ALGEBRA I HOMEWORK I

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Problem 1. *Show that $\mathcal{P}(X)$ is a monoid wrt the binary operation of intersection, with identity $X \in \mathcal{P}(X)$. Given $f : X \to Y$, show $f^* : \mathcal{P}(Y) \to \mathcal{P}(X)$ is a monoid homomorphism.*

Proof. Suppose we are given the fact that $\mathcal{P}(X)$ is a set, and is unique. Define a binary operation $\cap : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$ by $(A, B) \mapsto \{x \in X \mid x \in A \wedge x \in B\}$. Associativity of $\cap$ follows from the associativity of conjunction in logic which we will not prove. Since $A \cap X = X \cap A = A$ for all $A \subset X$, X is the identity element.

Define $f^* : A \mapsto f^{-1}(A)$. Since f is a function, $f^{-1}(Y) = X$, so the identity maps to the identity. Suppose $A, B \subset Y$. We claim $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$. Suppose $x \in f^{-1}(A \cap B)$. Then $f(x) \in A \cap B \subset A, B$ so $x \in f^{-1}(A)$ and $f^{-1}(B)$. Conversely, suppose $x \in f^{-1}(A) \cap f^{-1}(B)$. Then $f(x) \in A \cap B$. $\square$

Problem 2. *Let $S(X)$ the free monoid on X of finite sequences in X, with natural map $\delta : X \to S(X)$. Show for any monoid N and a function $f : X \to N$ there exists a unique monoid homomorphism $\phi_f : S(X) \to N$ such that $\phi_f \circ \delta = f$.*

Proof. The natural map $\delta : X \to S(X)$ is given by sending elements of x to the one-element sequence $(x) \in S(X)$. Suppose we have a monoid N and a function $f : X \to N$. Define $\phi_f : S(X) \to N$ as $(x_1, \ldots, x_n) \mapsto f(x_1) *_N \cdots *_N f(x_n)$, and the identity (empty sequence) maps to the identity of N. Then $\phi_f((x_1, \ldots, x_n, x_{n+1}, \ldots, x_m)) = f(x_1) *_N \cdots *_N f(x_n) *_N f(x_{n+1}) *_N \cdots *_N f(x_m) = \phi_f((x_1, \ldots, x_n)) *_N \phi_f((x_{n+1}, \ldots, x_m))$ so ϕ_f is a monoid homomorphism. Since $\delta(x) = (x)$, and $\phi_f((x)) = f(x)$, we have $\phi_f \circ \delta = f$. Suppose we have another monoid homomorphism $\phi'_f : S(X) \to N$ such that $\phi'_f \circ \delta = f$. We want to show that $\phi'_f = \phi_f$. By the commuting condition, we must have $\phi_f((x)) = \phi'_f((x))$ for all $x \in X$. Also since ϕ'_f is a monoid homomorphism, we must have $\phi'_f((x_1, \ldots, x_n)) = \phi'_f((x_1) \cdots (x_n)) = \phi'_f((x_1)) *_N \cdots *_N \phi'_f((x_n))$. But this is just $\phi_f((x_1)) *_N \cdots *_N \phi_f((x_n)) = \phi_f((x_1, \ldots, x_n))$, so $\phi'_f = \phi_f$. $\square$

Problem 3. *Prove or provide counterexample:*

1. *If $\mathrm{Aut}(G)$ cyclic then G abelian.*
2. *If G group and $H \leq G$ has finite index, then there exists $N \trianglelefteq G$ of finite index with $N \leq H$.*

Proof. 1. Consider the inner automorphism group $\mathrm{Inn}(G)$, which is a subgroup of $\mathrm{Aut}(G)$. This is the group of automorphisms of G defined by conjugation. Since subgroups of cyclic groups are cyclic we conclude that $\mathrm{Inn}(G)$ is also cyclic. Define a group homomorphism $\phi : G \to \mathrm{Inn}(G)$ by $g \mapsto \phi_g$ where $\phi_g(x) = gxg^{-1}$ for all $x \in G$. Since $e \mapsto \phi_e = 1_G$ and $gh \mapsto \phi_{gh} = \phi_g \circ \phi_h$, this is indeed a group homomorphism. Suppose $\phi_g = 1_G$. Then $gxg^{-1} = x$ for all $x \in G$, so $\ker \phi = Z(G)$

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where $Z(G) = \{z \in G \mid zg = gz \ \forall g \in G\}$. Since ϕ is surjective, by the first isomorphism theorem, we have $G/Z(G) \cong \mathrm{Inn}(G)$ which is cyclic, say $\langle gZ(G) \rangle$. It follows that any element of G is of the form $g^n z$ for $z \in Z(G)$ and some $n \in \mathbb{Z}$. Since $g \cdot g^n z = g^{n+1} z = g^n gz = g^n z \cdot g$ for all $z \in Z(G)$ and $n \in \mathbb{Z}$, we conclude that g itself is in $Z(G)$, so $gZ(G) = Z(G)$. Therefore $G/Z(G) \cong \{\bullet\}$, which implies $G = Z(G)$, i.e. G is abelian. □

Proof. 2. Suppose $[G : H] = n$. Write

$$G = \bigsqcup_{1 \le i \le n} x_i H$$

for $x_i \in G$. We define a group homomorphism $G \to S_{G/H}$, where $S_{G/H}$ is the symmetric group on the set of left cosets of H in G. Define it by $g \mapsto \phi_g$ where $\phi_g : G/H \to G/H$ is a function on the set G/H, given by $xH \mapsto gxH$. Suppose $gxH = gyH$. It is obvious that $xH = yH$, so ϕ_g is injective, thus bijective since $|G/H| = n < \infty$. Therefore ϕ_g is indeed an element of $S_{G/H}$. Now since $e \mapsto \phi_e = 1_{G/H}$ and $fg \mapsto \phi_{fg} = \phi_f \circ \phi_g$, it follows that $\phi : g \mapsto \phi_g$ is a group homomorphism.

We claim that $\ker \phi \le H$. Suppose $\phi_g = 1_{G/H}$, i.e. $gxH = xH$ for all $x \in G$. In particular, $gH = H$ must hold, so g must be in H. Therefore $\ker \phi \le H$. Also, by the first isomorphism theorem, $G/\ker \phi \cong \mathrm{im}(\phi) \le S_{G/H}$, so $\ker \phi$ is a finite index $(= |\mathrm{im}(\phi)|)$ normal subgroup of G which is also a subgroup of H. □

Problem 4. *Let $\phi : G \to G'$ be a group homomorphism.*

1. *Show Γ_ϕ is a subgroup of $G \times G'$.*
2. *Show ϕ factors as $p \circ i$ where $i : G \to H$ and $p : H \to G'$ are injective, surjective homomorphism resp.*

Proof. 1. Obviously a subset of $G \times G'$. The identity element of $G \times G'$ is $(e_G, e_{G'})$. Since ϕ is a group homomorphism, it sends identities to identities, so Γ_ϕ has the identity. Also, $(x, \phi(x)) \cdot (y, \phi(y)) = (xy, \phi(xy))$, so Γ_ϕ is multiplicatively closed. The inverse of $(x, \phi(x))$ is $(x^{-1}, \phi(x^{-1}))$. □

Proof. 2. We claim that $\phi : G \to G'$ factors as $G \xrightarrow{i} G \times G' \xrightarrow{p} G'$. Define $i : G \to G \times G'$ as $g \mapsto (g, \phi(g))$, whose kernel is trivial so is injective. This is a group homomorphism since it sends identity to identity, and preserves the group law. Now define $p : G \times G' \to G'$ as $(g, g') \mapsto g'$, the projection on the second coordinate. This too is a group homomorphism quite obviously, and is surjective by definition. Then we can observe that $(p \circ i)(x) = p(x, \phi(x)) = \phi(x)$, so $p \circ i = \phi$. □

Problem 5. *Prove*

1. *We can identify N_i with a normal subgroup of G_i.*
2. *The image of H in $G_1/N_1 \times G_2/N_2$ is the graph of an isomorphism $G_1/N_1 \xrightarrow{\sim} G_2/N_2$.*

Proof. 1. Denote by i_1, i_2 the inclusion maps $N_1 \hookrightarrow H$, $N_2 \hookrightarrow H$, respectively. Consider the following diagram

$$\begin{tikzcd}
G_2 \cong \{e_1\} \times G_2 \arrow[r, hook, "\iota_2"] & G_1 \times G_2 \arrow[r, "\pi_1"] & G_1 \\
& N_2 \arrow[u, hook] \arrow[ul, dashed, "\exists! p_2 \circ i_2"] &
\end{tikzcd}$$

where ι_i are inclusion maps from G_i to $G_1 \times G_2$. By definition, N_2 sent through π_1 is zero, so by the universal property of the kernel, there exists a unique morphism from N_2 to $\ker \pi_1 \cong G_2$ that makes the diagram commute. This morphism is injective since both $N_2 \to G_1 \times G_2$ and $G_2 \to G_1 \times G_2$ are. We want to show that this map is $p_2 \circ i_2$. To show this, it suffices to show that $\iota_2 \circ p_2 \circ i_2$ is the inclusion of N_2 into $G_1 \times G_2$.

Consider the following diagram

$$\begin{tikzcd}
& G_1 \times G_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
G_1 & N_2 \arrow[l, "0"] \arrow[r, "p_2 \circ i_2"'] \arrow[u, dashed, "\exists!"] & G_2 \arrow[ul, bend right, "\iota_2"']
\end{tikzcd}$$

where by the universal property of products, there exists a unique morphism from N_2 into $G_1 \times G_2$ that commutes with projections and arrows into G_i. First note that the inclusion $N_2 \to G_1 \times G_2$ commutes with other arrows by definition. To show that $\iota_2 \circ p_2 \circ i_2$ is the inclusion, we show it commutes with π_2 and $p_2 \circ i_2$. Consider $\pi_2 \circ \iota_2 \circ p_2 \circ i_2$. Since $\pi_2 \circ \iota_2 = 1_{G_2}$, this is just $p_2 \circ i_2$. The commutativity of the left hand side is obvious. Hence, $\iota_2 \circ p_2 \circ i_2$ is equal to the inclusion $N_2 \hookrightarrow G_1 \times G_2$, so by uniqueness we conclude that the injective morphism $N_2 \to G_2$ derived from the kernel is in fact $p_2 \circ i_2$. Since this is injective, we may identify N_2 with its image in G_2, and since $\operatorname{im}(p_2 \circ i_2) = p_2(N_2)$ where $N_2 \trianglelefteq H$ and p_2 surjective, is a normal subgroup in G_2. Vice versa for N_1 in G_1. $\square$

Proof. 2. The image of H in $G_1/N_1 \times G_2/N_2$ is (h_1N_1, h_2N_2) where $(h_1, h_2) \in H$. Since p_i are surjective, H surjects onto G_i. For us to define a function $G_1/N_1 \to G_2/N_2$, we need to check well-definedness. Suppose $h_1N_1 = h_1'N_1$. We want to show this implies $h_2N_2 = h_2'N_2$ for all $(h_1, h_2), (h_1', h_2') \in H$. Using the fact that $h_1^{-1}h_1' = (h_1^{-1}h_1', e_2) \in N_1 \subset H$, and $(h_1^{-1}h_1', h_2^{-1}h_2') \in H$, we conclude that $(h_1^{-1}h_1', h_2^{-1}h_2')(h_1^{-1}h_1', e_2)^{-1} = (e_1, h_2^{-1}h_2') \in H$. This is obviously in the kernel of p_1, so is in N_2. Hence $h_2^{-1}h_2' \in N_2$, so $G_1/N_1 \to G_2/N_2$ is well-defined. This also defines a homomorphism due to the group structure on H. We can make this construction backwards, $G_2/N_2 \to G_1/N_1$ which sends for $(h_1, h_2) \in H$ as h_2N_2 to h_1N_1, and it is obvious that these two homomorphisms are inverses of each other. Therefore $G_1/N_1 \xrightarrow{\sim} G_2/N_2$, and the image of H is the graph of this isomorphism. $\square$

Problem 6. *Prove the following*

1. $[G, G]$ *is a normal subgroup of* G *and* G^{ab} *is abelian*
2. *For any group homomorphism* $\phi : G \to A$ *with* A *abelian, there exists a unique morphism* $\overline{\phi} : G^{\mathrm{ab}} \to A$ *such that* $\phi = \overline{\phi} \circ \pi$.

Proof. 1. By definition $[G, G]$ is a subgroup of G. We want to show $[G, G]$ is invariant under conjugation. Consider $c \in [G, G]$ and any $g \in G$. Then $gcg^{-1}c^{-1} \in [G, G]$ by definition. Since $[G, G]$ is a subgroup, we have $gcg^{-1} \in [G, G]$, so $[G, G]$ is indeed invariant under conjugation. If we have $x[G, G]$ and $y[G, G]$, then since $x^{-1}y^{-1}xy \in [G, G]$ we have $x^{-1}y^{-1}xy[G, G] = [G, G]$ so it follows that $xy[G, G] = yx[G, G]$. $\square$

Proof. 2. By context we assume $\pi : G \to G/[G, G]$ is the canonical projection. Suppose c is any commutator in G. Then $\phi(c) = e_A$. Since $[G, G]$ is generated by the set of all commutators of G, it follows that $\phi([G,G]) = \{e_A\}$. Hence $[G,G] \subset \ker\phi$, so by the universal property of the quotient group there exists such unique $\overline{\phi} : G^{\mathrm{ab}} \to A$. $\square$