Hydrodynamics

Tutorial 6: dispersive waves

1 A wake of waves

Figure 1: **Duck waves.** A duck moving around in a pond produces a wave pattern in its wake (Jardin des Tuileries, Paris. Photograph AA).

An object moving at a free surface, such as a boat or an animal, generates a characteristic wave pattern illustrated figure 1. This pattern is confined in a wedge of half-opening angle θ, remarkably constant across the scales. In the following we set out to determine this angle..

1. The pulsation ω of a monochromatic water wave is given by:

$$\omega^2 = gk \tanh kh. \tag{1}$$

   Here k, g and h stand respectively for the wavenumber, the gravity and the water depth. Determine the phase velocity c.
2. Express the group velocity c_g as a function of c.
3. Find the asymptotic expressions of c and c_g in the deep water limit.

A key assumption in the determination of the wave pattern lies in the fact that the pattern appears *steady* or frozen from the point of view of an observer moving with the boat.

4. Find the phase velocity of a steady wavefront moving in the direction of the boat.
5. Steady wavefronts are such that the (prolongated) crests pass through the boat. Find the phase velocity of a steady wavefront whose direction of propagation forms an angle α with the velocity of the boat.
6. By considering all possible directions deduce the shape of the wave pattern produced by the boat.
7. Remembering that wavepackets propagate at the group velocity c_g refine your prediction and obtain the semi-angle characterizing *Kelvin's wedge* enclosing the wave pattern:

$$\theta = \arcsin\left(\tfrac{1}{3}\right). \tag{2}$$

2 Asymptotic shape of a wavetrain

The Korteweg-de Vries equation describes the propagation of wave by considering first-order dispersive effects:

$$\frac{\partial \eta}{\partial t} + c_0 \frac{\partial \eta}{\partial x} + \frac{c_0 h_0^2}{6}\frac{\partial^3 \eta}{\partial x^3} = 0. \tag{3}$$

In the following we consider a wavepacket generated near the origin at initial time, and we seek to determine the asymptotic shape of the wavetrain far from the origin. We assume that the free surface elevation is a known data and that the wave height decreases to zero at infinity:

$$\int_{-\infty}^{\infty} \eta \, \mathrm{d}x = A \quad \text{and} \quad \lim_{x\to\infty} \eta = 0. \tag{4}$$

1. Demonstrate that this equation describes dispersive waves (Hint: insert a wavelike disturbance and compute the phase velocity).
2. Rewrite this equation in a frame moving at velocity c_0, i.e. using $\xi = x - c_0 t$ and t.

In the following, we look for a *scale invariant* solution of the Korteweg-de Vries equation.

3. Writing $\alpha = c_0 h_0^2$, find the relations linking the scale factors.
4. Show that $\eta(\xi, t)$ can be represented as:

$$\eta(\xi, t) = \frac{1}{(\alpha t)^{1/3}} f(\mu), \tag{5}$$

   where μ is a similitude variable to be precised.
5. Insert this functional relation into Korteweg-de Vries equation and determine the self-similar shape of the free surface.

▷ **Airy's function.** The solution to:

$$y''(x) = xy(x) \tag{6}$$

is Airy's function $\mathrm{Ai}(x)$. This function satisfies $\lim_{x\to\infty} \mathrm{Ai}(x) = 0$ and $\int_{-\infty}^{\infty} \mathrm{Ai}(u)\,\mathrm{d}u = 1$.

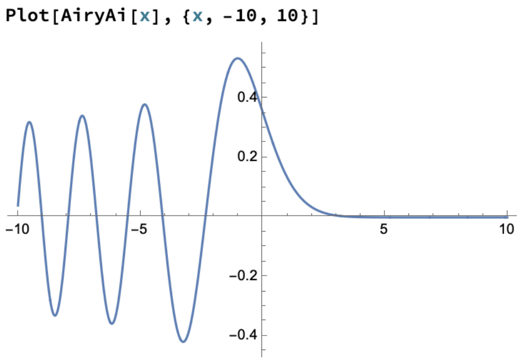

Figure 2: **Airy's function.**

References