Nevanlinna's First Main Theorem for Coherent Ideal Sheaves

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Introduction

Please feel free to interrupt if you find any errors!

In today's talk, we will extend the definitions of the proximity function, growth function and counting function from divisors to cycles of higher codimension and state the first main theorem of Nevanlinna theory via ideal sheaves.

Recalling some definitions

First, we will recall some definitions:

Definition (Counting functions for Divisors)

Let D be an effective divisor on $\mathbb{C}^m$ where $D = \sum_\lambda k_\lambda D_\lambda$. Define

$$n_k(t, D) := t^{2-2m} \int_{B(t) \cap (\sum_\lambda \min\{k, k_\lambda\} D_\lambda)} \alpha^{m-1},$$

where $\alpha := dd^c ||z||^2$ is a $(1,1)$-form on $\mathbb{C}^m$. Recall that the convention $\int_D \eta$ for a $2m-2$ form η on $\mathbb{C}^m$ is to calculate $\sum_\lambda k_\lambda \int_{D_\lambda} \eta$, assuming the k_λ are locally bounded Borel-measurable functions with compact support.

Using this, define $N_k(r, D) := \int_1^r \frac{n_k(t,D)}{t} \, dt$ for $r > 1$ and let $n = n_\infty$, $N = N_\infty$.

Recalling some definitions

Definition (Proximity function w.r.t. f and D)

Let $f : \mathbb{C}^m \to N$ be a meromorphic function and D a divisor on N such that the pullback of D is defined. Let (L, h) be a hermitian line bundle on N, and let σ be a holomorphic section such that $(\sigma) = D$ and $||\sigma||_h \leq 1$. Define

$$m_f(r, D) := \int_{||z||=r} \log \frac{1}{||\sigma \circ f(z)||_h} \gamma(z)$$

where $\gamma = d^c \log ||z||^2 \wedge \beta^{m-1}$ and $\beta = dd^c \log ||z||^2$.

Recalling some definitions

Definition (Order function of f w.r.t. L)

Let f and L be as before. Since L is equipped with a hermitian metric h, it defines a Chern form ω_L on N. Define

$$T_f(r, \omega_L) := \int_1^r \frac{dt}{t^{2m-1}} \int_{B(t)} f^* \omega_L \wedge \alpha^{m-1}$$

for $r \geq 1$.

Since the Chern form depends on the hermitian metric on the line bundle, a different hermitian metric h' gives rise to a different Chern form ω'_L. However, it is a fact that $T_f(r, \omega_L) - T_f(r, \omega'_L) = O(1)$ so we will write as $T_f(r, L)$ without ambiguity.

Recalling the First Main Theorem

Using terminology we defined, we may state:

Theorem (FMT for line bundles)

For $D \in |L|$ as before, we have

$$T_f(r, L) = N(r, f^*D) + m_f(r, D) + O(1).$$

Now recall the meromorphic (1-dimensional) First Main Theorem.

Theorem (FMT for meromorphic functions)

Let $f : \mathbb{C} \to \mathbb{P}^1$ be holomorphic and let $a \in \mathbb{P}^1$. Then we have

$$T(r, f) = N\left(r, \frac{1}{f - a}\right) + m\left(r, \frac{1}{f - a}\right) + O(1)$$

where the $O(1)$ term does not depend on r.

(I think this corresponds to the $L = \mathcal{O}(1)$ case, where $D = a$, instead of $\mathcal{O}(-1)$.)

Removing the basepoint-free assumption

For the statement of the Casorati-Weiestrass theorem we needed the space E to be basepoint-free. To extend this to arbitrary vector spaces, we need to extend the definition of the proximity function $m_f(r, D)$ to cycles of higher codimension.

Notice that we have a natural candidate to replace D; we can look at the ideal sheaves corresponding to these divisors. For example, we could replace a degree d hypersurface with its ideal sheaf $\mathcal{O}(-d)$.

Caution: non-effective divisors do not correspond to ideal sheaves. They do not form a (nontrivial) linear system in the first place.

Prelude–Coherent Sheaves

We will be solely interested in coherent sheaves.

Definition (Coherent sheaf)

Let X be a complex manifold. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *coherent* if

- there exists an affine open cover U_i of X such that there exists surjections $\mathcal{O}_X|_{U_i}^{\oplus n} \twoheadrightarrow \mathcal{F}|_{U_i}$ for some n, for each i, ($=$ is **finite type**)
- and for any open set $U \subset X$ and any $\mathcal{O}_X|_U$-module homomorphism $\varphi : \mathcal{O}_X|_U^{\oplus n} \to \mathcal{F}|_U$, the kernel $\ker\varphi$ is a finite type $\mathcal{O}_X|_U$-module.

In most cases, it is fine to think of coherent sheaves on X as an "abelian-categorification of vector bundles" on X.

Coherent Ideal Sheaves

From now on, things will get very messy (at least to me) because we will roam freely between both algebraic and analytic worlds. The definition below will be purely algebraic.

Definition (Projective variety, Hartshorne)

An integral, separated scheme X of finite type over $\mathbb{C}$ is called an (complex) algebraic variety. It is called projective if it is isomorphic to a (closed) subscheme of projective space.

When N is interpreted as a scheme, we assume N to be a projective variety in the sense above. As the definition implies, we will allow singularities and disallow nilpotent sections of the structure sheaf.

Coherent Ideal Sheaves

Lemma (2.4.1)

Let $\mathcal{I}$ be a coherent ideal sheaf of $\mathcal{O}_N$. There exists a very ample line bundle L on N such that $\Gamma(N, \mathcal{I} \otimes_{\mathcal{O}_N} L)$ generates $\mathcal{I}_x \otimes_{\mathcal{O}_{N,x}} L_x$ for each $x \in N$.

Proof.

To be absolutely rigorous, we need to define line bundles and ampleness in the scheme setting, but for the sake of time just think of these as rational analogues of the real deal.

If we interpret N as a scheme, consider an embedding $\iota : N \hookrightarrow \mathbb{P}^N$ (which exists since we assumed N to be projective) and the pullback sheaf $\iota^*\mathcal{O}(1) := \mathcal{L}$, which is very ample by definition. Serre's theorem (Theorem II.5.17, Hartshorne) asserts that there exists an integer n where $\mathcal{I} \otimes_{\mathcal{O}_N} \mathcal{L}^{\otimes n}$ is globally generated. Since pullbacks and tensor products commute, we may take $L := \mathcal{L}^{\otimes n}$.

$\square$

Coherent Ideal Sheaves

Continued.

Now interpret N as a complex manifold. If we consider the stalk $\mathcal{I}_x$ as a sheaf on a point x, we may consider the sheaf extended by zero to the entirety of N. Consider the exact sequence

$$0 \to \mathcal{K}_x \to \mathcal{I} \to \mathcal{I}_x \to 0$$

where the map $\mathcal{I} \to \mathcal{I}_x$ is the morphism induced by construction of direct limits, and $\mathcal{K}_x$ is the kernel.
Take your favorite positive line bundle L on N (which exists, take any ample line bundle). Twist enough such that the higher cohomology of

$$0 \to \mathcal{K}_x \otimes_{\mathcal{O}_N} L^{\otimes n} \to \mathcal{I} \otimes_{\mathcal{O}_N} L^{\otimes n} \to \mathcal{I}_x \otimes_{\mathcal{O}_N} L^{\otimes n} \to 0$$

vanishes. (In fact it suffices for $H^1(\mathcal{K}_x \otimes L^{\otimes n})$ to vanish) Therefore we obtain a surjection $\Gamma(\mathcal{I} \otimes L^{\otimes n}) \twoheadrightarrow \Gamma(\mathcal{I}_x \otimes L^{\otimes n})$. □

Coherent Ideal Sheaves

Continued.

Since $\mathcal{I}$ is coherent, there exists a neighborhood U_x of x such that $\Gamma(\mathcal{I} \otimes L^{\otimes n})$ generates $(\mathcal{I} \otimes L^{\otimes n})_y$ for every $y \in U$.[1] The U_x form an open cover for N, which is compact, so we can take a finite subcover, and take n to be the maximum twist among the finite open neighborhoods such that this holds. If needed, take n to be even bigger if $L^{\otimes n}$ does not turn out to be very ample. $\square$

[1] This part is quite dubious

The Proximity Function

The lemma we just proved asserts that there exists sections $\{\phi_j\}_{1 \le j \le \ell}$ of $\mathcal{I} \otimes L$ such that the stalks of the ϕ_j at x generate $\mathcal{I}_x \otimes L_x$, for all $x \in N$.

As $\mathcal{I}(N) \otimes_{\mathcal{O}(N)} L(N) \subset \mathcal{O}(N) \otimes_{\mathcal{O}(N)} L(N) \cong L(N)$, these sections are naturally interpreted as sections of $L(N)$ supported on the closed subscheme defined by $\mathcal{I}$. Since (unless the support is N) $\{\phi_j\}_{j \le \ell}$ cannot generate L, extend this to a basis of L, denoted by $\{\phi_j\}_{j \le \ell'}$ (which exists as very ample line bundles are globally generated).

The Proximity Function

Take a common trivialization of both sets of sectiosn, and define a function

$$d_{\mathcal{I}}(x) := \sqrt{\frac{\sum_{j=1}^{\ell} |\phi_j(x)|^2}{\sum_{j=1}^{\ell'} |\phi_j(x)|^2}}$$

for $x \in N$. The denominator is never zero, as L is basepoint-free.

This definition depends on the choice of the line bundle L, but there exists some $C > 0$ such that

$$C^{-1} d_{\mathcal{I}}(x) \leq d'_{\mathcal{I}}(x) \leq C d_{\mathcal{I}}(x)$$

for all $x \in N$, where $d'_{\mathcal{I}}$ is the function defined for another choice of L.

The Proximity Function

Now define the scheme $Y = (\mathrm{Supp}\mathcal{O}_N/\mathcal{I}, \mathcal{O}_N/\mathcal{I})$, which is a closed subscheme of N (since the sheaf is coherent). Denote $d_Y(x) := d_{\mathcal{I}}(x)$ as we defined earlier, and $\phi_{\mathcal{I}}(x) := -\log d_{\mathcal{I}}(x)$ the proximity potential of $\mathcal{I}$.

Let $f : \mathbb{C}^m \to N$ be a meromorphic mapping such that $f(\mathbb{C}^m) \nsubseteq \mathrm{Supp}\mathcal{O}_N/\mathcal{I}$. Then, $f^*\phi_{\mathcal{I}}(z) = \xi_1(z) - \xi_2(z)$ is written as the difference of two plurisubharmonic functions. (cf. 2.3.25, 2.3.27.)

The Proximity Function

Now we are ready to define the proximity function for ideal sheaves.

Definition (Proximity function for $\mathcal{I}$)

Let $\mathcal{I}$ be an ideal sheaf. Define

$$m_f(r, \mathcal{I}) := \int_{||z||=r} \phi_{\mathcal{I}} \circ f(z)\, \gamma(z).$$

The integral is finite and well-defined up to $O(1)$.[2]

[2] Why?

Order function of f w.r.t. $\mathcal{I}$

Using these definitions, define a current $\omega_{\mathcal{I},f} := 2dd^c \log \frac{1}{d_{\mathcal{I}} \circ f(z)}$ on $\mathbb{C}^m$.

Definition (Order function of f w.r.t. $\mathcal{I}$)

We define

$$T(r, \omega_{\mathcal{I},f}) := \int_1^r \frac{dt}{t} \int_{B(t)} \omega_{\mathcal{I},f}$$

to be the order function.

Counting function for $f^*\mathcal{I}$

First, I will write the definition:

Definition

Let Y denote the closed subscheme defined by the ideal sheaf $\mathcal{I}$. Define the counting function as

$$N(r, f^*\mathcal{I}) := \int_1^r \frac{dt}{t^{2m-1}} \int_{f^*Y \cap B(t)} \alpha^{m-1}$$

for $r \geq 1$.

There is some ambiguity in this; first, we did not define f^*Y precisely. The naive inverse image may consist of components of complex codimension greater than 1, but we do not want this. Therefore, we define f^*Y as only counting the components of codimension 1. Then the counting function is well-defined.

First Main Theorem

Theorem

Let $f : \mathbb{C}^m \to M$ be a meromorphic mapping, and $\mathcal{I}$ an ideal sheaf of M such that $f(\mathbb{C}^m) \nsubseteq \mathrm{Supp}\mathcal{O}_M/\mathcal{I}$. Then,

$$T(r, \omega_{\mathcal{I},f}) = N(r, f^*\mathcal{I}) + m_f(r, \mathcal{I}) - m_f(1, \mathcal{I}).$$

Finally, we will look at some properties of the newly defined proximity function.

Some properties of the proximity function

- $\mathcal{I} \subset \mathcal{J}$ implies $m_f(r, \mathcal{J}) \leq m_f(r, \mathcal{I}) + O(1)$. Equivalently, $Y \subset Z$ implies $m_f(r, Y) \leq m_f(r, Z) + O(1)$, which is intuitive.
- $m_f(r, I_1 \otimes I_2) = m_f(r, I_1) + m_f(r, I_2) + O(1)$. Geometrically, as $I \otimes J \cong IJ$ and $V(IJ) = V(I) \cup V(J)$, this makes sense.
- $m_f(r, I_1 + I_2) \neq \min\{m_f(r, I_1), m_f(r, I_2)\} + O(1)$. Again, geometrically $I + J$ corresponds to intersections, hence this also makes sense.

Thank You

Thank you for listening! Any (algebraic) questions?

Sorry that I'm still not used to plurisubharmonic functions and currents.
I'm catching up!