MAT-2201 2023: Second mandatory assignment

Hans Munthe-Kaas

October 5, 2023

Deadline: October 13 (you can hand in till October 15, 23:59).

Submission: You have to submit your answer in a *single* file. I accept PDF and Jupyter notebook. Nice handwriting is fine, but learning LaTeX is useful for anybody who want to write mathematics.

Score: All points count equally. You should have at least 70% score to pass.

Problem 1

Let ϵ be a positive real number and consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4+\epsilon \end{bmatrix}.$$

(a) Apply $PA = LU$ factorization (with partial pivoting) to the matrix A.

(b) Compute the infinity norm condition number of A.
What is $\text{cond}_\infty(A)$ when $\epsilon = 0.01$?

(c) Use the result from (a) to solve the system $Ax = b$ where

$$b = \begin{bmatrix} 2.9 \\ 6.2 \end{bmatrix}.$$

What is the solution when $\epsilon = 0.01$?

(d) Imagine that the numbers 2.9 and 6.2 in the vector b came from some measurement process, and that it turns out that a more precise measurement gives the vector

$$b = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

Repeat (c), but now with the new and improved vector b. What is the solution when $\epsilon = 0.01$?

(e) Now, denote the solution from (c) by x_a and the solution in (d) by x_0. We consider x_a as an approximate solution to the system from (d) with $\epsilon = 0.01$. To be specific, for the current problem, the system under consideration is $Ax = b$ with

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4.01 \end{bmatrix}, \qquad b = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

Compute the backward error $\|b - Ax_a\|$ and the forward error $\|x_0 - x_a\|$. Compute also the corresponding error magnification factor and compare with the condition number found in (b).

Problem 2

We will fit curves by Least Squares. To make this more fun, we use a real example, Hubbles Law for the expansion of the universe:

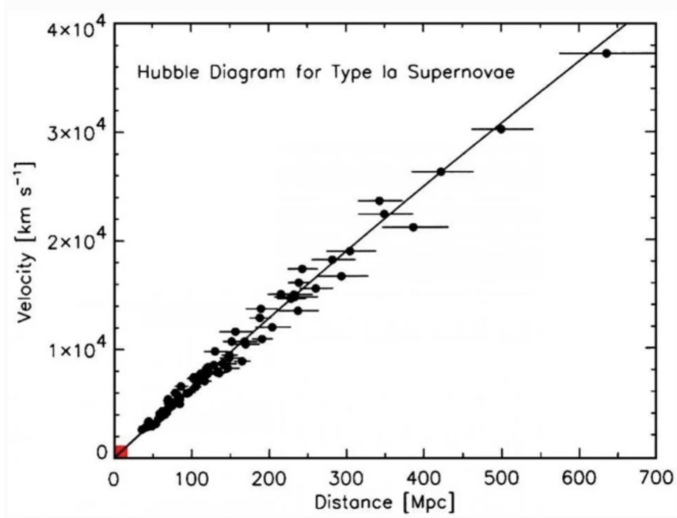
From the article on *Hubble's law* in Wikipedia: The law of expansion of the universe was predicted by Alexander Friedmann and discovered by Edwin Hubble in the 1920s. The great device for distance measures is 'Type 1A' supernovae, exploding stars having a very predictable energy which are used as 'cosmic standard candles' to measure distances. This gives a set of data (x_i, y_i) where x is distance from Earth and y is the velocity the object is moving away from us. Cosmologists are measuring distances in Megaparsecs (Mpc), which can be expressed in light years and kilometer as

$$1\text{Mpc} = 3.26 \cdot 10^6\text{ly} = 3.086 \cdot 10^{19} km.$$

The most distant observed galaxies have a *proper distance* of 9800Mpc, and they move away faster than the speed of light![1]

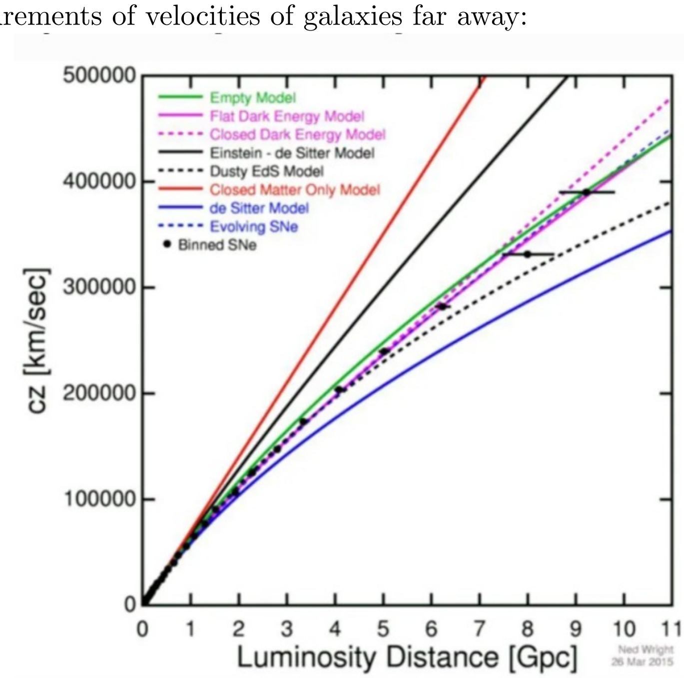
[1] Cosmological distance and speed is a tricky issue, 9800Mpc is 31 billion light years, which seems funny since the universe is 13.7 billion years old, but it is correct. The explanation has to do with the expansion of the universe, see: YouTube1 and YouTube2.

Measurements of velocities of galaxies in 'our region' of the universe:



Hubble conjectured that the data follows a straight line and the slope of this line is called the *Hubble constant*, H. Hubble thought the constant was $H \approx 500$(km/s)/Mpc. Current measurements indicate that $H \approx 70$(km/s)/Mpc. Note that H has dimension 1/time, and $1/H$ is called the *Hubble time*. In simple models $1/H$ is the age of the universe.

Measurements of velocities of galaxies far away:



For many years it was conjectured that the expansion of the universe should slow down with time, due to the gravitational pull. Thus one expected that far out (long time ago) the expansion parameter H should be larger than close to us. Cosmologists were shocked in 1998 when it was discovered that the opposite was the case, the expansion of the universe is accellerating!

For this exercise, we use the following 18 datapoints (1Gpc = 1000Mpc):

x (Gpc)	0.01	0.02	0.28	0.33	0.38	0.56	2.26	2.35	2.45
y (10^5km/s)	0.01	0.01	0.19	0.20	0.27	0.34	1.26	1.49	1.46
x (Gpc)	3.26	3.65	3.74	5.43	6.36	7.24	7.70	8.02	9.39
y (10^5km/s)	2.01	2.23	2.30	2.82	3.32	3.52	3.77	3.80	4.12

(a) Model 1: $y(x) = ax + b$. Fit this model to data with Least Squares. Compute the sum of squares (norm of residual). Find Hubbles constant as the slope of the curve.

(b) It is obvious that $y(0) = 0$. We should refine the model:
Model 2: $y(x) = ax$. Fit this model to data with Least Squares. Compute the sum of squares (norm of residual). Find Hubbles constant as the slope of the curve.

(c) To check if H is really a constant or not, we try to fit a quadratic curve with $y(0) = 0$.
Model 3: $y(x) = ax^2 + bx$. Fit this model to data with Least Squares. Compute the sum of squares (norm of residual). Find Hubbles 'constant' in our time, $H = y'(0)$. Find Hubbles 'constant' for the galaxies 9000 Mpc away.

(d) Model 4: $y(x) = ax^3 + bx^2 + cx$. Fit this model and compare with Model 2. Compute the sum of squares (norm of residual).

Problem 3

This is about divided differences and Newtons interpolation formula. In my lecture 21/9/23, I explained how to use use the divided difference table to compute the value of the interpolating polynomial $p(x)$ at arbitrary points by computing backwards in the difference table. I also explained how you can compute or incorporate information about (higher order) derivatives by repeating x-values, since

$$\begin{aligned} f[x_i] &= f(x_i) \\ f[x_i x_i] &= f'(x_i) \\ f[x_i x_i x_i] &= \frac{1}{2} f''(x_i) \\ f[x_i x_i x_i x_i] &= \frac{1}{6} f'''(x_i) \\ &\text{etc.} \end{aligned}$$

(a) Find the Newton form of the cubic (3'rd degree) polynomial $p(x)$ such that $p(0) = 0$, $p'(0) = 1$, $p(2) = 2$, $p'(2) = -1$.

(b) In the divided difference table in **(a)**, compute backwards to find $p(1)$ and $p'(1)$.

(c) You probably know the summation formula

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Well known is also the sum of squares formula

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Use divided differences to find a formula for

$$\sum_{i=1}^{n} i^3 = ?$$

Hint: the formula is a 4'th degree polynomial in n.

Problem 4

(a) Assume that Chebychev interpolation is used to find a fifth degree interpolating polynomial $Q_5(x)$ on the interval $[-1, 1]$ for the function $f(x) = e^x$. Use the interpolation error formula to find a worst-case estimate for the error $|e^x - Q_5(x)|$ that is valid for x throughout the interval $[-1, 1]$.

(b) Answer the same questions as in **(a)**, but for the interval $[0.6, 1.0]$.