MODULI SPACES OF SEMISTABLE SHEAVES

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ABSTRACT. This article serves both as a report for the 2024 summer College of Natural Sciences Undergraduate Research Internship program, and as an introduction to moduli spaces of semistable sheaves.

Moduli spaces of coherent sheaves on n-folds appear naturally in algebraic geometry. For $n = 3$, the moduli space of sheaves on a Calabi-Yau threefold provides a setting for calculating Donaldson-Thomas invariants. Moreover, the moduli spaces of sheaves for $n \leq 2$ provide a wide range of geometrically interesting examples.[HL10] We outline the proof of existence of the moduli space of semistable sheaves on polarized projective schemes with fixed Hilbert polynomials. All schemes are assumed to be noetherian, and base schemes are $\operatorname{Spec}\mathbb{C}$ otherwise stated.

1. Grothendieck's Quot Scheme

Roughly speaking, we want to construct a space whose points correspond to isomorphism classes of coherent sheaves on some X. A convenient way of doing this is to use the notion of a Quot scheme, which parametrizes quotients of sheaves. We first define the Quot functor, which sends a scheme to the set of equivalence classes of quotients having fixed Hilbert polynomial $P \in \mathbb{Q}[z]$ parametrized by that scheme. Recall that the Hilbert polynomial of a coherent sheaf F on a projective scheme with fixed ample line bundle $\mathcal{O}(1)$ is defined as $m \mapsto \chi(F(m))$ and encodes data such as rank and Chern classes.

Definition 1.1. Let S be a scheme over $\mathbb{C}$, and let X be projective over S with structure morphism f. Fix an f-ample line bundle $\mathcal{O}_X(1)$, and fix $\mathcal{H} \in \mathsf{Coh}_X$ and $P \in \mathbb{Q}[z]$. Define the Quot functor

$$\mathcal{Q}uot_{X/S}(\mathcal{H}, P) : \mathsf{Sch}_S^{\mathrm{op}} \to \mathsf{Set}$$

by sending an S-scheme T to the set of all T-flat quotients[1] $\mathcal{H}_T := \mathcal{O}_T \otimes_{\mathcal{O}_{X \times_S T}} \mathcal{H} \to F \in \mathsf{Coh}_{X \times_S T}$ with $\chi(F|_{X \times_S \operatorname{Spec}(\kappa(t))}(m)) = P(m)$ for all $t \in T$, and sending an S-morphism $g : T' \to T$ to the map that sends $\mathcal{H}_T \to F$ to $\mathcal{H}_{T'} \to h^*F$. Here, h is the dotted arrow in the following diagram

$$\begin{tikzcd}
 & X \arrow[dd] & \\
X \times_S T' \arrow[ur] \arrow[rr, dotted] \arrow[dd] & & X \times_S T \arrow[ul] \arrow[dd] \\
 & S & \\
T' \arrow[ur] \arrow[rr, "g"] & & T \arrow[ul]
\end{tikzcd}$$

which must be $\mathrm{id}_X \times g$.

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[1] Define quotients of $E \in \mathsf{Coh}_X$ to be equivalence classes of epimorphisms $q : E \to F$, where $q \sim q'$ whenever $\ker q = \ker q'$.

It is a theorem of Grothendieck that the Quot functor is in fact represented by a projective S-scheme. Due to its length, we will not state the proof here. Refer to [Nit05] for the full proof.

Theorem 1.2 (Grothendieck)**.** *The functor $\mathcal{Q}uot_{X/S}(\mathcal{H}, P)$ is represented by a projective S-scheme $\mathrm{Quot}_{X/S}(\mathcal{H}, P)$. In other words, the Quot functor is naturally isomorphic to $\mathrm{Hom}_{\mathsf{Sch}_S}(-, \mathrm{Quot}_{X/S}(\mathcal{H}, P)) : \mathsf{Sch}_S^{\mathrm{op}} \to \mathsf{Set}$. The representing scheme is called the Quot scheme.*

For the situation of our interest, it will suffice to let $S = \operatorname{Spec} \mathbb{C}$. Notice that in this case $\mathcal{Q}uot_{X/S}(\mathcal{H}, P)(S)$ becomes the set of all quotients of $\mathcal{H}$ having Hilbert polynomial P. By Theorem 1.2, it follows that quotients of $\mathcal{H}$ correspond to closed points of the scheme $\mathrm{Quot}_{X/S}(\mathcal{H}, P)$, which justifies the intuition that Quot schemes parametrize quotients of coherent sheaves having some fixed Hilbert polynomial. However note that this is not quite what we wanted initially; we want to find a scheme that parametrizes all coherent sheaves with Hilbert polynomial P on X, not just quotients of some sheaf. To remedy this, we consider sheaves F on X with Hilbert polynomial P, together with a choice of basis for $H^0(X, F(m))$ for some large enough m. This defines an epimorphism $\mathcal{H} := \mathcal{O}_X(-m)^{\oplus \dim_{\mathbb{C}} H^0(X, F(m))} \to F$, hence corresponds to a closed point of $\mathrm{Quot}_{X/\mathbb{C}}(\mathcal{H}, P)$. One last problem is that we do not want $\mathcal{H}$ to depend on the choice of F, as otherwise it would define a different Quot scheme. It turns out that we can make this possible via imposing a semistability condition.

Definition 1.3 (Semistable sheaves)**.** The reduced Hilbert polynomial $p(E)$ of a coherent sheaf E on a projective scheme X with fixed $\mathcal{O}_X(1)$ is the Hilbert polynomial of E divided out by its leading coefficient. Define E to be pure if $\dim F = \dim E$ for all nontrivial coherent subsheaves $F \subset E$. If E is a pure sheaf of the same dimension as X, and if for any proper subsheaf $F \subset E$ we have $p(F) \leq p(E)$, we call E semistable. Here, the ordering is lexicographic.

Theorem 3.3.7 of [HL10] implies that the family of semistable sheaves is bounded (whose definition we will not state). By Lemma 1.7.6 of the same reference, this implies the existence of some integer m such that every semistable sheaf F with Hilbert polynomial P is m-regular, i.e. $H^i(X, F(m-i)) = 0$ for all $i > 0$. Moreover, by Serre's vanishing theorem, one may choose m such that $H^i(X, F(m)) = 0$ for $i > 0$, which implies $\chi(F(m)) = \dim_{\mathbb{C}} H^0(X, F(m)) = P(m) =: n$. In this case, each $F(m)$ is globally generated, so one may consider a surjection $\rho : \mathcal{H} \to F$ where $\mathcal{H} = \mathbb{C}^n \otimes_{\mathbb{C}} \mathcal{O}_X(-m)$. This surjection is induced by composing an isomorphism $\mathbb{C}^n \cong H^0(X, F(m))$ with the evaluation map $H^0(X, F(m)) \otimes \mathcal{O}_X(-m) \to F$. Hence, each semistable sheaf F on X having Hilbert polynomial P defines a closed point of $\mathrm{Quot}_{X/\mathbb{C}}(\mathcal{H}, P)$. In fact, if we consider all closed points $[\mathcal{H} \to E] \in \mathrm{Quot}_{X/\mathbb{C}}(\mathcal{H}, P)$ where E is semistable and the composition $\mathbb{C}^n \xrightarrow{\sim} H^0(X, \mathcal{H}(m)) \to H^0(X, E(m))$ is an isomorphism, these points form an open subset $R \subset \mathrm{Quot}_{X/\mathbb{C}}(\mathcal{H}, P)$. This is because being semistable is an open property in flat families, due to Proposition 2.3.1 of [HL10]. Since choosing a basis for $H^0(X, F(m))$ has a GL_n-action, and orbits under this action are in the same isomorphism class, we must consider the quotient R/GL_n as our desired parameter space. In the following section, we will see how to define this quotient, and see that this indeed parametrizes sheaves in a suitable sense. Or in words we will define, it will be a *moduli space*.

2. The Moduli Space

We define a moduli functor $\mathcal{M}$ intended to parametrize the set of isomorphism classes of semistable sheaves with fixed Hilbert polynomial P. Unlike the Quot functor, in general, we cannot hope for $\mathcal{M}$ to be represented by a scheme. However, there always exists a projective scheme M that *corepresents* $\mathcal{M}$, which is a weaker notion of representability. We will call M the moduli space of semistable sheaves.

Definition 2.1 (Moduli functor)**.** Let X be projective over $\mathbb{C}$ with fixed ample line bundle $\mathcal{O}_X(1)$. Fix some $P \in \mathbb{Q}[z]$. Define $\mathcal{M}' : \mathsf{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathsf{Set}$ by sending $S \to \operatorname{Spec} \mathbb{C}$ to the set of isomorphism classes of S-flat families of semistable sheaves on X having Hilbert polynomial P. If $f : S' \to S$ is a $\mathbb{C}$-morphism, let $\mathcal{M}'(f)$ be the map that sends $[F]$ to $[(\mathrm{id}_X \times f)^* F]$. Furthermore, we impose another equivalence relation on the target category where $F \sim F'$ whenever $F \cong F' \otimes p^* L$ for some $L \in \operatorname{Pic}(S)$, where $p : X \times_{\mathbb{C}} S \to S$ is the projection. We define the functor $\mathcal{M}$ to be $\mathcal{M}'/\sim$.

As stated above, moduli functors in general are not representable. Hence we introduce the notion of corepresentability:

Definition 2.2. A functor $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ is corepresented by $F \in \mathcal{C}$ if there exists a natural transformation $\alpha : \mathcal{F} \Rightarrow h^F$ such that any natural transformation $\alpha' : \mathcal{F} \Rightarrow h^G$ factors uniquely through $\beta : h^F \Rightarrow h^G$.

$$\begin{tikzcd} \mathcal{F} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & h^F \arrow[d, Rightarrow, dashed, "\exists!"] \\ & h^G \end{tikzcd}$$

Such an α universally corepresents $\mathcal{F}$ if for any $\phi : h^T \Rightarrow h^F$, the fiber product $\mathcal{T} := h^T \times_{h^F} \mathcal{F}$ is corepresented by T.

$$\begin{tikzcd} \mathcal{T} \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, phantom, "\lrcorner", very near start] & h^T \arrow[d, Rightarrow] \\ \mathcal{F} \arrow[r, Rightarrow, "\alpha"'] & h^F \end{tikzcd}$$

Notice that corepresentability directly generalizes representability. One may also check that a corepresenting object is unique up to unique isomorphism; any two such functors must be isomorphic in the functor category, which implies there is a unique isomorphism between the two objects due to the Yoneda lemma. Using these definitions, we may state the following theorem:

Theorem 2.3 (Theorem 4.3.4 of [HL10])**.** *The functor $\mathcal{M}$ is universally corepresented by a projective $\mathbb{C}$-scheme M, whose closed points are in bijection with S-equivalence classes of semistable sheaves with Hilbert polynomial P. This scheme is called the moduli space of semistable sheaves. It follows that M also corepresents $\mathcal{M}'$.*

Two semistable sheaves having the same reduced Hilbert polynomial are said to be S-equivalent if the direct sums of the factors of their respective Jordan-Hölder filtration are isomorphic. It is thus the S-equivalence classes of semistable sheaves on X that we may parametrize with M.

3. Algebraic Group Actions

We address the problem of defining a quotient of R with respect to the GL_n-action.

Definition 3.1 (Good quotients)**.** Let G be an affine algebraic group acting on X. A morphism $\varphi : X \to Y$ is a good quotient if it satisfies the following:

1. φ is affine and G-invariant.
2. φ is surjective, and $U \subset Y$ is open if and only if $\varphi^{-1}(U) \subset X$ is open.
3. The composition $\mathcal{O}_Y \to \varphi_* \mathcal{O}_X \to (\varphi_* \mathcal{O}_X)^G$ is an isomorphism.
4. If W is a G-invariant closed subset of X, then its image is closed in Y. If W_1 and W_2 are disjoint invariant closed subsets of X, then the images are disjoint.

Denote a good quotient of X as $X /\!\!/ G$. We call φ a universal good quotient if $Y' \times_Y X \to Y'$ is a good quotient for any morphism $Y' \to Y$.

In general, a good quotient of a group action may not exist. It is a result of Geometric Invariant Theory that group actions by certain groups called reductive groups admit good quotients. Also, we will need a lot of definitions such as G-linearization of sheaves, and the semistable points $X^{\mathrm{ss}}(L)$ of X with respect to a G-linearized ample line bundle L. See [MFK94] for the definitions. Note that GL_n is in fact a reductive group, hence we may apply the results of GIT to our situation.

Theorem 3.2. *Let G be a reductive group acting on a projective scheme X with G-linearized ample line bundle L. Then there is a projective scheme Y and a morphism $\pi : X^{\mathrm{ss}}(L) \to Y$ such that π is a universal good quotient for the G-action.*

Using this result, we try to find a good quotient of R. Recall that R is the open subset of $\mathrm{Quot}_{X/\mathbb{C}}(\mathcal{H}, P)$ consisting of points $[\mathcal{H} \to E]$ with E semistable and $\mathbb{C}^n \xrightarrow{\sim} H^0(X, \mathcal{H}(m)) \to H^0(X, E(m))$ an isomorphism. Lemma 4.3.1 of [HL10] states that if $R \to M$ is a categorical quotient for the GL_n-action (which is true if the quotient is good), then M corepresents $\mathcal{M}'$, thus corepresents $\mathcal{M}$. Thus, the moduli problem amounts to finding a good quotient of R. To use the result above, we first find a linearized ample line bundle on R. This is achieved by defining $L_\ell = \det(p_*(\widetilde{F} \otimes q^*\mathcal{O}_X(\ell)))$ where $q^*\mathcal{H} \to \widetilde{F}$ is the universal quotient module pulled back via the projection $q : \mathrm{Quot}_{X/\mathbb{C}}(\mathcal{H}, P) \times X \to X$ and p is the other projection to $\mathrm{Quot}_{X/\mathbb{C}}(\mathcal{H}, P)$. For sufficiently large ℓ, this line bundle is very ample, and has an induced GL_n-linearization which we will not discuss here. By Theorem 4.3.3 of [HL10], it follows that for suitable fixed m and ℓ, we have $R = \overline{R}^{\mathrm{ss}}(L_\ell)$, where $\overline{R}$ is the closure of R in $\mathrm{Quot}_{X/\mathbb{C}}(\mathcal{H}, P)$. Thus, by Theorem 3.2, it follows that there exists a projective scheme $R/\!\!/G$ and a morphism $\pi : R \to R/\!\!/G$ such that this is a universal good quotient. This quotient universally corepresents the functor $\mathcal{M}$, hence by Theorem 2.3 is the moduli space of semistable sheaves on X.

References

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