4. Shared Graph Problems

Alice and Bob have to solve some problems on a graph $G = (V, E)$ with n vertices. Alice sees a subset of the edges $A \subseteq E$ and Bob sees the rest $B = E \backslash A$. They both see all the vertices of G and the unique vertex IDs from 1 to n. Observe that Alice and Bob do not like each other, so they want to communicate as little as possible. Help Alice and Bob to show the following:

Used notation (for notationally accurate reader): something like $f(n) = O(...)$ here means in rigorous notation $f(n) \in O(...)$). When something is deduced from $O(...)$ or $\Omega(...)$ in formulas, it means it works for any function set from this sets (e.g. $\frac{n^2}{n} = \frac{O(n^2)}{\Omega(n)} = O(n)$ means rigorously $\frac{n^2}{n} \in S, S = \{\text{functions f s. t. } f = h/g, h \in O(n^2), g \in \Omega(n)\}),$ and $S = O(n)$ as sets.

1. **They can compute the average degree of G with $O(\log n)$ communication complexity. [3]**

   Average degree $\bar{d} = \frac{2m}{n}$, where $m = |E| = |A| + |B|$ (edge counts to both ends). As Alice and Bob know n, they can then communicate their parts $\bar{d}_A = \frac{2|A|}{n}$ and $\bar{d}_B = \frac{2|B|}{n}$ and calculate result as a sum. Because $\bar{d}_A, \bar{d}_B \leq \frac{2m}{n} \leq \frac{n(n-1)}{n} = n - 1$, this communication can be done in $O(\log n)$ too.

2. **They can check whether G is connected with $O(n \log n)$ communication complexity. [5]**

   One of them, e.g. Alice, could send list of IDs of their vertices grouped by connected components (e.g. after IDs of one of components ended, there is 0 sent, and then IDs of the new component and so on). Each ID is coded in $\log n$, so there will be sent no more that $n \log n + n = O(n \log n)$.

   After Bob received this, he could check whether his edges connect components sent by Alice in such a way, that they all are connected (he can just iterate over his edges and merge components of Alice if his edge connects them, until all his edges end or all Alice components are connected in one).

   After that, in one bit ha can communicate result to Alice. The cost is still $O(n \log n)$.

3. **Show that the communication complexity for checking whether G is connected is at least $\Omega(n)$ by an appropriate reduction. [7]**

   Let's consider graphs constructed from skeleton G' (1). It's an appropriate (maximal and odd) number of connected pairs of vertices ($2\lfloor\frac{n-2}{4}\rfloor$ pairs, all remaining nodes are connected to the one additional last pair). For Ω–notation (from task description) we need to consider $n \to \infty$, so number of holes between pairs is $2\lfloor\frac{n-2}{4}\rfloor = \Omega(n)$

   Then except connecting pairs and the last clique edges, the graph G it could have also edges of two types (for Alice and Bob) between the consequent pairs of nodes (fig. 2).

   So, all these graphs with inserted in holes edges could be coded from the side of Alice and Bob as $2\lfloor\frac{n-2}{4}\rfloor$–bit binary strings, and the graph is connected if and only if, obviously, bit–wise disjunction of them is only–ones string
   (proof: $\rightarrow$: suppose have zero in some bit $\Rightarrow$ there both edges aren't present $\Rightarrow$ would be break in one of the holes.
   $\leftarrow$: we have at least one connection in every the hole $\Rightarrow$ corresp. bit is one and disjunction is one too).

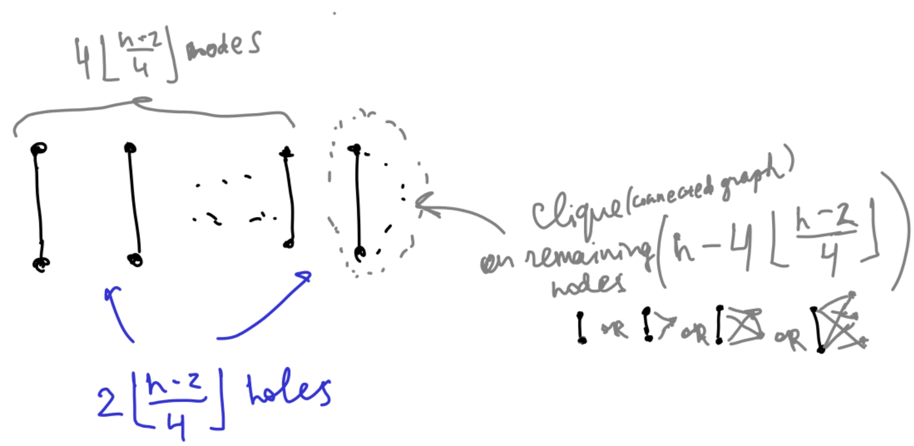

Figure 1: Skeleton G' of the graph

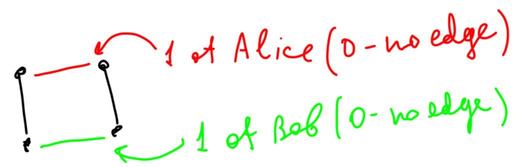

Figure 2: Edges of Alice and Bob.

This means, that our problem now is to understand if there are both zeros in some position of Alice and Bob strings.

Let's restrict ourselves to the arbitrary binary strings of length $2\lfloor\frac{n-2}{4}\rfloor$, which are concatenations of $\lfloor\frac{n-2}{4}\rfloor$–long x and $\bar{x}$ (i.e. $x \circ \bar{x}$). Denote Alica's string as x, Bob's - as $\bar{y}$ (bar is bit–wise negation). Number of such a strings is $2^{\lfloor\frac{n-2}{4}\rfloor}$ obviously.

Then we still need to know if $x \circ \bar{x}$ and $y \circ \bar{y}$ have simultaneous zeros in some position. But from Lemma 9.21 from lectures we know, that this is equivalent to knowing $x = y$. And for any x having $y = x$ and only for this will mean, that our graph (restricted to derived from skeleton G' and with antisymmetric edges (i.e. first x, then $\bar{x}$)) is connected. All these pairs create fooling set of the size $2^{\lfloor\frac{n-2}{4}\rfloor}$, and therefore from Lemma 9.18 the communication complexity is $\Omega(n)$.

Because we considered only subset of all possible graphs G and edges of Alice and Bob, and concluded, that even for this we need CC $\Omega(n)$, then for not lesser set of graphs and edges combinations the CC would be not less, because using as input the same data, we'll need at least the same CC, because Alice and Bob now will know less about the structure of graph and other side's edges (so to say restrictions deliver some additional information to Alice and Bob).

Now assume G is bipartite, that is, its nodes can be colored with 2 colors. Alice and Bob both know that G is bipartite.

4. **Show that they can find a valid 2-coloring of G with a communication complexity of $O(n \log n)$. [3]**

As in the subtask 2., Alice can send all her connected components, but now before sending she colors each of them in 2 colors and send first colored in 0, then in 1 (sending e.g. "$n+1$" after she sent all ones from the cluster colored in 1, then sending all colored in 1 and after that sends "0" showing that the cluster ended and e.g. "$n+2$" to show that her components ended).

Because we add only no more than n new signs-partitions "$n+1$", the overall communication complexity still will be $O(n \log n)$.

After receiving this, Bob goes through the components received from Alice, adding them to the whole graph (initially with empty set of edges). After adding the next component consequently, he adds his edges which are incident with vertices from the component, and colors not colored ends of this edge (if not colored) or inverts colors in one of the connected by edge components appropriately (if both components are colored already; after that these merged components are considered as one). If edge connects nodes from one component, nothing will change, as this component is already colored and edge can't ruin this coloring, as coloring of component is unique (accurate to the inversion).

Because the graph is bipartite, it's possible to color at each step and at each step the merged graph will be colored properly in 2 colors.

After all components from Alice are processed, Bob adds his remaining edges (if exist) and colors emerged his own components (which could be done as the graph is bipartite).

5. **Show that $\Omega(n)$ is a lower bound on the communication complexity by an appropriate reduction. [7]**

   Further N is the same as n.

   Similar to the subtask 3 we assume additionally that the skeleton (fig. 3) is known (groups of 6 nodes connected in 3 pairs each, the first and second nodes from all groups are connected (blue lines) in order to use reference color as a color (colors are in green) of the first (left–upper) node (as we can always invert colors, we use this reference for convenience to have the only one coloring, supposing w.l.o.g. that this reference node is colored to 0). All the other nodes (if remain, no more than 5; we need to consider $n \to \infty$ for communication cost order estimation) form full bipartite graph with one part from one node (only it if no nodes left).

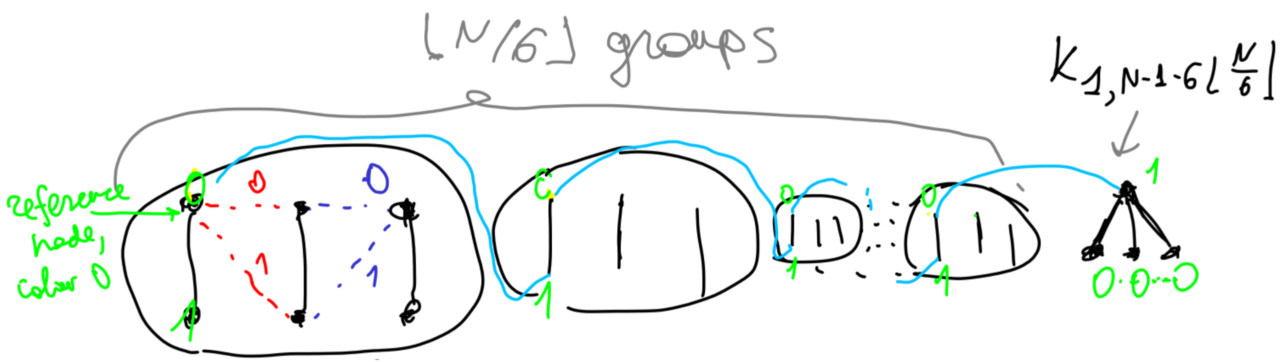

Figure 3: Skeleton G' of the graph

Alice and Bob both have $\lfloor N/6 \rfloor$–bit strings which code their edges inside groups as shown at the figure. At the next figure 4 we show proper colorings of these groups. There are only one coloring for 2–bit Alice-Bob combination, but $0-0$ and $1-1$ result in the same coloring, whereas any other will give different. So, we have presumably a fooling set, namely (x, x), where x is any bit–string of length $\lfloor N/6 \rfloor$. There are $2^{\lfloor N/6 \rfloor}$ such a strings, therefore $|S| = 2^{\lfloor N/6 \rfloor}$. So, if it's really a fooling set, communication cost would be from lecture Lemma 9.18 $\log |S| = \lfloor N/6 \rfloor = \Omega(N)$.

Let's prove, that it's a fooling set. We've shown, that $f((x, x))$ gets the upper coloring from fig. 4 for all 6–node groups. And this corresponds to bipartite coloring of the graph, which is unique for this graph (when fixing color of the reference node).

Let's assume we have (x, y) and (y, x), $y \neq x$. Then we will have different coloring, as in some bit they have different values. Therefore, coloring of at least one the groups will be as second or third at the fig- 4, therefore $f((x, y)) \neq f((x, x)) \neq f((y, x))$, but $f((x, x)) = f((y, y))$. So, it's really a fooling set. And we have needed estimate.

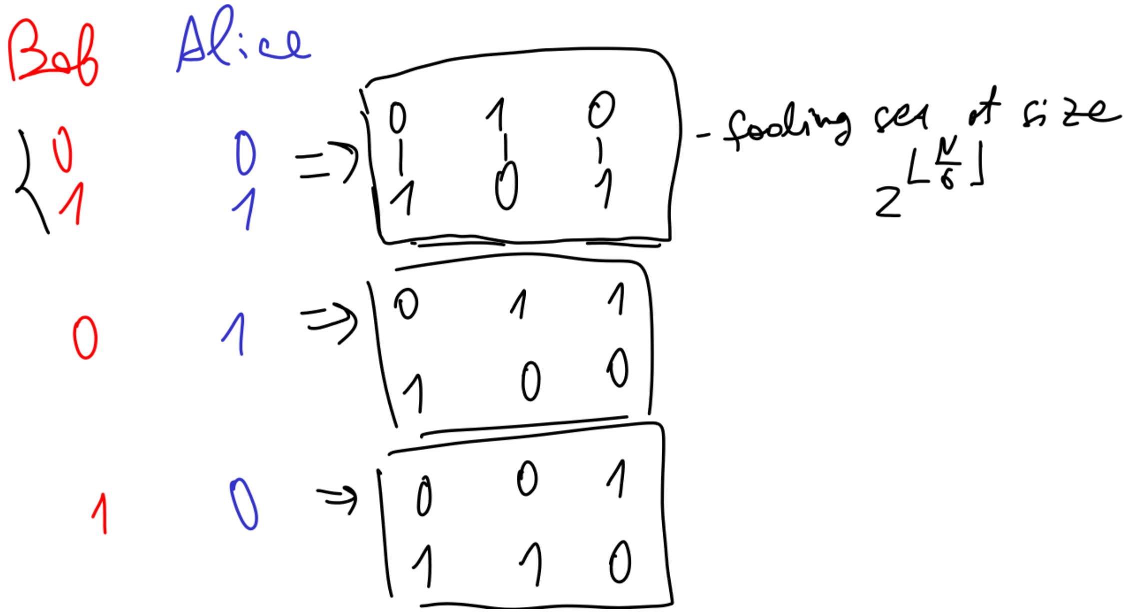

Figure 4: Possible colorings of the group depending on Alice and Bob's information

References