Physics of fluids & nonlinear physics

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Modelling fluids

Chapter 1

Fluid motion

The description of fluid flows can rapidly be obscured behind seemingly complex equations (e.g. Navier–Stokes). Even worse these equations can change depending on the context and problem considered. Actually, the governing equations for fluid motion express truly simple physical principles, namely **mass conservation** (the mass $m(t)$ of a fluid particle is constant), **momentum conservation** (a fluid particle's momentum obeys Newton's second law $m\gamma = \Sigma F$) and **energy conservation** (the energy of a fluid particle follows the first and second principle of thermodynamics). In this introduction we will review the derivation of the fluid mechanics equations by expressing these fundamental conservation principles.

1.1 Forces

As other continuum media, fluids carry force fields that determine their equilibrium (when the net force contribution is zero) or their motion. If each fluid particle is subject to *body forces*, it is also exposed to *surface forces* called **stresses**, as for example pressure (Fig. 1.1) that we now examine.

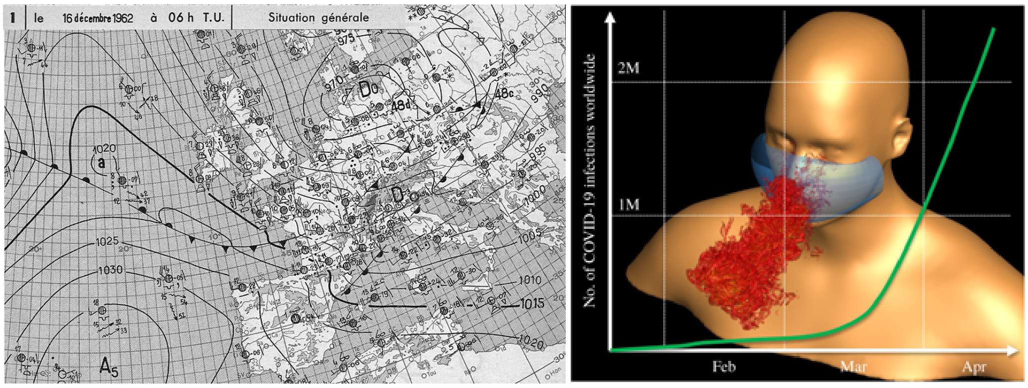

Figure 1.1: **Illustrations of pressure forces in daily-life phenomena.** Left: France isobaric map from December, 16 1962. The map reveals low-pressure area and anticyclones (high-pressure area). The clustering isobars near Corsica are a signature of the violent winds that swept the region (force 12 on the Beaufort scale i.e. the threshold for hurricanes for sailors ; 216 km/h at the cap Corse). Source: `http://tempetes.meteo.fr`. Right: a coughing event is associated with a violent lung compression. The resulting overpressure drives a rapid airflow that may torn and transport liquid droplets and aerosols (Mittal *et al.*, 2020).

1.1.1 Pressure

▷ **Archimedes principle without equation.** Consider a fluid at equilibrium in the gravity field (Fig. 1.3). Let's isolate now mentally a fluid portion. It experiences from the gravity field a force corresponding to its *weight* $\boldsymbol{P}$ pointing downwards. It also bears *pressure forces* from the surrounding fluid. Since the fluid portion is at equilibrium, the net pressure force has to balance the weight (equal in intensity but opposite in direction). Now if in our mind experiment we were to replace the fluid portion by a solid object, the resulting action of the external forces would not change: the net force exerted by pressure forces is still equal in intensity to the weight of the fluid "displaced" by the solid. The resultant of pressure forces corresponds to the *buoyancy* (Lighthill, 1986). Of course if the solid is denser (or less dense) than the surrounding fluid, the equilibrium would be lost and fluid/body motion would set in.

Pressure is a force per unit surface which is **normal** to the considered surface[1]. This type of distributed force in a fluid is a specificity of continuum media, and is referred to as **stress**. Here the corresponding stress expression is therefore:

$$\mathrm{d}\boldsymbol{f} = -p\,\boldsymbol{n}\,\mathrm{d}S. \tag{1.1}$$

The minus sign translate the state of compression in which fluids generally are (so that pressure is a positive quantity, unless in very specific cases of tensile solicitations of fluids).

▷ **Pressure as a body force.** On figure 1.2 we illustrate the action of pressure forces exerted on a small cylindrical fluid portion. The portion has a base S and a height $\mathrm{d}n$ leaning on two isobars p and $p+\mathrm{d}p$. The action of pressure forces on the side of the cylinder is zero by symmetry, so that the net pressure force is simply the sum of the contributions exerted on each bounding face : $S\,p$ and $-S\,(p+\mathrm{d}p)$ (let's count positively – and arbitrarily – the forces oriented along the pressure gradient), which is $-S\,\mathrm{d}p$. If we divide this force by the volume of the small element, $S\,\mathrm{d}n$, it appears that pressure forcescan be perceived as a body force of intensity $-\frac{\partial p}{\partial n}$ acting along ∇p. In other words pressure forces may be understood as body forces of intensity $-\nabla p$; this is the meaning of this term appearing in both Euler and the Navier–Stokes equations. This is

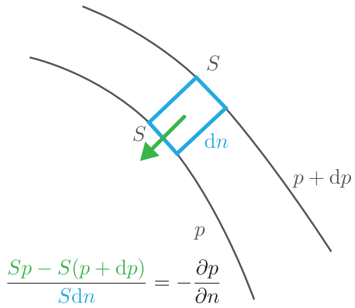

Figure 1.2: Writing down a force balance on a volume portion $S\,\mathrm{d}n$ leaning on two isobars (of level p and $p+\mathrm{d}p$), we see that the pressure action may be understood as a body force per unit volume of intensity $-\frac{\partial p}{\partial n}$.

here a physical interpretation of the divergence theorem which would have given directly:

$$-\iint_{\partial V} p\,\boldsymbol{n}\,\mathrm{d}S = -\iiint_V \nabla p\,\mathrm{d}V.$$

1.1.2 Stresses

In the general context of a flowing fluid, stress has no particular reason to be aligned with the normal – and, as a matter of fact, it is not. But multiplying the normal vector $\boldsymbol{n}$ with a scalar can only give another vector still aligned with $\boldsymbol{n}$, and using the cross product is of no help either because the cross product between any vector and $\boldsymbol{n}$ can only give a vector perpendicular to $\boldsymbol{n}$. So we need another mean to obtain a vector arbitrarily oriented from the unique knowledge of $\boldsymbol{n}$. The mathematical object allowing to perform this operation is the 2-rank tensor. On using Einstein notations, this gives:

$$\mathrm{d}f_i = \sigma_{ij} n_j\,\mathrm{d}S. \tag{1.2}$$

We have here to remember that this object is only a mean to obtain $\mathrm{d}\boldsymbol{f}$ not necessarily aligned with $\boldsymbol{n}$.
Of course it is possible to obtain the simple case of a stress aligned with $\boldsymbol{n}$ using this formalism. As an example, the stress tensor of a fluid at rest (éq. 1.1) is:

$$\sigma_{ij} = -p\,\delta_{ij}. \tag{1.3}$$

It is possible to show that this stress tensor $\boldsymbol{\sigma}$ is necessarily a symmetric tensor, i.e. $\sigma_{ij} = \sigma_{ji}$ (Batchelor, 1967). The demonstration's main idea is to write an angular momentum balance at the fluid particle level; the only dominant term in this equation involves the antisymmetric part of $\boldsymbol{\sigma}$. As it is not balanced by any term, it necessarily vanishes. There is one exception however: in the very particular case of a *moment density*, as in active matter or certain magnetic colloids, this term can be balanced and the tensor be non-symmetric (see for example the study of Soni *et al.*, 2019).

[1]We may see this feature as the *definition* of a fluid, i.e. a medium unable to resist shear (Prandtl & Tietjens, 1957). A shear stress would therefore unvariably set the fluid into motion. Actually Batchelor (1967, §1.3) argue that if the pressure force would not be aligned with the normal vector, equilibrium could not be achieved.

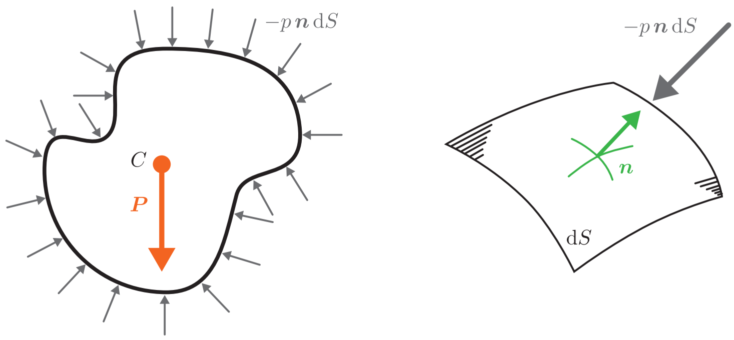

Figure 1.3: Left: when at equilibrium in the gravity field, every fluid portion experiences pressure forces that exactly balance the weight action $\boldsymbol{P}$. Right: the pressure stress is normal to each surface element.

1.1.3 Body forces

Fluids are also subject to more conventional body forces, which can be magnetic, electrostatic, gravity or result from non-inertial effects (think of centrifuge or Coriolis pseudo-forces). On noting $\boldsymbol{F}$ the body force **per unit mass** acting on the fluid particle level, we can write the net body force exerted on a fluid portion V as:

$$\iiint_V \rho \boldsymbol{F}\,\mathrm{d}V. \tag{1.4}$$

▹ **To summarize.** The net force acting on a fluid portion V is the sum of surface and body forces:

$$\oiint_{\partial V} \boldsymbol{\sigma}\cdot\boldsymbol{n}\,\mathrm{d}S + \iiint_V \rho\boldsymbol{F}\,\mathrm{d}V. \tag{1.5}$$

1.2 Fluid equilibrium

At equilibrium, pressure and body force exerted on any part V of a fluid balance each other:

$$\iiint_V \rho\boldsymbol{F}\,\mathrm{d}V - \iint_{\partial V} p\,\boldsymbol{n}\,\mathrm{d}S = \boldsymbol{0},$$

or, on using the divergence theorem:

$$\iiint_V \left(\rho\boldsymbol{F} - \nabla p\right)\,\mathrm{d}V = \boldsymbol{0}. \tag{1.6}$$

This relation being verified for *any* fluid domain, we necessarily get at the local level:

$$\rho\boldsymbol{F} = \nabla p. \tag{1.7}$$

And we recover here the pressure force expressed as a body force $-\nabla p$.
Remark: Only those density and force fields such that $\rho\boldsymbol{F}$ can be expressed as a gradient allow to reach hydrostatic equilibrium (counter-example: the baroclinic instability developing when iso-p differ from iso-ρ, or in other words as soons as pressure is not a simple function of ρ).

▹ **The case of conservative forces.** Conservative forces derive from the potential Ψ:

$$\boldsymbol{F} = -\nabla\Psi, \tag{1.8}$$

so that

$$-\rho\nabla\Psi = \nabla p, \tag{1.9}$$

and therefore

$$\nabla\rho\times\nabla\Psi = \boldsymbol{0}. \tag{1.10}$$

The iso-ρ (isopycnals) are then superimposed to equipotentials, themselved superimposed with isobars.
Consequence: in such a system, a free surface corresponding to $p = 0$ for example will also be an equipotential $\Psi = \text{const}$.

▷ **Example: the equilibrium of a rotating fluid.** Let's consider a rotating container filled with liquid. The container rotates at angular velocity Ω in the gravity field (the axis of rotation is vertical, i.e. directed along $\boldsymbol{e}_z$). In the rotating frame, each fluid particle experiences gravity but also the centrifuge pseudo-force:

$$\boldsymbol{F}_{\text{cent}} = -\rho\, \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})\,. \tag{1.11}$$

Here $\boldsymbol{\Omega} = \Omega\, \boldsymbol{e}_z$. We can then write:

$$\boldsymbol{F}_{\text{cent}} = \rho \Omega^2 r\, \boldsymbol{e}_r = -\rho \nabla \Psi_{\text{cent}}, \tag{1.12}$$

where

$$\Psi_{\text{cent}} = -\frac{1}{2}\Omega^2 r^2 = -\frac{1}{2}\Omega^2\left(x^2 + y^2\right). \tag{1.13}$$

Adding gravity effects with $\Psi_{\text{grav}} = gz$ we get

$$\Psi_{\text{tot}} = \Psi_{\text{grav}} + \Psi_{\text{cent}} = gz - \frac{1}{2}\Omega^2\left(x^2 + y^2\right). \tag{1.14}$$

We here remark that the equipotentials are paraboloids. As a result the free surface characterised with $p = 0$[2] will also adopt a paraboloidal shape:

$$z_{\text{surf}} = \frac{1}{2}\frac{\Omega^2}{g} r^2 + \text{const.} \tag{1.15}$$

▷ **Exercise: fluid planet equilibrium.** Consider a self-gravitating fluid sphere. The gravity force per unit masse $\boldsymbol{F}$ exerted on each fluid particle derives from the gravity potential Ψ such that:

$$\nabla^2 \Psi = 4\pi \mathcal{G}\rho, \tag{1.16}$$

where $\mathcal{G}$ is the universal gravitational constant. With the help of the hydrostatic equation, and supposing that the fluid has a constant density[3] ρ_0, show that the radial pressure profile satisfies:

$$p(r) = \tfrac{2}{3}\pi \mathcal{G}\rho_0^2\left(R^2 - r^2\right), \tag{1.17}$$

with R the planet radius.

1.3 Fluid motion

Now that we have described the forces at play in a fluid and the conditions for equilibrium, let's focus on fluid motion.
One technical issue with the simple conservation laws mentioned at the beginning of the chapter is that they are expressed at the fluid particle level. This seems quite natural, but it conflicts with our usual representation of space. Let's clarify what we mean by this by considering a typical fluid flow, for example the one around an airplane wing. Typically we are interested in estimating the forces exerted by the flowing fluid on a the plane, and this requires to build a knowledge of the stresses exerted on the wing. The pressure applied at one given point of the wing is ultimately a consequence of the conservation laws evoked earlier, but we clearly do not want to track the life and trajectory of every fluid particle[4] that will ever very shortly pass in the neighbourhood of the airplane to get this prediction! Rather we seek to make sure that the conservation laws are satisfied while looking at a fixed point of space (and therefore see quite a number of fluid particles passing there). In order to express the conservation laws governing the motion, we thus need to describe the variation of any quantity attached to a fluid particle, such as a concentration or its momentum.

[2] see chapter 2 for a discussion of the hypotheses behind this boundary condition.

[3] This is a really crude approximation on the density profile, which better describes incompressible (!) planets rather than gaseous ones or stars. The hydrostatics of stars can however be rationalised by taking into account the state law of polytropes, and possibly the radiation pressure adding up to the kinetic pressure. The reader interested in stellar hydrostatics may look at the *Lane-Emden equation* described e.g. in Chandrasekhar (1957, chap. IV).

[4] There is actually an alternate form of the fluid mechanics equations called *Lagrangian fluid mechanics* that exploit this viewpoint, but it gets quickly untractable and only a few specific flows can be described with this approach (Bennett, 2006).

1.3.1 Differentiation along motion

Let's take the example of a concentration field $c(\boldsymbol{x}, t)$. Following a fluid particle in its motion, we can write down how the concentration attached to it varies with time:

$$c(\boldsymbol{x} + \boldsymbol{u}\, \delta t, t + \delta t) - c(\boldsymbol{x}, t) = \delta t \underbrace{\left(\frac{\partial c}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, c\right)}_{\frac{Dc}{Dt}} + O\left(\delta t^2\right), \tag{1.18}$$

where we made the rate of concentration change $\frac{Dc}{Dt}$ appear. Note that this quantity differs from $\frac{\partial c}{\partial t}$ which would rather measure the variation of c at a fixed (*eulerian*) position of space, without following the fluid particle. The operator $\frac{D}{Dt}$ is called **particle derivative** (or material, or convective, or Lagrangian derivative) :

$$\frac{Dc}{Dt} = \frac{\partial c}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, c. \tag{1.19}$$

As an illustration, the equation governing the concentration field transported by a flowing fluid without considering diffusion effects is therefore simply:

$$\frac{Dc}{Dt} = 0 \quad \text{or} \quad \frac{\partial c}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, c = 0. \tag{1.20}$$

We note also that the *acceleration* of a fluid particle is simply $\frac{D\boldsymbol{u}}{Dt}$.

1.3.2 Diffusive and convective fluxes

When flowing, fluids transport mass, but also chemical species, energy and momentum. To describe the corresponding transport modes, we will use the notion of **flux** (of mass, momentum, energy). The vectorial flux $\boldsymbol{j}$ characterises the transfer of a quantity across an oriented surface $\delta A\, \boldsymbol{n}$ per unit time:

$$\boldsymbol{j} \cdot \boldsymbol{n}\, \delta A \tag{1.21}$$

1.3.3 Advection

The first transport mode for mass, momentum or energy is advection. Let's suppose that a given field, for example concentration c again, is **transported** with the fluid at velocity[5] $\boldsymbol{u}$, i.e. each fluid particle conserves its concentration. Consider now the **fixed** surface element δA represented figure 1.4. The matter quantity c flowing across the surface[6] during a short moment δt is $(c\, \boldsymbol{u} \cdot \boldsymbol{n})\, \delta A\, \delta t$. As a result the matter quantity transported across δA per unit surface and per unit time is $\boldsymbol{j}_{\text{adv}} \cdot \boldsymbol{n}$ where

$$\boldsymbol{j}_{\text{adv}} = c\, \boldsymbol{u} \tag{1.22}$$

is the matter **flux** associated with advection. Now if the surface element is **mobile** and moves at velocity $\boldsymbol{w}$, the advection flux

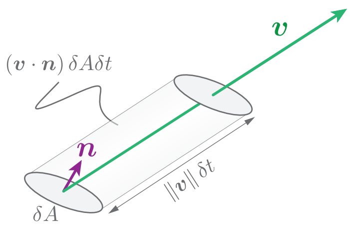

Figure 1.4: A surface portion δA of normal $\boldsymbol{n}$ is traversed by a fluid volume $(\boldsymbol{u} \cdot \boldsymbol{n})\, \delta A\, \delta t$ during δt. This volume is counted positively if $\boldsymbol{u}$ points towards the same half-space as $\boldsymbol{n}$ (in which case $\boldsymbol{u} \cdot \boldsymbol{n} > 0$), and negatively otherwise.

generalises to:

$$\boldsymbol{j}_{\text{adv}} = c\, (\boldsymbol{u} - \boldsymbol{w}) \tag{1.23}$$

We note that in the context of a **material domain**, i.e. moving with the same velocity as the fluid, we get $\boldsymbol{w} = \boldsymbol{u}$ and the advection flux cancels out by construction.

[5] The velocity $\boldsymbol{u}$ is understood as the average velocity of molecules in the vicinity of the considered point. With this definition we see that the velocity already incorporates diffusion effects. In mixtures chemical species usually have different velocities that need a careful treatment (Bird *et al.*, 2002, §17.7).

[6] We here make use of the fact that a slanted cylinder of length $\|\boldsymbol{u}\| \mathrm{d}t$ is the same as a right cylinder of same height $(\boldsymbol{u} \cdot \boldsymbol{n})$. This is Cavalieri's principle – which can also be demonstrated with a simple integration.

1.3.4 Diffusion

Now, even without any underlying flow, simple matter (species concentration), energy or momentum inhomogeneities will give rise to transfer spontaneously. These **diffusive** exchanges are quantified per unit surface and time with the **diffusive flux** $\boldsymbol{j}_{\text{diff}}$. Even if a detailed modelling of these exchanges is a complex feat, they can nonetheless be described with phenomenological relations (constrained with thermodynamical arguments) such as Fick's law for mass transport for example (see §1.5).

1.4 Balance equation for an integrated quantity

Fluxes characterise exchanges across surfaces. With their help we can determine the evolution of a quantity c contained in a domain in the most general way[7]:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V c\,\mathrm{d}V}_{\text{Variation}} = \underbrace{-\oiint_{\partial V} \boldsymbol{j}\cdot\boldsymbol{n}\,\mathrm{d}S}_{\text{Exchange}} + \underbrace{\iiint_V \varphi\,\mathrm{d}V}_{\text{Production}}, \tag{1.24}$$

so that the *variation* of an integrated quantity in V is given by a balance of entering/leaving quantity into/from the domain (*exchange*) and the possible *production* (or destruction) of c inside V. Before going deeper into the derivation of conservation laws, we have to precise the meaning of the integral derivation appearing in the left hand side. If the considered volume is fixed, the derivation process is easy as we can just swap derivation and integration. But if the domain is moving or deforming, care must be taken in writing this derivation.

1.4.1 Volume variation of a material domain and integral derivation

We will now give a meaning to the derivation of an integral performed over a deformable domain. To start with, let's consider the quite specific (but still really common) case of a **material** domain, i.e. we follow the same fluid particles though time. As time flows this domain may see its volume change as indicated on figure 1.5, hence as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \mathrm{d}V = \oiint_{\partial V} \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S \quad \left(= \iiint_V \nabla\cdot\boldsymbol{u}\,\mathrm{d}V\right). \tag{1.25}$$

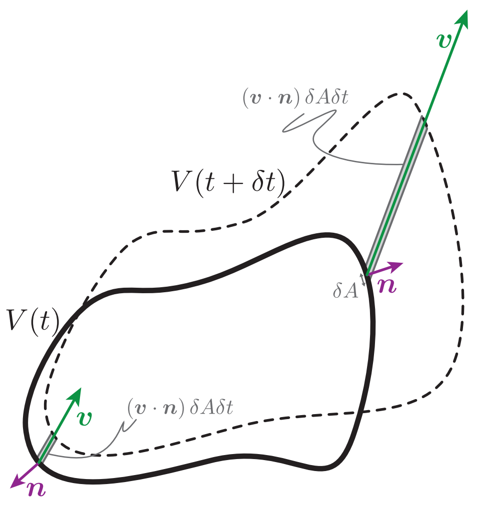

Figure 1.5: A material volume is transported by a velocity field $\boldsymbol{u}$. Each portion δA of the domain boundaries is advected with $\boldsymbol{u}$ and this yields a volume change $(\boldsymbol{u}\cdot\boldsymbol{n})\,\delta A\delta t$ during δt. The resulting total volume change rate $\frac{\mathrm{d}V}{\mathrm{d}t}$ is given by $\iint(\boldsymbol{u}\cdot\boldsymbol{n})\,\mathrm{d}A$.

[7] This relation is obtained on **physical** grounds.

▷ **Signification of the divergence.** The previous relation allows to shed light on the divergence of a velocity field $\boldsymbol{u}$. Actually, consider a material volume $\tau(t)$ constituted with the same fluid particles. The previous balance might be rewritten with the help of the divergence theorem as:

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \iint \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S \tag{1.26}$$

$$= \iiint \nabla\cdot\boldsymbol{u}\,\mathrm{d}V. \tag{1.27}$$

Thus in the limit where $\tau(t)$ is really small (in fact sufficiently small so that we can consider $\nabla\cdot\boldsymbol{u}$ constant throughout the domain), we can write:

$$\lim_{\tau\to 0}\frac{1}{\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \nabla\cdot\boldsymbol{u}. \tag{1.28}$$

The divergence of a velocity field can therefore be understood as the *rate of volume change of a fluid particle.*

▷ **Integral derivation.** We now have the toolset enabling the clarification of the derivation of the integral over a material domain $V(t)$. Let's focus on the variation of the following quantity:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)}\theta\,\mathrm{d}V.$$

To shed some light over this quantity, let's divide mentally the domain in a multitude of tiny cubes or fluid particles. As the fluid domain moves, the value of θ will change for all fluid particles at the rate $\frac{D\theta}{Dt}$. Moreover the integration element $\mathrm{d}V$ will change as well with the rate $(\nabla\cdot\boldsymbol{u})\,\mathrm{d}V$. In other words[8,9] :

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)}\theta\,\mathrm{d}V = \iiint_{V(t)}\frac{D\theta}{Dt}\,\mathrm{d}V + \iiint_{V(t)}\theta\,\underbrace{\frac{D\mathrm{d}V}{Dt}}_{(\nabla\cdot\boldsymbol{u})\mathrm{d}V} \tag{1.29}$$

1.4.2 Conservation of a quantity. Application to mass conservation

Now that the derivation of an integral has been elucidated, we are in a position to use equation (1.24) which expresses the general conservation of a quantity, for either a fixed or moving domain. Choosing one type of domain or another is largely a matter of context. For example we may consider a moving domain to establish the momentum conservation equation of a given fluid portion. A balance over a fixed domain may also present some interest, when designing for example the evolution of a quantity traversing a fixed mesh cell in a numerical code. Depending on the application, we will choose the more relevant viewpoint.
In order to clarify the use of balances over fixed or moving domains, let's now establish the mass conservation equation (without production nor destruction of mass), first in a fixed domain and then in a moving one.

1. **Mass conservation in a fixed domain** V_{fixed}. The balance equation (1.24) reads :

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_{\text{fixed}}}\rho\,\mathrm{d}V = -\oiint_{\partial V_{\text{fixed}}}\rho\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S. \tag{1.30}$$

The domain being fixed, we let the derivation enter in the integral:

$$\iiint_{V_{\text{fixed}}}\frac{\partial\rho}{\partial t}\,\mathrm{d}V = -\oiint_{\partial V_{\text{fixed}}}\rho\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S, \tag{1.31}$$

and, on applying the divergence theorem, we obtain:

$$\iiint_{V_{\text{fixed}}}\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u})\;\mathrm{d}V = 0. \tag{1.32}$$

[8] This relation is obtained on **mathematical** grounds.

[9] We note that this relation can directly be generalised to the case where the domain moves with a velocity differing from that of the fluid (fictitious velocity, flame propagation, balance over a domain moving with a wave). In this case, it suffices to replace $\boldsymbol{u}$ with the domain velocity $\boldsymbol{w}$.

Further noting that this balance is actually true for every possible domain, it follows that the integrand actually vanishes:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \boldsymbol{u}) = 0. \tag{1.33}$$

This is the **continuity equation**[10] that embodies mass conservation. This type of reasoning exploiting the validity of an integral expression for any volume to obtain a relation at the fluid particle (or *local*) level is very common.

2. **Mass conservation for a material domain** $V(t)$. This time there is no flux term because no fluid particle enters nor leaves the material domain, by definition. The balance equation (1.24) then reads:

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\iiint_{V(t)} \rho\,\mathrm{d}V}_{m(t)} = 0. \tag{1.34}$$

But as the domain is now moving, we have to apply the integral derivation procedure seen earlier:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \rho\,\mathrm{d}V = \iiint_{V(t)} \frac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{u}\,\mathrm{d}V = 0. \tag{1.35}$$

From the latter we recover again the continuity equation (1.33).

▷ **Conservation of a quantity per unit mass.** Thanks to the continuity relation it is possible to obtain a simplified expression for the transport of a quantity per unit mass ξ (i.e. such that the quantity associated with a fluid particle be $\rho\xi$):

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \rho\xi\,\mathrm{d}V = \iiint_{V(t)} \rho\frac{D\xi}{Dt}\,\mathrm{d}V. \tag{1.36}$$

We let the reader demonstrate this relation.

▷ **The incompressible fluid.** A recurring case of great practical value is that of an **incompressible evolution** where we do *not* suppose that the whole fluid a constant density, but rather that each fluid particle conserves its density. This implies:

$$\frac{D\rho}{Dt} = 0 \quad \text{and therefore} \quad \nabla\cdot\boldsymbol{u} = 0. \tag{1.37}$$

A velocity field $\boldsymbol{u}$ satisfying the zero divergence property qualifies as a *solenoidal* field.

1.4.3 Conservation of a possibly diffusing passive scalar. Convection-diffusion equation

Let's consider again a concentration field c advected in a fluid domain with the velocity field $\boldsymbol{u}$. Due to molecular thermal agitation, this field is also subject to diffusion phenomena characterised by the flux $\boldsymbol{j}_{\text{diff}}$. Here again we can obtain the evolution equation for the concentration by following two different routes:

1. By considering a **fixed domain** V_{fixed}. In this case, and in absence of any source/sink for the concentration, we will simply write that the total variation is given by the sum of the fluxes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_{\text{fixed}}} c\,\mathrm{d}V = -\oiint_{\partial V_{\text{fixed}}} (\boldsymbol{j}_{\text{conv}} + \boldsymbol{j}_{\text{diff}})\cdot\boldsymbol{n}\,\mathrm{d}S,$$

so that:

$$\iiint_{V_{\text{fixed}}} \frac{\partial c}{\partial t}\,\mathrm{d}V = -\iiint_{V_{\text{fixed}}} \nabla\cdot(\boldsymbol{j}_{\text{conv}} + \boldsymbol{j}_{\text{diff}})\,\mathrm{d}S.$$

Anticipating on §1.5 by writing the diffusive flux with Fick's law $\boldsymbol{j}_{\text{diff}} = -D\nabla c$ we obtain at the local level:

$$\frac{\partial c}{\partial t} + \nabla\cdot(c\boldsymbol{u}) = \nabla\cdot(D\nabla c) \tag{1.38}$$

For an incompressible evolution with a constant diffusion coefficient, this equation reduces to the classic **advection-diffusion equation**:

$$\frac{\partial c}{\partial t} + (\boldsymbol{u}\cdot\nabla)\,c = D\nabla^2 c. \tag{1.39}$$

[10] This denomination has been used for a long time, but is actually not really justifiable...

2. Or by considering a **material domain** V_{mat}. This time there is no convective flux by construction (see §1.3.3) but only a diffusive flux:
$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_{\mathrm{mat}}} c\,\mathrm{d}V = -\oiint_{\partial V_{\mathrm{mat}}} \boldsymbol{j}_{\mathrm{diff}}\cdot\boldsymbol{n}\,\mathrm{d}S,$$
so that, by deriving the integral over the material domain:
$$\iiint_{V_{\mathrm{mat}}} \frac{Dc}{Dt} + c\,(\nabla\cdot\boldsymbol{u})\ \mathrm{d}V = \iiint_{V_{\mathrm{mat}}} \nabla\cdot(D\nabla c)\,\mathrm{d}S.$$
This relation holds true for every possible domain, and as a result we retrieve the local form of equation (1.38).

1.4.4 Equation for momentum conservation

Momentum conservation for a fluid domain simply follows from Newton' second law $m\gamma = \Sigma \boldsymbol{F}$, never forget this! Let's write this law for a material domain $V(t)$:
$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_{\mathrm{mat}}} \rho u_i\,\mathrm{d}V = \oiint_{\partial V_{\mathrm{mat}}} \sigma_{ij}n_j\,\mathrm{d}S + \iiint_{V_{\mathrm{mat}}} \rho F_i\,\mathrm{d}V \tag{1.40}$$
or, going to the local level:
$$\rho\frac{Du_i}{Dt} = \frac{\partial\sigma_{ij}}{\partial x_j} + \rho F_i \tag{1.41}$$
That was quite simple. However there is a loophole in the previous expression as the stress tensor $\boldsymbol{\sigma}$ is here unknown (except in the case of limited interest of a static fluid, see equation (1.3)). Even if equation (1.41) holds true whatever the nature of the fluid (even for say exotic viscoelastic fluids), it is of limited use as long as this stress tensor is not clarified. We therefore have to make a connection between the stresses and the motion of the fluid, for we expect that the stresses will change as soon as the fluid starts to flow. This connection will me be made thanks to *constitutive laws* that will really characterise the fluid we are looking at.

1.5 Constitutive laws

The conservation laws expressed for a flowing fluid are expressed with fluxes and stresses whose precise form still remains to be determined. In order to make some progress in this determination, let's consider the following situation: without any macroscopic motion, a liquid contains a colourant with concentration c distributed inhomogeneously, with area more or less concentrated. Even in absence of a mean motion, the collisions between molecules will lead the colourant molecules to migrate randomly in the liquid. As a result, due to the inhomogeneous colourant distribution, there will be an excess of particles traversing ∂A that originate from a more concentrated area than the reverse. The consequence is a net transport of colourant molecules from more concentrated area to less concentrated ones, that will be active as long as the non-equilibrium situation persists: this is the **diffusion process**. It seems difficult to track the motion of each colourant particles from a statistical viewpoint. But from a *phenomenological* viewpoint, it is quite clear that diffusion will remain active only if a concentration gradient ∇c exists. The idea of **gradient-type laws** is to suppose that the components of the flux $\boldsymbol{j}_{\mathrm{diff}}$ depend linearly on ∇c, i.e. :
$$\left(\boldsymbol{j}_{\mathrm{diff}}\right)_i = k_{ij}\frac{\partial c}{\partial x_j}. \tag{1.42}$$
For an isotropic fluid, it seems reasonable to consider that in absence of any privileged axes the flux will be aligned with ∇c and of opposite sign (so as to restore equilibrium rather than to destroy it further!), so that :
$$\boldsymbol{j}_{\mathrm{diff}} = -k\nabla c, \tag{1.43}$$
and therefore:
$$k_{ij} = -k\delta_{ij}. \tag{1.44}$$
We here recover the general form of mass (Fick) and energy (Fourier) transport:
$$\begin{cases}\textbf{Fick}\text{'s law :} & \boldsymbol{j}_{\mathrm{mass}} = -D\nabla c & (1.45)\\ \textbf{Fourier}\text{'s law :} & \boldsymbol{j}_{\mathrm{energy}} = -k\nabla T & (1.46)\end{cases}$$

▷ **Momentum diffusion and viscous stress.** The case of momentum diffusion is a bit more complex due to the vectorial nature of $\rho\boldsymbol{u}$. To start with let's consider the simple situation of a parallel flow $\left(U(y), 0, 0\right)$. Due to intermolecular collisions, there is a vertical **diffusion** of momentum. As for mass transport, we suppose that the flux is linear with the momentum gradient, so that:

$$j_{\text{diff}} = -\mu \frac{\mathrm{d}U}{\mathrm{d}y}, \tag{1.47}$$

Note that this vertical flux of horizontal momentum has the following dimensions;

momentum per unit volume × volume/area/time

or $[\mathsf{ML}^{-1}\mathsf{T}^{-2}]$, the same dimension of a stress. This dimensional similitude is no coincidence: the transferred momentum is achieved with a stress, in accordance with Newton' second law. We may indeed rewrite (1.47) as:

$$\sigma_{xy} = \mu \frac{\mathrm{d}U}{\mathrm{d}y}, \tag{1.48}$$

which is the horizontal stress exerted on a surface with a vertical normal[11]. The proportionality coefficient μ characterising the efficiency of momentum transport is the *dynamic viscosity*.

▷ **General expression for the stress tensor of a Newtonian fluid.** In the context of a general flow, we can foresee that the stress tensor will depend on each component of the velocity gradient[12], so that:

$$\sigma_{ij} = -p\,\delta_{ij} + \alpha_{ijkl} \frac{\partial u_k}{\partial x_l}. \tag{1.49}$$

This form of the stress tensor is compatible with the static fluid stress tensor presented earlier: as soon as the velocity vanishes, the stress tensor adopts its static limit (1.3). But we can go further. If we decompose the velocity gradient tensor into a symmetric part $\boldsymbol{e}$ (associated with a deformation) and an antisymmetric part $\boldsymbol{\omega}$ (associated with solid body rotation) :

$$u_{i,j} = e_{i,j} + \omega_{i,j} \quad \text{with} \quad e_{i,j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{et} \quad \omega_{i,j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right),$$

Actually momentum diffusion (i.e. viscous stress) can only occur when the flow deviates from solid-body motion (translation or rotation): as a result $\boldsymbol{\sigma}$ cannot depend on $\boldsymbol{\omega}$ and:

$$\sigma_{ij} = -p\,\delta_{ij} + A_{ijkl} e_{kl}. \tag{1.50}$$

Additional considerations on the fluid isotropy that we will not develop here allow for a drastic reduction in the number of coefficients to finally get:

$$\sigma_{ij} = -p\,\delta_{ij} + 2\mu e_{ij} + \lambda\,(\nabla\cdot\boldsymbol{u})\,\delta_{ij}. \tag{1.51}$$

1.6 Navier-Stokes equation

Having elucidated the structure of the stress tensor (1.51) we can rewrite the equation for momentum conservation (1.41) :

$$\rho \frac{Du_i}{Dt} = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\mu e_{ij} + \lambda\,(\nabla\cdot\boldsymbol{u})\,\delta_{ij}\right) \tag{1.52}$$

This equation is the *Navier-Stokes equation*.
In the particular case (but of great practical interest) of an incompressible evolution with viscosity gradient, this equation simplifies to:

$$\rho \frac{Du_i}{Dt} = \rho f_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \tag{1.53}$$

or, in vectorial form:

$$\underbrace{\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}\right)}_{m\gamma} = \underbrace{-\nabla p + \mu\nabla^2\boldsymbol{u} + \rho\boldsymbol{f}}_{\Sigma F}. \tag{1.54}$$

[11] the reader will note the sign difference between the expressions (1.47) and (1.48) that originate from the notation convention for stresses.

[12] This is this linear dependence to shear that qualifies a fluid as Newtonian. But we can imagine (and actually there exists) more complex relations, of nonlinar nature, between stress and shear. **Rheology** is the discipline that studies these particular non-newtonian behaviours.

Chapter 2

Boundary conditions and interfaces

We have seen in the previous chapter how to express mass and momentum conservation at the fluid particle level, and we have derived the corresponding equations for fluid motion. These equations are naturally associated with **boundary conditions** that either express conservation laws (e.g. no mass flux at a boundary) or peculiar physical processes occurring at a surface (temperature or velocity continuity). In both cases these boundary conditions will be pivoting in the determination of the solution.

2.1 Fluxes at boundaries: impermeability and imbibition

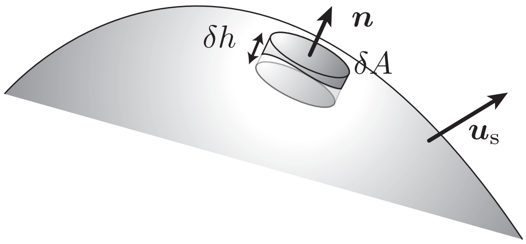

Figure 2.1: Balance over an elementary volume located across a solid boundary.

Let's consider a solid moving in a fluid at velocity $\boldsymbol{u}_\mathrm{s}$ (possibly time dependent) and let's conduct a mass balance on a small cylindrical volume of base δA and height δh located across the fluid and the solid (Fig. 2.1). Now let δh tends to 0 so that the mass element shrinks down to zero. From now on, the mass variation of the element should necessarily be zero, and this implies that the mass flux from the fluid has to be balanced by the mass flux from the solid side:

$$-\left(\boldsymbol{j}_\mathrm{fluid} \cdot \boldsymbol{n}_\mathrm{fluid}\right) \delta A - \left(\boldsymbol{j}_\mathrm{solid} \cdot \boldsymbol{n}_\mathrm{solid}\right) \delta A = 0. \tag{2.1}$$

Let's write arbitrarily $\boldsymbol{n} = \boldsymbol{n}_\mathrm{fluid} = -\boldsymbol{n}_\mathrm{solid}$ so that:

$$-\boldsymbol{j}_\mathrm{fluid} \cdot \boldsymbol{n} + \boldsymbol{j}_\mathrm{solid} \cdot \boldsymbol{n} = 0. \tag{2.2}$$

Without mixing phenomena, the fluid velocity $\boldsymbol{u}$ already takes into account diffusive effects (it is the chemical species velocity) and the mass flux reduces to the convective flux. Beware that as we are in the solid reference frame, the relative velocity of the fluid is $\boldsymbol{u} - \boldsymbol{u}_\mathrm{s}$, so that the mass flux on the fluid side is:

$$\boldsymbol{j}_\mathrm{fluid} = \rho\left(\boldsymbol{u} - \boldsymbol{u}_\mathrm{s}\right). \tag{2.3}$$

▷ **The impermeable wall.** A very common case is that of an **imperméable** solid in which the fluid cannot penetrate; the fluid mass flux within the solid is therefore zero and $\boldsymbol{j}_\mathrm{solid} = \boldsymbol{0}$. Mass conservation expressed at an impermeable boundary therefore reduces to:

$$\boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{u}_\mathrm{s} \cdot \boldsymbol{n} \tag{2.4}$$

This is the **impermeability condition** for an object (or a wall). As the name implies, it simply expresses the fact that the fluid cannot penetrate into the solid. This condition takes the form of a **continuity of normal velocities**.
Note: in the quite particular (but still very common!) case of a fixed solid object, this condition reduces to $\boldsymbol{u} \cdot \boldsymbol{n} = 0$.

▷ **Permeable wall.** The previous discussion naturally extends to the case of permeable walls. Those may correspond to biological tissues permeable to given solutes, to materials swollen by solvents, to terrains soaked by rain or to lifting sails or wings made porous in order to control the boundary layer (such as Cousteau and Malavard' turbosail seen in the lecture).
Now the mass flux is non-zero and its precise determination requires a knowledge of the flow *inside* the solid. Suppose however that the imbibition velocity $\boldsymbol{u}_{\text{imbib}}$ be constant and known (this corresponds for example to a suction with an imposed flowrate). The mass conservation at the permeable wall will then be written:

$$\rho(\boldsymbol{u} - \boldsymbol{u}_{\text{s}}) \cdot \boldsymbol{n} = \rho(\boldsymbol{u}_{\text{imbib}}) \cdot \boldsymbol{n}, \tag{2.5}$$

thus

$$\boldsymbol{u} \cdot \boldsymbol{n} = (\boldsymbol{u}_{\text{s}} + \boldsymbol{u}_{\text{imbib}}) \cdot \boldsymbol{n}. \tag{2.6}$$

▷ **Boundary conditions on a concentration field near a wall.** The boundary conditions to apply on the transport equation of a concentration field (1.38) can be obtainedd following the same principles. Imagine a concentration field transported with a fluid flow $\boldsymbol{u}$. Suppose also that an (impermeable) solid is moving in the fluid at velocity $\boldsymbol{u}_{\text{s}}$. In the solid reference frame the mass flux across any surface $\delta A\, \boldsymbol{n}$ is:

$$\boldsymbol{j}_{\text{mass}} = \boldsymbol{j}_{\text{conv}} + \boldsymbol{j}_{\text{diff}} = c\,(\boldsymbol{u} - \boldsymbol{u}_s) - D\nabla c. \tag{2.7}$$

The impermeability condition at the wall for the concentration field c is therefore:

$$c(\boldsymbol{u} - \boldsymbol{u}_{\text{s}}) \cdot \boldsymbol{n} - D\nabla c \cdot \boldsymbol{n} = 0, \tag{2.8}$$

because the mass flux inside the solid is zero. On using the impermeability condition on the velocity field (2.4), this relation reduces to:

$$\nabla c \cdot \boldsymbol{n} \equiv \frac{\partial c}{\partial n} = 0. \tag{2.9}$$

This Neumann condition is also called a **no-flux boundary condition.**

▷ **Mass transfer at an interface: evaporation.** One last example of boundary conditions arising from conservation considerations is the continuity of the mass flux across a liquid-gas interface, when the liquid is evaporating.
Let's write a mass balance on a fluid element analogous to the previous one: a small cylindrical element of base δA and height δh located across a moving interface with velocity $\boldsymbol{u}_{\text{i}}$. We let the height δh of the element tend to zero, so that the element mass equally tends to zero. The mass balance over this element is:

$$\boldsymbol{j}_{\text{liquid conv.}} = \boldsymbol{j}_{\text{gas conv.}} + \boldsymbol{j}_{\text{gas diff.}} \tag{2.10}$$

so that :

$$\rho_\ell\,(\boldsymbol{u}_\ell - \boldsymbol{u}_{\text{i}}) \cdot \boldsymbol{n} = \rho_v\,(\boldsymbol{u}_g - \boldsymbol{u}_{\text{i}}) \cdot \boldsymbol{n} - D\nabla\rho_v \cdot n. \tag{2.11}$$

In the case of a violent evaporation (e.g. combustion front), the flux is dominated by convection effects and

$$\boldsymbol{u}_g \sim \underbrace{\frac{\rho_\ell}{\rho_v}}_{\gg 1} \boldsymbol{u}_\ell$$

so that the (so-called Stefan) flow induced in the vapour is much stronger than that in the liquid. In the other limit where evaporation is very slow (case of a slowly drying water drop), it is rather the diffusive term that dominates.

2.2 Phenomenological conditions: adherence and continuity

In addition to the previous boundary conditions arising from conservation principles, fluids are also subject to other boundary conditions as well. The latter have been established and confirmed on experimental grounds, so there is a consensus about their relevance but not an exact demonstration. These phenomenological conditions are **field continuity conditions** at interfaces and apply on velocity, temperature etc.

▷ **A short history of adherence.** The adherence condition $\boldsymbol{u} = \boldsymbol{u}_s$ has an astonishing history full of twists and turns, which is accounted for in details in Goldstein (1950). At the XVIII$^{\text{th}}$ century, the theoretical description of potential flows (corresponding to the idealisation of perfect flow of fluid) was already well established, but the comparisons with experimental data were mediocre. Daniel Bernoulli was well aware of this fact and attributed the discrepancies between (ideal) predicted flows and those observed to some "adherence condition" that would prevail at the wall. Coulomb then demonstrated experimentally that an disk oscillating in a liquid was not particularly affected by a change in surface properties (smooth, rough or covered with grease). Therefore it appeared to Coulomb that the fluid velocity matched the disk velocity in its vicinity. In this vision the fluid has the same properties in every point of space; the flow just fulfils an additional condition at the wall.
But during the XIX$^{\text{th}}$ century alternate theories appeared. Girard proposed that the lqiuid layer adjacent to a wall had differing physical properties, leading it to adhere the solid. The next fluid layer (composed of "regular fluid") could then freely slip onto the first affixed layer. Navier also got interested into this problem and suggested a boundary condition involving a slip directly proportional to the shear stress $\beta u = \mu \frac{\partial u}{\partial n}$. Here the ratio μ/β has the dimension of a length: it is the *slip length*. From there a period of relative confusion ensued where famous theoreticians of the time (Poisson, Stokes) alternately adopted one or the other of the boundary conditions.
With time however, delicate experiments conducted by Couette or Maxwell advocated definitely for the adherence condition. Maxwell suggested on molecular dynamics considerations that if Navier's condition was effectively valid, the length over which slip occurred was so small – of the order of a few mean free paths – that considering it to be zero was a really reasonable hypothesis. This amounted to consider adherence at the wall: $\boldsymbol{u} = \boldsymbol{u}_s$ (Maxwell, 1879). This is in particular verified in the usual conditions of laboratory experiments, but not necessarily for rarefied gas flows (e.g. atmospheric reentry, hypersonic flight) or flows involving fluids with long polymer chains (microfluidic flows), as experiments confirm.
During the XX$^{\text{th}}$ century, the almost perfect agreement between experimental observations and theoretical predictions using adherence condition for a number of flows (Poiseuille flow, Couette flow, Stokes' sphere settlement in a viscous fluid, the instability threshold for the Taylor-Couette experiment etc) definitely settled the validity of this condition.
Moreover, the scaling laws for the drag on an object deduced from dimensional considerations based on the characteristic scales ρ, U, D and μ capture the evolution of aerodynamic forces over a wide range of scales. If another lengthscale (associated with slip) was relevant in the description of conventional flows, it would result in an alteration of the scaling laws (not observed).
We can understand adherence condition (continuity of $\boldsymbol{u}$ across an interface) but also the condition on temperature continuity as resulting from equilibrium at the molecular level. The extremely rapid transfers between neighbouring molecules induce a quasi-instantaneous equilibrium of mean momentum (velocity) and mean thermal agitation velocity (temperature) (Batchelor, 1967). At an interface between a medium I and a medium II (solid-fluid, or fluid-fluid), we will therefore get:

$$\boldsymbol{u}_{\mathrm{I}} = \boldsymbol{u}_{\mathrm{II}} \quad \text{and} \quad T_{\mathrm{I}} = T_{\mathrm{II}}. \tag{2.12}$$

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