DERIVED CATEGORIES OF K3 SURFACES

HOJIN LEE

Abstract. This senior thesis serves as an introduction to problems on the derived categories of K3 surfaces.

Contents

Schemes are an algebraic analogue of manifolds. However, there are two main differences between schemes and manifolds. The first is that their topologies differ. Schemes are equipped with a topology called the Zariski topology, roughly speaking, whose closed sets correspond to zero sets of polynomials. The Zariski topology is often non-Hausdorff, as can be seen from the attempt to separate two distinct points in $\mathbb{C}$ with complements of zero sets. Manifolds, on the other hand, are locally copies of $\mathbb{R}^n$ and hence share their nice topological properties. The second main difference is that schemes come equipped with something called a structure sheaf, which roughly corresponds to taking the ring of C^∞ functions on open subsets of a manifold. However, the structure sheaf (of a scheme) is in nature algebraic, compared to the ring of C^∞ functions on a smooth manifold, and this is due to the structure sheaf capturing subtle aspects such as multiplicity of a polynomial. For example, the order of vanishing is stated in terms of the length of a module. Many theorems in algebraic geometry, such as Bezout's theorem, require to count the multiplicity of a zero for it to work properly. It is because we cannot distinguish the zero set of f with the zero set of f^2 that we have to use the language of schemes which has a structure sheaf that keeps track of its functions, rather than manifolds, in algebraic geometry.

1. Sheaves

Definition 1.1 (Sheaves)**.** A sheaf F on a topological space X with values in the category of sets Set is a correspondence

$$F : U \mapsto F(U)$$

where U is an open subset of X, and $F(U)$ is a set together with restriction maps $\mathrm{res}_{U \to V} : F(U) \to F(V)$ whenever $V \subset U$. Moreover, the restriction map must be the identity if $U = V$, and if $W \subset V \subset U$, the restriction map from $U \to W$ must be equal to the composition of the restriction maps $U \to V$ and $V \to W$. On top of that, F must satisfy the following two properties:

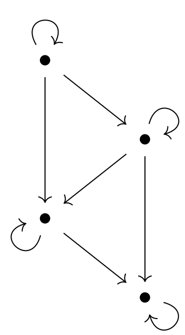
(1) If we have an open cover $\{U_i\}_{i \in I}$ of U, and two elements $s, t \in F(U)$ satisfy $\mathrm{res}_{U \to U_i}(s) = \mathrm{res}_{U \to U_i}(t)$ for all $i \in I$, then $s = t$.
(2) If we have an open cover $\{U_i\}_{i \in I}$ of U and elements $s_i \in F(U_i)_{i \in I}$ such that we have $\mathrm{res}_{U \to U_i \cap U_j}(s_i) = \mathrm{res}_{U \to U_i \cap U_j}(s_j)$ for all $i, j \in I$, then there exists some $s \in F(U)$ such that $\mathrm{res}_{U \to U_i}(s) = s_i$ for all $i \in I$.

Given two sheaves F, G on X, a sheaf morphism $\varphi : F \to G$ is defined to be a family of maps $\varphi_U : F(U) \to G(U)$ that commute with the restriction morphisms $\mathrm{res}_{U \to V}$, i.e., $\mathrm{res}_{U \to V} \circ \varphi_U = \varphi_V \circ \mathrm{res}_{U \to V}$.

This definition does not involve categories or functors, but is rather verbose. Also, in practice we must consider sheaves with values in various categories, so we want to refrain from using notions such as $s \in F(U)$. For tools such as étale cohomology, it is not sufficient to consider only the open sets of a topological space, but one must rather replace the domain with a category that roughly resembles $\mathrm{Open}(X)$, the category of open sets of X. This is quite far-fetched, so we will not introduce this in this article. To give a more algebraic description of sheaves, we will introduce limits and use them to give another definition of sheaves. In the process, we will explore how the notion of a limit can explain a variety objects in category theory defined by a universal property, such as products and kernels.

Definition 1.2 (Diagrams)**.** Let J be a small category, i.e., $\mathrm{Ob}(J)$ and $\mathrm{Mor}(J)$ are sets. Let $\mathcal{C}$ be an arbitrary category. A diagram of shape J in $\mathcal{C}$ is a functor $F : J \to \mathcal{C}$.

This definition is missing pragmatic details, as in practice there are only certain types of categories J we are interested in using. The category J is called the index category, and most often the morphisms between two objects of J will be finite. For an example of an index category one might use in practice, let J be the following category:



In the picture above, the objects are the four bullet points, and the arrows depicted are the only morphisms of J with the circular arrows being identity morphisms. Notice that this forces the diagram to commute. A diagram F of shape J in an arbitrary category $\mathcal{C}$ would look like the following:

$$\begin{tikzcd}
X \arrow[loop above] \arrow[dd, "f"'] \arrow[dr, "g"] & \\
& Y \arrow[loop right] \arrow[dl, "h"] \arrow[dd, "j"] \\
Z \arrow[loop left] \arrow[dr, "i"] & \\
& W \arrow[loop right]
\end{tikzcd}$$

Here, X, Y, Z, W are (not necessarily distinct) objects of $\mathcal{C}$, which correspond to the image of the bullet points under F. The morphisms f, g, h, i, j also correspond to the images of the morphisms of J under F. As seen above, one can think of J as a convenient way of capturing the essence of commutative diagrams in arbitrary categories.

Definition 1.3 (Cones)**.** Let $F : J \to \mathcal{C}$ be a diagram of shape J in $\mathcal{C}$. A cone to F in the category $\mathcal{C}$ is a pair of an object C, and a family of morphisms $\varphi_X : C \to F(X)$ in $\mathcal{C}$ for each object X of J. Moreover, the morphisms φ_X must be compatible with the J-shaped diagram in $\mathcal{C}$, i.e., for every morphism $f : X \to Y$ in J we must have $F(f) \circ \varphi_X = \varphi_Y$.

Using our example diagram, we may picture a cone $(C, \{\varphi_\bullet\})_{\bullet \in J}$ to F as follows:

$$\begin{tikzcd}
& X \arrow[dd, "f"'] \arrow[dr, "g"] & \\
C \arrow[ur, dotted, "\varphi_X"] \arrow[rr, dotted, "\varphi_Y"] \arrow[dr, dotted, "\varphi_Z"] \arrow[ddrr, dotted, "\varphi_W"'] & & Y \arrow[dl, "h"] \arrow[dd, "j"] \\
& Z \arrow[dr, "i"] & \\
& & W
\end{tikzcd}$$

By abuse of notation, we have written φ_X instead of $\varphi_\bullet$, since bullet points all look the same. The identity morphisms have also been omitted and will continued to be omitted. All arrows in the picture above commute. It is obvious why this is called a cone, as the commutative diagram literally looks like a cone.

Given a diagram $F : J \to \mathcal{C}$, there are plenty of cones to F one may think of. For trivial examples, consider letting $C = X$ in the above example, where $\varphi_X = \mathrm{id}_X$. The compatibility condition fixes all other φ's. Therefore, one is rarely interested in a cone itself, but a universal cone that is defined via a universal property among other cones. This leads to the definition of a limit in a category.

Definition 1.4 (Limit of a diagram)**.** Suppose $F : J \to \mathcal{C}$ is a diagram. The limit (L, ψ) of the diagram F is a cone to F in $\mathcal{C}$, such that for any other cone (C, φ) to F, there exists a unique morphism $u : C \to L$ that commutes with the families ψ and φ. More precisely, for every object $X \in J$ we must have $\varphi_X = \psi_X \circ u$.

For these reasons, limits are sometimes called universal cones. Using our example diagram, we may depict this as follows:

$$\begin{tikzcd}
& & X \arrow[dr, "g"] \arrow[dd, "f"] & \\
& L \arrow[ur, dotted] \arrow[rr, dotted] \arrow[dr, dotted] \arrow[ddrr, dotted] & & Y \arrow[dl, "h"] \arrow[dd, "j"] \\
C \arrow[ur, dotted, "\exists ! u"] \arrow[uurr, dotted] \arrow[urrr, dotted] \arrow[rr, dotted] \arrow[drrr, dotted] & & Z \arrow[dr, "i"] & \\
& & & W
\end{tikzcd}$$

As any other object defined by a universal property, there is no guarantee that a limit of a diagram must exist in an arbitrary category $\mathcal{C}$. All we know is that if a limit of a diagram exists, it must be unique by its universal nature. The problem of determining the existence of a limit in a category is very important, however, will not be treated in this article. Instead, we will assume that categories admit limits, and will reinterpret categorical objects in the limit language.

Example 1.5 (Products)**.** Let J be the following category:

$$\bullet \qquad \bullet$$

There are two objects, and there are no morphisms except the identity morphisms. Let $\mathcal{C}$ be a category that admits limits, and consider a diagram $F : J \to \mathcal{C}$ with image being the objects A and B. The limit of the diagram F is the categorical product $A \prod B$ together with its projections, which can be seen from the following diagram:

$$\begin{tikzcd}
& L \arrow[dl, "p_A"'] \arrow[dr, "p_B"] & \\
A & & B \\
& C \arrow[ul] \arrow[ur] \arrow[uu, dotted, "\exists !"] &
\end{tikzcd}$$

Similarly, if J is a category with only bullet points and no arrows between them, the limit of a diagram of shape J will be an arbitrary product.

Example 1.6 (Fiber products)**.** Let J be the following category:

$$\begin{tikzcd}
& \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

There are three objects, and the only morphisms are the identity morphisms together with the arrows depicted. If $F : J \to \mathcal{C}$ is a diagram, the limit of F is the fiber product. Again, this can be seen as follows:

$$\begin{tikzcd}
C \arrow[drr] \arrow[ddr] \arrow[dr, dotted, "\exists!"] & & \\
& L \arrow[r] \arrow[d] & E \arrow[d] \\
& N \arrow[r] & M
\end{tikzcd}$$

Notice how the compatibility condition in cones keeps the diagram a lot more tidier.

Example 1.7 (Equalizers and kernels)**.** Let J be the following category:

$$\begin{tikzcd}
\bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet
\end{tikzcd}$$

There are only two objects, as in the case of products. However, we have two morphisms from one object to another. Suppose we have a diagram $F : J \to \mathcal{C}$ which maps to the following:

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}$$

The limit of the diagram F is called the equalizer of the pair $f, g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Putting the cones into perspective,

$$\begin{tikzcd}
L \arrow[r] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
C \arrow[u, dotted, "\exists!"] \arrow[ur] & &
\end{tikzcd}$$

L, which is the equalizer, is universal among objects and morphisms to A such that their compositions with f and g result in the same morphism (hence the name equalizer). Notice how this could be a special case of the fiber product. There is an important special case of the equalizer, known as the kernel. This notion is valid in general for any category $\mathcal{C}$ which has zero morphisms. A zero object in a category $\mathcal{C}$ is an object that is both initial and terminal. In other words, it is an object 0 such that $\mathrm{Hom}_{\mathcal{C}}(0, X)$ and $\mathrm{Hom}_{\mathcal{C}}(X, 0)$ are singleton sets for every object X in $\mathcal{C}$. It can be easily seen that such an object, if it exists, is unique up to unique isomorphism. If $\mathcal{C}$ has a zero object, then for every pair X, Y of objects in $\mathcal{C}$, there exists a unique zero morphism $0 : X \to Y$ given by the composition of the unique morphisms $X \to 0$ and $0 \to Y$. Categories such as Top or Set do not have zero morphisms, as they do not have zero objects. Given any morphism $f : X \to Y$ in a category with zero objects, the equalizer of f and $0 : X \to Y$ is called the kernel of f. When $\mathcal{C} = \mathsf{Ab}$ or any other familiar abelian category, this notion of kernel coincides with the usual one.

Now that we have developed the notion of limits, especially equalizers, we may define sheaves in a more algebraic sense.

Definition 1.8 (Sheaves)**.** The category of open sets $\mathrm{Open}(X)$ of a topological space X is defined to have objects as open sets of X, and $\mathrm{Hom}(U, V)$ a singleton if $U \subset V$, and an empty set if otherwise. A sheaf F on a topological space X with values in a category $\mathcal{C}$ is a contravariant functor

$$F : \mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{C}$$

from the category of open sets of X to $\mathcal{C}$, such that for any open set $U \subset X$ and an open covering $\{U_i\}_{i \in I}$ of U, the first arrow in the following diagram

$$F(U) \longrightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{(i,j) \in I \times I} F(U_i \cap U_j)$$

is the equalizer of the second two. The first arrow is induced by the $F(U_i \hookrightarrow U)$,

$$\begin{tikzcd} & F(U_i) \\ F(U) \arrow[ur, "F(U_i \hookrightarrow U)"] \arrow[r, dotted, "\exists !"'] & \prod_{i \in I} F(U_i) \arrow[u, "p_i"'] \end{tikzcd}$$

which correspond to the restriction maps defined earlier. The second arrows are given by restricting $F(U_i)$ and $F(U_j)$ to $F(U_i \cap U_j)$, respectively. We assume that $\mathcal{C}$ has nice enough properties to admit all structures that are needed. Given two sheaves $F, G : \mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{C}$, a sheaf morphism $\varphi : F \to G$ is given by a natural transformation $F \Rightarrow G$.

Notice how using functorial notation automatically gives information about the restriction maps and satisfies all properties needed. Now that we have defined sheaves in a general matter, we introduce the types of sheaves that are used in practice. Most often we will consider sheaves with values in CRing or in abelian categories such as Ab or Mod_A, as sheaves with values merely in Set do not carry structure and are not suitable for homological calculations. For sheaves with values in an abelian category, one may define images and kernels of sheaf morphisms, but we will not go through the construction here and assume that such objects exist.

Example 1.9 (Sheaf of smooth functions)**.** Let M be a smooth manifold. Consider a correspondence $F : U \mapsto F(U)$ where $F(U)$ is the ring of C^∞ functions from U to $\mathbb{R}$, with addition and multiplication defined pointwise. The restriction maps $\mathrm{res}_{U \to V} : F(U) \to F(V)$ are given by sending f to $f|_V = f \circ i_{U \hookrightarrow V}$. Notice how the restriction maps are also ring homomorphisms. Thus the sheaf F of smooth functions on a manifold can be considered as a sheaf with values in CRing.

Example 1.10 (Constant sheaf)**.** Let X be any topological space, and consider the correspondence $F : U \mapsto A$ for A a fixed (nontrivial) abelian group where the restriction morphisms are all fixed to id_A. This is not quite a sheaf, as the sheaf conditions imply that any two elements $s, t \in F(\varnothing) = A$ must be equal. To see this, recall the (1) condition of a sheaf defined earlier, and consider the open cover $\{U_i\}_{i \in I}$ of $\varnothing$ indexed by $I = \varnothing$. It is vacuously true that for all $i \in I$, the restrictions of s and t to each U_i are the same. As we assumed that A is nontrivial, this is not true. However for every contravariant functor F, there is a unique sheaf F^+ associated to it called the sheafification, where there is a natural transformation $F \Rightarrow F^+$ such that for any natural transformation $F \Rightarrow G$ to a sheaf G, there exists a unique sheaf morphism $F^+ \to G$ that comutes with the natural transformation $F \Rightarrow F^+$. It is possible to construct this sheaf, but we will not go into details. The sheafification of the constant functor $F : U \mapsto A$ will be called the constant sheaf on X with value in A, denoted $\underline{A}$. For each open set $U \subset X$, the abelian group $\underline{A}(U)$ is the space of locally constant functions $f : U \to A$. In other words, if U has two connected components, then $\underline{A}(U) = A \oplus A$.

To study the local behavior of smooth functions on a manifold M near a point p, it suffices to consider the germ of smooth functions C_p^∞, defined as the space of pairs (f, U) where U is an open subset of M containing p and $f \in C^\infty(U)$, taken quotient by the equivalence relation $(f, U) \sim (g, V)$ if and only if $f|_{U \cap V} = g|_{U \cap V}$. The germ at p can equivalently be defined as the direct limit $\varinjlim_{U \ni p} C^\infty(U)$, which we will not define here. Rather, the structure of the direct limit should be understood through the construction of the germ of smooth functions. The direct limit is an instance of a colimit, which is obtained by reversing all the arrows involved in the definition of the limit of a diagram. This definition can be used in the context of sheaves, which gives an analogue of the notion of the germ of functions in a more general sense.

Definition 1.11 (Stalk of a sheaf)**.** Let $F : \mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{C}$ be a sheaf on X with values in $\mathcal{C}$, where $\mathcal{C}$ admits not only limits but also colimits. Then one may define the stalk of F at $p \in X$ as the direct limit $\varinjlim_{U \ni p} F(U)$.

2. Schemes

A smooth manifold M is covered by coordinate open covers, which come with homeomorphisms to an open subset of $\mathbb{R}^n$, or equivalently, $\mathbb{R}^n$ itself. Similarly, schemes are covered by open subsets called affine schemes, which are analogous to the role of $\mathbb{R}^n$ in defining smooth manifolds. Instead of $\mathbb{R}^n$, affine schemes are of the form $\operatorname{Spec} R$ for some commutative ring R, which we will explain the meaning of.

Definition 2.1 (Prime ideals)**.** Let R be a commutative ring. An ideal $\mathfrak{p} \subsetneq R$ is said to be a prime ideal if $ab \in \mathfrak{p}$ implies either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

Definition 2.2 (Maximal ideals)**.** Let R be a commutative ring. An ideal $\mathfrak{m} \subsetneq R$ is said to be a maximal ideal if $\mathfrak{m} \subsetneq I$ for any other ideal I implies $I = R$.

Notice that maximal ideals are automatically prime; if $ab \in \mathfrak{m}$ but neither a, b are in $\mathfrak{m}$, then the ideal $(\mathfrak{m}, a)$ is a proper ideal that properly contains $\mathfrak{m}$. Although we have defined prime and maximal ideals for arbitrary rings, the rings we will be most interested will be either a polynomial ring or a quotient of it. To see why this is the case, we introduce the spectrum of a ring. More precisely, we will introduce it as a topological space, but will later extend the definition to objects called locally ringed spaces.

Definition 2.3 (Spectrum of a ring, I)**.** Let R be a commutative ring. The topological space $\operatorname{Spec} R$ has points the prime ideals of R, and closed sets as the set of prime ideals containg $E \subset R$ for some E.

Example 2.4. Suppose $R = \mathbb{C}[x]$. The spectrum of R is the set of prime ideals of $\mathbb{C}[x]$, which consists of the zero ideal (0) and ideals of the form $(x - a)$ for $a \in \mathbb{C}$. Thus $\operatorname{Spec} \mathbb{C}[x]$ *almost* recovers $\mathbb{C}$ under the Zariski topology, with the exception of the zero ideal. One is tempted to get rid of the zero ideal by replacing the definition of Spec to only consider maximal ideals, but this method is not algebraically convenient as Spec cannot be seen as a contravariant functor from CRing to Top. In other words, inverse images of prime ideals under ring homomorphisms are prime ideals, whereas maximal ideals may no longer be maximal. A typical example is the inclusion $\mathbb{Z} \to \mathbb{Q}$ where the inverse image of (0) is clearly not maximal in $\mathbb{Z}$.

As mentioned above, for a ring homomorphism $\varphi : R \to S$, Spec induces a set-map $\varphi^* : \operatorname{Spec} S \to \operatorname{Spec} R$ by taking the inverse images of each prime ideal. An interesting fact is that this induced set-map is in fact a continuous one; the inverse image of open sets in $\operatorname{Spec} R$ are again open in $\operatorname{Spec} S$.

Example 2.5 (Affine n-space)**.** For a field k, consider the polynomial ring $k[x_1, \ldots, x_n]$ of n variables. The affine n-space over k is defined to be $\operatorname{Spec} k[x_1, \ldots, x_n]$. Suppose we have $k = \mathbb{C}$ and $n = 2$. Then as a topological space, the affine complex plane can be pictured as three layers; the top layer of maximal ideals, i.e., height 2 primes, the middle layer of height 1 primes, and the lowest layer of height 0 primes. In this case, there is only one height 0 prime, namely the zero ideal. $\operatorname{Spec} \mathbb{C}[x_1, x_2]$ has maximal ideals of the form $(x - a, y - b)$ for $a, b \in \mathbb{C}$, which are in correspondence with the complex plane $\mathbb{C}^2$ (beware that the picture is fundamentally incorrect though; complex dimensions are represented through real dimensions). These are the height 2 primes. The height 1 primes are of the form (f) where f is an irreducible polynomial in x and y, and the height 0 prime is the zero ideal. For a ring R, the dimension of R is defined to be the maximal length of a chain $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ of prime ideals. For topological spaces, we define its Krull dimension to be the maximal length of a chain $Z_0 \subsetneq \cdots \subsetneq Z_n$ of irreducible closed subsets. In the case of $\operatorname{Spec} \mathbb{C}[x_1, x_2]$, notice that the dimension defined on rings translates directly to the Krull dimension.

To put it in different words, the closed points of the topological space $\operatorname{Spec} A$ corresponds to the original space, and the other points are additional information.

Definition 2.6. A local ring is a ring with a unique maximal ideal.

Definition 2.7. Let R be a ring, and $S \subset R$ be a subset such that $1 \in S$, and $a, b \in S$ implies $ab \in S$. The quotient of $R \times S$ under the equivalence relation $(r, s) \sim (r', s')$ if and only if there exists some $s'' \in S$ such that $s''(rs' - r's) = 0$ is called the localization of R with respect to S, denoted R_S. This has a natural ring structure defined by $[(r, s)] + [(r', s')] = [(rs' + r's, ss')]$ and $[(r, s)] \times [(r', s')] = [(rr', ss')]$. Well-definedness is straightforward to verify.

We will introduce what are called locally ringed spaces, which is a class of objects that is more general than a scheme.

Definition 2.8. A locally ringed space is a pair $(X, \mathcal{O}_X)$ of a topological space X and a sheaf of rings $\mathcal{O}_X$ such that at every point $p \in X$, the stalk $\mathcal{O}_{X,p}$ is a local ring. The sheaf $\mathcal{O}_X$ is called the structure sheaf. A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair (φ, ψ) where $\varphi : X \to Y$ is a continuous function, and $\psi : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$ is a sheaf homomorphism which locally is a local ring homomorphism.

In fact, the pair (M, C^∞) of a smooth manifold and its sheaf of C^∞ functions actually forms a locally ringed space, where for each germ C^∞_p the unique maximal ideal corresponds to the ideal of functions vanishing at p. Keeping this definition in mind, we put additional structure on the spectrum of a ring defined earlier. More precisely, we will assign to it a structure sheaf.

Definition 2.9 (Spectrum of a ring, II)**.** Let R be a commutative ring. The spectrum of a ring is a pair $(X, \mathcal{O}_X)$ where the topological space X is given by the usual construction of Spec, and for every standard open subset D_f of X, the sheaf $\mathcal{O}_X$ assigns the ring A_f. Note that this is not exactly the construction, see Hartshorne's *Algebraic Geometry* for the preciese definitions.

The idea is that the ring A we start with, is the ring of global functions, and we make a topological space $\operatorname{Spec} A$ with its prime ideals. Then, we assign to open subsets suitable localizations of the ring A of global functions.

Definition 2.10 ($\mathcal{O}_X$-modules)**.** Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module F on X is a sheaf of abelian groups such that for every open subset $U \subset X$, the group $F(U)$ has an $\mathcal{O}_X(U)$-module structure. The $\mathcal{O}_X$-module homomorphisms are defined to respect the module structure.

Definition 2.11 (Schemes)**.** A scheme $(X, \mathcal{O}_X)$ is a locally ringed space such that there exists an open covering $\{U_i\}_{i \in I}$ of X such that $(U, \mathcal{O}_X|_U)$ is isomorphic to $\operatorname{Spec} A$ for some ring A, as a locally ringed space. These U_i are called affine open subsets. Often the scheme is denoted X for simplicity.

Definition 2.12 (Varieties)**.** A variety X over a field k is a reduced irreducible scheme of finite type over k. Here, a reduced scheme is a scheme whose stalks of structure sheaves are reduced rings, i.e., their nilradicals are zero. Irreducible means that the underlying topological space X of the scheme itself is irreducible, i.e., any two nonempty open subsets must have nonempty intersection. Some authors also include the condition of separability but we will not explain this here.

As seen above, (at least from what I see) algebraic geometry is trying to translate the language of classical geometry, for example differential geometry and complex geometry, into algebraic language involving sheaves. Thus it is helpful to think of, and to some degree valid, to think of schemes over an algebraically closed field of characteristic zero, as merely complex manifolds. Although this lacks technical detail, there is a principle called Serre's *géométrie algébrique et géométrie analytique* principle that bridges the two.

3. Sheaf cohomology

Suppose we are working with sheaves with values in a suitable abelian category. Given an exact sequence of sheaves $0 \to F \to G \to H \to 0$ on a topological space X, the assignment $F \mapsto F(X)$ is naturally a functor, which is called the global sections functor and denoted $\Gamma(-)$. In general, the global sections functor is only left exact, but not right exact. However, for such functors, there is a natural way to obtain a long exact sequence from the original short exact sequence. This approach is called the derived functor approach. For explaining sheaf cohomology, we will use direct construction together with some assumptions, rather than the general derived functor approach.

Before introducing sheaf cohomology, we must know what cohomology actually is about. One typically starts with the singular cohomology of CW complexes. Here, I will sketch the idea of singular cohomology. For simplicity, let's assume we are working with smooth manifolds. By looking at n-dimensional simplices and their continuous maps into our target manifold M, we may imagine the free abelian group generated by these maps. These groups are extremely large, and in general contain little useable information. However, we can define the boundary map from the n-th complex to the $n-1$th complex, which defines a chain complex. By taking the dual of this complex, and taking the successive quotients of images and kernels, we obtain singular cohomology. These connect with sheaf cohomology, in the form that the cohomology of constant sheaves with values in some A is in fact singular cohomology with coefficients in A.

From now on, assume we are working with sheaves on some topological space X with value in Ab. This will be general enough to cover sheaves with value in Vect or Mod (although the latter is rarely used in complex geometry).

Definition 3.1 (Injective sheaves)**.** A sheaf I is injective if for every injective sheaf homomorphism $P \to Q$, and a morphism $f : P \to I$, there exists a $g : Q \to I$ that commutes with the other morphisms.

We can generalize this to the notion of an injective object, which involves replacing the sheaves above with objects in some arbitrary category. These type of objects are used to form what is called an injective resolution of an object A, which is an exact sequence of the form

$$0 \to A \to I_0 \to I_1 \to \cdots$$

Another useful way to think of this is to view this as a complex that is quasi-isomorphic to $A[0]$. These resolutions are important in the theory of derived functors, as they are an example of an acyclic resolution. Any acyclic resolution gives rise to naturally isomorphic derived functors. In our case of sheaves, truncating A itself and applying the global sections functor $\Gamma(-)$ makes the above exact sequence no longer exact, but rather gives a complex of abelian groups (since we assumed we are working with sheaves of abelian groups). The i-th cohomology is called the i-th sheaf cohomology group, and is denoted $H^i(X, A)$, as one would write for singular cohomology with coefficients in A.

Sheaf cohomology is a powerful tool both in algebraic geometry and in complex analytic geometry. There is an especially powerful tool, called the exponential sheaf sequence

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0$$

defined on a complex manifold X. Here, $\underline{\mathbb{Z}}$ denotes the constant sheaf of $\mathbb{Z}$ on X, and $\mathcal{O}_X$ the sheaf of holomorphic functions on X, and $\mathcal{O}_X^\times$ the sheaf of invertible holomorphic functions. The first map is given by multiplying the constant functions by $2\pi i$, and the second map is the exponential map. By construction of sheaf cohomology, we can obtain a long exact sequence from this short exact seqeunce of sheaves. From this long exact sequence, and the fact that $H^1(X, \mathcal{O}_X^\times)$ is isomorphic to the Picard group $\mathrm{Pic}(X)$ of isomorphism classes of line bundles on X with group operator $\otimes$, we can obtain information about line bundles. The sections of the line bundles in the classical sense form a sheaf, and this is what we are talking about here. Another useful tool we obtain is the first Chern class, which corresponds to the connecting homomorphism with domain $H^1(X, \mathcal{O}_X^\times)$. Often for simple complex manifolds, we can explicitly calculate the sheaf cohomology groups thanks to the fact that constant sheaf cohomology is isomorphic to singular cohomology. The cell structure of complex manifolds can be simple.

4. Derived categories of coherent sheaves

It is helpful pedagogically to know that derived categories originated from extending Serre duality to more general situations. Also, to use derived categories as a tool one must be familiar with its precise definitions. However, as I am neither familiar of these two aspects, I will briefly sketch the definition of derived categories and how they are used in research. Then, I will introduce ongoing research topics on derived categories, regarding them as a type of invariant for varieties.

Fix an abelian category $\mathcal{A}$. Denote by $K(\mathcal{A})$ its category of chain complexes, up to chain homotopy. This is called the homotopy category of chain complexes. As we mentioned above earlier while defining sheaf cohomology, we are interested in the quasi-isomorphisms between chain complexes and want them to behave as isomorphisms in our new category. Unfortunately this does not happen, so we have to formally invert them, in analogy to the localization of a ring. We denote this category as $D(\mathcal{A})$, and call it the derived category of $\mathcal{A}$. Often we will be looking at the bounded derived category of $\mathcal{A}$, whose objects consist of chain complexes that are bounded, i.e., whose objects are all zero outside of some interval of indices. We will denote this specifically as $D^+(\mathcal{A})$.

Definition 4.1 (Coherent sheaves)**.** Let X be a scheme[1]. A sheaf F of $\mathcal{O}_X$-modules is coherent if over every affine open subset $U = \operatorname{Spec} A$, the sheaf $F|_U$ is isomorphic to the sheaf $\widetilde{M}$ associated to the finitely generated module $M = F(U)$ over A.

Coherent sheaves include all sheaves that we can think of as vector bundles. Not only that, this includes ideal sheaves and all sorts of not-so-nice sheaves. However, we are interested in them as they are suitable for homological manipulations, as we have tools to deal with finitely generated modules. One fact is that coherent sheaves may seem a very general notion, but they are not. For example, in the process of taking the sheaf cohomology of a coherent sheaf, it may be necessary to involve non-coherent sheaves in the injective resolution. For this reason, the precise construction of the derived category of coherent sheaves involves working with the category of quasicoherent sheaves, which is a relaxation of the finitely generated modules in the definition above to allow finitely presented modules too. The precise construction of the derived category of coherent sheaves is extremely sophisticated and out of the authors expertise, so I will use a heuristic that these categories are categories of complexes of coherent sheaves up to cohomology. We denote the (bounded) derived category of coherent sheaves on a scheme X as $D^{\mathrm{b}}(\mathsf{Coh}_X)$. In practice, we will only work with projective varieties X over $\mathbb{C}$. The derived category of a scheme X almost always refers to the bounded derived category of coherent sheaves on X.

Example 4.2 (Derived category of $\mathbb{P}^n$)**.** We first introduce the derived category of the simplest projective variety, namely $\mathbb{P}^n$ itself. It is known that $D^{\mathrm{b}}(\mathsf{Coh}_{\mathbb{P}^n})$ is generated by the sequence $\mathcal{O}(-n).\mathcal{O}(-n+1), \ldots, \mathcal{O}(-1), \mathcal{O}$ using the shift operations and taking cones. See [Cal05] for the proof.

The derived category is an interesting invariant of a (smooth) projective variety. The category of mere coherent sheaves, Coh_X on some projective variety X determines X entirely, up to isomorphism. Thus it follows that Coh_X and Coh_Y being equivalent for two projective varieties X and Y implies X and Y are isomorphic. However for the derived category $D^{\mathrm{b}}(\mathsf{Coh}_X)$ and $D^{\mathrm{b}}(\mathsf{Coh}_Y)$, their equivalences do not imply isomorphisms of X and Y. We will see examples of this phenomenon later, using the Fourier-Mukai functor (or transformation, whatever you like to call it). Meanwhile, for projective varieties X with canonical sheaf ω_X either ample or $\omega_X^\vee := \mathcal{H}om(\omega_X, \mathcal{O}_X)$ ample, Bondal and Orlov have provided a method of recovering the variety X entirely from its derived category $D^{\mathrm{b}}(\mathsf{Coh}_X)$. We call such X

[1]We have to assume that X is locally noetherian, which is a very mild assumption. However when working with sheaves on analytic spaces, the story is different. Even the fact that $\mathcal{O}_{\mathbb{C}^n}$ is coherent is far from trivial, and is called Oka's coherence theorem.

that have either ω_X ample resp. $\omega_X^\vee$ ample as general type, resp. Fano varieties. The construction roughly is proceeded as follows. Denote its derived category as $\mathcal{D}$. First, we identify skyscraper sheaves of closed points of X. This is done by defining a point object in $\mathcal{D}$, and showing that an object is a point object if and only if it is isomorphic to $\mathcal{O}_x[r]$, i.e., a shift of a skyscraper sheaf of a closed point of x. From this, we can collect the points of X. Next, one defines an invertible object in the category $\mathcal{D}$, and shows that an object is invertible if and only if it is isomorphic to a shift of some invertible sheaf on X. Now, using this we collect line bundles on X, from which we can define the Zariski topology on the closed points we obtained earlier. Furthermore, if either ω_X or $\omega_X^\vee$ is ample, then the form of autoequivalences of $\mathcal{D}$, i.e., the group of equivalences of $\mathcal{D}$ with itself with group operation being composition of functors, is completely determined. Explicitly, it is of the form $\mathrm{Aut}(X) \ltimes (\mathrm{Pic}(X) \oplus \mathbb{Z})$, where $\mathbb{Z}$ comes from the shift functor on $\mathcal{D}$. However, the two results stated above do not hold in the case $\omega_X \cong \mathcal{O}_X$. The projective varieties under the condition $\omega_X \cong \mathcal{O}_X$, together with some additional assumptions that rule out trivial cases, are called Calabi-Yau manifolds.

What makes Calabi-Yau manifolds so special? First, the canonical sheaf ω_X is a birational invariant of a projective variety X, meaning that we can differentiate varieties by their canonical ring $\bigoplus_i H^0(X, \omega_X^{\otimes i})$. However, these are all trivial for Calabi-Yau manifolds, by definition. Also, Calabi-Yau manifolds and their derived categories are some of the main objects of interest in homological mirror symmetry. By introducing works of Mukai, we will show that the results of Bondal and Orlov mentioned above fail to hold for Calabi-Yau manifolds in dimension 2, called K3 surfaces.

Example 4.3 (Fermat quartic, quintic threefold)**.** Consider the projective varieties $\mathrm{Proj}\,\mathbb{C}[x_0, x_1, x_2, x_3]/(x_0^4 + x_1^4 + x_2^4 + x_3^4)$ and $\mathrm{Proj}\,\mathbb{C}[x_0, x_1, x_2, x_3, x_4]/(x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5)$. These are the most basic examples of Calabi-Yau manifolds, each being of dimension two and three. The latter is called the quintic threefold for obvious reasons, and is an object of interest in enumerative geometry. The fact that these objects have trivial canonical bundles can be checked immediately from their degrees, and one can easily check that they satisfy the other conditions of Calabi-Yau manifolds as well.

5. Fourier-Mukai transforms and moduli spaces

Definition 5.1 (Higher direct image, derived pullback, derived tensor product)**.** Let $f : X \to Y$ be a morphism of schemes. The direct image functor f_* of a sheaf F on X, denoted f_*F, is a sheaf on Y defined by $f_*F(U) := F(f^{-1}(U))$ for every open subset $U \subset Y$. For suitable values, this functor is only left exact and its right derived functors $R^i f_*$ are called the higher direct image functors. Similarly for a sheaf G on Y, we define $f^{-1}G$ to be the sheafification of the association $U \mapsto \varinjlim_{V \supset f(U)} G(V)$. Then we may define f^*G as $f^{-1}G \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$, which is called the inverse image functor f^* or the pullback functor. In general this is only right exact, and we denote by $L_i f^*$ its left derived functors, called the derived pullback functors. Similarly, tensor products of sheaves (again with values in suitable categories) are in general only right exact, and their left derived functor is called the derived tensor product denoted $\otimes^L$.

Definition 5.2 (Fourier-Mukai transform)**.** Let X and Y be two smooth projective complex varieties. Fix an element $\mathcal{E} \in D^{\mathrm{b}}(\mathsf{Coh}_{X\times Y})$, called the kernel. The Fourier-Mukai transform with kernel $\mathcal{E}$ is a functor

$$\Phi_{\mathcal{E}} : D^{\mathrm{b}}(\mathsf{Coh}_X) \to D^{\mathrm{b}}(\mathsf{Coh}_Y)$$

given by $A \mapsto (p_Y)_*(E \otimes^L p_X^* A)$, where all operations are derived functors. Here, p_X and p_Y are projections from $X \times Y$ to X and Y, respectively, and the product of schemes is over the structure morphism to $\operatorname{Spec}\mathbb{C}$.

It turns out that equivalences of derived categories take the form of a Fourier-Mukai transform with some kernel. Therefore, to find examples of nonisomorphic varieties having equivalent derived categories, it is helpful to use the language of Fourier-Mukai transforms. From now on we give a rough sketch on the construction of the moduli space of sheaves. The moduli space of sheaves on a K3 surface, together with Fourier-Mukai transforms, will give examples of nonisomorphic K3 surfaces having equivalent derived categories.

What are moduli spaces of sheaves? Roughly speaking, we want to construct a space whose points correspond to isomorphism classes of coherent sheaves on some variety X. A convenient way of doing this is to use the notion of a Quot scheme, which parametrizes quotients of sheaves. To do this, we define the Quot functor.

Definition 5.3 (Hilbert polynomial of a coherent sheaf on a projective scheme)**.** Let X be a projective scheme with a fixed polarization $\mathcal{O}(1)$, i.e., with a fixed embedding into projective space. The Hilbert polynomial of a coherent sheaf F on X with respect to this polarization is defined as $m \mapsto \chi(F(m))$ for integers m. This is a polynomial in m with rational coefficients, and its coefficients and degree encode data such as rank and Chern classes of F.

Definition 5.4 (Quot functor)**.** Let S be a scheme over $\mathbb{C}$ and let X be projective over S with structure morphism f. Fix an f-ample line bundle $\mathcal{O}_X(1)$, and fix some coherent sheaf $\mathcal{H}$ on X and a polynomial $P \in \mathbb{Q}[z]$. Define

$$\mathcal{Q}uot_{X/S}(\mathcal{H}, P) : \mathsf{Sch}_S^{op} \to \mathsf{Set}$$

by sending an S-scheme T to the set of all T-flat quotients $\mathcal{H}_T := \mathcal{O}_T \otimes_{\mathcal{O}_{X\times_S T}} \mathcal{H} \to F$, with $\chi(F|_{X\times_S \operatorname{Spec}(\kappa(t))}(m)) = P(m)$ for all $t \in T$, and sending an S-morphism $g : T' \to T$ to the map that sends $\mathcal{H}_T \to F$ to $\mathcal{H}_{T'} \to h^*F$. Here, $h = \mathrm{id}_X \times g$. Here, a quotient is defined to be an equivalence class of epimorphisms, where two are equivalent whenever their kernels agree.

Theorem 5.5 (Grothendieck)**.** *The functor $\mathcal{Q}uot_{X/S}(\mathcal{H}, P)$ is represented by a projective S-scheme $\mathrm{Quot}_{X/S}(\mathcal{H}, P)$, called the Quot scheme.*

Since we are interested in schemes over $\mathbb{C}$, it suffices to let $S = \operatorname{Spec}\mathbb{C}$. In this case, $\mathcal{Q}uot_{X/S}(\mathcal{H}, P)(S)$ is the set of all quotients of $\mathcal{H}$ that have Hilbert polynomial P. It follows that quotients of $\mathcal{H}$ correspond to closed points of the scheme $\mathrm{Quot}_{X/S}(\mathcal{H}, P)$, which justifies the intuition that Quot schemes parametrize quotients of coherent sheaves having some fixed Hilbert polynomial.

However, note that this is not quite what we want. We want to find a scheme that parametrizes all coherent sheaves on X that have Hilbert polynomial P, not just quotients of some sheaf. The remedy is to consider sheaves F on X with Hilbert polynomial P, together with a choice of a basis for the vector space $H^0(X, F(m))$ for sufficiently large m. This will define an epimorphism from $\mathcal{H} :=$

$\mathcal{O}_X(-m)^{\oplus \dim_{\mathbb{C}} H^0(X,F(m))}$ to F, hence will correspond to a closed point of $\mathrm{Quot}_{X/\mathbb{C}}(\mathcal{H}, P)$. One last problem is that this $\mathcal{H}$ depends on the choice of the target F, which would make it impossible to parametrize all sheaves as quotients of a single $\mathcal{H}$. It turns out that looking at semistable sheaves fixes this problem.

Definition 5.6 (Semistable sheaves)**.** The reduced Hilbert polynomial $p(E)$ of a coherent sheaf E on a projective scheme X with fixed polarization $\mathcal{O}_X(1)$ is the Hilbert polynomial of E divided out by its leading coefficient. We call a sheaf E to be pure if $\dim F = \dim E$ holds for all nontrivial coherent subsheaves $F \subset E$. If E is a pure sheaf of dimension $\dim X$ and if for any proper subsheaf $F \subset E$ we have $p(F) \leq p(E)$, we call E semistable. The polynomial ordering is lexicographic.

Theorem 3.3.7 of [HL10] implies that the family of semistable sheaves is bounded, whose definition we will not state here. By Lemma 1.7.6 of the same reference, this implies the existence of some integer m such that every semistable sheaf F having Hilbert polynomial P is m-regular, i.e., $H^i(X, F(m-i)) = 0$ for all $i > 0$. Moreover by Serre's vanishing theorem, one may fix an m such that $H^i(X, F(m)) = 0$ for $i > 0$, which implies that $\chi(F(m)) = \dim_{\mathbb{C}} H^0(X, F(m)) = P(m) =: n$. In this case, each $F(m)$ is globally generated, so we can consider a surjection $\rho : \mathcal{H} \to F$ where $\mathcal{H} = \mathbb{C}^n \otimes_{\mathbb{C}} \mathcal{O}_X(-m)$. This surjection is induced by composing an isomorphism $\mathbb{C}^n \cong H^0(X, F(m))$ with the evaluation map $H^0(X, F(m)) \otimes \mathcal{O}_X(-m) \to F$. Hence, each semistable sheaf F on X that has Hilbert polynomial P defines a closed point of $\mathrm{Quot}_{X/\mathbb{C}}(\mathcal{H}, P)$. In fact, if we consider all closed points $[\mathcal{H} \to E]$ where E is semistable and the composition $\mathbb{C}^n \xrightarrow{\sim} H^0(X, \mathcal{H}(m)) \to H^0(X, E(m))$ is an isomorphism, these points form an open subset R of $\mathrm{Quot}_{X/\mathbb{C}}(\mathcal{H}, P)$. This is because semistability is an open condition for flat families. Finally, since choosing a basis for $H^0(X, F(m))$ has a GL_n-action that preserves isomorphism classes, we consider the quotient of R under this action to be our moduli space of (semistable) sheaves, having Hilbert polynomial P.

Moduli spaces, in general, are representing objects of what is called a moduli functor. However even in the most reasonable cases, moduli functors may not be representable. The next best notion is that of corepresentability, and a corepresenting object is called a coarse moduli space, opposed to the representing object case where we call the object a fine moduli space.

6. K3 surfaces

Definition 6.1 (K3 surface)**.** A K3 surface X is a two-dimensional complex variety with $\omega_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

We have mentioned an example, namely the Fermat quartic, above. In fact, every smooth quartic in $\mathbb{P}^3$ defines a K3 hypersurface. For a smooth projective variety X and a coherent sheaf E on X, we define the Mukai vector $v(E) \in H^{2*}(X; \mathbb{Q})$ as $\mathrm{ch}(E)\sqrt{\mathrm{td}(X)}$ using the Chern character and Todd class. For a $v = (v_i) \in \bigoplus H^{2i}(X; \mathbb{Q})$, we define the dual $v^\vee$ as $(-1)^i v_i$. The pairing $(v, w) = -\int_X v^\vee \cdot w$ defines a bilinear form on $\bigoplus H^{2i}(X; \mathbb{Q})$. From now on we denote by X a K3 surface. If E is a coherent sheaf on X with rank r and first and second Chern classes c_1 and c_2, we must have $v(E) = (r, c_1, c_1^2/2 - c_2 + r)$. Therefore, we can recover r, c_1 and c_2 from the Mukai vector $v(E)$. Fixing a Mukai vector fixes a Hilbert polynomial, which gives us a moduli space of sheaves. Denote this moduli space as $M(v)$.

Proposition 6.2. *The expected dimension of the stable locus M^s of $M(v)$ is $(v,v)+2$. Since the intersection form on K3 surfaces is even, the expected dimension must also be even.*

Note that the obstruction $\mathrm{Ext}^2(E,E)_0$ vanishes, because $\mathrm{Ext}^2(E,E)_0 \cong \mathrm{Hom}(E,E)_0^\vee = 0$ for any $E \in M^s$. Hence, M^s is smooth at every E. We will look at moduli spaces of dimension 0 and 2.

Theorem 6.3 (Zero-dimensional moduli spaces)**.** *Suppose $(v,v)+2=0$. If M^s is nonempty, then M consists of a single reduced point. In particular, $M^s = M$.*

Proof. Let $[E] \in M^s$ and $[F] \in M$ for semistable sheaves E and F. Since $\chi(E,F)$ depends only on the Chern classes of F, and E has the same Chern class, we have $\chi(F,E) = \chi(E,E) = -(v,v) = 2$. Due to Serre duality, we must have either $\mathrm{Hom}(F,E) \neq 0$ or $\mathrm{Hom}(E,F) \neq 0$. In both cases, it follows that $E \cong F$. □

Theorem 6.4 (Two-dimensional moduli spaces)**.** *Suppose $(v,v)+2=2$. If M^s has an irreducible component M_1 that is also a complete variety, then $M_1 = M^s = M$, i.e., M is irreducible and all semistable sheaves are stable. This implies that M is smooth and irreducible. In particular, M is a smooth surface.*

Hence, if the stable locus M^s has a nice component, the whole moduli space has very nice properties. We also have the following theorem:

Theorem 6.5 (Two-dimensional moduli spaces, again)**.** *Suppose $(v,v)+2=2$. Assume that $M^s = M$ and that M is a fine moduli space, which ensures there exists a universal family $\mathcal{E}$ over the product $M \times X$. Then M is a K3 surface, and its derived category is equivalent to that of X via the Fourier-Mukai transform $\Phi_{\mathcal{E}}$ having kernel $\mathcal{E}$.*

From this, it follows that if we find a two-dimensional moduli space of sheaves on a K3 surface, chances are that the two are not isomorphic but their derived categories are equivalent via $\Phi_{\mathcal{E}}$. In this case, we call the two Fourier-Mukai partners, and for special cases the explicit number of Fourier-Mukai partners is known. Thus, in general there is no way to recover a K3 surface using its derived category. Even the problem of determining the autoequivalence group is extremely hard, and only the case for K3 surfaces having Picard rank 1 is resolved using analytic methods. K3 surfaces can have any Picard number from 1 to 22, where 1 is the generic case.

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