COMPLEX MANIFOLDS REVIEW

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1. Holomorphic line bundles on $\mathbb{P}^n$

Definition 1. A **complex vector bundle of rank** r on a complex manifold X is a smooth manifold E together with a smooth map $\pi : E \to X$ such that each $\pi^{-1}(x)$ is a $\mathbb{C}$-vector space of dimension r. Additionally, there must be an open cover $\{U_i\}_{i\in I}$ of X together with homeomorphisms $\varphi_i : U_i \times \mathbb{C}^r \xrightarrow{\sim} \pi^{-1}(U_i)$, where $\pi|_{\pi^{-1}(U_i)} \circ \varphi_i$ agrees with the projection of $U_i \times \mathbb{C}^r$ onto U_i, and for each $x \in U_i$ we must have $\varphi_i(x, -) : \mathbb{C}^r \to \pi^{-1}(x)$ be a $\mathbb{C}$-linear isomorphism.

E is usually called the total space of the vector bundle.

Definition 2. A **holomorphic vector bundle of rank** r on a complex manifold X, is a complex manifold E together with a holomorphic map $\pi : E \to X$ which is also a complex vector bundle.

Note that there is another definition:

Definition 3. A **holomorphic vector bundle of rank** r on a complex manifold X, is a complex vector bundle $\pi : E \to X$ together with a local trivialization whose transition maps $g_{ij} : U_i \cap U_j \to \mathrm{GL}_r(\mathbb{C})$, $x \mapsto (\varphi_i^{-1}|_{\pi^{-1}(U_i\cap U_j)} \circ \varphi_j|_{U_i\cap U_j\times\mathbb{C}^r})(x, -)$ are holomorphic, i.e., g_{ij} are holomorphic maps between complex manifolds $U_i \cap U_j$ and $\mathrm{GL}_r(\mathbb{C})$. (These are naturally complex manifolds, as they are open subsets of complex manifolds.)

If a complex vector bundle satisfies definition 3, then it satisfies definition 2. I don't know if the converse holds, so for safety we will use definition 3.

Definition 4. A **holomorphic line bundle** is a holomorphic vector bundle of rank 1.

Now we will look at holomorphic line bundles on $\mathbb{P}^n$, which are probably the most important. We first define the tautological line bundle $\mathcal{O}(-1)$, from which it is possible to define all other $\mathcal{O}(n)$.

Definition 5. The total space of the **tautological line bundle** $\mathcal{O}_{\mathbb{P}^n}(-1)$ is a complex submanifold of $\mathbb{P}^n \times \mathbb{C}^{n+1}$, writing coordinates as $[x_0 : \cdots : x_n] \times (y_0, \dots, y_n)$, which is given by the equations $x_i y_j - x_j y_i = 0$ for all $i \neq j \in \{0, \dots, n\}$. The bundle projection $\pi : \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathbb{P}^n$ is given by restricting the projection $\pi_1 : \mathbb{P}^n \times \mathbb{C}^{n+1} \to \mathbb{P}^n$ to $\mathcal{O}_{\mathbb{P}^n}(-1) \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$.

Intuitively, the total space of the line bundle $\mathcal{O}(-1)$ is given by pairs $([\ell], z)$ where $[\ell] \in \mathbb{P}^n$, and z is the collection of points in ℓ.

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Example 6 (Total space of $\mathcal{O}_{\mathbb{P}^1}(-1)$)**.** According to the definition above, this is a complex submanifold of $\mathbb{P}^1 \times \mathbb{C}^2$ given by the equation $x_0y_1 - x_1y_0 = 0$. There are no other equations, since indices are in $\{0,1\}$. Is it believable that the $(y_0, y_1) \in \mathbb{C}^2$ that satisfy the equation above, are precisely the points on the line defined by (x_0, x_1)? Plug in $(y_0, y_1) = (x_0, x_1)$. Now plug in $(\lambda x_0, \lambda x_1)$ for whatever complex number λ you want. Since x_0 and x_1 are homogeneous coordinates, they cannot simultaneously be zero. Thus it follows that the set of $(\lambda x_0, \lambda x_1)$ in $\mathbb{C}^2$ indeed forms a (complex) line.

Now that we know the total space description of $\mathcal{O}_{\mathbb{P}^n}(-1)$, let's try to translate this into transition function language. To do this, we first need to find a trivializing open cover $\{U_i\}$ of the bundle. We claim that the standard open cover $\{U_i\}_{i=0,\dots,n}$ of $\mathbb{P}^n$ is a trivializing open cover for $\mathcal{O}(-1)$. (I'll omit the subscript $\mathbb{P}^n$ from now on.) Suppose we restrict to a standard open U_i, which is given as the nonvanishing of x_i, the ith homogeneous coordinate on $\mathbb{P}^n$. Then we can rewrite the defining equations to $y_j = \frac{x_j}{x_i} y_i$ for all $j \neq i$, as x_i is nonzero. Hence on U_i, $\mathcal{O}(-1)$ is given as U_i times the tuples $(\frac{x_0}{x_i}y_i, \frac{x_1}{x_i}y_i, \dots, y_i, \dots, \frac{x_n}{x_i}y_i) = y_i(\frac{x_0}{x_i}, \dots, 1, \dots, \frac{x_n}{x_i})$. Recall that the coordinates of $U_i \cong \mathbb{C}^n$ are given as $(\frac{x_0}{x_1}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i})$, i.e., $\mathcal{O}(-1)$ on U_i is just $U_i \times \mathbb{C}$, where $\mathbb{C}$ can be thought of as the i-th component of $\mathbb{C}^{n+1}$. Therefore, the standard opens U_i are indeed trivializations for $\mathcal{O}(-1)$!

Let's find the transition functions $g_{ij} : U_i \cap U_j \to \mathrm{GL}_1(\mathbb{C})$ of $\mathcal{O}(-1)$. Since $\mathrm{GL}_1(\mathbb{C})$ is just $\mathbb{C}^*$, the g_{ij} should just be holomorphic functions from $U_i \cap U_j$ to $\mathbb{C}^*$. To easily find the transition functions, you need to write down the local descriptions. Thankfully we did that right above. On each U_i, $\mathcal{O}(-1)$ is given by pairs $(\frac{x_0}{x_1}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}) \times y_i(\frac{x_0}{x_i}, \dots, 1, \dots, \frac{x_n}{x_i})$ in $U_i \times \mathbb{C}^{n+1}$, where y_i could be any complex number. Now pick any point $x = [x_0 : \cdots : x_n] \in U_i \cap U_j$. We must have $x_i, x_j \neq 0$. On the trivialization over U_i, the fibers are given by $y_i(\frac{x_0}{x_i}, \dots, 1, \dots, \frac{x_n}{x_i})$, and over U_j, they are given by $y_j(\frac{x_0}{x_j}, \dots, 1, \dots, \frac{x_n}{x_j})$. But for the same x, these two fibers must agree, by definition of a line bundle! Notice how multiplying $\frac{x_i}{x_j}$ to $y_i(\frac{x_0}{x_i}, \dots, 1, \dots, \frac{x_n}{x_i})$ yields $y_i(\frac{x_0}{x_j}, \dots, 1, \dots, \frac{x_n}{x_j})$. Hence putting $y_i = \frac{x_i}{x_j} y_j$ gives equality. In other words, $y_j = \frac{x_j}{x_i} y_i = \left(\frac{x_i}{x_j}\right)^{-1} y_i$. It follows that the transition function $g_{ij} : U_i \cap U_j \to \mathbb{C}^*$ is given by sending $[x_0 : \cdots : x_n]$ to $\left(\frac{x_i}{x_j}\right)^{-1}$. This is obviously holomorphic, since x_i, x_j are nonzero, and obviously well-defined as their ratio does not change.

Definition 7. The line bundles $\mathcal{O}(k)$, $k \in \mathbb{Z}$, on $\mathbb{P}^n$ are defined as the ones having trivializing open cover the standard opens $\{U_i\}_{i=0\dots,n}$ of $\mathbb{P}^n$, together with transition functions $g_{ij} : U_i \cap U_j \to \mathbb{C}^*$ given by $[x_0 : \cdots : x_n] \mapsto \left(\frac{x_i}{x_j}\right)^k$.

You may ask, *why define $\mathcal{O}(k)$ as the ones having transition function exponents k? Why not $-k$?* I think this is because if we define $\mathcal{O}(-1)$, and hence $\mathcal{O}(k)$ as above, then the global sections of $\mathcal{O}(k)$ for $k > 0$ become degree k homogeneous polynomials in $n+1$ variables. So it's less messier than having to say that the global sections of $\mathcal{O}(-k)$ are degree k polynomials.