COMMUTATIVE ALGEBRA HOMEWORK I

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I tend to use the conventions of Atiyah & Macdonald, since I have studied this first. Please bear with me.

Problem 1. *Compute the following:*

1. $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(n), \mathbb{Z}/(m))$ *and* $\mathrm{Hom}_{k[x]}(k[x]/(x^n), k[x]/(x^m))$.
2. $\mathbb{Z}/(n) \otimes_{\mathbb{Z}} \mathbb{Z}/(m)$ *and* $k[x]/(x^n) \otimes_{k[x]} k[x]/(x^m)$.
3. $k[x,y]/(y^2 - x^2(x+1)) \otimes_{k[x,y]} k[x,y]/(y)$.

Proof. 1. Consider the exact sequence

$$\mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/(n) \to 0$$

and apply $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/(m))$ to get

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(n), \mathbb{Z}/(m)) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/(m)) \xrightarrow{\times n} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/(m)).$$

Using the identity $\mathrm{Hom}_A(A, M) \cong M$, we conclude

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(n), \mathbb{Z}/(m)) \cong \ker\left(\mathbb{Z}/(m) \xrightarrow{\times n} \mathbb{Z}/(m)\right).$$

For $\overline{k} \in \mathbb{Z}/(m)$ to be in the kernel, $kn = mz$ must hold for some integer z. Let $d = \gcd(m,n)$ and write $m = dm'$, $n = dn'$. Then $k = m'z/n'$. There are d possibilities of z, namely $0, n', \dots, (d-1)n'$. It follows that the kernel is a subgroup of order d of $\mathbb{Z}/(m)$, which is isomorphic to $\mathbb{Z}/(d)$.

Similarly, consider the exact sequence

$$k[x] \xrightarrow{\times x^n} k[x] \to k[x]/(x^n) \to 0$$

and apply $\mathrm{Hom}_{k[x]}(-, k[x]/(x^m))$ to get

$$0 \to \mathrm{Hom}_{k[x]}(k[x]/(x^n), k[x]/(x^m)) \to k[x]/(x^m) \xrightarrow{\times x^n} k[x]/(x^m),$$

again using the identity $\mathrm{Hom}_A(A, M) \cong M$. In the case $n \geq m$, the kernel $\ker\left(k[x]/(x^m) \xrightarrow{\times x^n} k[x]/(x^m)\right)$ is just $k[x]/(x^m)$. If $n < m$, then for $f + (x^m) \in k[x]/(x^m)$ to be in the kernel, f must have degree $< m - n$ coefficients all zero. Denote as P the $k[x]$-module of all such $f + (x^m)$. Then the $k[x]$-module homomorphism $k[x]/(x^n) \xrightarrow{\times x^{m-n}} P$ is bijective, so $P \cong k[x]/(x^n)$. Therefore we may conclude

$$\mathrm{Hom}_{k[x]}(k[x]/(x^n), k[x]/(x^m)) \cong k[x]/(x^{\min(m,n)}).$$

$\square$

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Proof. 2. We first prove the identity

$$R/I \otimes_R R/J \cong R/(I+J).$$

Define $\varphi : R/I \times R/J \to R/(I+J)$ by $(r_1 + I, r_2 + J) \mapsto r_1 r_2 + I + J$. Suppose $r_1 - r_1' \in I$ and $r_2 - r_2' \in J$. Since $r_1 r_2 - r_1' r_2' = r_2(r_1 - r_1') + r_1'(r_2 - r_2') \in I + J$, this map is well-defined. Also, this map is R-bilinear, which yields $\exists! \widetilde{\varphi} : (R/I) \otimes_R (R/J) \to R/(I+J)$ such that $(r_1 + I) \otimes (r_2 + J) \mapsto r_1 r_2 + I + J$. Note that $(r_1 + I) \otimes (r_2 + J) = r_1 r_2 (1 + I) \otimes (1 + J)$, so $\widetilde{\varphi}$ is bijective, hence an isomorphism.

Using this identity, we conclude $\mathbb{Z}/(n) \otimes_{\mathbb{Z}} \mathbb{Z}/(m) \cong \mathbb{Z}/((n) + (m))$, where $(n) + (m) = (\gcd(m,n))$. Write $d = \gcd(m,n)$, then this is $\mathbb{Z}/(d)$.

As above, we have

$$k[x]/(x^n) \otimes_{k[x]} k[x]/(x^m) \cong k[x]/((x^n) + (x^m)),$$

where $(x^n) + (x^m) = (x^{\min(m,n)})$. Hence this is $k[x]/(x^{\min(m,n)})$. □

$k[x]/(x^2) \oplus k[x]/(x+1)$

Proof. 3. Using the identity proved in 2, this is isomorphic to $k[x,y]/(y, y^2 - x^2(x+1))$. ~~I do not know how to further simplify this.~~ □

Problem 2. *Let A be a ring, let $I_1, \ldots, I_n$ ideals of A s.t. $I_i + I_j = A$ for $i \neq j$. Show $A/(\bigcap_k I_k) \cong \prod_k A/I_k$.*

Proof. Define $\phi : A \to \prod_k A/I_k$ by $a \mapsto (a + I_1, \ldots, a + I_n)$. We claim this homomorphism is surjective. To show this, it is enough to find $x_k \in A$ such that $\phi(x_k) = (I_1, \ldots, 1 + I_k, \ldots, I_n)$ for each $1 \leq k \leq n$. Since $I_k + I_j = A$ for all $j \neq k$, ~~there exists $u_j \in I_j$ such that $u_k + u_j = 1$ for a fixed $u_k \in I_k$. Take $x_k = 1 - u_k$. Then for all j, we have $x_k = u_j$, so $(x_k + I_1, \ldots, x_k + I_k, \ldots, x_k + I_n) = (I_1, \ldots, 1 + I_k, \ldots, I_n)$. Surjectivity follows obviously.~~ Also, the kernel of ϕ is a such that $a \in I_k$ for all k, i.e. is $\bigcap_k I_k$. By the first isomorphism theorem we have $A/(\bigcap_k I_k) \cong \prod_k A/I_k$. □

$\exists\, u_j \in I_j,\ u_{kj} \in I_k$ s.t.
$u_j + u_{kj} = 1$, i.e.,
$u_j = 1 - u_{kj}$.
Let $x_k = \prod_{j \neq k} u_j$,
which is in all I_j, $j \neq k$,
but modulo I_k is 1.

Problem 3. *Prove the following are equivalent:*

1. *A contains a nontrivial idempotent*
2. *$A \cong A_1 \times A_2$ for some nonzero rings A_1 and A_2*

If $e \in A$ is a nontrivial idempotent, describe localization of A with respect to $\{e\}$.

Proof. $(\Rightarrow)$ Suppose $e \in A$ such that $e^2 = e$, and $e \neq 0, 1$. Consider the ring homomorphism $A \to Ae \times A(1-e)$ given by $a \mapsto (ae, a(1-e))$, where $Ae = \{ae \mid a \in A\}$, and similarly for $A(1-e)$. Both have multiplicative identity e, $1-e$, respectively. If $ae = a(1-e) = 0$, then $ae + a - ae = a = 0$, so the map is injective. Also, for any $(ae, b(1-e))$ we have $ae + b(1-e) \mapsto (ae, b(1-e))$. Therefore $A \cong Ae \times A(1-e)$.

$(\Leftarrow)$ Take $(1,0) \in A_1 \times A_2$.

We now describe the localization of A with respect to $\{e\}$. We claim that $A_{\{e\}} \cong Ae$. Define a ring homomorphism $Ae \to A_{\{e\}}$ given by $ae \mapsto ae/e$. Since $a/e = ae/e$ for all $a \in A$, it follows that e/e is the multiplicative identity of $A_{\{e\}}$. Since $e \mapsto e/e$, this maps 1 to 1. Suppose $ae/e = 0$. Then $ae = 0$, so the kernel is trivial. Also, for any $a/e \in A_{\{e\}}$, if we send $ae \mapsto ae/e$, then $ae/e = a/e$, so this map is surjective. Therefore $A_{\{e\}} \cong Ae$. □

Problem 4. *Identify associated primes of a finitely generated abelian group, viewed as a $\mathbb{Z}$-module, in terms of the usual structure theory of finitely generated abelian groups.*

Proof. In the general case, the group is isomorphic to some $\mathbb{Z}^n \oplus \mathbb{Z}/p_1^{n_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_k^{n_k}\mathbb{Z}$, where p_i are prime numbers, not necessarily distinct. We first prove a preliminary result:

$$\mathrm{Ass}(M \oplus N) = \mathrm{Ass}(M) \cup \mathrm{Ass}(N) \text{ for } A\text{-modules } M \text{ and } N.$$

Suppose $\mathfrak{p} \in \mathrm{Ass}(M \oplus N)$. Then $\mathfrak{p} = \mathrm{Ann}(m, n)$ for some nonzero $(m, n) \in M \oplus N$. This means that elements of $\mathfrak{p}$ annihilate both m and n, so $\mathfrak{p} \subset \mathrm{Ann}(m) \cap \mathrm{Ann}(n)$. If $\mathfrak{p} = \mathrm{Ann}(m)$, then $\mathfrak{p} \in \mathrm{Ass}(M)$. Suppose $\mathfrak{p} \subsetneq \mathrm{Ann}(m)$. There exists some element $a \in \mathrm{Ann}(m)$ which is not in $\mathfrak{p}$. Then, $a(m, n) = (am, an) = (0, an) \neq 0$ since $a \notin \mathfrak{p} = \mathrm{Ann}(m, n)$. Hence $an \neq 0$, so $a \notin \mathrm{Ann}(n)$. From this, it follows that $\mathrm{Ann}(m) - \mathfrak{p} \subset \mathrm{Ann}(m) - \mathrm{Ann}(n)$, so $\mathrm{Ann}(m) \cap \mathrm{Ann}(n) \subset \mathfrak{p} \subset \mathrm{Ann}(m) \cap \mathrm{Ann}(n)$ which implies $\mathfrak{p} = \mathrm{Ann}(m) \cap \mathrm{Ann}(n)$. In this case, $\mathfrak{p}$ is either $\mathrm{Ann}(m)$ or $\mathrm{Ann}(n)$, but we assumed $\mathfrak{p} \subsetneq \mathrm{Ann}(m)$, so $\mathfrak{p} = \mathrm{Ann}(n)$. Hence $\mathfrak{p} \in \mathrm{Ass}(N)$. Therefore, $\mathrm{Ass}(M \oplus N) \subset \mathrm{Ass}(M) \cup \mathrm{Ass}(N)$. Conversely, if $\mathfrak{p}$ is either $\mathrm{Ann}(m)$ or $\mathrm{Ann}(n)$ for nonzero m, n, then it follows that $\mathfrak{p} = \mathrm{Ann}(m, 0)$ or $\mathrm{Ann}(0, n)$, which are both nonzero in $M \oplus N$. Thus the opposite inclusion holds, proving the result.

Using this, it suffices to find $\mathrm{Ass}(\mathbb{Z})$ and $\mathrm{Ass}(\mathbb{Z}/p^k\mathbb{Z})$. Since $\mathbb{Z}$ is torsion-free, the only associated prime of $\mathbb{Z}$ is (0). Now for $\mathbb{Z}/p^k\mathbb{Z}$, the prime ideals of $\mathbb{Z}$ that annihilates nonzero elements of $\mathbb{Z}/p^k\mathbb{Z}$ are (0) and (p).

Combining everything, the associated primes of $G \cong \mathbb{Z}^n \oplus \mathbb{Z}/p_1^{n_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_k^{n_k}\mathbb{Z}$ as a $\mathbb{Z}$-module are (0) and (p_i) for $1 \leq i \leq k$. Note that p_i may not be distinct. $\square$

Problem 5. *Let A noetherian. Prove the total quotient ring $Q(A)$ has finitely many maximal ideals.*

Proof. Recall $Q(A) = A_{S(A)}$ where $S(A)$ is the set of A-regular elements of A, i.e. the non zerodivisors of A. The following lemmas consist the proof of the Lasker-Noether theorem, which is a copy of Atiyah & Macdonald, pp.82-83.

Lemma 7.11. Every ideal in a noetherian ring is a finite intersection of irreducible ideals.

Proof. Suppose not; then the set of ideals in A for which the lemma is false is not empty, hence has a maximal element $\mathfrak{a}$. Since $\mathfrak{a}$ is reducible, we have $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ where $\mathfrak{b} \supset \mathfrak{a}$ and $\mathfrak{c} \supset \mathfrak{a}$. Hence each of $\mathfrak{b}, \mathfrak{c}$ is a finite intersection of irreducible ideals and therefore so is $\mathfrak{a}$; contradiction. $\square$

Lemma 7.12. In a noetherian ring, every irreducible ideal is primary.

Proof. Passing to quotient ring, ETS for zero ideal. Let $xy = 0$ with $y \neq 0$, and consider the chain of ideals $\mathrm{Ann}(x) \subset \mathrm{Ann}(x^2) \subset \cdots$. By ACC, this stabilizes at some n. It follows that $(x^n) \cap (y) = 0$, for if $a \in (y)$ then $ax = 0$, and if $a \in (x^n)$ then $a = bx^n$ so $bx^{n+1} = 0$, hence $b \in \mathrm{Ann}(x^{n+1}) = \mathrm{Ann}(x^n)$, so $bx^n = 0$. Thus $a = 0$. Since (0) is irreducible by hypothesis, and $(y) \neq 0$, we have $x^n = 0$, so (0) is primary. $\square$

Therefore, every (proper) ideal of a noetherian ring has a primary decomposition. In particular, the ideal (0) has a minimal primary decomposition, say $(0) = \bigcap_{i=1}^n \mathfrak{q}_i$. Denote $\mathfrak{p}_i := \sqrt{\mathfrak{q}_i}$. We claim that the set of zero divisors D of A is the union $\bigcup_{i=1}^n \mathfrak{p}_i$.

Suppose $x \in D$. Then, $xy = 0$ for some nonzero $y \in A$. Thus, $xy = 0 \in (0) = \bigcap_{i=1}^n \mathfrak{q}_i$, which implies $xy \in \mathfrak{q}_i$ for all i. By primary-ness, either $x \in \sqrt{\mathfrak{q}_i} = \mathfrak{p}_i$ or $y \in \mathfrak{q}_i$. If x is not in $\mathfrak{p}_i$ for all i, then $y \in \mathfrak{q}_i$ for all i, thus $y \in \bigcap_{i=1}^n \mathfrak{q}_i = (0)$ so $y = 0$. This contradicts our assumption that $y \neq 0$. Therefore, x must be in at least one $\mathfrak{p}_i$, which implies $D \subset \bigcup_{i=1}^n \mathfrak{p}_i$.

Conversely, suppose $x \in \bigcup_{i=1}^n \mathfrak{p}_i$. WLOG, suppose $x \in \mathfrak{p}_1$. Since we assumed the decomposition is minimal, we must have $(0) \neq \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_n$, so there exists some nonzero $y \in \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_n$. Suppose $x^k \in \mathfrak{q}_1$. Since $x^k y \in \bigcap_{i=1}^n \mathfrak{q}_i$, it follows that $x^k y = 0$. Suppose k is the minimal k such that $x^k \in \mathfrak{q}_1$, i.e. $x^k y = 0$. If $k = 1$, then $x \in D$. If $k > 1$, then $x \cdot x^{k-1} y = 0$, and by minimality $x^{k-1} y \neq 0$, so $x \in D$. Thus $\bigcup_{i=1}^n \mathfrak{p}_i \subset D$, proving our desired result.

Therefore, $D = \bigcup_{i=1}^n \mathfrak{p}_i$, and every prime ideal $\mathfrak{p}$ contained in D is contained in some $\mathfrak{p}_i$ by prime avoidance. Since prime ideals of $Q(A)$ are of the form $S(A)^{-1}\mathfrak{p}$ for primes $\mathfrak{p}$ of A contained in D, it follows that every prime ideal of $Q(A)$ is contained in one of the $S(A)^{-1}\mathfrak{p}_i$, so the $S(A)^{-1}\mathfrak{p}_i$ are the only possible maximal prime ideals of $Q(A)$. (Note that not all $S(A)^{-1}\mathfrak{p}_i$ may be maximal.) Since maximal ideals are automatically prime, it follows that the maximal primes coincide with the maximal ideals. Hence there are finitely many (at most n) of them. $\square$

Problem 6. *Give an example of a noetherian ring A and an ideal $I \subset A$ s.t. $\sqrt{I}$ is prime but I is not primary.*

Proof. Let $A = k[x, y]$ and $I = (x^2, xy)$ for k a field, as in Problem 7. A is noetherian by Hilbert's basis theorem. Then $\sqrt{I} = (x, xy) = (x)$, which is prime since x is obviously irreducible, and $k[x, y]$ is a UFD. But I itself is not primary since $xy \in I$, and $x \notin I$ but $y^n \notin I$ for all n. $\square$

Problem 7. *Let $A = k[x, y]$ and $I = (x^2, xy) \subset A$.*

1. *Show (x^2, xy, y^2) and (x^2, y) are (x, y)-primary.*
2. *Prove $I = (x) \cap (x^2, xy, y^2)$ and $I = (x) \cap (x^2, y)$ are minimal primary decompositions of I.*

In particular, minimal primary decompositions need not be unique.

Proof. 1. We will use the following lemma:

Lemma. If $\sqrt{I}$ is maximal then I is ($\sqrt{I}$-)primary.

Proof. Let $\sqrt{I} = \mathfrak{m}$. The image of $\mathfrak{m}$ in A/I is the nilradical, which is the intersection of all prime ideals of A/I. But primes of A/I correspond to primes of A containing I, which automatically contain $\sqrt{I} = \mathfrak{m}$. Thus the image of $\mathfrak{m}$ in A/I is the only prime ideal $\mathfrak{p}$ of A/I. Suppose a is not in the unique prime ideal, i.e. is not nilpotent. Then since the quotient ring by this prime is a field, $a + \mathfrak{p}$ has an inverse, say $b + \mathfrak{p}$ s.t. $ab + \mathfrak{p} = 1 + \mathfrak{p}$, so $ab - 1 =: n$ is nilpotent. Since $ab = 1 + n$, a unit plus a nilpotent, ab is a unit. Therefore there exists some $c \in A$ such that $abc = 1 = a(bc)$, so a is a unit. Thus every element of A/I is either a unit or a nilpotent, so every zero divisor in A/I is not a unit, hence is a nilpotent. Hence I is primary. $\square$

Using the lemma above, we show $\sqrt{I}$ is maximal. Since $x^2 \in (x^2, xy, y^2)$, x is in the radical. Same for y. Therefore $(x, y) \subset \sqrt{(x^2, xy, y^2)}$, but since the radical is a proper ideal we conclude that the radical must equal to (x, y), since (x, y) is maximal. To see this, check $k[x, y]/(x, y) \cong k$. Thus (x^2, xy, y^2) is (x, y)-primary.

Also, x and y are in the radical of (x^2, y), so $(x, y) = \sqrt{(x^2, y)}$ which implies (x^2, y) is (x, y)-primary. $\square$

Proof. 2. First, we must check the identities actually hold. Since $(x) \cap (x^2, xy, y^2) = (x^2, xy, xy^2) = (x^2, xy)$, and $(x) \cap (x^2, y) = (x^2, xy)$, both equalities are valid.

Above we have checked that (x), (x^2, xy, y^2) and (x^2, y) are primary ideals, so both are indeed primary decompositions. To check minimality, we have to show the radicals are distinct prime ideals, and no primary ideal contains another.

Since $\sqrt{(x)} = (x)$, $\sqrt{(x^2, xy, y^2)} = (x, y)$, these are distinct. Also, $(x) \nsubseteq (x^2, xy, y^2)$ since $x \notin (x^2, xy, y^2)$. Conversely, $(x^2, xy, y^2) \nsubseteq (x)$ since $y^2 \notin (x)$.

Also, $\sqrt{(x)} = (x)$ and $\sqrt{(x^2, y)} = (x, y)$, and $(x) \nsubseteq (x^2, y)$ since $x \notin (x^2, y)$ and $(x^2, y) \nsubseteq (x)$ since $y \notin (x)$. As we can see, primary decompositions, even though minimal, may not be unique.

(In the sense of the lecture notes, the primary decompositions are minimal since I itself is not primary, and there are two primary ideals in each.) $\square$

Problem 8. *Define* $\dim A = \dim \operatorname{Spec} A$*, the RHS being Krull dimension of a topological space. Is being of finite dimension a local property?*

Proof. We cite the example of Nagata: Let $A = k[x_1, x_2, \ldots]$ the polynomial ring over a field k with countably many indeterminates. Let $m_1, m_2, \ldots$ be an increasing sequence of positive integers such that $m_{i+1} - m_i > m_i - m_{i-1}$ for all $i > 1$. Let $\mathfrak{p}_i = (x_{m_i+1}, \ldots, x_{m_{i+1}})$ and let $S = A - \bigcup_i \mathfrak{p}_i$. This S is multiplicatively closed. Since each $S^{-1}\mathfrak{p}_i$ has height $m_{i+1} - m_i$, it follows that $\dim S^{-1}A = \infty$. It is known that this ring is noetherian; then since localization of noetherian rings are noetherian, and noetherian local rings have finite Krull dimension, this example serves as an example of an infinite dimensional ring, whose local rings are all finite dimensional. $\square$

COMMUTATIVE ALGEBRA HOMEWORK II

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Problem 1. *Let $A = k[x]_{(x)}$ be the local ring at (x) of the polynomial ring in one variable x over a field k. Find an A-module M that is not finitely generated, but such that M/xM is finitely generated.*

Proof. Consider $M = Q(A)$, the total ring of fractions of A. This is an A-module via $A \to Q(A)$. The elements of A are rational functions that are defined at $x = 0$. Note that since $Q(k[x]) = k(x)$ and since the localization of $k[x]$ at (x) is a subring of $k(x)$, it follows that $Q(A)$ is the function field $k(x)$, which contains x^{-i}, so a finite subset of $k(x)$, as a $k[x]_{(x)}$-module, cannot generate all of the x^{-i} for $i > 0$. Hence, this is not finitely generated as an A-module. On the other hand, $M/xM = 0$ since $M = Q(A) = k(x)$ is a field, and $x \in k(x)^\times$. □

Problem 2. *Show that the Jacobson radical of a ring A is $J(A) := \{a \in A \mid 1 + ab$ is a unit for every $b \in A\}$.*

Proof. Suppose $a \in \bigcap \mathfrak{m}$. Assume by contradiction that $1 + ab$ is not a unit. Then $1 + ab \in \mathfrak{m}$ for some maximal ideal. But, $a \in \mathfrak{m}$, so it follows that $ab \in \mathfrak{m}$, and this implies $1 \in \mathfrak{m}$, which is not possible. Hence $a \in J(A)$.

On the other hand, suppose $a \notin \bigcap \mathfrak{m}$. Then $a \notin \mathfrak{m}$ for some maximal ideal, so $(a, \mathfrak{m}) = (1)$. It follows that $ab + c = 1$ for some $c \in \mathfrak{m}$ and $b \in A$. Therefore $1 - ab = c$, where c is a nonunit. Thus $a \notin J(A)$. □

Problem 3. *Compute the normalization of $A = \mathbb{C}[x, y]/(y^2 - x^2(x + 1))$.*

Problem 4. *Let $A = k[x_1, \dots, x_n]$ be a polynomial ring over a field k. Show that any maximal ideal of A can be generated by n elements $f_1, \dots, f_n$ where $f_i \in k[x_1, \dots, x_i] \subset k[x_1, \dots, k_n]$ for each $i = 1, \dots, n$.*

Proof. We use induction on n. The base case $n = 1$, i.e. $k[x_1]$ is clear. Assume the result up to n. Consider the inclusion $k[x_1, \dots, x_n] \subset k[x_1, \dots, x_n][x_{n+1}]$, and suppose $\mathfrak{m}$ is a maximal ideal of $k[x_1, \dots, x_n][x_{n+1}]$. By Nullstellensatz, we know that $\mathfrak{m}^c := \mathfrak{m} \cap k[x_1, \dots, x_n]$ is a maximal ideal of $k[x_1, \dots, x_n]$, thus by induction hypothesis, generated by n elements. Now consider the ring $(k[x_1, \dots, x_n]/\mathfrak{m}^c)[x_{n+1}]$, which is isomorphic to $k[x_1, \dots, x_{n+1}]/\mathfrak{m}^c[x_{n+1}]$. Maximal ideals of this ring correspond to maximal ideals of $k[x_1, \dots, x_{n+1}]$ containing $\mathfrak{m}^c[x_{n+1}]$, in particular $\mathfrak{m}$. This is because the elements of $\mathfrak{m}^c[x_{n+1}]$ are polynomials of x_{n+1} with coefficients in $\mathfrak{m}^c$, hence in $\mathfrak{m}$. Therefore, $\mathfrak{m}$ corresponds to a maximal ideal of $(k[x_1, \dots, x_n]/\mathfrak{m}^c)[x_{n+1}]$, a polynomial ring over a field, thus generated by some $f = a_k x_{n+1}^k + \cdots + a_1 x_{n+1} + a_0$ for $a_i \in k[x_1, \dots, x_n]/\mathfrak{m}^c$. Thus the corresponding maximal ideal in $k[x_1, \dots, x_{n+1}]/\mathfrak{m}^c[x_{n+1}]$ is generated by the image of f under the quotient by $\mathfrak{m}^c[x_{n+1}]$. If we write $b_i \in k[x_1, \dots, x_n]$ as such that $b_i \mapsto a_i$ under the

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quotient by $\mathfrak{m}^c$, then it follows that $\mathfrak{m}$ is generated by $b_k x_{n+1}^k + \cdots + b_1 x_{n+1} + b_0$, and the n generators of $\mathfrak{m}^c \subset k[x_1, \ldots, x_n]$. Therefore $\mathfrak{m}$ is generated by $n+1$ elements.

□

Problem 5. *Prove that a ring A is Jacobson if and only if $\ell_f^*(\operatorname{Spm}(A_f)) \subset \operatorname{Spm} A$ for every $f \in A$.*

Proof. Suppose A is Jacobson. If $f = 0$, the result is immediate since $A_f = 0$. Suppose $f \neq 0$. Note that primes of A_f are in bijective correspondence with primes of A that do not contain f. This preserves inclusion, so it is enough to show that prime ideals that are maximal with respect to the condition of not containing f are maximal ideals in A. Let $\mathfrak{p}$ be a prime ideal of A that does not contain f, which is maximal among those not containing f. Since A is Jacobson, we have $\mathfrak{p} = \bigcap \mathfrak{m}$. Thus $f \notin \bigcap \mathfrak{m}$, so $f \notin \mathfrak{m}$ for some $\mathfrak{m}$. Since $\mathfrak{p} \subset \mathfrak{m}$ and we assumed $\mathfrak{p}$ to be maximal with respect to the condition of not containing f, we conclude $\mathfrak{p} = \mathfrak{m}$.

Now let $\mathfrak{p}$ be a prime ideal of A. For $f \in A \setminus \mathfrak{p}$, we have an inclusion $\mathfrak{p} \subset \mathfrak{m}_f$ where $\mathfrak{m}_f$ is a maximal prime ideal of A not containing f, which is maximal in A by assumption. Suppose $a \in \bigcap_{f \in A \setminus \mathfrak{p}} \mathfrak{m}_f$. Then $a \neq f$ for all $f \in A \setminus \mathfrak{p}$, so $a \in \mathfrak{p}$. Thus $\bigcap_{f \in A \setminus \mathfrak{p}} \mathfrak{m}_f \subset \mathfrak{p}$, where the opposite inclusion is obvious. Thus A is Jacobson. □

Problem 6. *Let A be a domain of dimension ≥ 1.*

(1) *Show that if A is Jacobson then $\operatorname{Spm} A$ is infinite.*
(2) *Suppose that $\dim A = 1$. Show that A is Jacobson if and only if $\operatorname{Spm} A$ is infinite. (Also, assume A is noetherian!)*

Proof. (1) Assume by contradiction that $\operatorname{Spm} A$ is finite, say $\mathfrak{m}_i$ for $i = 1, \ldots, n$. Since A is a domain of dimension ≥ 1, (0) cannot be maximal so the $\mathfrak{m}_i$ are nonzero. From each $\mathfrak{m}_i$ pick a nonzero element a_i. The product $\prod a_i$ is nonzero since A is a domain, and is in $\bigcap \mathfrak{m}_i$. However since A is a Jacobson domain, it follows that $(0) = \bigcap \mathfrak{m}$ for some maximal ideals, so $\bigcap \mathfrak{m}_i = (0)$, a contradiction. □

Proof. (2) We just proved the forward direction. Suppose $\operatorname{Spm} A$ is infinite. All primes of height 1 in A are maximal since $\dim A = 1$. Thus we have to prove for primes of height 0, namely the zero ideal. Suppose there is some $0 \neq f \in \bigcap \mathfrak{m}$. Then the ideal (f) has finitely many minimal primes since A is noetherian. These minimal primes are nonzero, hence automatically maximal by $\dim A = 1$. This is a contradiction to $|\operatorname{Spm} A| = \infty$, where all $\mathfrak{m}$ are minimal primes of f. Hence $(0) = \bigcap \mathfrak{m}$, so A is Jacobson. □

Problem 7. *Show that a localization of a normal domain is a normal domain.*

Problem 8. *Let A be a domain. Show that the following are equivalent:*

(1) *A is a normal domain.*
(2) *$A_{\mathfrak{p}}$ is a normal domain for all $\mathfrak{p} \in \operatorname{Spec} A$.*
(3) *$A_{\mathfrak{m}}$ is a normal domain for all $\mathfrak{m} \in \operatorname{Spm} A$.*

Problem 9. *We define a ring A to be normal if $A_{\mathfrak{p}}$ is a normal domain for all $\mathfrak{p} \in \operatorname{Spec} A$. Show that if a ring A is normal, then $A[x]$ is normal.*

Proof. We want to show that for every $\mathfrak{q} \in \operatorname{Spec} A[x]$, the local ring $(A[x])_{\mathfrak{q}}$ is a normal domain. Let $\mathfrak{p} := A \cap \mathfrak{q} \in \operatorname{Spec} A$ since it is the inverse image under $A \to A[x]$. Since A is normal, we have $A_{\mathfrak{p}}$ a normal domain. We show that $A_{\mathfrak{p}}[x]$ is also a normal domain, i.e. if D is a normal domain, then $D[x]$ is a normal domain. Suppose $f \in K(D[x])$ is integral over $D[x]$. We want to show that $f \in D[x]$. Since the ring $K(D)[x]$ contains $D[x]$, it follows that since f is a root of a monic polynomial with coefficients in $D[x]$, it is also a root of a monic polynomial with coefficients in $K(D)[x]$. Also, $K(D)[x]$ is normal since $K(D)$ is a field, so we have $f \in K(D)[x]$ integral over $D[x]$. Suppose we have $f^n + d_{n-1}f^{n-1} + \cdots + d_1 f + d_0 = 0$ for $d_i \in D[x]$. We may rewrite this by putting $x^N + f - x^N$ in place of f to get

$$(x^N + f - x^N)^n + d_{n-1}(x^N + f - x^N)^{n-1} + \cdots + d_1(x^N + f - x^N) + d_0 = 0$$

where by using the binomial expansion we can write this in the form

$$(x^N + f)^n + d'_{n-1}(x^N + f)^{n-1} + d'_{n-2}(x^N + f)^{n-2} + \cdots + d'_0 = 0$$

where $d'_i \in D[x]$. If we move d'_0 we get

$$(x^N + f)\left((x^N + f)^{n-1} + d'_{n-1}(x^N + f)^{n-2} + \cdots + d'_1\right) = -d'_0$$

which is of the form $GH = F$ for $G, H \in K(D)[x]$ and $F \in D[x]$, for monic F, G, H, assuming $N >> 0$. We may apply Eisenbud, Proposition 4.11. to conclude that the coefficients of G and H are integral over D. In particular, this implies that the coefficients of f are integral over D, and since D is a normal domain, we may conclude $f \in D[x]$. Therefore, we have proved that $A_{\mathfrak{p}}[x]$ is also a normal domain.

Now we claim that $(A[x])_{\mathfrak{q}} \cong S^{-1}(A_{\mathfrak{p}}[x])$ where S is the image of $A[x] \setminus \mathfrak{q}$ under the unique morphism $A[x] \to A_{\mathfrak{p}}[x]$ that sends 1 to 1 and x to x. By Eisenbud, Proposition 4.13., this would finish the proof. Suppose $\varphi : A[x] \to A_{\mathfrak{p}}[x] \to S^{-1}(A_{\mathfrak{p}}[x])$ is the composition of the obvious morphisms. The $\varphi(A[x] \setminus \mathfrak{q})$ are obviously units in the codomain, and elements of $S^{-1}(A_{\mathfrak{p}}[x])$ are of the form $(\overline{f}/1)(\varphi(s)/1)^{-1} = \varphi(f)\varphi(s)^{-1}$ for $f \in A[x]$ and $s \in A[x] \setminus \mathfrak{q}$. Therefore it suffices to show that if $\varphi(f) = 0$ for some $f \in A[x]$, we have $fs = 0$ for some $s \in A[x] \setminus \mathfrak{q}$. Suppose $f = a_0 + a_1 x + \cdots + a_n x^n$. Via φ this maps to $(\overline{a_0} + \overline{a_1}x + \cdots + \overline{a_n}x^n)/1 = 0$ in $S^{-1}(A_{\mathfrak{p}}[x])$. Thus we have $s(\overline{a_0} + \overline{a_1}x + \cdots + \overline{a_n}x^n) = 0$ for some $s \in S$. Since $A_{\mathfrak{p}}[x]$ is a domain, we must have $\overline{a_0} + \overline{a_1}x + \cdots + \overline{a_n}x^n = 0$, i.e. $\overline{a_i} = 0$ for all i. This means that there exists $s_i \in A \setminus \mathfrak{p}$ such that $s_i a_i = 0$ for each i. Therefore we conclude $\prod_i s_i \cdot f = 0$, and since $\prod_i s_i \in A \setminus \mathfrak{p} \subset A[x] \setminus \mathfrak{q}$, we may apply Atiyah & Macdonald, Corollary 3.2. to conclude that $(A[x])_{\mathfrak{q}} \cong S^{-1}(A_{\mathfrak{p}}[x])$. Since $\mathfrak{q} \in \operatorname{Spec} A[x]$ was arbitrary, we win. □

Problem 10. *Let A be a noetherian ring. Show that the following are equivalent:*

(1) *A is normal.*
(2) *A is reduced and integrally closed in its total ring of fractions.*
(3) *A is a finite product of normal domains.*

Proof. Suppose A is a noetherian normal ring. Since each $A_{\mathfrak{p}}$ is a domain, its nilradical is trivial, i.e. $\mathfrak{R}_{\mathfrak{p}} = 0$ for all $\mathfrak{p}$ by Atiyah & Macdonald, Corollary 3.12. It follows that $\mathfrak{R} = 0$, so A is reduced. Now suppose $x \in Q(A)$ is integral over A, say $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ for $a_i \in A$. Let $\mathfrak{p}$ a prime of A. Via the map $Q(A) \to Q(A) \otimes_A A_{\mathfrak{p}}$ given by $a \mapsto a \otimes 1$, send this equation to get a monic polynomial of $(x \otimes 1)$ with coefficients in $A_{\mathfrak{p}}$. Since $A \to A_{\mathfrak{p}}$ is flat, and $A \subset Q(A)$, we have $A \otimes_A A_{\mathfrak{p}} \subset Q(A) \otimes_A A_{\mathfrak{p}}$. Also, $A \otimes_A A_{\mathfrak{p}} \cong A_{\mathfrak{p}}$ and $Q(A) \otimes_A A_{\mathfrak{p}} \cong S(A)^{-1}A \otimes_A A_{\mathfrak{p}} \cong S(A)^{-1}A_{\mathfrak{p}}$, i.e. the localization of $A_{\mathfrak{p}}$ by the image of $S(A)$ through the map $A \to A_{\mathfrak{p}}$. Since $A_{\mathfrak{p}}$ is a normal domain, and since $x \otimes 1 \in S(A)^{-1}A_{\mathfrak{p}} \subset K(A_{\mathfrak{p}})$ is integral over $A_{\mathfrak{p}}$ it follows that $x \otimes 1 \in A_{\mathfrak{p}}$.

Thus, we may rewrite $x \otimes 1$ as $a \otimes (1/f)$ for some $a \in A$ and $f \in A \setminus \mathfrak{p}$. This implies that $fx - a$ maps to zero under the map $Q(A) \to Q(A) \otimes_A A_{\mathfrak{p}}$, since $(fx-a) \otimes 1 = (fx) \otimes 1 - a \otimes 1 = f(x \otimes 1) - f(a \otimes (1/f)) = 0$. Since $Q(A) \otimes_A A_{\mathfrak{p}} \cong Q(A)_{\mathfrak{p}} \cong S(A)^{-1} A_{\mathfrak{p}}$, this means that $(fx-a)/1$ is zero, i.e. $f'(fx-a) = 0$ in $A_{\mathfrak{p}}$ for some f' in the image of $S(A)$ in $A_{\mathfrak{p}}$, say $f' = s/1$ for $s \in S(A)$. This in turn implies that $f''s(fx-a) = 0$ for some $f'' \in A \setminus \mathfrak{p}$, where since $s \in S(A)$ this becomes $f''fx = f''a$ in A. Define $I = \{a \in A \mid ax \in A\}$, an ideal of A. Notice that $f''f \in I$, where $f'' \in A \setminus \mathfrak{p}$. Since $\mathfrak{p}$ was arbitrary, it follows that I does not contain any prime of A, which implies $I = (1)$. Thus $1x \in A$, so A is integrally closed in $Q(A)$.

Now suppose A is a noetherian ring which is reduced and integrally closed in its total ring of fractions. Denote by $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ its minimal prime ideals. Suppose $x \in A$ is a zerodivisor, i.e. $xy = 0$ for $y \neq 0$. Then $y \neq \mathfrak{p}_i$ for some i since $(0) = \bigcap \mathfrak{p}_i$. Hence, $x \in \mathfrak{p}_i$. Conversely, if $x \in \mathfrak{p}_i$ for some i, then consider the localization $A \to A_{\mathfrak{p}_i}$. Through this, x maps into $\mathfrak{p}_i A_{\mathfrak{p}_i}$, which is the only prime of $A_{\mathfrak{p}_i}$. Since $A_{\mathfrak{p}_i}$ is reduced, it follows that $\mathfrak{p}_i A_{\mathfrak{p}_i} = 0$, i.e. x maps to zero. Hence, there exists $y \in A \setminus \mathfrak{p}_i$ such that $xy = 0$. Therefore, the set of zerodivisors of A is precisely $\bigcup_{i=1}^n \mathfrak{p}_i$. With results of the last homework, together with the fact that the $\mathfrak{p}_i$ are minimal, we may conclude that the maximal ideals of $Q(A)$ are precisely $S(A)^{-1}\mathfrak{p}_i$, and by reducedness their intersection is zero. Since $S(A)^{-1}\mathfrak{p}_i + S(A)^{-1}\mathfrak{p}_j = (1)$ for any $i \neq j$, apply the Chinese remainder theorem to conclude that $Q(A) \cong \prod_{i=1}^n Q(A)/S(A)^{-1}\mathfrak{p}_i$. Note that $Q(A)/S(A)^{-1}\mathfrak{p}_i \cong S(A)^{-1}(A/\mathfrak{p}_i) \cong A_{\mathfrak{p}_i}/\mathfrak{p}_i A_{\mathfrak{p}_i}$ where $\mathfrak{p}_i A_{\mathfrak{p}_i} = 0$ (this is because $A_{\mathfrak{p}_i}$ is reduced, hence the nilradical $\mathfrak{p}_i A_{\mathfrak{p}_i} = 0$), so $Q(A) \cong \prod_{i=1}^n K(A_{\mathfrak{p}_i}) \cong \prod_{i=1}^n A_{\mathfrak{p}_i}$. Denote by $e_i = (0, \ldots, 1, \ldots, 0)$ the ith idempotent of $Q(A)$. Since A is integrally closed in $Q(A)$, and $e_i^2 - e_i = 0$, A must contain the e_i. Then $A \cong \prod_{i=1}^n Ae_i$, where for each i we have $\operatorname{Ann}(e_i) = \mathfrak{p}_i$, so $Ae_i \cong A/\mathfrak{p}_i$, i.e. $A \cong \prod_{i=1}^n A/\mathfrak{p}_i$. Since we have $A/\mathfrak{p}_i \subset A_{\mathfrak{p}_i}$ for each i, it follows that each $A/\mathfrak{p}_i$ is integrally closed in its field of fractions (since by assumption A is integrally closed in $Q(A)$, thus every element in $A_{\mathfrak{p}_i} \times \prod_{j \neq i}\{0\}$ is a solution of a monic polynomial over A, where the other $A/\mathfrak{p}_j$, $j \neq i$ are irrelevant), hence a normal domain. Thus A is a product of finitely many normal domains.

Suppose $A = \prod_{i=1}^n A_i$ is a finite product of normal domains. Since $\operatorname{Spec} A = \coprod_{i=1}^n \operatorname{Spec} A_i$, localization of A corresponds to localization at each A_i. By assumption these are normal domains, so A is normal. $\square$

COMMUTATIVE ALGEBRA HOMEWORK III

HOJIN LEE 2021–11045

Problem 1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Show that a finitely generated A-module M is flat if and only if $M/\mathfrak{m}^n M$ is flat as an $A/\mathfrak{m}^n$-module for every n.*

Proof. Suppose M is flat. It suffices to show that for $A/\mathfrak{m}^n$-modules, we have $- \otimes_{A/\mathfrak{m}^n} (M/\mathfrak{m}^n M)$ exact. Suppose we have some $A/\mathfrak{m}^n$-module N. Then $N \otimes_A M \cong N \otimes_{A/\mathfrak{m}^n} A/\mathfrak{m}^n \otimes_A M \cong N \otimes_{A/\mathfrak{m}^n} (M/\mathfrak{m}^n M)$, where since $- \otimes_A M$ is exact, this is also exact.

Now suppose $M/\mathfrak{m}^n M$ is flat for all n, and let I be an arbitrary ideal of A. Consider the following diagram in Mod_A

$$\begin{tikzcd}
{} \arrow[r, dashed] & I/(I \cap \mathfrak{m}^n) \arrow[r] & A/\mathfrak{m}^n \arrow[r] & A/(I + \mathfrak{m}^n) \arrow[r] & 0 \\
0 \arrow[r] & I \arrow[r] \arrow[u, "\mathrm{coker}"] & I \times A \arrow[r] \arrow[u, "\mathrm{coker}"] & A \arrow[r] \arrow[u, "\mathrm{coker}"] & 0 \\
0 \arrow[r] & I \cap \mathfrak{m}^n \arrow[r] \arrow[u, hook] & I \times \mathfrak{m}^n \arrow[r] \arrow[u, hook] & I + \mathfrak{m}^n \arrow[r] \arrow[u, hook] & 0 \\
 & & & 0 \arrow[u] \arrow[r, dashed, no head] & {}
\end{tikzcd}$$

where the maps in the bottom row are given by $a \mapsto (a, a)$ and $(a, b) \mapsto a - b$, respectively. The top row is defined by extending these maps, and exactness is obvious. Then from the Snake Lemma, we may obtain an SES

$$0 \to I/(I \cap \mathfrak{m}^n) \to A/\mathfrak{m}^n \to A/(I + \mathfrak{m}^n) \to 0.$$

Note that each term is annihilated by $\mathfrak{m}^n$, hence naturally a $A/\mathfrak{m}^n$-module. By assumption $- \otimes_{A/\mathfrak{m}^n} M/\mathfrak{m}^n M$ is exact, so by applying this we obtain

$$0 \to (I/I \cap \mathfrak{m}^n) \otimes_{A/\mathfrak{m}^n} M/\mathfrak{m}^n M \to M/\mathfrak{m}^n M \to A/(I + \mathfrak{m}^n) \otimes_{A/\mathfrak{m}^n} M/\mathfrak{m}^n M \to 0$$

where since $M/\mathfrak{m}^n M \cong A/\mathfrak{m}^n \otimes_A M$, the first part of the sequence is equivalent to

$$0 \to I/(I \cap \mathfrak{m}^n) \otimes_A M \to M/\mathfrak{m}^n M.$$

Now consider another SES

$$0 \to \mathfrak{m}^n \cap I \to I \to I/(I \cap \mathfrak{m}^n) \to 0$$

and apply $- \otimes_A M$ to get

$$(\mathfrak{m}^n \cap I) \otimes_A M \to I \otimes_A M \to I/(I \cap \mathfrak{m}^n) \otimes_A M \to 0.$$

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Now we show the kernel of $\varphi : I \otimes_A M \to M$ is trivial. Suppose k maps to zero under this map. Then considering the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
\mathfrak{m}^n \cap I \otimes_A M \arrow[r] & I \otimes_A M \arrow[r] \arrow[d, "\varphi"'] & I/(I \cap \mathfrak{m}^n) \otimes_A M \arrow[r] \arrow[d] & 0 \\
 & M \arrow[r] & M/\mathfrak{m}^n M &
\end{tikzcd}$$

it must also map to zero in $M/\mathfrak{m}^n M$, hence by injectivity, must map to zero in $I/(I \cap \mathfrak{m}^n) \otimes_A M$ via the top right horizontal arrow. By exactness of the top row, we conclude that k is in the image of $\mathfrak{m}^n \cap I \otimes_A M \to I \otimes_A M$. Since this holds for all n, an element that maps to k must be in $\mathfrak{m}^n \cap I \otimes_A M$ for all n. By Artin-Rees, we have $\mathfrak{m}^{n+N} \cap I = \mathfrak{m}^n(\mathfrak{m}^N \cap I) \subset \mathfrak{m}^n I$ for $n \geq 0$ and some large enough N. Hence $\ker \varphi \subset \bigcap_{n>>0} \operatorname{im}((\mathfrak{m}^n I) \otimes_A M \to I \otimes_A M) = \bigcap_{n>>0} \mathfrak{m}^n(I \otimes_A M) = 0$ by Krull's intersection theorem. Since I was an arbitrary ideal, it follows that M is flat.

□

Problem 2. *Let $\mathfrak{m}$ be a maximal ideal of A. Show $A \to \widehat{(A, \mathfrak{m})}$ factors through the localization map $A \to A_{\mathfrak{m}}$.*

Proof. It suffices to show that the map $A \to \widehat{(A, \mathfrak{m})}$ sends every element of $A - \mathfrak{m}$ to units. Suppose $s \in A - \mathfrak{m}$. This element is sent to $(s_0, s_1, s_2, \dots) \in \widehat{A}$ where $s_i \equiv s \mod \mathfrak{m}^i$, and $s_0 = s$. Since $\mathfrak{m}$ is maximal, we may find some $a \in A$ such that $as \equiv 1 \mod \mathfrak{m}$, hence $as \mapsto (1, as_1 + as_0 - 1, as_2, as_3, \dots) = (1, as_1, as_2, as_3, \dots)$, which is a unit since each term must be units. This is due to the compatibility conditions of the completion. Since as maps to a unit, it follows that s maps to a unit. Hence elements in $A - \mathfrak{m}$ are sent to units in $\widehat{A}$, thus the completion map factors through the canonical localization map $A \to A_{\mathfrak{m}}$. □

Problem 3.

Problem 4. *Let M, N be finitely generated modules over a Noetherian local ring A, such that $\widehat{M} \cong \widehat{N}$ as $\widehat{A}$-modules.*

1. *Show that* $\widehat{\operatorname{Hom}_A(M, N)} \cong \operatorname{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N})$.
2. *Let* $\widehat{\mathfrak{m}} \leq \widehat{A}$ *be the maximal ideal. Show that* $\widehat{\mathfrak{m}} \operatorname{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N})$ *consists of maps* $\widehat{M} \to \widehat{\mathfrak{m}}\widehat{N} \subset \widehat{N}$.
3. *Let* $\varphi \in \operatorname{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N})$ *be an isomorphism. Use Nakayama and (2) to show that if* $\varphi' \in \operatorname{Hom}_A(M, N)$ *differs from* φ *by an element of*
$$\widehat{\mathfrak{m}} \operatorname{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N}),$$
then φ' *is surjective.*
4. *Show that there exists* $\varphi' \in \operatorname{Hom}_A(M, N)$ *and* $\varphi'' \in \operatorname{Hom}_A(N, M)$ *that are surjective. Conclude that* φ' *is an isomorphism.*

Proof. (1) We first show that $\operatorname{Hom}_A(M, N)$ is finitely generated. Since M is finitely generated, we have $A^n \to M \to 0$ for some n. Apply $\operatorname{Hom}_A(-, N)$ to obtain $0 \to \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(A^n, N) \cong N^n$. Since $\operatorname{Hom}_A(M, N)$ is isomorphic to a submodule of a finitely generated module over a noetherian ring, it is also finitely generated. Thus we may conclude $\widehat{\operatorname{Hom}_A(M,N)} \cong \widehat{A} \otimes_A \operatorname{Hom}_A(M,N)$. Also using Eisenbud, Proposition 2.10, since M is finitely presented (since noetherian and finitely generated; consider kernel of $A^n \to M$) and since $\widehat{A}$ is flat over A, we may conclude that $\widehat{A} \otimes_A \operatorname{Hom}_A(M,N) \cong \operatorname{Hom}_{\widehat{A}}(\widehat{A} \otimes_A M, \widehat{A} \otimes_A N)$ which again is isomorphic to $\operatorname{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N})$. □

Proof. (2) Elements of $\operatorname{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N})$ are $\widehat{A}$-module homomorphisms from $\widehat{M}$ to $\widehat{N}$, so obviously the elements of $\widehat{\mathfrak{m}} \operatorname{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N})$ are the maps $\widehat{M} \to \widehat{\mathfrak{m}}\widehat{N}$. □

Proof. (3) I will assume the typo actually means $\varphi' \in \operatorname{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N})$. Suppose we have $\varphi' - \varphi \in \widehat{\mathfrak{m}} \operatorname{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N})$. By (2), this implies that $\varphi' - \varphi$ maps into $\widehat{\mathfrak{m}}\widehat{N}$. Thus, the induced maps $\overline{\varphi}, \overline{\varphi'} : \widehat{M}/\widehat{\mathfrak{m}}\widehat{M} \to \widehat{N}/\widehat{\mathfrak{m}}\widehat{N}$ agree, and since φ is an isomorphism, we must have $\overline{\varphi'}$ surjective. Consider the exact sequence $\widehat{M} \to \widehat{N} \to \operatorname{coker} \varphi' \to 0$, and apply $- \otimes_{\widehat{A}} \widehat{A}/\widehat{\mathfrak{m}}$ to get $\widehat{M}/\widehat{\mathfrak{m}}\widehat{M} \to \widehat{N}/\widehat{\mathfrak{m}}\widehat{N} \to \operatorname{coker} \varphi' \otimes_{\widehat{A}} \widehat{A}/\widehat{\mathfrak{m}} \to 0$ where $\operatorname{coker} \varphi' \otimes_{\widehat{A}} \widehat{A}/\widehat{\mathfrak{m}} \cong \operatorname{coker} \overline{\varphi'} = 0$. Thus $\operatorname{coker} \varphi' = \widehat{\mathfrak{m}} \operatorname{coker} \varphi'$, and we may apply Nakayama to conclude that φ' is surjective. □

Proof. (4) Since $\widehat{M}$ and $\widehat{N}$ are isomorphic, let $f : \widehat{M} \to \widehat{N}$ be an isomorphism. We claim that there exists some $g : M \to N$ such that $\widehat{g} - f \in \widehat{\mathfrak{m}} \operatorname{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N})$. Consider the ring $\widehat{\operatorname{Hom}_A(M,N)}/\widehat{\mathfrak{m}}\widehat{\operatorname{Hom}_A(M,N)}$. This is isomorphic to $\widehat{A}/\widehat{\mathfrak{m}} \otimes_{\widehat{A}} \widehat{\operatorname{Hom}_A(M,N)}$. Again, this is isomorphic to $\widehat{A}/\widehat{\mathfrak{m}} \otimes_{\widehat{A}} \widehat{A} \otimes_A \operatorname{Hom}_A(M,N) \cong \widehat{A}/\widehat{\mathfrak{m}} \otimes_A \operatorname{Hom}_A(M,N)$ since hom is finitely generated, as we have shown in (1). By Atiyah & Macdonald, Theorem 10.15 (iii), we have $\widehat{A}/\widehat{\mathfrak{m}} \cong A/\mathfrak{m}$, so this is isomorphic to $A/\mathfrak{m} \otimes_A \operatorname{Hom}_A(M,N) \cong \operatorname{Hom}_A(M,N)/\mathfrak{m} \operatorname{Hom}_A(M,N)$. Therefore, we have a surjection from $\operatorname{Hom}_A(M,N)$ to $\widehat{\operatorname{Hom}_A(M,N)}/\widehat{\mathfrak{m}}\widehat{\operatorname{Hom}_A(M,N)}$. Also, since f is induced by some element of $\operatorname{Hom}_A(M,N)$ by how we constructed the isomorphism in (1), we conclude that there exists some $g \in \operatorname{Hom}_A(M,N)$ such that $\widehat{g} - f \in \widehat{\mathfrak{m}} \operatorname{Hom}_{\widehat{A}}(\widehat{M}, \widehat{N})$. By (3), it follows that $\widehat{g} : \widehat{M} \to \widehat{N}$ is surjective. By considering the exact sequence $M \to N \to \operatorname{coker} g \to 0$ and taking inverse limits to get $\widehat{M} \to \widehat{N} \to \widehat{\operatorname{coker} g} \to 0$, we have $\widehat{\operatorname{coker} g} = 0$, which means that $\mathfrak{m} \operatorname{coker} g = \operatorname{coker} g$. Apply Nakayama to conclude that $\operatorname{coker} g = 0$, i.e. g is surjective. Take $\varphi' = g$.

For φ'', repeat the exact same process above, in the other direction, to obtain a surjective morphism $\varphi'' \in \operatorname{Hom}_A(N,M)$. By Eisenbud, Corollary 4.4a, surjective endomorphisms of finitely generated modules are isomorphisms. In this case $\varphi'' \circ \varphi'$ is a surjective endomorphism of M, where since M is a finitely generated A-module, it is an isomorphism. Then φ' must be injective, and since φ' was surjective to begin with, M and N are isomorphic. □

Problem 5. *Let A be a Noetherian ring, and let $\mathfrak{m} = (f_1, \dots, f_n) \leq A$. Show that*

$$\widehat{(A, \mathfrak{m})} \cong A[[x_1, \dots, x_n]]/(x_1 - f_1, \dots, x_n - f_n).$$

Proof. The ring $A[[x_1, \dots, x_n]]/(x_1 - f_1, \dots, x_n - f_n)$ is isomorphic to

$$A[x_1, \dots, x_n]/(x_1 - f_1, \dots, x_n - f_n) \otimes_{A[x_1, \dots, x_n]} A[[x_1, \dots, x_n]].$$

Recall that if M is a finite A-module where A is noetherian, we have $\widehat{M} \cong M \otimes_A \widehat{A}$. Here, $A[[x_1, \dots, x_n]]$ is the completion of $A[x_1, \dots, x_n]$ with respect to $(x_1, \dots, x_n)$, hence the original ring is isomorphic to the completion of $A[x_1, \dots, x_n]/(x_1 - f_1, \dots, x_n - f_n)$ at $(x_1, \dots, x_n)$, i.e. the completion of A at $(f_1, \dots, f_n) = \mathfrak{m}$. $\square$

Problem 6.

COMMUTATIVE ALGEBRA HOMEWORK IV

HOJIN LEE 2021–11045

Problem 1. *Let (a, b) be a regular sequence in a domain A. Show $ax - b$ generates a prime in $A[x]$.*

Proof. Since (a, b) is a regular sequence in A, it means that a is a non zerodivisor in A, and b is a non zerodivisor in $A/(a)$. We show that $(ax - b)$ is prime by showing that this is the kernel of the A-algebra homomorphism $\varphi : A[x] \to A_a$ given by $x \mapsto b/a$. Since the codomain is an integral domain, the result from this is immediate. First, we obviosuly have $(ax - b) \subset \ker \varphi$. Assume by contradiction that $(ax - b) \subsetneq \ker \varphi$, i.e. there exists some $g \in \ker \varphi - (ax - b)$. Take g to be of minimal degree among such g. Note that g cannot be a constant since a is not a zerodivisor. Thus we may write $g = c_n x^n + \cdots + c_0$ for $n > 0$. Now, c_n cannot be in (a) since otherwise, $d := c_n b/a$ is an element of A since a is a non zerodivisor, and $0 = g(b/a) = c_n(b/a)^n + c_{n-1}(b/a)^{n-1} + \cdots + c_0 = (d + c_{n-1})(b/a)^{n-1} + \cdots + c_0$. This implies that $(d + c_{n-1})x^{n-1} + \cdots + c_0$ is in $\ker \varphi$ of degree smaller than that of g, so $h \in (ax - b)$ by assumption. This would imply $g = h + c_n x^n - dx^{n-1} = h + (c_n/a)x^{n-1}(ax - b) \in (ax - b)$, a contradiction. But from $0 = c_n(b/a)^n + \cdots + c_0$ we have $-c_n b^n = c_{n-1}b^{n-1}a + \cdots + c_0 a^n \in (a)$, and since b is a non zerodivisor in $A/(a)$, we also have b^n a non zerodivisor in $A/(a)$, so it follows that $c_n \in (a)$, which is a contradiction to what we have just shown above. Hence $\ker \varphi = (ax - b)$, thus $(ax - b)$ is prime in $A[x]$. $\square$

Problem 2. *Let A be noetherian. Show TFAE:*

(1) *A is reduced.*
(2) *The following hold:*
   (a) *localization of A at primes of height 0 are regular.*
   (b) *all associated primes of A have height 0.*
(3) *(R_0) and (S_1) hold for A.*

Proof. Suppose A is reduced. Consider the localization of A at a prime $\mathfrak{p}$ of height 0. This $A_\mathfrak{p}$ has a unique prime ideal. Suppose we have $a/s \in \mathfrak{p}A_\mathfrak{p}$ a nilpotent. Then $a^n/s^n = 0$, i.e. $a^n s' = 0$ for some $n > 0$ and some $s' \in A - \mathfrak{p}$. It follows that as' is nilpotent in A, hence must be zero. Thus $a/s = 0/1$ in $A_\mathfrak{p}$, so we have $\mathfrak{p}A_\mathfrak{p} = 0$, i.e. $A_\mathfrak{p}$ is a field, thus regular. Now suppose $\mathfrak{p} \in \mathrm{Ass}(A)$, say $\mathfrak{p} = \mathrm{Ann}(a)$. Suppose $\mathfrak{q} \subsetneq \mathfrak{p}$. Since $a\mathfrak{p} = (0) \subset \mathfrak{q}$ and there exists some element in $\mathfrak{p} - \mathfrak{q}$, it follows that $a \in \mathfrak{q}$. But then $a \in \mathfrak{p}$, so $a^2 = 0$ which is nonsense since A is reduced. Hence associated primes of A are of height 0.

Assume (a) and (b) hold. We want to show that $A_\mathfrak{p}$ is a regular local ring for all $\mathfrak{p}$ of height 0. This is just (a). Now we want to show that $\operatorname{depth} A_\mathfrak{p} \geq \min\{1, \mathrm{ht}(\mathfrak{p})\}$ for any $\mathfrak{p}$. For primes of height 0, this is obvious. If $\mathrm{ht}(\mathfrak{p}) \geq 1$, then by (b) we have $\mathfrak{p}$ non-associated, so there exists at least one non-zerodivisor in $\mathfrak{p}$. This is because

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if $\mathfrak{p}$ consists of only zerodivisors, then it would be contained in some minimal prime by prime avoidance. By (b), this cannot happen. Hence depth $A_{\mathfrak{p}} \geq 1$.

Now suppose (R_0) and (S_1) hold for A. Again, as in the previous homework, we will use the fact that being reduced is a local property (Atiyah & Macdonald). Thus it is enough to showing $A_{\mathfrak{p}}$ being reduced for all primes $\mathfrak{p}$. We use induction on the height of $\mathfrak{p}$. For height 0 primes, this is immediate by R_0. Suppose $\mathrm{ht}(\mathfrak{p}) \geq 1$, and the result holds for every prime of height less than $\mathfrak{p}$. Then by S_1, we have depth $A_{\mathfrak{p}} \geq 1$, i.e. there is a non-zerodivisor $f \in \mathfrak{p}A_{\mathfrak{p}}$. Hence the localization map $A_{\mathfrak{p}} \to (A_{\mathfrak{p}})_f$ is injective. For an arbitrary ring R, we have $R \to \prod_{\mathfrak{p}} R_{\mathfrak{p}}$ injective, since being zero is a local property. Thus $(A_{\mathfrak{p}})_f$ is a subring of the product of localizations at prime ideals. The prime ideals of $(A_{\mathfrak{p}})_f$ correspond to prime ideals of $A_{\mathfrak{p}}$ not containing f, i.e. prime ideals of A contained in $\mathfrak{p}$ not containing f. (We are abusing notation for f, but the choices for f are obvious.) By induction hypothesis, the localizations of $(A_{\mathfrak{p}})_f$ are reduced, hence its product, hence $A_{\mathfrak{p}}$. End of proof. □

Problem 3. *Let $(A, \mathfrak{m}, \kappa)$ a regular local ring of dimension $d \geq 0$. Let $x_1, \ldots, x_d$ a regular system of parameters for A.*

(1) *Let $f \in \mathfrak{m}$. Let $a_1 \ldots, a_d \in A$ s..t $f = \sum_{j=1}^d a_j x_j$ for some $a_i \in A$. Show that*
$$(a_1 \bmod \mathfrak{m}, \ldots, a_d \bmod \mathfrak{m}) \in \kappa^d$$
is uniquely determined.

(2) *Let $1 \leq n \leq d$ and let $f_1, \ldots, f_n \in \mathfrak{m}$. Choose $a_{ij} \in A$ for $1 \leq i \leq n$ and $1 \leq j \leq d$ such that $f_i = \sum_j a_{ij} x_j$. Show that $A/(f_1, \ldots, f_n)$ is a regular local ring iff the matrix $(a_{ij} \bmod \mathfrak{m}) \in \mathrm{Mat}_{n\times d}(\kappa)$ has rank $d - \dim A/(f_1, \ldots, f_n)$.*

Proof. (1) Since A is regular local, we have $\mathfrak{m}/\mathfrak{m}^2 \cong \kappa^d$. Since $(x_1, \ldots, x_d) = \mathfrak{m}$, and the dimension of $\mathfrak{m}/\mathfrak{m}^2$ is d, the images $\overline{x_1}, \ldots, \overline{x_d}$ span $\mathfrak{m}/\mathfrak{m}^2$ hence form a basis for the d-dimensional κ-vector space $\mathfrak{m}/\mathfrak{m}^2$. Now consider $\overline{f} \in \mathfrak{m}/\mathfrak{m}^2$. Then $\overline{f}$ can be written uniquely as a κ-linear combination of the $\overline{x}_i$, say $\sum_i k_i \overline{x}_i$ with $k_i \in \kappa \cong A/\mathfrak{m}$. Hence if f can be written as $\sum_j a_j x_j$, then the coefficients a_j are determined uniquely up to modulo $\mathfrak{m}$. □

Proof. (2) By Bruns & Herzog, Proposition 2.2.4, $A/(f_1, \ldots, f_n)$ is a regular local ring if and only if $(f_1, \ldots, f_n)$ is generated by a subset of a regular system of parameters. Hence, we may assume $(f_1, \ldots, f_n) = (x_{s_1}, \ldots, x_{s_k})$ for some distinct $s_k \in \{1, \ldots, n\}$, $k \leq n$. Then since the x_i map to basis elements of the d-dimensional κ-vector space under the projection, it follows that the rank of the matrix $(a_{ij} \bmod \mathfrak{m})$ is just equal to k. But note that since the x_{s_i} form a subsequence of the regular sequence x_i, they too are regular, and the dimension of $A/(f_1, \ldots, f_n) = A/(x_{s_1}, \ldots, x_{s_k})$ is just $\dim A - k = d - k$. Hence $d - (d - k) = k$, which shows the forward direction. Conversely, if we denote $\dim A/(f_1, \ldots, f_n) = d - k$, the matrix has rank k so we may find k elements of the x_i that span the image $(f_1, \ldots, f_n)$ in $\mathfrak{m}/\mathfrak{m}^2$. Thus $(f_1, \ldots, f_n)$ is generated by a subset of the x_i of k elements, which implies that $A/(f_1, \ldots, f_n)$ is a regular local ring by the proposition mentioned. □

Problem 4. $(A, \mathfrak{m})$ *is a noetherian local ring. Prove or disprove:*

(1) *If $\dim A = 0$ then A is CM.*

(2) *If* $\dim A = 1$ *and* A *is reduced, then* A *is CM.*
(3) *If* $\dim A = 2$ *and* A *is normal, then* A *is CM.*
(4) *If* A *is a regular local ring, then* A *is CM.*
(5) *If* $f_1, \dots, f_n$ *is an* A*-sequence in* $\mathfrak{m}$*, then* A *is CM iff* $A/(f_1, \dots, f_n)$ *is.*

Proof. (1) Since $\operatorname{depth} A \leq \dim A = 0$, equality holds automatically. □

Proof. (2) We claim that exists an element $x \in \mathfrak{m}$ that is not a zerodivisor. Suppose not. Then $\mathfrak{m}$ lies in the set of zerodivisors, i.e. the union of the minimal primes of A. (This needs reducedness, I recall proving it in the first or second homework.) By prime avoidance, $\mathfrak{m}$ would be equal to one of the minimal primes, but since $\operatorname{ht}(\mathfrak{m}) = 1$ this cannot happen. Thus x is A-regular, and $\operatorname{depth} A = \dim A = 1$. □

Proof. (3) Since A is normal, (R_1) and (S_2) hold by Serre's criterion. Since A is local, we have $A \cong A_{\mathfrak{m}}$ where $\operatorname{ht}(\mathfrak{m}) = \dim A = 2$. Then by (S_2) we have $\operatorname{depth}(A_{\mathfrak{m}}) = \dim(A_{\mathfrak{m}})$, which implies $\operatorname{depth} A = \dim A$. Hence A is CM. □

Proof. (4) Since A is a noetherian local ring, it has finite dimension. Denote $n = \dim A$ and $\mathfrak{m} = (x_1, \dots, x_n)$ a minimal system of generators. By Bruns & Herzog, Proposition 2.2.5, the sequence $x_1, \dots, x_n$ is an A-regular sequence. Hence $\operatorname{depth} A \geq \dim A$, so A is CM. □

Proof. (5) A is CM $\Leftrightarrow \operatorname{depth} A = \dim A$, and $A/(f_1, \dots, f_n)$ is CM $\Leftrightarrow \operatorname{depth}(A/(f_1, \dots, f_n)) = \dim(A/(f_1, \dots, f_n)) = \dim A - n$ since the sequence is regular. Also, by the lecture notes, we have $\operatorname{depth}(A/(f_1, \dots, f_n)) = \operatorname{depth} A - n$, so A is CM if and only if $A/(f_1, \dots, f_n)$ is. □

Problem 5.

Problem 6. *Give an example of a noetherian local ring* A *of positive dimension and a zerodivisor* $f \in A$ *such that* $\dim A/f = \dim A - 1$.

Proof. Let $A = k[[x, y]]/(x^2, xy)$. A is a noetherian local ring since $k[[x, y]]$ is. Since $\overline{x}$ and $\overline{y}$ are both nonzero in A, and $\overline{xy} = 0$, $\overline{y}$ is a zerodivisor in A. Since modding out by nilpotents do not change the dimension, and $\overline{x}$ is nilpotent, we have $\dim A = \dim A/(\overline{x}) = \dim k[[y]] = 1$. But $A/(\overline{y}) = k[[x]]/(x^2)$, and again by modding out nilpotents we have $\dim A/(y) = \dim k[[x]]/(x) = \dim k = 0$. Taking $f = \overline{y}$, we have $\dim A/f = \dim A - 1$. □

COMMUTATIVE ALGEBRA HOMEWORK V

HOJIN LEE 2021–11045

Problem 1. *Let $\varphi : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a local homomorphism of noetherian local rings. Let A be regular and B CM. Prove TFAE:*

(1) *B is flat over A;*
(2) $\dim B - \dim A = \dim B/\mathfrak{m}B$.

Proof. Suppose B is flat over A, the result (1) $\Rightarrow$ (2) holds generally due to the following theorem in the lecture notes (under **§System of Parameters**)

Theorem. Let $\varphi : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a local homomorphism of noetherian local rings. Then $\dim B - \dim A \leq \dim B/\mathfrak{m}B$. Equality holds if B is flat.

I omit the proof since it is in the lecture notes.

To prove the converse, we use induction on $\dim A$. Suppose the result (2) $\Rightarrow$ (1) holds for every pair of rings $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ such that $\dim R < \dim A$ (of course assuming each is regular resp. CM.) For the base case $\dim A = 0$, this implies A being a field, and B, being an algebra over a field, is flat. Hence we may assume $\dim A > 0$ (which implies $\dim B > 0$, since we're assuming that (2) holds). By prime avoidance and dimension conditions, we may pick a non zerodivisor $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. We have $\dim(B/(x)) \leq \dim(A/(x)) + \dim B/\mathfrak{m}B$ as usual. Since x is a non zerodivisor in A, we have $\dim A/(x) = \dim A - 1$. By hypothesis, we have $\dim B = \dim A + \dim B/\mathfrak{m}B$, so $\dim B/(x) \leq \dim B - 1$. Krull's PIT is applicable, which tells us that $\dim B/(x) = \dim B - 1$, and thus the condition (2) holds for the pair $(A/(x), B/(x))$. Since $x \in \mathfrak{m} \setminus \mathfrak{m}^2$, $A/(x)$ is a regular local ring. Also since B is Cohen-Macaulay, by the unmixedness theorem, $\varphi(x) \in \mathfrak{n}$ cannot be contained in a minimal prime of B, otherwise the dimension would not drop by 1. This implies that $\varphi(x)$ is a non zerodivisor in B. Also we have $\operatorname{depth}(B/(x)) = \operatorname{depth}(B) - 1 = \dim B - 1 = \dim B/(x)$, so $B/(x)$ is Cohen-Macaulay. Hence $B/(x)$ is flat over $A/(x)$ by inductive hypothesis. Since x is both A and B-regular, by a corollary[1] in the lecture notes, $B/(x)$ being flat over $A/(x)$ is equivalent to B being flat over A, which is the desired result. $\square$

Problem 2. *Let k be a field, and $\varphi : k[x] \to k[x_1, \ldots, x_n]$ an injective k-algebra homomorphism. Show that φ is flat.*

Proof. $k[x]$ is a PID, and $k[x_1, \ldots, x_n]$ is a torsion-free module over $k[x]$ via φ. We show that torsion-free modules over a PID are flat, say $M \in \mathsf{Mod}_A$ is a torsion-free module over a PID A. Let I be a nonzero ideal of A. Then $I = (a)$ for some non zerodivisor a, hence we have a SES

$$0 \to A \xrightarrow{\times a} A \to A/(a) \to 0.$$

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[1] under §Local criterion for flatness

The associated LES of Tor w.r.t. M is

$$\cdots \to \operatorname{Tor}_A^1(M, A/(a)) \to M \xrightarrow{\times a} M \to M/(a)M \to 0,$$

so $\operatorname{Tor}_A^1(M, A/(a)) \cong \{m \in M \mid am = 0\}$. But since M is torsion-free, this vanishes. Since I was arbitrary, we have proved that M is flat over A. Directly apply this to $\varphi : k[x] \to k[x_1, \dots, x_n]$ to get the desired result. $\square$

Problem 3. *Let A be a noetherian domain.*

(1) *Show that A is a UFD if and only if every irreducible $p \in A$ is a prime.*
(2) *Show that A is a UFD if and only if every $\mathfrak{p} \in \operatorname{Spec} A$ with $\operatorname{ht}(\mathfrak{p}) = 1$ is principal.*
(3) *Let $p \in A$ be a nonzero prime. Show that A is a UFD if and only if A_p is a UFD.*

Proof. (1) Suppose A is a UFD, and let $ab \in (p)$, i.e. $ab = pc$ for some nonzero $a, b, c \in A$. Since A is a UFD, p must appear in the factorization of either a or b, so $a \in (p)$ or $b \in (p)$ must hold. Hence p is prime. Conversely, suppose every irreducible $p \in A$ is a prime. Define

$$\mathcal{C} := \{(a) \subset A \mid a \text{ is not a product of irreducibles}\}$$

partially ordered by inclusion. Assume by contradiction that $\mathcal{C} \neq \varnothing$. Since A is noetherian, $\mathcal{C}$ has a maximal element, say (a). Since a itself is not irreducible, we may write $a = bc$ for $b, c \in A$ both nonzero nonunits. This implies $(a) \subsetneq (b)$, since $a \in (b)$ and c is a nonunit. Similarly, $(a) \subsetneq (c)$ holds. By maximality of (a), both b and c are products of irreducibles, but this would imply $a = bc$ being a product of irreducibles, thus a contradiction. Hence $\mathcal{C} = \varnothing$, so every element of A has a factorization into irreducibles.

Suppose $a = u \prod_i p_i = v \prod_j q_j$ are two factorizations into irreducibles. Since the p_i are prime, we must have $p_1 | q_j$ for some j. WLOG let $j = 1$. Then $p_1 | q_1$. This means that $q_1 = kp_1$ for some $k \in A$, but since q_1 is irreducible, we must have $k \in A^\times$. Thus p_1 and q_1 are associates. Repeat this process for a/p_1, and so on to prove uniqueness. Thus A is a UFD. $\square$

Proof. (2) Suppose A is a UFD. Let $\mathfrak{p}$ be a height 1 prime of A. Since A is a domain, the maximal chain is $(0) \subsetneq \mathfrak{p}$. Pick some nonzero $x \in \mathfrak{p}$. Since A a UFD, x has a factorization into irreducibles, and x must have at least one irreducible factor (otherwise $\mathfrak{p} = A$). Denote that irreducible factor p. Then $(p) \subset \mathfrak{p}$, and by (1) (p) is a prime ideal. By height conditions, $(p) = \mathfrak{p}$, so it is principal.

Conversely suppose every height 1 prime is principal. Let $p \in A$ be an irreducible element. Since it is a nonunit, the ideal (p) is proper, hence contained in some prime ideal. Let $\mathfrak{p} \in V(p)^{\min}$. By Krull's PIT, $\mathfrak{p}$ is of height ≤ 1. Since A is a domain and p is nonzero, $\mathfrak{p}$ must be of height 1. By hypothesis, $\mathfrak{p}$ is principal, say $\mathfrak{p} = (p')$. Then $p = p'a$ for some $a \in A$, so either p' or a is a unit. p' is not a unit, so a must be a unit, thus $(p) = (p') = \mathfrak{p}$. Hence p is prime, so we may apply the converse of (1) to conclude that A is a UFD. $\square$

Proof. (3) Suppose A is a UFD. By (2) this is equivalent to every height 1 prime being principal. Let $\mathfrak{p}$ be a height 1 prime of A_p. Consider the prime $\mathfrak{q} \subset A$ that corresponds to $\mathfrak{p}$. By the inclusion preserving correspondence of primes, we conclude that $\mathfrak{q}$ is also of height 1, hence principal by (2) since A a UFD. Let

$\mathfrak{q} = (q)$. Note that $\mathfrak{p} = (q/1)$. Hence all height 1 primes of A_p are principal, thus A_p is also a UFD.

Conversely, suppose A is not a UFD. We want to show that A_p is not a UFD. By (1), not being a UFD is equivalent to having an irreducible element that is not prime. Say $q \in A$ is an irreducible element that is not prime. We claim that $q/1 \in A_p$ is an irreducible element that is not prime.

We first show irreducibility. Suppose $\frac{q}{1} = \frac{a}{p^m}\frac{b}{p^n}$. We want to show that one of the factors is a unit. If $m = n = 0$ then it trivially holds, so assume $m + n > 0$. Since $ab = qp^{m+n} \in (p)$ in A, and since p is prime in A, we must have either $a \in (p)$ or $b \in (p)$. WLOG we may write $a = pa'$, and plugging this into the identity above, we get $a'b = qp^{m+n-1}$. Repeating this process finitely many times, we may write $a = \alpha p^{m'}$ and $b = \beta p^{n'}$ where $m' + n' = m + n$, hence getting $\alpha\beta = q$. Since q is irreducible in A, we must have either α or β a unit. This implies either a/p^m or b/p^n a unit, which is the desired result. Hence $q/1$ is irreducible in A_p.

However, $q/1$ cannot be prime in A_p. Suppose it did; suppose $(q/1)$ is a prime ideal of A_p. We show this implies (q) is prime. Suppose $ab \in (q)$. Then $\frac{a}{1}\frac{b}{1} = \frac{ab}{1} \in (q/1)$, which is prime by assumption, so either $a/1$ or $b/1$ is in $(q/1)$. WLOG suppose $\frac{a}{1} = \frac{q}{1}\frac{c}{p^k}$ for some $c \in A$. If $k = 0$ then $a \in (q)$ automatically. Hence assume $k > 0$, which implies $ap^k = qc$. Hence $qc \in (p)$. If q were a multiple of p, then $q = pu$ for a unit u since q is irreducible, hence q is prime. Now suppose q is not a multiple of p. Then c must be in (p), and we may write $c = c'p^k$ to conclude that $a = qc'$. This implies $a \in (q)$, so (q) is prime. Thus $(q/1)$ prime implies (q) prime, which is not the case, so $q/1$ is not a prime in A_p. Hence A_p is not a UFD. $\square$

Problem 4. *Let A be a noetherian local ring. Show that A is artinian if there exists a nonzero injective A-module that is finitely generated.*

Proof. Noetherian rings of dimension 0 are artinian, so we show $\dim A = 0$. Assume by contradiction that $\dim A > 0$. Let M be a finitely generated injective A-module. If $\operatorname{Ass}(M) = \{\mathfrak{m}\}$, we have an injection $A/\mathfrak{m} \hookrightarrow M$. Since $\dim A > 0$, there exists some $\mathfrak{p} \subsetneq \mathfrak{m}$, and the natural projection $A \to A/\mathfrak{m}$ factors through $A/\mathfrak{p}$ to yield a map $A/\mathfrak{p} \to A/\mathfrak{m}$. Compose this with the injection to get a map $A/\mathfrak{p} \to M$. In the case $\operatorname{Ass}(M) \neq \{\mathfrak{m}\}$, take any $\mathfrak{p} \in \operatorname{Ass}(M)$. In both cases $\operatorname{Hom}_A(A/\mathfrak{p}, M) \neq 0$ and $\mathfrak{p} \subsetneq \mathfrak{m}$. Pick any $x \in \mathfrak{m} \setminus \mathfrak{p}$. Then consider the following diagram

$$\begin{tikzcd}
 & M & \\
0 \arrow[r] & A/\mathfrak{p} \arrow[u, "f"] \arrow[r, "\times x"'] & A/\mathfrak{p} \arrow[ul, dashed, "\exists"']
\end{tikzcd}$$

where the bottom row is injective, since x is nonzero in $A/\mathfrak{p}$. By the injectivity of M, for every $f : A/\mathfrak{p} \to M$ corresponds some morphism $A/\mathfrak{p} \to M$ that commutes, i.e. every $f \in \operatorname{Hom}_A(A/\mathfrak{p}, M)$ can be written as xg for some $g \in \operatorname{Hom}_A(A/\mathfrak{p}, M)$. Thus $\operatorname{Hom}_A(A/\mathfrak{p}, M) \subset x\operatorname{Hom}_A(A/\mathfrak{p}, M)$, and the opposite inclusion is immediate so equality holds. Since $\operatorname{Hom}_A(A/\mathfrak{p}, M)$ is finitely generated and $x \in \mathfrak{m}$, by Nakayama we have $\operatorname{Hom}_A(A/\mathfrak{p}, M) = 0$. Hence $\dim A$ must be 0, so A is artinian.[2] $\square$

[2]This proof based on this mathSE post

Problem 5. *Let $(A, \mathfrak{m}, k)$ be a noetherian local ring. Show that A is regular if $\operatorname{id}_A k < \infty$.*

Proof. We use induction on $d = \dim A$. Suppose $d = 0$. Then $\operatorname{id}_A k = \operatorname{depth} A \leq \dim A = 0$, so k is injective. Hence $\operatorname{Hom}_A(-, k)$ is exact, and we may apply this to the SES

$$0 \to \mathfrak{m} \to A \to k \to 0$$

to obtain

$$0 \to \operatorname{Hom}_A(k, k) \to \operatorname{Hom}_A(A, k) \to \operatorname{Hom}_A(\mathfrak{m}, k) \to 0.$$

Since $\operatorname{Hom}_A(k, k) = \operatorname{Hom}_k(k, k) \cong k$ and $\operatorname{Hom}_A(A, k) \cong k$, and we may write this as

$$0 \to k \to k \to \operatorname{Hom}_A(\mathfrak{m}, k) \to 0$$

where $\operatorname{Hom}_A(\mathfrak{m}, k) = 0$ since $\operatorname{Hom}_A(\mathfrak{m}, k)$ is naturally a k-module, together with additivity of k-dimensions. But from $\mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2 \to 0$ and consequently $0 \to \operatorname{Hom}_A(\mathfrak{m}/\mathfrak{m}^2, k) \to \operatorname{Hom}_A(\mathfrak{m}, k) = 0$, we have $\operatorname{Hom}_A(\mathfrak{m}/\mathfrak{m}^2, k) = \operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k) = 0$. If $\mathfrak{m} \neq \mathfrak{m}^2$, this cannot happen, so we must have $\mathfrak{m} = \mathfrak{m}^2$. Thus $\mathfrak{m} = 0$ by Nakayama, and $A = k$ is regular.

Now assume $d > 0$ and $\operatorname{id}_A k < \infty$. By inductive hypothesis, every noetherian local ring of dimension less than d is regular if the residue field has finite injective dimension. As in problem 1, by prime avoidance and $d > 0$, we may pick a non zerodivisor $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. Note that the residue field of $A/(x)$ is also k. We show that $\operatorname{id}_{A/(x)} k < \infty$, which would imply regularity of $A/(x)$, which in turn implies the regularity of A since x is A-regular.

By a lemma in the lecture notes,[3] we have isomorphisms

$$\operatorname{Ext}_A^{i+1}(k, \mathfrak{m}) \cong \operatorname{Ext}_{A/(x)}^{i}(k, \mathfrak{m}/(x)\mathfrak{m})$$

for all $i \geq 0$. By the same lemma, we have $\operatorname{Ext}_A^{i+1}(k, A) \cong \operatorname{Ext}_{A/(x)}^{i}(k, A/(x))$ for all $i \geq 0$. Also, since $\operatorname{id}_A k = \sup\{i \mid \operatorname{Ext}_A^i(k, k) \neq 0\} < \infty$, we must have $\operatorname{Ext}_A^i(k, k) = 0$ for all $i >> 0$. Consider the SES

$$0 \to \mathfrak{m} \to A \to k \to 0$$

and the LES of Ext given by

$$\cdots \to \operatorname{Ext}_A^i(k, k) \to \operatorname{Ext}_A^{i+1}(k, \mathfrak{m}) \to \operatorname{Ext}_A^{i+1}(k, A) \to \operatorname{Ext}_A^{i+1}(k, k) \to \cdots$$

where since $\operatorname{Ext}_A^i(k, k) = 0$ for all $i >> 0$, we have $\operatorname{Ext}_A^{i+1}(k, \mathfrak{m}) \cong \operatorname{Ext}_A^{i+1}(k, A)$ for all $i >> 0$. Combined with the results above, we have

$$\operatorname{Ext}_{A/(x)}^{i}(k, \mathfrak{m}/(x)\mathfrak{m}) \cong \operatorname{Ext}_{A/(x)}^{i}(k, A/(x))$$

for all $i >> 0$.

Now consider the SES

$$0 \to \mathfrak{m}/(x)\mathfrak{m} \to A/(x) \to k \to 0$$

and the associated LES of Ext w.r.t $A/(x)$ given by

$$\cdots \to \operatorname{Ext}_{A/(x)}^{i}(k, k) \to \operatorname{Ext}_{A/(x)}^{i+1}(k, \mathfrak{m}/(x)\mathfrak{m}) \to \operatorname{Ext}_{A/(x)}^{i+1}(k, A/(x)) \to \operatorname{Ext}_{A/(x)}^{i+1}(k, k) \to \cdots$$

[3] which is Bruns & Herzog, Lemma 3.1.16. Writing this since there is no full proof in lecture notes

where the middle arrow is an isomorphism by the naturality of the isomorphism of the lemma above, and the commutativity of the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m} \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{m}/(x)\mathfrak{m} \arrow[r] & A/(x) \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

Therefore we have $\operatorname{Ext}^i_{A/(x)}(k,k) = 0$ for all $i >> 0$, i.e. $\operatorname{id}_{A/(x)} k \leq \infty$. Since x is A-regular, $\dim A/(x) < \dim A$, and we may apply the inductive hypothesis to conclude that $A/(x)$ is regular. Hence A is regular.[4] $\square$

[4]This proof based on this mathSE post