5/4 Adeles and Ideles

- Notation. Let K be a number field, deg n over $\mathbb{Q}$. $n=r+2s$: $\{\sigma_1,\cdots,\sigma_n\}=\mathrm{Mor}(K,\mathbb{C})$, $\sigma_{r+i}=\overline{\sigma_{r+i+s}}$.

① Embedding $\Phi: K \hookrightarrow \Omega := \mathbb{R}^r \times \mathbb{C}^s$, $z \mapsto (\sigma_1(z),\cdots,\sigma_r(z)) \times (\sigma_{r+1}(z),\cdots,\sigma_{r+s}(z))$, $\mathcal{O}_K=\{\alpha\in K \mid \alpha \text{ is integral over } \mathbb{Z}\}$.

② $M_K := \{\text{primes of } K\} = \begin{cases} S_\infty = \{\text{archimedean primes}\} : \text{ex. } \|z\|=|\sigma_r(z)|_{\mathbb{R}},\ \|z\|=|\sigma_{r+1}(z)|_{\mathbb{C}}^2 \\ \text{nonarchimedean} : \|z\|_{\mathfrak{p}} = (\mathcal{O}_K/\mathfrak{p})^{-v_{\mathfrak{p}}(z)} \end{cases}$

③ For $v\in M_K$, $K_v :=$ completion of (K,v). $\mathcal{O}_v := \{\alpha\in K_v \mid \|\alpha\|\le 1\}$, $\mathcal{O}_v^\times = \{\alpha\in K_v \mid \|\alpha\|=1\}$.

$(v\in S_\infty \Rightarrow K_v=\mathbb{R} \text{ or } \mathbb{C},\ \mathcal{O}_v=K_v,\ \mathcal{O}_v^\times = K_v^\times)$

§ Classical theorems (of algebraic number theory)

① (Strong Approximation) Let $\|\cdot\|_0, \|\cdot\|_1, \cdots, \|\cdot\|_n \in M_K$ (inequivalent). Let $x_i\in K_{\mathfrak{p}_i}$, $\varepsilon_i>0$ for $i=1\sim n$. Then there is $y\in K$ s.t. $\|y-x_i\|_i<\varepsilon_i$ for $i=1\sim n$ and $\|y\|_\alpha\le 1$ for all the other values not equivalent to one of $\|\cdot\|_0, \|\cdot\|_1,\cdots,\|\cdot\|_n$.

② (Finiteness of class group) $\mathrm{Cl}\,\mathcal{O}_K := J_K/P_K$ where $J_K=\underset{\text{f.g. } \mathcal{O}_K\text{-submodule}}{\{\text{fractional ideals of } K\}}$, $P_K=\underset{\text{principal ideals}}{\{x\mathcal{O}_K\in I \mid x\in K\}}$

Then $\mathrm{Cl}(\mathcal{O}_K)$ is finite.

③ (Dirichlet Unit theorem) $U_K=\mathcal{O}_K^\times$ is a f.g. group of rank $r+s-1$.

④ (Product formula) For $\alpha\in K^\times$, $\prod_{v\in M_K}\|\alpha\|_v=1$.

- <u>RMK</u>. ① = generalization of CRT

②,③ = (geometry of numbers) significantly uses that '$\Phi(\mathcal{O}_K)$ is the lattice of Ω'

④ $\simeq$ degree of principal divisor is zero

§ Definition of Adeles and Ideles

① **Definition 22.1.** Let (X_i) be a family of topological spaces indexed by $i \in I$, and let (U_i) be a family of open sets $U_i \subseteq X_i$. The *restricted product* $\prod(X_i, U_i)$ is the topological space

$$\prod(X_i, U_i) := \left\{(x_i) \in \prod X_i : x_i \in U_i \text{ for almost all } i \in I\right\}$$

with the basis of open sets

$$\mathcal{B} := \left\{\prod V_i : V_i \subseteq X_i \text{ is open for all } i \in I \text{ and } V_i = U_i \text{ for almost all } i \in I\right\},$$

where *almost all* means all but finitely many.

$\Rightarrow \prod(X_i, U_i) = \varinjlim X_S$ where $X_S = \prod_{i \in S} X_i \times \prod_{i \notin S} U_i$, $S \subset I$ finite.

$\mathbb{A}_K := \prod(K_v, \mathcal{O}_v)$: topological ring, LCH.

* Basis of topology : $\prod$ (finitely many open ball) $\times \prod_v \mathcal{O}_v$.

② $\mathbb{I}_K = \prod(K_v^\times, \mathcal{O}_v^\times)$

* Basis of topology : $\prod$ (finitely many open balls) $\times \prod_v \mathcal{O}_v^\times$.

Then $\mathbb{I}_K$ is a LCH topological group (while $\mathbb{A}_K^\times$ is not a topological group)

③ Example. $S = S_\infty = \{$archimedean primes$\}$

$$\mathbb{A}_K^{S_\infty} = \prod_{v \in S_\infty} K_v \times \prod_{v \notin S_\infty} \mathcal{O}_v = \Omega \times \Delta \text{ where } \Delta := \prod_{v \notin S_\infty} \mathcal{O}_v .$$

$$\mathbb{I}_K^{S_\infty} = \prod_{v \in S_\infty} K_v^\times \times \prod_{v \notin S_\infty} \mathcal{O}_v^\times$$

④ Basic properties

- $\iota : K \to \mathbb{A}_K$, $\iota : K^\times \to \mathbb{I}_K$, $z \mapsto (z, z, z, \cdots)$.
- Since $\mathcal{O}_v^\times := \{x \in K_v \mid \|x\|_v = 1\}$, $\|\cdot\| : \mathbb{I}_K \to \mathbb{R}_{>0}$, $(z_v)_v \mapsto \prod_{v \in M_K} \|z_v\|_v$ is well-defined.

  $\iota(K) \subset \mathbb{I}' := \{z \in \mathbb{I}_K \mid \|z\| = 1\}$ (Product formula)
- $\mathbb{I} \twoheadrightarrow J_K$, $(x_v)_v \mapsto \prod_{p_v \notin S_\infty} p_v^{\mathrm{ord}_v x_v}$ (idèle to ideal)

- **RMK.** LCH abelian topological group $\Rightarrow$ Haar measure

• RMK. By Strong Approximation, $\mathbb{A}_K = \iota(K) + \mathbb{A}_K^{S_\infty}$.

But for ideles, $\mathbb{I}_K = \iota(K^\times)\mathbb{I}_K^{S_\infty} \Leftrightarrow \mathcal{O}_K$ is a PID : generally NOT true

(It's because $\mathbb{I}_K/\iota(K^\times)\mathbb{I}_K^{S_\infty} \simeq J_K/P_K$ under $\mathbb{I}_K \twoheadrightarrow J_K$)

• Prop. (Finiteness of ideal class group, reformulated)

$\mathbb{I}_K = \iota(K^\times)\mathbb{I}_K^S$ for some finite $S_\infty \subset S \subset M_K$.

pf. Take $S = S_\infty \cup \{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\}$ where $\mathfrak{p}_1^{\mathbb{Z}} \cdot \cdots \cdot \mathfrak{p}_k^{\mathbb{Z}}$ covers $\mathrm{Cl}\,\mathcal{O}_K$.

$(x_v)_v \in \mathbb{I}_K \Rightarrow \prod_v \mathfrak{p}_v^{\mathrm{ord}_v x_v} = \mathfrak{p}_1^{l_1} \cdots \mathfrak{p}_k^{l_k} \cdot (z)$ for some $(l_i) \in \mathbb{Z}^k$ and $z \in K^\times$.

Then $(x_v)_v = z \cdot (z^{-1}x_v)_v$: $z \in K^\times$, $(z^{-1}x_v)_v \in \mathbb{I}_K^S$. $\square$

§ Structural theorems of $\mathbb{A}_K, \mathbb{I}_K$.

Thm. (1) $\iota(K) \subset \mathbb{A}_K$, $\iota(K^\times) \subset \mathbb{I}_K$ are discrete.

(2) $\mathbb{A}_K/\iota(K)$, $\mathbb{I}_K^1/\iota(K^\times)$ are compact.

pf. ① (discreteness are easy)

Product formula shows that any $a \in K^\times$ has $v \in M_K$ s.t. $\|a\|_v \geq 1$.

② (Compactness of $\mathbb{A}_K/\iota(K)$)

$\mathbb{A}_K = \iota(K) + \mathbb{A}_K^{S_\infty} \Rightarrow \mathbb{A}_K/\iota(K) \simeq \mathbb{A}_K^{S_\infty}/\iota(\mathcal{O}_K)$

Recall $\Phi : K \hookrightarrow \Omega = \mathbb{R}^r \times \mathbb{C}^s$, $\mathbb{A}_K^{S_\infty} = \Omega \times \Delta$.

$\mathbb{A}_K^{S_\infty}/(\iota(\mathcal{O}_K) + 0 \times \Delta) = \Omega/\Phi(\mathcal{O}_K)$: vec. sp / full lattice $\therefore$ cpt.

$(\iota(\mathcal{O}_K) + 0 \times \Delta)/\iota(\mathcal{O}_K) = 0 \times \Delta/(0 \times \Delta \cap \iota(\mathcal{O}_K)) = 0 \times \Delta$: cpt (Tychonoff)

③ (Compactness of $\mathbb{I}_K^1/\iota(K^\times)$ $\leftrightarrow$ Unit theorem & Finiteness of ideal class group)

$M_K = \underbrace{S_\infty \sqcup T}_{S \text{ s.t. } \mathbb{I}_K = \mathbb{I}_K^S \iota(K^\times)} \sqcup S^c$

$\prod_{v\in M_K} K_v^\times = \Omega_1^\times \times \Omega_2^\times \times \Omega_3^\times$ | $\mathbb{I}^S = \Omega_1^\times \times \Omega_2^\times \times \Delta_3^\times$

$\prod_{v\in M_K} (\mathcal{O}_v)^\times = \Omega_1^\times \times \Delta_2^\times \times \Delta_3^\times$

$\Delta_2^\times \subset \Sigma_{12} \subset \Omega_2^\times$ $\quad \Sigma_{12} = \prod_{v\in T} \pi_v^{\mathbb{Z}} \mathcal{O}_v^\times : (\pi_v) = p_v^h$. Also $\Pi := \prod_{v\in T} \pi_v^{\mathbb{Z}} \le K^\times$

① $\mathbb{I}^1/K^\times = \mathbb{I}_1^S/K^S$ ($K^S := \mathbb{I}^S \cap K^\times$: S-units of K)

Since $\Pi \le K^S$, ets $\mathbb{I}_1^S/(\Pi\cdot\mathcal{O}_K^\times)(1\times\Delta_2\times\Delta_3)$

② $\mathbb{I}_1^S/(\Omega_1\times\Sigma_{12}\times\Delta_3)^1 \simeq \mathbb{I}^S/(\Omega_1\times\Sigma_{12}\times\Delta_3) = \Omega_2^\times/\Sigma_{12}$: compact

§ What happens to ideles in Galois extension of number fields?

$(G = Gal(L/K))$

* The key object of Class Field Theory is $\boxed{C_K = \mathbb{I}_K / K^\times}$.

$$\begin{array}{ccccccccc} 1 & \to & K^\times & \longrightarrow & \mathbb{I}_K & \to & C_K & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & L^\times & \longrightarrow & \mathbb{I}_L & \to & C_L & \to & 1 \end{array}$$

(injective verticals)

① $\mathbb{I}_L$ is a G-module with natural construction.

$a = (a_w)_w \in \mathbb{I}_L \Rightarrow \sigma a := (\sigma a_w)_{\sigma w} = (\sigma a_{\sigma^{-1} w})_w \in \mathbb{I}_L \quad (\|x\|_{\sigma w} := \|\sigma^{-1} x\|_w)$

② $(\mathbb{I}_L)^G = \mathbb{I}_K$ with natural embedding $\mathbb{I}_K \hookrightarrow \mathbb{I}_L$ ($*$ $w|v \Rightarrow K_v \hookrightarrow L_w$)

$a \in (a_w)_w \in (\mathbb{I}_L)^G \Leftrightarrow a_{\sigma w} = \sigma a_w$ for all $\sigma \in G$ and $w \Leftrightarrow a_w \in L_w{}^{G_w} = K_v$

③ G-module LES

$1 \to L^\times \to \mathbb{I}_L \to C_L \to 1 \Rightarrow 1 \to (L^\times)^G \to (\mathbb{I}_L)^G \to C_L{}^G \to H^1(G, L^\times) = 0$.

$\therefore C_L{}^G = C_K$

④ Prime ideal of K ramifies in L iff it divides $\mathrm{Disc}(\mathcal{O}_L/\mathcal{O}_K)$.

$\Rightarrow$ When one deals with $\mathbb{I}_L$, it is convinient to take S :
(1) contain all archimedean primes
(2) contain all ramifying primes
(3) $\mathbb{I}_L = \mathbb{I}_L^S \, \iota(L^\times)$

- <u>Thm.</u> (Hilbert–Noether) $H^1(G, L^\times) = 0$.

<u>pf.</u> Let $(a_\sigma)_{\sigma \in G} \in Z^1(G, L^\times)$: $a_{\tau\sigma} = (\tau a_\sigma) \cdot (a_\tau)$

Since elements of G are linearly independent over K, $\sum_{\sigma \in G} a_\sigma \cdot \sigma \neq 0$.

Take $c \in L^\times$ s.t. $b := \sum a_\sigma \cdot \sigma c \in L^\times$. Then

$\tau b = \sum_\sigma \tau a_\sigma (\tau \sigma c) = a_\tau^{-1} \sum_\sigma a_{\tau\sigma} (\tau\sigma c) = a_\tau^{-1} b$, $a_\tau = \dfrac{b}{\tau b} = \dfrac{\tau(b^{-1})}{b^{-1}}$.

$\therefore (a_\sigma)_{\sigma \in G} \in B^1(G, L^\times)$, $H^1(G, L^\times) = 0$. □

① Ideal description of CFT

L/K finite abelian

$\psi_{L/K}: I^S \to Gal(L/K)$

$p_1^{n_1}\cdots p_t^{n_t} \mapsto \prod (p_i, L/K)^{n_i}$

ex. When $K = \mathbb{Q}[\sqrt{m}]$,

$I^S \mapsto Gal(\mathbb{Q}[\sqrt{m}]/\mathbb{Q})$ is a Legendre symbol.

$$\begin{tikzcd}
I_{K'}^S \arrow[r, "\psi_{L/K'}"] \arrow[d, "N_m"'] \arrow[dr, phantom, "\circlearrowleft"] & Gal(L/K') \arrow[d, hook] \\
I_K^S \arrow[r, "\psi_{L/K}"] & Gal(L/K)
\end{tikzcd}$$

$$\underbrace{I_K^{S(\mathfrak{m})}/K_{\mathfrak{m},1}}\cdot N_m(I_L^{S(\mathfrak{m})}) \simeq Gal(L/K)$$

$C_{\mathfrak{m}} := I_K^{S(\mathfrak{m})}/K_{\mathfrak{m},1}$ (ray class group mod $\mathfrak{m}$)

In particular, there is $L_{\mathfrak{m}}$ (ray class field 〃)

$\varprojlim C_{\mathfrak{m}} \simeq Gal(K^{ab}/K)$

② Idèlic description of CFT

$\phi_K : \mathbb{I}_K \to Gal(K^{ab}/K)$

(a) $\phi_K(K^\times) = 1$

(b) L/K finite, abelian ($K \overset{fin}{\subset} L \subset K^{ab}$)

$\Rightarrow \phi_{L/K} : \mathbb{I}_K / K^\times (N_m \mathbb{I}_L) \simeq Gal(L/K)$

or, $\quad \mathbb{I}_K / K^\times (N_m \mathbb{I}_L) = C_K/N_m(C_L) = C_L^G/N_m C_L = H_T^0(G, C_L)$

$$\begin{tikzcd}
K_v^\times \arrow[r, "\phi_v"] \arrow[d] & Gal(L_w/K_v) \arrow[d, hook] \\
\mathbb{I}_K \arrow[r, "\phi_{L/K}"] & Gal(L/K)
\end{tikzcd}$$

local CFT $\Rightarrow$ global CFT

$H_T^*(G, \mathbb{I}_L) \simeq \bigoplus_v H_T^*(G^v, L^{v\times})$

(G^v, L^v means choice of $w \in M_L$ which devides v)

(* $H^0(G, C_L) = C_L^G$. But Tate? extends $0 \to C^0 \to C^1 \to \cdots$

$\Downarrow$

$C^1 \to C^0 \to C^0 \to C^1 \to C^2 \to \cdots$)