BIANCHI GROUP QUOTIENTS OF HYPERBOLIC 3-SPACE

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Abstract. This is my attempt at understanding the paper *On The Growth of the Number of Totally Geodesic Surfaces in some Hyperbolic 3-Manifolds*, by Junehyuk Jung. (arXiv:1801.00875)

The main theorem is:

Theorem 1. *Let $d > 0$ be a squarefree integer and $d \equiv 3 \mod 4$. Assume there does not exist an invariant of $\mathrm{Cl}(\mathbb{Q}(\sqrt{-d}))$ which is divisible by 4.*[1] *Let $\Gamma_d = \mathrm{PSL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-d})})$, and let $\xi(X) = \#$ of immersed totally geodesic surfaces in $\Gamma_d \backslash \mathbb{H}^3$ having area less than X. Then,*

$$\xi(X) = \frac{\tau(d)\pi}{4} \prod_p \left(1 - \frac{\chi_{-d}(p)^2 + \chi_{-d}(p)}{p^2} + \frac{1}{p^3}\right) X + o(X),$$

where $\tau(d)$ is the number of distinct positive integer divisors of n.

Here, an 'invariant' of the class group is short for an 'invariant factor'. I do not know much about the structure of class groups yet, so I cannot say much about this.

I've identified the proof of this main theorem to be in roughly four main steps:

(1) identify each totally geodesic surface in $\Gamma_d \backslash \mathbb{H}^3$ with a maximal Fuchsian subgroup of Γ_d,
(2) express that Fuchsian subgroup as a $\mathbb{Z}$-order of a quaternion algebra over $\mathbb{Q}$,
(3) use quaternion algebra theory to find the area of each totally geodesic surface; we obtain a formula for the area
(4) turn the area formula obtained as such into an asymptotic formula, via Wiener-Ikehara.

Thus, it seems that the geometric statements are translated into quaternion algebra theory, and the calculations are all done there. However, we must note that Step 1 relies on the assumption that there is no invariant factor of $\mathrm{Cl}(\mathbb{Q}(\sqrt{-d}))$ divisible by four, although the author speculates that the main theorem holds even without this assumption.

1. Preliminaries: A model of $\mathbb{H}^3$ via Hamiltonian quaternions

Before we do anything, we will understand how $\mathbb{H}^3$ is modelled. (Caution: $\mathbb{H}$ without a superscript will indicate the Hamiltonian quaternions.)

Definition 2. Define *upper half space* to be the topological space $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_{>0}$. Define the length element $ds^2 = \frac{|dx|^2 + dy^2}{y^2}$ which makes this into *hyperbolic 3-space*.

Analogous to the fact that $\partial \mathbb{H}^2 = S^1$ for hyperbolic 2-space, we can think of the *sphere at infinity* to be the set $\mathbb{C} \times \{0\} \cup \infty$. Using our upper half space model, we can think of geodesic lines as either half-circles or lines which meet the 'sphere' at infinity perpendicularly. Similarly, the geodesic planes are half-spheres or planes that meet the 'sphere' at infinity perpendicularly. Geometrically, it would be more appropriate to call the 'sphere at infinity' as the 'plane at infinity' in our upper half space model, as we can't really see the point at infinity. Overall, our situation is really similar to that of $\mathbb{H}^2$ so geometrically it would be easier to understand our situation as an analogue of the upper half plane model.

Now that we have defined our working model of $\mathbb{H}^3$, we will identify the orientation-preserving isometry group $\mathrm{Isom}^+(\mathbb{H}^3)$ of hyperbolic 3-space, without proof. Recall that $\mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}_2(\mathbb{R})$.

Date: September 10, 2025.

[1] This assumption is weakened (or equivalent) to assuming nonexistence of an element of order 4, in the published version

Consider the embedding $\mathbb{H}^3 \to \mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$ given by sending (x, y) to $x + yj$. Note that $y \in \mathbb{R}_{>0}$, and since $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$, the decomposition of $\mathbb{H}$ as above is valid. Using these quaternionic coordinates, we can define a left-action of $\mathrm{SL}_2(\mathbb{C})$ on $z \in \mathbb{H}^3$ given by $g \cdot z = (az + b)(cz + d)^{-1}$ for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Caution: the multiplication is no longer commutative, so we cannot write the action as fractions without ambiguity. Now, using some facts that are not very clear to me, and some tools like Iwasawa decomposition, we can prove that the orientation-preserving isometry group is indeed $\mathrm{PSL}_2(\mathbb{C})$.[2]

Definition 3 (Kleinian groups)**.** A Kleinian group Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$. For such groups, the quotient $\Gamma\backslash\mathbb{H}^3$ is Hausdorff, and the quotient map is a local isometry at all points with trivial stabilizer.

Definition 4 (Bianchi groups)**.** A Bianchi group is a group of the form $\mathrm{PSL}_2(\mathcal{O}_F)$ for F an imaginary quadratic number field.

Moreover, Bianchi groups are Kleinian. Finally, we have the following two-dimensional definition:

Definition 5 (Fuchsian groups)**.** A Fuchsian group is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ (or equivalently, of $\mathrm{Isom}^+(\mathbb{H}^2)$).

Now I will write down the following definition just in case:

Definition 6. A Fuchsian group Γ is called *elementary* if there exists a nonempty Γ-invariant set in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ that contains at most two points. Equivalently, the group is either cyclic or dihedral.

We are not interested in elementary Fuchsian groups. Nonelementary Fuchsian groups can be categorized by the set of their limit points $L(\Gamma) \subset \partial\mathbb{H}^2$ of the set Γz for $z \in \mathbb{H}^2$. If $L(\Gamma) = \partial\mathbb{H}^2$, then it is said to be a Fuchsian group of the first kind, and otherwise, it is called of the second kind. In this case, the limit set is a nowhere dense perfect subset of $\partial\mathbb{H}^2$, which is sort of a Cantor set.

Now, we have finally understood how the Bianchi groups Γ_d actually act on hyperbolic 3-space. We will look at the properties of their quotients.

2. Step 1: Identifying geodesic surfaces with maximal Fuchsian subgroups

Consider our manifold $\Gamma_d\backslash\mathbb{H}^3$, and let S be an immersed totally geodesic surface in it. Then the lifts of S to the whole $\mathbb{H}^3$ consists of a collection of hyperbolic planes, each of which bound a circle or a line $\mathcal{C}$ in $\partial\mathbb{H}^3 \supset \mathbb{C}$. Note that the group $\mathrm{Stab}(\mathcal{C}, \Gamma_d) := \{\gamma \in \Gamma_d \mid \gamma\mathcal{C} = \mathcal{C}\}$ is conjugate to an arithmetic group of (not necessarily orientation-preserving) isometries of $\mathbb{H}^2$. The group $\mathrm{Stab}^+(\mathcal{C}, \Gamma_d) := \{\gamma \in \mathrm{Stab}(\mathcal{C}, \Gamma_d) \mid \gamma \text{ preserves components of } \partial\mathbb{H}^3 \setminus \mathcal{C}\}$ is a maximal arithmetic Fuchsian subgroup of $\mathrm{PSL}_2(\mathbb{C})$.

2.1. Local and global Eichler symbols.

Note that local quadratic extensions are either ramified, inert, or split. Thus, we aim for a similar classification for quaternion orders. We first define local Eichler symbols and later give a global definition.

Let R be a *complete* DVR with $\mathfrak{p} \in \mathrm{Spm}\, R$, i.e., it is complete in its $\mathfrak{p}$-adic topology. Now let $F = \mathrm{Frac}\, R$, and let B be a quaternion algebra[3] over F. Let $O \subset B$ be an R-order,[4] and let $k = R/\mathfrak{p}$ be the residue field.

The k-algebra $O/\operatorname{rad} O$ is semisimple and has a standard involution. It is either k, a separable quadratic k-algebra, or a quaternion algebra over k. Denote $J = \operatorname{rad} O$.[5]

Definition 7 (Local Eichler symbol)**.** Define $(O/\mathfrak{p})$ to be

(1) $*$ if O/J is a quaternion algebra;
(2) 1 if $O/J \cong k \times k$;

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[2] I still do not understand the intuition behind $\mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}_2(\mathbb{C})$. I do roughly understand that $\mathrm{PSL}_2(\mathbb{R})$ is the group of isometries of $\mathbb{H}^2$ via the upper half plane model.

[3] An F-algebra where there exist $i, j \in B$ such that $1, i, j, ij$ form an F-basis for B, and $i^2 = a, j^2 = b, ji = -ij$ for some $a, b \in F^\times$.

[4] An R-order $O \subset B$ of a quaternion algebra over $\mathrm{Frac}\, R$ is a lattice that is also a subring of B. A (R-)lattice in a finite-dimensional F-vector space V is a finitely generated R-submodule $M \subset V$ with $MF = V$. When R is a PID, M being an R-lattice is equivalent to $M \cong R^n$.

[5] The radical of O is its Jacobson radical viewed as a ring; it is the intersection of all maximal left ideals.

(3) 0 if $O/J \cong k$;
(4) -1 if O/J is a separable quadratic field extension of k.

Now since we are working with $R = \mathbb{Z}$ and hence $F = \mathbb{Q}$, we need a global definition. Let R be a Dedekind domain. For a nonzero prime $\mathfrak{p} \subset R$, define the Eicher symbol at $\mathfrak{p}$ to be $(O/\mathfrak{p}) := (\widehat{O}/\mathfrak{p}\widehat{R})$, where the r.h.s. is the local Eichler symbol and $\widehat{O}$, $\widehat{R}$ are $\mathfrak{p}$-adic completions of O and R, respectively.[6]

If we assume that $|k| < \infty$ (which holds in any case we are interested in) then for $a \in R$, $(a/\mathfrak{p})$ denotes the generalized Kronecker symbol, and is defined to be 0, 1, -1 when $F[x]/(x^2 - a)$ is ramified, split, or inert, respectively. Furthermore when $\operatorname{char} k \neq 2$, this is just the Legendre symbol. Recall the discriminant quadratic form $\Delta : B \to F$ given by $\alpha \mapsto \operatorname{trd}(\alpha)^2 - 4\operatorname{nrd}(\alpha)$. We can calculate the symbol $(O/\mathfrak{p}) = \epsilon$ via $(\Delta(\alpha)/\mathfrak{p}) = \{0, \epsilon\}$ for $\alpha \in O$. Here, $\epsilon = -1, 0, 1$. We will use this later in area computations.

2.2. **Extended Bianchi groups.** We can consider the maximal discrete extension of Γ_d in $\mathrm{PSL}_2(\mathbb{C})$, and we will call these extended Bianchi groups, denoted B_d. More concretely, consider the set

$$L_d = \{\sigma = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{Mat}_{2\times 2}(\mathcal{O}_d) \mid \det(\sigma) = \epsilon N\},$$

where $\epsilon \in \mathcal{O}_d^\times$ and N is the norm of the ideal $(\alpha, \beta, \gamma, \delta)$ in $\mathcal{O}_d$. The image of this L_d in $\mathrm{PSL}_2(\mathbb{C})$ is B_d. A Fuchsian subgroup of B_d fixes a circle or a straight line C in $\partial\mathbb{H}^3$, and also preserves the two components of $\mathbb{C} \setminus C$. Such a Fuchsian subgroup is called maximal if it is not a proper subgroup of any other Fuchsian subgroup of B_d.

Theorem 8. *Suppose there does not exist an invariant factor of* $\mathrm{Cl}(\mathbb{Q}(\sqrt{-d}))$ *divisible by 4. Let* $d \equiv 3 \mod 4$. *Let* f *be a primitive integral indefinite hermitian binary form. Then* f *is* B_d*-equivalent to* $f_{m,c}(z, w) = |z|^2 + 2\operatorname{Re}(\frac{m}{\sqrt{-d}}\overline{z}w) + c|w|^2$ *for* $0 \leq m < d/2$ *and* $c \in \mathbb{Z}$.

Okay. I think I get the intuition. If we have a totally geodesic surface in $\Gamma_d\backslash\mathbb{H}^3$, lift this to $\mathbb{H}^3$. Then we get a geodesic surface S of $\mathbb{H}^3$. This bounds a circle or straight line in $\partial\mathbb{H}^3$, where $\mathrm{Stab}(S, \mathrm{PSL}_2(\mathcal{O}_d)) = \mathrm{Stab}(C, \mathrm{PSL}_2(\mathcal{O}_d))$ which turns out to be a Fuchsian group. (This is believable, as by definition we are looking at a group that fixes a hyperbolic surface.) So we can parametrize geodesic surfaces in the Bianchi orbifolds by looking at the circles and lines they (more precisely their lifts) bound in $\partial\mathbb{H}^3$. Each of these circles or lines are defined by an integral binary hermitian form, whose explicit forms we have just found above (put 1 in w). I don't understand the details though; especially I don't understand Lemma 3.2.

3. Step 2: Expressing the Fuchsian subgroup as a $\mathbb{Z}$-order of a $\mathbb{Q}$-quaternion algebra

For an indefinite quaternion algebra over $\mathbb{Q}$, and a $\mathbb{Z}$-order O of reduced discriminant D, we have a formula to compute the volume of $\Gamma^1(O)\backslash\mathbb{H}^2$, where $\Gamma^1(O) \subset \mathrm{PSL}_2(\mathbb{R})$ is the discrete group associated to the group $PO^1 = O^1/\{\pm 1\}$ of units of reduced norm 1.[7] Theorem 3.4. of the paper realizes the stabilizer group F_S of $a|z|^2 + 2\operatorname{Re}(B\overline{z}) + c = 0$ as $P\rho' M^1$, where everything is explicitly given. For an order O, the notation O^1 means the group consisting of $\gamma \in O^\times$ with reduced norm 1. The group $\Gamma^1(O)$ is defined to be $P\iota(O^1)$, where ι is defined explicitly.[8] The quaternion algebra used is $(-d, D/\mathbb{Q})$, and the $\mathbb{Z}$-order M is defined to be

$$M = \mathbb{Z}\left[1, \frac{\alpha_1(1+i)}{2}, \frac{\beta(i+1)}{2} + \frac{bi+j}{d_0\alpha_1}, \frac{-bd - bi - j + ij}{2d_0}\right]$$

for some integers $0 \leq \beta < a/d_0$ and $\alpha_1 | \frac{a}{d_0}$. The exact conditions for d_0 and a, b, c, d are in Theorem 3.4.

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6 The $\mathfrak{p}$-adic completion of R is straightforward. The $\mathfrak{p}$-adic completion of O is defined to be $O \otimes_R \widehat{R}$.

7 In Vulakh p 304, it is stated that the group $\mathrm{Stab}(C, B_d)$ can be identified with the group of B_d-units of the hermitian form f defining it.

8 The definition is explicit, but I dont get the intuition; see Voight's book

4. Step 3: Area computation through quaternion algebras

Using the explicit formula for areas of quotient surfaces, since above we've identified which quaternion algebra and $\mathbb{Z}$-order we're using, (we're using $(-d, D/\mathbb{Q})$ as our quaternion algebra and M as our $\mathbb{Z}$-order) we need to calculate the numbers $\lambda(O,p)$, and $[\mathbb{Z}_p^\times : \mathrm{nrd}(O_p^\times)]$. (The problem is, I cannot find the exact statement of Theorem 3.3 in the reference.) This is done in Lemma 3.5, and the first part is a calculation. To prove the second part, that $[\mathbb{Z}_p^\times : \mathrm{nrd}(M_p^\times)]$ is 2 if $p|(\frac{d}{d_0}, \frac{D}{d_0})$, and 1 otherwise (for an odd prime p) we first observe that $(\mathbb{Z}_p^\times)^2 \subset \mathrm{nrd}(M_p^\times)$. This is quite obvious, as $\mathrm{nrd} : \alpha \mapsto \alpha\overline{\alpha}$, and since M_p is a $\mathbb{Z}_p$-order, it contains $\mathbb{Z}_p$, so the image of nrd must contain the squares. Thus we have a chain of inclusions $(\mathbb{Z}_p^\times)^2 \subset \mathrm{nrd}(M_p^\times) \subset \mathbb{Z}_p^\times$. (I strongly believe that there is a typo in this part.)

Lemma 9. *For an odd prime p, $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2 \cong \mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2$. A fortiori, $\cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. I can't seem to find a nice proof of the first part. Maybe reading Serre's *A Course in Arithmetic* will help. I think the correct sketch is to consider the ring homomorphism $\mathbb{Z}_p \to \mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p$, consider its associated group homomorphism $\mathbb{Z}_p^\times \to \mathbb{F}_p^\times$, and show that the kernel of $\mathbb{Z}_p^\times \to \mathbb{F}_p^\times \to \mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2$ is just $(\mathbb{Z}_p^\times)^2$. The second part is well-known. ☐

Using this, we have $[\mathbb{Z}_p^\times : (\mathbb{Z}_p^\times)^2] = [\mathbb{Z}_p^\times : \mathrm{nrd}(M_p^\times)][\mathrm{nrd}(M_p^\times) : (\mathbb{Z}_p^\times)^2] = 2$, so depending on whether the inclusion $(\mathbb{Z}_p^\times)^2 \subset \mathrm{nrd}(M_p^\times)$ is strict or not, we have our desired result.

For $p = 2$, however, the above proof does not work, as the group $\mathbb{Z}_2^\times/(\mathbb{Z}_2^\times)^2 \cong V_4$ instead of $\mathbb{Z}/2\mathbb{Z}$.

Now, Theorem 3.7 gives a formula for the immersed totally geodesic surface $\mathcal{S}_{m,c,r}$. I have not gone through the detailed calculations.

5. Step 4: Turning the area formula into an asymptotic relation

Here is the formula:

Theorem 10. *For $0 \le m < d$, any $c \in \mathbb{Z}$ such that $m^2 > cd$, let $d_0 = d/(m,d)$ and $D = (m^2d - cd^2)/(m,d)^2$. Then the geodesic surface $\mathcal{S}_{m,c,r}$ corresponding to $\sigma_r^{-1}\mathcal{C}_{m,c}$ has area*

$$\mathrm{vol}(\mathcal{S}_{m,c,r}) = \frac{d\pi}{d_0^2}\frac{2^{-\omega((d/d_0,D))}}{3}\prod_{p|\frac{d}{d_0}}\frac{1-p^{-2}}{1-\left(\frac{D}{p}\right)p^{-1}}D\prod_{p|D,p\nmid d}(1+\chi_{-d}(p)p^{-1})$$

where $\omega(n)$ is the number of distinct prime factors of n, and χ_{-d} is the quadratic Dirichlet character associated to $\mathbb{Q}(\sqrt{-d})$.

Note that d is subject to our conditions imposed in the beginning. Finding a formula for $\#\{(m,c,r) \mid \mathrm{vol}(\mathcal{S}_{m,c,r}) < X\}$ w.r.t. X would finish our proof. We first define some numbers:

Definition 11. Let $C = \prod_p(1 - p^{-1} + (p + \chi_{-d}(p))^{-1})$, and $F(n) = n\prod_{p|n}(1 + \chi_{-d}(p)p^{-1})$.

Next, we prove the following:

Theorem 12. *Recall that τ was the number of divisors. We have*

$$\#\{(m,c,r) \mid \mathrm{vol}(\mathcal{S}_{m,c,r}) < X\} = \frac{3C\tau(d)}{2\pi}\left(\prod_{p|d}(1+p^{-2}+p^{-3}+\cdots)\right)X + o(X).$$

Proof. Recall that c was any integer; rewrite c as $\frac{d}{d_0}k + \kappa$ with $k \in \mathbb{Z}$ and $0 \le \kappa < \frac{d}{d_0} - 1$. Hence our problem is equivalent to finding

$$\#\{k \mid \mathrm{vol}(\mathcal{S}_{m,\frac{d}{d_0}k+\kappa,r}) < X\},$$

for fixed κ, r, and m, and then summing over the possible κ, r, m (of which there are only finitely many choices). We first need to prove the following lemma:

Lemma 13. *Let a be an integer such that if $p|a$, then $p|d$. (i.e., shares prime factors with d) Then for any integer r, we have*

$$\#\{n \equiv r \mod a \mid F(n) < X\} = \frac{C}{a}X + o(X).$$

This lemma depends on the value a.

Proof. Since $F(an+r) = (a,r)F\left(\frac{a}{(a,r)}n + \frac{r}{(a,r)}\right)$, we may assume $(a,r) = 1$.[9] Considering the Dirichlet characters $\psi_0, \dots, \psi_{\phi(a)-1}$ modulo a, define

$$D_F(s, \psi_j) = \sum_{n=1}^{\infty} \psi_j(n) F(n)^{-s}.$$

It turns out that $D_F(s, \psi_j)$ are holomorphic in $\mathrm{Re}(s) > 0$ except for $D_F(s, \psi_0)$, which has a simple pole at $s = 1$. Its residue is given by $\frac{\phi(a)C}{a}$. If we define $a(m)$ to be $\#\{n \equiv r \mod a : F(n) = m\}$, then the Dirichlet series

$$\sum_{m=1}^{\infty} \frac{a(n)}{m^s}$$

converges absolutely to $\frac{C}{a(s-1)}$ in $\mathrm{Re}(s) > 1$. We use the Weiner-Ikehara theorem:

Lemma 14. *Suppose* $\sum_{n=1}^{\infty} a(n)n^{-s}$, $a(n) \geq 0$ *converges to an analytic function in* $\mathrm{Re}(s) > b$ *with simple pole of residue* c *at* $s = b$. *Then* $\sum_{n \leq X} a(n) \sim \frac{c}{b} X^b$.

Here, we plug in $c = \frac{C}{a}$, and $b = 1$ to get $\sum_{n \leq X} a(n) \equiv \frac{C}{a} X + o(X)$, and since $\sum_{n \leq X} a(n)$ is just $\#\{n \equiv r \mod a : F(n) \leq X\}$, our result follows.[10] □

Now that we have this lemma, we go back to our problem of finding

$$\#\{k \mid \mathrm{vol}(\mathcal{S}_{m, \frac{d}{d_0}k + \kappa, r}) < X\}$$

for fixed κ, r, and m. Using the volume formula, this is equivalent to

$$\#\{k \mid \frac{d\pi 2^{-\omega(\gcd(\frac{d}{d_0}, \kappa))}}{3d_0^2} \prod_{p \mid \frac{d}{d_0}} \frac{1 - p^{-2}}{1 - \left(\frac{-\kappa}{p}\right) p^{-1}} \cdot F(d_0 dk - d_0^2 \kappa + \frac{m^2 d}{(m,d)^2}) < X\}$$

where by rephrasing our lemma above, we have that $\#\{k \mid F(ak + r) < X\} = \frac{C}{a} X + o(X)$ so we may take $a = d_0 d$ in the case above. Note that our a satisfies $p|a \Rightarrow p|d$, as by definition $d_0 = d/(m,d)$, so it must share prime factors with d. Hence we get the asymptotic formula

$$\frac{3Cd_0}{\pi d^2} 2^{\omega(\gcd(\frac{d}{d_0}, \kappa))} \prod_{p \mid \frac{d}{d_0}} \frac{1 - \left(\frac{-\kappa}{p}\right) p^{-1}}{1 - p^{-2}} X + o(X)$$

for each fixed κ, r, and m. First, sum over $0 \leq \kappa < \frac{d}{d_0}$. A long calculation shows that

$$\sum_{\kappa} 2^{\omega(\gcd(\frac{d}{d_0}, \kappa))} \prod_{d \mid \frac{d}{d_0}} \left(1 - \left(\frac{-\kappa}{p}\right) p^{-1}\right) = \frac{d}{d_0} \prod_{p \mid \frac{d}{d_0}} (1 + p^{-1}).$$

Note that this is not the whole formula, but only the part dependent on κ. So for each m and r, the formula simplifies to

$$\#\{c \mid \mathrm{vol}(\mathcal{S}_{m,c,r}) < X\} = \frac{3C}{\pi d} \prod_{p \mid \frac{d}{d_0}} \frac{1}{1 - p^{-1}} X + o(X).$$

Note as $\frac{d}{d_0} = \gcd(m, d)$, the product still depends on m. As $0 \leq m < d$, we sum $\prod_{p \mid \gcd(m,d)} \frac{1}{1-p^{-1}}$ to get $d \prod_{p|d} (1 - p^{-1} + (p-1)^{-1})$. Thus we obtain

$$\#\{m, c \mid \mathrm{vol}(\mathcal{S}_{m,c,r}) < X\} = \frac{3C}{\pi} \left(\prod_{p|d} (1 + p^{-2} + p^{-3} + \cdots) \right) X + o(X)$$

but as this does not depend on r, and we have $\frac{\tau(d)}{2}$ choices of r, our theorem is (almost) complete. □

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[9]I do NOT understand why this assumption is valid.

[10]If F takes values in $\mathbb{Z}$, there is no problem in this reasoning. However if not, the last equality may not hold.