COXETER GROUPS

A. H. LEE

1. Introduction

Before even stating the definition of Coxeter groups, it is a good idea to introduce why one would be interested in such groups. Coxeter groups are a natural language for crystal-like structures arising in geometry. An important subclass of these structures is the root systems of semisimple Lie algebras and their corresponding Weyl groups (which are Coxeter groups, too). There may be a deeper representation-theoretic reason that I am not aware of to why Coxeter groups are so important (eg. Iwahori-Hecke algebras, Kazhdan-Lusztig polynomials, etc.) in contrast to mere Weyl groups.

2. Basic definitions

Recall group presentations: we can describe abstract groups by specifying generators and the relations among them. (Warning: These are not unique in general.) For a finite[1] set S and any two $s \neq s' \in S$, impose a relation among these by $(ss')^{m_{ss'}} = 1$ where $m_{ss'}$ is the order of ss', or either ss' has infinite order (meaning that they do not admit a relation). Also, let $m_{ss} = 1$, i.e., $s^2 = 1$ for every $s \in S$. These generators and relations define a Coxeter group W. More precisely, we will call (W, S) a Coxeter system (or pair). If $|S| = \ell$, we will say that the system (W, S) has rank ℓ, and write $S = \{s_1, \ldots, s_\ell\}$. A Coxeter group W is irreducible if it cannot be written as a product of other nontrivial Coxeter groups.

A Coxeter matrix $M = (m_{ss'})$ is a symmetric matrix with 1 along the diagonals, and $m_{ss'} \in \{2, 3, \ldots, \infty\}$. Since in Coxeter systems we have $m_{ss'} = m_{s's}$, we can see that a Coxeter system gives rise to a Coxeter matrix. Later in 5.8 we will see that the converse also holds.

For each Coxeter system (W, S), we can assign a graph (with labelled edges) called a Coxeter graph Γ_W. Draw a node for each $s \in S$, and draw an edge from s to s' if $m_{ss'} > 2$, labelled by $m_{ss'}$. For $m_{ss'} = 3$ the label is often omitted.

Remark 2.1. A product of Coxeter systems is again a Coxeter system. If $W = \prod_\alpha W_\alpha$, then $\Gamma_W = \coprod_\alpha \Gamma_{W_\alpha}$, i.e., the disjoint union. A Coxeter group W is irreducible if and only if Γ_W is connected.

An important structure of Coxeter groups is the length function.

Definition 2.2. Let (W, S) be a Coxeter system. For every $w \in W$, define its length to be $\ell(w) = n$, the smallest natural number n such that w can be written as a product of n elements of the generating set S. If $w = s_1 \cdots s_n$ and $\ell(w) = n$, we call this a reduced decomposition.

Lemma 2.3. *Let $w, w' \in W$.*

(1) $\ell(w^{-1}) = \ell(w)$;
(2) $\ell(w) - \ell(w') \leq \ell(ww') \leq \ell(w) + \ell(w')$.

Proof. Suppose $\ell(w) = n$, i.e., we can write $w = s_1 \cdots s_n$. Then notice that we can write $w^{-1} = s_n \cdots s_1$, so we have $\ell(s^{-1}) \leq n = \ell(w)$. By a symmetric argument we can see that $\ell(w) \leq \ell(w^{-1})$, so the first result follows. For the second part, let $w = s_1 \cdots s_p$ and $w' = s'_1 \cdots s'_q$. Then we can write ww' as $s_1 \cdots s_p s'_1 \cdots s'_q$, so indeed $\ell(ww') \leq \ell(w) + \ell(w')$. Also from $\ell(ww'(w')^{-1}) \leq \ell(ww') + \ell((w')^{-1}) = \ell(ww') + \ell(w')$, we have $\ell(w) - \ell(w') \leq \ell(ww')$. $\square$

Date: September 25, 2025.

[1] In general, we do not need the finiteness of the generating set. However, we will restrict our attention to the case where generating sets are finite. In fact, there are not many practical examples that involve an infinite generating set.

3. Some examples

We give some examples of Coxeter groups.

Example 3.1. Consider the symmetric group S_n on n letters. It can be generated by elements of the form $(i, i+1)$ for $1 \leq i < n$, so let the generating set S consist of such elements. The Coxeter graph of (S_n, S) is given by:

$$\circ \text{———} \circ \text{———} \cdots \text{———} \circ \text{———} \circ$$

where there are $n-1$ dots, each corresponding to elements of S. Recall that an edge without a label means that the corresponding m is 3; this can be seen from the fact that $(i, i+1)(i+1, i+2) = (i, i+1, i+2)$, so is of order 3, but for non-adjacent permutations, their composition is of order 2. Order 2 edges are omitted from the diagram.

Example 3.2. Consider the group O_n of $n \times n$ signed permutation matrices, meaning that we allow ± 1 as nonzero entires in the permutation matrices. Then this group is isomorphic to $S_n \rtimes (\mathbb{Z}/(2))^n$ where S_n acts on $(\mathbb{Z}/(2))^n$ in the obvious way. Let $s_i = (i, i+1)$ for $1 \leq i < n$, and let s_n change the sign of the last column. Then $S = \{s_1, \ldots, s_n\}$ becomes a generating set for O_n, and we call the Coxeter system (O_n, S) the hyperoctahedral group. The Coxeter diagram is given as follows:

$$\circ \text{———} \circ \text{———} \cdots \text{———} \circ \overset{4}{\text{———}} \circ$$

where there are n nodes each corresponding to elements of S. This can be seen in similar methods as above. The last edge having order 4 can be seen by its matrix representation being $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and this has order 4.

Remark 3.3. The group S_n has a homomorphism to ± 1 given by $\tau \mapsto (-1)^{\ell(\tau)}$. We can generalize this notion to any Coxeter system, due to the existence of the length function. To show this is well-defined, it suffices to show that the relations are sent to 1, which is true because $(ss')^m \mapsto (-1)^{2m} = 1$.

4. Rank 2 Coxeter systems

We focus on the case $|S| = 2$. This is because in many cases, we can reduce the problem of general Coxeter systems to rank 2 cases. In fact this can be seen geometrically, as if we have two vectors in arbitrary-dimensional Euclidean space, the reflections they generate are totally determined by the 2-dimensional subspace they are contained in.

Theorem 4.1. *If (W, S) is a Coxeter system of rank 2, then either W is a finite dihedral group D_m or the infinite dihedral group $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/(2)$.*

Proof. Since (W, S) is rank 2 we can write $S = \{s, s'\}$ with one relation $(ss')^m = 1$. If $m = \infty$, then $W = \mathbb{Z}/(2) * \mathbb{Z}/(2)$, which is isomorphic to $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/(2)$. If $m < \infty$, if we let $a = ss'$ and $b = s$, then we have $\langle a \rangle = \mathbb{Z}/(m)$ which is normal in W as $s(ss')^j s^{-1} = (ss')^{-j}$. Also $\langle b \rangle \langle a \rangle = W$, so we have $W = \mathbb{Z}/(m) \rtimes \mathbb{Z}/(2) = D_m$. □

5. Geometric realization, canonical representation

Let V be an $\mathbb{R}$-vector space equipped with a symmetric bilinear form $\langle -, - \rangle$. Let $a \in V$ such that $\langle a, a \rangle \neq 0$. We can define a reflection S_a w.r.t. a, as the unique linear transformation fixing $\{x \in V \mid \langle x, a \rangle = 0\}$, which sends $a \mapsto -a$. In fact, we can explicitly write $s_a(x)$ as $x - \frac{2\langle x, a \rangle}{\langle a, a \rangle} a$.

We will geometrically realize Coxeter groups as being generated by such reflections. Let $V = \mathbb{R}^\ell$ for $\ell = |S|$ and let $e_1, \ldots, e_\ell$ be the standard basis. We want to define a symmetric bilinear form $\langle -, - \rangle$ on V such that s_{e_i} satisfy the Coxeter group relations.

Lemma 5.1. *The group $\mathrm{O}(2, \mathbb{R})$ consists of reflections and rotations.*

Proof. Let e_1, e_2 be the standard basis elements of $\mathbb{R}^2$. Let $\varphi \in \mathrm{O}(2, \mathbb{R})$. Since this preserves the standard inner product on $\mathbb{R}^2$, we have $\varphi(e_1) = ae_1 + be_2$ for $a^2 + b^2 = 1$, say $a = \cos\theta$ and $b = \sin\theta$. Also

$\varphi(e_2) = \pm(-be_1 + ae_2)$, so the matrix representation of φ is either $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, or $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$. The first one corresponds to a rotation of angle θ, and the latter one is a reflection through the line perpendicular to $(\cos(\frac{\theta+\pi}{2}), \sin(\frac{\theta+\pi}{2}))$. □

Corollary 5.2. *If α and β are unit vectors in a finite dimensional $\mathbb{R}$-vector space V, then the composition $s_\beta s_\alpha$ is a rotation through twice the angle between them.*

Proof. If $\alpha = \pm\beta$, then there is nothing to prove. Suppose not, then $s_\alpha s_\beta$ fixes an $(n-2)$-dimensional subspace $\alpha^\perp \cap \beta^\perp$ of V. Hence, we may reduce our situation to the 2-dimensional subspace of V containing both α and β. We claim that $\ker(1 - s_\beta s_\alpha) = 0$, which would imply that this is a rotation. Suppose otherwise, then there exists a nonzero vector x such that $x = s_\beta s_\alpha(x)$, i.e., $s_\alpha(x) = s_\beta(x)$. Writing down the equations, this means that $x - \frac{2\langle x,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha = x - \frac{2\langle x,\beta\rangle}{\langle\beta,\beta\rangle}\beta$, and since α and β are both assumed to be unit vectors, we have $\langle x,\alpha\rangle\alpha = \langle x,\beta\rangle\beta$. But this implies that α and β are parallel vectors, which is a contradiction to our assumption. Hence $s_\beta s_\alpha$ cannot be a reflection, so it must be a rotation of the 2-dimensional subspace of V containing α and β. With this knowledge, it suffices to find out the rotation angle of any single vector to determine the whole rotation. Take $\alpha \mapsto s_\beta s_\alpha(\alpha) = s_\beta(-\alpha)$, and it is not hard to see that this rotates α twice the angle between β and α. □

If two unit vectors α, β make an angle $\pi - \frac{\pi}{m}$, then $s_\beta s_\alpha$ is a rotation of angle $-\frac{2\pi}{m}$ so has finite order m. Also $\langle\alpha,\beta\rangle = -\cos(\frac{\pi}{m})$, so we make the following definition:

Definition 5.3. Let (W, S) be a Coxeter system of rank ℓ. The canonical bilinear form $B(-,-)$ on $V = \mathbb{R}^\ell$ is given as $B(e_s, e_{s'}) = -\cos(\frac{\pi}{m_{ss'}})$ for $m_{ss'} < \infty$, and -1 otherwise.

Note that in the case $s = s'$, $m_{ss'}$ can be considered as just 1, so it is natural to define $B(e_s, e_s) = 1$. We now define the canonical representation ρ of W. First on S, define a linear map in V as $\rho(s) : x \mapsto x - 2B(x, e_s)e_s$ for $s \in S$ and $x \in V$.

Lemma 5.4. *For each $s \in S$, the linear map $\rho(s)$ is a reflection of V that preserves the bilinear form $B(-,-)$.*

Proof. First note that $\rho(s)$ is indeed a reflection, as it sends $e_s \mapsto -e_s$, and it fixes the subspace $\{x \in V \mid B(x, e_s) = 0\}$. We calculate

$$B(\rho(s)(x), \rho(s)(y)) = B(x - 2B(x, e_s)e_s, y - 2B(y, e_s)e_s) \tag{5.1}$$
$$= B(x, y - 2B(y, e_s)e_s) - 2B(x, e_s)B(e_s, y - 2B(y, e_s)e_s) \tag{5.2}$$
$$= B(x, y) - 2B(y, e_s)B(x, e_s) - 2B(x, e_s)(B(e_s, y) - 2B(y, e_s)B(e_s, e_s)) \tag{5.3}$$
$$= B(x, y) - 2B(y, e_s)B(x, e_s) - 2B(x, e_s)B(e_s, y) + 4B(x, e_s)B(y, e_s) \tag{5.4}$$
$$= B(x, y). \tag{5.5}$$

□

Lemma 5.5. *The restriction of $B(-,-)$ to $\mathbb{R}e_s \oplus \mathbb{R}e_{s'}$ for $s \neq s'$ is positive semidefinite, and is positive-definite if and only if $m_{ss'} < \infty$.*

Proof. Let $x = ae_s + be_{s'}$ and $m = m_{ss'}$. Recall that a bilinear form is positive semidefinite if $B(x, x) \geq 0$ for all $x \in V$, and is positive definite if the equality holds only when $x = 0$. Plug in $x = ae_s + be_{s'}$ and expand:

$$B(x, x) = B(ae_s + be_{s'}, ae_s + be_{s'}) \tag{5.6}$$
$$= a^2 + 2B(ae_s, be_{s'}) + b^2 \tag{5.7}$$
$$= a^2 + b^2 + 2abB(e_s, e_{s'}). \tag{5.8}$$

For $m = \infty$, we have defined $B(e_s, e_{s'})$ to be -1, so in that case the above expression becomes $(a - b)^2 \geq 0$. This can be zero for any $a = b$, so in this case, $B(-,-)$ is strictly positive-semidefinite. Now suppose $m_{ss'} < \infty$. In that case we have $B(e_s, e_{s'}) = a^2 + b^2 - 2ab\cos(\frac{\pi}{m})$, and we can show that this equals $(a - b\cos(\frac{\pi}{m}))^2 + b^2\sin^2(\frac{\pi}{m})$. As $s \neq s'$ we must have $m > 1$, and thus for this expression to be zero, we must have $b = 0$ and $a = b\cos(\frac{\pi}{m}) = 0$. Hence in the case $m < \infty$, $B(-,-)$ is positive-definite. □

Lemma 5.6. *The order of the composition $\rho(s)\rho(s')$ is $m_{ss'}$.*

Proof. For $s = s'$, this is obvious. Now suppose $s \neq s'$, and suppose that $m_{ss'} < \infty$. Again, we restrict to the subspace $E = \mathbb{R}e_s \oplus \mathbb{R}e_{s'}$. As $m_{ss'} < \infty$, $B(-,-)$ defines an inner product on E, and since we know that $s_\beta s_\alpha$ has finite order m where $\langle \alpha, \beta \rangle = -\cos(\frac{\pi}{m})$, it indeed has finite order m. In the case $m_{ss'} = \infty$, let $x = e_s + e'_s$ and notice that $\rho(s)\rho(s')$ sends e_s to $e_s + 2x$. Repeating this indefinitely, e_s is never sent to e_s itself, so indeed the order is infinite. □

Proposition 5.7. *Let (W, S) be a Coxeter system, and $B(-,-)$ the associated canonical bilinear form. Let $\mathrm{O}(B) \subset \mathrm{GL}(V)$ be the subgroup preserving the bilinear form $B(-,-)$. Then there exists a unique homomorphism*

$$\rho : W \to \mathrm{O}(B)$$

which agrees with the correspondence we gave above on elements of $S \subset W$.

In fact, ρ is faithful and its image is discrete in $\mathrm{O}(B)$.

Corollary 5.8. *Any Coxeter matrix arises as the associated matrix of a Coxeter system.*

Proof. Let M be an $\ell \times \ell$ Coxeter matrix, and consider a set S of ℓ letters. Let F be the free group on S. Let N be the normal subgroup of F generated by elements of the form $(ss')^{m_{ss'}}$. Writing $p : F \to F/N$ the natural map, it turns out that $(F/N, p(S))$ is a Coxeter system with associated matrix M. Details are omitted. □

6. Application: finiteness criterion for Coxeter groups

We work towards developing a finiteness criterion for Coxeter groups. This is a step towards the classification of finite Coxeter groups.

Lemma 6.1. *Let (W, S) be an irreducible Coxeter system. If $E \subset V$ is a W-invariant proper subspace of V, then $E \subset V^\perp = \{x \in V \mid B(x, y) = 0 \ \forall y \in V\}$. Here, B is the canonical bilinear form of (W, S).*

Proof. We first claim that $e_s \notin E$ for every $s \in S$. Let $S' = \{s \in S \mid e_s \in E\}$, and denote $S'' = S \setminus S'$. We want to show that S' is empty. Suppose not, say we have some $s \in S'$. As we have assumed our W to be irreducible, there must exist a $t \in S''$ such that $B(e_s, e_t) \neq 0$. Then $\rho(t)(e_s) = e_s - 2B(e_s, e_t)e_t$, so $2B(e_s, e_t)e_t = e_s - \rho(t)(e_s)$. Since E is W-invariant, we must have $\rho(t)(e_s) \in E$, so in particular the r.h.s. must be in E. But then this implies that $e_t \in E$, which is a contradiction. Hence S' must be empty.

Now take an arbitrary $x \in E$. For each $s \in S$, we have $2B(x, e_s)e_s = x - \rho(s)x \in E$, and since $e_s \notin E$, the only possibility is for $B(x, e_s)$ to be zero. Since this holds for every e_s, we have $x \in V^\perp$. Therefore $E \subset V^\perp$. □

Corollary 6.2. *Again, let (W, S) be an irreducible Coxeter system.*

(1) *If $B(-,-)$ is degenerate, then ρ is not completely reducible.*[2]
(2) *If $B(-,-)$ is nondegenerate then ρ is irreducible.*

Proof. (1) Note that $V^\perp$ is a W-invariant subspace of V, as every $\rho(s)$ preserves $B(-,-)$. Also, we have $e_s \notin V^\perp$ as $B(e_s, e_s) = 1$ for every $s \in S$. If $B(-,-)$ is degenerate, then this means that $V^\perp$ is nontrivial, hence $0 \subsetneq V^\perp \subsetneq V$. If the representation $\rho : W \to \mathrm{O}(B)$ is completely reducible, then there exists a nontrivial proper W-invariant subspace $E \subsetneq V$ such that $V = V^\perp \oplus E$ as W-modules. But by 6.1, we have $E \subset V^\perp$, which is a contradiction.

(2) Notice that $B(-,-)$ being nondegenerate implies that $V^\perp = 0$, so again by 6.1, any proper W-invariant subspace of V must be zero. Thus ρ is irreducible. □

Corollary 6.3. *If (W, S) is an irreducible Coxeter system and $B(-,-)$ is degenerate, then W must be infinite.*

Proof. By Maschke's theorem, real representations of finite groups are completely reducible. But in this case, ρ isn't. Hence W must be infinite. □

[2] A.k.a. semisimple, i.e., it is a direct sum of irreps.

Hence we don't have finiteness in the degenerate case, so we have to bet on the nondegenerate case. In fact, we have a nice finiteness criterion given as follows:

Theorem 6.4. *Let (W, S) be an irreducible Coxeter system. Then W is finite if and only if $B(-,-)$ is positive-definite.*

References

[1] H. Hiller. *Geometry of Coxeter Groups.* Monographs and Studies in Mathematics. Pitman Pub., 1982.