Further examples of complex manifolds

Chapter 3

Complex Singularities

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Holomorphic line bundles

A **holomorphic line bundle** formalizes the intuitive idea of a *holomorphic family of complex lines*.

For example, the trivial $\mathbb{C} \times M$ where M is a complex manifold.

Holomorphic line bundles

Definition. First think of (L, π, M) such that

- The total space L which is a complex manifold
- The base M which also is a complex manifold
- The natural projection $\pi : L \to M$ which is holomorphic

note that (L, π, M) is not yet a holomorphic line bundle.

Holomorphic line bundles

(L, π, M) is called a **holomorphic line bundle** if for all $x \in M$ there exist

- a neighborhood U of x in M
- a biholomorphic map $\Psi : \pi^{-1}(U) \to \mathbb{C} \times U$

such that

- Each fiber $L_m := \pi^{-1}(m)$ for $m \in M$ is a complex 1-dim space
- $\mathrm{Proj} \circ \Psi = \pi$[1]
- The induced map $\Psi(m) : L_m \to \mathbb{C} \times \{M\}$ is a linear isomorphism

Ψ is called a *local trivialization* of L (over U)

[1] ?

Gluing maps of holomorphic line bundles

Recall how we defined a manifold M. For an open cover $(U_\alpha)_{\alpha \in A}$ of M, we may find trivializations Ψ_α over the U_α.
Therefore, we may define **gluing maps** on the overlaps $U_{\alpha\beta} := U_\alpha \cap U_\beta$ as

$$g_{\beta\alpha} : U_{\alpha\beta} \to \mathrm{Aut}(\mathbb{C}) \simeq \mathbb{C}^*$$

[2] which are holomorphic maps $m \mapsto g_{\beta\alpha}(m)$ determined by

[2] For rank m, $\mathrm{Aut}(\mathbb{C})$ is replaced with $\mathrm{GL}(m, \mathbb{C})$

Gluing maps of holomorphic line bundles

The gluing maps satisfy the *cocycle condition*

$$g_{\alpha\gamma}(m) \circ g_{\gamma\beta}(m) \circ g_{\beta\alpha}(m) = \mathbf{1}_{\mathbb{C}}$$

for all $\alpha, \beta, \gamma \in A$, $m \in U_{\alpha\beta\gamma}$.

Gluing maps of holomorphic line bundles

Note that the total space L can be defined as

$$\left(\coprod_{\alpha \in A} \mathbb{C} \times U_\alpha\right) / \sim$$

where $(z_\alpha, m_\alpha) \sim (z_\beta, m_\beta) \iff m_\alpha = m_\beta =: m$ and $z_\beta = (g_{\beta\alpha}(m))(z_\alpha)$.

Holomorphic section

Definition. A **holomorphic section** of a holomorphic line bundle $L \xrightarrow{\pi} M$ is a holomorphic map

$$u : M \to L$$

such that $u(m) \in L_m$ for all $m \in M$. Denote by $\mathcal{O}_M(L)$ the space of holomorphic sections of $L \to M$.

Every line bundle admits the *zero section* which associates to m the origin of L_m.

If a line bundle L is given by a gluing cocycle $g_{\beta\alpha}$ then a section is a collection of holomorphic $f_\alpha : U_\alpha \to \mathbb{C}$ where $f_\beta = g_{\beta\alpha} f_\alpha$.

The tautological line bundle

The tautological line bundle $\mathcal{T}_N$ over $\mathbb{P}^N$ is defined by

$$\mathcal{T}_N = \left\{[z, l] \in \mathbb{C}^{N+1} \times \mathbb{P}^N \mid z \in l\right\}$$

hence 'tautological'.
Task: prove $\mathcal{T}_N$ is a complex manifold and $(\mathcal{T}_N, \pi, \mathcal{P}^N)$ is a holomorphic line bundle by constructing holomorphic charts and local trivializations.

Tautological line bundle over $\mathbb{P}^1$

The projective line can be identified with the Riemann sphere.
The two open sets U_0 and U_1 on $\mathbb{P}^1$ correspond to the canonical charts

$$U_0 = V_N = S^2 \setminus \text{South Pole}, \quad U_1 = V_S = S^2 \setminus \text{North Pole}$$

with $z = z_1/z_0$ on V_N, and $\zeta = z_0/z_1$ on V_S related by $\zeta = 1/z$.
The translation function on U_{01} is given by

$$g_{10}(z) = g_{SN}(z) = z_1/z_0 = z$$

The total space is covered by $W_N = \pi^{-1}(U_N)$ and $W_S = \pi^{-1}(V_N)$ with coordinates given by (s, t) on W_N where $z_0 = s, z_1 = st$ and (u, v) on W_S where $z_0 = uv, z_1 = v$.

Tautological line bundle over $\mathbb{P}^1$

The transition map between the two coordinates is

$$(u, v) = (st, t^{-1})$$

Functorial operations on line bundles

Suppose we are given two holomorphic line bundles $L, \overline{L} \to M$ defined by the open cover (U_α) [3] and the holomorphic gluing cocycles

$$g_{\beta\alpha}, \overline{g}_{\beta\alpha} : U_{\alpha\beta} \to \mathbb{C}^*$$

The **dual** of L is the holomorphic line bundle L^* defined by

$$1/g_{\beta\alpha} : U_{\beta\alpha} \to \mathbb{C}^*$$

The **tensor product** of $L, \overline{L}$ is the line bundle $L \otimes \overline{L}$ defined by the gluing cocycle $g_{\beta\alpha}\overline{g}_{\beta\alpha}$.[4]

[3] meaning that the charts are the same?

[4] meaning of juxtaposition?

Functorial operations on line bundles

A **bundle morphism** $L \to \overline{L}$ is a holomorphic section of $\overline{L} \otimes L^*$, or equivalenty a holomorphic map $L \to \overline{L}$ such that for all $m \in M$ we have $\phi(L_m) \subset \overline{L}_m$, and the induced map $L_m \to \overline{L}_m$ is linear.[5]
Denote by $\mathrm{Pic}(M)$ the set of isomorphism classes of holomorphic lines bundles over M.[6]
The tensor product induces an abelian group structure on $\mathrm{Pic}(M)$.

[5] What is ϕ??
[6] Pic as in Picard group, with group operation $\otimes$

Divisors

A **divisor** is a formal linear combination over $\mathbb{Z}$ of codimension 1 subvarieties.

Examples

Suppose $f : \mathbb{D} \to \mathbb{C}$ is a holomorphic function such that $f^{-1}(0) = \{0\}$. The origin is a codimension one subvariety, so it defines a divisor (0) on $\mathbb{D}$. Define

$$(f)_0 = n(0)$$

where n is the multiplicity of 0. (or equivalenty, one plus the Milnor number of f at zero)

Examples

Suppose $f : \mathbb{D} \to \mathbb{C} \cup \{\infty\}$ a meromorphic function with zeros ζ_i with multiplicity n_i, and poles μ_j of order m_j.

$$(f)_0 = \sum_i n_i \zeta_i$$

while

$$(f)_\infty = \sum_j m_j \mu_j$$

The **principal divisor** (f) is $(f)_0 - (f)_\infty = (f)_0 - (1/f)_0$.
Note that if $g : \mathbb{D} \to \mathbb{C}$ is a nonvanishing holomorphic function, then $(gf) = (f)$.

Examples

M, a complex manifold. $f : M \to \mathbb{C} \cup \{\infty\}$ a meromorphic function (or equivalenty, a holomorphic function $f : M \to \mathbb{P}^1$) then

$$(f) = (f^{-1}(0)) - (f^{-1}(\infty))$$

Also a codimension 1 submanifold V of a complex manifold M defines a divisor on M.[7]

[7] ??

More on divisors

In general, a divisor is obtained by patching the principal divisors of a family of locally defined meromorphic functions.
It is described by an open cover (U_α) and a collection of meromorphic $f_\alpha : U_\alpha \to \mathbb{C} \cup \{\infty\}$ such that on $U_{\alpha\beta}$ the ratios f_α / f_β are nowhere vanishing holomorphic functions.
In other words, f_α and f_β have zeros and poles of same order on the overlaps.

More on divisors

A divisor is called **effective** if the f_α are holomorphic.
Denote by $\mathbf{Div}(M)$ the set of divisors on M, and by $\mathbf{PDiv}(M)$ the set of principal divisors.
To a divisor D with defining functions f_α one can associate a line bundle $[D]$ described by the gluing cocycle $g_{\beta\alpha} = f_\beta / f_\alpha$.

More on divisors

Denote by $\emptyset$ the divisor determined by the constant function 1.
If D, E are two divisors described by defining functions f_α and g_α, denote $D + E$ the divisor described by $f_\alpha g_\beta$.
Also denote by $-D$ the divisor described by $(1/f_\alpha)$.
Observe that $D + (-D) = \emptyset$.

More on divisors

Thus, $(\mathbf{Div}(M), +)$ is an abelian group, and **PDiv** is a subgroup.
Note that $[D + E] = [D] \otimes [E]$ and $[-D] = [D]^*$ in $\mathrm{Pic}(M)$.
Therefore the map $\mathbf{Div}(M) \to \mathrm{Pic}(M)$ defined by $D \mapsto [D]$ is an abelian group homomorphism.
Note that the kernel of this map is $\mathbf{PDiv}(M)$.[8] Thus

$$\mathbf{Div}(M)/\mathbf{PDiv}(M) \to \mathrm{Pic}(M)$$

is injective. This is also surjective when M is an algebraic manifold, by Hodge-Lefschetz.

[8] why?? What is the identity element of Pic(M)?

Further examples

Consider the tautological line bundle $\mathcal{T}_N \to \mathbb{P}^N$. Its dual is called the **hyperplane line bundle**, denoted H_N. If U_i is the standard atlas on $\mathbb{P}^N$, H_N is given by the gluing cocycle $g_{ji} = z_i/z_j$.
We claim that any linear function A from $\mathbb{C}^{N+1}$ to $\mathbb{C}$ defines a section of H.
Define

$$A_i : U_i \to \mathbb{C}, \quad A_i([z_0 : \cdots : z_N]) = \frac{1}{z_i} A(z_0, \cdots, z_N)$$

where $A_j = (z_i/z_j)A_i = g_{ji}A_i$ which proves our claim.[9]

[9] Need inspection. Recall definition of section.

Further examples

Any $P \in \mathcal{P}_{d,N}$[10] defines a holomorphic section of H^d, thus constructing an injection $\mathcal{P}_{d,N} \hookrightarrow \mathcal{O}_{\mathbb{P}^N}(H^d)$.
In fact, this is an isomorphism.

[10] Note that this implies $N+1$ variables.

Informal introduction

Consider two distinct lines passing the origin. In the origin, we cannot distinguish these two lines. However in the projectivized space, these two lines are distinct points.

We aim to formalize this distinction.

Blowup of M at m

Suppose M is a complex N-manifold, and $m \in M$. The **blowup** of M at m is the complex manifold $\widehat{M_m}$ constructed by

- choosing a neighborhood U of M biholomorphic to the unit open ball $B \subset \mathbb{C}^N$, setting $\widehat{U_m} := \beta_{N-1}^{-1}(B) \subset \mathcal{T}_{N_1}$[11]
- The blowdown map β_{N-1} establishes an isomorphism

$$\widehat{U_m} \setminus \mathbb{P}^{N-1} \simeq B \setminus \{0\} \simeq U \setminus \{m\}$$

Now glue $\widehat{U_m}$ to $M \setminus \{m\}$ using the blowdown map to obtain $\widehat{M_m}$.

[11] beta??

Blowup of M at m

Observe there exists a natural holomorphic map $\beta : \widehat{M_m} \to M$ called the **blowdown map**. The fiber $\beta^{-1}(m)$ is called the **exceptional divisor** and is a hypersurface isomorphic to $\mathbb{P}^{N-1}$, denoted by E. Observe that

$$\beta : \widehat{M_m} \setminus E \to M \setminus \{m\}$$

is biholomorphic.

Some examples

$\mathcal{T}_{N-1}$ is the blowup of $\mathbb{C}^N$ at the origin.
The blowup of M at m is diffeomorphic in an orientation-preserving fashion to the connected sum

$$M \# \overline{\mathbb{P}}^N$$

where $\overline{\mathbb{P}}^N$ is the oriented smooth manifold obtained by changing the canonical orientation of $\mathbb{P}^N$.[12]

[12] this is an exercise.

Proper transform

$m \in M$. S a closed subset of M. The **proper transform** of S in $\widehat{M_m}$ is the closure of $\beta^{-1}(S \setminus \{m\})$ in $\widehat{M_m}$, denoted by $\overline{S}_m$.
Example. Consider $S = \{z_0 z_1 = 0\} \subset M = \mathbb{C}^2$. The blowup $\widehat{M_0}$ is covered by [13]

[13] I really lost it here...