Recent Results in Art Galleries

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Invited Paper

Two points in a polygon are called visible if the straight line segment between them lies entirely inside the polygon. The ***art gallery problem*** *for a polygon P is to find a minimum set of points G in P such that every point of P is visible from some point of G. This problem has been shown to be NP-hard by Lee and Lin [71]. However, Chvátal showed that the number of points of G will never exceed $\lfloor n/3 \rfloor$ for a simple polygon of n sides [21]. This latter result is referred to as the art gallery theorem.*

Many variations on the art gallery problem have been studied, and work in this area has accelerated after the publication of the monograph of O'Rourke [92], which deals exclusively with this topic.

This paper provides an introduction to art gallery theorems, and surveys the recent results of the field. The emphasis is on the results rather than the techniques. In addition, this paper examines several new problems that have the same geometric flavor as art gallery problems.

I. Introduction

A. Definitions

This section contains necessary definitions, some background on art galleries, and a discussion of the scope of this paper. We begin with the definitions, following O'Rourke [92].

A *polygon* is generally defined as an ordered sequence of at least three points $v_1, v_2, \cdots, v_n$ in the plane, called vertices, and the n line segments $\overline{v_1v_2}, \overline{v_2v_3}, \cdots, \overline{v_{n-1}v_n}$, and $\overline{v_nv_1}$, called edges. A *simple* polygon is then a polygon with the constraint that nonconsecutive edges do not intersect. A simple polygon is a Jordan curve, and thus divides the plane into three subsets: the polygon itself, the (bounded) interior, and the (unbounded) exterior. However, we will henceforth use the term "polygon" to refer to "simple polygon plus interior." Polygons are thus closed, bounded sets in the plane.

A polygon P is said to be covered by a collection of subsets of P if the union of these subsets is exactly P. The collections of subsets is called a cover of P. A cover of P is called a decomposition if the intersection of each pair of subsets in the cover has zero area. A *triangulation* of a polygon is a decomposition of the polygon into triangles without adding vertices. This is done by chopping the polygon with *diagonals* (line segments between nonadjacent vertices). A polygon and one of its triangulations are shown in Fig. 1. A *triangulation graph* of a polygon P is the graph on the vertices of P where two vertices are joined if they share an edge, or are the endpoints of a diagonal in a fixed triangulation. The class of polygon triangulation graphs is the same as the class of *maximal outerplanar graphs.*

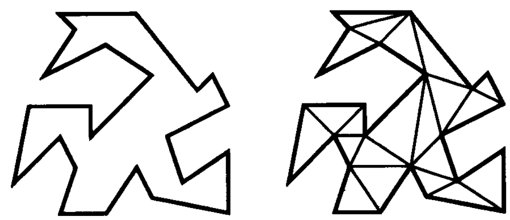

Fig. 1. A polygon and one of its triangulations.

Many algorithms in computational geometry incorporate a polygon triangulation step. Recently, Chazelle presented an algorithm that will compute a triangulation of a polygon in $O(n)$ time [15]; the algorithmic complexity results presented here have been reanalyzed in the light of this result. Consequently, the running times quoted in this paper often do not agree with that presented in the paper to which the algorithm is attributed. In particular, whenever a $\log \log n$ term is not present in a complexity result in this paper, but is in the original source, it is because Chazelle's algorithm has been substituted for the previously-best $O(n \log \log n)$ algorithm of Tarjan and van Wyk [117].

Let x and y be two points in a polygon P. We will say that x and y are visible if the line segment $\overline{xy}$ does not intersect the exterior of P. In Fig. 2, the point a is visible to b and c, but not d. Visible points are said to see one another. The set of all points of P visible from x is a polygon, called the visibility polygon of x, and is denoted $V(x, P)$.

We will distinguish some sets of points in a polygon by calling them guard sets. The individual elements of a guard

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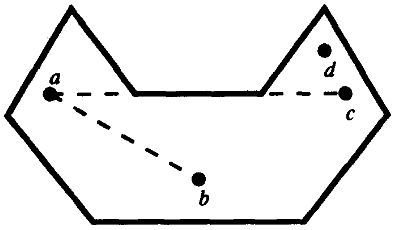

Fig. 2. Point a can see b and c, but not d.

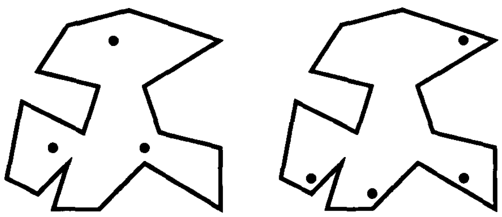

Fig. 3. A covering guard set; a hidden set.

set are called guards. If all of the points in a guard set are vertices of P, then G is called a vertex guard set, and the elements of G are called vertex guards. Otherwise G is called a point guard set, and its elements point guards. Other types of guards will be discussed later.

A guard set G is said to cover a polygon P if the collection of sets $\{V(g, P) \mid g \in G\}$ covers P. The points in the polyon on the left in Fig. 3 are a covering guard set. We will later see that the definition of covering for guard sets is a simple generalization of the usual definition of covering given above. The *art gallery problem* for a polygon P is to find a minimum-cardinality covering guard set G for P. It is so called because one envisions the polygon P as the floor plan of an art gallery, and the points of G as locations to place guards, so that every part of the art gallery is seen by at least one guard. We use $g(P)$ to denote the cardinality of a minimum covering guard set for the polygon P.

A concept similar to that of covering guard sets is *hidden sets*. A hidden set is a set of points H in a polygon such that no pair of points of H is visible. The points in the polygon on the right in Fig. 3 are a hidden set. Hidden sets are known in the mathematics literature as visually independent sets and are related to "property P_m" [65].

An *orthogonal polygon* is a polygon with edges that alternate between horizontal (zero slope) and vertical (infinite slope). Orthogonal polygons have also been called isothetic and rectilinear. Orthogonal polygons are an important subclass of polygons which arise in many computing applications, owing to the ease with which they are represented and manipulated, and to the design of many machines (such as image scanners and plotting devices) that are used in these applications. Restricting the polygons considered in the art gallery problem to orthogonal polygons creates an interesting subclass of problems, and has led to many results.

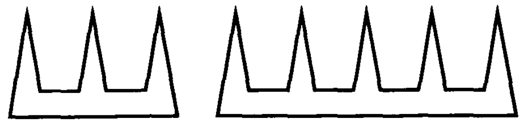

Fig. 4. Comb polygons.

B. Art Gallery History

The original art gallery problem, posed in conversation by Klee to Chvátal, is to find the smallest number of point guards necessary to cover any polygon of n vertices; this number will be denoted $g(n)$ (not to be confused with $g(P)$ as defined above). In terms of galleries, $g(n)$ is the minimum number of guards necessary to supervise any gallery with n walls.

Chvátal quickly proved that $g(n) = \lfloor n/3 \rfloor$, a result which has come to be known as the art gallery theorem [21]. First, he showed that $g(n) \geq \lfloor n/3 \rfloor$, by exhibiting the class of polygons now known as comb polygons; examples of comb polygons are shown in Fig. 4 for $n = 9$ and $n = 15$. Comb polygons exist for any n that is a multiple of 3. Each comb polygon requires $n/3$ guards, as no one guard can see into any two "teeth" (upward triangular regions) of the comb, and there are $n/3$ such teeth.

Next, Chvátal showed that $g(n) \leq \lfloor n/3 \rfloor$, by a relatively complex inductive argument on triangulation graphs of polygons. Fisk later gave the following more concise proof of this inequality [47]: First, triangulate the polygon. Next, three-color the vertices of the triangulation graph: assign each vertex one of three different colors, so that no two vertices which are adjacent in the graph have the same color. Each triangle of the graph, which corresponds to a triangle of the triangulation, will have one vertex of each color. Furthermore, every point of a triangle is visible to each vertex of that triangle. Therefore, choosing any of the three color classes will result in a set of vertices from which every point of every triangle, and thus every point of the polygon, is visible (i.e., each color class is a covering vertex guard set). By the pigeonhole principle, the smallest of these color classes will contain at most $\lfloor n/3 \rfloor$ vertices.

Later, Lee and Lin showed that the art gallery problem for polygons (given a polygon P, find the minimum number of guards necessary to cover P) is NP-hard [70], by reduction from Boolean three-satisfiability. Their result is for vertex guards, and this was extended to point guards by Aggarwal [2]. The reader unfamiliar with complexity theory is referred to the book of Garey and Johnson [50].

Although Lee and Lin's result implies that it is impractical to find a minimum set of guards (i.e., to find $g(P)$ guards) for a given polygon, Avis and Toussaint showed that it is possible to find a set of $g(n)$ guards for a polygon in polynomial time [7]. Algorithms for finding such guard sets are called guard placement algorithms. Most guard placement algorithms work by imitating upper-bound art gallery proofs, and Avis and Toussaint's algorithm is no exception, being an algorithmic imitation of Fisk's proof.

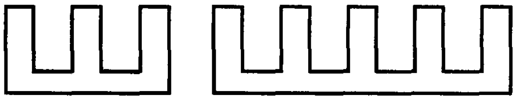

Fig. 5. Orthogonal comb polygons.

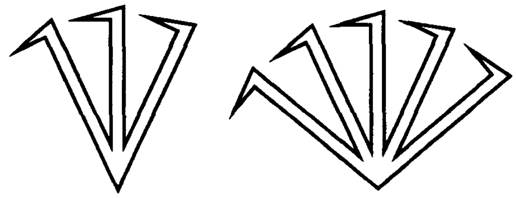

Fig. 6. Polygons requiring $\lfloor n/4 \rfloor$ edge guards.

C. Orthogonal Art Galleries

Kahn, Klawe, and Kleitman investigated the art gallery problem restricted to orthogonal polygons [63]. They exhibit the orthogonal comb polygons of Fig. 5, establishing that $orth(n) \geq \lfloor n/4 \rfloor$ ($orth(n)$ denotes the maximum number of guards necessary for any orthogonal polygon of n vertices). They prove a matching upper bound, $orth(n) \leq \lfloor n/4 \rfloor$, in the same manner as Fisk proved the original art gallery theorem, but they decompose the polygon into *convex quadrilaterals* rather than triangles, and then four-color the quadrilateralization graph, so that each quadrilateral has one vertex of each of the four colors. The bulk of their paper is devoted to proving that every orthogonal polygon has a decomposition into convex quadrilaterals.

Edelsbrunner, O'Rourke, and Welzl gave an $O(n)$ point guard placement algorithm for orthogonal polygons, based on L-shaped partitioning [37]. Lubiw and Sack and Toussaint have presented other linear placement algorithms based on quadrilateralization [76], [100], [103].

D. Importance of Art Galleries

Art gallery problems are studied by computing scientists because they are fundamental visibility problems, and visibility is a central issue in many computing applications. Application areas for visibility include robotics [69], [123], motion planning [75], [80], vision [113], [124], graphics [79], [17], CAD/CAM [12], [38], computer-aided architecture [34], [99], and pattern recognition [5], [118]. Other reasons that art gallery problems are studied are that they are a continuous form of classical facility-location problems, have a simple formulation, and require an interesting interplay of graph theory, geometry, and computing science in their solution.

The monograph of O'Rourke [91] is devoted to art gallery problems and contains well-written detailed expositions of the results mentioned above, and the techniques used in their proofs. Since the publication of this book, activity in art gallery problems has rapidly increased, yielding many new theorems and algorithms. This paper is an attempt to collect these recent results into one place. However, we do not intend this paper to be a tutorial on the proof techniques used, and hence provide only a few details about the methods.

E. Organization of Paper

The remainder of this paper is organized into six sections. Section II contains results about different types of guards. Section III contains covering results, and Section IV is about covering the outsides of polygons. Section V contains results on visibility graphs, and Section VI contains problems which are not strictly art gallery problems, but have the same geometric feel. Conclusions are drawn in Section VII.

II. Generalized Guards

In this section, we consider some variations of the art gallery problem that arise when specified subsets of the polygon, rather than just points, are allowed as elements of guard sets. A point will be called visible to such a subset if it is visible to some point in that subset. This notion of visibility from a subset is known as weak visibility, in contrast to *strong visibility*, where a point is called visible to a subset if it is visible to every point of the subset [6].

More formally, if R is a subset inside a polygon P, we let $V(R, P) = \{p \in P \mid \exists r \in R$ such that p and r are visible$\}$. We are still interested in finding minimum-cardinality covering guard sets, but now the individual guards will be various types of subsets. We are concerned only with the number of guards, and not with the sizes of the individual guards. Typical types of subsets used as guards are convex sets or polygon edges.

This branch of variations on the art gallery problem was started by Toussaint in 1981, when he asked how the art gallery theorem would change if guards were allowed to patrol individual edges of a polygon rather than continually standing at the same point. He wanted to know the guarding function $g^E(n)$, the minimum number of *edge guards* necessary to cover any polygon of n vertices.

Toussaint's conjecture is that if a small number of polygons are excluded, $g^E(n) = \lfloor n/4 \rfloor$. To lend weight to this conjecture, he exhibited the polygon class illustrated in Fig. 6, which establishes that $g^E(n) \geq \lfloor n/4 \rfloor$. Two types of polygons are known that require more than $\lfloor n/4 \rfloor$ edge guards. These polygons, proposed by Paige and Shermer, require $\lfloor (n+1)/4 \rfloor$ guards, and are shown in Fig. 7. However, these polygons are thought to be isolated exceptions, hence the qualification in Toussaint's conjecture.

O'Rourke was the first to make progress on Toussaint's conjecture. Although he was unable to establish an upper bound on $g^E(n)$, he was able to prove an upper bound on $g^M(n)$, the minimum number of *mobile guards* necessary for any polygon of n vertices [90]. Mobile guards are a slightly more general version of edge guards; each mobile guard can patrol either an edge or a diagonal of the polygon. Thus, every edge guard is a mobile guard, and $g^M(n) \leq g^E(n)$.

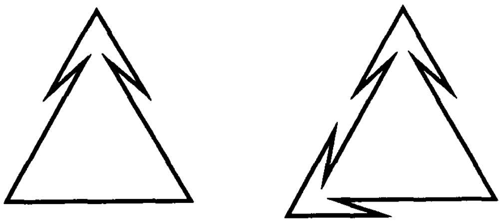

Fig. 7. The two polygons requiring $\lfloor (n+1)/4 \rfloor$edge guards.

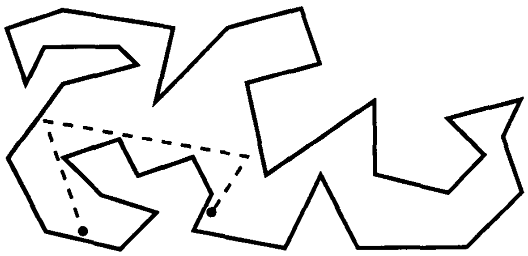

Fig. 8. A pair of L_3 visible points.

O'Rourke's result is that $g^M(n) = \lfloor n/4 \rfloor$. The polygons of Fig. 6 establish the lower bound, $g^M(n) \geq \lfloor n/4 \rfloor$, for mobile guards as well as edge guards, and the polygons of Fig. 7 are no longer exceptions. The matching upper bound, $g^M(n) \geq \lfloor n/4 \rfloor$, is proved by an induction argument on triangulation graphs, similar in technique to Chvátal's proof of the original art gallery theorem.

The similarity of the methods of Chvátal and O'Rourke is no accident. These methods, and the results that they prove, are special cases of more general methods and results, as Shermer later established [106], [107]. To state this result, we must first generalize our notion of visibility.

We will call two points in a polygon link-j visible (L_j visible) if they can be joined by a path of j or fewer line segments that lie entirely inside the polygon. Figure 8 shows a pair of points in a polygon that are L_3 visible. L_1 visibility is the usual visibility: two points are visible if they can be joined by a path of one segment lying inside the polygon. Using L_0 visibility, a point is visible only to itself. Thus, covering problems can be thought of as art gallery problems using L_0 visibility.

One way to define convexity is to call a set convex if each pair of points in the set is visible (the line segment between the points lies in the set). This definition is generalized by calling a set link-k convex (L_k convex) if every pair of points in that set is L_k visible. Thus, L_0-convex sets are points, and L_1-convex sets are convex sets. Link visibility and link convexity were introduced by Horn and Valentine [59], [120], and first studied in a computational setting by Suri [115].

Shermer considers the art gallery problem where L_k-convex subsets of P are allowed as guards, using L_j visibility. Let $g_j^k(n)$ be the associated guarding function: $g_j^k(n)$ is the minimum number of L_k-convex guards necessary to cover any polygon of n vertices, using $L_j$visibility. The essence of Shermer's result is that $g_j^k(n) = \lfloor n/(k+2j+1) \rfloor$. He also shows that this result also holds for more restricted classes of guards, and has an $O(n^2)$ placement algorithm for any of these guard classes.

Chvátal's theorem is the special case of this result when $k = 0$ (point guards) and $j = 1$ (normal visibility). O'Rourke's result is the special case $k = 1, j = 1$. Another by-product of this work is a tight bound of $\lfloor n/(j+1) \rfloor$ on the maximum size of maximum hidden sets in polygons of n vertices, using L_j visibility.

Using L_k-convex guards and L_j visibility in orthogonal polygons should also yield an interesting art gallery theorem. Let $orth_j^k(n)$ be the associated guarding function. Kahn, Klawe, and Kleitman's orthogonal polygon art gallery thoerem is the special case $j = 1$, $k = 0$: $orth_1^0(n) = \lfloor n/4 \rfloor$. Also, Aggarwal proved that $orth_1^1(n) = \lfloor (3n+4)/16 \rfloor$ [2]. Nothing else except the trivial $orth_j^k(n) \leq g_j^k(n)$ is known about $orth_j^k(n)$. Gewali and Ntafos have studied *orthogonal* L_j visibility (each of the j links must be horizontal or vertical), obtaining an $O(n^3)$ algorithm for finding a minimum point guard cover of orthogonal polygons that are reducible to *grids* [50]. A grid is a collection of intersecting horizontal and vertical line segments used as an abstraction of the structure of an orthogonal polygon.

Shermer investigated *diagonal* guards, and the associated guarding function $g^D(n)$ [109]. Diagonal guards are guards which are allowed to patrol a segment between *nonadjacent* vertices—mobile guards that cannot patrol edges of the polygon. He has shown that $\lfloor (2n+2)/7 \rfloor \leq g^D(n) \leq \lfloor (n-1)/3 \rfloor$; the upper bound is tight for triangulation graphs. (Note that O'Rourke uses the term *diagonal guard* for what we call a mobile guard.)

Finally, we return to Toussaint's original question about the function $g^E(n)$. O'Rourke established that the usual technique of reducing an art gallery problem to a problem on a triangulation graph is not strong enough to prove the conjecture that $\lfloor n/4 \rfloor$ edge guards are sufficient. However, Shermer recently showed that the usual technique does yield a proof that $\lfloor 3n/10 \rfloor$ edge guards are sufficient (except for $n = 3, 6$, or 13, where one extra guard may be required). Thus, aside from a small number of exceptions, $\lfloor n/4 \rfloor \leq g^E(n) \leq \lfloor 3n/10 \rfloor$.

Sack and Suri have given an $O(n)$ algorithm to detect if a given polygon can be guarded by one edge guard [101]; polygons so guardable are called weakly edge visible polygons. Ke gave an $O(n \log n)$ algorithm for the related problem of detecting if a given polygon can be guarded by one line-segment guard [66].

Conjecture 1 (Toussaint): Aside from a few small n, $g^E(n) = \lfloor n/4 \rfloor$.

Conjecture 2 (Toussaint): The minimum number of edge guards sufficient for any star-shaped polygon of n vertices is $\lfloor n/5 \rfloor$.

Open Problem 1: Find tight bounds on $orth_j^k(n)$.

Open Problem 2: Find tight bounds on $g^D(n)$.

III. Polygon Covering

In this section, we consider the complexity of covering polygons by various types of subpolygons. Covering is intimately related to guarding: guarding a polygon with guards chosen from some set S is the same as covering the polygon with visibility polygons of members of S. These visibility polygons are often interesting types of sets in their own right. For example, if S is the set of all points of some polygon P, the visibility polygons of S are the maximal star-shaped subsets of P.

Let $\mathcal{P}$ be some property that a set can have. We can then define the $\mathcal{P}$-cover decision problem as follows:

$\mathcal{P}$-cover

INSTANCE: A simple polygon P, and an integer m.

QUESTION: Can P be covered by m or fewer subpolygons, each with property $\mathcal{P}$?

For example, if $\mathcal{P}$ is the property of being convex, the $\mathcal{P}$-cover problem is the convex cover problem. This decision problem can be solved in polynomial time iff a minimum cover can be found in polynomial time. We similarly define the $\mathcal{P}$-guarding problem:

$\mathcal{P}$-guarding

INSTANCE: A simple polygon P, and an integer m.

QUESTION: Can P be covered by m or fewer subpolygons, each a visibility polygon of a subset of P with property $\mathcal{P}$?

We shall see that for most interesting properties $\mathcal{P}$, the $\mathcal{P}$-cover, and $\mathcal{P}$-guarding problems are NP-hard. However, contrary to what one would expect, there is a difficulty in placing many of the $\mathcal{P}$-cover and $\mathcal{P}$-guarding problems in NP. This is because the minimum covers of a polygon might possibly contain pieces that have vertices at coordinates taking a large number of bits to specify. O'Rourke has begun investigation of this difficulty for convex cover [89]. The only nontrivial $\mathcal{P}$-cover or $\mathcal{P}$-guarding problems known to be in NP are those $\mathcal{P}$-guarding problems for which there are only a polynomial number of possible guards (e.g. vertex guarding or diagonal guarding).

Many NP-hardness results for covering and guarding were first established for polygons with holes [93], [73], [76], [50], [78]. Later, Lee and Lin were the first to show that some $\mathcal{P}$-cover and $\mathcal{P}$-guarding problems are NP-hard in polygons without holes [70]. They showed NP-hardness for vertex guarding, edge guarding, and point guarding (star cover). Later, Culberson and Reckhow established NP-hardness for convex cover and similar problems where only the boundary or the vertices of the polygon must be covered [27], and Hecker and Herwig have shown that diagonal guarding, triangulation triangle guarding, and mobile guarding are NP-hard [56]. Shermer showed that, for any $k \geq 1$, L_k-convex cover is NP-hard, and for any $k \geq 0$ and $j \geq 1$, using L_j visibility, L_k-convex guarding is NP-hard [106]. Also, Pesant has started an investigation of the complexity of k-fold covering and guarding (where each point in the polygon must be covered by k or more sets or guards) [95].

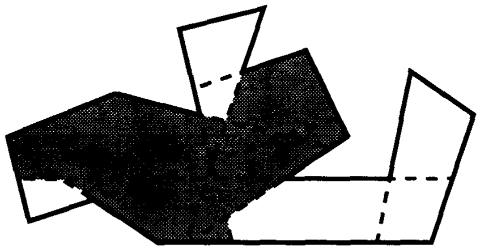

Fig. 9. A restricted star-shaped subpolygon.

Contrary to the hardness of covering, Chazelle and Dobkin were able to show that finding a minimum convex *partition* (cover by nonoverlapping convex pieces) of a polygon is polynomial, and exhibited an $O(n^3)$ algorithm for this problem [16]. This problem is NP-hard in polygons with holes [73]. Also, Keil found an $O(n^7 \log n)$ algorithm for partitioning a polygon into a minimum number of star-shaped polygons, where the vertices of each star-shaped polygon are constrained to be a subset of the vertices of the base polygon [67].

One approach to dealing with the difficulty of covering polygons is to devise algorithms to find solutions that approximate the optimal ones. Ghosh presented an $O(n^5 \log n)$ time algorithm that finds a set of vertex guards that has at most $O(\log n)$ times the minimum number of vertex guards [51]. Aggarwal, Ghosh, and Shyamasundar give an $O(n^4 \log n)$ algorithm to find a cover of a polygon by *restricted* star-shaped pieces that has at most $O(\log n)$ times the minimum number of such covering pieces [3]. A *restricted set* in a polygon P is a subpolygon that has only parts of edges of P or parts of extensions of edges of P as subpolygon edges; a restricted star-shaped set is shown in Fig. 9.

Another approach is to consider $\mathcal{P}$-cover problems for fixed small m; even this seems difficult. Lee and Preparata have given a linear algorithm for detecting if a polygon is star-shaped (star cover with $m = 1$) [71]. There is a trivial algorithm for detecting if a polygon is convex (convex cover with $m = 1$), and a linear algorithm by Shermer to determine if a polygon is the union of two convex polygons (convex cover with $m = 2$) [108]. Recently, Belleville gave an $O(n^2)$ algorithm to determine if a polygon is the union of two L_k-convex polygons (L_k-convex cover with $m = 2$), and an $O(n^4)$ algorithm to determine if a polygon is the union of two star-shaped polygons (star cover with $m = 2$). No other results of this type are known. In a more general setting, Breen has given characterizations of unions of two star-shaped sets [11], and Stamey and Marr have characterized unions of two convex sets [112].

Restricting the class of polygons to be covered to orthogonal polygons has led to many interesting problems. Most covering problems for orthogonal polygons are easy to place in NP. There are general results for covering

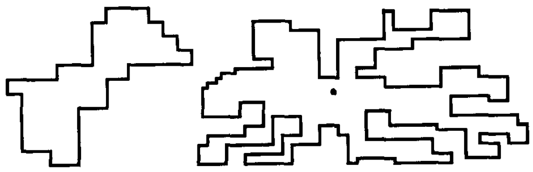

Fig. 10. An orthogonally convex polygon and orthogonally convex star.

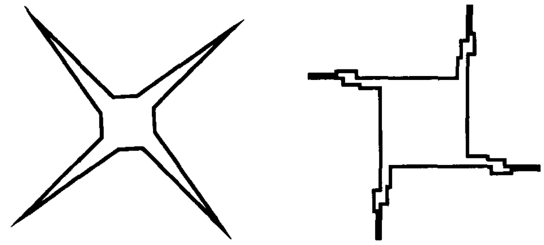

Fig. 11. Polygons that do not admit hidden vertex guard sets.

or partitioning orthogonal polygons with two types of subpolygons: rectangles and orthogonally convex stars.

Culberson and Reckhow have shown that rectangle cover (orthogonal) is NP-complete, even if only the boundary is to be covered [27]. In polygons with holes, Conn and O'Rourke have shown that covering either the boundary or the reflex vertices is NP-complete. However, they obtain an $O(n^{2.5})$ time algorithm for covering the convex vertices [22]. Also, Imai and Asano have given an $O(n^{1.5} \log n)$ algorithm to find a minimum partition of an orthogonal polygon with holes into a minimum number of rectangles [61], and Liou, Tan, and Lee have an $O(n)$ algorithm for this problem if the polygon has no holes [74].

Orthogonally convex polygons are polygons for which any vertical or horizontal line intersects the polygon in exactly one segment. Orthogonally convex stars are polygons formed as the union of a set of orthogonally convex polygons that have a common intersection. An orthogonally convex polygon and an orthogonally convex star are shown in Fig. 10. Motwani, Raghunathan, and Saran have shown that a minimum cover of an orthogonal polygon by orthogonally convex stars can be found in $O(n^{10})$ time [82], and this result was later improved to $O(n^8)$ time by Raghunathan [96]. Rawlins has shown that $\lfloor (n+4)/8 \rfloor$ such covering polygons are sometimes necessary and always sufficient [97].

Many polynomial algorithms have been found for minimum covering problems on restricted classes of orthogonal polygons or using other types of covering sets [98], [26], [68], [77], [48], [8], [14], [4], [82], [81]. Most of these algorithms proceed by dividing the polygons into rectangles by extending the edges, computing a graph expressing the visibility relationship between the rectangles so formed, and finding clique covers on this visibility graph (or a geometric equivalent of the above process). Motwani *et al.* have noted that the visibility graphs used in these covering algorithms are all subclasses of *perfect* graphs [81].

Related to minimum covering and guarding problems are problems of finding maximum hidden sets. Shermer has shown that finding a maximum hidden set in a polygon is NP-hard, even if the hidden set is restricted to the vertices [105]. He also proved that finding minimum *hidden guard sets* (guard sets that are also hidden sets) is NP-hard. Surprisingly, not all polygons have hidden *vertex* guard sets, as the two polygons of Fig. 11 illustrate. Determining if a polygon has a hidden vertex guard set is NP-complete. Shermer has also provided an $O(n)$ algorithm for determining if a polygon has a maximum hidden set of exactly two points [108], and Ghosh *et al.* have given an $O(n^2)$ algorithm to find the maximum hidden vertex set in a polygon weakly visible from an edge between two convex vertices [53].

Open Problem 3: What is the complexity of convex cover (or star cover) with $m = 3$? (Can a polynomial-time algorithm be found to determine if a polygon is the union of three convex (or three star-shaped) polygons?)

Open Problem 4: What is the complexity of finding a minimum vertex guard set in a star-shaped polygon?

Open Problem 5: What is the complexity of finding a minimum *partition* of a polygon into star-shaped polygons?

Open Problem 6: Is convex cover in NP?

IV. Holes, Arrangements, and Polygon Exteriors

By *arrangement* we mean a collection of objects, each with an orientation and position. In two dimensions, arrangements of lines and points have been well studied [36], and arrangements of other objects are beginning to be investigated. We will mainly be concerned with guarding the portions of the plane left uncovered by arrangements of various compact objects. We call this uncovered region the exterior of the arrangement, and define it to include its boundary. We then define exterior visibility analogously to the way we defined (interior) visibility: two points in the exterior of an arrangement are considered visible if the line segment between them contains only points in the exterior of the arrangement.

Polygons with holes are related to arrangements of polygons; the exterior of a polygon arrangement can be considered the interior of a polygon with holes without the surrounding polygon, or with an infinitely large surrounding polygon (see Fig. 12). Also, the exterior of a single polygon is the special case of polygon arrangement exteriors where there is only one polygon in the arrangement. Thus, in this section, we study the results on polygons with holes, polygon exteriors, and exteriors of arrangements.

Let $g(n,h)$ be the minimum number of point guards necessary for any polygon with n vertices and h holes. O'Rourke gave a proof that $g(n,h) \leq \lfloor (n+2h)/3 \rfloor$. Shermer established that $g(n,h) \geq \lfloor (n+h)/3 \rfloor$, and conjectures that this is the tight bound. He was able to prove this for $h = 1$, showing that $g(n,1) = \lfloor (n+1)/3 \rfloor$. These results are all found in [91]. All of the results above hold for vertex guards as well, and it is believed that the bounds on vertex guards are the same as those for point guards.

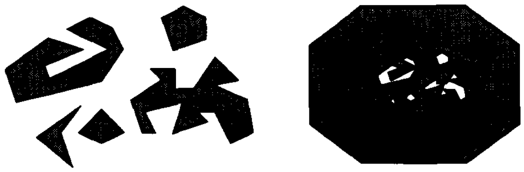

Fig. 12. Polygon arrangements are similar to polygons with holes.

Fig. 13. {Orthogonal polygons requiring $\lfloor 2n/7 \rfloor$ vertex guards.

Let $orth(n, h)$ be the minimum number of point guards necessary for any orthogonal polygon with n vertices and h orthogonal holes. Also, let $orth^V(n, h)$ be the minimum number of vertex guards in the same situation; $orth(n, h) \leq orth^V(n, h)$. O'Rourke's method extends to show that $orth^V(n, h) \leq \lfloor (n + 2h)/4 \rfloor$. O'Rourke conjectured that $orth(n, h)$ was independent of h: $orth(n, h) = \lfloor n/4 \rfloor$ [91], which Aggarwal established for $h = 1$ and $h = 2$ [2]. Hoffmann recently proved the full conjecture [58]. He proceeds by showing that any orthogonal polygon with holes can be divided into $\lfloor n/4 \rfloor$ orthogonal star-shaped polygons, each with at most 16 vertices. This is the only general tight bound known on any type of guarding in polygons with holes. Hoffmann also claims an $O(n^{1.5} \log^2 n \log \log n)$ guard placement algorithm.

Hoffmann also partially addresses the vertex guard problem, considering $orth^V(n, \cdot)$, the maximum of $orth^V(n, h)$ over all h. Using the polygons illustrated in Fig. 13, he establishes that $orth^V(n, \cdot) \geq \lfloor 2n/7 \rfloor$. This disproves an earlier conjecture of Aggarwal that $orth^V(n, \cdot) = \lfloor 3n/11 \rfloor$.

Czyzowicz *et al.* consider the problem of guarding *rectangular art galleries* [29]. A rectangular art gallery is formed by subdividing a rectangle into any number of smaller aligned rectangles (rooms). Any two rooms that share an edge have a door between them. A rectangular art gallery is shown in Fig. 14. Guards are allowed to stand either in a room or at a door between two rooms. Czyzowicz *et al.* proved that, in rectangular art galleries with r rooms, the maximum number of guards needed is $\lceil r/2 \rceil$. This many guards are needed by the rectangular gallery of r rooms constructed by dividing the outer rectangle by only vertical line segments. Their result extends to the case where the outer shape is any orthogonal polygon of n vertices: $\lceil (r + \frac{n}{2} - 2)/2 \rceil = \lceil (2r + n - 4)/4 \rceil$ guards are required. If the outer orthogonal polygon has h holes and n vertices, $\lceil (2r + n - 2h - 4)/2 \rceil$ guards are required.

Fejes Tóth considered the problem of guarding an arrangement of convex sets [46]. He found that, aside from a few small n, $4n - 7$ point guards were sufficient for any arrangement of n pairwise-*noncrossing* convex sets. Convex sets A and B are said to cross if $A \setminus B$ or $B \setminus A$ is disconnected (see Fig. 15). Examples of arrangements

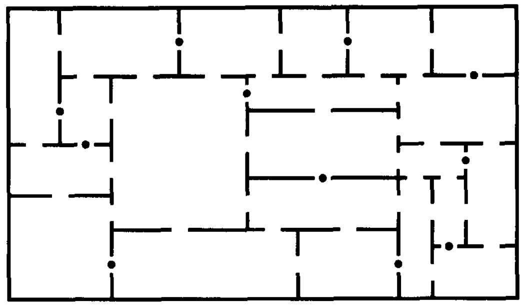

Fig. 14. A rectangular art gallery with guards.

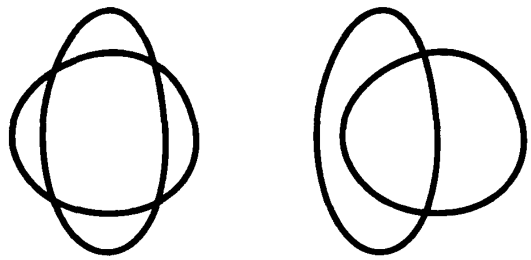

Fig. 15. Crossing and noncrossing convex sets.

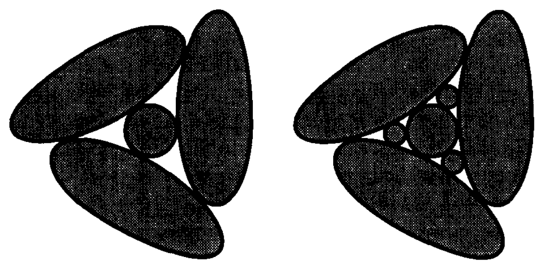

Fig. 16. Arrangements requiring $4n - 7$ guards.

requiring $4n - 7$ guards are shown in Fig, 16; each "triangular" region in the exterior of each arrangement is shaped so as to require two guards. Recently, Urrutia and Zaks rediscovered this result [119], and obtained bounds for this problem in higher dimensions.

Fejes Tóth also considered arrangements of circles. As any pair of circles is noncrossing, the $4n - 7$ upper bound on point guards holds. However, no circle arrangments require this many guards. In fact, Fejes Tóth was able to show that no circle arrangements require more than $2n$ point guards, and that the only arrangements requiring this many are those in which the circles are centered on a line, with consecutive circles just touching, as in Fig. 17. Any other arrangements (of more than two circles) require at most $2n - 2$ point guards. This result was rediscovered (for pairwise-disjoint circles) independently by Czyzowicz *et al.* and Coullard *et al.* [31], [23]. Although these results, and the ones for convex sets, are for guarding the boundary of the arrangement, the same proofs hold for guarding the entire exterior.

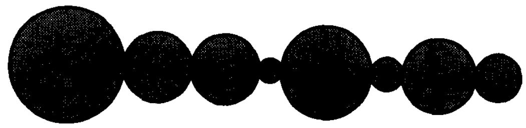

Fig. 17. An arrangementof circles requiring $2n$ point guards.

Czyzowicz, Rivera-Campo, and Urrutia have also investigated arrangements of triangles or rectangles [30]. They show that $\lfloor(4n+4)/3\rfloor$ point guards are sufficient to guard any arrangement of n disjoint triangles. However, this bound does not seem to be tight, and they conjecture that there is some constant c for which every arrangement requires only $n+c$ guards. In the special case of disjoint *homothetic* triangles (triangles whose edges form three sets of parallel segments), they obtain such a bound, showing that $n+1$ guards suffice. They also exhibit a class of arrangements, similar to Fig. 16 but consisting of triangles, that requires $n-1$ guards.

Their work on rectangle arrangements also yielded special-case and general results. The general result is that $\lfloor(4n+4)/3\rfloor$ point guards are sufficient to guard any arrangement of n disjoint isothetic rectangles. For the special case where the rectangles are restricted to all have the same width, they show that $n+2$ guards suffice. They also exhibit a class of equal-width rectangle arrangements requiring $n-1$ guards. Everett and Toussaint have improved the $n+2$ bound to n for the case of equal-size squares and have given an $O(n \log n)$ placement algorithm for this many guards [45].

Many people have studied art gallery questions on arrangments of disjoint line segments. For this problem, O'Rourke proved a tight bound of $\lfloor 2n/3\rfloor$ for point guards, and Boenke and Shermer established a tight bound of n for vertex guards [91]. Recently, attention has been placed on seeing the "boundary" of the arrangement (all of the points on the segments) rather than the entire plane. This leads to interesting variations, because segments, unlike strictly convex sets, need only be seen from one side (as in Fig. 18). For this type of guarding, Czyzowicz *et al.* show an upper bound of $\lceil 2n/3\rceil - 3$ for point guards, which they improve to $\lceil(n+1)/2\rceil$ when the segments are all either vertical or horizontal [31], [33]. Also, Lenhart and Jennings obtain a tight bound of $\lfloor n/2\rfloor$ when the guards are allowed to patrol entire segments of the arrangement [72], [62].

Czyzowicz *et al.* have also studied another notion of guarding for arrangements, suggested by Santoro. We will call an arrangement protected by a set of guards if each set of the arrangement contains a point that is seen by some guard (Czyzowicz *et al.* use the term *guarded* for this, and use *illumination* for our notion of guarding). Thus, if any set is removed, some guard would notice it. Czyzowicz *et al.* have proved that any arrangement of n line segments can be protected by $\lceil n/2\rceil$ points, and that there are such arrangements requiring $\lfloor(2n-3)/5\rfloor$ point guards to protect [33]. They have also shown the following upper bounds for protection using point guards: $\lfloor 2n/3\rfloor$ for disjoint convex sets, $\lceil n/2\rceil$ for disjoint isothetic rectangles, $\lceil 4n/7\rceil$ for disjoint triangles, and $\lceil n/2\rceil$ for homothetic triangles [32]. The rectangle and homothetic triangle bounds are almost tight; there are examples for both that require $\lfloor n/2\rfloor$ guards.

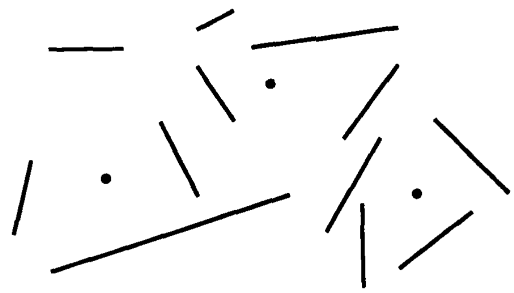

Fig. 18. A set of points guarding the boundary of an arrangement of segments.

We now turn our attention to visibility outside of a single polygon, and a variation where we consider visibility both inside and outside of a polygon. The art gallery problem for the exterior of a single polygon has been called the fortress problem: the polygon represents a fortress, and we wish to see any enemy that approaches. Similarly, when one wishes to simultaneously see both the inside and the outside of a polygon, then the problem is called the prison-yard problem: one must watch both for people on the outside trying to break in and for people on the inside trying to break out. These problems were suggested by Malkelvitch and Wood.

The outsides of polygons are much the same as the insides of polygons, and thus the fortress problem is similar to the art gallery problem. Aggarwal and O'Rourke showed the maximum number of point guards necessary for the fortress problem is $\lceil n/3\rceil$, and Shermer showed that all but two of these guards can be located at the polygon vertices [91]. If all guards must be located at vertices, O'Rourke and Wood showed that $\lceil n/2\rceil$ is the maximum necessary (convex polygons require this many) [91].

All of the above results proceed by "turning the polygon inside out." One method of doing this involves placing points l and r below and (respectively) to the left and right of the polygon and then connecting these points to each other and to a, the highest point on the polygon. Then the vertex a is split in two to give a new polygon whose interior is approximately the exterior of the original polygon (see Fig. 19; cf. Fig. 12). If the original polygon had n vertices, the new polygon has $n+3$ vertices. This construction can be used to convert many interior (art gallery) bounds to exterior (fortress) bounds—interior bounds of $f(n)$ become exterior bounds of $f(n+3)$. This procedure does not work for guards restricted to geometric features of the original polygon, as the new polygon has vertices and edges that are not vertices or edges of the original polygon. None of the bounds given by this procedure are known to be tight.

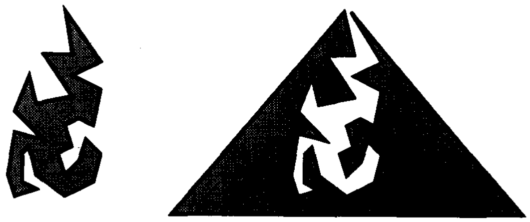

Fig. 19. Turning a polygon inside out.

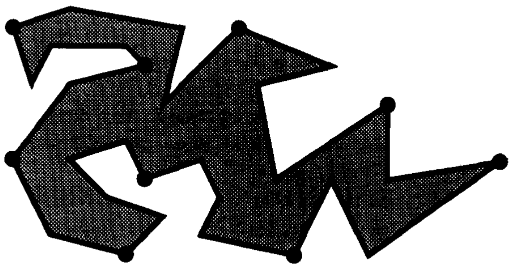

Fig. 20. A prison yard: guards see both interior and exterior.

We end this section with one of the most exciting new developments in art galleries. Recently, Kleitman and Füredi proved the celebrated conjecture of O'Rourke that the maximum number of vertex guards necessary for a prison yard is $\lceil n/2 \rceil$ [49]. They also show a bound of $\lfloor n/2 \rfloor$ for nonconvex polygons. The $\lceil n/2 \rceil$ bound is interesting in that it is the same as the vertex-guard bound for the fortress problem; in some sense you get the inside guarded for free. A polygon with a set of prison-yard guards is shown in Fig. 20.

Conjecture 3: $g(n,h) = g^V(n,h) = \lfloor (n+h)/3 \rfloor$.

Conjecture 4: $orth^V(n,h) = \lfloor (n+h)/4 \rfloor$.

Conjecture 5 (Hoffman): $orth^V(n,\cdot) = \lfloor 2n/7 \rfloor$.

Conjecture 6 (Czyzowicz, Rivera-Campo, and Urrutia): There is a constant c such that $n + c$ point guards are sufficient to guard every arrangement of n disjoint triangles.

Conjecture 7 (Czyzowicz, Rivera-Campo, Urrutia, and Zaks): There is a constant c such that $n/2 + c$ point guards are sufficient to see the entire "boundary" of any arrangement of n disjoint line segments.

Open Problem 7: Find tight bounds on the number of guards required to protect any arrangement of n disjoint line segments.

V. Visibility Graphs

In 1983, Avis and ElGindy introduced the concept of *visibility graphs* [5]. The visibility graph of a polygon is the graph of the visibility relation on the vertices of P. More formally, the visibility graph $VG(P)$ of a polygon

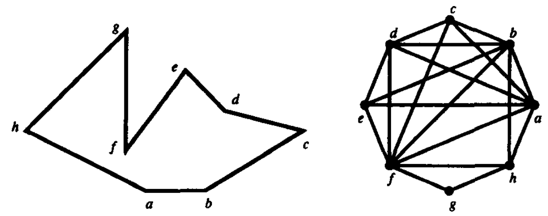

Fig. 21. A polygon and its visibility graph.

P is defined as:

$$
\begin{aligned}
VG(P) &= (V, E), \quad \text{where} \\
V &= \{v \mid v \text{ is a vertex of } P\} \\
E &= \{(u,v) \mid \text{ the vertices } u \text{ and } v \text{ are visible in } P\}.
\end{aligned}
$$

A polygon and its visibility graph are shown in Fig. 21.

There are three fundamental problems concerning visibility graphs: computation, characterization or recognition, and reconstruction. The only one of these problems satisfactorily solved is the problem of visibility graph computation: given a polygon P, compute $VG(P)$. Avis and ElGindy originally gave an $O(n^2)$ algorithm for this problem [5], [40], and more recently Hershberger presented an $O(|E|)$ algorithm ($|E|$ being the number of edges in the resulting graph) [57]. The $O(|E|)$ algorithm, although having the same asymptotic time bound in the worst case ($|E|$ can be as large as $(n^2 - n)/2$), represents a significant improvement over the $O(n^2)$ algorithm in many cases ($|E|$ can be as small as $2n - 2$).

The visibility graph characterization problemis to find a set of graph-theory properties that exactly define visibility graphs of polygons. Such a characterization would almost certainly lead to an algorithm for the visibility graph recognition problem: given a graph G, is G the visibility graph of some polygon? Unfortunately, visibility graphs have defied characterization. However, some partial characterizations have been obtained. Ghosh gave a set of necessary conditions for visibility graphs [52], and Everett later showed that these can be checked for any graph in $O(n^3)$ time [42]. However, Everett also showed that Ghosh's conditions are not sufficient to characterize visibility graphs. Other necessary conditions for visibility graphs have been given by Coullard and Lubiw [24] and Vinay and Veni Madhavan [121].

Everett and Corneil have tried several other approaches to the visibility graph characterization problem. One of these approaches, currently a popular means of graph characterization, is to find a complete set of forbidden minors of a graph. However, this approach does not apply to visibility graphs, as they can contain large cliques, and thus have no forbidden minors. Everett and Corneil instead concentrate on forbidden *induced* subgraphs [44]. (An induced subgraph of a graph G is formed by taking *any* subset of the vertices of G, and then *all* edges of G whose endpoints are both in the selected vertex set. Ordinary subgraphs may omit some of the edges of G between selected vertices.) They showed that the Grötsch graph (the graph of Fig. 22) is a forbidden

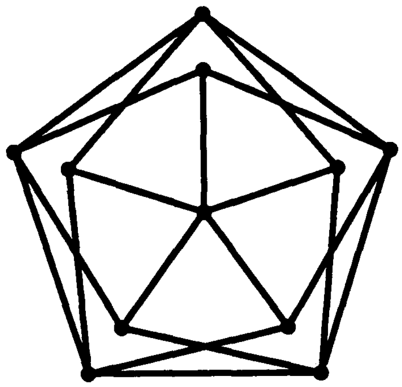

Fig. 22. A forbidden induced subgraph forvisibility graphs.

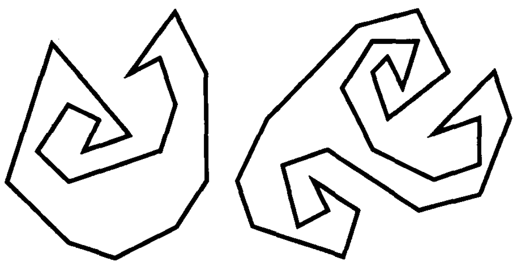

Fig. 23. A spiral and a 2-spiral.

induced subgraph of visibility graphs, and extended this graph to an infinite family of minimal forbidden graphs, quashing hopes for a characterization of visibility graphs via forbidden induced subgraphs. However, they continued examining different known classes of graphs for forbidden subgraphs of visibility graphs. They showed that no tree, or any convex bipartite graph, is forbidden. ElGindy had previously shown that no maximal outer planar graph (polygon triangulation graph) is forbidden [39]. Everett and Corneil also found a chordal forbidden graph, and Shermer found a bipartite forbidden graph [109].

Another approach to characterization of visibility graphs of polygons is to try to characterize visibility graphs of restricted classes of polygons. Everett and Corneil have characterized the visibility graphs of *spiral* and partially characterized *2-spiral* polygons [43]. A spiral polygon is a polygon whose reflex vertices form a single chain on the polygon boundary, while a 2-spiral polygon's reflex vertices can be grouped into two chains (see Fig. 23). Convex polygons (0-spirals) have visibility graphs that are cliques, and Everett and Corneil showed that the visibility graphs of spirals are *interval graphs*, and that the visibility graphs of 2-spirals are *perfect graphs*, provided that the strong perfect graph conjecture is true. They also supply a linear recognition algorithm for visibility graphs of spirals.

The reconstruction problem for a visibility graph is to find a polygon P such that $VG(P)$ is the given graph. This has also proved to be a tough problem, and thus several

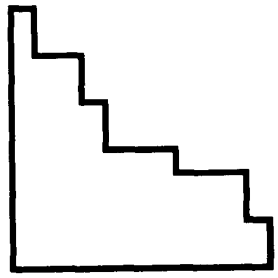

Fig. 24. A staircase polygon.

researchers have tried variations or special cases. ElGindy gave an $O(n \log n)$ reconstruction algorithm for polygons whose visibility graph is a triangulation graph [39], and Everett and Corneil gave a reconstruction algorithm for visibility graphs of spiral polygons [43]. Abello and Eğecioğlu studied the reconstruction of visibility graphs of *staircase polygons*: orthogonal polygons with only one bottom and one left edge (see Fig. 24) [1]. O'Rourke considered reconstructing only the convex hull of a polygon, given both its internal and external visibility graphs [92]. Most recently, the first general reconstruction result was obtained by Coullard and Lubiw, who presented an $O(|E|)$ algorithm for the reconstruction problem where the distances between each visible pair of vertices is given [24].

Visibility graphs have also been studied for objects other than polygons. Most notable are the visibility graph computation algorithms for arrangements of segments or polygons. Most recently, Overmars and Welzl gave an $O(|E| \log n)$ time algorithm for finding visibility graphs of arrangments of segments [94], Ghosh and Mount gave an optimal $O(|E| + n \log n)$ algorithm for polygon or segment arrangements [55], and Kapoor and Maheshwari have another segment algorithm with the same time complexity [64]. Visibility graphs have also been studied for arrangements of vertical line segments, where visibility is allowed only in the horizontal direction. A full characterization of such graphs, as well as a linear reconstruction algorithm, was given (independently) by Wismath [122] and Tamassia and Tollis [116].

Visibility graphs capture some, but not all, of the structure of visibility in polygons. The two most fundamental visibility properties that a polygon can have are convexity and a star shape. Given $VG(P)$, we can determine if P is convex. However, we cannot determine if P is star-shaped; Fig. 25 shows two polygons, one star-shaped and one not, with the same visibility graph.

A structure that does capture the full visibility structure of a polygon, at the expense of being infinite, is the *point visibility graph*. Point visibility graphs were introduced by Shermer as a unifying framework for visibility problems [106], [105]. Given a polygon P, the point visibility graph

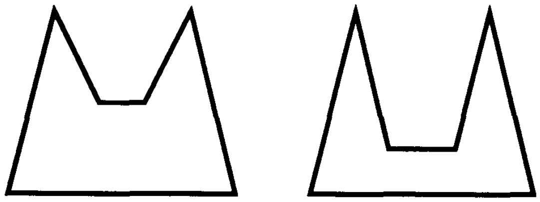

Fig. 25. Two polygons with the same visibility graph.

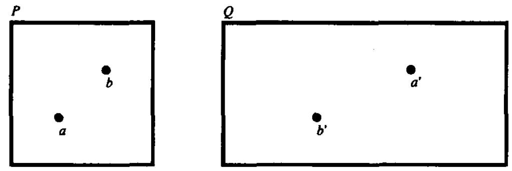

Fig. 26. A nonhomeomorphic polygon isomorphism.

$PVG(P)$ of P is defined as follows:

$$PVG(P) = (V, E), \text{ where}$$
$$V = \{p \mid p \text{ is a point in } P\}$$
$$E = \{(p,q) \mid \text{the points } p \text{ and } q \text{ are visible in} P\}.$$

PVG's are continuous graphs in the sense defined by Nash-Williams [84]. For any polygon P, $PVG(P)$ has an uncountable number of vertices, and each vertex has uncountable degree; thus, PVG's are unsuitable for direct use in computation.

However, many visibility problems on a polygon P can be expressed as graph theory problems on $PVG(P)$. For example, the art gallery problem for P is the dominating set problem for $PVG(P)$. Several other previously studied computational geometry problems (finding hidden sets, link distance, link center, link radius, link diameter) are graph problems on PVG's (independent sets, graph distance, center, radius, diameter). Motivated by this, we will use the term *pure visibility problem* to denote any problem on a polygon P that can be solved by mapping the problem onto $PVG(P)$, solving a graph problem, and mapping the solution back to P. In constrast, a *hybrid visibility problem* is one in which other information, such as metric or temporal information, is needed. Several hybrid visibility problems will be discussed in the next section.

Graph problems on PVG's (visibility problems on polygons) and the same problems on finite graphs tend to have the same complexity: either both are NP-hard or both are polynomial. For instance, finding a maximum hidden set is NP-hard, as is finding a maximum independent set in a finite graph; finding link distance in a polygon and finding graph distance in a finite graph are both polynomial.

In addition, PVG's provide a method for turning many graph-theory properties and problems into geometric properties and problems. One such interesting graph-theory property is *isomorphism* between two graphs. The analogous geometric property is *polygon isomorphism*: two polygons are isomorphic if they have a visibility-preserving one-to-one mapping between their points. Counterintuitively, it is possible that this mapping is not continuous, in the sense that points that are "next to" one another in one polygon may not be next to one another after the mapping (in topological terms, the isomorphism need not be a homeomorphism). Such an isomorphism is illustrated in Fig. 26; each point (x, y) of P is mapped to the point $(2x, y)$ of Q, except that the point $a = (x_a, y_a)$ of P is mapped to $a' = (2x_b, y_b)$ in Q and $b = (x_b, y_b)$ of P is mapped to $b' = (2x_a, y_a)$ of Q. This mapping is an isomorphism, as it is one-to-one, and visibility is preserved (every pair of points is visible in both polygons). However, points next to a in P do not map to points next to a' in Q, so the mapping is not a homeomorphism.

All convex polygons are isomorphic, as their PVG is a complete graph. Surprisingly, it is also true that all polygons with one reflex vertex are isomorphic. Shermer and Mac Donald have found an $O(n^2)$ algorithm to determine if two spiral polygons are isomorphic [111]. The complexity of detecting isomorphism of two simple polygons has not been determined; in fact, it is not even known if this problem is decidable.

No characterizations of point-visibility graphs are known. Paralleling the investigation of visibility graphs, Shermer has found forbidden induced subgraphs (including some bipartite graphs) for point visibility graphs [109]. Note that any graph forbidden for PVG's is also forbidden for visibility graphs, as any visibility graph is an induced subgraph of some PVG.

Open Problem 8: Find a characterization of visibility graphs that can be checked in polynomial time.

Open Problem 9: Given a visibility graph G and a Hamiltonian cycle of G, find a polynomial time algorithm to construct a polygon P with $VG(P) = G$ and the given cycle corresponding to the boundary of P.

Open Problem 10: Characterize PVG's of polygons.

Conjecture 8: Polygon isomorphism is not decidable.

VI. Hybrid Visibility Problems

As defined in the preceding section, a hybrid visibility problem is a problem that involves not only visibility, but also other geometric or conceptual properties. In this section, we review several such problems that have the same spirit as art gallery problems. The majority of these hybrid problems for which results are known are concerned with visibility and metric information: finding shortest paths from which polygons are covered.

Chin and Ntafos defined a watchman route for a polygon P as a closed walk in P such that every point of P is visible from some point of the walk [20]. The watchman route problem is then to find a shortest watchman route for a given polygon. A polygon with a shortest watchman route is shown in Fig. 27.

Chin and Ntafos proved that the watchman route problem is NP-hard if the given polygon has holes [20]. They also showed that it remains NP-hard if the polygon and holes are restricted to be convex, or orthogonal. In the same

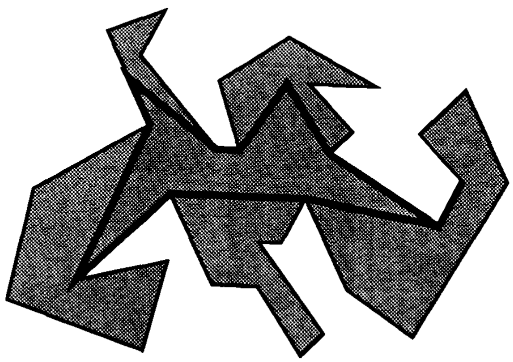

Fig. 27. A polygon with a shortest watchman route.

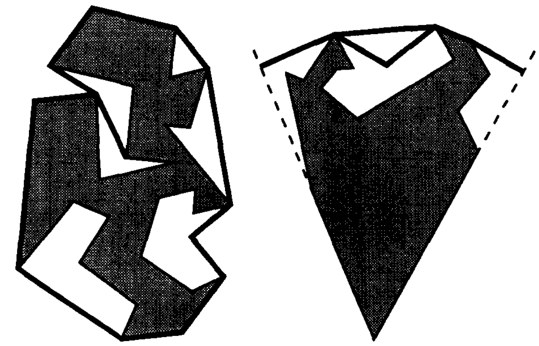

Fig. 28. The two types of external watchman routes.

paper, they give an $O(n)$ algorithm for finding a shortest watchman route in an orthogonal polygon (without holes). Their algorithm introduced to computational geometry a technique now common in the solution of similar problems: "unrolling" a polygon or triangulation by reflecting it about one or more of its edges. This is an extension of the classical method of finding the shortest path between two points, where the path must touch a given line.

Chin and Ntafos have also given an $O(n^4)$ algorithm for solving the watchman route problem in simple polygons [18]. However, the algorithm solves the problem only if given a starting point s that is required to be on the watchman route. This is not a serious limitation; a point on the minimum watchman route in most polygons can be found in $O(n)$ time. Furthermore, any polygon P with $g_1^1(P) > 1$ (P cannot be covered by one convex guard) has a minimum watchman route containing a reflex vertex of the polygon. Thus, if $g_1^1(P) > 1$, by using brute force we can find a minimum route without a given starting point in $O(n^5)$ time: apply the $O(n^4)$ algorithm $O(n)$ times (once with each reflex vertex as s). Only the case of $g_1^1(P) = 1$ (one convex guard suffices) remains to be solved to obtain a polynomial algorithm for the general problem. Related to this problem is the work of Ke, who gives an algorithm for finding the shortest segment from which a polygon is visible [66].

Ntafos and Gewali gave an $O(n^4)$ algorithm for the watchman route problem (without a given starting point) in the exterior of a polygon [88]. External watchman routes can take two forms: either they completely encircle the polygon or they don't (see Fig. 28).

Ntafos and Gewali show that the case where the polygon is encircled by the route can be easily handled with the aid of the internal watchman route algorithm. Their solution for other cases also uses the internal watchman route algorithm, but requires a much trickier analysis. They also show that external watchman routes for weakly externally visible polygons (a class including monotone and star-shaped) or orthogonal polygons can be found in $O(n)$ time.

Nilsson and Wood have investigated the problem of finding routes for *several* watchmen in spiral polygons: each point of the polygon must be visible to at least one watchman at some point of his route [85]. A polygon with routes for three watchmen is shown in Fig. 29. Nilsson and Wood show that they can find a minimum-length set of routes for m watchmen in spiral polygons in $O(n^2m)$ time and $O(n^2)$ space, using dynamic programming. They specify the algorithm for minimizing the sum of the lengths of the routes but essentially the same algorithm works for minimizing the length of the maximum-length route.

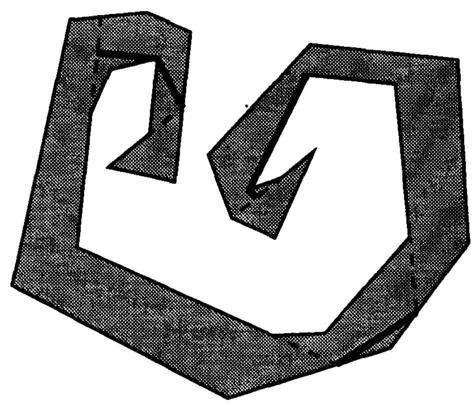

Fig. 29. Routes for three watchmen in a spiral polygon.

Ntafos has also considered a generalization of the watchman route problem called the robber route problem [86]. Given a polygon P, a starting point s on the boundary of P, a set of points (called threats) $T \subset P$, and a set of partial edges (called sights) S of P, a robber route for this situation is a path beginning and ending at s that sees all of the sights and is not seen by any of the threats. He solves this problem by first reducing it to a robber route problem with no threats, and from there to a watchman route problem. Assuming that both $|T|$ and $|S|$ are $O(n)$, each of the reductions takes less time than solving the resulting watchman route problem, and therefore Ntafos's algorithm solves this problem in $O(n^4)$ time for simple polygons and $O(n)$ time for orthogonal polygons.

A watchman route can be considered a route inside a polygon that visits the visibility polygons of each vertex. A generalization of this is to find a route in a polygon

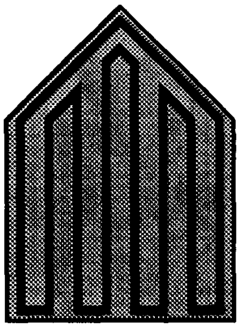

Fig. 30. A d-sweeper route in a polygon.

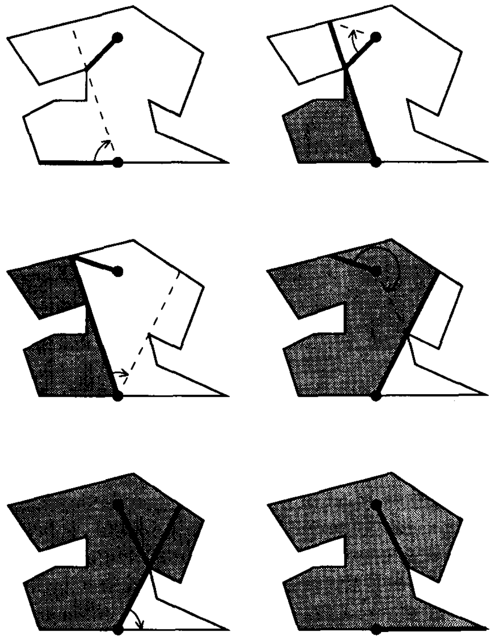

Fig. 31. A searchlight schedule.

that visits each of an arbitrary collection of subpolygons. Chin and Ntafos call such routes safari routes if they are allowed to enter the subpolygons, and zoo-keeper routes if they are restricted to touch only the boundaries of the subpolygons—the sets are envisioned as cages in which animals that need to be fed are kept. Chin and Ntafos studied the minimum-length zoo-keeper and safari route problems for convex subpolygons. They showed that, in general, both problems are NP-hard. However, if the cages are restricted to touch the boundary of the containing polygon, polynomial algorithms are possible. Let n denote the total number of sides in the polygon and the subpolygons, and k the number of subpolygons. Chin and Ntafos give an $O(n \log^k n)$ algorithm for the zoo-keeper route problem [19], and Ntafos gave an $O(n^3)$ algorithm for the safari route problem [87]. Czyzowicz *et al.* gave an $O(n)$ algorithm for the special case of these problems where the set of subpolygons to visit is the set of edges of the polygon [28]; this result was also obtained by Burkard, Rote, Yao, and Yu, who solved the problem of finding a shortest path visiting every edge of a polygonal path in d dimensions, for any $d \geq 2$ [13].

Ntafos used his safari route algorithm to obtain polynomial-time approximation algorithms for variations of the watchman route problem using d visibility [87]. Two points in a polygon are called d-visible if they are both visible (in the usual sense) and the distance between them is no greater than d. Ntafos considers both finding routes from which the boundary of the polygon is d-visible (the d-watchman route problem) and finding routes from which every point in the polygon is d-visible (the d-sweeper problem). A d-watchman route is much like a watchman route, roughly following the outline of the polygon. A d-sweeeper route, however, may need to wiggle around in the center of the polygon, as in Fig. 30.

Sugihara, Suzuki, and Yamashita have investigated a different type of hybrid visibility problem, called searchlight scheduling [114]. A *searchlight* is a type of guard that can see in only one direction at any one time, but may vary the direction of visibility continuously over time. Searchlights, like guards, cannot see through the exterior of the polygon. Consider a polygon containing several searchlights, with an intruder in the polygon who is allowed to move continuously with any speed. The searchlight scheduling problem for such an instance is to find a schedule for the (angular) motion of the searchlights so that the intruder, no matter what path he takes, is detected in a finite amount of time. The intruder is detected at a point in time if any of the visibility rays of the searchlights at that time contain the intruder's location. Figure 31 shows a polygon with two searchlights, and a searchlight schedule; in each subfigure, the dark area represents the area that the intruder cannot be in.

Sugihara, Suzuki, and Yamashita obtained necessary and sufficient conditions for a polygon with one or two searchlights to have a schedule. They also show a method to reduce any searchlight scheduling problem that has a searchlight on the polygon boundary to several simpler searchlight scheduling problems.

Another similar scheduling problem is *searching* in a polygon: schedule the motion of a set of point guards in a polygon to detect an intruder in finite time (the intruder behaves as in searchlight scheduling). Crass, Suzuki, and Yamashita have studied this problem for *k-searchers*, which are moving points that can see along k rays (which can rotate, like searchlights), and *∞-searchers*, which are moving points that have a 360° field of vision [25]. They have given some necessary conditions and some sufficient conditions for a polygon to be searchable by one k-searcher or ∞-searcher. Icking and Klein have given an algorithm to determine if a polygon can be searched by two guards that move along the boundary of the polygon, each keeping the other guard in view [60].

Open Problem 11: Find a polynomial-time algorithm to find a shortest watchman route in a simple polygon without a specified starting point.

Conjecture 9 (Chin and Ntafos): There exists an $O(n^2)$ time algorithm to find a shortest watchman route in a simple polygon, given a starting point s.

Open Problem 12: Find a polynomial-time algorithm to find a shortest watchman *path* (not route) in a polygon.

Open Problem 13: Characterize the three-searchlight instances of the searchlight scheduling problem that have a schedule.

Open Problem 14 (Crass, Suzuki, and Yamashita): What is the complexity of determining if a simple polygon is searchable by one searcher (k-searcher for a given k or ∞-searcher)?

Conjecture 10 (Crass, Suzuki, and Yamashita): Any polygon that is searchable by a single ∞-searcher is also searchable by a 2-searcher.

VII. Conclusion

Visibility plays an important role in many computing applications. We have reviewed the recent results in art gallery theorems, an area that deals with the combinatorial aspects of visibility. We have also reviewed many related visibility issues, both algorithmic and combinatorial, and presented several open problems in each major area surveyed.

Despite the many recent advances, such as the prisonyard theorem and the theorem for point guards in orthogonal polygons with holes, several fundamental questions remain unanswered: Can a characterization of visibility graphs be found? What is the art gallery theorem for general polygons with holes? What is the edge guard art gallery theorem? These questions, and other more numerous but less central ones, ensure that the current level of research in art galleries and related topics will continue. Ideally, a coherent theory of visibility will emerge, with a battery of techniques applicable to any practical visibility problems that arise.

Also, new variations of visibility are continually proposed and investigated. Recent work with alternative defintions of visibility and metavisibility include that of Breen [10], Dean, Lingas, and Sack [35], Estivill-Castro and Raman [41], Munro, Overmars, and Wood [83], Rawlins [97], and Schuierer, Rawlins, and Wood [104]. These new definitions bring up interesting "metaquestions" such as: Under which definitions of visibility does a certain theorem hold? What types of relations can reasonably be called a "visibility"? How does visibility theory relate to complexity theory, graph theory, and topology?

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