$K_0(\mathsf{Var})$ AND STABLE BIRATIONAL EQUIVALENCE

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Abstract. These are supplementary notes for a talk I will give at a seminar on Feb 14, 2025. (Obviously I won't go through all the details.)

1. Introduction

Larsen and Lunts provide a precise relation between the Grothendieck group of varieties and the ring of stable birational equivalence classes of varieties over the field of complex numbers.

Definition 1.1 (Varieties)**.** There are a *variety* of definitions of what a variety is. In the original text, a variety over a fixed (not necessarily algebraically closed) field k is a geometrically reduced separated scheme of finite type over k. Notice that we left out irreducibility from the definition.

Remark 1.2. One must note that subtle technicalities arise regarding the notion of being reduced. In general, the product of schemes that are reduced over k need not be reduced; this holds when k is a perfect field.[1] Instead, if we assume that X and Y are *geometrically reduced* schemes over any field k, then it holds that $X \times_k Y$ is also geometrically reduced. See Stacks, 035Z.

1.1. The Grothendieck ring $K_0(\mathsf{Var}_k)$ of k-varieties.

The additive group structure of $K_0(\mathsf{Var}_k)$ is generated by isomorphism classes $[X]$ of varieties over some fixed field k, subject to the scissor relation

$$[X \setminus Y] = [X] - [Y]$$

whenever Y is a closed subvariety[2] of X. The multiplicative monoid structure is given by

$$[X] \cdot [Y] = [X \times_k Y]$$

where it is clear that the point class $\operatorname{Spec} k$ acts as the identity.

1.2. Stable birational equivalence.

Suppose we are given two irreducible varieties X and Y. We say that X and Y are stably birational if $X \times_k \mathbb{P}^m$ is birational to $Y \times_k \mathbb{P}^n$ for some $m, n \geq 0$. If we denote the stable birational equivalence class of X as (X), then we may give the set of such classes a monoid structure similarly by setting $(X) \cdot (Y) = (X \times_k Y)$. Again, notice that the point class acts as the identity. Denote this monoid SB. Instead of imposing a scissor relation, we will be interested in the monoid ring $\mathbb{Z}[\mathrm{SB}]$.

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[1] Being perfect means that for a reduced commutative k-algebra A, given any field extension K/k the ring $A \otimes_k K$ must also be reduced. Being geometrically reduced is a stronger condition than being reduced; it means that for any field extension, the base change is also reduced.

[2] In general, taking the reduced scheme structure for Y does not guarantee Y being geometrically reduced. I'm not sure how to fix this; maybe restrict k to perfect fields?

2. The Theorem

2.1. **Statement of the theorem.** Denote by $\mathbb{L} = [\mathbb{A}^1_k]$ the class of the affine line in $K_0(\mathsf{Var}_k)$, and call this the Lefschetz motive. Larsen and Lunts construct an isomorphism of rings

$$K_0(\mathsf{Var}_\mathbb{C})/(\mathbb{L}) \xrightarrow{\sim} \mathbb{Z}[\mathrm{SB}]$$

where $(\mathbb{L})$ is the ideal in $K_0(\mathsf{Var}_\mathbb{C})$ generated by $\mathbb{L}$. This isomorphism sends a class $[X]$ of a smooth complete variety X to its stable birational class (X).

2.2. **Proof outline.** The proof of this theorem heavily depends on the fact $k = \mathbb{C}$ (or by the Lefschetz principle, that k is an algebraically closed field of characteristic zero.) This is because the proof involves resolution of singularities. It is interesting to see that analogues of this theorem hold over $k = \mathbb{F}_q$.

Theorem 2.1. *Let $k = \mathbb{C}$. Let G be a commutative monoid, and $\mathbb{Z}[G]$ its monoid ring. Let M be the monoid of isomorphism classes of smooth complete irreducible varieties, where the product is defined as usual. If $f : M \to G$ is a monoid homomorphism that satisfies*

(1) $f([X]) = f([Y])$ *if X and Y are birational,*
(2) $f([\mathbb{P}^n]) = 1$ *for all $n \geq 0$,*

then there exists a unique ring homomorphism $g : K_0(\mathsf{Var}_\mathbb{C}) \to \mathbb{Z}[G]$ which agrees with f on elements of M.

Sketch. For each isomorphism class $[X]$ in $K_0(\mathsf{Var}_\mathbb{C})$, we construct the element $g([X])$ so that it agrees with $f([X])$. We define g so that it is well-defined on the class $[X]$, it preserves the relations of $K_0(\mathsf{Var}_\mathbb{C})$, and it preserves the multiplication. This is done by induction on the dimension n of the variety; more precisely, the construction of $g([X])$ is given a priori, and we verify that the construction satisfies the desired properties using the induction hypothesis. This defines the map f for the classes of smooth complete irreducible varieties, and this is enough since these are the generators of $K_0(\mathsf{Var}_\mathbb{C})$.[3] Additionally, the induction step requires the existence of smooth compactifications, which also is due to Hironaka's theorem and depends on the characteristic being zero. □

Now if we let $G = \mathrm{SB}$, the monoid of stable birational equivalence classes of varieties (over $\mathbb{C}$), we obtain a surjective monoid homomorphism $f_{\mathrm{SB}} : M \to \mathrm{SB}$ that satisfies the two conditions of the theorem above. This gives rise to a surjective ring homomorphism $g_{\mathrm{SB}} : K_0(\mathsf{Var}_\mathbb{C}) \to \mathbb{Z}[\mathrm{SB}]$.

Corollary 2.2. *Let $X_1, \dots, X_m$, and $Y_1, \dots, Y_n$ be smooth complete varieties. Suppose we have $\sum_i m_i[X_i] = \sum_j n_j[Y_j]$ in $K_0(\mathsf{Var}_\mathbb{C})$. Then in fact, $m = n$, and (after suitably renumbering) the X_i and Y_i are stably birational, and $m_i = n_i$.*

Proof. Send the equation through g_{SB}. □

Since this ring homomorphism is surjective, we would like to find out its kernel. It turns out that the kernel is just the principal ideal generated by the Lefschetz motive, as mentioned above.

Proposition 2.3. *The kernel of $\varphi := g_{\mathrm{SB}} : K_0(\mathsf{Var}_\mathbb{C}) \to \mathbb{Z}[\mathrm{SB}]$ is generated by $\mathbb{L}$.*

[3] We will not prove this fact.

First step. First we show that indeed $(\mathbb{L}) \subset \ker\varphi$. Since $[\mathbb{P}^1] = [\mathbb{P}^0] + [\mathbb{A}^1]$, we have $\varphi(\mathbb{L}) = 0$. Conversely, if some element is in the kernel of φ, express it as a linear combination $\sum_i m_i[X_i] + \sum_j n_j[Y_j]$ of classes of smooth complete varieties where $m_i > 0$ and $n_j < 0$. By using the corollary above, we conclude that the number of indices i and j must coincide, and that each corresponding X_i and Y_i must be stably birational. Therefore, for any element in the kernel of φ, we can write it as a linear combination $\sum_i m_i([X_i] - [Y_i])$ where X_i and Y_i are stably birational. Hence it suffices to show that each $[X_i] - [Y_i]$ is some multiple of $\mathbb{L}$ in $K_0(\mathsf{Var}_{\mathbb{C}})$.

Notice for $X \times \mathbb{P}^n$, as $[\mathbb{P}^n] = [\mathbb{P}^0] + [\mathbb{A}^1] + \cdots + [\mathbb{A}^n]$, we have $[X \times \mathbb{P}^n] = [X] \cdot [\mathbb{P}^n] = [X] \cdot (1 + \mathbb{L} + \cdots + \mathbb{L}^n)$. Henceforth we may replace $[X \times \mathbb{P}^n]$ with $[X]$ since they differ by an element in $\ker\varphi$, and we may replace the stable birational pair X_i and Y_i with a birational pair X and Y. □

To actually conclude the proof, we need a lemma whose proof I will not provide:

Lemma 2.4. *If X and Y are birational varieties over $\mathbb{C}$, then either X is a blowup of Y along a smooth center $Z \subset Y$ or the other way around.*

Concluding the proof. If X is a blowup of Y along a smooth Z with exceptional divisor E, we have $E \cong Z \times \mathbb{P}^\ell$ for some ℓ and we may write $[X] - [Y] = [X] - ([X] - [E] + [Z]) = [E] - [Z] = (\mathbb{L} + \mathbb{L}^2 + \cdots + \mathbb{L}^\ell) \cdot [Z] \in (\mathbb{L})$. □

Alas, we obtain a nice result considering smooth complete varieties over $\mathbb{C}$:

Corollary 2.5. *Let X and Y be smooth complete varieties over $\mathbb{C}$. Then X and Y are stably birational if and only if $[X] \equiv [Y]$ modulo $\mathbb{L}$ in $K_0(\mathsf{Var}_{\mathbb{C}})$.*

3. Counting rational points over $\mathbb{F}_q$

As we have heavily relied on Hironaka's resolution of singularities, the proof for this theorem does not work over positive characteristic. Nonetheless, we may assume that everything we know holds over $\mathbb{C}$, holds over $\mathbb{F}_q$. Suppose we are working over a finite field $\mathbb{F}_q$ and we have the analogous isomorphism

$$K_0(\mathsf{Var}_{\mathbb{F}_q})/(\mathbb{L}) \xrightarrow{\sim} \mathbb{Z}[\mathrm{SB}].$$

Consider the set map $K_0(\mathsf{Var}_{\mathbb{F}_q}) \to \mathbb{Z}/q\mathbb{Z}$ which sends an isomorphism class $[X]$ of a variety X to the number of its $\mathbb{F}_q$-points $X(\mathbb{F}_q)$, modulo q. It is believable that this map is well-defined, as the number of rational points of $X \setminus Y$ for a subvariety would be the number of rational points of X minus that of Y. Notice that the Lefschetz motive $\mathbb{L}$ has q rational points, so this set map actually factors through $K_0(\mathsf{Var}_{\mathbb{F}_q})/(\mathbb{L}) \to \mathbb{Z}/q\mathbb{Z}$. As in the $\mathbb{C}$ case, we will assume that the Grothendieck ring is generated by smooth complete varieties over $\mathbb{F}_q$. Then we may use Corollary 2.5. to conclude that if X and Y are smooth complete varieties over $\mathbb{F}_q$ which are stably birational, then their isomorphism classes are equivalent modulo $\mathbb{L}$, hence the number of their $\mathbb{F}_q$-rational points modulo q are equal. This is in fact a theorem:

Theorem 3.1. *The number of $\mathbb{F}_q$-rational points modulo q is a stable birational invariant for smooth complete varieties over $\mathbb{F}_q$.*