The geometry of blowups

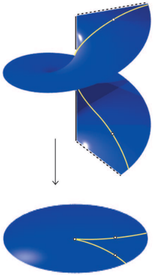
Further material

Complex Singularities

July 12, 2023

Introduction

The blowup is the most simple and typical case of a birational map[1] that is not an isomorphism.
It is the typical method of *resolving singularities*.



[1] a rational map such that its inverse is also rational

Resolution of singularities

Suppose X is an algebraic set with singularities. We want to find a manifold X' such that there exists a map $\pi : X' \to X$ which parametrizes X. For example, recall the difference between *regular surfaces* and *parametrized surfaces*.

This is called **resolving singularities**.

Some examples of resolutions

Example

Consider the cylinder $x^2 + y^2 = 1$ in $\mathbb{A}^3$, and the map from this cylinder to $X = \{(x, y, z) \in \mathbb{A}^3 \mid x^2 + y^2 - z^2 = 0\}$ given by

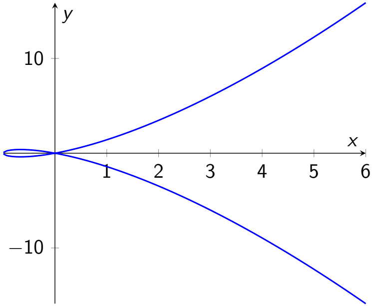
$$\pi : (x, y, z) \mapsto (xz, yz, z)$$

In this case, we already knew X'. However, in further examples we try to construct resolutions. The main technique is **blowing up** points.

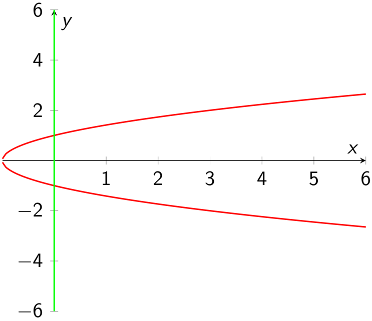
Blowing up the singularity of $y^2 = x^3 + x^2$

Substituting $y = tx$, we get the equation $x^2(t^2 - (x+1)) = 0$ which yields two nonsingular curves.

$y^2 = x^3 + x^2$



$t^2 = x + 1$ with $x = 0$



Blowing up the singularity of $y^2 = x^3 + x^2$

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Blowing up points in higher dimensions

Example

Consider the cone $x^2 + y^2 = z^2$ in $\mathbb{A}^3$. This has a singularity at O, so we try to resolve it by substituting $x = zs$ and $y = zt$. We get $z^2(s^2 + t^2 - 1) = 0$, which again leads to two nonsingular varieties. We call $z = 0$ the exceptional plane.

Singular varieties that need more than one blowup to resolve them

Example

Consider the curve $y^8 = z^5$ in $\mathbb{A}^2$. Let $z = yt$ to get $y^5(t^5 - y^3) = 0$. This is not yet nonsingular. Therefore, we take $t^5 - y^3 = 0$ and blow it up again. Let $y = ts$, to get $t^3(s^3 - t^2) = 0$. Blow up one more time to get nonsingular varieties.

Formal definition

I could not find a suitable formal definition of a blowup of general spaces...

Blowups of projective spaces

Consider $\mathbb{P}^n$ and $\mathbb{P}^{n-1}$ with coordinates $(x_0 : \cdots : x_n)$ and $(y_1 : \cdots : y_n)$ respectively. For points $x = (x_0 : \cdots : x_n)$ and $y = (y_1 : \cdots : y_n)$, denote $(x, y) \in \mathbb{P}^n \times \mathbb{P}^{n-1}$ as $(x_0 : \cdots : x_n : y_1 : \cdots : y_n)$.

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Consider the closed subvariety $\Pi \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$ defined by

$$\{(x, y) \in \mathbb{P}^n \times \mathbb{P}^{n-1} \mid x_i y_j = x_j y_i \quad \text{for} \quad i, j = 1, \ldots, n\}$$

Blowups of projective spaces

Definition

The map $\sigma : \Pi \to \mathbb{P}^n$ defined by restricting the first projection $\mathbb{P}^n \times \mathbb{P}^{n-1} \to \mathbb{P}^n$ is called the **blowup** of $\mathbb{P}^n$ centered at $\xi = (1 : 0 : \cdots : 0) \in \mathbb{P}^n$.

An exercise of the text

Problem

Prove that the blowup of the complex manifold M at a point m is diffeomorphic in an orientation preserving manner to the connected sum

$$M\#\overline{\mathbb{P}}^N$$

where $\overline{\mathbb{P}}^N$ is the oriented smooth manifold obtained by changing the canonical orientation of $\mathbb{P}^N$.