The Riemann mapping theorem

Chapter 8

Complex Function Theory 2

July 19, 2023

Introduction

In the previous lecture, we have examined various examples of conformal maps between subsets of $\mathbb{C}$.

In this lecture, we will study the existence of conformal maps from Ω to $\mathbb{D}$ for various Ω.

Introduction

Notice that for $\Omega = \mathbb{C}$, there cannot be a conformal map $F\colon \Omega \to \mathbb{D}$ since if so, by Liouville's theorem this would be a constant.

Also since a conformal map is a homeomorphism, and $\mathbb{D}$ is simply connected, we must have Ω also simply connected.

Statement of the Riemann mapping theorem

Theorem 3.1 (Riemann mapping theorem)

Suppose Ω is a proper subset of $\mathbb{C}$ and is simply connected. If $z_0 \in \Omega$, there exists a unique conformal map $F : \Omega \to \mathbb{D}$ such that $F(z_0) = 0$ and $F'(z_0) > 0$.

Corollary 3.2

Any two proper simply connected open subsets in $\mathbb{C}$ are conformally equivalent.

Riemann mapping theorem

Suppose F and G are both conformal maps from Ω to $\mathbb{D}$ that satisfy the conditions. Then $F \circ G^{-1} \in \mathrm{Aut}(\mathbb{D})$, so $(F \circ G^{-1})(z) = e^{i\theta} z$ and since $(F \circ G^{-1})'(0) > 0$ we have $e^{i\theta} = 1$, so $F = G$. Thus uniqueness follows.

We now prove existence of such conformal maps.

Idea of the proof

Consider all injective holomorphic maps $f\colon \Omega \to \mathbb{D}$ with $f(z_0) = 0$. From these, we construct an f so that its image fills all of $\mathbb{D}$ by making $f'(z_0)$ as large as possible.

Some definitions

Definition

Let Ω be an open subset of $\mathbb{C}$. A family $\mathcal{F}$ (not necessarily closed) of holomorphic functions on Ω is said to be **normal** if every sequence in $\mathcal{F}$ has a subsequence that converges u.o.c on Ω.

Definition

$\mathcal{F}$ is said to be **uniformly bounded on compact subsets of** Ω if for each compact set $K \subset \Omega$, there exists $B > 0$ such that $|f(z)| \leq B$ for all $z \in K$ and $f \in \mathcal{F}$.

Some definitions

Definition

$\mathcal{F}$ is **equicontinuous** on a compact K if for all $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $z, w \in K$ and $|z - w| < \delta$, then $|f(z) - f(w)| < \epsilon$ for all $f \in \mathcal{F}$.

A theorem on $\mathcal{F}$

Theorem 3.3 (Montel's theorem)

Suppose $\mathcal{F}$ is a family of holomorphic functions on Ω that is uniformly bounded on compact subsets of Ω. Then,

- $\mathcal{F}$ is equicontinuous on every compact subset of Ω.
- $\mathcal{F}$ is a normal family.

Before we prove this, we introduce the notion of an **exhaustion**. A sequence $\{K_l\}_{l=1}^{\infty}$ of compact subsets of Ω is called an exhaustion if

- K_l is contained in K_{l+1}° for all l
- Any compact $K \subset \Omega$ is contained in K_l for some l. In particular, $\Omega = \bigcup_{l=1}^{\infty} K_l$.

A lemma

To prove Montel's theorem, we introduce a lemma.

Lemma 3.4

Any open set Ω in the complex plane has an exhaustion.

Proof

If Ω is bounded, let K_l be the set of all points in Ω at distance $\geq 1/l$ from the boundary of Ω. (More rigorously, let $K_l = \Omega - \cup_{z \in \partial\Omega} B(z, 1/l)$) If Ω is not bounded, let K_l be defined the same, but additionally we require $|z| \leq l$ for all $z \in K_l$. Such K_l form an exhaustion.

Proof of Montel's theorem

Now we prove Montel's theorem.

Proof of equicontinuity of $\mathcal{F}$

Choose an arbitrary compact $K \subset \Omega$ and $f \in \mathcal{F}$. Choose $r > 0$ such that $D_{3r}(z) \subset \Omega$ for all $z \in K$. Conclude that $|f(z) - f(w)| < C|z - w|$ for all $z, w \in K$ with $|z - w| < r$, and all $f \in \mathcal{F}$ by using Cauchy's integral formula on $D_{2r}(w)$.

Proof of Montel's theorem

Proof of normality of $\mathcal{F}$

Let $\{f_n\}_{n=1}^{\infty}$ be any sequence in $\mathcal{F}$ and K a compact subset of Ω. Choose a sequence of points $\{w_j\}_{j=1}^{\infty}$ that is dense in Ω. Since $\mathcal{F}$ is uniformly bounded, there is a convergent subsequence of $\{f_n(w_1)\}$. Rename such f_n as $f_{n,1}$, which becomes a subsequence of $\{f_n\}$. Now from this subsequence $\{f_{n,1}\}$ we extract another subsequence $\{f_{n,2}\}$ such that $\{f_{n,1}(w_2)\}$ converges. Name this $\{f_{n,2}\}$, and so on.

Proof of Montel's theorem

Proof of normality of $\mathcal{F}$

Let $g_n = f_{n,n}$. Notice that $g_n(w_j)$ converges for each j. Now we show that g_n converges uniformly on K.

Fix an arbitrary $\epsilon > 0$, and pick $\delta > 0$ such that $|z - z_0| < \delta$ implies $|f(z) - f(z_0)| < \epsilon$ for all $f \in \mathcal{F}$ and all $z, z_0 \in \Omega$. (Such a δ exists since we assumed $\mathcal{F}$ equicontinuous on Ω.)

Consider the open cover $\{D(w_j, \delta)\}_{j=1}^{\infty}$ of K. This is indeed an open cover since we assumed $\{w_j\}$ dense in Ω. (In fact it is also an open cover of Ω.) Since we assumed K is compact, there must be a finite subcover, hence there exists some $J > 0$ such that $\{D(w_j, \delta)\}_{j=1}^{J}$ that covers K.

Proof of Montel's theorem

Proof of normality of $\mathcal{F}$

Choose some N such that $n, m > N$ implies $|g_m(w_j) - g_n(w_j)| < \epsilon$ for all $j = 1, \ldots, J$. Together with the fact that $z \in K$ implies $z \in D(w_j, \delta)$ for some $j \leq J$, we have

$$|g_n(z) - g_m(z)| \leq |g_n(z) - g_n(w_j)| + |g_n(w_j) - g_m(w_j)| + |g_m(w_j) - g_m(z)| < 3\epsilon$$

Hence $\{g_n\}$, a subsequence of $\{f_n\}_{n=1}^{\infty}$, converges uniformly on K.

Proof of Montel's theorem

Proof of normality of $\mathcal{F}$

Since our choice of K was arbitrary, we have proved that every sequence in $\mathcal{F}$ has a subsequence that converges u.o.c on Ω.

End of proof! Or is it...?

Note that our proof that the subsequence converges depended on K. We have to find a subsequence that converges on *every* compact subset of Ω. In other words, the subsequence we have proved to converge uniformly on K may fail to converge on other compact subsets of Ω.

Proof of Montel's theorem

Proof of normality of $\mathcal{F}$

By Lemma 3.4, Ω has an exhaustion $K_1 \subset K_2 \subset \cdots \subset K_l \subset \cdots$. Suppose $\{g_{n,1}\}$ is a subsequence of $\{f_n\}$ that converges uniformly on K_1. (The existence of such subsequence is what we showed in previous slides.) From $\{g_{n,1}\}$ extract a subsequence $\{g_{n,2}\}$ that converges uniformly on K_2, and so on. Then $\{g_{n,n}\}$ is a subsequence of $\{f_n\}$ that converges uniformly on every K_l, and since the K_l exhaust Ω, $\{g_{n,n}\}$ converges uniformly on any compact subset of Ω.

A proposition

Proposition 3.5

If Ω is a connected open subset of $\mathbb{C}$ and $\{f_n\}$ is a sequence of injective holomorphic functions on Ω that converges u.o.c. on Ω to f, then f is either injective, or a constant.

Proof

Suppose f is not injective, so $\exists z_1, z_2 \in \Omega$ such that $f(z_1) = f(z_2)$. Define $g_n(z) = f_n(z) - f_n(z_1)$ and note that g_n has z_1 as its only zero. $\{g_n\}$ converges u.o.c on Ω to $g(z) = f(z) - f(z_1)$. Note that if g is not identically zero, then z_2 is an isolated zero of g. Take γ as a small circle centered at z_2, isolating the zero. Derive a contradiction by using the argument principle on g and g'.

Proof of the Riemann mapping theorem

Step 1

Assume Ω is a simply connected proper open subset of $\mathbb{C}$. Choose $\alpha \notin \Omega$ and define $f(z) = \log(z - \alpha)$. Note that f is injective. Pick $w \in \Omega$ and observe $f(z) \neq f(w) + 2\pi i$ for all $z \in \Omega$. In fact, $f(z)$ and $f(w) + 2\pi i$ can always be separated by a fixed ball centered at $f(w) + 2\pi i$. Now define $F(z) = 1/(f(z) - (f(w) + 2\pi i))$. $F : \Omega \to F(\Omega)$ is a conformal map. Also by the fact that $f(z)$ and $f(w) + 2\pi i$ are separated, the image $F(\Omega)$ is bounded. Therefore we may translate and rescale F such that $0 \in F(\Omega) \subset \mathbb{D}$.

Proof of the Riemann mapping theorem

Step 2

By step 1, assume Ω is an open subset of $\mathbb{D}$ containing zero. Consider the family $\mathcal{F}$ of all injective holomorphic functions on Ω that map into $\mathbb{D}$ and fix the origin.

Note that $\mathcal{F}$ is nonempty and is obviously uniformly bounded. Observe that $|f'(0)|$ are uniformly bounded for $f \in \mathcal{F}$ by the Cauchy inequality[a].

[a] Check if you have time

Proof of the Riemann mapping theorem

Step 2

Let $s = \sup_{f \in \mathcal{F}} |f'(0)|$ and choose $\{f_n\} \subset \mathcal{F}$ such that $|f_n'(0)| \to s$ as $n \to \infty$. By Montel's theorem, $\{f_n\}$ has a subsequence that converges uniformly on compact sets to some $f \in H(\Omega)$. Note that $s \geq 1$ so f cannot be constant. Therefore f is injective by Proposition 3.5. Also we have $|f(z)| < 1$ by the maximum modulus principle, and also $f(0) = 0$ so we have $f \in \mathcal{F}$ with $|f'(0)| = s$.

Proof of the Riemann mapping theorem

Step 3

Now since we have found a holomorphic injective map $f : \Omega \to \mathbb{D}$, we only have to prove f is surjective. We prove this by contradiction.

Suppose f is not surjective. Therefore there exists some $\alpha \in \mathbb{D} \setminus f(\Omega)$. Since Ω is simply connected, so is $U := (\psi_\alpha \circ f)(\Omega)$. Note that $0 \notin U$ because ψ_α interchanges 0 and α. Thus it is possible to define a square root function on U by $g(w) := \exp(\frac{1}{2}\log(w))$.

Proof of the Riemann mapping theorem

Step 3

Define $F := \psi_{g(\alpha)} \circ g \circ \psi_\alpha \circ f$. Note that $F \in \mathcal{F}$, since F is holomorphic, maps into $\mathbb{D}$ and 0 to 0, and is injective. Now if $h(z) := z^2$, we have $f = \psi_\alpha^{-1} \circ h \circ \psi_{g(\alpha)}^{-1} \circ F := \Phi \circ F$. Φ maps $\mathbb{D}$ into $\mathbb{D}$ with $\Phi(0) = 0$ and is not injective. By the Schwarz lemma, we conclude $|\Phi'(0)| < 1$.

Observe $f'(0) = \Phi'(0)F'(0)$, so $|f'(0)| < |F'(0)|$. Since $|f'(0)|$ was maximal in $\mathcal{F}$, this is a contradiction. Therefore f must be surjective.