# MAXIMAL FUCHSIAN SUBGROUPS OF THE $d = 2$ BIANCHI GROUP

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Abstract. Let $\Gamma$ denote the $d = 2$ Bianchi group $\mathrm{PSL}(2, \mathbb{Z}[\sqrt{-2}])$. We give an explicit description of all conjugacy classes of maximal nonelementary Fuchsian subgroups of $\Gamma$ as integral orders of certain indefinite quaternion algebras over $\mathbb{Q}$. Using this description, we also provide the covolumes corresponding to each conjugacy class. As an application, we compute the limit $\lim_{x\to\infty} \frac{\Pi(x)}{x}$ where $\Pi(x)$ counts the number of primitive totally geodesic immersed surfaces in the manifold $\Gamma\backslash\mathbb{H}^3$ with area less than $x$.

## 1. Introduction

We identify the ideal points $\partial\mathbb{H}^3$ of hyperbolic 3-space $\mathbb{H}^3$ with the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Under this identification, the action of $\mathrm{PSL}(2, \mathbb{C})$ on the boundary $\mathbb{C}$ can be described as linear fractional transformations. Let $\mathcal{O}_2$ denote the ring $\mathbb{Z}[\sqrt{-2}]$ and let $\Gamma$ be the $d = 2$ Bianchi group $\mathrm{PSL}(2, \mathcal{O}_2)$. To avoid cumbersome repetition, we will call a

$$\text{maximal nonelementary Fuchsian subgroup of } \Gamma$$

an *$F$-subgroup*. In this paper, we provide an explicit description of the conjugacy classes of $F$-subgroups of $\Gamma$ as $\mathbb{Z}$-orders in certain indefinite quaternion algebras, and through this, we provide the volume formulae corresponding to each $F$-subgroup.

Such an explicit description is possible because up to conjugacy, every $F$-subgroup can be described as a subgroup of $\Gamma$ which, through the action described above, stabilizes a unique circle $\mathcal{C}$ in $\mathbb{C}$ whose explicit equation is known. The equation of such a circle can be represented uniquely in a *reduced form*

$$A|z|^2 + B\overline{z} + \overline{B}z + C = 0,$$

where $A, C \in \mathbb{Z}$ and $B \in \mathcal{O}_2$, and furthermore

$$\gcd(A, \mathrm{Re}(B), \mathrm{Im}(B), C) = 1.$$

For a reduced form, we define its discriminant $D$ to be $B\overline{B} - AC$, which should be a positive integer. The explicit equations defining the circles $\mathcal{C}$ corresponding to each $\mathrm{PGL}(2, \mathcal{O}_2)$-conjugacy class of $F$-subgroups can be found using the results of [Vul91].

On the other hand, results of [JM96] provide the exact number $n_2(D)$ of the $\Gamma$-conjugacy classes of $F$-subgroups whose corresponding circles $\mathcal{C}$ have discriminant $D$, and these numbers depend only on the congruence class of $D$ modulo some fixed integer. Note that $n_2(D) < \infty$ is ensured by [MR91]. Since these numbers agree with the number of formulae found above, all conjugacy classes of $F$-subgroups are fixed under the $\mathrm{PGL}(2, \mathcal{O}_2)$-action.

*Remark* 1. As mentioned in [JM96], such an agreement may not occur for other Bianchi groups in general.

Using these explicit equations, we then find a description of their corresponding $F$-subgroups as $\mathbb{Z}$-orders in quaternion algebras. Let

$$Q = (-2, D)_{\mathbb{Q}}$$

be the indefinite quaternion algebra over $\mathbb{Q}$ with standard basis $1, i, j, ij$ where $i^2 = -2$, $j^2 = D$ and $ij = -ji$. Here, $D \in \mathbb{Z}_{>0}$ is the discriminant of the $\mathcal{C}$ corresponding to the $F$-subgroup we will realize. It is possible to find a $\mathbb{Z}$-order $\mathcal{M}$ in $Q$ with respect to some matrix representation

$$\rho : Q \otimes \mathbb{R} \to \mathrm{M}_2(\mathbb{C})$$

such that $\rho(\mathcal{M}^1)$ becomes the $F$-subgroup we wanted to represent. We denote as $\mathcal{M}^1$ the group of elements of reduced norm 1 in $\mathcal{M}$.

**Theorem 2.** *Let $D$ be the discriminant of the equation of the circle corresponding to an $F$-subgroup. Then each conjugacy class of $F$-subgroups can be represented as the group of reduced norm 1 elements of the following $\mathbb{Z}$-orders in $Q$:*

(1) $\mathbb{Z}[1, i, j, ij]$ *for any positive integer $D$,*
(2) $\mathbb{Z}[1, i, \frac{1+j}{2}, \frac{i+ij}{2}]$ *whenever* $D \equiv 1 \mod 4$,
(3) $\mathbb{Z}[1, i, \frac{3i+j}{4}, \frac{2+ij}{4}]$ *whenever* $D \equiv 2 \mod 16$,
(4) $\mathbb{Z}[1, i, \frac{i+j+ij}{4}, \frac{2+2i+ij}{4}]$ *whenever* $D \equiv 6 \mod 16$,
(5) $\mathbb{Z}[1, i, \frac{i+j}{2}, \frac{ij}{2}]$ *whenever* $D \equiv 2 \mod 4$,
(6) $\mathbb{Z}[1, i, \frac{1+j+ij}{2}, \frac{i+ij}{2}]$ *whenever* $D \equiv 3 \mod 4$.

Furthermore, this explicit description allows us to compute the covolumes of the corresponding $F$-subgroups, which are given below. Throughout the paper, the symbol $\left(\frac{-2}{p}\right)$ is defined to be zero for $p = 2$, and will denote the Legendre symbol for $p > 2$.

**Theorem 3.** *Let $\mathcal{M}_{(i)}$ be the $(i)$-th $\mathbb{Z}$-order in Theorem 2. For simplicity, let $F(D)$ denote $D \prod_{p|D} \left(1 + \left(\frac{-2}{p}\right) p^{-1}\right)$ for positive integers $D$. Then the covolumes of the groups $\mathcal{M}^1_{(i)}/\{\pm 1\} \subset \mathrm{PSL}(2, \mathbb{R})$ are given as $c\pi F(D)$, where the values of $c$ in each case are given as follows:*

$\mathcal{M}_{(1)}$. $c = 1$ *when* $D \equiv 0 \mod 8$ *and* $c = 2$ *otherwise,*
$\mathcal{M}_{(2)}$. $c = 1$ *when* $D \equiv 1 \mod 8$ *and* $c = \frac{1}{3}$ *when* $D \equiv 5 \mod 8$,
$\mathcal{M}_{(3)}, \mathcal{M}_{(4)}$. $c = \frac{1}{6}$ *regardless of the congruence class of $D$,*
$\mathcal{M}_{(5)}$. $c = \frac{1}{2}$,
$\mathcal{M}_{(6)}$. $c = 1$ *when* $D \equiv 3 \mod 8$, *and* $c = \frac{1}{3}$ *when* $D \equiv 7 \mod 8$.

*Remark* 4. Let $K = \mathbb{Q}(\sqrt{-d})$ for positive squarefree $d$. The $\mathbb{Z}$-orders and corresponding covolumes of maximal nonelementary Fuchsian subgroups of $\mathrm{PSL}(2, \mathcal{O}_K)$ are given in [MR91] for the $d = 1$ case, and in [Jun19] for $d \equiv 3 \mod 4$ under the assumption that the ideal class group of $K$ does not contain any element of order 4. Furthermore a prime geodesic theorem for such $d$ has been given in [Jun19]. However, the precise methods of [Jun19] except Lemma 11 cannot be extended to the $d = 2$ case, as there are orders with basis elements having denominators a higher power of 2 (namely, $\mathcal{M}_{(3)}$ and $\mathcal{M}_{(4)}$). We also give a detailed explanation on the 2-adic reduced norm group calculation in Section 3.2 which complements the volume formula discussion of both [MR91] and [Jun19].

A simple application of the results of Theorem 3 and Lemma 11 yields an asymptotic formula for the growth of primitive immersed totally geodesic surfaces in the hyperbolic 3-manifold $\Gamma\backslash\mathbb{H}^3$.

**Theorem 5.** *Let $\Pi(x)$ denote the number of primitive immersed totally geodesic surfaces in $\Gamma\backslash\mathbb{H}^3$ with area less than $x$. Then,*

$$\lim_{x\to\infty} \frac{\Pi(x)}{x} = \frac{45}{16\pi} \cdot \prod_{p>2} \left( 1 - \frac{1}{p} + \frac{1}{p + \left(\frac{-2}{p}\right)} \right) \tag{1.1}$$

*where $\left(\frac{-2}{p}\right)$ is the Legendre symbol.*

Together with the results of [Jun19] and [MR91], we have obtained a list of precise prime geodesic theorems for surfaces in the $d = 1, 2$ and $d \equiv 3 \mod 4$ Bianchi orbifolds under the assumption that the ideal class group of $\mathbb{Q}(\sqrt{-d})$ contains no order 4 element.

*Remark* 6. The most general classification result for Bianchi groups, to date, is in [Vul17] which assumes that the class group contains no order 4 element. Furthermore, for $d \not\equiv 3 \mod 4$, one can see for example through the results of [JM96] that already the $d = 5$ case can have up to 9 conjugacy classes having the same discriminant, and moreover the congruence condition on the discriminant is given modulo 200. Hence, even for small $d$ it may be impractical to find the $\mathbb{Z}$-orders and covolumes using the methods of this paper.

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## 2. Obtaining the Integral Orders

We start with a list of explicit formulae corresponding to $\mathrm{PGL}(2, \mathcal{O}_2)$-conjugacy classes of $F$-subgroups. This is a direct consequence of the classification result in [Vul91].

**Proposition 7** ([Vul91], Theorem 3)**.** *Each $\mathrm{PGL}(2, \mathcal{O}_2)$-conjugacy class of $F$-subgroups of $\Gamma$ is represented uniquely by the group $\mathrm{Stab}(f_{k,c}(z), \Gamma)$ with $c \in \mathbb{Z}$, where the $f_{k,c}(z)$ are given as $(k)$:*

*(1)* $|z|^2 + c$, *(2)* $2|z|^2 + \overline{z} + z + 2c$,

*(3)* $4|z|^2 - \sqrt{-2}\overline{z} + \sqrt{-2}z + 4c$, *(4)* $4|z|^2 + (2 - \sqrt{-2})\overline{z} + (2 + \sqrt{-2})z + 4c$,

*(5)* $2|z|^2 - \sqrt{-2}\overline{z} + \sqrt{-2}z + 2c$, *(6)* $2|z|^2 + (1 - \sqrt{-2})\overline{z} + (1 + \sqrt{-2})z + 2c$,

*all presented in reduced form. Here,* $\mathrm{Stab}(f, \Gamma)$ *is the subgroup of* $\Gamma$ *whose elements stabilize the circle* $\mathcal{C}$ *defined by* $f$*.*

Observe that the discriminant $D$ in each case is $-c$, $1-4c$, $2-16c$, $6-16c$, $2-4c$, and $3-4c$, respectively. Furthermore from the results of [JM96], one can calculate that the numbers of $\Gamma$-conjugacy classes $n_2(D)$ are given as

$$n_2(D) = \begin{cases} 1 & \text{if } D \equiv 0 \bmod 4, \\ 3 & \text{if } D \equiv 2 \text{ or } 6 \bmod 16, \\ 2 & \text{otherwise,} \end{cases}$$

and these numbers agree with the number of formulae in 7 having corresponding discriminants. To find an explicit description of the $\mathbb{Z}$-orders associated to the stabilizers of each of the six equations, we start with the simplest case of $|z|^2 + c$ and use computations in this case to obtain a description of other cases.

**Lemma 8.** *Let* $Q$ *denote the quaternion algebra* $(-2, D)_{\mathbb{Q}}$ *with standard basis* $1$, $i$, $j$, $ij$*. Define a matrix embedding* $\rho : Q \otimes \mathbb{R} \to \mathrm{M}_2(\mathbb{C})$ *of* $Q$ *by*

$$i \mapsto \begin{bmatrix} \sqrt{-2} & 0 \\ 0 & -\sqrt{-2} \end{bmatrix}, \quad j \mapsto \begin{bmatrix} 0 & D \\ 1 & 0 \end{bmatrix}, \quad ij \mapsto \begin{bmatrix} 0 & D\sqrt{-2} \\ -\sqrt{-2} & 0 \end{bmatrix}.$$

*Let* $\mathcal{M}$ *be the* $\mathbb{Z}$*-order* $\mathbb{Z}[1, i, j, ij]$ *inside* $Q$*. Then, the group* $P\rho(\mathcal{M}^1) := \rho(\mathcal{M}^1)/\{\pm 1\} \subset \Gamma$ *consists precisely of elements whose action stabilizes the circle* $|z|^2 - D = 0$*.*

*Proof.* First notice that elements of the form

$$\begin{bmatrix} a & Db \\ \bar{b} & \bar{a} \end{bmatrix} \in \mathrm{PSL}(2, \mathbb{C})$$

fix the circle $|z|^2 - D = 0$. More explicitly, $z$ is sent to

$$z' := \frac{az + Db}{\bar{b}z + \bar{a}}$$

and it is straightforward to verify that $|z'|^2 - D = 0$. Given an element $\alpha := t + xi + yj + zij$ in $\mathcal{M}$, we have

$$\rho(\alpha) = \begin{bmatrix} t + x\sqrt{-2} & Dy + Dz\sqrt{-2} \\ y - z\sqrt{-2} & t - x\sqrt{-2} \end{bmatrix}$$

and observe that the image of reduced norm 1 elements are precisely inside $\mathrm{PSL}(2, \mathbb{C})$. Furthermore as $t, x, y, z \in \mathbb{Z}$, we conclude that

$$P\rho(\mathcal{M}^1) = \mathrm{Stab}(|z|^2 - D, \Gamma).$$

□

Next, we find some $T \in \mathrm{PSL}(2, \mathbb{C})$ whose action on $\partial \mathbb{H}^3$ sends the circle $f_{1,c}$ to $f_{k,c}$. This allows us to easily describe the stabilizer groups using the fact that

$$\mathrm{Stab}(f_{k,c}, \mathrm{PSL}(2, \mathbb{C})) = T^{-1} \cdot \mathrm{Stab}(f_{1,c}, \mathrm{PSL}(2, \mathbb{C})) \cdot T,$$

and that taking the intersection with $\Gamma$ yields $\mathrm{Stab}(f_{k,c}, \Gamma)$. Notice that this group is precisely $\rho(\mathcal{M}^1_{(k)})/\{\pm 1\}$ according to our notation.

**Lemma 9.** *Let $A|z|^2 + B\overline{z} + \overline{B}z + C = 0$ be in reduced form. Then the element*

$$T = \begin{bmatrix} \sqrt{A} & \frac{B}{\sqrt{A}} \\ 0 & \frac{1}{\sqrt{A}} \end{bmatrix} \in \mathrm{PSL}(2, \mathbb{C})$$

*sends the circle to the one defined by $|z|^2 = D$, where $D$ is the discriminant of the reduced form.*

*Proof.* By direct calculation, verify that $(Az + B)(A\overline{z} + \overline{B}) - D = 0$. □

The embedding $\rho$, as defined in Lemma 8, sends reduced norm 1 elements of the form $t + xi + yj + zij$ for $t, x, y, z \in \mathbb{R}$ to elements of the group $\mathrm{Stab}(f_{1,c}, \mathrm{PSL}(2, \mathbb{C}))$. Thus we may conjugate the matrix representation $\rho$ to obtain another matrix representation $\rho' = T^{-1}\rho T$ of $Q$, and this gives rise to $\mathrm{Stab}(f_{k,c}, \mathrm{PSL}(2, \mathbb{C}))$. Explicitly, $\rho'$ is given as follows:

$$\rho'(i) = \begin{bmatrix} \sqrt{-2} & \frac{2\sqrt{-2}B}{A} \\ 0 & -\sqrt{-2} \end{bmatrix}, \quad \rho'(j) = \begin{bmatrix} -B & \frac{D-B^2}{A} \\ A & B \end{bmatrix}, \quad \rho'(ij) = \begin{bmatrix} B\sqrt{-2} & \frac{D+B^2}{A}\sqrt{-2} \\ -A\sqrt{-2} & -B\sqrt{-2} \end{bmatrix}.$$

To find the $\mathbb{Z}$-order of $Q$ corresponding to $\mathrm{Stab}(f_{i,c}, \Gamma)$, it suffices to find the conditions on $t$, $x$, $y$, $z$ such that the matrix

$$\rho'(t + xi + yj + zij)$$

has entries in $\mathcal{O}_2$. As $A$ appears in the denominator and hence affects the integrality conditions, we will split cases according to when $A = 2$, and when $A = 4$.

2.1. **The orders $\mathcal{M}_{(2)}$, $\mathcal{M}_{(5)}$, $\mathcal{M}_{(6)}$.** The element $t + xi + yj + zij$ via $\rho'$ is sent to

$$\begin{bmatrix} t + x\sqrt{-2} - yB + zB\sqrt{-2} & xB\sqrt{-2} + y\frac{D-B^2}{2} + z\frac{D+B^2}{2}\sqrt{-2} \\ 2y - 2z\sqrt{-2} & t - x\sqrt{-2} + yB - z\sqrt{-2}B \end{bmatrix}$$

where we calculate each case for $B = 1$, $-\sqrt{-2}$ and $1 - \sqrt{-2}$. Each case yields the matrices

$$\begin{bmatrix} t - y + (x + z)\sqrt{-2} & y\frac{D-1}{2} + (x + \frac{D+1}{2}z)\sqrt{2} \\ 2y - 2z\sqrt{-2} & t + y - (x + z)\sqrt{-2} \end{bmatrix}$$

for $B = 1$,

$$\begin{bmatrix} t + 2z + (x + y)\sqrt{-2} & 2x + y\frac{D+2}{2} + z(\frac{D-2}{2})\sqrt{-2} \\ 2y - 2z\sqrt{-2} & t - 2z - (x + y)\sqrt{-2} \end{bmatrix}$$

for $B = -\sqrt{-2}$ and

$$\begin{bmatrix} (t - y + 2z) + (x + y + z)\sqrt{-2} & 2x + y\frac{D+1}{2} + 2z + (x + y + z\frac{D-1}{2})\sqrt{-2} \\ 2y - 2z\sqrt{-2} & t + y - 2z - (x + y + z)\sqrt{-2} \end{bmatrix}$$

for $B = 1 - 2\sqrt{-2}$, respectively. It is immediate that in all three case we must have $t, x, y, z \in \frac{1}{2}\mathbb{Z}$, whence we may rewrite $t + xi + yj + zij$ as

$$\frac{1}{2}(a + bi + cj + dij)$$

for $a, b, c, d \in \mathbb{Z}$.

For the first case $\mathcal{M}_{(2)}$, we obtain the two integral relations

$$a + c \equiv b + d \equiv 0 \mod 2$$

and hence the order has a $\mathbb{Z}$-basis consisting of the elements

$$\frac{1+j}{2},\ \ \frac{i}{2},\ \ j,\ \ ij.$$

This can be seen immediately by replacing each $a, b, c, d$ by integer parameters. Through identical methods, in the case $\mathcal{M}_{(5)}$ we obtain the relations

$$a \equiv b + c \equiv 0 \mod 2$$

and for $\mathcal{M}_{(6)}$ we have

$$a + c \equiv b + c + d \equiv 0 \mod 2,$$

so in each case we have $\mathbb{Z}$-bases consisting of the elements

$$1,\ \ \frac{i+j}{2},\ \ j,\ \ \frac{ij}{2}$$

and

$$\frac{1+j+ij}{2},\ \ \frac{i+ij}{2},\ \ j,\ \ ij,$$

respectively. Following the convention of [MR91] which includes 1 and $i$ as generating elements of the $\mathbb{Z}$-orders, we obtain

$$\mathbb{Z}\left[1, i, \frac{1+j}{2}, \frac{i+ij}{2}\right], \quad \mathbb{Z}\left[1, i, \frac{i+j}{2}, \frac{ij}{2}\right], \quad \mathbb{Z}\left[1, i, \frac{1+j+ij}{2}, \frac{i+ij}{2}\right] \tag{2.1}$$

as explicit descriptions of $\mathcal{M}_{(2)}$, $\mathcal{M}_{(5)}$, and $\mathcal{M}_{(6)}$, respectively.

2.2. **The orders $\mathcal{M}_{(3)}$, $\mathcal{M}_{(4)}$.** For $\mathcal{M}_{(3)}$, the element $t + xi + yj + zij$ is sent to

$$\begin{bmatrix} t + 2z + (x+y)\sqrt{-2} & x + y\frac{D+2}{4} + z\frac{D-2}{4}\sqrt{-2} \\ 4y - 4z\sqrt{-2} & t - 2z - (x+y)\sqrt{-2} \end{bmatrix}$$

and we obtain $t \in \frac{1}{2}\mathbb{Z}$ and $x, y, z \in \frac{1}{4}\mathbb{Z}$, keeping in mind the congruence relations of $D$ enforced by the equations in 7. Writing $t + xi + yj + zij$ as

$$\frac{a}{2} + \frac{1}{4}(bi + cj + dij)$$

for $a, b, c, d \in \mathbb{Z}$, we obtain the integral relations

$$b + c \equiv 0 \mod 4$$

and

$$a + d \equiv 0 \mod 2,$$

which leads to a $\mathbb{Z}$-basis consisting of the elements

$$\frac{2+ij}{4},\quad \frac{ij}{2},\quad \frac{i+3j}{4},\quad j.$$

By a change of $\mathbb{Z}$-basis we obtain an explicit description

$$\mathcal{M}_{(3)} = \mathbb{Z}\left[1, i, \frac{3i+j}{4}, \frac{2+ij}{4}\right]. \tag{2.2}$$

For $\mathcal{M}_{(4)}$, we obtain

$$\begin{bmatrix} t-2y+2z+(x+y+2z)\sqrt{-2} & x+\frac{D-2}{4}y+2z+(x+y+\frac{D+2}{4}z)\sqrt{-2} \\ 4y-4z\sqrt{-2} & t+2y-2z-(x+y+2z)\sqrt{-2} \end{bmatrix}$$

from which follows the integral relations

$$a+c+d \equiv 0 \mod 2,$$

and

$$b+c+2d \equiv 0 \mod 4,$$

following the convention of the case $\mathcal{M}_{(3)}$. This leads to a $\mathbb{Z}$-basis consisting of the elements

$$\frac{2+2i+ij}{4},\quad \frac{i+j+ij}{4},\quad \frac{ij}{2},\quad i$$

and after a change of basis, leads to the explicit description

$$\mathcal{M}_{(4)} = \mathbb{Z}\left[1, i, \frac{i+j+ij}{4}, \frac{2+2i+ij}{4}\right]. \tag{2.3}$$

## 3. Volume Formulae

Let $N(\mathcal{M})$ denote the absolute value of the reduced discriminant of the order $\mathcal{M}$. From [Voi21b] we obtain a volume formula for any $\mathbb{Z}$-order $\mathcal{M}$ in $Q$ as

$$\mathrm{vol}(P\rho(\mathcal{M}^1)\backslash\mathbb{H}^2) = \frac{\pi N(\mathcal{M})}{3}\prod_{p\mid N(\mathcal{M})}\lambda(\mathcal{M},p)\cdot[\mathbb{Z}_p^\times : \mathrm{nrd}(\mathcal{M}_p^\times)]^{-1} \tag{3.1}$$

where $P\rho(\mathcal{M}^1)$ is understood to be in $\mathrm{PSL}(2,\mathbb{R})$, and the symbol $\lambda(\mathcal{M},p)$ is defined to be

$$\lambda(\mathcal{M},p) = \frac{1-p^{-2}}{1-\left(\frac{\mathcal{M}}{p}\right)p^{-1}}$$

where $\left(\frac{\mathcal{M}}{p}\right)$ is the Eichler symbol of $\mathcal{M}$ at $p$. Let $\mathcal{O}$ be the $\mathbb{Z}$-order $\mathbb{Z}[1,i,j,ij]$ in $Q$. By standard computations, the absolute value of the reduced discriminant of $\mathcal{O}$ is $8D$. Using the relation

$$N(\mathcal{O}) = [\mathcal{M}:\mathcal{O}]\cdot N(\mathcal{M}),$$

we obtain the reduced discriminants of the remaining cases $\mathcal{M}$ of Theorem 2. For the orders (2), (5), and (6), we have $[\mathcal{M}:\mathcal{O}] = 4$ and hence $N(\mathcal{M}) = 2D$. For (3) and (4) we have $[\mathcal{M}:\mathcal{O}] = 16$, and hence $N(\mathcal{M}) = \frac{D}{2}$.

Observe that for $p \neq 2$, the $\mathbb{Z}_p$-order $\mathcal{M}_p$ for orders $\mathcal{M}$ of Theorem 2 are equal to $\mathcal{O}_p$ as all denominators that occur are powers of 2 and hence invertible in such $\mathbb{Z}_p$. Let $\alpha = t + xi + yj + zij \in \mathcal{O}_p$. As

$$\mathrm{nrd}(\alpha) \equiv t^2 + 2x^2 \mod p,$$

for any $p \neq 2$ we may always find $\alpha$ such that $\mathrm{nrd}(\alpha) \notin (\mathbb{Z}_p^\times)^2$. To see this, consider the cases where 2 is and is not a quadratic residue modulo $p$, separately. If 2 is a quadratic residue modulo $p$, the expression is equivalent to a sum of two squares, and modulo $p$ this attains all possible values. If 2 is not a quadratic residue modulo $p$, the value 2 itself would suffice.

Together with the inclusion of the groups

$$(\mathbb{Z}_p^\times)^2 \subset \mathrm{nrd}(\mathcal{O}_p^\times) \subset \mathbb{Z}_p^\times,$$

and the fact that for $p \neq 2$ the subgroup $(\mathbb{Z}_p^\times)^2$ of squares in $\mathbb{Z}_p^\times$ is of index 2, we conclude that

$$[\mathbb{Z}_p^\times : \mathrm{nrd}(\mathcal{M}_p^\times)] = 1$$

for every $p \neq 2$.

A simplification can be obtained also for the Eichler symbols at $p \neq 2$. As $\mathcal{M}_p = \mathcal{O}_p$ we have

$$\left(\frac{\mathcal{M}}{p}\right) = \left(\frac{\mathcal{O}}{p}\right),$$

and furthermore by [Voi21a] the value of $\left(\frac{\mathcal{O}}{p}\right)$ can be obtained by computing the Kronecker symbol

$$\left(\frac{\Delta(\alpha)}{p}\right)$$

for $\alpha \in \mathcal{O}$ where

$$\Delta(\alpha) := \mathrm{trd}(\alpha)^2 - 4\,\mathrm{nrd}(\alpha).$$

Namely, for a placeholder $\epsilon \in \{-1, 0, 1\}$, the Eichler symbol of an order $\mathcal{M}$ is

$$\left(\frac{\mathcal{M}}{p}\right) = \epsilon \text{ if and only if } \left(\frac{\Delta(\alpha)}{p}\right) = 0 \text{ or } \epsilon$$

for every $\alpha \in \mathcal{M}$. As in our case

$$\Delta(\alpha) = -8x^2 + 4Dy^2 + 8Dz^2,$$

and since we are looking at primes $p \neq 2$ that divide $D$, the corresponding Kronecker symbol turns out to be

$$\left(\frac{-2x^2}{p}\right) = 0 \text{ or } \left(\frac{-2}{p}\right)$$

depending on the value of $x$ modulo $p$. Therefore we may replace the Eichler symbol $\left(\frac{\mathcal{M}}{p}\right)$ with the Legendre symbol $\left(\frac{-2}{p}\right)$ in the case $p \neq 2$, and the expression of $\lambda(\mathcal{M}, p)$ simplifies into

$$\lambda(\mathcal{M}, p) = 1 + \left(\frac{-2}{p}\right) p^{-1}$$

by cancelling out the term $1 - \left(\frac{-2}{p}\right) p^{-1}$. In summary, Equation 3.1 simplifies into

$$\mathrm{vol}(P\rho(\mathcal{M}^1)\backslash\mathbb{H}^2) = \frac{\pi N(\mathcal{M}) \cdot \lambda(\mathcal{M}, 2)}{3 \cdot [\mathbb{Z}_2^\times : \mathrm{nrd}(\mathcal{M}_2^\times)]} \prod_{p | N(\mathcal{M})} \left(1 + \left(\frac{-2}{p}\right) p^{-1}\right) \tag{3.2}$$

whenever $2|N(\mathcal{M})$, noticing that $\left(\frac{-2}{2}\right) = 0$, and into

$$\mathrm{vol}(P\rho(\mathcal{M}^1)\backslash\mathbb{H}^2) = \frac{\pi N(\mathcal{M})}{3} \prod_{p | N(\mathcal{M})} \left(1 + \left(\frac{-2}{p}\right) p^{-1}\right) \tag{3.3}$$

when $2 \nmid N(\mathcal{M})$. Therefore, except the cases (3) and (4) of Theorem 2 where 2 does not divide $N(\mathcal{M})$, we must calculate both the Eichler symbols at $p = 2$, and the indices of the reduced norm groups at $p = 2$ to obtain the precise volume formulae.

3.1. **Computing the Eichler symbol of $\mathcal{M}$ at $p = 2$.** We compute the Eichler symbols of the orders $\mathcal{M}$ of (1), (2), (5) and (6) in Theorem 2 by computing $\left(\frac{\Delta(\alpha)}{2}\right)$ for $\alpha \in \mathcal{M}$. Recall that we denote the (i)-th order of Theorem 2 as $\mathcal{M}_{(\mathrm{i})}$. For $\alpha = t + ix + jy + ijz \in \mathcal{M}_{(1)}$, we have

$$\Delta(\alpha) = -8x^2 + 4Dy^2 + 8Dz^2$$

which is divisible by 2 regardless of the value of $D$ and hence $\left(\frac{\Delta(\alpha)}{2}\right) = 0$. Therefore,

$$\lambda(\mathcal{M}_{(1)}, 2) = \frac{3}{4}.$$

For $\alpha = t + xi + y\frac{1+j}{2} + z\frac{i+ij}{2} \in \mathcal{M}_{(2)}$ we have

$$\Delta(\alpha) = -8x^2 - 8xz - 2z^2 + Dy^2 + 2Dz^2,$$

but as $D \equiv 1 \mod 4$ this value is congruent to $Dy^2$ modulo 8. Hence the symbol may depend on whether $D \equiv 1$ or 5 mod 8. In the case $D \equiv 1 \mod 8$, this becomes

$$\Delta(\alpha) \equiv y^2 \mod 8,$$

and $\left(\frac{\Delta(\alpha)}{2}\right)$ can either be 0 or 1, depending on the value of $y$. When $D \equiv 5 \mod 8$, the symbol $\left(\frac{\Delta(\alpha)}{2}\right)$ is 0 or $-1$ depending on the value of $y$. In summary,

$$\lambda(\mathcal{M}_{(2)}, 2) = \begin{cases} \frac{3}{2} & \text{when } D \equiv 1 \mod 8, \\ \frac{1}{2} & \text{when } D \equiv 5 \mod 8. \end{cases}$$

For $\alpha = t + xi + y\frac{i+j}{2} + z\frac{ij}{2} \in \mathcal{M}_{(5)}$, we have

$$\Delta(\alpha) = -2(2x + y)^2 + Dy^2 + 2Dz^2,$$

but in this case $D \equiv 2 \mod 4$, so $\Delta(\alpha)$ is even. Therefore,

$$\lambda(\mathcal{M}_{(5)}, 2) = \frac{3}{4}.$$

For $\alpha = t + xi + y\frac{1+j+ij}{2} + z\frac{i+ij}{2} \in \mathcal{M}_{(6)}$, we have

$$\Delta(\alpha) \equiv (D-2)y^2 + 4yz + 2z^2 \mod 8,$$

which is even when $y$ is. However if $y$ is odd, then depending on whether $D \equiv 3$ or $7 \mod 8$, the value of $\Delta(\alpha)$ can be either

$$\Delta(\alpha) \equiv y^2 + 4yz + 2z^2, \text{ or } 5y^2 + 4yz + 2z^2,$$

modulo 8. First assume that $D \equiv 3 \mod 8$. In the case that $z$ is even, we obtain

$$\Delta(\alpha) \equiv y^2 \mod 8,$$

which is 1 modulo 8 as $y$ is odd. In the case that $z$ is odd, we obtain

$$\Delta(\alpha) \equiv (y + 2z)^2 - z^2 - z^2 \equiv -1 \mod 8,$$

keeping in mind that the only odd square modulo 8 is 1. Hence $\Delta(\alpha) \equiv \pm 1 \mod 8$, and

$$\lambda(\mathcal{M}_{(6)}, 2) = \frac{3}{2} \quad \text{when } D \equiv 3 \mod 8.$$

On the other hand, for $D \equiv 7 \mod 8$, we have either

$$\Delta(\alpha) \equiv 5y^2 \equiv -3 \mod 8$$

or

$$\Delta(\alpha) \equiv (2y + z)^2 + y^2 + z^2 \equiv 3 \mod 8,$$

depending on whether $z$ is even or odd. Hence $\Delta(\alpha) \equiv \mp 3 \mod 8$, and

$$\lambda(\mathcal{M}_{(6)}, 2) = \frac{1}{2} \quad \text{when } D \equiv 7 \mod 8.$$

This completes the computation of the Eichler symbols at $p = 2$.

3.2. **Computing the Index of the Reduced Norm Group at $p = 2$.** Unlike odd primes $p$ where the index of the square unit group in $\mathbb{Z}_p^\times$ is of index 2, the square unit group in $\mathbb{Z}_2^\times$ is of index 4, and more precisely $\mathbb{Z}_2^\times/(\mathbb{Z}_2^\times)^2$ is isomorphic to the Klein four group $V_4$. This is because there are three intermediate subgroups

$$1, 3 + 8\mathbb{Z}_2, \quad 1, 5 + 8\mathbb{Z}_2, \quad 1, 7 + 8\mathbb{Z}_2$$

of index 2 between $(\mathbb{Z}_2^\times)^2 = 1 + 8\mathbb{Z}_2$ and $\mathbb{Z}_2^\times = 1 + 2\mathbb{Z}_2$. Therefore $\mathrm{nrd}(\mathcal{M}_2^\times)$ attaining a nonsquare invertible value modulo 8 is not enough to ensure that $\mathrm{nrd}(\mathcal{M}_2^\times) = \mathbb{Z}_2^\times$, and we must calculate all possible values of nrd modulo 8.

Let $\alpha = t + xi + yj + zij \in \mathcal{M} = \mathcal{M}_{(1)}$. It follows that

$$\mathrm{nrd}(\alpha) = t^2 + 2x^2 + 7Dy^2 + 6Dz^2,$$

and we examine the values modulo 8. In the case that $D \equiv 0 \mod 8$, this becomes

$$t^2 + 2x^2 \mod 8,$$

and for this to be a unit in $\mathbb{Z}_2$ we must have $t$ odd. Hence in this case the group $\mathrm{nrd}(\mathcal{M}_2^\times) = 1, 3 + 8\mathbb{Z}_2$ and is of index 2 in $\mathbb{Z}_2^\times$. When $D \not\equiv 0 \mod 8$, however, the reduced norm of $1 + 2j$ is

$$1 + 4D \mod 8$$

and notice that for odd $D$ this value is always 5 modulo 8. Also, the reduced norm of $1 + ij$ is

$$1 + 6D \mod 8,$$

which for $D \equiv 2, 6 \mod 8$ turns out to be 5 mod 8. For $D \equiv 4 \mod 8$ notice that the reduced norm of $1 + ij$ is

$$1 + 7D \equiv 5 \mod 8.$$

In summary,

$$[\mathbb{Z}_2^\times : \mathrm{nrd}(\mathcal{M}_2^\times)] = \begin{cases} 2 & \text{when } D \equiv 0 \mod 8 \\ 1 & \text{otherwise.} \end{cases}$$

For $\mathcal{M}_{(2)}, \mathcal{M}_{(5)}$ and $\mathcal{M}_{(6)}$, as the reduced norms of the elements

$$1 + i, \ \ 1 + j, \ \ 1 + 2j \ \ \text{and} \ \ 1 + ij$$

are

$$3, \ \ 1 + 7D, \ \ 1 + 4D \ \ \text{and} \ \ 1 + 6D \ \ \text{modulo} \ \ 8,$$

respectively, one can verify that in each order there are at least two elements whose reduced norms modulo 8 are distinct and are among $3, 5$ and $7$, regardless of the congruence class of $D$ modulo 8. Hence the index $[\mathbb{Z}_2^\times : \mathrm{nrd}(\mathcal{M}_2^\times)]$ is always 1 in these cases. These results combined with Section 3.1 prove Theorem 3.

*Remark* 10. The analysis of this section complements the proof of Lemma 3.7 of [Jun19], where the values of $\mathrm{nrd}(\mathcal{M}_2^\times)$ were computed modulo only up to 4. It is likely that this also caused the error in Humbert's formula as mentioned in [MR91]. Nonetheless, the lemma of [Jun19] stays valid which can be seen by further computation.

3.3. **Counting Prime Geodesic Surfaces.** Using the volume formulae of Theorem 3, it is possible to prove a prime geodesic theorem for surfaces in the hyperbolic 3-manifold $\Gamma\backslash\mathbb{H}^3$ using a simple analytic lemma given below:

**Lemma 11** ([Jun19], 3.10)**.** *For any integer $r$, the function w.r.t. $X$*

$$\#\{D \equiv r \mod 2^a \mid F(D) < X\}$$

*is asymptotic to $\frac{C}{2^a} X$ as $X \to \infty$, where the constant $C = \prod_p \left(1 - \frac{1}{p} + \frac{1}{p + \left(\frac{-2}{p}\right)}\right)$, and $F(D)$ is as in Theorem 3.*

Let $\Pi(x)$ denote the number of primitive immersed totally geodesic surfaces in the manifold $\Gamma\backslash\mathbb{H}^3$ having area less than $x$. As such surfaces are in correspondence with $F$-subgroups of $\Gamma$, we may write the function $\Pi(x)$ as

$$\Pi(x) = \sum_{k=1}^{6} \#\{D \mid \mathrm{vol}(\mathcal{M}^1_{(k)}/\{\pm 1\}) < x\}$$

where by the results of Theorem 3, we can write each summand in the form of

$$\#\{D \mid \mathrm{vol}(\mathcal{M}^1_{(k)}/\{\pm 1\}) < x\} = \sum_i \#\{D \equiv r_i \mod 2^{a_i} \mid c_i \cdot \pi F(D) < x\}$$

for corresponding values of $r_i, a_i$ and $c_i$. Applying Lemma 11, we have

$$\#\{D \equiv r_i \mod 2^{a_i} \mid c_i \cdot \pi F(D) < x\} \sim \frac{C}{\pi 2^{a_i} c_i} x$$

as $x \to \infty$, and the sum of the corresponding asymptotic coefficients yields

$$\Pi(x) \sim \frac{45C}{16\pi} x.$$

This proves the identity 1.1.

## References


[JM96] D. G. James and C. Maclachlan. Fuchsian subgroups of bianchi groups. *Transactions of the American Mathematical Society*, 348(5):1989–2002, 1996.

[Jun19] Junehyuk Jung. On the growth of the number of primitive totally geodesic surfaces in some hyperbolic 3-manifolds. *Journal of Number Theory*, 202:160–175, 2019.

[MR91] C. Maclachlan and A. W. Reid. Parametrizing fuchsian subgroups of the bianchi groups. *Canadian Journal of Mathematics*, 43(1):158–181, 1991.

[Voi21a] John Voight. *Quaternion orders: second meeting*, pages 393–411. Springer International Publishing, Cham, 2021.

[Voi21b] John Voight. *Volume formula*, pages 731–741. Springer International Publishing, Cham, 2021.

[Vul91] L. Ya. Vulakh. Classification of maximal fuchsian subsgroups of some bianchi groups. *Canadian Mathematical Bulletin*, 34(3):417–422, 1991.

[Vul17] L. Vulakh. *Maximal Fuchsian Subgroups of Extended Bianchi Groups*, pages 297–310. 10 2017.



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