Investigation into the Concavity of the Jensen Park Wind Farm Model

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The Park model is used for a rough estimate how much power increase is possible with stationary axial induction control and to demonstrate that the state-of-the-art-greedy approach of operating wind farms necessarily leads to sub-optimal results.

From numerical solving the farm equation with the MATLAB function *fmincon* we know that all elements and the eigenvalues of the approximated Hessian matrices are negative for the considered cases of two to ten turbines. We also see that the on-diagonal elements of the approximated Hessian matrix are larger than the sum of the off-diagonal elements. Below is outlined how we can analytically show that the elements of the Hessian matrix are negative. However, showing that the wind farm function is concave is future work.

The power of the i^{th} turbine in a wind farm with n_{WT} turbines is

$$P_{WTi}(a_i) = 0.5\rho_a A_R c_P(a_i) U_i^3(a_i). \tag{1}$$

with air density ρ_a, rotor surface A_R, wind at the turbine U_i and power coefficient c_P depending on axial induction factor a_i

$$c_P(a_i) = 4a_i(1-a_i)^2. \tag{2}$$

Wind speed V_{Ri} at turbine i depends on the free-flow wind V_∞, the thrust coefficient of all upstream turbines, i.e. all turbines from the set $\mathcal{W} = \{1, 2, ..., j, ..., i, .., n_{WT}\}$, and coefficient $c_{i,j}$

$$U_i = V_\infty \left(1 - 2\sqrt{\sum_{j \in \mathcal{W}, x_i > x_j} (a_j c_{ji})^2}\right) \tag{3}$$

depending on rotor diameter D_j, turbine position x_i and wake expansion coefficient β

$$c_{ji} = \left(\frac{D_j}{D_j + 2\beta(x_i - x_j)}\right)^2$$

The sum term for the wake interaction from equation (3) can be reformulated as a vector multiplication for the i^{th} turbine

$$\sum_{j \in \mathcal{W}, x_i > x_j} (a_j c_{ji})^2 = \boldsymbol{E}_i \boldsymbol{C}_{\text{square},\boldsymbol{n_{WT}}} \boldsymbol{a}_{\text{square},\boldsymbol{n_{WT}}} = \boldsymbol{c}_{\text{square},\boldsymbol{i}} \boldsymbol{a}_{\text{square},\boldsymbol{n_{WT}}}$$

where the selector vector $\boldsymbol{E}_i \in \mathbb{R}^{1\times n_{WT}}$ is used to select the i^{th} row from the matrix

$$\boldsymbol{C}_{\text{square},\boldsymbol{n_{WT}}} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ c_{12}^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{1n_{WT}}^2 & c_{2n_{WT}}^2 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{n_{WT}}$$

and the squares of the axial induction factors form the vector

$$\boldsymbol{a}_{\text{square},\boldsymbol{n_{WT}}} = [a_1^2 \; a_2^2 \; \cdots \; a_{n_{WT}}^2]^T \in \mathbb{R}^{n_{WT}\times 1}$$

This results in the wake influence function W defined as

$$W(\boldsymbol{a_i}) = (1 - 2(\boldsymbol{c}_{\text{square},\boldsymbol{i}}\boldsymbol{a}_{\text{square},\boldsymbol{i}})^{0.5})^3 = (1 - 2(\boldsymbol{c}_{\text{square},\boldsymbol{i}}(\boldsymbol{a_i} \cdot \boldsymbol{a_i}))^{0.5})^3 = (V_W(\boldsymbol{a_i}))^3 \quad (4)$$

with $\boldsymbol{a_i} = [a_1 \; a_2 \; \cdots \; a_i]^T \in \mathbb{R}^{i\times 1}$. We introduce the function $V_W(\boldsymbol{a_i})$ for simpler notation in the derivative terms. The total farm power P_{WF} can hence be optimized by changing the axial induction factors, which are the square roots of the elements in vector $\boldsymbol{a}_{\text{square},\boldsymbol{n_{WT}}}$. The farm power is a product of three terms: a constant term, the cube of the freestream wind speed, which is treated here as well as a constant, and the sum of the power coefficients and the wake influence function at the turbines. Both the power coefficient function c_P from equation (2) and the wake interaction function W from equation (4) depend on the axial induction coefficients. The power coefficient of the j^{th} turbine depends on the axial induction of this turbine. For the wake influence function, the induction factors of all turbines upstream of this turbine need to be considered. Therefore, there is obviously no correction for the first turbine that experiences the freestream wind, resulting in the last row of $\boldsymbol{C_{n_{WT}}}$ to contain only zeros. The wake of the last turbine in the row does not influence any other turbine, leading to the first column of $\boldsymbol{C_{n_{WT}}}$ to contain only zeros. This results in

$$P_{WF}(\boldsymbol{a_{n_{WT}}}) = \underbrace{0.5\rho_a A_R V_\infty^3}_{K_V} \sum_{i=1}^{n_{WT}} c_P(a_i) W(\boldsymbol{a_{i-1}}).$$

The optimal axial induction factor vector $\boldsymbol{a}$ for all turbines can be hence found by solving the optimization problem

$$\max_{\boldsymbol{a_{n_{WT}}}} P_{WF}(\boldsymbol{a_{n_{WT}}}) = \max_{\boldsymbol{a_{n_{WT}}}} K_V \sum_{i=1}^{n_{WT}} c_P(a_i) W(\boldsymbol{a_{i-1}}), \quad (5)$$

i.e, by deriving the roots of the derivative

$$\left[\frac{\partial P_{WF}(\boldsymbol{a_{n_{WT}}})}{\partial a_1} \; \frac{\partial P_{WF}(\boldsymbol{a_{n_{WT}}})}{\partial a_2} \; \dots \; \frac{\partial P_{WF}(\boldsymbol{a_{n_{WT}}})}{\partial a_{n_{WT}}} \right] \stackrel{!}{=} \mathbf{0}.$$

With the partial derivative for the j^{th} turbine influencing the output of the j^{th} turbine through its axial induction factor and the outputs of the $(j+1)^{\text{th}}$ to $n_{\text{WT}}^{\text{th}}$ turbines through its wake

$$\frac{\partial P_{WF}(\boldsymbol{a_{n_{WT}}})}{\partial a_j} = K_V \left(\frac{\partial c_P(a_j)}{\partial a_j} W(\boldsymbol{a_{j-1}}) + \sum_{i=j+1}^{n_{WT}} c_P(a_i) \frac{\partial W(\boldsymbol{a_{i-1}})}{\partial a_j} \right)$$

the second partial derivatives are

$$\frac{\partial^2 P_{WF}(\boldsymbol{a_{n_{WT}}})}{\partial a_j^2} = K_V \left(\frac{\partial^2 c_P(a_j)}{\partial a_j^2} W(\boldsymbol{a_{j-1}}) + \sum_{i=j+1}^{n_{WT}} c_P(a_i) \frac{\partial^2 W(\boldsymbol{a_{i-1}})}{\partial a_j^2} \right) \tag{6}$$

and

$$\frac{\partial^2 P_{WF}(\boldsymbol{a_{n_{WT}}})}{\partial a_j \partial a_l} = K_V \left(\frac{\partial c_P(a_j)}{\partial a_j} \frac{\partial W(\boldsymbol{a_{j-1}})}{\partial a_l} + \sum_{i=\max(j,l)+1}^{n_{WT}} \left(c_P(a_i) \frac{\partial^2 W(\boldsymbol{a_{i-1}})}{\partial a_j \partial a_l} \right) \right) \tag{7}$$

For the off-diagonals, only the wake terms for the turbines downstream of both the j^{th} and the l^{th} turbine are calculated due to the derivative terms otherwise being equal to zero. The first and second derivatives of the power coefficient as defined in equation (2) are

$$\frac{\partial c_P(a_i)}{\partial a_i} = 4(3a_i^2 - 4a_i + 1) = 4(1 - 3a_i)(1 - a_i) \tag{8}$$

and

$$\frac{\partial^2 c_P(a_i)}{\partial a_i^2} = 4(6a_i - 4) \tag{9}$$

with the axial induction limited to the physically meaningful set $a_i \in [0, 0.5]$ and hence the only optimum at the Betz limit with $a^* = 1/3$.

The derivative of the wake reduction is calculated as

$$\begin{aligned}\frac{\partial W(\boldsymbol{a_i})}{\partial a_j} =& \frac{\partial}{\partial a_j} \left(1 - 2(\boldsymbol{c}_{\text{square},\boldsymbol{i}} \boldsymbol{a}_{\text{square},\boldsymbol{i}})^{0.5}\right)^3 \\ =& -6a_j c_{ji}^2 \left(1 - 2(\boldsymbol{c}_{\text{square},\boldsymbol{i}} \boldsymbol{a}_{\text{square},\boldsymbol{i}})^{0.5}\right)^2 (\boldsymbol{c}_{\text{square},\boldsymbol{i}} \boldsymbol{a}_{\text{square},\boldsymbol{i}})^{-0.5} \\ =& -6a_j c_{ji}^2 (V_W(\boldsymbol{a_i}))^2 f_{\text{sqr}}(\boldsymbol{a_i})\end{aligned} \tag{10}$$

with $f_{\text{sqr}}(\boldsymbol{a_i}) = (\boldsymbol{c}_{\text{square},\boldsymbol{i}} \boldsymbol{a}_{\text{square},\boldsymbol{i}})^{-0.5}$. The second derivatives of the wake reduction are calculated as

$$\begin{aligned}\frac{\partial^2 W(\boldsymbol{a_i})}{\partial a_j^2} =& 6a_j^2 c_{ji}^2 (V_W(\boldsymbol{a_i}))^2 (f_{\text{sqr}}(\boldsymbol{a_i}))^3 + 24a_j^2 c_{ji}^2 V_W(\boldsymbol{a_i}) (f_{\text{sqr}}(\boldsymbol{a_i}))^2 \\ & - 6c_{ji} (V_W(\boldsymbol{a_i}))^2 f_{\text{sqr}}(\boldsymbol{a_i})\end{aligned} \tag{11}$$

and

$$\frac{\partial^2 W(\boldsymbol{a_{i-1}})}{\partial a_j \partial a_l} = a_j^2 c_{ji}^2 a_l^2 c_{li}^2 \left(6(f_{\text{sqr}}(\boldsymbol{a_i}))^3 - 24 f_{\text{sqr}}(\boldsymbol{a_i})\right) \tag{12}$$

To get some intuition for the wind farm function of the Park model, we consider the simple example of an array consisting only of two turbines with

$$P_{WF2}([a_1, a_2]) = K_V (4a_1(1 - a_1)^2 + 4a_2(1 - a_2)^2 (1 - 2a_1 c_{12})^3).$$

The power of the second turbine P_{WT2} is influenced by how much energy is taken out of the wind at the first turbine and hence its wake, $W(a_1)$, and the expansion factor c_{12}. For the case of two turbines there is, as expected, $a_2^* = 1/3$, as this fullfills

$$\begin{aligned}\left.\frac{\partial P_{WF2}(\boldsymbol{a}_2)}{\partial a_2}\right|_{a_2=a_2^*} &\overset{!}{=} 0 \\ &= K_V(4(1-3a_2)(1-a_2))((1-2a_1c_{12})^3).\end{aligned}$$

Hence a_1^* can be calculated depending on the wake interaction coefficient c_{12} as

$$\begin{aligned}\left.\frac{\partial P_{WF2}(\boldsymbol{a}_2)}{\partial a_1}\right|_{a_1=a_1^*,a_2=a_2^*} &\overset{!}{=} 0 \\ &= K_V((4(1-3a_1)(1-a_1) - 24a_2(1-a_2)^2(c_{12}(1-2a_1c_{12})^2))) \\ &= K_V(a_1^2(12-24c_{12}^3c_{p2}^*) + a_1(-16+24c_{12}^2c_{p2}) + 4 - 6c_{12}c_{p2})\end{aligned}$$

with $c_{p2}^* = c_p(a_2^*)$ resulting in a polynomial with the two roots, 0.242 and 1.016, and therefore $a_1^* = 0.242$. We checked that both MATLAB optimizers $fmincon$ and $fminsearch$ converge to this value.

We get the second derivative terms as

$$\begin{aligned}\left.\frac{\partial^2 P_{WF2}(\boldsymbol{a}_2)}{\partial a_1^2}\right|_{a_1=a_1^*,a_2=a_2^*} &= K_V(2a_1(12-24c_{12}^3c_{p2}^*) - 16 + 24c_{12}^2c_{p2}) \approx -8.675K_V \\ \left.\frac{\partial^2 P_{WF2}(\boldsymbol{a}_2)}{\partial a_1 \partial a_2}\right|_{a_1=a_1^*,a_2=a_2^*} &= 4K_V(3a_2^2-4a_2+1)(-24c_{12}^3a_1^2+24c_{12}^2a_1-6c_{12}) = 0 \\ \left.\frac{\partial^2 P_{WF2}(\boldsymbol{a}_2)}{\partial a_1^2}\right|_{a_1=a_1^*,a_2=a_2^*} &= 4K_V(6a_2-4)(1-2a_1c_{12})^3 \approx -4.765K_V\end{aligned}$$

and hence the eigenvalues of the Hessian are negative and the Park model for two turbines is concave.

For the three turbine, configuration it can be derived that the optimal thrust coefficients are $a_3^* = 0.333$, $a_2^* = 0.23$, and $a_1^* = 0.21$. However, the equations are more involved. The wind farm power is

$$P_{WF3}(a_1,a_2,a_3) = K_V(c_p(a_1) + c_p(a_2)(1-2a_1c_{12})^3 + c_p(a_3)(1-2(a_1^2c_{13}^2 + a_2^2c_{23}^2)^{0.5})^3).$$

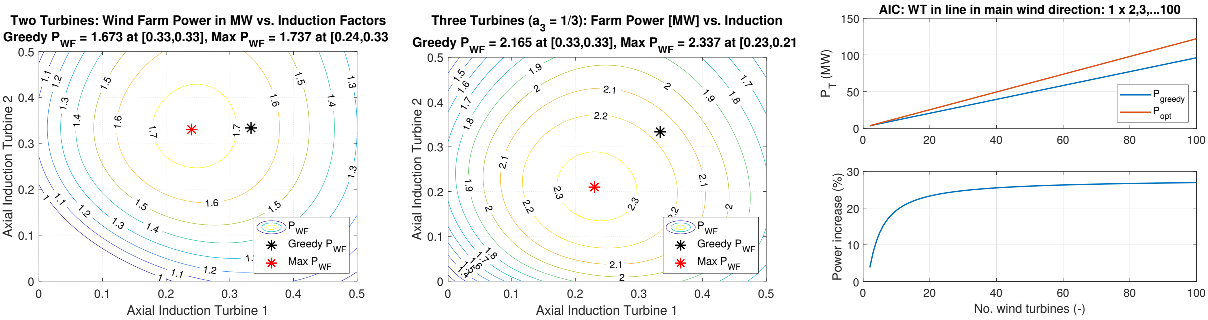
The partial derivatives are

$$\begin{aligned}\frac{\partial P_{WF3}(a_1,a_2,a_3)}{\partial a_3} =& K_V(4(3a_3^2-4a_3+1)(1-2(a_1^2c_{13}^2+a_2^2c_{23}^2)^{0.5})^3) \\ \frac{\partial P_{WF3}(a_1,a_2,a_3)}{\partial a_2} =& K_V(4(3a_2^2-4a_2+1)(1-2a_1c_{12})^3 \\ & - 6c_p(a_3)a_2c_{23}^2(1-2(a_1^2c_{13}^2+a_2^2c_{23}^2)^{0.5})^2(a_1^2c_{13}^2+a_2^2c_{23}^2)^{-0.5}) \\ \frac{\partial P_{WF3}(a_1,a_2,a_3)}{\partial a_1} =& K_V(4(3a_1^2-4a_1+1) - 6c_p(a_2)c_{12}(1-2a_1c_{12})^2 \\ & - 6c_p(a_3)a_1c_{13}(1-2(a_1^2c_{13}^2+a_2^2c_{23}^2)^{0.5})^2(a_1^2c_{13}^2+a_2^2c_{23}^2)^{-0.5})\end{aligned}$$

As before, the axial induction factor of the most downstream turbine is the solution of $\partial c_p(a_3)/\partial a_3 = 0$ and therefore $a_3^* = 1/3$. For the other two more involved terms, we did not calculate an analytical solution, but only calculated the solutions with the MATLAB optimizers and verified that including them in the first partial derivatives yields results close to zero.

Figure 1 shows the farm power for a grid of possible combinations of a_1 and a_2 for an array of two turbine and three turbines and the optimal combination of this axial induction factors to visualize the concavity of the Park model. It also shows how, according to the Park model, the power output would increase with increasing number of turbines in the array. This clearly exaggerates the improvement possible and shows the limitation of this simple model.

Note that the Jensen model assumes that the turbines all face the wind and hence is not suitable for calculating wake effects of turbines with intentional yaw misalignment used in WRC. For this, the Bastankhah Gaussian wake model is used.



(a) Two-turbine array

(b) Three-turbine array

(c) AIC power increase up to 100 turbines

Figure 1: Wind farm power for arrays with two and three turbines with axial induction factor a_1 and a_2 of the first two turbines and power yield improvement per turbine number according to the Park model