Convexity

Chapter 3

Functional Analysis

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Dual space

Definition

The **dual space** of a TVS X is the vector space X^* whose elements are the continuous linear functionals on X.

Note that addition and scalar multiplication are defined in X^* by $(\Lambda_1 + \Lambda_2)x = \Lambda_1 x + \Lambda_2 x$ and $(\alpha\Lambda)x = \alpha \cdot \Lambda x$. Thus it is clear that X^* is indeed a vector space.

Some terminology

An additive functional Λ on a complex vector space X is called real-linear (resp. complex-linear) if $\Lambda(\alpha x) = \alpha \Lambda x$ for all $x \in X$ and for every real (resp. complex) scalar α.

Note that if u is the real part of a complex-linear functional f on X, then u is real-linear and $f(x) = u(x) - iu(x)$ because $z = \mathrm{Re} z - i\mathrm{Re}(iz)$ for all $z \in \mathbb{C}$. Conversely, if $u : X \to \mathbb{R}$ is real-linear on a complex vector space X and if f is defined the same, then f is also complex-linear.

Thus we may conclude that a complex-linear functional on X is an element of X^* if and only if its real part is continuous, and that every continuous real-linear $u : X \to \mathbb{R}$ is the real part of a unique $f \in X^*$.