Elliptic Functions

Chapter 9

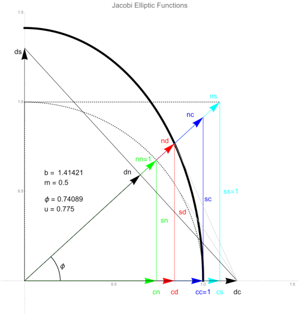
Complex Function Theory 2

July 21, 2023

Introduction

The theory of elliptic functions grew out of the study of elliptic integrals. These are integrals of the form $\int R(x, \sqrt{P(x)})\, dx$ where R is rational and P is a polynomial of degree 3 or 4.

These integrals arose in computing the arc-length of an ellipse.



Elliptic functions

We are interested in meromorphic functions f on $\mathbb{C}$ that have two periods, i.e. there exist two nonzero complex ω_1, ω_2 such that $f(z+\omega_1) = f(z)$, and $f(z+\omega_2) = f(z)$ for all z. We call such functions doubly periodic.

Note that we are only interested in the case $\omega_2/\omega_1 \notin \mathbb{R}$, since if not, the function would have a simple period or be constant.

Elliptic functions

Let $\tau = \omega_2/\omega_1$. Interchange ω_1 and ω_2, if necessary, to make $\operatorname{Im}(\tau) > 0$. Observe that f has periods ω_1 and ω_2 if and only if $F(z) = f(\omega_1 z)$ has periods 1 and τ, and f is meromorphic if and only if F is.

Therefore we assume that f is a meromorphic function on $\mathbb{C}$ with periods 1 and τ where $\operatorname{Im}(\tau) > 0$.

Elliptic functions

Successive applications of the periodicity conditions yield

$$f(z+n+m\tau) = f(z)$$

for all $n, m \in \mathbb{Z}$ and all $z \in \mathbb{C}$. Therefore it is natural to consider the **lattice** in $\mathbb{C}$ defined by

$$\Lambda = \{n + m\tau \mid n, m \in \mathbb{Z}\}.$$

We say 1 and τ generate Λ. Note that f is invariant under translations by elements of Λ.

Associated to the lattice Λ is the **fundamental parallelogram** defined by $P_0 = \{z \in \mathbb{C} \mid z = a + b\tau \text{ where } 0 \le a < 1 \text{ and } 0 \le b < 1\}$. This parallelogram is fundamental in the sense that if we understand the behavior of f in P_0, then we may understand the whole behavior of f.

Elliptic functions

We define z and w to be congruent modulo Λ if $z = w + n + m\tau$ for some $n, m \in \mathbb{Z}$, and write $z \sim w$.

Proposition 1.1

Suppose f is a meromorphic function with two periods 1 and τ which generate the lattice Λ. Then

- Every point in $\mathbb{C}$ is congruent to a unique point in the fundamental parallelogram.
- Every point in $\mathbb{C}$ is congruent to a unique point in any given parallelogram.
- The lattice Λ provides a disjoint covering of the complex plane, in the sense $\mathbb{C} = \bigcup_{n,m\in\mathbb{Z}}(n + m\tau + P_0)$.
- The function f is completely determined by its values in any period parallelogram.

Liouville's theorems

Theorem 1.2

An entire doubly periodic function is constant.

Proof

Note that the function is completely determined by its values on P_0, which is bounded, thus the function is bounded, hence constant.

Thus a doubly periodic function must be at most meromorphic.

Elliptic function

Definition

A nonconstant doubly periodic meromorphic function is called an elliptic function.

Note that an elliptic funciton has only finitely many zeros and poles in any parallelogram.

Elliptic function

Theorem 1.3

The total number of poles of an elliptic function in P_0 is always ≥ 2.

Proof

First assume f has no poles on ∂P_0. By the residue theorem, the contour integral along ∂P_0 is zero, hence the number of poles cannot be 1. Theorem 1.2 tells us that the number of poles cannot be zero.

If f has a pole on ∂P_0, choose a small $h \in \mathbb{C}$ such that if $P = h + P_0$, then f has no poles on ∂P. Argue as above.

The order of an elliptic function

Definition

The total number of poles (counted with multiplicities) of an elliptic function is called its order.

Theorem 1.4

Every elliptic function of order m has m zeros in P_0.

Proof

Assume that f has no zeros or poles on ∂P_0. Use the argument principle $\int_{\partial P_0} f'/f\,dz = 2\pi i(N_z - N_p)$. By periodicity, the integral on the left becomes zero.

In the case where there is a pole or zero on ∂P_0, again shift P_0 a little.

Next steps

We have studied about the properties of elliptic functions. But, does such an elliptic function exist? If so, what does it look like?

The Weierstrass $\wp$ function

In this section, we provide an example of an elliptic function. To do this, we try first to make a function with period 1 and poles at the integers. Note that $F(z) = \sum_{n\in\mathbb{Z}} 1/(z+n)$ does not converge absolutely, so we have to define $F(z) = 1/z + \sum_{n=1}^{\infty}(1/(z+n) + 1/(z-n))$.

We take this idea to the doubly periodic case. Note that $\sum_{\omega\in\Lambda} 1/(z+\omega)^2$ does not converge. To overcome this, define as

$$\frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right]$$

where $\Lambda^* = \Lambda \setminus 0$. We have subtracted $1/\omega^2$ to make the sum converge.

The Weierstrass $\wp$ function

Lemma 1.5

The two series

$$\sum_{(n,m)\neq(0,0)} \frac{1}{(|n|+|m|)^r} \quad \text{and} \quad \sum_{n+m\tau\in\Lambda^*} \frac{1}{|n+m\tau|^r}$$

converge if $r > 2$.

Try to prove yourself.

The Weierstrass $\wp$ function

Definition

The Weierstrass $\wp$ function is defined as

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right] \\ &= \frac{1}{z^2} + \sum_{(n,m) \neq (0,0)} \left[\frac{1}{(z+n+m\tau)^2} - \frac{1}{(n+m\tau)^2} \right].\end{aligned}$$

We claim that $\wp$ is a meromorphic funciton with two poles at the lattice points.

The Weierstrass $\wp$ function

Theorem 1.6

The function $\wp$ is an elliptic function of periods 1 and τ, and double poles at the lattice points.[a]

[a] double poles?

Proof

We have already proved that $\wp$ is an elliptic function with double poles. Thus we show $\wp$ has periods 1 and τ. To show this, note that $\wp'(z) = -2\sum 1/(z+n+m\tau)^3$, so $\wp'$ is periodic with periods 1 and τ. Thus $\wp(z+1) = \wp(z) + a$ and $\wp(z+\tau) = \wp(z) + b$ hold. Together with the fact that $\wp$ is even, plug in $z = -1/2$ and $-\tau/2$ to conclude the proof.

Properties of $\wp$

First, $\wp$ is even so $\wp'$ must be odd. In addition we know that $\wp'(1/2) = \wp'(\tau/2) = 0$. Note that $(1+\tau)/2$ is also a zero of $\wp'$. Since $\wp'$ is elliptic and has order 3, the three points $1/2, \tau/2, (1+\tau)/2$ are the only roots of $\wp'$ in the fundamental parallelogram, with multiplicity 1.

From this, we may conclude that $\wp(z) = \wp(1/2)$ has a double root at $z = 1/2$. Also since $\wp$ has order 2, there cannot be other roots inside the fundamental parallelogram.

Properties of $\wp$

Theorem 1.7

The function $(\wp')^2$ is the cubic polynomial in $\wp$,

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$$

where $e_1 = \wp(1/2), e_2 = \wp(\tau/2), e_3 = \wp((1+\tau)/2)$.

Proof

The only roots of $F(z) = (\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$ in the fundamental parallelogram have multiplicity 2, and are at the points $1/2, \tau/2, (1+\tau)/2$. Also $(\wp')^2$ has double roots at these points. Note that $(\wp')^2/F$ is holomorphic and doubly periodic, thus is constant. Find this constant by expanding $\wp$ and $\wp'$ near zero. $\square$

Every elliptic function arises from $\wp$

Theorem 1.8

Every elliptic function f with periods 1 and τ is a rational function of $\wp$ and $\wp'$.

Lemma 1.9

Every even elliptic function F with periods 1 and τ is a rational function of $\wp$.