INTERSECTION THEORY

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1. March 5^{th}

Let X be compact and smooth. Then $Z_1, Z_2 \subset X$ represent cohomology classes via Poincaré duality, so we may use the cup product in cohomology to define an intersection product on homology. If Z_1 and Z_2 meet transversally, then by definition $\dim Z_1 + \dim Z_2 - \dim X = \dim Z_1 \frown Z_2$. If $Z_1 = Z_2$, then $Z_1 \frown Z_2$ is the Euler class of the normal bundle.[1] Also if Z_2 and Z_2' are homologous, then they define the same intersection multiplicities.

Our geometric setting is as follows: We have an algebraic scheme X together with two closed subschemes Z_1 and Z_2 by viewing them as Chow cycles $[Z_1], [Z_2] \in A_*(X)$. Note that in this process, we decompose Z_i into its irreducible components and add them up with suitable multiplicities.

Definition 1.1. The category of affine schemes over k, denoted Aff_k, is the opposite category $\mathsf{CAlg}_k^{\mathrm{op}}$ of the category of commutative k-algebras.

We now explain how to construct a scheme by gluing affine schemes along affine subschemes. Suppose we have $\{U_i\}$ an open cover of X, where each $U_i = \operatorname{Spec} A_i$. Also denote $U_i \cap U_j = \operatorname{Spec} A_{ij}$. We must have ring maps $\psi_{ij} : A_i \to A_{ij}$, $\varphi_{ij} : A_{ij} \to A_{ji}$ satisfying

(1) $\varphi_{ij} = \varphi_{ji}^{-1}$,

(2) φ_{ij} induces a lifting

$$\begin{tikzcd}
A_{ij} \otimes_{A_i} A_{ik} \arrow[r, dashed, "\widetilde{\varphi_{ij}}"] & A_{ji} \otimes_{A_j} A_{jk} \\
A_{ij} \arrow[u, "\psi_{ik}"] \arrow[r, "\varphi_{ij}"] & A_{ji} \arrow[u, "\psi_{jk}"']
\end{tikzcd}$$

(3) we have the cocycle condition

$$\begin{tikzcd}
A_{ij} \otimes_{A_i} A_{ik} \arrow[rr, "\widetilde{\varphi_{ik}}"] \arrow[dr, "\widetilde{\varphi_{ij}}"'] & \circlearrowleft & A_{ki} \otimes_{A_k} A_{kj} \\
& A_{ji} \otimes_{A_j} A_{jk} \arrow[ur, "\widetilde{\varphi_{jk}}"'] &
\end{tikzcd}$$

Then the $U_i = \operatorname{Spec} A_i$ glue together to define a new scheme X. We define the structure sheaf as follows:

(1) For an affine open $V \hookrightarrow U_i$, via functoriality, if $A_i \to B_i$ then we have $\mathcal{O}_X(U) = B_i$.

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[1] Check!

(2) For an arbitrary open $V \hookrightarrow U_i$, we have $\mathcal{O}_X(V) := \{(s_i \in \mathcal{O}_X(V_i))_i \mid V_i \subset V \text{ affine, such that } s_i = s_j \text{ on } V_i \cap V_j\}$.
(3) For an arbitrary[2] $V \subset X$, define $\mathcal{O}_X(V) := \{(s_i \in \mathcal{O}_X(V \cap U_i))_i \mid \varphi_{ij}(s_i) = s_j\}$.

1.1. **Properties of Schemes.** A scheme is reduced if $\mathcal{O}_X(U)$ is reduced for all $U \subset X$ open. A scheme is irreducible if its underlying topological space is. X is reduced and irreducible if and only if $\mathcal{O}_X$ is integral, i.e. $\mathcal{O}_X(U)$ is an integral domain for all $U \subset X$. In this case, we call X a variety.[3]

1.2. **Chow Group.** Define $Z_k(X)$ the free abelian group generated by k-dimensional subvarieties.

1.3. **Application.** We will count curves. To be precise, we will show that the number of degree d rational curves[4] passing through more than $3d-1$ points on the plane is zero, passing through less than $3d-1$ points is infinite, and passing through $3d-1$ points is N_d where it is given by

$$N_1 = 1, \quad N_d = \sum_{d_1, d_2 > 0, d_1 + d_2 = d} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right).$$

The N_d are called **Gromov-Witten invariants**. We want to understand them via intersection theory.

1.4. **The Moduli Space of Curves of Genus Zero.** The genus zero rational nodal curves are of the form of the union of finitely many $\mathbb{P}^1$'s where

(1) two $\mathbb{P}^1$'s can meet at at most 1 point, called a *node*,
(2) around a node, locally it looks like the zero set of xy, where $x = 0$ and $y = 0$ are the two $\mathbb{P}^1$'s,
(3) the dual graph has vertices $\mathbb{P}^1$'s and edges as nodes,
(4) and $g = 0$ means that the dual graph is a tree. (Think of Riemann Surfaces.)

Suppose we have a curve C with k marked points $x_1, \ldots, x_k$. If the x_i are distinct, and are not nodes, we call C prestable. If $C \xrightarrow{\sim} C'$ takes x_i to x_i' for all i, then it is an isomorphism of $(C, x_1, \ldots, x_k) \to (C', x_1', \ldots, x_k')$. Note that the only automorphism of $\mathbb{P}^1$ is the identity. Denote by m_k the moduli space of prestable curves. Then this is an algebraic stack which is smooth, and we know its dimension is $h^1(T_C) - h^0(T_C) + k = g - 3 + k = k - 3$, using Riemann-Roch (if C is smooth). Here, the $+k$ term is due to the marked points being able to mode 1-dimensionally.

We say $(C, x_1, \ldots, x_k)$ is **stable** if each $\mathbb{P}^1$ component has # of marked points on $\mathbb{P}^1$ + # of nodes ≥ 3. Denote by $\overline{\mathcal{M}}_k \subset m_k$ the open substack of stable curves. Then $\overline{\mathcal{M}}_k$ is a Deligne-Mumford stack, so its dimension is nonnegative.[5] Since it is an open substack of a smooth stack, $\overline{\mathcal{M}}_k$ itself is smooth.

[2] open?
[3] One may add conditions such as finite type, which is implied in Fulton.
[4] Curves birationally equivalent to a line; in this case the projective line
[5] ?

2. March 7th

2.1. **The Chow Group.** Define $A_k(X)$, $k \geq 0$ for a scheme X as the quotient of the free abelian groups $Z_k(X)$ generated by k-dimensional subvarieties of X, by the subgroup $W_k(X)$ whose elements are called rational equivalences. The elements of $W_k(X)$ are determined by a finite collection $(W_i, f_i : W_i \to \mathbb{P}^1)_i$ where the W_i are $(k+1)$-dimensional subvarieties of X, and f is a nonconstant morphism. Define

$$\sum_i [\mathrm{div}(W_i, f_i)] := \sum_i \sum_{Z_i \subset W_i \ k\text{-dim'l subvar.}} \mathrm{ord}_{Z_i} f_i \cdot [Z_i] \in Z_k(X)$$

where $\mathrm{ord}_{Z_i} f_i$ is the order of vanishing of f_i along Z_i. Given W, a $(k+1)$-dimensional variety, and $Z \subset W$ a k-dimensional subvariety, $\mathrm{ord}_Z : R(W)^\times \to \mathbb{Z}$ is a group homomorphism where $R(W)$ is the k-algebra of rational functions on W, following Fulton's notation.

Now, what makes the correspondence $X \mapsto A_k(X)$ a functor? It turns out, this doesn't work for all morphisms $f : X \to Y$. Assume we are working over $k = \mathbb{C}$. Then, Grothendieck-Riemann-Roch tells us that for a proper morphism $f : X \to Y$ between smooth X, Y, and $K(X)$ the abelian group generated by coherent sheaves on X under the relation $E_1 - E_2 + E_3 \in \sim \Leftrightarrow 0 \to E_1 \to E_2 \to E_3 \to 0$ exact.[6] Then, the statement is that

$$\begin{array}{ccc} K(X) & \xrightarrow{\mathrm{td}(T_X)\cdot \mathrm{ch}(-)} & A_*(X) \\ {\scriptstyle f_*}\big\downarrow & & {\scriptstyle f_*}\big\downarrow \\ K(Y) & \xrightarrow{\mathrm{td}(T_Y)\cdot \mathrm{ch}(-)} & A_*(Y) \end{array}$$

In other words, for $E \in K(X)$ we have

$$\mathrm{ch}(f_* E) \cdot \mathrm{td}(T_Y) = f_*(\mathrm{ch}(E) \cdot \mathrm{td}(T_X)).$$

We do not require X and Y to be proper. Assume $Y = \bullet$. Then f is proper. The f_* between the K-groups[7] means the Euler characteristic of $\alpha \in K(X)$ and f_* between the A-groups[8] means integration. Note that we have coefficients in $\mathbb{Q}$.[9]

Let X be a (nonsingular?) curve, and $L \in K(X)$ a line bundle on X. Then, the LHS of GRR is $h^0(X, L) - h^1(X, L)$, and the RHS is $\int_X \mathrm{td}(T_X) \cdot \mathrm{ch}(L) = \int_X (1 + \frac{1}{2} c_1(T_X))(1 + c_1(L)) = \int_X \frac{1}{2} c_1(T_X) + c_1(L)$ where since $\int_X c_1(T_X)$ is the Euler characteristic of X (which is $2 - 2g$), we have $1 - g + \deg L$.

To further specialize, consider the case where X is an elliptic curve, and let $p \neq q \in X$ be points on X. Let $L_1 = \mathcal{O}_X$, and $L_2 = \mathcal{O}(p - q)$ be degree zero line bundles on X. Then we have $h^0(L_1) = 1$ and $h^0(L_2) = 0$.[10]

2.2. **Applications of GRR.** The computation of virtual dimensions of Gromov-Witten invariants and Donaldson-Thomas invariants uses GRR.

Let $f : X \hookrightarrow Y$ be a closed regular embedding, where $\dim f = \dim Y - \dim X$. Define the intersection functor $X \frown : A_k(Y) \to A_{k - \dim f}(X)$ so we have $A_k(Y) \xrightarrow{X \frown} A_{k-\dim f}(X) \xrightarrow{f_*} A_{k - \dim f}(Y) \to H_{2(k - \dim f)}(Y)$

[6] TODO: refine the definition of $K(X)$

[7] Grothendieck groups; K-theory

[8] The Chow groups

[9] Why?

[10] Why?

3. March 12th

Counting rational curves on the plane. We briefly review the last lecture. Denote N_d the number of degree d rational curves in the plane passing through $3d-1$ points on the plane.

Theorem 3.1. *Kontsevich–Manin, Ruan–Tian The number N_d of degree d rational curves passing through $3d-1$ points on $\mathbb{P}^2$ is given by*

$$N_d = \sum_{d_1,d_2>0, d_1+d_2=d} N_{d_1}N_{d_2}\left(d_1^2d_2^2\binom{3d-4}{3d_1-2} - d_1^3d_2\binom{3d-4}{3d_1-1}\right).$$

Note that for $d>2$, the degree d rational curves must be singular. Recall that we introduced m_k, the moduli space of prestable rational curves with k marked points, and its open substack $\overline{\mathcal{M}}_k$ of stable curves. Then m_k is a smooth algebraic stack of dimension $k-3$, and is compact. For example, $\mathbb{C}^n - 0/\mathbb{C}^\times$ is open in $\mathbb{C}^n/\mathbb{C}^\times$, but is compact in projective space.[11]

Exercise 3.1. *Using stability conditions and some facts in complex analysis, prove that $\overline{\mathcal{M}_3} \cong \{\bullet\}$. This means that every $(C, x_1, x_2, x_3) \in \overline{\mathcal{M}_k}$ is isomorphic to $(\mathbb{P}^1, 0, 1, \infty)$, and there does not exist any automorphism of this.*

Exercise 3.2. *Prove that $\overline{\mathcal{M}} \cong \mathbb{P}^1$.*

The Stabilization Map. We have a map $\mathfrak{m}_k \to \overline{\mathcal{M}}_k$ called the stabilization map. This map, composed with the inclusion map, yields the identity.[12] Suppose $C \in \mathfrak{m}_k$ is a connected curve. (Note that we define specialization map only when $k \geq 3$, otherwise the codomain is empty.) We have *unstable* components $\mathbb{P}^1 \subset C$ with either

(1) one node (rational tail),
(2) one node and one marked point,
(3) two nodes (rational bridge).

If a component has a marked point, then that component is stable.[13]

For an algebraic stack X, we can define $Z_k(X)/W_k(X)$ by replacing varieties by reduced and irreducible (=integral) substacks. However, this definition contradicts some intuition. For example, consider weighted projective space. A remedy for Deligne-Mumford stacks was found by Gillet & Vistoli by taking $A_k(X)_\mathbb{Q} := (Z_k/W_k)\otimes_\mathbb{Z}\mathbb{Q}$, removing torsion. However, Artin stacks still have problems. Proper pushforward expected, or intersection of degree d in weighted projective space $1/2$[14]

Kresch introduced a good notion of Chow groups with $\mathbb{Z}$-coefficients having functoriality, but a representation of an element of the Chow group may not be geometric. Note that we always assume the base field to be $\mathbb{C}$.

3.1. Moduli of Maps.

Fix a smooth projective variety X, called the **target space**. Also fix numerical data, the genus $g=0$, and some homology class $d \in H_2(X,\mathbb{Z})$.[15] Then a rational stable map of degree d is a pair (C,f) where $C \in m_k$, $f: C \to X$ such that $f_*[C] = d$ such that each $\mathbb{P}^1 \subset C$ has at least 3

[11] ?
[12] ?
[13] ?
[14] ????
[15] Note that we are working over $\mathbb{C}$, the second homology always is $\mathbb{Z}$.

special points (nodes + markings) if $f|_{\mathbb{P}^1}$ is constant. We define an isomorphism $\varphi : (C, f) \to (C', f')$ to be

$$\begin{tikzcd} C \arrow[rr, "\sim"] \arrow[dr, "f"'] & & C' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

The stability condition is equivalent to $\omega_C = T_C^\vee$, the cotangent bundle, if C is smooth. The line bundle $\omega_C(x_1 + \cdots + x_k) \otimes f^*\mathcal{O}_X(3)$ is ample on C. Q. Why 3? A. To cover all cases. Q. Depends on g, k, etc.? A. No.

A collection of (C, f) with isomorphisms is a Deligne-Mumford stack and it is denoted by $\overline{\mathcal{M}}_k(X, d)$, the moduli space of stable maps.

Example 3.2. Let $X = \bullet$, then $\overline{\mathcal{M}}_k(X, 0) \cong \overline{\mathcal{M}}_k$. This space is compact, is a Deligne-Mumford stack, but importantly not a smooth Deligne-Mumford stack.

3.2. **Fundamental Classes.** Assume $\overline{\mathcal{M}}_k(X, d)$ to be smooth and no obstruction bundle.[16] Then its $\mathbb{C}$-dimension is $c_1(T_X) \frown d + \dim X - 3 + k$, where $\dim X - 3 + k = \dim m_k$. Also, Grothendieck-Riemann-Roch tells us that $c_1(T) \frown d + \dim X$ is equal to $h^0(C, f^*T_X) - h^1(C, f^*T_X)$. Denote this $\mathbb{C}$-dimension as $\mathbf{vd}$, the virtual dimension.

When $X = \mathbb{P}^2$, there is a canonical isomorphism $H_2(X) \cong \mathbb{Z}$. In this case, $\overline{\mathcal{M}}_{k,d} := \overline{\mathcal{M}}_k(\mathbb{P}^2, d)$ is smooth (but may not be a scheme) so we have its fundamental class $[\overline{\mathcal{M}}_{k,d}] \in A_{\mathbf{vd}} \to H_{2\mathbf{vd}}(\overline{M}_{k,d})$ where $\mathbf{vd}$ is what we defined above.

In general, we have a natural virtual fundamental class

$$[\overline{\mathcal{M}}_k(X, d)]^{\mathbf{vir}} \in A_{\mathbf{vd}}(\overline{\mathcal{M}}_k(X, d)).$$

Over $\overline{\mathcal{M}}_k(X, d)$, there are k many maps $\mathrm{ev}_i : \overline{\mathcal{M}}_k(X, d) \to X$ given by

$$(C, x_1, \ldots, x_k, f) \mapsto f(x_i)$$

called the evaluation map. For $\alpha_i \in H^*(X)$ with $\sum_i \deg \alpha_i = 2\mathbf{vd}$, we may define

$$\int_{[\overline{\mathcal{M}}_k(X,d)]^{\mathbf{vir}}} \smile_i \mathrm{ev}_i^*(\alpha_i)$$

as

$$\deg\left(\smile_i \mathrm{ev}_i^*(\alpha_i) \frown [\overline{\mathcal{M}}_k(X, d)]^{\mathbf{vir}}\right) \in \mathbb{Q}.$$

[17] What does this mean? If there are $X_i \subset X$ whose Poincaré duals are equal to α_i, the integration of the product of pullback of α_i via evaluation map over the virtual fundamental class of the moduli space of stable maps represents the number of maps $f : C \to X$ such that $f(x_i) \in X_i$.

Let P be the Poincaré dual of a point homology class, so $P \in H^4(\mathbb{P}^2)$. We will just call this the point class. Then we have

$$\int_{[\overline{\mathcal{M}}_{k,d}]} \prod_i \mathrm{ev}_i^*(P)$$

[16] ?

[17] in $\mathbb{Q}$??

where $[\overline{\mathcal{M}}_{k,d}]^{\mathbf{vir}} \in H_{6d-2+2k}(\overline{\mathcal{M}}_{k,d})$ and $\prod_i \mathrm{ev}_i^*(P) \in H^{4k}(\overline{\mathcal{M}}_{k,d})$, so this is non-trivial when $4k = 6d-2+2k$, i.e. when $k = 3d-1$. Then define

$$N_d := \int_{\overline{\mathcal{M}}_{3d-1,d}} \prod_i \mathrm{ev}_i^*(P) \in \mathbb{Q}.$$

This counts degree d rational curves passing through $3d-1$ points on the plane. Note that this also counts curves with nodal singularities.

4. March 14th

We want to prove that the number of degree d rational curves passing through $3d-1$ points on the plane, is

$$N_d = \int_{[\overline{\mathcal{M}}_{3d-1}(\mathbb{P}^2,d)]} \prod_{i=1}^{3d-1} \mathrm{ev}_i^*(P)$$

where $P \in H^4(\mathbb{P}^2)$ is the point class defined in the previous lecture.

The string equation. If $\overline{\mathcal{M}}_k(X,d) \neq \emptyset$, then for $1 \in H^0(X)$,

$$\int_{[\overline{\mathcal{M}}_{k+1}(X,d)]^{\mathrm{vir}}} \prod_{i=1}^{k} \mathrm{ev}_i^*(\alpha_i) \cdot \mathrm{ev}_{k+1}^*(1) = 0.$$

The divisor equation. If $\overline{\mathcal{M}}_k(X,d) \neq \emptyset$ and $D \subset X$ defined by $f = 0$ for $f \in \mathcal{O}_X$. Think of D as a cohomology class in $H^2(X)$. Then,

$$\int_{[\overline{\mathcal{M}}_{k+1}(X,d)]^{\mathrm{vir}}} \prod_{i=1}^{k} \mathrm{ev}_i^*(\alpha_i) \cdot \mathrm{ev}_{k+1}^*(D) = (D \cap d) \int_{[\overline{\mathcal{M}}_{k}(X,d)]^{\mathrm{vir}}} \prod_{i=1}^{k} \mathrm{ev}_i^*(\alpha_i)$$

and the map $\pi : \overline{\mathcal{M}}_{k+1}(X,d) \to \overline{\mathcal{M}}_k(X,d)$ is flat. Also, $\pi^*[\overline{\mathcal{M}}_k(X,d)]^{\mathrm{vir}} = [\overline{\mathcal{M}}_{k+1}(X,d)]^{\mathrm{vir}}$. Intersection theory proves these results for arbitrary X.

In the case $X = \mathbb{P}^2$, the virtual class becomes actually the fundamental class, and the genus is determined by the degree. Then N_d counts nodal curves. Here, we fix divisors to mean Cartier divisors.

One Axiom of CohFT. The fiber product

$$[\overline{\mathcal{M}}_{k_1+\{n\}}(X,d) \times_{\mathbb{P}^2} \overline{\mathcal{M}}_{k_2+\{n\}}(X,d)]^{\mathrm{vir}} \subset \overline{\mathcal{M}}_{k_1+k_2}$$

where $k_1+\{n\}$ means n is another marked point. Consider the map $i : \overline{\mathcal{M}}_1 \times \overline{\mathcal{M}}_2 \to \overline{\mathcal{M}}_1 \times \overline{\mathcal{M}}$ where the first product is the fiber product, and the second one is the cartesian product. Then, $i_*[\overline{\mathcal{M}}_{k_1+\{n\}}(X,d) \times_{\mathbb{P}^2} \overline{\mathcal{M}}_{k_2+\{n\}}(X,d)]^{\mathrm{vir}} = e(\mathrm{ev}_n^*(\Delta)) \cap ([\overline{\mathcal{M}}_{k_1+\{n\}}(X,d)]^{\mathrm{vir}} \times [\overline{\mathcal{M}}_{k_2+\{n\}}(X,d)]^{\mathrm{vir}})$, where e is the Euler class. In the case of $\mathbb{P}^2$, the virtual class is equal to the fundamental class.

Proof. Consider $g : W \to \mathbb{P}^1$ given by $W = \overline{\mathcal{M}}_{3d,d} \xrightarrow{\text{forget } f} m_{3,d} \xrightarrow{\text{forget } x_1,\ldots,x_{3d-4}} m_4 \xrightarrow{\text{stab}} \overline{\mathcal{M}}_4 = \mathbb{P}^1$. Consider $[\mathrm{div}(W,g)] \in W_{6d-1}(\overline{\mathcal{M}}_{3d,d})$. Prove that this is $[g^{-1}(0)] - [g^{-1}(\infty)]$![18] For a k-dimensional scheme X, assume we have an irreducible decomposition of $X_{\mathrm{red}} = X_1 \cup \cdots \cup X_\ell$ where X_{red} is the reduced scheme associated to X.[19] Since the X_i are varieties, we may define the local rings $A_i := \mathcal{O}_{X_i,X}$. Then, define the cycle associated to X as $[X] := \sum_i \mathrm{length}_{A_i}(A_i)[X_i]$.

[18] TODO

[19] definition?

Take $\alpha = \prod_{i=1}^{3d-4} \mathrm{ev}_{x_i}^*(P) \cdot \mathrm{ev}_0^*(P) \cdot \mathrm{ev}_1^*(P) \cdot \mathrm{ev}_\infty^*(H) \cdot \mathrm{ev}_x^*(H)$ where $\overline{\mathcal{M}}_4 = \{(C, 0, 1, \infty, x)\}$. Then $\alpha \in H^{12d-4}(\overline{\mathcal{M}}_{3d,d})$. Then since $[\mathrm{div}(W, g)] \in W_{6d-1}(\overline{\mathcal{M}}_{3d,d})$, we have $\deg(\alpha \frown [g^{-1(0)}]) = \deg(\alpha \frown [g^{-1}(\infty)])$.

Now we compute $\deg(\alpha \frown [g^{-1}(0)])$. Note that

$$g^{-1}(0) = \bigsqcup_{d_1+d_2=d, I \cup J = \{x_1, \ldots, x_{3d-4}\}} \overline{\mathcal{M}}_{I \cup \{0,x,n\},d_1} \times_{\mathbb{P}^2} \overline{\mathcal{M}}_{J \cup \{1,\infty,n\},d_2}$$

which gives us a cycle decomposition

$$[g^{-1}(0)] = \sum_{d_1+d_2=d, I \cup J = \{x_1, \ldots, x_{3d-4}\}} [\overline{\mathcal{M}}_{I \cup \{0,x,n\},d_1} \times_{\mathbb{P}^2} \overline{\mathcal{M}}_{J \cup \{1,\infty,n\},d_2}].$$

On $\overline{\mathcal{M}}_{I \cup \{0,x,n\},d_1} \times_{\mathbb{P}^2} \overline{\mathcal{M}}_{J \cup \{1,\infty,n\},d_2}$, α restricts to

$$\left(\prod_{i \in I} \mathrm{ev}_i^*(P) \smile \mathrm{ev}_0^*(P) \cdot \mathrm{ev}_x^*(H)\right) \cdot \left(\prod_{i \in J} \mathrm{ev}_i^*(P) \smile \mathrm{ev}_1^*(P) \cdot \mathrm{ev}_\infty^*(H)\right)$$

which we will write as $\alpha_I \cdot \alpha_J$.[20]

Therefore, we have

$$\deg((\alpha_I \cdot \alpha_J) \frown [\overline{\mathcal{M}}_I \times_{\mathbb{P}^2} \overline{\mathcal{M}}_J])$$

$$= \sum_{j=0}^{2} \deg(\alpha_I \, \mathrm{ev}_n^*(H^j) \frown [\overline{\mathcal{M}}_I]) \cdot \deg(\alpha_J \, \mathrm{ev}_n^*(H^{2-j}) \frown [\overline{\mathcal{M}}_J]),$$

where the equality holds by the CohFT axiom, since the fiber product becomes just the product.[21]

□

5. March ?

Exercise 5.1. *Here are some exercises:*

1. *Show $A_k\mathbb{A}^n = 0$ for $k \neq n$, and $A_n\mathbb{A}^n = Z_n\mathbb{A}^n = \mathbb{Z}$.*
2. *Show $A_k\mathbb{P}^n = \mathbb{Z}$ if $0 \leq k \leq n$, since $\mathbb{P}^n - \mathbb{P}^{n-1} = \mathbb{A}^n$ for all n.*
3. *Let H be $V(f)$ for a homogeneous f of degree d in $\mathbb{P}^n$. Show that $[H] = d[\mathbb{P}^{n-1}] \in A_{n-1}(\mathbb{P}^n)$, and show $A_{n-1}(\mathbb{P}^n - H) = \mathbb{Z}/d\mathbb{Z}$.*

6. May 7th

7. May 9th

7.1. **Gromov–Witten invariants.**

Consider a 2-term complex $\mathbb{E}^\vee : [T \to E]$ where T is the zeroth term. Let M be a space (scheme/DM stack etc.) with a perfect obstruction theory $\mathbb{E} \to \mathbb{L}_M$.

Definition 7.1. The virtual fundamental class of M is $[M]^{\mathrm{vir}} := 0_E^![C_{\mathrm{BF}}] = 0_{[E/T]}^![C_{M/\mathrm{pt}}] \in A_{\mathrm{rank}\,E}(M)_{\mathbb{Q}}$ where C_{BF} is the Behrend-Fantechi cone.

[20] What does · mean?

[21] ?

We may view this as an Euler class (without cutout triple.)[22]

Consider C_{BF} as a space $\pi : C_{\mathrm{BF}} \to M$ with a τ-tautological section $\mathcal{O}_{C_{\mathrm{BF}}} \to \pi^*E$. Then $(C_{\mathrm{BF}}, \pi^*E, \tau)$ cuts out $Z(\tau) = M \subset C_{\mathrm{BF}}$. Then we have $[M]^{\mathrm{vir}} = e(\pi^*E, \tau) \in A_{\operatorname{rank}\mathbb{E}}(M)$.

8. May 14th

8.1. **Quantum Lefschetz for $g = 0$.** Refined Gysin Map.

[22] ?