COHOMOLOGY THEORIES AND POWER SERIES

HOJIN LEE

Abstract. Following the axiomatization of cohomolgy by Eilenberg and Steenrod, removing the axiom that the higher cohomology groups of a point vanishes results in what is called a generalized cohomology theory. In this talk, I will introduce examples of generalized cohomology theories that admit a definition of Chern classes, and show that they can be assigned an invariant in the form of a power series in two variables called a formal group law. Formal group laws are important in algebraic number theory, and this connection between cohomology and number theory leads to the research area of chromatic homotopy theory.

1. What is a cohomology theory?

1.1. Multiplicative cohomology theories.

Definition 1.1 (Category of CW pairs)**.** We define a category of CW pairs, denoted $\mathsf{CWPairs}$. Objects are pairs (X, A) of CW complexes where A is a subcomplex of X. The morphisms $(X, A) \to (Y, B)$ are continuous maps $f : X \to Y$ where $f(A) \subset B$. (Note that f does not need to be *cellular*, i.e., $f(A^n) \subset B^n$ for all n. Also, we do not require the CW complexes to be finite.)

Definition 1.2 (Cohomology functors and natural transformations)**.** First we define a functor $\rho : \mathsf{CWPairs} \to \mathsf{CWPairs}$ given by sending (X, A) to $(A, \varnothing)$ and $f : (X, A) \to (Y, B)$ to $f|_A : (A, \varnothing) \to (B, \varnothing)$. Using this, define the cohomology functors $E^n : \mathsf{CWPairs}^{\mathrm{op}} \to \mathsf{Ab}$ and natural transformations $\delta^n : E^n \circ \rho \Rightarrow E^{n+1}$ for all $n \in \mathbb{Z}$. We will denote $E^n(f)$ as f^* for short.

Given this data $\{E^n, \delta^n\}_{n\in\mathbb{Z}}$, we will provide the axioms that these must satisfy in order to be called a cohomology theory.

Definition 1.3 (Generalized Eilenberg-Steenrod axioms)**.** Generalized cohomology theories must satisfy the following axioms:

(1) (Exactness Axiom) For any $(X, A) \in \mathsf{CWPairs}$, we have an exact sequence

$$\cdots \to E^{n-1}(A) \xrightarrow{\delta^{n-1}} E^n(X, A) \xrightarrow{j^*} E^n(X) \xrightarrow{i^*} E^n(A) \xrightarrow{\delta^n} E^{n+1}(X, A) \to \cdots$$

where $E^*(X, \varnothing)$ is denoted as $E^*(X)$ for simplicity, and $i : (A, \varnothing) \hookrightarrow (X, \varnothing)$, $j : (X, \varnothing) \hookrightarrow (X, A)$ are inclusions.

(2) (Homotopy Axiom) If there exists a homotopy $h : (X \times I, A \times I) \to (Y, B)$ of two morphisms f and g in $\mathsf{CWPairs}$, then $f^* = g^*$ on all cohomology groups.

(3) (Excision Axiom) The inclusion $k : (A, A \cap B) \hookrightarrow (A \cup B, B)$ induces an isomorphism k^* on all cohomology groups.

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Furthermore, if $E^n(\mathrm{pt}) = 0$ for all $n \neq 0$, then E^* is said to be an ordinary cohomology theory. Without this condition, E^* is said to be a generalized cohomology theory, or an extraordinary cohomology theory if it is not ordinary. In fact, all ordinary cohomology theories are equivalent in some sense (as can be seen from de Rham and singular cohomology being isomorphic). So in general, $E^*(\mathrm{pt})$ has a structure of a graded abelian group. This is called the coefficient group of the cohomology theory E. Recall that familiar cohomology groups all had a $E^*(\mathrm{pt})$-module structure.

Example 1.4. Given a CW complex X, the continuous maps $\Delta^n \to X$ together with differentials ∂ form a chain complex of abelian groups. Taking the homology of the dual complex gives us singular cohomology. Given a smooth manifold M, the vector spaces of differential n-forms $\Omega^n(M)$ together with the exterior derivative d naturally form a cochain complex. Taking the homology gives us de Rham cohomology. Both theories qualify as ordinary cohomology theories.

Note, however, that both singular and de Rham cohomology have additional structure on the cohomology groups given by cup products and wedge products, respectively. We will also generalize this structure to a general product structure for cohomology theories.

Definition 1.5 (Multiplicative structure on cohomology)**.** Fix a generalized cohomology theory E^* (we will omit the δ). Let m and n be integers. For all $(X, A), (Y, B) \in \mathsf{CWPairs}$, suppose we have homomorphisms $\mu_{m,n} : E^m(X, A) \otimes E^n(Y, B) \to E^{m+n}(X \times Y, A \times Y \cup X \times B)$ such that

(1) $\mu_{m,n}$ are natural in (X, A) and (Y, B),
(2) and the following diagrams commute:

$$\begin{tikzcd}
E^m(A) \otimes E^n(Y,B) \arrow[r, "\delta^m \otimes 1"] \arrow[d, "\mu_{m,n}"'] & E^{m+1}(X,A) \otimes E^n(Y,B) \arrow[dd] \\
E^{m+n}(A \times Y, A \times B) \arrow[d, equal, "k^*"'] & \\
E^{m+n}(A \times Y \cup X \times B, X \times B) \arrow[r, "\delta^{m+n}"'] & E^{m+n+1}((X,A) \times (Y,B))
\end{tikzcd}$$

$$\begin{tikzcd}
E^m(X,A) \otimes E^n(B) \arrow[r, "(-1)^m \otimes \delta^n"] \arrow[d, "\mu_{m,n}"'] & E^m(X,A) \otimes E^{n+1}(Y,B) \arrow[dd, "\mu_{m,n+1}"] \\
E^{m+n}(X \times B, A \times B) \arrow[d, equal, "k^*"'] & \\
E^{m+n}(A \times Y \cup X \times B, A \times Y) \arrow[r, "\delta^{m+n}"'] & E^{m+n+1}((X,A) \times (Y,B))
\end{tikzcd}$$

We will denote the family of $\mu_{m,n}$ as $\mu : E^* \otimes E^* \to E^*$. If μ furthermore satisfies

(1) $\mu \circ (\mathrm{id}_{E^*} \otimes \mu) = \mu \circ (\mu \otimes \mathrm{id}_{E^*})$
(2) there exists $1 \in E^0(\mathrm{pt})$ such that $\mu(1 \otimes x) = \mu(x \otimes 1) = x$ for all $x \in E^n(X, A)$,

(3) $\mu = T^* \circ \mu \circ t$ where $t : E^m(X,A) \otimes E^n(Y,B) \to E^n(Y,B) \otimes E^m(X,A)$ given by $u \otimes v \mapsto (-1)^{mn} v \otimes u$ and $T : (X,A) \times (Y,B) \to (Y,B) \times (X,A)$ given by $(x,y) \mapsto (y,x)$,

then we will call E^* together with μ a multiplicative cohomology theory. This gives a graded-commutative ring structure on the cohomology groups $E^*(X)$. Throughout the rest of this talk, a *cohomology theory* will refer to a generalized cohomology theory with multiplicative structure, unless otherwise specified.

2. Topological K-theory

Now we will construct an example of a cohomology theory that is truly general, i.e., its coefficient ring has nontrivial graded structure. We will do this by considering vector bundles on a space. In general, there is a theory of k-bundles for any topological field k, but for simplicity we will focus on the $k = \mathbb{C}$ case for compact spaces.

Definition 2.1. Let X be compact Hausdorff. Consider the category of vector bundles on X, and define a free abelian group with generators the isomorphism classes $[V]$ of vector bundles on X, modulo the subgroup generated by $[V]-[E]+[F]$ for every exact sequence $0 \to V \to E \to F \to 0$ of vector bundles. Denote this abelian group as $K^0(X)$.

If X is a point, vector bundles over X are just vector spaces. Since the dimension classifies isomorphism classes of vector bundles, $K^0(X) = \mathbb{Z}$. The result is same when X is contractible. Now already, this K^0 resembles cohomology as it is contravariant and homotopy invariant. To actually construct a cohomology theory, we construct higher K^n and connecting homomorphisms. This is a bit technical, so I won't go into great detail.

Definition 2.2. Let X be compact Hausdorff. Define $\widehat{\mathrm{GL}}(C(X)) = \bigcup_n \mathrm{GL}(n, C(X))$ where $C(X) = C(X;\mathbb{C})$ and $\mathrm{GL}(n, C(X))$ is the group of $n \times n$ invertible matrices with entries in $C(X)$. Impose a relation $\sim$ on $\widehat{GL}$ by letting $S \sim \mathrm{diag}(S, 1)$ for $S \in \mathrm{GL}(n, C(X))$, for all n. Define $\mathrm{GL}(C(X))$ to be $\widehat{\mathrm{GL}}(C(X))$ modulo $\sim$. Similarly, define $\widehat{GL}(C(X))_0 = \bigcup_n \mathrm{GL}(n, C(X))_0$ where the subscript denotes the component of the identity, and let $\mathrm{GL}(C(X))_0$ be $\widehat{\mathrm{GL}}(C(X))_0$ modulo $\sim$.

It turns out that $\mathrm{GL}(C(X))_0$ is a normal subgroup of $\mathrm{GL}(C(X))$.

Definition 2.3. Let X be compact Hausdorff. Define $K^{-1}(X)$ as the quotient group $\mathrm{GL}(C(X))/\mathrm{GL}(C(X))_0$.

In fact, algebraic K-theory stems from the observation[1] that $K_0(C(X)) = K^0(X)$, and furthermore we can replace $C(X)$ with any ring R in the definition of K^{-1} to get K_1.

Example 2.4 (K^{-1} of a point)**.** For a point, we have $\mathrm{GL}(C(\mathrm{pt})) = \mathrm{GL}(\mathbb{C})$. Since $\mathrm{GL}(n, \mathbb{C})$ is connected for all n, K^{-1} of a point is trivial.

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[1]To elaborate, this is related to the Serre-Swan theorem that for a compact Hausdorff space X, the category of finitely generated projective $C(X;k)$-modules is equivalent to the category of k-vector bundles over X. In other words, the study of vector bundles can be done through algebra and vice versa.

Again, we observe that $K^{-1}(-)$ is a contravariant functor on the category of compact Hausdorff spaces. This is essentially because the functor $C(-;\mathbb{C})$ is contravariant. It is also true that K^{-1} is homotopy invariant. (Warning: we have defined the K-groups only for compact spaces, and naively extending this definition to noncompact spaces results in it failing to satisfy the Eilenberg-Steenrod axioms. For example, the correct K^{-1} for $\mathbb{R}$ is $\mathbb{Z}$, but $K^{-1}(\mathrm{pt}) = *$. We cannot argue that $K^{-1}(\mathbb{R}) = K^{-1}(\mathrm{pt})$ just because they are homotopy equivalent.) After defining K^{-1}, we can define a connecting homomorphism $\delta : K^{-1} \to K^0$, hence the index -1. However, this process is too tedious so I will leave it out.

An observation is that for every compact Hausdorff space X, the group $K^0(X)$ contains a summand of $\mathbb{Z}$ generated by the trivial bundles. This is similar to the case of reduced singular cohomology, where H^0 always contained a summand of $\mathbb{Z}$. By considering pointed spaces (X, x_0) and the inclusion $j : x_0 \hookrightarrow X$, we define the reduced K-groups to be the kernel of $j^* : K^*(X) \to K^*(x_0)$, where $* = 0$ or -1. Denote this group by $\widetilde{K}^*(X)$. Using this, we can extend our definition of K-groups to locally compact Hausdorff spaces X, where $K^*(X)$ is defined to be $\widetilde{K}^*(X^+)$ where X^+ is the one-point compactification of X, and the basepoint is taken to be the point at infinity. (If X is already compact, the one-point compactification adds a disjoint point.) Using this definition, we may define $K^{-n}(X) = K^0(X \times \mathbb{R}^n)$ for all $n > 0$ and any locally compact Hausdorff X. A striking fact is that the K^0 and K^{-1} groups are essentially the entirety of the K-groups.

Theorem 2.5 (Bott periodicity)**.** *Let X be locally compact Hausdorff. Then, $K^*(X) \cong K^{*+2}(X)$.*

Below are some computations of the complex K-theory of spaces:

Example 2.6. Let $X = \mathrm{pt}$. As we have seen above, $K^0(X) = \mathbb{Z}$ and $K^{-1}(X)$ is trivial. Thus as a graded abelian group, $K^*(X) = \bigoplus_{n\in\mathbb{Z}} \mathbb{Z}$. If we also consider the ring structure, this becomes $\mathbb{Z}[x, x^{-1}]$ for a formal variable x in degree 2. (This x is also called a *Bott element*.)

Example 2.7. Let $X = \mathbb{R}^n$. As $K^0(\mathbb{R}^n) = K^{-n}(\mathrm{pt})$, this is $\mathbb{Z}$ when n is even, and trivial when n is odd. Also $K^{-1}(\mathbb{R}^n) = K^0(\mathbb{R}^{n+1})$, this is trivial when n is even and $\mathbb{Z}$ when n is odd.

Example 2.8. Let $X = S^n$. As this is the one-point compactification of $\mathbb{R}^n$, its reduced K-theory is the K-theory of $\mathbb{R}^n$. Therefore, $K^0(S^n)$ is $\mathbb{Z}^2$ when n is even, and $\mathbb{Z}$ when n is odd. Similarly, $K^{-1}(S^n)$ is trivial when n is even, and $\mathbb{Z}$ when n is odd.

In the case above, using the fact that $\mathbb{CP}^1$ is homeomorphic to S^2 we can explicitly find what a generator of $K^0(\mathbb{CP}^1)$ looks like. The result is that the class $[\mathcal{O}(-1)]$ together with the trivial generator $[\mathcal{O}]$ generate the K^0 as an abelian group.

3. Complex orientation

For even better algebraic properties, we will restrict our attention to cohomology theories that are *complex oriented*. This condition is not restrictive though; both singular cohomology and complex K-theory are complex oriented. These cohomology theories have a nice theory of Chern classes, which turn out to be extremely interesting and connects algebraic topology with number theory in a surprising way.

We first review Chern classes in the context of complex geometry. If X is a complex manifold, one can consider complex line bundles L together with a hermitian metric h on its fibers. Using this data, we can define a $(1,1)$-form $\omega_{(L,h)}$ on X, which defines a class $[\omega_{(L,h)}] \in H^2(X, \mathbb{C})$ in singular cohomology. The form depends on the choice of hermitian metric h, but the class itself is independent of any choice of hermitian structure. Hence we may write $[\omega_L]$ without ambiguity. We define this class to be the first Chern class $c_1(L)$ of a line bundle L on X.

Proposition 3.1. *Let X be a complex manifold and L_1, L_2 be complex line bundles on X. Then $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$.*

Thus the (first) Chern class sends tensor products to addition of classes. It turns out that we can construct a theory of Chern classes in the context of generalized cohomology theories having an additional structure, called *complex orientation*. We will not go into great detail of this definition, and it is enough to think of this condition as something ensuring a nice definition of Chern classes.

Definition 3.2 (Complex oriented cohomology theories)**.** A cohomology theory E is complex orientable if the morphism

$$i^* : E^2(BU(1)) \to E^2(S^2)$$

is surjective, where $i : S^2 \to BU(1)$ is a representative of $1 \in \pi_2(BU(1)) = \mathbb{Z}$. It turns out that this is equivalent to the corresponding morphism in reduced cohomology to be surjective.

A complex orientation on E is an element c_1^E in the second reduced cohomology of $BU(1)$, such that the pullback via i is the multiplicative identity of the ring $E(\mathrm{pt})$. This is also called the *first Chern class*.

Example 3.3. Recall the case of ordinary cohomology. Complex line bundles L on X are classified by maps $\varphi : X \to BU(1)$, and the pullback of the universal class via this map is the first Chern class of L.

In fact, there are suitable definitions of c_i^E for $i > 1$ on the cohomology of $BU(n)$, so pulling back these classes result in the definition of i-th Chern classes for all i, up to the rank n of the vector bundle. By a not-so-hard argument, we can also find out that the tensor products of line bundles arise as a power series of individual Chern classes.

Theorem 3.4. *Given a complex oriented cohomology theory E, there exists a power series $f \in E^*(\mathrm{pt})[[u, v]]$ in two variables such that for any two line bundles L_1, L_2 on X, we have*

$$c_1(L_1 \otimes L_2) = f(c_1(L_1), c_1(L_2)).$$

Corollary 3.5. *As tensor products are associative and commutative, it follows that such f must satisfy*

(1) $f(0, v) = v$,
(2) $f(u, v) = f(v, u)$,
(3) $f(u, f(v, w)) = f(f(u, v), w)$.

In other words, f is a formal group law over $E^*(\mathrm{pt})$.

Example 3.6. As Chern classes for ordinary cohomology was additive, we have $f = u + v$. However this is not the case for complex K-theory. By definition, (all) Chern classes are in even degree cohomology, so in the case of K-theory we may consider Chern classes to be sitting in K^0. Recall that K^0 is the group of classes of vector bundles on the space, thus Chern classes have a simple description. In fact, if L is a complex line bundle on X, its first Chern class is $[L]-[\mathcal{O}]$. Using this, we try to calculate $c_1(L_1 \otimes L_2)$. It suffices to calculate the class $[L_1 \otimes L_2]$, which is $[L_1][L_2]$ by definition of the multiplicative structure on K-theory (it is derived from $\otimes$ on K^0). Thus, we have $c_1(L_1 \otimes L_2) = [L_1][L_2] - [\mathcal{O}] = (c_1(L_1) + [\mathcal{O}])(c_1(L_2) + [\mathcal{O}]) - [\mathcal{O}] = u + v + uv$ where u and v are the respective first Chern classes. Hence the formal group law for K-theory is $u + v + uv$.

4. Complex cobordism

Compare the Eilenberg-Maclane spectra $K(G, n)$ of a group G, and the sphere spectrum $\mathbb{S}$. The former has a nice homotopy theory (by definition), but does not behave well homologically. The latter is well-behaved with respect to homology, but its homotopy groups are the stable homotopy groups of spheres. It turns out that a cohomology theory called *complex cobordism* bridges these two cases. Not only that, it turns out that complex cobordism is universal among complex oriented cohomology theories in the fact that it is initial. I will not give the full detailed definition.

Definition 4.1 (Complex cobordism)**.** For X a smooth manifold, let $\Omega^k(X)$ denote the set of cobordism classes of proper complex oriented maps $Y \to X$ of codimension k. The notion of maps being cobordant is a suitable generalization of the notion of cobordant manifolds, where submanifolds are generalized to maps.

With additional structure, complex cobordism becomes a complex oriented cohomology theory. The following property is what makes them interesting:

Proposition 4.2. *Let E^* be any complex oriented cohomology theory. There exists a unique map $\Omega^* \to E^*$ that sends elements of $\Omega^*(X)$ represented by $f : Y \to X$ to the fundamental class $[Y]$ in $E^*(X)$. In other words, Ω^* is initial among all such theories.*

As complex cobordism is universal among complex oriented cohomology theories, we may expect its formal group law to have some universal property among other formal group laws. This is indeed true; its formal group law turns out to be defined over what is called a Lazard ring, and this is a famous theorem of Quillen. (Quillen showed that the coefficient ring of Ω^* is $\mathbb{L}$.)

Definition 4.3 (Lazard ring)**.** Consider the polynomial ring $R = \mathbb{Z}[\{a_{ij}\}_{i,j\in\mathbb{N}}]$ on countably many variables, and consider the ideal I generated by the relations among a_{ij} such that $f := \sum_{i,j} a_{ij} u^i v^j$ becomes a formal group law for the variables u and v. The quotient ring $\mathbb{L} = R/I$ is called the Lazard ring. The formal group law f over $\mathbb{L}$ is universal in the sense that for any other ring R and a formal group law g over it, there exists a unique homomorphism $\varphi : \mathbb{L} \to R$ such that $g = \varphi(f)$.

Theorem 4.4 (Lazard)**.** $\mathbb{L} \cong \mathbb{Z}[\{x_i\}_{i\in\mathbb{N}}]$.

5. Further topics: chromatic homotopy theory

By results above, the (1-dimensional) formal group law of a (complex oriented) cohomology theory serves as an algebraic invariant of the cohomology theory itself. As such, we may try to classify all cohomology theories using properties of formal groups. Although it is not a complete invariant, there is a notion of the height of a formal group law, which induces a filtration of cohomology theories. This is called the *chromatic filtration*. Here is a table of (known) complex oriented cohomology theories, sorted by its height (also called chromatic level).

Chromatic filtration	E^*	coefficient ring for fixed p
0	ordinary cohomology	any abelian group
1	topological K-theory	$\mathbb{Z}[t, t^{-1}]$ where $\deg(t) = 2$
2	elliptic cohomology	varies
n	n-th Morava K-theory $K(n)$	$\mathbb{F}_p[v_n, v_n^{-1}]$ where $\deg(v_n) = 2(p^n - 1)$
∞	complex cobordism	$\mathbb{L}$

In more technical terms, *chromatic homotopy theory* is the study of complex-oriented cohomology theories by viewing them as sheaves over the moduli stack $\mathcal{M}_{\text{fg}}$ of (1-dimensional) formal group laws. There is a method of constructing cohomology theories from formal group laws, called the Landweber exact functor theorem. Using this, we can create a lot of cohomology theories, including elliptic cohomology whose formal group law comes from elliptic curves, and Morava K-theory comes from directly generalizing topological K-theory, hence the similar coefficient ring. In the table above, you can see that $K(n)$ also has some form of Bott periodicity going on, with period $2(p^n - 1)$. The name 'chromatic' comes from the spectrum decomposition of electromagnetic waves; they are waves (hence periodic) and can be classified by their periods. Of course this is not a mathematically precise analogy.

One may wonder about the relations of such complex oriented cohomology theory, defined in the context of CW complexes, has with the cohomology theories appearing in algebraic geometry. In general, algebraic varieties cannot be interpreted as a topological space in the geometric sense, so the cohomology theories defined on them differ drastically from the ones in this talk. However, there are indeed formal similarities, where the Weil cohomology theories (nontrivial examples include ℓ-adic cohomology and crystalline cohomology) are analogous to the *ordinary* cohomology theories defined in the CW sense. Cohomology theories for varieties analogous to the extraordinary cohomology theories are mostly conjectural as far as I know.