COMMUTATIVE ALGEBRA HOMEWORK IV

HOJIN LEE 2021–11045

Problem 1. *Let (a, b) be a regular sequence in a domain A. Show $ax - b$ generates a prime in $A[x]$.*

Proof. Since (a, b) is a regular sequence in A, it means that a is a non zerodivisor in A, and b is a non zerodivisor in $A/(a)$. We show that $(ax - b)$ is prime by showing that this is the kernel of the A-algebra homomorphism $\varphi : A[x] \to A_a$ given by $x \mapsto b/a$. Since the codomain is an integral domain, the result from this is immediate. First, we obviosuly have $(ax - b) \subset \ker \varphi$. Assume by contradiction that $(ax - b) \subsetneq \ker \varphi$, i.e. there exists some $g \in \ker \varphi - (ax - b)$. Take g to be of minimal degree among such g. Note that g cannot be a constant since a is not a zerodivisor. Thus we may write $g = c_n x^n + \cdots + c_0$ for $n > 0$. Now, c_n cannot be in (a) since otherwise, $d := c_n b/a$ is an element of A since a is a non zerodivisor, and $0 = g(b/a) = c_n(b/a)^n + c_{n-1}(b/a)^{n-1} + \cdots + c_0 = (d + c_{n-1})(b/a)^{n-1} + \cdots + c_0$. This implies that $(d + c_{n-1})x^{n-1} + \cdots + c_0$ is in $\ker \varphi$ of degree smaller than that of g, so $h \in (ax - b)$ by assumption. This would imply $g = h + c_n x^n - dx^{n-1} = h + (c_n/a)x^{n-1}(ax - b) \in (ax - b)$, a contradiction. But from $0 = c_n(b/a)^n + \cdots + c_0$ we have $-c_n b^n = c_{n-1} b^{n-1} a + \cdots + c_0 a^n \in (a)$, and since b is a non zerodivisor in $A/(a)$, we also have b^n a non zerodivisor in $A/(a)$, so it follows that $c_n \in (a)$, which is a contradiction to what we have just shown above. Hence $\ker \varphi = (ax - b)$, thus $(ax - b)$ is prime in $A[x]$. $\square$

Problem 2. *Let A be noetherian. Show TFAE:*

1. *A is reduced.*
2. *The following hold:*
   - (a) *localization of A at primes of height 0 are regular.*
   - (b) *all associated primes of A have height 0.*
3. *(R_0) and (S_1) hold for A.*

Proof. Suppose A is reduced. Consider the localization of A at a prime $\mathfrak{p}$ of height 0. This $A_\mathfrak{p}$ has a unique prime ideal. Suppose we have $a/s \in \mathfrak{p}A_\mathfrak{p}$ a nilpotent. Then $a^n/s^n = 0$, i.e. $a^n s' = 0$ for some $n > 0$ and some $s' \in A - \mathfrak{p}$. It follows that as' is nilpotent in A, hence must be zero. Thus $a/s = 0/1$ in $A_\mathfrak{p}$, so we have $\mathfrak{p}A_\mathfrak{p} = 0$, i.e. $A_\mathfrak{p}$ is a field, thus regular. Now suppose $\mathfrak{p} \in \mathrm{Ass}(A)$, say $\mathfrak{p} = \mathrm{Ann}(a)$. Suppose $\mathfrak{q} \subsetneq \mathfrak{p}$. Since $a\mathfrak{p} = (0) \subset \mathfrak{q}$ and there exists some element in $\mathfrak{p} - \mathfrak{q}$, it follows that $a \in \mathfrak{q}$. But then $a \in \mathfrak{p}$, so $a^2 = 0$ which is nonsense since A is reduced. Hence associated primes of A are of height 0.

Assume (a) and (b) hold. We want to show that $A_\mathfrak{p}$ is a regular local ring for all $\mathfrak{p}$ of height 0. This is just (a). Now we want to show that $\operatorname{depth} A_\mathfrak{p} \geq \min\{1, \mathrm{ht}(\mathfrak{p})\}$ for any $\mathfrak{p}$. For primes of height 0, this is obvious. If $\mathrm{ht}(\mathfrak{p}) \geq 1$, then by (b) we have $\mathfrak{p}$ non-associated, so there exists at least one non-zerodivisor in $\mathfrak{p}$. This is because

Date: May 22, 2024.

if $\mathfrak{p}$ consists of only zerodivisors, then it would be contained in some minimal prime by prime avoidance. By (b), this cannot happen. Hence depth $A_{\mathfrak{p}} \geq 1$.

Now suppose (R_0) and (S_1) hold for A. Again, as in the previous homework, we will use the fact that being reduced is a local property (Atiyah & Macdonald). Thus it is enough to showing $A_{\mathfrak{p}}$ being reduced for all primes $\mathfrak{p}$. We use induction on the height of $\mathfrak{p}$. For height 0 primes, this is immediate by R_0. Suppose $\operatorname{ht}(\mathfrak{p}) \geq 1$, and the result holds for every prime of height less than $\mathfrak{p}$. Then by S_1, we have depth $A_{\mathfrak{p}} \geq 1$, i.e. there is a non-zerodivisor $f \in \mathfrak{p}A_{\mathfrak{p}}$. Hence the localization map $A_{\mathfrak{p}} \to (A_{\mathfrak{p}})_f$ is injective. For an arbitrary ring R, we have $R \to \prod_{\mathfrak{p}} R_{\mathfrak{p}}$ injective, since being zero is a local property. Thus $(A_{\mathfrak{p}})_f$ is a subring of the product of localizations at prime ideals. The prime ideals of $(A_{\mathfrak{p}})_f$ correspond to prime ideals of $A_{\mathfrak{p}}$ not containing f, i.e. prime ideals of A contained in $\mathfrak{p}$ not containing f. (We are abusing notation for f, but the choices for f are obvious.) By induction hypothesis, the localizations of $(A_{\mathfrak{p}})_f$ are reduced, hence its product, hence $A_{\mathfrak{p}}$. End of proof. □

Problem 3. *Let $(A, \mathfrak{m}, \kappa)$ a regular local ring of dimension $d \geq 0$. Let $x_1, \dots, x_d$ a regular system of parameters for A.*

(1) *Let $f \in \mathfrak{m}$. Let $a_1 \dots, a_d \in A$ s..t $f = \sum_{j=1}^{d} a_j x_j$ for some $a_i \in A$. Show that*
$$(a_1 \bmod \mathfrak{m}, \dots, a_d \bmod \mathfrak{m}) \in \kappa^d$$
is uniquely determined.

(2) *Let $1 \leq n \leq d$ and let $f_1, \dots, f_n \in \mathfrak{m}$. Choose $a_{ij} \in A$ for $1 \leq i \leq n$ and $1 \leq j \leq d$ such that $f_i = \sum_j a_{ij} x_j$. Show that $A/(f_1, \dots, f_n)$ is a regular local ring iff the matrix $(a_{ij} \bmod \mathfrak{m}) \in \operatorname{Mat}_{n \times d}(\kappa)$ has rank $d - \dim A/(f_1, \dots, f_n)$.*

Proof. (1) Since A is regular local, we have $\mathfrak{m}/\mathfrak{m}^2 \cong \kappa^d$. Since $(x_1, \dots, x_d) = \mathfrak{m}$, and the dimension of $\mathfrak{m}/\mathfrak{m}^2$ is d, the images $\overline{x_1}, \dots, \overline{x_d}$ span $\mathfrak{m}/\mathfrak{m}^2$ hence form a basis for the d-dimensional κ-vector space $\mathfrak{m}/\mathfrak{m}^2$. Now consider $\overline{f} \in \mathfrak{m}/\mathfrak{m}^2$. Then $\overline{f}$ can be written uniquely as a κ-linear combination of the $\overline{x}_i$, say $\sum_i k_i \overline{x}_i$ with $k_i \in \kappa \cong A/\mathfrak{m}$. Hence if f can be written as $\sum_j a_j x_j$, then the coefficients a_j are determined uniquely up to modulo $\mathfrak{m}$. □

Proof. (2) By Bruns & Herzog, Proposition 2.2.4, $A/(f_1, \dots, f_n)$ is a regular local ring if and only if $(f_1, \dots, f_n)$ is generated by a subset of a regular system of parameters. Hence, we may assume $(f_1, \dots, f_n) = (x_{s_1}, \dots, x_{s_k})$ for some distinct $s_k \in \{1, \dots, n\}$, $k \leq n$. Then since the x_i map to basis elements of the d-dimensional κ-vector space under the projection, it follows that the rank of the matrix $(a_{ij} \bmod \mathfrak{m})$ is just equal to k. But note that since the x_{s_i} form a subsequence of the regular sequence x_i, they too are regular, and the dimension of $A/(f_1, \dots, f_n) = A/(x_{s_1}, \dots, x_{s_k})$ is just $\dim A - k = d - k$. Hence $d - (d - k) = k$, which shows the forward direction. Conversely, if we denote $\dim A/(f_1, \dots, f_n) = d - k$, the matrix has rank k so we may find k elements of the x_i that span the image $(f_1, \dots, f_n)$ in $\mathfrak{m}/\mathfrak{m}^2$. Thus $(f_1, \dots, f_n)$ is generated by a subset of the x_i of k elements, which implies that $A/(f_1, \dots, f_n)$ is a regular local ring by the proposition mentioned. □

Problem 4. $(A, \mathfrak{m})$ *is a noetherian local ring. Prove or disprove:*

(1) *If $\dim A = 0$ then A is CM.*

(2) *If* $\dim A = 1$ *and* A *is reduced, then* A *is CM.*
(3) *If* $\dim A = 2$ *and* A *is normal, then* A *is CM.*
(4) *If* A *is a regular local ring, then* A *is CM.*
(5) *If* $f_1, \dots, f_n$ *is an* A*-sequence in* $\mathfrak{m}$*, then* A *is CM iff* $A/(f_1, \dots, f_n)$ *is.*

Proof. (1) Since $\operatorname{depth} A \leq \dim A = 0$, equality holds automatically. □

Proof. (2) We claim that exists an element $x \in \mathfrak{m}$ that is not a zerodivisor. Suppose not. Then $\mathfrak{m}$ lies in the set of zerodivisors, i.e. the union of the minimal primes of A. (This needs reducedness, I recall proving it in the first or second homework.) By prime avoidance, $\mathfrak{m}$ would be equal to one of the minimal primes, but since $\operatorname{ht}(\mathfrak{m}) = 1$ this cannot happen. Thus x is A-regular, and $\operatorname{depth} A = \dim A = 1$. □

Proof. (3) Since A is normal, (R_1) and (S_2) hold by Serre's criterion. Since A is local, we have $A \cong A_{\mathfrak{m}}$ where $\operatorname{ht}(\mathfrak{m}) = \dim A = 2$. Then by (S_2) we have $\operatorname{depth}(A_{\mathfrak{m}}) = \dim(A_{\mathfrak{m}})$, which implies $\operatorname{depth} A = \dim A$. Hence A is CM. □

Proof. (4) Since A is a noetherian local ring, it has finite dimension. Denote $n = \dim A$ and $\mathfrak{m} = (x_1, \dots, x_n)$ a minimal system of generators. By Bruns & Herzog, Proposition 2.2.5, the sequence $x_1, \dots, x_n$ is an A-regular sequence. Hence $\operatorname{depth} A \geq \dim A$, so A is CM. □

Proof. (5) A is CM $\Leftrightarrow \operatorname{depth} A = \dim A$, and $A/(f_1, \dots, f_n)$ is CM $\Leftrightarrow \operatorname{depth}(A/(f_1, \dots, f_n)) = \dim(A/(f_1, \dots, f_n)) = \dim A - n$ since the sequence is regular. Also, by the lecture notes, we have $\operatorname{depth}(A/(f_1, \dots, f_n)) = \operatorname{depth} A - n$, so A is CM if and only if $A/(f_1, \dots, f_n)$ is. □

Problem 5.

Problem 6. *Give an example of a noetherian local ring* A *of positive dimension and a zerodivisor* $f \in A$ *such that* $\dim A/f = \dim A - 1$.

Proof. Let $A = k[[x, y]]/(x^2, xy)$. A is a noetherian local ring since $k[[x, y]]$ is. Since $\overline{x}$ and $\overline{y}$ are both nonzero in A, and $\overline{xy} = 0$, $\overline{y}$ is a zerodivisor in A. Since modding out by nilpotents do not change the dimension, and $\overline{x}$ is nilpotent, we have $\dim A = \dim A/(\overline{x}) = \dim k[[y]] = 1$. But $A/(\overline{y}) = k[[x]]/(x^2)$, and again by modding out nilpotents we have $\dim A/(y) = \dim k[[x]]/(x) = \dim k = 0$. Taking $f = \overline{y}$, we have $\dim A/f = \dim A - 1$. □