ALGEBRAIC TOPOLOGY

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1. Cohomology Groups

1.1. **Cohomology in Terms of Homology.** Given a chain complex $C_\bullet$ of free abelian groups, dualize this chain complex via $\mathrm{Hom}(-, G)$ to get a cochain complex. Define $H^n(C_\bullet; G)$ to be the nth homology of this cochain complex. We construct a homomorphism

$$h : H^n(C_\bullet; G) \to \mathrm{Hom}(H_n(C_\bullet), G)$$

from the nth cohomology of $C_\bullet$ to the hom group from the nth homology of $C_\bullet$ to G. Prior to this, we define a homomorphism

$$h' : \ker(\delta^n) \to \mathrm{Hom}(H_n(C_\bullet), G)$$

where δ^n is the map $C_n^\vee \to C_{n+1}^\vee$.

Suppose we have some $\varphi \in \ker(\delta^n)$. By definition, $\varphi \in C_n^\vee$, which means that it is some homomorphism $\varphi : C_n \to G$. Also by definition we have that $\delta^n(\varphi) = 0$, but $\delta^n(\varphi) = \varphi \circ \partial_{n+1}$ where ∂ is the differential in the original chain complex. Since $\varphi \circ \partial_{n+1} = 0$, it follows that $\varphi(\mathrm{im}(\partial_{n+1})) = 0$, where since $\mathrm{im}(\partial_{n+1}) = B_n$, we have $\varphi(B_n) = 0$. Now if we consider the restriction $\varphi|_{Z_n} : Z_n \hookrightarrow C_n \xrightarrow{\varphi} G$, it still holds that B_n is sent to zero. By the universal property of quotient groups, we conclude there is a unique morphism $Z_n/B_n \to G$ that commutes with other morphisms.

$$\begin{tikzcd}
C_n \arrow[r, "\varphi"] & G \\
Z_n \arrow[u, hook] \arrow[r, two heads] & Z_n/B_n \arrow[u, dashed, "\exists!"']
\end{tikzcd}$$

Call this $\overline{\varphi}$. This $h' : \varphi \mapsto \overline{\varphi}$ is obviously a homomorphism. Now, we want to show that it sends elements of $\mathrm{im}(\delta^{n-1}) \subset \ker(\delta^n)$ to zero. For φ to be in $\mathrm{im}(\delta^{n-1})$ means that

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there exists some $\psi : C_{n-1} \to G$ such that $\psi \circ \partial_n = \varphi$. In this case, $\overline{\varphi}(Z_n/B_n) = \varphi(Z_n)$, but since $\psi \circ \partial_n = \varphi$, we have $\varphi(Z_n) = \psi(\partial_n(Z_n)) = \psi(0) = 0$. Therefore $\overline{\varphi} = 0$, so h' indeed sends $\mathrm{im}(\delta^{n-1})$ to zero. Again, by the universal property of quotient groups, there exists a unique morphism $h : \ker(\delta^n)/\operatorname{im}(\delta^{n-1}) \to \mathrm{Hom}(H_n(C_\bullet), G)$ that commutes with h'. Since $\ker(\delta^n)/\operatorname{im}(\delta^{n-1})$, this is our desired $h : H^n(C_\bullet; G) \to \mathrm{Hom}(H_n(C_\bullet), G)$.

Consider the SES $0 \to Z_n \to C_n \xrightarrow{\partial_n} B_{n-1} \to 0$, which splits since B_{n-1} is free. Therefore, there exists a projection $p : C_n \to Z_n$ such that $p|_{Z_n} = \mathrm{id}_{Z_n}$. Suppose we are given some $\overline{\varphi} \in \mathrm{Hom}(H_n(C_\bullet), G)$. Then $\overline{\varphi} : H_n(C_\bullet) \to G$, and precompose with $C_n \xrightarrow{p} Z_n \xrightarrow{\pi} Z_n/B_n = H_n(C_\bullet)$ to get a morphism $C_n \to G$. This morphism vanishes on B_n, by definition. Vanishing on B_n means that, if pulled back via $\partial_{n+1} : C_{n+1} \to C_n$, the resulting morphism vanishes, i.e. this morphism is in the kernel of δ^n. Therefore, we have a map from $\mathrm{Hom}(H_n(C_\bullet), G)$ to $\ker(\delta^n)$. Since this map is given by $\overline{\varphi} \mapsto \overline{\varphi} \circ \pi \circ p$, we can check that extending via $Z_n \hookrightarrow C_n$ gives $\overline{\varphi} \circ \pi$ since p restricts to the identity, and by the universal property, we conclude that the unique map determined by $\overline{\varphi} \circ \pi \circ p$ via h' is just $\overline{\varphi}$ itself. Therefore, not only is h' surjective, it actually has a section, which gives rise to a section of h. Therefore, we have a split exact sequence

$$0 \to \ker h \to H^n(C_\bullet; G) \xrightarrow{h} \mathrm{Hom}(H_n(C_\bullet), G) \to 0.$$

From this, we have $H^n(C_\bullet; G) \cong \ker h \oplus \mathrm{Hom}(H_n(C_\bullet), G)$.

Now we wish to find out what $\ker h$ actually is. To do this, we first consider the SES of chain complexes

$$0 \to Z_\bullet \to C_\bullet \to B_{\bullet-1} \to 0$$

where the differentials of $Z_\bullet$ and $B_\bullet$ are both given as the zero map. Dualize both the SES and the differentials via $\mathrm{Hom}(-, G)$ to get another SES of (co)chain complexes

$$0 \to B^\vee_{\bullet-1} \to C^\vee_\bullet \to Z^\vee_\bullet \to 0.$$

Exactness is guaranteed by $C_n \cong Z_n \oplus B_{n-1}$ and $\mathrm{Hom}(A \oplus B, G) \cong \mathrm{Hom}(A, G) \times \mathrm{Hom}(B, G)$. Now from this SES, we get a LES of homology given by

$$\cdots \to B^\vee_{n-1} \to H^n(C_\bullet; G) \to Z^\vee_n \xrightarrow{i^\vee_n} B^\vee_n \to \cdots$$

where the indices increase, since we dualized the chain complexes. Note that the map $i^\vee_n : Z^\vee_n \to B^\vee_n$ is the dual of the inclusion $i : B_n \to Z_n$. From this LES, we can extract a SES

$$0 \to \mathrm{coker}(i^\vee_{n-1}) \to H^n(C_\bullet; G) \to \ker(i^\vee_n) \to 0$$

but note that elements of $\ker(i^\vee_n)$ are homomorphisms $Z_n \to G$ that vanish on B_n, i.e. is $\mathrm{Hom}(Z_n/B_n, G) = \mathrm{Hom}(H_n(C_\bullet), G)$. The map h we have defined earlier corresponds to this surjective map. Therefore we have $H^n(C_\bullet; G) \cong \mathrm{coker}(i^\vee_{n-1}) \oplus \mathrm{Hom}(H_n(C_\bullet), G)$ so far. We wish to find out the term $\mathrm{coker}(i^\vee_{n-1})$.

1.2. **Ext Groups as Derived Functors of Hom.** Before we find out $\mathrm{coker}(i^\vee_{n-1})$, we first define $\mathrm{Ext}^1(A, G)$ for an abelian group A. Consider an exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to A \to 0$$

where F_i are free abelian groups. Apply the functor $\mathrm{Hom}(-, G)$ to this sequence to obtain

$$0 \to \mathrm{Hom}(A, G) \to \mathrm{Hom}(F_0, G) \to \mathrm{Hom}(F_1, G) \to \mathrm{Hom}(F_2, G) \to \cdots$$

where the first three maps are exact, because $\mathrm{Hom}(-, G)$ is left-exact. If we truncate $\mathrm{Hom}(A, G)$, we obtain a complex

$$0 \to \mathrm{Hom}(F_0, G) \to \mathrm{Hom}(F_1, G) \to \mathrm{Hom}(F_2, G) \to \cdots .$$

We define $\mathrm{Ext}^1(A, G)$ to be the first (co)homology of this complex. It is a fact of homological algebra that the Ext group does not depend on the free resolution of A. Note that the zeroth (co)homology is isomorphic to the kernel of the second map, which is isomorphic to $\mathrm{Hom}(A, G)$. Also, via the connecting homomorphism, given a SES

$$0 \to A \to B \to C \to 0$$

of abelian groups, we may consider the SES of free resolutions

$$0 \to F_\bullet(A) \to F_\bullet(B) \to F_\bullet(C) \to 0$$

and dualize and consider the LES of (co)homology to get a LES

$$0 \to C^\vee \to B^\vee \to A^\vee \to \mathrm{Ext}^1(C, G) \to \mathrm{Ext}^1(B, G) \to \mathrm{Ext}^1(A, G) \to \cdots .$$

(Note that if we define $\mathrm{Ext}^n(A, G)$ to be the n-th cohomology of the dualized free resolution, then the LES naturally extends to Ext groups for $n > 1$.)

Another property of Ext groups is that for free groups F, we have $\mathrm{Ext}^1(F, G) = 0$. To see this, take the free resolution $0 \to F \to F \to 0$ of F, and dualize to get $0 \to F^\vee \to F^\vee \to 0$, so when the first $F^\vee$ is truncated, all positive cohomology vanishes. A more general fact about Ext, is that for projective groups P, we have $\mathrm{Ext}^i(P, G) = 0$ for all $i > 0$. The projective condition is a generalization of freeness, so obviously free groups are projective. Projective groups are defined to satisfy the lifting property, and using this one may prove that in fact Ext groups defined by using projective resolutions are the same as the ones using free resolutions.

Using this algebraic machinery, consider the SES

$$0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1}(C_\bullet) \to 0$$

and take the LES

$$\begin{aligned} &0 \to \mathrm{Hom}(H_{n-1}(C_\bullet), G) \to \mathrm{Hom}(Z_{n-1}, G) \xrightarrow{i_{n-1}^\vee} \mathrm{Hom}(B_{n-1}, G) \\ &\longrightarrow \mathrm{Ext}^1(H_{n-1}(C_\bullet), G) \to \mathrm{Ext}^1(Z_{n-1}, G) \to \mathrm{Ext}^1(B_{n-1}, G) \to \cdots \end{aligned}$$

Note that Z_{n-1} and B_{n-1} are subgroups of a free abelian group, so are free abelian, hence their higher Ext groups vanish. Therefore, $\mathrm{coker}(i_{n-1}^\vee)$ turns out to be $\mathrm{Ext}^1(H_{n-1}(C_\bullet), G)$. From this, we have the Universal Coefficient Theorem:

Theorem 1.1. *Let $C_\bullet$ be a chain complex of free abelian groups. Then,*

$$H^n(C_\bullet; G) \cong \mathrm{Ext}^1(H_{n-1}(C_\bullet), G) \oplus \mathrm{Hom}(H_n(C_\bullet), G).$$

In other words, the cohomology groups, which are the homology groups of the dual complex, are determined entirely by homology groups of the original complex.

1.3. **Cohomology of Spaces.** Now that we have studied how to work with chains of abelian groups and their cohomology, we return to topology. Recall that we have defined the singular n-chains $C_n(X)$ to be the free abelian groups generated by continuous functions $\sigma : \Delta^n \to X$. Using this, we define the singular n-cochains $C^n(X; G) := \mathrm{Hom}(C_n(X), G)$, and the coboundary maps δ as the dual of ∂. Define nth cohomology to be the nth cohomology of this cochain complex. Via the universal coefficient theorem, we have

$$H^n(X; G) \cong \mathrm{Ext}^1(H_{n-1}(X), G) \oplus \mathrm{Hom}(H_n(X), G).$$

We plug in $n = 0$ to get $H^0(X; G) \cong \mathrm{Hom}(H_0(X), G)$. Since $H_0(X)$ correspond to the path components of X, and elements of the zeroth cohomology are locally constant functions (functions such that their values on each $[v_0, v_1] \to X$ are zero when taken the boundary) intuitively this quite makes sense.

Plug in $n = 1$ to get $H^1(X; G) \cong \mathrm{Ext}^1(H_0(X), G) \oplus \mathrm{Hom}(H_1(X), G)$ where the group $H_0(X)$ is free on the path components of X, so the Ext vanishes, yielding $H^1(X; G) \cong \mathrm{Hom}(H_1(X), G)$.

Note that the algebraic objects above are all abelian groups, but we may loosen this condition to all R-modules, R a commutative ring. Not just any commutative ring, but a PID since we want R-submodules of free R-modules to be free. This holds if and only if R is a PID. Considering this, we may take R to be a field F, and in this case since all F-modules are free, via the universal coefficient theorem we have $H^n(X;F) \cong \mathrm{Hom}_{\mathsf{Vect}_F}(H_n(X;F),F)$. Note that in the case $F = \mathbb{F}_p$ or $F = \mathbb{Q}$, it holds that $\mathrm{Hom}_{\mathsf{Vect}_F}(H_n(X;F),F) = \mathrm{Hom}_{\mathsf{Ab}}(H_n(X;F),F)$. Beware, this does not hold for arbitrary fields, such as $\mathbb{F}_{p^2}$.

We define reduced cohomology groups, just as we defined reduced homology groups. Suppose we have a chain complex $C_\bullet(X)$, and augment it with the degree map ϵ to get $\cdots \to C_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$. Dualize this chain complex via $\mathrm{Hom}(-,G)$ and take the homology, and the resulting groups are the reduced cohomology groups. Note that $\widetilde{H}^n(X;G)$ agrees with $H^n(X;G)$ for $n > 0$, and for $n = 0$ we have $\widetilde{H}^0(X;G) \cong \mathrm{Ext}^1(\widetilde{H}_{-1}(X),G) \oplus \mathrm{Hom}(\widetilde{H}_0(X),G)$ where since $\widetilde{H}_{-1}(X) = \mathbb{Z}/\operatorname{im}(\epsilon)$ which is zero since ϵ is surjective, we have $\widetilde{H}^0(X;G) \cong \mathrm{Hom}(\widetilde{H}_0(X),G)$. Heuristically, the elements of $\widetilde{H}^0(X;G)$ are the locally constant functions from X to G, modded out by globally constant functions.

Now we define relative cohomology, just as we defined relative homology. Recall that for a pair (X,A), we have define the relative chain complex by the SES

$$0 \to C_n(A) \to C_n(X) \to C_n(X,A) \to 0$$

where $C(X,A)$ is just $C(X)/C(A)$, and the inclusion map is given by composing $\sigma : \Delta^n \to A$ with $i : A \to X$ the inclusion. This short exact sequence splits, because we can define a retract $C_n(X) \to C_n(A)$ given by restricting $\sigma : \Delta^n \to X$ to $\sigma : \Delta^n \to A$, depending on the image of σ. Since split exact sequences are preserved by additive functors, and $\mathrm{Hom}(-,G)$ is additive, we get the dual SES

$$0 \to C^n(X,A;G) \to C^n(X;G) \to C^n(A;G) \to 0.$$

From this, we may obtain the long exact sequence of cohomology groups

$$\cdots \to H^n(X,A;G) \to H^n(X;G) \to H^n(A;G) \xrightarrow{\delta} H^{n+1}(X,A;G) \to \cdots$$

where δ is the connecting homomorphism obtained in the construction of the long exact sequence of homology.

1.4. **Induced Homomorphisms.** Recall that a continuous map $f : X \to Y$ induced a chain map $f_\# : C_n(X) \to C_n(Y)$. The dual of this map is called the cochain map, $f^\# : C^n(Y;G) \to C^n(X;G)$. The relation $f_\#\partial = \partial f_\#$ dualizes to $\delta f^\# = f^\#\delta$.

1.5. **Homotopy Invariance and Excision.** As in homology, homotopic maps f,g from X to Y define the same homomorphisms $f^* = g^*$ on cohomology. Again, as in homology, this comes from the fact that $f_\#$ and $g_\#$ are chain homotopic, and dualizing the relations. Chain homotopic chain maps induce the same homomorphisms on homology. Also if we have subspaces $Z \subset A \subset X$ where $\overline{Z} \subset A^\circ$, the inclusion $i : (X - Z, A - Z) \to (X,A)$ induces isomorphisms on cohomology groups.

1.6. **Axioms for Cohomology.** A reduced cohomology theory is a sequence of contravariant functors from CW complexes to abelian groups, with natural coboundary homomorphisms $\widetilde{h}^n(A) \to \widetilde{h}^{n+1}(X/A)$ for CW pairs (X,A), such that

(1) If $f \simeq g$, then $f^* = g^*$.
(2) For each CW pair (X,A) there exists a long exact sequence of cohomology.
(3) If $X = \vee_\alpha X_\alpha$, then $\widetilde{h}^n(X) \cong \prod_\alpha(X_\alpha)$, similar to how homology is a covariant functor from the category of pointed CW complexes to abelian groups and preserved coproducts. In this case, it is a product.

1.7. **Simplicial and Cellular Cohomology.** If X is a Δ-complex and A a subcomplex, the simplicial chains $\Delta_n(X, A)$ dualize to simplicial cochains $\Delta^n(X, A; G) := \mathrm{Hom}(\Delta_n(X, A), G)$. The resulting cohomology groups are defined to be the simplicial cohomology groups, denoted $H^n_\Delta(X, A; G)$. Also, the cellular cochain complex $H^n(X^n, X^{n-1}; G)$ is isomorphic to the dual of the cellular chain complex. We will not prove this here.

1.8. **Mayer-Vietoris.** Suppose $X = A^\circ \cup B^\circ$. Then consider the SES

$$0 \to C^\bullet(A + B; G) \to C^\bullet(A; G) \oplus C^\bullet(B; G) \to C^\bullet(A \cap B; G) \to 0$$

and the LES of cohomology induced by this chain is the Mayer-Vietoris sequence.

2. Cup Product

Definition 2.1. Suppose $\varphi \in C^k(X; R)$ and $\psi \in C^l(X; R)$. Define $\varphi \smile \psi \in C^{k+l}(X; R)$ to be a cochain such that for $\sigma : \Delta^{k+l} \to X$ we have

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|_{[v_0,\dots,v_k]})\psi(\sigma|_{[v_k,\dots,v_{k+l}]}).$$

Lemma 2.2. *Suppose we have $\varphi \in C^k(X; R)$ and $\psi \in C^l(X; R)$. Then we have $\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^k \varphi \smile \delta\psi$.*

Proof. Calculation. □

Using this cup product on cochains and the lemma, we may conclude that this induces a product on cohomology, the cup product. The product is associative and distributive.

3. Poincaré Duality

Theorem 3.1. *Let M be a* ***closed*** *connected n-manifold.*

1. *If M is R-orientable, then the map $H_n(M; R) \to H_n(M|x; R) \cong R$ is an isomorphism for all $x \in M$. (This implies that the top homology is isomorphic to R.)*
2. *If M is not R-orientable, the map $H_n(M; R) \to H_n(M|x; R) \cong R$ is injective, with image $\{r \in R \mid 2r = 0\}$ for all $x \in M$.*
3. *The homology vanishes for $i > n$.*

An element of $H_n(M; R)$ whose image in the local homology $H_n(M|x; R)$ is a generator for all x is called a fundamental class for M with coefficients in R. Denote by $[M]$, but note that this need not be unique. This fundamental class exists if M is closed and R-orientable. This follows from the following lemma:

Lemma 3.2. *Let M be an n-manifold. Let $A \subset M$ be a compact subset. Then*

1. *If $x \mapsto \alpha_x$ is a section of the covering space $M_R \to M$, then there is a unique class $\alpha_A \in H_n(M|A; R)$ whose image in $H_n(M|x; R)$ is α_x for all $x \in A$.*
2. *$H_i(M|A; R) = 0$ for $i > n$.*

Proof. Observe if the lemma is true for compact sets A, B and $A \cap B$, it holds for $A \cup B$ too. ($A \cap B$ being compact follows from Hausdorffness of the manifold, actually.) Consider the MV sequence

$$0 \to H_n(M|A \cup B) \to H_n(M|A) \oplus H_n(M|B) \to H_n(M|A \cap B) \to \cdots$$

where the first map is $\alpha \mapsto (\alpha, -\alpha)$ and the second map is $(\alpha, \beta) \mapsto \alpha + \beta$. □