Hydrodynamics

Tutorial 5: Saint-Venant equations

1 Dam break

We set out to describe the dynamics resulting from the rupture of a dam. The liquid contained is supposed to have a initial depth noted h_0 and no velocity, and occupies $x < 0$. The dam is located at $x = 0$ and breaks at $t = 0$. In the problem we model the evolution of the liquid with the Saint-Venant equations.
The Saint-Venant equations describing the motion of a thin inertial liquid film read:

$$\begin{cases} h_t + (hu)_x = 0 & \text{(1a)} \\ u_t + uu_x + gh_x = 0 & \text{(1b)} \end{cases}$$

1. We start by considering a still liquid (depth h_0 and no velocity). By linearising the Saint-Venant equations around this base flow, show that they admit plane wave solutions propagating at $\pm c$. Give the value of the celerity c. Does it depend on the wavelength?
2. Show that the (full) Saint-Venant equations can be rewritten as:

$$\left(\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right)(u \pm 2c) = 0, \tag{2}$$

3. Deduce that some quantities (called the *Riemann invariants*) are preserved along the curves $\frac{\mathrm{d}x}{\mathrm{d}t} = u \pm c$ (called *characteristics*) noted respectively C^+ and C^- in the following.
4. Represent the characteristics for the region far from the dam, where the water is undisturbed. Supposing that the set of characteristics cross each other, deduce the value of h and u there, and the real shape of the characteristics.
5. Show that this region is bounded by $x < -c_0 t$.
6. Supposing that the C^+ characteristics enter into the domain $x > -c_0 t$, show that u and h are constants along the C^- curves, and that these are lines.
7. At $t = 0$ the fluid only occupies the region $x < 0$ so the C^- characteristics must all come from zero and satisfy

$$\frac{x}{t} = u - c \tag{3}$$

   but also $u + 2c = 2c_0$.
8. Deduce that the shape and velocity of the water satisfy at all instants:

$$\begin{cases} h(x,t) = \dfrac{h_0}{9}\left(2 - \dfrac{x}{c_0 t}\right)^2 & \text{(4a)} \\ u(x,t) = \dfrac{2}{3}\left(c_0 + \dfrac{x}{t}\right) & \text{(4b)} \end{cases}$$

9. Compute the real shape of the characteristics and conclude on the validity of the assumptions.

References