Convexity

Chapter 3

Functional Analysis

July 14, 2023

Dual space

Definition

The **dual space** of a TVS X is the vector space X^* whose elements are the continuous linear functionals on X.

Note that addition and scalar multiplication are defined in X^* by $(\Lambda_1 + \Lambda_2)x = \Lambda_1 x + \Lambda_2 x$ and $(\alpha\Lambda)x = \alpha \cdot \Lambda x$. Thus it is clear that X^* is indeed a vector space.

Some terminology

An additive functional Λ on a complex vector space X is called real-linear (resp. complex-linear) if $\Lambda(\alpha x) = \alpha \Lambda x$ for all $x \in X$ and for every real (resp. complex) scalar α.

Note that if u is the real part of a complex-linear functional f on X, then u is real-linear and $f(x) = u(x) - iu(x)$ because $z = \mathrm{Re} z - i\mathrm{Re}(iz)$ for all $z \in \mathbb{C}$. Conversely, if $u : X \to \mathbb{R}$ is real-linear on a complex vector space X and if f is defined the same, then f is also complex-linear.

Thus we may conclude that a complex-linear functional on X is an element of X^* if and only if its real part is continuous, and that every continuous real-linear $u : X \to \mathbb{R}$ is the real part of a unique $f \in X^*$.

Dominated extension theorems

Theorem

Suppose M is a subspace of a real vector space X, $p : X \to \mathbb{R}$ satisfies $p(x+y) \leq p(x) + p(y)$ and $p(tx) = p(x)$ for $x, y \in X$ and $t \geq 0$, and $f : M \to \mathbb{R}$ is linear and $f(x) \leq p(x)$ on M. Then, there exists a linear $\Lambda : X \to \mathbb{R}$ such that $\Lambda x = f(x)$ for $x \in M$, and $-p(-x) \leq \Lambda x \leq p(x)$ for all $x \in X$.

Dominated extension theorems

Proof.

Suppose $M \neq X$. Choose $x_1 \in X \setminus M$, and define $M_1 = \{x + tx_1 \mid x \in M, \quad t \in \mathbb{R}\}$. Check that M_1 is a vector space. Since $f(x) + f(y) = f(x+y) \leq p(x+y) \leq p(x - x_1) + p(x_1 + y)$, we have $f(x) - p(x - x_1) \leq p(y + x_1) - f(y)$ for $x, y \in M$. Let α be the least upper bound of $f(x) - p(x - x_1)$ as x goes through M. Then, $f(x) - \alpha \leq p(x - x_1)$ for all $x \in M$, and $f(y) + \alpha \leq p(y + x_1)$ for all $y \in M$.

Now we define f_1 on M_1 by $f_1(x + tx_1) = f(x) + t\alpha$ for $x \in M$ and $t \in \mathbb{R}$. Then, $f_1 = f|_M$ and f_1 is linear on M_1. Also $f_1 \leq p$ on M_1.[a]

[a]??

Dominated extension theorems

Proof.

Now we conclude the proof by transfinite induction. Let $\mathcal{P}$ be the collection of all pairs (M', f') where M' is a subspace of X containing M, and f' is a linear functional on M' extending f, and satisfying $f' \leq p$ on M'. Impose a partial ordering by $(M', f') \leq (M'', f'')$ if $M' \subset M''$ and $f' = f''$ on M'. By Hausdorff's maximality theorem, there exists a maximal totally ordered subcollection $\Omega \subset \mathcal{P}$.

Let Φ be the collection of all M' such that $(M', f') \in \Omega$. This makes Φ a totally ordered set, and the union $\tilde{M}$ of all members of Φ is a subspace of X.[a] Note that if $x \in \tilde{M}$ then $x \in M'$ for some $M' \in \Phi$, so we define $\Lambda x = f'(x)$ where f' is the pair function of M'.

[a] Why?

Dominated extension theorems

Proof.

Note that Λ is well-defined on $\tilde{M}$.[a] Also Λ is linear, and $\Lambda \leq p$. Now if $\tilde{M}$ were a proper subspace of X, then we could extend $\tilde{M}$, which contradicts the maximality of Ω. Thus $\tilde{M} = X$ and we have $-p(-x) \leq -\Lambda(-x) = \Lambda x$ for all $x \in X$. □

[a] Not so clear..

Dominated extension theorems

Here is another dominated extension theorem.

Theorem

Suppose M is a subspace of a vector space X. Let p be a seminorm on X, and f be a linear functional on M such that $|f(x)| \leq p(x)$ for all $x \in M$. Then f extends to a linear functional Λ on X such that $|\Lambda x| \leq p(x)$ for all $x \in X$.

Proof.

If the scalar field is $\mathbb{R}$, then this follows by the previous theorem since $p(-x) = p(x)$. Now assume the scalar field to be $\mathbb{C}$. Put $u = \mathrm{Re} f$. By the previous theorem, there is a real-linear U on X such that $U = u$ on M and $U \leq p$ on X.

Dominated extension theorems

Proof.

Let Λ be the complex-linear functional on X whose real part is U. Now it follows that $\Lambda = f$ on M. Finally, to each $x \in X$ there is an $\alpha \in \mathbb{C}$, $|\alpha| = 1$ such that $\alpha\Lambda x = |\Lambda x|$. Hence

$$|\Lambda x| = \Lambda(\alpha x) = U(\alpha x) \leq p(\alpha x) = p(x)$$

□

Corollary

If X is a normed space and $x_0 \in X$, then there exists $\Lambda \in X^*$ such that $\Lambda x_0 = ||x_0||$ and $|\Lambda x| \leq ||x||$ for all $x \in X$.

Dominated extension theorems

Proof.

For $x_0 = 0$, take $\Lambda = 0$. If $x_0 \neq 0$, apply the previous theorem with $p(x) = ||x||$ and M the one-dimensional space generated by x_0, and $f(\alpha x_0) = \alpha ||x_0||$. $\square$

The separation theorem

Theorem

Suppose A, B are disjoint nonempty convex sets in a TVS X.

(a) *If A is open, then there exist $\Lambda \in X^*$ and $\gamma \in \mathbb{R}$ such that $Re\Lambda x < \gamma \leq Re\Lambda y$ for every $x \in A$ and $y \in B$.*

(b) *If A is compact, B is closed and X is locally convex, then there exist $\Lambda \in X^*$, $\gamma_1 \in \mathbb{R}$ and $\gamma_2 \in \mathbb{R}$ such that $Re\Lambda x < \gamma_1 < \gamma_2 < Re\Lambda y$ for all $x \in A$, $y \in B$.*

Note that if the scalar field is $\mathbb{R}$, then $\mathrm{Re}\Lambda = \Lambda$.

The separation theorem

Proof.

We let the scalar field be $\mathbb{R}$ since for $\mathbb{C}$ there is a unique complex-linear functional on X whose real part is the desired real-linear functional.

(a) Fix $a_0 \in A$ and $b_0 \in B$. Put $x_0 = b_0 - a_0$ and put $C = A - B + x_0$.
T $\square$

Weak topologies

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a set X, and assume $\mathcal{T}_1 \subset \mathcal{T}_2$. Then we say $\mathcal{T}_1$ is weaker than $\mathcal{T}_2$, or stronger vice versa.

Notice that the topology of a compact Hausdorff space has rigidity; It cannot be weakened without losing the Hausdorff axiom, and cannot be strengthened without losing compactness.

Fact

If $\mathcal{T}_1 \subset \mathcal{T}_2$ on X, and if $\mathcal{T}_1$ is Hausdorff and $\mathcal{T}_2$ is compact, then $\mathcal{T}_1 = \mathcal{T}_2$.

To show this, consider a $\mathcal{T}_2$-closed $F \subset X$. Thus F is compact in $\mathcal{T}_2$, from which it follows that it is compact in $\mathcal{T}_1$, hence is $\mathcal{T}_1$-closed. Therefore $\mathcal{T}_1 \supset \mathcal{T}_2$, so the two are equal.

Weak topologies

Here is another fact.

Fact

Consider the quotient topology $\mathcal{T}_N$ of X/N, where $E \in \mathcal{T}_N$ if $\pi^{-1}(E) \in \mathcal{T}$. By definition, $\mathcal{T}_N$ is the strongest topology on X/N that makes π continuous, and is the weakest one that makes π an open mapping.

Weak topologies

Definition

Suppose X is a set and $\mathcal{F}$ is a nonempty family of mappings $f : X \to Y_f$ where each Y_f is a topological space. Let $\mathcal{T}$ be the topology generated by subbases $f^{-1}(V)$ for $f \in \mathcal{F}$ and V open in Y_f. Then $\mathcal{T}$ is the weakest topology on X that makes each f continuous. This $\mathcal{T}$ is called the **weak topology** on X induced by $\mathcal{F}$, or the $\mathcal{F}$-topology of X.

Example

The product topology $\mathcal{T}$ of $X = \prod X_\alpha$ is the $\{\pi_\alpha\}$-topology of X, the weakest one that makes each projection map continuous.

Weak topologies

Fact

Suppose $\mathcal{F}$ is a family of mappings $f : X \to Y_f$, where X is a set and each Y_f is a Hausdorff space. If $\mathcal{F}$ separates points on X, then the $\mathcal{F}$-topology of X is Hausdorff.

Proof.

Suppose $p \neq q$ are points of X. Then $f(p) \neq f(q)$ for some $f \in \mathcal{F}$, so $f(p)$ and $f(q)$ have disjoint neighborhoods in Y_f such that their inverse images are open and disjoint. Thus X in the weak topology is Hausdorff. □

A metrization theorem

Theorem

If X is a compact topological space, and if some sequence $\{f_n\}$ of continuous real-valued functions separates points on X, then X is metrizable.

Proof.

Let $\mathcal{T}$ be the topology of X. Without loss of generality, assume $|f_n| \leq 1$ for all n. Let $\mathcal{T}_d$ be the topology induced on X by the metric $d(p, q) = \sum_{n=1}^{\infty} 2^{-n}|f_n(p) - f_n(q)|$. (Check this is indeed a metric.) Note that d is continuous on $X \times X$ by the Weierstrass p-test. The open balls in this metric are $\mathcal{T}$-open, hence $\mathcal{T}_d \subset \mathcal{T}$. Recall the fact on slide 14 to conclude that $\mathcal{T} = \mathcal{T}_d$. Therefore X is metrizable. □

A Lemma

Lemma

Suppose $\Lambda_1, \ldots, \Lambda_n$ *and* Λ *are linear functionals on a vector space* X*. Let* $N = \{x \in X \mid \Lambda_1 x = \cdots = \Lambda_n x = 0\}$*. Then the following are equivalent:*

a *There are scalars* $\alpha_1, \ldots, \alpha_n$ *such that* $\Lambda = \alpha_1 \Lambda_1, \ldots, \alpha_n \Lambda_n$*.*

b *There exists a* $\gamma < \infty$ *such that* $|\Lambda x| \leq \gamma \max_{1 \leq i \leq n} |\Lambda_i x|$ *for all* $x \in X$*.*

c $\Lambda x = 0$ *for every* $x \in N$*.*

A Lemma

Proof.

If you think a while, it is clear that a implies b, b implies c. Thus we prove c implies a. Let Φ be the scalar field. Define $\pi : X \to \Phi^n$ by $pi(x) = (\Lambda_1 x, \ldots, \Lambda_n x)$. Note that $\pi(x) = \pi(x')$ implies $\Lambda x = \Lambda x'$. Therefore the linear functional f on $\pi(X)$ is well-defined by $f(\pi(x)) = \Lambda x$. By extension theorems, extend f to a linear functional F on Φ^n. This means there exist $\alpha_i \in \Phi$ such that $F(u_1, \ldots, u_n) = \alpha_1 u_1 + \cdots + \alpha_n u_n$. Therefore $\Lambda x = F(\pi(x)) = F(\Lambda_1 x, \ldots, \Lambda_n x) = \sum_{i=1}^n \alpha_i \Lambda_i x$. □

A theorem on the topology of dual spaces

Theorem

Suppose X is a vector space, and X' is a vector space of linear functionals on X such that $\Lambda x_1 \neq \Lambda x_2$ for some $\Lambda \in X'$ whenever $x_1 \neq x_2$ (i.e. X' is a separating vector space.) Then the topology $\mathcal{T}'$ of X', when applied to X, makes X into a locally convex space whose dual space is X'.