Analytical Proof of Concavity of the Jensen Park Wind Farm Model

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The Park model is used for a rough estimate how much power increase is possible with stationary axial induction control and to demonstrate that the state-of-the-art-greedy approach of operating wind farms necessarily leads to sub-optimal results. The power of the i^{th} turbine in a wind farm with n_{WT} turbines is

$$P_{WTi}(a_i) = 0.5\rho_a A_R c_P(a_i) U_i^3(a_i). \tag{1}$$

with air density ρ_a, rotor surface A_R, wind at the turbine U_i and power coefficient c_P depending on axial induction factor a_i

$$c_P(a_i) = 4a_i(1-a_i)^2. \tag{2}$$

Wind speed V_{Ri} at turbine i depends on the free-flow wind V_∞, the thrust coefficient of all upstream turbines, i.e. all turbines from the set $\mathcal{W} = \{1, 2, ..., j, ..., i, .., n_{WT}\}$, and coefficient $c_{i,j}$

$$U_i = V_\infty \left(1 - 2\sqrt{\sum_{j\in\mathcal{W}, x_i > x_j} (a_j c_{ji})^2}\right) \tag{3}$$

depending on rotor diameter D_j, turbine position x_i and wake expansion coefficient β

$$c_{ji} = \left(\frac{D_j}{D_j + 2\beta(x_i - x_j)}\right)^2$$

The sum term for the wake interaction from equation (3) can be reformulated as a vector multiplication for the i^{th} turbine

$$\sum_{j\in\mathcal{W}, x_i > x_j} (a_j c_{ji})^2 = \boldsymbol{E}_i \boldsymbol{C}_{\text{square},\boldsymbol{n_{WT}}} \boldsymbol{a}_{\text{square},\boldsymbol{n_{WT}}} = \boldsymbol{c}_{\text{square},\boldsymbol{i}} \boldsymbol{a}_{\text{square},\boldsymbol{n_{WT}}}$$

where the selector vector $\boldsymbol{E}_i \in \mathbb{R}^{1\times n_{WT}}$ is used to select the i^{th} row from the matrix

$$\boldsymbol{C}_{\text{square},\boldsymbol{n_{WT}}} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ c_{12}^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{1n_{WT}}^2 & c_{2n_{WT}}^2 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{n_{WT}}$$

and the squares of the axial induction factors form the vector

$$\boldsymbol{a}_{\text{square},\boldsymbol{n_{WT}}} = [a_1^2 \; a_2^2 \; \cdots \; a_{n_{WT}}^2]^T \in \mathbb{R}^{n_{WT} \times 1}$$

This results in the wake influence function W defined as

$$W(\boldsymbol{a_i}) = (1 - 2(\boldsymbol{c}_{\text{square},\boldsymbol{i}} \boldsymbol{a}_{\text{square},\boldsymbol{i}})^{0.5})^3 = (1 - 2(\boldsymbol{c}_{\text{square},\boldsymbol{i}}(\boldsymbol{a_i} \cdot \boldsymbol{a_i}))^{0.5})^3 \tag{4}$$

with $\boldsymbol{a_i} = [a_1 \; a_2 \; \cdots \; a_i]^T \in \mathbb{R}^{i \times 1}$. The total farm power P_{WF} can hence be optimized by changing the axial induction factors, which are the square roots of the elements in vector $\boldsymbol{a}_{\text{square},\boldsymbol{n_{WT}}}$. The farm power is a product of three terms: a constant term, the cube of the freestream wind speed, which is treated here as well as a constant, and the sum of the power coefficients and the wake influence function at the turbines. Both the power coefficient function c_P from equation (2) and the wake interaction function W from equation (4) depend on the axial induction coefficients. The power coefficient of the j^{th} turbine depends on the axial induction of this turbine. For the wake influence function, the induction factors of all turbines upstream of this turbine need to be considered. Therefore, there is obviously no correction for the first turbine that experiences the freestream wind, resulting in the last row of $\boldsymbol{C_{n_{WT}}}$ to contain only zeros. The wake of the last turbine in the row does not influence any other turbine, leading to the first column of $\boldsymbol{C_{n_{WT}}}$ to contain only zeros. This results in

$$P_{WF}(\boldsymbol{a_{n_{WT}}}) = \underbrace{0.5\rho_a A_R V_\infty^3}_{K_V} \sum_{j=1}^{n_{WT}} c_P(a_j) W(\boldsymbol{a_{j-1}}).$$

The optimal axial induction factor vector $\boldsymbol{a}$ for all turbines can be hence found by solving the optimization problem

$$\max_{\boldsymbol{a_{n_{WT}}}} P_{WF}(\boldsymbol{a_{n_{WT}}}) = \max_{\boldsymbol{a_{n_{WT}}}} K_V \sum_{i=1}^{n_{WT}} c_P(a_i) W(\boldsymbol{a_{i-1}}), \tag{5}$$

i.e, by simply deriving the roots of the derivative

$$\frac{\partial P_{WF}(\boldsymbol{a_{n_{WT}}})}{\partial \boldsymbol{a}} = \left[\frac{\partial P_{WF}(\boldsymbol{a})}{\partial a_1} \; \frac{\partial P_{WF}(\boldsymbol{a})}{\partial a_2} \; ... \; \frac{\partial P_{WF}(\boldsymbol{a})}{\partial a_{n_{WT}}} \right] \stackrel{!}{=} \boldsymbol{0}.$$

With the partial derivative for the i^{th} turbine

$$\frac{\partial P_{WF}(\boldsymbol{a_i})}{\partial a_i} = K_V \left(\frac{\partial c_P(a_i)}{\partial a_i} W(\boldsymbol{a_{i-1}}) + \underbrace{c_P(a_i) \frac{\partial W(\boldsymbol{a_{i-1}})}{\partial a_i}}_{=0} \right)$$

the second partial derivatives are

$$\frac{\partial^2 P_{WF}(\boldsymbol{a_i})}{\partial a_i^2} = K_V \frac{\partial^2 c_P(a_i)}{\partial a_i^2} W(\boldsymbol{a_{i-1}}) \tag{6}$$

and

$$\frac{\partial^2 P_{WF}(\boldsymbol{a_i})}{\partial a_i \partial a_j} = K_V \frac{\partial c_P(a_i)}{\partial a_i} \frac{\partial W(\boldsymbol{a_{i-1}})}{\partial a_j} \tag{7}$$

The first and second derivatives of the power coefficient as defined in equation (2) are

$$\frac{\partial c_P(a_i)}{\partial a_i} = 4(3a_i^2 - 4a_i + 1) = 4(1 - 3a_i)(1 - a_i) \tag{8}$$

and

$$\frac{\partial^2 c_P(a_i)}{\partial a_i^2} = 4(6a_i - 4) \tag{9}$$

with the axial induction limited to the physically meaningful set $a_i \in [0, 0.5]$ and hence the only optimum at the Betz limit with $a^* = 1/3$.

The derivative of the wake reduction is calculated as

$$\begin{aligned} \frac{\partial W(\boldsymbol{a_{i-1}})}{\partial a_j} &= \frac{\partial}{\partial a_j} \left(1 - 2(\boldsymbol{c}_{\text{square},\boldsymbol{i}} \boldsymbol{a}_{\text{square},\boldsymbol{n_{WT}}})^{0.5}\right)^3 && (10) \\ &= -6a_j c_{ji}^2 \left(1 - 2(\boldsymbol{c}_{\text{square},\boldsymbol{i}} \boldsymbol{a}_{\text{square},\boldsymbol{n_{WT}}})^{0.5}\right)^2 (\boldsymbol{c}_{\text{square},\boldsymbol{i}} \boldsymbol{a}_{\text{square},\boldsymbol{n_{WT}}})^{-0.5} && (11) \end{aligned}$$

It is possible to show that equations (6) and (7) are negative for all axial induction factors within physically meaningful bounds for a physically sensible wake expansion coefficient β and distances between turbines. We show that in both equations one product term is always negative and the other positive to demonstrate that the Park model is a concave function.

The second derivative of the power coefficient, quation (9) is negative for all axial induction factors within $a_i \in [0, 0.5]$. Therefore, equation (6) will be negative if the wake influence function, i.e., the term $W(\boldsymbol{a_i})$, is positive. This positivity indicates that the wind speed is reduced, but not reversed. Consequently, in a meaningful deficit model, this wind term is indeed positive, which ensures that equation (6) is in fact negative. We can also show this mathematically: This means that the product $\boldsymbol{c}_{\text{square},\boldsymbol{i}} \boldsymbol{a}_{\text{square},\boldsymbol{i}}$ has to be smaller than 0.25, as the term $(1 - 2(\boldsymbol{c}_{\text{square},\boldsymbol{i}} \boldsymbol{a}_{\text{square},\boldsymbol{i}})^{0.5})$ has to be positive. We can pull the maximum value for $\boldsymbol{a}_{\text{square},\boldsymbol{i}} = 0.25$ out and deduce that the magnitude of the vector $\boldsymbol{c}_{\text{square},\boldsymbol{n_{WT}}}$ has to be smaller than 1. With the selected values for $\beta = 0.075$ and the distance between the turbines multiple values of $5D$ this is indeed the case: We find an upper bound for the series $\sum_{i=1}^{\infty} \left(\frac{1}{1+0.75i}\right)^4$ by initially pulling out the first terms, which is $(1/1.75)^4 \approx 0.107$.

Leveraging the Riemann Zeta function function $\zeta(4) = \sum_{i=1}^{\infty} \frac{1}{i^4} = \pi^4/90$, the remaining series is approximated by

$$\sum_{i=3}^{\infty} \left(\frac{1}{1 + 0.75i}\right)^4 \leq \frac{256}{81} \sum_{i=2}^{\infty} \frac{1}{i^4} = \frac{256}{81} \left(\frac{\pi^4}{90} - 1\right) \approx 0.260$$

Therefore, the total sum is bounded by 0.367 and hence converges to a value below 1. With this, equation (6), which gives the diagonal elements of the Hessian matrix, is negative.

The optimization problem in equation (5) is hence concave if the partial derivative term in equation (7), which gives the off-diagonal elements of the Hessian matrix, is also negative. This is indeed the case, as the derivative of the power coefficient, equation (8), is positive and the derivative of the wake reduction function, equation (10), is negative for all axial induction factors $a_i \in [0, 0.5]$.

To get some intuition for this, we consider the simple example of an array consisting only of two turbines. The power of the second turbine P_{WT2} is influenced by how much energy is taken out of the wind at the first turbine and hence its wake, $W(a_1)$, and the expansion factor c_{12}. For the case of two turbines there is, as expected, $a_2^* = 1/3$, as this fullfills

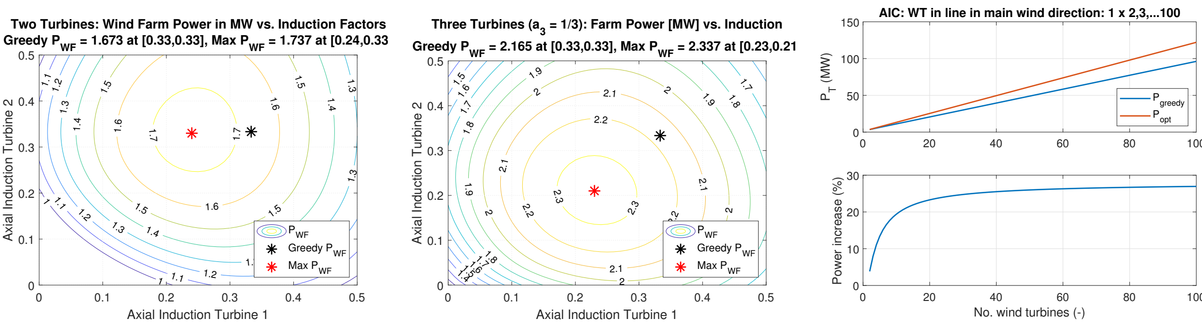
$$\begin{aligned}\left.\frac{\partial P_{WF}(a_1,a_2)}{\partial a_2}\right|_{a_2=a_2^*} &\stackrel{!}{=} 0 \\ &= K_V(4(1-3a_2)(1-a_2))(1-2a_1c_{12}).\end{aligned}$$

Hence a_1^* can be calculated depending on the wake interaction coefficient c_{12} as

$$\left.\frac{\partial P_{WF}(a_1,a_2)}{\partial a_1}\right|_{a_1=a_1^*,a_2=a_2^*} \stackrel{!}{=} 0$$
$$= K_V(4(1-3a_1)(1-a_1) - 4a_2(1-a_2)^2(c_{12}(1-2a_1c_{12})^2))$$

with $a_1^* = 0.24$. For the three turbine, configuration it can be derived similarly that the optimal thrust coefficients are $a_3^* = 0.33$, $a_2^* = 0.23$, and $a_1^* = 0.21$. Figure 1 shows the farm power for a grid of possible combinations of a_1 and a_2 for an array of two turbine and three turbines and the optimal combination of this axial induction factors to visualize the concavity of the Park model. It also shows how, according to the Park model, the power output would increase with increasing number of turbines in the array. This clearly exaggerates the improvement possible and shows the limitation of this simple model.

Note that the Jensen model assumes that the turbines all face the wind and hence is not suitable for calculating wake effects of turbines with intentional yaw misalignment used in WRC. For this, the Bastankhah Gaussian wake model is used.



(a) Two-turbine array

(b) Three-turbine array

(c) AIC power increase up to 100 turbines

Figure 1: Wind farm power for arrays with two and three turbines with axial induction factor a_1 and a_2 of the first two turbines and power yield improvement per turbine number according to the Park model