Étale cohomology I

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The aim of this series of talks is to get us used to the techniques of étale cohomology, especially ℓ-adic cohomology and to use it to prove some neat results. These are mostly based on Milne's notes.

1 Étale sites

A *site* is a categorical generalization of topological spaces. The most familiar example is that of the Zariski site of a scheme X, given by its Zariski open subsets. Let S be a fixed scheme.

1.1 Big étale site

The big étale site of S is the category of schemes over S, with an étale cover for each object. This is big in the sense that it may not be a small category.

1.2 Small étale site

The small étale site of S is the (full) subcategory of the big étale site of S, restricting objects to include only étale morphisms $U \to S$ for U an affine scheme.

1.3 Étale morphisms

There are a bunch of equivalent formal definitions of an étale morphism between schemes, but intuitively, it is an analogue of a local homeomorphism in the CW sense. (Caveat: this is only valid for characteristic zero.) First we define local étale morphisms, i.e., étale ring homomorphisms. Let $f : R \to S$ be a morphism of local rings. Then, it is étale if it is flat and unramified. This means that the functor $- \otimes_R S$ is exact, the image of the maximal ideal of R generates the maximal ideal of S, and the residue field $k(B)$ is a finite separable extension of $k(A)$.

A morphism $f : X \to Y$ is étale at $x \in X$ if it is flat and unramitied at x. This f is called étale if it is étale for all points. Here are some examples and nonexamples.

From the viewpoint that étale morphisms are local isomorphisms, if we have a smooth morphism f between complex algebraic varieties, it is étale when it has nonzero Jacobian. This is in spirit of the implicit funciton theorem.

Another important example is the case of a finite separable field extension $K \to L$ and the corresponding scheme map $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$, which is étale. In fact, a scheme X over a field K is étale over K if and only if it is the disjoint union of $\mathrm{Spec}(K_i)$ where the K_i are finite separable extensions of K.

Now, a list of examples from MathSE which we won't prove in detail:

1. $\mathrm{Spec}\mathbb{Q}[T]/(T^2 - 4) \to \mathrm{Spec}\mathbb{Q}$
2. $\mathrm{Spec}\mathbb{Q}[T]/(T^2 + 4) \to \mathrm{Spec}\mathbb{Q}$
3. $\mathrm{Spec}\mathbb{F}_{p^2} \to \mathrm{Spec}\mathbb{F}_p$

the above morphisms are all étale. For 1 and 2, the spaces over $\mathrm{Spec}\mathbb{Q}$ are just two-dimensional reduced $\mathbb{Q}$-algebras; for 3, $\mathbb{F}_{p^2}$ is a finite separable extension of $\mathbb{F}_p$.

Here are a list of nonexamples, from the same MathSE post:

1. $\mathbb{A}^1 \to \mathrm{Spec}(\mathbb{C}[X,Y]/(Y^2 - X^3)$ given by $t \mapsto (t^2, t^3)$ is not étale at the origin.
2. $\mathrm{Spec}\mathbb{Q}[T]/(T^2) \to \mathrm{Spec}\mathbb{Q}$ is not étale due to the $\mathbb{Q}$-algebra not being reduced, so it cannot be represented as a finite product of field extensions of $\mathbb{Q}$.

Credit: here[1]

1.4 Unambiguity of the étale site

Since we have defined two types of étale sites, there is ambiguity in the terminology of "the" étale site of a scheme. However, our purpose of defining the étale site is to do sheaf cohomology on it, and in this case there is no difference in the two.

Theorem. For abelian sheaves defined on the big étale site over a scheme, its cohomology groups agree with those of the sheaf restricted to the small étale site. In fact, this holds for Zariski sites, too. Thus we will use the term "the étale site".

Note, however, that there are technical differences outside the realm of abelian sheaf cohomology. If we have base scheme S, then sometimes it is useful to view a scheme X over S as a functor of points

$$\mathrm{Hom}_{\mathsf{Sch}_S}(-, X) : \mathsf{AffSch}_S^{\mathrm{op}} \to \mathsf{Set},$$

and this becomes a sheaf on the (big or small?) Zariski site. However this fails to be a sheaf on the small étale site, and requires the whole big étale site.

[1] https://math.stackexchange.com/questions/178121/intuition-for-%C3%A9tale-morphisms

1.5 Étale sheaf cohomology

After defining the étale topology, the construction of sheaf cohomology is practically the same as the usual one on the Zariski topology. Instead, we define sheaves as a contravariant functor to Ab on the étale site, and check sheaf conditions for étale coverings. The étale sheaf cohomology groups are the (right) derived functors of the global sections functor.

2 The étale fundamental group

We will work with the analogy of an étale morphism being as sort of a covering map. Given a scheme X over a field k, we will define its fundamental group purely algebraically. Recall that in the CW setting, the deck transformations of the universal covering recovers the fundamental group. As we cannot define π_1 using homotopy equivalence of loops, we turn to coverings.

Consider the category FinEt_X where objects are finite étale morphisms $Y \to X$, and the morphisms are over X. Pick a geometric point $\overline{x}$ of X and consider the functor $F : \mathsf{FinEt}_X \to \mathsf{Set}$ sending Y to $\mathsf{Hom}_X(\overline{x}, Y)$. If F is representable, namely we have some finite étale covering $U \to X$ such that $F(Y)$ agrees with $\mathrm{Hom}_X(U, Y)$, then U would have the desired properties of a universal covering of Y. However, F is usually not representable.

For example, consider the affine line $\mathbb{A}^1$ over an algebraically closed field k of characteristic zero. The finite étale coverings of $\mathbb{A}^1 - 0$ are precisely the maps $t \mapsto t^n : \mathbb{A}^1 - 0 \to \mathbb{A}^1 - 0$ for $n \geq 1$. This differs from the analytic case, as the universal covering of $\mathbb{C} - 0$ can be realized through the exponential function.

However, the functor F is pro-representable, which means that there exists a directed system $(X_i)_{i \in I}$ of finite étale coverings of X, such that

$$F = \varinjlim_{i \in I} \mathrm{Hom}_X(X_i, -).$$

Although the system isn't really a space, we will call it the universal covering space of X. For example, the finite étale coverings of $\mathbb{A}^1 - 0$ form a directed system partially ordered by divisibility of the positive n.

When choosing a universal covering of X, we choose each X_i to have degree equal to the order of $\mathrm{Aut}_X(X_i)$. A map $X_j \to X_i$ for $i \leq j$ induces a homomorphism $\mathrm{Aut}_X(X_j) \to \mathrm{Aut}_X(X_i)$, and using this we may define the étale fundamental group

$$\pi_1(X, \overline{x}) := \varprojlim_i \mathrm{Aut}_X(X_i)$$

as the inverse limit of finite groups. Notice for $\mathbb{A}^1 - 0$ we have $\pi_1(\mathbb{A}^1 - 0, \overline{x}) = \varprojlim_n \mathbb{Z}/n\mathbb{Z} \cong \widehat{\mathbb{Z}}$, the profinite integers.

For $\mathbb{P}^1$ over an algebraically closed field, any finite étale covering π has to be an isomorphism. This is because of the Riemann-Hurwitz formula, π can be either degree 1 or zero, but degree zero maps are obviously not étale. Hence π

must be an isomorphism. It follows that the π_1 is 1, and the same applies to $\mathbb{A}^1$.

Now we look at the étale fundamental group of a field spectrum Spec$k$. We know that the étale covers should be the spectrum of a finite product of finite separable extensions of k. Choosing a geometric point of Spec$K$ is choosing an algebraically closed field K which is separable over k. If we define

$$F : \mathsf{Et}_k \to \mathsf{Set}$$

as $F(A) = \mathrm{Hom}_k(A, K)$ from the étale k-algebras to Set, it turns out that the directed system $(k_i)_{i\in I}$ of finite Galois extensions of k ind-represents F, that is,

$$F = \varinjlim_{i\in I} \mathrm{Hom}_k(-, k_i).$$

Then it turns out that the inverse limit $\mathrm{Aut}_k(k_i)$ is just the Galois group $\mathrm{Gal}(k^s/k)$, where k^s is the separable closure of k in the algebraic closure K.

In general, if X is a connected algebraic variety over k, and if the base change X_{k^s} is also connected, then we have an exact sequence of groups

$$1 \to \pi_1(X_{k^s}, \overline{x}) \to \pi_1(X, \overline{x}) \to \mathrm{Gal}(k^s/k) \to 1.$$

3 Next time: Hilbert Theorem 90, cohomology of $\mathbb{G}_m$ and a step towards ℓ-adic cohomology

One can ask if the vector bundles (locally free $\mathcal{O}_X$-modules of rank n) in the Zariski topology, agree with that of the vector bundles in the étale topology. In fact, they are the same. This is due to vector bundles of rank n being classified by $H^1(X, \mathrm{GL}_n)$, which are isomorphic in both topologies. Together with Brauer groups, we will go into detail the next time.

We will also need the knowledge of the étale cohomology groups of the multiplicative group scheme $\mathbb{G}_m$ to compute $H^r_{\text{ét}}(X, \mu_n)$ via the Kummer sequence. This in turn will lead to ℓ-adic cohomology.