Sorbonne Université
Faculté des Sciences

Year 2022-2023
Physics of Fluids & Nonlinear Physics

Hydrodynamics

Tutorial 3: viscous flows

1 Taylor' scraper (1962)

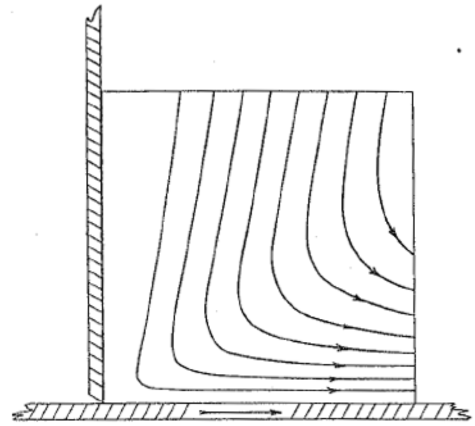

Fig. 1. Streamlines of flow of viscous fluid for $\alpha = \frac{1}{2}\pi$.

Figure 1: **Forced flow in a corner of angle α.** Streamlines of the viscous flow induced by the displacement of the bottom plate at velocity U (here $\alpha = \frac{\pi}{2}$).

1. Write down the equations and boundary conditions describing this problem in native variables (pressure/velocity).
2. We look for a solution to this flow by introducing the streamfunction ψ such that $u = -\nabla \times \left(\psi(r,\theta)e_z\right)$. Show that the Stokes equations can be reduced to:
$$\nabla^4\psi = 0 \tag{1}$$
3. Show that the boundary conditions for ψ are (supposing that the plate advances at velocity $-U$):
$$\begin{cases} \dfrac{\partial\psi}{\partial r} = 0 \quad \text{et} \quad \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} = U & \text{en } \theta = 0 \qquad \text{(2a)} \\[2ex] \dfrac{\partial\psi}{\partial r} = 0 \quad \text{et} \quad \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} = 0 & \text{en } \theta = \alpha \qquad \text{(2b)} \end{cases}$$
4. We look for a solution ψ under the following form:
$$\psi(r,\theta) = rf(\theta) \tag{3}$$
Justify this choice.
5. Introducing the differential operator $\mathcal{D} = \frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + \mathcal{I}$, where $\mathcal{I}$ is the identity operator, show that:
$$\mathcal{D}^2 f = 0. \tag{4}$$
Deduce the general form of $f(\theta)$:
$$f(\theta) = A\sin\theta + B\cos\theta + C\theta\sin\theta + D\theta\cos\theta \tag{5}$$
6. Explicit the boundary conditions for $f(\theta)$.
7. By application of these boundary conditions, show:
$$\left(A,B,C,D\right) = \left(\alpha^2, 0, -\alpha + \sin\alpha\cos\alpha, -\sin^2\alpha\right) \times \frac{U}{\alpha^2 - \sin^2\alpha}. \tag{6}$$

8. Represent with Python these streamlines for $\alpha = \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{4}, \pi$.
9. Estimate the order of magnitude the neglected acceleration term in the Stokes approximation as a function of ρ, U et r. Evaluate as well the order of magnitude of the viscous force. Deduce a region of validity for this approximation as a function of μ, ρ et U.
10. Retrieve the expression for the normal stress exerted on the scraper found by Talor.
11. Discuss his conclusion on the way painters use their scraper to clean their palette.

2 Flow motion induced by microorganisms

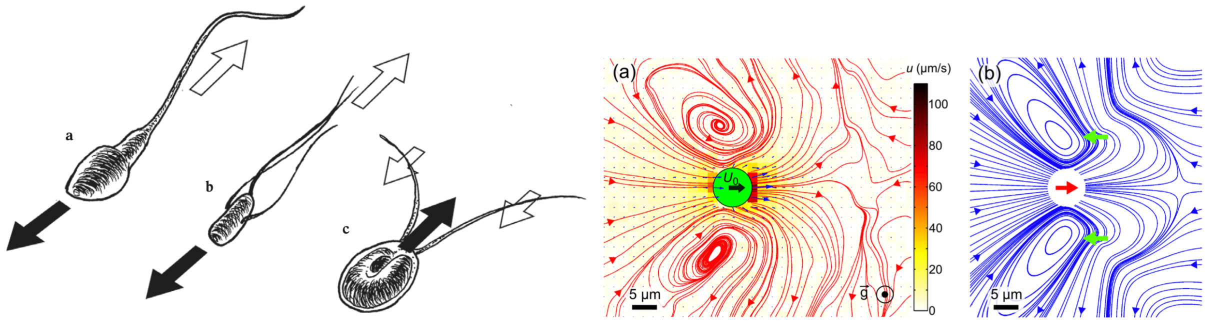

Figure 2: **Swimming microorganisms**. Left: several microorganisms swimming: spermatozoon, bacterium, biflagellated alga (Lauga, 2020). Right: the flow motion measured around a swimming alga (red lines) can be well approximated with a superposition of Stokeslets (Drescher *et al.*, 2010).

1. Represent with Python the streamlines around a single Stokeslet.
2. Using the superposition principle, try to reproduce the flow pattern around a swimming biflagellated alga (hint: consider three Stokeslets as indicated in the figure, of net zero force).

Appendix: differential operators in cylindrical coordinates

Let $f(r, \theta, z)$ be a scalar function of space and $u(r, \theta, z) = (u_r(r, \theta, z), u_\theta(r, \theta, z), u_z(r, \theta, z))$ a vector field. We define:

$$\nabla f = \begin{pmatrix} \dfrac{\partial f}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial f}{\partial \theta} \\ \dfrac{\partial f}{\partial z} \end{pmatrix}, \quad \nabla \cdot u = \frac{1}{r}\frac{\partial}{\partial r}\left(r u_r\right) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}, \quad \Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

36

ON SCRAPING VISCOUS FLUID FROM A PLANE SURFACE

REPRINTED FROM

Miszellaneen der Angewandten Mechanik (Festschrift Walter Tollmien), edited by M. Schäfer, Akademie-Verlag, Berlin (1962), pp. 313–15

The stream function ψ, representing two-dimensional fluid flow, satisfies $\nabla^4\psi = 0$ when the flow is so slow that inertia is negligible. Simple solutions of this equation expressed in polar co-ordinates can be found in the form $\psi = r^n f(\Theta)$ but they seldom have much physical interest except as terms in series expansions. When n is greater than 1 they usually represent flow in a corner produced by agents which can only be described by other types of solution. When n is less than 1 the flow velocity near the origin is infinite. The case when $n = 1$ has physical significance because it can represent flow of a viscous fluid when a flat scraper moves over a flat sheet pushing fluid before it.

The general solution of $\nabla^4\psi = 0$ when $n = 1$ is

$$\psi = r(A\cos\Theta + B\sin\Theta + C\Theta\cos\Theta + D\Theta\sin\Theta). \tag{1}$$

If (1) represents the flow in front of a scraper moving with velocity U over a flat surface (taken as $\Theta = 0$) it is convenient to superpose a velocity $-U$ parallel to $\Theta = 0$ so as to reduce the motion to steady flow. The radial and tangential velocities are

$$u = -\frac{1}{r}\frac{\partial\psi}{\partial\Theta} \quad \text{and} \quad v = \frac{\partial\psi}{\partial r}.$$

If the scraper is inclined at angle α to the flat plate the boundary conditions are

$$\psi = 0,\ \frac{1}{r}\frac{\partial\psi}{\partial\Theta} = U \quad \text{at} \quad \Theta = 0 \quad \text{and} \quad \psi = \frac{\partial\psi}{\partial\Theta} = 0 \quad \text{at} \quad \Theta = \alpha.$$

Using these conditions (1) becomes

$$\psi = \frac{Ur}{\alpha^2 - \sin^2\alpha}\left\{\alpha^2\sin\Theta - \sin^2\alpha\,\Theta\cos\Theta - \left(\frac{\alpha\sin^3\alpha + \alpha^2\cos\alpha - \cos\alpha\sin^2\alpha}{\sin\alpha + \alpha\cos\alpha}\right)\Theta\sin\Theta\right\}. \tag{2}^*$$

* *Editor's note.* Fig. 1, which shows the streamlines for the case $\alpha = \frac{1}{2}\pi$, is taken from an article by Sir Geoffrey Taylor entitled 'Similarity solutions of hydrodynamic problems' published in *Aeronautics and Astronautics* (Pergamon Press, 1960) and not reprinted in these volumes.

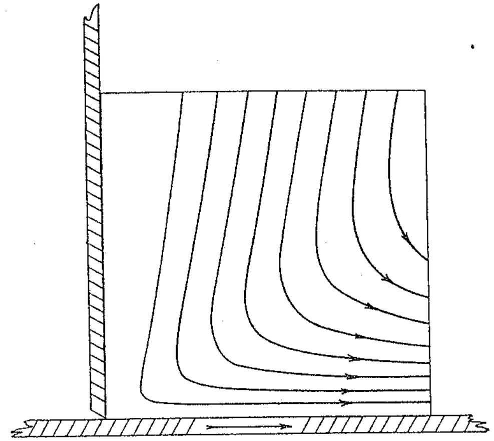

Fig. 1. Streamlines of flow of viscous fluid for $\alpha = \frac{1}{2}\pi$.

If P and S are the stresses normal and tangential to the scraper at distance r from the point of contact and μ is the viscosity

$$P = \frac{2\mu U}{r}\left(\frac{\alpha\sin\alpha}{\alpha^2 - \sin^2\alpha}\right), \tag{3}$$

$$S = \frac{2\mu U}{r}\,\frac{\sin\alpha - \alpha\cos\alpha}{\alpha^2 - \sin^2\alpha}. \tag{4}$$

When α is small these tend to $P = 6\mu U/r\alpha^2$ and $S = 2\mu U/r\alpha$ which are the values that can be found using the approximate equations of lubrication theory.

Resolving the stress in direction L at right angle to the plate and D parallel with it,

$$L = P\cos\alpha + S\sin\alpha = \frac{2\mu U}{r}\left(\frac{\sin^2\alpha}{\alpha^2 - \sin^2\alpha}\right),$$

$$D = P\sin\alpha - S\cos\alpha = \frac{2\mu U}{r}\left(\frac{\alpha - \sin\alpha\cos\alpha}{\alpha^2 - \sin^2\alpha}\right).$$

Values of L, D, P and S divided by $2\mu U/r$ for various values of α are given in table 1, and are displayed in fig. 2. It will be seen that D decreases as α increases, and attains its least value $2\mu U/\pi r$ when $\alpha = \pi$. The most interesting and perhaps unexpected feature of the calculations is that L does not change sign in the range $0 < \alpha < \pi$. In the range $\frac{1}{2}\pi < \alpha < \pi$ the contribution to L due to normal stress is of opposite sign to that due to tangential stress, but the latter is the greater. The palette knives used by artists for removing paint from their palettes are very flexible scrapers. They can therefore only be used at such an angle that P is small and as will be seen in the figure this occurs only when α is nearly 180°. In fact artists instinctively hold their palette knives in this position.

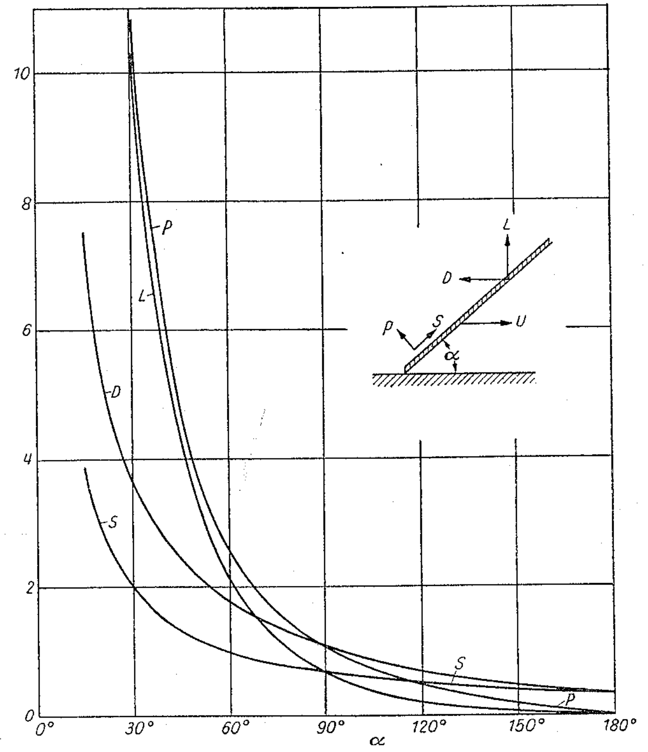

Fig. 2. Values of P, S, L and D divided by $2\mu U/r$.

Table 1

$\alpha°$	$\frac{Pr}{2\mu U}$	$\frac{Sr}{2\mu U}$	$\frac{Lr}{2\mu U}$	$\frac{Dr}{2\mu U}$
0	∞	∞	∞	∞
15	43	3·82	42	7·5
30	10·8	2·03	10·3	3·67
45	4·8	1·31	4·30	2·44
60	2·61	0·98	2·15	1·77
75	1·61	0·80	1·19	1·36
90	1·07	0·68	0·68	1·07
105	0·73	0·60	0·38	0·85
120	0·50	0·53	0·21	0·70
135	0·33	0·47	0·10	0·56
150	0·20	0·42	0·04	0·46
165	0·08	0·37	0·014	0·38
180	0	0·32	0	0·32

A plasterer on the other hand holds a smoothing tool so that α is small. In that way he can get the large values of L/D which are needed in forcing plaster from protuberances to hollows.

Though the fluid velocity is everywhere finite the stress becomes infinite at $r = 0$ in this solution. In fact in any real situation continuous contact between scraper and plate along a line will not occur so that infinite stress at $r = 0$ will be relieved over a region comparable with the width of the gap.