More on Theta Functions and the Generating Function

Chapter 10

Complex Function Theory 2

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The modular character of Θ

Recall that $\wp$ and E_k were reflected by

$$\tau \mapsto \tau + 1 \quad \text{and} \quad \tau \mapsto -1/\tau.$$

Since $\Theta(z \mid \tau + 1) \neq \Theta(z \mid \tau)$, let $T : \tau \mapsto \tau + 2 \quad$ and $\quad S : \tau \mapsto -1/\tau$.

We will study the transformation of $\Theta(z \mid \tau)$ under the mapping $\tau \mapsto -1/\tau$.

The modular character of Θ

Theorem 1.6

If $\tau \in \mathbb{H}$, then

$$\Theta(z \mid -1/\tau) = \sqrt{\frac{\tau}{i}} e^{\pi i \tau z^2} \Theta(z\tau \mid \tau) \quad \text{for all } z \in \mathbb{C}.$$

where the branch is cut so that $\sqrt{\tau/i}$ is defined on $\mathbb{H}$, and is positive when $\tau = it, t > 0$.

Corollaries

Corollary 1.7

If $\tau \in \mathbb{H}$, then $\theta(-1/\tau) = \sqrt{\tau/i}\,\theta(\tau)$.

Corollaries

Corollary 1.8

If $\tau \in \mathbb{H}$, then

$$\begin{aligned}\theta(1-1/\tau) &= \sqrt{\frac{\tau}{i}} \sum_{n=-\infty}^{\infty} e^{\pi i (n+1/2)^2 \tau} \\ &= \sqrt{\frac{\tau}{i}} (2e^{\pi i \tau/4})\end{aligned}$$

The second identity means that $\theta(1-1/\tau) \sim \sqrt{\tau/i} 2e^{i\pi\tau/4}$ as $\operatorname{Im}(\tau) \to \infty$.

The Dedekind eta function

Definition

The Dedekind eta function is defined for $\mathrm{Im}(\tau) > 0$ by

$$\eta(\tau) = e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})$$

Proposition 1.9

If $\mathrm{Im}(\tau) > 0$, then $\eta(-1/\tau) = \sqrt{\tau/i}\,\eta(\tau)$.

Generating functions

Given a sequence $\{F_n\}_{n=0}^{\infty}$, we define its generating function as

$$F(x) = \sum_{n=0}^{\infty} F_n x^n$$

Example

The partition function. If n is a positive integer, let $p(n)$ denote the number of ways n can be written as a sum of positive integers. Now consider the sequence $\{p(n)\}$.

For $|x| < 1$, we can write $1/(1 - x^k) = \sum_{m=0}^{\infty} x^{km}$ and multiply these to obtain $p(n)$ as the coefficient of x^n.

The partition sequence

Theorem 2.1

If $|x| < 1$, then $\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$.

We may obtain this result by using the fact that $\frac{1}{1-x^k} = \sum_{m=0}^{\infty} x^{km}$. (Try yourself!)

The partition sequence

Now we consider $p_o(n)$, the number of partitions of n into odd parts, and $p_u(n)$, the number of partitions of n into unequal parts.

Remarkably, $p_o = p_u$.

The partition sequence

Note that $\prod(1+x^n)$ is the generating function for p_u, and $\prod 1/(1-x^{2n-1})$ is the generating function for p_o. To prove this, use the facts that

$$\prod_{n=1}^{\infty}(1+x^n)\prod_{n=1}^{\infty}(1-x^n)=\prod_{n=1}^{\infty}(1-x^{2n})$$

and

$$\prod_{n=1}^{\infty}(1-x^{2n})\prod_{n=1}^{\infty}(1-x^{2n-1})=\prod_{n=1}^{\infty}(1-x^n)$$

Deeper into the partiton sequence

Let $p_{e,u}(n)$ denote the number of partitions of n into an even number of unequal parts, and $p_{o,u}(n)$ denote the number of partitions of n into an odd number of unequal parts.

Definition

Integers of the form $k(3k+1)/2$ are called **pentagonal numbers**

It is know that $p_{e,u} = p_{o,u}$ unless n is a pentagonal number. If n is a pentagonal number, then $p_{e,u}(n) - p_{o,u}(n) = (-1)^k$ for $n = k(3k+1)/2$.

Deeper into the partition sequence

Proposition 2.2

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{k=-\infty}^{\infty}(-1)^k x^{k(3k+1)/2}$$

Deeper into the partition sequence

Proof

First observe that

$$\prod_{n=1}^{\infty}(1-x^n)=\sum_{n=1}^{\infty}[p_{e,u}(n)-p_{o,u}(n)]x^n$$

[a] If we set $x=e^{2\pi iu}$, then

$$\prod_{n=1}^{\infty}(1-e^{2\pi inu})=\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}e^{2\pi iz})(1+q^{2n-1}e^{-2\pi iz})$$

where $q=e^{3\pi iu}$ and $z=1/2+u/2$. Note that by Thm 1.3, the product is

$$\sum_{n=-\infty}^{\infty}e^{3\pi in^2u}(-1)^n e^{2\pi inu/2}$$

[a] ?