

Hydrodynamics

Tutorial 2: boundary conditions, capillarity & adhesion

1 Impermeable obstacles

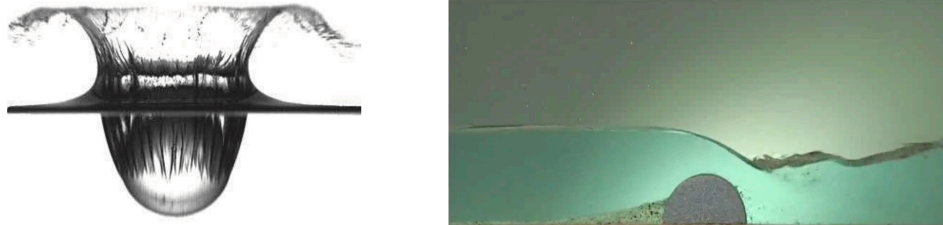


Figure 1: **Liquid deflection.** Left: an hydrophobic bead impacts a water pool at $\approx 5 \text{ m}\cdot\text{s}^{-1}$ and produces a splash (liquid corolla) on impact (Eggers, 2007). Right: a flow over an obstacle in a wave tank gives rise to the *hydraulic jump* phenomenon (Science Education Resource Center at Carleton College¹).

▷ **Impact.** A spherical obstacle of radius R impacts a liquid tank at velocity $-U_{\text{impact}}\mathbf{e}_z$. On impact the fluid is violently set into motion by the object.

1. Supposing that the object preserves its velocity during impact, determine the position C of the sphere centre through time. Consider that at initial time $t = 0$, C is located at $(x, y, z) = (0, 0, 0)$. Deduce the equation of the sphere surface for all time in the laboratory (i.e. fixed) frame.
2. Obtain the expression for the normal vector \mathbf{n} at each point of the sphere in cartesian coordinates.
3. Noting the fluid velocity $\mathbf{u} = (u, v, w)$, determine the expression for the impermeability condition at the sphere surface.

▷ **Flow over an obstacle.** We now consider the flow of a river over an obstacle, such as the one illustrated 1. For simplification purposes, we take the flow as uniform far upstream, $\mathbf{u} = U\mathbf{e}_x$, and translation invariant (that is, 2D) in the transverse direction. The vertical direction is given by \mathbf{e}_y and we note the velocity components $\mathbf{u} = (u, v)$. The bottom height is given by the function $y = f(x)$.

4. Determine the vector normal to the river bottom.
5. Express the impermeability condition at the bottom.
6. Sketch schematically the vertical velocity near the bottom in the context of the localised obstacle (think of a Gaussian hump). Comment.

2 Flow around a cylinder with suction

We consider in this exercise the structure of the flow around a cylinder able to exert a suction (or blowing) on the surrounding fluid, as a rudimentary model of a *turbosail* (see lecture).

In the remaining, we consider the flow as translation-invariant (2D), incompressible and potential. We note $\mathbf{u} = (u, v)$ the components of the velocity field and we introduce the streamfunction ψ such that:

$$\begin{cases} u = \frac{\partial \psi}{\partial y} \\ v = -\frac{\partial \psi}{\partial x} \end{cases} \quad (1)$$

$$\quad (2)$$

Depending on the context we will use cartesian coordinates (x, y) or polar ones (r, θ) .

1. Show that a flow describe with such a streamfunction automatically satisfies mass conservation.
2. Show that the potential flow condition implies $\Delta\psi = 0$.

¹<https://serc.carleton.edu/NAGTWorkshops/geomorph/emriver/index.html>

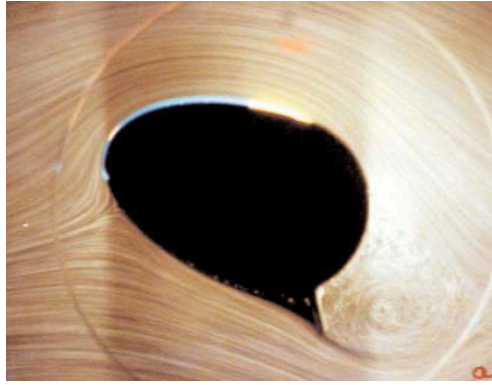


Figure 2: **Turbosail model at ONERA.** The suction through the cylinder boundary allows the flow to reattach on the cylinder.

3. Verify that $\psi_{\text{unif}} = ay$ and $\psi_{\text{source}} = b\theta$ are elementary solutions of $\Delta\psi = 0$. What are the associated flow patterns?
4. What is the streamfunction associated with a uniform flow $\mathbf{u} = (U, 0)$?
5. More complex multipolar solutions can be built from the previous elementary solutions. Compute the expression for the dipolar fields $\psi_{\parallel} = \frac{\partial}{\partial x}\psi_{\text{source}}$ and $\psi_{\perp} = \frac{\partial}{\partial y}\psi_{\text{source}}$. Represent the corresponding streamlines with Python.
6. Show that it is possible to build the inviscid potential flow around a cylinder of radius R as the superposition of ψ_{unif} and of a dipolar contribution.
7. Propose a model for the suction effect, which is characterised with a constant suction velocity $-u_{\text{suction}}\mathbf{e}_r$ at the wall.
8. Represent with Python the streamlines corresponding to $u_{\text{suction}}/U = 0, 0.1, 1$.

3 Confined diffusion of a chemical species

We aim to characterise the impact of walls or boundaries on the diffusion of a contaminant. In order to simplify the study we suppose that diffusion acts along a single axis (this may be seen as a case of translation-invariance along two directions, or of diffusion in a tube). We note $c(x, t)$ the studied concentration field.

We show that a solution to the diffusion equation

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} \quad (3)$$

that exhibit the *scale invariance* property is:

$$c(x, t) = \frac{M}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right) \quad (4)$$

1. Verify that this field is actually a solution to the diffusion equation.
2. Integrate the total concentration field for a given time. What is the meaning of the constant M ?
3. Plot with Python the evolution of this field for different instants (we'll set $M = 1$ and $D = 1$).

As the diffusion equation is linear, any weight sum of elementary solutions will also be a solution. In the following we propose to use this property to build fields that comply with the boundary conditions of the problem. This approach is reminiscent of the use of “image charges” in electrostatics to model equipotential surfaces (boundary “metallisation”).

▷ **Diffusion near an impermeable wall** A quantity M of passive scalar is deposited at a distance ℓ from an impermeable surface located at $x = 0$.

4. What is the boundary condition for the concentration field at $x = 0$?
5. By placing a fictitious concentration field symmetric with respect to the wall, show how to build a concentration field respecting the impermeability condition. Give the expression for this field and represent it with Python.

▷ **Diffusion near an absorbing wall** We now coat the previous wall with a reactant that recombines with the concentration field so that the effective absorbing condition $c(0, t) = 0$ is now verified.

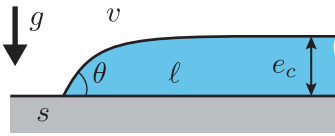
6. Propose a strategy to find the evolution of the concentration field in these conditions.
7. What is the concentration flux towards the wall?

▷ **Confined diffusion** We now consider the diffusion of a contaminant when confined between two impermeable walls located at $x = \pm \ell$. The contaminant is initially deposited at $x = 0$.

8. Is it sufficient to add three elementary solutions to model the influence of the walls? To answer this question it will be helpful to estimate the flux at each wall.
9. To circumvent this issue, we propose to model the field as a superposition of an infinite sum of elementary solutions located at $x = 2j\ell$, $j \in \mathbb{Z}$.
10. In order to determine the weight of each term, write the zero-flux condition at $x = \ell$.
11. Plot with Python the evolution of the field for different instants.
12. What is the asymptotic value of the concentration in the gap?

4 Liquid puddles

A liquid laid over a rigid substrate in the gravity field spontaneously adopts a puddle shape, as illustrated in the facing figure. The liquid has a density ρ , a surface tension γ with the air, and presents a contact angle θ with the solid. We denote the gravity with g .



We look for the limit thickness e_c of the puddle far from the edges.

1. Represent graphically the forces acting on the puddle portion illustrated on the figure.
2. Write the equilibrium condition for the puddle, and deduce the thickness e_c as a function of ρ , g , θ and γ .
3. What is the asymptotic value of e_c in the limit where $\theta \ll 1$?

5 Capillary adhesion of a sphere

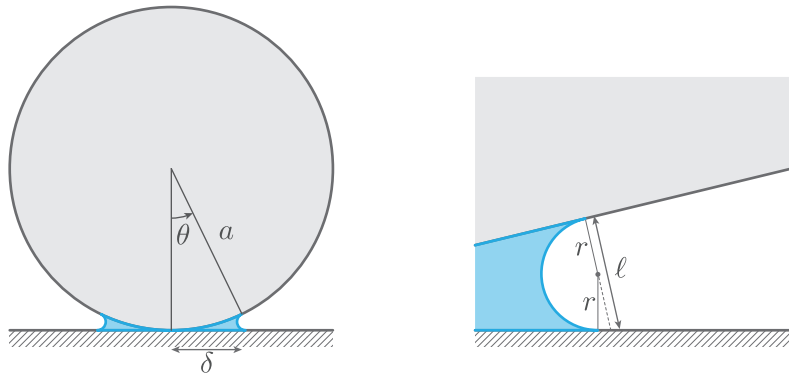


Figure 3: **Capillary adhesion of a sphere.** Left: a sphere of radius a adheres to a plane thanks to a small meniscus of radius δ . Right: the meniscus exhibits a curvature radius r much smaller than the sphere radius.

We consider in this problem the capillary adhesion undergone by a solid sphere of radius a contacting a rigid plane. The sphere adhesion is induced by the presence of a small meniscus made of a perfectly wetting liquid (zero contact angle). The meniscus presents a radial extension δ that will be supposed much smaller than the radius of the sphere (see figure 3). We look for a determination of the vertical component of the capillary force exercée by the meniscus on the sphere, and henceforth noted F_{cap} . We note the density ρ and the surface tension of the liquid with air γ . The liquid will be considered quiescent throughout the exercise and we will neglect the influence of gravity.

1. Justify rapidly why the liquid viscosity cannot enter in the determination of F_{cap} . Then show using dimensional analysis that:

$$F_{\text{cap}} = \gamma a \mathcal{F}(\varepsilon),$$

where ε is a small nondimensional parameter which will be explicitated. In this exercise we are interested in the limit $\varepsilon \ll 1$.

2. As the contact angle between the liquid and the solids (both the plane and the bead) is zero, the interface is sharply curved, as indicated figure 3. We suppose in particular that the curvature radius r represented on the sketch is much smaller than the interface' second curvature radius. As a result we suppose that the meniscus displays a single radius

of curvature r in first approximation. Without calculus, give the sign of the pressure difference between the meniscus and the air $\Delta p = p_{\text{ménisque}} - p_{\text{atm}}$; is the meniscus in a state of higher or lower pressure with respect to the atmosphere? Explain.

3. In order to track the meniscus extension we introduce the angle θ represented on figure 3. We remind that $\delta \ll a$ and therefore $\theta \ll 1$. Give the relation linking δ to a and θ in first approximation.

The force exerted by the liquid meniscus on the bead can be decomposed into two contributions: one due to Laplace pressure, and a second corresponding to the contact line action. Let's focus on the first contribution to start with. In the remaining we will not consider the influence of atmospheric pressure².

4. Noting S_{sphere} the portion of the sphere wetted by the meniscus, give the formal expression for the pressure force exerted by the meniscus onto the sphere.
5. Show that we can replace this expression with:

$$\iint_{S_{\text{disc}}} \Delta p e_z dS,$$

where S_{disc} now represents the surface of the plane wetted by the meniscus.

6. On considering that $\ell \simeq 2r$ (see figure 3), propose an estimation of the curvature radius r as a function of a and of θ to first order in θ (we remind that $\cos \theta = 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 + O(\theta^6)$ for $\theta \ll 1$).
7. Deduce the expression of the pressure contribution in the capillary adhesion force to first order.
8. Explain why the contact line contribution is negligible in this problem.
9. Conclude on the value of the adhesion force, and comment on the dependence with the meniscus volume. What is the limit of the function $\mathcal{F}(x)$ of question 1 when x tends to 0?

6 Curvature of a pendent drop



In order to develop a device for surface tension measurement, we wish to write a numerical code to compute the shape of an axisymmetric pendent drop. To do so it is necessary to know the curvature $\kappa(z)$ of the drop, and to link it to the drop profile $r(z)$ (the equation for the interface is then $r = r(z)$).

1. Explain why working directly with $r(z)$ is not a good strategy. How does this quantity behave near the drop bottom?

We thus work in the following with the variable $q(z) = r^2(z)$.

2. Propose a function $\mathcal{S}(r, z)$ that vanishes at the free surface and involves $q(z)$.
3. Compute the normalised gradient of this function. What is the meaning of this field restriction at the drop free surface?
4. Show that the drop curvature $\kappa(z)$ can be expressed as:

$$\kappa(z) = \frac{4q'(z)^2 - 4q(z)(-2 + q''(z))}{(4q(z) + q'(z)^2)^{3/2}}. \quad (5)$$

Appendix: differential operators in cylindrical coordinates

Let $f(r, \theta, z)$ be a scalar function of space and $u(r, \theta, z) = (u_r(r, \theta, z), u_\theta(r, \theta, z), u_z(r, \theta, z))$ a vector field. We define:

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial r} \\ \frac{1}{r} \frac{\partial f}{\partial \theta} \\ \frac{\partial f}{\partial z} \end{pmatrix}, \quad \nabla \cdot u = \frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}, \quad \Delta f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

References

EGGERS, JENS 2007 Fluid dynamics: Coupling the large and the small. *Nat Phys* 3 (3), 145–146.

²In fact we can show that atmospheric pressure contribution naturally cancels out: the intensity of the adhesion force does not depend on it.