Conformal equivalence and the Schwarz lemma

Chapter 8

Complex Function Theory 2

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Some definitions

Definition

A bijective holomorphic $f\colon U \to V$ is called a **conformal** map (or **biholomorphism**). Given such f, we say U and V are **conformally equivalent**, or simply **biholomorphic**.

Note that the inverse of f is automatically holomorphic. We turn this into a proposition.

A proposition

Proposition 1.1

If $f\colon U \to V$ is holomorphic and injective, then $f'(z) \neq 0$ for all $z \in U$. In particular, the inverse of f defined on its range is holomorphic, and thus the inverse of a conformal map is also holomorphic.

Thus, U, V are conformally equivalent if and only if there exist holomorphic f, g such that $g \circ f = \mathrm{Id}_U$ and $f \circ g = \mathrm{Id}_V$.

Note that in this text, we require a conformal map to be injective.

The disc and upper half-plane

Definition

The **upper half plane** $\mathbb{H}$ is defined as

$$\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

We will show that $\mathbb{H}$ is conformally equivalent to $\mathbb{D}$. The conformal maps are given by $F(z) = \frac{i-z}{i+z}$ and $G(w) = i\frac{1-w}{1+w}$.

The disc and upper half-plane

Theorem 1.2

The map $F : \mathbb{H} \to \mathbb{D}$ is a conformal map, with inverse $G : \mathbb{D} \to \mathbb{H}$.

The disc and upper half-plane

Note that F can be extended to $\partial\mathbb{H} = \mathbb{R}$. Take $x \in \mathbb{R}$ and observe $F(x)$. It is easy to check $|F(x)| = 1$.

Furthermore, we may write

$$F(x) = \frac{i - x}{i + x} = \frac{1 - x^2}{1 + x^2} + i\frac{2x}{1 + x^2}$$

and reparametrize x as $\tan t$ for $(-\pi/2, \pi/2)$. Verify that $F(x) = e^{i2t}$.

Fractional linear transformations

Mappings of the form

$$z \mapsto \frac{az+b}{cz+d}$$

for complex a, b, c, d with nonzero determinant are referred to as fractional linear transformations.

This transformation is also referred to as a Möbius transformation. These are conformal maps from the Riemann sphere to itself. These transformations also explain 'reflection' with respect to a circle, or a sphere.

Further examples

Here we provide examples of conformal mappings.

Translations and dilations

Example 1

The translation $z \mapsto z + h$ is a conformal map from $\mathbb{C}$ to itself. Note that for real h, this also suffices to be a conformal map of $\mathbb{H}$.

The map $z \mapsto cz$ is called a dilation if $c > 0$, and a rotation if $|c| = 1$. If $c < 0$, then it is a dilation by $|c|$ together with a π-rotation.

The power map

Example 2

For a positive integer n, the map $z \mapsto z^n$ is conformal from $S = \{z \in \mathbb{C} \mid 0 < \arg(z) < \pi/n\}$ to $\mathbb{H}$. The inverse is $w \mapsto w^{1/n}$ with the principal branch of the logarithm.

Upper half disc to first quadrant

Example 3

Define $f(z) = (1+z)/(1-z)$. This takes the upper half disc $\{x+iy \mid x^2+y^2 < 1 \text{ and } y > 0\}$ conformally to $\{u+iv \mid u > 0 \text{ and } v > 0\}$. Verify the inverse map is given by $g(w) = (w-1)/(w+1)$, and both are holomorphic.

The logarithm

Example 4

The map $z \mapsto \log z$ with the branch cut $i\mathbb{R}^-$, takes $\mathbb{H}$ to the strip $\{u + iv \mid u \in \mathbb{R}, 0 < v < \pi\}$. Check that the inverse map is $w \mapsto e^w$.

The logarithm

Example 5

Note that $z \mapsto \log z$ also defines a conformal map from the upper half disc to $\{u + iv \mid u < 0, 0 < v < \pi\}$.

The exponential map

Example 6

The map $z \mapsto e^{iz}$ takes $\{x + iy \mid -\pi/2 < x < \pi/2, y > 0\}$ to $\{u + iv \mid u^2 + v^2 < 1, u > 0\}$.

The function $f(z) = -1/2(z + 1/z)$

Example 7

$f(z) = -1/2(z + 1/z)$ is a conformal map from the upper half disc to the upper half plane. Examine the boundary behavior of f.

$f(z) = \sin z$

$\sin z$ takes $\mathbb{H}$ conformally onto the half strip $\{x + iy \mid -\pi/2 < x < \pi/2, y > 0\}$. Verify this by letting $\zeta = e^{iz}$, then $\sin z = -1/2(i\zeta + 1/i\zeta)$ together with example 6, a rotation by i, and example 7.

Introduction

The Dirichlet problem in the open set Ω consists of solving both $\nabla^2 u = 0$ in Ω, and $u = f$ on $\partial\Omega$.

In this section, we connect the Dirichlet problem to conformal maps.

Ω is a strip

We first solve when Ω is a strip. To do this, we introduce a lemma.

Lemma 1.3

Let U, V be open sets of $\mathbb{C}$, and $F : V \to U$ a holomorphic function. If $u : U \to \mathbb{C}$ is harmonic, then $u \circ F$ is harmonic on V.

Ω is a strip

Back to the problem; let $\Omega = \{x + iy \mid x \in \mathbb{R}, 0 < y < 1\}$. Suppose we have two boundary functions f_0, f_1 defined on $\mathbb{R}$. We ask for a solution $u(x, y)$ in Ω such that $u(x, 0) = f_0(x)$ and $u(x, 1) = f_1(x)$.

Assume f_0, f_1 are continuous and vanish at infinity.

Overview of proof

$$
\begin{tikzcd}
\boxed{\text{Strip}} \arrow[r, "\text{conformal map}"] & \boxed{\text{Unit Disk}} \arrow[r, "\text{Poisson kernel}"] & \boxed{\text{Solution}}
\end{tikzcd}
$$

Overview of proof

$$\boxed{\text{Strip}} \xrightarrow{\text{conformal map}} \boxed{\text{Unit Disk}} \xrightarrow{\text{Poisson kernel}} \boxed{\text{Solution}}$$

We introduce the mappings $F : \mathbb{D} \to \Omega$ and $G : \Omega \to \mathbb{D}$, defined by

$$F(w) = \frac{1}{\pi} \log \left(i \frac{1-w}{1+w} \right) \quad \text{and} \quad G(z) = \frac{1-e^{\pi z}}{i+e^{\pi z}}$$

Note that F and G are conformal and inverse to one another.

Dirichlet problem in the unit disc

Observe the behavior of $F(e^{i\phi})$ when ϕ varies from $-\pi$ to 0, and 0 to π. Notice that for negative ϕ, the value spans $i+\infty$ from $i-\infty$, and for positive ϕ, it spans $\mathbb{R}$.

We define

$$\tilde{f}_1(\phi) = f_1(F(e^{i\phi}) - i) \quad \text{whenever} - \pi < \phi < 0$$

and

$$\tilde{f}_0(\phi) = f_0(F(e^{i\phi})) \quad \text{whenever } 0 < \phi < \pi.$$

Dirichlet problem in the unit disc

Recall that we have defined f_0 and f_1 to vanish at infinity. Therefore we may extend $\tilde{f}_1$ and $\tilde{f}_2$ continuously to $\partial\mathbb{D}$. Name this function $\tilde{f}$. Now we solve the Dirichlet problem in the unit disc by the Poisson integral

$$\tilde{u} = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-\phi)\tilde{f}(\phi)\,d\phi$$

where $w = re^{i\theta}$ and $P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2}$. If we put $u(z) = \tilde{u}(G(z))$, then by Lemma 1.3 this is harmonic in Ω. (The tedious calculations are omitted here.)

The Schwarz lemma

The Schwarz lemma is a simple result showing the rigidity of holomorphic functions.

Lemma 2.1

Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then

- $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$
- If $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then f is a rotation.
- $|f'(0)| \leq 1$, and f is a rotation if equality holds.

Automorphisms of the disc

Using the Schwarz lemma, we determine the automorphisms of the disc.

A conformal map from an open set Ω to itself is called an **automorphism**, denoted $\mathrm{Aut}(\Omega)$.

We may construct a group structure on $\mathrm{Aut}(\Omega)$ by taking the group operation to be the composition of maps.

The maps ψ_α

Recall the automorphisms of the form

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}$$

where $\alpha \in \mathbb{C}$ with $|\alpha| < 1$.

Exercise

Prove ψ_α is indeed an automorphism of $\mathbb{D}$.

What do automorphisms of $\mathbb{D}$ look like?

Theorem 2.2

All automorphisms f of $\mathbb{D}$ are of the form

$$f(z) = e^{i\theta} \frac{\alpha - z}{1 - \overline{\alpha} z}$$

for $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$.

What do automorphisms of $\mathbb{D}$ look like?

Corollary 2.3

The only automorphisms of $\mathbb{D}$ that fix the origin are the rotations.

Note that for any $\alpha, \beta \in \mathbb{D}$, there is a $\psi \in \mathrm{Aut}(\mathbb{D})$ mapping α to β. An example is $\psi_\beta \circ \psi_\alpha$.

The structure of Aut($\mathbb{D}$)

Aut($\mathbb{D}$) is *almost* isomophic to SU(1,1), the group of all 2×2 matrices that preserve the Hermitian form

$$\langle Z, W \rangle = z_1 \overline{w_1} - z_2 \overline{w_2}$$

on $\mathbb{C}^2 \times \mathbb{C}^2$ for $Z = (z_1, z_2)$ and $W = (w_1, w_2)$.

Automorphisms of $\mathbb{H}$

Now we determine the automorphisms of $\mathbb{H}$, using the knowledge of $\mathrm{Aut}(\mathbb{D})$.

Define $\Gamma : \mathrm{Aut}(\mathbb{D}) \to \mathrm{Aut}(\mathbb{H})$ as $\Gamma(\phi) = F^{-1} \circ \phi \circ F$, where $F : \mathbb{H} \to \mathbb{D}$ is the conformal map defined earlier. Note that Γ is a group isomorphism.

Exercise

Verify that Γ is indeed a group isomorphism.

Automorphisms of $\mathbb{H}$

Recall that $F(z) = \frac{i-z}{i+z}$, $G(w) = i\frac{1-w}{1+w}$ and $\phi : \mathbb{D} \to \mathbb{D}$ is of the form $\phi(z) = e^{i\theta}\frac{\alpha - z}{1-\overline{\alpha}z}$.

Exercise

Check that all elements of $\mathrm{Aut}(\mathbb{H})$ are of the form $z \mapsto \frac{az+b}{cz+d}$ for real a, b, c, d such that $ad - bc = 1$, by calculating $(G \circ \phi \circ F)(z)$.

The special linear group

Definition

Special linear group

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \quad \text{and} \quad \det(M) = ad - bc = 1 \right\}$$

Definition

Given $M \in \mathrm{SL}_2(\mathbb{R})$ we define f_M as

$$f_M(z) = \frac{az + b}{cz + d}$$

Again, automorphisms of $\mathbb{H}$

Theorem 2.4

Every automorphism of $\mathbb{H}$ takes the form f_M for some $M \in \mathrm{SL}_2(\mathbb{R})$. Conversely, every map of this form is an automorphism of $\mathbb{H}$.

Note that $\mathrm{Aut}(\mathbb{H})$ is not isomorphic to $\mathrm{SL}_2(\mathbb{R})$ since M and $-M$ give rise to the same $f_M = f_{-M}$. Thus we obtain a new group $\mathrm{PSL}_2(\mathbb{R})$, called the **projective special linear group**.

$\mathrm{PSL}_2(\mathbb{R})$

Conclusion

$\mathrm{Aut}(\mathbb{H}) \simeq \mathrm{PSL}_2(\mathbb{R})$