COMMUTATIVE ALGEBRA HOMEWORK II

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Problem 1. *Let $A = k[x]_{(x)}$ be the local ring at (x) of the polynomial ring in one variable x over a field k. Find an A-module M that is not finitely generated, but such that M/xM is finitely generated.*

Proof. Consider $M = Q(A)$, the total ring of fractions of A. This is an A-module via $A \to Q(A)$. The elements of A are rational functions that are defined at $x = 0$. Note that since $Q(k[x]) = k(x)$ and since the localization of $k[x]$ at (x) is a subring of $k(x)$, it follows that $Q(A)$ is the function field $k(x)$, which contains x^{-i}, so a finite subset of $k(x)$, as a $k[x]_{(x)}$-module, cannot generate all of the x^{-i} for $i > 0$. Hence, this is not finitely generated as an A-module. On the other hand, $M/xM = 0$ since $M = Q(A) = k(x)$ is a field, and $x \in k(x)^\times$. $\square$

Problem 2. *Show that the Jacobson radical of a ring A is $J(A) := \{a \in A \mid 1 + ab$ is a unit for every $b \in A\}$.*

Proof. Suppose $a \in \bigcap \mathfrak{m}$. Assume by contradiction that $1 + ab$ is not a unit. Then $1 + ab \in \mathfrak{m}$ for some maximal ideal. But, $a \in \mathfrak{m}$, so it follows that $ab \in \mathfrak{m}$, and this implies $1 \in \mathfrak{m}$, which is not possible. Hence $a \in J(A)$.

On the other hand, suppose $a \notin \bigcap \mathfrak{m}$. Then $a \notin \mathfrak{m}$ for some maximal ideal, so $(a, \mathfrak{m}) = (1)$. It follows that $ab + c = 1$ for some $c \in \mathfrak{m}$ and $b \in A$. Therefore $1 - ab = c$, where c is a nonunit. Thus $a \notin J(A)$. $\square$

Problem 3. *Compute the normalization of $A = \mathbb{C}[x, y]/(y^2 - x^2(x + 1))$.*

Problem 4. *Let $A = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. Show that any maximal ideal of A can be generated by n elements $f_1, \ldots, f_n$ where $f_i \in k[x_1, \ldots, x_i] \subset k[x_1, \ldots, k_n]$ for each $i = 1, \ldots, n$.*

Proof. We use induction on n. The base case $n = 1$, i.e. $k[x_1]$ is clear. Assume the result up to n. Consider the inclusion $k[x_1, \ldots, x_n] \subset k[x_1, \ldots, x_n][x_{n+1}]$, and suppose $\mathfrak{m}$ is a maximal ideal of $k[x_1, \ldots, x_n][x_{n+1}]$. By Nullstellensatz, we know that $\mathfrak{m}^c := \mathfrak{m} \cap k[x_1, \ldots, x_n]$ is a maximal ideal of $k[x_1, \ldots, x_n]$, thus by induction hypothesis, generated by n elements. Now consider the ring $(k[x_1, \ldots, x_n]/\mathfrak{m}^c)[x_{n+1}]$, which is isomorphic to $k[x_1, \ldots, x_{n+1}]/\mathfrak{m}^c[x_{n+1}]$. Maximal ideals of this ring correspond to maximal ideals of $k[x_1, \ldots, x_{n+1}]$ containing $\mathfrak{m}^c[x_{n+1}]$, in particular $\mathfrak{m}$. This is because the elements of $\mathfrak{m}^c[x_{n+1}]$ are polynomials of x_{n+1} with coefficients in $\mathfrak{m}^c$, hence in $\mathfrak{m}$. Therefore, $\mathfrak{m}$ corresponds to a maximal ideal of $(k[x_1, \ldots, x_n]/\mathfrak{m}^c)[x_{n+1}]$, a polynomial ring over a field, thus generated by some $f = a_k x_{n+1}^k + \cdots + a_1 x_{n+1} + a_0$ for $a_i \in k[x_1, \ldots, x_n]/\mathfrak{m}^c$. Thus the corresponding maximal ideal in $k[x_1, \ldots, x_{n+1}]/\mathfrak{m}^c[x_{n+1}]$ is generated by the image of f under the quotient by $\mathfrak{m}^c[x_{n+1}]$. If we write $b_i \in k[x_1, \ldots, x_n]$ as such that $b_i \mapsto a_i$ under the

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quotient by $\mathfrak{m}^c$, then it follows that $\mathfrak{m}$ is generated by $b_k x_{n+1}^k + \cdots + b_1 x_{n+1} + b_0$, and the n generators of $\mathfrak{m}^c \subset k[x_1, \ldots, x_n]$. Therefore $\mathfrak{m}$ is generated by $n+1$ elements.

□

Problem 5. *Prove that a ring A is Jacobson if and only if $\ell_f^*(\operatorname{Spm}(A_f)) \subset \operatorname{Spm} A$ for every $f \in A$.*

Proof. Suppose A is Jacobson. If $f = 0$, the result is immediate since $A_f = 0$. Suppose $f \neq 0$. Note that primes of A_f are in bijective correspondence with primes of A that do not contain f. This preserves inclusion, so it is enough to show that prime ideals that are maximal with respect to the condition of not containing f are maximal ideals in A. Let $\mathfrak{p}$ be a prime ideal of A that does not contain f, which is maximal among those not containing f. Since A is Jacobson, we have $\mathfrak{p} = \bigcap \mathfrak{m}$. Thus $f \notin \bigcap \mathfrak{m}$, so $f \notin \mathfrak{m}$ for some $\mathfrak{m}$. Since $\mathfrak{p} \subset \mathfrak{m}$ and we assumed $\mathfrak{p}$ to be maximal with respect to the condition of not containing f, we conclude $\mathfrak{p} = \mathfrak{m}$.

Now let $\mathfrak{p}$ be a prime ideal of A. For $f \in A \setminus \mathfrak{p}$, we have an inclusion $\mathfrak{p} \subset \mathfrak{m}_f$ where $\mathfrak{m}_f$ is a maximal prime ideal of A not containing f, which is maximal in A by assumption. Suppose $a \in \bigcap_{f \in A \setminus \mathfrak{p}} \mathfrak{m}_f$. Then $a \neq f$ for all $f \in A \setminus \mathfrak{p}$, so $a \in \mathfrak{p}$. Thus $\bigcap_{f \in A \setminus \mathfrak{p}} \mathfrak{m}_f \subset \mathfrak{p}$, where the opposite inclusion is obvious. Thus A is Jacobson. □

Problem 6. *Let A be a domain of dimension ≥ 1.*

(1) *Show that if A is Jacobson then $\operatorname{Spm} A$ is infinite.*
(2) *Suppose that $\dim A = 1$. Show that A is Jacobson if and only if $\operatorname{Spm} A$ is infinite. (Also, assume A is noetherian!)*

Proof. (1) Assume by contradiction that $\operatorname{Spm} A$ is finite, say $\mathfrak{m}_i$ for $i = 1, \ldots, n$. Since A is a domain of dimension ≥ 1, (0) cannot be maximal so the $\mathfrak{m}_i$ are nonzero. From each $\mathfrak{m}_i$ pick a nonzero element a_i. The product $\prod a_i$ is nonzero since A is a domain, and is in $\bigcap \mathfrak{m}_i$. However since A is a Jacobson domain, it follows that $(0) = \bigcap \mathfrak{m}$ for some maximal ideals, so $\bigcap \mathfrak{m}_i = (0)$, a contradiction. □

Proof. (2) We just proved the forward direction. Suppose $\operatorname{Spm} A$ is infinite. All primes of height 1 in A are maximal since $\dim A = 1$. Thus we have to prove for primes of height 0, namely the zero ideal. Suppose there is some $0 \neq f \in \bigcap \mathfrak{m}$. Then the ideal (f) has finitely many minimal primes since A is noetherian. These minimal primes are nonzero, hence automatically maximal by $\dim A = 1$. This is a contradiction to $|\operatorname{Spm} A| = \infty$, where all $\mathfrak{m}$ are minimal primes of f. Hence $(0) = \bigcap \mathfrak{m}$, so A is Jacobson. □

Problem 7. *Show that a localization of a normal domain is a normal domain.*

Problem 8. *Let A be a domain. Show that the following are equivalent:*

(1) *A is a normal domain.*
(2) *$A_\mathfrak{p}$ is a normal domain for all $\mathfrak{p} \in \operatorname{Spec} A$.*
(3) *$A_\mathfrak{m}$ is a normal domain for all $\mathfrak{m} \in \operatorname{Spm} A$.*

Problem 9. *We define a ring A to be normal if $A_\mathfrak{p}$ is a normal domain for all $\mathfrak{p} \in \operatorname{Spec} A$. Show that if a ring A is normal, then $A[x]$ is normal.*

Proof. We want to show that for every $\mathfrak{q} \in \operatorname{Spec} A[x]$, the local ring $(A[x])_\mathfrak{q}$ is a normal domain. Let $\mathfrak{p} := A \cap \mathfrak{q} \in \operatorname{Spec} A$ since it is the inverse image under $A \to A[x]$. Since A is normal, we have $A_\mathfrak{p}$ a normal domain. We show that $A_\mathfrak{p}[x]$ is also a normal domain, i.e. if D is a normal domain, then $D[x]$ is a normal domain. Suppose $f \in K(D[x])$ is integral over $D[x]$. We want to show that $f \in D[x]$. Since the ring $K(D)[x]$ contains $D[x]$, it follows that since f is a root of a monic polynomial with coefficients in $D[x]$, it is also a root of a monic polynomial with coefficients in $K(D)[x]$. Also, $K(D)[x]$ is normal since $K(D)$ is a field, so we have $f \in K(D)[x]$ integral over $D[x]$. Suppose we have $f^n + d_{n-1}f^{n-1} + \cdots + d_1 f + d_0 = 0$ for $d_i \in D[x]$. We may rewrite this by putting $x^N + f - x^N$ in place of f to get

$$(x^N + f - x^N)^n + d_{n-1}(x^N + f - x^N)^{n-1} + \cdots + d_1(x^N + f - x^N) + d_0 = 0$$

where by using the binomial expansion we can write this in the form

$$(x^N + f)^n + d'_{n-1}(x^N + f)^{n-1} + d'_{n-2}(x^N + f)^{n-2} + \cdots + d'_0 = 0$$

where $d'_i \in D[x]$. If we move d'_0 we get

$$(x^N + f)\left((x^N + f)^{n-1} + d'_{n-1}(x^N + f)^{n-2} + \cdots + d'_1\right) = -d'_0$$

which is of the form $GH = F$ for $G, H \in K(D)[x]$ and $F \in D[x]$, for monic F, G, H, assuming $N >> 0$. We may apply Eisenbud, Proposition 4.11. to conclude that the coefficients of G and H are integral over D. In particular, this implies that the coefficients of f are integral over D, and since D is a normal domain, we may conclude $f \in D[x]$. Therefore, we have proved that $A_{\mathfrak{p}}[x]$ is also a normal domain.

Now we claim that $(A[x])_{\mathfrak{q}} \cong S^{-1}(A_{\mathfrak{p}}[x])$ where S is the image of $A[x] \setminus \mathfrak{q}$ under the unique morphism $A[x] \to A_{\mathfrak{p}}[x]$ that sends 1 to 1 and x to x. By Eisenbud, Proposition 4.13., this would finish the proof. Suppose $\varphi : A[x] \to A_{\mathfrak{p}}[x] \to S^{-1}(A_{\mathfrak{p}}[x])$ is the composition of the obvious morphisms. The $\varphi(A[x] \setminus \mathfrak{q})$ are obviously units in the codomain, and elements of $S^{-1}(A_{\mathfrak{p}}[x])$ are of the form $(\overline{f}/1)(\varphi(s)/1)^{-1} = \varphi(f)\varphi(s)^{-1}$ for $f \in A[x]$ and $s \in A[x] \setminus \mathfrak{q}$. Therefore it suffices to show that if $\varphi(f) = 0$ for some $f \in A[x]$, we have $fs = 0$ for some $s \in A[x] \setminus \mathfrak{q}$. Suppose $f = a_0 + a_1 x + \cdots + a_n x^n$. Via φ this maps to $(\overline{a_0} + \overline{a_1}x + \cdots + \overline{a_n}x^n)/1 = 0$ in $S^{-1}(A_{\mathfrak{p}}[x])$. Thus we have $s(\overline{a_0} + \overline{a_1}x + \cdots + \overline{a_n}x^n) = 0$ for some $s \in S$. Since $A_{\mathfrak{p}}[x]$ is a domain, we must have $\overline{a_0} + \overline{a_1}x + \cdots + \overline{a_n}x^n = 0$, i.e. $\overline{a_i} = 0$ for all i. This means that there exists $s_i \in A \setminus \mathfrak{p}$ such that $s_i a_i = 0$ for each i. Therefore we conclude $\prod_i s_i \cdot f = 0$, and since $\prod_i s_i \in A \setminus \mathfrak{p} \subset A[x] \setminus \mathfrak{q}$, we may apply Atiyah & Macdonald, Corollary 3.2. to conclude that $(A[x])_{\mathfrak{q}} \cong S^{-1}(A_{\mathfrak{p}}[x])$. Since $\mathfrak{q} \in \operatorname{Spec} A[x]$ was arbitrary, we win. □

Problem 10. *Let A be a noetherian ring. Show that the following are equivalent:*

(1) *A is normal.*
(2) *A is reduced and integrally closed in its total ring of fractions.*
(3) *A is a finite product of normal domains.*

Proof. Suppose A is a noetherian normal ring. Since each $A_{\mathfrak{p}}$ is a domain, its nilradical is trivial, i.e. $\mathfrak{N}_{\mathfrak{p}} = 0$ for all $\mathfrak{p}$ by Atiyah & Macdonald, Corollary 3.12. It follows that $\mathfrak{N} = 0$, so A is reduced. Now suppose $x \in Q(A)$ is integral over A, say $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ for $a_i \in A$. Let $\mathfrak{p}$ a prime of A. Via the map $Q(A) \to Q(A) \otimes_A A_{\mathfrak{p}}$ given by $a \mapsto a \otimes 1$, send this equation to get a monic polynomial of $(x \otimes 1)$ with coefficients in $A_{\mathfrak{p}}$. Since $A \to A_{\mathfrak{p}}$ is flat, and $A \subset Q(A)$, we have $A \otimes_A A_{\mathfrak{p}} \subset Q(A) \otimes_A A_{\mathfrak{p}}$. Also, $A \otimes_A A_{\mathfrak{p}} \cong A_{\mathfrak{p}}$ and $Q(A) \otimes_A A_{\mathfrak{p}} \cong S(A)^{-1}A \otimes_A A_{\mathfrak{p}} \cong S(A)^{-1}A_{\mathfrak{p}}$, i.e. the localization of $A_{\mathfrak{p}}$ by the image of $S(A)$ through the map $A \to A_{\mathfrak{p}}$. Since $A_{\mathfrak{p}}$ is a normal domain, and since $x \otimes 1 \in S(A)^{-1}A_{\mathfrak{p}} \subset K(A_{\mathfrak{p}})$ is integral over $A_{\mathfrak{p}}$ it follows that $x \otimes 1 \in A_{\mathfrak{p}}$.

Thus, we may rewrite $x \otimes 1$ as $a \otimes (1/f)$ for some $a \in A$ and $f \in A \setminus \mathfrak{p}$. This implies that $fx - a$ maps to zero under the map $Q(A) \to Q(A) \otimes_A A_{\mathfrak{p}}$, since $(fx-a)\otimes 1 = (fx)\otimes 1 - a \otimes 1 = f(x\otimes 1) - f(a \otimes (1/f)) = 0$. Since $Q(A)\otimes_A A_{\mathfrak{p}} \cong Q(A)_{\mathfrak{p}} \cong S(A)^{-1}A_{\mathfrak{p}}$, this means that $(fx-a)/1$ is zero, i.e. $f'(fx-a) = 0$ in $A_{\mathfrak{p}}$ for some f' in the image of $S(A)$ in $A_{\mathfrak{p}}$, say $f' = s/1$ for $s \in S(A)$. This in turn implies that $f''s(fx-a) = 0$ for some $f'' \in A \setminus \mathfrak{p}$, where since $s \in S(A)$ this becomes $f''fx = f''a$ in A. Define $I = \{a \in A \mid ax \in A\}$, an ideal of A. Notice that $f''f \in I$, where $f'' \in A \setminus \mathfrak{p}$. Since $\mathfrak{p}$ was arbitrary, it follows that I does not contain any prime of A, which implies $I = (1)$. Thus $1x \in A$, so A is integrally closed in $Q(A)$.

Now suppose A is a noetherian ring which is reduced and integrally closed in its total ring of fractions. Denote by $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ its minimal prime ideals. Suppose $x \in A$ is a zerodivisor, i.e. $xy = 0$ for $y \neq 0$. Then $y \neq \mathfrak{p}_i$ for some i since $(0) = \bigcap \mathfrak{p}_i$. Hence, $x \in \mathfrak{p}_i$. Conversely, if $x \in \mathfrak{p}_i$ for some i, then consider the localization $A \to A_{\mathfrak{p}_i}$. Through this, x maps into $\mathfrak{p}_i A_{\mathfrak{p}_i}$, which is the only prime of $A_{\mathfrak{p}_i}$. Since $A_{\mathfrak{p}_i}$ is reduced, it follows that $\mathfrak{p}_i A_{\mathfrak{p}_i} = 0$, i.e. x maps to zero. Hence, there exists $y \in A \setminus \mathfrak{p}_i$ such that $xy = 0$. Therefore, the set of zerodivisors of A is precisely $\bigcup_{i=1}^n \mathfrak{p}_i$. With results of the last homework, together with the fact that the $\mathfrak{p}_i$ are minimal, we may conclude that the maximal ideals of $Q(A)$ are precisely $S(A)^{-1}\mathfrak{p}_i$, and by reducedness their intersection is zero. Since $S(A)^{-1}\mathfrak{p}_i + S(A)^{-1}\mathfrak{p}_j = (1)$ for any $i \neq j$, apply the Chinese remainder theorem to conclude that $Q(A) \cong \prod_{i=1}^n Q(A)/S(A)^{-1}\mathfrak{p}_i$. Note that $Q(A)/S(A)^{-1}\mathfrak{p}_i \cong S(A)^{-1}(A/\mathfrak{p}_i) \cong A_{\mathfrak{p}_i}/\mathfrak{p}_i A_{\mathfrak{p}_i}$ where $\mathfrak{p}_i A_{\mathfrak{p}_i} = 0$ (this is because $A_{\mathfrak{p}_i}$ is reduced, hence the nilradical $\mathfrak{p}_i A_{\mathfrak{p}_i} = 0$), so $Q(A) \cong \prod_{i=1}^n K(A_{\mathfrak{p}_i}) \cong \prod_{i=1}^n A_{\mathfrak{p}_i}$. Denote by $e_i = (0, \dots, 1, \dots, 0)$ the ith idempotent of $Q(A)$. Since A is integrally closed in $Q(A)$, and $e_i^2 - e_i = 0$, A must contain the e_i. Then $A \cong \prod_{i=1}^n Ae_i$, where for each i we have $\operatorname{Ann}(e_i) = \mathfrak{p}_i$, so $Ae_i \cong A/\mathfrak{p}_i$, i.e. $A \cong \prod_{i=1}^n A/\mathfrak{p}_i$. Since we have $A/\mathfrak{p}_i \subset A_{\mathfrak{p}_i}$ for each i, it follows that each $A/\mathfrak{p}_i$ is integrally closed in its field of fractions (since by assumption A is integrally closed in $Q(A)$, thus every element in $A_{\mathfrak{p}_i} \times \prod_{j \neq i}\{0\}$ is a solution of a monic polynomial over A, where the other $A/\mathfrak{p}_j$, $j \neq i$ are irrelevant), hence a normal domain. Thus A is a product of finitely many normal domains.

Suppose $A = \prod_{i=1}^n A_i$ is a finite product of normal domains. Since $\operatorname{Spec} A = \coprod_{i=1}^n \operatorname{Spec} A_i$, localization of A corresponds to localization at each A_i. By assumption these are normal domains, so A is normal. $\square$