ALGEBRAIC GEOMETRY FALL FINAL

HOJIN LEE

The solution always assumes k is algebraically closed.

Problem 1. *Let X and Y be smooth projective over k. Let $p_X : X \times_k Y \to X$ and $p_Y : X \times_k Y \to Y$ be the two projections. Is it always true that*

$$\operatorname{Pic}(X \times_k Y) \simeq p_X^* \operatorname{Pic}(X) \times p_Y^* \operatorname{Pic}(Y)?$$

Prove or provide a counterexample.

Proof. We provide a counterexample. Let X and Y both be an elliptic curve C over k with a fixed point P, and rename p_X and p_Y as p_1, p_2, respectively. Let $l = C \times P$ and $m = P \times C$, and denote $\Delta \subset C \times_k C$ the diagonal. The self intersection of l and m are both zero since l, m are fibers of p_1, p_2, and two fibers of the same map are linearly equivalent, and two distinct fibers do not meet. To calculate Δ^2, we use the fact that $\Delta^2 = \deg_\Delta(\mathcal{L}(\Delta) \otimes \mathcal{O}_\Delta) = \deg_\Delta \mathcal{N}_{\Delta/C\times C} = \deg_C(\Omega_{C/k}^\vee) = 2 - 2g = 0$. Also, $l \cdot m = l \cdot \Delta = m \cdot \Delta = 1$ since they meet at one point, so l, m and Δ are nontrivial elements of $\operatorname{Num}(C \times C)$.

Now we show l, m and Δ are linearly independent in $\operatorname{Num}(C \times C)$. Let $pl + qm + r\Delta = 0$ for integers p, q, r. Take the intersection with l to get $q + r = 0$. Again take intersection with m to get $p + r = 0$. Finally take intersection with Δ to get $p + q = 0$, so this implies $p = q = r = 0$. Hence l, m, Δ indeed are linearly independent, so the rank of $\operatorname{Num}(C \times C)$ is at least 3. However, the group $p_1^* \operatorname{Pic}(C) \oplus p_2^* \operatorname{Pic}(C)$ modulo numerical equivalence is generated by l and m, so is of rank 2. Hence the isomorphism given in the problem cannot exist for a product of elliptic curves. □

Problem 2. *Give an example of a smooth projective X/k and a divisor $D \in \operatorname{Div}(X)$ satisfying the condition.*

(a) *D is effective but not basepoint-free.*
(b) *D is basepoint-free but not ample on X.*
(c) *D is ample on X but not very ample.*

Proof. (a) Suppose X is an elliptic curve and $D = P$ a point in X. D is obviously effective. By Riemann-Roch, we have $\ell(D) - \ell(K - D) = 2 - g = 1$. Also, $\deg D = 1 > 0 = 2g - 2 = \deg K$, which implies $\deg(K - D) < 0$. Therefore $\ell(K - D) = 0$ by Hartshorne, Lemma IV.1.2. Hence $\ell(D) = 1$, so $\dim |D| = \ell(D) - 1 = 0$. Hence there is only one element in $|D|$, namely D, so the complete linear system has D as a basepoint. □

Proof. (b) Let X be a smooth projective curve of genus 0, and $D = P - Q$ for two points P, Q in X. Then $\deg D = 0 \geq 2g$, so by Hartshorne, Corollary IV.3.2, $|D|$ has no basepoints. However, by Hartshorne, Corollary IV.3.3, $\deg D = 0$ implies D not ample. □

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Proof. (c) Let X be an elliptic curve. Let $D = P + Q$ for points P, Q in X. Then $\deg D = 2 > 0$, so D is ample by Hartshorne, Corollary IV.3.3. However, by Riemann-Roch, we have $\ell(D) - \ell(K - D) = 3 - g = 2$ where $\ell(K - D) = 0$ by the same proof as in (a). Also $|D|$ has no basepoints since $\deg D = 2 \geq 2 = 2g$. Therefore $|D|$ defines a morphism of X into $\mathbb{P}_k^{\ell(D)-1} = \mathbb{P}_k^1$. This obviously cannot be a closed immersion, so D cannot be very ample. □

Problem 3. *Let $X \subset \mathbb{P}_k^n$ be an irreducible smooth projective hypersurface of degree d. Show that the canonical sheaf of X is trivial if and only if $d = n + 1$.*

Proof. By Hartshorne, Proposition II.8.20, we have that $\omega_X \simeq \omega_{\mathbb{P}_k^n} \otimes \mathcal{L} \otimes \mathcal{O}_X$ where $\mathcal{L}$ is the associated invertible sheaf of X in $\mathbb{P}_k^n$. Since $\omega_{\mathbb{P}_k^n} \simeq \mathcal{O}_{\mathbb{P}_k^n}(-n-1)$, and $\mathcal{L} \simeq \mathcal{O}_X(d)$, we have $\omega_{\mathbb{P}_k^n} \otimes \mathcal{L} \otimes \mathcal{O}_X \simeq \mathcal{O}_X(-n-1) \otimes \mathcal{O}_X(d) \simeq \mathcal{O}_X(d-n-1)$. Hence ω_X is trivial if and only if $d = n + 1$. □

Problem 4. *Let X/k be a smooth projective surface.*

(a) *Let H be an ample divisor. For any $0 \nsim D \in \operatorname{Div}(X)$, show that if $D \cdot H = 0$ then $\ell(D) = 0$.*
(b) *Let C be a smooth irreducible projective curve in X. Show that if $C \cdot C < 0$ then C is rigid in the sense that $|C| = 0$.*

Proof. (a) We show the contrapositive. Suppose $\ell(D) > 0$. Then $|D| \neq \emptyset$ so D is linearly equivalent to an effective divisor. Since H is ample, by Nakai-Moishezon we have $D \cdot H > 0$. □

Proof. (b) Again, we show the contrapositive. Suppose $|C| \neq 0$. This means that there is some effective $C' \sim C$ where $C' \neq C$. Therefore $C \cdot C = C \cdot C'$, and since $C \neq C'$ where C is irreducible, C' does not have any irreducible component in common with C. This implies the intersection number is nonnegative, so we must have $C \cdot C' \geq 0$. Hence $C \cdot C \geq 0$. □

Problem 5. *Let X/k be a smooth projective quartic surface in $\mathbb{P}_k^3$.*

(a) *Compute the numbers $\dim_k H^q(X, \Omega_X^p)$ for $0 \leq p, q \leq 2$.*
(b) *Show that $\chi(\mathcal{L}(D)) = \frac{1}{2}D^2 + 2$ for each $D \in \operatorname{Div}(X)$.*
(c) *Show that the projection map $\operatorname{Pic}(X) \to \operatorname{Num}(X)$ is an isomorphism.*
(d) *Show that for each $d \geq 1$, there are at most finitely many smooth rational projective curves of degree d on X.*

Proof. (a) Write $h^{p,q} = \dim_k H^q(X, \Omega_X^p)$. Since X is a smooth quartic in $\mathbb{P}_k^3$, by Problem 3, we have $\omega_X = \Omega_X^2 \simeq \mathcal{O}_X$. By Serre duality, $\dim_k H^0(X, \Omega_X^0) = \dim_k H^2(X, \Omega_X^2)$ and since $\Omega_X^0 \simeq \mathcal{O}_X \simeq \Omega_X^2$, we have $h^{0,0} = h^{0,2} = h^{2,0} = h^{2,2} = \dim_k H^0(X, \mathcal{O}_X) = 1$. Next, consider the SES on $\mathbb{P}_k^3$ given by

$$0 \to \mathcal{O}_{\mathbb{P}_k^3}(-4) \to \mathcal{O}_{\mathbb{P}_k^3} \to i_*\mathcal{O}_X \to 0$$

where $i : X \hookrightarrow \mathbb{P}_k^3$. Take the LES of cohomology to get

$$\cdots \to H^1(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}_k^3}) \to H^1(\mathbb{P}_k^3, i_*\mathcal{O}_X) \to H^2(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}_k^3}(-4)) \to \cdots$$

and note that $H^1(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}_k^3}) = H^2(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}_k^3}(-4)) = 0$ by Hartshorne, Theorem III.5.1. Since $0 = H^1(\mathbb{P}_k^3, i_*\mathcal{O}_X) \simeq H^1(X, \mathcal{O}_X)$, we have $h^{0,1} = h^{2,1} = 0$. Also consider the following SES on X

$$0 \to \mathcal{O}_X(-4) \to \Omega_{\mathbb{P}_k^3}^1 \otimes \mathcal{O}_X \to \Omega_X^1 \to 0$$

of Hartshorne, Theorem II.8.17 and again derive the LES

$$\cdots \to H^0(X, \Omega^1_{\mathbb{P}^3_k} \otimes \mathcal{O}_X) \to H^0(X, \Omega^1_X) \to H^1(X, \mathcal{O}_X(-4)) \to \cdots$$

where $H^1(X, \mathcal{O}_X(-4)) \simeq H^1(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(-4)) = 0$ by Hartshorne, Theorem III.5.1. By the LES of cohomology of the SES of Hartshorne, Theorem II.8.13 given by

$$0 \to \Omega^1_{\mathbb{P}^3_k} \to \mathcal{O}_{\mathbb{P}^3_k}(-1)^{\oplus 4} \to \mathcal{O}_{\mathbb{P}^3_k} \to 0,$$

since $H^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(-1)^{\oplus 4}) = H^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(-1))^{\oplus 4} = 0$ we know that

$$0 = H^0(\mathbb{P}^3_k, \Omega^1_{\mathbb{P}^3_k}) \simeq H^0(X, \Omega^1_{\mathbb{P}^3_k} \otimes \mathcal{O}_X).$$

Therefore $H^0(X, \Omega^1_X) = 0$, so $h^{1,0} = h^{1,2} = 0$. (Serre duality is used in obvious places.) Now we are left with $h^{1,1} = \dim_k H^1(X, \Omega^1_X)$. Note that $\chi(\Omega^1_X) = \dim_k H^0(X, \Omega^1_X) - \dim_k H^1(X, \Omega^1_X) + \dim_k H^2(X, \Omega^1_X)$ by dimensional vanishing. Also we just calculated $h^{1,0} = h^{1,2} = 0$, so it turns out that $h^{1,1} = -\chi(\Omega^1_X)$.

From now on for the sake of brevity, I will denote $P = \mathbb{P}^3_k$. I follow the proof techniques of the midterm exam, problem 6. Consider the SES

$$0 \to \mathcal{O}_X(-4) \to \Omega^1_P \otimes \mathcal{O}_X \to \Omega^1_X \to 0.$$

It follows that $\chi(\Omega^1_P \otimes \mathcal{O}_X) = \chi(\mathcal{O}_X(-4)) + \chi(\Omega^1_X)$. Thus we must calculate both $\chi(\Omega^1_P \otimes \mathcal{O}_X)$ and $\chi(\mathcal{O}_X(-4))$. First we calculate $\chi(\Omega^1_P \otimes \mathcal{O}_X)$. To do this, we tensor the SES

$$0 \to \mathcal{O}_P(-4) \to \mathcal{O}_P \to \mathcal{O}_X \to 0$$

with Ω^1_P to get

$$0 \to \Omega^1_P(-4) \to \Omega^1_P \to \Omega^1_P \otimes \mathcal{O}_X \to 0.$$

Take the Euler characteristics to get

$$\chi(\Omega^1_P) = \chi(\Omega^1_P \otimes \mathcal{O}_X) + \chi(\Omega^1_P(-4))$$

where we know that $\chi(\Omega^1_P) = -1$ from the SES mentioned above of Hartshorne, Theorem II.8.13. Now twist that SES by $\mathcal{O}(-4)$ to get

$$0 \to \Omega^1_P(-4) \to \mathcal{O}_P(-5)^{\oplus 4} \to \mathcal{O}_P(-4) \to 0$$

so $\chi(\mathcal{O}_P(-5)^{\oplus 4}) = \chi(\Omega^1_P(-4)) + \chi(\mathcal{O}_P(-4))$. By the formulas I proved in the midterm exam, $\chi(\mathcal{O}_P(-5)) = -4$, and $\chi(\mathcal{O}_P(-4)) = -1$. Hence $\chi(\Omega^1_P(-4)) = 1 - 4 \times 4 = -15$. Thus $-1 = \chi(\Omega^1_P \otimes \mathcal{O}_X) - 15$, so $\chi(\Omega^1_P \otimes \mathcal{O}_X) = 14$.

Now we calculate $\chi(\mathcal{O}_X(-4))$. By twisting

$$0 \to \mathcal{O}_P(-4) \to \mathcal{O}_P \to \mathcal{O}_X \to 0$$

we get

$$0 \to \mathcal{O}_P(-8) \to \mathcal{O}_P(-4) \to \mathcal{O}_X(-4) \to 0$$

so $\chi(\mathcal{O}_X(-4)) = \chi(\mathcal{O}_P(-4)) - \chi(\mathcal{O}_P(-8))$. Again by the formulas, $\chi(\mathcal{O}_P(-4)) = -1$ and $\chi(\mathcal{O}_P(-8)) = -\binom{7}{4} = -35$. Thus $\chi(\mathcal{O}_X(-4)) = -1 + 35 = 34$. Therefore, $\chi(\Omega^1_P \otimes \mathcal{O}_X) = \chi(\mathcal{O}_X(-4)) + \chi(\Omega^1_X)$, so $\chi(\Omega^1_X) = 14 - 34 = -20$. This implies $h^{1,1} = 20$. □

Proof. (b) By Riemann-Roch, we have $\chi(\mathcal{L}(D)) = \frac{1}{2} D \cdot (D - K) + \chi(\mathcal{O}_X)$. We have shown that the canonical sheaf of X is trivial, hence $K \sim 0$ which implies $D \cdot K = 0$. Also, from

$$0 \to \mathcal{O}_P(-4) \to \mathcal{O}_P \to \mathcal{O}_X \to 0$$

we have $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_P) - \chi(\mathcal{O}_P(-4)) = 1 - (-1)$, again from the formulas. Therefore $\chi(\mathcal{L}(D)) = \frac{1}{2}D^2 + 2$. $\square$

Proof. (c) Suppose that $\mathcal{O}_X \not\simeq \mathcal{L}(D) \in \operatorname{Pic}^n X$. Let H be an ample divisor on X, then since $\mathcal{L}(D) \in \operatorname{Pic}^n X$ we have $D \cdot H = 0$. By Problem 4(a), this implies $\ell(D) = H^0(X, \mathcal{L}(D)) = 0$. The same holds for $-D$, so by Serre duality, we have $\dim_k H^2(X, \mathcal{L}(D)) = \dim_k H^0(X, \mathcal{L}(-D)) = 0$. Therefore with dimensional vanishing we conclude that $\chi(\mathcal{L}(D)) = -\dim_k H^1(X, \mathcal{L}(D)) \leq 0$. By (b), this means that $\frac{1}{2}D^2 + 2 \leq 0$, so $D^2 \leq -4$, which is a contradiction to our assumption that $D \cdot E = 0$ for all divisors E. Hence nontrivial $\mathcal{L}(D)$ cannot be in $\operatorname{Pic}^n X$, so the projection $\operatorname{Pic} X \to \operatorname{Num} X$ is an isomorphism. $\square$

Proof. (d) We use the fact that $\operatorname{Num} X$ is finitely generated abelian by Néron-Severi, without proof. On the other hand, $\operatorname{Num} X$ is torsion-free by construction, since if $nD \sim 0$ for some $\mathcal{L}(D) \in \operatorname{Pic} X$, $nD \cdot E = n(D \cdot E) = 0$ so $D \cdot E = 0$ for all $E \in \operatorname{Div} X$, hence $\mathcal{L}(D)$ would be in $\operatorname{Pic}^n X$. Thus $\operatorname{Num} X$ is free abelian of finite rank. By (c), $\operatorname{Num} X$ is isomorphic to $\operatorname{Pic} X$, so the Picard group is also free abelian of finite rank.

Rational curves on X are curves of genus zero, so (if they exist) we may consider them as effective divisors D on X. Also by the adjunction formula, we have $-2 = D \cdot (D + K)$ for K a canonical divisor on X, which is trivial. Therefore $D^2 = -2 < 0$, so if they exist, they are rigid by Problem 4(b). Hence distinct smooth rational curves on X cannot be linearly equivalent to each other, so an element of $\operatorname{Pic} X$ uniquely determines a rational curve (again, if such curve exists). Now since we know that $\operatorname{Pic} X$ is finitely generated, there are finitely many effective divisor classes of fixed degree d, and since each class determines at most one rational curve, there are finitely many rational curves of fixed degree d on X for each d. $\square$

Thank you for your time and effort.