PROVING SERRE DUALITY

ANTHONY HOJIN LEE

I will follow Hartshorne's proof of Serre duality for coherent sheaves on projective schemes. I will try my best to go beyond Hartshorne's book and provide details about things that may be confusing in the first read, possibly excluding some routine homological algebra.

1. Serre duality

Serre duality claims that there exists a duality theorem for a sheaf denoted ω°, and later on we will see that this sheaf exists for projective schemes. We will moreover show that this ω° is isomorphic to the canonical line bundle in our case of interest.

Definition 1.1. Let X be a complete scheme of dimension n over a field k. A *dualizing sheaf* ω_X° on X is a coherent $\mathcal{O}_X$-module together with a k-linear map $t : H^n(X, \omega_X^\circ) \to k$, such that for all coherent $\mathcal{O}_X$-modules $\mathcal{F}$ on X, the natural pairing

$$\operatorname{Hom}_X(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X^\circ)$$

composed with t is a perfect pairing, i.e., it gives rise to a k-vector space isomorphism $\operatorname{Hom}_X(\mathcal{F}, \omega_X^\circ) \xrightarrow{\sim} H^n(X, \mathcal{F})^\vee$.

Remark 1.2. The natural pairing above is given by sending $\varphi \in \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\circ)$ to the k-linear map $H^n(X, \mathcal{F}) \to H^n(X, \omega_X^\circ)$ induced by φ. That the map is k-linear is obvious when one accepts the fact that $H^n(-)$ is a functor from $\mathsf{Mod}_{\mathcal{O}_X}$ to $\mathsf{Mod}_{\mathcal{O}_X(X)}$. (If you read Hartshorne's book carefully, you may notice that cohomology for $\mathcal{O}_X$-modules was never even defined, and it was only defined for sheaves of abelian groups. We define cohomology as the right derived functors of the global sections functor $\Gamma(-) : \mathsf{Mod}_{\mathcal{O}_X} \to \mathsf{Mod}_{\mathcal{O}_X(X)}$ instead of the one taking values in Ab. Nonetheless the fact that applying the forgetful functor to Ab results in the usual abelian cohomology groups is proven in Hartshorne, so there is no problem in our definition. That the map is k-linear follows from this observation together with the remark below.)

Remark 1.3. The cohomology groups that appear above all have the structure of a k-vector space. As we take injective resolutions of $\mathcal{O}_X$-modules and their global sections, it suffices to show that the global sections of $\mathcal{O}_X$ has a k-algebra structure, which will induce a k-module structure on all cohomology groups. This is trivially true, as by definition X comes with a structure morphism $f : X \to \operatorname{Spec} k$, which includes a sheaf (of rings) morphism $f^\# : \mathcal{O}_{\operatorname{Spec} k} \to f_*\mathcal{O}_X$. Taking global sections gives a ring homomorphism $k \to \mathcal{O}_X(X)$, which naturally endows $\mathcal{O}_X(X)$ with a k-algebra structure. In fact, it follows that every group of local sections of $\mathcal{O}_X$ has a k-algebra structure which can be used to see that $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\circ)$ has a k-module structure.

Remark 1.4. Let us go in-depth about the conditions in the definition above. We have seen that there is a (k-bilinear) pairing, using the fact that X is a k-scheme. We have not seen how the condition of X being complete and the sheaf of modules being coherent are used, yet. Recall that X being complete means that the structure morphism $X \to \operatorname{Spec} k$ is proper, and this implies that the cohomology groups of coherent $\mathcal{O}_X$-modules are finite dimensional k-vector spaces. By definition of a perfect pairing, it follows that $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\circ)$ must also be finite dimensional for there to exist one. This is indeed true; for $\mathcal{F}$ and $\mathcal{G}$ coherent sheaves over a noetherian scheme (which X is, being finite type over a field) the internal hom $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is coherent, hence its global section $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a finite dimensional k-vector space.

Remark 1.5. Just in case, I will include the proof that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a coherent $\mathcal{O}_X$-module over a noetherian scheme X. As far as I recall, Hartshorne's book defines coherence for sheaves over a scheme restrictively, but in general coherence can be defined over ringed spaces, and one must be careful while dealing with the notion of coherence outside the category of schemes (cf. Oka's coherence theorem). As coherence is a local property, and since we're working with noetherian schemes, the general definition of an $\mathcal{O}_X$-module $\mathcal{F}$ being coherent reduces to the existence of an exact sequence of the form $\mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus m} \to \mathcal{F} \to 0$. Applying $\mathcal{H}om(-, \mathcal{G})$, this becomes yet another exact sequence $0 \to \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(\mathcal{O}_X^{\oplus m}, \mathcal{G}) \to \mathcal{H}om(\mathcal{O}_X^{\oplus n}, \mathcal{G})$, and this is essentially because $\mathcal{H}om(-,-)$ serves as the internal hom of the tensor category $(\mathsf{Mod}_{\mathcal{O}_X}, \mathcal{O}_X, \otimes)$, that is, it admits a hom-tensor adjunction, hence preserves limits (I won't prove this here). In fact from this data we can obtain moreover that the sequence $0 \to \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{G}^{\oplus m} \to \mathcal{G}^{\oplus n}$ is exact, and since $\mathcal{G}^{\oplus m} \to \mathcal{G}^{\oplus n}$ is in Coh_X, an abelian (full) subcategory of $\mathsf{Mod}_{\mathcal{O}_X}$, we may conclude $\mathcal{H}om(\mathcal{F}, \mathcal{G}) \in \mathsf{Coh}_X$.

Proposition 1.6 (Dualizing sheaf is unique)**.** *Let X be as above. Then, a dualizing sheaf for X, if exists, is unique. More precisely, suppose ω, t is one and ω', t' is another. Then there exists a unique isomorphism $\varphi : \omega \to \omega'$ such that $t = t' \circ H^n(\varphi)$.*

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Proof. Consider the isomorphism $\mathrm{Hom}_{\mathcal{O}_X}(\omega, \omega') \xrightarrow{\sim} H^n(X, \omega)^\vee$. As $t : H^n(X, \omega) \to k$ is an element of $H^n(X, \omega)^\vee$, there exists a unique element $\varphi : \omega \to \omega'$ that corresponds to t under the isomorphism above, i.e., such that $t' \circ H^n(\varphi) = t$. Similarly, there exists a unique $\psi : \omega' \to \omega$ such that $t \circ H^n(\psi) = t'$. Combining the two, we have $t \circ H^n(\psi) \circ H^n(\varphi) = t \circ H^n(\psi \circ \varphi) = t$. As id_ω also satisfies $t \circ \mathrm{id} = t$, we have $\psi \circ \varphi = \mathrm{id}_\omega$. By a symmetric argument we may conclude that φ is an isomorphism, unique by construction. □

Until now, we have required X to be merely complete. However, from now on we will assume moreover that X is projective over k. This is possible because we will later see that all complete curves are in fact projective. We will prepare some lemmas to prove the existence of a dualizing sheaf for projective schemes.

Lemma 1.7 (Serre duality for $P = \mathbb{P}^N_k$)**.** *There exists an isomorphism $t : H^N(P, \omega_P) \xrightarrow{\sim} k$ and this induces a perfect pairing*

$$\mathrm{Hom}_P(\mathcal{F}, \omega_P) \times H^N(X, \mathcal{F}) \to H^N(P, \omega_P) \xrightarrow{t} k$$

of finite-dimensional k-vector spaces, where the pairing is the obvious one and $\mathcal{F}$ is any coherent $\mathcal{O}_P$-module. Moreover, for every $i \geq 0$ there exists isomorphisms

$$\mathrm{Ext}^i_P(\mathcal{F}, \omega_P) \xrightarrow{\sim} H^{N-i}(P, \mathcal{F})^\vee$$

that are natural in $\mathcal{F}$ and agree with the prior perfect pairing when $i = 0$.

Proof. I'll follow the book and write X instead of P. Since $\omega_X \cong \mathcal{O}(-n-1)$ we have $H^n(X, \omega_X) = H^n(X, \mathcal{O}(-n-1)) \cong k$ by knowledge of cohomology on projective space. Next we verify that the pairing $\mathrm{Hom}(\mathcal{F}, \omega) \times \mathrm{Hom}(X, \mathcal{F}) \to H^n(X, \omega) \cong k$ is perfect for all coherent $\mathcal{F}$. First we verify the case $\mathcal{F} = \mathcal{O}(q)$ for some integer q. In this case, $\mathrm{Hom}(\mathcal{F}, \omega) \cong \mathrm{Hom}(\mathcal{O}(q), \omega) \cong \mathrm{Hom}(\mathcal{O}_X, \omega(-q))$ (as tensoring by an invertible sheaf is an equivalence of categories, so is fully faithful) which is just $H^0(X, \omega(-q)) = H^0(X, \mathcal{O}(-n-1-q))$. The pairing is immediate from III.5.1. Next, the case $\mathcal{F} = \bigoplus_i \mathcal{O}(q_i)$ for finite indices i and integers q_i is immediate from previous observations. Now for arbitrary coherent sheaves first note that they admit a locally free resolution $\mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$, where each $\mathcal{E}$ is a direct sum of some $\mathcal{O}(q_i)$. Recall that $\mathcal{F}(n)$ is generated by global sections for all n large enough; this means that we have a surjection $\mathcal{O}_X^{\oplus k} \to \mathcal{F}(n)$. As tensoring preserves cokernels (or kernels too in this case) we have a surjection $\mathcal{E}_0 := \mathcal{O}_X(-n)^{\oplus k} \to \mathcal{F}$. The kernel of this surjection is a coherent subsheaf of $\mathcal{E}_0$, hence admits another surjection of the same type. In fact, we can extend this process indefinitely to the left, but it suffices to consider only up to the second term in our case. If we apply the functors $\mathrm{Hom}(-, \omega_X)$ and $H^n(X, -)^\vee$ to the two term resolution, we obtain an isomorphism between the two terms involving the locally free sheaves, and hence an isomorphism $H^n(X, \mathcal{F})^\vee \cong \mathrm{Hom}(\mathcal{F}, \omega)$ which is moreover functorial in $\mathcal{F}$. To show the second part, it suffices to show that $\mathrm{Ext}^i(-, \omega)$ and $H^{n-i}(X, \mathcal{F})^\vee$ are both coeffaceable contravariant δ-functors. I will not prove this here as it is straightforward. □

Lemma 1.8. *Let $P := \mathbb{P}^N_k$ and let $\mathcal{F}$ be a coherent sheaf on P. Then there exists some n_0 such that for all $n \geq n_0$, the sheaf $\mathcal{F}(n)$ is globally generated.*

Proof. Fix a standard affine open U_i of P, and choose a finite set of generators $(s_{ij}) \in M_i = \mathcal{F}(U_i)$ where M_i is a finitely generated $\mathcal{O}(U_i)$-module. Again, fix some element s_{ij}. We first show that there exists some n such that $x_i^n s_{ij}$ extends to a global section t_{ij} of $\mathcal{F}(n)$, where $x_i^n s_{ij} \in (\mathcal{F} \otimes \mathcal{O}(n))(U_i)$ and x_i is interpreted as the restriction of the global section $x_i \in \Gamma(P, \mathcal{O}(1))$ to $U_i = \{p \in P \mid (x_i)_p \notin \mathfrak{m}_p \mathcal{O}(1)_p\}$. Consider another affine open U_k, and consider the restriction of s_{ij} to $U_i \cap U_k$ which is an element of the module $(M_k)_{x_i}$. By definition, there exists some n such that the restriction of s_{ij} is of the form $\frac{t_k}{x_i^n}$ for some $t_k \in M_k$, i.e., the section t_k over U_k restricts to $x_i^n s_{ij}$ on $U_i \cap U_k$. Similarly there exist $t_\ell \in M_\ell$ that restrict to $x_i^m s_{ij}$ on $U_i \cap U_\ell$, for some m. By suitable multiplication of x_i to t_k and t_ℓ, we may assume that $m = n$. Thus we have proved the existence of sections t_k and t_ℓ of $\mathcal{F}$ over U_k and U_ℓ that restrict to $x_i^n s_{ij}$ on $U_i \cap U_k \cap U_\ell$. Hence the section $t_k - t_\ell$ on $U_k \cap U_\ell$, when restricted to U_i, vanishes; this means that there exists some n' such that $x_i^{n'}(t_k - t_\ell) = 0$ in $\mathcal{F}(U_k \cap U_\ell)$. The number n' depends on k and ℓ, but since there are only finitely many affine opens, we may take a single natural number m' such that the affine sections $x_i^{m'} t_k$ agree on every overlap. These sections glue to yield a global section t_{ij} of $\mathcal{F}(n + m')$ which restricts to $x_i^{n+m'} s_{ij}$ on U_i. Now choose a single integer n such that this works for every (s_{ij}). The map induced by multiplication with x_i^n is an isomorphism of $\mathcal{F}(U_i)$ onto $\mathcal{F}(n)(U_i)$, so the sections $x_i^n s_{ij}$ generate $\mathcal{F}(n)(U_i)$. Do this for every affine open cover U_i to find a collection of global sections that generate $\mathcal{F}(n)$ (which is moreover finite). □

Lemma 1.9. *Let X be a closed subscheme of codimension r of P. Then $\mathcal{E}xt^i_P(\iota_* \mathcal{O}_X, \omega_P) = 0$ for all $i < r$. Here, ω_P is the canonical line bundle of P.*

Proof. Note that the literature may abbreviate $\iota_* \mathcal{O}_X$ where $\iota : X \hookrightarrow P$ as $\mathcal{O}_X$, which is a coherent $\mathcal{O}_P$-module due to being a closed subscheme of a noetherian scheme. (cf. II.5.9. By definition of schemes, the pushforward sheaf attains an $\mathcal{O}_P$-module structure via the ring map $\iota^\# : \mathcal{O}_P \to \iota_* \mathcal{O}_X$.) Write $\mathcal{F}^i := \mathcal{E}xt^i_P(\mathcal{O}_X, \omega_P)$. We first show that $\mathcal{F}^i$ are coherent for all $i > 0$ (note we have already shown the coherence of $\mathcal{H}om$ in Remark 1.5.). It suffices to show coherence of $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})$ for coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ on a noetherian scheme. Again, the question is local so affine locally we have $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})|_U = \mathcal{E}xt^i(\mathcal{F}|_U, \mathcal{G}|_U)$, so the result follows by commutative algebra. Now since these $\mathcal{F}^i$ are coherent sheaves on projective space, suitable twists result in them being globally generated by Lemma 1.8. This implies that for nonzero $\mathcal{F}^i$, there exists an integer

$q_0 >> 0$ such that $\mathcal{F}^i(q)$ has global sections for all $q \geq q_0$, hence the group of global sections of $\mathcal{F}^i(q)$ is not trivial. The contrapositive tells us that it is enough to see $\mathcal{F}^i(q)$ does not have any global sections for all $q >> 0$ to show that $\mathcal{F}^i$ is zero. Since $\mathcal{E}xt^i_P(\mathcal{O}_X, \omega_P) \otimes \mathcal{O}(q) = \mathcal{E}xt^i_P(\mathcal{O}_X(-q), \omega_P)$ its global sections are $\operatorname{Ext}^i_P(\mathcal{O}_X(-q), \omega_P)$. By Serre duality for P, this is isomorphic to $H^{N-i}(P, \mathcal{O}_X(-q))^\vee$ where $i < r$ means $N - i > N - r = \dim X$, and the cohomology groups whose degree exceed the dimension are zero. As this does not depend on q, we may conclude that the $\mathcal{F}^i$ are zero for all $i < r$. $\square$

Lemma 1.10. *Let $\omega_X^\circ := \mathcal{E}xt^r_P(\mathcal{O}_X, \omega_P)$. Then for any (not necessarily coherent) $\mathcal{O}_X$-module $\mathcal{F}$ we have a k-vector space isomorphism*

$$\operatorname{Hom}_X(\mathcal{F}, \omega_X^\circ) \cong \operatorname{Ext}^r_P(\iota_*\mathcal{F}, \omega_P)$$

which is natural in $\mathcal{F}$.

Proof. Recall that ω_P is the canonical sheaf on P. As $\operatorname{Ext}^i_P(\iota_*\mathcal{F}, \omega_P) = (R^i \operatorname{Hom}_P(\iota_*\mathcal{F}, -))(\omega_P) = h^i(\operatorname{Hom}_P(\iota_*\mathcal{F}, \mathcal{I}^\bullet))$ for an injective $\mathcal{O}_P$-resolution of ω_P, it suffices to calculate $\operatorname{Hom}_P(\iota_*\mathcal{F}, \mathcal{I})$. We claim that this is equal to $\operatorname{Hom}_X(\mathcal{F}, \iota^*\mathcal{H}om_P(\iota_*\mathcal{O}_X, \mathcal{I}))$. First we show that $\operatorname{Hom}_P(\iota_*\mathcal{F}, \mathcal{I}) \cong \operatorname{Hom}_X(\mathcal{F}, \iota^*\mathcal{I})$. As $\iota_*\mathcal{F}$ is supported on X, $\operatorname{Hom}_P(\iota_*\mathcal{F}, \mathcal{I}) = \operatorname{Hom}_P(\iota_*\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}_P} \iota_*\mathcal{O}_X) = \operatorname{Hom}_P(\iota_*\mathcal{F}, \iota_*\iota^*\mathcal{I})$, which is $\operatorname{Hom}_X(\mathcal{F}, \iota^*\mathcal{I})$ since ι_* is fully faithful. Now we show $\iota^*\mathcal{I} = \iota^*\mathcal{H}om_P(\iota_*\mathcal{O}_X, \mathcal{I})$. Since $\iota_*\mathcal{H}om_X(\mathcal{O}_X, \iota^*\mathcal{I}) = \iota_*\mathcal{H}om_X(\iota^*\iota_*\mathcal{O}_X, \iota^*\mathcal{I}) = \mathcal{H}om_P(\iota_*\mathcal{O}_X, \iota_*\iota^*\mathcal{I}) = \mathcal{H}om_P(\iota_*\mathcal{O}_X, \mathcal{I})$, and $\iota_*\mathcal{H}om_X(\mathcal{O}_X, \iota^*\mathcal{I}) = \iota_*\iota^*\mathcal{I}$, by applying ι^* on both sides we have our desired result. If we denote $\mathcal{J}^i = \iota^*\mathcal{H}om_P(\iota_*\mathcal{O}_X, \mathcal{I}^i)$, we have $\operatorname{Ext}^i_P(\iota_*\mathcal{F}, \omega_P) = h^i(\operatorname{Hom}_X(\mathcal{F}, \mathcal{J}^\bullet))$. Since $\operatorname{Hom}_X(-, \mathcal{J}^i) = \operatorname{Hom}_P(\iota_*(-), \mathcal{I}^i)$ and both ι_* and $\operatorname{Hom}_P(-, \mathcal{I}^i)$ are exact, the $\mathcal{O}_X$-modules $\mathcal{J}^i$ are also injective.

Now since $\mathcal{H}om_P(\iota_*\mathcal{O}_X, \mathcal{I}^i)$ is supported on X, this is essentially the same as its pullback so we may calculate $\mathcal{E}xt^i_P(\iota_*\mathcal{O}_X, \omega_P)$ by taking cohomology of the chain $\mathcal{J}^\bullet$. By Lemma 1.9., the cohomology of $\mathcal{J}^\bullet$ vanishes for $i < r$. Using this information, we may consider the following diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & & \operatorname{im} f_0 \arrow[d] & \operatorname{im} f_1 \arrow[d] & & & \operatorname{im} f_{r-2} \arrow[d] & \operatorname{im} f_{r-1} \arrow[d] & & \\
0 \arrow[r] & \mathcal{J}^0 \arrow[r, "f_0"] & \mathcal{J}^1 \arrow[r, "f_1"] \arrow[d] & \mathcal{J}^2 \arrow[r] \arrow[d] & \cdots \arrow[r] & & \mathcal{J}^{r-1} \arrow[r, "f_{r-1}"] \arrow[d] & \mathcal{J}^r \arrow[r, "f_r"] \arrow[d] & \mathcal{J}^{r+1} \arrow[r] & \cdots \\
 & & \operatorname{im} f_1 \arrow[d] & \operatorname{im} f_2 \arrow[d] & & & \operatorname{im} f_{r-1} \arrow[d] & \operatorname{im} f_r \arrow[d] & & \\
 & & 0 & 0 & & & 0 & 0 & &
\end{tikzcd}$$

where every arrow before $\mathcal{J}^{r+1}$ is exact. As $\operatorname{im} f_0 = \mathcal{J}^0$, and it is injective, the first vertical SES splits and $\operatorname{im} f_1$ as a direct summand of an injective module is itself injective again. Then we may repeat this step up to $\mathcal{J}^r$ and conclude that all $\operatorname{im} f_r$ are injective $\mathcal{O}_X$-modules and the sequence is split exact up to the r-th term. More precisely, our situation is as follows:

$$0 \longrightarrow \mathcal{J}^0 \xrightarrow{f_0} \mathcal{J}_1^1 \oplus \mathcal{J}_2^1 \xrightarrow{(f_1,0)} \mathcal{J}_1^2 \oplus \mathcal{J}_2^2 \xrightarrow{(f_2,0)} \mathcal{J}_1^3 \oplus \mathcal{J}_2^3 \longrightarrow \cdots$$

Here, $\mathcal{J}_1^i = \operatorname{im} f_i$. Note that the maps for $\mathcal{J}_2^i$ are all zero up to r because of the exactness of $\mathcal{J}^i$. Also the subcomplex $\mathcal{J}_1^i$ is exact since the maps are just f_i. We extend this decomposition to the whole of $\mathcal{J}^i$ by letting $\mathcal{J}_1^i$ be zero for $i > r$, and the rest becomes $\mathcal{J}_2^i$. Thus we have a decomposition $\mathcal{J}^\bullet = \mathcal{J}_1^\bullet \oplus J_2^\bullet$ of the entire chain complex. Recall that $\operatorname{Ext}^i_P(\iota_*\mathcal{F}, \omega_P) = h^i(\operatorname{Hom}_P(\iota_*\mathcal{F}, \mathcal{J}^\bullet))$. Using the fact that $\operatorname{Hom}_P(\iota_*\mathcal{F}, \mathcal{J}^\bullet) = \operatorname{Hom}_P(\iota_*\mathcal{F}, \mathcal{J}_1^\bullet \oplus \mathcal{J}_2^\bullet) = \operatorname{Hom}_P(\iota_*\mathcal{F}, \mathcal{J}_1^\bullet) \oplus \operatorname{Hom}_P(\iota_*\mathcal{F}, \mathcal{J}_2^\bullet)$ and that we are interested in the r-th homology, the LHS is equal to $\operatorname{Ext}^r_P(\iota_*\mathcal{F}, \omega_P)$ and the RHS is equal to $\operatorname{Ext}^0_X(\mathcal{F}, \omega_X^\circ)$. This is because $\mathcal{J}_1^\bullet$ contributes nothing, and $h^r(\mathcal{J}_2^\bullet) = \omega_X^\circ$ which is followed by an injective resolution of it. (Hence we have to shift it r down.) Therefore we have our desired isomorphism $\operatorname{Hom}_X(\mathcal{F}, \omega_X^\circ) = \operatorname{Ext}^r_P(\iota_*\mathcal{F}, \omega_P)$. $\square$

Remark 1.11. We have exploited the fact that the natural map $\iota^*\iota_*\mathcal{F} \to \mathcal{F}$ is an isomorphism, which does not hold in every case. Let us see in detail why this holds. For a general morphism of schemes $f : X \to Y$, we always have an adjunction $f^* \dashv f_*$, and consequently a counit $f^*f_* \Rightarrow \operatorname{id}_{\mathsf{Mod}_{\mathcal{O}_X}}$. The counit is a natural isomorphism if and only if the right adjoint f_* is fully faithful,[1] which is indeed true in our case as ι is a closed immersion.

Now we have all the ingredients to prove the existence of a dualizing sheaf for projective varieties.

Proposition 1.12 (Existence of ω_X°)**.** *Let X be projective over k. Then X has a dualizing sheaf.*

[1]Need verification

Proof. Embed X as a closed subscheme of some $P := \mathbb{P}^N_k$, let $n := \dim X$, $r := N - n$ and define $\omega_X^\circ := \mathcal{E}xt^r_P(\mathcal{O}_X, \omega_P)$. By Lemma 1.10., we have an isomorphism $\operatorname{Hom}_X(\mathcal{F}, \omega_X^\circ) \cong \operatorname{Ext}^r_P(\iota_*\mathcal{F}, \omega_P)$ for any $\mathcal{O}_X$-module $\mathcal{F}$. Moreover if $\mathcal{F}$ is coherent, by Serre duality for P we have an isomorphism $\operatorname{Ext}^r_P(\iota_*\mathcal{F}, \omega_P) \cong H^n(P, \mathcal{F})^\vee$. Therefore we have obtained an isomorphism $\operatorname{Hom}_X(\mathcal{F}, \omega_X^\circ) \cong H^n(X, \mathcal{F})^\vee$ which is natural in $\mathcal{F} \in \mathsf{Coh}_X$. Taking $\mathcal{F} = \omega_X^\circ$, denote by $t \in H^n(X, \omega_X^\circ)^\vee$ the unique element corresponding to the identity in $\operatorname{Hom}_X(\omega_X^\circ, \omega_X^\circ)$ under the isomorphism we just obtained. It is clear that (ω_X°, t) is a dualizing sheaf. □

Theorem 1.13 (Serre duality for projective schemes)**.** *Let X be an equidimensional Cohen-Macaulay projective scheme of dimension n over an algebraically closed field k. Fix a dualizing sheaf ω_X° and a very ample sheaf $\mathcal{O}(1)$ on X. The following are true:*

- *there are isomorphisms*
$$\theta^i : \operatorname{Ext}^i(\mathcal{F}, \omega_X^\circ) \to H^{n-i}(X, \mathcal{F})^\vee$$
which are natural in $\mathcal{F}$ where $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module,
- *moreover if $\mathcal{F}$ is locally free then we have $H^i(X, \mathcal{F}(-q)) = 0$ for $i < n$ and all $q >> 0$.*

Proof. For the first bullet point, this is essentially the same as in the case of $\mathbb{P}^n$, and then showing that the functors are universal. Now let $\mathcal{F}$ be locally free. Denote the embedding as $\iota : X \hookrightarrow \mathbb{P}^N_k$. Pick a closed point $x \in X$; since $\mathcal{F}$ is locally free, near x it can be written as a direct sum of $\mathcal{O}_{X,x}$. As the depth of direct sums of modules is equal to the minimum of the summands, its depth is just equal to that of $\mathcal{O}_{X,x}$, which is n as we assumed X to be Cohen-Macaulay of equidimension n. If we let $A = \mathcal{O}_{P,x}$, the local ring in projective space, this is a regular local ring of dimension n. By the Auslander-Buchsbaum formula we have $\operatorname{pd}_A \mathcal{F}_x = N - n$, so $\mathcal{E}xt^i_P(\mathcal{F}, -) = 0$ for all $i > N - n$. Now as $H^i(X, \mathcal{F}(-q))^\vee \cong \operatorname{Ext}^{N-i}_P(\mathcal{F}, \omega_P(q)) \cong H^0(P, \mathcal{E}xt^{N-i}_P(\mathcal{F}, \omega_P(q)))$ for $q >> 0$ (cf. III.6.9.) this is zero by above. Hence $H^i(X, \mathcal{F}(-q)) = 0$ for all $i < n$ and $q >> 0$. □

Corollary 1.14. *Let X be as above. For locally free sheaves $\mathcal{F}$ on X we have isomorphisms*
$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X^\circ)^\vee$$
which are natural in $\mathcal{F}$.

Proof. By III.6.3., we have $H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X^\circ)^\vee \cong \operatorname{Ext}^{n-i}_X(\mathcal{O}_X, \mathcal{F}^\vee \otimes \omega_X^\circ)^\vee$, which is $\operatorname{Ext}^{n-i}_X(\mathcal{F}, \omega_X^\circ)^\vee$ by III.6.7., and this is $H^i(X, \mathcal{F})^{\vee\vee} \cong H^i(X, \mathcal{F})$. □

To actually use Serre duality in proving theorems such as Riemann-Roch, we must first show that in nice cases the dualizing sheaf ω_X° is isomorphic to the canonical sheaf ω_X. This is true for nonsingular projective varieties. We will only state the theorems and not prove them as they are quite involved. However this does not devalue the theorem; the fact that we can identify the dualizing sheaf is perhaps one of the most important pieces in proving Riemann-Roch.

Theorem 1.15. *Let X be a closed subscheme of $P = \mathbb{P}^N_k$ which is a lci of codimension r. Denote by $\mathcal{I}$ its ideal sheaf. Then we have $\omega_X^\circ \cong \omega_P \otimes \bigwedge^r(\mathcal{I}/\mathcal{I}^2)^\vee$ where ω_X° is defined as in Lemma 1.10.*

Proof. Omitted. □

Corollary 1.16. *If X is a nonsingular projective variety, then $\omega_X^\circ \cong \omega_X$.*

Proof. $\omega_X \cong \omega_P \otimes \bigwedge^r(\mathcal{I}/\mathcal{I}^2)^\vee$ by II.8.20. □