Penrose Tilings

SEGL Seminars

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What are tilings?

Fundamentally, **tiling** is nothing more than an equivalence relation on the space $\mathbb{R}^k$. Each equivalence class is referred to as a **tile**.

Thus, while tiling itself has no significant meaning, interesting properties emerge when additional conditions that tiles must satisfy are imposed.

What are tilings?

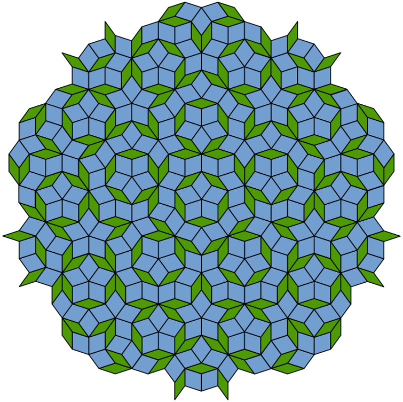
The following are some intuitively meaningful constraints that can be applied to tiling.

Simple tiling refers to a finite set of tile types, meaning that space is divided into a finite number of shapes. In this case, each of these shapes is called a **prototile**.

A tiling is said to be **periodic** if it is invariant under some (nontrivial) translation of the plane. Otherwise, it is **aperiodic**.

What is a Penrose tiling?

A Penrose tiling is a simple tiling generated by P where elements are $2\pi/5$-rotations of two rhombuses.



What is a Penrose tiling?

In fact, all Penrose tilings are aperiodic, which is remarkable since it has very few prototiles. There are many proofs of the Penrose tiling being aperiodic, but we will not go into the details.

The pentagrid

What de Bruijn noticed was that by certain rules, one may find five set of lines that are 'parallel'.

This can be defomed into a pentagrid, while maintaining the intersection points.

Each intersection point corresponds to a rhombus.

The pentagrid

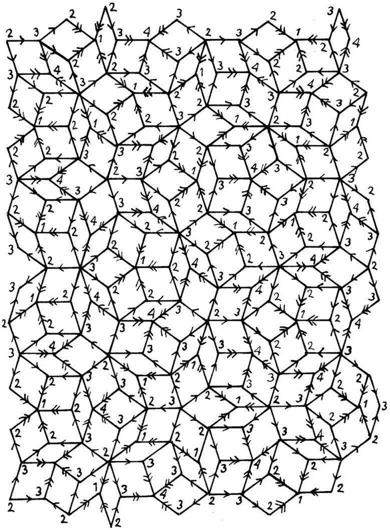


Fig. 2. An AR-pattern (for the indices 1, 2, 3, 4 see Section 6).

The pentagrid

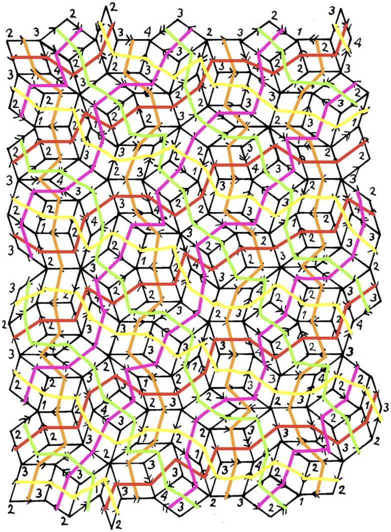
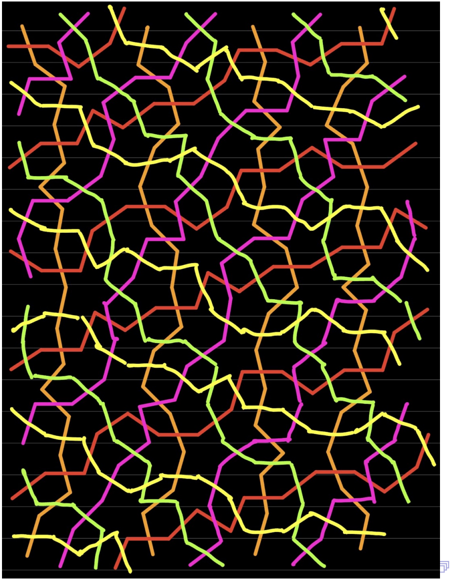


Fig. 2. An AR-pattern (for the indices 1, 2, 3, 4 see Section 6).

The pentagrid



The pentagrid

From now on, you may consider we are dealing with tilings of $\mathbb{C}$, rather than $\mathbb{R}^2$.

We may denote the offset of each one of the five families of the pentagrid as γ_i, for $i = 0, 1, 2, 3, 4$. Additionally, we assume $\sum \gamma_i = 0$.[1]

[1] The paper assumes this, and uses it when proving that patterns generated by the pentagrid indeed tile the plane.

Singular cases

Either when $\gamma_i = 0$ for all i, or for three of the five i, we get a **singular** case.

In the first case, there are 10 possibilites. In the latter, there are two.

The integer values K_j

In the paper, de Bruijn introduced the values K_j, which allows us to find a Penrose tiling from the offsets, i.e. γ values, given some $z \in \mathbb{C}$.

$$K_j(z) = \left\lceil \mathrm{Re}(z\zeta^{-j}) + \gamma_j \right\rceil$$

for $j = 0, 1, 2, 3, 4$ and $\zeta = e^{2\pi i/5}$.

Visualization of this process will be introduced later on.

The complex number ξ

From γ_i, we may calculate a single complex number ξ.

$$\xi = \sum_j \gamma_j \zeta^{2j}$$

It is the result of de Bruijn that ξ, except for singular values, completely determines a Penrose tiling.

To summarize, every Penrose pattern determines a pentagrid. Conversely, every nonsingular pentagrid determines a Penrose pattern.

Overall Process

$$\xi \longrightarrow \gamma_0, \ldots, \gamma_4 \longrightarrow \text{Pentagrid} \longrightarrow \begin{matrix}\text{Penrose} \\ \text{Tiling}\end{matrix}$$

From each intersection point, z_0, of a line of r-th grid and a line of s-th grid, we calculate

$$\left(K_0\left(z_0\right), \ldots, K_4\left(z_0\right)\right) + \epsilon_1\left(\delta_{0r}, \ldots, \delta_{4r}\right) + \epsilon_2\left(\delta_{0s}, \ldots, \delta_{4s}\right)$$

where $\epsilon_1, \epsilon_2 \in \{0, 1\}$. Then, for each vector $(k_0, \ldots, k_4)$ we calculated above, we assign a complex number

$$f\left(k_0, \ldots, k_4\right) = \sum_j k_j \zeta^j$$

and these four complex numbers form a tile of Penrose tiling.

Optimization

- Since $f\left(k_0, k_1, \ldots, k_4\right) = f\left(0, k_1 - k_0, \ldots, k_4 - k_0\right)$, we can memorize it and use again.
- To minimize errors from floating point calculation, we didn't used complex number in Python.
- We pre-calculated $\cos\left(\frac{2k\pi}{5}\right)$ and $\sin\left(\frac{2k\pi}{5}\right)$ since they are used a lot to compute ζ^j.

Tiling spaces

First let's define more precisely what "tiling" we are dealing with.

Definition

The term **tile** refers to a subset of $\mathbb{R}^k$ that is compact, connected, and its closure is itself. A **tiling** of $\mathbb{R}^k$ refers to the partitioning of $\mathbb{R}^k$ into the tiles. A **simple tiling** is a tiling that satisfies the following three conditions:

1. Each tile is a polytope.
2. There is a finite collection $\{p_i\}_{i=1}^n$ of polytopes such that every tile of the tiling is a translated copy of some p_i.
3. If two tiles meet, they meet across the entire face.

Therefore, we can consider the set of all tilings of $\mathbb{R}^k$, denoted as $\mathcal{T}$. Let us now introduce the following definition to establish a metric between two tilings.

Definition

For $\varepsilon > 0$, we say $T_1,\ T_2 \in \mathcal{T}$ are ε-*close* if they agree on a ball of radius ε^{-1} around the origin, up to a further translation by ε or less.

This naturally induces a metric on $\mathcal{T}$.

Tiling spaces

We are now ready to define the **tiling space** X_T of $T \in \mathcal{T}$.

Definition

The **orbit** of a tiling $T \in \mathcal{T}$ refers to the equivalence class of T under the translation. In other words, it refers the following set:

$$\mathcal{O}_T := \left\{ T + x \mid x \in \mathbb{R}^k \right\}$$

The **tiling space** X_T is the closure of $\mathcal{O}_T$ in the tiling metric.

The set X_T is closed under translation and complete under the tiling metric, so it is natural to refer to it using the term "space".

Tiling spaces

Why tiling space?

Fact

X_T is compact if T is a simple tiling.

Fact

For any $T' \in X_T$, $X_T = X_{T'}$.

Fact

If T is invariant under translation by a lattice L (which has the same meaning as periodic), $X_T \cong \mathbb{R}^2/L \cong \mathbb{T}^2$.

Fact

X_T can always be viewed as inverse limits.

Next steps

Calculating the cohomology of a space describable by an inverse limit is a standard procedure.

Thus, our objectives include gaining a comprehensive understanding of fundamental cohomology concepts, particularly Čech cohomology and PE (Pattern-Equivariant) cohomology in modern tiling theory.

In addition, we intend to delve deeper into the cut-and-project method of generating aperiodic tilings and explore the connections between tiling theory and lattice theory.

Thank You