< 2025.5.13 Stiefel–Whitney Class and Spin Structure >

* Outlines

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III. Application of Spin Structure

- Brief Introduction to Floer Cohomology
- Orientation of Moduli Space

* References

[1] J. Milnor and J. Stasheff, Characteristic Classes (Annals of Mathematics Studies 202)

[2] H. B. Lawson and M. Michelson, Spin Geometry (Princeton Mathematical Series 38)

I. Stiefel–Whitney Class

Stiefel-Whitney cohomology classes of a vector bundle is characterized by 4 axioms.

Let $\xi: E \to B$ be a real dim n vector bundle

Axiom 1 $\quad w_i(\xi) \in H^i(B(\xi); \mathbb{Z}/2\mathbb{Z}) \quad i = 0, 1, 2, \cdots$

$$w_0(\xi) = 1 \in H^0(B(\xi); \mathbb{Z}/2\mathbb{Z})$$

$$w_i(\xi) = 0 \text{ for } i > n$$

Axiom 2 (Naturality) for a bundle map $f: \xi \to \eta$

$$w_i(\xi) = f^* w_i(\eta)$$

Axiom 3 (The Whitney product Thm)

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \cup w_{k-i}(\eta)$$

Axiom 4 For the line bundle γ_1^1 over $\mathbb{P}^1$, $w_1(\gamma) \neq 0$.

Assume the existence of $w_i(\xi)$ satisfying Axiom 1–4.

Def total Stiefel–Whitney class of ξ:

$$w(\xi) := 1 + w_1(\xi) + \cdots + w_n(\xi) + 0 + \cdots$$

Properties of S-W class :

① For trivial bundle ε, $w_i(\varepsilon \oplus \eta) = w_i(\eta)$

② ξ has k cross section which nowhere lin. indep,

$$w_{n-k+1}(\xi) = \cdots = w_n(\xi) = 0$$

⇝ S-W class captures how a real vec. bundle twists over the base space

Example : S-W class of the tangent bundle of $\mathbb{R}P^n$.

Let a : generator of $H^1(\mathbb{R}P^n ; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}_2[x]/(x^{n+1})$.

$(a \in H^1(\mathbb{R}P^n ; \mathbb{Z}/2\mathbb{Z}))$

<u>Observation 1</u> $w(\gamma_n^1) = 1 + a$ (γ_n^1 : canonical line bundle of $\mathbb{R}P^n$).

$\because$ For $i : \mathbb{P}^1 \hookrightarrow \mathbb{P}^n$, $i^* w_1(\gamma_n^1) = w_1(\gamma_1^1) \neq 0$

$\Rightarrow w_1(\gamma_n^1) \neq 0$ i.e. $w_1(\gamma_n^1) = a$.

<u>Observation 2</u> $T(\mathbb{R}P^n) \cong \mathrm{Hom}(\gamma_n^1, \gamma^\perp)$

($\gamma^\perp$: orthogonal comp. of γ_n^1 in $\mathbb{P}^n \times \mathbb{R}^{n+1}$).

$\because$ $T\mathbb{P}^n$ = Set of all pairs $\{(x, v), (-x, -v)\}$

$\Updownarrow$

linear mapping $l : \mathbb{R} \to \mathbb{R}^n$ s.t. $l(x) = v$.

$\Rightarrow T\mathbb{R}P^n \cong \mathrm{Hom}(\gamma_n^1, \gamma^\perp)$.

By obs 2, $T\mathbb{R}P^n \oplus \varepsilon^1 \cong \mathrm{Hom}(\gamma_n^1, \gamma^\perp) \oplus \mathrm{Hom}(\gamma_n^1, \gamma_n^1)$

$$\begin{aligned}
&\cong \mathrm{Hom}(\gamma_n^1, \gamma^\perp \oplus \gamma_n^1) \cong \mathrm{Hom}(\gamma_n^1, \varepsilon^{n+1}) \\
&\cong \mathrm{Hom}(\gamma_n^1, \varepsilon^1 \oplus \cdots \oplus \varepsilon^1) \\
&\cong \mathrm{Hom}(\gamma_n^1, \varepsilon^1) \oplus \cdots \oplus \mathrm{Hom}(\gamma_n^1, \varepsilon^1) \\
&\cong \gamma_n^1 \oplus \cdots \oplus \gamma_n^1
\end{aligned}$$

$\Rightarrow w(T\mathbb{R}P^n) \underbrace{\cancel{w(\varepsilon^1)}}_{1} = (w(\gamma_n^1))^{n+1} = (1+a)^{n+1}$

$\therefore w(\mathbb{R}P^n) = (1+a)^{n+1}$

§ Existence of S-W class

n-dim real vector bundle $\xi : E \to B$

$\rightsquigarrow$ $O(n)$ - principle bundle $P_0(E) \to B$

$\rightsquigarrow$ map $f : B \to BO(n)$ ($BO(n)$: classifying space).

$\rightsquigarrow$ induced map $f^* : H^\bullet(BO(n); \mathbb{Z}/2\mathbb{Z}) \to H^\bullet(B; \mathbb{Z}/2\mathbb{Z})$

$$\begin{gathered} \cong \\ \mathbb{Z}_2[w_1, w_2, \cdots, w_n] \quad (w_k \in H^k(BO(n); \mathbb{Z}/2\mathbb{Z})) \end{gathered}$$

Then let $w_k(\xi) := f^*(w_k)$ and

$$w(\xi) := \sum_{k=0}^{n} w_k(\xi) \quad (w_0(\xi) = 1).$$

$\Rightarrow$ $w_k(\xi)$ satisfies Axiom 1-4

<u>Thm</u> ξ is orientable iff $w_1(\xi) = 0$.

pf) To prove this, define w_1 alternatively

- $P_0(E)/SO(n)$: 2-fold covering of B.

  $COV_2(B)$: set of equi. class of 2-fold covering of B.

$$\begin{array}{ccccccc} H^1(B; \mathbb{Z}_2) & \xrightarrow[(\because UCT)]{\cong} & Hom(H_1(B); \mathbb{Z}_2) & \xrightarrow{\cong} & Hom(\pi_1(B); \mathbb{Z}_2) & \xrightarrow{\cong} & COV_2(B) \\ \cup & & & & & & \cup \\ w_1'(E) & & & & & & P_0(E)/SO(n) \end{array}$$

For this $w_1'(E)$,

$$\left. \begin{array}{l} (1)\ w_1' f^* = f^* w_1' \quad (\because P_0(f^*E) = f^* P_0(E)) \\ (2)\ w_1'(EO(n)) = w_1 \quad (\because \text{if not, } \forall \text{ bundle are orientable}) \end{array} \right\} \Rightarrow w_1' = w_1$$

$w_1(E) = 0$ iff $E \to B$ orientable is clear.

II. Spin Structure

§ Definition of Spin Structure

For real vector bundle $\xi : E \to B$ (dim n).

ξ induces a $SO(n)$-principle bundle $P_{SO}(E) \to B$ iff it is orientable.

<u>Def</u> $Spin(n)$: universal cover of $SO(n)$

Since $\pi_1(SO(n); \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, $Spin(n) \longrightarrow SO(n)$ is 2-fold covering.

<u>Def</u> If there exists an $Spin(n)$-principle bundle $P_{spin}(E) \longrightarrow B$ and the bundle map $f : P_{spin}(E) \to P_{SO}(E)$

s.t. the following diagram commutes:

$$
\begin{tikzcd}
Spin(n) \arrow[rr, "\pi"] & & SO(n) \\
P_{spin}(E) \arrow[rr, "f"] \arrow[dr] & & P_{SO}(E) \arrow[dl] \\
& B &
\end{tikzcd}
$$

We say $E \to B$ is spin bundle and $(P_{spin}(E), f)$ is its spin structure

Alternative definition of the spin structure:

<u>Def</u> A spin structure of oriented vec. bundle $E \to B$

is a homotopy class of a trivialization of E over $X^{(1)}$

which can be extended to $X^{(2)}$.

Thm ① ξ is orientable iff $w_1(\xi) = 0$

② ξ is spin iff $w_1(\xi) = 0$ and $w_2(\xi) = 0$.

pf) ① : in page 4

② : more complicate...

Geometric interpretation of $w_1 = 0$ and $w_2 = 0$:

- $E \to B$ orientable

$\Leftrightarrow$ f^*E is trivial over S^1 for any conti. $f : S^1 \to B$

- $E \to B$ spin

$\Leftrightarrow$ f^*E is trivial

for any compact surface Σ & conti. $f : \Sigma \to B$.

only if $n \geq 3$

II. Application of Spin Structure

§ Brief Introduction to Floer Cohomology

Basic Settings:

(M, ω) : $2n$-dim sympl. mfd, L: Lagr. submfd, J: almost complex structure

ϕ_H^1 : small Hamiltonian perturbation.

Idea of Floer Cohomology (Technical parts are omitted).

- Cochain complex: $CF(L, \phi_H^1(L)) := \bigoplus_{p \in L \cap \phi_H^1(L)} \Lambda \cdot p$

  (Λ: Novikov field. not important in this talk).

- Floer differential $\partial : CF(L, \phi_H^1(L)) \to CF(L, \phi_H^1(L))$.

$$\partial(p) := \sum_{\substack{q \in L \cap \phi_H^1(L) \\ [u] : \mathrm{ind}([u]) = 1}} \left(\# \underset{\sim\sim\sim\sim\sim}{\mathcal{M}(p, q ; [u] ; J)}\right) T^{\omega([u])} q$$

We have interest in $\mathcal{M}(p, q ; [u] ; J)$ in this talk.

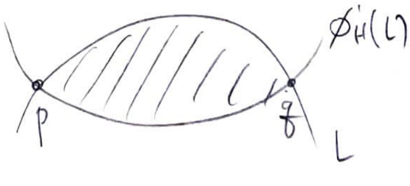
$u : \mathbb{R} \times [0,1] \to M$: J-holomorphic strip

$$\frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} = 0.$$

s.t. $\begin{cases} u(s,0) \in L, \ u(s,1) \in \phi_H^1(L) & \forall s \in \mathbb{R} \\ \lim_{s \to +\infty} u(s,t) = p, \ \lim_{s \to -\infty} u(s,t) = q & \forall t \in [0,1] \end{cases}$

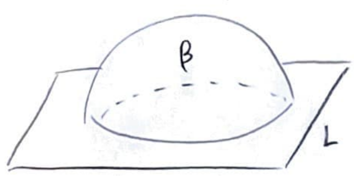
$\rightsquigarrow \mathcal{M}(p, q ; [u], J)$: moduli sp. of J-holomorphic strip

$[u] \in \pi_2(M, L)$



$\Rightarrow$ In unobstructed case, $\partial^2 = 0$ then $HF(L, L) := H^*(CF(L, \phi_H^1(L)), \partial)$.

moduli space of J-holo. strip can be consider as a moduli sp. of J-holomorphic disc $u : (D^2, \partial D^2) \to (M, L)$.



$[u] = \beta \in \pi_2(M, L)$.

$\rightsquigarrow$ moduli of the above disc : $\mathcal{M}(M, L; \beta)$.

<u>Thm</u> $\mathcal{M}(M, L; \beta)$ is orientable if L is a spin mfd.

(Sketch of proof).

- For $u \in \mathcal{M}(M, L; \beta)$:

  Let $E = u^* TM$ over D^2

  $F = (u|_{\partial D^2})^* TL$ over ∂D^2 (real subbundle of $u^* TM|_{\partial D^2}$)

  trivialization of F over $\partial D^2 \rightsquigarrow$ orientation at u

- If L spin, orientation do not change in loop.

  For loop of J-holo. disc γ, there is a J-holo. map

  $\tilde{u} : (D^2 \times S^1, \partial D^2 \times S^1) \to (M, L)$.

  $\rightsquigarrow (\tilde{u}|_{\partial D^2 \times S^1})^* TL$ is trivial over $\partial D^2 \times S^1$ ($\Leftarrow L$: spin).

  $\Rightarrow$ Consistent orientation i.e $\mathcal{M}(M, L; \beta)$ orientable

<u>RMK</u> Thm holds when L is relative spin too.