<2025. 4. 10. An Introduction to Toric Varieties>

* Outlines

I. Toric Varieties

II. Symplectic Toric Manifolds

III. Delzant's Construction
(Correspondence between toric varieties and sympl. toric mfds)

* References

[1] M. Audin, The Topology of Torus Actions on Symplectic Manifolds
(Progress in Mathematics 93)

[2] A. Silva, Lectures on Symplectic Geometry
(Lecture Notes in Mathematics 1764)

I. Toric Varieties

§ Basic notations for toric varieties

real torus $T^N := \{(t_1, \cdots, t_N) \in \mathbb{C}^N \mid |t_i| = 1\}$

Complex torus $T^N_{\mathbb{C}} := \{(t_1, \cdots, t_N) \in \mathbb{C}^N \mid t_i \neq 0\} = (\mathbb{C}^*)^N$

These two groups act on $\mathbb{C}^N$ by trivial way:

$(t_1, \cdots, t_N) \cdot (z_1, \cdots, z_N) = (t_1 z_1, \cdots, t_N z_N)$

Def $(e_1, \cdots, e_N)$: canonical basis in any of the sp. $\mathbb{Z}^N \subset \mathbb{Q}^N \subset \mathbb{R}^N \subset \mathbb{C}^N$

for $I \subset \{1, 2, \cdots, N\}$,

Coord. subsp. $e_I := \{z \mid j \notin I \Rightarrow z_j = 0\}$ $(\mathbb{C}^{|I|} \subset \mathbb{C}^N)$

Corresponding complex torus $T_I := \{t \mid j \notin I \Rightarrow |t_j| = 1\}$

open cone $e_I' := \{z \mid j \notin I \Leftrightarrow z_j = 0\}$ $((\mathbb{C}^*)^{|I|} \subset \mathbb{C}^N)$

$\Rightarrow$ In $T^N_{\mathbb{C}} \curvearrowright \mathbb{C}^N$, $\begin{cases} z \in e_I \text{ iff } \operatorname{Stab}(z) \supset T_I \\ z \in e_I' \text{ iff } \operatorname{Stab}(z) = T_I \end{cases}$

Let $\pi : \mathbb{Z}^N \to \mathbb{Z}^n$ be a linear map s.t. $\pi \otimes \mathbb{Q}$ surjective

$\mathbb{K} \subset \mathbb{Z}^N$: $\ker(\pi : \mathbb{Z}^N \to \mathbb{Z}^n)$

$K \subset T^N$: $\ker(\pi : \mathbb{R}^N/\mathbb{Z}^N \to \mathbb{R}^n/\mathbb{Z}^n)$

$K_{\mathbb{C}} \subset T^N_{\mathbb{C}}$: complexification of K

Def $z \in \mathbb{C}^N$ lies in a

principal orbit if $K_{\mathbb{C}} \cap T_I = \{(1, \cdots, 1)\}$ $\forall I$

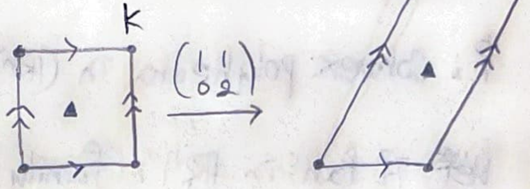
singular orbit if $\exists I$ s.t. $\dim(K_{\mathbb{C}} \cap T_I) > 0$

exceptional orbit if $\exists I$ s.t. $K_{\mathbb{C}} \cap T_I$ is a fin. gp.

Prop points lie in a singular orbit of $K_{\mathbb{C}} \subseteq \{z \in e_I \mid \mathbb{K} \otimes \mathbb{C} \cap e_I \neq 0\}$

Example

$\pi : \mathbb{Z}^2 \to \mathbb{Z}^2$: linear map given by $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$



$\mathbb{K} = 0,\ K = \{(\varepsilon, \varepsilon) \mid \varepsilon = \pm 1\} \subset T^2$

$K_{\mathbb{C}} \subseteq T^N_{\mathbb{C}} = \{(\varepsilon\ \varepsilon) \mid \varepsilon = \pm 1\} \subset T^2_{\mathbb{C}}$

$\Rightarrow K_{\mathbb{C}} \curvearrowright \mathbb{C}^2$ has no singular orbit but exceptional orbit $\{(0.0)\}$

§ Fans and Toric Varieties

P: convex polyhedra in $(\mathbb{R}^n)^\vee$, Γ: face of P

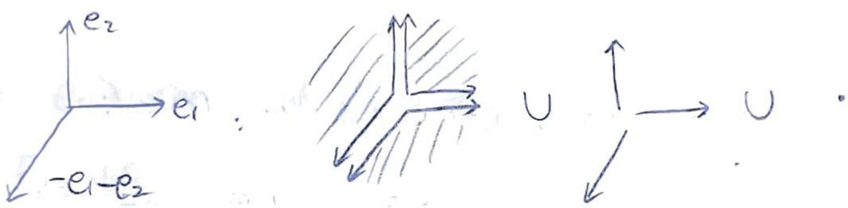
<u>Def</u> A fan in $\mathbb{R}^n$: family of convex polyhedral cones in $\mathbb{R}^n$

having vertex 0, generating by integer vectors s.t.

(1) any face of a cone in Σ is a cone in Σ

(2) the intersection of two cones in Σ is a face of each of them.

ex)



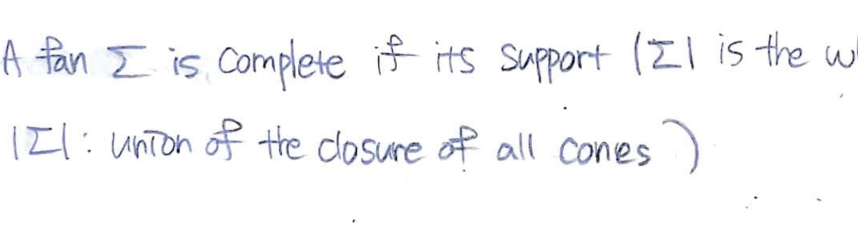
<u>Def</u> tangent cone of Γ: $\sigma_\Gamma := \bigcup_{r\ge 0} r\cdot(P-m)$ for any $m\in\Gamma$

convex dual cone for cone σ:

$\check{\sigma} := \{ x\in E \mid \varphi(x)\ge 0 \ \forall \varphi\in\sigma \}$

associated fan for P: $\Sigma(P) := \{\check{\sigma}_\Gamma\}$

ex) P $\qquad$ $\{\sigma_\Gamma\}$ $\qquad$ $\{\check{\sigma}_\Gamma\} = \Sigma(P)$



<u>Def</u> A fan Σ is complete if its support $|\Sigma|$ is the whole sp $\mathbb{R}^n$

($|\Sigma|$: union of the closure of all cones)

<u>prop</u> P is compact $\Leftrightarrow$ $\Sigma(P)$ is complete

pf) easy.

§ Construct a toric varieties

$\Sigma(k)$: k-skelleton (set of all k-dim cones)

$\Sigma(1) = (x_1, \cdots x_N)$ for the given fan Σ

$$\pi : \mathbb{Z}^N \longrightarrow \mathbb{Z}^n$$
$$(e_1 \cdots e_N) \longmapsto (\lambda_1 \cdots \lambda_N)$$

$\langle x_I$: Cone of Σ, generated by $\{\lambda_i\}_{i \in I}$.

Assumptions

(1) $\pi \otimes \mathbb{Q} : \mathbb{Q}^N \longrightarrow \mathbb{Q}^n$ surjective (i.e. Σ contains a dim n cone)

(2) $\langle x_I \in \Sigma \Rightarrow e_I \cap k \otimes \mathbb{C} = \{0\}$ (Condition to avoid a complizated singularities)

Def Σ : given fan

$$u_\Sigma = \mathbb{C}^N - \bigcup_{\{I \mid \langle \lambda_I \notin \Sigma\}} e_I \quad \left(\text{analogy of } u_\pi = \mathbb{C}^N - \bigcup_{k \cap e_I \neq 0} e_I\right)$$

Then, the space $X_\Sigma := u_\Sigma / K_\mathbb{C}$ is the toric variety associated with Σ.

RMK Assumption (2) $\Rightarrow u_\Sigma \subset u_\pi$

$\Rightarrow K_\mathbb{C}$ induced by π acts on u_Σ without singular orbits.

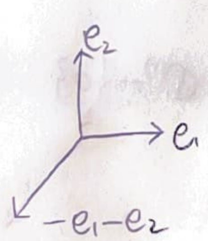
$\Rightarrow$ at worst, X_Σ is an orbifold

(singularities caused by exceptional orbits)

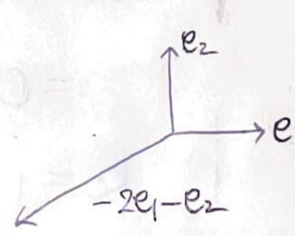
Q. When X_Σ be a smooth mfd?

Def A fan Σ is smooth if each of its cones is generated by a part of a $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^n$.

ex)



smooth



nonsmooth

RMK Σ smooth $\Rightarrow$ Assumptions (1) and (2).

$\because$ (1): trivial

(2): Let $\langle \lambda_I$: dim k cone. wlog let $I = \{1, 2, \cdots k\}$

$$e_I \cap \mathbb{K} \otimes \mathbb{C} = \{ \sum_{i=1}^{k} \alpha_i e_i \mid \pi(\Sigma \alpha_i e_i) = 0, \alpha_i \in \mathbb{C} \}$$
$$= \{ \sum_{i=1}^{k} \alpha_i e_i \mid \Sigma \alpha_i \lambda_i = 0, \alpha_i \in \mathbb{C} \}$$
$$= 0.$$

Prop X_Σ smooth $\Leftrightarrow$ Σ smooth

pf) X_Σ smooth iff $K_\mathbb{C}$ has no exceptional orbits.

i.e. $\langle \lambda_I \in \Sigma \Rightarrow K \cap T_I = \{1\}$

K contains a nontrivial elt means

$\exists z \in \mathbb{Q}^N$ s.t. z nonintegral, $\pi(z) \in \mathbb{Z}^n$.

Let $I = \{1, \cdots, n\}$

$\Rightarrow K \cap T_I$ nontrivial fin. gp. iff

$\exists$ relatively prime $a_1 \cdots a_n$ and $m \geq 2$ s.t. $\pi\left(\sum_{i=1}^{n} \frac{a_i}{m} e_i\right) \in \mathbb{Z}^n$

Now consider $\sum_{i=1}^{n} a_i \lambda_i$ in $\mathbb{Z}^n$. $\lightning$

Examples

(1) noncompact cases.

Σ: 1st octant in $\mathbb{R}^n$	Σ': 1st octant in $\mathbb{R}^n$, except n-dim cone
$\Rightarrow N = n, \pi = id$	$\Rightarrow N = n, \pi = id$
$\Rightarrow U_\Sigma = \mathbb{C}^n - \bigcup_{\{I \mid \langle \lambda_I \notin \Sigma\}} e_I$	$\Rightarrow U_{\Sigma'} = \mathbb{C}^n - \bigcup_{\{I \mid \langle \lambda_I \notin \Sigma'\}} e_I$
$= \mathbb{C}^n$,	$= \mathbb{C}^n - e_{\{1,\cdots,n\}} = \mathbb{C}^n - \{0\}$
$K_\mathbb{C} = 1$	$K_\mathbb{C} = 1$
$\Rightarrow X_\Sigma = \mathbb{C}^n$	$\Rightarrow X_{\Sigma'} = \mathbb{C}^n - \{0\}$

(2) smooth case.

Σ : Complete fan generated by $x_i = e_i \, (i=1,\dots,n)$, $x_{n+1} = -(e_1+\dots+e_n)$

$\Rightarrow U_\Sigma = \mathbb{C}^{n+1} - \{0\}$,

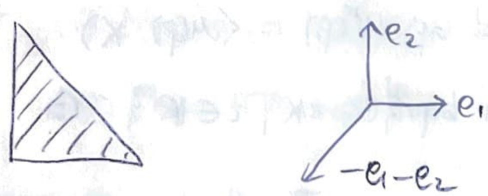
$\mathbb{K} = \{(m,\dots,m) \in \mathbb{Z}^{n+1} \mid m \in \mathbb{Z}\}$

$K = \{(t,\dots,t) \in T^N \mid t \in \mathbb{C}, |t| = 1\}$

$K_\mathbb{C} = \{(z,\dots,z) \in T_\mathbb{C}^N \mid z \in \mathbb{C}^*\}$

$\Rightarrow X_\Sigma = U_\Sigma / K_\mathbb{C} \cong \mathbb{C}^{n+1} - \{0\} / \mathbb{C}^* = \mathbb{C}P^n$

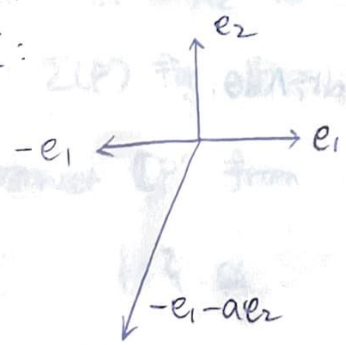
ex) P $\qquad$ $\Sigma(P)$ $\qquad$ $X_{\Sigma(P)}$



$\mathbb{C}P^2$

(3) nonsmooth case.

Σ :



$\Rightarrow U_\Sigma = (\mathbb{C}^2 - \{0\}) \times (\mathbb{C}^2 - \{0\})$

$\mathbb{K} = \langle (1,0,1,0), (1,a,0,1) \rangle$

$\left.\begin{array}{l} K \cong T^2 \\ K_\mathbb{C} \cong (\mathbb{C}^*)^2 \end{array}\right\}$ depend on a

$K_\mathbb{C} \curvearrowright (\mathbb{C}^*)^4$ as $(u,v)\cdot(z_1,z_2,z_3,z_4) = (u^a v z_1, u z_2, v z_3, u z_4)$

$X_\Sigma = U_\Sigma / K_\mathbb{C} =: \mathcal{H}_a$ (Hirzebruch surface)

II. Symplectic Toric Manifolds

§ Preliminaries in symplectic geometry

<u>Def</u> The 2-form $\omega \in \Omega^2(M)$ is symplectic if ω is closed and ω_p is skew-symmetric bilinear $\forall p \in M$.

(M, ω) is a sympl. mfd. if M is a mfd and ω is a sympl. form

<u>Def</u> Symplectic G-action $\psi : G \to \mathrm{Sympl}(M, \omega)$ is a hamiltonian action if $\exists\ \mu : M \to \mathfrak{g}^*$ s.t.

(1) $d\mu^X = i_{X^\#}\omega$ $\forall X \in \mathfrak{g}$
$\begin{cases} \mu^X : M \to \mathbb{R} \text{ defined by } \mu^X(p) := \langle \mu(p), X \rangle \\ X^\# : \text{v.f. on } M \text{ gen. by } \{\exp tX \mid t \in \mathbb{R}\} \subseteq G \end{cases}$

(2) $\mu \circ \psi_g = Ad_g^* \circ \mu \quad \forall g \in G$

Then we call μ is a moment map.

ex) $(\mathbb{C}^n, \omega)$: $\omega = \sum dx_i \wedge dy_i = \frac{i}{2} \sum dz_i \wedge d\bar{z}_i = \sum r_i dr_i \wedge d\theta_i$

$S^1 \curvearrowright (\mathbb{C}^n, \omega)$ as multiplying $t \in S^1$

$\Rightarrow$ This action is hamiltonian with moment map

$$\mu : \mathbb{C}^n \longrightarrow \mathbb{R}$$
$$z \longmapsto +\frac{|z|^2}{2} + \text{const.}$$

$\because d\mu = +\frac{1}{2} d(\sum r_i^2)$

$$X^\# = \frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2} + \cdots + \frac{\partial}{\partial\theta_n}$$

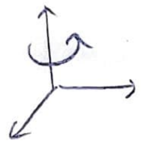
$$i_{X^\#}\omega = +\sum r_i dr_i = +\frac{1}{2}\sum dr_i^2 = d\mu$$

<u>Def</u> Compact, connected sympl. mfd (M^{2n}, ω) is a symplectic toric manifold if there is an effective hamiltonian action $T^n \curvearrowright (M^{2n}, \omega)$ and with a corresponding moment map $\mu : M \to \mathbb{R}^n$.

§ Symplectic reduction

- dim k symmetry group actions reduce the dim of phase sp. by $2k$.

ex) dim 3 Configuration sp. 1 particle $\Rightarrow$ 6-dim phase sp.



dim 1 rotational symm $\Rightarrow$ $\vec{x} \times \vec{p_z}$ conserved

$\Rightarrow$ 2-dim are reduced, then 4-dim lefts.

Thm (M, ω, G, μ): hamiltonian G-space

If G acts freely on level set of μ, there is a reduction (M_{red}, ω_{red})

s.t. (1) $M_{red} = \mu^{-1}(a)/G$

(2) $i^*\omega = \pi^*\omega_{red}$ for $i: \mu^{-1}(a) \to M$, $\pi: \mu^{-1}(a) \to M_{red}$

Motivating Example for III

Recall) $\mathbb{C}P^n = X_\Sigma$ for Σ made by $(\lambda_1, \cdots, \lambda_n, \lambda_{n+1}) = (e_1, \cdots, e_n, -(e_1 + \cdots + e_n))$

$\Sigma = \Sigma(P)$ for $P = \{(z_1, \cdots, z_n) \mid z_i \geq 0 \ \forall i,\ z_1 + \cdots + z_n \leq 1\}$

Construct $\mathbb{C}P^n$ from $\mathbb{C}^* \curvearrowright \mathbb{C}^{n+1} - \{0\}$

$$\left.\begin{aligned} \mathbb{C}^{n+1} - \{0\} &\cong S^{2n+1} \times (0, +\infty) \\ \mathbb{C}^* &\cong \underbrace{S^1}_{\text{hamiltonian level set}} \times \underbrace{(0, +\infty)}_{\text{image of hamiltonian map}} \end{aligned}\right] \text{compatible}$$

$$\mathbb{C}P^n \left\langle \begin{aligned} &\cong \mathbb{C}^{n+1} - \{0\} / \mathbb{C}^* \left(= U_\Sigma / K_c \ : \text{as a toric variety}\right) \\ &\cong S^{2n+1}/S^1, \text{ remained hamiltonian action} \end{aligned}\right.$$

III. Delzant's Construction

Thm (Delzant)

Toric manifolds are classified by Delzant polytopes.

(Delzant polytopes = primitive compact convex polyhedra)

$\qquad\qquad\qquad\qquad\updownarrow \qquad\qquad \updownarrow$

$\qquad\qquad\qquad$ smooth $\quad$ complete $\qquad$ (in fan)

$$\left\{\begin{array}{c}\text{toric manifolds}\\ (M^{2n},\omega,T^n,\mu)\end{array}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{array}{c}\text{Delzant polytopes}\\ P\end{array}\right\} \left(\longleftrightarrow \left\{\begin{array}{c}\text{smth, complete fan}\\ \Sigma\end{array}\right\}\right)$$

$\xrightarrow{①}$: moment map ($P=\mu(M^{2n})$)

$\xleftarrow{②}$: Delzant construction

Now we will follow the process of Delzant construction.

Step 1 Construct a K-action on U_Σ

$T^N \curvearrowright \mathbb{C}^N$: hamiltonian map with the moment map $\mu: \mathbb{C}^N \to (\mathbb{R}^N)^* = t^*$

$$\mu(z_1,\cdots,z_N) = \frac{1}{2}(|z_1|^2,\cdots,|z_N|^2)$$

Let k: Lie alg. for $K \subset T$

$\Rightarrow \mu': U_\Sigma \hookrightarrow U_\pi \hookrightarrow \mathbb{C}^N \xrightarrow{\mu} t^* \xrightarrow{p} k^*$ (p induced by $K \hookrightarrow T$)

is the moment map for $K \curvearrowright U_\Sigma$

Step 2 Get X_Σ by symplectic reduction

$K \curvearrowright U_\Sigma$ has no singular orbit $\Rightarrow \mu'(U_\Sigma)$ consists of regular values.

Prop $\forall\, z \in \mu'(U_\Sigma)$, the inclusion $i: \mu'^{-1}(z) \hookrightarrow U_\Sigma$

induces a homeomorphism $\mu'^{-1}(z)/K \to U_\Sigma/K_{\mathbb{C}} = X_\Sigma$.

$\Rightarrow$ We can endow a reduced sympl. form ω_z on X_Σ

and hamiltonian action of n-dim torus $Q = T^N/K$ (so dose μ)

$\Rightarrow$ We get toric mfd $(X_\Sigma, \omega_z, Q, \mu)$

Now, we need to check whether ①∘② = id.

<u>Prop</u> Σ is the fan associated to the polyhedra $P_\xi = \mu(X_\Sigma)$, for any ξ.

pf) Let the reduced moment map $\mu : X_\Sigma \to \mathfrak{q}^*$ ($\mathfrak{q}$: Lie alg. of Q)

Consider the exact seq.

$$0 \longrightarrow \mathbb{K}\otimes\mathbb{R} \longrightarrow \mathbb{R}^N \xrightarrow{\pi} \mathbb{R}^n \longrightarrow 0$$

$$0 \longrightarrow \mathfrak{k} \longrightarrow \mathfrak{t} \longrightarrow \mathfrak{q} \longrightarrow 0$$

$\Updownarrow$ dual

$$\begin{array}{ccccccccc} 0 & \to & \mathfrak{q}^* & \to & \mathfrak{t}^* & \xrightarrow{p} & \mathfrak{k}^* & \to & 0 \\ & & & & \cup & & \cup & & \\ & & & & \zeta_0 & \longmapsto & \xi & & \end{array}$$

$$P_\xi = \mu'(X_\Sigma) = p^{-1}(\xi) \cap \operatorname{Im}\mu = p^{-1}(\xi) \cap \{\varphi : \mathfrak{t} \to \mathbb{R} \mid \varphi(e_i) \ge 0 \ \forall i\}$$

$$= \{\varphi \in \mathfrak{t}^* \mid \varphi(e_i) \ge 0,\ \varphi|_{\mathfrak{k}} = \zeta_0|_{\mathfrak{k}}\}$$

$\Rightarrow$ for $\varphi \in P_\xi$, $\varphi - \zeta_0$ vanishes on $\mathfrak{k}$

$\Rightarrow$ $\langle \varphi - \zeta_0, e_i \rangle$ depends only on $\pi(e_i) = x_i$

i.e. $P_\xi = \{\psi \in \mathfrak{q}^* \mid \psi(\lambda_i) \ge \zeta_0(e_i)\}$

(λ_i : normal direction of $(n-1)$-faces of P_ξ)

$\therefore$ Cones of P_ξ are only depend on λ_is

and they are exactly Σ.