Completeness

Chapter 2

Functional Analysis

July 13, 2023

The Banach-Steinhaus theorem

Theorem

(Banach-Steinhaus) Suppose X, Y are TVSs. Let Γ be a collection of continuous linear maps from $X \to Y$. Let B be the set of all $x \in X$ such that $\Gamma(x) = \{\Lambda x \mid \Lambda \in \Gamma\}$ are bounded in Y. Now if B is of the second category in X, then $B = X$ and Γ is equicontinuous.

The Banach-Steinhaus theorem

Proof.

Pick balanced neighborhoods[a] W, U of 0_Y such that $\overline{U} + \overline{U} \subset W$.[b] Put $E = \bigcap_{\Lambda \in \Gamma} \Lambda^{-1}(\overline{U})$. Now if $x \in B$, then by definition $\Gamma(x)$ must be bounded so $\Gamma(x) \subset nU$ for some n. This implies $x \in nE$.[c] Consequently, $B \subset \bigcup_n nE$.

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[a] Recall that S is balanced if $aS \subset S$ for all scalars $|a| \leq 1$. Also recall that every TVS has a balanced local base, so it is enough to prove for balanced neighborhoods of 0_Y.

[b] Why is this possible? Need regularity condition?

[c] Direct proof?

The Banach-Steinhaus theorem

Proof.

Since B is of the second category, at least one of the nE must also be of the second category.[a] Also recall that $x \mapsto nx$ is a homeomorphism of X to X, from which it follows that E is itself of the second category in X. By definition of E, E is closed, thus must have an interior point. This is because any closed set whose interior is empty is of the first category.

[a] Suppose all nE are not of the second category, i.e. of the first category. Then by properties on page 43, $\bigcup_n nE$ must also be of the first category, and cannot contain B as a subset.

The Banach-Steinhaus theorem

Proof.

Pick $x \in E^\circ$. Consider the translated set $x - E$. Since the interior is nonempty, and contains zero, it contains a neighborhood V of 0_X. From $V \subset x - E$ deduce $\Lambda(V) \subset \Lambda x - \Lambda(E) \subset \overline{U} - \overline{U}$ considering the definition of E. Since by assumption U was balanced, we have that $\Lambda(V) \subset W$ for every $\Lambda \in \Gamma$. Thus, the collection Γ is equicontinuous. $\square$

A corollary

Theorem

If Γ is a collection of continuous linear maps from an F-space X into a TVS Y, and if the $\Gamma(x)$ are bounded in Y for every $x \in X$, then Γ is equicontinuous.

Proof.

Since F-spaces are of the second category (spaces such that their topology is induced by a complete invariant metric), $B = X$ is of the second category and the proof follows. $\square$

A special yet familiar case

Let X and Y be Banach spaces, and suppose that $\Gamma(x)$ is bounded, i.e. $\sup_{\Lambda\in\Gamma}||\Lambda x|| < \infty$ for all $x \in X$. Using previous theorems, we conclude that there exists $M < \infty$ such that $||\Lambda x|| \leq M$ whenever $||x|| \leq 1$, for all $\Lambda \in \Gamma$. Hence it follows that $||\Lambda x|| \leq M||x||$ for all $x \in X$, $\Lambda \in \Gamma$.[1]

[1] Not so clear...

Limits of continuous linear maps

Theorem

Suppose X, Y are TVSs. Let $\{\Lambda_n\}$ be a sequence of continuous linear maps from X to Y. Then,

- *If $C = \{x \in X \mid \{\Lambda_n x\}$ is Cauchy in $Y\}$ and C is of the second category in X, then $C = X$.*
- *If $L = \{x \in X \mid \Lambda x := \lim_{n\to\infty} \Lambda_n x$ exists$\}$ and if L is of the second category in X, and Y is an F-space, then $L = X$ and Λ is continuous.*

Limits of continuous linear maps

Proof.

(a) Cauchy sequences are bounded, so we may apply the Banach-Steinhaus theorem to the family $\{\Lambda_n\}$ and conclude that $\{\Lambda_n\}$ is equicontinuous. It is clear that C is a subspace of X. However, if we assume that $\overline{C}$ is a *proper* subspace of X, it must have empty interior, thus being of the first category. Since $C \subset \overline{C}$, it follows that C is also of the first category, which is a contradiction to what we assumed. Therefore $\overline{C}$ must be X, i.e. C is dense in X. Now we must show that $C = X$. We show this by showing $X \subset C$.

Limits of continuous linear maps

Proof.

Fix some $x \in X$. Let W be a neighborhood of 0_Y. Recall the definition of equicontinuity; since $\{\Lambda_n\}$ is equicontinuous, there is a neighborhood V of 0_X such that $\Lambda_n(V) \subset W$ for $n = 1, 2, 3, \ldots$. Also since C is dense in X, $C \cap (x + V)$ must be nonempty. Thus pick some $x' \in C \cap (x + V)$, and choose n, m large enough that $\Lambda_n x' - \Lambda_m x' \in W$. Also note that the following holds:

$$(\Lambda_n - \Lambda_m)x = \Lambda_n(x - x') + (\Lambda_n - \Lambda_m)x' + \Lambda_m(x' - x)$$

Since $x - x' \in V$ and we have $\Lambda_n(V) \subset W$ for all n, we may conclude that $\Lambda_n x - \Lambda_m x \in W + W + W$. Thus $\{\Lambda_n x\}$ is a Cauchy sequence in Y, which implies $x \in C$. Therefore, $X \subset C$. □

Limits of continuous linear maps

Proof.

(b) Recall the definition of a F-space.[a] Therefore all Cauchy sequences of Y converge in Y, implying $L = C$. Hence by (a), we have $L = X$. Suppose V, W are as in the proof of (a). Then $\Lambda_n(V) \subset W$ holds for all n, which implies $\Lambda(V) \subset \overline{W}$. Therefore Λ is continuous.[b] □

[a] Topology is induced by a complete invariant metric
[b] Why..????

A corollary

Theorem

If $\{\Lambda_n\}$ is a sequence of continuous linear mappings from an F-space X into a TVS Y, and if $\Lambda x = \lim_{n\to\infty} \Lambda_n x$ exists for all $x \in X$, then Λ is continuous.

A corollary

Proof.

Note that $\{\Lambda_n\}$ is equicontinuous by Thm 2.6. Choose neighborhoods of 0_X, 0_Y as V, W, respectively, such that $\Lambda_n(V) \subset W$ for all n. Thus $\Lambda(V) \subset \overline{W}$, so Λ is continuous. □

A variant of the Banach-Steinhaus theorem

This variant of the Banach-Steinhaus theorem applies the category argument to a *compact set*.

Theorem

Suppose X, Y are TVSs. Let $K \subset X$ be a compact convex set. Γ is a collection of continuous linear mappings $X \to Y$. Suppose the orbits

$$\Gamma(x) = \{\Lambda x \mid \Lambda \in \Gamma\}$$

are bounded in Y for all $x \in K$. Then, there exists a bounded set $B \subset Y$ such that $\Lambda(K) \subset B$ for all $\Lambda \in \Gamma$.

A variant of the Banach-Steinhaus theorem

Proof.

Let $B = \bigcup_{x \in K} \Gamma(x)$. Pick balanced neighborhoods W, U of 0_Y such that $\overline{U} + \overline{U} \subset W$. Put $E := \bigcap_{\Lambda \in \Gamma} \Lambda^{-1}(\overline{U})$. If $x \in K$, then $\Gamma(x) \in nU$ for some n, so $x \in nE$. Therefore, $K = \bigcup_{n=1}^{\infty}(K \cap nE)$. If $K \cap nE$ (which is closed) has empty interior in K for all n, then each must be of the first category, hence K must be of the first category. However, this is contradictory to our assumption that K is a compact convex set, thus of the second category. Therefore for some n, $K \cap nE$ must have nonempty interior relative to K.

A variant of the Banach-Steinhaus theorem

Proof.

Now we fix n such that $K \cap nE$ has a nonempty interior in K. Pick some $x_0 \in (K \cap nE)^\circ$. Also choose a balanced neighborhood V of 0_X such that $K \cap (x_0 + V) \subset nE$.[a] Now, fix a $p > 1$ such that $K \subset x_0 + pV$.[b] Choose any $x \in K$, and let $z = (1 - p^{-1})x_0 + p^{-1}x$. z is in K since by assumption, K is convex. Also note that $z - x_0 = p^{-1}(x - x_0) \in V$, hence $z \in x_0 + V \subset nE$. Since $\Lambda(nE) \subset n\overline{U}$ for all $\Lambda \in \Gamma$ and since $x = pz - (p-1)x_0$, we have $\Lambda x = pn\overline{U} - (p-1)n\overline{U} \subset pn(\overline{U} + \overline{U}) \subset pnW$. Recall how we defined B. Thus $B \subset pnW$, so B is bounded. □

[a] Note that this is possible since we have chosen an interior point.

[b] This is possible since K is compact.

The open mapping theorem

Theorem

(Open mapping theorem) Suppose $\Lambda : X \to Y$ is continuous and linear where X is an F-space and Y is a TVS. If $\Lambda(X)$ is of the second category in Y, then Λ is an open mapping, and Y is an F-space.

The Open Mapping Theorem

Proof.

Let V be a neighborhood of 0_X. We must show that $\Lambda(V)$ contains a neighborhood of 0_Y.
Since we assumed X is an F-space, let d be an invariant metric on X compatible with $\mathcal{T}_X$. Define $V_n = \{x \mid d(x, 0) < 2^{-n}r\}$ for $n = 0, 1, 2, \ldots$ where $r > 0$ is chosen such that $V_0 \subset V$. We will prove that for some neighborhood W of 0_Y, $W \subset \overline{\Lambda(V_1)} \subset \Lambda(V)$ holds.

The Open Mapping Theorem

Proof.

Note that $V_1 \supset V_2 - V_2$. This implies

$$\overline{\Lambda(V_1)} \supset \overline{\Lambda(V_2) - \Lambda(V_2)} \supset \overline{\Lambda(V_2)} - \overline{\Lambda(V_2)}$$

If we show $\overline{\Lambda(V_2)}$ has nonempty interior, then we can prove $W \subset \overline{\Lambda(V_1)}$.[a] Since V_2 is a neighborhood of 0_X, we have $\Lambda(X) = \bigcup_{k=1}^{\infty} k\Lambda(V_2)$.[b] Therefore, at least one $k\Lambda(V_2)$ is of the second category in Y. Since $y \mapsto ky$ is a homeomorphism of Y to itself, $\Lambda(V_2)$ is also of the second category in Y. Therefore its closure must have nonempty interior. Thus we have proved $W \subset \overline{\Lambda(V_1)}$. Now we must prove $\overline{\Lambda(V_1)} \subset \Lambda(V)$.

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[a] How?

[b] Why?

The Open Mapping Theorem

Proof.

To prove $\overline{\Lambda(V_1)} \subset \Lambda(V)$, fix $y_1 \in \overline{\Lambda(V_1)}$. Assume $n \geq 1$ and y_n is chosen in $\overline{\Lambda(V_n)}$. Since $\overline{\Lambda(V_{n+1})}$ contains a neighborhood of 0_Y, we have that $(y_n - \overline{\Lambda(V_{n+1})}) \cap \Lambda(V_n) \neq \emptyset$. Therefore there exists some $x_n \in V_n$ such that $\Lambda x_n \in y_n - \overline{\Lambda(V_{n+1})}$. If you put $y_{n+1} = y_n - \Lambda x_n$, then $y_{n+1} \in \overline{\Lambda(V_{n+1})}$. Thus we may find a sequence y_n.

The Open Mapping Theorem

Proof.

Since $d(x_n, 0) < 2^{-n} r$ for $n = 1, 2, 3, \ldots$, the sum $x_1 + \cdots + x_n$ forms a Cauchy sequence. Since X is complete, this converges to some $x \in X$ such that $d(x, 0) < r$. Therefore $x \in V$. Also, $\sum_{n=1}^{m} \Lambda x_n = y_1 - y_{m+1}$, and $y_{m+1} \to 0$ as $m \to \infty$, so we conclude that $y_1 = \Lambda x \in \Lambda(V)$. This proves that Λ is an open mapping.

The Open Mapping Theorem

Proof.

Finally, we must show that Y is an F-space. To do this, we show that Y is homeomorphic to some F-space. Theorem 1.41 shows that X/N is an F-space if N is the kernel of $\Lambda : X \to Y$. Thus we define $f : X/N \to Y$ as $f(x+N) = \Lambda x$ for $x \in X$. Hence, f is an isomorphism. Suppose $V \subset Y$ is open. Then we have $f^{-1}(V) = \pi(\Lambda^{-1}(V))$ where π is the canonical projection. Since π is an open map, it is clear that f is continuous. Conversely, assume that $E \subset X/N$ is open. Then $f(E) = \Lambda(\pi^{-1}(E))$ which is open, thus f is an open map. Therefore, f is a homeomorphism between X/N and Y, thus Y is an F-space. $\square$

Some corollaries of the open mapping theorem

Corollary

(a) *If Λ is a continuous linear mapping of an F-space X onto an F-space Y, then Λ is open.*

(b) *If Λ satisfies (a) and is injective, then $\Lambda^{-1} : Y \to X$ is continuous.*

(c) *If X, Y are Banach spaces, and if $\Lambda : X \to Y$ is continuous, linear, bijective, then there exist positive real numbers a and b such that $a||x|| \leq ||\Lambda x|| \leq b||x||$ for all $x \in X$.*

(d) *If $\mathcal{T}_1 \subset \mathcal{T}_2$ are vector topologies on X, and if both $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ are F-spaces, then $\mathcal{T}_1 = \mathcal{T}_2$.*

Some corollaries of the open mapping theorem

Proof.

(a) follows from the open mapping theorem and Baire's theorem. (b) is an immediate consequence of (a). (c) follows from (b), since the inequalities express the continuity of Λ and Λ^{-1}. (d) is obtained by applying (b) to the identity mapping $\iota : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$. $\square$

Graphs of functions

Definition

If X, Y are sets, and $f : X \to Y$, then the **graph** of f is the set of all points $(x, f(x)) \in X \times Y$. If X, Y are topological spaces, then $X \times Y$ is given the product topology. In this case, if $f : X \to Y$ is continuous, then its graph is closed in $X \times Y$. For linear mappings between F-spaces, this necessary condition is also sufficient.

Graphs of functions

Theorem

If X is a topological space, Y is Hausdorff, and $f : X \to Y$ is continuous, then the graph G of f is closed in Y.

Proof.

Let Ω denote the complement of G in $X \times Y$. Fix some $(x_0, y_0) \in \Omega$. Then it is clear that $y_0 \neq f(x_0)$. Therefore y_0 and $f(x_0)$ have disjoint neighbrhoods V, W, respectively, in Y. Since f is continuous, x_0 has a neighborhood U such that $f(U) \subset W$. Therefore, $(x_0, y_0) \in U \times V \subset \Omega$. Therefore Ω is open. □

The closed graph theorem

Theorem

Suppose X, Y are F-spaces, $\Lambda : X \to Y$ is linear, and its graph $G = \{(x, \Lambda x) \mid x \in X\}$ is closed in $X \times Y$. Then Λ is continuous.

Proof.

Note that $X \times Y$ is an F-space, with incucing metric

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

and obvious vector operations. Since Λ is linear, G is a subspace of $X \times Y$. Closed subsets of complete metric spaces are complete, so G is an F-space. Now define $\pi_1 : G \to X$, $\pi_2 : X \times Y \to Y$ by $\pi_1(x, \Lambda x) = x$ and $\pi_2(x, y) = y$. π_1 is a continuous linear bijective map of G to X. By the open mapping theorem, $\pi_1^{-1} : X \to G$ is continuous, therefore $\Lambda = \pi_2 \circ \pi_1^{-1}$ is continuous. $\square$

How to verify G is closed

Often, the condition that G is closed is verified by showing that if $\{x_n\}$ a sequence in X, and $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} \Lambda x_n$ exist, then $y = \Lambda x$. This implies that G is closed.

Bilinear mappings

Definition

Suppose X, Y, Z are vector spaces and $B : X \times Y \to Z$. For each $x \in X$ and $y \in Y$, associate the maps $B_x : Y \to Z$ and $B^y = X \to Z$ defined by $B_x(y) = B(x, y) = B^y(x)$. B is **bilinear** if B_x and B_y are linear for all x, y.

Definition

If X, Y, Z are TVSs and if every B_x, B^y is continuous, then B is **separately continuous**.

Note that if B is continuous then B is separately continuous. We study when the converse holds, i.e. when B being separately continuous implies continuity.

Bilinear mappings

Theorem

Suppose $B : X \times Y \to Z$ is bilinear, and separately continuous. Suppose X is an F-space and Y, Z are TVSs. Then $B(x_n, y_n) \to B(x_0, y_0)$ whenever $x_n \to x_0$ and $y_n \to y_0$. If Y is metrizable, it follows that B is continuous.

Proof.

Let U, W be neighborhoods of 0_Z such that $U + U \subset W$. Define $b_n(x) = B(x, y_n)$. Since B is continuous as a function of y, we have $\lim_{n\to\infty} b_n(x) = B(x, y_0)$ for all $x \in X$. Therefore $\{b_n(x)\}$ is a bounded subset of Z for each x. Since each b_n is a continuous linear mapping of the F-space X, from the Banach-Steinhaus theorem it follows that $\{b_n\}$ is equicontinuous. Hence there exists a neighborhood V of 0_X such that $b_n(V) \subset U$.

Bilinear mappings

Proof.

Also note that $B(x_n, y_n) - B(x_0, y_0) = b_n(x_n - x_0) + B(x_0, y_n - y_0)$. If n is large enough, then $x_n \in x_0 + V$ so that $b_n(x_n - x_0) \in U$, and $B(x_0, y_n - y_0) \in U$ since B is continuous in y and $B(x_0, 0) = 0$. Hence $B(x_n, y_n) - B(x_0, y_0) \in U + U \subset W$ for large enough n.

Now, suppose Y is metrizable. Then so is $X \times Y$, so the continuity of B follows from what we proved. □