THE A–D–E CLASSIFICATION

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Abstract. In this talk, I will introduce the simply laced Dynkin diagrams and the so-called ADE classification of various mathematical objects via such diagrams. Some instances that fall under this classification include simply laced Lie algebras, finite subgroups of the spatial rotation group and SU_2, and consequently the du Val singularities in algebraic geometry.

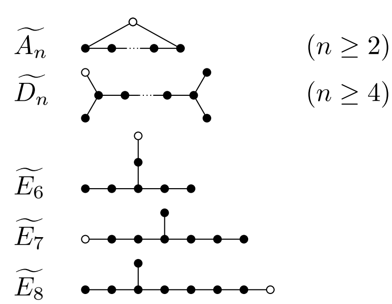
"If we needed to make contact with an alien civilization and show them how sophisticated our civilization is, perhaps showing them Dynkin diagrams would be the best choice!"
–P. Etingof *et al*, in *Introduction to representation theory.*

1. Some graph theory

Definition 1.1. Recall that a graph is a collection of vertices and edges. We will not allow loops (edges from a vertex to itself) or multiple edges, and will only consider connected graphs with finitely many vertices. Given such a graph G, we can define its adjacency matrix A_G, which is defined as $A_{ij} = 1$ if vertices i and j have an edge, and 0 otherwise. Notice that A_G is symmetric, so its eigenvalues are real. If the largest eigenvalue of A_G is less than 2, we will call the graph a Dynkin diagram, and if it is equal to 2, we will call it an extended Dynkin diagram. (I will omit the words 'simply laced' as we will not consider multiple edges.)

Proposition 1.2. *Let G be as above. The (extended) Dynkin diagrams are classified up to 2 infinite families, and 3 exceptional diagrams.*

Proof. We will use the fact that the extended Dynkin diagrams are classified as follows, without proof:



Here, the subscripts are the number of nodes that are filled in. These are precisely the connected graphs with no multiple edges whose maximum eigenvalue is 2. (I

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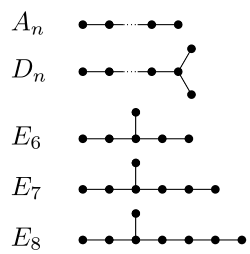
will use the term 'eigenvalue of a graph' to denote the eigenvalues of its adjacency matrix.) This can be shown not so difficultly using basic graph theory, but I won't go into details.

We will use the fact that the largest eigenvalue of a subgraph is less than or equal to the largest eigenvalue of the original graph. In particular, this implies that any graph G that has maximum eigenvalue less than 2 cannot contain the graphs listed above. This greatly restricts the possibilities, as not containing $\widetilde{A_n}$ ensures it is a tree, and not containing $\widetilde{D_4}$ eliminates any degree 4 or higher vertices. Also, if it exists, a degree 3 vertex must be unique, since otherwise it implies containing $\widetilde{D_n}$ for some n. (Here, the degree of a vertex is the number of edges containing it.) In conclusion, the only possibilities are as follows:



Notice how they're just the extended Dynkin diagrams, minus the hollow vertex (hence the name). □

These Dynkin diagrams are not so interesting as graphs themselves, however, these graphs appear naturally in a wide range of classification problems which make them interesting.

2. Classification of simply laced root systems

To see where the seemingly arbitrary names A_n, D_n and E_6, E_7, E_8 come from, we study the classification of simple Lie algebras via root systems. We will be classifying abstract Lie algebras, as opposed to the real Lie algebras occuring as the tangent space at the origin of a C^∞ Lie group. Let k be an algebraically closed field of characteristic zero.

Definition 2.1 (Lie algebras and morphisms)**.** A Lie algebra $\mathfrak{g}$ is a k-vector space equipped with a binary operation

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

called the Lie bracket, which is k-bilinear, alternating, and satisfies the Jacobi identity:

$$\sum_{\sigma \in A_3} [x_{\sigma(1)}, [x_{\sigma(2)}, x_{\sigma(3)}]] = 0$$

for all vectors $x_1, x_2, x_3 \in \mathfrak{g}$. A Lie algebra homomorphism between two Lie algebras is a k-linear map that preserves the Lie bracket.

We try to classify the finite dimensional Lie algebras. As opposed to vector spaces which are classified by dimension, there can be different Lie bracket structures on the same underlying vector space. As a trivial example, the k-vector space $\mathrm{Mat}_n(k)$ of $n \times n$ matrices can be either equipped with a Lie bracket $[X, Y] := XY - YX$, or it can be defined to be identically zero. In general, the classification problem is too messy, but the following theorem asserts that we can decompose finite dimensional Lie algebras into a 'messy part' and a 'nice part', where it is possible to classify the latter.

Theorem 2.2 (Levi decomposition)**.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then the following sequence*

$$0 \to \mathrm{rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}_{\mathrm{ss}} \to 0$$

is split. (This also holds for nonclosed fields of characteristic zero.)

Here, $\mathrm{rad}(\mathfrak{g})$ is the unique maximal solvable ideal (whatever that means) of $\mathfrak{g}$, and $\mathfrak{g}_{\mathrm{ss}}$ is the quotient, which is a semisimple subalgebra of $\mathfrak{g}$. Being semisimple means that its only solvable ideal is zero. In fact, we can say more:

Theorem 2.3. *Semisimple Lie algebras are direct sums of simple Lie algebras. (A Lie algebra is simple if it has no nontrivial ideals*[1]*.)*

Therefore we have reduced our attention to the simple Lie algebras. These can be classified using (irreducible) root systems. To be precise, simple Lie algebras admit a root space decomposition via something called a *Cartan subalgebra*, which is analogous to the diagonal part in the Jordan canonical form of linear algebra. (This is where the algebraically closed condition is used.) Although the roots in this sense are *a priori* linear functionals on the Cartan subalgebra, they correspond uniquely to a root system defined in Euclidean space as follows:[2]

Definition 2.4 (Root system)**.** A rank n root system is a finite set Φ of nonzero vectors in $\mathbb{R}^n$ such that:

(1) $\mathrm{span}\Phi = \mathbb{R}^n$,
(2) the only scalar multiples of $\alpha \in \Phi$ are itself and $-\alpha$,
(3) Φ is invariant under reflections by the hyperplanes of its roots,
(4) for all $\alpha, \beta \in \Phi$, $2\langle \alpha, \beta\rangle / ||\alpha||^2$ is an integer.

The root system is reducible if there is a partition of Φ into $\Phi_1 \cup \Phi_2$ such that elements of Φ_1 and Φ_2 are orthogonal. Otherwise, it is irreducible.

The classification of irreducible root systems gives rise to a complete classification of simple Lie algebras. To each root system, one can draw a Dynkin diagram placing the roots as vertices, and the number of edges corresponding to the angle between the roots. There are four infinite families A_n, B_n, C_n, D_n of irreducible root systems and hence four families of Lie algebras $\mathfrak{sl}_{n+1}, \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{so}_{2n}$, called the classical Lie algebras, and five other exceptional root systems corresponding to Lie algebras $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ of dimension 14, 52, 78, 133, and 248, respectively.

Example 2.5 (A_n and $\mathfrak{sl}_{n+1}$)**.** The root system A_2 is generated by combinations of the vectors $(1, 0)$ and $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ in $\mathbb{R}^2$. The Lie algebra $\mathfrak{sl}_3$ consists of 3×3 traceless matrices, with Cartan subalgebra $\mathfrak{h}$ as the diagonal matrices. Then, there are six roots $\pm\alpha, \pm\beta, \pm\gamma$ on $\mathfrak{h}^*$ that satisfy $\alpha + \beta = \gamma$, so $\mathfrak{sl}_3$ indeed corresponds to A_2.

This is broader than the ADE classification we are interested in. In particular, the Dynkin diagrams involve multiple edges. To make our Dynkin diagram single-edged, it suffices to restrict our attention to root systems where every root has the

[1] subspaces closed under the Lie bracket

[2] The fact that the vector space is defined over $\mathbb{R}$ is irrelevant; this is just to help visualize the roots. Although it is unfortunate that we cannot visualize D_n.

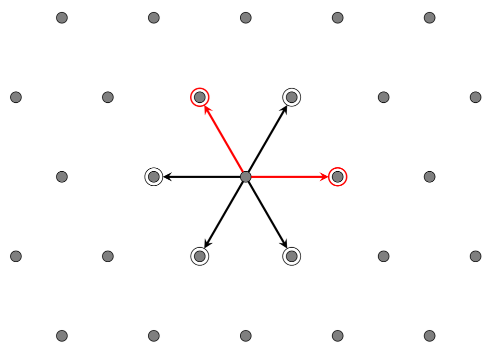


FIGURE 1. A_2

same length. These are called simply laced root systems, and they are precisely the A_n, D_n, E_6, E_7, E_8. Thus, the simple Lie algebras that are classified by simply laced root systems are $\mathfrak{sl}_{n+1}, \mathfrak{so}_{2n}, \mathfrak{e}_6, \mathfrak{e}_7$, and $\mathfrak{e}_8$.

3. Finite subgroups of SU_2 and SO_3

The classification of (simply laced) root systems and consequently simple Lie algebras is perhaps one of the most important instances of the ADE classification. However, there are various classification problems that are less abstract than Lie groups that also follow the ADE classification. Consider the matrix group $\mathrm{SO}_3(\mathbb{R})$ of 3×3 orthogonal real matrices with determinant 1. Using the natural action of SO_3 on the unit sphere S^2 and using the orbit-stabilizer theorem, the classification of finite subgroups basically boils down to a combinatorial problem. Again this is tedious so I'll state the results.

We have an infinite family of rotations $\mathbb{Z}/n\mathbb{Z}$ given by the symmetries of a cone with an n-gon base, an infinite family of dihedral groups D_n of the symmetries of an n-gon, and three exceptional groups given by the symmetries of Platonic solids. There are only three exceptional groups because the cube and octahedron, the dodecahedron and icosahedron are mutually dual polytopes and have the same symmetry groups. (In familiar terms, the groups correspond to A_4, S_4 and A_5.) Notice the tetrahedron is self-dual. In fact, via the 2-1 covering $\mathrm{SU}_2 \to \mathrm{SO}_3$, the finite subgroups of SU_2 follow the same classification.

To see that the groups mentioned actually correspond to the ADE type graphs, we lift it to SU_2 to make its representations easier to handle, and then find the McKay graph $M(G)$ via the character table of the inclusion $G \to \mathrm{SU}_2$. This is done similarly to the Dynkin diagrams drawn from root systems. Then, it follows that the McKay graphs correspond to the extended Dynkin diagrams as expected. This is part of what's called the McKay correspondence.

The classification of subgroups of SU_2 also has some consequences in algebraic geometry. Let X be a complex surface and $p \in X$ a singular point. If, locally at p, X is isomorphic to $\mathbb{C}^2/\Gamma$ for some finite subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$, then X is said to have a du Val singularity at p. Note that finite subgroups of SL_2 are conjugate to finite subgroups of SU_2. One can draw exceptional divisors of the minimal resolution of a du Val singularity, and surprisingly they form a simply laced Dynkin

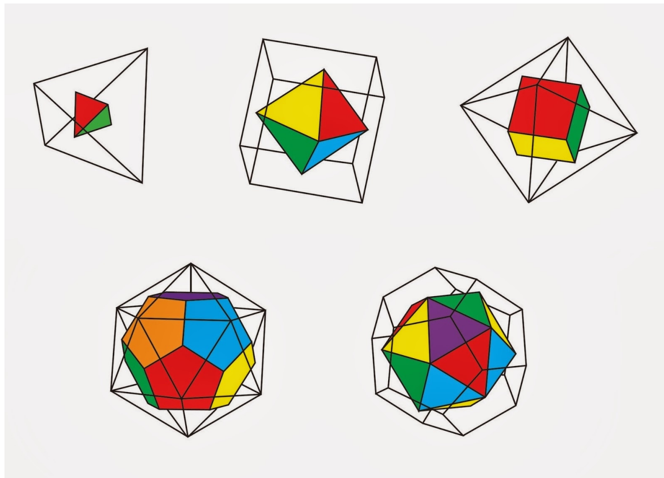
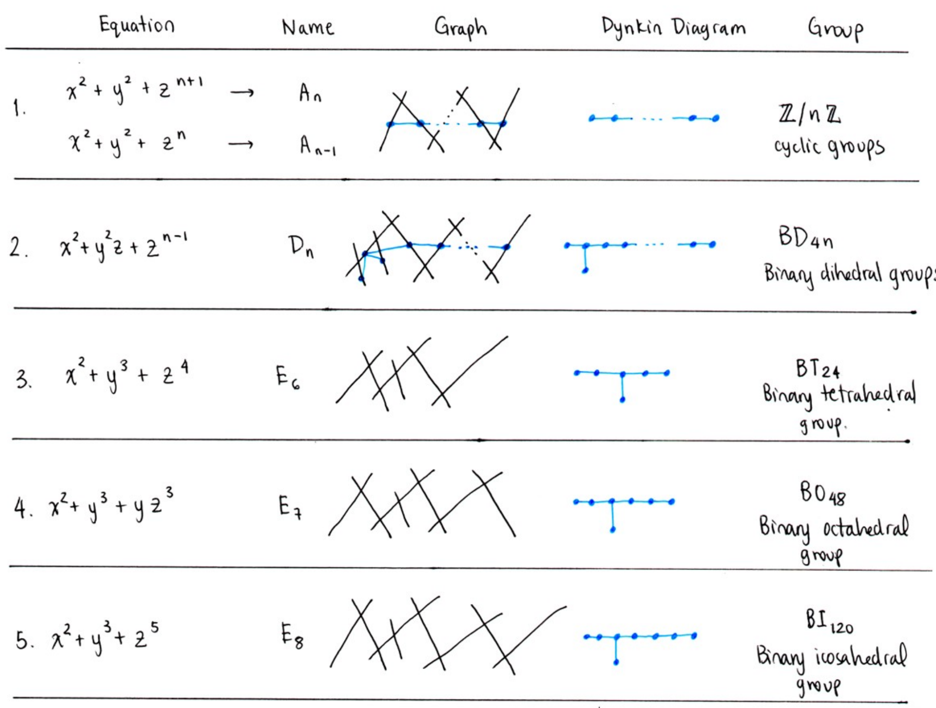


FIGURE 2. Platonic solids and their duals, stolen from the internet



	Equation		Name	Graph	Dynkin Diagram	Group
1.	$x^2+y^2+z^{n+1}$	$\longrightarrow$	A_n			$\mathbb{Z}/n\mathbb{Z}$ cyclic groups
	$x^2+y^2+z^n$	$\longrightarrow$	A_{n-1}			
2.	$x^2+y^2z+z^{n-1}$		D_n			BD_{4n} Binary dihedral groups
3.	$x^2+y^3+z^4$		E_6			BT_{24} Binary tetrahedral group.
4.	$x^2+y^3+yz^3$		E_7			BO_{48} Binary octahedral group
5.	$x^2+y^3+z^5$		E_8			BI_{120} Binary icosahedral group

FIGURE 3. Again stolen from the internet

diagram. Vertices correspond to the (-2)-curves, with an edge connecting them if they intersect.

4. More examples

In my last talk, I proved the result that there are 27 lines on every smooth cubic surface. I received a question about the meaning of the number 27; at that time I did not know much. However after further investigation, I found out that the number 27 in this sense is incredibly special, as the E_6 Lie algebra is deeply related to the lines on a cubic, and hence the number 27. So in some sense you could say that the number 27 came from E_6. (For example, the smallest dimension of the irreps of E_6 is 27.)

Recall the parameter space U of smooth cubics; if we follow a closed loop in the parameter space, this defines a permutation on the 27 lines. If the parameter space is simply connected, then this monodromy action must be trivial. On the other hand, since it is a permutation, it must be a subgroup of S_{27}. Surprisingly, the actual monodromy group is neither trivial nor the whole permutation group, but actually isomorphic to the reflection group generated by the root system corresponding to E_6.

$$\text{Image of } \pi_1(U, x_0) \to \text{Perm}(27 \text{ lines})?$$

The rank 8 lattice generated by the roots of E_8 give the optimal solution to the sphere packing problem in 8 dimensions. The packing is obtained by placing radius $1/\sqrt{2}$ spheres centered on each lattice point. As such, *exceptional* results in mathematics have some exceptional Lie algebra lurking behind them, I guess.