Camera models and calibration

Read tutorial chapter 2 and 3.1
http://www.cs.unc.edu/~marc/tutorial/

Szeliski's book pp.29-73

Schedule (tentative)

#	date	topic
1	Sep.19	Introduction and geometry
2	Sep.26	**Camera models and calibration**
3	Oct.3	Invariant features
4	Oct.10	Optical flow & Particle Filters
5	Oct.17	Model fitting (RANSAC, EM, …)
6	Oct.24	Multiple-view geometry
7	Oct.31	Image segmentation
8	Nov.7	Stereo Matching & MVS
9	Nov.14	Structure-from-Motion & SLAM
10	Nov.21	Specific object recognition
11	Nov.28	Shape from X
12	Dec.5	Object category recognition
13	Dec.12	Tracking
14	Dec.19	Research Overview & Lab tours

Brief geometry reminder

2D line-point coincidence relation: $\mathbf{l}^{\mathsf{T}}\mathbf{x} = 0$

Point from lines: $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$ Line from points: $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$

3D plane-point coincidence relation: $\pi^{\mathsf{T}}\mathbf{X} = 0$

Point from planes: $\begin{bmatrix} \pi_1^{\mathsf{T}} \\ \pi_2^{\mathsf{T}} \\ \pi_3^{\mathsf{T}} \end{bmatrix} \mathbf{X} = 0$ Plane from points: $\begin{bmatrix} \mathbf{X}_1^{\mathsf{T}} \\ \mathbf{X}_2^{\mathsf{T}} \\ \mathbf{X}_3^{\mathsf{T}} \end{bmatrix} \pi = 0$

3D line representation:
(as two planes or two points)

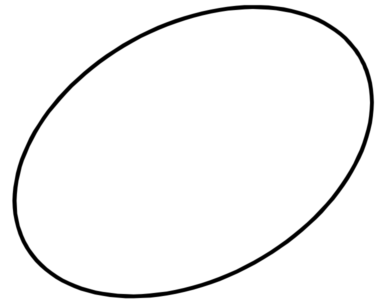
$$\begin{bmatrix} \mathrm{P}^{\mathsf{T}} \\ \mathrm{Q}^{\mathsf{T}} \end{bmatrix} [\mathrm{A}\ \mathrm{B}] = 0_{2\mathrm{x}2}$$

2D Ideal points $(x_1, x_2, 0)^{\mathsf{T}}$ 3D Ideal points $(X_1, X_2, X_3, 0)^{\mathsf{T}}$

2D line at infinity $\mathbf{l}_\infty = (0,0,1)^{\mathsf{T}}$ 3D plane at infinity $\Pi_\infty = (0,0,0,1)^{\mathsf{T}}$

Conics

Curve described by 2nd-degree equation in the plane



$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

or *homogenized* $x \mapsto x_1/x_3,\ y \mapsto x_2/x_3$

$$ax_1^{\ 2} + bx_1x_2 + cx_2^{\ 2} + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

or in matrix form

$$\mathrm{x}^{\mathsf{T}}\mathbf{C}\,\mathrm{x} = 0 \quad \text{with} \quad \mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

5DOF: $\{a : b : c : d : e : f\}$

Five points define a conic

For each point the conic passes through

$$ax_i^2 + bx_i y_i + cy_i^2 + dx_i + ey_i + f = 0$$

or

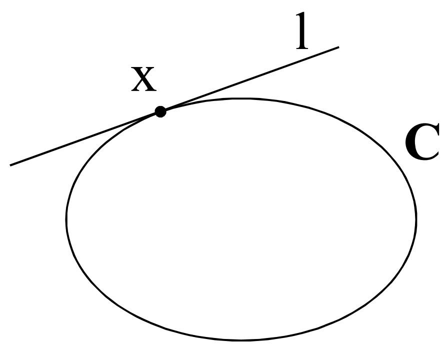
$$\left(x_i^2, x_i y_i, y_i^2, x_i, y_i, 1\right)\mathbf{c} = 0 \qquad \mathbf{c} = \left(a, b, c, d, e, f\right)^{\mathsf{T}}$$

stacking constraints yields

$$\begin{bmatrix} x_1^2 & x_1 y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3 y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4 y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5 y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \mathbf{c} = 0$$

Tangent lines to conics

The line l tangent to $\mathbf{C}$ at point x on $\mathbf{C}$ is given by $\mathrm{l}=\mathbf{C}\mathrm{x}$

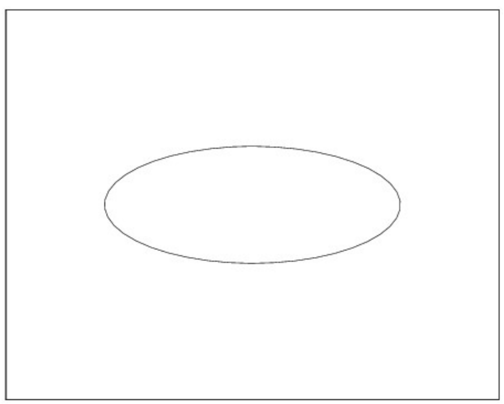


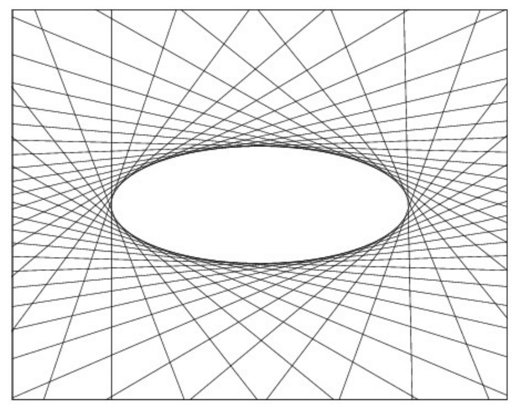
Dual conics

A line tangent to the conic $\mathbf{C}$ satisfies $\mathbf{l}^{\mathsf{T}}\mathbf{C}^{*}\mathbf{l}=0$

In general ($\mathbf{C}$ full rank): $\mathbf{C}^{*}=\mathbf{C}^{-1}$

Dual conics = line conics = conic envelopes

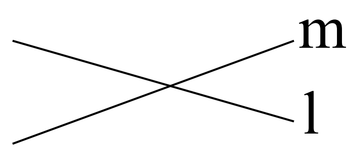




Degenerate conics

A conic is degenerate if matrix $\mathbf{C}$ is not of full rank

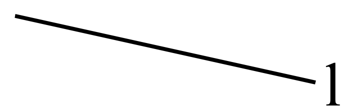
e.g. two lines (rank 2)



$$\mathbf{C} = \mathrm{l}\mathrm{m}^{\mathsf{T}} + \mathrm{m}\mathrm{l}^{\mathsf{T}}$$

e.g. repeated line (rank 1)

$$\mathbf{C} = \mathrm{l}\mathrm{l}^{\mathsf{T}}$$



Degenerate line conics: 2 points (rank 2), double point (rank1)

Note that for degenerate conics $\left(\mathbf{C}^{*}\right)^{*} \neq \mathbf{C}$

Quadrics and dual quadrics

$$X^T Q X = 0$$ (Q : 4x4 symmetric matrix)

$$Q = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \\ \circ & \circ & \circ & \bullet \end{bmatrix}$$

- 9 d.o.f.
- in general 9 points define quadric
- det Q=0 ↔ degenerate quadric
- tangent plane $\pi = QX$

$$\pi^T Q^* \pi = 0$$

- relation to quadric $Q^* = Q^{-1}$ (non-degenerate)

2D projective transformations

Definition:

A *projectivity* is an invertible mapping h from P^2 to itself such that three points x_1,x_2,x_3 lie on the same line if and only if $h(x_1),h(x_2),h(x_3)$ do.

Theorem:

A mapping $h:P^2 \rightarrow P^2$ is a projectivity if and only if there exist a non-singular 3x3 matrix $\mathbf{H}$ such that for any point in P^2 represented by a vector x it is true that $h(x)=\mathbf{H}x$

Definition: Projective transformation

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \quad x' = \mathbf{H}\,x$$

8DOF

projectivity=collineation=projective transformation=homography

Transformation of 2D points, lines and conics

For a point transformation

$$\mathbf{x}' = \mathbf{H}\,\mathbf{x}$$

Transformation for lines

$$\mathbf{l}' = \mathbf{H}^{-T}\,\mathbf{l}$$

Transformation for conics

$$\mathbf{C}' = \mathbf{H}^{-T}\mathbf{C}\mathbf{H}^{-1}$$

Transformation for dual conics

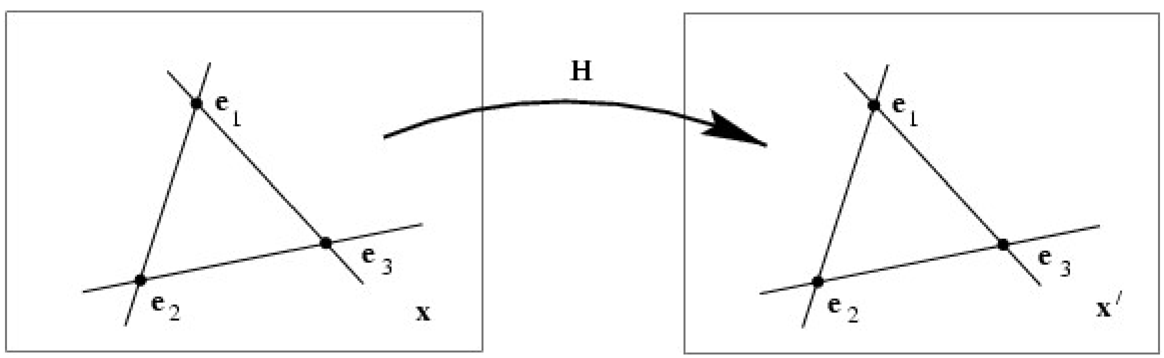
$$\mathbf{C}'^{*} = \mathbf{H}\mathbf{C}^{*}\mathbf{H}^{T}$$

Fixed points and lines

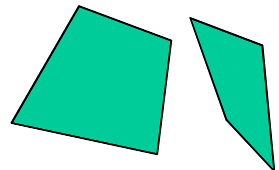
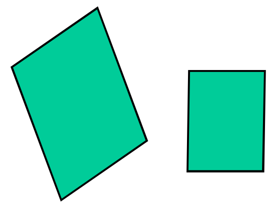
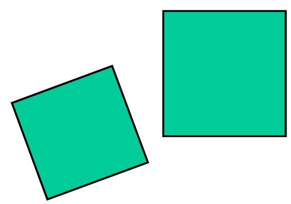
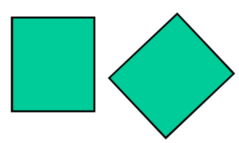
$$\mathbf{H}\,\mathrm{e} = \lambda\,\mathrm{e}$$ (eigenvectors $\mathbf{H}$ =fixed points)

(λ_1=$\lambda_2 \Rightarrow$ pointwise fixed line)

$$\mathbf{H}^{-\mathrm{T}}\,\mathrm{l} = \lambda\,\mathrm{l}$$ (eigenvectors $\mathbf{H}^{-\mathrm{T}}$ =fixed lines)



Hierarchy of 2D transformations

		transformed squares	invariants
Projective 8dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, order of contact (intersection, tangency, inflection, etc.), cross ratio
Affine 6dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallellism, ratio of areas, ratio of lengths on parallel lines (e.g midpoints), linear combinations of vectors (centroids). **The line at infinity $\mathbf{l}_\infty$**
Similarity 4dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratios of lengths, angles. **The circular points I,J**
Euclidean 3dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		lengths, areas.

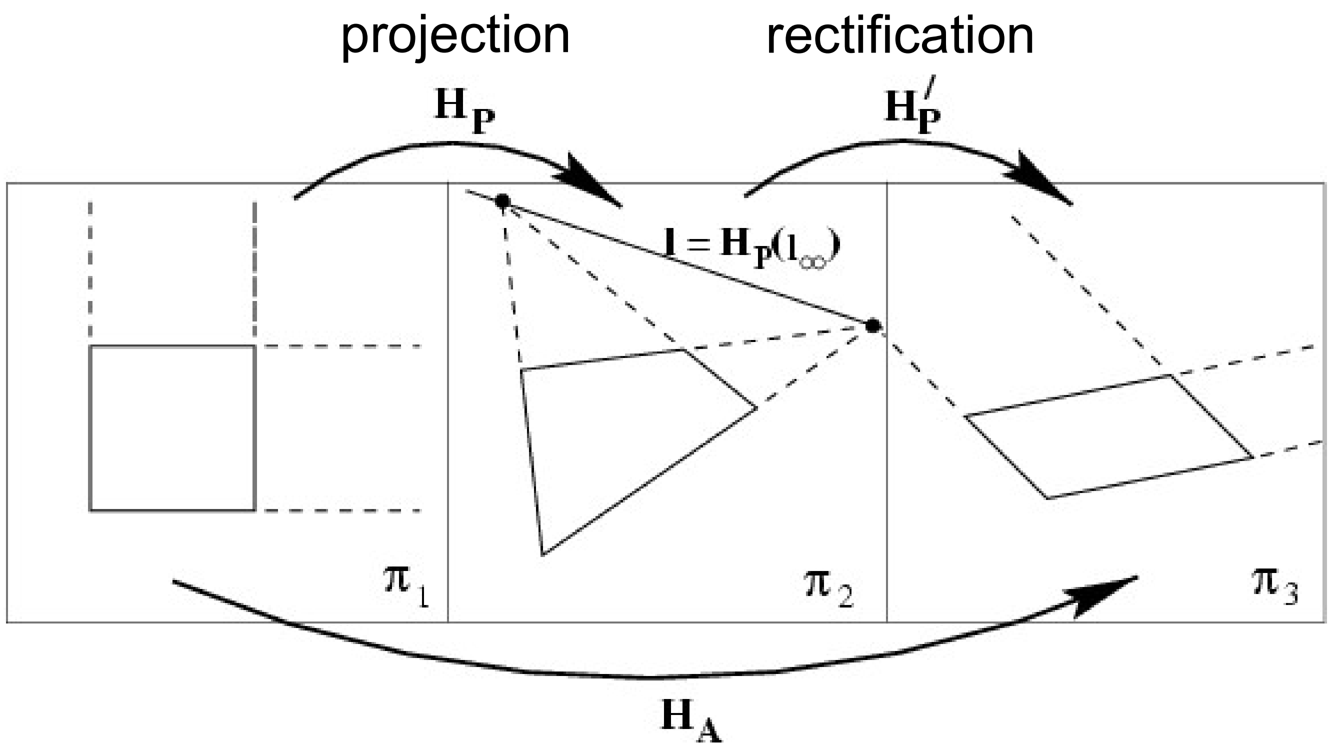
The line at infinity

$$\mathbf{l}'_\infty = \mathbf{H}_A^{-\mathsf{T}}\,\mathbf{l}_\infty = \begin{bmatrix} \mathbf{A}^{-\mathsf{T}} & 0 \\ -\mathbf{t}^{\mathsf{T}}\mathbf{A}^{-\mathsf{T}} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{l}_\infty$$

The line at infinity l_∞ is a fixed line under a projective transformation H if and only if H is an affinity

Note: not fixed pointwise

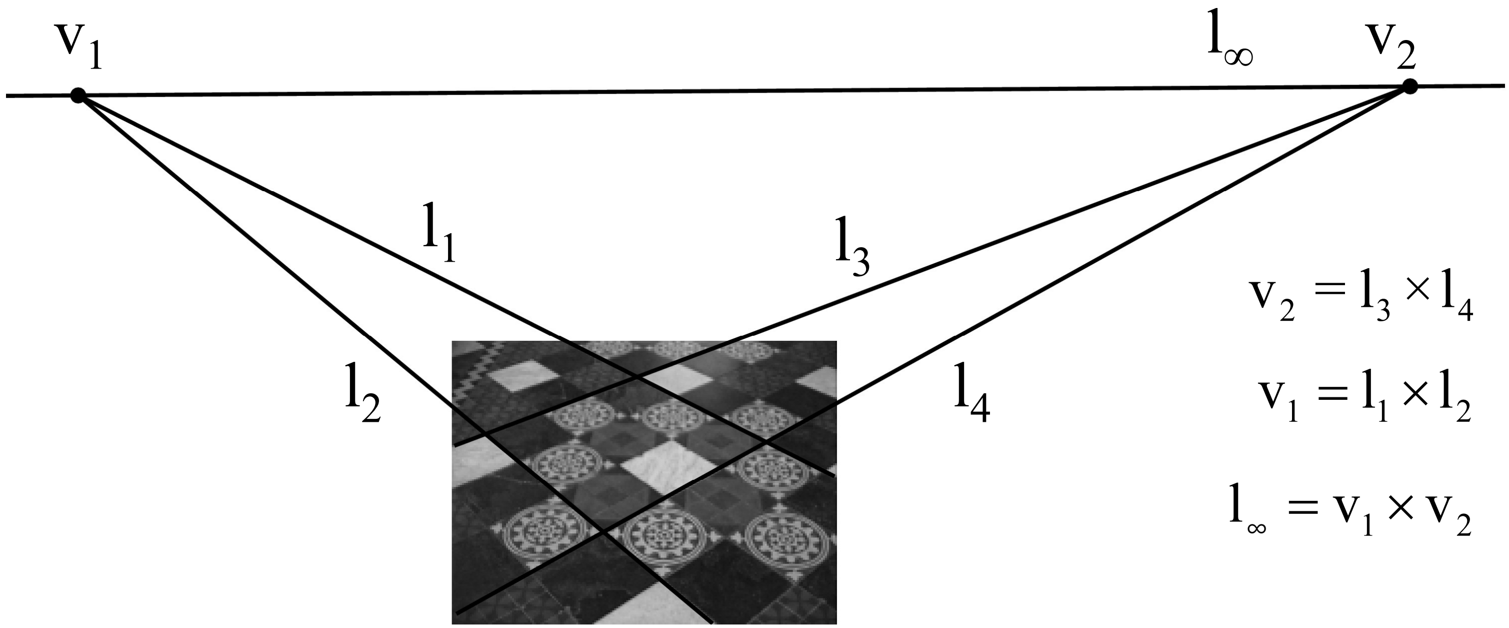
Affine properties from images



$$\mathbf{H'_P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$$

$$\mathbf{l}_\infty = [l_1 \quad l_2 \quad l_3]^T, l_3 \neq 0$$

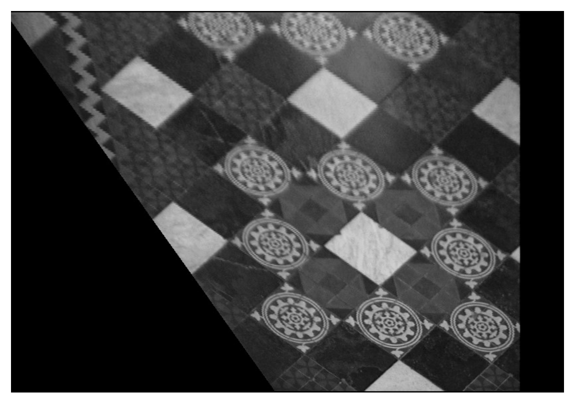
Affine rectification



$$v_2 = l_3 \times l_4$$

$$v_1 = l_1 \times l_2$$

$$l_\infty = v_1 \times v_2$$



The circular points

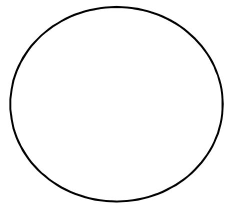
$$\mathrm{I} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \qquad \mathrm{J} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

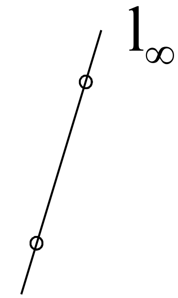
$$\mathrm{I}' = \mathbf{H}_S\, \mathrm{I} = \begin{bmatrix} s\cos\theta & s\sin\theta & t_x \\ -s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = se^{i\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \mathrm{I}$$

The circular points I, J are fixed points under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

The circular points

"circular points"





$$x_1^2 + x_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$
$$x_3 = 0$$

$$x_1^2 + x_2^2 = 0$$

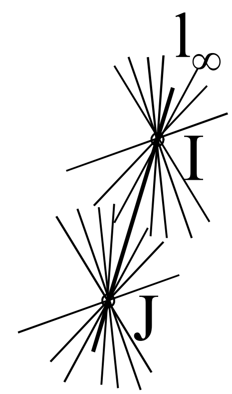
$$\mathrm{I} = (1, i, 0)^{\mathsf{T}}$$
$$\mathrm{J} = (1, \text{-}i, 0)^{\mathsf{T}}$$

Algebraically, encodes orthogonal directions

$$\mathrm{I} = (1,0,0)^{\mathsf{T}} + i(0,1,0)^{\mathsf{T}}$$

Conic dual to the circular points

$$\mathbf{C}^*_\infty = \mathrm{IJ}^\top + \mathrm{JI}^\top \qquad \mathbf{C}^*_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



$$\mathbf{C}^*_\infty = \mathbf{H}_S \mathbf{C}^*_\infty \mathbf{H}_S^\top$$

The dual conic $\mathbf{C}^*_\infty$ is fixed conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

Note: $\mathbf{C}^*_\infty$ has 4DOF

l_∞ is the nullvector

Angles

Euclidean: $\mathbf{l} = (l_1, l_2, l_3)^\mathsf{T} \qquad \mathbf{m} = (m_1, m_2, m_3)^\mathsf{T}$

$$\cos\theta = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}$$

Projective: $$\cos\theta = \frac{\mathbf{l}^\mathsf{T}\, \mathbf{C}_\infty^{*}\, \mathbf{m}}{\sqrt{(\mathbf{l}^\mathsf{T}\, \mathbf{C}_\infty^{*}\, \mathbf{l})(\mathbf{m}^\mathsf{T}\, \mathbf{C}_\infty^{*}\, \mathbf{m})}}$$

$\mathbf{l}^\mathsf{T}\, \mathbf{C}_\infty^{*}\, \mathbf{m} = 0$ (orthogonal)

Transformation of 3D points, planes and quadrics

For a point transformation (cfr. 2D equivalent)

$$X' = \mathbf{H}\,X \qquad (x' = \mathbf{H}\,x)$$

Transformation for planes

$$\pi' = \mathbf{H}^{-T}\,\pi \qquad (l' = \mathbf{H}^{-T}\,l)$$

Transformation for quadrics

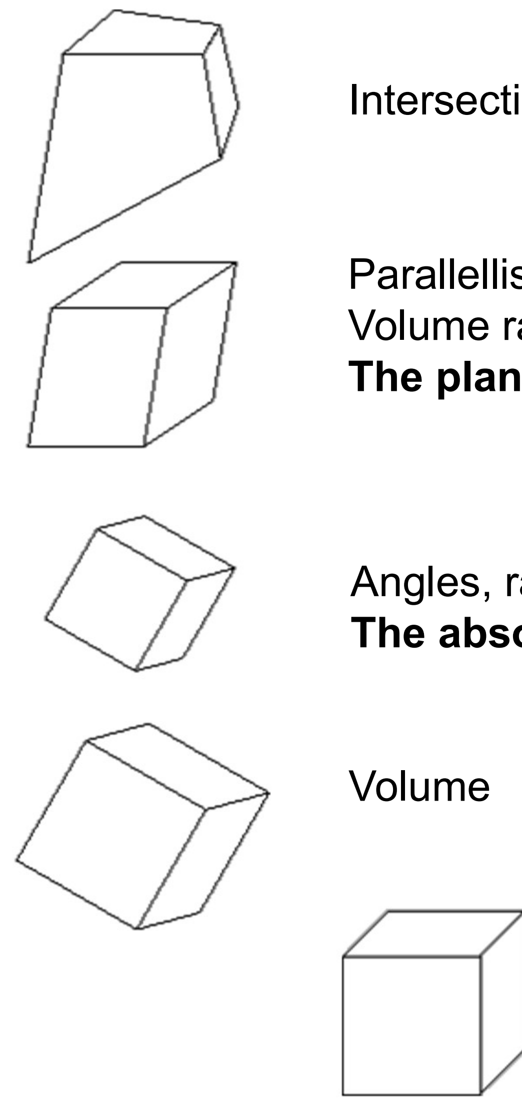
$$Q' = H^{-T} Q H^{-1} \qquad (\mathbf{C}' = \mathbf{H}^{-T}\mathbf{C}\mathbf{H}^{-1})$$

Transformation for dual quadrics

$$Q'^{*} = H Q^{*} H^{T} \qquad (\mathbf{C}'^{*} = \mathbf{H}\mathbf{C}^{*}\mathbf{H}^{T})$$

Hierarchy of 3D transformations

Projective 15dof	$\begin{bmatrix} A & t \\ v^T & v \end{bmatrix}$	Intersection and tangency
Affine 12dof	$\begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix}$	Parallellism of planes, Volume ratios, centroids, **The plane at infinity π_∞**
Similarity 7dof	$\begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix}$	Angles, ratios of length **The absolute conic Ω_∞**
Euclidean 6dof	$\begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$	Volume



The plane at infinity

$$\pi'_\infty = \mathbf{H}_A^{-\mathsf{T}} \pi_\infty = \begin{bmatrix} \mathbf{A}^{-\mathsf{T}} & 0 \\ -\mathrm{t}^{\mathsf{T}} \mathbf{A}^{-\mathsf{T}} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \pi_\infty$$

The plane at infinity π_∞ is a fixed plane under a projective transformation H iff H is an affinity

1. canonical position $\pi_\infty = (0,0,0,1)^{\mathsf{T}}$
2. contains directions $\mathrm{D} = (X_1, X_2, X_3, 0)^{\mathsf{T}}$
3. two planes are parallel $\Leftrightarrow$ line of intersection in π_∞
4. line // line (or plane) $\Leftrightarrow$ point of intersection in π_∞

The absolute conic

The absolute conic Ω_∞ is a (point) conic on π_∞.

In a metric frame:

$$\left.\begin{array}{c} X_1^2 + X_2^2 + X_3^2 \\ X_4 \end{array}\right\} = 0$$

or conic for directions: $(X_1, X_2, X_3)\mathrm{I}(X_1, X_2, X_3)^\mathsf{T}$

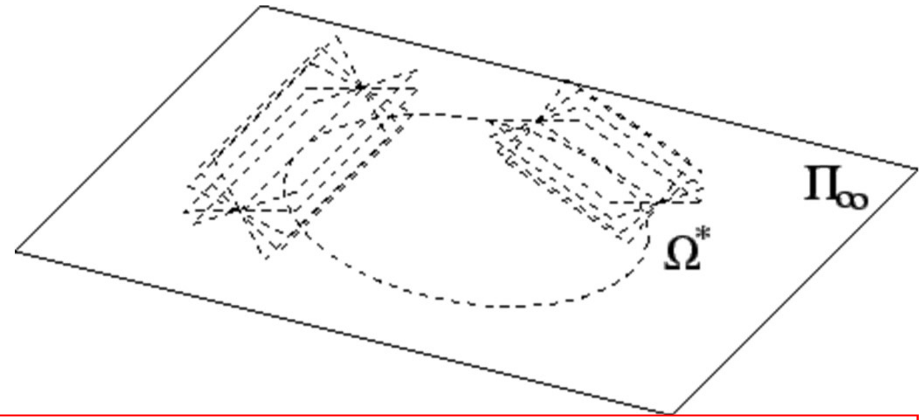
(with no real points)

> The absolute conic Ω_∞ is a fixed conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

1. Ω_∞ is only fixed as a set
2. Circle intersect Ω_∞ in two circular points
3. Spheres intersect π_∞ in Ω_∞

The absolute dual quadric

$$\Omega^*_\infty = \begin{bmatrix} \mathrm{I} & 0 \\ 0^\mathsf{T} & 0 \end{bmatrix}$$



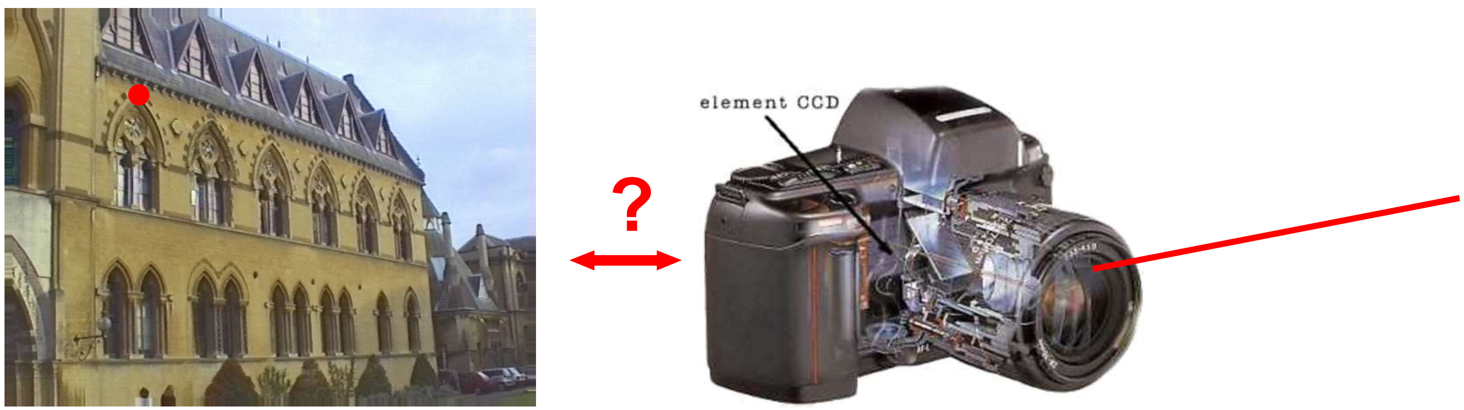
The absolute dual quadric Ω^*_∞ is a fixed conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

1. 8 dof
2. plane at infinity π_∞ is the nullvector of Ω_∞
3. Angles:

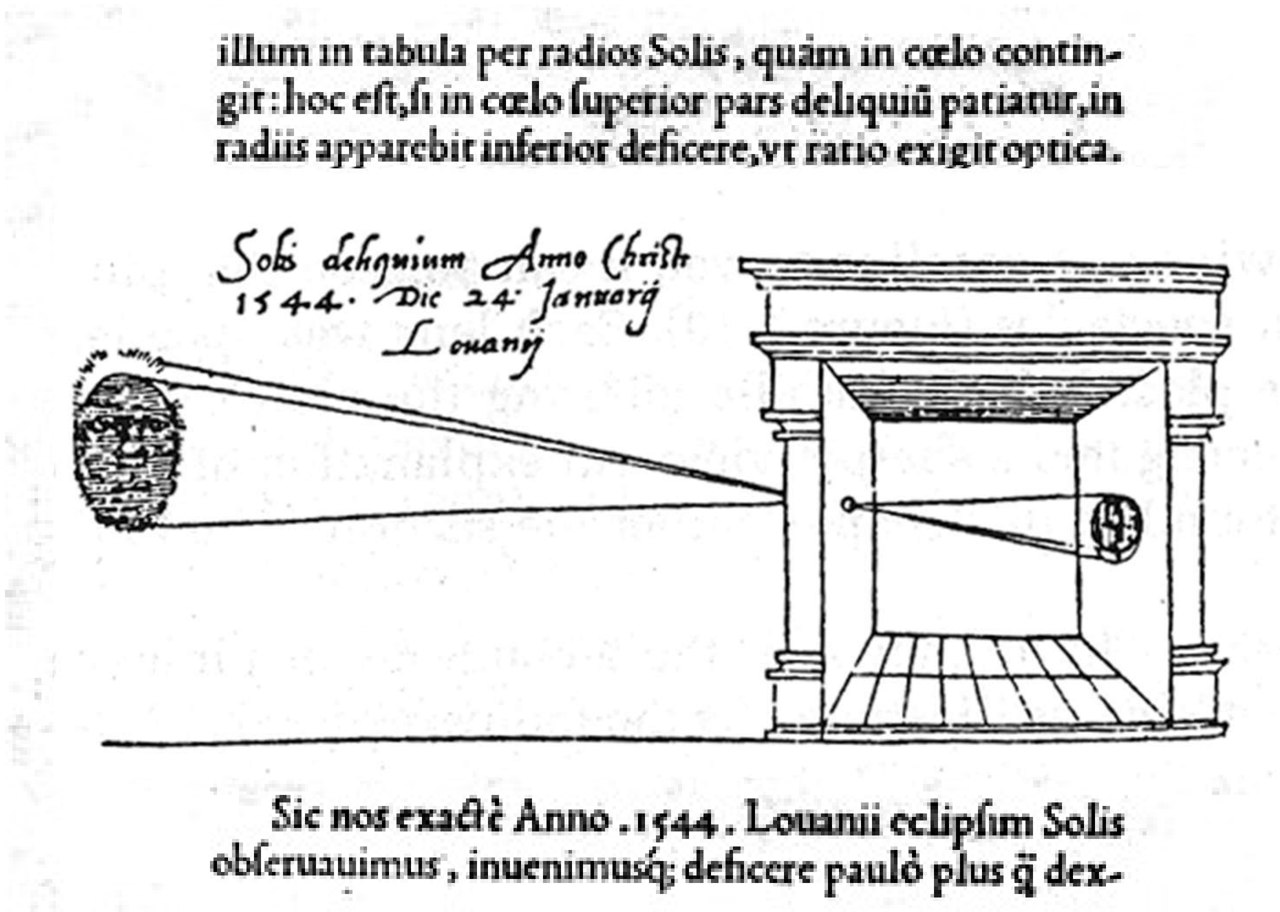
$$\cos\theta = \frac{\pi_1^\mathsf{T}\Omega^*_\infty\pi_2}{\sqrt{\left(\pi_1^\mathsf{T}\Omega^*_\infty\pi_1\right)\left(\pi_2^\mathsf{T}\Omega^*_\infty\pi_2\right)}}$$

Camera model

Relation between pixels and rays in space



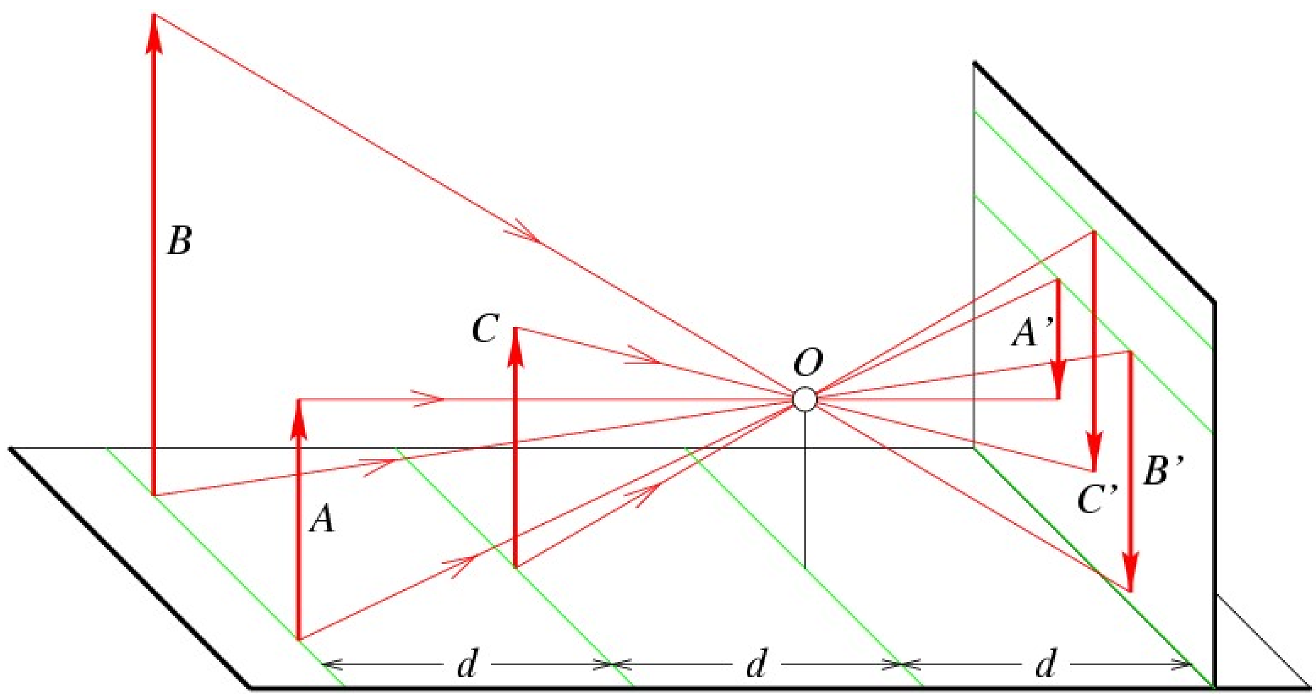
Pinhole camera


illum in tabula per radios Solis, quàm in cœlo contingit: hoc est, si in cœlo superior pars deliquiũ patiatur, in radiis apparebit inferior deficere, vt ratio exigit optica.

Sic nos exactè Anno .1544. Louanii eclipsim Solis obseruauimus, inuenimusq; deficere paulò plus q̃ dex-

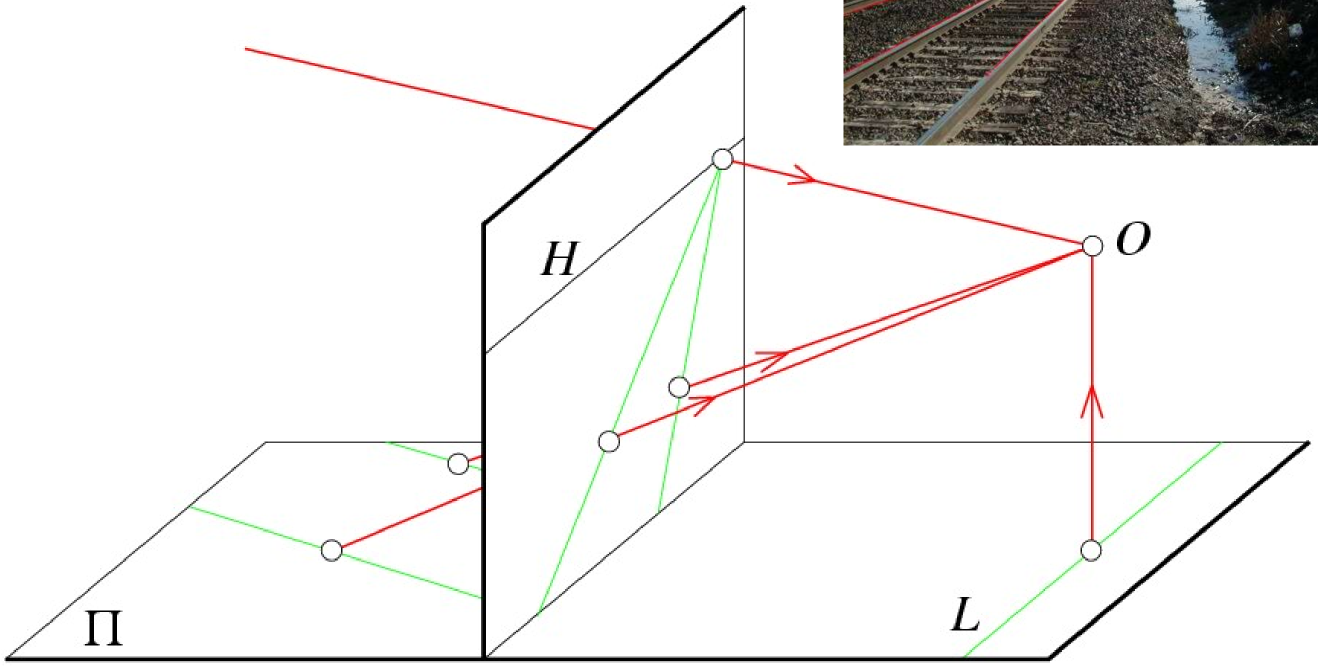
Gemma Frisius, 1544

Distant objects appear smaller

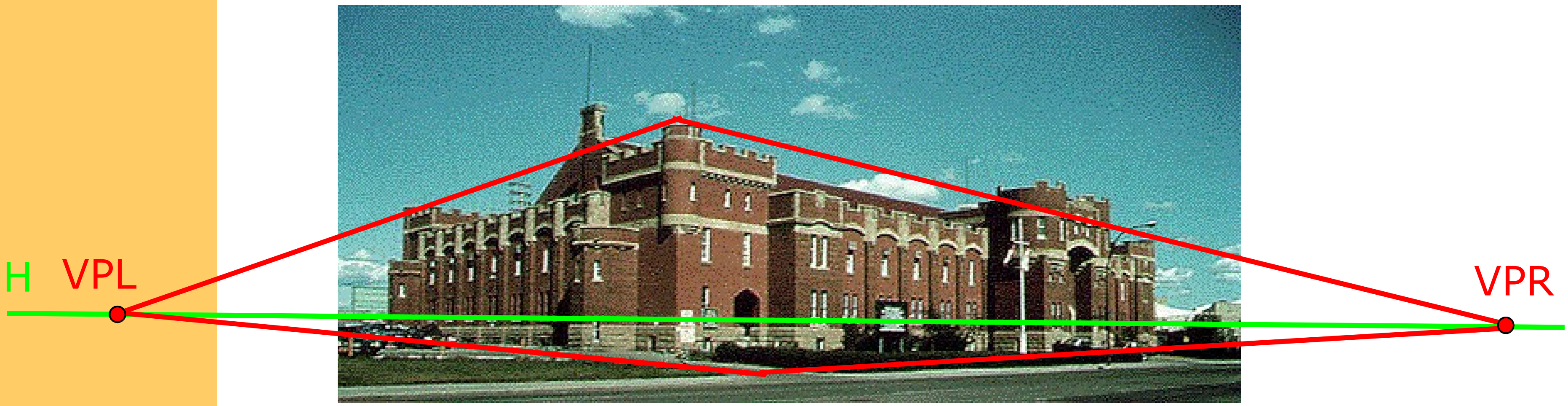


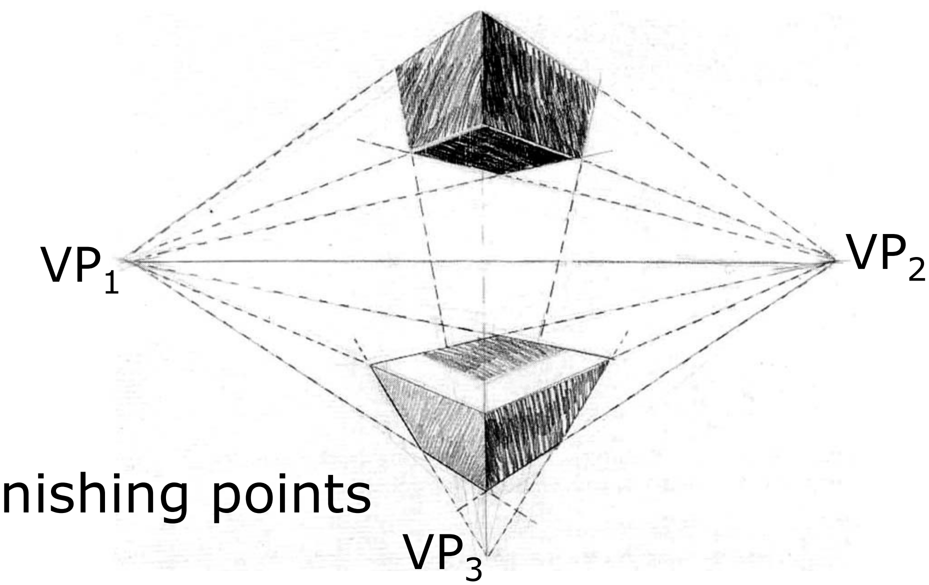
Parallel lines meet

- vanishing point



Vanishing points

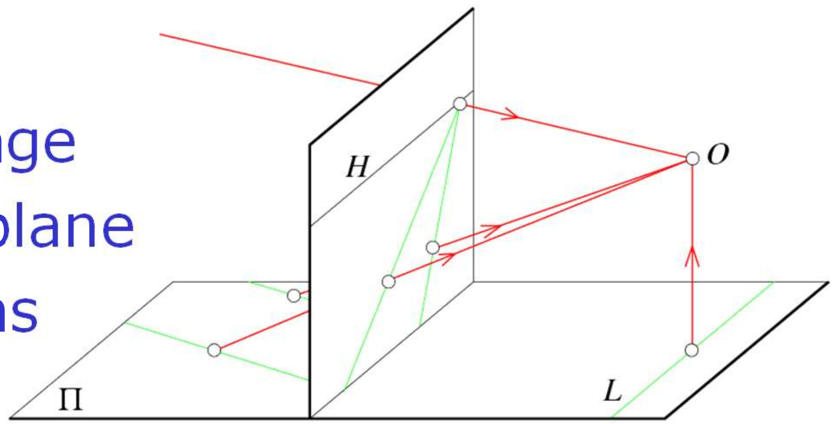




To different directions
correspond different vanishing points

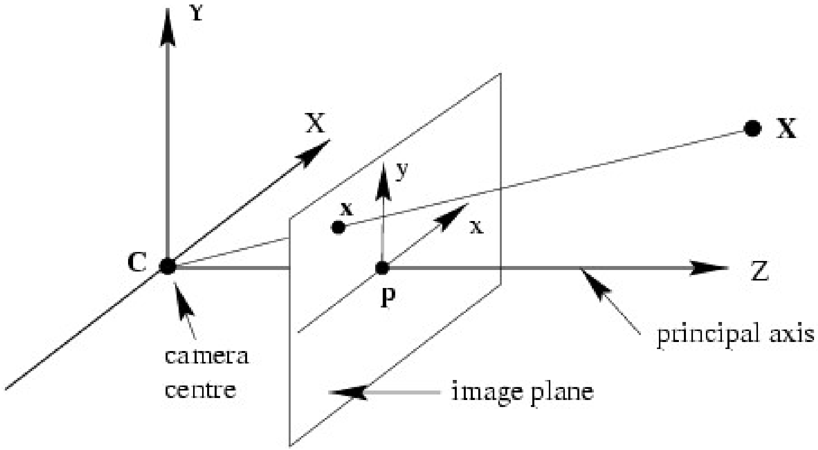
Geometric properties of projection

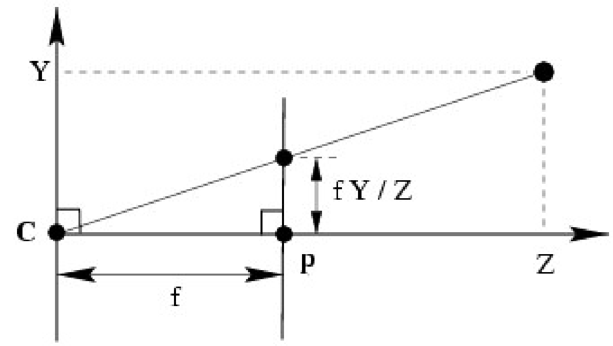
- Points go to points
- Lines go to lines
- Planes go to whole image
  or half-plane
- Polygons go to polygons



- Degenerate cases:
  - line through focal point yields point
  - plane through focal point yields line

Pinhole camera model



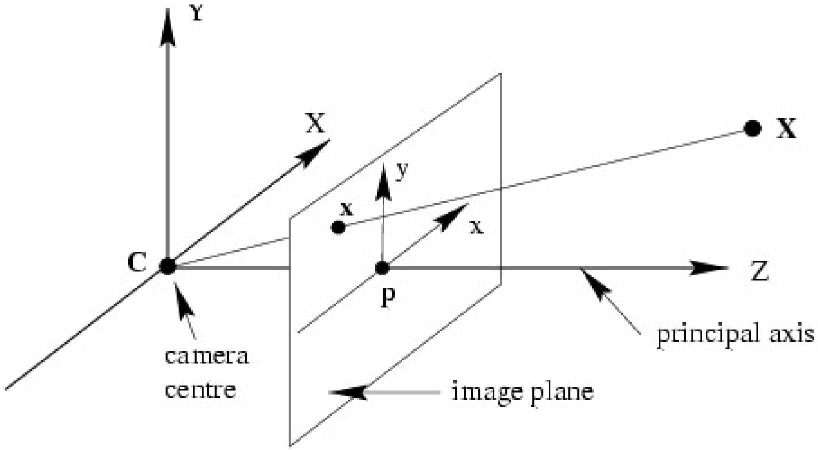


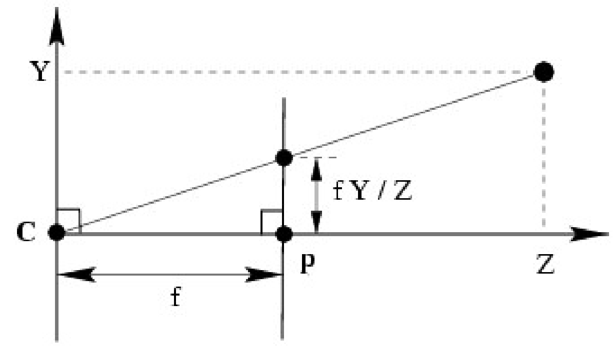
$$(X, Y, Z)^T \mapsto (fX/Z, fY/Z)^T$$

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

linear projection in homogeneous coordinates!

Pinhole camera model



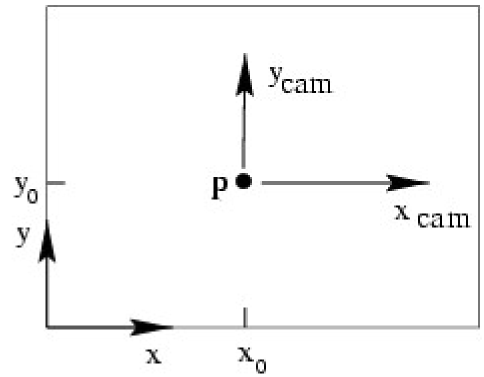


$$\begin{pmatrix} fX\\ fY\\ Z\end{pmatrix} = \begin{bmatrix} f & & & 0\\ & f & & 0\\ & & 1 & 0\end{bmatrix}\begin{pmatrix}X\\Y\\Z\\1\end{pmatrix} = \begin{bmatrix} f & & \\ & f & \\ & & 1\end{bmatrix}\begin{bmatrix}1 & & & 0\\ & 1 & & 0\\ & & 1 & 0\end{bmatrix}\begin{pmatrix}X\\Y\\Z\\1\end{pmatrix}$$

$$x = PX$$

$$P = \mathrm{diag}(f, f, 1)\left[I \mid 0\right]$$

Principal point offset

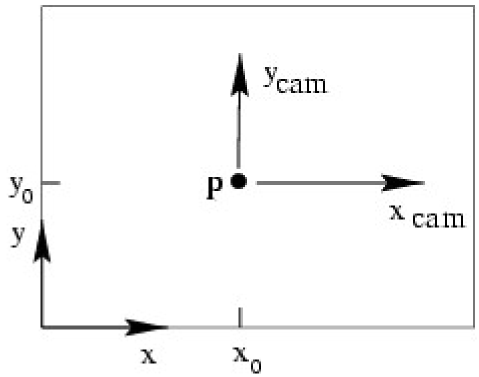


$$(X,Y,Z)^T \mapsto (fX/Z+p_x, fY/Z+p_y)^T$$

$(p_x, p_y)^T$ principal point

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} fX+Zp_x \\ fY+Zp_y \\ Z \end{pmatrix} = \begin{bmatrix} f & & p_x & 0 \\ & f & p_y & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Principal point offset

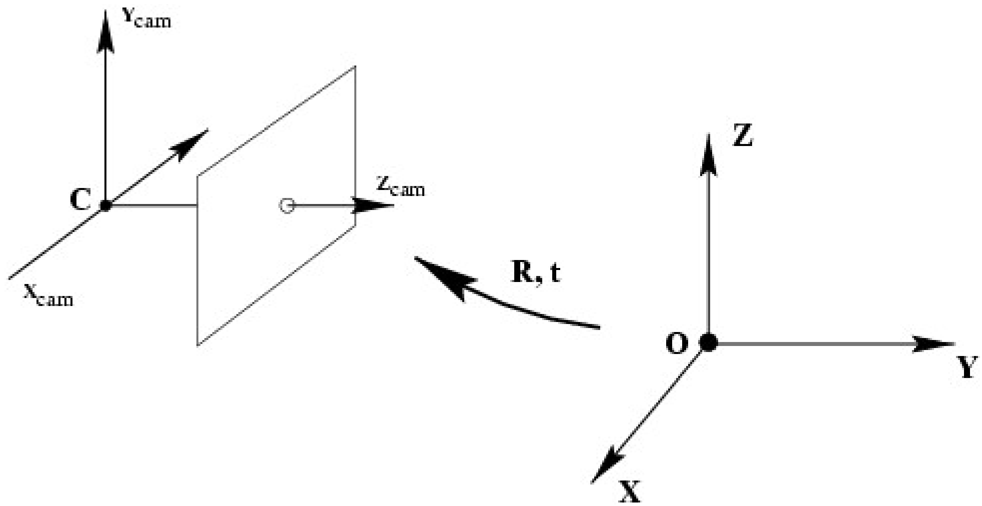


$$\begin{pmatrix} fX + Zp_x \\ fY + Zp_y \\ Z \end{pmatrix} = \begin{bmatrix} f & & p_x & 0 \\ & f & p_y & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\mathrm{x} = \mathrm{K}[\mathrm{I} \mid 0]\mathrm{X}_{\mathrm{cam}}$$

$$K = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$$ calibration matrix

Camera rotation and translation



$$\tilde{X}_{cam} = R\left(\tilde{X} - \tilde{C}\right)$$

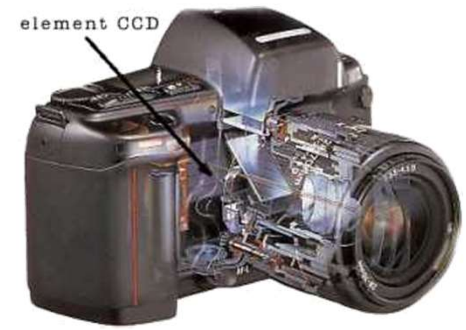
$$X_{cam} = \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} X$$

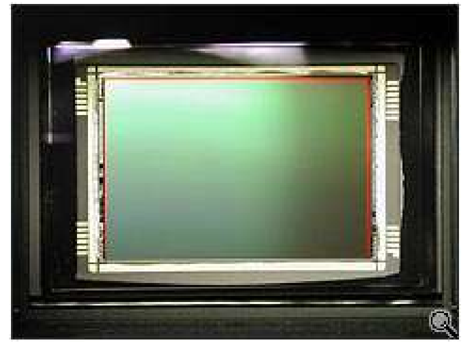
$$\mathbf{x} = \mathbf{K}[I \mid 0]X_{cam}$$

$$x = KR\left[I \mid -\tilde{C}\right]X$$

$$x = PX \qquad P = K[R \mid t] \qquad t = -R\tilde{C}$$

CCD camera





$$K = \begin{bmatrix} \alpha m_x & & p_x \\ & \alpha m_y & p_y \\ & & 1 \end{bmatrix} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$$

General projective camera

$$K = \begin{bmatrix} \alpha_x & s & p_x \\ & \alpha_y & p_y \\ & & 1 \end{bmatrix}$$

$$\mathrm{P} = \underbrace{\mathrm{KR}}_{\text{non-singular}}\left[\mathrm{I} \mid \quad \tilde{\mathrm{C}}\right]$$ 11 dof (5+3+3)

$$\mathrm{P} = \mathrm{K}\left[\underbrace{\mathrm{R} \mid \mathrm{t}}\right]$$

intrinsic camera parameters

extrinsic camera parameters

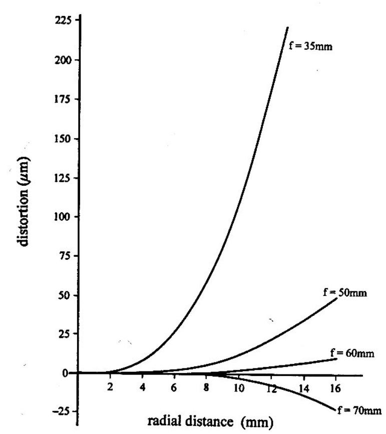
Radial distortion

- Due to spherical lenses (cheap)
- Model:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f_x & s & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix} R \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{R}^\top & -\mathbf{R}^\top \mathrm{t} \\ 0_3^\top & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \right)$$

$$R \qquad (x, y) = (1 + K_1(x^2 + y^2) + K_2(x^2 + y^2)^2 + \ldots) \begin{bmatrix} x \\ y \end{bmatrix}$$

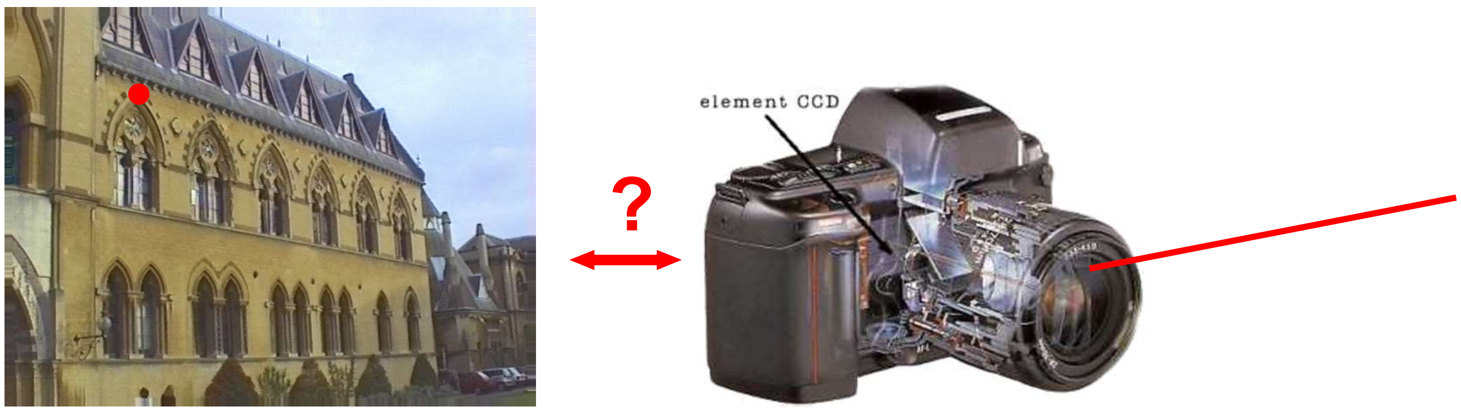
straight lines are not straight anymore



http://foto.hut.fi/opetus/260/luennot/11/atkinson_6-11_radial_distortion_zoom_lenses.jpg

Camera model

Relation between pixels and rays in space



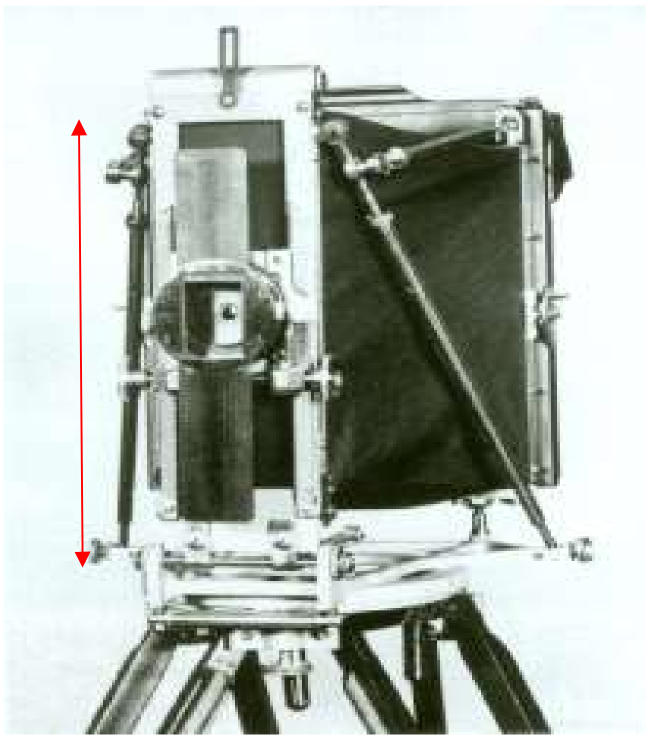
Projector model

Relation between pixels and rays in space

(dual of camera)

(main geometric difference is vertical principal point offset to reduce keystone effect)

Meydenbauer camera



vertical lens shift
to allow direct
ortho-photographs

Fig. 5: The principle of »Plane-Table Photogrammetry«
(after an instructional poster of Meydenbauer's institute)

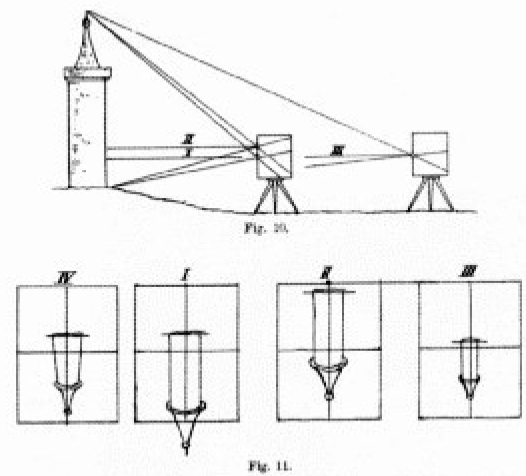
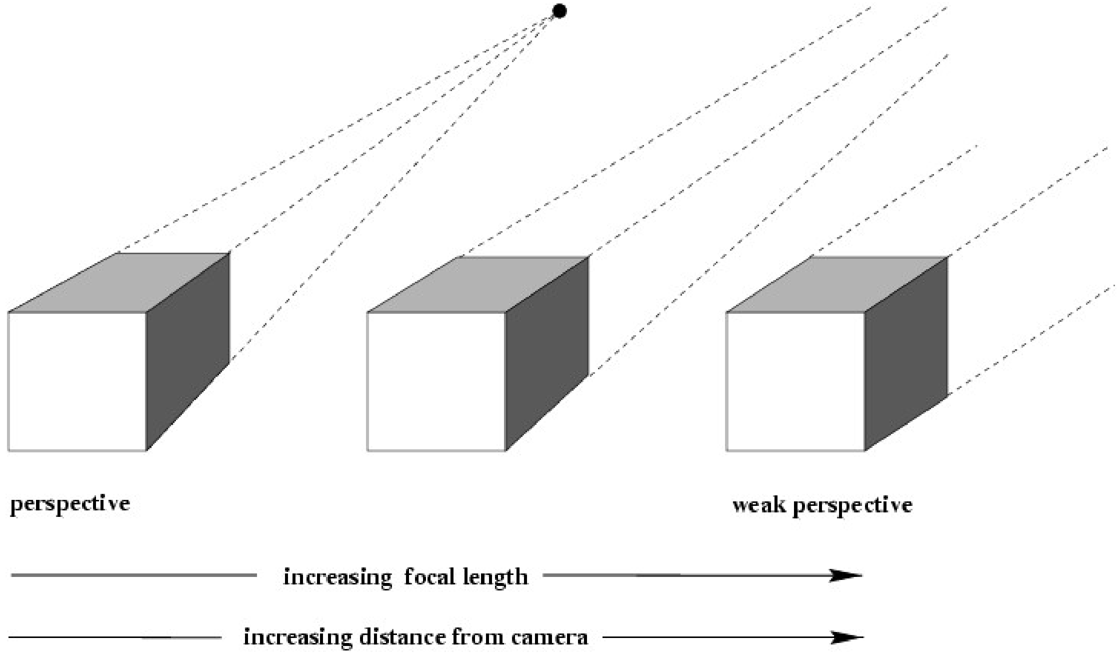


Fig. 6: The effect of a vertical shift of the camera lens;
the position II makes the best use of the image format
(after Meydenbauer's textbook from 1912)

Affine cameras



Action of projective camera on points and lines

projection of point

$$\mathrm{x} = \mathrm{P}\mathrm{X}$$

forward projection of line

$$\mathrm{X}(\mu) = \mathrm{P}(\mathrm{A} + \mu\mathrm{B}) = \mathrm{PA} + \mu\mathrm{PB} = \mathrm{a} + \mu\mathrm{b}$$

back-projection of line

$$\Pi = \mathrm{P}^{\mathrm{T}}\mathrm{l}$$

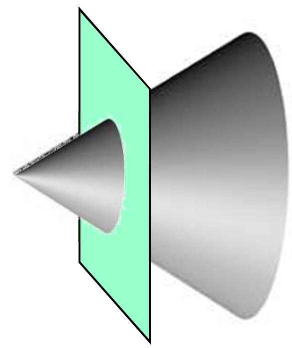
$$\Pi^{\mathrm{T}}\mathrm{X} = \mathrm{l}^{\mathrm{T}}\mathrm{P}\mathrm{X} \qquad \left(\mathrm{l}^{\mathrm{T}}\mathrm{x} = 0;\ \mathrm{x} = \mathrm{P}\mathrm{X}\right)$$

Action of projective camera on conics and quadrics

back-projection to cone

$$Q_{co} = P^T C P \qquad x^T C x = X^T P^T C P X = 0$$

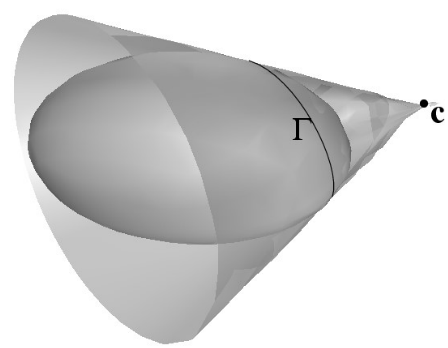
$$(x = PX)$$



projection of quadric

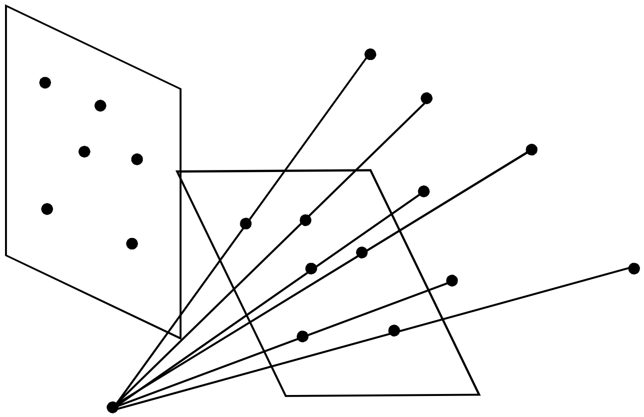
$$C^* = P Q^* P^T \qquad \Pi^T Q^* \Pi = l^T P Q^* P^T l = 0$$

$$(\Pi = P^T l)$$



Resectioning

$$X_i \leftrightarrow x_i \qquad P\,?$$



Direct Linear Transform (DLT)

$$\mathrm{x}_i = \mathrm{P}\mathrm{X}_i \qquad [\mathrm{x}_i]_\times \mathrm{P}\mathrm{X}_i \qquad \mathbf{P} = \begin{bmatrix} \mathbf{P}^{1\top} \\ \mathbf{P}^{2\top} \\ \mathbf{P}^{3\top} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \mathbf{0}^\top & -w_i\mathbf{X}_i{}^\top & y_i\mathbf{X}_i{}^\top \\ w_i\mathbf{X}_i{}^\top & \mathbf{0}^\top & -x_i\mathbf{X}_i{}^\top \\ -y_i\mathbf{X}_i{}^\top & x_i\mathbf{X}_i{}^\top & \mathbf{0}^\top \end{bmatrix}}_{\text{rank-2 matrix}} \begin{pmatrix} \mathbf{P}^1 \\ \mathbf{P}^2 \\ \mathbf{P}^3 \end{pmatrix} = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{0}^\top & -w_i\mathbf{X}_i{}^\top & y_i\mathbf{X}_i{}^\top \\ w_i\mathbf{X}_i{}^\top & \mathbf{0}^\top & -x_i\mathbf{X}_i{}^\top \end{bmatrix} \begin{pmatrix} \mathbf{P}^1 \\ \mathbf{P}^2 \\ \mathbf{P}^3 \end{pmatrix} = \mathbf{0}$$

$$\mathtt{A}_i\mathtt{p} = \mathtt{0}$$

$$\mathbf{A}\mathtt{p} = \mathtt{0} \qquad \mathbf{A} = \begin{bmatrix} \mathtt{A}_1 \\ \vdots \\ \mathtt{A}_n \end{bmatrix}$$

Direct Linear Transform (DLT)

$$\mathrm{A}\mathrm{p} = 0$$

Minimal solution

P has 11 dof, 2 independent eq./points

$\Rightarrow$ 5½ correspondences needed (say 6)

Over-determined solution

$n \geq 6$ points

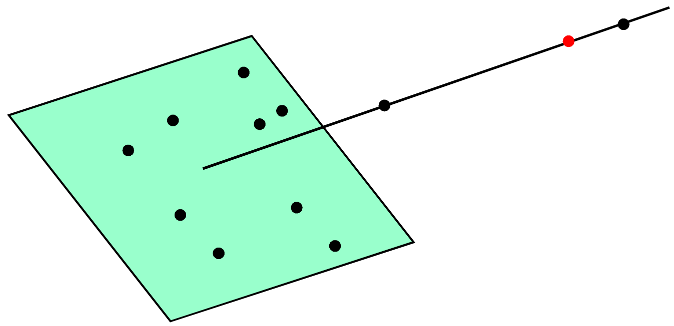
minimize $\|\mathrm{Ap}\|$ subject to constraint

$\|\mathrm{p}\| = 1$

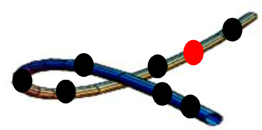
$\longrightarrow$ **use SVD**

Degenerate configurations

(i) Points lie on <u>plane</u> or single line passing through projection center



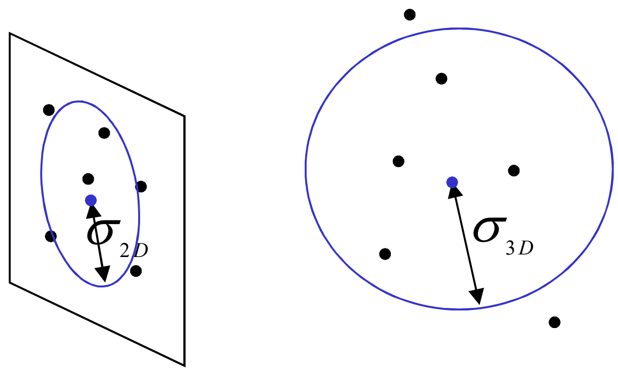
(ii) Camera and points on a twisted cubic



Data normalization

Scale data to values of order 1

1. **move center of mass to origin**
2. **scale to yield order 1 values**



$$\tilde{\mathtt{x}} = \mathtt{T}\mathtt{x} \qquad \tilde{\mathtt{X}} = \mathtt{U}\mathtt{X}$$

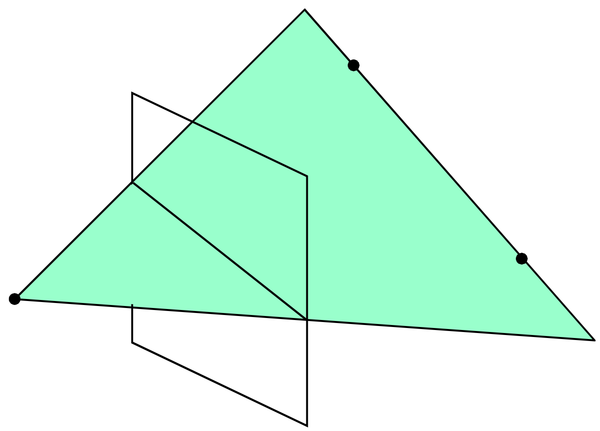
$$\mathtt{T} = \begin{bmatrix} \sigma_{2D} & 0 & \bar{x} \\ 0 & \sigma_{2D} & \bar{y} \\ 0 & 0 & 1 \end{bmatrix}^{-1} \qquad \mathtt{U} = \begin{bmatrix} \sigma_{3D} & 0 & 0 & \bar{X} \\ 0 & \sigma_{3D} & 0 & \bar{Y} \\ 0 & 0 & \sigma_{3D} & \bar{Z} \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}$$

Line correspondences

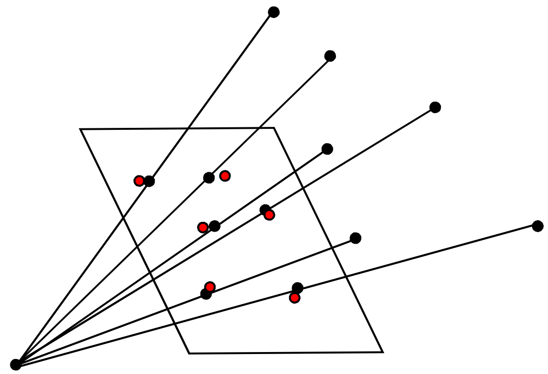
Extend DLT to lines

$$\Pi = P^T l_i \qquad \text{(back-project line)}$$

$$l_i^T P X_{1i} \qquad l_i^T P X_{2i} \qquad \text{(2 independent eq.)}$$



Geometric error



$$\sum_i d(\mathbf{x}_i, \hat{\mathbf{x}}_i)^2$$

$$\min_{\mathrm{P}} \sum_i d(\mathbf{x}_i, \mathrm{P}\mathbf{X}_i)^2$$

Gold Standard algorithm

Objective

Given n≥6 2D to 3D point correspondences $\{X_i \leftrightarrow x_i'\}$, determine the Maximum Likelyhood Estimation of P

Algorithm

(i) Linear solution:

(a) Normalization: $\tilde{X}_i = UX_i \quad \tilde{x}_i = Tx_i$

(b) DLT

(ii) Minimization of geometric error: using the linear estimate as a starting point minimize the geometric error:

$$\min_{P} \sum_i d(\tilde{\mathbf{x}}_i, \tilde{P}\tilde{\mathbf{X}}_i)^2$$

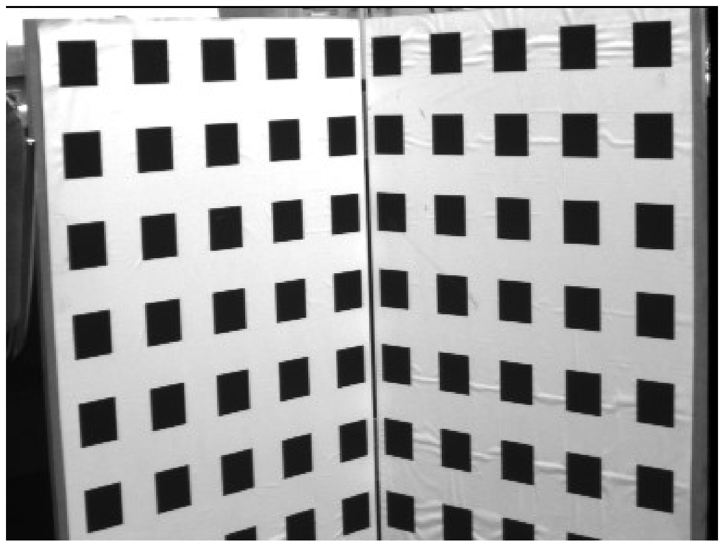
(iii) Denormalization: $P = T^{-1}\tilde{P}U$

Calibration example

(i) Canny edge detection
(ii) Straight line fitting to the detected edges
(iii) Intersecting the lines to obtain the images corners

typically precision <1/10

(H&Z rule of thumb: 5n constraints for n unknowns)



	f_y	f_x/f_y	skew	x_0	y_0	residual
linear	1673.3	1.0063	1.39	379.96	305.78	0.365
iterative	1675.5	1.0063	1.43	379.79	305.25	0.364

Errors in the image (standard case)

$$\sum_i d(\mathbf{x}_i, \hat{\mathbf{x}}_i)^2 \qquad \hat{x}_i = PX_i$$

Errors in the world

$$\sum_i d(\mathbf{X}_i, \widehat{\mathbf{X}}_i)^2 \qquad x_i = P\widehat{X}_i$$

Errors in the image and in the world

$$\sum_{i=1}^{n} d_{\mathrm{Mah}}(\mathbf{x}_i, \mathrm{P}\widehat{\mathbf{X}}_i)^2 + d_{\mathrm{Mah}}(\mathbf{X}_i, \widehat{\mathbf{X}}_i)^2$$

$$\widehat{X}_i$$

Restricted camera estimation

Find best fit that satisfies

- skew s is zero
- pixels are square
- principal point is known
- complete camera matrix K is known

$$\mathrm{K} = \begin{bmatrix} \alpha_x & s & x_0 \\ & \alpha_y & y_0 \\ & & 1 \end{bmatrix}$$

Minimize geometric error
→impose constraint through parametrization

Minimize algebraic error
→assume map from param q → P=K[R|-RC], i.e. p=g(q)
→minimize ||Ag(q)||

Restricted camera estimation

$$\mathrm{K} = \begin{bmatrix} \alpha_x & s & x_0 \\ & \alpha_y & y_0 \\ & & 1 \end{bmatrix}$$

Initialization

- Use general DLT
- Clamp values to desired values, e.g. s=0, $\alpha_x = \alpha_y$

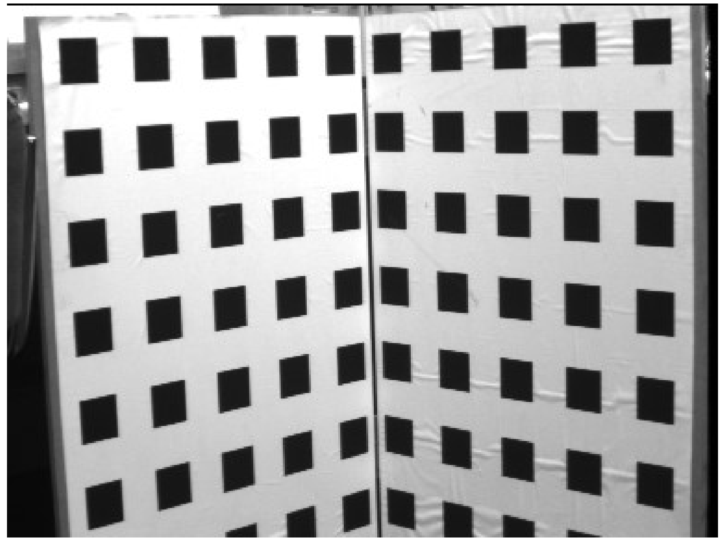
Note: can sometimes cause big jump in error

Alternative initialization

- Use general DLT
- Impose soft constraints

$$\sum_i d(\mathbf{x}_i, \mathrm{P}\mathbf{X}_i)^2 + ws^2 + w(\alpha_x - \alpha_y)^2$$

- gradually increase weights



	f_y	f_x/f_y	skew	x_0	y_0	residual
algebraic	1633.4	1.0	0.0	371.21	293.63	0.601
geometric	1637.2	1.0	0.0	371.32	293.69	0.601

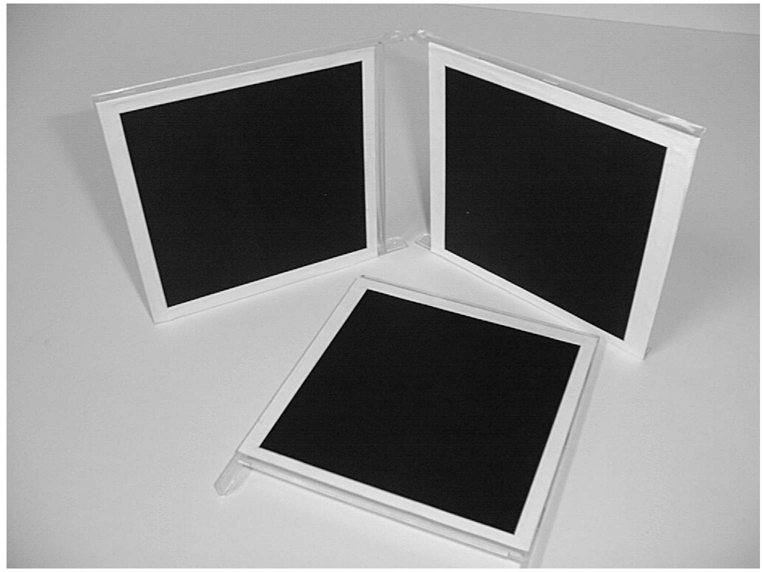
	f_y	f_x/f_y	skew	x_0	y_0	residual
linear	1673.3	1.0063	1.39	379.96	305.78	0.365
iterative	1675.5	1.0063	1.43	379.79	305.25	0.364

Image of absolute conic

$$
\begin{aligned}
\omega^* &= \mathbf{P}\Omega^*\mathbf{P}^\top \\
&= \mathbf{K}\mathbf{R}\begin{bmatrix} \mathbf{I} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{I} \\ 0 \end{bmatrix}\mathbf{R}^\top\mathbf{K}^\top \\
&= \mathbf{K}\mathbf{K}^\top
\end{aligned}
$$

$$
\omega = \mathbf{K}^{-1}\mathbf{K}^{-\top}
$$

A simple calibration device



(i) compute H for each square
(corners $\rightarrow$ (0,0),(1,0),(0,1),(1,1))
(ii) compute the imaged circular points $H(1,\pm i,0)^T$
(iii) fit a conic to 6 circular points
(iv) compute K from ω through cholesky factorization

(≈ Zhang's calibration method)

Some typical calibration algorithms

Tsai calibration

Tsai, Roger Y. (1986) "An Efficient and Accurate Camera Calibration Technique for 3D Machine Vision," *Proceedings of IEEE Conference on Computer Vision and Pattern Recognition*, Miami Beach, FL, 1986, pp. 364–374.

Tsai, Roger Y. (1987) "A Versatile Camera Calibration Technique for High-Accuracy 3D Machine Vision Metrology Using Off-the-Shelf TV Cameras and Lenses," *IEEE Journal of Robotics and Automation*, Vol. RA–3, No. 4, August 1987, pp. 323–344.

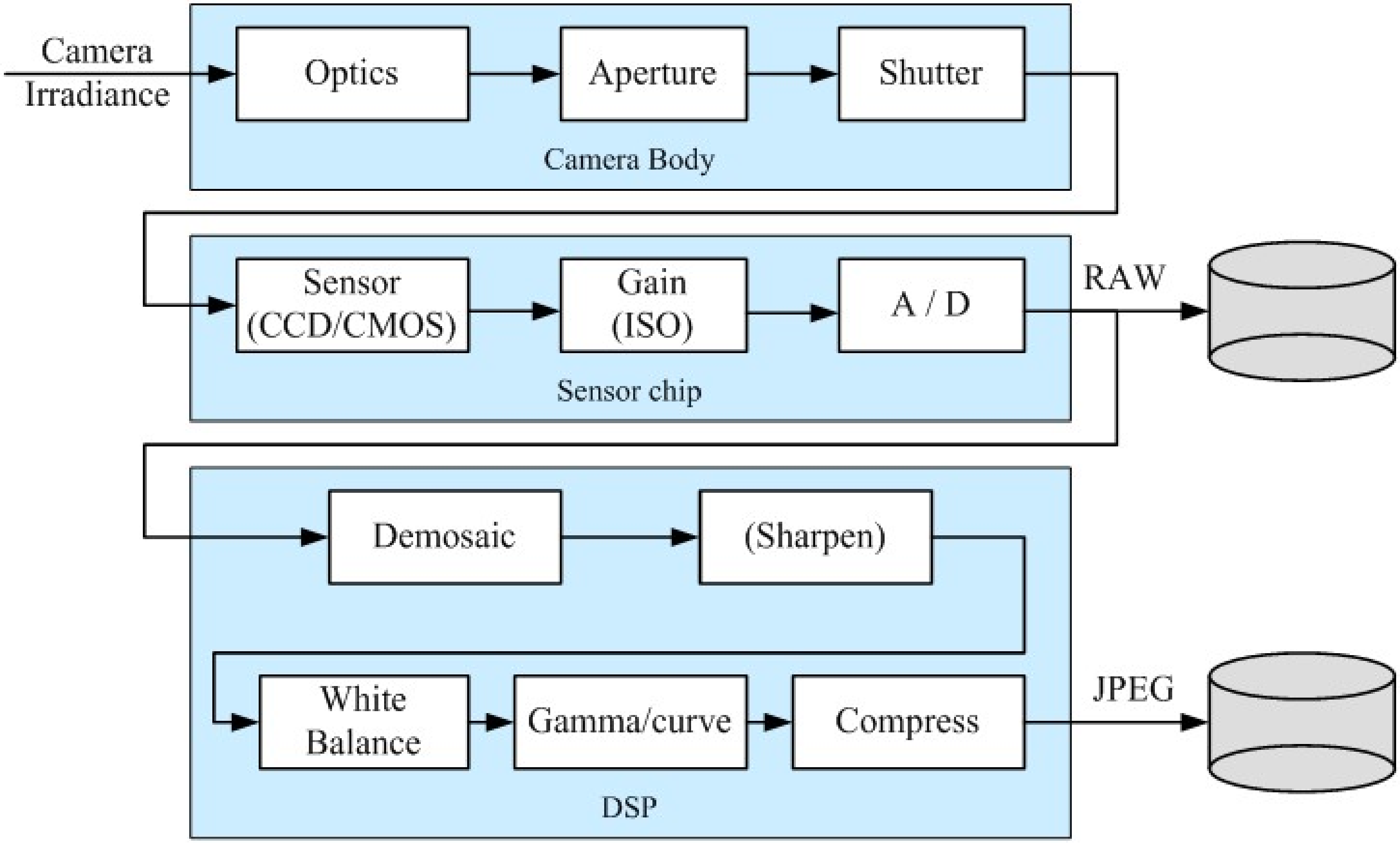
Zhangs calibration

http://research.microsoft.com/~zhang/calib/

Z. Zhang. A flexible new technique for camera calibration. IEEE Transactions on Pattern Analysis and Machine Intelligence, 22(11):1330-1334, 2000.

Z. Zhang. Flexible Camera Calibration By Viewing a Plane From Unknown Orientations. International Conference on Computer Vision (ICCV'99), Corfu, Greece, pages 666-673, September 1999.

http://www.vision.caltech.edu/bouguetj/calib_doc/



from Szeliski's book

Next week:
Image features