REPORT FOR THE 2023 FALL UNDERGRADUATE INTERNSHIP PROGRAM

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ABSTRACT. This is the report for the 2023 fall semester undergraduate internship program, in which I studied about Coxeter polytopes in hyperbolic d-space. This report consists of what I have learned about this subject throughout the semester, focused on the work of Vinberg, Lanner, Esselmann, and Kaplinskaya, and will also serve as an introduction to this subject. This article requires no prior knowledge to any of the subject matter. I thank Professor Gye-Seon Lee who introduced me to this wonderful field of mathematics.

1. Introduction to Hyperbolic Geometry

A hand-waiving explanation of hyperbolic geometry is that it is sort the opposite of spherical geometry, having constant negative curvature instead of positive curvature. Just like S^d is defined to be the space $|x| = 1$ for $x \in \mathbb{R}^{d+1}$, analogously we work in $\mathbb{R}^{d+1}$ to define hyperbolic space as a sphere of imaginary radius. To do this, we must introduce a new notion of inner product. Thus prior to defining hyperbolic space, we first define Lorentzian space and the Lorentzian inner product.

Suppose $x, y \in \mathbb{R}^{d+1}$. The Lorentzian inner product of x, y, denoted $x \circ y$, is defined to be

$$x \circ y = -x_1y_1 + x_2y_2 + \cdots + x_{d+1}y_{d+1},$$

where the subscripts are coordinates with respect to the standard basis of $\mathbb{R}^{d+1}$. The Lorentzian norm is defined as the complex number

$$||x|| = (x \circ x)^{1/2}$$

for $x \in \mathbb{R}^{d+1}$. With respect to this norm, consider the sphere of unit imaginary radius

$$F^d = \{x \in \mathbb{R}^{d+1} \mid ||x||^2 = -1\}.$$

Since F^d has two connected components, we select the component where $x_1 > 0$, called the positive sheet of F^d. The positive sheet of F^d, together with the Lorentzian inner product, is our model of hyperbolic d-space. (Note that there are other models of hyperbolic space, but they are essentially the same thing.) The reason why we use d to denote dimension instead of the customary n is to reserve n for the number of facets of a polytope, which we will define later.

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The metric $d_H(x, y)$ for $x, y \in \mathbb{H}^d$ is defined to be the unique nonnegative number that satisfies

$$x \circ y = -\cosh d_H(x, y).$$

This is indeed a metric, see [5] for the proof.

Next, we define hyperplanes in hyperbolic space. For $e \in \mathbb{R}^{d+1}$ satisfying $||e|| > 0$, we define the hyperplane determined by e as

$$H_e = \{x \in \mathbb{H}^d \mid x \circ e = 0\}.$$

Note that vectors parallel to e define the same hyperplane, so we may assume $||e|| = 1$. Intuitively, hyperplanes bisect hyperbolic space. Denote by

$$H_e^- = \{x \in \mathbb{H}^d \mid x \circ e \leq 0\},$$

and define H_e^+ analogously. Given $e, f \in \mathbb{R}^{d+1}$ where $||e|| = ||f|| = 1$, the hyperplanes H_e and H_f intersect if and only if $|e \circ f| < 1$, are parallel if and only if $|e \circ f| = 1$, and are ultraparallel if and only if $|e \circ f| > 1$. If the two intersect, their dihedral angle $\widehat{H_e H_f}$ is given by $\cos \widehat{H_e H_f} = |e \circ f|$.

2. Reflections, Coxeter Polytopes and Coxeter Diagrams

As in familiar Euclidean space, we may define a reflection in hyperbolic space, with respect to a hyperplane. For each hyperplane H_e, there exists a unique isometry in $\mathrm{Isom}(\mathbb{H}^d)$ of order 2 that leaves H_e fixed. We will call such an isometry a reflection with respect to the hyperplane H_e.

We define a convex d-polytope in $\mathbb{H}^d$ to be a subset of the form $P = \bigcap_{i \in I} H_i^-$ where I is a finite index set consisting of $i \in \mathbb{R}^{d+1}$ such that $||i|| = 1$. Also, we require P to contain a nonempty open subset of $\mathbb{H}^d$. In this article, we are mainly interested in compact polytopes, so we assume that P is bounded. A d-polytope P is bounded by $(d-1)$-polytopes, which are again bounded by $(d-2)$-polytopes and so on. We call bounding polytope 'faces', where the highest dimensional $(d-1)$ faces are called facets.

Suppose P satisfies the above conditions. If for all pairs of H_i they either do not meet, or have dihedral angle of the form π/k for $k \in \mathbb{Z}_{\geq 2}$, we call P a Coxeter d-polytope. Some literature define P together with its reflection group generated by its hyperplanes as Coxeter polytopes. Such P are the fundamental domains of the discrete groups generated by reflections in the H_i, and conversely, each discrete reflection group has a Coxeter polytope as its fundamental domain. For each Coxeter polytope bounded by n hyperplanes, its associated Gram matrix is defined to be a symmetric $n \times n$ matrix $G = \{g_{ij}\}$ where $g_{ii} = 1$, and g_{ij} for $i \neq j$ is $-\cos(\pi/m_{ij})$ if $\widehat{H_i H_j} = \pi/m_{ij}$, -1 if H_i and H_j are parallel, and $-\cosh d_H(H_i, H_j)$ if H_i and H_j are ultraparallel. Note that Coxeter polytopes are very essential in that they can tile space, which is ensured by definition.

We may encode the data of a Coxeter polytope P in a Coxeter diagram. A Coxeter diagram is a labeled graph, where each vertex corresponds to facets of P. Vertices of the Coxeter diagram of P are connected by a line labeled $k-2$ if the dihedral angle of the corresponding facets is π/k. For $k=3$ it is customary to draw a single line. For $k=2$, there is no line joining the vertices. If the two facets do not meet, then the corresponding vertices are joined by a dotted line.

3. Compact Coxeter d-Polytopes with $d+k$ facets

Coxeter polytopes for Euclidean and spherical space are defined analogously as hyperbolic Coxeter polytopes. The classification of all such Coxeter polytopes was accomplished by Coxeter [2]. However, the classification of hyperbolic Coxeter polytopes is not finished. By work of Vinberg [6], it is known that there does not exist compact hyperbolic Coxeter d-polytopes for $d \geq 30$. The highest dimension of known examples as of January 2024 is $d=8$, and between $d=9$ and 29, we do not know about the existence or nonexistence of such polytopes.

One way of classifying such polytopes is classifying them by the number n of facets. Obviously, a d-polytope must have at least $n=d+1$ facets, so it is natural to think of d polytopes with $n=d+k$ facets, for $k \geq 1$. From now on, 'polytope' implies being compact hyperbolic Coxeter.

3.1. d-polytopes with $d+1$ facets.

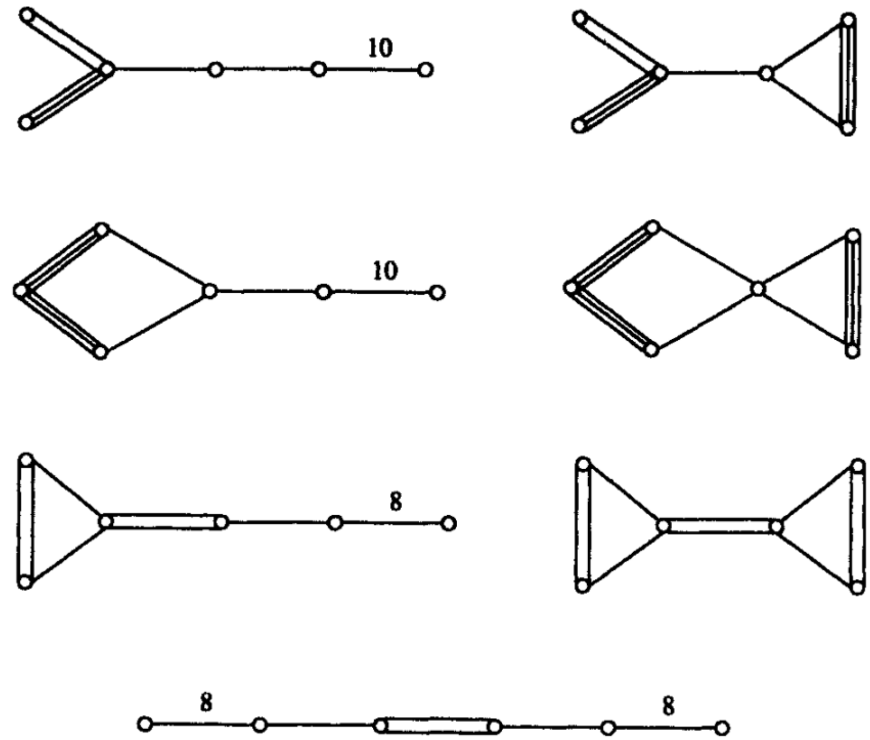
Such polytopes are completely classified by Lannér, and are called simplices. Their Coxeter diagrams are called Lannér diagrams. They exist only up to $d=4$, and their Coxeter diagrams are in [3], see Table 1.

3.2. d-polytopes with $d+2$ facets.

Such polytopes are completely classified by Kaplinskaya and Esselmann. Kaplinskaya [4] has classified polytopes having combinatorial type of a product of a line segment and a simplex. We call such polytopes prisms.

Prisms exist only for $d \leq 5$, since simplices exist only for $d \leq 4$ as mentioned above. Kaplinskaya used the classification of simplices to find all prisms.

The work of Esselmann [3] completes the classification of d-polytopes with $d+2$ facets by classifying all polytopes of combinatorial type the product of two simplices for $d=4$, and proving nonexistence for $d \geq 5$. This exhausts all possibilities, since polytopes in $d=3$ are all classified by work of Andreev, and polytopes in $d=2$ are classified by work of Poincaré. Also by work of Vinberg, every compact Coxeter polytope is simple (i.e. every k-face belongs to $d-k$ facets) and simple polytopes with $d+2$ facets are products of two simplices. The list is as follows:



We introduce how Essselmann [3] completed this classification. Esselmann uses a tool called a Gale transform, which describes a polytope by transforming its vertices into vectors in dimensions lower than the original polytope. More specifically, the data of inclusion relations of faces of P (called the face-lattice) can be determined by unit vectors in $\mathbb{R}^{n-d-1}$. Hence for d-polytopes having $n = d+2$ facets, their combinatorial data can be read off the Gale transformation of their $d+2$ vertices which are $d+2$ unit vectors in $\mathbb{R}$. Hence each entry is either $+1$ or -1. Esselmann then finds conditions on the multiplicities of each $+1$ and -1, denoted x and y, by assigning Lannér diagrams of order x and y and finds Coxeter diagrams S that contain such Lannér diagrams of order x and y. (To be precise, we need additional conditions on S but we do not explain this.)

For S a Coxeter diagram, denote by $C_k(S)$ the set of all Coxeter diagrams $S' \supset S$, $|S'| = |S| + k$ such that

- S' is connected,
- S' does not contain a parabolic subdiagram, and
- S' contains exactly the Lannér diagrams already contained in S.

Here, a Coxeter diagram is parabolic if it has determinant zero but every proper subdiagram is spherical. A Coxeter diagram is spherical if the corresponding Gram matrix is positive definite.

A vertex $v \in S$ is called an open vertex if there exists $S' \in C_1(S)$ such that v is adjacent to the vertex $S' \setminus S$. It turns out that one of the two Lannér subdiagrams must have an open vertex, hence we have 7 possibilities. Esselmann goes through each possibility, and proves that there exist only the polytopes listed above.

3.3. **d-polytopes with $d+3$ facets.** Such polytopes are also completely classified, and there is a visual list of all such diagrams for $d = 4, 5, 6, 8$ in Anna Felikson's homepage. It is known that such polytopes exist only for $d \leq 8$, $d \neq 7$. The only known example for $d = 8$ by Bugaenko is in fact the only one existing. As in the case above, the main strategy is to use Gale transformations, and restrict possible combinatorial types via previously known results.

3.4. **d-polytopes with $d+4$ facets and beyond.** From $d+4$ and higher, there is known complete classification as of January 2024. Partial results for $n = d+4$, $d = 4$ and 5 are known due to Felikson, Tumarkin and Burcroff. However, for $n = d+4$, it is known that they do not exist for $d > 7$. There is an example in $d = 7$ due to Bugaenko, which is unique. Also, there are known examples for $n = d+5$ and $d+6$.

4. Allowing Ideal and Hyperideal Vertices

In defining polytopes, we may allow vertices to be in the boundary of hyperbolic space. Here we are not using the definition of hyperbolic space we introduced earlier, as by definition there is no boundary. However, as a subset of $\mathbb{R}^{n+1}$, we may consider the projectivization of the cone defined by $\{x \in \mathbb{R}^{d+1} \mid ||x|| = 0\}$ as its boundary. Precisely, we define a hyperbolic d-polytope to be a polytope P in an affine chart of $\mathbb{RP}^d$ such that every facet of P intersects $\mathbb{H}^d$. Thus we are allowing ideal vertices (vertices on the boundary of $\mathbb{H}^d$) and hyperideal vertices (vertices outside of $\overline{\mathbb{H}^d}$). We may define Coxeter polytopes analogously, which vastly increases the complexity of the theory.

In particular, Choi, Lee and Marquis [1] use the technique of truncating a hyperideal vertex of Coxeter polytopes, and describes 2-perfect hyperbolic Coxeter truncation polytopes P of dimension $d \geq 4$. For a Coxeter d-polytope P which has at least one hyperideal vertex, for each hyperideal vertex v take the dual hyperplane H_v defined by the hyperbolic quadratic form. Then H_v intersects perpendicularly all the edges containing v. Truncate all the hyperideal vertices of P via their dual hyperplanes to obtain a finite volume polytope. Here, a 2-perfect polytope is a polytope where each edge meets hyperbolic space.

5. Conclusion

This internship was my first encounter with hyperbolic geometry and combinatorics, and was tough. However I was able to learn how research progresses in modern day combinatorics and geometry, and some useful techniques. Also, I find the problem of classifying hyperbolic Coxeter polytopes to be very essential and beautiful. By truncating a polytope with hyperideal vertices as in [1], I hoped to find new examples of finite volume polytopes. Although I could not find new examples, I hope to find some in the near future.

References

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