

*Physics of fluids & nonlinear physics*

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# Contents

<b>Contents</b>	<b>3</b>
<b>1 Fluid motion</b>	<b>7</b>
1.1 Forces . . . . .	7
1.1.1 Pressure . . . . .	7
1.1.2 Stresses . . . . .	8
1.1.3 Body forces . . . . .	9
1.2 Fluid equilibrium . . . . .	9
1.3 Fluid motion . . . . .	10
1.3.1 Differentiation along motion . . . . .	11
1.3.2 Diffusive and convective fluxes . . . . .	11
1.3.3 Advection . . . . .	11
1.3.4 Diffusion . . . . .	12
1.4 Balance equation for an integrated quantity . . . . .	12
1.4.1 Volume variation of a material domain and integral derivation . . . . .	12
1.4.2 Conservation of a quantity. Application to mass conservation . . . . .	13
1.4.3 Conservation of a possibly diffusing passive scalar. Convection-diffusion equation . . . . .	14
1.4.4 Equation for momentum conservation . . . . .	15
1.5 Constitutive laws . . . . .	15
1.6 Navier-Stokes equation . . . . .	16
<b>2 Boundary conditions and interfaces</b>	<b>17</b>
2.1 Fluxes at boundaries: impermeability and imbibition . . . . .	17
2.2 Phenomenological conditions: adherence and continuity . . . . .	18
2.3 Where is the interface? . . . . .	19
2.3.1 Kinematic boundary condition. . . . .	19
2.3.2 Surface geometry I. Normals. . . . .	20
2.4 Capillarity . . . . .	20
2.4.1 Cohesion . . . . .	21
2.4.2 Surface tension . . . . .	21
2.4.3 Surface geometry II. Curvature. . . . .	22
2.5 Young-Laplace's pressure jump . . . . .	23
2.5.1 Surface tension? . . . . .	25
2.5.2 Stress (dis-)continuity . . . . .	25
2.6 Equilibrium shape for a meniscus . . . . .	26
<b>3 Viscous flows</b>	<b>29</b>
3.1 Low-Re number flows . . . . .	30
3.2 Stokes flow's properties . . . . .	30
3.3 Moving in a viscous world . . . . .	31
3.3.1 Point force induced flow: the Stokeslet . . . . .	32
3.3.2 The motion of slender objects . . . . .	33
<b>Bibliography</b>	<b>35</b>



## **Modelling fluids**



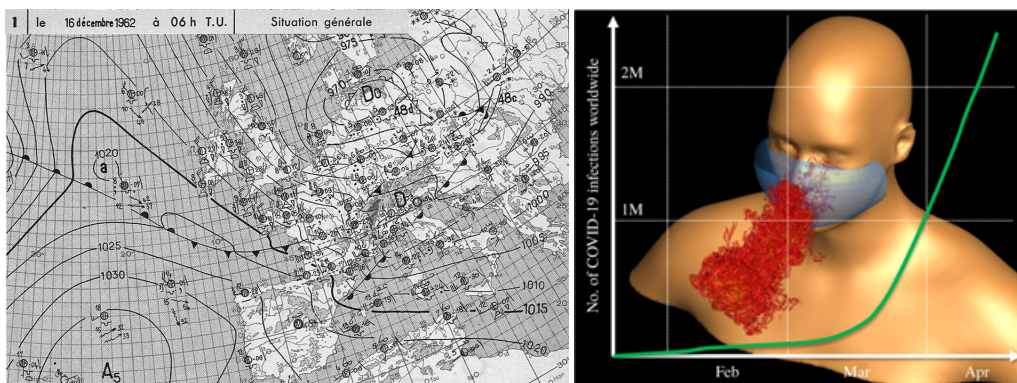
# Chapter 1

## Fluid motion

The description of fluid flows can rapidly be obscured behind seemingly complex equations (e.g. Navier–Stokes). Even worse these equations can change depending on the context and problem considered. Actually, the governing equations for fluid motion express truly simple physical principles, namely **mass conservation** (the mass  $m(t)$  of a fluid particle is constant), **momentum conservation** (a fluid particle's momentum obeys Newton's second law  $m\gamma = \Sigma F$ ) and **energy conservation** (the energy of a fluid particle follows the first and second principle of thermodynamics). In this introduction we will review the derivation of the fluid mechanics equations by expressing these fundamental conservation principles.

### 1.1 Forces

As other continuum media, fluids carry force fields that determine their equilibrium (when the net force contribution is zero) or their motion. If each fluid particle is subject to *body forces*, it is also exposed to *surface forces* called **stresses**, as for example pressure (Fig. 1.1) that we now examine.



Pressure is a force per unit surface which is **normal** to the considered surface<sup>1</sup>. This type of distributed force in a fluid is a specificity of continuum media, and is referred to as **stress**. Here the corresponding stress expression is therefore:

$$d\mathbf{f} = -p \mathbf{n} dS. \quad (1.1)$$

The minus sign translate the state of compression in which fluids generally are (so that pressure is a positive quantity, unless in very specific cases of tensile solicitations of fluids).

▷ **Pressure as a body force.** On figure 1.2 we illustrate the action of pressure forces exerted on a small cylindrical fluid portion. The portion has a base  $S$  and a height  $dn$  leaning on two isobars  $p$  and  $p + dp$ . The action of pressure forces on the side of the cylinder is zero by symmetry, so that the net pressure force is simply the sum of the contributions exerted on each bounding face :  $S p$  and  $-S (p + dp)$  (let's count positively – and arbitrarily – the forces oriented along the pressure gradient), which is  $-S dp$ . If we divide this force by the volume of the small element,  $S dn$ , it appears that pressure forces can be perceived as a body force of intensity  $-\frac{\partial p}{\partial n}$  acting along  $\nabla p$ . In other words pressure forces may be understood as body forces of intensity  $-\nabla p$ ; this is the meaning of this term appearing in both Euler and the Navier–Stokes equations. This is

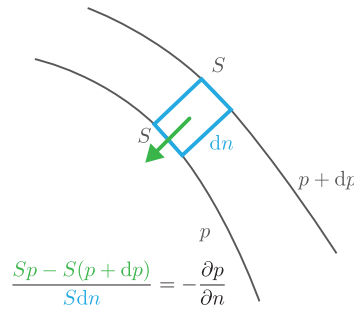


Figure 1.2: Writing down a force balance on a volume portion  $S dn$  leaning on two isobars (of level  $p$  and  $p + dp$ ), we see that the pressure action may be understood as a body force per unit volume of intensity  $-\frac{\partial p}{\partial n}$ .

here a physical interpretation of the divergence theorem which would have given directly:

$$-\iint_{\partial V} p \mathbf{n} dS = -\iiint_V \nabla p dV.$$

### 1.1.2 Stresses

In the general context of a flowing fluid, stress has no particular reason to be aligned with the normal – and, as a matter of fact, it is not. But multiplying the normal vector  $\mathbf{n}$  with a scalar can only give another vector still aligned with  $\mathbf{n}$ , and using the cross product is of no help either because the cross product between any vector and  $\mathbf{n}$  can only give a vector perpendicular to  $\mathbf{n}$ . So we need another mean to obtain a vector arbitrarily oriented from the unique knowledge of  $\mathbf{n}$ . The mathematical object allowing to perform this operation is the 2-rank tensor. On using Einstein notations, this gives:

$$df_i = \sigma_{ij} n_j dS. \quad (1.2)$$

We have here to remember that this object is only a mean to obtain  $d\mathbf{f}$  not necessarily aligned with  $\mathbf{n}$ .

Of course it is possible to obtain the simple case of a stress aligned with  $\mathbf{n}$  using this formalism. As an example, the stress tensor of a fluid at rest (éq. 1.1) is:

$$\sigma_{ij} = -p \delta_{ij}. \quad (1.3)$$

It is possible to show that this stress tensor  $\boldsymbol{\sigma}$  is necessarily a symmetric tensor, i.e.  $\sigma_{ij} = \sigma_{ji}$  (Batchelor, 1967). The demonstration's main idea is to write an angular momentum balance at the fluid particle level; the only dominant term in this equation involves the antisymmetric part of  $\boldsymbol{\sigma}$ . As it is not balanced by any term, it necessarily vanishes. There is one exception however: in the very particular case of a *moment density*, as in active matter or certain magnetic colloids, this term can be balanced and the tensor be non-symmetric (see for example the study of Soni *et al.*, 2019).

<sup>1</sup>We may see this feature as the *definition* of a fluid, i.e. a medium unable to resist shear (Prandtl & Tietjens, 1957). A shear stress would therefore unvariably set the fluid into motion. Actually Batchelor (1967, §1.3) argue that if the pressure force would not be aligned with the normal vector, equilibrium could not be achieved.



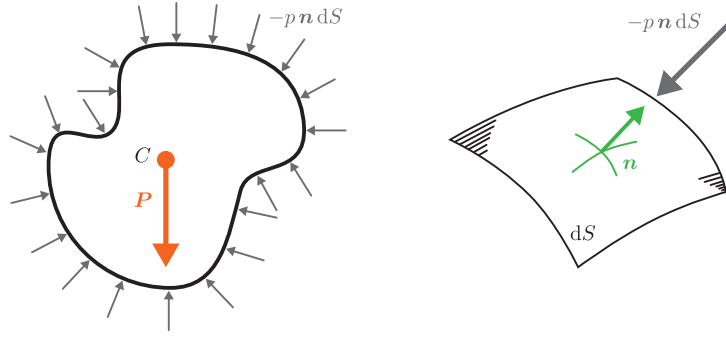


Figure 1.3: Left: when at equilibrium in the gravity field, every fluid portion experiences pressure forces that exactly balance the weight action  $P$ . Right: the pressure stress is normal to each surface element.

### 1.1.3 Body forces

Fluids are also subject to more conventional body forces, which can be magnetic, electrostatic, gravity or result from non-inertial effects (think of centrifuge or Coriolis pseudo-forces). On noting  $\mathbf{F}$  the body force **per unit mass** acting on the fluid particle level, we can write the net body force exerted on a fluid portion  $V$  as:

$$\iiint_V \rho \mathbf{F} dV. \quad (1.4)$$

▷ **To summarize.** The net force acting on a fluid portion  $V$  is the sum of surface and body forces:

$$\oint_{\partial V} \boldsymbol{\sigma} \cdot \mathbf{n} dS + \iiint_V \rho \mathbf{F} dV. \quad (1.5)$$

## 1.2 Fluid equilibrium

At equilibrium, pressure and body force exerted on any part  $V$  of a fluid balance each other:

$$\iiint_V \rho \mathbf{F} dV - \iint_{\partial V} p \mathbf{n} dS = \mathbf{0},$$

or, on using the divergence theorem:

$$\iiint_V (\rho \mathbf{F} - \nabla p) dV = \mathbf{0}. \quad (1.6)$$

This relation being verified for *any* fluid domain, we necessarily get at the local level:

$$\rho \mathbf{F} = \nabla p. \quad (1.7)$$

And we recover here the pressure force expressed as a body force  $-\nabla p$ .

**Remark:** Only those density and force fields such that  $\rho \mathbf{F}$  can be expressed as a gradient allow to reach hydrostatic equilibrium (counter-example: the baroclinic instability developing when iso- $p$  differ from iso- $\rho$ , or in other words as soon as pressure is not a simple function of  $\rho$ ).

▷ **The case of conservative forces.** Conservative forces derive from the potential  $\Psi$ :

$$\mathbf{F} = -\nabla \Psi, \quad (1.8)$$

so that

$$-\rho \nabla \Psi = \nabla p, \quad (1.9)$$

and therefore

$$\nabla \rho \times \nabla \Psi = \mathbf{0}. \quad (1.10)$$

The iso- $\rho$  (isopycnals) are then superimposed to equipotentials, themselves superimposed with isobars.

**Consequence:** in such a system, a free surface corresponding to  $p = 0$  for example will also be an equipotential  $\Psi = \text{const.}$

▷ **Example: the equilibrium of a rotating fluid.** Let's consider a rotating container filled with liquid. The container rotates at angular velocity  $\Omega$  in the gravity field (the axis of rotation is vertical, i.e. directed along  $\mathbf{e}_z$ ). In the rotating frame, each fluid particle experiences gravity but also the centrifuge pseudo-force:

$$\rho \mathbf{F}_{\text{cent}} = -\rho \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}). \quad (1.11)$$

Here  $\boldsymbol{\Omega} = \Omega \mathbf{e}_z$ . We can then write:

$$\rho \mathbf{F}_{\text{cent}} = \rho \Omega^2 r \mathbf{e}_r = -\rho \nabla \Psi_{\text{cent}}, \quad (1.12)$$

where

$$\Psi_{\text{cent}} = -\frac{1}{2} \Omega^2 r^2 = -\frac{1}{2} \Omega^2 (x^2 + y^2). \quad (1.13)$$

Adding gravity effects with  $\Psi_{\text{grav}} = gz$  we get

$$\Psi_{\text{tot}} = \Psi_{\text{grav}} + \Psi_{\text{cent}} = gz - \frac{1}{2} \Omega^2 (x^2 + y^2). \quad (1.14)$$

We here remark that the equipotentials are paraboloids. As a result the free surface characterised with  $p = 0^2$  will also adopt a paraboloidal shape:

$$z_{\text{surf}} = \frac{1}{2} \frac{\Omega^2}{g} r^2 + \text{const.} \quad (1.15)$$

▷ **Exercise: fluid planet equilibrium.** Consider a self-gravitating fluid sphere. The gravity force per unit masse  $\mathbf{F}$  exerted on each fluid particle derives from the gravity potential  $\Psi$  such that:

$$\nabla^2 \Psi = 4\pi G \rho, \quad (1.16)$$

where  $G$  is the universal gravitational constant. With the help of the hydrostatic equation, and supposing that the fluid has a constant density<sup>3</sup>  $\rho_0$ , show that the radial pressure profile satisfies:

$$p(r) = \frac{2}{3} \pi G \rho_0^2 (R^2 - r^2), \quad (1.17)$$

with  $R$  the planet radius.

### 1.3 Fluid motion

Now that we have described the forces at play in a fluid and the conditions for equilibrium, let's focus on fluid motion.

One technical issue with the simple conservation laws mentioned at the beginning of the chapter is that they are expressed at the fluid particle level. This seems quite natural, but it conflicts with our usual representation of space. Let's clarify what we mean by this by considering a typical fluid flow, for example the one around an airplane wing. Typically we are interested in estimating the forces exerted by the flowing fluid on a the plane, and this requires to build a knowledge of the stresses exerted on the wing. The pressure applied at one given point of the wing is ultimately a consequence of the conservation laws evoked earlier, but we clearly do not want to track the life and trajectory of every fluid particle<sup>4</sup> that will ever very shortly pass in the neighbourhood of the airplane to get this prediction! Rather we seek to make sure that the conservation laws are satisfied while looking at a fixed point of space (and therefore see quite a number of fluid particles passing there). In order to express the conservation laws governing the motion, we thus need to describe the variation of any quantity attached to a fluid particle, such as a concentration or its momentum.

<sup>2</sup>see chapter 2 for a discussion of the hypotheses behind this boundary condition.

<sup>3</sup>This is a really crude approximation on the density profile, which better describes incompressible (!) planets rather than gaseous ones or stars. The hydrostatics of stars can however be rationalised by taking into account the state law of polytropes, and possibly the radiation pressure adding up to the kinetic pressure. The reader interested in stellar hydrostatics may look at the *Lane-Emden equation* described e.g. in Chandrasekhar (1957, chap. IV).

<sup>4</sup>There is actually an alternate form of the fluid mechanics equations called *Lagrangian fluid mechanics* that exploit this viewpoint, but it gets quickly untractable and only a few specific flows can be described with this approach (Bennett, 2006).

### 1.3.1 Differentiation along motion

Let's take the example of a concentration field  $c(\mathbf{x}, t)$ . Following a fluid particle in its motion, we can write down how the concentration attached to it varies with time:

$$c(\mathbf{x} + \mathbf{u} \delta t, t + \delta t) - c(\mathbf{x}, t) = \delta t \underbrace{\left( \frac{\partial c}{\partial t} + (\mathbf{u} \cdot \nabla) c \right)}_{\frac{Dc}{Dt}} + O(\delta t^2), \quad (1.18)$$

where we made the rate of concentration change  $\frac{Dc}{Dt}$  appear. Note that this quantity differs from  $\frac{\partial c}{\partial t}$  which would rather measure the variation of  $c$  at a fixed (*eulerian*) position of space, without following the fluid particle. The operator  $\frac{D}{Dt}$  is called **particle derivative** (or material, or convective, or Lagrangian derivative) :

$$\frac{Dc}{Dt} = \frac{\partial c}{\partial t} + (\mathbf{u} \cdot \nabla) c. \quad (1.19)$$

As an illustration, the equation governing the concentration field transported by a flowing fluid without considering diffusion effects is therefore simply:

$$\frac{Dc}{Dt} = 0 \quad \text{or} \quad \frac{\partial c}{\partial t} + (\mathbf{u} \cdot \nabla) c = 0. \quad (1.20)$$

We note also that the *acceleration* of a fluid particle is simply  $\frac{D\mathbf{u}}{Dt}$ .

### 1.3.2 Diffusive and convective fluxes

When flowing, fluids transport mass, but also chemical species, energy and momentum. To describe the corresponding transport modes, we will use the notion of **flux** (of mass, momentum, energy). The vectorial flux  $\mathbf{j}$  characterises the transfer of a quantity across an oriented surface  $\delta A \mathbf{n}$  per unit time:

$$\mathbf{j} \cdot \mathbf{n} \delta A \quad (1.21)$$

### 1.3.3 Advection

The first transport mode for mass, momentum or energy is advection. Let's suppose that a given field, for example concentration  $c$  again, is **transported** with the fluid at velocity<sup>5</sup>  $\mathbf{u}$ , i.e. each fluid particle conserves its concentration. Consider now the **fixed** surface element  $\delta A$  represented figure 1.4. The matter quantity  $c$  flowing across the surface<sup>6</sup> during a short moment  $\delta t$  is  $(c \mathbf{u} \cdot \mathbf{n}) \delta A \delta t$ . As a result the matter quantity transported across  $\delta A$  per unit surface and per unit time is  $\mathbf{j}_{\text{adv}} \cdot \mathbf{n}$  where

$$\mathbf{j}_{\text{adv}} = c \mathbf{u} \quad (1.22)$$

is the matter **flux** associated with advection. Now if the surface element is **mobile** and moves at velocity  $\mathbf{w}$ , the advection flux

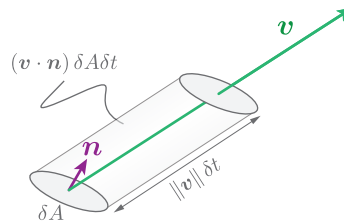


Figure 1.4: A surface portion  $\delta A$  of normal  $\mathbf{n}$  is traversed by a fluid volume  $(\mathbf{u} \cdot \mathbf{n}) \delta A \delta t$  during  $\delta t$ . This volume is counted positively if  $\mathbf{u}$  points towards the same half-space as  $\mathbf{n}$  (in which case  $\mathbf{u} \cdot \mathbf{n} > 0$ ), and negatively otherwise.

generalises to:

$$\mathbf{j}_{\text{adv}} = c (\mathbf{u} - \mathbf{w}) \quad (1.23)$$

We note that in the context of a **material domain**, i.e. moving with the same velocity as the fluid, we get  $\mathbf{w} = \mathbf{u}$  and the advection flux cancels out by construction.

<sup>5</sup>The velocity  $\mathbf{u}$  is understood as the average velocity of molecules in the vicinity of the considered point. With this definition we see that the velocity already incorporates diffusion effects. In mixtures chemical species usually have different velocities that need a careful treatment (Bird *et al.*, 2002, §17.7).

<sup>6</sup>We here make use of the fact that a slanted cylinder of length  $\|\mathbf{u}\| \delta t$  is the same as a right cylinder of same height  $(\mathbf{u} \cdot \mathbf{n})$ . This is Cavalieri's principle – which can also be demonstrated with a simple integration.

### 1.3.4 Diffusion

Now, even without any underlying flow, simple matter (species concentration), energy or momentum inhomogeneities will give rise to transfer spontaneously. These **diffusive** exchanges are quantified per unit surface and time with the **diffusive flux**  $\mathbf{j}_{\text{diff}}$ . Even if a detailed modelling of these exchanges is a complex feat, they can nonetheless be described with phenomenological relations (constrained with thermodynamical arguments) such as Fick's law for mass transport for example (see §1.5).

## 1.4 Balance equation for an integrated quantity

Fluxes characterise exchanges across surfaces. With their help we can determine the evolution of a quantity  $c$  contained in a domain in the most general way<sup>7</sup>:

$$\underbrace{\frac{d}{dt} \iiint_V c \, dV}_{\text{Variation}} = - \underbrace{\oint_{\partial V} \mathbf{j} \cdot \mathbf{n} \, dS}_{\text{Exchange}} + \underbrace{\iiint_V \varphi \, dV}_{\text{Production}}, \quad (1.24)$$

so that the *variation* of an integrated quantity in  $V$  is given by a balance of entering/leaving quantity into/from the domain (*exchange*) and the possible *production* (or destruction) of  $c$  inside  $V$ . Before going deeper into the derivation of conservation laws, we have to precise the meaning of the integral derivation appearing in the left hand side. If the considered volume is fixed, the derivation process is easy as we can just swap derivation and integration. But if the domain is moving or deforming, care must be taken in writing this derivation.

### 1.4.1 Volume variation of a material domain and integral derivation

We will now give a meaning to the derivation of an integral performed over a deformable domain. To start with, let's consider the quite specific (but still really common) case of a **material** domain, i.e. we follow the same fluid particles though time. As time flows this domain may see its volume change as indicated on figure 1.5, hence as:

$$\frac{d}{dt} \iiint_V dV = \oint_{\partial V} \mathbf{u} \cdot \mathbf{n} \, dS \quad \left( = \iiint_V \nabla \cdot \mathbf{u} \, dV \right). \quad (1.25)$$

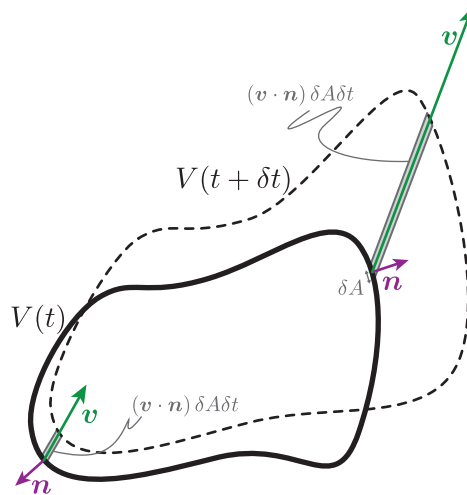


Figure 1.5: A material volume is transported by a velocity field  $\mathbf{u}$ . Each portion  $\delta A$  of the domain boundaries is advected with  $\mathbf{u}$  and this yields a volume change  $(\mathbf{u} \cdot \mathbf{n}) \delta A \delta t$  during  $\delta t$ . The resulting total volume change rate  $\frac{dV}{dt}$  is given by  $\oint (\mathbf{u} \cdot \mathbf{n}) \, dA$ .

<sup>7</sup>This relation is obtained on **physical** grounds.

▷ **Signification of the divergence.** The previous relation allows to shed light on the divergence of a velocity field  $\mathbf{u}$ . Actually, consider a material volume  $\tau(t)$  constituted with the same fluid particles. The previous balance might be rewritten with the help of the divergence theorem as:

$$\frac{d\tau}{dt} = \iint \mathbf{u} \cdot \mathbf{n} dS \quad (1.26)$$

$$= \iiint \nabla \cdot \mathbf{u} dV. \quad (1.27)$$

Thus in the limit where  $\tau(t)$  is really small (in fact sufficiently small so that we can consider  $\nabla \cdot \mathbf{u}$  constant throughout the domain), we can write:

$$\lim_{\tau \rightarrow 0} \frac{1}{\tau} \frac{d\tau}{dt} = \nabla \cdot \mathbf{u}. \quad (1.28)$$

The divergence of a velocity field can therefore be understood as the *rate of volume change of a fluid particle*.

▷ **Integral derivation.** We now have the toolset enabling the clarification of the derivation of the integral over a material domain  $V(t)$ . Let's focus on the variation of the following quantity:

$$\frac{d}{dt} \iiint_{V(t)} \theta dV.$$

To shed some light over this quantity, let's divide mentally the domain in a multitude of tiny cubes or fluid particles. As the fluid domain moves, the value of  $\theta$  will change for all fluid particles at the rate  $\frac{D\theta}{Dt}$ . Moreover the integration element  $dV$  will change as well with the rate  $(\nabla \cdot \mathbf{u}) dV$ . In other words<sup>8,9</sup>:

$$\frac{d}{dt} \iiint_{V(t)} \theta dV = \iiint_{V(t)} \frac{D\theta}{Dt} dV + \iiint_{V(t)} \theta \underbrace{\frac{DdV}{Dt}}_{(\nabla \cdot \mathbf{u})dV} \quad (1.29)$$

#### 1.4.2 Conservation of a quantity. Application to mass conservation

Now that the derivation of an integral has been elucidated, we are in a position to use equation (1.24) which expresses the general conservation of a quantity, for either a fixed or moving domain. Choosing one type of domain or another is largely a matter of context. For example we may consider a moving domain to establish the momentum conservation equation of a given fluid portion. A balance over a fixed domain may also present some interest, when designing for example the evolution of a quantity traversing a fixed mesh cell in a numerical code. Depending on the application, we will choose the more relevant viewpoint.

In order to clarify the use of balances over fixed or moving domains, let's now establish the mass conservation equation (without production nor destruction of mass), first in a fixed domain and then in a moving one.

1. **Mass conservation in a fixed domain**  $V_{\text{fixed}}$ . The balance equation (1.24) reads :

$$\frac{d}{dt} \iiint_{V_{\text{fixed}}} \rho dV = - \oint_{\partial V_{\text{fixed}}} \rho \mathbf{u} \cdot \mathbf{n} dS. \quad (1.30)$$

The domain being fixed, we let the derivation enter in the integral:

$$\iiint_{V_{\text{fixed}}} \frac{\partial \rho}{\partial t} dV = - \oint_{\partial V_{\text{fixed}}} \rho \mathbf{u} \cdot \mathbf{n} dS, \quad (1.31)$$

and, on applying the divergence theorem, we obtain:

$$\iiint_{V_{\text{fixed}}} \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right) dV = 0. \quad (1.32)$$

<sup>8</sup>This relation is obtained on **mathematical** grounds.

<sup>9</sup>We note that this relation can directly be generalised to the case where the domain moves with a velocity differing from that of the fluid (fictitious velocity, flame propagation, balance over a domain moving with a wave). In this case, it suffices to replace  $\mathbf{u}$  with the domain velocity  $\mathbf{w}$ .

Further noting that this balance is actually true for every possible domain, it follows that the integrand actually vanishes:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \quad (1.33)$$

This is the **continuity equation**<sup>10</sup> that embodies mass conservation. This type of reasoning exploiting the validity of an integral expression for any volume to obtain a relation at the fluid particle (or *local*) level is very common.

2. **Mass conservation for a material domain**  $V(t)$ . This time there is no flux term because no fluid particle enters nor leaves the material domain, by definition. The balance equation (1.24) then reads:

$$\frac{d}{dt} \underbrace{\iiint_{V(t)} \rho \, dV}_{m(t)} = 0. \quad (1.34)$$

But as the domain is now moving, we have to apply the integral derivation procedure seen earlier:

$$\frac{d}{dt} \iiint_{V(t)} \rho \, dV = \iiint_{V(t)} \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} \, dV = 0. \quad (1.35)$$

From the latter we recover again the continuity equation (1.33).

▷ **Conservation of a quantity per unit mass.** Thanks to the continuity relation it is possible to obtain a simplified expression for the transport of a quantity per unit mass  $\xi$  (i.e. such that the quantity associated with a fluid particle be  $\rho\xi$ ):

$$\frac{d}{dt} \iiint_{V(t)} \rho \xi \, dV = \iiint_{V(t)} \rho \frac{D\xi}{Dt} \, dV. \quad (1.36)$$

We let the reader demonstrate this relation.

▷ **The incompressible fluid.** A recurring case of great practical value is that of an **incompressible evolution** where we do *not* suppose that the whole fluid a constant density, but rather that each fluid particle conserves its density. This implies:

$$\frac{D\rho}{Dt} = 0 \quad \text{and therefore} \quad \nabla \cdot \mathbf{u} = 0. \quad (1.37)$$

A velocity field  $\mathbf{u}$  satisfying the zero divergence property qualifies as a *solenoidal* field.

### 1.4.3 Conservation of a possibly diffusing passive scalar. Convection-diffusion equation

Let's consider again a concentration field  $c$  advected in a fluid domain with the velocity field  $\mathbf{u}$ . Due to molecular thermal agitation, this field is also subject to diffusion phenomena characterised by the flux  $\mathbf{j}_{\text{diff}}$ . Here again we can obtain the evolution equation for the concentration by following two different routes:

1. By considering a **fixed domain**  $V_{\text{fixed}}$ . In this case, and in absence of any source/sink for the concentration, we will simply write that the total variation is given by the sum of the fluxes:

$$\frac{d}{dt} \iiint_{V_{\text{fixed}}} c \, dV = - \oint_{\partial V_{\text{fixed}}} (\mathbf{j}_{\text{conv}} + \mathbf{j}_{\text{diff}}) \cdot \mathbf{n} \, dS,$$

so that:

$$\iiint_{V_{\text{fixed}}} \frac{\partial c}{\partial t} \, dV = - \iiint_{V_{\text{fixed}}} \nabla \cdot (\mathbf{j}_{\text{conv}} + \mathbf{j}_{\text{diff}}) \, dV.$$

Anticipating on §1.5 by writing the diffusive flux with Fick's law  $\mathbf{j}_{\text{diff}} = -D\nabla c$  we obtain at the local level:

$$\frac{\partial c}{\partial t} + \nabla \cdot (c\mathbf{u}) = \nabla \cdot (D\nabla c) \quad (1.38)$$

For an incompressible evolution with a constant diffusion coefficient, this equation reduces to the classic **advection-diffusion equation**:

$$\frac{\partial c}{\partial t} + (\mathbf{u} \cdot \nabla) c = D\nabla^2 c. \quad (1.39)$$

<sup>10</sup>This denomination has been used for a long time, but is actually not really justifiable...

2. Or by considering a **material domain**  $V_{\text{mat}}$ . This time there is no convective flux by construction (see §1.3.3) but only a diffusive flux:

$$\frac{d}{dt} \iiint_{V_{\text{mat}}} c \, dV = - \oint_{\partial V_{\text{mat}}} \mathbf{j}_{\text{diff}} \cdot \mathbf{n} \, dS,$$

so that, by deriving the integral over the material domain:

$$\iiint_{V_{\text{mat}}} \frac{Dc}{Dt} + c (\nabla \cdot \mathbf{u}) \, dV = \iiint_{V_{\text{mat}}} \nabla \cdot (D \nabla c) \, dS.$$

This relation holds true for every possible domain, and as a result we retrieve the local form of equation (1.38).

#### 1.4.4 Equation for momentum conservation

Momentum conservation for a fluid domain simply follows from Newton's second law  $m\gamma = \Sigma \mathbf{F}$ , never forget this! Let's write this law for a material domain  $V(t)$ :

$$\frac{d}{dt} \iiint_{V_{\text{mat}}} \rho u_i \, dV = \oint_{\partial V_{\text{mat}}} \sigma_{ij} n_j \, dS + \iiint_{V_{\text{mat}}} \rho F_i \, dV \quad (1.40)$$

or, going to the local level:

$$\rho \frac{Du_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho F_i \quad (1.41)$$

That was quite simple. However there is a loophole in the previous expression as the stress tensor  $\sigma$  is here unknown (except in the case of limited interest of a static fluid, see equation (1.3)). Even if equation (1.41) holds true whatever the nature of the fluid (even for say exotic viscoelastic fluids), it is of limited use as long as this stress tensor is not clarified. We therefore have to make a connection between the stresses and the motion of the fluid, for we expect that the stresses will change as soon as the fluid starts to flow. This connection will be made thanks to *constitutive laws* that will really characterise the fluid we are looking at.

## 1.5 Constitutive laws

The conservation laws expressed for a flowing fluid are expressed with fluxes and stresses whose precise form still remains to be determined. In order to make some progress in this determination, let's consider the following situation: without any macroscopic motion, a liquid contains a colourant with concentration  $c$  distributed inhomogeneously, with area more or less concentrated. Even in absence of a mean motion, the collisions between molecules will lead the colourant molecules to migrate randomly in the liquid. As a result, due to the inhomogeneous colourant distribution, there will be an excess of particles traversing  $\partial A$  that originate from a more concentrated area than the reverse. The consequence is a net transport of colourant molecules from more concentrated area to less concentrated ones, that will be active as long as the non-equilibrium situation persists: this is the **diffusion process**. It seems difficult to track the motion of each colourant particles from a statistical viewpoint. But from a *phenomenological* viewpoint, it is quite clear that diffusion will remain active only if a concentration gradient  $\nabla c$  exists. The idea of **gradient-type laws** is to suppose that the components of the flux  $\mathbf{j}_{\text{diff}}$  depend linearly on  $\nabla c$ , i.e. :

$$(\mathbf{j}_{\text{diff}})_i = k_{ij} \frac{\partial c}{\partial x_j}. \quad (1.42)$$

For an isotropic fluid, it seems reasonable to consider that in absence of any privileged axes the flux will be aligned with  $\nabla c$  and of opposite sign (so as to restore equilibrium rather than to destroy it further!), so that :

$$\mathbf{j}_{\text{diff}} = -k \nabla c, \quad (1.43)$$

and therefore:

$$k_{ij} = -k \delta_{ij}. \quad (1.44)$$

We here recover the general form of mass (Fick) and energy (Fourier) transport:

$$\left\{ \begin{array}{ll} \text{Fick's law :} & \mathbf{j}_{\text{mass}} = -D \nabla c \\ \text{Fourier's law :} & \mathbf{j}_{\text{energy}} = -k \nabla T \end{array} \right. \quad (1.45)$$

$$(1.46)$$

▷ **Momentum diffusion and viscous stress.** The case of momentum diffusion is a bit more complex due to the vectorial nature of  $\rho \mathbf{u}$ . To start with let's consider the simple situation of a parallel flow  $(U(y), 0, 0)$ . Due to intermolecular collisions, there is a vertical **diffusion** of momentum. As for mass transport, we suppose that the flux is linear with the momentum gradient, so that:

$$j_{\text{diff}} = -\mu \frac{dU}{dy}, \quad (1.47)$$

Note that this vertical flux of horizontal momentum has the following dimensions;

momentum per unit volume  $\times$  volume/area/time

or  $[\text{ML}^{-1}\text{T}^{-2}]$ , the same dimension of a stress. This dimensional similitude is no coincidence: the transferred momentum is achieved with a stress, in accordance with Newton's second law. We may indeed rewrite (1.47) as:

$$\sigma_{xy} = \mu \frac{dU}{dy}, \quad (1.48)$$

which is the horizontal stress exerted on a surface with a vertical normal<sup>11</sup>. The proportionality coefficient  $\mu$  characterising the efficiency of momentum transport is the *dynamic viscosity*.

▷ **General expression for the stress tensor of a Newtonian fluid.** In the context of a general flow, we can foresee that the stress tensor will depend on each component of the velocity gradient<sup>12</sup>, so that:

$$\sigma_{ij} = -p \delta_{ij} + \alpha_{ijkl} \frac{\partial u_k}{\partial x_l}. \quad (1.49)$$

This form of the stress tensor is compatible with the static fluid stress tensor presented earlier: as soon as the velocity vanishes, the stress tensor adopts its static limit (1.3). But we can go further. If we decompose the velocity gradient tensor into a symmetric part  $\mathbf{e}$  (associated with a deformation) and an antisymmetric part  $\boldsymbol{\omega}$  (associated with solid body rotation):

$$u_{i,j} = e_{i,j} + \omega_{i,j} \quad \text{with} \quad e_{i,j} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad \text{et} \quad \omega_{i,j} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right),$$

Actually momentum diffusion (i.e. viscous stress) can only occur when the flow deviates from solid-body motion (translation or rotation): as a result  $\boldsymbol{\sigma}$  cannot depend on  $\boldsymbol{\omega}$  and:

$$\sigma_{ij} = -p \delta_{ij} + A_{ijkl} e_{kl}. \quad (1.50)$$

Additional considerations on the fluid isotropy that we will not develop here allow for a drastic reduction in the number of coefficients to finally get:

$$\sigma_{ij} = -p \delta_{ij} + 2\mu e_{ij} + \lambda (\nabla \cdot \mathbf{u}) \delta_{ij}. \quad (1.51)$$

## 1.6 Navier-Stokes equation

Having elucidated the structure of the stress tensor (1.51) we can rewrite the equation for momentum conservation (1.41):

$$\rho \frac{Du_i}{Dt} = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} (2\mu e_{ij} + \lambda (\nabla \cdot \mathbf{u}) \delta_{ij}) \quad (1.52)$$

This equation is the *Navier-Stokes equation*.

In the particular case (but of great practical interest) of an incompressible evolution with viscosity gradient, this equation simplifies to:

$$\rho \frac{Du_i}{Dt} = \rho f_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \quad (1.53)$$

or, in vectorial form:

$$\underbrace{\rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right)}_{m\gamma} = - \underbrace{\nabla p + \mu \nabla^2 \mathbf{u}}_{\Sigma F} + \rho \mathbf{f}. \quad (1.54)$$

<sup>11</sup>the reader will note the sign difference between the expressions (1.47) and (1.48) that originate from the notation convention for stresses.

<sup>12</sup>This is this linear dependence to shear that qualifies a fluid as Newtonian. But we can imagine (and actually there exists) more complex relations, of nonlinear nature, between stress and shear. **Rheology** is the discipline that studies these particular non-newtonian behaviours.



## Chapter 2

# Boundary conditions and interfaces

We have seen in the previous chapter how to express mass and momentum conservation at the fluid particle level, and we have derived the corresponding equations for fluid motion. These equations are naturally associated with **boundary conditions** that either express conservation laws (e.g. no mass flux at a boundary) or peculiar physical processes occurring at a surface (temperature or velocity continuity). In both cases these boundary conditions will be pivoting in the determination of the solution.

### 2.1 Fluxes at boundaries: impermeability and imbibition

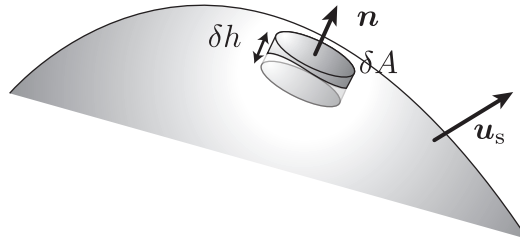


Figure 2.1: Balance over an elementary volume located across a solid boundary.

Let's consider a solid moving in a fluid at velocity  $\mathbf{u}_s$  (possibly time dependent) and let's conduct a mass balance on a small cylindrical volume of base  $\delta A$  and height  $\delta h$  located across the fluid and the solid (Fig. 2.1). Now let  $\delta h$  tends to 0 so that the mass element shrinks down to zero. From now on, the mass variation of the element should necessarily be zero, and this implies that the mass flux from the fluid has to be balanced by the mass flux from the solid side:

$$-(\mathbf{j}_{\text{fluid}} \cdot \mathbf{n}_{\text{fluid}}) \delta A - (\mathbf{j}_{\text{solid}} \cdot \mathbf{n}_{\text{solid}}) \delta A = 0. \quad (2.1)$$

Let's write arbitrarily  $\mathbf{n} = \mathbf{n}_{\text{fluid}} = -\mathbf{n}_{\text{solid}}$  so that:

$$-\mathbf{j}_{\text{fluid}} \cdot \mathbf{n} + \mathbf{j}_{\text{solid}} \cdot \mathbf{n} = 0. \quad (2.2)$$

Without mixing phenomena, the fluid velocity  $\mathbf{u}$  already takes into account diffusive effects (it is the chemical species velocity) and the mass flux reduces to the convective flux. Beware that as we are in the solid reference frame, the relative velocity of the fluid is  $\mathbf{u} - \mathbf{u}_s$ , so that the mass flux on the fluid side is:

$$\mathbf{j}_{\text{fluid}} = \rho (\mathbf{u} - \mathbf{u}_s). \quad (2.3)$$

► **The impermeable wall.** A very common case is that of an **impermeable** solid in which the fluid cannot penetrate; the fluid mass flux within the solid is therefore zero and  $\mathbf{j}_{\text{solid}} = \mathbf{0}$ . Mass conservation expressed at an impermeable boundary therefore reduces to:

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{u}_s \cdot \mathbf{n} \quad (2.4)$$

This is the **impermeability condition** for an object (or a wall). As the name implies, it simply expresses the fact that the fluid cannot penetrate into the solid. This condition takes the form of a **continuity of normal velocities**.

Note: in the quite particular (but still very common!) case of a fixed solid object, this condition reduces to  $\mathbf{u} \cdot \mathbf{n} = 0$ .

▷ **Permeable wall.** The previous discussion naturally extends to the case of permeable walls. Those may correspond to biological tissues permeable to given solutes, to materials swollen by solvents, to terrains soaked by rain or to lifting sails or wings made porous in order to control the boundary layer (such as Cousteau and Malavard' turbosail seen in the lecture). Now the mass flux is non-zero and its precise determination requires a knowledge of the flow *inside* the solid. Suppose however that the imbibition velocity  $\mathbf{u}_{\text{imbib}}$  be constant and known (this corresponds for example to a suction with an imposed flowrate). The mass conservation at the permeable wall will then be written:

$$\rho(\mathbf{u} - \mathbf{u}_s) \cdot \mathbf{n} = \rho(\mathbf{u}_{\text{imbib}}) \cdot \mathbf{n}, \quad (2.5)$$

thus

$$\mathbf{u} \cdot \mathbf{n} = (\mathbf{u}_s + \mathbf{u}_{\text{imbib}}) \cdot \mathbf{n}. \quad (2.6)$$

▷ **Boundary conditions on a concentration field near a wall.** The boundary conditions to apply on the transport equation of a concentration field (1.38) can be obtained following the same principles. Imagine a concentration field transported with a fluid flow  $\mathbf{u}$ . Suppose also that an (impermeable) solid is moving in the fluid at velocity  $\mathbf{u}_s$ . In the solid reference frame the mass flux across any surface  $\delta A \mathbf{n}$  is:

$$\mathbf{j}_{\text{mass}} = \mathbf{j}_{\text{conv}} + \mathbf{j}_{\text{diff}} = c(\mathbf{u} - \mathbf{u}_s) - D\nabla c. \quad (2.7)$$

The impermeability condition at the wall for the concentration field  $c$  is therefore:

$$c(\mathbf{u} - \mathbf{u}_s) \cdot \mathbf{n} - D\nabla c \cdot \mathbf{n} = 0, \quad (2.8)$$

because the mass flux inside the solid is zero. On using the impermeability condition on the velocity field (2.4), this relation reduces to:

$$\nabla c \cdot \mathbf{n} \equiv \frac{\partial c}{\partial n} = 0. \quad (2.9)$$

This Neumann condition is also called a **no-flux boundary condition**.

▷ **Mass transfer at an interface: evaporation.** One last example of boundary conditions arising from conservation considerations is the continuity of the mass flux across a liquid-gas interface, when the (single-phase) liquid is evaporating. Let's write a mass balance on a fluid element analogous to the previous one: a small cylindrical element of base  $\delta A$  and height  $\delta h$  located across a moving interface with velocity  $\mathbf{u}_i$ . We let the height  $\delta h$  of the element tend to zero, so that the element mass equally tends to zero. The mass balance over this element is:

$$\mathbf{j}_{\text{liquid conv.}} \cdot \mathbf{n} = \left( \mathbf{j}_{\text{gas conv.}} + \mathbf{j}_{\text{gas diff.}} \right) \cdot \mathbf{n} \quad (2.10)$$

so that :

$$\rho_\ell (\mathbf{u}_\ell - \mathbf{u}_i) \cdot \mathbf{n} = \rho_v (\mathbf{u}_g - \mathbf{u}_i) \cdot \mathbf{n} - D\nabla \rho_v \cdot \mathbf{n}. \quad (2.11)$$

In the case of a violent evaporation (e.g. combustion front), the flux is dominated by convection effects and

$$\mathbf{u}_g \sim \underbrace{\frac{\rho_\ell}{\rho_v}}_{\gg 1} \mathbf{u}_\ell$$

so that the (so-called Stefan) flow induced in the vapour is much stronger than that in the liquid. In the other limit where evaporation is very slow (case of a slowly drying water drop), it is rather the diffusive term that dominates.

## 2.2 Phenomenological conditions: adherence and continuity

In addition to the previous boundary conditions arising from conservation principles, fluids are also subject to other boundary conditions as well. The latter have been established and confirmed on experimental grounds, so there is a consensus about their relevance but not an exact demonstration. These phenomenological conditions are **field continuity conditions** at interfaces and apply on velocity, temperature etc.

▷ **A short history of adherence.** The adherence condition  $\mathbf{u} = \mathbf{u}_s$  has an astonishing history full of twists and turns, which is accounted for in details in Goldstein (1950). At the XVIII<sup>th</sup> century, the theoretical description of potential flows (corresponding to the idealisation of perfect flow of fluid) was already well established, but the comparisons with experimental data were mediocre. Daniel Bernoulli was well aware of this fact and attributed the discrepancies between (ideal) predicted flows and those observed to some “adherence condition” that would prevail at the wall. Coulomb then demonstrated experimentally that a disk oscillating in a liquid was not particularly affected by a change in surface properties (smooth, rough or covered with grease). Therefore it appeared to Coulomb that the fluid velocity matched the disk velocity in its vicinity. In this vision the fluid has the same properties in every point of space; the flow just fulfils an additional condition at the wall.

But during the XIX<sup>th</sup> century alternate theories appeared. Girard proposed that the liquid layer adjacent to a wall had differing physical properties, leading it to adhere the solid. The next fluid layer (composed of “regular fluid”) could then freely slip onto the first affixed layer. Navier also got interested into this problem and suggested a boundary condition involving a slip directly proportional to the shear stress  $\beta \mathbf{u} = \mu \frac{\partial \mathbf{u}}{\partial n}$ . Here the ratio  $\mu/\beta$  has the dimension of a length: it is the *slip length*. From there a period of relative confusion ensued where famous theoreticians of the time (Poisson, Stokes) alternately adopted one or the other of the boundary conditions.

With time however, delicate experiments conducted by Couette or Maxwell advocated definitely for the adherence condition. Maxwell suggested on molecular dynamics considerations that if Navier’s condition was effectively valid, the length over which slip occurred was so small – of the order of a few mean free paths – that considering it to be zero was a really reasonable hypothesis. This amounted to consider adherence at the wall:  $\mathbf{u} = \mathbf{u}_s$  (Maxwell, 1879). This is in particular verified in the usual conditions of laboratory experiments, but not necessarily for rarefied gas flows (e.g. atmospheric reentry, hypersonic flight) or flows involving fluids with long polymer chains (microfluidic flows), as experiments confirm (Hénot *et al.*, 2018).

During the XX<sup>th</sup> century, the almost perfect agreement between experimental observations and theoretical predictions using adherence condition for a number of flows (Poiseuille flow, Couette flow, Stokes’ sphere settlement in a viscous fluid, the instability threshold for the Taylor-Couette experiment etc) definitely settled the validity of this condition.

Moreover, the scaling laws for the drag on an object deduced from dimensional considerations based on the characteristic scales  $\rho$ ,  $U$ ,  $D$  and  $\mu$  capture the evolution of aerodynamic forces over a wide range of scales. If another lengthscale (associated with slip) was relevant in the description of conventional flows, it would result in an alteration of the scaling laws (not observed).

We can understand adherence condition (continuity of  $\mathbf{u}$  across an interface) but also the condition on temperature continuity as resulting from equilibrium at the molecular level. The extremely rapid transfers between neighbouring molecules induce a quasi-instantaneous equilibrium of mean momentum (velocity) and mean thermal agitation velocity (temperature) (Batchelor, 1967). At an interface between a medium I and a medium II (solid-fluid, or fluid-fluid), we will therefore get:

$$\mathbf{u}_I = \mathbf{u}_{II} \quad \text{and} \quad T_I = T_{II}. \quad (2.12)$$

## 2.3 Where is the interface?

A wide range of flows are associated with interfaces: avalanches, rivers, oceans, emulsion droplets, rain or ink drops etc. As we shall see, an interface between a liquid and another fluid is not just an immaterial boundary geometrically delimiting portions of space occupied by fluids. An interface is an **active region** where surface tension forces act, in some way analogous to those exerted by a taut membrane.

But before precisising the nature of these forces acting on the interface, we first need to be able to know where the interface is! Indeed, an interface is constantly moving by definition and we have to start tracking the interface through time and space. This is the purpose of the *kinematic boundary condition* which will allow us to follow the interface in its motion.

### 2.3.1 Kinematic boundary condition.

In order to describe the motion of any given interface, as a free surface for example, let’s introduce an arbitrary function  $\mathcal{S}(\mathbf{x}, t)$  that vanishes on the interface at all times: for each point  $\mathbf{x}_{\text{interf}}$  of the interface,  $\mathcal{S}(\mathbf{x}_{\text{interf}}, t) = 0$ . Between two successive instants, an interface point is displaced with a distance  $\delta \mathbf{x} = \mathbf{u}_i \delta t$  and:

$$\underbrace{\mathcal{S}(\mathbf{x}_{\text{interf}} + \mathbf{u}_i \delta t, t + \delta t)}_0 \approx \underbrace{\mathcal{S}(\mathbf{x}_{\text{interf}}, t) + \delta t \left( \frac{\partial \mathcal{S}}{\partial t} + (\mathbf{u}_i \cdot \nabla) \mathcal{S} \right)}_{\frac{D_i \mathcal{S}}{Dt}} = 0. \quad (2.13)$$

To get the interface velocity  $\mathbf{u}_i$ <sup>13</sup> we just need to express the mass flux boundary condition at the interface. If there is no mass transfer at the interface (such as e.g. evaporation), the latter reads:

$$\rho (\mathbf{u} - \mathbf{u}_i) \cdot \mathbf{n} = 0 \quad \text{thus} \quad \mathbf{u} \cdot \mathbf{n} = \mathbf{u}_i \cdot \mathbf{n}, \quad (2.14)$$

so that the kinematic boundary condition describing the motion of a free surface simply reads:

$$\frac{D\mathcal{S}}{Dt} = 0 \quad (2.15)$$

▷ **Example: description of a free surface.** Consider a free surface such as the ocean's, characterised<sup>14</sup> with  $z = \zeta(x, y, t)$ . The surface is described at each instant by the vanishing of the function<sup>15</sup>:

$$\mathcal{S}(\mathbf{x}, t) = z - \zeta(x, y, t). \quad (2.16)$$

The kinematic boundary condition describing the position of the free surface is therefore:

$$\frac{D\mathcal{S}}{Dt} = 0 \quad \text{hence} \quad \frac{\partial \zeta}{\partial t} = u_z - u_x \frac{\partial \zeta}{\partial x} - u_y \frac{\partial \zeta}{\partial y} \quad (2.17)$$

### 2.3.2 Surface geometry I. Normals.

In a number of problems involving free surfaces, deformable and/or complex boundaries, the expression for the normal vector has to be provided. A systematic procedure to obtain the expression for this vector stems in remarking that the free surface coincides with a  $\mathcal{S}$ -isosurface. As a result the normal vector is necessarily parallel with the **gradient** of  $\mathcal{S}$ :

$$\mathbf{n} = \frac{\nabla \mathcal{S}}{\|\nabla \mathcal{S}\|} \quad (2.18)$$

▷ **Example: free surface normal.** Let's consider the previous example again. We have:

$$\nabla \mathcal{S} = \begin{pmatrix} -\frac{\partial \zeta}{\partial x} \\ -\frac{\partial \zeta}{\partial y} \\ 1 \end{pmatrix},$$

so

$$\mathbf{n} = \frac{1}{\sqrt{1 + \frac{\partial \zeta^2}{\partial x^2} + \frac{\partial \zeta^2}{\partial y^2}}} \begin{pmatrix} -\frac{\partial \zeta}{\partial x} \\ -\frac{\partial \zeta}{\partial y} \\ 1 \end{pmatrix}. \quad (2.19)$$

We note the the direction of this normal is quite arbitrary and chosen by convention (usually, the convention is that of exterior-pointing normal. However this only makes sense if the very notion of exterior makes sense!).

## 2.4 Capillarity

Having described kinematically interfaces (i.e. how to locate them) along with the boundary conditions resulting from mass conservation and molecular equilibria (adherence condition), we shall now get interested into the dynamical aspects of interfaces and more particularly to the phenomenon of *capillarity* (de Gennes *et al.*, 2015).

An interface between two immiscible fluids, as that between air and water is, is a material surface separating the two media. But actually such an interface is in reality way more than a simple geometric place delimiting two portions of space. An interface **bears forces** whose manifestation is visible in daily-life examples, such as the clogging of wet hairs, the spooling of spider thread within glue droplets or the sustentation of small aquatic animals such as the Gerris (Fig. 2.2).

<sup>13</sup>We note that in the expression of  $\mathcal{S}$ , only the normal component  $\mathbf{u}_i \cdot \mathbf{n}$  of  $\mathbf{u}_i$  at the interface is involved.

<sup>14</sup>We here suppose that the interface is single valued (no wave breaking!).

<sup>15</sup>We could imagine other functions as well, e.g.  $\mathcal{S}(\mathbf{x}, t) = \tanh(z - \zeta(x, y, t)) \dots$

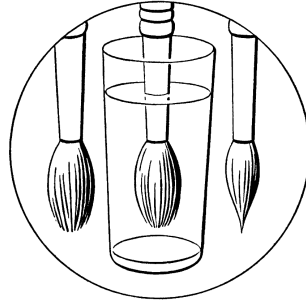


Fig. 1

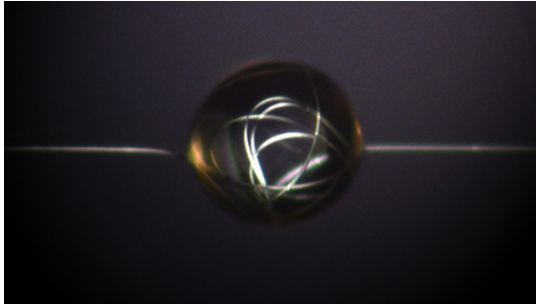


Figure 2.2: Top: when pulled out of water, a wet brush has its bristles stuck altogether, but those are neatly separated if the brush is dry or fully immersed (Boys, 1890). Bottom left: a drop deposited on a sufficiently thin fibre such as the spider capture silk induces coiling and spooling within the droplet (Elettro *et al.*, 2016). Bottom right: the Gerris, an aquatic insect, can literally walk on water although it is denser (photograph by Ryoichi on Flickr). These examples are manifestations of surface tension exerted at liquid-gas interfaces.

### 2.4.1 Cohesion

The microscopic origin of surface tension is rooted in **cohesion** effects in liquids. Indeed, atoms or molecules that constitute matter interact with each other with forces of changing nature. Repulsive at very short distance, the forces become attractive at longer range. In the context of a gas, the intermolecular distances are far too large for these interactions to have any notable effect. But this is not true for dense phases such as liquids where molecules attract each other. This matter cohesion phenomenon is responsible for the rounded shape of droplets, the curvature of menisci or that of soap films. The history of the link between attractive forces and surface tension is already present in the writings of Newton, but this link was not really elucidated until Laplace in 1805 (de Laplace, 1805). With the unique hypothesis of attractive forces between two matter particles decaying with distance, Laplace succeeded in finding with differential calculus all equilibria shapes of interfaces. He also explained the capillary rise in tubes, a phenomenon resisting understanding so far (Rowlinson, 2005). He further evidenced a pressure jump phenomenon (normal stress) proportional to curvature, as we shall now see.

### 2.4.2 Surface tension

In parallel to Laplace's works on cohesion and capillarity, Thomas Young also managed to find in 1805 a result on the pressure jump across interfaces identical to Laplace's finding. But he followed an entirely different route (Young, 1805). On the basis of experimental observations of free surface deformations similar to those of elastic membrane, Young postulated the existence of a tension force localised at the liquid surface, and analogue to membrane tension – such as the one developing in a stretched balloon. Following this reasoning, a small, flat and rectangular surface element  $\delta S = \delta \ell_1 \delta \ell_2$  such as the one illustrated figure 2.3 would be subject to a total force:

$$(\gamma \delta \ell_2 - \gamma \delta \ell_2) \mathbf{e}_1 + (\gamma \delta \ell_1 - \gamma \delta \ell_1) \mathbf{e}_2 = \mathbf{0}.$$

As a matter of fact, each force contribution is balanced by another because the surface is flat. But we feel here that this result may change as soon as the surface will be curved. Before proceeding in the estimation of the surface tension force for a general surface, we have to characterise the curvature of a surface.

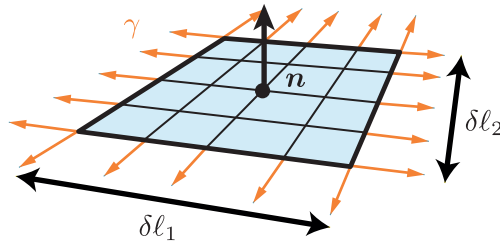


Figure 2.3: A flat interface portion is subject to a tension force at its periphery.

### 2.4.3 Surface geometry II. Curvature.

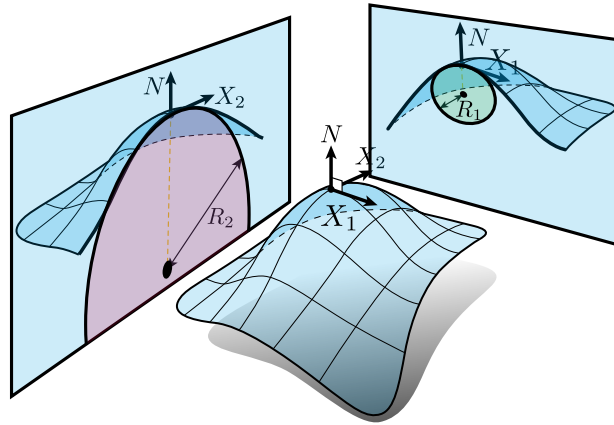


Figure 2.4: A curved surface portion intersected with a plane displays different radii of curvature depending on the plane orientation.

Let's consider the portion of curved interface represented figure 2.4. The curvature of the surface on a given point is a geometric quantity that can be built as follows. Start by identifying the normal  $\mathbf{n}$  to the surface at the chosen point. Consider now two perpendicular planes that each contain  $\mathbf{n}$ . Each plane will intersect the surface following a curve which may be approximated by a circle in the vicinity of the chosen point. Of course the circle radius will differ in each plane in the general case: we will note it for example  $R_1$  in the plane  $(\mathbf{e}_1, \mathbf{n})$  and  $R_2$  in the plane  $(\mathbf{e}_2, \mathbf{n})$ . The **mean** (or arithmetic) **curvature**  $\kappa$  of the surface at the considered point is defined as the sum of the two circle curvatures :

$$\kappa \equiv \frac{1}{R_1} + \frac{1}{R_2}. \quad (2.20)$$

This definition calls for two remarks. The first one regards the unicity of  $\kappa$ 's definition. Indeed we might envisage another set of two orthogonal planes, each of them also containing  $\mathbf{n}$  (obtained via a simple rotation around  $\mathbf{n}$ ). We would then get two other values for  $R_1$  and  $R_2$ . Remarkably, it can be shown in surface geometry that if  $R_1$  and  $R_2$  are not defined in an univocal way, the sum  $\frac{1}{R_1} + \frac{1}{R_2}$  is independent of the planes choice! The curvature  $\kappa$  is a surface **invariant** (except in ill-defined cases, such as surfaces with cusps or corners – in which case the curvature diverges). The second remark is about the curvature sign. Curvature radii are signed quantities, considered positive by convention if the curvature centre is inside the surface, and negative otherwise (to differentiate what is inside from what is outside, we still use the exterior normal convention). It is therefore possible to have a surface of negative curvature.

▷ **Curvature of a sphere.** To illustrate this notion, let's consider a simple sphere of radius  $R$ . At a given point of the surface, the normal vector is given by  $\mathbf{e}_r$  and the two perpendicular planes intersecting the sphere will be two (exact) circles each of radius  $R$ . As a consequence we will have:

$$\kappa_{\text{sphere}} = \frac{2}{R}. \quad (2.21)$$

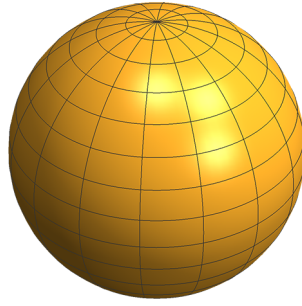


Figure 2.5: A sphere.

▷ **Extension to general surfaces.** The previous approach, although visual and making direct contact with the meaning of curvature, suffers from some shortcomings. Indeed, beyond elementary shapes (cylinder, cone, axisymmetric surfaces etc) it appears difficult to identify the relevant curvature radii for more general surfaces, such as the wavefield described by equation (2.16).

Fortunately there exists an alternate definition for the curvature that comes from surface differential geometry. This alternate definition allows for an easy computation of curvature in every situations:

$$\kappa \equiv \nabla_s \cdot \mathbf{n}. \quad (2.22)$$

Here the operator  $\nabla_s$  represents the surface divergence. In practice we will calculate the (full) divergence of the vector field  $\mathbf{n}$  defined in every point of space, and then simply consider the restriction of this field at the interface. Now an example to shed some light over this definition.

▷ **Curvature of a sphere (again).** Let's consider again the sphere example. In order to determine  $\mathbf{n}$ , we introduce the "colour function"  $\mathcal{S}(r) = r - R$ . Its normalised gradient gives  $\mathbf{n} = \mathbf{e}_r$  in every point of space – and in particular at the sphere's surface. The divergence of  $\mathbf{e}_r$  reads *in every point of space*:

$$\nabla \cdot \mathbf{e}_r = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \times 1) = \frac{2}{r} \quad (2.23)$$

The restriction of this field *to the sphere's surface* gives the curvature:

$$\kappa_{\text{sphere}} = \nabla \cdot \mathbf{e}_r|_{r=R} = \frac{2}{r} \Big|_{r=R} = \frac{2}{R}. \quad (2.24)$$

▷ **Curvature of a wavefield.** Let's go back to the wavefield equation (2.16). The divergence of the normal vector field (2.19) in each point of space is:

$$\kappa = -\frac{\mathcal{A}_{\parallel} \zeta}{\sqrt{1 + \zeta_x^2 + \zeta_y^2}} + \frac{\zeta_x (\zeta_x \zeta_{xx} + \zeta_y \zeta_{yx}) + \zeta_y (\zeta_x \zeta_{xy} + \zeta_y \zeta_{yy})}{(1 + \zeta_x^2 + \zeta_y^2)^{3/2}}, \quad (2.25)$$

where  $\mathcal{A}_{\parallel} \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$  represents the horizontal Laplace operator. We denote in the previous expression differentiation with indices, so that  $\zeta_{yy} = \frac{\partial^2 \zeta}{\partial y^2}$ . We see that it would have been difficult to establish this relation with circles tangencing the surface! We finally note that in the limit of slender slope, where  $\zeta_x, \zeta_y$  are *a priori* much smaller than 1, the expression simplifies significantly to become

$$\kappa \simeq -\mathcal{A}_{\parallel} \zeta \quad \text{in the limit of slender slopes.} \quad (2.26)$$

Yet another manifestation of Laplace operator!

## 2.5 Young-Laplace's pressure jump

Having clarified the notion of surface curvature, we can go back to the estimation of the effective surface tension force exerted on a *curved* interface portion. To do so we consider the elementary surface portion described in figure 2.6. This elementary surface is subtended by two elementary arcs of curvilinear lengths  $\partial \ell_1$  and  $\partial \ell_2$ . To first order the surface is then given by

$$\partial S = \partial \ell_1 \partial \ell_2.$$



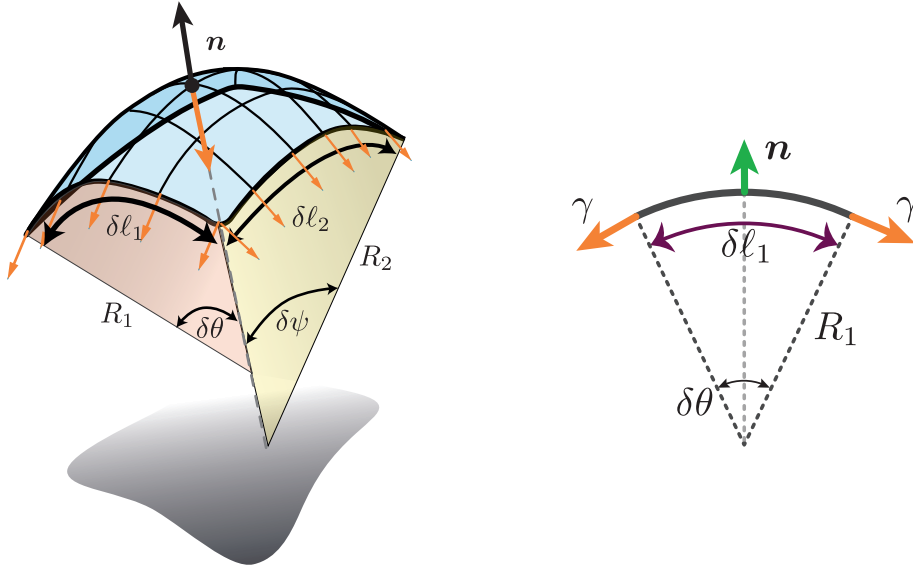


Figure 2.6: Left: a portion of curved interface is subjected to an effective force that depends both on surface tension and on geometry  $-\gamma\kappa \mathbf{n} \delta S$ . Right: sketch of the forces acting on a surface slice showing that contributions along the normal add up.

These two elementary arcs are linked to the two curvature radii  $R_1$  and  $R_2$  by the infinitesimal angles  $\delta\theta$  and  $\delta\psi$  represented on figure 2.6 :

$$\begin{cases} \delta\ell_1 = R_1 \delta\theta \\ \delta\ell_2 = R_2 \delta\psi \end{cases} \quad (2.27a)$$

$$(2.27b)$$

To quantify the action of the forces, start by considering the cut indicated on figure 2.6 (right). The two contributions of intensity  $\gamma$  are each inclined by  $\delta\theta/2$  with respect to the tangent at the considered point. As a result, the projection along the tangent  $\mathbf{t}$  and the normal  $\mathbf{n}$  are:

$$\begin{cases} \text{projection along } \mathbf{t} : & \gamma \cos(\delta\theta/2) - \gamma \cos(\delta\theta/2) = 0 \\ \text{projection along } \mathbf{n} : & -\gamma \sin(\delta\theta/2) - \gamma \sin(\delta\theta/2) \simeq -2\gamma \delta\theta/2 = -\gamma \delta\theta \end{cases} \quad (2.28)$$

$$(2.29)$$

This is true for each cut along the path  $\delta\ell_2$ , so that the contribution of surface tension for these two opposite sides is  $-\gamma \delta\theta \delta\ell_2$  along  $\mathbf{n}$ . We can repeat this line of reasoning for the two other sides, and we obtain the total contribution of surface tension on this small curved element:

$$\delta f_{\text{cap}} = -\gamma (\delta\theta \delta\ell_2 + \delta\psi \delta\ell_1) \mathbf{n}. \quad (2.30)$$

We can reexpress this relation with the help of the curvature radii (2.27):

$$\delta f_{\text{cap}} = -\gamma \left( \frac{1}{R_1} \delta\ell_1 \delta\ell_2 + \frac{1}{R_2} \delta\ell_1 \delta\ell_1 \right) \mathbf{n}, \quad (2.31)$$

so that finally:

$$\delta f_{\text{cap}} = -\gamma \kappa \mathbf{n} \delta S. \quad (2.32)$$

In other words a small curved surface element will be accelerated towards its curvature centre with a force of intensity proportional to both surface tension and curvature. The force exerted on an element therefore depends on its geometry!

Let's stop for a moment and discuss the consequences of such a force. From the previous expression we might be tempted to conclude that each element of a water droplet is accelerated towards its centre, and that the droplet should collapse onto itself. To understand how droplets and other interfaces can withstand this force and remain at equilibrium, we have to remember that droplets are made of a liquid (e.g. water) that is hardly compressible. In practice this incompressibility constraint is guaranteed<sup>16</sup> with the help of a pressure field that will balance the effect of surface tension. Actually, a pressure field is present

<sup>16</sup>Note: even in compressible fluids (ex: bubble) a pressure field is established to counterbalance the effect of capillary force.



in the liquid, even at equilibrium. As seen in chapter 1 the pressure force exerted onto the small considered surface element will be:

$$\delta \mathbf{f}_{\text{pressure}} = (p_{\text{int}} - p_{\text{ext}}) \mathbf{n} \delta S.$$

The interface portion equilibrium condition will then be:

$$\delta \mathbf{f}_{\text{cap}} + \delta \mathbf{f}_{\text{pressure}} = \mathbf{0}, \quad (2.33)$$

so that

$$\Delta p = p_{\text{int}} - p_{\text{ext}} = \gamma \kappa. \quad (2.34)$$

To ensure equilibrium, a **pressure jump** will be established across an interface. This surprising result, due to Young and Laplace, implies that a millimetric rain droplet can sustain overpressure of  $\gamma \kappa \simeq 7 \times 10^{-2} \times \frac{2}{10^{-3}} = 140$  Pa with respect to the atmosphere, and a micron-sized fog droplet will endure an overpressure of  $\simeq 7 \times 10^{-2} \times \frac{2}{10^{-6}} = 1.4 \times 10^5$  Pa, thus 1.4 bars more than atmospheric pressure!

### 2.5.1 Surface tension?

We followed in the previous analysis Young's viewpoint according to which the interface would be taut by a constant surface tension  $\gamma$ . We also recovered Laplace's result of the pressure jump when traversing a curved interface. Laplace did not suppose a distributed surface tension but only attractive forces between molecules. But as we have seen in the previous part, it is possible to switch mathematically from a tangent force representation to an effective normal force, and the reverse. Our demonstration is in fact the expression of a generalisation of Frénet's formulae to three dimensions:

$$\oint_C \mathbf{p} d\ell = - \iint_A \kappa \mathbf{n} dS, \quad (2.35)$$

where  $\mathbf{p}$  is a vector tangent to the interface and  $C$  a closed curve over the interface (Tryggvason *et al.*, 2011). From this, we may wonder if surface tension exists *really*, or if it is not simply a convenient mathematical trick to describe the attractive interactions between molecules. It is now possible to use dynamic molecular simulations to answer this question. As shown

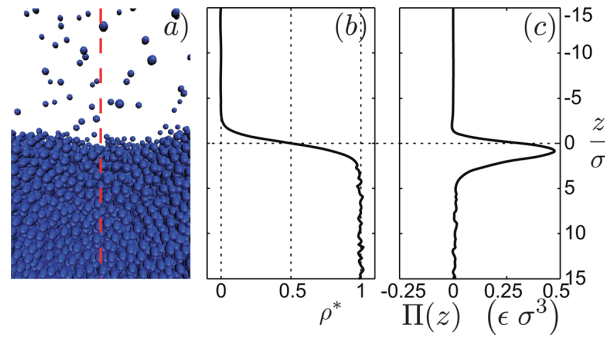


Figure 2.7: Dynamic molecular simulations show that in the vicinity of an interface, molecules are farther apart than in the liquid bulk. This depletion is associated with an increase of attractive force between molecules at the surface. The macroscopic consequence of this increase is the emergence of surface tension (Marchand *et al.*, 2011).

in figure 2.7, there exists a force parallel to the interface that stems from a depletion in near surface molecules: surface tension has therefore a physical reality! (Berry, 1971; Marchand *et al.*, 2011).

### 2.5.2 Stress (dis-)continuity

Thanks to the notion of *stress*, we may easily extend the previous result to the case of moving interfaces. To do so let's consider an elementary volume in the shape of a camembert box leaning on each side of the interface, see figure 2.1. Now we let the volume thickness tend to 0, and so will its mass – and therefore its momentum. As a result the sum of the forces acting on this volume must be zero as well. We have already seen that the net sum of interfacial stresses is  $-\gamma \kappa \mathbf{n} \delta S$ . The stress exerted by the exterior medium is  $\boldsymbol{\sigma}_{\text{ext}} \mathbf{n}_{\text{ext}} \delta S$ , and similarly for the stress exerted by the interior medium  $\boldsymbol{\sigma}_{\text{int}} \mathbf{n}_{\text{int}} \delta S$ , with  $\mathbf{n}_{\text{ext}} = -\mathbf{n}_{\text{int}} = \mathbf{n}$ . The condition for interface equilibrium will thus be:

$$(\boldsymbol{\sigma}_{\text{ext}} - \boldsymbol{\sigma}_{\text{int}}) \mathbf{n} = \gamma \kappa \mathbf{n}. \quad (2.36)$$

This is a really general relation that expresses a **normal stress jump** at the interface. So in the static case where  $\sigma_{ij} = -p\delta_{ij}$ , this relation becomes:

$$\Delta p = p_{\text{int}} - p_{\text{ext}} = \gamma\kappa, \quad (2.37)$$

which is naturally the Laplace pressure jump seen in (2.34).

## 2.6 Equilibrium shape for a meniscus

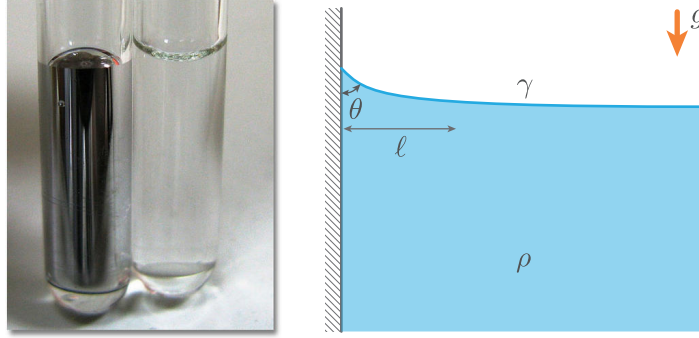


Figure 2.8: Left: mercury or water contained in test tubes exhibit sharply different menisci (Creative Commons). Right: equilibrium profile for a meniscus, parameterised by the wetting angle at the wall.

We now have all the keys to determine interface shapes under the influence of surface tension. As an illustration we will consider in this last part the equilibrium shape for a meniscus. A meniscus is the particular shape adopted by a liquid-gas interface near a wall (figure 2.8 left). Actually at the contact of a wall, an interface displays a *contact angle* that depends on the chemical nature of the liquid and of the wall: water or alcohol on glass or metal will have contact angles smaller than  $\frac{\pi}{2}$  (*wetting* or *hydrophilic* case) whereas water on wax, on a non-stick frying pan or mercury over glass will display contact angles larger than  $\frac{\pi}{2}$ . The corresponding case will qualify as *non-wetting* or *hydrophobic*.

▷ **Equation for the meniscus.** At the wall contact, the interface forms an angle  $\theta$ . Yet far from the wall the interface is back to planar under the influence of gravity. In between, the interface will adopt a shape of a membrane “taut” by the action of surface tension – and under the influence of gravity. Over which lengthscale  $\ell$  does this matching take place? We can give a first estimation with dimensional analysis. Here  $\ell = f(\rho, \gamma, g)$ , and as  $[\gamma] = \text{MT}^{-2}$  a direct application of  $\pi$  theorem informs us that this matching length will be proportional to:

$$\ell_{\text{gc}} = \sqrt{\frac{\gamma}{\rho g}}. \quad (2.38)$$

This is the *gravity-capillary length* (sometimes simply coined capillary length).

Let’s now determine the equation driving the shape for the interface. To this end, let’s introduce first the  $z$  coordinate whose origin will be at the interface height very far from the wall. We will monitor the interface height at each point with the function  $z = h(x)$ . At the interface level the pressure jump relation (2.34) allows us to write:

$$p(z = h) - P_{\text{atm}} = \gamma\kappa, \quad (2.39)$$

where  $p(z = h)$  is the pressure in the liquid near the interface, and  $P_{\text{atm}}$  is the atmospheric pressure. Before going further, let’s remark that the interface curvature centre is *outside* (of the liquid): the pressure jump will therefore be negative, which implies that the liquid pressure will be *lower* than atmospheric pressure.

In the liquid, hydrostatics applies, so  $p(z) = P_{\text{atm}} - \rho g z$ . We therefore have:

$$-\rho g h = \gamma\kappa, \quad (2.40)$$

or

$$h = -\ell_{\text{gc}}^2 \kappa. \quad (2.41)$$

This is the interface equation.

▷ **Equilibrium profile in the slender slope limit.** In general this equation is quite difficult to solve because of its nonlinearities. Indeed, the interface curvature reads:

$$\kappa(x) = -\frac{b''}{(1 + b'^2)^{3/2}}. \quad (2.42)$$

But in the slender slope limit, corresponding to contact angles close to  $\frac{\pi}{2}$  – and anyway always true sufficiently far from the wall –,  $b' \ll 1$  and  $\kappa \simeq -b''$ . As a result (2.41) admits exponential solutions of the type  $e^{x/\ell_{gc}}$  and  $e^{-x/\ell_{gc}}$ . Only the latter solution allows to recover a flat profile at infinity. In the slender slope limit, the interface is therefore described with:

$$b(x) = Ae^{-x/\ell_{gc}}, \quad (2.43)$$

where  $A$  is a constant set by the boundary conditions. In the slender slope limit, we will have near the wall  $b' = -(\frac{\pi}{2} - \theta)$ , so:

$$b(x) = \ell_{gc}(\frac{\pi}{2} - \theta)e^{-x/\ell_{gc}}. \quad (2.44)$$

▷ **Meniscus weight.** If we were to suppress the wall in figure 2.8 right, the interface would naturally get back to the position  $z = 0$ . A question is therefore: what bears the meniscus weight? (because it is not the liquid pressure far below). The only remaining candidate is the wall: the wall exerts a force of intensity  $\gamma$  parallel to the interface and therefore lifts the liquid. In vertical projection, this force has a contribution  $F_{cap} = \gamma \cos \theta$ , which has to balance the meniscus weight.

We can verify this fact by calculus with the help of equation (2.41). Let's introduce the angle  $\phi$  that the interface forms locally with the horizontal ( $\phi = \theta - \pi/2$ ). Whenever we follow an elementary interface portion  $ds$ , this angle changes by an amount  $d\phi$  such that  $ds = R d\phi = -\frac{1}{\kappa} d\phi$ . From this relation we deduce  $\kappa = -\frac{d\phi}{ds}$ . The equation for the meniscus then becomes:

$$-\rho g b(s) = -\gamma \frac{d\phi}{ds}. \quad (2.45)$$

On remarking that  $\cos \phi = \frac{dx}{ds}$ , we see that the meniscus weight can be written as:

$$P = -\rho g \int_0^\infty b(x) dx = -\rho g \int_0^\infty b(s) \cos \phi ds. \quad (2.46)$$

Thus, by multiplying the interface equation by  $\cos \phi$  we get:

$$\underbrace{-\rho g \int_0^\infty b(s) \cos \phi ds}_P = -\gamma \int_0^\infty \cos \phi \frac{d\phi}{ds} ds = -\gamma [\sin \phi]_0^\infty = \gamma \sin \phi_0 = \underbrace{-\gamma \cos \theta}_{-F_{cap}}. \quad (2.47)$$



## Chapter 3

# Viscous flows

Whenever the viscous diffusion timescale  $\rho L^2/\mu$  is short with respect to inertial  $L/U$  or imposed  $\omega^{-1}$  timescales, flows will be dominated by viscosity. This situation occurs of course if the viscosity of the material is high, e.g. with mud, magma or glass melt (figure 3.1) but not only. Glaciers were reported to flow as early as 1873 (Aitken, 1873). These so-called *rivers of ice* indeed flow, albeit very slowly – typically less than a metre a day – making inertial phenomena irrelevant (figure 3.1; see also the slow motion footage taken by BBC Earth Lab <https://youtu.be/ghC-Ut0fW4o>). At very small scales where bacteria and algae evolve, diffusion competes and usually overcomes inertial effects. As a result, microorganisms living at such scales have evolved non-intuitive strategies to move as we will see next.

In all these examples the ratio between the diffusive and inertial timescales – the Reynolds number  $Re = \rho UL/\mu$  – is much smaller than one. Fluid motion can still be described with the Navier-Stokes equations in this context, but we will now see that the condition  $Re \ll 1$  implies that some terms have not the same order of magnitude than others. Exploiting the low-Re number hypothesis will enable us to obtain the relevant equation for viscous flow motion, the **Stokes equation**.

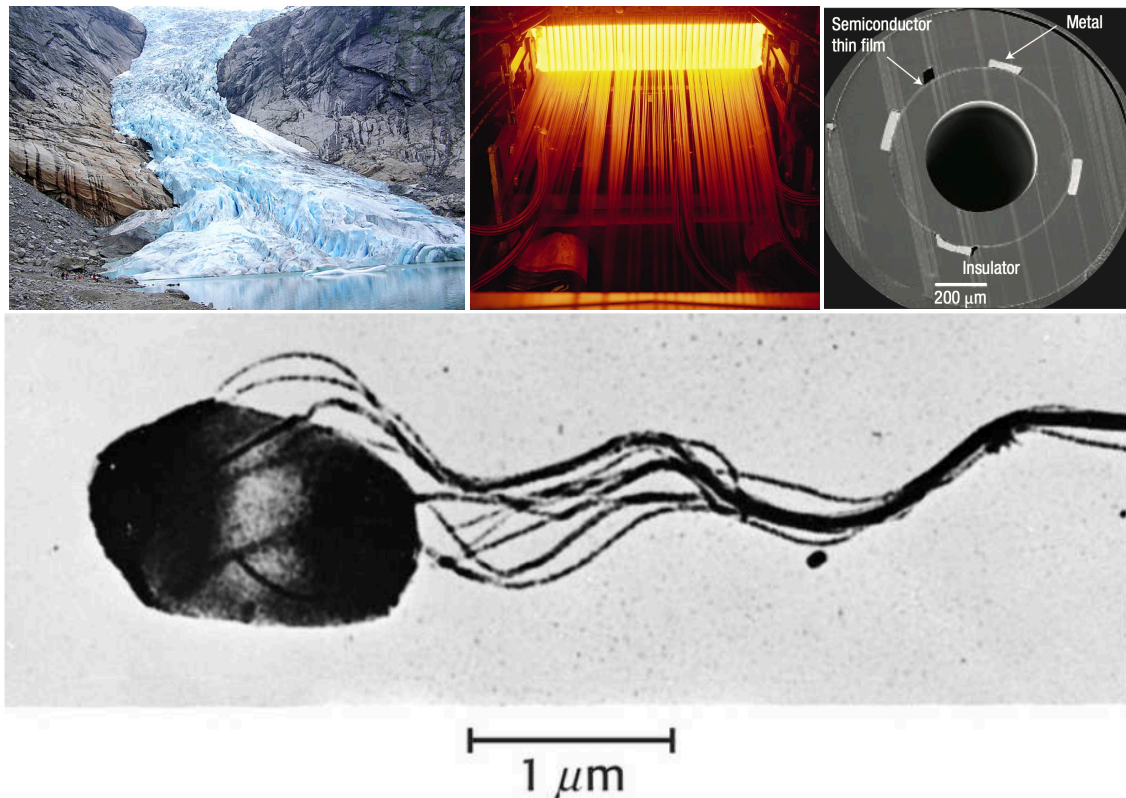


Figure 3.1: **Flows dominated by viscosity.** Top left : the Briksdalsbreen glacier in Norway is slowly flowing into a lake (photograph by vicrogo, public domain). Top middle: glass fibre manufacturing (photograph Saint-Gobain). Top right: modern optical fibre drawing process allow to produce multilayered fibres (Abouraddy *et al.*, 2007). Bottom: micron-sized salmonellae swim with the help of a bundle of rotating helicoidal flagellae (Elgeti *et al.*, 2015).

Organism	length	velocity	frequency	Re	Re <sub>ω</sub>
<b>Bacterium</b>	10 μm	10 μm/s	100 Hz	10 <sup>-4</sup>	10 <sup>-2</sup>
<b>Spermatozoon</b>	100 μm	100 μm/s	10 Hz	10 <sup>-2</sup>	10 <sup>-1</sup>
<b>Ciliate</b>	100 μm	1 mm/s	10 Hz	10 <sup>-1</sup>	10 <sup>-1</sup>
Tadpole	1 cm	10 cm/s	10 Hz	10 <sup>3</sup>	10 <sup>3</sup>
Small fish	10 cm	10 cm/s	10 Hz	10 <sup>4</sup>	10 <sup>5</sup>
Penguin	1 m	1 m/s	1 Hz	10 <sup>6</sup>	10 <sup>6</sup>
Sperm whale	10 m	1 m/s	0.1 Hz	10 <sup>7</sup>	10 <sup>7</sup>

Table 3.1: **Reynolds number for different living organisms.** Bacteria, spermatozoa and ciliates are all characterised by Reynolds numbers Re and Re<sub>ω</sub> much smaller than unity. The flowing fluid in their vicinity is therefore accurately described with the Stokes equation. Data from Lauga (2020).

### 3.1 Low-Re number flows

Whenever fluids have a high viscosity  $\mu$ , or flow at small scale  $L$  or with a low velocity  $U$ , the equations describing their motion can be greatly simplified. This can be seen by non dimensionalising the equations of motions with the natural scales of the problem  $\mathbf{U} = U\mathbf{u}$ ,  $\mathbf{X} = L\mathbf{x}$  (here we denote dimensioned quantities with capital letters). The pressure can be made dimensionless with a natural viscous scale  $P = \mu \frac{U}{L} p$  and finally if there is a imposed timescale  $\omega^{-1}$  (e.g. the inverse of the swimming frequency) we may write  $T = \omega^{-1}t$ .

$$\text{Re}_\omega \frac{\partial \mathbf{u}}{\partial t} + \text{Re} (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \Delta \mathbf{u}. \quad (3.1)$$

Here, two Reynolds numbers appear:

$$\text{Re}_\omega = \frac{\rho L^2 \omega}{\mu} \quad \text{and} \quad \text{Re} = \frac{\rho UL}{\mu}, \quad (3.2)$$

which each may be interpreted as a ratio of timescales as seen in the introduction of this chapter. Note that for e.g. flagellae propelled microorganisms, the oscillatory Reynolds number Re<sub>ω</sub> involves the relevant velocity scale  $L\omega$  for the fluid set into motion by the oscillating flagella.

In equation (3.1) each variable has been rescaled with its expected range of variation range. Therefore each of the force terms (right hand side) is  $O(1)$  while the unsteady and convective part of momentum variation (left hand side) are respectively of order Re<sub>ω</sub> and Re. For flows without an imposed frequency (flowing glass melts or glacier flow), these two numbers will be identical. But if the flow is produced by an oscillating object, they can significantly differ. Table 3.1 reports an estimation of these two Reynolds numbers for a range of organisms living in aqueous environments, where it can be seen that the double condition Re<sub>ω</sub> ≪ 1, Re ≪ 1 is fulfilled in the realm of microorganisms.

In this limit, the Navier-Stokes equation reduces to the much simpler **Stokes equation**:

$$\nabla p = \Delta \mathbf{u} \quad \text{or, in its dimensioned version:} \quad \nabla P = \mu \Delta \mathbf{U}. \quad (3.3)$$

▷ **Cauchy equation.** The Stokes equation may also be rewritten as the following *Cauchy equation*:

$$\nabla \cdot \boldsymbol{\sigma} = \mathbf{0}. \quad (3.4)$$

With this formulation it becomes apparent that viscous flows are *force free*: in absence of significant inertia, the forces balance each other.

### 3.2 Stokes flow's properties

▷ **Linearity.** The Stokes equation (3.3) is **linear**: this means that elementary solutions can be used to construct more involved ones, either as a weighted sum of individual singular solutions or as a convolution integral between an appropriate Green's function and data boundaries. For example, the knowledge of the velocity  $\mathbf{U}$  at the boundary  $S$  of a fluid domain allows to express the fluid velocity  $\mathbf{u}$  at any point  $\mathbf{r}$  of the domain as:

$$u_i(\mathbf{r}) = \int_S G_{ij}(\mathbf{r}|\mathbf{r}_0) U_j(\mathbf{r}_0) dS(\mathbf{r}_0). \quad (3.5)$$



Similarly a knowledge of the *forces*<sup>17</sup>  $\mathbf{F}$  at the domain boundary would lead to the following formal form for the solution:

$$u_i(\mathbf{r}) = \int_S g_{ij}(\mathbf{r}|\mathbf{r}_0) F_j(\mathbf{r}_0) dS(\mathbf{r}_0). \quad (3.6)$$

A practical consequence of these relations is the **unicity** of the Stokes flow solution: the boundary conditions uniquely determine the solution. This contrasts with the multitude of solutions that can be encountered for higher Reynolds number and originate (mathematically) from the nonlinear term of the Navier-Stokes equation.

▷ **Reversibility.** Another quite surprising property of Stokes flow is their **reversibility**. This counter-intuitive phenomenon



Figure 3.2: **Stokes flow kinematic reversibility.** Left: a drop of dye is injected in a quiescent viscous liquid filling the space between two concentric cylinders. Middle: on rotating the inner cylinder with a handle, the dye is stretched and stirred so that it becomes barely visible. Right: reversing the cylinder motion allows to relocate the dye's drop at its initial position almost perfectly (from G.I. Taylor's *Low-Reynolds-Numbers Flows* movie © National Committee for Fluid Mechanics Films / Education Development Center).

is illustrated on figure 3.2. A drop of dye mixed by differential rotation in a Taylor-Couette apparatus can be “unmixed” by reversing the boundary velocity. From a mathematical point of view, this reversal property can be understood by noting that changing the sign of the boundary velocity in (3.5) simply changes the sign of the solution. Alternatively it can also be noted that the transformation  $(\mathbf{u}, p) \rightarrow (-\mathbf{u}, -p)$  also yields a solution of the Stokes equation (3.3) (provided that the boundary conditions are transformed as well).

▷ **A paradox?** From a physical point of view, this reversibility is more troublesome: if diffusion is associated with irreversible microscopic phenomena, how can diffusion-dominated flows be reversible? Actually the reversal illustrated figure 3.2 is purely *kinematic*: while the velocity fields have been reversed and the drop of dye retrieved its overall initial position, molecular diffusion has acted on the microscopic scale as evidenced by the slightly smeared aspect of the final drop, heat has been produced by viscous dissipation and the entropy of the final state is indeed higher than that of the starting state.

### 3.3 Moving in a viscous world

A paradigm for motion in viscous fluids is the settlement of a sphere, first investigated by Stokes. Arguably lengthy calculations (see tutorial) allow to obtain the expression for the velocity and pressure field around a sphere of radius  $R$  settling steadily at velocity  $-\mathbf{V}^\infty$  in a quiescent viscous fluid in its reference frame:

$$\left\{ \begin{array}{l} u_i = -\frac{3R}{4} V_j^\infty \left( \frac{\partial_{ij}}{r} + \frac{r_i r_j}{r^3} \right) - \frac{3R^3}{4} V_j^\infty \left( \frac{\partial_{ij}}{3r^3} - \frac{r_i r_j}{r^5} \right), \end{array} \right. \quad (3.7a)$$

$$\left\{ \begin{array}{l} p - p_\infty = -\frac{3\mu R}{2} \frac{V_j^\infty r_j}{r^3}. \end{array} \right. \quad (3.7b)$$

These expressions allow to evaluate the stresses at the sphere surface and to deduce the well-known **Stokes drag**  $\mathbf{F} = 6\pi\mu R \mathbf{V}^\infty$  exerted on the sphere.

An alternative and very useful viewpoint is to present these results in terms of the force  $\mathbf{f} = -\mathbf{F}$  exerted by the sphere on the fluid:

$$u_i = \underbrace{\frac{1}{8\pi\mu R} f_j \left( \frac{\partial_{ij}}{r} + \frac{r_i r_j}{r^3} \right)}_{\text{Stokeslet contribution}} + \frac{R^2}{8\pi\mu R} f_j \left( \frac{\partial_{ij}}{3r^3} - \frac{r_i r_j}{r^5} \right). \quad (3.8)$$

<sup>17</sup> *Mixed boundary conditions*, constituted of velocity data on part of the boundaries, and forces data on other, can be treated with the same procedure.

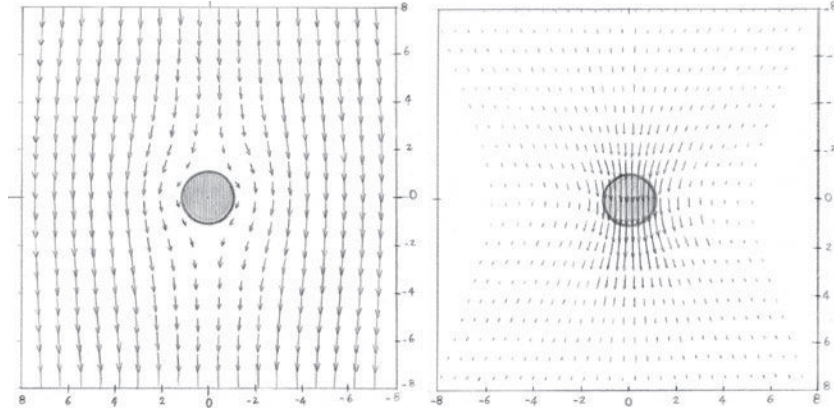


Figure 3.3: **Spheres in viscous flows.** Left: A fixed sphere deflects the surrounding flowing fluid. Right: A moving sphere in a still environment pushes the fluid in its vicinity (Guazzelli & Morris, 2011).

Interestingly if we were to shrink the size of the sphere to 0 while keeping the force constant, the only remaining term in the flow field would be the first one. This so-called Stokeslet contribution is of fundamental importance in suspension dynamics, bacteria hydrodynamics and more generally in the modelling of viscous flows.

### 3.3.1 Point force induced flow: the Stokeslet

The Stokeslet is a fundamental solution for the Stokes equation, and describes the flow that would be induced by a point force  $\mathbf{f} \delta(\mathbf{x} - \mathbf{x}_0)$  located at  $\mathbf{x} = \mathbf{x}_0$ . The corresponding velocity and pressure fields therefore satisfy the following *forced Stokes equation*:

$$-\nabla p + \mu \Delta \mathbf{u} + \mathbf{f} \delta(\mathbf{x} - \mathbf{x}_0) = \mathbf{0}. \quad (3.9)$$

Note that in this expression  $\mathbf{f}$  is a constant vector.

The flow field solution is termed **Stokeslet** and is characterized by:

$$\mathbf{u}_{\text{stokeslet}}|_i(\mathbf{x}) = \frac{1}{8\pi\mu} S_{ij}(\mathbf{x}|\mathbf{x}_0) f_j, \quad (3.10)$$

with  $S_{ij}$  being the *Oseen-Burgers tensor* defined as:

$$S_{ij} = \frac{\delta_{ij}}{r} + \frac{r_i r_j}{r^3}. \quad (3.11)$$

Let's now see in more details how this solution is constructed. To do so it will prove useful to first introduce the Green's function for Laplace equation  $g(\mathbf{x}|\mathbf{y})$ .

▷ **Green's function for Laplace equation.** The Green's function  $g(\mathbf{x}|\mathbf{y})$  for Laplace equation is the function satisfying:

$$\Delta g(\mathbf{x}|\mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}). \quad (3.12)$$

It is an harmonic function of space except at the point  $\mathbf{x} = \mathbf{y}$  where it is singular. Symmetry considerations on this function suggest that it only depends on the radius  $r = \|\mathbf{x} - \mathbf{y}\|$ . On integrating over a small ball containing the singularity we get:

$$\oint \frac{\partial g}{\partial r} dS = 1,$$

so that the Green's function for Laplace equation is:

$$g(\mathbf{x}|\mathbf{y}) = -\frac{1}{4\pi r}. \quad (3.13)$$



▷ **Stokeslet obtention.** With the help of the Green's function for Laplace equation, the divergence of the forced Stokes equation (3.9) may be written as

$$\Delta \left( p - \mathbf{f} \cdot \nabla \left( -\frac{1}{4\pi r} \right) \right) = 0. \quad (3.14)$$

The maximum principle for harmonic functions allows us to directly write the pressure as:

$$p = \mathbf{f} \cdot \nabla \left( -\frac{1}{4\pi r} \right). \quad (3.15)$$

Injecting this form for the pressure in equation (3.9) we obtain:

$$\mu \Delta u_i = \frac{1}{4\pi} \left( \underbrace{\delta_{ij} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_k} \left( \frac{1}{r} \right)}_{\Delta} - \underbrace{\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \left( \frac{1}{r} \right)}_{\nabla \nabla} \right) f_j. \quad (3.16)$$

From the structure of this relationship, the adventurous reader might tempt to look for a solution of the form:

$$u_i = \frac{1}{4\pi} \left( \delta_{ij} \Delta \mathcal{H} - \frac{\partial^2 \mathcal{H}}{\partial x_i \partial x_j} \right) f_j. \quad (3.17)$$

Note that this velocity field is solenoidal, as:

$$u_{i,i} = \frac{1}{4\pi} (\Delta \mathcal{H}_{,j} - \Delta \mathcal{H}_{,j}) f_j \equiv 0. \quad (3.18)$$

Injecting (3.17) into (3.16) we get:

$$\frac{1}{4\pi} (\Delta - \nabla \nabla) \left( \mu \Delta \mathcal{H} - \left( \frac{1}{r} \right) \right) = 0. \quad (3.19)$$

This reduces to the following Poisson equation for  $\mathcal{H}$ <sup>18</sup>:

$$\mu \Delta \mathcal{H} = \frac{1}{r} \quad \text{with solution:} \quad \mathcal{H} = \frac{1}{2\mu} r. \quad (3.20)$$

Noting that  $r = (r_k r_k)^{1/2}$ , we deduce:

$$r_{,i} = \frac{r_{k,i} r_k}{(r_m r_m)^{1/2}} \equiv \frac{x_i}{r} \quad \text{and similarly} \quad r_{,ij} = \frac{\delta_{ij}}{r} - \frac{r_i r_j}{r^3}, \quad (3.21)$$

to finally obtain the expression of the Stokeslet velocity field (3.10) we were looking for:

$$\mathbf{u}_{\text{stokeslet}} = \frac{1}{8\pi\mu} \mathbf{S} \mathbf{f}. \quad (3.22)$$

### 3.3.2 The motion of slender objects

We have seen that in the limit of radius shrinking down to zero, a sphere applying a force to a viscous fluid generates a Stokeslet flow. But looking back at the full expression for the flow set into motion by a sphere of finite size (3.8), it is apparent that the total contribution is actually composed of two parts: a Stokeslet and higher-order singularity – a dipole. Without entering into the details of the derivation, [Hancock \(1953\)](#) and [Lighthill \(1975\)](#) proposed to describe the fluid flows around more general, and slender, objects, as integrals of Stokeslets and dipole contribution. More precisely it appears that the force exerted by a viscous fluid on a very long cylinder of radius  $R$  and length  $L$  depends on the orientation of the flow. If the flow is perpendicular to the filament axis, the force exerted on the cylinder per unit length is:

$$f_{\perp} \simeq c_{\perp} u_{\perp} \quad \text{with} \quad c_{\perp} = \frac{4\pi\mu}{\ln(L/R)}. \quad (3.23a)$$

<sup>18</sup>Note that we did not consider the integration constants because they would not appear in the velocity field expression anyway.

And similarly, if the flow is now parallel to the fibre:

$$f_{\parallel} \simeq c_{\parallel} u_{\parallel} \quad \text{with} \quad c_{\parallel} = \frac{2\pi\mu}{\ln(L/R)}. \quad (3.23b)$$

Note that there is a factor 2 between  $f_{\perp}$  and  $f_{\parallel}$ . This drag anisotropy has consequences on the settling of slender objects that we shall not detail here, but the interested reader will find detailed and useful accounts in [Duprat & Stone \(2016\)](#) or [Lauga \(2020\)](#) for example.

# Bibliography

- ABOURADDY, A. F., BAYINDIR, M., BENOIT, G., HART, S. D., KURIKI, K., ORF, N., SHAPIRA, O., SORIN, F., TEMELKURAN, B. & FINK, Y. 2007 Towards multimaterial multifunctional fibres that see, hear, sense and communicate. *Nat. Mater.* **6** (5), 336–347.
- AITKEN, J. 1873 Glacier motion. *Nature* **7** (172), 287–288.
- BATCHELOR, G. K. 1967 *An introduction to fluid dynamics*. Cambridge University Press.
- BENNETT, A. 2006 *Lagrangian fluid dynamics*. Cambridge University Press.
- BERRY, M V 1971 The molecular mechanism of surface tension. *Physics Education* **6** (2), 79–84.
- BIRD, R. B., STEWART, W. E. & LIGHTFOOT, E. N. 2002 *Transport phenomena (2nd edition)*. John Wiley & Sons.
- BOYS, C. V. 1890 *Soap bubbles and the forces which mould them*. Thomas Y. Crowell Company.
- CHANDRASEKHAR, S. 1957 *An introduction to the study of stellar structure*. Courier Corporation.
- DE GENNES, P.-G., BROCHARD-WYART, F. & QUÉRÉ, D. 2015 *Gouttes, bulles, perles et ondes*. Belin.
- DUPRAT, C. & STONE, H. A. 2016 *Fluid-Structure Interactions in Low-Reynolds-Number Flows*. Royal Society of Chemistry.
- ELETTRO, H., NEUKIRCH, S., VOLLRATH, F. & ANTKOWIAK, A. 2016 In-drop capillary spooling of spider capture thread inspires hybrid fibers with mixed solid–liquid mechanical properties. *Proc. Natl Acad. Sci. U.S.A.* **113** (22), 6143–6147.
- ELGETI, J., WINKLER, R. G. & GOMPPER, G. 2015 Physics of microswimmers – single particle motion and collective behavior: a review. *Rep. Prog. Phys.* **78** (5), 056601.
- GOLDSTEIN, S. 1950 *Modern developments in fluid dynamics: an account of theory and experiment relating to boundary layers, turbulent motion and wakes*. Clarendon Press.
- GUAZZELLI, E. & MORRIS, J. F. 2011 *A physical introduction to suspension dynamics*. Cambridge University Press.
- HANCOCK, G. J. 1953 The self-propulsion of microscopic organisms through liquids. *Proc. Roy. Soc. Lond. A* **217** (1128), 96–121.
- HÉNOT, M., GRZELKA, M., ZHANG, J., MARIOT, S., ANTONIUK, I., DROCKENMULLER, E., LÉGER, L. & RESTAGNO, F. 2018 Temperature-controlled slip of polymer melts on ideal substrates. *Phys. Rev. Lett.* **121** (17).
- DE LAPLACE, P. S. 1805 *Traité de mécanique céleste. Supplément au livre X*. Courcier, Paris.
- LAUGA, É. 2020 *The fluid dynamics of cell motility*. Cambridge University Press.
- LIGHTHILL, J. 1986 An informal introduction to theoretical fluid mechanics. *The Institute of Mathematics & its Applications Monograph Series, Oxford: Clarendon Press, 1986*.
- LIGHTHILL, M. J. 1975 *Mathematical Biofluidynamics*. SIAM.
- MARCHAND, A., WEIJS, J. H., SNOEIJER, J. H. & ANDREOTTI, B. 2011 Why is surface tension a force parallel to the interface? *Am. J. Phys.* **79** (10), 999–1008.

- MAXWELL, J. C. 1879 VII. on stresses in rarified gases arising from inequalities of temperature. *Philos. Trans. R. Soc.* **170**, 231–256.
- MITTAL, R., NI, R. & SEO, J.-H. 2020 The flow physics of COVID-19. *J. Fluid Mech.* **894**.
- PRANDTL, L. & TIETJENS, O. K. G. 1957 *Applied hydro-and aeromechanics*. Dover publications.
- ROWLINSON, J. S. 2005 *Cohesion: a scientific history of intermolecular forces*. Cambridge University Press.
- SONI, V., BILILIGN, E. S., MAGKIRIADOU, S., SACANNA, S., BARTOLO, D., SHELLEY, M. J. & IRVINE, W. T. M. 2019 The odd free surface flows of a colloidal chiral fluid. *Nat. Phys.* **15**, 1188–1194.
- TRYGGVASON, G., SCARDOVELLI, R. & ZALESKI, S. 2011 *Direct numerical simulations of gas–liquid multiphase flows*. Cambridge University Press.
- YOUNG, THOMAS 1805 An essay on the cohesion of fluids. *Philos. Trans. R. Soc.* **95**, 65–87.

# Index

Archimedes principle, 7

boundary conditions

adherence, 19

evaporation, 18

impermeability, 17

kinematic, 19

no-flux, 18

conservation law, 12

mass conservation, 14

momentum conservation, 15

constitutive laws, 15

continuity equation, 14

curvature, 22

divergence (meaning), 13

flux, 11

advective, 11

diffusive, 12

Green's function, 32

Laplace pressure jump, 25

meniscus, 26

normal vector, 20

Stokes drag, 31

Stokes equation, 30

Stokes reversibility, 31

Stokeslet, 32

surface tension, 20

velocity (definition), 11

viscosity, 16