COMMUTATIVE ALGEBRA HOMEWORK V

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Problem 1. *Let $\varphi : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a local homomorphism of noetherian local rings. Let A be regular and B CM. Prove TFAE:*

(1) *B is flat over A;*
(2) $\dim B - \dim A = \dim B/\mathfrak{m}B$.

Proof. Suppose B is flat over A, the result (1) $\Rightarrow$ (2) holds generally due to the following theorem in the lecture notes (under **§System of Parameters**)

Theorem. Let $\varphi : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a local homomorphism of noetherian local rings. Then $\dim B - \dim A \leq \dim B/\mathfrak{m}B$. Equality holds if B is flat.

I omit the proof since it is in the lecture notes.

To prove the converse, we use induction on $\dim A$. Suppose the result (2) $\Rightarrow$ (1) holds for every pair of rings $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ such that $\dim R < \dim A$ (of course assuming each is regular resp. CM.) For the base case $\dim A = 0$, this implies A being a field, and B, being an algebra over a field, is flat. Hence we may assume $\dim A > 0$ (which implies $\dim B > 0$, since we're assuming that (2) holds). By prime avoidance and dimension conditions, we may pick a non zerodivisor $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. We have $\dim(B/(x)) \leq \dim(A/(x)) + \dim B/\mathfrak{m}B$ as usual. Since x is a non zerodivisor in A, we have $\dim A/(x) = \dim A - 1$. By hypothesis, we have $\dim B = \dim A + \dim B/\mathfrak{m}B$, so $\dim B/(x) \leq \dim B - 1$. Krull's PIT is applicable, which tells us that $\dim B/(x) = \dim B - 1$, and thus the condition (2) holds for the pair $(A/(x), B/(x))$. Since $x \in \mathfrak{m} \setminus \mathfrak{m}^2$, $A/(x)$ is a regular local ring. Also since B is Cohen-Macaulay, by the unmixedness theorem, $\varphi(x) \in \mathfrak{n}$ cannot be contained in a minimal prime of B, otherwise the dimension would not drop by 1. This implies that $\varphi(x)$ is a non zerodivisor in B. Also we have $\operatorname{depth}(B/(x)) = \operatorname{depth}(B) - 1 = \dim B - 1 = \dim B/(x)$, so $B/(x)$ is Cohen-Macaulay. Hence $B/(x)$ is flat over $A/(x)$ by inductive hypothesis. Since x is both A and B-regular, by a corollary[1] in the lecture notes, $B/(x)$ being flat over $A/(x)$ is equivalent to B being flat over A, which is the desired result. $\square$

Problem 2. *Let k be a field, and $\varphi : k[x] \to k[x_1, \ldots, x_n]$ an injective k-algebra homomorphism. Show that φ is flat.*

Proof. $k[x]$ is a PID, and $k[x_1, \ldots, x_n]$ is a torsion-free module over $k[x]$ via φ. We show that torsion-free modules over a PID are flat, say $M \in \mathsf{Mod}_A$ is a torsion-free module over a PID A. Let I be a nonzero ideal of A. Then $I = (a)$ for some non zerodivisor a, hence we have a SES

$$0 \to A \xrightarrow{\times a} A \to A/(a) \to 0.$$

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[1]under §Local criterion for flatness

The associated LES of Tor w.r.t. M is

$$\cdots \to \mathrm{Tor}^1_A(M, A/(a)) \to M \xrightarrow{\times a} M \to M/(a)M \to 0,$$

so $\mathrm{Tor}^1_A(M, A/(a)) \cong \{m \in M \mid am = 0\}$. But since M is torsion-free, this vanishes. Since I was arbitrary, we have proved that M is flat over A. Directly apply this to $\varphi : k[x] \to k[x_1, \ldots, x_n]$ to get the desired result. $\square$

Problem 3. *Let A be a noetherian domain.*

(1) *Show that A is a UFD if and only if every irreducible $p \in A$ is a prime.*
(2) *Show that A is a UFD if and only if every $\mathfrak{p} \in \operatorname{Spec} A$ with $\mathrm{ht}(\mathfrak{p}) = 1$ is principal.*
(3) *Let $p \in A$ be a nonzero prime. Show that A is a UFD if and only if A_p is a UFD.*

Proof. (1) Suppose A is a UFD, and let $ab \in (p)$, i.e. $ab = pc$ for some nonzero $a, b, c \in A$. Since A is a UFD, p must appear in the factorization of either a or b, so $a \in (p)$ or $b \in (p)$ must hold. Hence p is prime. Conversely, suppose every irreducible $p \in A$ is a prime. Define

$$\mathcal{C} := \{(a) \subset A \mid a \text{ is not a product of irreducibles}\}$$

partially ordered by inclusion. Assume by contradiction that $\mathcal{C} \neq \varnothing$. Since A is noetherian, $\mathcal{C}$ has a maximal element, say (a). Since a itself is not irreducible, we may write $a = bc$ for $b, c \in A$ both nonzero nonunits. This implies $(a) \subsetneq (b)$, since $a \in (b)$ and c is a nonunit. Similarly, $(a) \subsetneq (c)$ holds. By maximality of (a), both b and c are products of irreducibles, but this would imply $a = bc$ being a product of irreducibles, thus a contradiction. Hence $\mathcal{C} = \varnothing$, so every element of A has a factorization into irreducibles.

Suppose $a = u \prod_i p_i = v \prod_j q_j$ are two factorizations into irreducibles. Since the p_i are prime, we must have $p_1 | q_j$ for some j. WLOG let $j = 1$. Then $p_1 | q_1$. This means that $q_1 = kp_1$ for some $k \in A$, but since q_1 is irreducible, we must have $k \in A^\times$. Thus p_1 and q_1 are associates. Repeat this process for a/p_1, and so on to prove uniqueness. Thus A is a UFD. $\square$

Proof. (2) Suppose A is a UFD. Let $\mathfrak{p}$ be a height 1 prime of A. Since A is a domain, the maximal chain is $(0) \subsetneq \mathfrak{p}$. Pick some nonzero $x \in \mathfrak{p}$. Since A a UFD, x has a factorization into irreducibles, and x must have at least one irreducible factor (otherwise $\mathfrak{p} = A$). Denote that irreducible factor p. Then $(p) \subset \mathfrak{p}$, and by (1) (p) is a prime ideal. By height conditions, $(p) = \mathfrak{p}$, so it is principal.

Conversely suppose every height 1 prime is principal. Let $p \in A$ be an irreducible element. Since it is a nonunit, the ideal (p) is proper, hence contained in some prime ideal. Let $\mathfrak{p} \in V(p)^{\min}$. By Krull's PIT, $\mathfrak{p}$ is of height ≤ 1. Since A is a domain and p is nonzero, $\mathfrak{p}$ must be of height 1. By hypothesis, $\mathfrak{p}$ is principal, say $\mathfrak{p} = (p')$. Then $p = p'a$ for some $a \in A$, so either p' or a is a unit. p' is not a unit, so a must be a unit, thus $(p) = (p') = \mathfrak{p}$. Hence p is prime, so we may apply the converse of (1) to conclude that A is a UFD. $\square$

Proof. (3) Suppose A is a UFD. By (2) this is equivalent to every height 1 prime being principal. Let $\mathfrak{p}$ be a height 1 prime of A_p. Consider the prime $\mathfrak{q} \subset A$ that corresponds to $\mathfrak{p}$. By the inclusion preserving correspondence of primes, we conclude that $\mathfrak{q}$ is also of height 1, hence principal by (2) since A a UFD. Let

$\mathfrak{q} = (q)$. Note that $\mathfrak{p} = (q/1)$. Hence all height 1 primes of A_p are principal, thus A_p is also a UFD.

Conversely, suppose A is not a UFD. We want to show that A_p is not a UFD. By (1), not being a UFD is equivalent to having an irreducible element that is not prime. Say $q \in A$ is an irreducible element that is not prime. We claim that $q/1 \in A_p$ is an irreducible element that is not prime.

We first show irreducibility. Suppose $\frac{q}{1} = \frac{a}{p^m}\frac{b}{p^n}$. We want to show that one of the factors is a unit. If $m = n = 0$ then it trivially holds, so assume $m + n > 0$. Since $ab = qp^{m+n} \in (p)$ in A, and since p is prime in A, we must have either $a \in (p)$ or $b \in (p)$. WLOG we may write $a = pa'$, and plugging this into the identity above, we get $a'b = qp^{m+n-1}$. Repeating this process finitely many times, we may write $a = \alpha p^{m'}$ and $b = \beta p^{n'}$ where $m' + n' = m + n$, hence getting $\alpha\beta = q$. Since q is irreducible in A, we must have either α or β a unit. This implies either a/p^m or b/p^n a unit, which is the desired result. Hence $q/1$ is irreducible in A_p.

However, $q/1$ cannot be prime in A_p. Suppose it did; suppose $(q/1)$ is a prime ideal of A_p. We show this implies (q) is prime. Suppose $ab \in (q)$. Then $\frac{a}{1}\frac{b}{1} = \frac{ab}{1} \in (q/1)$, which is prime by assumption, so either $a/1$ or $b/1$ is in $(q/1)$. WLOG suppose $\frac{a}{1} = \frac{q}{1}\frac{c}{p^k}$ for some $c \in A$. If $k = 0$ then $a \in (q)$ automatically. Hence assume $k > 0$, which implies $ap^k = qc$. Hence $qc \in (p)$. If q were a multiple of p, then $q = pu$ for a unit u since q is irreducible, hence q is prime. Now suppose q is not a multiple of p. Then c must be in (p), and we may write $c = c'p^k$ to conclude that $a = qc'$. This implies $a \in (q)$, so (q) is prime. Thus $(q/1)$ prime implies (q) prime, which is not the case, so $q/1$ is not a prime in A_p. Hence A_p is not a UFD. □

Problem 4. *Let A be a noetherian local ring. Show that A is artinian if there exists a nonzero injective A-module that is finitely generated.*

Proof. Noetherian rings of dimension 0 are artinian, so we show $\dim A = 0$. Assume by contradiction that $\dim A > 0$. Let M be a finitely generated injective A-module. If $\operatorname{Ass}(M) = \{\mathfrak{m}\}$, we have an injection $A/\mathfrak{m} \hookrightarrow M$. Since $\dim A > 0$, there exists some $\mathfrak{p} \subsetneq \mathfrak{m}$, and the natural projection $A \to A/\mathfrak{m}$ factors through $A/\mathfrak{p}$ to yield a map $A/\mathfrak{p} \to A/\mathfrak{m}$. Compose this with the injection to get a map $A/\mathfrak{p} \to M$. In the case $\operatorname{Ass}(M) \neq \{\mathfrak{m}\}$, take any $\mathfrak{p} \in \operatorname{Ass}(M)$. In both cases $\operatorname{Hom}_A(A/\mathfrak{p}, M) \neq 0$ and $\mathfrak{p} \subsetneq \mathfrak{m}$. Pick any $x \in \mathfrak{m} \setminus \mathfrak{p}$. Then consider the following diagram

$$\begin{tikzcd} & M & \\ 0 \arrow[r] & A/\mathfrak{p} \arrow[u, "f"] \arrow[r, "\times x"'] & A/\mathfrak{p} \arrow[ul, dashed, "\exists"'] \end{tikzcd}$$

where the bottom row is injective, since x is nonzero in $A/\mathfrak{p}$. By the injectivity of M, for every $f : A/\mathfrak{p} \to M$ corresponds some morphism $A/\mathfrak{p} \to M$ that commutes, i.e. every $f \in \operatorname{Hom}_A(A/\mathfrak{p}, M)$ can be written as xg for some $g \in \operatorname{Hom}_A(A/\mathfrak{p}, M)$. Thus $\operatorname{Hom}_A(A/\mathfrak{p}, M) \subset x\operatorname{Hom}_A(A/\mathfrak{p}, M)$, and the opposite inclusion is immediate so equality holds. Since $\operatorname{Hom}_A(A/\mathfrak{p}, M)$ is finitely generated and $x \in \mathfrak{m}$, by Nakayama we have $\operatorname{Hom}_A(A/\mathfrak{p}, M) = 0$. Hence $\dim A$ must be 0, so A is artinian.[2] □

[2]This proof based on this mathSE post

Problem 5. *Let $(A, \mathfrak{m}, k)$ be a noetherian local ring. Show that A is regular if $\operatorname{id}_A k < \infty$.*

Proof. We use induction on $d = \dim A$. Suppose $d = 0$. Then $\operatorname{id}_A k = \operatorname{depth} A \leq \dim A = 0$, so k is injective. Hence $\operatorname{Hom}_A(-, k)$ is exact, and we may apply this to the SES

$$0 \to \mathfrak{m} \to A \to k \to 0$$

to obtain

$$0 \to \operatorname{Hom}_A(k, k) \to \operatorname{Hom}_A(A, k) \to \operatorname{Hom}_A(\mathfrak{m}, k) \to 0.$$

Since $\operatorname{Hom}_A(k, k) = \operatorname{Hom}_k(k, k) \cong k$ and $\operatorname{Hom}_A(A, k) \cong k$, and we may write this as

$$0 \to k \to k \to \operatorname{Hom}_A(\mathfrak{m}, k) \to 0$$

where $\operatorname{Hom}_A(\mathfrak{m}, k) = 0$ since $\operatorname{Hom}_A(\mathfrak{m}, k)$ is naturally a k-module, together with additivity of k-dimensions. But from $\mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2 \to 0$ and consequently $0 \to \operatorname{Hom}_A(\mathfrak{m}/\mathfrak{m}^2, k) \to \operatorname{Hom}_A(\mathfrak{m}, k) = 0$, we have $\operatorname{Hom}_A(\mathfrak{m}/\mathfrak{m}^2, k) = \operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k) = 0$. If $\mathfrak{m} \neq \mathfrak{m}^2$, this cannot happen, so we must have $\mathfrak{m} = \mathfrak{m}^2$. Thus $\mathfrak{m} = 0$ by Nakayama, and $A = k$ is regular.

Now assume $d > 0$ and $\operatorname{id}_A k < \infty$. By inductive hypothesis, every noetherian local ring of dimension less than d is regular if the residue field has finite injective dimension. As in problem 1, by prime avoidance and $d > 0$, we may pick a non zerodivisor $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. Note that the residue field of $A/(x)$ is also k. We show that $\operatorname{id}_{A/(x)} k < \infty$, which would imply regularity of $A/(x)$, which in turn implies the regularity of A since x is A-regular.

By a lemma in the lecture notes,[3] we have isomorphisms

$$\operatorname{Ext}_A^{i+1}(k, \mathfrak{m}) \cong \operatorname{Ext}_{A/(x)}^{i}(k, \mathfrak{m}/(x)\mathfrak{m})$$

for all $i \geq 0$. By the same lemma, we have $\operatorname{Ext}_A^{i+1}(k, A) \cong \operatorname{Ext}_{A/(x)}^{i}(k, A/(x))$ for all $i \geq 0$. Also, since $\operatorname{id}_A k = \sup\{i \mid \operatorname{Ext}_A^i(k, k) \neq 0\} < \infty$, we must have $\operatorname{Ext}_A^i(k, k) = 0$ for all $i >> 0$. Consider the SES

$$0 \to \mathfrak{m} \to A \to k \to 0$$

and the LES of Ext given by

$$\cdots \to \operatorname{Ext}_A^i(k, k) \to \operatorname{Ext}_A^{i+1}(k, \mathfrak{m}) \to \operatorname{Ext}_A^{i+1}(k, A) \to \operatorname{Ext}_A^{i+1}(k, k) \to \cdots$$

where since $\operatorname{Ext}_A^i(k, k) = 0$ for all $i >> 0$, we have $\operatorname{Ext}_A^{i+1}(k, \mathfrak{m}) \cong \operatorname{Ext}_A^{i+1}(k, A)$ for all $i >> 0$. Combined with the results above, we have

$$\operatorname{Ext}_{A/(x)}^{i}(k, \mathfrak{m}/(x)\mathfrak{m}) \cong \operatorname{Ext}_{A/(x)}^{i}(k, A/(x))$$

for all $i >> 0$.

Now consider the SES

$$0 \to \mathfrak{m}/(x)\mathfrak{m} \to A/(x) \to k \to 0$$

and the associated LES of Ext w.r.t $A/(x)$ given by

$$\cdots \to \operatorname{Ext}_{A/(x)}^{i}(k, k) \to \operatorname{Ext}_{A/(x)}^{i+1}(k, \mathfrak{m}/(x)\mathfrak{m}) \to \operatorname{Ext}_{A/(x)}^{i+1}(k, A/(x)) \to \operatorname{Ext}_{A/(x)}^{i+1}(k, k) \to \cdots$$

[3] which is Bruns & Herzog, Lemma 3.1.16. Writing this since there is no full proof in lecture notes

where the middle arrow is an isomorphism by the naturality of the isomorphism of the lemma above, and the commutativity of the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m} \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{m}/(x)\mathfrak{m} \arrow[r] & A/(x) \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

Therefore we have $\operatorname{Ext}^i_{A/(x)}(k,k) = 0$ for all $i >> 0$, i.e. $\operatorname{id}_{A/(x)} k \leq \infty$. Since x is A-regular, $\dim A/(x) < \dim A$, and we may apply the inductive hypothesis to conclude that $A/(x)$ is regular. Hence A is regular.[4] □

[4]This proof based on this mathSE post