Complex Manifolds

Chapter 1

Complex Singularities

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Introduction

In this course, we will explain how to extract **topological information** about a **complex manifold** by studying the holomorphic maps

$$f : X \to T, \dim T = 1$$

and their critical points.

Complex n-manifold

A complex n-manifold is...

...a locally compact Hausdorff[1] space X together with

- Open cover U_α of X
- Homeomorphisms $h_\alpha : U_\alpha \to \mathcal{O}_\alpha$ where $\mathcal{O}_\alpha \subset \mathbb{C}^n$ is open
- Change of coordinate maps are *biholomorphic*

biholomorphism: $\phi : U \subset \mathbb{C}^n \to V \subset \mathbb{C}^n$ such that U, V are open and ϕ, ϕ^{-1} are holomorphic.

Note that the charts map from X to $\mathbb{C}^n$, not the other way.

[1] Other definitions also assume 2nd countability.

Holomorphic functions on X

A function $f : X \to \mathbb{C}$ is said to be **holomorphic** if $f|_{U_\alpha} \circ h_\alpha^{-1} : \mathcal{O}_\alpha \to \mathbb{C}$ is holomorphic.

Maps $f : X \to \mathbb{C}^m$ are **holomorphic** if each component function is holomorphic (in the sense defined above).

Similarly, if Y is a complex m-manifold with a holomorphic atlas (V_i, g_i) and if $F : X \to Y$ is continuous, then F is said to be **holomorphic** if for every i, the map $g_i \circ F : F^{-1}(V_i) \to g_i(U_i) \subset \mathbb{C}^m$ is holomorphic (in the sense defined above).

cf. Do Carmo, defining differentiable functions between surfaces

Further definitions

For a complex n-manifold X, we mean by **local coordinates near** x a biholomorphic map from a neighborhood of x onto an open subset of $\mathbb{C}^n$.

Remark: $\mathbb{C}^n = (z_1, \cdots, z_n)$ with $z_k = x_k + iy_k$ is equipped with a canonical orientation given by

$$dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$$

and every biholomorphic map preserves this orientation. Therefore, every complex manifold has a natural orientation.

It is clear when we think about $\mathbb{C}^1$, the complex plane has an obvious orientation (counterclockwise on the complex plane).

Further definitions

Suppose $F : X \to \mathbb{C}^m$, $F = (F_1, \ldots, F_m)$ is a holomorphic map. A point $x \in X$ is regular if there exists local coordinates $(z_1, \ldots, z_n)$ near x such that the Jacobian matrix

$$\left(\frac{\partial F_i}{\partial z_j}(x)\right)_{1 \leq i \leq m, 1 \leq j \leq n}$$

has maximal rank.[2] This extends to holomorphic maps $F : X \to Y$. A point $x \in X$ which is not regular is called critical.[3] A point $y \in Y$ is said to be a regular value of F if $F^{-1}(y)$ are regular points.

[2] ?

[3] Respect to which F?

Further definitions

A critical point $x \in X$ is **nondegenerate** if there exists local coordinates near x and $F(x)$ respectively such that F can be locally described as a function[4] $u(z)$ and the Hessian has nonzero determinant.

A holomorphic map $F : X \to Y$ is a **Morse map** if

- $\dim Y = 1$
- All critical points of F are nondegenerate
- For all critical values $y \in Y$, $F^{-1}(y)$ contains a unique critical point.

Example $f = z_1^2 + \cdots + z_n^2$ is Morse. $\dim \mathbb{C}$ is obviously 1. The only critical point is 0, for which the Hessian is $(2\delta_{ij})$ which has determinant $2^n \neq 0$. The third condition is clear.

[4] Holomorphic?

Projective space

N-dimensional complex projective space $\mathbb{P}^N$: The quotient of $\mathbb{C}^{N+1} \setminus \{0\}$ modulo the equivalence relation

$$u \sim v \Leftrightarrow \exists \lambda \in \mathbb{C}^*; v = \lambda u$$

The natural projection $\pi : \mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{P}^N$ is defined by sending elements to their (equivalence) classes.

$\mathbb{P}^N$ is given a topology by having U open iff $\pi^{-1}(U)$ is open in $\mathbb{C}^{N+1} \setminus \{0\}$.
Example $\mathbb{P}^1$ is called the Riemann sphere.

Projective space

Canonical holomorphic atlas on $\mathbb{P}^n$ consists of the charts $(U_i, \phi_i)_{i \leq N}$ where

$$U_i = \{[z_0 : \cdots : z_N] \mid z_i \neq 0\}$$

and $\phi_i : U_i \to \mathbb{C}^N$ given by

$$\phi([z_0 : \cdots : z_n]) = (\zeta_1, \ldots, \zeta_N)$$

Read the text for the definition of ζ_k.

Each U_i is biholomorphic to $\mathbb{C}^N$, and $\mathbb{P}^N \setminus U_i = \{[z_0 : \cdots : z_N] \mid z_i = 0\}$ can be identified with $\mathbb{P}^{N-1}$.[5]

[5] Why is this important?

Submanifolds

X, a complex n-manifold. A **codimension** k **submanifold** of X is a closed subset $Y \subset X$ such that for every point $y \in Y$ there exists an open neighborhood $U_y \subset X$, and local holomorphic coordinates $(z_1, \ldots, z_n)$ on U_Y such that

- $z_1(y) = \cdots = z_n(y) = 0$
- $y' \in U_y \cap Y \iff z_1(y') = \cdots = z_k(y\prime) = 0$

The codimension k submanifolds are complex $n - k$-manifolds.[6]

[6]Proof?

Implicit Function Theorem

Theorem. If $F : X \to Y$ is a holomorphic map, dim $Y = k$, and $y \in Y$ is a regular value of F, then the fiber $F^{-1}(y)$ is a codimension k submanifold of X.
This is an easy way of obtaining a submanifold.[7]
We omit the proof of this theorem.

[7] cf. Do Carmo, Diff. Geo. of Curves and Surfaces

Sard's Theorem

Theorem. If $F : X \to Y$ is a holomorphic map, then the set of critical points has measure zero.
Applying this to the IFT, we conclude that almost all fibers $F^{-1}(y)$ are smooth submanifolds, or in other words the generic fiber is smooth.
Again, we omit the proof.[8]

[8] However, it is worthwhile to note that one of the most famous proofs of this theorem by J. Milnor depends on the second-countability of X, which in this case is omitted.

Sard's Theorem

Using this, we may regard X as a union of the fibers $F^{-1}(t)$ for $t \in T$. We show that the understanding of the changes in the topology and geometry of $F^{-1}(T)$ as t approaches critical values leads to nontrivial conclusions.

Algebraic Manifolds

An **algebraic manifold** is a smooth algebraic variety. They are constructed by the closed subsets

$$V_P = \{[z_0 : \cdots : z_N] \in \mathbb{P}^N \mid P(z_0, \ldots, z_N) = 0\}$$

for $P \in \mathcal{P}_{d,N}$, which is the space of degree d homogeneous polynomials in $N + 1$ variables over the base field $\mathbb{C}$.

V_P are called hypersurfaces of degree d.

Algebraic Manifolds

You may have noticed that the V_p may fail to be manifolds. However using Sard's theorem we claim that *almost all* V_P are codimension-1 submanifolds of $\mathbb{P}^N$.

Algebraic Manifolds

Claim. For almost all $P \in \mathcal{P}_{d,N}$, V_p is a codimension 1 submanifold of $\mathbb{P}^N$.
Proof. Consider the holomorphic map

$$F : X \to \mathbb{P}(d, N), \quad ([\mathbf{z}], [P]) \mapsto [P]$$

where $X = \{([\mathbf{z}], [P]) \in \mathbb{P}^N \times \mathbb{P}(d, N) \mid P(\mathbf{z}) = 0\}$.[9] We know X is a smooth manifold by the implicit function theorem. Then we may apply Sard's theorem to $F^{-1}(P) = V_P$, which shows that almost all of them are smooth.

[9] $\mathbb{P}(d, N) = \mathcal{P}^*_{d,N}/\sim$ where $P \sim Q$ iff $\lambda P = Q$ for some $\lambda \in \mathbb{C}$.

The case $d = 1$

If $d = 1$ then all polynomials are linear. The zero set of linear polynomials are hyperplanes. In this case, the hyperplane V_P completely determines the image of P in $\mathbb{P}(1, N)$.[10] Therefore, the space $\mathbb{P}(1, N)$ can be identified with the set of vector hyperplanes in $\mathbb{P}^N$.

$\mathbb{P}(1, N)$ is called the dual of $\mathbb{P}^N$ and is denoted by $\check{\mathbb{P}}^N$.

[10] This is because V_P are invariant to scalar multiples of P.

Projective Variety

Given a set $(P_s)_{s\in S}$ of homogeneous polynomials in $N+1$ variables over $\mathbb{C}$, define

$$V(S)=\bigcup_{s\in S} V_{P_s}$$

$V(S)$ is called a **projective variety**.
Often, a projective variety is a smooth submanifold. Chow's theorem states that an analytic subspace of complex projective space closed in the usual topology is an algebraic subvariety.

Projections

Suppose X is a smooth, degree d curve[11] in $\mathbb{P}^2$. Note that lines in $\mathbb{P}^2$ are hyperplanes. Fix a point $C \in \mathbb{P}^2$ and a line $L \subset \mathbb{P}^2 \setminus \{C\}$.

For any $p \in \mathbb{P}^2 \setminus \{C\}$, denote $[Cp]$ the unique projective line determined by C and p.

Denote by $f(p)$ the intersection of $[Cp]$ and L. Then f is holomorphic, and is called the **projection** from C to L.

[11] meaning that it has codimension 1, therefore having dimension 1

Projections

Recall $X \subset \mathbb{P}^2$. If $C \notin X$ then the restriction $f|_X : X \to L$ is a holomorphic map. Its critical points are $p \in X$ such that $[Cp]$ is tangent to X.[12]

[12] Proof?

Projections

Now suppose C is a point at infinity, i.e. the line $\{[0, z_1, z_2] \mid (z_1, z_2) \in \mathbb{C}^2\} \subset \mathbb{P}^2$. It is good practice to understand why the set of points at infinity is a line in projective space.
Since X is of degree d, every line in $\mathbb{P}^2$ intersects X in d points (counting multiplicities).

Projections

The **dual** of the center C is the line $\check{C} \in \check{\mathbb{P}}^2$ consisting of all affine hyperplanes in $\mathbb{P}^2$ passing through C.

The **dual** of X is the closed set $\check{X} \subset \check{\mathbb{P}}^2$ consisting of all the lines in $\mathbb{P}^2$ tangent to X.

$\check{X}$ is a (possibly) singular curve in $\mathbb{P}^2$, i.e. it can be described as the zero locus of a homogeneous polynomial.[13]

[13] ?

Projections

A critical point of f corresponds to a line through C (point in $\check{C}$) which is tangent to X (belongs to $\check{X}$). Thus the expected number of critical points is the expected number of intersection points between $\check{X}$ and $\check{C}$. This is precisely the degree of $\check{X}$.[14]

[14] ?

A Historical Remark

Consider the affine curve $\{(x, y) \in \mathbb{C}^2 \mid y^2 = x(x-1)(x-t)\}$. Now identify the complex (affine) plane $\mathbb{C}^2$ with $\mathbb{P}^2 \setminus \{z_0 = 0\}$ by taking $x = z_1/z_0$ and $y = z_2/z_0$. This leads to the cubic

$$z_2^2 z_0 = z_1(z_1 - z_0)(z_1 - tz_0)$$

in $\mathbb{P}^2$ which can be regarded as the closure of the graph of the original function not mentioned here.[15]

[15] How to prove that this is the closure?