ALGEBRA I HOMEWORK III

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Problem 1. *Let $n \in \mathbb{N}$. Write down a formula for the order of the conjugacy class of an element of S_n with cycle type $\{k_1, \ldots, k_m\}$.*

Proof. By the lecture notes, we know that elements of S_n are conjugate if and only if they have the same cycle type. Assume $k_1 \leq k_2 \leq \cdots \leq k_m$. We wish to find the number of distinct elements of S_n that have the same cycle type. To do this, we partition a permutation sequence of 1 through n into m parts via $k_1, \ldots, k_m$, and divide out $n!$ by the number of duplicates. For each cycle of length k_i, there are k_i ways to write the same k_i-cycle, so we must first divide out by $\prod_{i=1}^m k_i$. Also, if we write a_i as the number of elements equal to i in the cycle type, we can find out that $\prod_{i=1}^m k_i = \prod_i (i)^{a_i}$. Also, for each i-cycle there are $a_i!$ ways to rearrange each i-cycle, so we must divide out by $\prod_i a_i!$. Combining these, we conclude that the order of the conjugacy class is $n!/\prod_i (i)^{a_i}(a_i!)$, where a_i is the number of i's in the cycle type. $\square$

Problem 2. *Let $n \in \mathbb{N}$. Solve the following:*

1. *Show $S_n = \langle (12), (13), \ldots, (1n) \rangle$.*
2. *Show $S_n = \langle (12), (23), \ldots, (n-1, n) \rangle$.*
3. *Show $S_n = \langle (12), (1 \cdots n) \rangle$.*
4. *Show that if n is prime, $\sigma \in S_n$ any n-cycle, and $\tau \in S_n$ any transposition, then $S_n = \langle \sigma, \tau \rangle$.*

Proof. 1. Elements of S_n have a cycle decomposition, and any cycle $(a_1 a_2, \ldots, a_k) = (a_1 a_k)(a_1 a_{k-1}) \cdots (a_1 a_2)$. Therefore S_n is generated by transpositions, and it suffices to show that we can make any transposition with the given $(1k)$. Suppose we have a transposition (ab). Then, this is equal to $(1a)(1b)(1a)$, since $a \mapsto 1 \mapsto b$ and vice versa. Therefore, any transposition can be made, so S_n can be generated. $\square$

Proof. 2. Since $(k\ \ k+1) = (1\ \ k+1)(1\ \ k)(1\ \ k+1)$, we have $(23) = (12)(13)(12)$, $(34) = (13)(14)(13)$ and so on. Therefore we have $(13) = (12)(23)(12)$, $(14) = (13)(34)(13)$ and so on. Therefore, $(1k) \in \langle (12), (23), \ldots, (n-1, n) \rangle$ for all k, so it is S_n indeed. $\square$

Proof. 3. Note that $(1 \cdots n)(12)(1 \cdots n)^{-1} = (23)$, and in general $(1 \cdots n)(k\ \ k+1)(1 \cdots n)^{-1} = (k+1\ \ k+2)$. Therefore we may obtain $(12), (23), \ldots, (n-1, n)$ which generated S_n as we showed above. $\square$

Proof. 4. Suppose we have σ any n-cycle, and $\tau = (ab)$. Suppose $\sigma^k(a) = b$ for some k. Then since $\langle \sigma^k \rangle \leq \langle \sigma \rangle$, the order of σ^k must divide n, and since $\sigma^k \neq e$, we conclude that σ^k is of order n. Since n is prime, we conclude that σ^k is an n-cycle, since the order of an element equals the lcm of each cycle. Therefore,

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by replacing σ with σ^k it suffices to show that $(ab), (abc\cdots)$ generates $S_{\{a,b,c,\ldots\}}$. Under the bijective correspondence $1 = a$, $2 = b$, $3 = c$ and so on, this is equivalent to showing the permutation group on $\{1, 2, \ldots, n\}$, i.e. S_n, is generated by (12) and $(1\cdots n)$. This is what we showed above. $\square$

Problem 3. *Let $n \geq 3$.*

1. *For each $j \in J = \{1, \ldots, n\}$, let H_j be the stabilizer of j in A_n. Show that $[A_n : H_j] = n$, and that $H_j \cong A_{n-1}$ for all $j \in J_n$.*
2. *Suppose $H \leq A_n$ is a subgroup of index n. Show that $H \cong A_{n-1}$ by showing that the left translation action of A_n on A_n/H induces an isomorphism $H_1 \cong H$.*

Proof. 1. Elements of H_j are elements of A_n that leave j fixed, i.e. even permutations on the set $J \setminus \{j\}$. This is exactly A_{n-1}, so $H_j \cong A_{n-1}$. This automatically implies $[A_n : H_j] = n$. $\square$

Proof. 2. Consider the homomorphism $\psi : A_n \to S_{A_n/H}$ given by $x \mapsto \varphi_x$ where $\varphi_x(aH) = xaH$. Since $xaH = xbH$ implies $aH = bH$, φ_x is an injective map from $S_{A_n/H}$ to itself, thus bijective, and indeed $\varphi_x \in S_{A_n/H}$. In the previous homework, we showed that A_n is simple for $n \geq 5$, using the fact that it is generated by 3-cycles, all 3-cycles are conjugate, and every nontrivial normal subgroup contains a 3-cycle. (Frankly, I have only showed it for A_5, but this generalizes easily.) By the previous homework, we know that for a finite group, the whole group cannot be the union of the conjugates of a proper subgroup. Therefore, we know that $A_n \neq \bigcup_{a \in A_n} aHa^{-1}$, so there must exist some $x \in A_n$ such that $x \notin aHa^{-1}$ for all $a \in A_n$. Then, it follows that φ_x is not the identity, since $x \notin aHa^{-1} \Rightarrow xaH \neq aH \Rightarrow \varphi_x \neq \mathrm{id}_{S_{A_n/H}}$. Therefore the homomorphism ψ is not trivial, so $\ker\psi$ is a proper normal subgroup of A_n.

For $n \geq 5$, this implies that $\ker\psi = \{e\}$, i.e. ψ is injective. Then A_n can be identified with its image $\psi(A_n)$ in $S_{A_n/H}$, and same for $H \leq A_n$. Note that under the left multiplication action of A_n on A_n/H, the stabilizer of $H \in A_n/H$ is precisely φ_h for $h \in H$. Therefore, $\psi(H) \leq \psi(A_n)$ is the stabilizer subgroup in $S_{A_n/H}$ of H in A_n/H, hence is the stabilizer of H in $A_n \cong \psi(A_n) \leq S_{A_n/H}$. By the problem above, $\psi(H) \cong A_{n-1}$, so $H \cong A_{n-1}$.

Now we treat the cases $n = 3, 4$. For $n = 3$, subgroups of index 3 must have order 1, which is the trivial element. Obviously $e \cong A_2$. Now for $n = 4$, subgroups of index 4 must have 3 elements. Groups of order 3 are isomorphic to $\mathbb{Z}/3\mathbb{Z}$, which is again isomorphic to A_3. $\square$

Problem 4. *Let H be a simple group of order 60. Show that $H \cong A_5$ and compute $|\operatorname{Syl}_p(H)|$ for every prime p.*

Proof. We show that $H \leq A_6$. By the Sylow theorems, we have $n_5 \equiv 1 \bmod 5$, and $n_5 | 12$. The only possibilities are $n_5 = 1$ or $n_5 = 6$. Since H is assumed to be simple, $n_5 = 1$ cannot happen, so $n_5 = 6$. Also, we know that all Sylow 5-subgroups of H are conjugate. Therefore, we may consider a group homomorphism $\psi : H \to S_{\mathrm{orb}(K)}$ where K is a Sylow 5-subgroup of H, and $\mathrm{orb}(K)$ is the orbit set of K under conjugation by elements of H. The homomorphism is given by $h \mapsto \varphi_h$, where $\varphi_h(aKa^{-1}) = haKa^{-1}h^{-1}$ is a set map from $S_{\mathrm{orb}(K)}$ to itself. This set map is obviously injective, hence bijective since $|\operatorname{orb}(K)| = 6 < \infty$. Thus indeed $\varphi_h \in S_{\mathrm{orb}(K)}$.

Now we show that ψ is nontrivial. It suffices to show that there exists some $h \in H$ such that $\varphi_h \neq \mathrm{id}_{S_{\mathrm{orb}(K)}}$. For this to happen, it suffices to find some h where $hKh^{-1} \neq K$. Now if $hKh^{-1} = K$ for all $h \in H$, K would be a normal subgroup of H. But we assumed that H is simple, so this cannot happen, and such h exists. Therefore the homomorphism ψ is nontrivial, and the kernel of this homomorphism cannot be the entirety of H.

We know that the kernel of a group homomorphism is a normal subgroup of the domain group. Hence, it follows that $\ker \psi$ is a proper normal subgroup of H. Since we assumed H to be simple, the only such group is $\{e\}$, so ψ is in fact injective. Therefore we may view H to be a subgroup of $S_{\mathrm{orb}(K)} \cong S_6$. Not only that, if we consider the composition $H \xrightarrow{\psi} S_{\mathrm{orb}(K)} \xrightarrow{\mathrm{sgn}} \{\pm 1\}$, the kernel of this homomorphism cannot be trivial, since otherwise H injects into $\{\pm 1\}$ which is nonsense ($|H| = 60$...) Therefore, the kernel is a nontrivial normal subgroup of H. Since H is simple, this means that the kernel is H, so in fact every element of $\psi(H)$ has sign $+1$, which implies that $\psi(H) \leq A_6$. Now since $H \cong \psi(H)$, $\psi(H)$ is an order 60 subgroup of A_6, hence of index 6. From the problem above, it follows that $\psi(H) \cong A_5$, i.e. $H \cong A_5$.

As we have shown above, $n_5 = 6$. The remaining numbers are n_2 and n_3. We must have $n_3 \equiv 1 \bmod 3$ and $n_3 | 20$, so $n_3 = 1, 4, 10$. Also A_5 is simple so $n_3 = 1$ cannot happen. The only possibilities are $n_3 = 4, 10$. Note that Sylow 3-subgroups are of order 3, and we know that order 3 elements of A_5 are precisely the 3-cycles. Note that

$$(123), (124), (125), (134), (135), (145), (234), (235), (245), (345)$$

are all 3-cycles that generate distinct subgroups of order 3. Hence $n_3 = 10$. Now, $n_2 \equiv 1 \bmod 2$, and $n_2 | 15$ so $n_2 = 1, 3, 5, 15$, again excluding 1 to get $n_2 = 3, 5, 15$. Suppose $n_2 = 15$. Since Sylow 2-subgroups are of order 4, we conclude that A_5 has 3×15 elements that are of order either 2 or 4. But this cannot happen since $n_3 = 10$, so there are at least 20 elements of A_5 that have order 3. Therefore, either $n_2 = 3$ or 5. However, using the fact that subgroups of order 4 of A_5 must be contained in Sylow 2-subgroups, we find 5 distinct subgroups of A_5 of order 4:

$$\{e, (12)(34), (13)(24), (14)(23)\}$$
$$\{e, (12)(35), (13)(25), (15)(23)\}$$
$$\{e, (12)(45), (14)(25), (15)(24)\}$$
$$\{e, (13)(45), (14)(35), (15)(34)\}$$
$$\{e, (23)(45), (24)(35), (25)(34)\}$$

Therefore these subgroups themselves are the Sylow 2-subgroups, and $n_2 = 5$. $\square$

Problem 5. *Solve the following:*

1. *Compute the subgroup lattice of A_4.*
2. *Show that S_n are solvable for $n \leq 4$.*

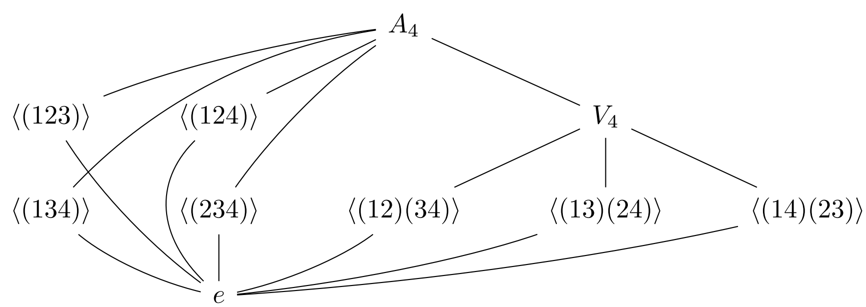
Proof. 1. Elements of A_4:

$$\{e, (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}$$

The possible orders of subgroups are divisors of 12, namely 1,2,3,4,6. The order 2 subgroups are the ones generated by (12)(34), (13)(24) and (14)(23), since these are the only elements of A_4 of order 2. The order 3 subgroups are the ones generated by (123), (124), (134) and (234), again because nontrivial elements of subgroups of order 3 have order 3, and we know what the order 3 elements of A_4 look like. Also, the only order 4 subgroup of A_4 is $\{e, (12)(34), (13)(24), (14)(23)\}$ since either $n_2 = 1$ or 3, but if $n_2 = 3$ then there must be 9 elements of order either 2 or 4, which is not the case. For simplicity, denote this group as V_4.

Now we find subgroups of order 6. Elements of such subgroup must have order either 2 or 3. Therefore, we check what the group generated by, say $(ab)(cd)$ and (abc) is. Note that $(ab)(cd)(abc) = (bdc)$, $(abc)(ab)(cd) = (acd)$, and $(bdc)(acd) = (abd)$. Therefore, the group generated by $(ab)(cd)$ and (abc) contains all 3-cycles, so it clearly cannot be of order 6. By plugging numbers into a, b, c, d, we may conclude that no subgroup of order 6 of A_4 exists.

Therefore, the subgroup lattice looks like this:[1]



□

Proof. 2. Since A_n are index 2 subgroups of S_n, they are normal, and the quotient is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Thus, it suffices to show that A_n are solvable for $n \leq 4$. Since $A_3 \cong \mathbb{Z}/3\mathbb{Z}$, it is trivially solvable. Hence we prove for A_4. Consider the chain $e \leq \langle(12)(34)\rangle \leq V_4 \leq A_4$ of subgroups of A_4. They are of order 1, 2, 4, 12, respectively, so the quotients are isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$, which are abelian. Thus A_4 is solvable. □

Problem 6.

1. *Show that the permutation action of A_n on J_n is $(n-2)$-transitive for all $n \geq 3$.*
2. *For $n \geq 5$, classify all subgroups of S_n whose permutation action on J_n is $(n-2)$-transitive.*

Proof. 1. Show that the action of A_n on $J_n^{[n-2]}$ is transitive. Here, $J_n^{[n-2]}$ is the set of sequences of $n-2$ elements of $J_n = \{1, 2, \ldots, n\}$, all distinct. Suppose we have an element $(a_1, \ldots, a_{n-2})$ of $J_n^{[n-2]}$. Consider a permutation σ that sends i to $\sigma(i) = a_i$, for $1 \leq i \leq n-2$. There are two such permutations, depending on whether $\sigma(n) = a_n$ or a_{n-1}. Now in one case it is odd, and one case it is even, so there is exactly one such permutation σ in A_n. Therefore any element of $J_n^{[n-2]}$ can be written as $(a_1, \ldots, a_{n-2}) = \sigma(1, 2, \ldots, n-2)$ where $\sigma \in A_n$. □

Proof. 2. We classify subgroups of S_n ($n \geq 5$) whose permutation action on J_n is $(n-2)$-transitive. First, the order of $J_n^{[n-2]}$ is $\binom{n}{2} \times (n-2)! = n!/2$, so for the action

[1] I know it looks hideous but I had to do it to fit this in the page

to be $(n-2)$-transitive the subgroup must have at least $n!/2$ elements. Above we have shown that A_n acts $(n-2)$-transitively on J_n, so A_n is such a subgroup. We show that the only subgroup of S_n of index 2 is A_n. Suppose $A_n \neq H \leq S_n$ where $[S_n : H] = 2$. Since $A_n \neq H$, H cannot contain all 3-cycles of S_n, say $(abc) \notin H$. (If H contained all 3-cycles, then it would properly contain A_n, which implies $[S_n : H] = 1$.) This implies $(abc)^{-1} = (acb) \notin H$, so $H, (abc)H, (acb)H$ are three distinct cosets of H, contradictory to the assumption that $[S_n : H] = 2$. Thus A_n is the only such subgroup. Trivially, S_n is also an $(n-2)$-transitive subgroup of S_n. $\square$