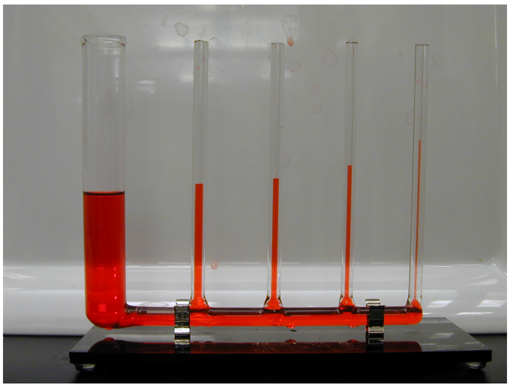
Hydrodynamics
Tutorial 1: fluid motion

1 Dimensional analysis



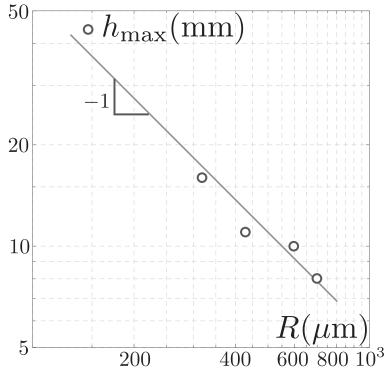


Figure 1: **Maximal ascension.** Left: the maximal height for capillary ascension depends on tube radius R. Right: maximal height h observed for the capillary rise of ethanol in tubes of different radii R (data from the authors).

▷ **Imbibition.** A narrow capillary tube is brought into contact with a wetting liquid. The liquid spontaneously rises in the tube up to a height h (figure 1). This height depends a priori on the *surface tension* γ ($[\gamma] = \mathsf{MT}^{-2}$), gravity g, density ρ and on the tube radius R :

$$h = f(\gamma, \rho, g, R) \tag{1}$$

1. Using the characteristic scales ρ, g and R, show that the previous functional relation can be rewritten as:

$$\pi = \mathcal{F}(\pi_1), \tag{2}$$

where π corresponds to the nondimensional height (the observable), and π_1 to the non-dimensional surface tension. Write the expression for π and π_1 (that we will take proportional to h and γ respectively).

2. Experiments and physical analysis show that $\mathcal{F}(x) = 2x$. What is the scaling law of h with respect to R ? Is it compatible with the experimental results reported 1 ?

▷ **Molecular diffusion.** A drop of dye is delicately deposited in a liquid. Due to constant molecular motion and collisions, the area expands by *diffusion* – a process whose efficiency is characterised with the diffusion coefficient D (of dimension $[D] = \mathsf{L}^2\mathsf{T}^{-1}$).

3. Using dimensional analysis show that the dye drop spreads following a square root law at long times $R(t) \propto t^{1/2}$.

▷ **Turbulent diffusion.** *(from Eggers & Fontelos (2015)).* We consider again the previous experiment but now the liquid is vigorously stirred, so as as to create turbulent motions stirring and mixing the dye. This process is a priori much more efficient than simple molecular diffusion, so that we neglect the latter in the following. The stirring intensity is characterised with ε, the energy quantity per unit time and mass in the liquid.

4. what is the dimension of ε ?

5. Show that the drop area now grows according to

$$R(t) = A\left(\varepsilon t^3\right)^{1/2}, \tag{3}$$

where A is a dimensionless universal constant. This result is the signature of a process much more efficient than molecular diffusion, and known as *Richardson's law* (Richardson, 1926; Eggers & Fontelos, 2015).

2 Tumbling cards

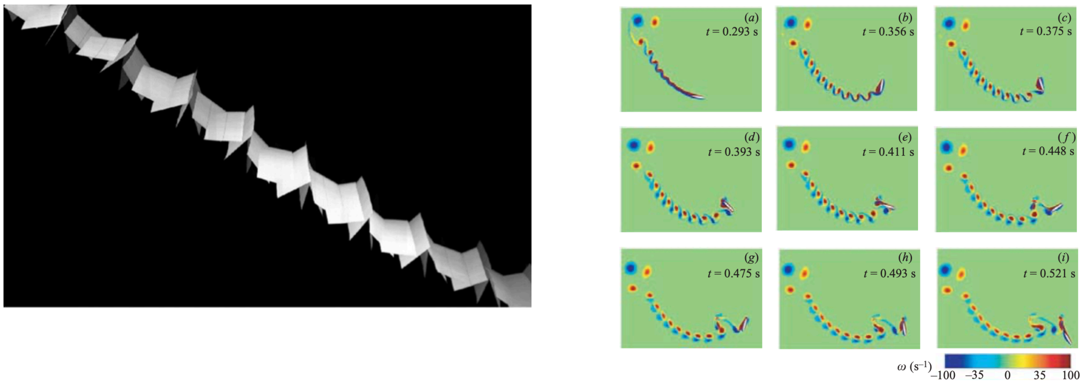


Figure 2: **Tumbling cards.** Left: chronophotograph of a falling and tumbling card. Right: numerical simulations of the process reveal an intricate vortex wake associated with boundary layer detachment.

Freely falling cards or metro tickets naturally tumble when falling. The observable frequency of tumbling is the result from an intricate process coupling the card flight with a complex vortex wake, see Fig. 2. Here we will see how experiments and dimensional analysis can help us in obtaining quantitative predictions about the frequency.

1. By cutting out cards from a piece of paper, observe the tumbling phenomenon.

2. Identify the key parameters influencing the frequency Ω *a priori*. We will denote the length of the card l, its width w and thickness t. The density of the card will be denoted ρ_c and that of air ρ_a. Are there other relevant parameters?

3. From the experiments, observe qualitatively the independence of Ω with l.

4. Experiments further reveal that the tumbling is insensitive to the nondimensional air viscosity (Reynolds or Archimedes number). For fixed density ratios, show the nondimensional frequency ultimately depend on a single nondimensional parameter.

5. Using the dataset given as supplementary material, find the scaling law linking these two parameters. Conclude on the tumbling frequency scaling law.

6. In order to shed further light on this scaling, further experiments reveal that the descent velocity V follow the scaling $V \sim \Omega w$. Using this result, show that an order of magnitude balance between weight and drag allows to recover the tumbling frequency scaling.

3 Starting plane shear flow

We are interested here in the setting up of a fluid flowing in between two parallel plates of infinite extension when one of them is suddenly set into motion (Batchelor, 1967; Ockendon & Ockendon, 1995). The plates are separated with a distance h. At $t = 0$ the upper plate is abruptly set into motion at velocity $\boldsymbol{U} = (U, 0, 0)$. The fluid, at rest until that moment, starts progressively to move due to momentum diffusion until it reaches the Couette steady-state. Note also that there is no imposed pressure gradient and that momentum diffusion is the only cause for fluid motion. The dynamic viscosity of the fluid is noted μ, the kinematic viscosity ν and we neglect the action of gravity. We will consider an translation-invariant evolution along the two directions parallel to the plates (this amounts to consider a **parallel** flow) and we will also suppose that the flow is **incompressible**[1].

1. Propose an estimation of the order of magnitude of the viscous shear stress exerted on one of the plate in the steady limit, along with an estimation of the typical transient timescale.

[1] Over very short times of the order h/c with c the sound celerity in the fluid, this hypothesis can be invalidated. But we can put figures to get an idea by taking e.g. water as a working fluid ($c \simeq 1500\mathrm{m}\cdot\mathrm{s}^{-1}$) and $h = 1$cm as the distance between the plates. The acoustic timescale. is then of the order of 7 μs to be compared with the diffusive timescale exceeding a minute (7 orders of magnitude apart!). Moreover it is quite possible that in the experimental setup the starting of the plate cannot be considered impulsive over the acoustic timescale.

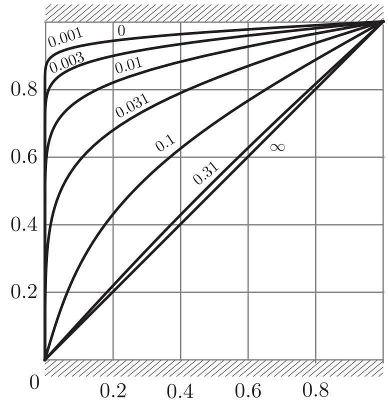


Figure 3: **Transient of a Couette flow**. Several successive velocity profiles are shown here as a function of the nondimensional time $\bar{t} = \nu t/h^2$.

2. Write down the equations expressing mass and momentum conservation (using scalar projections), along with the boundary conditions and the initial condition of the problem. Beware of neglecting the unsteady terms.
3. Nondimensionalise the equations using the natural scales of the problem by setting $y = h\bar{y}$, $u = U\bar{u}$ and $t = h^2\bar{t}/\nu$.
4. In the steady limit, determine the flow profile $\bar{u}_{\text{couette}}(\bar{y})$ along with its gradient, and deduce the dimensioned value of the wall shear stress both at the bottom and at the top plate.

We now seek to describe the onset of this flow with a solution of the form:

$$\bar{u}(\bar{y}, \bar{t}) = \bar{u}_{\text{couette}}(\bar{y}) + \bar{u}_{\text{unst}}(\bar{y}, \bar{t}) \tag{4}$$

5. Injecting the profile (4) in the equations for motion, obtain a new set of equations and boundary conditions for $\bar{u}_{\text{unst}}(\bar{y}, \bar{t})$. What is the initial condition for $\bar{u}_{\text{unst}}(\bar{y}, 0)$?
6. We look for a solution by using the variable separation technique, i.e. we pose $\bar{u}_{\text{unst}}(\bar{y}, \bar{t}) = f(\bar{y})g(\bar{t})$. Obtain the equations governing $f(\bar{y})$ and $g(\bar{t})$ along with the general form of the solutions.
7. Show that the application of the boundary conditions imposes a quantisation condition on the solutions. Deduce the form of the velocity profile as a Fourier series whose coefficients are yet to be determined.
8. Demonstrate the orthogonality relation:

$$\int_0^1 \sin(n\pi x)\sin(m\pi x)\,\mathrm{d}x = \begin{cases} 0 & \text{if } n \neq m \\ \frac{1}{2} & \text{if } n = m \end{cases}$$

9. Exploit the orthogonality relation so as to determine the flow transient, then show that the dimensioned expression for the velocity field is

$$u(y, t) = \frac{Uy}{h} + \sum_{n=0}^{\infty} \frac{2U}{n\pi}(-1)^n \sin\left(\frac{n\pi y}{h}\right) e^{-\frac{n^2\pi^2}{h^2}\nu t} \tag{5}$$

10. What is the asymptotic solution? What is the first correction for long times?

Short numerical project & A primer on scale invariance

Design a numerical code to capture the evolution of the velocity profile with time. Pick a method of your choice (it can rely on the Fourier solution just obtained, or on solving numerically the diffusion equation with e.g. a finite difference method). Observe the limitation of the numerical approach for short times.

▷ **A glimpse on boundary layers and scale invariance.**

1. Try to obtain an asymptotic solution for very short times $\bar{t} = O(\varepsilon)$, with $\varepsilon \ll 1$.

2. We want to shed a closer look to what happens near the starting wall. In order to do so, introduce a "zoomed" scale $\tilde{y} = \varepsilon_y(1 - \bar{y})$. What value of ε_y allows to recover all the terms of the diffusion equation? Rewrite the boundary conditions using these variables.

3. Looking at the numerical solution, it appears that during the whole transient, the solution looks similar but stretched as time flows. This prompts the idea of looking for scale invariant solutions, – that keep on being solutions if time or space is stretched appropriately. To start this investigation, let's introduce "stretched" space and time variables : $\tilde{y} \longrightarrow y^\star y$, $\bar{t} \longrightarrow t^\star t$ and $\bar{u} \longrightarrow u^\star u$, where the starred factors are positive stretching factors, which can take any value. Insert these new variables in the equations and boundary conditions. Is it possible to impose constraints on the scaling factors so that the equations and boundary conditions look the same in the stretched space?

4. Just as the equations and boundary conditions exhibit scale invariance, the solution of the problem is also expected to exhibit the same scale invariance. Show that the solution $u^\star u = f(y^\star y, t^\star t)$ can be rewritten as $u = g(y/\sqrt{t})$.

5. Represent the numerical solution for different times as a function of $\eta = y/\sqrt{t}$.

6. Show that the solution for the problem is $\bar{u}(\bar{y}, \bar{t}) = \mathrm{erfc}\left(\frac{1-\bar{y}}{2\sqrt{\bar{t}}}\right)$. Compare the theoretical prediction to the numerical results.

4 Poiseuille flow in a tube of arbitrary shape

A fluid flows in a tube of uniform section (not necessarily circular) under the action of a constant pressure gradient $-\frac{\partial p}{\partial x}$. We suppose that the flow is fully established (no x dependency) and parallel ($v = w = 0$). For given tube section A and pressure gradient, we look for the tube geometry that minimises the total viscous force exerted on the wall (Ockendon & Ockendon, 1995).

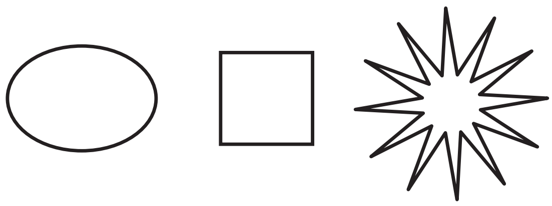


Figure 4: A family of tubes of constant section, but with different shapes.

1. Show the relation:
$$\mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = c.$$
What is the the meaning of c here?

2. Show that the total viscous force exerted on the wall takes the following expression:
$$\oiint_{\partial D} \mu \frac{\partial u}{\partial n}\, \mathrm{d}S.$$

3. On using Green's formulae:
$$\iiint_V \Psi \nabla^2 \varphi\, \mathrm{d}V = -\iiint_V \nabla\Psi \cdot \nabla\varphi\, \mathrm{d}V + \oiint_{\partial V} \Psi \frac{\partial \varphi}{\partial n}\, \mathrm{d}S,$$
Answer the question.

4. Show that we could retrieve this result in a blink by conducting a force balance on a fluid portion.

References

Batchelor, G. K. 1967 *An introduction to fluid dynamics*. Cambridge University Press.

Eggers, Jens & Fontelos, Marco Antonio 2015 *Singularities: formation, structure, and propagation*. Cambridge University Press.

Ockendon, Hilary & Ockendon, John R 1995 *Viscous flow*. Cambridge University Press.

Richardson, Lewis Fry 1926 Atmospheric diffusion shown on a distance-neighbour graph. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character* **110** (756), 709–737.