# SOLUTIONS MANUAL TO CA, ATIYAH-MACDONALD

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Rings are always assumed to be commutative with unity.

Contents



## 1. Rings and Ideals

**Problem 1.1.** *Let $x$ be a nilpotent of $A$. Show that $1+x$ is a unit. Deduce that the sum of a nilpotent and a unit is a unit.*

*Proof.* Recall that a unit of a ring is an element $x$ such that there exists $y$ such that $xy = 1$. Notice that $(1+x)(x^{n-1} - x^{n-2} + x^{n-3} - \cdots \pm 1) = x^n \pm 1 = \pm 1$ if $x^n = 0$. We may use the same logic for any unit $u$ so that $(u+x)(x^{n-1} - ux^{n-2} + u^2x^{n-3} - \cdots \pm u^n) = x^n \pm u^{n+1} = u^{n+1}$. Therefore $u+x$ is also a unit. □

**Problem 1.2.** *Let $A$ be a ring, $A[x]$ the ring of polynomials. Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove*

1. *$f$ is a unit in $A[x]$ iff $a_0$ a unit in $A$ and $a_1, \cdots, a_n$ are nilpotent.*
2. *$f$ nilpotent iff $a_i$ are nilpotent.*
3. *$f$ a zero divisor iff there exists $a \neq 0$ in $A$ such that $af = 0$.*
4. *$f$ is primitive if $(a_0, a_1, \ldots, a_n) = (1)$. Prove if $f, g \in A[x]$ then $fg$ primitive iff $f$ and $g$ are primitive.*

*Proof.* 1. Suppose $f$ is a unit. Then, there exists some $g = b_0 + b_1x + \cdots + b_mx^m$ such that $fg = 1$. Writing out the coefficients, it follows that $fg = a_0b_0 + \cdots + (a_nb_{m-1} + a_{n-1}b_m)x^{n+m-1} + a_nb_mx^{n+m} = 1$. Therefore, $a_0$ is a unit, and $a_nb_m = a_nb_{m-1} + a_{n-1}b_m = 0$. We show by induction on $r$, that $a_n^{r+1}b_{m-r} = 0$. For the base case, take $r = 0$, which we just verified. Next, suppose this holds for all $0 \leq r \leq k$. Then, since we know that $a_nb_{m-k-1} + a_{n-1}b_{m-k} + \cdots + a_{n-k-1}b_m = 0$, if we multiply this with $a_n^{k+1}$ we get $a_n^{k+2}b_{m-k-1} + a_n^{k+1}a_{n-1}b_{m-k} + \cdots + a_n^{k+1}a_{n-k-1}b_m = 0$. By assumption, all terms except the first one vanish, so we have $a_n^{k+2}b_{m-k-1} = 0$. Put $k = m-1$ to get $a_n^{m+1}b_0 = 0$, where since $b_0$ is a unit, we may conclude $a_n^{m+1} = 0$. Hence $a_n$ is nilpotent. Notice that

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$-a_nx^ng$ is also nilpotent in $A[x]$, and by Exercise 1 we know that the sum of a nilpotent and a unit is a unit. Therefore, $1-a_nx^ng = (a_0+a_1x+a_2x^2+\cdots+a_{n-1}x^{n-1})g = u$ is a unit in $A[x]$. If we write its inverse as $u'$, then $(a_0+a_1x+\cdots+a_{n-1}x^{n-1})u'g = 1$, where we may use the above results to conclude again that $a_{n-1}$ is a nilpotent.

Conversely, suppose that $a_0$ is a unit in $A$, and $a_1,\ldots,a_n$ are nilpotent. It is clear that each $a_ix^i$ is nilpotent, and the sum of a unit and nilpotent is a unit, so it follows that $f$ is a unit in $A[x]$. □

*Proof.* 2. Suppose $f$ is nilpotent, i.e. $f^k = 0$ for some $k$. Then $f^k = a_n^kx^{nk}+\cdots = 0$, so its leading coefficient is $a_n^k = 0$, hence $a_n$ is nilpotent. Since $a_nx^n$ is also nilpotent, we have $f - a_nx^n = a_{n-1}x^{n-1}+\cdots$ and this also must be nilpotent since the sum of nilpotent elements is again nilpotent. Thus repeat this process until you exhaust all $a_i$.

Conversely, if all $a_i$ are nilpotent, then all $a_ix^i$ are nilpotent, and $f$ is a finite sum of nilpotent elements hence nilpotent itself. □

*Proof.* 3. Suppose $f$ is a zero-divisor, i.e. there exists some nonzero $g$ such that $fg = 0$. Choose such $g = b_0+b_1x+\cdots+b_mx^m$ of least degree $m$. Then we have $a_nb_m = 0$. Therefore, the polynomial $a_ng$ is of degree $< m$, but since $a_ngf = 0$, we must have $a_ng = 0$ or else it would be a contradiction to our minimal degree assumption. This implies $a_nx^ng = 0$, so $fg = (a_0+a_1x+\cdots+a_{n-1}x^{n-1}+a_nx^n)g = (a_0+a_1x+\cdots+a_{n-1}x^{n-1})g = 0$. Now argue as above with our new polynomial of degree $n-1$ to conclude that $a_{n-r}g = 0$ for all $0 \le r \le n$. This implies $a_ib_m = 0$ for all $a_i$, so indeed $b_mf = 0$ where we assumed $g$ to be of least degree, hence $b_m \neq 0$. The converse is trivial. □

*Proof.* 4. Let $f = a_0+a_1x+\cdots+a_nx^n$ and $g = b_0+b_1x+\cdots+b_mx^m$. Then $fg = \sum_{k=0}^{m+n}\left(\sum_{i+j=k}a_ib_j\right)x^k$. Since we assumed $fg$ to be primitive, it follows that there exists $c_k$ in $A$ such that $\sum_{k=0}^{m+n}c_k\left(\sum_{i+j=k}a_ib_j\right) = 1$. Rearranging this equality into terms of $a_i$ and $b_j$ respectively would show that $f$ and $g$ are primitive.

To prove the converse, we first assume $f$ and $g$ are primitive, and derive a contradiction when we assume $fg$ is primitive. Write $fg = c_0+c_1x+\cdots+c_{m+n}x^{m+n}$. Then this means that $(c_0,c_1,\ldots,c_{m+n}) \neq A$ is a proper ideal. Denote as $\mathfrak{m}$ the maximal ideal in $A$ containing this ideal, and denote $K = A/\mathfrak{m}$. Consider the canonical map $\phi : A[x] \to K[x]$ given by $x \mapsto x$ and the projection of $A$ to $A/\mathfrak{m}$. Notice that $\phi(fg) = 0$ by how we defined $\mathfrak{m}$. In contrast, since $f$ and $g$ are both primitive, $\phi(f) \neq 0 \neq \phi(g)$. This is because their coefficients generate $A$. If either one is sent to zero, then this would be contradictory to our assumption that both are primitive since then their coefficients would generate an ideal contained in $\mathfrak{m}$. Therefore we have $\phi(fg) = \phi(f)\phi(g) = 0$ but $\phi(f),\phi(g)$ both nonzero. This cannot happen in $K[x]$ which is a unique factorization domain. This proof is based on this MathSE answer. □

**Problem 1.3.** *Generalize the results of Exercise 2 to a polynomial ring $A[x_1,\ldots,x_r]$ in several indeterminates.*

*Proof.* 1. Suppose $f$ is a unit in $A[x_1,\ldots,x_r] = (A[x_1,\ldots,x_{r-1}])[x_r]$. Using results of Exercise 2, if we write $f = a_0+a_1x_r+a_2x_r^2+\cdots+a_nx_r^n$, we know that $a_0$ is a unit in $A[x_1,\ldots,x_{r-1}]$ and $a_1,a_2,\ldots,a_n$ are nilpotents in $A[x_1,\ldots,x_{r-1}]$. Since each $a_1,a_2,\ldots,a_n$ can be written as polynomials in $r-1$ indeterminates and they are nilpotent, we conclude that every coefficient is also a nilpotent by what we will prove below. Turning our attention to $a_0$, since $a_0$ is a unit in $A[x_1,\ldots,x_{r-1}]$ we can repeat this process finitely many times until we reach the case we treated in Exercise 2.

Conversely, suppose the constant term is a unit in $A$, and all other coefficients are nilpotent. Nilpotent coefficients multiplied to indeterminates result in again nilpotent elements in the polynomial ring, hence by Exercise 1, we conclude that $f$ is a sum of a unit and a nilpotent, hence a unit. □

*Proof.* 2. Suppose $f \in A[x_1, \dots, x_r]$ is nilpotent. Write $f = a_0 + a_1h_1 + a_2h_2 + \cdots + a_nh_n$ where $h_i$ are products of distinct combinations of indeterminates $x_1, \dots, x_r$. Since $f$ is nilpotent we have $f^k = a_n^k h_n^k + \cdots = 0$, so its leading coefficient $a_n$ is nilpotent. It follows that $-a_nh_n$ is nilpotent in $A[x_1, \dots, x_r]$, so we may repeat this process for $f - a_nh_n$ and so on.

Conversely, suppose every $a_i$ is nilpotent. Then every $a_ih_i$ is nilpotent, so $f$ must also be nilpotent. □

*Proof.* 3. Suppose $f \in A[x_1, \dots, x_r]$ is a zero divisor. Then there exists nonzero $a \in A[x_1, \dots, x_{r-1}]$ such that $af = 0$, by Exercise 2. If we write $f = a_0 + a_1x_r + \cdots + a_nx_r^n$, it follows that $aa_i = 0$ for all $0 \le i \le n$. By assumption we have $f$ nonzero, so we cannot have $a_i$ zero for all $i$. Hence, $a$ must be a zero divisor in $A[x_1, \dots, x_{r-1}]$, so there must be some nonzero $b \in A[x_1, \dots, x_{r-2}]$ such that $ba = 0$, again by Exercise 2. Repeat this finitely many times. The converse is obvious. □

*Proof.* 4. Let $f = a_0 + a_1p_1 + \cdots + a_np_n$ and $g = b_0 + b_1p_1 + \cdots + b_mp_m$ where the $p_i$ are distinct combinations of indeterminates $x_1, \dots, x_r$. Using this, proceed with the same technique used in Exercise 2. □

**Problem 1.4.** *In the ring $A[x]$, the Jacobson radical is equal to the nilradical.*

*Proof.* Recall that the Jacobson radical is the intersection of all the maximal ideals of $A[x]$. The nilradical is the intersection of all the prime ideals of $A[x]$. The inclusion of the nilradical in the Jacobson radical is obvious since maximal ideals are prime ideals. Now suppose that $f = \sum a_ix^i$ is an element of the Jacobson radical. By Proposition 1.9., $1 - fg$ is a unit in $A[x]$ for any $g$. Hence, if we set $g = x$, $1 - xf$ is a unit in $A[x]$ and by Exercise 2 this means that the $a_i$, $i \ge 0$ are nilpotents. Again by Exercise 2 this implies $f$ being nilpotent. Therefore, $f$ is an element of the nilradical. □

**Problem 1.5.** *Let $A$ be a ring and $A[[x]]$ the ring of formal power series $f = \sum_{n=0}^{\infty} a_nx^n$ with coefficients in $A$. Show*

1. *$f$ a unit in $A[[x]]$ iff $a_0$ a unit in $A$.*
2. *If $f$ is nilpotent, then $a_n$ is nilpotent for all $n \ge 0$. Is the converse true?*
3. *$f$ belongs to the Jacobson radical of $A[[x]]$ iff $a_0$ belongs to the Jacobson radical of $A$.*
4. *The contraction of a maximal ideal $\mathfrak{m}$ of $A[[x]]$ is a maximal ideal of $A$, and $\mathfrak{m}$ is generated by $\mathfrak{m}^c$ and $x$.*
5. *Every prime ideal of $A$ is the contraction of a prime ideal of $A[[x]]$.*

*Proof.* 1. Suppose $f$ is a unit in $A[[x]]$. Then there exists some $g \in A[[x]]$ such that $fg = 1$. By comparing coefficients it follows immediately that $a_0$ is a unit in $A$. Conversely, suppose $a_0$ is a unit of $A$. Then there exists some $b_0 \in A$ such that $a_0b_0 = 1$. Define $b_i$ such that for $g = \sum_{i=0}^{\infty} b_ix^i$ we have $fg = 1$. □

*Proof.* 2. Suppose $f$ is nilpotent. Then $f^k = 0$ for some $k$, so $a_0^k = 0$. Hence $a_0$ is nilpotent, so $f - a_0 = \sum_{n=1}^{\infty} a_nx^n$ is also nilpotent. Thus for any $a_n$, $n \ge 0$, it is nilpotent.

The converse is not true. Define $a_n$ to be required to be raised at an increasing degree in order to be zero. □

*Proof.* 3. For every maximal ideal $\mathfrak{m} \subset A$, construct the formal power series ring $\mathfrak{m}[[x]]$ with coefficients in $\mathfrak{m}$. This is an ideal of $A[[x]]$ since multiplication is given by coefficient operations, and the coefficients of elements of $\mathfrak{m}[[x]]$ are all in $\mathfrak{m}$, which is an ideal. Now suppose we had an ideal $I$ of $A[[x]]$ strictly containing $\mathfrak{m}[[x]]$. This means that there exists some $f \in I$ that has some coefficient $a$ in $A \setminus \mathfrak{m}$. By subtracting irrelevant terms, we may assume $f = ax^k$ for some $k$. The important fact is that we may invert elements of the formal power ring by the identity

$$(1+x)^{-1} = \sum_{i=0}^{\infty} (-1)^i x^i.$$

Hence plug in $x^k - 1$, and we may invert $f$ to obtain $a \in I$. Then $\mathfrak{m}[[x]] \subset (\mathfrak{m}, a, x) \subset I$, but since $(\mathfrak{m}, a) = A$ by maximality, we have $A[[x]] \subset I$ so $I = A[[x]]$. Hence $\mathfrak{m}[[x]]$ is indeed maximal. Since $f$ is in the Jacobson radical of $A[[x]]$, it follows that $f \in \bigcap_{\mathfrak{m} \subset A} \mathfrak{m}[[x]]$ by what we proved above. Hence $a_0$, as a coefficient of $f$, must be in $\bigcap_{\mathfrak{m} \subset A} \mathfrak{m}$ which is the Jacobson radical of $A$.

Conversely, suppose $a_0$ is in the Jacobson radical of $A$. Then, for any $a \in A$, $1 - a_0 a$ must be a unit of $A$. Suppose $g = b_0 + b_1 x + \cdots$. Then $1 - fg = 1 - (a_0 + a_1 x + \cdots)(b_0 + b_1 x + \cdots) = 1 - a_0 b_0 + x(\cdots)$, so by what we proved above, $1 - fg$ is a unit in $A[[x]]$. Since $g$ was arbitrary, we conclude that $f$ belongs to the Jacobson radical of $A[[x]]$. □

*Proof.* 4. Denote $\phi : A \to A[[x]]$ that sends $a$ to $(a, 0, 0, \ldots)$. Suppose $\mathfrak{m} \subset A[[x]]$ is maximal. Then the contraction $\mathfrak{m}^c = \phi^{-1}(\mathfrak{m})$. To show $\mathfrak{m}^c$ is maximal, suppose $\mathfrak{m}^c \subsetneq I \subset A$. Then, since $\phi$ is injective it follows that $\phi(\mathfrak{m}^c) \subsetneq \phi(I)$ in $A[[x]]$. Hence, $(\phi(\mathfrak{m}^c), x) \subsetneq (\phi(I), x)$ in $A[[x]]$. Now we prove $\mathfrak{m} \subset (\phi(\mathfrak{m}^c), x)$.

Suppose not, i.e. there exists some $f \in \mathfrak{m}$ such that some coefficient of $f$ is not in $\mathfrak{m}^c$. By subtracting suitable terms and dividing by suitable powers of $x$, we may assume that the first coefficient is not in $\mathfrak{m}^c$. This implies $\phi^{-1}(f) \notin \mathfrak{m}^c$, but since $\phi^{-1}(\mathfrak{m}) = \mathfrak{m}^c$ this cannot happen. Hence $\mathfrak{m} \subset (\phi(\mathfrak{m}^c), x)$, and consequently $\mathfrak{m} \subsetneq (\phi(I), x)$. Since $\mathfrak{m}$ is maximal, we have $(\phi(I), x) = A[[x]]$ and thus $I = A$. Therefore $\mathfrak{m}^c$ is maximal.

To show that $\mathfrak{m}$ is generated by $\mathfrak{m}^c$ and $x$, we show the opposite inclusion of $\mathfrak{m} \supset (\phi(\mathfrak{m}^c), x)$. This follows from the fact that $\phi(\mathfrak{m}^c) = \phi(\phi^{-1}(\mathfrak{m})) \subset \mathfrak{m}$ so $(\phi(\mathfrak{m}^c), x) \subset (\mathfrak{m}, x)$. If $x \notin \mathfrak{m}$, then this would mean $\mathfrak{m} \subsetneq (\mathfrak{m}, x)$ which contradicts the maximality assumption. Hence, $x \in \mathfrak{m}$ and $(\mathfrak{m}, x) = \mathfrak{m}$, so it follows that $(\phi(\mathfrak{m}^c), x) \subset \mathfrak{m}$. Together with the inclusion proved above, we conclude that $\mathfrak{m} = (\phi(\mathfrak{m}^c), x) = (\mathfrak{m}^c, x)$. □

*Proof.* 5. Suppose $\mathfrak{p} \subset A$ is a prime ideal. Consider the ring $\mathfrak{p}[[x]]$. Suppose $fg \in \mathfrak{p}[[x]]$. Then by definition, it holds that all coefficients of $fg$ are in $P$. If we write $f = a_0 + a_1 x + \cdots$ and $g = b_0 + b_1 x + \cdots$, then $fg = a_0 b_0 + (a_1 b_0 + a_0 b_1)x + (a_2 b_0 + a_1 b_1 + a_0 b_2)x^2 + \cdots$ where every coefficient is an element of $\mathfrak{p}$. Starting from the first coefficient, we have either $a_0 \in \mathfrak{p}$ or $b_0 \in \mathfrak{p}$. From this we may move on to the second coefficient and also decide either $a_i$ or $b_i$, $i = 1, 2$ are elements of $\mathfrak{p}$. If we repeat this process indefinitely, it follows that either every $a_i \in \mathfrak{p}$ or every $b_i \in \mathfrak{p}$, or both. Hence either $f$ or $g$ is in $\mathfrak{p}[[x]]$, so $\mathfrak{p}[[x]]$ is a maximal ideal of $A[[x]]$. It is obvious that $\mathfrak{p}$ is a contraction of $\mathfrak{p}[[x]]$. □

**Problem 1.6.** *A ring $A$ is such that every ideal not contained in the nilradical contains a nonzero idempotent. Prove that the nilradical and Jacobson radical of $A$ are equal.*

*Proof.* The inclusion of the nilradical into the Jacobson radical is trivial. We show that the Jacobson radical is contained in the nilradical. Suppose $a \in A$ is an element of the Jacobson radical. We prove by contradiction; assume $a$ is not nilpotent. Then, it follows that $(a)$ is an ideal of $A$ that is not contained in the nilradical. Hence, $(a)$ contains some nonzero idempotent $e$, say $e = ra$ for some $r \in A$. Then it follows that $ra(ra - 1) = 0$. Since $a$ is in the Jacobson radical, we must have $1 - ax$ a unit for every $x \in A$. In this case $x = r$, so $ra - 1$ is a unit. Hence $ra = 0$, but $e \neq 0$ by assumption. Therefore $a$ must be nilpotent. □

**Problem 1.7.** *Let $A$ be a ring in which every element $x$ satisfies $x^n = x$ for some $n > 1$, dependent on $x$. Show that every prime ideal in $A$ is maximal.*

*Proof.* Suppose $\mathfrak{p} \subset A$ is a prime ideal. If $\mathfrak{p} \subsetneq I \subset A$ for an ideal $I$, pick some element $a \in I \setminus \mathfrak{p}$. Then it follows that $\mathfrak{p} \subsetneq (\mathfrak{p}, a) \subset A$, and suppose $a^n = a$. Hence $a^{n-1}(a-1) = 0$. Since we picked $a \in I \setminus \mathfrak{p}$, it follows that $a^2 \in I \setminus \mathfrak{p}$; suppose not, that $a^2 \in \mathfrak{p}$. Then by the property of prime ideals, $a \in \mathfrak{p}$ which is a contradiction. We may use induction on this fact to conclude that $a^{n-1} \in I \setminus \mathfrak{p}$. Then since $0 \in \mathfrak{p}$ and $a^{n-1} \notin \mathfrak{p}$, we must have $a - 1 \in \mathfrak{p}$ by prime property. Therefore, if we write $a = 1 + p$ for $p \in \mathfrak{p}$, we have $(\mathfrak{p}, a) = (\mathfrak{p}, 1 + p) = (\mathfrak{p}, 1) = A$. Therefore $A \subset I \subset A$, and $I = A$ hence $\mathfrak{p}$ is maximal. □

**Problem 1.8.** *Let $A$ be a nonzero ring. Show that the set of prime ideals of $A$ has minimal elements with respect to inclusion.*

*Proof.* For integral domains, this is just $(0)$. In the general case, consider the set $S$ of prime ideals of $A$. If this is empty, then it is vacuously true. Suppose the set is nonempty, and suppose it has a non-terminating descending chain of prime ideals

$$\cdots, \subsetneq \mathfrak{p}_{-3} \subsetneq \mathfrak{p}_{-2} \subsetneq \mathfrak{p}_{-1} \subsetneq \mathfrak{p}_0 \subset A.$$

Note that $\mathfrak{P} := \bigcap_i \mathfrak{p}_i$ is an ideal, and we want to show that it is prime. Suppose $ab \in \mathfrak{P}$ but neither $a \in \mathfrak{P}$ nor $b \in \mathfrak{P}$. Then we can find some $\mathfrak{p}_i$ such that both $a, b \in A \setminus \mathfrak{p}_i$. Note that $A \setminus \mathfrak{p}_i$ is a multiplicatively closed set; if $x, y \notin \mathfrak{p}_i$, then $xy \notin \mathfrak{p}_i$. This is just the contrapositive of the definition of a prime ideal. Hence $ab \in A \setminus \mathfrak{p}_i$, which is a contradiction. Therefore, we must have either $a \in \mathfrak{P}$ or $b \in \mathfrak{P}$. Thus $\mathfrak{P}$ is a prime ideal. The descending chain we provided was arbitrary, so every non-terminating descending chain has a lower bound, namely $\mathfrak{P}$. (Beware, $\mathfrak{P}$ is dependent on the choice of the chain.) Hence, by Zorn's Lemma, every such chain has a minimal prime ideal.

If the chain terminates in finite steps in the first place, then pick the smallest prime ideal; there is nothing else to show. □

**Problem 1.9.** *Let $\mathfrak{a}$ be an ideal $\neq (1)$ in a ring $A$. Show that $\mathfrak{a} = r(\mathfrak{a})$ iff $\mathfrak{a}$ is an intersection of prime ideals.*

*Proof.* Suppose $\mathfrak{a} = \bigcap_i \mathfrak{p}_i$, and suppose $x^n \in \mathfrak{a}$. Then $x^n \in \mathfrak{p}_i$ for all $i$, where for each $i$ we have $x \in \mathfrak{p}_i$. Therefore $x \in \bigcap_i \mathfrak{p}_i = \mathfrak{a}$, so $r(\mathfrak{a}) = \mathfrak{a}$. Conversely, suppose $\mathfrak{a}$ is equal to its radical. We claim that the radical of $\mathfrak{a}$ is the intersection $\mathfrak{P} := \bigcap_i \mathfrak{p}_i$ of all prime ideals $\mathfrak{a} \subset \mathfrak{p}_i$. By definition of $r(\mathfrak{a})$, we have $r(\mathfrak{a}) \subset \mathfrak{P}$ since if $x \notin \mathfrak{P}$, then there exists some $\mathfrak{p}_i$ such that $x \in A \setminus \mathfrak{p}_i$, so $x^k \in A \setminus \mathfrak{p}_i$ for all $k > 0$, hence $x \notin r(\mathfrak{a})$. Thus, for every $a \notin r(\mathfrak{a})$ we must find a prime ideal containing $r(\mathfrak{a})$ and not containing $a$. This would imply $a \notin r(\mathfrak{a}) \Rightarrow a \notin \mathfrak{P}$, which is equivalent to showing $\mathfrak{P} \subset r(\mathfrak{a})$.

Since $r(\mathfrak{a})$ is an ideal of $A$, we may consider the quotient ring $A/r(\mathfrak{a})$, and the projection $\phi : A \to A/r(\mathfrak{a})$. Fix an element $x \notin r(\mathfrak{a})$ and consider the localization $\psi : A/r(\mathfrak{a}) \to (A/r(\mathfrak{a}))_{[x]}$ where $[x]$ is the class of $x$ in $A/r(\mathfrak{a})$. Note that $(A/r(\mathfrak{a}))_{[x]} \neq 0$, since otherwise, we would have $0 = 1$ which means $0 = [x]^n = [x^n]$ for some $n$, so $x^n \in r(\mathfrak{a})$ which implies $x^{nm} \in \mathfrak{a}$ which implies $x \in r(\mathfrak{a})$, contrary to our assumption. Since $(A/r(\mathfrak{a}))_{[x]}$ is nonzero, it has a maximal ideal $\mathfrak{m}$, which is consequently prime. The inverse image $\mathfrak{p}_x := (\psi \circ \phi)^{-1}(\mathfrak{m})$ is a prime ideal of $A$ containing $r(\mathfrak{a})$ and not containing $x$. Therefore we have proved $\mathfrak{P} \subset r(\mathfrak{a})$, which combined with the above gives $\mathfrak{P} = r(\mathfrak{a})$. This proof is based on this MathSE answer. □

**Problem 1.10.** *Let $A$ be a ring, $\mathfrak{R}$ its nilradical. Show that the following are equivalent:*

1. *$A$ has exactly one prime ideal;*
2. *every element of $A$ is either a unit or nilpotent;*
3. *$A/\mathfrak{R}$ is a field.*

*Proof.* Suppose $A$ has exactly one prime ideal. Recall that $\mathfrak{R} = \bigcap_i \mathfrak{p}_i$. Since $A$ has one prime ideal, $\mathfrak{R} = \mathfrak{p}$. Thus, if $a \in \mathfrak{p}$ then it is automatically a nilpotent. Suppose $a \in A \setminus \mathfrak{p}$. If $a$ is not a unit, then $(\mathfrak{p}, a)$ is an ideal strictly containing $\mathfrak{p}$, but not containing a unit, hence a proper ideal of $A$. This cannot happen since $\mathfrak{p}$ is the unique prime ideal, hence maximal.

Suppose that every element of $A$ is either a unit or a nilpotent. The nonzero elements of $A/\mathfrak{R}$ are of the form $a+\mathfrak{R}$ where $a$ is a unit in $A$. Suppose $ab = 1$, then $(a+\mathfrak{R})(b+\mathfrak{R}) = ab + \mathfrak{R} = 1 + \mathfrak{R}$. Therefore $A/\mathfrak{R}$ is a field.

Suppose $A/\mathfrak{R}$ is a field. If $A = 0$, then there is nothing to show. Suppose $A \neq 0$. There exists a maximal ideal $\mathfrak{m} \subset A$. Since maximal ideals are prime, and $\mathfrak{R} = \bigcap_i \mathfrak{p}_i$, we have $\mathfrak{R} \subset \mathfrak{m}$. Suppose $x \notin \mathfrak{R}$. Since $A/\mathfrak{R}$ is a field, $x + \mathfrak{R}$ has a multiplicative inverse $x^{-1} + \mathfrak{R}$. Therefore the ideal $(x, \mathfrak{R}) = (x(x^{-1} + r), \mathfrak{R}) = (1, \mathfrak{R}) = A$ for some $r \in \mathfrak{R}$. Thus $x$ cannot be in $\mathfrak{m}$ since otherwise, $\mathfrak{m}$ would be the whole ring. Therefore we have proved $\mathfrak{m} \subset \mathfrak{R}$, so $\mathfrak{m} = \mathfrak{R} = \bigcap_i \mathfrak{p}_i$. Having prime ideals other than $\mathfrak{m}$ would not make sense. □

**Problem 1.11.** *In a Boolean ring $A$ show that*

1. $2x = 0$ *for all* $x \in A$*;*
2. *every prime ideal* $\mathfrak{p}$ *is maximal, and* $A/\mathfrak{p}$ *is a field with two elements*
3. *every finitely generated ideal in* $A$ *is principal.*

*Proof.* 1. If $A = 0$ then there is nothing to prove. Suppose $0 \neq 1 \in A$. Consider $1 + 1 \in A$. Since $x^2 = x$ for all $x \in A$, it follows that $(1 + 1)^2 = 1 + 1$. The LHS is equal to $1^2 + 2 + 1^2 = 1 + 2 + 1$. Therefore it follows that $2 = 0$. The rest follows. □

*Proof.* 2. The first fact follows directly from Exercise 7. Since $\mathfrak{p}$ is maximal, $A/\mathfrak{p}$ is indeed a field. For every $x \in A$, we have $x(1 - x) = 0$, so either $x \in \mathfrak{p}$, or if $x \notin \mathfrak{p}$ then by the equation above we have $1 - x \in \mathfrak{p}$. Thus the quotient ring $A/\mathfrak{p}$ is a field with two elements. □

*Proof.* 3. Let $I \subset A$ be an ideal generated by 2 elements, say $I = (x, y)$. Then consider the element $z = x+y+xy \in I$. It follows that $zx = x+xy+xy = x$, and $zy = xy+y+xy = y$. Hence if $a \in (x, y)$, then $a$ must be of the form $a_1x + a_2y$ for $a_i \in A$, which is just $za$. Therefore $a \in (z)$. Conversely, if $b \in (z)$ then it is obvious that $b \in (x, y)$. Thus $(x, y) = (x+y+xy)$. Use induction on the number of generators, see this MathSE answer for the original proof. □

**Problem 1.12.** *A local ring contains no idempotent* $\neq 0, 1$.

*Proof.* Suppose $A$ is a local ring and $\mathfrak{m}$ its unique maximal ideal. Suppose $e \neq 0, 1$ is an idempotent. Then, $e(e - 1) = 0$, and since $e \neq 0, 1$ it follows that $e$ and $e - 1$ are both zero divisors. Ideals generated by zero divisors cannot contain a unit, hence $e$ and $e - 1$ both must be in $\mathfrak{m}$, but then $e - (e - 1) = 1 \in \mathfrak{m}$ which doesn't make sense. □

**Problem 1.13.** *Let $K$ be a field and let $\Sigma$ be the set of all irreducible monic polynomials $f$ in one indeterminate with coefficients in $K$. Let $A$ be the polynomial ring over $K$ generated by indeterminates $x_f$, one for each $f \in \Sigma$. Let $\mathfrak{a}$ be the ideal of $A$ generated by the polynomials $f(x_f)$ for all $f \in \Sigma$. Show that $\mathfrak{a} \neq (1)$.*

*Proof.* Suppose $1 \in \mathfrak{a}$. Then there must exist some finite sum $\sum_{i=1}^{n} r_i f_i(x_i) = 1$ for $r_i \in A$. Since $A$ is a polynomial ring over $K$, and every $f_i$ is monic, it follows that all $r_i$ must be zero, which implies $0 = 1$. □

**Problem 1.14.** *In a ring $A$, let $\Sigma$ be the set of all ideals in which every element is a zero divisor. Show that the set $\Sigma$ has maximal elements and that every maximal element of $\Sigma$ is a prime ideal. Hence the set of zero divisors in $A$ is a union of prime ideals.*

*Proof.* Order $\Sigma$ by set-theoretic inclusion. If we write $S$ as the set of all zero divisors of $A$, then every chain of $\Sigma$ is bounded above by $S$.[1] □

[1]?

**Problem 1.15.** *Let $A$ be a ring and let $X$ be the set of all prime ideals of $A$. For each subset $E$ of $A$, let $V(E)$ denote the set of all prime ideals of $A$ which contain $E$. Prove that*

1. *if $\mathfrak{a}$ is the ideal generated by $E$, then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.*
2. *$V(0) = X$, $V(1) = \emptyset$*
3. *if $(E_i)$ is any family of subsets of $A$, then*
$$V(\bigcup_i E_i) = \bigcap_i V(E_i).$$
4. *$V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{ab}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for any ideals $\mathfrak{a}, \mathfrak{b}$ of $A$.*

*Proof.* 1. $V(\mathfrak{a}) \subset V(E)$ is obvious. Suppose $\mathfrak{p} \in V(E)$ i.e. $\mathfrak{p} \supset E$. Then $\mathfrak{p} \supset (E)$ since $\mathfrak{p}$ is an ideal. Now suppose $\mathfrak{p} \in V(\mathfrak{a})$. Then $\mathfrak{p} \supset \mathfrak{a}$, and if $x \in A$ is such that $x^n \in \mathfrak{a}$, then since $\mathfrak{a} \subset \mathfrak{p}$ we have $x \in \mathfrak{p}$. Hence $r(\mathfrak{a}) \subset \mathfrak{p}$. Conversely, suppose $\mathfrak{p} \in V(r(\mathfrak{a}))$. Then obviously $\mathfrak{p} \supset r(\mathfrak{a}) \supset \mathfrak{a}$. □

*Proof.* 2. $V(0) = \{\mathfrak{p} \in X \mid \mathfrak{p} \supset 0\} = X$, $V(1) = \{\mathfrak{p} \in X \mid \mathfrak{p} \supset (1)\} = \emptyset$. □

*Proof.* 3. Suppose $\mathfrak{p} \in V(\bigcup E_i)$. Then $\mathfrak{p} \supset \bigcup E_i$, so $\mathfrak{p} \supset E_i$ for all $i$, hence $\mathfrak{p} \in \bigcap V(E_i)$. Conversely, if $\mathfrak{p} \in \bigcap V(E_i)$, then $\mathfrak{p} \supset E_i$ for all $i$, so $\mathfrak{p} \supset \bigcup E_i$ and thus $\mathfrak{p} \in V(\bigcup E_i)$. □

*Proof.* 4. We first prove $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$. Suppose $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$. Then, $\mathfrak{p} \supset \mathfrak{a} \cap \mathfrak{b}$. Since $\mathfrak{a} \supset \mathfrak{ab}$ and $\mathfrak{b} \supset \mathfrak{ab}$, it follows that $\mathfrak{a} \cap \mathfrak{b} \supset \mathfrak{ab}$. Hence $\mathfrak{p} \supset \mathfrak{ab}$, so $\mathfrak{p} \in V(\mathfrak{ab})$. Conversely, suppose $\mathfrak{p} \supset \mathfrak{ab}$. We need to show that $\mathfrak{p} \supset \mathfrak{a} \cap \mathfrak{b}$. If we have some $x \in \mathfrak{a} \cap \mathfrak{b}$, then it follows that $x \in \mathfrak{ab}$, so $\mathfrak{p} \supset \mathfrak{a} \cap \mathfrak{b}$.[2]

Now suppose $\mathfrak{p} \supset \mathfrak{ab}$. Suppose we have $a, b \notin \mathfrak{p}$ where $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Then $ab \notin \mathfrak{p}$, so $\mathfrak{p} \supsetneq \mathfrak{ab}$. Hence $\mathfrak{p} \supset \mathfrak{a}$ or $\mathfrak{p} \supset \mathfrak{b}$. Conversely, suppose $\mathfrak{p} \supset \mathfrak{a}$ or $\mathfrak{p} \subset \mathfrak{b}$. Without loss of generality, suppose $\mathfrak{p} \supset \mathfrak{a}$. Then $\mathfrak{p} \supset \mathfrak{a} \supset \mathfrak{ab}$. □

**Problem 1.16.** *Draw pictures of* $\operatorname{Spec}\mathbb{Z}$, $\operatorname{Spec}\mathbb{R}$, $\operatorname{Spec}\mathbb{C}[x]$, $\operatorname{Spec}\mathbb{R}[x]$, $\operatorname{Spec}\mathbb{Z}[x]$.

*Proof.* □

**Problem 1.17.** *For each $f \in A$, let $X_f$ denote the complement of $V(f)$ in $X = \operatorname{Spec} A$. The sets $X_f$ are open. Show that the form a basis of open sets for the Zariski topology, and that*

1. *$X_f \cap X_g = X_{fg}$;*
2. *$X_f = \emptyset \Leftrightarrow f$ is nilpotent;*
3. *$X_f = X \Leftrightarrow f$ is a unit;*
4. *$X_f = X_g \Leftrightarrow r((f)) = r((g))$;*
5. *$X$ is quasicompact*
6. *More generally, each $X_f$ is quasicompact*
7. *An open subset of $X$ is quasicompact if and only if it is a finite union of sets $X_f$.*

*Proof.* First we show that the $X_f$ form a basis. To do this, for any open set in the topology, and any point in that open set, we must find a basis element that contains the point and is contained in the open set. For a ring $A$, suppose we have any subset $E \subset A$. Then the open set corresponding to $E$ is $X - V(E)$ where $X = \operatorname{Spec} A$. Suppose $X - V(E)$ is nonempty, so we have some $\mathfrak{p} \in X - V(E)$. What this means is that $\mathfrak{p}$ does not contain $E$ as a subset. We want to find some $X_f = X - V(f)$ that contains $\mathfrak{p}$, and is contained in $X - V(E)$, in other words we want to find some $f \in A$ such that $f \notin \mathfrak{p}$, and that $V(E) \subset V(f)$. For this to happen, we must find $f$ in $E$. Since $\mathfrak{p}$ does not contain $E$, there exists some element in $E$ that is not contained in $\mathfrak{p}$. Pick such element as $f$, and then we are done. □

[2] ?

*Proof.* 1. $X_f = X - V(f)$, and $X_g = X - V(g)$ so $X_f \cap X_g = X - (V(f) \cup V(g)) = X - V(fg) = X_{fg}$. $\square$

*Proof.* 2. $X_f = \emptyset \Leftrightarrow V(f) = X \Leftrightarrow f \in \mathfrak{R}$ where $\mathfrak{R}$ is the nilradical. $\square$

*Proof.* 3. $X_f = X \Leftrightarrow V(f) = \emptyset \Leftrightarrow (f) = A$. $\square$

*Proof.* 4. $X_f = X_g \Leftrightarrow V(f) = V(g)$ which means that $\mathfrak{p} \supset (f)$ if and only if $\mathfrak{p} \supset (g)$. Recall that the radical of an ideal is the intersection of all prime ideals containing that ideal, so it follows that $r((f)) = r((g))$. Conversely, suppose $r((f)) = r((g))$. Then if $\mathfrak{p} \supset (f)$, then $\mathfrak{p} \supset r((f)) = r((g)) \supset (g)$, and vice versa. $\square$

*Proof.* 5. Since we proved that the $X_f$ form a basis for the Zariski topology, we may cover $X$ with $X_{f_i}$. Then it follows that $X = \bigcup_i X_{f_i} = \bigcup_i (X - V(f_i)) = X - \bigcap_i V(f_i)$, so $\bigcap_i V(f_i) = V(\bigcup_i (f_i)) = \emptyset$. Thus, no prime ideal of $A$ contains $\bigcup_i (f_i)$, hence $\bigcup_i (f_i) = (1)$. Now consider the sum $\sum_i (f_i)$ of the ideals $(f_i)$. By definition, $\sum_i (f_i)$ contains $\bigcup_i (f_i)$ and its elements are finite linear combinations of elements of $(f_i)$. Therefore, $\sum_i (f_i) = (1)$, so there exists a finite equation $\sum_j g_j f_j = 1$. Hence the $f_j$ generate the unit ideal, so $\bigcup_j X_{f_j} = \bigcup_j (X - V(f_j)) = X - \bigcap_j V(f_j) = X - V(\bigcup_j (f_j)) = X - V(\sum_j (f_j)) = X - V((1)) = X$. $\square$

*Proof.* 6. We show each $X_f$ is quasicompact. By the same logic as above, we may cover $X_f$ with basic open sets. Suppose we are given an open covering $X_f = \bigcup_i X_{f_i}$ by basic open sets $X_{f_i}$. Since $X_f \cap X_g = X_{fg}$, it suffices to show for $X_f = \bigcup_i X_{ff_i}$. This is equivalent to $V(f) = V(\bigcup_i (ff_i)) = V(\sum_i (ff_i))$, which is again equivalent to $r((f)) = r(\sum_i (ff_i))$. Thus it follows that $f^n = \sum_{i=1}^k a_i (ff_i)$ for $a_i \in A$ and some $n > 0$. We claim that $r((f)) = r(\sum_{i=1}^k (ff_i))$. Suppose some $a \in r((f))$. Then $a^m = rf$, so $a^{mn} = (rf)^n \in \sum_{i=1}^k a_i (ff_i)$, so $a \in r(\sum_{i=1}^k (ff_i))$. Conversely, if $a \in r(\sum_{i=1}^k (ff_i))$ then $a^m = \sum_{i=1}^k a_i (ff_i) \in (f)$ so $a \in r((f))$. $\square$

*Proof.* 7. Suppose $X$ is quasicompact. We may cover $X$ with basic open sets, then there must be a finite subcover. Conversely, suppose $X = \bigcup_{i=1}^n X_{f_i}$. Since each $X_{f_i}$ is quasicompact, it follows that $X$ is also quasicompact. $\square$

**Problem 1.18.** *Show that*

1. *the set* $\{x\}$ *is closed in* Spec $A$ *iff* $\mathfrak{p}_x$ *is maximal;*
2. $\overline{\{x\}} = V(\mathfrak{p}_x)$*;*
3. $y \in \overline{\{x\}}$ *iff* $\mathfrak{p}_x \subset \mathfrak{p}_y$*;*
4. $X$ *is a* $T_0$ *space.*

*Proof.* 1. Suppose $\{x\}$ is closed in Spec $A$. This means that $\{x\} = V(E)$ for some $E$, so there is a unique prime ideal $\mathfrak{p}_x$ containing $E$. This means that $\mathfrak{p}_x$ is maximal. Conversely, suppose $\mathfrak{p}_x$ is maximal. Then $V(\mathfrak{p}_x)$ is a point, which is closed. $\square$

*Proof.* 2. The closure of $x$ is the intersection of all closed sets containing $x$. If a prime $\mathfrak{p}$ contains the prime ideal corresponding to $x$, then $\mathfrak{p}$ is in the closure of $x$. This is just $V(\mathfrak{p}_x)$. $\square$

*Proof.* 3. This follows directly from the above. $\square$

*Proof.* 4. A $T_0$ space is where $x \neq y$ implies there exists a closed set separating $x$ and $y$. More precisely, there either exists a closed set containing $x$ and not containing $y$, and vice versa. Since the closure of a point is definitely contained in any closed set containing the point, we may rephrase this into $x \neq y$ implies $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$. Take the contrapositive, and this becomes $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$ implies $x = y$. By using the result of 3, note that the condition implies $\mathfrak{p}_x = \mathfrak{p}_y$, which is just $x = y$. Thus $X$ is indeed a $T_0$ space. $\square$

**Problem 1.19.** *Show that* $\operatorname{Spec} A$ *is irreducible if and only if the nilradical of* $A$ *is a prime ideal.*

*Proof.* Denote as $\mathfrak{R}$ the nilradical. Suppose $X$ is irreducible, and suppose $ab \in \mathfrak{R}$. This means that for some $n > 0$ we have $(ab)^n = 0$. Suppose both $X_a$ and $X_b$ are nonempty. Then by irreducibility, $X_a \cap X_b = X_{ab} = X_0 = X - V((0)) \neq \emptyset$, but this is absurd since every prime ideal must contain the zero ideal. Therefore either $X_a$ or $X_b$ must be empty, hence $V(a) = V(0)$ or $V(b) = V(0)$ which is equivalent to $a \in \mathfrak{R}$ or $b \in \mathfrak{R}$. Thus the nilradical is prime.

Conversely, suppose the nilradical is prime. Suppose $X_a \cap X_b = \emptyset$. Then, $X_a \cap X_b = X_{ab} = \emptyset = X_0$, so $ab \in \mathfrak{R}$. By the prime condition, we must have either $a \in \mathfrak{R}$ or $b \in \mathfrak{R}$. Thus, either $X_a = \emptyset$ or $X_b = \emptyset$. Hence $X$ is irreducible. □

**Problem 1.20.** *Let* $X$ *be a topological space.*

1. *If* $Y$ *is an irreducible subspace of* $X$*, then* $\overline{Y}$ *in* $X$ *is irreducible.*
2. *Every irreducible subspace of* $X$ *is contained in a maximal irreducible subspace.*
3. *The maximal irreducible subspaces of* $X$ *are closed and cover* $X$*. (They are called the irreducible components.) What are the irreducible components of a Hausdorff space?*
4. *If* $A$ *is a ring and* $X = \operatorname{Spec} A$*, then the irreducible components of* $X$ *are the closed sets* $V(\mathfrak{p})$ *where* $\mathfrak{p}$ *is the minimal prime ideal of* $A$*.*

*Proof.* 1. We show the contrapositive. Suppose $\overline{Y} \subset X$ is not irreducible. Then there exists some nonempty $U \cap \overline{Y}$ and $V \cap \overline{Y}$ for $U, V$ open sets in $X$ such that $U \cap V \cap \overline{Y} = \emptyset$. Since $U \cap \overline{Y}$ is nonempty, there exists some $p \in U \cap \overline{Y}$. Since $p \in \overline{Y}$, this implies that any neighborhood of $p$ intersects $Y$. Since $U$ is a neighborhood of $p$, this implies $U \cap Y \neq \emptyset$. The same holds for $V \cap Y \neq \emptyset$. Since $U \cap V \cap Y \subset U \cap V \cap \overline{Y} = \emptyset$ and both $U \cap Y$ and $V \cap Y$ are nonempty, $Y$ is not irreducible. □

*Proof.* 2. Suppose we have some irreducible subspace $A \subset X$. Consider the partially ordered set $\{A_\alpha\}_\leq$ of irreducible subspaces containing $A$, ordered by inclusion. We show every totally ordered subset has an upper bound. Given $\{A_i\}_{i \in I}$ a totally ordered subset, we claim that $\bigcup_i A_i$ is an upper bound. If $U, V$ are nonempty open subsets of $\bigcup_i A_i$, there exists $i, j$ such that $U \cap A_i \neq \emptyset$ and $V \cap A_j \neq \emptyset$. Without loss of generality, we may assume $A_j \subset A_i$. Then by irreducibility of $A_i$, it follows that $U \cap A_i$ and $V \cap A_i$ must intersect. Hence, $U \cap \bigcup_i A_i$ and $V \cap \bigcup_i A_i$ must intersect, showing irreducibility of $\bigcup_i A_i$. Therefore, every totally ordered subset has an upper bound. By Zorn's lemma, the partially ordered set $\{A_\alpha\}_\leq$ has a maximal element. □

*Proof.* 3. Suppose $M$ is a maximal irreducible subspace. Then, $M \subset \overline{M} = M$, so $M$ is indeed closed. Now consider singleton sets $\{x\}$ for $x \in X$. Singleton sets are irreducible in the subspace topology, obviously. By 1, we know that $\overline{\{x\}}$ is irreducible, which is contained in some maximal irreducible subspace, so the maximal irreducible subspaces cover $X$. For a Hausdorff space, if a subspace has more than one element, then it cannot be irreducible, hence the irreducible components are singletons. □

*Proof.* 4. [3] □

**Problem 1.21.** *Let* $\phi : A \to B$ *be a ring homomorphism and let* $X = \operatorname{Spec} A$*, and* $Y = \operatorname{Spec} B$*.* $\phi$ *induces a mapping* $\phi^* : Y \to X$*. Show that*

1. *If* $f \in A$ *then* $(\phi^*)^{-1}(X_f) = Y_{\phi(f)}$*.*
2. *If* $\mathfrak{a}$ *is an ideal of* $A$*, then* $(\phi^*)^{-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$*.*
3. *If* $\mathfrak{b}$ *is an ideal of* $B$*, then* $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{b}^c)$*.*

---
[3] TODO

4. *If $\phi$ is surjective, then $\phi^*$ is a homeomorphism opf $Y$ onto the closed subset $V(\ker\phi)$ of $X$.*
5. *If $\phi$ is injective, then $\phi^*(Y)$ is dense in $X$. More precisely, $\phi^*(Y)$ is dense in $X$ iff $\ker\phi \subset \mathfrak{R}$.*
6. *Let $\psi : B \to C$ be another ring homomorphism. Then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.*
7. *Let $A$ be an integral domain with just one nonzero prime ideal $\mathfrak{p}$. Let $K$ be the field of fractions of $A$. Let $B = (A/\mathfrak{p}) \times K$. Define $\phi : A \to B$ by $\phi(x) = (\overline{x}, x)$ where $\overline{x}$ is the image of $x$ in $A/\mathfrak{p}$. Show that $\phi^*$ is bijective, but not a homeomorphism.*

*Proof.* 1. ($\subset$) Suppose $\mathfrak{p} \in (\phi^*)^{-1}(X_f)$. Then, $\phi^*(\mathfrak{p}) \in X_f$, and $\phi^{-1}(p) \in X - V(f)$, so $\phi^{-1}(\mathfrak{p}) \notin V(f)$. This means that $\phi^{-1}(\mathfrak{p}) \not\supseteq (f)$, which implies the existence of some $rf \notin \phi^{-1}(\mathfrak{p})$. It follows that $\phi(rf) = \phi(r)\phi(f) \notin \mathfrak{p}$, so $(\phi(f)) \not\subseteq \mathfrak{p}$. Therefore $\mathfrak{p} \notin V(\phi(f))$ hence $\mathfrak{p} \in Y - V(\phi(f))$.

($\supset$) Suppose $\mathfrak{p} \in Y - V(\phi(f))$. Then $\mathfrak{p} \notin V(\phi(f))$ thus $(\phi(f)) \not\subseteq \mathfrak{p}$. This means that $\phi(f) \notin \mathfrak{p}$, since otherwise, it would imply $(\phi(f)) \subset \mathfrak{p}$. Again, $\phi(f) \notin \mathfrak{p}$ implies $f \notin \phi^{-1}(\mathfrak{p})$, which follows from the contrapositive. Therefore the ideal $(f)$ is not contained in $\phi^{-1}(\mathfrak{p})$, so $\phi^{-1}(\mathfrak{p}) = \phi^*(\mathfrak{p}) \notin V(f)$. Thus $\phi^*(\mathfrak{p}) \in X_f$, so $\mathfrak{p} \in (\phi^*)^{-1}(X_f)$. □

*Proof.* 2. ($\subset$) Suppose $\mathfrak{p} \in (\phi^*)^{-1}(V(\mathfrak{a}))$. Then, $\phi^*(\mathfrak{p}) \in V(\mathfrak{a})$. By definition, $\phi^{-1}(\mathfrak{p}) \supset \mathfrak{a}$. so $\mathfrak{p} \supset \phi(\mathfrak{a})$. Since $\mathfrak{p}$ is an ideal, we have $\mathfrak{p} \supset \mathfrak{a}^e$. Therefore $\mathfrak{p} \in V(\mathfrak{a}^e)$.

($\supset$) Conversely, suppose $\mathfrak{p} \in V(\mathfrak{a}^e)$. Then $\mathfrak{p} \supset \mathfrak{a}^e \supset \phi(\mathfrak{a})$. Then, $\phi^{-1}(\mathfrak{p}) \supset \phi^{-1}(\phi(\mathfrak{a})) \supset \mathfrak{a}$, so $\phi^*(\mathfrak{p}) \in V(\mathfrak{a})$. Therefore, $\mathfrak{p} \in (\phi^*)^{-1}(V(\mathfrak{a}))$. □

*Proof.* 3. ($\subset$) Suppose $\mathfrak{p} \in \phi^*(V(\mathfrak{b}))$. Then, $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$ for some $\mathfrak{q} \supset \mathfrak{b}$. Since $\phi^{-1}(\mathfrak{q}) \supset \phi^{-1}(\mathfrak{b})$, it follows that $\mathfrak{p} \supset \phi^{-1}(\mathfrak{b})$. Therefore we have $\mathfrak{p} \in V(\phi^{-1}(\mathfrak{b})) = V(\mathfrak{b}^c)$. Thus, $\phi^*(V(\mathfrak{b})) \subset V(\mathfrak{b}^c)$. Since $V(\mathfrak{b}^c)$ is closed, $\overline{\phi^*(V(\mathfrak{b}))} \subset V(\mathfrak{b}^c)$ follows.

($\supset$) Since $\overline{\phi^*(V(\mathfrak{b}))}$ is closed, we may assume $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{a})$ for some $\mathfrak{a} \subset A$. By what we proved in 2, we have $(\phi^*)^{-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$. Therefore, $V(\mathfrak{a}^e) = (\phi^*)^{-1}(\overline{\phi^*(V(\mathfrak{b}))}) \supset (\phi^*)^{-1}(\phi^*(V(\mathfrak{b}))) \supset V(\mathfrak{b})$. From this, it follows that $\mathfrak{a}^e \subset r(\mathfrak{a}^e) \subset r(\mathfrak{b})$. We claim that $r(\mathfrak{a}) \subset r(\mathfrak{b}^c)$.

Suppose $a \in r(\mathfrak{a})$. Then, $a^m \in \mathfrak{a}$ for some $m > 0$. Therefore, $\phi(a^m) \in \mathfrak{a}^e \subset r(\mathfrak{b})$, so $\phi(a^m)^n \in \mathfrak{b}$ for some $n > 0$. It follows that $a^{mn} \in \mathfrak{b}^c$, so $a \in r(\mathfrak{b}^c)$.

Using this, we conclude that $r(\mathfrak{a}) \subset r(\mathfrak{b}^c)$, i.e. $V(\mathfrak{b}^c) \subset V(\mathfrak{a}) = \overline{\phi^*(V(\mathfrak{b}))}$. This proof is based on this MathSE answer. □

*Proof.* 4. We want to show that the map $\phi^* |_{V(\ker\phi)}: Y \to V(\ker\phi)$ given by $\mathfrak{q} \mapsto \phi^*(\mathfrak{q})$ is a homeomorphism. If $\mathfrak{q}$ is an ideal of $Y$, then $(0) \subset \mathfrak{q}$, so $\ker\phi \subset \phi^{-1}(\mathfrak{q}) = \phi^*(\mathfrak{q})$. Thus, the map above is valid.

**Surjectivity.** Suppose $\mathfrak{p} \in V(\ker\phi)$. We claim that $\phi(\mathfrak{p})$ is a prime ideal of $B$. (We use the fact that the surjective image of an ideal is an ideal, without proof.) Suppose $ab \in \phi(\mathfrak{p})$. Then, $ab = \phi(p)$ for some $p \in \mathfrak{p}$. Also since $\phi$ is surjective, we may find some $x, y \in A$ such that $\phi(x) = a$ and $\phi(y) = b$. Therefore, $\phi(xy - p) = 0$, so we have $xy - p \in \ker\phi \subset \mathfrak{p}$. Thus, $xy \in \mathfrak{p}$, and since $\mathfrak{p}$ is a prime ideal, we either have $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Therefore we either have $a \in \phi(\mathfrak{p})$ or $b \in \phi(\mathfrak{p})$, so $\phi(\mathfrak{p})$ is indeed a prime ideal of $B$. By surjectivity of $\phi$, we have $\phi^*(\phi(\mathfrak{p})) = \mathfrak{p}$.

**Injectivity.** $\phi^{-1}(\mathfrak{p}) = \phi^{-1}(\mathfrak{q})$ implies $\mathfrak{p} = \mathfrak{q}$ since $\phi$ is surjective.

Continuity of the map follows from basic topology. Now we show that the original map, hence the restricted map, is an open map. It is enough to show for basic open sets, so we investigate the image of $Y_b$ for $b \in Y$. Suppose $\mathfrak{p} \in Y_b = Y - V(b)$. The image (of the original $\phi^*$, not the restricted one) is $\phi^*(Y_b) = \{\phi^{-1}(\mathfrak{p}) \mid \mathfrak{p} \not\supseteq (b)\}$. Let $b = \phi(a)$ for some $a \in A$. $\mathfrak{p} \not\supseteq (b)$ implies some $rb \notin \mathfrak{p}$. Let $\phi(r') = r$, then this implies $r'a \notin \phi^{-1}(\mathfrak{p})$, hence $\phi^{-1}(\mathfrak{p}) \not\supseteq (a)$. Thus, the image $\phi^*(Y_b)$ is just the prime ideals of $A$ that do not contain $(a)$, which is equal to $X_a$. Therefore $\phi^*$ is an open map, hence its restriction is too. Therefore $\phi^* |_{V(\ker\phi)}$ is a homeomorphism.

In particular, consider the map $\phi : A \mapsto A/\mathfrak{R}$. By the results above, $\phi^*$ induces a homeomorphism of $\operatorname{Spec} A/\mathfrak{R}$ onto $V(\ker\phi) = V(\mathfrak{R}) = V(0) = \operatorname{Spec} A$. $\square$

*Proof.* 5. Suppose $\phi$ is injective. By the result of 3, we know that $\overline{\phi^*(V(0))} = \overline{\phi^*(Y)} = V((0)^c) = V(\ker\phi) = V(0) = X$, so $\phi^*(Y)$ is dense in $X$.

More precisely, suppose $\overline{\phi^*(Y)} = X$. By 3, $\overline{\phi^*(Y)} = \overline{\phi^*(V(\mathfrak{R}_{\mathfrak{B}}))} = V({\mathfrak{R}_{\mathfrak{B}}}^c) = X = V(0)$, so the radical of $\phi^{-1}(\mathfrak{R}_{\mathfrak{B}})$ is the nilradical $\mathfrak{R}_A$ of $A$. If $a \in \ker\phi$, then $\phi(a) = 0 \in \mathfrak{R}_B$ so $a \in \phi^{-1}(\mathfrak{R}_B) \subset \mathfrak{R}_A$, as desired.

Conversely, suppose $\ker\phi \subset \mathfrak{R}_A$. Then $\overline{\phi^*(Y)} = \overline{\phi^*(\mathfrak{R}_B)} = V(\mathfrak{R}_B^c) = V(\phi^{-1}(\mathfrak{R}_B))$. Suppose $a \in \phi^{-1}(\mathfrak{R}_B)$. Then $\phi(a) \in \mathfrak{R}_B$, so $\phi(a)^n = \phi(a^n) = 0$ for some $n > 0$. Therefore $a^n \in \ker\phi \subset \mathfrak{R}_A$, so $a^{nm} = 0$ for some $m > 0$. It follows that $a \in \mathfrak{R}_A$, so we have $\phi^{-1}(\mathfrak{R}_B) \subset \mathfrak{R}_A$, and thus $X = V(\mathfrak{R}_A) \subset V(\phi^{-1}(\mathfrak{R}_B))$. Hence $V(\phi^{-1}(\mathfrak{R}_B)) = \overline{\phi^*(Y)} = X$. $\square$

*Proof.* 6. Suppose $\mathfrak{p}$ a prime ideal of $C$. Then $(\psi\circ\phi)^*(\mathfrak{p}) = (\psi\circ\phi)^{-1}(\mathfrak{p}) = \phi^{-1}\circ\psi^{-1}(\mathfrak{p}) = \phi^{-1}(\psi^{-1}(\mathfrak{p})) = (\phi^*\circ\psi^*)(\mathfrak{p})$. $\square$

*Proof.* 7. We use the fact that prime ideals of $R_1 \times R_2$ are of the form $P_1 \times R_2$ and $R_1 \times P_2$, without proof. Since $\mathfrak{p}$ is maximal in $A$, we know that $A/\mathfrak{p}$ is a field. Therefore, $\operatorname{Spec} B$ as a set is $\{0 \times K, (A/\mathfrak{p}) \times 0\}$. Also, $\operatorname{Spec} A = \{0, \mathfrak{p}\}$. Since $\phi^{-1}(0 \times K) = \mathfrak{p}$, and $\phi^{-1}((A/\mathfrak{p}) \times 0) = 0$, $\phi^*$ is obviously a bijection. However, the prime $(A/\mathfrak{p}) \times 0$ is maximal in $B$, hence closed in $\operatorname{Spec} B$, but $(0)$ is not closed in $\operatorname{Spec} A$. Therefore $\phi^*$ cannot be a homeomorphism. $\square$

**Problem 1.22.** *Let $A = \prod_{i=1}^n A_i$ be the direct product of rings $A_i$. Show that* $\operatorname{Spec} A$ *is the disjoint union of open an closed subspaces $X_i$, where $X_i$ is canonically homeomorphic with* $\operatorname{Spec} A_i$.

*Conversely, let $A$ be any ring. Show that the following are equivalent:*

1. $X = \operatorname{Spec} A$ *is disconnected.*
2. $A \simeq A_1 \times A_2$ *where neither of the rings $A_1, A_2$ is the zero ring.*
3. $A$ *contains an idempotent* $\neq 0, 1$.

*Proof.* 1. Let $B = A_j$, and consider $\phi : A \to B$ the canonical projection. By 21.4, $\phi^*$ is a homeomorphism of $\operatorname{Spec} B$ onto the closed subset $V(\ker\phi) \subset \operatorname{Spec} A$. Since $V(\ker\phi) = V(0 \times \prod_{i\neq j} A_i) = X_j$, all we have to do is show that each $X_j$ are disjoint, and their union is $\operatorname{Spec} A$. Since the prime ideals of $A$ are of the form $P_j \times \prod_{i\neq j} A_i$, a prime of $A$ must be contained in $X_j$. Also, a prime cannot be contained in more than one $X_j$, since then it would imply the prime is $A$, which is absurd. Therefore $\operatorname{Spec} A$ is the disjoint union of spaces homeomorphic to $\operatorname{Spec} A_j$. (Note that this generalizes to infinite products.) $\square$

*Proof.* 2. We show $(1 \Rightarrow 3 \Rightarrow 2)$.

Suppose $X = \operatorname{Spec} A$ is disconnected, and suppose $X = M \sqcup N$ for two nonempty clopen $M$ and $N$. Let $M = V(\mathfrak{a})$ and $N = V(\mathfrak{b})$ for ideals $\mathfrak{a}, \mathfrak{b} \subsetneq A$. Since $M \cap N = \emptyset$, we have $V(\mathfrak{a}) \cap V(\mathfrak{b}) = V(\mathfrak{a}+\mathfrak{b}) = \emptyset$, thus $\mathfrak{a}+\mathfrak{b} = (1)$. Also since $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\cap\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = X$, we know that $r(\mathfrak{a}\mathfrak{b}) = \mathfrak{R}$, the nilradical of $A$. Now suppose we have $a + b = 1$ for $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Since $ab \in \mathfrak{R}$, the identity $1 = (a+b)^n = a^n + b^n + ab(\cdots)$ implies that $a^n + b^n$ is a unit in $A$, by Exercise 1. Hence for some nonzero $u \in A$ we must have $u(a^n + b^n) = 1$. From this, it follows that $ua^n \cdot 1 = ua^n(ua^n + ub^n) = u^2a^{2n} + u^2(ab)^n = u^2a^{2n}$, hence $ua^n(ua^n - 1) = 0$. Similarly, we have $ub^n(ub^n - 1) = 0$. From the fact that $\mathfrak{a}, \mathfrak{b} \subsetneq A$, it follows that neither $ua^n$ nor $ub^n$ can be 1. From this, and the identity $ua^n + ub^n = 1$, neither $ua^n$ nor $ub^n$ can be zero. Thus $A$ contains nonzero nonunit idempotents.

Now, for the sake of brevity call the idempotent element $e$. Obviously, $e + (1-e) = 1$, so $(e) + (1-e) = (1)$. Also $e(e-1) = 0$, so by Proposition 1.10 we conclude that $A \simeq A_1 \times A_2$ for $A_1 = A/(e)$ and $A_2 = A/(1-e)$. $(2 \Rightarrow 1)$ follows directly from what we proved in 1. $\square$

**Problem 1.23.** *Let $A$ be a Boolean ring, and let $X = \operatorname{Spec} A$.*

1. *For each $f \in A$, the set $X_f$ is both open and closed in $X$.*
2. *Let $f_1, \dots, f_n \in A$. Show that $X_{f_1} \cup \dots \cup X_{f_n} = X_f$ for some $f \in A$.*
3. *The sets $X_f$ are the only subsets of $X$ which are both open and closed.*
4. *$X$ is a compact Hausdorff space.*

*Proof.* 1. Since $A$ is Boolean, every element is idempotent. By what we proved in Exercise 22, every nonzero nonunit element $f$ induces a decomposition of $\operatorname{Spec} A$ into $V(f) \sqcup V(1-f)$ where $X_f = X - V(f) = V(1-f)$. Therefore every $X_f$ is clopen. □

*Proof.* 2. $\bigcup_{i=1}^n X_{f_i} = \bigcup_{i=1}^n (X - V(f_i)) = X - \bigcap_{i=1}^n V(f_i) = X - V(\sum_{i=1}^n (f_i))$. By Exercise 11, we know that finitely generated ideals in a Boolean ring are principal. Therefore we may assume $\sum_{i=1}^n (f_i) = (f)$ for some $f \in A$. It follows that $\bigcup_{i=1}^n X_{f_i} = X_f$. □

*Proof.* 3. The proof is given in the text. □

*Proof.* 4. Suppose $\mathfrak{p} \neq \mathfrak{q} \in X$. Then either $\mathfrak{p} - \mathfrak{q}$ or $\mathfrak{q} - \mathfrak{p}$ is nonempty. Without loss of generality, say $f \in \mathfrak{p} - \mathfrak{q}$. Then $\mathfrak{p} \in V(f)$ and $\mathfrak{q} \in X_f$. By 3, $X_f$ is clopen in $X$, and hence $V(f)$ is also clopen. The $X_f$ and $V(f)$ are our desired neighborhoods. (Note that $f$ is nonzero since all prime ideals share 0.) □

*Problems 24 to 28 do not require a proof.*

## 2. Modules

**Problem 2.1.** *Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if $m, n$ are coprime.*

*Proof.* The $\mathbb{Z}$-module $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$ is generated by elements of the form $x \otimes y$ where $x \in \mathbb{Z}/m\mathbb{Z}$ and $y \in \mathbb{Z}/n\mathbb{Z}$. Since $m, n$ are coprime, there exists integers $a, b$ such that $am + bn = 1$. Hence, $bn = 1 - am = 1 \in \mathbb{Z}/m\mathbb{Z}$ and $am = 1 - bn = 1 \in \mathbb{Z}/n\mathbb{Z}$. Therefore $x \otimes y = (bnx) \otimes y = (bx) \otimes (ny) = (bx) \otimes 0 = (bx) \otimes (0 \cdot 1) = (bx) \otimes (0 \cdot am) = (mbx) \otimes (0 \cdot a) = 0 \otimes 0$. Therefore, all generators of $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$ are equal to $0 \otimes 0$, so it is zero. □

**Problem 2.2.** *Let $A$ be a ring, $\mathfrak{a}$ an ideal, $M$ an $A$-module. Show that $(A/\mathfrak{a}) \otimes_A M$ is isomorphic to $M/\mathfrak{a}M$.*

*Proof.* Consider the exact sequence $\mathfrak{a} \to A \to A/\mathfrak{a} \to 0$. Tensoring preserves right-exactness, so $\mathfrak{a} \otimes_A M \to A \otimes_A M \to (A/\mathfrak{a}) \otimes_A M \to 0$ is also exact. By the first isomorphism theorem, $(A/\mathfrak{a}) \otimes_A M$ is isomorphic to $(A \otimes_A M)/\operatorname{im}(i \otimes 1_M)$ where $i : \mathfrak{a} \hookrightarrow A$. Note that $A \otimes_A M$ is isomorphic to $M$ via the isomorphism $\phi(a \otimes m) = am$. Also $\phi \circ (i \otimes 1_M) : \mathfrak{a} \otimes_A M \to M$ is given by $(a \otimes m) \to am$, so $\operatorname{im}(i \otimes 1_M)$ is isomorphic to $\mathfrak{a}M$. Therefore we conclude $(A/\mathfrak{a}) \otimes_A M \simeq M/\mathfrak{a}M$. □

**Problem 2.3.** *Let $A$ be a local ring, $M$ and $N$ finitely generated $A$-modules. Prove that if $M \otimes N = 0$, then $M = 0$ or $N = 0$.*

*Proof.* Let $\mathfrak{m}$ be the maximal ideal, $k = A/\mathfrak{m}$. Denote $M_k = k \otimes_A M$, which is isomorphic to $M/\mathfrak{m}M$ by Exercise 2. By Nakayama, $M_k = 0$ implies $M = 0$. Suppose $M \otimes_A N = 0$. Then by extension of scalars, $(M \otimes_A N)_k = 0$. Note that $(M \otimes_A N)_k = (M \otimes_A N) \otimes_A k \simeq M \otimes_A k \otimes_A N$. Also, $k \simeq k \otimes_k k$ since $k$ is a $k$-module over itself. Hence, $(M \otimes_A N)_k \simeq M \otimes_A (k \otimes_k k) \otimes_A N = M \otimes_A [(k \otimes_k k) \otimes_A N]$. Since $k$ is a $(k, A)$-bimodule, this is isomorphic to $M \otimes_A [k \otimes_k (k \otimes_A N)]$, which again is isomorphic to $(M \otimes_A k) \otimes_k (k \otimes_A N)$. This is just $M_k \otimes_k N_k$. Since $M_k$ and $N_k$ are both $k$-vector spaces, $M_k \otimes_k N_k = 0$ implies either $M_k = 0$ or $N_k = 0$. (Think of the contrapositive. If neither were zero, then there must be some nonzero element in $M_k \otimes_k N_k$.) Hence either $M = 0$ or $N = 0$. □

**Problem 2.4.** *Let $M_i$, $i \in I$ be any family of $A$-modules, and let $M$ be their direct sum. Prove that $M$ is flat iff each $M_i$ is flat.*

*Proof.* **Tensor Products over Arbitrary Direct Sums.** In advance to solving the problem, we first prove that $(\bigoplus M_i) \otimes N \simeq \bigoplus(M_i \otimes N)$ for modules $M_i$, $N$, over the same ring. Define a map $\phi : (\bigoplus M_i) \times N \to \bigoplus(M_i \otimes N)$ by $\phi((m_i)_{i\in I}, n) = (m_i \otimes n)_{i\in I}$. This map is well-defined since only finitely many $m_i$ are nonzero, hence only finitely many $m_i \otimes n$ are nonzero. Also this map is bilinear, which is easy to check. Therefore, by the universal property of the tensor product, there exists a unique homomorphism $\widetilde{\phi} : (\bigoplus M_i) \otimes N \to \bigoplus(M_i \otimes N)$ given by $(m_i)_{i\in I} \otimes n \mapsto (m_i \otimes n)_{i\in I}$. To show $\widetilde{\phi}$ is an isomorphism, we find its inverse.

Now consider $i_k : M_k \hookrightarrow \bigoplus M_i$ the canonical injection. Define $f_k : M_k \times N \to (\bigoplus M_i) \otimes N$ by $f_k(m_k, n) = i_k(m_k) \otimes n$. Again, $f_k$ is bilinear so there exists a unique homomorphism $\widetilde{f}_k : M_k \otimes N \to (\bigoplus M_i) \otimes N$ that satisfies $\widetilde{f}_k(m_k \otimes n) = i_k(m_k) \otimes n$. Now we define $\psi : \bigoplus(M_i \otimes N) \to (\bigoplus M_i) \otimes N$ by $(m_i \otimes n_i)_{i\in I} \mapsto \sum_{i\in I} \widetilde{f}_i(m_i \otimes n_i)$. The sum is finite, so the map is well-defined. Suppose we have $(m_i)_{i\in I} \otimes n \in (\bigoplus M_i) \otimes N$. Then, $(m_i)_{i\in I} \otimes n \mapsto (m_i \otimes n)_{i\in I} \mapsto \sum_{i\in I} \widetilde{f}_i(m_i \otimes n)$ where $\widetilde{f}_k(m_k \otimes n) = i_k(m_k) \otimes n$, so $\sum_{i\in I} \widetilde{f}_i(m_i \otimes n) = (m_i)_{i\in I} \otimes n$. Similarly, suppose we have $(\sum_k m_i \otimes n_k)_{i\in I} \in \bigoplus(M_i \otimes N)$. Since $(\sum_k m_i \otimes n_k)_{i\in I} = (m_i \otimes \sum_k n_k)_{i\in I}$, via $\psi$ this gets sent to $\left(\sum_{i\in I} i_i(m_i)\right) \otimes (\sum_k n_k) = (m_i)_{i\in I} \otimes (\sum_k n_k)$. Again, via $\widetilde{\phi}$, this gets sent to $(m_i \otimes \sum_k n_k)_{i\in I} = (\sum_k m_i \otimes n_k)_{i\in I}$, which is what we started with. Therefore $\widetilde{\phi}$ is indeed an isomorphism.

**Direct Sum of Short Exact Sequences.** A sequence of direct sums of modules (over the same index set) is exact if and only if its summands are part of an exact sequence. The maps of the exact sequence are defined termwise, so exactness of the sequence implies termwise exactness, and vice versa. Maps between arbitrary direct sums are well-defined by the nature of direct sums. (Elements of direct sums are collections of finitely many nonzero elements.)

**Main Proof.** Recall that $M$ is flat if $- \otimes_A M$ is exact, by definition. Suppose we are given a SES $0 \to N' \to N \to N'' \to 0$ of $A$-modules. $M$ is flat if and only if $0 \to N' \otimes M \to N \otimes M \to N'' \otimes M \to 0$ is exact. By what we proved above, we know that tensor products distribute over arbitrary direct sums, so we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N' \otimes M \arrow[r] \arrow[d, equal] & N \otimes M \arrow[r] \arrow[d, equal] & N'' \otimes M \arrow[r] \arrow[d, equal] & 0 \\
& \bigoplus_{i\in I}(N' \otimes M_i) \arrow[r, dashed] & \bigoplus_{i\in I}(N \otimes M_i) \arrow[r, dashed] & \bigoplus_{i\in I}(N'' \otimes M_i) &
\end{tikzcd}$$

where the vertical arrows are isomorphisms, and hence we may lift the sequence to a SES of direct sums over $I$. As we mentioned above, sequence of direct sums are exact if and only if each summand is exact, hence each SES $0 \to N' \otimes M_i \to N \otimes M_i \to N'' \otimes M_i \to 0$ must be exact. This is equivalent to every $M_i$ being flat. $\square$

**Problem 2.5.** *Let $A[x]$ be the ring of polynomials in one indeterminate over a ring $A$. Prove that $A[x]$ is a flat $A$-algebra.*

*Proof.* Note that $A[x] = \bigoplus_{i=0}^{\infty} A_i$ where $A_i$ is the $A$-module of degree $i$ polynomials, $A_0$ just being $A$. The $A$-algebra structure on $A[x]$ is given by polynomial multiplication. By Exercise 4, $A[x]$ is flat if and only if each $A_i$ is flat. Therefore it suffices to show that for all $A_i$ and for any SES $0 \to N' \to N \to N'' \to 0$, the sequence $0 \to N' \otimes A_i \to N \otimes A_i \to N'' \otimes A_i \to 0$ is exact. Notice that as an $A$-module, every $A_i$ is just isomorphic to $A$, and for an $A$-module $M$ it holds that $A \otimes_A M \simeq M$ canonically, so there is really nothing to show because tensoring with $A_i$ does not change the sequence. Thus all such sequences are trivially exact, hence $A[x]$ is a flat $A$-algebra. $\square$

**Problem 2.6.** *For any $A$-module, let $M[x]$ denote the set of all polynomials in $x$ with coefficients in $M$. Show that $M[x]$ is an $A[x]$-module. Show that $M[x] \cong A[x] \otimes_A M$.*

*Proof.* $M[x]$ is obviously an abelian group under addition, and obviously an $A[x]$-module. (The axioms of a module hold almost directly.) Since $A[x] = \bigoplus_{i=0}^{\infty} A_i$ where $A_i$ are degree $i$ polynomials with coefficients in $A$, $A[x] \otimes_A M \simeq \bigoplus_{i=0}^{\infty}(A_i \otimes_A M)$. Also, since $M[x] = \bigoplus_{i=0}^{\infty} M_i$, where $M_i$ are degree $i$ polynomials with coefficients in $M$, to show $A[x] \otimes_A M \simeq M[x]$ it suffices to show $A_i \otimes_A M \simeq M_i$ for each $i$. Construct $\phi : A_i \times M \to M_i$ by $\phi(ax^i, m) = amx^i$. This map is obviously $A$-bilinear, hence there exists a unique $\widetilde{\phi} : A_i \otimes_A M \to M_i$ given by $ax^i \otimes m \mapsto amx^i$. Now we define an inverse $\psi : M_i \to A_i \otimes_A M$ by $mx^i \mapsto x^i \otimes m$. Suppose we have some $ax^i \otimes m \in A_i \otimes_A M$. Then, $ax^i \otimes m \mapsto amx^i \mapsto x^i \otimes am = ax^i \otimes m$. Also, given $mx^i \in M_i$, $mx^i \mapsto x^i \otimes m \mapsto mx^i$, so $\widetilde{\phi}$ is indeed an isomorphism. Therefore $A[x] \otimes_A M \simeq \oplus_{i=0}^{\infty} M_i = M[x]$. $\square$

**Problem 2.7.** *Let $\mathfrak{p}$ be a prime ideal in $A$. Show that $\mathfrak{p}[x]$ is a prime ideal in $A[x]$. If $\mathfrak{m}$ is a maximal ideal in $A$, is $\mathfrak{m}[x]$ a maximal ideal in $A[x]$?*

*Proof.* Since $\mathfrak{p}$ is a prime ideal, $A/\mathfrak{p}$ is an integral domain. Consider the projection map $\pi : A[x] \to (A/\mathfrak{p})[x]$ defined by sending each coefficient through the canonical projection map $A \to A/\mathfrak{p}$. Then $\ker \pi = \mathfrak{p}[x]$, so by the first isomorphism theorem we have $A[x]/\mathfrak{p}[x] \simeq (A/\mathfrak{p})[x]$. The ring $(A/\mathfrak{p})[x]$ has no nonzero zero divisors (look at the highest order terms) thus it is an integral domain. Therefore $\mathfrak{p}[x]$ is a prime ideal.

On the other hand, $\mathfrak{m}[x]$ need not be (in fact cannot be!) maximal in $A[x]$. Consider the ideal $\mathfrak{m} \cup (A[x] \setminus A)$. One can verify that this is an ideal of $A[x]$ which properly contains $\mathfrak{m}[x]$, which is properly contained in $A[x]$. Or, just use the reasoning above. $(A/\mathfrak{m})[x]$ is definitely not a field. $\square$

**Problem 2.8.**

1. *If $M$ and $N$ are flat $A$-modules, then so is $M \otimes_A N$.*
2. *If $B$ is a flat $A$-algebra and $N$ is a flat $B$-module, then $N$ is flat as an $A$-module.*

*Proof.* 1. Suppose we are given an arbitrary SES $0 \to P' \to P \to P'' \to 0$ of $A$-modules. Since $M$ is flat, $0 \to P' \otimes_A M \to P \otimes_A M \to P'' \otimes_A M \to 0$ is exact. Also $N$ is flat, so $0 \to (P' \otimes_A M) \otimes_A N \to (P \otimes_A M) \otimes_A N \to (P'' \otimes_A M) \otimes_A N \to 0$ is exact. By associativity of the tensor product, we conclude that $M \otimes_A N$ is also flat. $\square$

*Proof.* 2. We work with the same notation as above. Since $B$ is a flat $A$-algebra, we have $0 \to P' \otimes_A B \to P \otimes_A B \to P'' \otimes_A B \to 0$. This is a SES of $B$-modules, so tensoring with $N$ results in $0 \to (P' \otimes_A B) \otimes_B N \to (P \otimes_A B) \otimes_B N \to (P'' \otimes_A B) \otimes_B N \to 0$. By associativity, and the fact that $B \otimes_B N \cong N$, we conclude that $N$ is flat as an $A$-module. $\square$

**Problem 2.9.** *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of $A$-modules. If $M'$ and $M''$ are finitely generated, then so is $M$.*

*Proof.* Since $M'$ and $M''$ are finitely generated, we may assume there exists surjective maps from $A^n$ and $A^m$, respectively. Suppose $(1, \dots, 1) \in A^m$ maps to $x_1 + \cdots + x_m \in M''$, $x_i$ generators of $M''$. (Please bear with the abuse of notation.) Since $M \to M''$ is surjective, there exist $y_1, \dots, y_m \in M$ that map to the $x_i$, accordingly. Then we may define a map from $A^m \to M$ given by $(1, \dots, 1) \mapsto y_1 + \cdots + y_m$, which makes the upper right triangle commutative.

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0 \\
0 \arrow[r] & A^n \arrow[r] \arrow[u, two heads] & A^{n+m} \arrow[r] & A^m \arrow[r] \arrow[u, two heads] \arrow[ul, dashed] & 0
\end{tikzcd}$$

By precomposing with $A^{n+m} \to A^m$, we have found a map from $A^{n+m} \to M$. This map is surjective by the short five lemma, so $M$ is finitely generated. $\square$

**Problem 2.10.** *Let $A$ be a ring, $\mathfrak{a}$ an ideal contained in the Jacobson radical of $A$; let $M$ be an $A$-module and $N$ a finitely generated $A$-module, and let $u : M \to N$ be a homomorphism. If the induced homomorphism $M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective, then $u$ is surjective.*

*Proof.* Suppose the induced homomorphism $\widetilde{u}$ is surjective. Then for any $n \in N$, we have some $m \in M$ such that $u(m) + \mathfrak{a}N = n + \mathfrak{a}N$, so $n - u(m) = \sum_{i \in I} a_i n_i$ for $a_i \in \mathfrak{a}$ and $n_i \in N$, $I$ finite. It follows that $N \subset u(M) + \mathfrak{a}N$. Hence we have $N = u(M) + \mathfrak{a}N$, from which $u(M) = N$ follows from Corollary 2.7. □

**Problem 2.11.** *Let $A \neq 0$ be a ring. Show that $A^m \cong A^n$ implies $m = n$.*

1. *If $\phi : A^m \to A^n$ is surjective, then $m \geq n$.*
2. *If $\phi : A^m \to A^n$ is injective, is it always the case that $m \leq n$?*

*Proof.* Let $\phi : A^m \to A^n$ be an isomorphism. Denote by $\phi^{-1} : A^n \to A^m$ its inverse. Now, consider the two maps $1_k \otimes \phi : k \otimes_A A^m \to k \otimes_A A^n$ and $1_k \otimes \phi^{-1} : k \otimes_A A^n \to k \otimes_A A^m$ where $k = A/\mathfrak{m}$ for $\mathfrak{m}$ a maximal ideal of $A$. Since $k \otimes_A (\bigoplus A) \cong \bigoplus (k \otimes_A A) \cong \bigoplus k$, the tensor products are $k$-vector spaces of dimension $m$ and $n$, respectively. The maps sends $u \otimes a \mapsto u \otimes \phi(a) \mapsto u \otimes \phi^{-1}(\phi(a))$ and vice versa, so the two $k$-vector spaces must be isomorphic. Hence $m = n$. □

*Proof.* 1. If $\phi : A^m \to A^n$ is surjective, tensoring with $k$ will also yield a surjective map $1_k \otimes \phi : k \otimes_A A^m \to k \otimes_A A^n$. This is a surjective linear map of finite dimensional vector spaces, so $m \geq n$. □

*Proof.* 2. This is in fact true. To prove by contradiction, suppose that $m > n$. Considering $\phi$ as an $A$-linear map from $A^m$ to itself by viewing $A^n$ as the sub $A$-module (not a subring!) of $A^m$ obatined by restricting to the first $n$ coordinates, by the Cayley-Hamilton theorem $\phi$ must satisfy a polynomial equation, say $\phi^k + a_{k-1}\phi^{k-1} + \cdots + a_1\phi + a_0 = 0$. If $a_0 = 0$, we may lower the degree of the polynomial, using injectivity of $\phi$. Hence, we may assume the polynomial equation is of minimal degree, thus $a_0 \neq 0$. However, sending $(0, 0, \ldots, 1)$ through $\phi$ results in $a_0 = 0$, which is a contradiction. This proof is based on this beautiful MathOF answer. □

**Problem 2.12.** *Let $M$ be a finitely generated $A$-module and $\phi : M \to A^n$ a surjective homomorphism. Show that $\ker \phi$ is finitely generated.*

*Proof.* Suppose $e_i$, $i = 1, \ldots, n$ are standard generators of $A^n$. Since $\phi$ is surjective, there exists $u_i \in M$ such that $\phi(u_i) = e_i$. Define a map $\psi : A^n \to M$ given by $e_i \mapsto u_i$. By the property of free modules, it follows that $\phi \circ \psi = 1_{A^n}$. Now, define a map $f : \ker \phi \oplus A^n \to M$ by sending $(p, a)$ to $p + \psi(a) \in M$. Then, sending this through $\phi$, we get $\phi(p + \psi(a)) = \phi(p) + \phi(\psi(a)) = a$. Therefore, we conclude that $\phi \circ f = p$ where $p : \ker \phi \oplus A^n \to A^n$ is the projection map. Thus the squares of the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & \ker\phi \arrow[r] \arrow[d, equal] & \ker\phi \oplus A^n \arrow[r, "p"] \arrow[d, dashed, "f"] & A^n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker\phi \arrow[r] & M \arrow[r, "\phi"] & A^n \arrow[r] & 0
\end{tikzcd}$$

By the short five lemma, $M \cong \ker \phi \oplus A^n$, and since $M$ is finitely generated, say $A^m$ surjects onto $M$, then compose this with the canonical projection $M \to \ker \phi$ to conclude that $\ker \phi$ is finitely generated. □

**Problem 2.13.** *Let $f : A \to B$ be a ring homomorphism, and let $N$ be a $B$-module. Regarding $N$ as an $A$-module by restriction of scalars, form the $B$-module $N_B = B \otimes_A N$. Show $g : N \to N_B$ which maps $y$ to $1 \otimes y$ is injective and that $g(N)$ is a direct summand of $N_B$.*

*Proof.* Define $g : N \to N_B$ by $y \mapsto 1 \otimes y$, and $p : N_B \to N$ by $b \otimes y \mapsto by$. The map $p$ is a well-defined $A$-module homomorphism since the map $B \times N \to N$ given by $(b, y) \mapsto by$ is $A$-bilinear by viewing both $B$ and $N$ as $A$-modules through $f$. Then $p \circ g = 1_N$, so $g$ must be injective. From this, consider the following diagram of $A$-modules

$$\begin{tikzcd}
0 \arrow[r] & \ker p \arrow[r] \arrow[d, equal] & \ker p \oplus \operatorname{im} g \arrow[r] \arrow[d] & \operatorname{im} g \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker p \arrow[r] & N_B \arrow[r, "p"] & N \arrow[r] & 0
\end{tikzcd}$$

where $\operatorname{im} g \cong N$ by injectivity of $g$. Define the map $\ker p \oplus \operatorname{im} g \to N_B$ by $(x, y) \mapsto x + y$. The left square obviously commutes. Since $p(x + y) = p(y)$, and $g^{-1}(y) = p(y)$, the right square also commutes, so by the short five lemma, we conclude that $N_B \cong \ker p \oplus \operatorname{im} g$ as $A$-modules. The isomorphism does not hold as $B$-modules; see this MathSE answer. $\square$

**Problem 2.15.** *Show that every element of $M$ can be written in the form $\mu_i(x_i)$ for some $i \in I$ and some $x_i \in M_i$. Show also that if $\mu_i(x_i) = 0$ then there exists $j \geq i$ such that $\mu_{ij}(x_i) = 0$.*

*Proof.* 1. Every element of $M$ is of the form $\sum m_i + D$. If all $m_i = 0$ then the result is vacuously true. Otherwise, since the indices where $m_i \neq 0$ is finite, by the directed property of the index set there exists some $j \in I$ such that $j \geq i$ for all $i$. Now define $x^i = \mu_{ij}(m_i) \in M_j$. Then, $x := \sum_i x_i \in M_j$, and note that $(\sum_i m_i) - x = \sum(m_i - x^i) = \sum(m_i - \mu_{ij}(m_i)) \in D$. Since $\mu_j(x) = x + D = \sum m_i + D$, end of proof. $\square$

*Proof.* 2. Suppose $\mu_i(x_i) = 0$. Note that this implies $x_i \in D$, so $x_i = \sum_{j,k}(m_j - \mu_{jk}(m_j))$ is a finite sum. Since $x_i \in M_i$, the terms in $M_i$ must add up to $x_i$ [4] $\square$

**Problem 2.16.** *Show that the direct limit is characterized up to isomorphism by the following property: Let $N$ be an $A$-module and for each $i \in I$ let $\alpha_i : M_i \to N$ be an $A$-module homomorphism such that $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $i \leq j$. Then there exists a unique homomorphism $\alpha : M \to N$ such that $\alpha_i = \alpha \circ \mu_i$ for all $i \in I$.*

*Proof.* We will use a more down-to-earth definition of a direct limit: Define $M$ as $\bigoplus M_i / \sim$ where $m_i \sim m_j$ iff there exists some $k$ such that $\mu_{ik}(m_i) = \mu_{jk}(m_j)$.

**Existence.** Given some $\overline{m_i} \in M$, define $\alpha : M \to N$ as $\overline{m_i} \mapsto \alpha_i(m_i)$. This map is well-defined since given any other $m_j$ in the equivalence class of $m_i$, there exists some $m_k$ such that $m_i \mapsto m_k$ and $m_j \mapsto m_k$. By the assumptions on $\alpha_i$, we know that $\alpha_i(m_i) = \alpha_k(m_k) = \alpha_j(m_j)$, so $\alpha$ is indeed well-defined.

**Uniqueness.** By the universal property of the coproduct, maps $\mu_i : M_i \to M$ factor uniquely through the canonical injection maps to yield a map $\bigoplus M_i \to M$. This unique map is just the projection to the equivalence classes, which is surjective. Call this map $\mu$. Hence if[5] $\square$

**Problem 2.17.** *Let $(M_i)_{i \in I}$ be a family of submodules of an $A$-module, such that for each pair of indices $i, j \in I$ there exists $k \in I$ such that $M_i + M_j \subset M_k$. Define $i \leq j$ to mean $M_i \subset M_j$ and let $\mu_{ij} : M_i \to M_j$ be the embedding of $M_i$ in $M_j$. Show that $\varinjlim M_i = \sum M_i = \bigcup M_i$. In particular, any $A$-module is the direct limit of its finitely generated submodules.*

*Proof.* We show $\varinjlim M_i \cong \bigcup M_i$ using the characterization above. Suppose we have any $A$-module $M$, and maps $\alpha_i : M_i \to N$ such that $\alpha_i = \alpha_j \circ \mu_{ij}$ for all $i \leq j$. We show that there exists a unique $A$-module homomorphism $\alpha : \bigcup M_i \to N$ such that $\alpha_i = \alpha \circ \mu_i$ for all $i \in I$. Note that $\bigcup M_i$ is a module; given two elements, there exists some $M_i$ containing both. Take $x \in \bigcup M_i$. Then, $x$ is in some $M_j$. Define $\alpha : M \to N$ as $\alpha(\overline{x}) = \alpha_j(x)$.

[4] TODO: I cannot see how the two definitions of direct limits are equivalent.
[5] TODO

We show this is well-defined. Suppose $x$ is also in some $M_k$. By hypothesis, there exists some index $l$ such that $j, k \leq l$, and $\alpha_j(x) = \alpha_\ell(\mu_{j\ell}(x))$ and $\alpha_k(x) = \alpha_\ell(\mu_{k\ell}(x))$ and since $\mu_{j\ell}(x) = \mu_{k\ell}(x)$, so it follows that the two are the same.

Next, we show $\alpha$ is a homomorphism. Suppose we have $m_j \in M_j$ and $m_k \in M_k$. We want to show that $\alpha(\overline{m_j}+\overline{m_k}) = \alpha(\overline{m_j})+\alpha(\overline{m_k})$. By definition, $\alpha(\overline{m_j}+\overline{m_k}) = \alpha_\ell(m_j+m_k)$ for some $j, k \leq \ell$, and $\alpha_\ell$ is obviously a homomorphism. Again, $\alpha(\overline{m_j}) = \alpha_j(m_j)$ and since $\mu_{j\ell}$ is the inclusion, $\alpha_j(m_j) = \alpha_\ell(m_j)$ where $m_j$ is viewed as an element of $M_\ell$. Do the same for $m_k$ to conclude that $\alpha$ is indeed a homomorphism.

Next, suppose we have another map $\beta : \bigcup M_i \to N$ that satisfies all the conditions. Since $\beta \circ \mu_i(x) = \alpha \circ \mu_i(x)$ for all $x \in M_i$, for all $i$, it follows that $\alpha = \beta$.

It is obvious that $\bigcup M_i \subset \sum M_i$. Suppose we have a finite sum $\sum_i m_i$. Then by the property of directed sets, there exists some $j$ such that $i \leq j$ for all $i$. Then, the sum is in $M_j$. Hence $\sum M_i = \bigcup M_i$.

Now suppose the $M_i$ are the finitely generated submodules of some $A$-module $M$. $\bigcup M_i \subset M$ is obvious, and if we have some $m \in M$, then the module generated by $m$ is a finitely generated submodule of $M$. Thus $M = \varinjlim M_i$. □

**Problem 2.18.** *Show that $\Phi$ defines a unique homomorphism $\phi = \varinjlim \phi_i : M \to N$ such that $\phi \circ \mu_i = \nu_i \circ \phi_i$ for all $i \in I$.*

*Proof.* Uniqueness is ensured by Exercise 16. If we define $\alpha_i = \nu_i \circ \phi_i : M_i \to N$, then $\alpha_j \circ \mu_{ij} = \nu_j \circ \phi_j \circ \mu_{ij} = \nu_j \circ \nu_{ij} \circ \phi_i = \nu_i \circ \phi_i$. □

**Problem 2.19.** *solved*

**Problem 2.20.** *solved*

**Problem 2.21.** *solved*

**Problem 2.22.** *Let $(A_i, \alpha_{ij})$ be a direct system of rings and let $\mathfrak{N}_i$ be the nilradical of $A_i$. Show that $\varinjlim \mathfrak{N}_i$ is the nilradical of $\varinjlim A_i$.*

*Proof.* First, we must justify seeing $\varinjlim \mathfrak{N}_i$ as a subset of $\varinjlim A_i$. Denote by $\iota_i$ the inclusion of $\mathfrak{N}_i$ into $A_i$. These $\iota_i$ form a direct limit homomorphism. By problem 19, since each is injective, the direct limit $\iota = \varinjlim \iota_i$ is also injective. Hence $\varinjlim \mathfrak{N}_i$ can be embedded into $\varinjlim A_i$ via $\iota$. Suppose $\overline{n_i} \in \varinjlim \mathfrak{N}_i$. Then $n_i \in \mathfrak{N}_i$ so $n_i^k = 0$ for some $k > 0$. Also $\iota_i(n_i) \in A_i$, and $\iota(\overline{n_i}) = \overline{\iota_i(n_i)}$ and since $\overline{\iota_i(n_i)}^k = \overline{\iota_i(n_i^k)} = 0$, we have $\iota(\overline{n_i})$ in the nilradical of $\varinjlim A_i$.

Now suppose $\overline{n_i}$ is in the nilradical of $\varinjlim A_i$. Then $\overline{n_i}^k = \overline{n_i^k} = 0$ for some $k > 0$. We know that for some $i \leq j$, $\alpha_{ij}(n_i^k) = 0$. Denote $\alpha_{ij}(n_i) = n_j$, then it follows that $n_j^k = 0$, thus $n_j$ is in the nilradical of $A_j$. This means that there exists some $r_j \in \mathfrak{N}_j$ such that $\iota_j(r_j) = n_j$. Then it follows that $\iota(\overline{r_j}) = \overline{\iota_j(r_j)} = \overline{n_j} = \overline{n_i}$, which proves that $\overline{n_i}$ can be realized as $\overline{r_j}$ in the limit of the nilradicals.

Note that showing the nilradical of a ring being trivial is not strong enough to imply that the ring is an integral domain; the ring $\mathbb{Z} \times \mathbb{Z}$ has trivial nilradical, but has obvious zero divisors. Hence, we show the contrapositive. Suppose $\overline{a_i}, \overline{a_j} \in \varinjlim A_i$ are nonzero and $\overline{a_i} \cdot \overline{a_j} = 0$. Now, pick some index $k$ such that $i, j \leq k$. Then $\alpha_{ik}(a_i)$ and $\alpha_{jk}(a_j)$ is in the zero class of $\varinjlim A_i$. Hence, by exercise 15, there exists some $l \in I$ where both are realized as zero. Then we have $\alpha_{il}(a_i) \cdot \alpha_{jl}(a_j) = 0$, and since both elements are nonzero, this means that $A_l$ has nonzero zero divisors. Thus we have proved the contrapositive. □

**Problem 2.23.** *Nothing to prove.*

**Problem 2.24.** *If $M$ is an $A$-module, TFAE:*

1. *$M$ is flat*
2. $\mathrm{Tor}_n^A(M, N) = 0$ *for all $n > 0$ and all $A$-modules $N$*

3. $\operatorname{Tor}_1^A(M, N) = 0$ *for all $A$-modules $N$.*

*Proof.* Given any $A$-module $N$, take its free resolution. Since any module $N$ has a set of (possibly infinite) generators, we may consider the projection from the free module generated by the generators of $N$ onto $N$. This projection has a kernel, which is also a module so we may again consider the set of generators of the kernel, and construct another projection onto the kernel. Extend this indefinitely to obtain a free resolution of a module, which is an exact sequence. Suppose we have a free resolution $\cdots \to F_2 \to F_1 \to F_0 \to N \to 0$. Since $M$ is flat, by tensoring we obtain an exact sequence $\cdots \to F_2 \otimes M \to F_1 \otimes M \to F_0 \otimes M \to N \otimes M \to 0$. Since $\operatorname{Tor}_n^A(M, N)$ is the $n$-th homology of the chain complex obtained by discarding $N \otimes M$, for every $n > 0$ we can see that the Tor group must vanish by exactness. This automatically implies 3. To show 3 implies 1, take any exact sequence $0 \to N' \to N \to N'' \to 0$ and take the LES of Tor. Since $\operatorname{Tor}_1^A = 0$, we obtain a SES $0 \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0$, $M$ is flat. $\square$

**Problem 2.25.** *Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence with $N''$ flat. Show $N'$ is flat iff $N$ is flat.*

*Proof.* Suppose $M$ is any $A$-module. From the LES of Tor, we may obtain

$$\cdots \to \operatorname{Tor}_{n+1}(M, N'') \to \operatorname{Tor}_n(M, N') \to \operatorname{Tor}_n(M, N) \to \operatorname{Tor}_n(M, N'') \to \cdots$$

Using the fact that $\operatorname{Tor}_i(A, B) \cong \operatorname{Tor}_i(B, A)$ for all modules (over the same commutative ring) without proof, it follows that $\operatorname{Tor}_n(M, N') \cong \operatorname{Tor}_n(M, N)$, for $n > 0$, in particular $n = 1$. Thus it follows that $N'$ is flat if and only if $N$ is flat.

Important note: This holds only when the third term is flat. In other words, $N'$ and $N$ both being flat may not imply $N''$ being flat. For example, consider the SES of $\mathbb{Z}$-modules $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$. $\square$

**Problem 2.26.** *Let $N$ be an $A$-module. Then $N$ is flat if and only if $\operatorname{Tor}_1(A/\mathfrak{a}, N) = 0$ for all finitely generated ideals $\mathfrak{a}$ in $A$.*

*Proof.* The forward direction is trivial. Now $N$ is flat if $\operatorname{Tor}_1(M, N) = 0$ for all finitely generated $A$-modules $M$; here is the reason. Any $A$-module $M$ is the direct limit of its finitely generated submodules, say $M = \varinjlim M_i$. Consider a SES of $A$-modules $0 \to P \to Q \to N \to 0$, and tensor with some $M_i$. If $\operatorname{Tor}_1(M_i, N) = 0$ for all $i$, then the sequence $0 \to P \otimes M_i \to Q \otimes M_i \to N \otimes M_i \to 0$ would be exact for all $i$. Pass to the direct limit to obtain a SES $0 \to \varinjlim(P \otimes M_i) \to \varinjlim(Q \otimes M_i) \to \varinjlim(N \otimes M_i) \to 0$, where since direct limits commute with tensor products, we conclude that $\operatorname{Tor}_1(M, N) = 0$ indeed. Thus, it is enough to show that $\operatorname{Tor}_1(M, N) = 0$ for finitely generated $M$.

Now suppose $M$ is finitely generated with generators $x_1, \ldots, x_n$. Denote by $M_i = \langle x_1, \ldots, x_i \rangle$, the submodule of $M$ generated by elements up to $i$. For each $i > 1$, we have SESs $0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0$, where $M_i/M_{i-1}$ is generated by $x_i$, so is of the form $A/\mathfrak{a}$ for an ideal $\mathfrak{a} \subset A$ thought of as the relations on $x_i$. Now, if each $\langle x_i \rangle$ is flat, then by Exercise 25 (for an SES $0 \to N' \to N \to N'' \to 0$ with $N''$ flat, $N$ is flat iff $N'$ is flat) this implies $M_2$ is flat (from the SES $0 \to M_1 \to M_2 \to M_2/M_1 \to 0$) and $M_3$ flat from the next SES and so on up to implying $M_n = M$ is flat. Therefore, it is enough to show $\operatorname{Tor}_1(A/\mathfrak{a}, N) = 0$ for all ideals $\mathfrak{a} \subset A$.

Now as an $A$-module, ideals $\mathfrak{a}$ are again a direct limit of its finitely generated submodules $\mathfrak{a}_i$. Thus, by assumption we have $0 \to \mathfrak{a}_i \otimes N \to A \otimes N \to (A/\mathfrak{a}_i) \otimes N \to 0$ for every $i$. Passing through the direct limit, we get $0 \to \mathfrak{a} \otimes N \to A \otimes N \to (A/\mathfrak{a}) \otimes N \to 0$, which implies $\operatorname{Tor}_1(A/\mathfrak{a}, N) = 0$ for all ideals $\mathfrak{a} \subset A$. $\square$

**Problem 2.27.** *Prove the following are equivalent:*

1. *$A$ is absolutely flat*
2. *Every principal ideal is idempotent*

3. *Every finitely generated ideal is a direct summand of $A$*

*Proof.* Suppose $A$ is absolutely flat. Let $x \in A$. Consider the following SES of $A$-modules:

$$0 \to (x) \to A \to A/(x) \to 0.$$

Tensor with $A/(x)$ as an $A$-module to get another SES

$$0 \to (x) \otimes_A (A/(x)) \to A \otimes_A (A/(x)) \to (A/(x)) \otimes_A (A/(x)) \to 0$$

and by Exercise 2 we know each are isomorphic to $(x)/(x)^2$, $A/(x)$, and $(A/(x))/(x)(A/(x))$ respectively. Now the map $A \to A/(x)$ is given by $a \mapsto \overline{a}$, the canonical projection, so it follows that the map $A \otimes_A (A/(x)) \to (A/(x)) \otimes_A (A/(x))$ is given by $a \otimes \overline{b} \mapsto \overline{a} \otimes \overline{b}$, and the isomorphisms are given by $a \otimes \overline{b} \mapsto \overline{ab}$, and $\overline{a} \otimes \overline{b} \mapsto \overline{ab} + (x)(A/(x)) = \overline{ab} + 0$, respectively. Hence passing through the isomorphisms, we conclude that the induced map from $A/(x)$ to $(A/(x))/(x)(A/(x)) = A/(x)$ is the identity map. Therefore, rewriting the second SES above passed through the isomorphisms, we get

$$0 \to (x)/(x)^2 \to A/(x) \xrightarrow{=} A/(x) \to 0$$

so we have $(x)/(x)^2 = 0$. Since $x$ was any element of $A$, we conclude that every principal ideal of $A$ is idempotent.

The proof of (ii) $\Rightarrow$ (iii) is provided in the text.

Suppose every finitely generated ideal is a direct summand of $A$. Exercise 26 implies that if $\mathrm{Tor}_1(A/\mathfrak{a}, N) = 0$ for all finitely generated ideals $\mathfrak{a}$, then $N$ is flat. Note that $A$, as an $A$-module, is flat. Therefore $\mathrm{Tor}_1(A, N) = 0$ for any $N$ by Exercise 24. Since $A \cong \mathfrak{a} \oplus (A/\mathfrak{a})$ and since direct sums commute with Tor, we have $\mathrm{Tor}_1(\mathfrak{a}, N) \oplus \mathrm{Tor}_1(A/\mathfrak{a}, N) = 0$, which implies $\mathrm{Tor}_1(A/\mathfrak{a}, N) = 0$. Therefore any $A$-module $N$ is flat, so $A$ is absolutely flat. □

**Problem 2.28.** *A Boolean ring is absolutely flat. The ring in which every element $x$ satisfies $x^n = x$ for some $n > 1$ is absolutely flat. The image of an absolutely flat ring is absolutely flat. If a local ring is absolutely flat, then it is a field. If $A$ is absolutely flat, every nonunit in $A$ is a zero divisor.*

*Proof.* [6]

□

## 3. Rings and Modules of Fractions

**Problem 3.1.** *Let $S$ be a multiplicatively closed subset of a ring $A$, and let $M$ be a finitely generated $A$-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.*

*Proof.* $S^{-1}M = 0$ if and only if $\mathrm{Ann}(S^{-1}M) = S^{-1}A$ and since $M$ finitely generated, $\mathrm{Ann}(S^{-1}M) = S^{-1}\,\mathrm{Ann}(M)$. Note that $S^{-1}A = S^{-1}\,\mathrm{Ann}(M)$ if and only if $\mathrm{Ann}(M) \cap S \neq \emptyset$, i.e. iff there exists some $s \in S$ that annihilates $M$. □

**Problem 3.2.** *Let $\mathfrak{a}$ be an ideal of $A$, and let $S = 1 + \mathfrak{a}$. Show that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$. Use this result and Nakayama's lemma to give a proof of (2.5).*

*Proof.* Suppose $x$ is an element of $S^{-1}\mathfrak{a}$. Then we may write $x$ as $a/s$ for some $a \in \mathfrak{a}$, and some $s \in S$ by identifying the denominators. Note that $x$ is an element of the Jacobson radical of $S^{-1}A$ if and only if $1 + xy$ is a unit in $S^{-1}A$ for any $y \in S^{-1}A$. Let $y = b/t$ for $b \in A$ and $t \in S$, again by identifying the denominators. Then $1 + xy = 1 + (ab)/(st) = (ab + st)/(st)$. If $ab + st \in 1 + \mathfrak{a} = S$, then this automatically admits an inverse. This is indeed true, since $st \in S$ so $st = 1 + a'$ for some $a' \in \mathfrak{a}$, and since $a \in \mathfrak{a}$ we have $ab \in \mathfrak{a}$, so $ab + st = 1 + a' + ab \in 1 + \mathfrak{a} = S$.

[6] TODO

The statement of (2.5) is: Let $M$ be a finitely generated $A$-module and let $\mathfrak{a}$ be an ideal of $A$ such that $\mathfrak{a}M = M$. Then there exists $x \equiv 1 \bmod \mathfrak{a}$ such that $xM = 0$.

Nakayama's lemma states that if $M$ is a finitely generated $A$-module, and $\mathfrak{a}$ is an ideal of $A$ contained in the Jacobson radical, then $\mathfrak{a}M = M$ implies $M = 0$.

Suppose $M = \mathfrak{a}M$. Then $S^{-1}M = (S^{-1}\mathfrak{a})(S^{-1}M)$, and since $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$ we may use Nakayama's lemma to conclude that $S^{-1}M = 0$. By Exercise 1, this holds if and only if there exists some $s \in S$ such that $sM = 0$. Now since $s \equiv 1 \bmod \mathfrak{a}$, let $x = s$. □

**Problem 3.3.** *Let $A$ be a ring, and let $S$ and $T$ be two multiplicatively closed subsets of $A$, and let $U$ be the image of $T$ in $S^{-1}A$. Show that $(ST)^{-1}A$ and $U^{-1}(S^{-1}A)$ are isomorphic.*

*Proof.* We assume the definition of $ST$ is $\{st \mid s \in S, t \in T\}$. Define a ring homomorphism $g : A \to U^{-1}(S^{-1}A)$ by $a \mapsto (a/1)/(1/1)$. We check if this map satisfies the conditions of Corollary 3.2; we want to show for any $st \in ST$, $g(st)$ is a unit, and $g(a) = 0$ implies $ast = 0$ for some $st \in ST$, and every element of $U^{-1}(S^{-1}A)$ is of the form $g(a)/g(st)$.

Firstly, $g(st) = (st/1)/(1/1)$, which has inverse $(1/s)/(t/1)$. The calculation is straightforward. Next, suppose $g(a) = (a/1)/(1/1) = 0$. This means that for some $(t/1)$, we have $(t/1)(a/1) = 0$ in $S^{-1}A$. Thus, $ast = 0$ in $A$ for some $s \in S$ and $t \in T$. Finally, every element of $U^{-1}(S^{-1}A)$ is of the form $(a/s)/(t/1)$, and $g(a)/g(st) = (a/1)/(1/1) \times [(st/1)/(1/1)]^{-1}$ where the inverse of $(st/1)/(1/1)$ is $(1/s)/(t/1)$ as stated above. Hence, $g(a)/g(st) = (a/1)/(1/1) \times (1/s)/(t/1) = (a/s)/(t/1)$. Therefore $g$ satisfies the conditions of Corollary 3.2, and thus implies the existence of a unique isomorphism from $(ST)^{-1}A$ to $U^{-1}(S^{-1}A)$. Hence the two are isomorphic. □

**Problem 3.4.** *Let $f : A \to B$ be a homomorphism of rings and let $S$ be a multiplicatively closed subset of $A$. Let $T = f(S)$. Show that $S^{-1}B$ and $T^{-1}B$ are isomorphic as $S^{-1}A$-modules.*

*Proof.* Before proving this, we must first define $S^{-1}B$. We define this as $B \times S$ modulo the equivalence relation $(b_1, s_1) \sim (b_2, s_2)$ iff there exists some $s \in S$ such that $s \cdot (s_2 \cdot b_1 - s_1 \cdot b_2) = 0$, where $s \cdot b = f(s)b$. In other words, $B$ must be localized by $S$ as an $A$-module via $f$. The definition of $T$ is straightforward. Now, define $g : B \times S \to B \times T$ by $(b, s) \mapsto (b, f(s))$. We show that this preserves the equivalence relation. By definition, if $(b_1, s_1) \sim (b_2, s_2)$ then it follows that $(b_1, f(s_1)) \sim (b_2, f(s_2))$. Now suppose $(b_1, t_1) \sim (b_2, t_2)$ in $B \times T$. Since $T = f(S)$, there exist $s_1$ and $s_2$ in $S$ that map to $t_1$ and $t_2$, respectively. Therefore, $t(b_1 f(s_2) - b_2 f(s_1))$ holds, and again there exists some $s \in S$ that maps to $t$. Then it follows that $(b_1, s_1) \sim (b_2, s_2)$ for any $s_1, s_2, s$ that map to the required elements in $T$. Therefore the equivalence relation is preserved, hence the two are isomorphic as $S^{-1}A$-modules. □

**Problem 3.5.** *Let $A$ be a ring. Suppose that for each prime ideal $\mathfrak{p}$, the local ring $A_\mathfrak{p}$ has no nonzero nilpotent element. Show that $A$ has no nonzero nilpotent element. If each $A_\mathfrak{p}$ is an integral domain, is $A$ necessarily an integral domain?*

*Proof.* **Elementary Proof.** Suppose $a \in A$ is a nonzero nilpotent, say $a^n = 0$ for $n > 1$. Consider $\mathrm{Ann}(a)$, the annihilator ideal of $a$. Since $a$ is nonzero, this ideal is proper, thus contained in some prime ideal $\mathfrak{p}$. Given any $s \in A_\mathfrak{p}$, the element $a/s$ is in fact a nilpotent. We want to show this is nonzero. To do this, we show that $s'a \neq 0$ for all $s' \in A - \mathfrak{p}$, which is in fact true since $\mathfrak{p}$ contains the annihilator ideal of $a$. Hence $a/s$ is a nonzero nilpotent in $A_\mathfrak{p}$.

**Nilradical Proof.** Suppose for each $\mathfrak{p}$ the local rings have trivial nilradicals. By Corollary 3.12, the nilradical of $A_\mathfrak{p}$ is $\mathfrak{N}_\mathfrak{p}$ where $\mathfrak{N}$ is the nilradical of $A$. Being trivial is a local property, so $\mathfrak{N} = 0$.

**Counterexample.** Consider $\mathbb{Z}\times\mathbb{Z}$. Every prime ideal of this ring is of the form $(p)\times\mathbb{Z}$ or $\mathbb{Z}\times(p)$ for $p$ prime or zero. Since $(\mathbb{Z}\times\mathbb{Z})_{\mathfrak{p}} \cong \mathbb{Z}_{(p)}$ for some $p$, and $\mathbb{Z}$ is an integral domain, so its localizations are subrings of its field of fractions, hence domains. However $\mathbb{Z}\times\mathbb{Z}$ is obviously not an integral domain. □

**Problem 3.6.** *Let $A$ be a ring $\neq 0$ and let $\Sigma$ be the set of all multiplicatively closed subsets $S$ of $A$ such that $0\notin S$. Show that $\Sigma$ has maximal elements, and that $S\in\Sigma$ is maximal if and only if $A-S$ is a minimal prime ideal of $A$.*

*Proof.* **Existence of Maximal Elements.** Partially order $\Sigma$ by inclusion. Given a chain $S_i$ of multiplicatively closed subsets, consider the union $\bigcup S_i$. If $x,y\in\bigcup S_i$, then $x,y\in S_j$ for some $j$, hence $xy\in S_j\subset\bigcup S_i$. Therefore $\bigcup S_i$ is a multiplicatively closed set, thus every chain has an upper bound. Since $\{1\}\in\Sigma$, by Zorn's Lemma $\Sigma$ has maximal elements.

**Second Part.** Suppose $S\in\Sigma$ is maximal. Denote $\mathfrak{p}=A-S$. We want to show $\mathfrak{p}$ is a minimal prime ideal. First, we show $\mathfrak{p}$ is an ideal of $A$. $0\in\mathfrak{p}$ by definition. Now we show $\mathfrak{p}$ is additive. Suppose $a,b\in\mathfrak{p}$. Now, we may consider a multiplicatively closed set containing both $S$ and $a$ by $S'=\{sa^n \mid s\in S, n\in\mathbb{Z}_{\geq 0}\}$. Since $a$ is not in $S$, and $S'$ contains $a$, $S'$ properly contains $S$. By maximality of $S$, $S'$ cannot be in $\Sigma$, i.e. $0\in S'$. Thus $sa^n=0$ for some $s\in S$ and some $n\geq 1$. Analogously, $tb^m=0$ for some $t\in S$ and some $m\geq 1$. Therefore if we let $N=m+n$, by the binomial theorem we have $st(a+b)^N=0$. Now interpreting the multiplicatively closed condition as $xy\notin S$ implying $x\notin S$ or $y\notin S$, and since $st(a+b)^N=0\notin S$ but $st\in S$, we must have $(a+b)^N\notin S$. Thus $(a+b)^N\in\mathfrak{p}$. Writing $(a+b)^N=(a+b)(a+b)^{N-1}$, if $a+b\in\mathfrak{p}$ we are done. Otherwise, $a+b\in S$, and $(a+b)^N\notin S$ so $(a+b)^{N-1}\notin S$, in other words $(a+b)^{N-1}\in\mathfrak{p}$. Repeat finitely many times to conclude that $a+b\in\mathfrak{p}$. This is contrary to our assumption that $a+b\notin\mathfrak{p}$, so it follows that $a+b\in\mathfrak{p}$ in the first place. Now we show for any $a\in A$ and any $p\in\mathfrak{p}$, we have $ap\in\mathfrak{p}$. As above, denote $S'=\{sp^n \mid s\in S, n\geq 0\}$ which contains $S$ and $p\notin S$, thus properly containing $S$. Hence $sp^n=0$ for some $n>0$, $s\in S$, so $sp^na^n=0\notin S$ for any $a\in A$. Therefore $(ap)^n\notin S$, i.e. $(ap)^n\in\mathfrak{p}$. If $ap\in\mathfrak{p}$ we are done. Otherwise, $(ap)^{n-1}\notin S$, so $(ap)^{n-1}\in\mathfrak{p}$. Repeat finitely many times to get $ap\in\mathfrak{p}$, so $ap\in\mathfrak{p}$ in the first place. Thus $\mathfrak{p}$ is indeed an ideal of $A$. It is also prime since it is the complement of a multiplicatively closed subset. Now if $\mathfrak{p}'\subsetneq\mathfrak{p}$, then $A-\mathfrak{p}\subsetneq A-\mathfrak{p}'$, i.e. $S\subsetneq A-\mathfrak{p}'$. Since $A-\mathfrak{p}'$ is a multiplicatively closed subset of $A$ not containing 0, by maximality of $S$ this cannot happen. Thus $\mathfrak{p}$ is indeed a minimal prime ideal.

Conversely, suppose $A-S$ is a minimal prime ideal of $A$. Write $\mathfrak{p}=A-S$. We want to show that $A-\mathfrak{p}$ is a maximal multiplicatively closed subset of $A$ not containing 0. Since $A-\mathfrak{p}$ is multiplicatively closed and does not contain zero, it is contained in some maximal $S'$. Hence it follows that $A-S'\subset\mathfrak{p}$, where since $S'$ is maximal, $A-S'$ is a minimal prime ideal. Now since $\mathfrak{p}$ is minimal too, we must have $A-S'=\mathfrak{p}=A-S$, i.e. $S=S'$. So $S=A-\mathfrak{p}$ is a maximal element of $\Sigma$. □

**Problem 3.7.** *Prove that*

1. *$S$ is saturated iff $A-S$ is a union of prime ideals*
2. *If $S$ is any multiplicatively closed subset of $A$, there is a unique smallest saturated multiplicatively closed subset $\overline{S}$ containing $S$, and that $\overline{S}$ is the complement in $A$ of the union of the prime ideals which do not meet $S$.*

*If $S=1+\mathfrak{a}$, find $\overline{S}$.*

*Proof.* 1.

($\Rightarrow$) Suppose $S$ is saturated. For every $x\in A-S$, we want to find a prime ideal $\mathfrak{q}$ such that $x\in\mathfrak{q}\subset A-S$. Since units are in $S$ ($\because$ if $u\cdot v=1\in S$ then $u\in S$) and $x$ is not in $S$, $x$ is a nonunit. Since $x/1$ is a nonunit too ($\because$ if $x/1\cdot a/s=1$ then $axs'=ss'\in S$ for some $s'\in S$ so $ax\in S$ so $x\in S$, but this is not the case) there exists a prime ideal $\mathfrak{p}$ of $S^{-1}A$

containing $(x/1)$, i.e. $x/1 \in \mathfrak{p} \subset S^{-1}A$. Denote $\phi : A \to S^{-1}A$ the canonical map. $\phi^{-1}(\mathfrak{p})$ is a prime ideal of $A$ containing $x$. Also, prime ideals of $S^{-1}A$ are in 1-1 correspondence with prime ideals of $A$ that do not meet $S$, so $\phi^{-1}(\mathfrak{p})$ does not meet $S$. Let $\mathfrak{q}_x = \phi^{-1}(\mathfrak{p})$. It follows that $A - S = \bigcup_{x \in A-S} \mathfrak{q}_x$.

($\Leftarrow$) Suppose $A - S$ is a union of prime ideals, say $\bigcup \mathfrak{p}_i$. We want to show that $xy \notin S$ if and only if $x \notin S$ or $y \notin S$. Suppose $xy \notin S$. Then $xy \in \bigcup \mathfrak{p}_i$, so $xy \in \mathfrak{p}_i$ for some $i$. So either $x \in \mathfrak{p}_i$ or $y \in \mathfrak{p}_i$, which implies either $x \notin S$ or $y \notin S$. Conversely, if $x \notin S$ or $y \notin S$, then $x \in \bigcup \mathfrak{p}_i$ or $y \in \bigcup \mathfrak{p}_i$ so either $x \in \mathfrak{p}_i$ or $y \in \mathfrak{p}_j$. Either way, $xy \in \mathfrak{p}_k$ for some $k = i$ or $k = j$, so $xy \in \bigcup \mathfrak{p}_i$, hence $xy \notin S$. □

*Proof.* 2. $\overline{S}$ is the intersection of saturated multiplicatively closed subsets containing $S$, where by 1 it follows that saturated sets are of the form $A - \bigcup \mathfrak{p}$. Thus, for a saturated set to contain $S$, all $\mathfrak{p}$ must not meet $S$. Hence $\overline{S} = A - \bigcup_{\mathfrak{p} \cap S = \emptyset} \mathfrak{p}$. $\overline{S}$ is unique by construction. $\overline{S}$ is saturated since given $xy \in \overline{S}$, we have $xy \in S'$ for all saturated $S'$ containing $S$, and $x, y \in S'$ for all $S'$, hence $x, y \in \overline{S}$. Also, if $x, y \in \overline{S}$ then $xy \in S'$ for all $S'$ so $\overline{S}$ is indeed saturated. □

*Proof.* Let $S = 1 + \mathfrak{a}$. By (ii), $\overline{S}$ is the complement of the union of prime ideals not meeting $1 + \mathfrak{a}$. Now if $\mathfrak{p}$ meets $1 + \mathfrak{a}$, then this is equivalent to $\mathfrak{p} + \mathfrak{a} \supset (1)$, so $\mathfrak{p} + \mathfrak{a} = A$. Hence otherwise, $\mathfrak{p} + \mathfrak{a} \subsetneq A$. Then we may write $\overline{S} = A - \bigcup_{\mathfrak{p}+\mathfrak{a} \subsetneq A} \mathfrak{p}$. Now, we claim that $\bigcup_{\mathfrak{p}+\mathfrak{a} \subsetneq A} = \bigcup_{\mathfrak{a} \subset \mathfrak{m}} \mathfrak{m}$ where $\mathfrak{m}$ are maximal ideals of $A$. Suppose $\mathfrak{p} + \mathfrak{a} \subsetneq A$. Then $\mathfrak{p} + \mathfrak{a}$ is a proper ideal, thus contained in some maximal ideal $\mathfrak{m}$. It follows that $\mathfrak{a} \subset \mathfrak{m}$. Conversely, if $\mathfrak{m} \supset \mathfrak{a}$, then obviously $\mathfrak{m} = \mathfrak{m} + \mathfrak{a} \subsetneq A$, and since $\mathfrak{m}$ is prime, it is a prime ideal satisfying the condition on the left hand side. Therefore $\overline{S} = A - \bigcup_{\mathfrak{a} \subset \mathfrak{m}} \mathfrak{m}$. □

**Problem 3.8.** *Let $S, T$ be multiplicatively closed subsets of $A$, such that $S \subset T$. Let $\phi : S^{-1}A \to T^{-1}A$ be the homomorphism which maps each $a/s \in S^{-1}A$ to $a/s$ considered as elements of $T^{-1}A$. Show that the following are equivalent:*

1. *$\phi$ bijective*
2. *For each $t \in T$, $t/1$ a unit in $S^{-1}A$*
3. *For each $t \in T$ there exists $x \in A$ such that $xt \in S$*
4. *$T$ is contained in the saturation of $S$*
5. *Every prime ideal which meets $T$ also meets $S$*

*Proof.* Suppose $\phi$ is bijective. Then $t/1 \in S^{-1}A$ maps to $t/1$ in $T^{-1}A$, where it is a unit. Suppose $t/1 \cdot u = 1$ in $T^{-1}A$. Since $\phi$ is bijective, there exists a $v \in S^{-1}A$ that maps to $u$. Therefore $\phi(t/1)\phi(v) = \phi(t/1 \cdot v) = 1$, and since $\phi$ is a bijective ring homomorphism we must have $t/1 \cdot v = 1$, thus $t/1$ is a unit in $S^{-1}A$.

Now suppose $t/1$ is a unit in $S^{-1}A$ for each $t$. Then there exists some $a/s \in S^{-1}A$ such that $t/1 \cdot a/s = 1$, hence $s'(at - s) = 0$ in $A$ for some $s' \in S$. Thus $as't = ss' \in S$, let $x = as' \in A$.

We want to show $T \subset \overline{S}$, where $\overline{S} = A - \bigcup \mathfrak{p}$ for $\mathfrak{p}$ not meeting $S$, by Exercise 7.(ii). In other words, we want to show that $T \cap \bigcup \mathfrak{p} = \emptyset$ for $\mathfrak{p}$ not meeting $S$. This is equivalent to showing $T \cap \mathfrak{p} = \emptyset$ for all $\mathfrak{p}$ not meeting $S$. Thus, we want to show if $\mathfrak{p}$ does not meet $S$, then $\mathfrak{p}$ does not meet $T$. Suppose the contrary; suppose $a \in \mathfrak{p} \cap T$. Then by assumption, there exists some $x \in A$ such that $ax \in S$, and since $ax \in \mathfrak{p}$ too, it follows that $ax \in \mathfrak{p} \cap S$.

Suppose $T \subset \overline{S}$. Then $\overline{S} = A - \bigcup \mathfrak{p}$ where the $\mathfrak{p}$ are prime ideals not meeting $S$. Since $\overline{S}$ contains $T$, this implies that prime ideals that do not meet $S$, also do not meet $T$.

**Injectivity.** We show $\ker \phi = 0$. Suppose $a/s = 0$ in $T^{-1}A$. We want to show $a/s = 0$ in $S^{-1}A$. Now this is equivalent to $s'a = 0$ for some $s' \in S$. Assume the contrary, that $\nexists s'$ such that $s'a = 0$. This implies that $\mathrm{Ann}(a) \cap S = \emptyset$. Before continuing, we prove the following claim:

**Claim.** If $I$ is an ideal of $A$, and $S$ is a multiplicatively closed subset such that $I \cap S = \emptyset$, then there exists a prime ideal $\mathfrak{p}$ containing $I$ such that $\mathfrak{p} \cap S = \emptyset$.

**Proof.** Suppose $I \cap S = \emptyset$. This means that $I \subset A \setminus S$, so we may consider the ideal $S^{-1}I \subset S^{-1}A$. Since $I$ does not meet $S$, $S^{-1}I$ does not contain any units, so is a proper ideal. Therefore it is contained in some maximal ideal $\mathfrak{m}$. Consider the inverse image $f^{-1}(\mathfrak{m})$ where $f$ is the canonical map $A \to S^{-1}A$. Then, $f^{-1}(\mathfrak{m})$ is a prime ideal of $A$ containing $I$ (since $f(I) \subset \mathfrak{m}$) which does not meet $S$ since $\mathfrak{m}$ is proper.

Using this claim, we conclude that there exists some prime ideal $\mathfrak{p}$ of $A$ that contains $\mathrm{Ann}(a)$, and satisfies $\mathfrak{p} \cap S = \emptyset$. Now by assumption, every prime ideal meeting $T$ meets $S$, so $\mathfrak{p} \cap S = \emptyset$ implies $\mathfrak{p} \cap T = \emptyset$. It follows that $\mathrm{Ann}(a) \cap T = \emptyset$, so $at \neq 0$ for all $t \in T$.

**Surjectivity.** We want to show for every $a/t \in T^{-1}A$, $a/t = a'/s'$ for some $a' \in A$ and $s' \in S$. Note that it is enough to show for all $t \in T$, there exists an $a \in A$ such that $at \in S$. (Then, given any $x/t \in T^{-1}A$, this is equal to $xa/ta$ which is in $S^{-1}A$.) Suppose not. Suppose for some $t \in T$ and for all $a \in A$, we have $at \notin S$. This implies that $(t) \cap S = \emptyset$ where $(t)$ is the ideal in $A$ generated by $t$. Thus, by our claim above, there exists some prime ideal $\mathfrak{p}$ containing $(t)$, such that $\mathfrak{p} \cap S = \emptyset$. Then by assumption this implies $\mathfrak{p} \cap T = \emptyset$, which is obviously wrong since $t \in \mathfrak{p}$. Contradiction; $\phi$ must be surjective. □

**Problem 3.9.** *Show that every minimal prime ideal of $A$ is contained in $D$. Prove that*

1. *$S_0$ is the largest multiplicatively closed subset of $A$ for which the homomorphism $A \to S_0^{-1}A$ is injective.*
2. *Every element in $S_0^{-1}A$ is either a zerodivisor or a unit.*
3. *Every ring in which every non unit is a zerodivisor is equal to its total ring of fractions.*

*Proof.* The question asking if all minimal primes are in $D$ is equivalent to that asking if all maximal multiplicatively closed subsets contain $A - D = S_0$. Suppose not, there exists some maximal $S$ (not containing zero) that did not contain $S_0$. Then there exists some $a \in S_0$ such that $a \notin S$. Then think of the set $\{a^n s \mid n \geq 0, s \in S\}$ which is multiplicatively closed and properly contains $S$. By maximality, $a^n s = 0$ for some $n > 0$ and some $s \in S$. Since $a \in S_0$, which is the set of non-zerodivisors, $a^n$ also must not be a zero divisor. However $s \neq 0$, so $a^n$ is a zero divisor, which is a contradiction. □

*Proof.* 1. First, we show $\phi : A \to S_0^{-1}A$ is indeed injective. If $a/1 = 0$ in $S_0^{-1}A$, then there exists some $s \in S_0$ such that $as = 0$. Now if $a \neq 0$, then $s$ must be a zero divisor, which is not the case. So $a = 0$, $\phi$ is injective.

Now we show that $S_0$ is the largest such set. If $S_0$ were to contain more elements, then the additional elements would be zero divisors, say $s$. Suppose $as = 0$ for some nonzero $a \in A$. Then $a/1 = 0$ since $as = 0$, but $a \neq 0$. Thus $\phi$ cannot be injective. □

*Proof.* 2. Suppose $a/s \in S_0^{-1}A$ is a nonunit. Therefore, $a \notin S_0$. ($\because a \in S_0 \Rightarrow a/s$ unit) Thus $a$ is a zero divisor, say $ab = 0$ for nonzero $b$. Then $a/s \cdot b/1 = 0$, so $a/s$ is a zero divisor. Thus nonunits are zero divisors. □

*Proof.* 3. Suppose we have a ring $A$ such that all nonunits are zero divisors. Since we know that $A \to S_0^{-1}A$ is injective, we show it is surjective. Given any $a/s$, we want to show this is equal to some $a'/1$, i.e. $(a - a's)s' = 0$ for some $s' \in S_0$. Then $s'$ is a unit, so we have $a = a's$. Also $s$ is a unit, so $a' = at$ for some $t \in S$ such that $ts = 1$. □

**Problem 3.10.** *Let $A$ be a ring.*

1. *If $A$ is absolutely flat and $S$ is any multiplicatively closed subset of $A$, then $S^{-1}A$ is absolutely flat.*
2. *$A$ is absolutely flat if and only if $A_{\mathfrak{m}}$ is a field for each maximal ideal $\mathfrak{m}$.*

*Proof.* 1. Before we prove this, we first prove the following claim:

**Claim.** For any $S^{-1}A$-module $P$, $P$ is isomorphic to $S^{-1}P$, the localization of $P$ as an $A$-module with respect to the canonical homomorphism $A \mapsto S^{-1}A$.

**Proof.** We construct an $A$-bilinear map $S^{-1}A \times P \to P$ where the first $P$ is viewed as an $A$-module. Denote ring action by $A$ as $a \cdot p = (a/1)p$. Define $\phi(a/s, p) = (a/s)p$. Then for any $k \in A$, we have $(ak/s, p) \mapsto (ak/s)p$, and $(a/s, k \cdot p) \mapsto (a/s)(k/1)p = (ak/s)p = k \cdot ((a/s)p)$. Thus $\phi$ is $A$-bilinear, so we consider the factorization via tensor product to obtain $\overline{\phi} : S^{-1}A \otimes_A P \to P$ given by $(a/s) \otimes p \mapsto (a/s)p$. Also define a map $\psi : P \to S^{-1}A \otimes_A P$ given by $p \mapsto (1/1) \otimes p$. Then observe $(a/s) \otimes p \mapsto (a/s)p \mapsto (1/1) \otimes (a/1)(1/s)p = a \cdot ((1/1) \otimes (1/s)p)$. Now if we have $x, y \in P$ and $s \in S$, then $s \cdot x = y$ if and only if $x = (1/s)y$. To see this, $s \cdot x = y$ is just $(s/1)x = y$, so we may multiply both sides by $1/s$ to get $x = (1/s)y$. Conversely, if $x = (1/s)y$ then multiply both sides by $s/1$ to get $(s/1)x = y$, which is just $s \cdot x = y$. Using this, we conclude that since $s \cdot ((1/1) \otimes (1/s)p) = (s/1)((1/1) \otimes (1/s)p) = (1/1) \otimes p$, we must have $(1/1) \otimes (1/s)p = (1/s)((1/1) \otimes p) = (1/s) \otimes p$. Therefore, $a \cdot ((1/1) \otimes (1/s)p) = a \cdot ((1/s) \otimes p) = (a/1)(1/s) \otimes p = (a/s) \otimes p$. Also $p \mapsto (1/1) \otimes p \mapsto p$, so the two maps are mutual inverses, and since $S^{-1}A \otimes_A P \cong S^{-1}P$ as $S^{-1}A$-modules by Proposition 3.5, we conclude $S^{-1}P \cong P$.

Now we return to our main problem. To show $S^{-1}A$ is absolutely flat, we must show $- \otimes_{S^{-1}A} M$ is exact for any $S^{-1}A$-module $M$. By what we proved above, $M \cong S^{-1}M$ where the latter $M$ is viewed as an $A$-module. So $- \otimes_{S^{-1}A} M \cong - \otimes_{S^{-1}A} S^{-1}M \cong S^{-1}(-) \otimes_{S^{-1}A} S^{-1}M$ where $(-)$ is also viewed as an $A$-module in the sense of the claim above. Now this is just $S^{-1}(- \otimes_A M)$, where since $A$ is absolutely flat, and since $S^{-1}(-)$ is exact, this is exact. □

*Proof.* 2. Suppose $A$ is absolutely flat. Let $\mathfrak{n}$ be the unique maximal ideal of $A_\mathfrak{m}$ and consider $x \in \mathfrak{n}$. We have a SES $0 \to (x) \to A_\mathfrak{m} \to A_\mathfrak{m}/(x) \to 0$ of $A_\mathfrak{m}$-modules where $(x)$ is the ideal of $A_\mathfrak{m}$ generated by $x$. Denote by $k = A_\mathfrak{m}/\mathfrak{n}$ the residue field. Tensor the SES with $k$ as an $A$-module to get $0 \to (x) \otimes_A k \to A_\mathfrak{m} \otimes_A k \to (A_\mathfrak{m}/(x)) \otimes_A k \to 0$. ($\because A$ is absolutely flat) Then using the identity $M \otimes_A (A/\mathfrak{a}) \cong M/\mathfrak{a}M$, we obtain a SES of $A$-modules $0 \to (x)/\mathfrak{n}(x) \to k \to k \to 0$, since $(x) \otimes A_\mathfrak{m}/\mathfrak{n} \cong (x)/\mathfrak{n}(x)$ and $A_\mathfrak{m} \otimes k \cong A_\mathfrak{m}/\mathfrak{n}A_\mathfrak{m} \cong A_\mathfrak{m}/\mathfrak{n} = k$, and $(A_\mathfrak{m}/(x)) \otimes (A_\mathfrak{m}/\mathfrak{n}) \cong (A_\mathfrak{m}/\mathfrak{n})/(A_\mathfrak{m}/\mathfrak{n}) \cdot (x) \cong A_\mathfrak{m}/\mathfrak{n} = k$. Note that an $A$-module homomorphism $k \to k$ can be viewed as a $k$-module homomorphism, since by construction of $k$, $k$ has an $A$-module structure given by the canonical map $A \to A_\mathfrak{m} \to A_\mathfrak{m}/\mathfrak{n}$. Therefore $k \to k$ is a surjective $k$-module homomorphism, i.e. a surjective linear map between 1-dimensional $k$-vector spaces. By linear algebra, this is an isomorphism. This implies that $\operatorname{im}((x)/\mathfrak{n}(x) \to k) = 0$, so $\ker((x)/\mathfrak{n}(x) \to k) = (x)/\mathfrak{n}(x) = 0$, i.e. $(x) = \mathfrak{n}(x)$. Since $\mathfrak{n}$ is an ideal of $A$, it is obviously finitely generated (by 1) and the Jacobson radical is just $\mathfrak{n}$, so we may use Nakayama's Lemma to conclude that $\mathfrak{n} = 0$. Thus the unique maximal ideal of $A_\mathfrak{m}$ is zero, which implies $A_\mathfrak{m}$ is a field.

Conversely, suppose $A_\mathfrak{m}$ is a field for all $\mathfrak{m}$. Suppose $M$ is any $A$-module. Then $M_\mathfrak{m}$ is an $A_\mathfrak{m}$-module, which is hence free. Since free modules are flat, we conclude that $M_\mathfrak{m}$ is flat for all $\mathfrak{m}$. Since flatness is a local property, $M$ is a flat $A$-module. Since $M$ was arbitrary, we conclude that $A$ is absolutely flat. □

**Problem 3.11.** *Prove the following are equivalent:*

1. $A/\mathfrak{R}$ *is absolutely flat*
2. *Every prime ideal of* $A$ *is maximal*
3. $\operatorname{Spec} A$ *is* $T_1$
4. $\operatorname{Spec} A$ *is Hausdorff*

*If these conditions are satisfied, show that* $\operatorname{Spec} A$ *is compact and totally disconnected.*

*Proof.* Suppose $A/\mathfrak{R}$ is absolutely flat. By Exercise 10 (ii), this implies that $(A/\mathfrak{R})_\mathfrak{m}$ is a field for every maximal $\mathfrak{m}$. The prime ideals of $(A/\mathfrak{R})_\mathfrak{m}$ correspond to the prime ideals of $A/\mathfrak{R}$ which are contained in $\mathfrak{m}$, but since each $(A/\mathfrak{R})_\mathfrak{m}$ is a field, there is only one prime ideal of $A/\mathfrak{R}$ contained in $\mathfrak{m}$ for each $\mathfrak{m}$. Therefore prime ideals of $A/\mathfrak{R}$ are maximal. Note that prime ideals of $A/\mathfrak{R}$ correspond to prime ideals of $A$ which contain $\mathfrak{R}$, but since

$\mathfrak{N}$ is the intersection of every prime ideal of $A$, we conclude that prime ideals of $A/\mathfrak{N}$ correspond to prime ideals of $A$. Hence every prime ideal of $A$ is maximal.

Suppose every prime ideal of $A$ is maximal. Then, $\mathfrak{p} = V(\mathfrak{p})$, so $\operatorname{Spec} A$ is $T_1$.

Suppose $X := \operatorname{Spec} A$ is $T_1$. Pick distinct points $\mathfrak{p} \neq \mathfrak{q} \in \operatorname{Spec} A$. Then WLOG pick $f \in \mathfrak{p}$ such that $f \notin \mathfrak{q}$. Note that prime ideals of $A_\mathfrak{p}$ correspond to prime ideals contained in $\mathfrak{p}$, which is just $\mathfrak{p}$ since otherwise there would be non closed points. Thus $A_\mathfrak{p}$ has a unique prime ideal $\mathfrak{p}A_\mathfrak{p}$. Also, the nilradical of $A_\mathfrak{p}$ coincides with this unique prime ideal, since the nilradical is the intersection of all prime ideals. Hence, $f/1 \in A_\mathfrak{p}$ is contained in $\mathfrak{p}A_\mathfrak{p}$, so is nilpotent, i.e. $sf^n = 0$ for some $s \in A - \mathfrak{p}$ and some $n \geq 0$. Now, $\mathfrak{q} \in X - V(f)$, and $\mathfrak{p} \in X - V(s)$. Also, $(X - V(f)) \cap (X - V(s)) = X - (V(f) \cup V(s))$. We claim that $X = V(f) \cup V(s)$, i.e. every prime ideal of $A$ either contains $f$ or $s$. Suppose not, then $f, s$ are in the complement of the prime ideal, which is a multiplicatively closed set $S$. Then $sf^n = 0$ must be in $S$, but it isn't since it's zero. Contradiction. Thus the given neighborhoods of $\mathfrak{p}$ and $\mathfrak{q}$ are disjoint, thus $\operatorname{Spec} A$ is Hausdorff.

Suppose $\operatorname{Spec} A$ is Hausdorff. Then obviously every prime ideal is maximal, hence following the reasoning of $1 \Rightarrow 2$ we conclude that $A/\mathfrak{N}$ is absolutely flat.

□

*Proof.* We have proved quasicompactness of $\operatorname{Spec} A$, together with Hausdorffness we have compactness. To prove total disconnectedness, we prove that every basic open set is closed. Suppose we have any basic open set $X_f$. It is enough to show that $V(f)$ is open. Suppose we have $\mathfrak{p} \in V(f)$. Then $f \in \mathfrak{p}$, and $f$ is nilpotent in $A_\mathfrak{p}$ as shown above. Therefore there exists some $s \in A - \mathfrak{p}$ such that $sf^n = 0$ for some $n > 0$. Then $\mathfrak{p} \in X_s = X - V(s)$, and $X_s \cap X_f$ also as above. Since $X_s$ is a neighborhood of $\mathfrak{p}$ not contained in $X_f$, we may take all such open sets and union them to conclude that $V(f)$ is open. □

**Problem 3.12.** *Show that the torsion elements of $M$ form a submodule of $M$. Show that*

1. *If $M$ is any $A$-module then $M/T(M)$ is torsion-free*
2. *If $f : M \to N$ is a module homomorphism, then $f(T(M)) \subset T(N)$.*
3. *If $0 \to M' \to M \to M''$ is exact, then $0 \to T(M') \to T(M) \to T(M'')$ is too.*
4. *If $M$ is any $A$-module, then $T(M)$ is the kernel of the mapping $x \mapsto 1 \otimes x$ of $M$ into $K \otimes_A M$ where $K$ is the field of fractions of $A$.*

*Proof.* If $m, n$ are torsion elements of $M$, then say $am = 0$ and $bn = 0$ for nonzero $a, b \in A$. Then $(m + n)ab = 0$ and $ab \neq 0$ since $A$ is an integral domain. Hence $m + n$ is a torsion element too. 0 is obviously a torsion element. The ring action is inherited from $M$, so $T(M)$ is a submodule of $M$. □

*Proof.* 1. Suppose an element $m + T(M)$ of $M/T(M)$ has torsion, say $am \in T(M)$ for some nonzero $a \in A$. Then this implies that $bam = 0$ for some nonzero $b \in A$, again $ab \neq 0$ so $m \in T(M)$ in the first place. Thus $T(M/T(M)) = 0$. □

*Proof.* 2. Suppose $m \in T(M)$. Consider $f(m)$, if $am = 0$ for some nonzero $a$, then $f(am) = af(m) = 0$ which implies $f(m) \in T(N)$. □

*Proof.* 3. **Injectivity.** Suppose $f : M' \to M$ is injective. Consider the map $m + T(M') \mapsto f(m) + T(M)$. First, this map is well-defined since if $m$ has torsion in $M'$, then its image, whatever $f$ is, must have torsion in $M$. Suppose $f(m) \in T(M)$. Then $af(m) = 0$ for some nonzero $a$, so $am = 0$, so $m \in T(M')$.

**Image = Kernel.** Denote $g : M \to M''$. By definition of the induced map, im $\subset$ ker. Suppose $k + T(M) \mapsto g(k) + T(M'') = 0$. Then $g(k) \in T(M'')$, so $ag(k) = 0$ for some nonzero $a$. Then $g(ak) = 0$, so $ak \in \ker g = \operatorname{im} f$. Therefore there exists some $m' \in M'$ such that $f(m') = ak$, so ker $\subset$ im. Thus the sequence is exact. □

*Proof.* 4. Note that $K = S^{-1}A$ for $S = A - \{0\}$. Thus, $K \otimes_A M = S^{-1}A \otimes_A M \cong S^{-1}M$. Also, $x \mapsto 1 \otimes x \mapsto 1/1 \otimes x \mapsto x/1$ via canonical maps and isomorphism. Hence if $x/1 = 0$ in $S^{-1}M$, this implies $sx = 0$ for some $s \in A - \{0\}$. Therefore $x$ is indeed in $T(M)$. The opposite direction is obvious, if $x \in T(M)$ then $ax = 0$ for some nonzero $a$, then $1 \otimes x = 1/a \otimes ax = 0$. □

**Problem 3.13.** *Let $S$ be a multiplicatively closed subset of an integral domain $A$. Show that $T(S^{-1}M) = S^{-1}(TM)$. Show TFAE:*

1. *$M$ torsion-free*
2. *$M_{\mathfrak{p}}$ torsion-free for all $\mathfrak{p}$*
3. *$M_{\mathfrak{m}}$ torsion-free for all $\mathfrak{m}$*

*Proof.* Suppose $(m/s)a = 0$ for some $a \neq 0$. Then $am/s = 0$, so $s'am = 0$ for some $s' \in S$. $A$ integral domain, so $as' \neq 0$. Therefore $m \in TM$. Hence, $m/s \in S^{-1}(TM)$. Conversely, if $m/s \in T(S^{-1}M)$ then $m \in TM$ so $am = 0$ for $a \neq 0$, hence $a(m/s) = 0$ so $m/s \in T(S^{-1}M)$. □

*Proof.* $M$ torsion free $\Rightarrow T(M) = 0$, so $(T(M))_{\mathfrak{p}} = T(M_{\mathfrak{p}}) = 0$ for all $\mathfrak{p}$. 2 to 3 is obvious. Suppose $M_{\mathfrak{m}}$ torsion free for all $\mathfrak{m}$. Then $(T(M))_{\mathfrak{m}} = T(M_{\mathfrak{m}}) = 0$ for all $\mathfrak{m}$. Being zero is a local property; $TM = 0$. □

**Problem 3.14.** *Let $M$ be an $A$-module and $\mathfrak{a}$ an ideal. Suppose $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \supset \mathfrak{a}$. Prove that $M = \mathfrak{a}M$.*

*Proof.* Note that maximal ideals containing $\mathfrak{a}$ correspond to maximal ideals of $A/\mathfrak{a}$. Consider the $A/\mathfrak{a}$-module $M/\mathfrak{a}M$. Its localizations at maximal ideals $\mathfrak{m}$ look like $M_{\mathfrak{m}}/(\mathfrak{a}M)_{\mathfrak{m}}$. Since $M_{\mathfrak{m}} = 0$ for all such $\mathfrak{m}$, we conclude that $(M/\mathfrak{a}M)_{\mathfrak{m}} = 0$ for all maximal ideals, and by local properties this implies $M/\mathfrak{a}M = 0$. □

**Problem 3.15.** *Let $A$ be a ring and let $F$ be the $A$-module $A^n$. Show that every set of $n$ generators of $F$ is a basis of $F$. Deduce that every set of generators of $F$ has at least $n$ elements.*

*Proof.* We follow the proof of the hint. Suppose $F = \langle x_1, \ldots, x_n \rangle$. Suppose $\phi : e_i \mapsto x_i$. Then $\phi$ is surjective, so let $N = \ker \phi$. We want to show $\phi$ is an isomorphism, i.e. is injective and surjective. By local properties, it is enough to show this locally, thus on a local ring $A$. Denote $k = A/\mathfrak{m}$ the residue field. Since $F$ is a free $A$-module, it is flat, and consider the SES $0 \to N \to F \xrightarrow{\phi} F \to 0$ and tensor with $k$. Since $\mathrm{Tor}_1(k, F) = 0$, we have a SES $0 \to k \otimes N \to k \otimes F \to k \otimes F \to 0$ where $F = A^n$ so $k \otimes F = k^n$. Hence we have $0 \to k \otimes N \to k^n \to k^N \to 0$, where the last map is a surjective linear transformation between $n$-dimensional vector spaces, hence is also injective. Therefore $k \otimes N = 0$, and since $k = A/\mathfrak{m}$, we have $(A/\mathfrak{m}) \otimes N \cong N/\mathfrak{m}N = 0$. In local rings the Jacobson radical is just $\mathfrak{m}$, so we have $N = 0$ by Nakayama since $N$ is finitely generated (Exercise 2.12). Every set of generators of $F$ has at least $n$ elements; suppose not, then the kernel of $\phi$ cannot be zero; take the contrapositive of what we just proved. □

**Problem 3.16.** *Let $B$ be a flat $A$-algebra. Then TFAE:*

1. *$\mathfrak{a}^{ec} = \mathfrak{a}$ for all ideals of $A$*
2. *$\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective*
3. *For every maximal ideal $\mathfrak{m}$ we have $\mathfrak{m}^e \neq (1)$*
4. *If $M$ is any nonzero $A$-module then $M_B \neq 0$*
5. *For every $A$-module $M$, the mapping $x \mapsto 1 \otimes x$ of $M$ into $M_B$ is injective*

*Proof.* If $\mathfrak{a}^{ec} = \mathfrak{a}$ for all ideals, then $\mathfrak{p}^{ec} = \mathfrak{p}$ too. By 3.16, $\mathfrak{p}$ is a contraction of a prime ideal of $B$. Therefore the map $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjecitve.

Suppose $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective. Take $\mathfrak{m} \in \operatorname{Spec} A$ to be maximal. Then since the map is surjective, there is a prime ideal $\mathfrak{n} \subset B$ such that $\phi^{-1}(\mathfrak{n}) = \mathfrak{m}$. Therefore $\phi(\phi^{-1}(\mathfrak{n})) = \phi(\mathfrak{m})$, and $\phi(\phi^{-1}(\mathfrak{n})) \subset \mathfrak{n}$, so $\mathfrak{m}^e$ is a proper ideal.

Suppose $\mathfrak{m}^e$ is proper for every maximal ideal $\mathfrak{m}$ of $A$. WTS if $M$ is any nonzero $A$-module, then $M_B \neq 0$. Pick some nonzero element $x \in M$ and let $M' = Ax$. One may consider the sequence $0 \to M' \to M$, and since $B$ is flat over $A$ we have $0 \to M'_B \to M_B$. Therefore it is enough to show $M'_B \neq 0$. Now since $M' = Ax$, it is isomorphic to $A/\mathfrak{a}$ for some ideal $\mathfrak{a}$ of $A$. Therefore $M'_B = M' \otimes_A B = (A/\mathfrak{a}) \otimes_A B \cong B/\mathfrak{a}B$, where $\mathfrak{a}B = \mathfrak{a}^e$. Since $\mathfrak{a} \subset \mathfrak{m}$ for some maximal ideal, we have $\mathfrak{a}^e \subset \mathfrak{m}^e \neq (1)$, so $M'_B \neq 0$.

Suppose $M_B \neq 0$. Consider the exact sequence $0 \to M' \to M \to M_B$ where $M'$ is the kernel of the map $x \mapsto 1 \otimes x$ from $M$ to $M_B$. To show this is injective, it is enough to show that $M' = 0$. Since $B$ is flat, we have an exact sequence $0 \to M'_B \to M_B \to (M_B)_B$. By Exercise 2.13, the map $M_B \to (M_B)_B$ is injective. Therefore $M'_B = 0$, i.e. $M' \otimes_A B = 0$. Again, the map $M' \to M' \otimes_A B$ is injective, so $M' = 0$.

Suppose $x \mapsto 1 \otimes x$ of $M \to M_B$ is injective for any $A$-module $M$. Take $M = A/\mathfrak{a}$. Then, $A/\mathfrak{a} \to B \otimes_A (A/\mathfrak{a}) \cong A/\mathfrak{a}B = A/\mathfrak{a}^e$ is injective as $A$-modules. If we denote $\phi : A \to B$, then this implies that if $\phi(k) \in \mathfrak{a}^e$, then $k \in \mathfrak{a}$. Thus if $k \in \mathfrak{a}^{ec}$, then $k \in \mathfrak{a}$, so $\mathfrak{a}^{ec} \subset \mathfrak{a}$. The opposite inclusion $\mathfrak{a}^{ec} \supset \mathfrak{a}$ is obvious, so we have equality. Since $\mathfrak{a}$ was arbitrary, this holds for every ideal. □

**Problem 3.17.** *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be ring homomorphisms. If $g \circ f$ is flat and $g$ is faithfully flat, then $f$ is flat.*

*Proof.* Suppose we have a sequence $0 \to M \to N$ of $A$-modules. Since $g \circ f$ is flat, we have $0 \to M \otimes_A C \to N \otimes_A C$. Now, this is equivalent to $0 \to M \otimes_A B \otimes_B C \to N \otimes_A B \otimes_B C$. For faithfully flat homomorphisms, they not only preserve injectivity, but if the tensored sequence is injective, it implies that the original sequence was also injective. So we have $0 \to M \otimes_A B \to N \otimes_A B$, so $f$ is indeed flat.

To prove the fact that we used, suppose we have a $B$-module homomorphism $u : P \to Q$ with kernel $K$. Then we have $0 \to K \to P \xrightarrow{u} Q$. Consider the tensor product with $C$ (which is faithfully flat), and suppose $u \otimes_B 1_C$ is injective. Thus, in the sequence $0 \to K \otimes_B C \to P \otimes_B C \to Q \otimes_B C$, we must have $K \otimes_B C = K_C = 0$. Now in Exercise 16, we proved that for faithfully flat $B$ over $A$, we must have $M_B \neq 0$ for any nonzero $A$-module $M$. Since $K_C = 0$ and $C$ is faithfully flat over $B$, we must have $K = 0$. Therefore $u$ is injective. □

**Problem 3.18.** *Let $f : A \to B$ be a flat homomorphism of rings, let $\mathfrak{q}$ be a prime ideal of $B$ and let $\mathfrak{p} = \mathfrak{q}^c$. Then $f^* : \operatorname{Spec} B_{\mathfrak{q}} \to \operatorname{Spec} A_{\mathfrak{p}}$ is surjective.*

*Proof.* [7] □

**Problem 3.19.** *Prove the following:*

1. *$M \neq 0$ iff $\operatorname{Supp} M \neq \emptyset$*
2. *$V(\mathfrak{a}) = \operatorname{Supp}(A/\mathfrak{a})$*
3. *If $0 \to M' \to M \to M'' \to 0$ is exact, then $\operatorname{Supp} M = \operatorname{Supp} M' \cup \operatorname{Supp} M''$.*
4. *If $M = \sum M_i$, then $\operatorname{Supp} M = \bigcup \operatorname{Supp} M_i$.*
5. *If $M$ is finitely generated, then $\operatorname{Supp} M = V(\operatorname{Ann} M)$.*
6. *If $M, N$ are finitely generated, then $\operatorname{Supp}(M \otimes_A N) = \operatorname{Supp} M \cap \operatorname{Supp} N$*
7. *If $M$ is finitely generated and $\mathfrak{a}$ an ideal of $A$, then $\operatorname{Supp}(M/\mathfrak{a}M) = V(\mathfrak{a} + \operatorname{Ann}(M))$.*
8. *If $f : A \to B$ is a ring homomorphism and $M$ is a finitely generated $A$-module, then $\operatorname{Supp}(B \otimes_A M) = (f^*)^{-1}(\operatorname{Supp} M)$.*

*Proof.* 1. $M \neq 0 \Leftrightarrow M_{\mathfrak{p}} \neq 0$ for some $\mathfrak{p} \Leftrightarrow \operatorname{Supp} M \neq \emptyset$. □

[7] TODO.

*Proof.* 2. Suppose $\mathfrak{p} \in V(\mathfrak{a})$. Then $\mathfrak{p} \supset \mathfrak{a}$. WTS $(A/\mathfrak{a})_\mathfrak{p}$ has a nonzero element. Take $s' \in A - \mathfrak{p}$, and consider the element $(s' + \mathfrak{a})/s$ in the local module, for $s \in A - \mathfrak{p}$. This element is zero if and only if for some $s'' \in A - \mathfrak{p}$, we have $s's'' \in \mathfrak{a}$. However, since both elements are in $A - \mathfrak{p}$ which is multiplicatively closed, we have $s's'' \in A - \mathfrak{p}$ for every element. Since $\mathfrak{a} \subset \mathfrak{p}$, this cannot be contained in $\mathfrak{a}$, so is nonzero.

Conversely, suppose we have some $\mathfrak{p} \subset A$ such that $(A/\mathfrak{a})_\mathfrak{p} \neq 0$. Since $\mathfrak{a}$ is proper, we have an element $(1 + \mathfrak{a})/1$ in $(A/\mathfrak{a})_\mathfrak{p}$ which is nonzero. This means that $1 \cdot s' \notin \mathfrak{a}$ for every $s' = A - \mathfrak{p}$, so $\mathfrak{a} \cap (A - \mathfrak{p}) = \emptyset$. Therefore, $\mathfrak{a} \subset \mathfrak{p}$, so $\mathfrak{p} \in V(\mathfrak{a})$. □

*Proof.* 3. Since localization is exact, we may consider the exact sequence $0 \to M'_\mathfrak{p} \to M_\mathfrak{p} \to M''_\mathfrak{p} \to 0$. If $\mathfrak{p} \in \operatorname{Supp} M$, then $M_\mathfrak{p} \neq 0$, so either $M'_\mathfrak{p} \neq 0$, or $M''_\mathfrak{p} \neq 0$. Hence $\mathfrak{p} \in \operatorname{Supp} M' \cup \operatorname{Supp} M''$. Proceed the same for the opposite inclusion. □

*Proof.* 4. $M = M_i + \sum_{j \neq i} M_j$, and since localization commutes with finite sums, we have $M_\mathfrak{p} = (M_i)_\mathfrak{p} + (\sum_{j \neq i} M_j)_\mathfrak{p}$. If $\mathfrak{p} \in \bigcup \operatorname{Supp} M_i$, then $\mathfrak{p} \in \operatorname{Supp} M_i$ for some $i$, and thus $(M_i)_\mathfrak{p} \neq 0$, so $M_\mathfrak{p} \neq 0$. Conversely, suppose $\mathfrak{p} \in \operatorname{Supp} M$. Since $M = \sum M_i$, we may regard $M$ as the direct limit of its submodules $M_i$, thus $M = \varinjlim M_i$. Then $M_\mathfrak{p} = (\varinjlim M_i)_\mathfrak{p} \neq 0$, where the localization functor $\underline{\mathrm{Mod}}_A \to \underline{\mathrm{Mod}}_{S^{-1}A}$ is left adjoint to the forgetful functor $\underline{\mathrm{Mod}}_{S^{-1}A} \to \underline{\mathrm{Mod}}_A$. (Forget the $S^{-1}A$-action via the canonical homomorphism $A \to S^{-1}A$.) Therefore, $(\varinjlim M_i)_\mathfrak{p} = \varinjlim (M_i)_\mathfrak{p} \neq 0$, which implies some $(M_i)_\mathfrak{p} \neq 0$ (just think of the converse). Hence $\mathfrak{p} \in \bigcup \operatorname{Supp} M_i$.

**Localization functor is left adjoint.**[8] □

*Proof.* 5. Suppose $\mathfrak{p} \in \operatorname{Supp} M$. Thus $M_\mathfrak{p} \neq 0$. WTS $\mathfrak{p}$ contains $\operatorname{Ann} M$. Suppose not, i.e. there exists some $a \in \operatorname{Ann} M$ such that $a \notin \mathfrak{p}$. In other words, $a \in A - \mathfrak{p}$, so $am = 0$ for all $m \in M$, which implies $M_\mathfrak{p} = 0$ which is a contradiction. Thus $\mathfrak{p}$ must contain $\operatorname{Ann} M$.

Conversely, suppose $\mathfrak{p} \notin \operatorname{Supp} M$. Then $M_\mathfrak{p} = 0$, thus $\operatorname{Ann}(M_\mathfrak{p}) = A_\mathfrak{p} = (\operatorname{Ann} M)_\mathfrak{p}$. This implies that $\operatorname{Ann} M \cap (A - \mathfrak{p}) \neq \emptyset$, so $\operatorname{Ann} M \not\subseteq \mathfrak{p}$, thus $\mathfrak{p} \notin V(\operatorname{Ann} M)$. □

*Proof.* 6. Suppose $\mathfrak{p} \in \operatorname{Supp}(M \otimes_A N)$. Thus $M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p} \neq 0$, so both $M_\mathfrak{p}$ and $N_\mathfrak{p}$ cannot be zero, hence $\mathfrak{p} \in \operatorname{Supp} M \cap \operatorname{Supp} N$.

Conversely, suppose $\mathfrak{p} \notin \operatorname{Supp}(M \otimes_A N)$. Then $M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p} = 0$. By Exercise 2.3, since $A_\mathfrak{p}$ is a local ring and $M_\mathfrak{p}$ and $N_\mathfrak{p}$ are both finitely generated ($\because$ localization preserves exactness; in particular surjectivity, and commutes with arbitrary direct sums) so we may conclude that either $M_\mathfrak{p} = 0$ or $N_\mathfrak{p} = 0$. Hence $\mathfrak{p} \notin \operatorname{Supp} M \cap \operatorname{Supp} N$. □

*Proof.* 7. Suppose $\mathfrak{p} \in \operatorname{Supp}(M/\mathfrak{a}M)$, i.e. $(M/\mathfrak{a}M)_\mathfrak{p} \neq 0$. WTS $\mathfrak{p}$ contains the ideal $\mathfrak{a} + \operatorname{Ann}(M)$. Suppose not; suppose there exists some $l \in \mathfrak{a} + \operatorname{Ann}(M)$ which is not contained in $\mathfrak{p}$, i.e. $l \in A - \mathfrak{p}$. An element of $(M/\mathfrak{a}M)_\mathfrak{p}$ is of the form $(k + \mathfrak{a}M)/s$ for $k \in M$ and $s \in A - \mathfrak{p}$. Suppose $l = a + n$ for $a \in \mathfrak{a}$ and $n \in \operatorname{Ann} M$. Then $kl = ka + kn = ka \in \mathfrak{a}M$, so the localized module $(M/\mathfrak{a}M)_\mathfrak{p} = 0$, which is a contradiction. Therefore $\mathfrak{p}$ must contain $\mathfrak{a} + \operatorname{Ann} M$.

Conversely, suppose $\mathfrak{p} \supset \mathfrak{a} + \operatorname{Ann} M$. Consider an element $(ms + \mathfrak{a}M)/t$ for $s, t \in S = A - \mathfrak{p}$, $0 \neq m \in M$. Then $mss' \neq 0$ for any $s' \in S$ since $ss' \in A - \mathfrak{p}$ is neither an annihilator nor an element of $\mathfrak{a}$. Therefore $\mathfrak{p} \in \operatorname{Supp}(M/\mathfrak{a}M)$. □

*Proof.* 8. Suppose $\mathfrak{q} \subset B$ is a prime, and let $\mathfrak{p} = f^{-1}(\mathfrak{q}) \subset A$, which is also a prime. Consider the localization at $\mathfrak{q}$ of the $B$-module $B \otimes_A M$. Note that $(B \otimes_A M)_\mathfrak{q} \cong B_\mathfrak{q} \otimes_B (B \otimes_A M) \cong (B_\mathfrak{q} \otimes_B B) \otimes_A M \cong B_\mathfrak{q} \otimes_A M$. Now consider the following diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A_\mathfrak{p} \arrow[r, dashed, "\exists! \tilde{f}"'] & B_\mathfrak{q} \end{tikzcd}$$

[8] TODO; don't want to focus too much on abstract nonsense right now

where the vertical arrows are the canonical homomorphisms. If we let $S = A - \mathfrak{p}$ and $T = B - \mathfrak{q}$, it follows that elements of $S$, say $s$, are mapped to $f(s)$, and via the canonical homomorphism, again are mapped to $f(s)/1$ in $B_\mathfrak{q}$. Note that $f(s) \in T$, since $s \in S$ so $s \notin \mathfrak{p} = f^{-1}(\mathfrak{q})$, so $f(s) \notin \mathfrak{q}$ which is just $f(s) \in T$. Therefore, elements of $S$ are indeed mapped to units, then by Proposition 3.1 the map factors uniquely through the canonical map $A \to A_\mathfrak{p}$. Denote this map by $\widetilde{f}$. By uniqueness, we can in fact conclude that $\widetilde{f}(a/s) = f(a)/f(s)$. Therefore, $B_\mathfrak{q}$ is naturally an $A_\mathfrak{p}$-module via $\widetilde{f}$. Hence, $B_\mathfrak{q} \otimes_A M \cong B_\mathfrak{q} \otimes_{A_\mathfrak{p}} A_\mathfrak{p} \otimes_A M \cong B_\mathfrak{q} \otimes_{A_\mathfrak{p}} M_\mathfrak{p}$. Now if $M_\mathfrak{p} = 0$, then it immediately follows that $B_\mathfrak{q} \otimes_{A_\mathfrak{p}} M_\mathfrak{p} \cong (B \otimes_A M)_\mathfrak{q} = 0$, so $\mathfrak{q} \notin (f^*)^{-1}(\operatorname{Supp} M)$ implies $f^*(\mathfrak{q}) = \mathfrak{p} \notin \operatorname{Supp} M$ which in turn implies $\mathfrak{q} \notin \operatorname{Supp}(B \otimes_A M)$.

Now conversely, suppose $\mathfrak{q} \notin \operatorname{Supp}(B \otimes_A M)$, i.e. $B_\mathfrak{q} \otimes_{A_\mathfrak{p}} M_\mathfrak{p} = 0$. We want to show that $M_\mathfrak{p} = 0$. Since $A_\mathfrak{p}$ and $B_\mathfrak{q}$ are both local rings, denote their maximal ideals as $\mathfrak{m}$, $\mathfrak{n}$, and their residue fields as $K$, $L$, respectively. By tensoring $B_\mathfrak{q} \otimes_{A_\mathfrak{p}} M_\mathfrak{p} = 0$ with $B_\mathfrak{q}/\mathfrak{n}$ as a $B_\mathfrak{q}$-module, we get $B_\mathfrak{q}/\mathfrak{n} \otimes_{A_\mathfrak{p}} M_\mathfrak{p} = L \otimes_{A_\mathfrak{p}} M_\mathfrak{p} = 0$. Recall the map $\widetilde{f} : A_\mathfrak{p} \to B_\mathfrak{q}$. By properties of the quotient ring, $\widetilde{f}$ induces a map $K \to B_\mathfrak{q}$ given by $a/s + \mathfrak{m} \mapsto f(a)/f(s)$. Compose this with the canonical map $B_\mathfrak{q} \to L$ to get a field homomorphism from $K$ to $L$, again which is given by $a/s + \mathfrak{m} \mapsto f(a)/f(s) + \mathfrak{n}$. Since a homomorphism of a field into a ring is either injective or zero, and the element $1/1 + \mathfrak{m}$ maps to $1/1 + \mathfrak{n}$, the map $K \to L$ is injective. Now via this map, $L$ is a $K$-module, so again $L \otimes_{A_\mathfrak{p}} M_\mathfrak{p} \cong L \otimes_K K \otimes_{A_\mathfrak{p}} M_\mathfrak{p} = L \otimes_K ((A_\mathfrak{p}/\mathfrak{m}) \otimes_{A_\mathfrak{p}} M_\mathfrak{p}) \cong L \otimes_K (M_\mathfrak{p}/\mathfrak{m}M_\mathfrak{p}) = 0$. Note that $M_\mathfrak{p}/\mathfrak{m}M_\mathfrak{p}$ is an $(A_\mathfrak{p}/\mathfrak{m})$-module, i.e. a $K$-vector space. By Exercise 2.27, $K$ is absolutely flat if and only if every principal ideal is idempotent. Note that $K$ is a field, so its only ideals are $(0)$ and $(1)$, both of which are idempotent. Therefore $M_\mathfrak{p}/\mathfrak{m}M_\mathfrak{p}$ is a flat $K$-module, hence tensoring $0 \to K \to L$ with this results in $0 \to M_\mathfrak{p}/\mathfrak{m}M_\mathfrak{p} \to L \otimes_K (M_\mathfrak{p}/\mathfrak{m}M_\mathfrak{p}) = 0$, so $M_\mathfrak{p}/\mathfrak{m}M_\mathfrak{p} = 0$, i.e. $M_\mathfrak{p} = \mathfrak{m}M_\mathfrak{p}$. Since $M$ is a finitely generated $A$-module, we have some surjection $A^n \to M \to 0$. Since localization is exact, we also have $(A^n)_\mathfrak{p} \cong (A_\mathfrak{p})^n \to M_\mathfrak{p} \to 0$ because the localization functor is left adjoint, hence preserves colimits. Therefore $M_\mathfrak{p}$ is a finitely generated $A_\mathfrak{p}$-module, and since $A_\mathfrak{p}$ is a local ring with unique maximal ideal $\mathfrak{m}$, its Jacobson radical is just $\mathfrak{m}$, so we may use Nakayama to conclude that $M_\mathfrak{p} = 0$. Therefore $\mathfrak{p} = f^{-1}(\mathfrak{q}) \notin \operatorname{Supp} M$, i.e. $f^*(\mathfrak{q}) \notin \operatorname{Supp} M$, which is $\mathfrak{q} \notin (f^*)^{-1}(\operatorname{Supp} M)$ as desired.

□

**Problem 3.20.** *Show that*

1. *Every prime ideal of $A$ is contracted iff $f^*$ is surjective*
2. *Every prime ideal of $B$ is an extended ideal $\Rightarrow$ $f^*$ injective.*

*Is the converse of 2. true?*

*Proof.* 1. Suppose every prime ideal of $A$ is contracted. Let $\mathfrak{p}$ be a prime ideal of $A$. Suppose $\mathfrak{p} = f^{-1}(I)$ for some ideal $I$ of $B$. Let $S = A - \mathfrak{p}$, and $T = f(S)$. Now since $S$ is multiplicatively closed, $T$ is too. Also, $T$ is disjoint from $I$, since if it were not, then there would exist an element both of $S$, and of $\mathfrak{p}$, which is not possible. Now consider the ideals of $B$ containing $I$, and that are disjoint from $T$. We may (partially) order them via inclusion. Suppose we have a chain $\mathfrak{q}_i$ of ideals of $B$ containing $I$ which are disjoint from $T$. Since the ideal $\sum \mathfrak{q}_i$ contains all $\mathfrak{q}_i$, and is disjoint from $T$, so this is indeed an upper bound. Since $I$ is an element of the poset, it is nonempty, and thus via Zorn's Lemma we conclude that there exists a maximal element $\mathfrak{m}$ that contains $I$ and is disjoint from $T$. We want to show this maximal $\mathfrak{m}$ is a prime ideal.

Suppose $ab \in \mathfrak{m}$. By definition $ab \notin T$, and since $T$ is multiplicative we have either $a \notin T$ or $b \notin T$. WLOG $a \notin T$. We claim that $a \in \mathfrak{m}$. If not, then $(\mathfrak{m}, a)$ is an ideal containing $\mathfrak{m}$ properly, and is disjoint with $T$, but this cannot happen since we chose $\mathfrak{m}$ to be maximal. Therefore $a$ must be in $\mathfrak{m}$, and thus $\mathfrak{m}$ is a prime ideal.

Now all we have to show is that $f^{-1}(\mathfrak{m}) = f^{-1}(I) = \mathfrak{p}$. Since $I \subset \mathfrak{m}$, we have $f^{-1}(I) \subset f^{-1}(\mathfrak{m})$. Now if $a \notin f^{-1}(I)$, then $a \in S$, which means that $f(a) \in T$, and thus $f(a) \notin I$. Therefore $a \notin f^{-1}(I)$. Hence $f^{-1}(\mathfrak{m}) = f^{-1}(I) = \mathfrak{p}$. Therefore, the induced map $f^*$ is surjective. The opposite direction is trivial. □

*Proof.* 2. Suppose $f^{-1}(\mathfrak{b}) = f^{-1}(\mathfrak{b}')$ for prime ideals of $B$. Since every prime ideal is extended, say $\mathfrak{b} = \mathfrak{a}^e$, and $\mathfrak{b}' = (\mathfrak{a}')^e$. By assumption we have $\mathfrak{a}^{ec} = (\mathfrak{a}')^{ec} \subset A$. Extend these to get $\mathfrak{a}^{ece} = (\mathfrak{a}')^{ece} = \mathfrak{b} = \mathfrak{b}'$.

The converse does not hold; consider the inclusion of rings $\phi : k[t^2, t^3] \to k[t]$ where $k$ is a field. Clearly, the prime ideal $(t)$ of $k[t]$ is not an extension of any ideal of $k[t^2, t^3]$. We will show that the induced map is injective. Consider the ideal $I = (t^2, t^3)$ of $k[t]$. We first show $k \subset k[t]/Ik[t]$. Since $Ik[t] \subset k[t^2, t^3]$, there are no elements of $Ik[t]$ of degree 1. Also, since $Ik[t] \cap k[t^2, t^3] = I$, $I$ does not contain elements of degree zero. Hence $k \subset k[t]/Ik[t]$. Consider the restriction map $k[t]/Ik[t] \to k$ given by killing $t$. Since $t$ is a nilpotent in $k[t]/Ik[t]$ (to be precise, $t + Ik[t]$ is) we conclude that $\operatorname{Spec} k$ is homeomorphic to $\operatorname{Spec} k[t]/Ik[t]$ by the first isomorphism theorem, and the fact that modding out nilpotents does not change spec. Using this fact, we show that $f^*$ is indeed injective.

Suppose $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are two primes of $k[t]$ that both contract to $\mathfrak{q}$, a prime of $k[t^2, t^3]$. We want to show that $\mathfrak{p}_1 = \mathfrak{p}_2$. Suppose not. Then WLOG we may pick some $f \in \mathfrak{p}_1 - \mathfrak{p}_2$. Then we have $If \subset \mathfrak{p}_1 \cap k[t^2, t^3]$ where $If = \{if \mid i \in I\}$. Now since $\mathfrak{p}_1$ and $\mathfrak{p}_2$ must agree at least on $k[t^2, t^3]$, we can say that $\mathfrak{p}_1 \cap k[t^2, t^3] \subset \mathfrak{p}_2$. Therefore $If \subset \mathfrak{p}_2$. Since by assumption $f \notin \mathfrak{p}_2$, so we may conclude $I \subset \mathfrak{p}_2$. Now since $\mathfrak{p}_1 \neq \mathfrak{p}_2$ and they both contain $I$, they must correspond to distinct prime ideals of $k[t]/Ik[t]$. However, as we have seen above, $\operatorname{Spec} k[t]/Ik[t] \simeq \operatorname{Spec} k$ which is a point, thus impossible. Hence $\mathfrak{p}_1 = \mathfrak{p}_2$, so $f^*$ is injective indeed. Thus we have found a counterexample. □

**Problem 3.21.**

1. *Show* $\phi^* : \operatorname{Spec} S^{-1}A \to \operatorname{Spec} A$ *is a homeomorphism of* $\operatorname{Spec} S^{-1}A$ *onto its image*
2. *Show that* $S^{-1}f^* : \operatorname{Spec} S^{-1}B \to \operatorname{Spec} S^{-1}A$ *is the restriction of* $f^*$ *to* $S^{-1}Y$ *and that* $S^{-1}Y = (f^*)^{-1}(S^{-1}X)$.
3. *Show* $\overline{f}^*$ *is the restriction of* $f^*$ *to* $V(\mathfrak{b})$
4. *Show* $(f^*)^{-1}(\mathfrak{p})$ *is naturally homeomorphic to* $\operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}) = \operatorname{Spec}(k(\mathfrak{p}) \otimes_A B)$ *where* $k(\mathfrak{p})$ *is the residue field of* $A_\mathfrak{p}$.

*Proof.* 1. By construction, $\phi^*$ is continuous. We need to show it is injective, and is an open map. To show it is injective, suppose we have two prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of $S^{-1}A$ whose inverse images in $A$ are the same. Suppose $\mathfrak{p} \neq \mathfrak{q}$. Then WLOG we have some $a/s \in \mathfrak{p} - \mathfrak{q}$, so $a/1 \in \mathfrak{p}$, but since $a/s \notin \mathfrak{q}$, and also $s/1 \notin \mathfrak{q}$, so $a/1 \notin \mathfrak{q}$. Hence $a/1 = \phi(a) \in \mathfrak{p} - \mathfrak{q}$, but then $a \in \phi^{-1}(\mathfrak{p})$, but not in $\phi^{-1}(\mathfrak{q})$. This is a contradiction to our assumption that they have same inverse images, so indeed $\phi^*$ is injective.

Now we show $\phi^*$ is an open map. It is enough to check openness on basis elements. Denote $Y = \operatorname{Spec} S^{-1}A$, and $X = \operatorname{Spec} A$. Suppose we have some basic open set $Y_{h/s} = \operatorname{Spec} S^{-1}A - V(h/s) = Y_{h/1}$ of $Y$. We claim that $\phi^*(Y_{h/1}) = \operatorname{im} \phi^* \cap X_h$. Suppose $\mathfrak{p}$ is a prime of $S^{-1}A$ that does not contain $h/1$. Then, it follows that $\phi^*(\mathfrak{p})$ is a prime of $A$ that does not contain $h$. Therefore one inclusion holds. Conversely, suppose $\mathfrak{p}$ is a prime of $A$ that does not contain $h$, which is in the image of $\phi^*$. This means that $\mathfrak{p}$ does not contain $h$, and also does not meet $S$. Therefore, $\mathfrak{p}$ corresponds to a prime of $S^{-1}A$ that does not contain $h/1$. Therefore it is true that $\phi^*$ sends basic open sets to basic open sets of $\operatorname{im} \phi^*$.

In particular, primes of $A_f$ correspond to primes not meeting $\{1, f, f^2, \ldots\}$, i.e. primes not containing $f$. This is just $X_f$. □

*Proof.* 2. Primes of $S^{-1}B$ correspond to primes of $B$ not meeting $f(S)$, and via $f^{-1}$, they correspond to primes (not all) of $A$ not meeting $S$. Thus they correspond to primes of $S^{-1}A$. The middle part is just $f^*$, so $S^{-1}f^*$ is indeed the restriction of $f^*$ to $S^{-1}Y$.

As above, we know that $f^*(S^{-1}Y) \subset S^{-1}X$, so $S^{-1}Y \subset (f^*)^{-1}(S^{-1}X)$. Conversely, if a prime is in $(f^*)^{-1}(S^{-1}X)$, then it means that the image through $f^*$ is in $S^{-1}X$, i.e. its pullback is a prime of $A$ not meeting $S$. Obviously the prime does not meet $f(S)$, so is in $S^{-1}Y$. □

*Proof.* 3. [9] □

*Proof.* 4.

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r, "f^*"] & \operatorname{Spec} A \\ \operatorname{Spec} B_{\mathfrak{p}} \arrow[u, hook] \arrow[r, dashed] & \operatorname{Spec} A_{\mathfrak{p}} \arrow[u, hook] \\ \operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}) \arrow[u, hook] \arrow[r, dashed, "f^*|"] & \operatorname{Spec}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}) \arrow[u, hook] \end{tikzcd}$$

The image of $\operatorname{Spec}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})$ in $\operatorname{Spec} A_{\mathfrak{p}}$ corresponds to the maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$, thus corresponds to the point $\mathfrak{p} \in \operatorname{Spec} A$. Therefore, $(f^*)^{-1}(\mathfrak{p}) = (f^*|)^{-1}(0) = \operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})$, since 0 is the only prime ideal.

Note that $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} \cong (A/\mathfrak{p}) \otimes_A B_{\mathfrak{p}} \cong (A/\mathfrak{p}) \otimes_A A_{\mathfrak{p}} \otimes_A B \cong (A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}) \otimes_A B = k(\mathfrak{p}) \otimes_A B$. □

**Problem 3.22.** *Let $A$ be a ring and $\mathfrak{p}$ a prime ideal of $A$. Then the canonical image of* $\operatorname{Spec} A_{\mathfrak{p}}$ *in* $\operatorname{Spec} A$ *is equal to the intersection of all open neighborhoods of $\mathfrak{p}$ in* $\operatorname{Spec} A$.

*Proof.* The canonical image consists of prime ideals of $A$ that are contained in $\mathfrak{p}$. Note that the intersection of all open neighborhoods of $\mathfrak{p}$ is just the intersection of all basic open sets $X_f$ that contain $\mathfrak{p}$. For $X_f$ to contain $\mathfrak{p}$ we must have $f \notin \mathfrak{p}$, i.e. $f \in S = A - \mathfrak{p}$. WTS $\bigcap_{f\in S} X_f =$ set of prime ideals in $\mathfrak{p}$. If a prime ideal is contained in $\mathfrak{p}$, then it is in all $X_f$ since $f \notin \mathfrak{p}$. Conversely, if a prime ideal does not contain $f$ for all $f \in S$, then it is contained in $\mathfrak{p}$. □

**Problem 3.23.** *Let $A$ ring, $X = \operatorname{Spec} A$ and $U$ a basic open set in $X$.*

1. *If $U = X_f$, show $A(U) = A_f$ depends only on $U$, not $f$.*
2. *Let $U' = X_g$ such that $U' \subset U$. Show $g^n = uf$ for some $n > 0$ and some $u \in A$, and define a homomorphism $\rho : A(U) \to A(U')$ by $a/f^m \mapsto au^m/g^{mn}$. Show $\rho$ depends only on $U$ and $U'$.*
3. *If $U = U'$ then $\rho$ is the identity*
4. *If $U \supset U' \supset U''$ are basic open sets then the diagram commutes*
5. *Let $x = \mathfrak{p} \in X$, show that $\varinjlim_{U\ni x} A(U) \cong A_{\mathfrak{p}}$*

*Proof.* 1. Suppose $U = X_f = X_g$. WTS $A_f \cong A_g$. Since $X_f = X_g$ implies $\sqrt{(f)} = \sqrt{(g)}$, it follows that $f^n = ag$ for some $n > 0$ and some $a \in A$, and $g^m = bf$ for some $m > 0$ and some $b \in A$. Consider the following diagram

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d] & A \arrow[d] \\ A_f \arrow[r, dashed, no head] & A_g \end{tikzcd}$$

where the vertical arrows are canonical homomorphisms. Since $f^n/1$ is invertible in $A_f$, thus $ag/1 = a/1 \cdot g/1$ is invertible in $A_f$, say $a/1 \cdot g/1 \cdot u = 1$ in $A_f$. Then $g/1 \cdot (a/1 \cdot u) = 1$,

[9] TODO: not at all rigorous! Check MathSE

so $g/1$ is a unit in $A_f$. It follows that all $g^n/1$ are units in $A_f$. By the universal property of localization, the map $A \to A_f$ factors uniquely through $A \to A_g$, and vice versa. If we take the map to be $a/f^n \mapsto a/g^n$, then this satisfies all the desired properties, hence by uniqueness is 'the' isomorphism between $A_f$ and $A_g$. Thus $A(U)$ depends only on $U$. $\square$

*Proof.* 2. Suppose $X_g \subset X_f$, i.e. $V(f) \subset V(g)$. This means that $\sqrt{(g)} \subset \sqrt{(f)}$, so since $(g) \subset \sqrt{(g)} \subset \sqrt{(f)}$, we have $g \in \sqrt{(f)}$, i.e. $g^n = uf$ for some $n > 0$ and $u \in A$. Now consider the map $A \to A_g$ given by the canonical homomorphism. We want to show that $f$ maps to a unit; as above, since $g^n/1$ is a unit, and $g^n = uf$, we have $uf/1$ a unit in $A_g$, which implies $f/1$ is indeed a unit in $A_g$. Therefore all $f^n/1$ are units in $A_g$, thus by the universal property of localization, the map $A \to A_g$ factors through the canonical homomorphism $A \to A_f$ to yield a map $A_f \to A_g$. Now if we define $\rho : a/f^m \mapsto au^m/g^{mn}$, since this commutes with $A \to A_f$ and $A \to A_g$, by the uniqueness, the map $A_f \to A_g$ is given by this. By what we showed in (i), this map is unique (up to isomorphism of (co)domain). $\square$

*Proof.* 3. If $U = U'$, then $\rho$ is an isomorphism by what we showed above. Therefore, up to isomorphism, $\rho$ is the identity map. $\square$

*Proof.* 4. Suppose $X_f \supset X_g \supset X_h$. Then $\sqrt{(f)} \supset \sqrt{(g)} \supset \sqrt{(h)}$, so suppose $h^n = ug$, $g^m = vf$ and $h^\ell = wf$. The map $A_f \to A_h$ is given by $a/f^k \mapsto aw^k/h^{\ell k}$, the map $A_f \to A_g$ by $a/f^k \mapsto av^k/g^{mk}$ and $A_g \to A_h$ is given by $a/g^j \mapsto au^j/h^{nj}$. Composing, we get $a/f^k \mapsto av^k/g^{mk} \mapsto av^k u^{mk}/h^{nmk}$, and on the other hand we have $a/f^k \mapsto aw^k/h^{\ell k}$. Now by the relations, we have $h^{nm} = u^m g^m = u^m vf$, and $h^\ell = wf$. Therefore, $(u^m v)^k h^{k\ell} - w^k h^{nmk} = (u^m v)^k w^k f^k - w^k (u^m v)^k f^k = 0$, so indeed the maps commute. $\square$

*Proof.* 5. First, we show that $\varinjlim_{U \ni x} A(U)$ is valid, i.e. the $A(U)$ for $U$ containing $x$ form a direct system, where the homomorphisms are the restriction maps defined in (ii). Suppose we have $X_f$ and $X_g$ basic opens containing $\mathfrak{p}$. Since $f, g \notin \mathfrak{p}$, by primeness it follows that $fg \notin \mathfrak{p}$ too. Also, we know that $X_f \cap X_g = X_{fg}$, so indeed the system is directed (there exists some $X_h$ such that $X_f$ and $X_g$ both restrict to). Also, we need to show that the restrictions commute whenever $X_f \subset X_g \subset X_h$, but we showed this in (iv). Therefore the limit $\varinjlim_{U \ni x} A(U)$ indeed exists.

Now for every $A(X_f) = A_f$, we construct a map $\alpha_f : A_f \to A_\mathfrak{p}$ by factoring the canonical homomorphism $A \to A_\mathfrak{p}$ via the canonical homomorphism $A \to A_f$. This is possible since 1 and $f^n$ map to units in $A_\mathfrak{p}$. Note that the restriction homomorphism from any $A_g$ to $A_f$ (where $X_g \supset X_f$ of course) may also be obtained by factoring the canonical homomorphism $A \to A_f$ through the canonical homomorphism $A \to A_g$. Since $X_g \supset X_f$, we have $f^n = ga$ for some $a \in A$, where $f^n/1 = ga/1$ is a unit in $A_f$, hence the $g^k$ maps to units in $A_f$. Using this, we may consider the following diagram

$$\begin{tikzcd}
 & A_\mathfrak{p} & \\
 & A \arrow[u, "\eta_\mathfrak{p}"'] \arrow[dl, "\eta_g"'] \arrow[dr, "\eta_f"] & \\
A_g \arrow[uur, bend left, dashed, "\alpha_g"] \arrow[rr, dashed, "\rho_{gf}"] & & A_f \arrow[uul, bend right, dashed, "\alpha_f"']
\end{tikzcd}$$

where the triangles commute. Therefore, $\alpha_f \circ \rho_{gf} \circ \eta_g = \alpha_f \circ \eta_f = \eta_\mathfrak{p}$, thus $\alpha_f \circ \rho_{gf}$ commutes with $\eta_g$ and $\eta_\mathfrak{p}$. By uniqueness of $\alpha_g$, we may conclude $\alpha_g = \alpha_f \circ \rho_{gf}$. By Exercise 2.16, there exists a unique homomorphism $\alpha : \varinjlim_{U \ni x} A(U) \to A_\mathfrak{p}$ such that $\alpha_f = \alpha \circ \rho_f$ for all $f$. Here, $\rho_f$ is the restriction of $\bigoplus_{g \notin \mathfrak{p}} A_g \to \varinjlim_{U \ni x} A(U)$ to $A_f$. We claim that $\alpha$ is an isomorphism.

**Surjectivity.** Suppose we have any $a/s \in A_\mathfrak{p}$. Consider $U = X_s$ which is a neighborhood of $\mathfrak{p}$, and consider $A(X_s) = A_s$. Since $\alpha_s = \alpha \circ \rho_s$, and since $\alpha_s(a/s) = a/s$, we may conclude that $\rho_s(a/s)$ maps via $\alpha$ to $a/s$. Therefore $\alpha$ is surjective.[10]

**Injectivity.** Suppose $\alpha(s) = 0$ for some $s \in \varinjlim_{U \ni x} A(U)$. By Exercise 2.15, $s = \rho_i(s_i)$ for some $i$. Therefore, $\alpha(\rho_i(s_i)) = \alpha_i(s_i) = 0$, where $\alpha_i : A_i \to A_\mathfrak{p}$ is given as above. Also, $\alpha_i(s_i) = 0$ implies that $\rho_{ij}(s_i) = 0$ for some $j$. Therefore, if any $s_k$ is in the same equivalence class of $s_i$ in the limit, they both restrict to 0 for some $\ell \geq k, j$. Hence if $\alpha(s) = 0$, then $s = 0$. □

**Problem 3.24.** *Show that the presheaf above has the following property: Let $(U_i)_{i\in I}$ be a covering of $X$ by basic open sets. For each $i \in I$, let $s_i \in A(U_i)$ such that for each pair $i, j$ the images of $s_i$ and $s_j$ in $A(U_i \cap U_j)$ are equal. Then there exists a unique $s \in A = A(X)$ whose image in $A(U_i)$ is $s_i$ for all $i \in I$.*

*Proof.* [11] □

**Problem 3.25.** *Show $h^*(T) = f^*Y \cap g^*Z$.*

*Proof.* By Exercise 21, the fiber $(h^*)^{-1}(\mathfrak{p}) = \operatorname{Spec}(k \otimes_A (B \otimes_A C))$ where $k = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. Since $\mathfrak{p}$ is in the image of $h^*$ if and only if the fiber is nonempty, this means that $k \otimes_A (B \otimes_A C) \neq 0$. The same holds for $f$ and $g$ too; $\mathfrak{p}$ is in the image of $f^*$ if and only if $k \otimes_A B \neq 0$ and is in the image of $g^*$ if and only if $k \otimes_A C \neq 0$. Since $k \otimes_A (B \otimes_A C) \cong B \otimes_A k \otimes_A C \cong B \otimes_A k \otimes_k k \otimes_A C$, and both $k \otimes_A B$ and $k \otimes_A C$ are $k$-vector spaces, this is nonzero if and only if $B \otimes_A k$ and $C \otimes_A k$ are both nonzero. Hence $\mathfrak{p} \in h^*(T)$ if and only if $\mathfrak{p} \in f^*Y \cap g^*Z$. □

**Problem 3.26.** *Show that $f^*(\operatorname{Spec} B) = \bigcap_\alpha f^*_\alpha(\operatorname{Spec}(B_\alpha))$.*

*Proof.* Suppose $\mathfrak{p} \in \operatorname{Spec} A$. Then $(f^*)^{-1}(\mathfrak{p}) = \emptyset$ iff $\operatorname{Spec}(B \otimes_A k) = \emptyset$ iff $\varinjlim(B_\alpha \otimes_A k) = 0$ iff $B_\alpha \otimes_A k = 0$ for some $\alpha$ iff $(f^*_\alpha)^{-1}(\mathfrak{p})$ for some $\alpha$. Therefore, $\mathfrak{p} \in f^*(\operatorname{Spec} B)$ iff $(f^*)^{-1}(\mathfrak{p}) \neq \emptyset$ iff $(f^*_\alpha)^{-1}(\mathfrak{p}) \neq \emptyset$ for all $\alpha$ iff $\mathfrak{p} \in \bigcap_\alpha f^*_\alpha(\operatorname{Spec} B_\alpha)$. □

**Problem 3.27.**

1. *Let $f_\alpha : A \to B_\alpha$ be a family of $A$-algebras, and let $f : A \to B$ be their tensor product. Show that the equality of Exercise 26 holds.*
2. *Let $f_\alpha : A \to B_\alpha$ a finite family of $A$-algebras and let $B = \prod_\alpha B_\alpha$. Define $f : A \to B$ by $x \mapsto (f_\alpha(X))$. Show $f^*(\operatorname{Spec} B) = \bigcup_\alpha f^*_\alpha(\operatorname{Spec} B_\alpha)$.*
3. *Show constructible topology on $X = \operatorname{Spec} A$ is finer than Zariski topology.*
4. *Show $X$ endowed with constructible topology is quasicompact.*

*Proof.* 1. Name the index set of $\alpha$ as $\Lambda$. Recall that $B = \varinjlim_{J \subset \Lambda} B_J$ where the direct limit runs over all finite subsets $J$ of $\Lambda$, and $B_J = \bigotimes_{\alpha \in J} B_\alpha$. By Exercise 26, we have $f^*(\operatorname{Spec} B) = \bigcap_{J \subset \Lambda} f^*_J(\operatorname{Spec} B_J)$, again where $J$ runs over finite subsets of $\Lambda$. By Exercise 25, $f^*_J(\operatorname{Spec} B_J) = \bigcap_{\alpha \in J} f^*_\alpha(\operatorname{Spec} B_\alpha)$, so combining these together, we have $f^*(\operatorname{Spec} B) = \bigcap_{\alpha \in \Lambda} f^*_\alpha(\operatorname{Spec} B_\alpha)$. □

*Proof.* 2. Suppose $\mathfrak{p} \in \operatorname{Spec} A$. Then $\mathfrak{p} \in f^*(\operatorname{Spec} B)$ iff $(f^*)^{-1}(\mathfrak{p})$ nonempty, which is $\operatorname{Spec}(k \otimes_A B)$ with $k$ the residue field of $\mathfrak{p}$. Now since $\prod_\alpha B_\alpha = \bigoplus_\alpha B_\alpha$ since finite, $k \otimes_A B = \bigoplus(k \otimes_A B_\alpha)$ and $\operatorname{Spec}(k \otimes_A B) = \bigsqcup_\alpha \operatorname{Spec}(k \otimes_A B_\alpha)$ by Exercise 1.22. Thus, iff $(f^*_\alpha)^{-1}(\mathfrak{p})$ is nonempty for some $\alpha$, i.e. $\mathfrak{p} \in f^*_\alpha(\operatorname{Spec} B_\alpha)$ for some $\alpha$. Therefore, $f^*(\operatorname{Spec} B) = \bigcup_\alpha f^*_\alpha(\operatorname{Spec} B_\alpha)$. □

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[10] Is $\alpha_s$ really the inclusion?

[11] TODO: Just calculation I think...

*Proof.* 3. As we have shown above, arbitrary intersections and finite unions of sets of the form $f^*(\operatorname{Spec} B)$ for some $f : A \to B$, again is of the same form. The closed sets of the Zariski topology are given by $V(E)$ for $E \subset A$. Now since we may write any $V(E)$ as $\bigcap_{f \in E} V(f)$, it is enough to check if the sets $V(f)$ are closed in the constructible topology. Consider the canonical homomorphism $\phi : A \to A/(f)$, then the image of $\phi^*$ corresponds to prime ideals of $A$ containing the ideal $(f)$. Note that if a prime contains $f$, then it contains $(f)$, and vice versa. Therefore, the image of $\phi^*$ corresponds to $V(f)$, thus the sets are indeed closed in the constructible topology. Now a basic open set $X_f$ of $X = \operatorname{Spec} A$ in the Zariski topology corresponds to prime ideals of $A$ not containing powers of $f$, or equivalently, prime ideals not containing the ideal $(f)$. Since we showed that Zariski-closed sets are constructibly closed, this means that Zariski-open sets are also constructibly open. However, the basic open set $X_f$ is both closed and open in the constructible topology, since it corresponds to the image of $\psi^*$ where $\psi : A \to A_f$ is the canonical localization map. Basic open sets in the Zariski topology are in general not closed. Since we have showed that the Zariski topology is coarser than the constructible topology, this shows that it is (in general) *strictly coarser*, i.e. the constructible topology is in general strictly finer than the Zariski topology. □

*Proof.* 4. [12] □

**Problem 3.28.**

1. *For each $g \in A$, the set $X_g$ is both open and closed in the constructible topology*
2. *$C'$ the smallest topology on $X$ for which $X_g$ are clopen; Show that it is Hausdorff*
3. *Show that $X_C \to X_{C'}$ is a homeomorphism; i.e. the constructible topology is the coarsest topology for the sets $X_g$ to be clopen*
4. *$X_C$ is compact Hausdorff and totally disconnected*

*Proof.* 1. We have showed this above. □

*Proof.* 2. Suppose we have $\mathfrak{p} \neq \mathfrak{q} \in X$, and pick two open neighborhoods $X_f$ and $X_g$. Then $X_f - X_{fg}$ and $X_g - X_{fg}$ are disjoint neighborhoods of the points, so is Hausdorff. □

*Proof.* 3. [13] □

*Proof.* 4. By iv) of the previous exercise and iii), it is compact and Hausdorff. For every $\mathfrak{q}$ different from $\mathfrak{p}$, we may find neighborhoods of $\mathfrak{q}$ not containing $\mathfrak{p}$. These neighborhoods are unions of basic opens, which are closed. Therefore, the union of all such neighborhoods is closed, and is $X - \{\mathfrak{p}\}$, so $\mathfrak{p}$ is clopen. Therefore it is totally disconnected. □

**Problem 3.29.** *Let $f : A \to B$ a ring homomorphism. Show $f^* : \operatorname{Spec} B \to \operatorname{Spec} A$ is a continuous closed mapping for the constructible topology.*

*Proof.* [14] □

**Problem 3.30.** *Show Zariski topology and constructible topology on* Spec $A$ *are same iff $A/\mathfrak{R}$ is absolutely flat.*

*Proof.* [15] □

[12]TODO
[13]TOOD
[14]TODO
[15]TODO

## 4. Primary Decomposition

**Problem 4.1.** *If $\mathfrak{a}$ has primary decomposition then* $\mathrm{Spec}(A/\mathfrak{a})$ *has finitely many irreducible components.*

*Proof.* The irreducible components of $\mathrm{Spec}(A/\mathfrak{a})$ correspond to minimal prime ideals of $A$ containing $\mathfrak{a}$, which correspond to minimal elements of $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ where $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$, $\mathfrak{p}_i := r(\mathfrak{q}_i)$. Therefore it must be finite. □

**Problem 4.2.** *If $\mathfrak{a} = r(\mathfrak{a})$ then $\mathfrak{a}$ has no embedded prime ideals.*

*Proof.* Suppose $\mathfrak{a}$ has a primary decomposition, or else this exercise is meaningless. Write $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$. Then $r(\mathfrak{a}) = \bigcap_{i=1}^n r(\mathfrak{q}_i) = \bigcap_{i=1}^n \mathfrak{p}_i = \mathfrak{a}$. We want to show $\mathfrak{a}$ has no embedded prime ideals, i.e. every prime $\mathfrak{p}_i$ is minimal. If we had some $\mathfrak{p}_i \subsetneq \mathfrak{p}_j$, then $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{p}_i = \bigcap_{i \neq j} \mathfrak{p}_i$, which contradicts the uniqueness of the primary decomposition. Therefore every associated prime is minimal, i.e. no embedded primes. □

**Problem 4.3.** *If $A$ is absolutely flat, every primary ideal is maximal.*

*Proof.* Recall that $A$ is absolutely flat if and only if every principal ideal is idempotent. Suppose we have a primary $\mathfrak{q} \subset A$. Since primary ideals are proper, we have some $f \in A \backslash \mathfrak{q}$. Since every principal ideal is idempotent, we have $\langle f \rangle = \langle f^2 \rangle$ which implies $f = af^2$ for some $a \in A$. Therefore we have $f(1 - af) = 0 \in \mathfrak{q}$, where $f \notin \mathfrak{q}$, so $(1 - af)^n \in \mathfrak{q}$ for some $n > 0$. This means that $\overline{1} - \overline{af}$ is nilpotent in the quotient $A/\mathfrak{q}$. Since $\overline{af} = \overline{1} - (\overline{1} - \overline{af})$, and $\overline{1}$ is a unit, we conclude that $\overline{af}$ is a unit in $A/\mathfrak{q}$. Thus $\overline{f}$ is a unit in $A/\mathfrak{q}$. Since the choice of $f$ was arbitrary, this holds for any nonzero element in $A/\mathfrak{q}$, i.e. every nonzero element of $A/\mathfrak{q}$ is a unit. Therefore $A/\mathfrak{q}$ is a field, and $\mathfrak{q}$ must be maximal. □

**Problem 4.4.** *In $\mathbb{Z}[t]$, the ideal $\mathfrak{m} = (2, t)$ is maximal and the ideal $\mathfrak{q} = (4, t)$ is $\mathfrak{m}$-primary, but is not a power of $\mathfrak{m}$.*

*Proof.* We first show $\mathfrak{m}$ is maximal. Consider the homomorphism $\phi : \mathbb{Z}[t] \to \mathbb{F}_2$ defined by $f \mapsto f(0) \mod 2$. The kernel of this is $f \in \mathbb{Z}[t]$ such that $f(0)$ is divisible by 2, which is of the form $f = tg + 2k$ for $g \in \mathbb{Z}[t]$ and $k \in \mathbb{Z}$. Therefore $\ker \phi \subset \mathfrak{m}$. Surely, $\ker \phi \supset \mathfrak{m}$ and $\phi$ is onto, so $\mathfrak{m}$ is maximal. Now we show the radical of $\mathfrak{q}$ is $\mathfrak{m}$.

Suppose we have $f \in \mathfrak{m}$, of the form $f = tg + 2h$ for $g, h \in \mathbb{Z}[t]$. Elements of $\mathfrak{q}$ are of the form $tg + 4h$ for $g, h \in \mathbb{Z}$, so we have $f^2 = t^2 g^2 + 4tgh + 4h^2$, which is in $\mathfrak{q}$. Therefore $\mathfrak{m} \subset r(\mathfrak{q})$. By maximality, it follows that $\mathfrak{m} = r(\mathfrak{q})$, and by Proposition 4.2 we have $\mathfrak{q}$ primary. Now elements such as $t + 4$ are in $\mathfrak{q}$ but not in $\mathfrak{m}^2$. Elements of $\mathfrak{m}^2$ are of the form $(tg + 2k)(th + 2l) = t^2 hg + 2t(kh + lg) + 4lk$ where $g, h \in \mathbb{Z}[t]$ and $k, l \in \mathbb{Z}$, so $t + 4$ cannot be in this. □

**Problem 4.5.** *In the polynomial ring $K[x, y, z]$ where $K$ is a field and $x, y, z$ are independent indeterminates, let $\mathfrak{p}_1 = (x, y)$, $\mathfrak{p}_2 = (x, z)$, $\mathfrak{m} = (x, y, z)$; $\mathfrak{p}_1, \mathfrak{p}_2$ are prime and $\mathfrak{m}$ is maximal. Let $\mathfrak{a} = \mathfrak{p}_1 \mathfrak{p}_2$. Show that $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a reduced primary decomposition of $\mathfrak{a}$. Which components are isolated and which are embedded?*

*Proof.* Denote $A = K[x, y, z]$. It is clear that $A/\mathfrak{p}_1 \cong K[z]$ and $A/\mathfrak{p}_2 \cong K[y]$, which are both integral domains since $K$ is a field. Therefore the $\mathfrak{p}_i$ are prime. Also, $A/\mathfrak{m} \cong K$ so $\mathfrak{m}$ is maximal. We first show $\mathfrak{p}_1 \mathfrak{p}_2 = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$.

Note that $\mathfrak{p}_1 \mathfrak{p}_2 = (x, y)(x, z) = (x^2, xy, xz, yz)$. Also, if $f \in \mathfrak{p}_1 \mathfrak{p}_2$ then it is quite clear that $f \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. Conversely, suppose $f \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. Since $f \in \mathfrak{p}_1$ it follows that $f = Px + Qy$. Also, since $f \in \mathfrak{p}_2$ we must have $f = Px + Q'zy$. Since $f \in \mathfrak{m}^2 = (x^2, y^2, z^2, xy, yz, xz)$, it follows that $P$ also has either $x, y, z$ so $f$ is either in $(x^2, zy)$ or $(xy, zy)$ or $(xz, zy)$ which are all in $\mathfrak{p}_1 \mathfrak{p}_2$.

By Proposition 4.2, $\mathfrak{m}^2$ is $\mathfrak{m}$-primary so indeed it is a primary decomposition. The radicals are clearly all distinct, and $(x, y)$ does not contain $(x, z) \cap (x^2, y^2, z^2, xy, yz, xz)$, for example $z^2$, and same for $(x,z)$, and also $(x,y)\cap(x,z)$ is not contained in $\mathfrak{m}^2$, for example $x$. Therefore it is a reduced primary decomposition. The associated primes are $\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{m}$ so the isolated primes are $\mathfrak{p}_1$ and $\mathfrak{p}_2$ where $\mathfrak{m}$ is embedded.

The geometric picture of $\mathfrak{a}$ is the $y$ axis together with the $z$ axis, and the origin. Each corresponds to $\mathfrak{p}_2$, $\mathfrak{p}_1$ and $\mathfrak{m}$, respectively. $\square$

**Problem 4.6.** *Let $X$ be an infinite compact Hausdorff space, $C(X)$ the ring of real valued continuous functions on $X$. Is the zero ideal decomposable in this ring?*

*Proof.* Suppose $(0)=\bigcap_{i=1}^n \mathfrak{q}_i$, a primary decomposition. Let $\mathfrak{p}_i := r(\mathfrak{q}_i)$. Since the $\mathfrak{p}_i$ are prime, they are contained in maximal ideals $\mathfrak{m}_i$, respectively. Let $\mathfrak{m}_i$ correspond to functions vanishing at $x_i$, respectively. Since $X$ is infinite, there exists some $x\in X$ such that $x\neq x_i$ for all $i$. Let $U$ and $V$ be neighborhoods of $x$ and $x_i$, respectively, such that $U\cap V=\emptyset$. Now take an open cover of $X$, which has a finite subcover. If cover elements have either $x$ or $x_i$, subtract the points to get open sets. Together with these open sets, and add $U$ and $V$ to get a finite open cover of $X$. Take a partition of unity subordinate to this open cover, then there exists $f,g\in C(X)$ where $f(x)\neq 0$ and $f=0$ outside $U$, and $g(x_i)\neq 0$ for all $i$ and $g=0$ outside $V$. It follows that $fg=0$, and since $g$ does not vanish on each $x_i$, it follows that $g\notin\mathfrak{m}_i$ for all $i$. This implies that $g^m\notin\mathfrak{m}_i$ for all $m>0$, which implies $g^m\notin\mathfrak{q}_i$ for all $i$, and since 0 is in each primary ideal, we have $f\in\mathfrak{q}_i$ for all $i$. Therefore $f\in\bigcap_{i=1}^n\mathfrak{q}_i=(0)$, but by definition $f\neq 0$. Contradiction. $\square$

**Problem 4.7.** *Prove the following*

1. $\mathfrak{a}[x]=\mathfrak{a}^e$
2. *If $\mathfrak{p}$ prime in $A$, then $\mathfrak{p}[x]$ is a prime ideal in $A[x]$.*
3. *If $\mathfrak{q}$ is $\mathfrak{p}$-primary in $A$, then $\mathfrak{q}[x]$ is $\mathfrak{p}[x]$-primary.*
4. *If $\mathfrak{a}=\bigcap_{i=1}^n\mathfrak{q}_i$ is a minimal primary decomposition, then $\mathfrak{a}[x]=\bigcap_{i=1}^n\mathfrak{q}_i[x]$ is a minimal primary decomposition in $A[x]$.*
5. *If $\mathfrak{p}$ is a minimal prime ideal of $\mathfrak{a}$, then $\mathfrak{p}[x]$ is a minimal prime ideal of $\mathfrak{a}[x]$.*

*Proof.* 1. Suppose $f=a_0+a_1x+\cdots+a_nx^n\in\mathfrak{a}[x]$. If we denote $\phi: A\to A[x]$ the inclusion, then $f=\phi(a_0)+\phi(a_1)x+\cdots+\phi(a_n)x^n\in\phi(\mathfrak{a})A[x]$. The converse is also quite obvious. $\square$

*Proof.* 2. Note that $A[x]/\mathfrak{p}[x]\cong(A/\mathfrak{p})[x]$ via 1st isomorphism. It suffices to show that $(A/\mathfrak{p})[x]$ is a domain. Since $A/\mathfrak{p}$ is a domain, it has no nonzero zerodivisors. For $f$ and $g$ in $(A/\mathfrak{p})[x]$ to be zero divisors, the leading terms must be zero divisors, but this cannot happen. Hence it is an integral domain, so $\mathfrak{p}[x]$ is prime. $\square$

*Proof.* 3. Suppose $\mathfrak{q}$ is a primary ideal of $A$ such that $r(\mathfrak{q})=\mathfrak{p}$. We first show $\mathfrak{q}[x]$ is a primary ideal of $A[x]$. Suppose $f=a_0+a_1x+\cdots+a_nx^n$ and $g=b_0+b_1x+\cdots+b_mx^m$, and $fg\in\mathfrak{q}[x]$. Since $fg=\sum_{i=0}^{n+m}\sum_{j+k=i}a_jb_kx^i$, it follows that every $\sum_{j+k=i}a_jb_k\in\mathfrak{q}$. $\square$

## 5. Integral Dependence and Valuations

## 6. Chain Conditions

## 7. Noetherian Rings

## 8. Artin Rings

## 9. Discrete Valuation Rings and Dedekind Domains

## 10. Completions

## 11. Dimension Theory