and the orientation-preserving diffeomorphism $\psi\colon \Sigma_0 \times I \to \Sigma_1 \times I$.

$$\begin{tikzcd}
& \Sigma_0 \times I \arrow[dd, dashed, "\psi"] & \\
\Sigma_0 \arrow[ur, "i_0"] \arrow[dr, "i_0'"'] & & \Sigma_1 \arrow[ul, "i_1"'] \arrow[dl, "i_1'"] \\
& \Sigma_1 \times I &
\end{tikzcd}$$

By composition with the projection $\Sigma_1 \times I \to \Sigma_1$, we get the diagram defining the homotopy between $f$ and $g$. $\square$

These considerations allow us to define the following class of bordisms, ensuring it is not equal to the one arising from $\mathrm{id} \sqcup \mathrm{id}$.

**(The twist bordism).** The twist diffeomorphism of manifolds $\sigma\colon \Sigma \amalg \Sigma' \to \Sigma' \amalg \Sigma$ defines a cobordism in $\mathbf{Bord}(n)$ which we'll denote as $T_{\Sigma,\Sigma'}\colon \Sigma \amalg \Sigma' \to \Sigma' \amalg \Sigma$.

**Proposition 2.25.** *The category* $\boldsymbol{Bord}(n)$ *has a symmetric monoidal structure* $(\boldsymbol{Bord}(n), \amalg, \varnothing, T)$

## 2.3 Topological Quantum Field Theories

**Definition 2.26.** An *$n$-dimensional topological quantum field theory* is a symmetric monoidal functor from $(\mathbf{Bord}(n), \amalg, \varnothing, T)$ to $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \sigma)$

Historically, the first explicit axiomatization of TQFTs was given by Atiyah in 1988 [Ati88]. Note that in Atiyah we do not yet encounter the explicit notion of bordisms as we defined them.

Although the interest in the topic mainly comes from physics, we will, as mentioned in the introduction, ignore that perspective entirely.

We here give a slightly refined version (which can be found in [Koc03]) of the original axioms, stated in terms of bordisms.

**Definition 2.27 (Axiomatization of a TQFT).** An *$n$-dimensional topological quantum field theory over a field* $\Bbbk$ consists of:

- a $\Bbbk$-vector space $Z(\Sigma)$ associated to each closed oriented $(n-1)$-dimensional manifold $\Sigma$
- a $\Bbbk$-linear map $Z(M)\colon Z(\Sigma_0) \to Z(\Sigma_1)$ associated to each $n$-dimensional bordism $M$ from $\Sigma_0$ to $\Sigma_1$

satisfying the following axioms:

A1 Two equivalent bordisms $M \cong N$ have the same image through $Z$, namely $Z(M) = Z(N)$

A2 The cylinder bordism $M = \Sigma \times I\colon \Sigma \to \Sigma$ is mapped to the identity

$$Z(\Sigma \times I) = \mathrm{id}_{Z(\Sigma)}\colon Z(\Sigma) \to Z(\Sigma)$$

A3 The glueing of two bordisms is mapped to the composition of their images, meaning that for two composable morphisms $M$, $N$, we have

$$Z(MN) = Z(N) \circ Z(M)$$

A4 The disjoint union of two bordisms is mapped to the tensor product of their images, meaning

$$Z(M \amalg N) = Z(M) \otimes Z(N)$$

A5 The empty manifold $\varnothing$ is mapped to the ground field $\Bbbk$. It follows from AA2 that the empty bordism $\varnothing \times I$ is sent to the identity $\mathrm{id}_{\Bbbk}$.

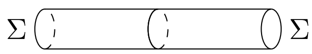
*Remark* 2.28. It seems natural to interpret these axioms in the language of category theory. Indeed, Axiom A1 well defines a map from $\mathbf{Bord}(n)$ to $\mathbf{Vect}_{\Bbbk}$ and Axioms A2 and A3 guarantee such a map is really a functor. Axioms A4 and A5 preserve the monoidal structure. To get a *symmetric* monoidal functor, as defined in 2.26, we would have to add a sixth axiom:

(A6) The twist bordism is mapped to the twist map of vector spaces, meaning

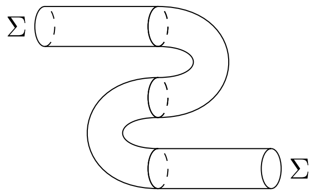
$$Z(T_{M,N}) = \sigma_{Z(M),Z(N)}$$

**(A brief digression on dualizability).** We now leave some space to discuss an important property regarding TQFTs, starting with what is known as the *snake decomposition of a cylinder*.

Take any closed $(n-1)$-dimensional manifold $\Sigma$ and consider the correspondig bordism $\Sigma \times I$. There are various ways in which we can decompose it. We can just cut it in the center




or we can bend it before cutting it, obtaining something like



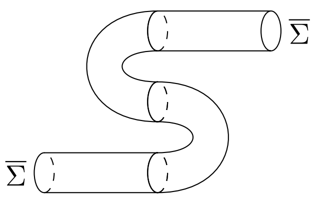

Notice how, after the cutting, the middle copy of $\Sigma$ is equipped with its opposite orientation for reasons we discussed in Example 2.10. We will denote it as $\overline{\Sigma}$. This type of decomposition is called the *snake decomposition* and can be written as a composition of arrows as

$$\Sigma \cong \Sigma \sqcup \varnothing \xrightarrow{\mathrm{id}_\Sigma \sqcup H} \Sigma \sqcup \overline{\Sigma} \sqcup \Sigma \xrightarrow{N \sqcup \mathrm{id}_\Sigma} \varnothing \sqcup \Sigma \cong \Sigma$$

where $N\colon \Sigma \sqcup \overline{\Sigma} \to \varnothing$ and $H\colon \varnothing \to \overline{\Sigma} \sqcup \Sigma$ are the "bent cylinders". We can then state that:

$$\mathrm{id}_\Sigma \cong (N \sqcup \mathrm{id}_\Sigma) \circ (id_\Sigma \sqcup H)$$

The same is true when taking the dual snake (i.e. the one obtained from a bordism $\overline{M}\colon \overline{\Sigma} \to \overline{\Sigma}$). In that case we'd have that $\mathrm{id}_{\overline{\Sigma}}$ is isomorphic to:




$$(\mathrm{id}_{\overline{\Sigma}} \sqcup N) \circ (H \sqcup \mathrm{id}_{\overline{\Sigma}})\colon \overline{\Sigma} \to \overline{\Sigma} \sqcup \Sigma \sqcup \overline{\Sigma} \to \overline{\Sigma}$$

An interesting result appears when evaluating a TQFT on these bordisms. Considering any $Z\colon \mathbf{Bord}(n) \to \mathbf{Vect}_{\Bbbk}$ and naming $V = Z(\Sigma)$, $\overline{V} = Z(\overline{\Sigma})$ we have

$$Z(\mathrm{id}_\Sigma) = \mathrm{id}_V \colon V \to V \qquad Z(N) = \beta\colon V \otimes \overline{V} \to \Bbbk \qquad Z(H) = \gamma\colon \Bbbk \to \overline{V} \otimes V$$

Through functoriality of $Z$, the isomorphisms above define the following linear maps:

$$\mathrm{id}_V = (\beta \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \gamma)\colon V \cong V \otimes \Bbbk \to V \otimes \overline{V} \otimes V \to \Bbbk \otimes V \cong V$$

$$\mathrm{id}_{\overline{V}} = (\mathrm{id}_{\overline{V}} \otimes \beta) \circ (\gamma \otimes \mathrm{id}_{\overline{V}})\colon \overline{V} \cong \Bbbk \otimes \overline{V} \to \overline{V} \otimes V \otimes \overline{V} \to \overline{V} \otimes \Bbbk \cong \overline{V}$$

with which we can state and prove the following proposition.

**Proposition 2.29.** *The image vector spaces in a TQFT are necessarily of finite dimension.*

*Proof.* Let $\gamma(1_{\Bbbk}) = \sum_{i=1}^n \overline{v}_i \otimes v_i$ for $\overline{v}_i \in \overline{V}$ and $v_i \in V$. Evaluating the composition $(\beta \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \gamma)$ on a vector $v \in V$ gives:

$$v \mapsto v \otimes \sum_{i=1}^n \overline{v}_i \otimes v_i \mapsto \sum_{i=1}^n \beta(v \otimes \overline{v}_i) v_i$$

Since this composition is equal to $\mathrm{id}_V$ we have, for every $v \in V$, the equivalence $v = \sum_{i=1}^n \beta(v \otimes \overline{v}_i) v_i$. This is equal to saying that every $v \in V$ can be written as a $\Bbbk$-linear combination of $\{v_1, \ldots, v_n\}$, hence $V$ is a vector space of finite dimension. By evaluating the other composition in $\overline{v}$ and applying the same argument, we conclude that $\overline{V}$ is also finite-dimensional. □

These maps further imply a canonical duality between the two spaces. One can prove that $\overline{V}$ can be identified with the dual of $V$.

**Proposition 2.30.** *Let $\Sigma$ be a closed $(n-1)$-manifold and $Z\colon$ **Bord**$(n) \to \mathbf{Vect}_{\Bbbk}$ a TQFT. Let $V = Z(\Sigma)$ and $\overline{V} = Z(\overline{\Sigma})$. Then:*

1. *$\overline{V}$ is canonically isomorphic to $V^*$*
2. *$V$ is canonically isomorphic to $\overline{V}^*$*

We omit the proof of such a result. Instead, we notice how in Atiyah's original axiomatization, this was an explicit axiom:

(A7) The image vector space $V$ of a closed manifold $\Sigma$ comes equipped with a nondegenerate pairing with $\overline{V} = Z(\overline{\Sigma})$

These observations are fundamental for understanding the core structure of the category we are working within. Indeed, they are a concrete instance of the categorical concept of *dualizability*, the main definitions of which where given in Section 1.3.

We now give a couple of examples of some simple TQFTs.

Example 2.31 (A trivial TQFT). We can define a *trivial TQFT* by assigning a field $\Bbbk$ to every $(n-1)$-dimensional closed manifold and $\mathrm{id}_{\Bbbk}$ to every $n$-dimensional bordism.

Example 2.32 (A 2-dimensional TQFT computing the genus). Consider a $\Bbbk$-vector space $V = \langle v_1, v_2 \rangle$ generated by two vectors. We can define a 2-dimensional TQFT $Z\colon \mathbf{Bord}(2) \to \mathbf{Vect}_{\Bbbk}$ by the following assigments:

The circle $\mathbb{S}^1$ is sent to the vector space $V$.

$Z($  $)\colon \Bbbk \to V$

$$1_{\Bbbk} \mapsto v_1$$

$Z($  $)\colon V \to \Bbbk$

$$v_0 \mapsto 0$$
$$v_1 \mapsto 1$$

$Z($  $)\colon V \otimes V \to V$

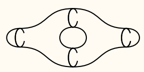
$$v_0 \otimes v_0 \mapsto v_1$$
$$v_0 \otimes v_1 \mapsto v_0$$
$$v_1 \otimes v_0 \mapsto v_0$$
$$v_1 \otimes v_1 \mapsto v_1$$

$Z($  $)\colon V \to V \otimes V$

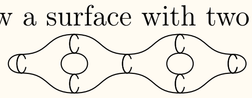
$$v_0 \mapsto v_0 \otimes v_1 + v_1 \otimes v_0$$
$$v_1 \mapsto v_1 \otimes v_1 + v_0 \otimes v_0$$

As we will prove in the end, defining $Z$ on these four bordisms is more than enough to get a complete definition of the TQFT. Now, let us understand through some explicit computations what this functor has to do with the genus of surfaces. Take for example the sphere $\mathbb{S}^2$, which can be seen as . Then we have

$$Z(\mathbb{S}^2)\colon 1_{\Bbbk} \mapsto v_1 \mapsto 1\,(=2^0)$$

Similarly, a torus $\mathbb{T}$ can be seen as , which means we'll compute

$$Z(\mathbb{T})\colon 1_{\Bbbk} \mapsto v_1 \mapsto v_0 \otimes v_0 + v_1 \otimes v_1 \mapsto v_1 + v_1 \mapsto 2\,(=2^1)$$

Consider now a surface with two holes $2\mathbb{T}^2$ which we can draw by glueing the above bordisms as . We get

$$Z(2\mathbb{T}^2)\colon 1_{\Bbbk} \mapsto v_1 \mapsto v_0 \otimes v_0 + v_1 \otimes v_1 \mapsto 2v_1 \mapsto 2(v_0 \otimes v_0 + v_1 \otimes v_1) \mapsto 4v_1 \mapsto 4\,(=2^2)$$

We see how we have defined a mapping $Z$ that assigns to each closed two-dimensional surface $\Sigma$ a linear map $Z(\Sigma)\colon \Bbbk \to \Bbbk$ that, evaluated at $1_{\Bbbk}$, returns $2^g$, where $g$ is the genus of the surface $\Sigma$.

### 2.3.1 A category of $n$TQFTs

Remembering how we ended our first chapter and having just seen how $n$TQFT correspond to symmetric monoidal functors from $(\mathbf{Bord}(n), \amalg, \varnothing, T)$ to $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \sigma)$, we can define a category of $n$-dimensional topological quantum field theories as follows.

**Definition 2.33.** The category of $n$-dimensional TQFTs is the functor category

$$\mathbf{nTQFT}_{\Bbbk} := \mathbf{SymMonCat}(\mathbf{Bord}(n), \mathbf{Vect}_{\Bbbk})$$

Arrows in such a category will then be monoidal natural transformations between such functors. More precisely, given two $n$-dimensional TQFTs (i.e two symmetric monoidal functors $Z\colon \mathbf{Bord}(n) \to \mathbf{Vect}_{\Bbbk}$, $Z'\colon \mathbf{Bord}(n) \to \mathbf{Vect}_{\Bbbk}$), a natural transformation between them is then defined by $\Bbbk$-linear maps $Z(\Sigma) \to Z'(\Sigma)$ for $\Sigma$ object in $\mathbf{Bord}(n)$.

# Chapter 3

# The two dimensional case

We now consider the category $\mathbf{Bord}(2)$. Recall that its objects are closed oriented 1-dimensional manifolds and its arrows are the equivalence classes of oriented 2-dimensional bordisms. Our goal is to describe its elementary pieces and how they interact with each other. To do so, we are going to define a *normal form* of such bordisms, which will facilitate their comparison.

We begin by adressing the existence of a bordism between closed oriented 1-dimensional manifolds. As for the topological underlying structure, it is a known result that all closed connected one dimensional manifolds are diffeomorphic to a circle $\mathbb{S}^1$. Any object in $\mathbf{Bord}(2)$ is then diffeomorphic to a disjoint union of circles (each with its own orientation). Given two 1-dimensional manifolds, $\Sigma_0$ with $m$ connected components and $\Sigma_1$ with $n$ connected components, we can define a bordism between the two by considering the 2-dimensional manifold $M$ comprising of $m+n$ "spheres with a hole".

**Proposition 3.1 (Existence of 2D Bordisms).** *For any two closed oriented 1-dimensional manifolds, there exists an oriented bordisms between them.*

*Remark* 3.2. While this result may seem straightforward in dimension 2, it is not trivial in general. For example, when considering oriented 0-manifolds, there exists a bordisms between two of them if and only if the "sums of the signed points" are equal between the two.

As defined, the category $\mathbf{Bord}(2)$ contains many diffeomorphic but distinct objects, which become redundant when studying its structure. To get a more essential representation, we take its skeleton.

**Definition 3.3 (Skeletons of a category).** Let $\mathcal{C}$ be a category. A *skeleton* of $\mathcal{C}$ is any full subcategory $\mathcal{S}$ such that each object of $\mathcal{C}$ is isomorphic in $\mathcal{C}$ to *exactly* one object of $\mathcal{S}$.

**(Properties of skeletons).** Any two skeletons $\mathcal{S}$, $\mathcal{S}'$ of the same category $\mathcal{C}$ are always isomorphic. We'll then speak of *the* skeleton of a category. The inclusion $\mathcal{S} \hookrightarrow \mathcal{C}$ defines an equivalence of categories. We have that two categories are equivalent if and only if their skeletons are isomorphic.

In the case of **Bord**$(2)$ we firstly need to understand how the isomorphism classes of its objects are defined.

**(Inverse bordism).** Let $M\colon \Sigma_0 \to \Sigma_1$ be an $n$-dimensional oriented bordism. The bordism $M^{-1}\colon \Sigma_1 \to \Sigma_0$ is an inverse to $M$ if $MM^{-1}$ is the identity bordism on $\Sigma_0$ and $M^{-1}M$ is the identity bordism on $\Sigma_1$.

**Lemma 3.4.** *Let $\Sigma_0$, $\Sigma_1$ be two closed oriented 1-dimensional manifolds. We have that $\Sigma_0$ and $\Sigma_1$ are diffeomorphic if and only if there exists an invertible morphism between them.*

Again, we recall that every object of **Bord**$(2)$ is diffeomorphic to a disjoint union of circles, where each components carries its own orientation (either clockwise or counter-clockwise). However, given two copies of $\mathbb{S}^1$ with opposite orientations we can always define, by reflection, an orientation preserving diffeomorphism between them. This implies they belong to the same iso class. Therefore when defining the skeleton of **Bord**$(2)$, the orientation becomes irrelevant. The only invariant for an object is its number of connected components.

**Definition 3.5 (Skeleton of Bord(2)).** Let $(\mathbf{n})$ denote the disjoint union of $n$ copies of $\mathbb{S}^1$. Then the objects

$$\{(\mathbf{0}), (\mathbf{1}), (\mathbf{2}), (\mathbf{3}), \dots\} = \{\varnothing, \mathbb{S}^1, \mathbb{S}^1 \amalg \mathbb{S}^1, \mathbb{S}^1 \amalg \mathbb{S}^1 \amalg \mathbb{S}^1, \dots\}$$

together with all the possible morphisms between them, form a skeleton of **Bord**$(2)$. By abuse of notation, from now on, we we'll refer to this category as **Bord**$(2)$.

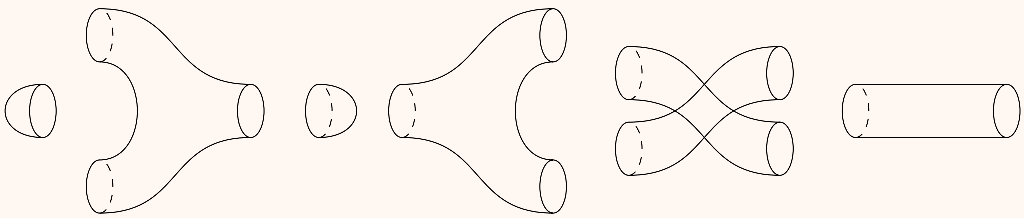
## 3.1 Generators

We now analyze the morphisms in detail.

**Definition 3.6 (Generators for a monoidal category).** A *generating set* for a monoidal category $(\mathcal{C}, \square, I)$ is a set of arrows $S \subset \mathrm{Arr}(\mathcal{C})$ such that every arrow in $\mathcal{C}$ can be obtained from the ones in $S$ by composition or through the monoidal product $\square$.

In the case of 2-dimensional bordisms, for example, the identity over the disjoint union of two circles ($\mathrm{id}_{(\mathbf{2})}$) is not a generator, since it decomposes as the disjoint union of two identities over a circle ($\mathrm{id}_{(\mathbf{1})} \amalg \mathrm{id}_{(\mathbf{1})}$). On the contrary, the twist bordism $T$ cannot be described as a disjoint union of identity bordisms. The exchange of components described by the twist makes it impossible to define a diffeomorphism to the identity on $\mathbb{S}^1 \amalg \mathbb{S}^1$.

**Theorem 3.7 (Generators of Bord(2)).** *The six bordisms (respectively: cap,* *pants, cocap, copants, twist and cylinder)*



*form a generating set for the monoidal category* $(\textbf{\textit{Bord}}(2), \amalg, \varnothing)$.

The proof of this theorem relies on Morse theory, for which we refer the reader to [Hir12, ch. 6 and 9]. However, in dimension 2, we can also appeal to the topological classification of surfaces, which provides a more elementary argument. To avoid losing the main focus, we will follow this second approach. In higher dimensions this is not possilble and we will need to rely on Morse theory.

**(A preliminary recollection: genus and Euler characteristic of a surface).** We recall that the *genus* of a compact, connected surface is intuitively the "number of holes". In the case of a surface *with* boundary we define its genus to be the genus of the closed surface obtained by "sewing" its boundary components. However, in our setting, the genus doesn't provide enough informations to be able to distinguish our surfaces (for example both the cylinder and the pair of pants will have genus 0); for this reason we consider the *Euler characteristic*. We hence recall that the Euler characteristic of a surface with boundary $M$ is

$$\chi(M) = 2 - 2g - p$$

where $g$ is the genus of $M$ and $p$ the number of connected components of its boundary.

We now recall the classical classification of surfaces without boundary.

*Two connected compact oriented surfaces without boundary are diffeomorphic if and only if they have the same genus (or equivalentely the same Euler characteristic).*

Since bordisms are surfaces *with* boundary we need an adapted version of the result.

**Lemma 3.8 (Topological classification of surfaces with boundary).** *Two connected compact oriented surfaces with oriented boundary are diffeomorphic if and only if they have the same genus (or equivalentely the same Euler characteristic), the same number of in-boundaries and the same number of out-boundaries.*

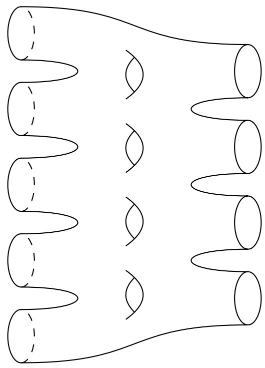
To keep track of such invariants we define the normal form in the following way.

**Definition 3.9 (Normal form of connected bordisms).** We define the normal form of a connected bordism $M\colon (\mathbf{m}) \to (\mathbf{n})$ with $g$ holes as the composition of

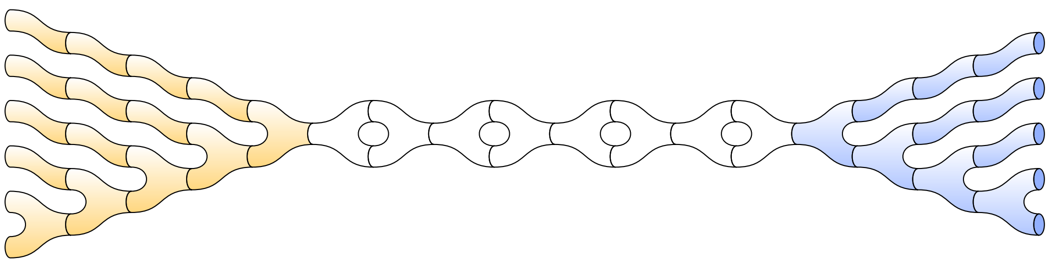
- an *in-part* $M_{in}\colon (\mathbf{m}) \to (\mathbf{1})$, consisting of $m-1$ copies of pair of pants . If $m = 0$, the in-part consists of a single cap .

- a *topological part* $M_{mid} \colon (\mathbf{1}) \to (\mathbf{1})$, consisting of $g$ pair of copants and $g$ pair of pants, arranged together to define $g$ holes .
- an *out-part* $M_{out} \colon (\mathbf{1}) \to (\mathbf{n})$, consisting of $n-1$ copies of pair of copants . If $n = 0$, the in-part consists of a single cocap .

To understand how we are supposed to attach such bordisms let's see a visual example. Consider a bordism with $m = 5$ in-boundary components, $n = 4$ out-boundary components and $g = 4$ holes.



We can then decompose it to obtain its normal form as:



The above definition leads to the following statement.

**Lemma 3.10.** *Every connected 2-dimensional bordism can be obtained through composition and disjoint union of the elementary bordisms , , , , .*

Let us now consider the case of non connected bordisms. While it is true that a non connected manifold is the disjoint union of connected surfaces, this fact alone is not enough to prove Theorem 3.7. The category of bordisms considers not only the topology of bordisms but also the permutations on their boundary components.

The main example of this fact can be seen by considering the twist. As a smooth manifold, the twist is the disjoint union of two cylinders. However, as a bordism, it is not isomorphic to the disjoint union of two identity morphisms since it flips the two boundaries. This problem can be solved by introducing permutations that let us untwist the bordisms. We now describe such procedure.

Let $M: (\mathbf{m}) \to (\mathbf{n})$ be a non connected bordism. For simplicity, assume $M$ has exactly two connected components, $M_1$ and $M_2$. The in-boundary $(\mathbf{m})$ will then be comprised of the in-boundaries of $M_1$ and $M_2$, not necessarily appearing in two distinct contiguous groups. We "reorder" such boundaries through a diffeomorphism $(\mathbf{m}) \to (\mathbf{m})$, which gives rise to a bordism $S: (\mathbf{m}) \to (\mathbf{m})$. By construction, the in-boundary of the composition $SM$ is exactly the disjoint union of the in-boundaries of its connected components $(SM)_1$, $(SM)_2$.

Applying the same method to the out-boundary we get a permutation bordism $T: (\mathbf{n}) \to (\mathbf{n})$ such that the composition $SMT: (\mathbf{m}) \to (\mathbf{n})$ is then exactly the disjoint union of the original connected components $M_1$ and $M_2$. The intial bordism $M$ then factors as $S^{-1}(SMT)T^{-1}$, leading to the following lemma.

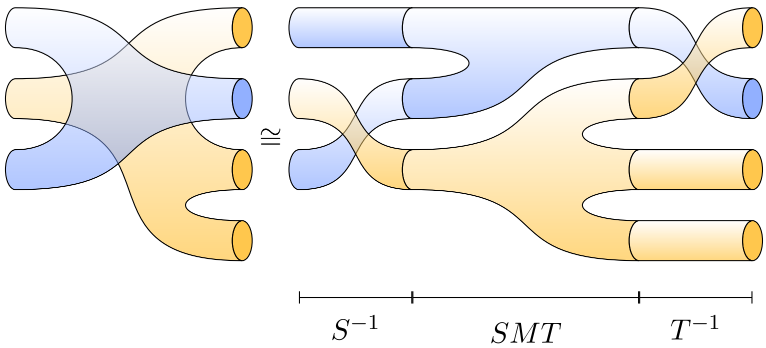
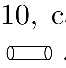



Figure 3.1: An illustration of the procedure used to prove Lemma 3.11. The two connected components $M_1$, $M_2$ are depicted in different colors.

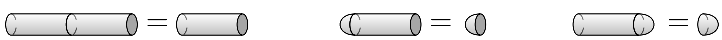
**Lemma 3.11.** *Every 2-dimensional bordism factors as a permutation bordism, followed by a disjoint union of connected bordisms, followed by a permutation bordism.*

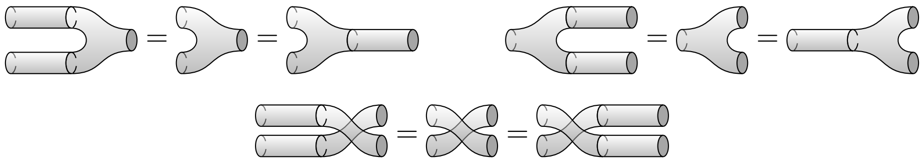
We are now ready to assemble these lemmas into a complete proof of Theorem 3.7. By Lemma 3.11, we can factor any bordism into two permutation bordisms and a disjoint union of connected bordism. Each of these connected components, by Lemma 3.10, can then be written in terms of the elementary bordisms , , , , . On the other hand, permutation bordisms can be obtained through composition and disjoint union of twist bordisms  (and, again, cylinders).

## 3.2 Relations

We descrive the interactions between the generators by giving a set of relations. Their validity follows from the Classification Lemma 3.8. Since the bordisms on both side of each relation have genus 0 and equal number of in and out boundaries, they are diffeomorphic hence equal in the **Bord**(2) category.

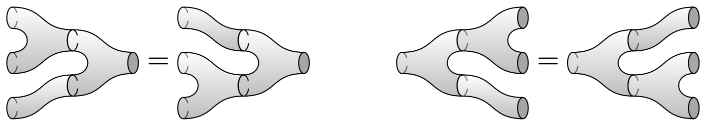
**(Identity relations).** Saying the cylinder is the identity bordism in dimension 2, means that composing it with any other bordism doesn't modify such bordism.





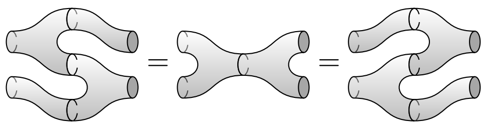
**(Handle cancellation).** Given a pair of pants (equivalentely a pair of copants), sewing a cap (equivalentely a cocap) on one of its legs, results in a cylinder.

**(Associativity and coassociativity).** When composing two pairs of pants (equivalentely pair of copants), it doesn't matter to which leg we attach the second pair.

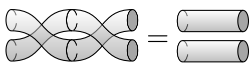


**(Commutativity and cocommutativity).** Attaching a twist to the legs of a pair of pants (equivalentely a pair of copants), gives a bordism diffeomorphic to the pair itself.

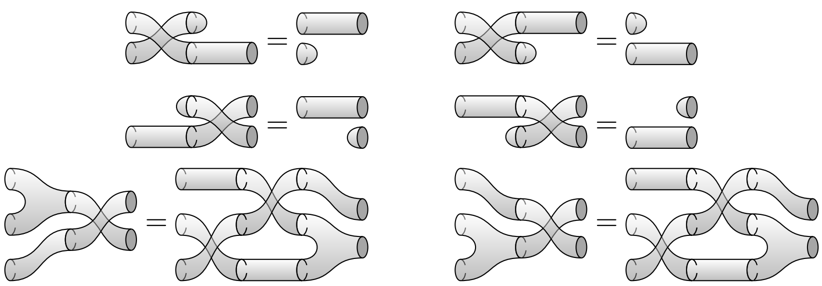
**(Frobenius relation).** When composing a pair of pants and a pair of copants, the following equivalences hold.

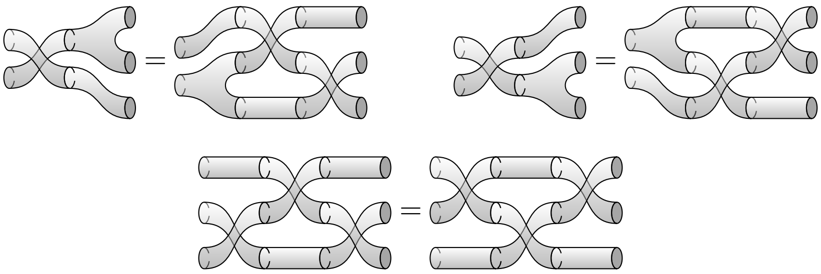


**(Inverse of the twist).** The twist bordism is its own inverse, meaning that composing two twists gives an identity bordism on (**2**).



**(Naturality of the twist).** For any pair of bordisms, the operations of applying the twist and taking the disjoint union commute, meaning that applying them in different order yields the same result.





To show the sufficiency of these relations we will describe a procedure that converts any connected bordism already decomposed in elementary pieces into its normal form. We proceed by induction on the number of twist bordisms, starting with the base case of zero twists.

Consider a decomposition of a connected surface $M$ with $m$ in-boundaries, $n$ out-boudaries and of genus $g$. Its Euler characteristic is given by

$$\chi(M) = 2 - 2g - m - n$$

Suppose such decomposition is comprised of $a$ pants , $b$ copants , $p$ caps  and $q$ cocaps . We can compute the Euler characteristic of such elementary pieces

$\chi($  $) = -1 = \chi($  $) \qquad \chi($  $) = 1 = \chi($  $)$

and by additivity of the Euler characteristic, state that $\chi(M) = p + q - a - b$ and obtain

$$2 - 2g - m - n = p + q - a - b$$

On the other hand we can sum the number of in boundaries and out boudaries to get

$$m = 2a + b + q \qquad n = a + 2b + p \quad \Rightarrow \quad a + q + n = b + p + m$$

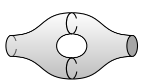
Combining the two equations and solving for $a$ and $b$ we get

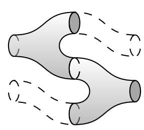
$$a = g + m - 1 + p \qquad b = g + n - 1 + q$$

Keeping in mind the definition of the normal form given in 3.9, we begin by describing how to move $m - 1$ copies of  to the left (to form the *in-part* of the normal form). To do so we use the above relations, depending on what we meet on the left of our pair of pants.

If we encounter a cap on one of the legs, the handle cancellation relation yelds an equivalence with a cylinder, which can hence be omitted. By assumption, this will happen exactly $p$ times, leaving us with only $g + m - 1$ copies of pants.

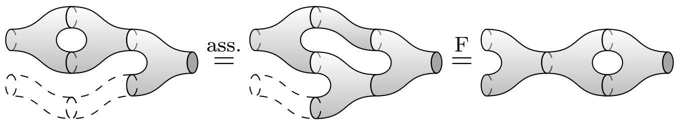
If instead we encounter a pair a copants, this can happen in two different ways:





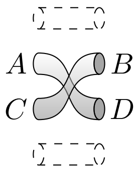
In the first case, we have no relation to apply and we produce a “hole”. Being $M$ a surface of genus $g$, this will happen exactly $g$ times. In the second case by Frobenius relation we can simply move  to the left.

Since the first case we just treated produces a “locked” structure, we still have one last case to consider. If we have such structure to the left of , we can use the associativity relation and the Frobenius relation to move the pair of pants to the left.



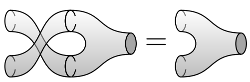

An analogous process can be described for the copies of , of which $q$ will vanish, $g$ will lock with $g$ copies of pants and $n-1$ will move to the right to form the *out-part* of the normal form.

Let us now consider a decomposition of a connected surface $M\colon (\mathbf{m}) \to (\mathbf{n})$ where twist bordisms appear. Pick any twist morphism $T$ and label its four boundaries. Without loss of generalities[1] we assume all pieces parallel to $T$ are cylinders. Having assumed $M$ to be connected, some of these boundaries must be connected to each other.

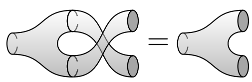



Suppose $A$ and $C$ are connected to each other. Then, to the left of the twist, we have a connected surface with at least one less twist than the original bordism $M$. By induction we can assume such surface can be brought to normal form and we can hence rearrange its outpart to get the twist to align with a copy of . Clearly, by cocommutativity, we are able to remove the twist. The same argument applies when assuming $B$ and $D$ are connected.

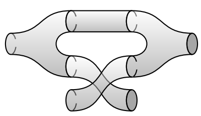
$A$ and $C$ connected:



$B$ and $D$ connected:

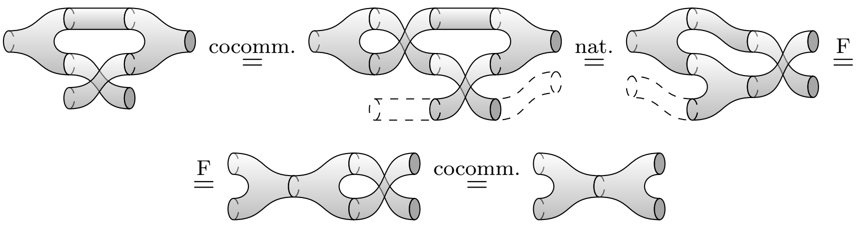


Suppose $A$ and $B$ are connected to each other and consider the (connected) surface connecting the two. Having assumed all pieces parallel to the twist to be cylinders, we can disconnect such surface and separately consider the one to the left and the one to the right of the twist. Again, such regions will be comprised of at least one less twist than $M$ and by induction they can be reduced to their normal forms. This procedure results in the following configuration.

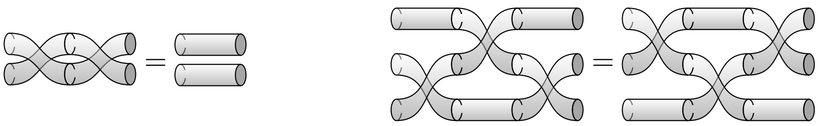


[1]Thanks to the identity relations we can always insert identities where needed.

Through cocommutativity and the Frobenius relation, as depicted below, we are able to eliminate the twist morphism. The same argument applies when assuming $C$ and $D$ to be connected.

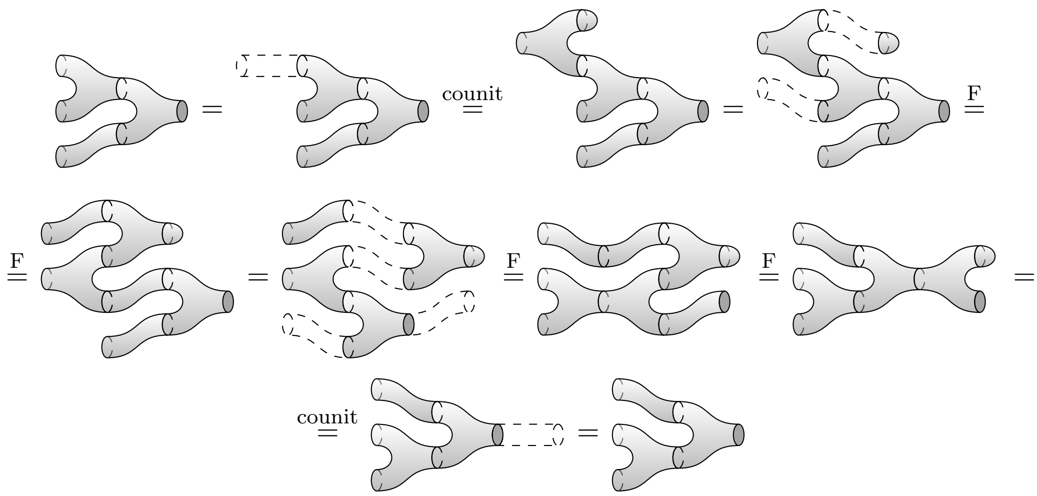



So far, we have only considered connected surfaces. We now adress the non-connected case. As we already did when proving Lemma 3.11 we now that given any bordism $M$ we are able to define a normal form as the composition of a permutation bordism ($S^{-1}$), a disjoint union of connected components ($SMT$) and another permutation bordism ($T^{-1}$). Starting from $M$ and observing that $S^{-1}S = \text{id}$ and $TT^{-1} = id$, we can get $M = S^{-1}SMTT^{-1}$ through naturality of the twist and it being its own inverse.



By the preceding argument for connected surfaces we can now reduce each connected component of $SMT$ to its normal form.

*Remark* 3.12 (A minimal set of relations.). The set relation we gave is not minimal, but rather convenient. Indeed we can find some implications hold between them. For example we can show that the Frobenius relation together with the handle cancellation imply associativity and coassociativity.



# Chapter 4

# Frobenius algebras

## 4.1 Three equivalent characterizations

We begin by recalling the essential algebraic notions, which will be necessary to give a proper definition of a Frobenius algebra. Through the chapter, we let $\Bbbk$ denote a field.

**Definition 4.1 (Algebras over a field).** A $\Bbbk$-*algebra* (i.e. an algebra over a field $\Bbbk$) is a $\Bbbk$-vector space $A$ equipped with two $\Bbbk$-linear maps

$$\mu\colon A \otimes A \to A \qquad \eta\colon \Bbbk \to A$$

such that the following diagrams commute.

$$\begin{tikzcd} & A \otimes A \otimes A \arrow[dl, "\mu\otimes\mathrm{id}_A"'] \arrow[dr, "\mathrm{id}_A\otimes\mu"] & \\ A \otimes A \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\ & A & \end{tikzcd}$$

$$\begin{tikzcd} & A \otimes A \arrow[dr, "\mu"] & & A \otimes A \arrow[dl, "\mu"'] & \\ \Bbbk \otimes A \arrow[ur, "\eta\otimes\mathrm{id}_A"] \arrow[rr, "\sim"'] & & A & & A \otimes \Bbbk \arrow[ul, "\mathrm{id}_A\otimes\eta"'] \arrow[ll, "\sim"] \end{tikzcd}$$

Note that, in the above diagram, the maps $\Bbbk \otimes A \to A$ and $A \otimes \Bbbk \to A$ represent the canonical scalar multiplication. We call $\mu$ the *multiplication* and $\eta$ the *unit map.*

This definition is equivalent to the standard notion of a (unital associative) $\Bbbk$-algebra from linear algebra: a $\Bbbk$-vector space equipped with an associative and unital bilinear product. Concretely, we can see that:

1. *Distributivity* follows from requiring $\mu$ to be $\Bbbk$-linear.

2. *Associativity* and *compatibility with scalar product* follow from the two commutative diagrams.

3. The *unit element* is defined by the image of $1 \in \Bbbk$ through $\eta$.

*Remark* 4.2. It is no coincidence that these diagrams seem similar to the ones appearing at the beginning of our thesis in 1.1. $\Bbbk$-algebras are indeed internal monoids in the monoidal category $(\mathbf{Vect}_\Bbbk, \otimes, \Bbbk)$, while (strict) monoidal categories are monoids in $(\mathbf{Cat}, \times, \mathbf{1})$. We'll define what this means in the next chapter. (See Definition 5.2)

**Definition 4.3 (Homomorphisms of $\Bbbk$-algebras).** Let $(A, \mu, \eta)$, $(A', \mu', \eta')$ be $\Bbbk$-algebras. A $\Bbbk$-*algebra homomorphism* $\varphi\colon A \to A'$ is a $\Bbbk$-linear map that preserves the multiplication and unit structures. Specifically, the following diagrams are required to commute:

$$\begin{tikzcd} A \otimes A \arrow[r, "\varphi\otimes\varphi"] \arrow[d, "\mu"] & A' \otimes A' \arrow[d, "\mu'"] \\ A \arrow[r, "\varphi"] & A' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "\varphi"] & & A' \\ & \Bbbk \arrow[ul, "\eta"] \arrow[ur, "\eta'"'] & \end{tikzcd}$$

We can then denote with $\mathbf{Alg}_\Bbbk$ the category of $\Bbbk$-algebras and $\Bbbk$-algebra homomorphisms.

We are now ready to state our first definition of a Frobenius algebra.

**Definition 4.4.** A *Frobenius algebra* is a finite dimensional $\Bbbk$-algebra $A$ equipped with a linear form $\varepsilon\colon A \to \Bbbk$, called the *Frobenius form*, whose kernel $\{x \in A \mid \varepsilon(a) = 0\}$ contains no nontrivial left ideals (i.e. only contains trivial ideals).

Being a Frobenius algebra is a structure, not a property. That is, a given algebra can admit multiple Frobenius structures, which must be specified each time to avoid ambiguity. Thus, it is a slight abuse of notation to say "$A$ is a Frobenius algebra"; nevertheless, we will do so from time to time, and what we mean is that we are assuming a Frobenius structure has been (or can be) chosen on $A$.

*Remark* 4.5. The condition that $\ker(\varepsilon)$ has no nontrivial left ideal is equivalent to it not containing nontrivial *principal* left ideals. This is equivalent to saying that $\varepsilon(Ax) = 0$ implies $x = 0$ for every $x \in A$.

Moreover, this condition can be expressed equivalently for right (principal) ideals.

We now remember that any linear form $\Lambda\colon A \to \Bbbk$ canonically defines a bilinear pairing $p\colon A \otimes A \to \Bbbk$ by composing with the multiplication on $A$. More precisely, we can send each $x \otimes y \in A \otimes A$ into $(\Lambda \circ \mu)(x \otimes y) = \Lambda(xy)$. By associativity of $\mu$, the pairing is itself associative, meaning $p(xy \otimes z) = p(x \otimes yz)$ for all $x, y, z \in A$.

On the other hand, given an associative pairing $p\colon A \otimes A \to \Bbbk$, we can always canonically define a linear form $\Lambda\colon A \to \Bbbk$ by fixing $1_A$ as one of the entries. More precisely, we set $\Lambda(a) = p(1_A \otimes a) = p(a \otimes 1_A)$.

One can easily check that these constructions are inverse to each other, giving a one-to-one correspondence between the two. A less obvious result is the following.

**Lemma 4.6.** *Let $A$ be a finite-dimensional $\Bbbk$-algebra. An associative pairing $A \otimes A \to \Bbbk$ is non-degenerate if and only if the corresponding linear form $A \to \Bbbk$ is a Frobenius form.*

This directly leads to an equivalent characterisation of a Frobenius algebra.

**Definition 4.7.** A *Frobenius algebra* is a finite-dimensional $\Bbbk$-algebra $A$ equipped with a non-degenerate associative pairing $\beta\colon A \otimes A \to \Bbbk$, called the *Frobenius pairing*.

A non-degenerate pairing $\beta\colon A \otimes A \to \Bbbk$ induces canonical $\Bbbk$-linear isomorphisms $A \xrightarrow{\sim} A^*$ and $A^* \xrightarrow{\sim} A$. The associativity of $\beta$ is equivalent to the A-linearity of these isomorphisms (with respect to the appropriate left or right module structure). The two equivalent characterisations following from such properties won't be of use for our thesis; nonetheless, we include them for completeness.

**Definition 4.8.** A *Frobenius algebra* is a finite-dimensional $\Bbbk$-algebra $A$ equipped with a left $A$-linear isomorphism to its dual.

**Definition 4.9.** A *Frobenius algebra* is a finite-dimensional $\Bbbk$-algebra $A$ equipped with a right $A$-linear isomorphism to its dual.

*Remark* 4.10. In these three definitions above, we required $A$ to be finite-dimensional. This is actually redundant, since with an argument analogous to the one provided in Proposition 2.29 we can show that the existence of a non-degenerate associative pairing implies the finite-dimensionality of $A$. The existence of a copairing $\gamma$ is indeed another instance of the notion of *dualizability*.

Our discussion has so far focused on algebras. We now introduce their dual concept by reversing all the arrows.

**Definition 4.11 (Coalgebras over a field).** A $\Bbbk$-coalgebra (i.e. a coalgebra over a field $\Bbbk$) is a $\Bbbk$-vector space $A$ equipped with two $\Bbbk$-linear maps

$$\delta\colon A \to A \otimes A \qquad \varepsilon\colon A \to \Bbbk$$

such that the following diagrams commute

$$\begin{tikzcd}
& A \otimes A \otimes A & \\
A \otimes A \arrow[ur, "\delta\otimes\mathrm{id}_A"] & & A \otimes A \arrow[ul, "\mathrm{id}_A\otimes\delta"'] \\
& A \arrow[ul, "\delta"] \arrow[ur, "\delta"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& A \otimes A \arrow[dl, "\varepsilon\otimes\mathrm{id}_A"'] & & A \otimes A \arrow[dr, "\mathrm{id}_A\otimes\varepsilon"] & \\
\Bbbk \otimes A & & A \arrow[ll, "\sim"'] \arrow[ul, "\delta"'] \arrow[ur, "\delta"] \arrow[rr, "\sim"] & & A \otimes \Bbbk
\end{tikzcd}$$

We call $\delta$ the *comultiplication* and $\varepsilon$ the *counit map*.

**Definition 4.12 (Homomorphisms of $\Bbbk$-coalgebras).** Let $(A, \delta, \varepsilon)$, $(A', \delta', \varepsilon')$ be $\Bbbk$-coalgebras. A $\Bbbk$-*coalgebra homomorphism* $\phi\colon A \to A'$ is a $\Bbbk$-linear map that preserves the comultiplication and counit structures. Concretely, the following diagrams are required to commute.

$$\begin{tikzcd} A \otimes A \arrow[r, "\phi\otimes\phi"] & A' \otimes A' \\ A \arrow[u, "\delta"] \arrow[r, "\phi"] & A' \arrow[u, "\delta'"'] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "\phi"] \arrow[dr, "\varepsilon"'] & & A' \arrow[dl, "\varepsilon'"] \\ & \Bbbk & \end{tikzcd}$$

We can hence state the last equivalent characterisation.

**Definition 4.13.** A *Frobenius algebra* is a $\Bbbk$-vector space $A$ equipped with both an algebra $(A, \mu, \eta)$ and a coalgebra $(A, \delta, \varepsilon)$ structure, such that the Frobenius relation, which corresponds to the commutativity of the diagrams below, holds.

$$\begin{tikzcd} A \otimes A \arrow[r, "\delta\otimes\mathrm{id}_A"] \arrow[d, "\mu"] & A \otimes A \otimes A \arrow[d, "\mathrm{id}_A\otimes\mu"] \\ A \arrow[r, "\delta"] & A \otimes A \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \arrow[r, "\mathrm{id}_A\otimes\delta"] \arrow[d, "\mu"] & A \otimes A \otimes A \arrow[d, "\mu\otimes\mathrm{id}_A"] \\ A \arrow[r, "\delta"] & A \otimes A \end{tikzcd}$$

Proving the equivalence of this last definition is not immediate when using this notation. To help ourselves in handling this proof, we will introduce *graphical calculus* in Section 4.2.

Finally, we define what it means to ask a Frobenius algebra to be symmetric. To do so, we recall the three characterisations (even if we've not proved their equivalence yet) we'll work with from now on and state the notion of symmetry for each of these frameworks.

**Definition 4.14 (Frobenius algebras).** A *Frobenius algebra* is a (finite dimensional) $\Bbbk$-algebra $A$ equippped with either:

(i) A linear form $\varepsilon\colon A \to \Bbbk$ whose kernel contains no nontrivial (left) ideals.
[See Definition 4.4]

(ii) A non-degenerate associative pairing $\beta\colon A \otimes A \to \Bbbk$ (and a copairing $\gamma\colon \Bbbk \to A \otimes A$).
[See Definition 4.7]

(iii) A coalgebra structure $(A, \delta, \varepsilon)$ satisfying the Frobenius relation.
[See Definition 4.13]

**Definition 4.15 (Commutative Frobenius algebras).** A Frobenius algebra $A$ is *commutative* if its underlying algebra structure is commutative.

For completeness, and to make sure the reader does not confuse the two structures, we define another important class of Frobenius algebras.

**Definition 4.16 (Symmetric Frobenius algebras).** A Frobenius algebra $A$ is *symmetric* if one of the following (equivalent) conditions holds:

1. The Frobenius form $\varepsilon\colon A \to \Bbbk$ is central, meaning that $\varepsilon(ab) = \varepsilon(ba)$ for every $a, b \in A$.

2. The pairing $\beta\colon A \otimes A \to \Bbbk$ is symmetric, meaning $\beta(a \otimes b) = \beta(b \otimes a)$ for every $a, b \in A$.

The difference between a commutative and a symmetric Frobenius algebra becomes clear: a *commutative Frobenius algebra* is indeed just a Frobenius algebra whose underlying algebra is commutative. We then have that all commutative Frobenius algebras are symmetric, while the converse does not hold.

We remind that since being a Frobenius algebra is a structure, the property of symmetry relates to such structure. We could then have two Frobenius structure on an algebra $A$ and have only one of them to be symmetric.

*Remark* 4.17. We previously mentioned a fourth characterisation of Frobenius algebra [See Definition 4.8]. The notion of symmetry can also be formulated in that context by asking for the left (resp. right) $A$-linear isomorphism to be also right (resp. left) $A$-linear.

## 4.2 Rigorous doodles

This section aims to define a graphical language that will help us see more clearly the equivalence between the various definitions of Frobenius algebras.

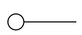
Through this section, we'll adopt these drawings[1] as formal notations corresponding to $\Bbbk$-linear maps $A^m \to A^n$. The tensor product of two maps will correspond to placing the diagrams *in parallel* (one above the other), while the composition will be represented by connecting them *in series* (by simply connecting the wires).

### 4.2.1 From pairing to Frobenius relation

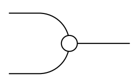
Let us start by "translating" the definition through copairing of a Frobenius algebra using such *wires* notation.

**(Frobenius algebra).** A *Frobenius algebra* is a finite dimensional $\Bbbk$-algebra $A$ equipped with a non-degenerate associative pairing $\beta\colon A \otimes A \to \Bbbk$, called the *Frobenius pairing*.

We start by drawing the maps defining a $\Bbbk$-algebra:



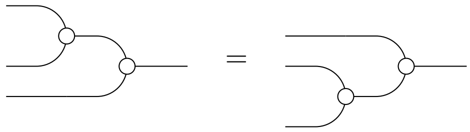
unit
$\eta\colon \Bbbk \to A$



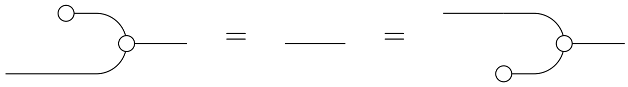
multiplication
$\mu\colon A \otimes A \to A$

[1] The notation used is mainly inspired to [Sob15]

This allows us to express its axioms graphically:



(associativity of $\mu$)

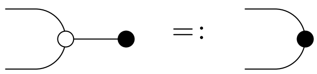


(unit axiom)

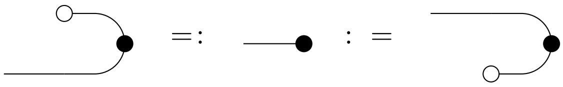
We now need to define a Frobenius form $\varepsilon\colon A \to \Bbbk$, which will induce a Frobenius pairing $\beta\colon A \otimes A \to \Bbbk$.

Frobenius form $\varepsilon\colon A \to \Bbbk$ Frobenius pairing $\beta\colon A \otimes A \to \Bbbk$

Their relation can be explained graphically as follows.

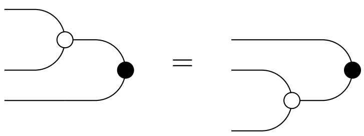


(defining $\beta$ from $\varepsilon$)



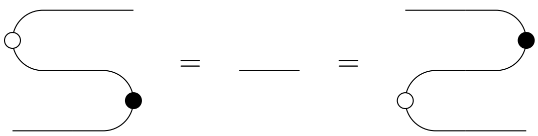
(defining $\varepsilon$ from $\beta$)

Combining the associativity of the multiplication and the first of the above relations we obtain the associativity for the pairing.



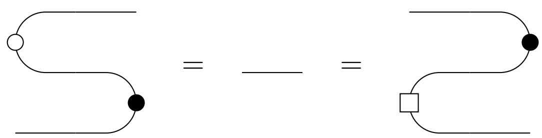
(associativity of $\beta$)

Finally to obtain a Frobenius algebra, we need the pairing to be non-degenerate. This means asking for the existence of a *copairing* $\gamma\colon \Bbbk \to A \otimes A$ such that the compositions $(\mathrm{id}_A \otimes \beta) \circ (\gamma \otimes \mathrm{id}_A)$ and $(\beta \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \gamma)$ equal the identity. We can translate such requirements in graphical terms.



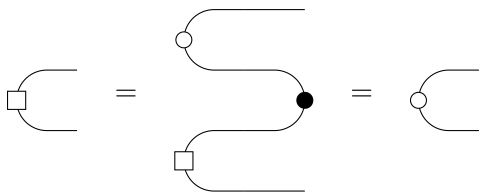
(non-degeneracy of $\beta$ or *snake relation*)

**(Uniqueness of the copairing).** We can prove that the copairing $\gamma$ is unique, just through graphical calculus (we'll still include the commutative diagrams on the side, to emphasise how the two approaches correspond). Suppose there exist two different copairings $\gamma$, $\xi$ satisfying such a non-degeneracy condition. We distinguish them graphically by drawing a white circle for $\gamma$ and a white square for $\xi$. We then have:



$$\begin{tikzcd} & A\otimes A\otimes A \arrow[dr, "\mathrm{id}_A\otimes\beta"] & \\ A \arrow[ur, "\gamma\otimes\mathrm{id}_A"] \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\mathrm{id}_A\otimes\xi"'] & & A \\ & A\otimes A\otimes A \arrow[ur, "\beta\otimes\mathrm{id}_A"'] & \end{tikzcd}$$

By considering the tensor of the two copairings and composing it with $\mathrm{id}_A\otimes\beta\otimes\mathrm{id}_A$, we obtain, through the above relations, the following:



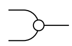
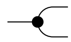
and the corresponding commutative diagram below.

$$\begin{tikzcd} & A\otimes A \arrow[d, "\gamma\otimes\mathrm{id}_A\otimes\mathrm{id}_A"] \arrow[dr, bend left, "\mathrm{id}_A\otimes\mathrm{id}_A"] & \\ \Bbbk \arrow[ur, bend left, "\xi"] \arrow[r, "\gamma\otimes\xi"] \arrow[dr, bend right, "\gamma"'] & A\otimes A\otimes A\otimes A \arrow[r, "\mathrm{id}_A\otimes\beta\otimes\mathrm{id}_A"] & A\otimes A \\ & A\otimes A \arrow[u, "\mathrm{id}_A\otimes\mathrm{id}_A\otimes\xi"] \arrow[ur, bend right, "\mathrm{id}_A\otimes\mathrm{id}_A"'] & \end{tikzcd}$$

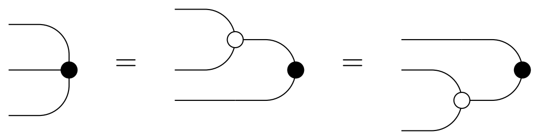
We conclude $\gamma=\xi$, proving uniqueness. We can see, just by this small example, how the graphical approach is way more straightforward compared to the diagrammatic one, which easily becomes messy.

We are now ready to prove the first equivalence:

**Proposition 4.18.** *Let $(A,\mu,\eta)$ be a $\Bbbk$-algebra equipped with a non-degenerate associative pairing $\beta\colon A\otimes A\to\Bbbk$. Then $A$ acquires a structure of coalgebra $(A,\delta,\varepsilon)$, such that the Frobenius relation holds.*

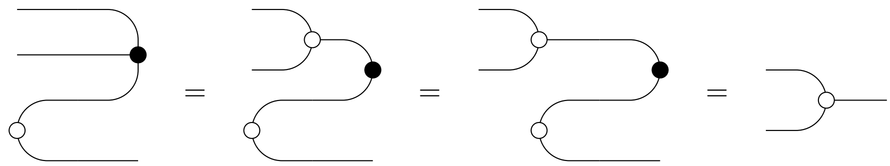
**(Defining a comultiplication).** Since the multiplication is defined as  , we would want the comultiplication to be something like  satisfying the coalgebra axioms.

Before doing so, by recalling the associativity of the pairing $\beta$, we define $\phi=(\mu\otimes\mathrm{id}_A)\circ\beta=(\mathrm{id}_A\otimes\mu)\circ\beta$. Looking at its graphical representation, we'll call it the *trident map*.

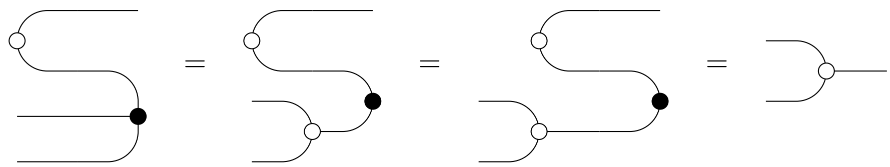


(trident map)

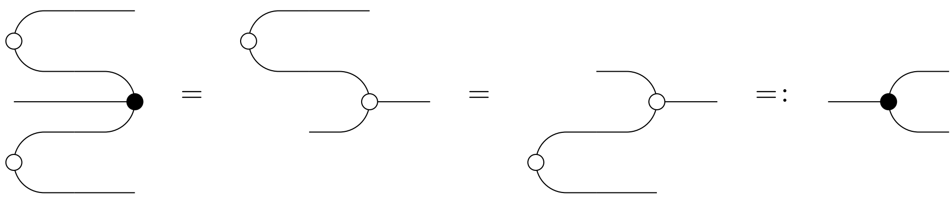
We then show how we can express the multiplication through the *trident map* (and the copairing):



and

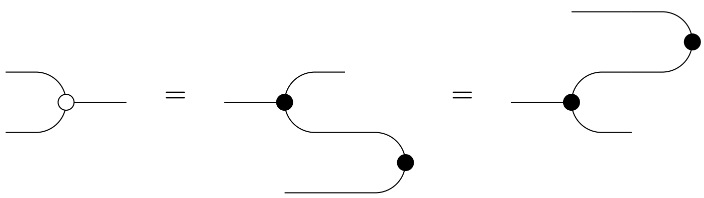


This gives the following equivalence



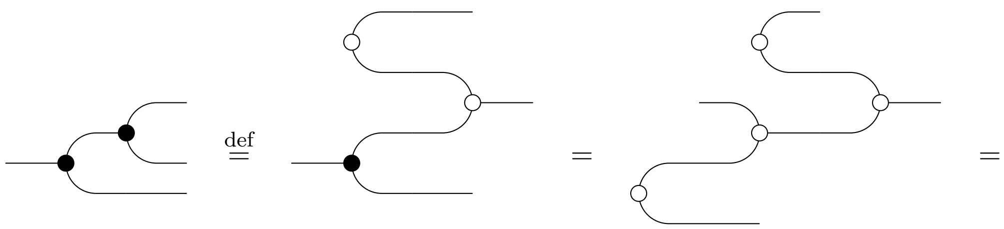
through which we finally define a *comultiplication* $\delta\colon A \to A \otimes A$ in terms of $\mu$ and $\gamma$.

The converse also holds: the multiplication can be written in terms of comultiplication and pairing. We omit an explicit proof as it follows the same pattern as the previous one.

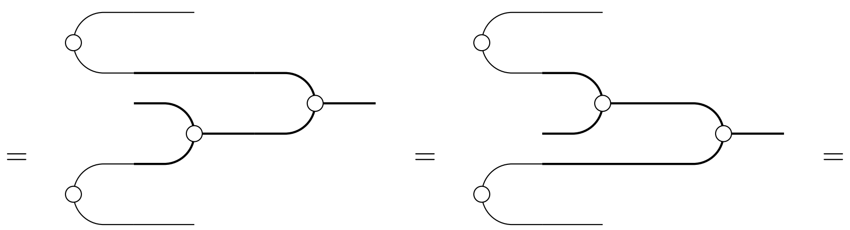


**(Defining a coalgebra).** To show that this map, together with the Frobenius form $\varepsilon\colon A \to \Bbbk$, defines a coalgebra, we need to prove the coassociativity and counit axioms.

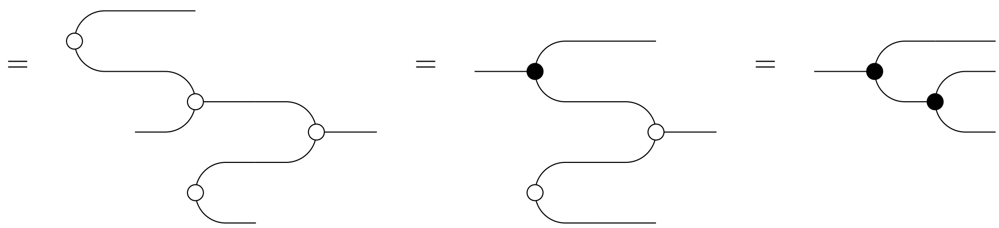
For the coassociativity, we start by unwinding the two comultiplications



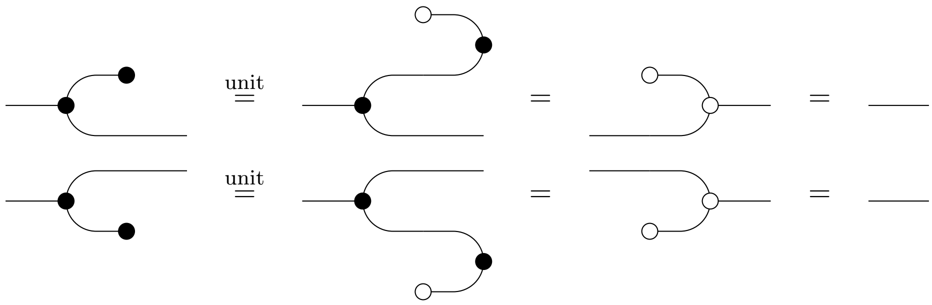
Then, using the associativity of multiplication, we get



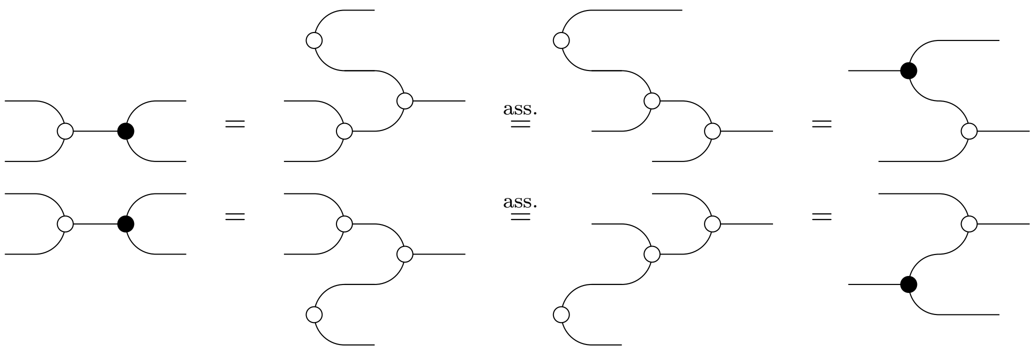
in which we can identify the definition of comultiplication, hence rewrite it as



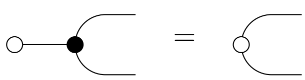
We now prove the counit axioms:



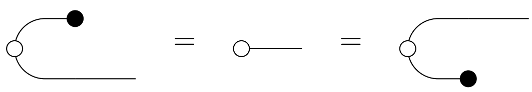
**(Frobenius relation).** Lastly, we show that the Frobenius relation is satisfied.



**(Uniqueness of the comultiplication).** To simplify some of the next drawings, we notice that the relation between $\gamma$ (copairing) and $\eta$ (unit) is dual to the relation between the Frobenius pairing and the Frobenius form.

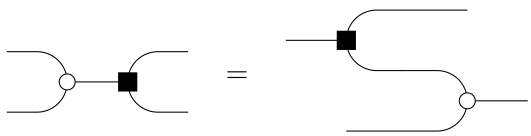


and

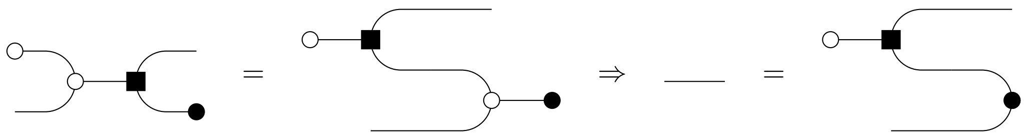


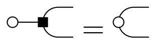
(The first follows from the definition of comultiplication and unit axioms. The second from the counit axiom.)

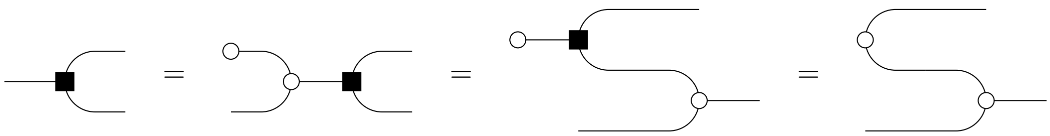
Suppose there exists another comultiplication $\omega\colon A \to A \otimes A$, with $\varepsilon$ as counit and satisfying the Frobenius relation. We distinguish it from $\delta$ by representing it with a black square.



By composing this equation on both sides with unit and counit, we get



By uniqueness of the copairing we can say that  and so we have



which corresponds to the definition of $\delta$. We have proved that the comultiplication is unique.

### 4.2.2 From Frobenius relation to pairing

Again, we begin by rephrasing the following definition of Frobenius algebra in terms of *wires*.

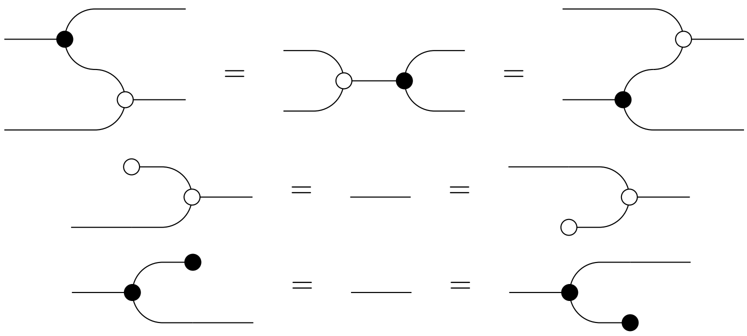
**(Frobenius algebra).** A *Frobenius algebra* is a $\Bbbk$-vector space $A$ equipped with both an algebra $(A, \mu, \eta)$ and a coalgebra $(A, \delta, \varepsilon)$ structure, such that the Frobenius relation, which corresponds to the commutativity of the diagrams below, holds.

$$\begin{tikzcd} A \otimes A \arrow[r, "\delta\otimes\mathrm{id}_A"] \arrow[d, "\mu"] & A \otimes A \otimes A \arrow[d, "\mathrm{id}_A\otimes\mu"] \\ A \arrow[r, "\delta"] & A \otimes A \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \arrow[r, "\mathrm{id}_A\otimes\delta"] \arrow[d, "\mu"] & A \otimes A \otimes A \arrow[d, "\mu\otimes\mathrm{id}_A"] \\ A \arrow[r, "\delta"] & A \otimes A \end{tikzcd}$$

Let us consider a vector space $A$ equipped with

$\eta\colon \Bbbk \to A$ $\qquad$ $\mu\colon A \otimes A \to A$ $\qquad$ $\varepsilon\colon A \to \Bbbk$ $\qquad$ $\delta\colon A \to A \otimes A$

such that the following three relations are satisfied.



(Frobenius relation)

(unit axiom)

(counit axiom)

We now prove the other side of the equivalence

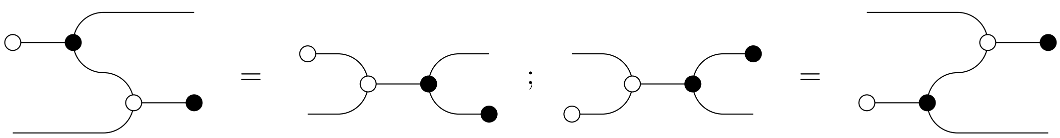
**Proposition 4.19.** *Let $(A, \mu, \eta, \delta\varepsilon)$ be a both a $\Bbbk$-algebra and a $\Bbbk$-coalgebra satisfying the Frobenius relation. Then $A$ is equipped with a non-degenerate associative pairing $\beta\colon A \otimes A \to \Bbbk$.*

By defining the maps

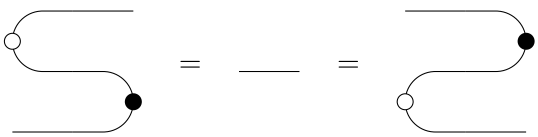
$\beta\colon = \varepsilon \circ \mu$ $\gamma\colon = \delta \circ \eta$

we obtain a Frobenius pairing on the algebra $A$.

**(Non-degeneracy of the pairing).** Taking the Frobenius relation, we can compose both sides with $\eta$ and $\varepsilon$ as depicted below.

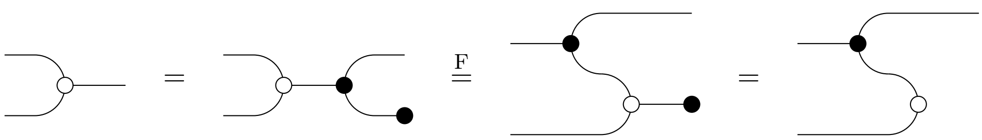


By definition of $\beta$ and $\gamma$ and applying the unit and counit axioms we obtain

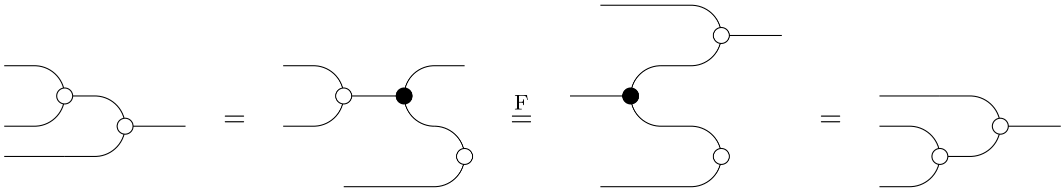


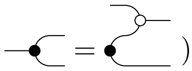
proving the non-degeneracy of $\beta\colon A \otimes A \to \Bbbk$.

**(Algebra and coalgebra structure).** By proving the associativity and coassociativity axioms, we can state $A$ is equipped with both an algebra and a coalgebra structure. We begin by proving $\mu\colon A \otimes A \to A$ is associative. From the Frobenius relation, we have the following equivalences.

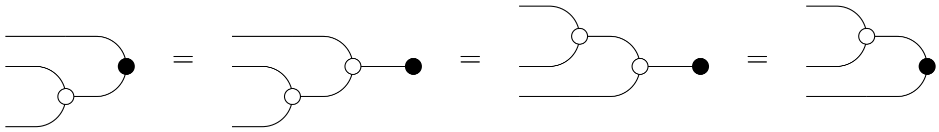


Through this equivalent representation of $\mu$, we are now able to prove associativity.



Similarly (by noticing  ) we are able to prove coassociativity for $\delta$.

**(Associativity of the pairing).** The associativity of $\mu$ is key for proving $\beta$ is associative too.



*Remark* 4.20. Through such wires, we are now able to see the difference between commutative and symmetric Frobenius algebras.

A commutative Frobenius algebra will satisfy commutativity and cocommutativity:

On the other hand, a symmetric Frobenius algebra satisfies the following

## 4.3 A category of Frobenius algebras

Having established various characterisations of Frobenius algebras in Section 4.1, we now focus on the morphisms between them. To do so, we will refer to the third (see 4.14) characterisation of a Frobenius algebra.

**Definition 4.21 (Frobenius algebra homomorphisms).** Let $(A, \eta, \mu, \varepsilon, \delta)$ and $(A', \eta', \mu', \varepsilon', \delta')$ be two Frobenius algebra. A *Frobenius algebra homomorphism* $\varphi\colon A \to A'$ is a $\Bbbk$-linear map which is both a $\Bbbk$-algebra homomorphism and a $\Bbbk$-coalgebra homomorphism.

In particular, a Frobenius algebra homomorphism preserves the Frobenius form (i.e. $\varepsilon' = \varphi(\varepsilon)$). Together with such morphisms, Frobenius algebras then form a category, which we name $\mathbf{FAlg}_{\Bbbk}$.

We now investigate the structure of this category.

**Proposition 4.22.** *The tensor product of two Frobenius algebras is again a Frobenius algebra, giving* $\mathbf{FAlg}_{\Bbbk}$ *a monoidal structure.*

For a proof of this result, we refer to the general statements given further in the text in Lemma 5.7 and Lemma 5.20.

Commutative Frobenius algebras form a full subcategory, which we denote as $\mathbf{cFAlg}_{\Bbbk}$.

# Chapter 5

# Drawing conclusions

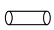
## 5.1 The main equivalence

In general, we can define a monoidal functor by just evaluating it on the generators of the source category. In our case, where a two-dimensional TQFT is a symmetric monoidal functor $Z\colon \mathbf{Bord}(2) \to \mathbf{Vect}_{\Bbbk}$, this implies that $Z$ is entirely described by its evaluation on the generating morphisms (and the objects of the skeleton).

Following the definition of a TQFT we then choose a $\Bbbk$-vector space $A$ as the image of the circle

$$(\mathbf{1}) \mapsto A$$

and, by functoriality,

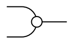
 $\mapsto [\mathrm{id}_A\colon A \to A]$

Being $Z$ a symmetric monoidal functor, it also follows that

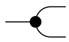
$(\mathbf{n}) \mapsto A \otimes \ldots \otimes A = A^{\otimes n}$ and  $\mapsto [\sigma\colon A \otimes A \to A \otimes A]$

The images of the remaining generators are defined as follows

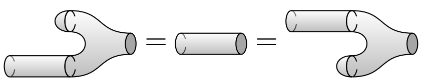
 $\mapsto [\eta\colon \Bbbk \to A]$ 

 $\mapsto [\mu\colon A \otimes A \to A]$ 

 $\mapsto [\varepsilon\colon A \to \Bbbk]$ 

 $\mapsto [\delta\colon A \to A \otimes A]$ 

The relations in $\mathbf{Bord}(2)$ then become relations between these $\Bbbk$-linear maps. For example, for the handle cancellation, we have:

 becomes 

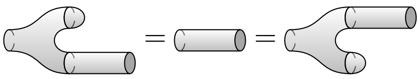
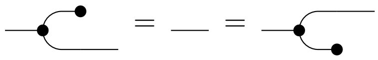
By "slimming" all the relations in $\mathbf{Bord}(2)$, it should becomes obvious there exists some kind of relation between topological quantum field theories and the Frobenius structure we studied in the previous chapter. These considerations lead us to the main theorem regarding the classification of 2-dimensional TQFTs.

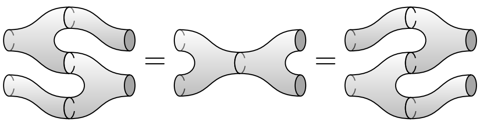
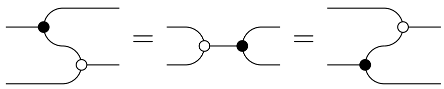
**Theorem 5.1.** *There is a canonical equivalence of categories between 2-dimensional TQFTs taking values in* $\mathbf{Vect}_{\Bbbk}$ *and commutative Frobenius* $\Bbbk$*-algebras.*

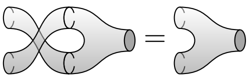
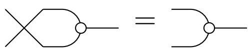
$$\mathbf{2TQFT}_{\Bbbk} \cong \mathbf{cFAlg}_{\Bbbk}$$

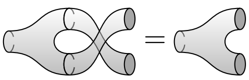
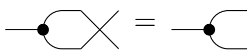
Thanks to our journey through the previous chapters the proof becomes straightforward. We can now explicitly build the correspondence, for both objects and arrows.

Indeed, given an object $Z$ in $\mathbf{2TQFT}_{\Bbbk}$ we get the axioms defining a structure of commutative Frobenius algebra on the vector space $Z((\mathbf{1})) = A$.

 becomes 

 becomes 

 becomes 

 becomes 

Conversely, given a commutative Frobenius algebra $(A, \eta, \mu, \varepsilon, \delta)$, we are able to determine a TQFT by evaluation on the generating morphisms as described above.

These assignments define a bijection. Starting with a symmetric monoidal functor $Z$ we obtain a commutative Frobenius algebra $A = Z((\mathbf{1}))$. From such algebra we then define a TQFT by mapping $(\mathbf{1})$ onto it, which exactly corresponds to the monoidal functor we started with.

To check this defines an equivalence between the two categories we must also consider what happens to morphisms. By definition, an arrow $u\colon Z \to Z'$ between two 2-dimensional TQFTs is a monoidal natural transformations between the two functors. In particular, the structure of $\mathbf{Bord}(2)$, implies that these natural transformations consist of $\Bbbk$-linear maps $A^{\otimes n} \to A'^{\otimes n}$, where $A = Z((\mathbf{1}))$, $A' = Z'((\mathbf{1}))$, compatible with all the arrows in $\mathbf{Bord}(2)$. Furthermore, by asking $u$ to be monoidal, these arrows are just the $n$-th tensor power of the $\Bbbk$-linear arrow $u_1\colon A \to A'$. Since every arrow in $\mathbf{Bord}(2)$ is built by tensoring and gluing the generators, the compatibility of $u$ with the arrows in $\mathbf{Bord}(2)$ is entirly determined by the corresponding diagrams.

$$\begin{tikzcd}
\Bbbk \arrow[r, equal] \arrow[d, "\eta"] & \Bbbk \arrow[d, "\eta'"] \\
A \arrow[r, "u_1"] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \arrow[r, "u_1 \otimes u_1"] \arrow[d, "\mu"] & A' \otimes A' \arrow[d, "\mu'"] \\
A \arrow[r, "u_1"] & A'
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "u_1"] \arrow[d, "\varepsilon"] & A' \arrow[d, "\varepsilon'"] \\
\Bbbk \arrow[r, equal] & \Bbbk
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "u_1"] \arrow[d, "\delta"] & A' \arrow[d, "\delta'"] \\
A \otimes A' \arrow[r, "u_1 \otimes u_1"] & A' \otimes A'
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes A \arrow[r, "u_1 \otimes u_1"] \arrow[d, "\sigma"] & A' \otimes A' \arrow[d, "\sigma'"] \\
A \otimes A \arrow[r, "u_1 \otimes u_1"] & A' \otimes A'
\end{tikzcd}$$

Let us now go back to the definitions of algebra and coalgebra homomorphisms (see Definition 4.3 and Definition 4.12). We notice that the commutative diagrams there are formally identical to the naturality condition imposed on the monoidal transformation $u\colon Z \to Z'$. We then have proved that each monoidal transformation in $\mathbf{Bord}(2)$ defines a morphism of Frobenius algebras, hence (by symmetry of the functors $Z$ and $Z'$) of commutative ones.

By applying the same argument backwards we reach the converse implication, thus completing the proof of the categorical equivalence.

## 5.2 A broader result

Let us put everything in the right context by giving some more definitions from category theory.

**Definition 5.2 (Internal monoids).** Let $(\mathcal{C}, \square, I)$ be a strict monoidal category. An *internal monoid* in $\mathcal{C}$ is an object $M$ equipped with two morphisms

$$\mu\colon M \square M \to M \quad \text{(multiplication)} \qquad\qquad \eta\colon I \to M \quad \text{(unit)}$$

such that the following diagrams commute

$$\begin{tikzcd}
& M \square M \square M \arrow[dl, "\mu \square \mathrm{id}_M"'] \arrow[dr, "\mathrm{id}_M \square \mu"] & \\
M \square M \arrow[dr, "\mu"'] & & M \square M \arrow[dl, "\mu"] \\
& M &
\end{tikzcd}$$

$$\begin{tikzcd}
& M \square M \arrow[dr, "\mu"] & & M \square M \arrow[dl, "\mu"'] & \\
I \square M \arrow[ur, "\eta \square \mathrm{id}_M"] \arrow[rr, "="'] & & M & & M \square I \arrow[ul, "\mathrm{id}_M \square \eta"'] \arrow[ll, "="]
\end{tikzcd}$$

The attentive reader has surely noticed that this definition is given on a strict monoidal category. The reason behind this choice is again Theorem 1.12. The general definition is not so different, we report it below for completeness[1].

[1]And also because in categories like $(\mathbf{Cat}, \times, \mathbf{1})$, $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk)$ we do not actually have strict unitality.

**(Internal monoids).** Let $(\mathcal{C}, \square, I, \alpha, l, r)$ be a monoidal category. An *internal monoid* in $\mathcal{C}$ is an object $M$ together with two morphisms

$$\mu\colon M \square M \to M \quad \text{(multiplication)} \qquad \eta\colon I \to M \quad \text{(unit)}$$

such that the following diagrams commute

$$\begin{tikzcd}
(M \square M) \square M \arrow[rr, "\alpha"] \arrow[d, "\mu\square\mathrm{id}_M"] & & M \square (M \square M) \arrow[d, "\mathrm{id}_M\square\mu"] \\
M \square M \arrow[dr, "\mu"'] & & M \square M \arrow[dl, "\mu"] \\
& M &
\end{tikzcd}$$

$$\begin{tikzcd}
& M \square M \arrow[dr, "\mu"] & & M \square M \arrow[dl, "\mu"'] & \\
I \square M \arrow[ur, "\eta\square\mathrm{id}_M"] \arrow[rr, "l_M"'] & & M & & M \square I \arrow[ul, "\mathrm{id}_M\square\eta"'] \arrow[ll, "r_M"]
\end{tikzcd}$$

By reversing the arrows, we also give the definition of *internal comonoid.* The definition extends analogously to the case of weak monoidal categories.

**Definition 5.3 (Internal comonoids).** Let $(\mathcal{C}, \square, I)$ be a strict monoidal category. An *internal comonoid* in $\mathcal{C}$ is an object $M$ equipped with two morphisms

$$\delta\colon M \to M \square M \quad \text{(comultiplication)} \qquad \varepsilon\colon M \to I \quad \text{(counit)}$$

such that the following diagrams commute

$$\begin{tikzcd}
& M \square M \square M & \\
M \square M \arrow[ur, "\delta\square\mathrm{id}_M"] & & M \square M \arrow[ul, "\mathrm{id}_M\square\delta"'] \\
& M \arrow[ul, "\delta"] \arrow[ur, "\delta"'] &
\end{tikzcd}$$

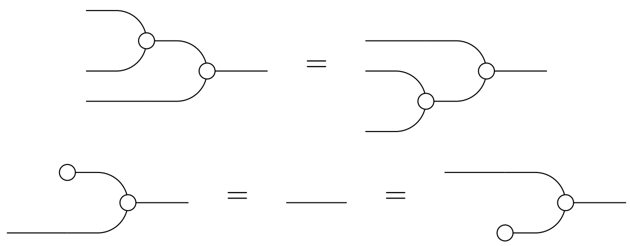
$$\begin{tikzcd}
& M \square M \arrow[dl, "\varepsilon\square\mathrm{id}_M"'] & & M \square M \arrow[dr, "\mathrm{id}_M\square\varepsilon"] & \\
I \square M & & M \arrow[ul, "\delta"'] \arrow[ur, "\delta"] \arrow[ll, "="] \arrow[rr, "="'] & & M
\end{tikzcd}$$

Both these structures can be grafically represented, similarly to what we did in the case of $\Bbbk$-Frobenius algebras.

**(Internal monoids).** Let $(\mathcal{C}, \Box, I)$ be a monoidal category. An *internal monoid* in $\mathcal{C}$ is an object $M$ equipped with two morphisms

$\mu\colon M \,\Box\, M \to M$ $\qquad\qquad$ $\eta\colon I \to M$

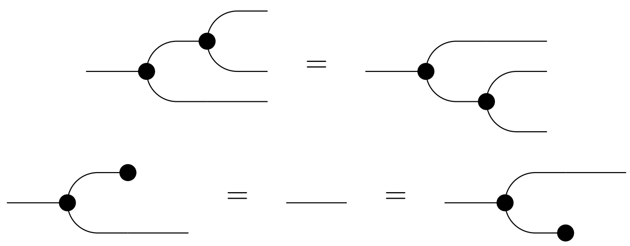
such that the following relations hold



**(Internal comonoids).** Let $(\mathcal{C}, \Box, I)$ be a monoidal category. An *internal comonoid* in $\mathcal{C}$ is an object $M$ equipped with two morphisms

$\delta\colon M \to M \,\Box\, M$ $\qquad\qquad$ $\epsilon\colon M \to I$

such that the following relations hold



Example 5.4. An internal monoid in $(\mathbf{Set}, \times, \{*\})$ is a monoid. An internal comonoid in $(\mathbf{Set}, \times, \{*\})$ is a comonoid.

An internal monoid in $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk)$ is a $\Bbbk$-algebra. An internal comonoid in $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk)$ is a $\Bbbk$-coalgebra.

An internal monoid in $(\mathbf{Cat}, \times, \mathbf{1})$, where $\mathbf{1}$ is a category with only one object, is a monoidal category. Although we have not done so, we can also define a comonoidal category, which will exactly be the comonoid in $(\mathbf{Cat}, \times, \mathbf{1})$

As we did for monoidal categories and $\Bbbk$-algebras we are able to define morphisms between internal monoids and comonoids.

**Definition 5.5 (Homomorphisms of internal monoids).** Let $(\mathcal{C}, \square, I)$ be a strict monoidal category. An *homomorphisms of internal monoids* $\varphi\colon (M, \mu, \eta) \to (M', \mu', \eta')$ is a morphism from $M$ to $M'$ such that the following diagrams commute.

$$\begin{tikzcd} M \square M \arrow[r, "\varphi"] \arrow[d, "\mu"] & M' \square M' \arrow[d, "\mu'"] \\ M \arrow[r, "\varphi"] & M' \end{tikzcd} \qquad \begin{tikzcd} M \arrow[rr, "\varphi"] & & M' \\ & I \arrow[ul, "\eta"] \arrow[ur, "\eta'"'] & \end{tikzcd}$$

**Definition 5.6 (Homomorphisms of internal comonoids).** Let $(\mathcal{C}, \square, I)$ be a strict monoidal category. An *homomorphisms of internal comonoids* $\phi\colon (M, \delta, \varepsilon) \to (M', \delta', \varepsilon')$ is a morphism from $M$ to $M'$ such that the following diagrams commute.

$$\begin{tikzcd} M \square M \arrow[r, "\phi"] & M' \square M' \\ M \arrow[r, "\phi"] \arrow[u, "\delta"] & M' \arrow[u, "\delta'"] \end{tikzcd} \qquad \begin{tikzcd} M \arrow[rr, "\phi"] \arrow[dr, "\varepsilon"'] & & M' \arrow[dl, "\varepsilon'"] \\ & I & \end{tikzcd}$$

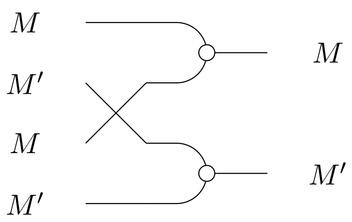
It is rather easy to check how composition of two morphisms of internal monoids (resp. comonoids) is again a morphism of internal monoids (resp. comonoids). Given a monoidal category $(\mathcal{C}, \square, I)$, we can define a new category $\mathbf{Mon}(\mathcal{C})$ where objects are internal monoids in $\mathcal{C}$ and arrows are the homomorphisms between them.

**Lemma 5.7.** *Let $(\mathcal{C}, \square, I, \beta)$ be a symmetric monoidal category. Then $\mathbf{Mon}(\mathcal{C})$ can be canonically equipped with a monoidal structure.*

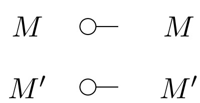
*Proof.* Let us take two internal monoids $(M, \mu, \eta)$, $(M', \mu', \eta')$ and define a structure of monoidal product on $M \square M'$. To do so, we define the multiplication $\mu \boxtimes \mu$ as $(\mu \square \mu') \circ (\mathrm{id}_M \square \beta \square \mathrm{id}_{M'})$.

$$\begin{tikzcd} (M \square M') \square (M \square M') \arrow[r, "\mu \boxtimes \mu'"] \arrow[d, "\mathrm{id}_M \square \beta \square \mathrm{id}_{M'}"'] & M \square M' \\ M \square M \square M' \square M' \arrow[ur, "\mu \square \mu'"'] & \end{tikzcd}$$

We define the unit by just taking the monoidal product $\eta \square \eta'$ and check that $(M \square M', \mu \boxtimes \mu', \eta \square \eta')$ satisfies the internal monoid axioms. To do so we can rely on a graphical representation of multiplication and unit.

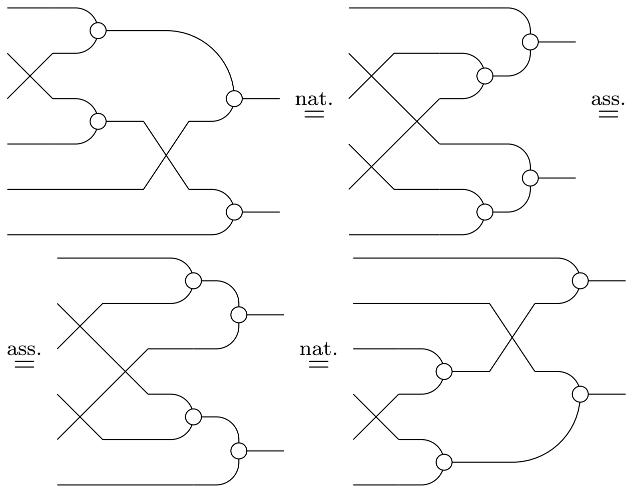


(multiplication)



(unit)

We hence have the associativity.

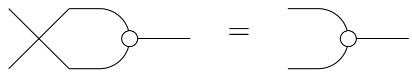



□

**Definition 5.8 (Commutative internal monoids).** Let $(\mathcal{C}, \square, I, \beta)$ be a symmetric monoidal category. An internal monoid $(M, \mu, \eta)$ in $\mathcal{C}$ is *commutative* if its multiplication is compatible with the twist map. Equivalentely we are asking for the following diagram to commute.

$$\begin{tikzcd}
M \square M \arrow[rr, "\mu"] \arrow[dr, "\beta"'] & & M \\
& M \square M \arrow[ur, "\mu"'] &
\end{tikzcd}$$

The above requirement can be represented through wires as

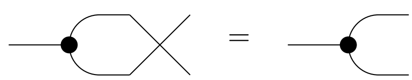



Similarly, we can define an analogous notion for internal comonoids.

**Definition 5.9 (Cocommutative internal comonoids).** Let $(\mathcal{C}, \square, I, \beta)$ be a symmetric monoidal category. An internal comonoid $(M, \delta, \varepsilon)$ in $\mathcal{C}$ is *cocommutative* if its comultiplication is compatible with the twist map. Equivalentely we are asking for the following diagram to commute.

$$\begin{tikzcd}
M \arrow[rr, "\delta"] \arrow[dr, "\delta"'] & & M \square M \\
& M \square M \arrow[ur, "\beta"'] &
\end{tikzcd}$$

which can be represented graphically as



**Lemma 5.10.** *Let $(\mathcal{C}, \square, I)$, $(\mathcal{D}, \boxtimes, J)$ be two monoidal categories and let $F \colon \mathcal{C} \to \mathcal{D}$ be a (lax) monoidal functor between the two. Then, given an internal monoid $(M, \mu, \eta)$ in $\mathcal{C}$, its image $(F(M), F(\mu), F(\eta))$ defines an internal monoid in $\mathcal{D}$.*

*Proof.* The proof of this statement follows by applying Definition 1.6 to the internal monoid structure. □

We can now construct an equivalence relation in some way analogous to our main one. This will help us generalize it later on.

**Definition 5.11 (The simplex category).** The *simplex category*[2] $\Delta$ is the category defined by taking

- finite totally ordered sets $[n] \coloneqq \{0, 1, 2, \ldots, n-1\}$ as objects
- order preserving maps as arrows

We will then have $[0] = \varnothing$, $[1] = \{0\}$, $[2] = \{0, 1\}$ and so on. It is worth noticing that, in the category $\Delta$, $[0]$ is an initial object and $[1]$ is terminal object.

**Definition 5.12 (Ordinal sum).** The *ordinal sum* is a bifunctor $+ \colon \Delta \times \Delta \to \Delta$ defined in the following way.

**(Objects:).** Consider any two objects $[m]$, $[n]$. Their ordinal sum is defined as the object $[m+n]$ with inclusions given by:

$$\begin{aligned} [m] &\to [m+n] \\ i &\mapsto i \end{aligned} \qquad\qquad \begin{aligned} [n] &\to [m+n] \\ i &\mapsto m+i \end{aligned}$$

**(Arrows:)** Consider any two arrows $f \colon [m] \to [n]$, $f' \colon [m'] \to [n']$. Their ordinal sum is $f + f' \colon [m+m'] \to [n+n']$ assigning

$$i \mapsto \begin{cases} f(i) & \text{for } i = 0, \ldots, m-1 \\ n + f'(i-m) & \text{for } i = m, \ldots, n-1 \end{cases}$$

Similarly to the sum we are accostumed to, the ordinal sum we just defined has $[0] = \varnothing$ as the neutral element.

Furthermore, this operation defines a monoidal structure $(\Delta, +, [0])$. We hence investigate generators and relations starting from some known properties of $\Delta$ [3].

ld reference

**Lemma 5.13.** *Every arrow in $\Delta$ can be obtained as a composition of*

$$\begin{aligned} \delta^n_k \colon [n] &\to [n+1] \quad \textit{for } k = 0, \ldots n, \textit{ called } \text{(co)face maps} \\ i &\mapsto \begin{cases} i & \textit{for } i < k \\ i+1 & \textit{for } i \geq k \end{cases} \end{aligned}$$

[2] This structure is sometimes referred to as the *skeletal augmented simplex category*.

[3] Alternatively we could completely define $\Delta$ graphically, without reliying on simplicial relations. Such approach can be found in [Koc03]

$$\sigma_k^n\colon [n+2] \to [n+1] \quad \textit{for } k = 0, \dots n, \textit{ called } \text{(co)degeneracy maps}$$

$$i \mapsto \begin{cases} i & \textit{for } i \leq k \\ i-1 & \textit{for } i > k \end{cases}$$

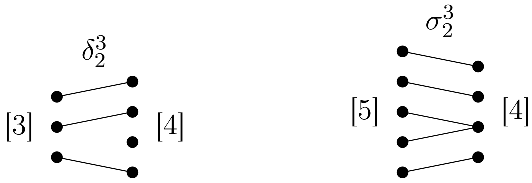
*satisfying the so called* (co)simplicial identities.

$$\delta_i^{n+1} \circ \delta_j^n = \delta_{j+1}^{n+1} \circ \delta_i^n \quad \textit{for } i \leq j \tag{5.1}$$

$$\sigma_j^n \circ \sigma_i^{n+1} = \sigma_i^n \circ \sigma_{j+1}^{n+1} \quad \textit{for } i \leq j \tag{5.2}$$

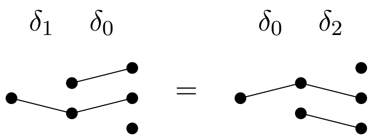
$$\sigma_j^n \circ \delta_i^{n+1} = \begin{cases} \delta_i^n \circ \sigma_{j-1}^{n-1} & \textit{for } i < j \\ \mathrm{id} & \textit{for } i = j, i = j+1 \\ \delta_{i-1}^n \circ \sigma_j^{n-1} & \textit{for } i > j+1 \end{cases} \tag{5.3}$$

But what does this lemma have to do with internal monoids? The relation between the two becomes clear when we represent the two maps and their relations graphically. We will represent each $[n]$ as a column of $n$ black dots, stacked one over the other from the bottom (element 0) to the top (element $n-1$). We begin by drawing the two maps defined in the above lemma for the case $n = 3$, $k = 2$.

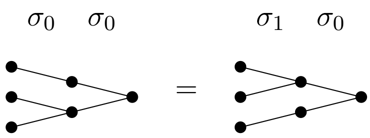



Then, the relations between the two can be depicted as follows.

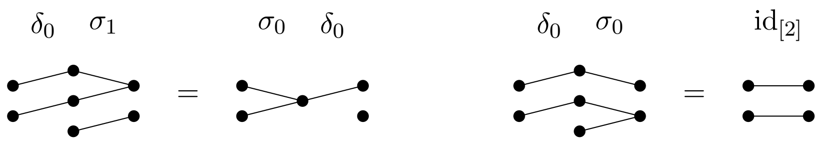
- We compute the first relation (5.1) for $n = 2$, $i = 0$, $j = 1$

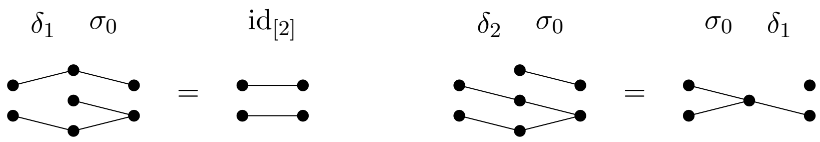



- We compute the second relation (5.2) for $n = i = j = 0$.




- And we draw four different cases for the last relation

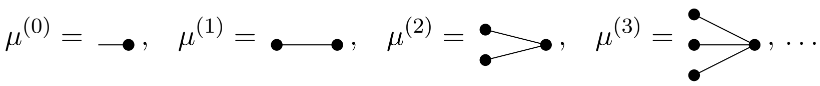





Since [1] is a terminal object in $\Delta$ we have a unique arrow

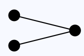
$$\mu^{(n)} \colon [n] \to [1] \qquad \text{for every } [n] \in \mathrm{Obj}(\Delta)$$

These arrows can be drawn as




We can hence rewrite the above lemma in the following way:

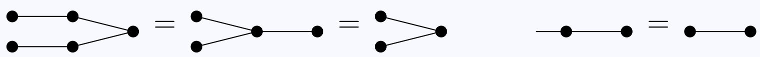
**Lemma 5.14.** *The monoidal category* $(\Delta, +, [0])$ *is completely generated in the sense of Definition 3.6 by the morphisms*

 *and* 

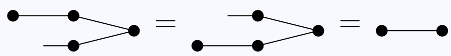
$$\mu^{(2)} \colon [2] \to [1] \qquad\qquad \mu^{(0)} \colon [0] \to [1]$$

*satisfying the following relations:*

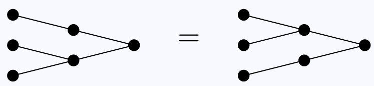
- *identity relations*




- *unitality*




- *associativity*




These axioms carry a striking resemblance to the ones we drew above, when describing the notion of internal monoid. We can hence state that $([1], \mu^{(2)}, \mu^{(0)})$ is a monoid in the monoidal category $(\Delta, +, [0])$. Since this monoid completely generates the whole category we have the following lemma.

**Lemma 5.15.** *The simplex category* $(\Delta, +, [0])$ *is the free monoidal category on a single monoid, namely* $([1], \mu^{(2)}, \mu^{(0)})$.

More is actually true.

**Theorem 5.16.** *Given a monoidal category $(\mathcal{C}, \square, I)$ there is a canonical equivalence of categories*

$$\mathbf{MonCat}(\Delta, \mathcal{C}) \cong \mathbf{Mon}(\mathcal{C})$$

*Proof.* To prove the statement holds we need to define the functors in both directions and check they really are inverses to each other.

We start from left to right, defining a functor $\phi\colon \mathbf{MonCat}(\Delta, \mathcal{C}) \to \mathbf{Mon}(\mathcal{C})$. Given a monoidal functor $F\colon \Delta \to \mathcal{C}$, we can just evaluate it on the monoid $([1], \mu^{(2)}, \mu^{(0)})$. By Lemma 5.10, we obtain an internal monoid $(F([1]), F(\mu^{(2)}), F(\mu^{(0)}))$ in $\mathcal{C}$. Now, let $u\colon F \Rightarrow G$ be a monoidal natural transformation and consider its component $u_{[1]}\colon F([1]) \to G([1])$. The naturality of $u$ with respect to the generating maps of $\Delta$, ensures the compatibility of $u_{[1]}$ with the monoid structure. This means that $u_{[1]}$ defines a morphism of internal monoids in $\mathcal{C}$.

Conversly, given a monoid $(M, \mu, \eta)$ in $\mathcal{C}$, we define a monoidal functor $F_M\colon \Delta \to \mathcal{C}$ as follows

$$\begin{aligned} F_M\colon \Delta &\to \mathcal{C} \\ [1] &\mapsto M \\ (\mu^{(2)}\colon [2] \to [1]) &\mapsto (\mu\colon M \,\square\, M \to M) \\ (\mu^{(0)}\colon [0] \to [1]) &\mapsto (\eta\colon I \to M) \end{aligned}$$

This association properly defines a functor because the relations of internal monoid are the same holding between $\mu^{(0)}$ and $\mu^{(2)}$. Given an homomorphism of monoids $\varphi\colon (M, \mu, \eta) \to (M', \mu', \eta')$, we then define a natural transformation $u\colon F_M \Rightarrow F_{M'}$ as:

$$u_1 = \phi\colon F_M([1]) = M \to F_{M'}([1]) = M'$$

$$u_n = \underbrace{u_1 \,\square \cdots \square\, u_1}_{n \text{ times}}$$

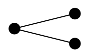
This determines a monoidal natural transformation between the two functors, therefore a well defined functor $\psi\colon \mathbf{Mon}(\mathcal{C}) \to \mathbf{MonCat}(\Delta, \mathcal{C})$.

Finally we want to check the two functors just defined are inverses to each other. We hence take a natural transformation $u\colon F \Rightarrow G$ in $\mathbf{MonCat}(\Delta, \mathcal{C})$ and compute its image through $\phi$. We obtain the homomorphism of monoids

$$F(u) = u_1\colon (F([1]), F(\mu^{(2)}), F(\mu^{(0)})) \to (G([1]), G(\mu^{(2)}), G(\mu^{(0)}))$$

to which we apply the functor $\psi$. Its image will be a natural transformation between the functors $F_{F([1])}$ and $F_{G([1])}$. Let us notice how, by definition of $\psi$, the functor $F_{F([1])}$ sends $[1]$ to $F([1])$, $\mu^{(2)}$ to $F(\mu^{(2)})$ and $\mu^{(0)}$ to $F(\mu^{(0)})$. We then have $F_{F([1])} = F$ and, by the same argument, $F_{G([1])} = G$. The natural transformation between the two is then equal to the one defined monoidally on $u_1$, hence $u\colon F \Rightarrow G$ itself. $\square$

*Remark* 5.17. The equivalence also holds when considering internal comonoids. In such case we'll need to consider the category $\Delta^{op}$, generated by

 and 

$$\mu^{(2)}\colon [2] \to [1] \qquad\qquad \mu^{(0)}\colon [0] \to [1]$$

satisfying the relations specular to the ones in Lemma 5.14. We then have the canonical equivalence of categories

$$\mathbf{MonCat}(\Delta^{op}, \mathcal{C}) \cong \mathbf{Comon}(\mathcal{C})$$

We still have not highlighted the connection between these general results and the equivalence we stated in Theorem 5.1. To do so we just need to choose the right monoidal category. Indeed by taking $(\mathcal{C}, \Box, I)$ equal to $(\mathbf{Vect}_\Bbbk, \otimes, \Bbbk)$ we obtain

$$\mathbf{MonCat}(\Delta, \mathbf{Vect}_\Bbbk) \cong \mathbf{Mon}(\mathbf{Vect}_\Bbbk) = \mathbf{Alg}_\Bbbk$$

where algebras finally appear. In this case we are still considering *all* algebras. To restrict to the category $\mathbf{cFAlg}_\Bbbk$ we need to define some more internal structures.

**Definition 5.18 (Internal Frobenius objects).** Let $(\mathcal{C}, \Box, I)$ be a monoidal category. An *internal Frobenius object* in $\mathcal{C}$ is an object $M$ equipped with four morphisms

$$\mu\colon M \,\Box\, M \to M \quad \text{(multiplication)} \qquad\qquad \eta\colon I \to M \quad \text{(unit)}$$

$$\delta\colon M \to M \,\Box\, M \quad \text{(comultiplication)} \qquad\qquad \varepsilon\colon M \to I \quad \text{(counit)}$$

such that the following diagrams commute

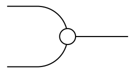
$$\begin{tikzcd}
 & M \,\Box\, M \arrow[dr, "\mu"] & & M \,\Box\, M \arrow[dl, "\mu"'] & \\
I \,\Box\, M \arrow[ur, "\eta \Box \mathrm{id}_M"] \arrow[rr, "="'] & & M & & M \,\Box\, I \arrow[ul, "\mathrm{id}_M \Box \eta"'] \arrow[ll, "="]
\end{tikzcd}$$

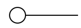
$$\begin{tikzcd}
 & M \,\Box\, M \arrow[dl, "\varepsilon \Box \mathrm{id}_M"'] & & M \,\Box\, M \arrow[dr, "\mathrm{id}_M \Box \varepsilon"] & \\
I \,\Box\, M & & M \arrow[ul, "\delta"'] \arrow[ur, "\delta"] \arrow[ll, "="] \arrow[rr, "="'] & & M \,\Box\, I
\end{tikzcd}$$

$$\begin{tikzcd}
 & M \,\Box\, M \,\Box\, M \arrow[dr, "\mu \Box \mathrm{id}_M"] & \\
M \,\Box\, M \arrow[ur, "\mathrm{id}_M \Box \delta"] \arrow[r, "\mu"] \arrow[dr, "\delta \Box \mathrm{id}_M"'] & M \arrow[r, "\delta"] & M \,\Box\, M \\
 & M \,\Box\, M \,\Box\, M \arrow[ur, "\mathrm{id}_M \Box \mu"'] &
\end{tikzcd}$$

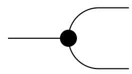
As we did for monoids and comonoids we can give a graphical representation of these objects.

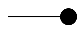
**(Internal Frobenius objects).** Let $(\mathcal{C}, \square, I)$ be a monoidal category. An *internal Frobenius object* in $\mathcal{C}$ is an object $M$ equipped with four morphisms





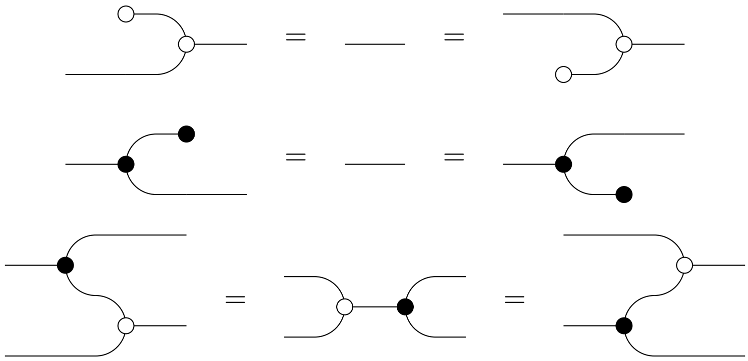
$\mu \colon M \square M \to M$ $\qquad$ $\eta \colon I \to M$





$\delta \colon M \to M \square M$ $\qquad$ $\epsilon \colon M \to I$

such that the following relations hold



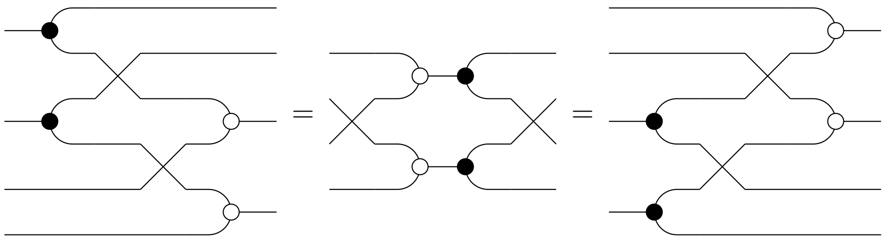
**Lemma 5.19.** *An internal Frobenius object $M$ is both an internal monoid and an internal comonoid.*

*Proof.* Let $(M, \mu, \eta, \delta, \varepsilon)$ be a Frobenius object. By proving $\mu$ and $\delta$ are associatives, we obtain an internal monoid $(M, \mu, \eta)$ and an internal comonoid $(M, \delta, \varepsilon)$. The proof of this fact is equivalent to the graphical proof of the equivalence of Frobenius algebras we gave in the previous chapter (see Proposition 4.19). $\square$

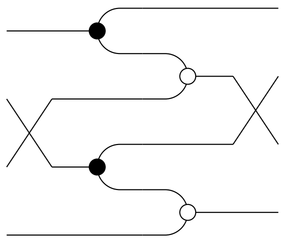
Following this lemma, we define *homomorphisms of internal Frobenius objects* to be morphisms $\varphi \colon (M, \mu, \eta, \delta, \varepsilon) \to (M', \mu', \eta', \delta', \varepsilon')$ that are homomorphisms of both the underlying monoid structure and the comonoid one. In this way, given any monoidal category $(\mathcal{C}, \square, I)$, we obtain a category $\mathbf{Frob}(\mathcal{C})$ whose objects are internal Frobenius objects in $\mathcal{C}$ and arrows are the homomorphisms between them.

**Lemma 5.20.** *Let $(\mathcal{C}, \square, I, \beta)$ be a symmetric monoidal category. Then* $\mathbf{Frob}(\mathcal{C})$ *can be canonically equipped with a monoidal structure.*

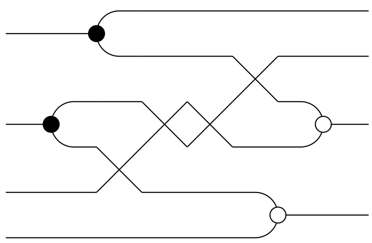
*Proof.* Let us take two internal Frobenius object $(M, \mu, \eta, \delta, \varepsilon)$, $(M', \mu', \eta', \delta', \varepsilon')$ and prove the multiplication defined in Lemmma 5.7 together with the analogous comultiplication define an internal Frobenius object $M \,\Box\, M'$. What we need to do is proving the Frobenius relation holds.



We prove the left-side of the equivalence, starting from the configuration in the middle. By applying the Frobenius relation we have



By naturality of the twist, this is equivalent to

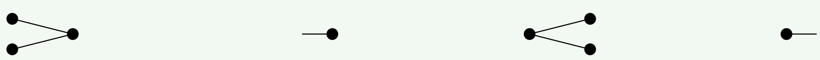


The twists in the middle vanish and we conclude our proof. □

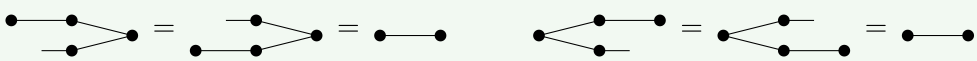
This framework allows us to define an analogue of the simplex category $\Delta$ in the case of internal Frobenius objects.

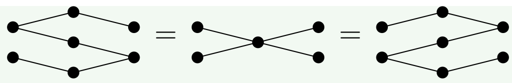
**Definition 5.21 (A free monoidal category on an internal Frobenius object).** Let $(\mathcal{F}, \Box, 0)$ be the monoidal category whose objects are the monoidal powers of a given object $F$ and whose arrows are generated in the sense of Definition 3.6 by

$\mu\colon F \,\Box\, F \to F$ $\quad\quad$ $\eta\colon 0 \to F$ $\quad\quad$ $\delta\colon F \to F \,\Box\, F$ $\quad\quad$ $\varepsilon\colon F \to 0$



satisfying the following relations:





We this definition we are able to extend Theorem 5.16 to the Frobenius case.

**Theorem 5.22.** *Given a monoidal category* $(\mathcal{C}, \Box, I)$ *there is a canonical equivalence of categories*

$$\mathbf{MonCat}(\mathcal{F}, \mathcal{C}) \cong \mathbf{Mon}(\mathcal{C})$$

Since we are interested in symmetric monoidal categories, it makes sense to understand how the Frobenius structure interacts with the given twist.

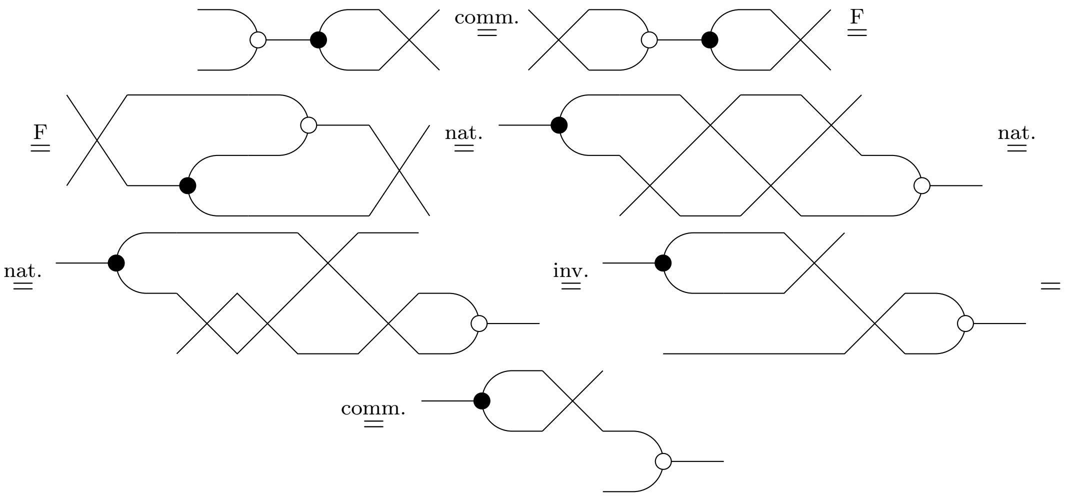
**Definition 5.23 (Commutative Frobenius objects).** An internal Frobenius object $(M, \mu, \eta, \delta, \varepsilon)$ in a symmetric monoidal category $(\mathcal{C}, \Box, I, \beta)$ is *commutative* if the twist commutes with the multiplication $\mu$.

**Definition 5.24 (Cocommutative Frobenius objects).** An internal Frobenius object $(M, \mu, \eta, \delta, \varepsilon)$ in a symmetric monoidal category $(\mathcal{C}, \Box, I, \beta)$ is *cocommutative* if the twist cocommutes with the comultiplication $\delta$.

**Lemma 5.25.** *An internal Frobenius object in a symmetric monoidal category is commutative if and only if it is cocommutative.*

*Proof.* Let $(\mathcal{C}, \Box, I, \beta)$ be a symmetric monoidal category. Assume $(M, \mu, \eta, \delta, \varepsilon)$ is a commutative Frobenius object in $\mathcal{C}$ and consider the composition $\beta \circ \delta$. Having already proved in Section 4.2.1 the uniqueness of the comultiplication, we can prove $\beta \circ \delta = \delta$ by showing that $\beta \circ \delta$ has counit $\varepsilon$ and satisfies the Frobenius relation. The first property follows by naturality of the twist:

Finally we prove the Frobenius relation holds.




$\square$

Having defined commutative internal Frobenius objects, we observe that they form a subcategory $\mathbf{cFrob}(\mathcal{C})$. At this point we notice that when trying to extend the equivalence to the commutative Frobenius case, the required relations are exactly the ones we enumerated in Section 3.2: $\mathbf{Bord}(2)$ is the free symmetric monoidal category on a commutative Frobenius object. We therefore conclude by stating the following equivalence.

**Theorem 5.26.** *Given a monoidal category $(\mathcal{C}, \Box, I)$ there is a canonical equivalence of categories*

$$\mathbf{SymMonCat}(\mathbf{Bord}(2), \mathcal{C}) \cong \mathbf{cFrob}(\mathcal{C})$$

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