An Invitation to ∞-Categories

250401

Reference]

[Lurie] Higher Topos Theory

[Töen] Derived Algebraic Geometry

§1. Model Categories.

<u>Def</u> A model category is a Category $\mathcal{C}$ w/ fibrations, Cofibrations, weak equivalences satisfying

Axiom 1. has small limit & Colimits

2. 2-of-3 axiom on weak equivalences

3. $f: X \to Y$ be retract of $g: X' \to Y'$ then it preserves all the above 3 classes

4. (Extension axiom)

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

for ① i Cofibration, p trivial fibration

or ② i trivial Cofibration, p fibration

5. (Factorization axiom)

$$X \xrightarrow{\text{cofib}} S \xrightarrow{\text{triv. fib}} Y \quad \& \quad X \xrightarrow{\text{triv. cofib}} T \xrightarrow{\text{fib.}} Y$$

<u>Def</u> A cylinder object of X is

$$\begin{tikzcd} X \amalg X \arrow[r, "\text{Cofib}", "i"'] \arrow[rr, bend right, "\text{fold map.}"'] & C \arrow[r, "\text{weak eq}", "j"'] & X \end{tikzcd}$$

$f, g: X \to Y$ are homotopic if $\exists$ Cylinder object C s.t.

$$\begin{tikzcd} X \amalg X \arrow[rr] \arrow[dr, "(f,g)"'] & \arrow[d, phantom, "\circlearrowright"] & C \arrow[dl] \\ & Y & \end{tikzcd}$$

<u>Def</u> homotopy Category of $\mathcal{C}$ is

Objects : fibrant Cofibrant objects

Morphism : homotopy equivalence classes.

<u>Thm</u> (Quillen). $h\mathcal{C} \simeq W^{-1}\mathcal{C}$.

Examples.

1. $\mathcal{C} = \mathrm{Top}$: W = weak eq. Fib = Serre fibrations
Cofib = retract of relative cell complexes

2. $\mathcal{C} = \mathrm{Ch}(\underline{R\text{-mod}})$: W = weak eq. ↱ Hovey.
projective ① Fib = degreewise surjective, Cofib
injective ② Cofib = " injective, Fib.) 2 models

Note that they have equivalent homotopy category, namely $D(R)$.
Quillen's thm formalizes the fact : $D(R)$ is equivalent to studying injective / projective complexes
" injective / projective resolutions".

3. $\mathcal{C} = \mathrm{sSet} := \mathrm{Fun}(\Delta^{op}, \mathrm{Set})$
W = weak equivalence, Cofibration = monomorphism
fibration = Kan fibrations :
$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

<u>Note</u> Δ^n is the simplicial set $\mathrm{Hom}(-, [n])$.
Λ^n_i is the ith horn · generated by non-degenerate $(n-1)$-face except the ith one.

<u>Remark</u> (Geometric Realization)
X be a compactly generated Hausdorff space.
$C^\cdot$ be cosimplicial space s.t. $C^n = \{(x_i) \in [0,1]^{n+1} \mid \sum x_i = 1\}$
Define $(\mathrm{Sing}_{C^\cdot}(X))_n := \mathrm{Hom}(C^n, X)$ s.t. $\mathrm{Sing}_{C^\cdot}(X) \in \mathrm{sSet}$.
i.e. $\mathrm{Sing}_{C^\cdot} : \mathrm{Sp} \to \mathrm{sSet}$ is a functor.
this has a left adjoint, denoted $S \mapsto |S|$, called the "geometric realization".

* For experts : $|S| := \int^{n \in \Delta} S_n \times C^n$ <u>"Coend"</u>.

Def (Quillen adjunction)

$\mathcal{C}, \mathcal{D}$ be model categories w/ $g : \mathcal{C} \rightleftarrows \mathcal{D} : f$. adjoint

f, g are called Quillen adjunction if

① f preserves fib & trivial fib or ② g preserves cofib & trivial cofib.

By fibrant (or cofibrant replacement), we get

$$\mathbb{L}g : h\mathcal{C} \rightleftarrows h\mathcal{D} : \mathbb{R}f.$$

Thm. This is an adjoint in homotopy categories.

If they are moreover equivalences, we call them Quillen equivalence.

Examples (Exercise ?).

1. $|\cdot| : sSet \rightleftarrows Top : Sing.$ is a Quillen equivalence.

2. $R' \otimes_R - : Ch(R\text{-mod}) \longrightarrow Ch(R'\text{-mod})$ is left Quillen (endowed w/ projective model structure).

§2. Why ∞-Categories ?

Idea. We want to remember All the data on Hom.

i.e. Want to view Hom not only as a set, but as a "space"

Example. $Ch(R\text{-Mod})$

For $C^\cdot, D^\cdot \in Ch(R\text{-Mod})$, $\mathrm{Hom}_{Ch(R\text{-Mod})}(C^\cdot, D^\cdot)$ is a priori a set.

But... define $\underline{\mathrm{Hom}}^i(C^\cdot, D^\cdot) := \prod_{k \in \mathbb{Z}} \mathrm{Hom}_R(C^i, D^{k+i})$

& $d_i : \underline{\mathrm{Hom}}^i(C^\cdot, D^\cdot) \longrightarrow \underline{\mathrm{Hom}}^{i+1}(C, D^\cdot)$ as

$$\{f_k\}_{k \in \mathbb{Z}} \mapsto \{d_D \circ f_k - (-1)^i f_k \circ d_C\}_k$$

then $\underline{\mathrm{Hom}}^i(C^\cdot, D^\cdot) \in Ch(k\text{-Mod})$.

Exercise. Check $\exists$ $\underline{\mathrm{Hom}}(C^\cdot, D^\cdot) \otimes \underline{\mathrm{Hom}}(D^\cdot, E^\cdot) \to \underline{\mathrm{Hom}}(C^\cdot, E^\cdot)$
by Composition.

Thus, in fact $Ch(R\text{-Mod})$ is a Category "enriched in $Ch(k\text{-mod})$"
* It's a dg category.

Example. Cat_{cat}. (2-Category)

Consider Category of (small) Categories. Morphisms are functors.
But we know that there are natural transformations as
"morphism of functors"

Thus, in fact Cat_{cat} is a Category "enriched in Cat"

We formalize these:

__Def__ $M, \otimes, I, \alpha, \lambda, \rho$ be a Monoidal Category
(I: unit, $\alpha_{A,B,C} : A \otimes (B \otimes C) \xrightarrow{\cong} (A \otimes B) \otimes C$ natural, $\lambda_A : I \otimes A \xrightarrow{\cong} A$ natural, $\rho_A : A \otimes I \xrightarrow{\cong} A$
satisfying pentagon / identity triangle axioms)

A __Category enriched over M__ is $\mathcal{C}$ w/
objects : $ob(\mathcal{C})$
Morphisms : $Map(a,b) \in M$ for $\forall a.b \in \mathcal{C}$ w/
① $\exists$ identity $id_a : I \to Map(a,a)$ in M
② $\circ_{abc} : Map(b,c) \otimes Map(a,b) \to Map(a,c)$ "Composition"

w/ pentagon / identity triangle axioms Coherently stated
using α & λ, ρ.

__Philosophy__ We should NOT say two functors are the "same"
the Correct notion is that they are (naturally) equivalent.
(by Some Coherent higher homotopy)

Example. Fundamental Groupoid. $\Pi_{\leq\infty} X$. (all morphisms are equivalences)

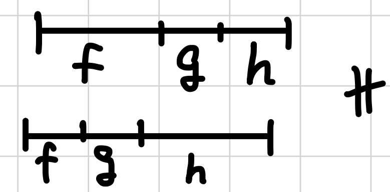
An attempt to view space as a category. X be topological sp.

Points of X are objects, paths are 1-morphisms, homotopies are 2-morphisms ...

Note that associativity is NOT strict.

$x \overset{f}{\text{---}} y \overset{g}{\text{---}} z \overset{h}{\text{---}} w$ $\qquad f\circ(g\circ h)$:

$(f\circ g)\circ h$:



However, there is a natural equivalence connecting them.

Considering higher morphisms, it is incredibly tricky to keep track of all homotopy coherence relations.

§3. Defining ∞-Categories.

Attempt 1. Define n-category as a category enriched in $\underbrace{(n-1)\text{-Cat}}$ Category of

$\Rightarrow$ define ∞-category as sort of their "limit". (inductively)

Problem: the associativity relations are strict : NOT what we want

Attempt 2. Define n-category as a category enriched in $\underbrace{n\text{-Category}}_{\text{of } (n-1)\text{-Cat.}}$

Problem: it's circular

So we need to define ∞-category encoding all homotopy coherence data, intrinsically. Inspired by the above example, (A space encodes all higher morphisms as homotopies)

Attempt 3. Define $(\infty,1)$-category as a category enriched in Top.

We call this "topological category".

Good! ((∞, n) category means all $\geq n$-morphisms are equivalences.)

But then one may give...

Attempt 4 Define $(\infty,1)$-Category as a Category enriched in sSet.
We Call this "Simplicial Category".

Since we know Top & sSet are Quillen-equivalent, they are Quillen-equivalent.
(I haven't introduced model structures on topological / simplicial cat, but for those who are interested, look up Bergner model structure)

Attempt 5 Define $(\infty,1)$-Category as a Category enriched in $Ch(k\text{-Mod})$
We Call this "dg Category".

This is related to the simplicial Category by Dold-Kan.

Thm (Dold-Kan Correspondence) $Ch_{\geq 0}(Ab) \cong sAb$.

Now we want a model independent description of a $(\infty,1)$-Category.

Final Attempt An $(\infty,1)$-Category is a Weak Kan Complex.

What an abrupt definition!? But we will see what this means...

Def A weak Kan complex is a simplicial set w/ the following lifting property:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur] & \end{tikzcd} \quad \text{for } \forall\, 0<i<n.$$

§4. Relation with other Categories

Def (Nerve Construction)
$\mathcal{C}$ be a classical Category Define $N(\mathcal{C})_n := \mathrm{Fun}([n], \mathcal{C})$

i.e. $N(\mathcal{C})_n = \{C_0 \to \cdots \to C_n\}$ w/ face & degeneracy maps
Composing adjacent two insert id.

this defines a simplicial set $N(\mathcal{C})$ called the nerve of a category.

<u>Thm</u>. TFAE (1) Simplicial set $K \cong N(\mathcal{C})$ for some category $\mathcal{C}$
(2) <u>Unique</u> inner horn liftings.

In particular, a classical category can viewed as ∞-category by nerve construction.

<u>Rmk</u>. What does horn liftings mean?

Λ^2_2 :
$$\begin{tikzcd} & c_1 \arrow[dr] & \\ c_0 \arrow[ur, dashed] \arrow[rr] & & c_2 \end{tikzcd}$$
extends to
$$\begin{tikzcd} & c_1 \arrow[dr] & \\ c_0 \arrow[ur] \arrow[rr] & & c_2 \end{tikzcd}$$
$\Rightarrow$ "existence of htpy inverse"

Λ^2_1 :
$$\begin{tikzcd} & c_1 \arrow[dr] & \\ c_0 \arrow[ur] \arrow[rr, dashed] & & c_2 \end{tikzcd}$$
"

$\Rightarrow$ "existence of composition up to htpy"

Thus, if $\mathcal{C}$ is a groupoid, we'll have all horn liftings
i.e. $N(\mathcal{C})$ is a Kan complex.

<u>Topological Category</u> $\mathcal{C}$

<u>Def</u> Define its 0-truncated category $\pi_0\mathcal{C}$ by replacing morphism space with its π_0.

In particular, let $\mathcal{H}$ be the htpy category of CW cpx category.
By Whitehead's thm, $\mathcal{H} \cong W^{-1}(CG)$ inverting weak equivalences.

<u>Def</u> Define its homotopy category $h\mathcal{C}$ by replacing morphism space with its class in $\mathcal{H}$.

Similar definitions for simplicial category $\mathcal{C}$.

<u>Next time</u>

Relations btwn ∞-category & simplicial / topological Cat.
+ Mapping spaces of ∞-category.