Critical points contain nontrivial information

Chapter 2

Complex Singularities

July 6, 2023

Introduction

We will show that the critical points determine most of the topological properties of

$$f : \Sigma \to T$$

where Σ and T are complex curves.

The Milnor number

Consider a holomorphic function $f : \mathbb{D} \to \mathbb{C}$ where $f(0) = 0$. We may rewrite f as

$$z^k(a_k + a_{k+1}z + \cdots)$$

for a positive integer k such that $a_k \neq 0$. We call k the **multiplicity** of $z_0 = 0$ in the fiber $f^{-1}(0)$.

Additionally, if $f'(0) = 0$, then $k \geq 2$, and $k - 1$ is called the **Milnor number** of the critical point, denoted by $\mu(f, 0)$.

We define the Milnor number of a regular point to be zero.

The Milnor number

Claim. 0 is a nondegenerate critical point if and only if its Milnor number μ is 1.

Proof. Suppose 0 is a nondegenerate critical point of f. Since it is nondegenerate, k cannot exceed 2, but also since it is a critical point k must be greater than 1. Therefore $k = 2$, so its Milnor number is 1. Conversely, suppose $\mu = 1$. Then it follows that $k = 2$, so 0 is a nondegenerate critical point.

Tougeron's determinacy theorem

Lemma. Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function with $\mu = \mu(f, 0) > 0$. Then there exist small open neighborhoods U, Z of $0 \in \mathbb{D}$ and a biholomorphic map $\phi : U \to Z$ such that

$$f(\phi(z)) = z^{\mu+1}$$

for all $z \in U$.

Proof. Note that for $\mu = 0$ this is just the implicit function theorem. Since we can write $f(z) = z^{\mu+1} g(z)$ where $g(z) \neq 0$ is a holomorphic function, we can find a small open neighborhood V of 0, and a holomorphic function $r : V \to \mathbb{C}$ such that $g(z) = (r(z))^{\mu+1}$. This means that $r(z) = g(z)^{1/\mu+1}$.[1]

[1] Well-defined?

Tougeron's determinacy theorem

Define $u := zr(z)$. Also recall that $g(z) = r(z)^{\mu+1}$, where you can plug in $r(z) = u/z$ to get $g(z) = (u/z)^{\mu+1}$. Now from $f(z) = z^{\mu+1}$ we conclude that $f(z) = u^{\mu+1}$.

The power map $u \to u^k$

The power map $u \to u^k$ defines a k-**sheeted branched cover** of $\mathbb{D}$ over itself (except zero).

Note that there is a **branching at zero**, meaning that the fiber over zero is different from fibers over other points.

It is more clear in polar expressions:

$$R^k e^{ki\theta}$$

for $0 \leq \theta \leq 2\pi$, $R > 0$ obviously covers $\mathbb{D}$ k times. However, for $R = 0$ this fails to cover $\mathbb{D}$.

Milnor numbers and fibers

We claim that the Milnor number $k-1$ is equal to the number of points in a *general* fiber minus the number of points in the *singular* fiber.[2]

For the power map $u \to u^k$, the number of points in a general fiber is k, and the singular fiber contains a single point 0.

[2] Is there any rigorous criteria?

Milnor numbers and fibers

If X and Y are complex 1-manifolds, any holomorphic function $f : X \to Y$ can be *locally* described as a holomorphic function $f : \mathbb{D} \to \mathbb{C}$.

Therefore, we may define the Milnor number for holomorphic functions between complex manifolds.

Now Tougeron's determinacy theorem tells us that *any* singular point of such holomorphic has a branching behaviour. Moreover, note that critical points are isolated (for nonconstant f), so that if X is compact then f may only have finitely many critical points.†

A brief digression on a proof

†Denote the critical points of f as $C \subset X$. Since critical points are isolated by the identity theorem, we may find open neighborhoods U_c for $c \in C$ such that $U_c \cap C = \{c\}$. Also C is closed in X since it doesn't have limit points. Then $\{U_c\}_{c \in C} \cup \{X - C\}$ is an open cover of X, which is compact, thus must have a finite subcover. Therefore C must be a finite set.

Milnor numbers and fibers

Moreover, if X is compact then only finitely many Milnor numbers of $f : X \to Y$ are nonzero.

Now, how are these Milnor numbers related to the topology of $f : \Sigma \to T$?

The Riemann-Hurwitz Theorem

Suppose that Σ and T are two compact complex curves. Suppose $f : \Sigma \to T$ is a nonconstant holomorphic map.

Note that Σ and T are topologically 2-dimensional closed oriented manifolds (known as a *Riemann surface*).

It is well known that such surfaces are completely determined, up to *homeomorphism*, by their Euler characteristic χ.

The Riemann-Hurwitz Theorem

The **Riemann-Hurwitz Theorem** states that $\chi(\Sigma)$ can be completely determined by *mild* global information of $f : \Sigma \to T$ and *detailed* local information of f, assuming that we know $\chi(T)$.

Here, the global information corresponds to the degree of f, and the local information corresponds to the Milnor numbers of the critical points of f.

Proof of the Riemann-Hurwitz Theorem

Theorem. $f : \Sigma \to T$ where f is a holomorphic function between compact complex curves. Suppose $\deg f = d > 0$. Then

$$\chi(\Sigma) = d\chi(T) - \sum_{p \in \Sigma} \mu(f, p)$$

Proof. We know that f has at most finitely many critical points in Σ. Therefore, we may denote the critical values of f as $t_1, \ldots, t_n$. Find a triangulation $\mathcal{T}$ of T having the critical values among its vertices. By definition we have

$$\chi(T) = \#V - \#E + \#F$$

Proof of the Riemann-Hurwitz Theorem

Now we define $\mu(t) = \sum_{p \in f^{-1}(t)} \mu(f, p)$, the sum of Milnor numbers of the fiber of $t \in T$. Observe that $\mu(t) = 0$ if and only if t is a regular value. Also refering to Fig 2.2, it is clear that

$$\mu(t_0) = \lim_{t \to t_0} \# f^{-1}(t) - \# f^{-1}(t_0) = d - \# f^{-1}(t_0)$$

for any $t_0 \in T$ where $\#$ is the counting function.

Intuitively, the Milnor number of a point represents how *far* the function is from its full degree.

Proof of the Riemann-Hurwitz Theorem

Since f is onto[3] we can lift $\mathcal{T}$ to a triangulation $\mathcal{T}' = f^{-1}(\mathcal{T})$ of Σ. Since critical points are isolated, we deduce that $\#E' = d\#E$ and $\#F' = d\#F$.[4]

Moreover, using $\mu(t_0) = d - \#f^{-1}(t_0)$ we deduce

$$\#V' = d\#V - \sum_{t \in T} \mu(t) = d\#V - \sum_{p \in \Sigma} \mu(f, p)$$

from which we can easily conclude that

$$\chi(\Sigma) = d\chi(T) - \sum_{p \in \Sigma} \mu(f, p).$$

[3] I have no idea why.

[4] Rigorous proof?

Nondegenerate case

Corollary. Suppose $f : \Sigma \to \mathbb{P}^1$ is a holomorphic map[5] which has *only nondegenerate critical points*. If ν is the number of those points, then $\chi(\Sigma) = 2\deg f - \nu$.

Proof. Based on results of homology, the Euler characteristic of $\mathbb{P}^1$ is 2. Recall that Milnor numbers of nondegenerate critical points were 1.

[5] Recall that $\mathbb{P}^1$ is a complex 1-manifold, where charts are given by the canonical open coverings.

Genus formula

We apply the Riemann-Hurwitz theorem on a classical problem.

Suppose $P \in \mathcal{P}_{d,2}$, and let $X = V_P$. We already know that for *generic* P, the set V_P is a compact 1-dimensional submanifold of $\mathbb{P}^2$. Its topological type is completely described by its genus, which is given below.

Genus formula. For generic $P \in \mathcal{P}_{d,2}$, the curve V_P is a Riemann surface of genus

$$g(V_P) = \frac{(d-1)(d-2)}{2}.$$

Proof of the genus formula

We use the Corollary derived earlier, together with projections from V_P to $\mathbb{P}^1$ as holomorphic maps.

Fix a line $L \subset \mathbb{P}^2$ and a point $C \in \mathbb{P}^2 \setminus V_P$. Recall the definition of a projection map $f : X \to L$.