ALGEBRAIC GEOMETRY FALL MIDTERM

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Forenote: I provide the statements of lecture note propositions in the end of the article.

Problem 1. *Let X be a separated noetherian scheme, possibly of infinite dimension. Must there be an integer $N \geq 0$ such that $H^i(X, \mathcal{F}) = 0$ for all $i > N$ and all quasicoherent $\mathcal{O}_X$-modules $\mathcal{F}$?*

Proof. Since X is noetherian, it can be covered by a finite number, say n affine open sets U_k. By Hartshorne, Theorem III.4.5, the Čech cohomology groups of $\mathcal{F}$ with affine open cover $\mathfrak{U} = \{U_k\}_{k=1}^n$ are isomorphic to the usual cohomology groups. Thus for $i \geq n$, $H^i(X, \mathcal{F})$ vanishes by construction of the Čech complex. Take $N = n - 1$. $\square$

Problem 2. *Let X be a noetherian scheme of finite dimension. Prove that X is affine if and only if $\mathcal{O}_X$ is ample and $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$.*

Proof. Suppose X is affine. Since any invertible sheaf on an affine scheme is ample (Hartshorne, II, Example 7.4.2), and since $\mathcal{O}_X$ is an invertible sheaf viewed as a module over itself, $\mathcal{O}_X$ must be ample. Moreover, the fact that $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$ follows from Problem 1, since X is noetherian and affine, hence separated (Hartshorne, Proposition II.4.1).

Conversely, suppose $\mathcal{O}_X$ is ample and $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$. By Serre's affineness criterion (Hartshorne, Theorem III.3.7), it is enough to show that $H^i(X, \mathcal{F}) = 0$ for all quasicoherent $\mathcal{F}$ and all $i > 0$. Suppose $\dim X = n$. By Grothendieck's vanishing theorem (Hartshorne, Theorem III.2.7), we have $H^i(X, \mathcal{F}) = 0$ for all $i > n$. We show $H^i(X, \mathcal{F}) = 0$ for $0 < i \leq n$ by descending induction on n.

Define $B := \bigcup_{U \subset X} \mathcal{F}(U)$, and A as the set of all finite subsets of B. For each $\alpha \in A$, denote $\mathcal{F}_\alpha$ the subsheaf of $\mathcal{F}$ generated by sections in α. We have $\lim_{\rightarrow} \mathcal{F}_\alpha = \mathcal{F}$ where each $\mathcal{F}_\alpha$ is coherent since it is locally generated by finite sections on a noetherian scheme. By **Proposition A**, cohomology commutes with direct limits on a noetherian space, so it suffices to show vanishing for coherent $\mathcal{F}$. By definition of ampleness, there exists some $n_0 > 0$ such that for every $N \geq n_0$, the sheaf $\mathcal{F} \otimes \mathcal{O}_X^N = \mathcal{F}$ is generated by its global sections, i.e. there is a surjection $\bigoplus_I \mathcal{O}_X \twoheadrightarrow \mathcal{F}$. This yields a SES

$$0 \longrightarrow \mathcal{R} \longrightarrow \bigoplus_I \mathcal{O}_X \longrightarrow \mathcal{F} \longrightarrow 0$$

of coherent $\mathcal{O}_X$-modules on X where $\mathcal{R}$ is the kernel. Taking the LES of cohomology we have $H^k(X, \mathcal{F}) \simeq H^{k+1}(X, \mathcal{R})$ for $k > 0$ since $H^i(X, \bigoplus_I \mathcal{O}_X) \simeq \bigoplus_I H^i(X, \mathcal{O}_X) = 0$ for $i > 0$, by hypothesis and noetherian condition. Since

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$H^{n+1}(X,\mathcal{R}) = 0$ by induction hypothesis we have $H^n(X,\mathcal{F}) = 0$, and since $\mathcal{F}$ was an arbitrary coherent $\mathcal{O}_X$-module this concludes our proof. $\square$

Problem 3. *Let X be a connected topological space. Prove that X is irreducible if and only if $H^1_Y(X,\underline{\mathbb{Z}}) = 0$ for all closed $Y \subset X$, where $\underline{\mathbb{Z}}$ denotes the constant sheaf.*

Proof. First note that for irreducible X, the constant sheaf $\underline{\mathbb{Z}}$ is flasque by considering continuous maps $U \to \mathbb{Z}$ must be constant. Then, use **Proposition B** to show that $H^i_Y(X,\underline{\mathbb{Z}}) = 0$ for all $i > 0$, and all closed $Y \subset X$.

On the other hand, suppose there exist disjoint nonempty open subsets U, V of X. We may assume U, V are connected by choosing components if needed. **Proposition C** yields an exact sequence

$$0 \to H^0_Y(X,\underline{\mathbb{Z}}) \to H^0(X,\underline{\mathbb{Z}}) \to H^0(U \cup V, \underline{\mathbb{Z}}|_{U\cup V}) \to H^1_Y(X,\underline{\mathbb{Z}}) \to \cdots$$

where $Y = X - U - V$ is a nonempty closed subset of X. Since $H^0(X,\underline{\mathbb{Z}}) \simeq \mathbb{Z}$ and $H^0(U\cup V,\underline{\mathbb{Z}}|_{U\cup V}) \simeq \mathbb{Z} \oplus \mathbb{Z}$, by tensoring with $\mathbb{Q}$ and using a dimension argument, we conclude $H^0(X,\underline{\mathbb{Z}}) \to H^0(U\cup V,\underline{\mathbb{Z}}|_{U\cup V})$ cannot be surjective. By exactness, the kernel of $H^0(U\cup V,\underline{\mathbb{Z}}|_{U\cup V}) \to H^1_Y(X,\underline{\mathbb{Z}})$ is not the entirety of $H^0(U\cup V,\underline{\mathbb{Z}}|_{U\cup V})$ which implies $H^1_Y(X,\underline{\mathbb{Z}}) \neq 0$. $\square$

Problem 4. *Compute the given cohomology groups.*

Proof. (a) Since X is an affine noetherian scheme, we may use the result of Problem 1. In this case, we may cover by $n = 1$ affine opens, hence $H^1(X,\mathcal{O}_X^\times) = 0$. $\square$

Proof. (b) Again we use Theorem III.4.5 of Hartshorne. Write $\mathbb{A}^2_k = \operatorname{Spec} k[x_1,x_2]$. Then we may cover X by affine opens $D(x_1)$ and $D(x_2)$. Then the Čech complex is $C^0 = \Gamma(D(x_1),\mathcal{O}_X) \times \Gamma(D(x_2),\mathcal{O}_X)$, $C^1 = \Gamma(D(x_1x_2),\mathcal{O}_X)$, $C^2 = 0$ and so on. Thus we may calculate the cokernel of

$$k[x_1,x_1^{-1},x_2] \times k[x_1,x_2,x_2^{-1}] \to k[x_1,x_2,x_1^{-1},x_2^{-1}]$$

where the map is given by $(f,g) \mapsto g - f$. Note that $k[x_1,x_1^{-1},x_2]$ is a k-vector space generated by $x_1^i x_2^j$ for $j \geq 0$, and $k[x_1,x_2,x_2^{-1}]$ is generated by $x_1^i x_2^j$ for $i \geq 0$. Also $k[x_1,x_2,x_1^{-1},x_2^{-1}]$ is generated by $x_1^i x_2^j$ for all i,j. Thus the image of this map must be generated by $x_1^i x_2^j$ where either $i \geq 0$ or $j \geq 0$. Therefore, the cokernel is isomorphic to the k-vector space generated by $x_1^i x_2^j$ where $i,j < 0$. This is our desired $H^1(X,\mathcal{O}_X)$. $\square$

Proof. (c) Write $\mathbb{A}^3_k = \operatorname{Spec} k[x_1,x_2,x_3]$. We may cover X with three affine opens $D(x_1)$, $D(x_2)$ and $D(x_3)$. Since $\Omega^1_{X/k}$ is a quasicoherent $\mathcal{O}_X$-module, we may use Hartshorne, Theorem III.4.5 to conclude that $H^3(X,\Omega^1_{X/k}) = 0$. $\square$

Problem 5. *Prove that if X is a noetherian scheme and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then $H^n(X,\mathcal{F}) = 0$ for all $n > \dim \mathit{Supp}\mathcal{F}$. Use this to show that if $(A,\mathfrak{m})$ is a noetherian local ring and M is a finitely generated A-module, then $H^n_{\mathfrak{m}}(M) = 0$ for $n > \dim M$.*

Proof. By **Proposition D**, $\operatorname{Supp}\mathcal{F}$ is closed in X. Let $Z = \operatorname{Supp}\mathcal{F}$ and consider the inclusion $j : Z \hookrightarrow X$. We have $j_* j^{-1}\mathcal{F} \simeq \mathcal{F}$ as abelian sheaves since the stalks are canonically isomorphic, by definition of Z as the support. Thus, together with **Proposition E**, we have

$$H^n(X,\mathcal{F}) \simeq H^n(X, j_*j^{-1}\mathcal{F}) \simeq H^n(Z, \mathcal{F}|_Z)$$

for all $n \geq 0$. By Grothendieck's vanishing theorem, $H^n(Z,\mathcal{F}|_Z) = 0$ for all $n > \dim Z = \dim \operatorname{Supp}\mathcal{F}$.

Recall that we defined the dimension of an A-module M as the dimension of $\operatorname{Supp}M = \operatorname{Supp}\widetilde{M}$ as a subspace of $\operatorname{Spec} A$. In this case, let $X = \operatorname{Spec} A$ and let $\mathcal{F} = \widetilde{M}$ be the coherent $\mathcal{O}_X$-module associated to M. Letting $P = V(\mathfrak{m})$, we have $H^i_{\mathfrak{m}}(M) \simeq H^i_P(X,\mathcal{F})$ for all $i \geq 0$ by **Proposition F**. By **Propositon C**, there exists a long exact sequence

$$\cdots \to H^i(X,\mathcal{F}) \to H^i(X-P, \mathcal{F}|_{X-P}) \to H^{i+1}_P(X,\mathcal{F}) \to \cdots$$

where from the fact that $H^i(X,\mathcal{F}) = 0$ for $i > \dim M$, we have $H^i(X-P,\mathcal{F}|_{X-P}) \simeq H^{i+1}_P(X,\mathcal{F})$ for $i > \dim M$. By removing P from $\operatorname{Supp}\widetilde{M}$, the dimension of the support decreases by 1, hence we have $H^i(X-P,\mathcal{F}|_{X-P}) = 0$ for $i > \dim M - 1$ by what we proved above. Also note that $H^{\dim M+1}_P(X,\mathcal{F}) = 0$ since $H^{\dim M}(X - P, \mathcal{F}|_{X-P}) = H^{\dim M+1}(X,\mathcal{F}) = 0$. This implies $H^i_P(X,\mathcal{F}) \simeq H^i_{\mathfrak{m}}(M) = 0$ for $i > \dim M$. □

Problem 6. *Compute the given Euler characteristics.*

Proof. (a) Say $P = \operatorname{Proj}\mathbb{C}[x_0,x_1,x_2,x_3]$. We first claim the following:

1. $H^0(P,\mathcal{O}_P(n))$ is a $\mathbb{C}$-vector space of dimension ${}_{n+3}C_n$ for $n \geq 0$.
2. $H^3(P,\mathcal{O}_P(n))$ is a $\mathbb{C}$-vector space of dimension ${}_{-n-1}C_{-n-4}$ for $n \leq -4$.
3. $H^i(P,\mathcal{O}_P(n)) = 0$ otherwise.

Note that $H^0(P,\mathcal{O}_P(n)) \simeq \Gamma(P,\mathcal{O}_P(n))$. For $n \geq 0$, the global section of $\mathcal{O}_P(n)$ is generated by $x_0^i x_1^j x_2^k x_3^l$ where $i+j+k+l = n$, each nonnegative. This proves the first claim. To prove the second, we follow the proof of Hartshorne, Theorem III.5.1. For $n \leq -4$, considering the Čech complex, $H^3(P,\mathcal{O}_P(n))$ is generated by negative monomials $x_0^i x_1^j x_2^k x_3^l$ where $i+j+k+l = n$. This amounts to finding nonnegative solutions of $a+b+c+d = -n-4$, which proves our second claim. The third claim follows from Hartshorne, Theorem III.5.1 and affine cover vanishing. Thus, it follows that $\chi(\mathcal{O}_P(n)) = \frac{(n+1)(n+2)(n+3)}{6}$ for all $n \in \mathbb{Z}$. □

Proof. (b) First let $n = 0$. Consider the exact sequence

$$0 \to \Omega^1_P \to \bigoplus_4 \mathcal{O}_P(-1) \to \mathcal{O}_P \to 0$$

of Hartshorne, Theorem II.8.13. Since $\chi(\bigoplus_4 \mathcal{O}_P(-1)) = \chi(\Omega^1_P) + \chi(\mathcal{O}_P)$ which holds by the rank-nullity theorem, by what we proved in (a) we have $0 = \chi(\Omega^1_P) + 1$. Therefore $\chi(\Omega^1_P) = -1$. Now suppose $n = -6$. By twisting the exact sequence above we obtain

$$0 \to \Omega^1_P(-6) \to \bigoplus_4 \mathcal{O}_P(-7) \to \mathcal{O}_P(-6) \to 0$$

from which we deduce $\chi(\bigoplus_4 \mathcal{O}_P(-7)) = \chi(\Omega^1_P(-6)) + \chi(\mathcal{O}_P(-6))$. Note that $\chi(\bigoplus_4 \mathcal{O}_P(-7)) = 4\chi(\mathcal{O}_P(-7))$ since cohomology commutes with direct sums in this case. Therefore we have $\chi(\Omega^1_P(-6)) = -80 + 10 = -70$. □

Proof. (c) By Hartshorne, Theorem II.8.17, we have a SES

$$0 \to \mathcal{J}/\mathcal{J}^2 \to \Omega^1_P \otimes \mathcal{O}_X \to \Omega^1_X \to 0$$

where $\mathcal{J}$ is the ideal sheaf of $X \subset P$. Since X is a degree 6 hypersurface, we have $\mathcal{J}/\mathcal{J}^2 = \mathcal{O}_X(-6)$. Therefore $\chi(\Omega^1_P \otimes \mathcal{O}_X) = \chi(\mathcal{O}_X(-6)) + \chi(\Omega^1_X)$. Also, from the exact sequence

$$0 \to \Omega^1_P(-6) \to \Omega^1_P \to \Omega^1_P \otimes \mathcal{O}_X \to 0$$

obtained by tensoring

$$0 \to \mathcal{O}_P(-6) \to \mathcal{O}_P \to \mathcal{O}_X \to 0$$

with Ω^1_P, we have $\chi(\Omega^1_P) = \chi(\Omega^1_P \otimes \mathcal{O}_X) + \chi(\Omega^1_P(-6))$ where from the calculations above, it follows that $\chi(\Omega^1_P \otimes \mathcal{O}_X) = 69$. Now we must calculate $\chi(\mathcal{O}_X(-6))$. Twisting the exact sequence above, we obtain

$$0 \to \mathcal{O}_P(-12) \to \mathcal{O}_P(-6) \to \mathcal{O}_X(-6) \to 0$$

from which we obtain $\chi(\mathcal{O}_X(-6)) = \chi(\mathcal{O}_P(-6)) - \chi(\mathcal{O}_P(-12))$. By calculating, it follows that $\chi(\mathcal{O}_X(-6)) = -10 + 165 = 155$. Therefore, we have $\chi(\Omega^1_X) = 69 - 155 = -86$. $\square$

List of propositions from the lectures:

Proposition A. *September 13th. Let X be a noetherian topological space. If $(\mathcal{F}_\alpha)_{\alpha \in A}$ is a direct system of abelian sheaves on X, then there exists a natural isomorphism*

$$\varinjlim H^i(X, \mathcal{F}_\alpha) \simeq H^i(X, \varinjlim \mathcal{F}_\alpha)$$

for all $i \geq 0$.

Proposition B. *September 18th. If $\mathcal{F}$ is flasque, then $H^i_Z(X, \mathcal{F}) = 0$ for all $i > 0$.*

Proposition C. *September 18th. Let $U = X \setminus Z$. For any $\mathcal{F} \in \underline{Ab}_X$, there exists a long exact sequence*

$$0 \longrightarrow H^0_Z(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(U, \mathcal{F}|_U) \longrightarrow H^1_Z(X, \mathcal{F}) \longrightarrow \cdots .$$

Proposition D. *September 18th. Let X be a noetherian scheme, and $\mathcal{F} \in \underline{Coh}_X$. Then Supp$\mathcal{F}$ is closed.*

Proposition E. *September 13th. Let $j : Y \hookrightarrow X$ be a closed embedding of topological spaces. Let $\mathcal{F} \in \underline{Ab}_Y$. Then $H^i(Y, \mathcal{F}) \simeq H^i(X, j_*\mathcal{F})$ for all $i \geq 0$.*

Proposition F. *September 18th. Let A be a noetherian ring and $M \in \underline{Mod}_A$, $\mathfrak{a}$ an ideal of A. Then,*

$$H^i_{\mathfrak{a}}(M) \simeq H^i_{V(\mathfrak{a})}(\operatorname{Spec} A, \widetilde{M})$$

for all $i \geq 0$.