# PLAYING WITH SURFACES

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## 1. There are 27 lines on a smooth cubic surface

Here is the statement:

**Theorem 1.1.** *Any smooth cubic surface in $\mathbb{P}^3_{\mathbb{C}}$ contains exactly 27 lines.*

I'm pretty sure this holds over any algebraically closed field of characteristic zero, but since we have to use the Implicit Function Theorem we will be working over $\mathbb{C}$.

## 2. Mumford's Proof

Mumford proves the theorem by first showing that every smooth cubic contains the same number of lines, and then calculates the easiest case which is the Fermat cubic. The idea is to construct a space whose points correspond to objects we are interested in.

2.1. **Parameter space of smooth cubic surfaces in $\mathbb{P}^3$.** First, let's look at an arbitrary cubic homogeneous polynomial in four variables:

$$f = \sum_{|\alpha|=3} c_\alpha X^\alpha$$

The possible partitions of 3 are $(3, 0, 0, 0)$, $(2, 1, 0, 0)$ and $(1, 1, 1, 0)$, each of which there are 4, 12, and 4 for a total of 20 possible terms $X^\alpha$. As the zero set of polynomials is not affected by homothety, and we are by no means interested in the zero polynomial, the space of solutions to cubic forms can be identified with the 19-dimensional projective space $\mathbb{P}^{19}$. Also if we denote by $\Delta$ the points $c \in \mathbb{P}^{19}$ where the corresponding cubic surfaces are singular, the smooth cubics are parametrized by $U := \mathbb{P}^{19} - \Delta$.

2.2. **Parameter space of lines in $\mathbb{P}^3$.** Consider the set of all lines $\ell \subset \mathbb{P}^3$. Denote this set $\mathbb{G}(1, 3)$, which is called the (projective) Grassmannian of lines in $\mathbb{P}^3$. We will show that this is naturally a subset of $\mathbb{P}^5$, and furthermore has the structure of a variety. To do this, we will first construct a map

$$\mathbb{G}(1, 3) \to \mathbb{P}^5$$

sending $\ell$ to a point in $\mathbb{P}^5$.

By definition of projective spaces, a line $\ell$ in $\mathbb{P}^3$ corresponds to a 2-dimensional $\mathbb{C}$-subspace of $\mathbb{C}^4$. If we pick two basis elements of this subspace, say $v_1$ and $v_2$, then sending $\ell$ to $v_1 \wedge v_2 \in \wedge^2\mathbb{C}^4 \cong \mathbb{C}^6$ defines a map from $\mathbb{G}(1, 3)$ to $\mathbb{P}^5$. This is well-defined as $\operatorname{Span} v_i = \operatorname{Span} w_i$ if and only if $v_1 \wedge v_2$ and $w_1 \wedge w_2$ are linearly dependent, and as $v_1$ and $v_2$ span a two dimensional space its wedge is always nonzero. This map is moreover injective, by the same fact.

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**Proposition 2.1.** *The image of* $\mathbb{G}(1,3)$ *in* $\mathbb{P}^5$ *is algebraic.*

*Proof sketch.* Notice that the image consists of elements of the form $\overline{v \wedge w}$ for $v, w \in \mathbb{C}^4$. Thus, we want to find when an element $\overline{u} \in \mathbb{P}^5$ is equal to some $\overline{v \wedge w}$. Now, if we view $u$ as an element of $\wedge^2\mathbb{C}^4$, this is in fact equivalent to the linear map $(-) \wedge u : \mathbb{C}^4 \to \wedge^3\mathbb{C}^4$ having rank 2.[1] If we fix bases of $\mathbb{C}^4$ and $\wedge^3\mathbb{C}^4$, this is again equivalent to the $3 \times 3$ minors of the matrix vanishing. As these are polynomials, we conclude that the set of viable $u$ is determined by a system of polynomial equations, hence is algebraic. □

2.3. **The incidence variety.** We will identify $\mathbb{G}(1,3)$ with its image in $\mathbb{P}^5$. Now that we have our parameter spaces, we will define a subspace

$$I \subset U \times \mathbb{G}(1,3)$$

as the pairs $(X, \ell)$ where $X$ is a smooth cubic, and $\ell$ is a line contained in $X$. We need to prove two propositions about the geometry of $I$.

**Proposition 2.2.** *$I$ is an algebraic set, i.e., is given by polynomial equations.*

*Proof.* Let's focus on a point $(X, \ell) \in I$, where we may assume that $\ell \subset \mathbb{P}^3$ is given by the equation $x_2 = x_3 = 0$, by suitably renaming coordinates of $\mathbb{P}^3$. Now $\ell$, as a point of $\mathbb{G}(1,3)$, can be given by the subspace

$$\mathrm{Span}_{\mathbb{C}}\{(1,0,0,0),(0,1,0,0)\} \subset \mathbb{C}^4.$$

Note that we may consider a neighborhood of $\mathbb{G}(1,3)$ that contains $\ell$, by considering all subspaces

$$\mathrm{Span}_{\mathbb{C}}\{(1,0,a_2,a_3),(0,1,b_2,b_3)\} \subset \mathbb{C}^4$$

by varying the parameters $a_2, a_3, b_2, b_3 \in \mathbb{C}$. Now, recall that the space of nonsingular cubic forms had coordinates $c_\alpha$. The necessary and sufficient condition for $(c_\alpha, a_2, a_3, b_2, b_3) \in U \times \mathbb{G}(1,3)$ to be contained in $I$ is obviously that

$$s(1,0,a_2,a_3) + t(0,1,b_2,b_3) \in Z(f_{c_\alpha})$$

for all $s, t \in \mathbb{C}$, where $f_{c_\alpha}$ is the cubic form corresponding to $c_\alpha$. This means that

$$\sum_\alpha c_\alpha s^{\alpha(0)} t^{\alpha(1)} (sa_2 + tb_2)^{\alpha(2)} (sa_3 + tb_3)^{\alpha(3)} = 0$$

for all $s, t \in \mathbb{C}$. This must be a degree 3 homogeneous polynomial in $s$ and $t$, so we could rewrite this equation according to the degree of $s$, where $F_i(c_\alpha, a_2, a_3, b_2, b_3)$ are the coefficients of $s^i$. Hence $I$ is defined as the zero set of four equations $F_i$ for $0 \le i \le 3$. □

Now you may question where this proof is heading towards. By nature of the definition of $I$, it is equipped with a projection map to its first factor in $U$. If we are given a smooth cubic, i.e., a point $X$ in $U$, the fiber at $X$ of the projection is exactly the set of lines contained in $X$. First, as above, given a point $(X, \ell)$ we want to show that there is a neighborhood of $X$ and a neighborhood of $\ell$ such that the four polynomials that cut out $I$ locally actually cut out a graph of a differentiable function $U \to \mathbb{G}(1,3)$. To do this, we show that the Jacobian $J := \frac{\partial(F_0,F_1,F_2,F_3)}{\partial(a_2,a_3,b_2,b_3)}$ is invertible at $a_2, a_3, b_2, b_3$, i.e., at $\ell$. We will take the Implicit Function Theorem in several complex variables as granted.

[1] For a simple case, let $u = e_3 \wedge e_4$ and try sending $e_i$ through it. The image is 2-dimensional.

**Proposition 2.3.** *$J$ is invertible at $\ell$.*

*Proof.* I will state the results rather than going through calculations. The first column of the matrix $J$ are the $(s,t)$-coefficients of

$$s\frac{\partial f_c}{\partial x_2}(s,t,0,0)$$

where $f_c$ is the cubic form defining $X$, with coefficients of $\mathbb{P}^3$ given as $x_i$. The other columns are

$$s\frac{\partial f_c}{\partial x_3}(s,t,0,0), \quad t\frac{\partial f_c}{\partial x_2}(s,t,0,0), \quad t\frac{\partial f_c}{\partial x_3}(s,t,0,0).$$

Assume by contradiction that $J$ is not invertible, then we would have a relation

$$L_2\frac{\partial f_c}{\partial x_2}(s,t,0,0) + L_3\frac{\partial f_c}{\partial x_3}(s,t,0,0) = 0$$

as polynomials in $s$ and $t$, where $L_2$ and $L_3$ are some linear combinations of $s$ and $t$. We may decompose the whole equation into linear factors (we're working with two variables over an algebraically closed field) and claim that there exist a common linear factor of $\frac{\partial f_c}{\partial x_2}(s,t,0,0)$ and $\frac{\partial f_c}{\partial x_2}(s,t,0,0)$. Recall that $\ell = \{(s:t:0:0)\}$, so this means that there is a point $p \in \ell$ such that $\frac{\partial f_c}{\partial x_2}(p) = \frac{\partial f_c}{\partial x_3}(p) = 0$. Since we also have $f_c(s,t,0,0) = 0$ for all $s$ and $t$ we automatically have $\frac{\partial f_c}{\partial x_i}(p) = 0$ for all $0 \le i \le 3$, which implies that $p \in \ell \subset X$ is a singular point of $X$. This is contrary to our assumption that $X$ is nonsingular, hence $J$ must be invertible. □

With these propositions, we may finish the proof.

**Proposition 2.4.** *The number of lines contained in a smooth cubic surface is locally constant on $U$.*

*Proof.* Fix a smooth cubic surface $X \in U$, and consider lines $\ell \subset \mathbb{P}^3$. If $\ell \subset X$, then we know that there is a neighborhood $V_\ell \times W_\ell$ of $(X,\ell) \in I$ such that it is the graph of a function $U \to \mathbb{G}(1,3)$. This implies that every cubic in $V_\ell$ contains exactly one line in the neighborhood $\ell \in W_\ell \subset \mathbb{G}(1,3)$. Now if $\ell$ is not contained in $X$, then $(X,\ell) \notin I$. Since $I$ is algebraic, we know that it is closed, so there exist a neighborhood $V_\ell \times W_\ell$ of $(X,\ell)$ such that no cubic of $V_\ell$ contains any line.

For our fixed $X$, we may vary $\ell \subset \mathbb{P}^3$ to cover $\mathbb{G}(1,3)$ by the open neighborhoods $W_\ell$ defined as above. As $\mathbb{G}(1,3)$ is projective, it is compact, and we have a finite subcover, say by the $W_{\ell_i}$. Then we can define $V := \bigcap_i V_{\ell_i}$ which is still a neighborhood of $X \in U$, such that every $Y \in V$ contains the same number of lines. Namely, the number of lines is the number of open neighborhoods $W_\ell$ containing a line $\ell$ in $X$. Hence the number of lines is finite, and is locally constant on the space $U$ of smooth cubic surfaces. □

As $U$ is the complement of $\Delta$ in $\mathbb{P}^{19}$, it is believable that $U$ is connected. The only case that this may not hold is when $\Delta$ is 19-dimensional, but its codimension should be at least 1 so this does not happen. Hence the number of lines is globally constant, so we can pick our favorite smooth cubic surface and compute the number of lines by hand.

**Proposition 2.5.** *The Fermat cubic $V(x_0^3 + x_1^3 + x_2^3 + x_3^3) \subset \mathbb{P}^3$ contains 27 lines.*

Proving this is straightforward calculation, which is tedious but not really illuminating so I won't prove it here. In fact, we can use our observations to prove that any smooth cubic surface is isomorphic to $\mathbb{P}^2$ blown up at 6 suitable points. Some texts prove the 27 lines theorem by proving this first.

(Proceed if time permits)

## 3. The geometry of blowups of the plane–calculating $N_3 = 12$

We can say much about geometric problems regarding the projective plane via the blowing up technique. Another important problem in enumerative geometry is to determine the number $N_d$ of degree $d$ rational plane curves passing through $3d-1$ general points in the plane. Here the term general means that we will not consider special configurations of the points. We won't actually *define* $N_d$ so here you go...

**Theorem 3.1** (Belief)**.** $N_d \in \mathbb{N}$

Using our intuition, $N_1 = 1$ is trivial. The fact that $N_2 = 1$ is also not that tricky, as the space of quadratic forms can be identified with $\mathbb{P}^5$, and the condition of passing through a point determines a hyperplane in $\mathbb{P}^5$. Hence passing through 5 points corresponds to 5 hyperplanes, so you end up with a single quadratic form. But from $N_3$, it becomes less trivial.

Suppose we have 8 points in the plane. We can consider the space of cubic forms

$$\sum_{|\alpha|=3} c_\alpha X^\alpha$$

where $\alpha$ ranges over the partions of 3 over 3 variables. Since we have $(3,0,0)$, $(2,1,0)$ and $(1,1,1)$, each having 3, 6, and 1 case each, we can identify the space with $\mathbb{P}^9$. As we have specified 8 points, this constrains the possible forms into a projective line, i.e., a pencil of cubics. As $\mathbb{P}^1$ is just a 2-dimensional vector space over $\mathbb{C}$, pick any two cubic forms $p$ and $q$ that generate this space. In other words, every cubic that passes through the 8 points can be represented as some linear combination $\lambda p + \mu q = 0$. Since we are interested in the best cases only, we may take both $p$ and $q$ to be nonsingular. We will try to define a morphism to $\mathbb{P}^1$ with coordinates $\lambda$ and $\mu$, such that its fiber is the zero set of $\lambda p + \mu q = 0$.

Notice how specifying another point in the plane determines a single element in the space $\lambda p + \mu q$ which passes it, due to dimension counting. Thus, it seems to be possible to define a map

$$\mathbb{P}^2 \to \mathbb{P}^1$$

sending a point in the plane to the $[\lambda : \mu]$ passing through it. However, by Bézout's theorem, $p$ and $q$ must meet at 9 points in the plane, obviously including the 8 points we already specified. This means that $p$ and $q$ simultaneously vanish at 9 points in the plane, so in this case $\lambda$ and $\mu$ aren't specified, and instead we have a rational map

$$\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$$

undefined at the 9 points where $p$ and $q$ meet. However, we may blow up the rational map at the 9 points to obtain an honest morphism $X \to \mathbb{P}^1$. This way, we can finally represent the zero set of every $\lambda p + \mu q = 0$ in $\mathbb{P}^2$ as the fiber of $X \to \mathbb{P}^1$.

**Theorem 3.2** (Degree-genus formula)**.** *A smooth curve of degree $d$ in $\mathbb{P}^2$ has genus $g = \frac{(d-1)(d-2)}{2}$.*

As the smoothness of cubic curves are determined by a finite set of equations, we can say that all but finite elements of $\lambda p + \mu q$ are nonsingular. This means that almost all fibers are curves of genus 1. However, we want to find the number of rational curves, which are curves of genus 0. The degree-genus formula forces such rational curves to be singular. Hence it suffices to find the number of singular fibers of this pencil. Cubic curves have two singularities which are called nodal and cuspidal singularities, but the latter rarely occurs so there is no problem in assuming that all singularities that occur are nodal.[2]

The topological Euler characteristics of a nonsingular curve and a nodal curve are zero and one, respectively. If every fiber of $X \to \mathbb{P}^1$ were nonsingular, then $X$ would have zero Euler characteristic. However, this is not the case. As $X$ is $\mathbb{P}^2$ blown up at 9 points, we may directly find its Euler characteristic via cell decompositions. As $\mathbb{P}^2 = \mathbb{P}^0 \cup \mathbb{A}^1 \cup \mathbb{A}^2$, its $\chi$ is 3. Also the process of blowing up a point involves removing a point and adding a projective line, it increases the Euler characteristic by 1. Hence $\chi(X) = 12$, so there are 12 nonsingular fibers. Hence $N_3 = 12$.

**Theorem 3.3** (Kontsevich–Manin, '94)**.**

$$N_d = \sum_{d_A + d_B = d} N_{d_A} N_{d_B} d_A^2 d_B \left( d_B \binom{3d-4}{3d_A - 2} - d_A \binom{3d-4}{3d_A - 1} \right)$$

[2]In fact, this is what's called a Lefschetz pencil. As we're looking at general pencils in $\mathbb{P}^9$, we can assume that the pencil is Lefschetz.