COMMUTATIVE ALGEBRA HOMEWORK III

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Problem 1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Show that a finitely generated A-module M is flat if and only if $M/\mathfrak{m}^n M$ is flat as an $A/\mathfrak{m}^n$-module for every n.*

Proof. Suppose M is flat. It suffices to show that for $A/\mathfrak{m}^n$-modules, we have $- \otimes_{A/\mathfrak{m}^n} (M/\mathfrak{m}^n M)$ exact. Suppose we have some $A/\mathfrak{m}^n$-module N. Then $N \otimes_A M \cong N \otimes_{A/\mathfrak{m}^n} A/\mathfrak{m}^n \otimes_A M \cong N \otimes_{A/\mathfrak{m}^n} (M/\mathfrak{m}^n M)$, where since $- \otimes_A M$ is exact, this is also exact.

Now suppose $M/\mathfrak{m}^n M$ is flat for all n, and let I be an arbitrary ideal of A. Consider the following diagram in Mod_A

$$\begin{tikzcd}
{} \arrow[r, dashed] & I/(I \cap \mathfrak{m}^n) \arrow[r] & A/\mathfrak{m}^n \arrow[r] & A/(I + \mathfrak{m}^n) \arrow[r] & 0 \\
0 \arrow[r] & I \arrow[r] \arrow[u, "\text{coker}"] & I \times A \arrow[r] \arrow[u, "\text{coker}"] & A \arrow[r] \arrow[u, "\text{coker}"] & 0 \\
0 \arrow[r] & I \cap \mathfrak{m}^n \arrow[r] \arrow[u, hook] & I \times \mathfrak{m}^n \arrow[r] \arrow[u, hook] & I + \mathfrak{m}^n \arrow[r] \arrow[u, hook] & 0 \\
 & & & 0 \arrow[u] \arrow[r, dashed, no head] & {}
\end{tikzcd}$$

where the maps in the bottom row are given by $a \mapsto (a, a)$ and $(a, b) \mapsto a - b$, respectively. The top row is defined by extending these maps, and exactness is obvious. Then from the Snake Lemma, we may obtain an SES

$$0 \to I/(I \cap \mathfrak{m}^n) \to A/\mathfrak{m}^n \to A/(I + \mathfrak{m}^n) \to 0.$$

Note that each term is annihilated by $\mathfrak{m}^n$, hence naturally a $A/\mathfrak{m}^n$-module. By assumption $- \otimes_{A/\mathfrak{m}^n} M/\mathfrak{m}^n M$ is exact, so by applying this we obtain

$$0 \to (I/I \cap \mathfrak{m}^n) \otimes_{A/\mathfrak{m}^n} M/\mathfrak{m}^n M \to M/\mathfrak{m}^n M \to A/(I + \mathfrak{m}^n) \otimes_{A/\mathfrak{m}^n} M/\mathfrak{m}^n M \to 0$$

where since $M/\mathfrak{m}^n M \cong A/\mathfrak{m}^n \otimes_A M$, the first part of the sequence is equivalent to

$$0 \to I/(I \cap \mathfrak{m}^n) \otimes_A M \to M/\mathfrak{m}^n M.$$

Now consider another SES

$$0 \to \mathfrak{m}^n \cap I \to I \to I/(I \cap \mathfrak{m}^n) \to 0$$

and apply $- \otimes_A M$ to get

$$(\mathfrak{m}^n \cap I) \otimes_A M \to I \otimes_A M \to I/(I \cap \mathfrak{m}^n) \otimes_A M \to 0.$$

Date: April 27, 2024.

Now we show the kernel of $\varphi : I \otimes_A M \to M$ is trivial. Suppose k maps to zero under this map. Then considering the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
\mathfrak{m}^n \cap I \otimes_A M \arrow[r] & I \otimes_A M \arrow[r] \arrow[d, "\varphi"'] & I/(I \cap \mathfrak{m}^n) \otimes_A M \arrow[r] \arrow[d] & 0 \\
 & M \arrow[r] & M/\mathfrak{m}^n M &
\end{tikzcd}$$

it must also map to zero in $M/\mathfrak{m}^n M$, hence by injectivity, must map to zero in $I/(I \cap \mathfrak{m}^n) \otimes_A M$ via the top right horizontal arrow. By exactness of the top row, we conclude that k is in the image of $\mathfrak{m}^n \cap I \otimes_A M \to I \otimes_A M$. Since this holds for all n, an element that maps to k must be in $\mathfrak{m}^n \cap I \otimes_A M$ for all n. By Artin-Rees, we have $\mathfrak{m}^{n+N} \cap I = \mathfrak{m}^n(\mathfrak{m}^N \cap I) \subset \mathfrak{m}^n I$ for $n \geq 0$ and some large enough N. Hence $\ker \varphi \subset \bigcap_{n>>0} \operatorname{im}((\mathfrak{m}^n I) \otimes_A M \to I \otimes_A M) = \bigcap_{n>>0} \mathfrak{m}^n (I \otimes_A M) = 0$ by Krull's intersection theorem. Since I was an arbitrary ideal, it follows that M is flat.

$\square$

Problem 2. *Let $\mathfrak{m}$ be a maximal ideal of A. Show $A \to \widehat{(A, \mathfrak{m})}$ factors through the localization map $A \to A_{\mathfrak{m}}$.*

Proof. It suffices to show that the map $A \to \widehat{(A, \mathfrak{m})}$ sends every element of $A - \mathfrak{m}$ to units. Suppose $s \in A - \mathfrak{m}$. This element is sent to $(s_0, s_1, s_2, \ldots) \in \widehat{A}$ where $s_i \equiv s \mod \mathfrak{m}^i$, and $s_0 = s$. Since $\mathfrak{m}$ is maximal, we may find some $a \in A$ such that $as \equiv 1 \mod \mathfrak{m}$, hence $as \mapsto (1, as_1 + as_0 - 1, as_2, as_3, \ldots) = (1, as_1, as_2, as_3, \ldots)$, which is a unit since each term must be units. This is due to the compatibility conditions of the completion. Since as maps to a unit, it follows that s maps to a unit. Hence elements in $A - \mathfrak{m}$ are sent to units in $\widehat{A}$, thus the completion map factors through the canonical localization map $A \to A_{\mathfrak{m}}$. $\square$

Problem 3.

Problem 4. *Let M, N be finitely generated modules over a Noetherian local ring A, such that $\widehat{M} \cong \widehat{N}$ as $\widehat{A}$-modules.*

1. *Show that* $\widehat{\operatorname{Hom}_A(M, N)} \cong \operatorname{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N})$.
2. *Let* $\widehat{\mathfrak{m}} \leq \widehat{A}$ *be the maximal ideal. Show that* $\widehat{\mathfrak{m}} \operatorname{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N})$ *consists of maps* $\widehat{M} \to \widehat{\mathfrak{m}}\widehat{N} \subset \widehat{N}$.
3. *Let* $\varphi \in \operatorname{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N})$ *be an isomorphism. Use Nakayama and (2) to show that if* $\varphi' \in \operatorname{Hom}_A(M, N)$ *differs from* φ *by an element of*
$$\widehat{\mathfrak{m}} \operatorname{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N}),$$
then φ' *is surjective.*
4. *Show that there exists* $\varphi' \in \operatorname{Hom}_A(M, N)$ *and* $\varphi'' \in \operatorname{Hom}_A(N, M)$ *that are surjective. Conclude that* φ' *is an isomorphism.*

Proof. (1) We first show that $\operatorname{Hom}_A(M, N)$ is finitely generated. Since M is finitely generated, we have $A^n \to M \to 0$ for some n. Apply $\operatorname{Hom}_A(-, N)$ to obtain $0 \to \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(A^n, N) \cong N^n$. Since $\operatorname{Hom}_A(M, N)$ is isomorphic to a submodule of a finitely generated module over a noetherian ring, it is also finitely generated. Thus we may conclude $\widehat{\mathrm{Hom}_A(M,N)} \cong \widehat{A} \otimes_A \mathrm{Hom}_A(M,N)$. Also using Eisenbud, Proposition 2.10, since M is finitely presented (since noetherian and finitely generated; consider kernel of $A^n \to M$) and since $\widehat{A}$ is flat over A, we may conclude that $\widehat{A} \otimes_A \mathrm{Hom}_A(M,N) \cong \mathrm{Hom}_{\widehat{A}}(\widehat{A} \otimes_A M, \widehat{A} \otimes_A N)$ which again is isomorphic to $\mathrm{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N})$. □

Proof. (2) Elements of $\mathrm{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N})$ are $\widehat{A}$-module homomorphisms from $\widehat{M}$ to $\widehat{N}$, so obviously the elements of $\widehat{\mathfrak{m}}\,\mathrm{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N})$ are the maps $\widehat{M} \to \widehat{\mathfrak{m}}\widehat{N}$. □

Proof. (3) I will assume the typo actually means $\varphi' \in \mathrm{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N})$. Suppose we have $\varphi' - \varphi \in \widehat{\mathfrak{m}}\,\mathrm{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N})$. By (2), this implies that $\varphi' - \varphi$ maps into $\widehat{\mathfrak{m}}\widehat{N}$. Thus, the induced maps $\overline{\varphi}, \overline{\varphi'} : \widehat{M}/\widehat{\mathfrak{m}}\widehat{M} \to \widehat{N}/\widehat{\mathfrak{m}}\widehat{N}$ agree, and since φ is an isomorphism, we must have $\overline{\varphi'}$ surjective. Consider the exact sequence $\widehat{M} \to \widehat{N} \to \operatorname{coker}\varphi' \to 0$, and apply $- \otimes_{\widehat{A}} \widehat{A}/\widehat{\mathfrak{m}}$ to get $\widehat{M}/\widehat{\mathfrak{m}}\widehat{M} \to \widehat{N}/\widehat{\mathfrak{m}}\widehat{N} \to \operatorname{coker}\varphi' \otimes_{\widehat{A}} \widehat{A}/\widehat{\mathfrak{m}} \to 0$ where $\operatorname{coker}\varphi' \otimes_{\widehat{A}} \widehat{A}/\widehat{\mathfrak{m}} \cong \operatorname{coker}\overline{\varphi'} = 0$. Thus $\operatorname{coker}\varphi' = \widehat{\mathfrak{m}} \operatorname{coker}\varphi'$, and we may apply Nakayama to conclude that φ' is surjective. □

Proof. (4) Since $\widehat{M}$ and $\widehat{N}$ are isomorphic, let $f : \widehat{M} \to \widehat{N}$ be an isomorphism. We claim that there exists some $g : M \to N$ such that $\widehat{g} - f \in \widehat{\mathfrak{m}}\,\mathrm{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N})$. Consider the ring $\widehat{\mathrm{Hom}_A(M,N)}/\widehat{\mathfrak{m}}\widehat{\mathrm{Hom}_A(M,N)}$. This is isomorphic to $\widehat{A}/\widehat{\mathfrak{m}} \otimes_{\widehat{A}} \widehat{\mathrm{Hom}_A(M,N)}$. Again, this is isomorphic to $\widehat{A}/\widehat{\mathfrak{m}} \otimes_{\widehat{A}} \widehat{A} \otimes_A \mathrm{Hom}_A(M,N) \cong \widehat{A}/\widehat{\mathfrak{m}} \otimes_A \mathrm{Hom}_A(M,N)$ since hom is finitely generated, as we have shown in (1). By Atiyah & Macdonald, Theorem 10.15 (iii), we have $\widehat{A}/\widehat{\mathfrak{m}} \cong A/\mathfrak{m}$, so this is isomorphic to $A/\mathfrak{m} \otimes_A \mathrm{Hom}_A(M,N) \cong \mathrm{Hom}_A(M,N)/\mathfrak{m}\,\mathrm{Hom}_A(M,N)$. Therefore, we have a surjection from $\mathrm{Hom}_A(M,N)$ to $\widehat{\mathrm{Hom}_A(M,N)}/\widehat{\mathfrak{m}}\widehat{\mathrm{Hom}_A(M,N)}$. Also, since f is induced by some element of $\mathrm{Hom}_A(M,N)$ by how we constructed the isomorphism in (1), we conclude that there exists some $g \in \mathrm{Hom}_A(M,N)$ such that $\widehat{g} - f \in \widehat{\mathfrak{m}}\,\mathrm{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N})$. By (3), it follows that $\widehat{g} : \widehat{M} \to \widehat{N}$ is surjective. By considering the exact sequence $M \to N \to \operatorname{coker} g \to 0$ and taking inverse limits to get $\widehat{M} \to \widehat{N} \to \widehat{\operatorname{coker} g} \to 0$, we have $\widehat{\operatorname{coker} g} = 0$, which means that $\mathfrak{m} \operatorname{coker} g = \operatorname{coker} g$. Apply Nakayama to conclude that $\operatorname{coker} g = 0$, i.e. g is surjective. Take $\varphi' = g$.

For φ'', repeat the exact same process above, in the other direction, to obtain a surjective morphism $\varphi'' \in \mathrm{Hom}_A(N, M)$. By Eisenbud, Corollary 4.4a, surjective endomorphisms of finitely generated modules are isomorphisms. In this case $\varphi'' \circ \varphi'$ is a surjective endomorphism of M, where since M is a finitely generated A-module, it is an isomorphism. Then φ' must be injective, and since φ' was surjective to begin with, M and N are isomorphic. □

Problem 5. *Let A be a Noetherian ring, and let $\mathfrak{m} = (f_1, \ldots, f_n) \leq A$. Show that*

$$\widehat{(A, \mathfrak{m})} \cong A[[x_1, \ldots, x_n]]/(x_1 - f_1, \ldots, x_n - f_n).$$

Proof. The ring $A[[x_1, \ldots, x_n]]/(x_1 - f_1, \ldots, x_n - f_n)$ is isomorphic to

$$A[x_1, \ldots, x_n]/(x_1 - f_1, \ldots, x_n - f_n) \otimes_{A[x_1,\ldots,x_n]} A[[x_1, \ldots, x_n]].$$

Recall that if M is a finite A-module where A is noetherian, we have $\widehat{M} \cong M \otimes_A \widehat{A}$. Here, $A[[x_1, \ldots, x_n]]$ is the completion of $A[x_1, \ldots, x_n]$ with respect to $(x_1, \ldots, x_n)$, hence the original ring is isomorphic to the completion of $A[x_1, \ldots, x_n]/(x_1 - f_1, \ldots, x_n - f_n)$ at $(x_1, \ldots, x_n)$, i.e. the completion of A at $(f_1, \ldots, f_n) = \mathfrak{m}$. $\square$

Problem 6.