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HOJIN LEE

This will be mainly about the notion of stability of sheaves. I will try to focus on the material of the book *The Geometry of Moduli Spaces of Sheaves* by D. Huybrechts and M. Lehn, but will include some material not covered in it. The book constructs coherent sheaves on surfaces, but for the sake of motivation I will start with bundles on curves. The notion of stability will be introduced, and stability conditions make the moduli space behave nicely. I do not know how exactly this makes the moduli space of sheaves well behaved, but I hope to know the answers in the upcoming weeks.

Throughout this talk, *vector bundle* means a locally free sheaf of finite rank, i.e. the sheaf of sections. I will prove the existence and uniqueness of the Harder-Narasimhan filtration for vector bundles on smooth projective curves. I will not cover the HN filtration for arbitrary coherent sheaves, or higher dimension analogues (probably next time?)

1. Harder–Narasimhan Filtration for Curves

Let X be a smooth projective curve over $\mathbb{C}$. If E is a vector bundle, we have two main invariants: the rank, and the degree. We define the rank and degree for coherent sheaves altogether, since even if we are mainly interested in vector bundles, coherent sheaves pop up from time to time. We need some preliminary definitions:

Definition 1.1. The *dimension* of a coherent sheaf is the dimension of its support.

Definition 1.2. Fix an ample line bundle $\mathcal{O}(1)$ on X. The *Hilbert polynomial* of a coherent sheaf E on X is given by

$$m \mapsto \chi(E \otimes \mathcal{O}(m))$$

where χ is the sheafy Euler characteristic. Denote by $\alpha_i(E)$ the ith coefficient of the Hilbert polynomial of E times $i!$.

Definition 1.3. Let E be a coherent sheaf of dimension $d = \dim X$. Define the *rank* of E to be

$$\operatorname{rank} E := \alpha_d(E)/\alpha_d(\mathcal{O}_X).$$

The *degree* of E is defined as

$$\deg E := \alpha_{d-1}(E) - \operatorname{rank}(E) \cdot \alpha_{d-1}(\mathcal{O}_X).$$

In general, the rank and degree may be rational, but for vector bundles on a smooth projective variety, both are integers.[1]

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[1] According to HL, by HRR, one has $\deg E = \deg \det E$

Lemma 1.4. *If $0 \to E' \to E \to E'' \to 0$ is a SES of coherent sheaves on X, then $\operatorname{rank} E = \operatorname{rank} E' + \operatorname{rank} E''$ and $\deg E = \deg E' + \deg E''$.*

Proof. Both follow directly from the additivity of the Hilbert polynomial on SES's. □

Definition 1.5. We define the *slope* of a nonzero coherent sheaf E of dimension $d = \dim(X)$ to be

$$\mu(E) = \deg E / \operatorname{rank} E \in \mathbb{Q} \cup \{+\infty\}.$$

When E is torsion, we define $+\infty$. Slope is not defined for the zero sheaf.

The following is a technical lemma we will need later on:

Lemma 1.6. *(See-saw property.) Suppose we have a SES*

$$0 \to A \to E \to B \to 0$$

of coherent sheaves on X. Then $\mu(A) < \mu(E)$ iff $\mu(E) < \mu(B)$, and $\mu(A) > \mu(E)$ iff $\mu(E) > \mu(B)$.

Proof. This follows from direct calculation, or one may proceed as follows: Consider the points $Z(E) := (-\deg E, \operatorname{rank} E)$ for a coherent sheaf E.[2] Notice that $Z(E) = Z(A) + Z(B)$. Conclude from drawing a parallelogram. □

Next, we define what it means for a vector bundle to be (semi)stable.

Definition 1.7. A vector bundle E is *stable of slope* λ if for all nonzero subbundles $A \hookrightarrow E$, we have $\mu(A) < \mu(E) = \lambda$. E is *semistable of slope* λ if $\mu(A) \leq \mu(E) = \lambda$. By convention, the zero sheaf is semistable of every slope.

Definition 1.8. Let E be a vector bundle on X. A filtration

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_m = E$$

is a Harder-Narasimhan filtration if:

(1) each quotient bundle E_i/E_{i-1} is semistable of slope λ_i,
(2) the λ_i are strictly decreasing.

2. Existence of HN Filtration

Lemma 2.1. *Let E be a vector bundle on X. There exists an integer d such that for every coherent subsheaf $F \subset E$, we have $\deg F \leq d$.*

Proof. Induction on rank of E; suppose the result holds for all vector bundles of smaller rank. If E a line bundle, then every $F \subset E$ either a line bundle of smaller degree or zero. In this case take $d = \max\{\deg E, 0\}$, thus holds for base case. If $\operatorname{rank} E > 1$, consider the following SES of vector bundles[3]

$$0 \to L \to E \to E' \to 0$$

where L is the line subbundle of E determined by some rational section of E. If $F \subset E$ is any coherent subsheaf, consider the SES

$$0 \to F' \to F \to F'' \to 0$$

[2] I speculate that this notion is used when defining other stability conditions in higher dimensions

[3] E' is also a vector bundle since the first map is a fiberwise embedding

where $F' = F \cap L$ and $F'' \subset E'$.[4] Since L and E' have smaller rank than E, we may use the induction hypothesis to conclude that

$$\deg F = \deg F' + \deg F'' \leq d' + d''$$

and then $d = d' + d''$ is our desired d. □

Theorem 2.2. *There exists a Harder-Narasimhan filtration for a vector bundle E on X.*

Proof. We use induction on $\operatorname{rank} E$; i.e. every bundle of smaller rank admits a HN filtration. Let $S = \{\mu(E') \mid 0 \neq E' \subset E\}$. By Lemma 2.1., we know that the degree of subbundles of E are bounded above. Since nonzero vector bundles are torsion free, there must be a maximum slope. Denote λ as the largest slope, and choose a subbundle E' whose rank is the largest among slope λ subbundles. This implies E' is semistable of slope λ, since otherwise it would contradict maximality of λ.

Let $E'' = E/E'$, which is a vector bundle of rank smaller than E.[5] By induction hypothesis, E'' admits a HN filtration, say

$$0 = E''_0 \subsetneq \cdots \subsetneq E''_m = E''$$

where $\lambda_i = \mu(E''_i/E''_{i-1})$ satisfy $\lambda_1 > \cdots > \lambda_m$. For $0 \leq i \leq m$, denote $\overline{E}''_i \subset E$ the inverse image of E''_i under the map $E \to E''$, where $\overline{E}''_0 = E'$. We claim that

$$0 \subsetneq E' = \overline{E}''_0 \subsetneq \cdots \subsetneq \overline{E}''_m = E$$

is a HN filtration of E. The successive quotients E' and E''_i/E''_{i-1} have slopes λ and λ_i, so it suffices to show that $\lambda > \lambda_1$. Suppose not, i.e. $\lambda \leq \lambda_1$. Then in the SES

$$0 \to E' \to \overline{E}''_1 \to E''_1 \to 0$$

we have $\mu(E') = \lambda$ and $\mu(E''_1) = \lambda_1$. By the see-saw property, we must have $\mu(\overline{E}''_1) \geq \lambda$, but since by maximality of λ we cannot have $> \lambda$, so this implies $\mu = \lambda$. However, since E' was chosen to be maximal among subbundles of E having slope λ, also this cannot happen. Hence $\lambda > \lambda_1$, and we have a HN filtration. □

3. Uniqueness of HN Filtration

Lemma 3.1. *Let E be a semistable vector bundle of slope λ. For any vector bundle surjection $E \to E''$, we have $\mu(E'') \geq \lambda$.*

Proof. From the SES $0 \to E' \to E \to E'' \to 0$ and semistability of E, we have $\mu(E') \leq \mu(E) = \lambda$, and by the see-saw property we have our desired result. □

Proposition 3.2. *Let E, F be semistable vector bundles of slope λ and λ'. If $\lambda > \lambda'$, then every map $f : E \to F$ is trivial.*

Proof. If $f \neq 0$, the image $\operatorname{im} f$ is a nonzero coherent subsheaf of F, hence a vector bundle of rank > 0. Then we have $\mu(E) \leq \mu(\operatorname{im} f) \leq \mu(F)$, i.e. nonsense. The first inequality follows from Lemma 3.1, and second inequality follows from semistability of F. □

Theorem 3.3. *A HN filtration for a vector bundle E on X is unique.*

[4] This is from another lecture note; I suppose $F \cap L$ is defined by the inclusions $L \subset E$ and $F \subset E$, the rest follows

[5] Quotients are bundles for subbundles!

Proof. We use induction on $\operatorname{rank} E$, i.e. assume the result holds for smaller rank. Suppose we have two HN filtrations of E:

$$0 = E_0 \subsetneq \cdots \subsetneq E_m = E$$
$$0 = E'_0 \subsetneq \cdots \subsetneq E'_n = E$$

where the quotients have slope $\lambda_1 > \cdots > \lambda_m$ and $\lambda'_1 > \cdots > \lambda'_n$. If we show $E_1 = E'_1$, then the rest follows by inductive hypothesis on

$$0 = E_1/E_1 \subsetneq E_2/E_1 \subsetneq \cdots \subsetneq E_m/E_1 = E/E_1$$
$$0 = E'_1/E'_1 \subsetneq E'_2/E'_1 \subsetneq \cdots \subsetneq E'_n/E'_1 = E/E'_1.$$

Suppose $\lambda_1 \neq \lambda'_1$, WLOG $\lambda_1 > \lambda'_1$. This implies $\lambda_1 > \lambda'_i$ for $1 \leq i \leq n$. By Proposition 3.2, it follows that $\operatorname{Hom}(E_1, E'_i/E'_{i-1}) = 0$, which implies $\operatorname{Hom}(E_1, E) = 0$. This is false since $E_1 \subset E$. Therefore we must have $\lambda_1 = \lambda'_1$.

This implies $\lambda_1 > \lambda'_i$ for $i > 1$, so $\operatorname{Hom}(E_1, E/E'_1) = 0$. In particular,

$$0 \to E_1 \to E \to E/E'_1 \to 0$$

must be zero, so $E_1 \subset E'_1$. Just as we did, we may show the opposite inclusion, so $E_1 = E'_1$ indeed. $\square$