Derived Alg. Geo.
: Introduction

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References]

[1] A.Khan, An Introduction to Derived Algebraic Geometry.

[2] Kerodon

[3] Lurie, Derived Algebraic Geometry

[4] Lurie, Higher Algebra

[5] Lurie, Higher Topos Theory

Introduction to Derived AG

§0. Motivation

We start w/ Bezout's theorem.

$C, C' \subseteq \mathbb{P}^2$. If they meet transversally, $C \cap C'$ has $m \cdot n$ points.

But, in general $[C]\cdot[C'] > [C \cap C']$

For strict inequality, we define intersection multiplicities as

$\sum (-1)^i \dim \mathrm{Tor}_i^{\mathcal{O}_{\mathbb{P}^n,p}}(\mathcal{O}_{C,p}, \mathcal{O}_{C',p}) \qquad (C, C' \subset \mathbb{P}^n)$

(Beware that it is not just $\dim_{\mathbb{C}} \mathcal{O}_{C\cap C',p}$. it involves higher Tor)

this is because $\otimes_{\mathcal{O}_{\mathbb{P}^n,p}}$ is NOT exact.

$\rightsquigarrow$ Why don't we view $\mathcal{O}_{C,p} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathbb{P}^n,p}} \mathcal{O}_{C',p}$ as a ring itself
(capturing all higher Tor)

Example. (What scheme theory cannot see)

In $\mathbb{A}^2$, $(x=0) \cap (y=0) = \operatorname{Spec} \mathbb{C}[x,y]/(x,y)$: 1-pt.

"$\int$"

$(x=0) \cap (x=0) = \operatorname{Spec} \mathbb{C}[x,y]/(x,x) \simeq \operatorname{Spec} \mathbb{C}[y]$

dimension jumped...

We want "$\mathbb{C}[x,y]/(x,x) \neq \mathbb{C}[x,y]/(x)$".

A categorification of ring handles such phenomena:

i.e. category $\mathcal{C}$ equipped w/ $\mathcal{C}\times\mathcal{C} \xrightarrow{+} \mathcal{C}$, $\mathcal{C}\times\mathcal{C} \xrightarrow{\cdot} \mathcal{C}$, etc..

(Decategorification will be given as π_0 (isomorphism classes))

Example. $R \longrightarrow R/(x-y)$ for categorified ring R.

Imposing "$x=y$" at categorical level $\rightsquigarrow$ generated by R & $\overset{x}{\bullet} \overset{\cong}{\frown} \overset{y}{\bullet}$

Note that if R was a groupoid, this doesn't change π_0.

However, we've added "additional iso." s.t. the ring itself is distinguished.

Example. $R/x \ncong R/(x,x) \ncong R/(x,x,x) \cdots$

Example. For $R = R'/(x-y)$, $\mathcal{C}'$ be discrete categorical ring w/ $\pi_0(\mathcal{C}') \cong R'$.

then $\mathcal{C} = \langle \mathcal{C}', x \overset{\simeq}{=} y \rangle$

$\mathcal{C}$ carries ① Objects $\pi_0 \cong R'/(x-y) = R = \mathrm{Tor}^0(R', R)$

② Automorphisms $\pi_1 = \ker(R' \xrightarrow[(x-y)]{} R') = \mathrm{Tor}^1(R', R)$

So $\mathcal{C}$ carries Tor^0 & Tor^1 information.

What about higher Tor in Serre's intersection formula?

$\rightsquigarrow$ $\mathcal{C}$ needs to be "higher category".

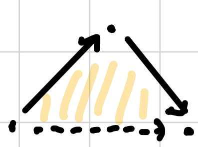
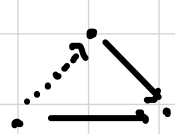
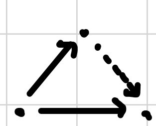
So we introduce "∞-categorical rings". — cdga (Computation)
— Simplicial ring (TV)
— E_∞ - rings (Lurie, SAG)

§1. ∞- categories.

∞-category is a category "homotopically enriched" in ∞-groupoids
$\cong$ Spaces
$\cong$ Kan cpx

Formal Defn: ∞-category is a weak Kan cpx.

i.e simplicial set w/ inner horn lifting property

 vs  &  $\rightsquigarrow$ 1-morphisms being equivalence

· all horn lifting property = Kan cpx "=" ∞-grpd

* limit / colimit / initial, final ...
even fibrations & localization ... are similarly defined as in classical categories.

Be aware that everything is defined "homotopically"

Ex) $pt \times_X pt = \Omega X$, not a point. (use homotopy fibered product).

explicitly, $F: I \to \mathcal{C}$ say $X \cong \lim F_i$ if

$$\mathrm{Map}_{\mathcal{C}}(Y, X) \longrightarrow \mathrm{Map}_{\mathrm{Fun}(I,\mathcal{C})}(\overline{Y}, F) \text{ is a weak equiv.}$$

$*$ Kan extension.

$$\begin{tikzcd} \mathcal{C}^\circ \arrow[r, "F_0"] \arrow[d, hook] & \mathcal{D} \arrow[d, "p"] \\ \mathcal{C} \arrow[r] \arrow[ur, dotted, "F"] & \mathcal{D}' \end{tikzcd}$$

be commutative diagram of ∞-Categories

p : inner fibration

F is a p-left Kan extension of F_0 if $\forall C \in \mathcal{C}$

$$\begin{tikzcd} \mathcal{C}^\circ \times_{\mathcal{C}} \mathcal{C}_{/C} = (\mathcal{C}^\circ_{/C}) \arrow[r, "F_C"] \arrow[d, hook] & \mathcal{D} \arrow[d, "p"] \\ (\mathcal{C}^\circ_{/C})^{\triangleright} \arrow[r] \arrow[ur] & \mathcal{D}' \end{tikzcd}$$

exhibits $F(C)$ as p-Colimit of F_C.
(Most cases, $p : \mathcal{D} \to *$)

<u>Fact</u> If $\mathcal{D} \xrightarrow{p} \mathcal{D}'$ is a CoCartesian fibration + "functor among fiber preserves colimit"
$\Rightarrow$ left p-Kan extension exists.

Example 1)

$$\begin{tikzcd} 1 \arrow[r, "X"] \arrow[d, hook] & \mathcal{C} \\ 0 \to 1 \arrow[ur, dotted] \end{tikzcd}$$

left Kan extension is $X \mapsto (0 \to X)$

2)

$$\begin{tikzcd} \begin{smallmatrix} \bullet \to \bullet \\ \downarrow \quad \\ \bullet \quad \end{smallmatrix} \arrow[r] \arrow[d] & \mathcal{C} \\ \begin{smallmatrix} \bullet \to \bullet \\ \downarrow \quad \downarrow \\ \bullet \to \bullet \end{smallmatrix} \arrow[ur, dotted] \end{tikzcd}$$

left Kan extension is the pushout diagram.

Heuristically, left (right) Kan extension is constructed by Colimit (limit).

$\mathcal{C}^\circ \hookrightarrow \mathcal{C}$ induces $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{C}^\circ, \mathcal{D})$

left Kan extension is its left adjoint, mapping $F_0 \mapsto F$.

"Construction": $F(X) = \varinjlim_{Y \to X} F_0(Y)$. (classical Case)

We've seen such Constructions many, many times...
(inverse image sheaf, shrieck functor, $\cdots$)

§2. ∞-enhancements of Classical AG

Def Category $\mathcal{C}$ is an algebraic category if

$\mathcal{C} \cong \mathrm{Fun}_{\Pi}(F^{op}, \mathrm{Set})$ where F: admit finite coproducts

↳ means mapping finite coproducts to finite products.

Example) $\mathrm{Set} \cong \mathrm{Fun}_{\Pi}(\mathrm{Fin}^{op}, \mathrm{Set})$

$\mathrm{Mod}_R \cong \mathrm{Fun}_{\Pi}(\mathrm{FFree}_R^{op}, \mathrm{Set})$ $\left(\because \right.$ $r \in \mathrm{Hom}_{\mathrm{FFree}^{op}}(R, R) \cong R$ induces R-action; $\Delta \in \mathrm{Hom}_{\text{"}}(R, R^{\oplus 2})$ induces $+ : M \times M \to M$

$\mathrm{CAlg}_R \cong \mathrm{Fun}_{\Pi}(\mathrm{Poly}_R^{op}, \mathrm{Set})$ $\left(\because \right.$ $r \in \mathrm{Hom}_{\mathrm{Poly}}(R[x], R[x])$ induces R-action; $f \mapsto f(xy) \in \mathrm{Hom}_{\text{"}}(R[x], R[x,y])$ induces $\times : A \times A \to A$. $f(x,y) \mapsto f(y,x)$ is id. $\rightsquigarrow$ Commutative

Def $\mathrm{Anim}(\mathcal{C}) := \mathrm{Fun}_{\Pi}(F^{op}, \mathrm{Grpd}_\infty)$

above example $\rightsquigarrow$ Grpd_∞, $D(R)_{\geq 0}$, $d\mathrm{CAlg}_R$ (∞-Cat. enhancements)

$D(R)_{\geq 0}$: ∞-Cat. localizations of $D(R)$ via quasi-iso.

$d\mathrm{CAlg}_R$: ∞-Cat. localizations of SCAlg_R via weak equiv

Fact $\mathrm{Anim}(\mathcal{C})$ is freely generated by F under filtered colimits & geometric realizations

it is the "universal" one among them.

$\Rightarrow$ $\Omega_{-/R} : \mathrm{CAlg}_R \longrightarrow \mathrm{CAlgMod}_R$ is enhanced to

$A \longmapsto (A, \Omega_{A/R})$

$\Omega^{anim}_{-/R} : d\mathrm{CAlg}_R \longrightarrow \mathrm{Anim}(\mathrm{CAlgMod}_R)$

$A \longmapsto (A, \underbrace{\mathbb{L}_{A/R}}_{\in D(A)_{\geq 0}})$

Def A derived stack is a functor $d\mathrm{CAlg}_R \longrightarrow \mathrm{Grpd}_\infty$ satisfying "étale descent condition."

Étale descent Condition. for $A \to B$

$$F(A) \to F(B) \rightrightarrows F(B \otimes^{\mathbb{L}}_A B) \overset{\to}{\rightrightarrows} F(B \otimes^{\mathbb{L}}_A B \otimes^{\mathbb{L}}_A B) \overset{\to}{\underset{\to}{\rightrightarrows}} \cdots$$

is a limit diagram. Why ∞-many terms ? (Heuristics)

⓪ $U_i \subset X$ "glue" : they just glue.
① Functions "glue" : sufficient to check on $U_i \cap U_j$
② sheaves (to sets) "glue" : " on $U_i \cap U_j$ & $U_i \cap U_j \cap U_k$
0-trunc. (cocycle cond.)
③ sheaves (to Grpd) "glue" : " on $U_i \cap U_j$ & $U_i \cap U_j \cap U_k$
1-trunc. (up to equiv. cocycle cond.)
(prestack) & $U_i \cap U_j \cap U_k \cap U_l$
Cocycle cond. on equiv. (1-morphisms)
⋮

thus, "higher structures" need "higher gluing data"

$\rightarrow$ ∞-structures need "∞ depth" gluing data.

* This "depth of gluing" or "truncatedness" heuristics is measured by the algebraicity n for n-Artin stacks

e.g. derived stack X is 0-Artin (derived alg. space) if
$\Delta: X \to X \times X$ is schematic & mono. & $\exists$ étale atlas $U \twoheadrightarrow X$
where U is a derived sch.

In particular, derived schemes are 0-Artin
Note that derived schemes take values in sets = 0-truncated Grpd_∞

Further, derived stack X is n-Artin if
$\Delta: X \to X \times X$ is $\underbrace{(n-1)\text{-Artin}}$ & $\exists$ smooth atlas $U \twoheadrightarrow X$ where
U is a derived sch.
this condition relates to
the "truncatedness" in the entries of the étale descent seq

* Derived DM stacks : 1-Artin w/ étale derived sch. atlas.

A Quick Excursion to Stable ∞-Categories

Def An ∞-Category $\mathcal{C}$ is stable if

① $\exists$ zero object $0 \in \mathcal{C}$ ② $\forall$ morphism has fiber / Cofiber.

③ Fiber seq = Cofiber seq.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$

exhibits $X = \Omega Y$, $Y = \Sigma X$.

Fact these form a ∞-functor $\Omega, \Sigma : \mathcal{C} \to \mathcal{C}$.

For Stable ∞-Cat $\mathcal{C}$, Ω & Σ are mutual equivalence inverses.

Def $X \mapsto X[n]$ denote Σ^n & $X \mapsto X[-n]$ denote Ω^n.

Def $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in $h\mathcal{C}$ is called the distinguished Δ if

$\exists$ diagram $\Delta^1 \times \Delta^2 \to \mathcal{C}$ as

$$\begin{tikzcd} X \arrow[r, "\tilde{f}"] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[r] \arrow[d, "\tilde{g}"] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r, "\tilde{h}"'] & W \end{tikzcd}$$

s.t. $\pi_0(\tilde{f}) = f$

$\pi_0(\tilde{g}) = g$

$h = (W \cong X[1]) \circ \pi_0(\tilde{h})$

outer $\square$

Fact Under above defn, $h\mathcal{C}$ is a triangulated cat.

We can always form a stable ∞-Category out of an ∞-Cat w/ zero obj & admitting finite limit / Colimits by

$$\mathcal{C}^{st} := \varprojlim \left(\cdots \to \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \right)$$

For $D(A)_{\geq 0}$, let $D(A)$ be the corr stable ∞-Cat.

Fact $D(A)$ has the t-structure $D(A)_{\geq 0}$, $D(A)_{< 0}$.

This $D(A)$ will play the role of the triangulated cat. $DCoh(R)$

e.g. $Perf(A) \subseteq D(A)$ be full subcat. gen. from A by finite lim/colim & direct summands

Fact $dCAlg_R \longrightarrow Cat_\infty$ is a sheaf.
$A \longmapsto D(A)$

Def For derived stack X, the stable ∞-cat. of quasi coh. sheaves $D_{qc}(X)$ is the right Kan extension of $A \mapsto D(A)$.

Why use stable ∞-Cat?

① Triangulated Category behaves badly : the formation of Cone is not functorial.

Ex)

$$\begin{tikzcd} k \arrow[r, "f"] \arrow[d] & 0 \arrow[r] \arrow[d] & k[1] \arrow[r, "id"] \arrow[d, shift right, "0"'] \arrow[d, shift left, "id"] & k[1] \arrow[d] \\ 0 \arrow[r, "g"'] & k[1] \arrow[r, "id"'] & k[1] \arrow[r] & 0 \end{tikzcd}$$

both rows are dist. Δ in $D(k)$

Why does this happen? Mapping Cones are unique up to non-unique iso.
Indeed, $D(R)$ only captures "up to weak equivalence" i.e. "up to homotopy" in the model cat. of sSet sense.
For functoriality, we need to capture its full homotopy data.

② $X \mapsto D_{qc}(X)$ satisfies descent. (induces sheaf of ∞-Cat.)
this doesn't hold for triangulated Cat. "Complexes do not satisfy descent".
This is a major reason why e.g. Cotangent cpx is most natural when considered as $\in D_{qc}(X)$
Descent allows to bypass all "gluing" issues.

③ Hidden Smoothness

Def $f : X \to Y$ be morph. of derived Artin stacks.
f is homotopically smooth if $\pi_0(f)$ is locally f.p & $\mathbb{L}_{X/Y}$ Perfect

Fact X : smooth, proper sch. $\left[\begin{array}{l} \mathcal{M}_{Perf}(X) := Map(X, \mathcal{M}_{Perf}) \\ \mathcal{M}_{Vect}(X) \end{array}\right.$ is homotopically smooth