Errera Rules !

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This is the internship report that summarizes the four month (April-July) spent by Antoine Lizée as an intern of the Dumais Laboratory, in the Department of Organismic and Evolutionary Biology of Harvard University.

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Less personal thoughts go toward two institutions that make my internship easier:

- The chair 'X Saint-Gobain' for their financial support.
- The developing team of Geogebra, which has developed an incredibly simple yet powerful geometrical tool for free, which helped me during my reflection.

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Introduction

1. Plant cell division, rules and geometry

1.I. Errera's rule

Cell division in plant tissues is one of the - few? - Biological processes which do not make a physicist mad by its complexity or unpredictability. Indeed, since the end of the 19th century, two statements concerning cell division in plant tissues have proven out to be so well followed by nature that they have reached the golden state of 'laws':

- Sach's rule states that the division leads to two daughter cells with equal volume. REF
- Errera's rule (1888) states that the dividing wall takes the shape a soap film would take under the same conditions. You can observe the results of cross observations between real cell shapes and soap films in figure 1. REF

These two rules and especially Errera's rule characterize an optimization process which enables the cell to conserve time, material and energy. Indeed, its surface tension gives the soap film an optimal shape with regards to its area. The cell 'chooses' the plane of division that minimize its area under the constraint of volume. Although the molecular processes underneath this optimization is not yet well understood, these rules have been tested and approved to a large extend – despite several exceptions stored by botanists.

1.II. Geometry and Probability

This large verification of what I will name from now "Errera's rules" has been carried out mainly on 2D tissues whose third dimension is just an extrusion of the 2D pattern, like epidermal tissues. For these cell "tilings", every calculation can be done in the plane, and Errera's rules applied to 2D patterns lead to one beautiful geometric rule: the geometrical result of this optimization is that the dividing wall, in theory, must be an arc of circle with right angles at its connections with the existing walls of the mother cell. My worst misnomer will be to recall this particular shape for the dividing wall by "Errera shape".

There can be several planes of divisions that have this Errera shape. These planes are all local minima in terms of area, and if the volume ratio of division is V/2, then there is at most one dividing plane for each couple of edges of the mother cell. The strict optimization of the process would lead to the cell choosing the global minimum, i.e. the wall that has the lower area among the several planes with Errera shape satisfying this volume constraint. Recent observations developed in Dumais Laboratory REF has shed light on a probabilistic behavior, stating that the dividing wall will finally be chosen among the candidates with a probability relying on the difference in area between this final wall and the global minimum. This dependence seems to be exponential.

2. My work in the Dumais Laboratory

I arrived in the Dumais Laboratory just after the publishing of the paper that shed light on the elegant probabilistic observation cited above. My work was to explore new way of analyzing and comparing these rules with real phenomenon. As a matter of fact, and because all started with MATLAB, I have worked mostly with this powerful tool I needed first to master – I have very little formation in computing and had never worked with MATLAB before. Sebastien Besson, a (French) postdoc who had worked a lot on this project before me, have spent several years building a set of MATLAB function and programs to numerize, create, compute, merge, divide and more broadly to study plant tissues. I have created other Matlab programs in this environment, in order to broaden this study of cell tissues and to carry on new simulations. My work can be organized in two parts:

1. The creation of tools to study more deeply division of plant cell on flat tissues through the simulation of tilings. To achieve this goal, I have created a tool of quantitative comparison between different tilings, to compare simulation and reality more accurately than usual observations. Additionally, my main achievement is the creation of a program that computes analytically the different results of Errera's division.

2. The research of solutions that would be the product of Errera's division on a cone. This is another way to explore Errera's rules of division, with a different approach. Indeed, the tissues at work are conic rather than flat and thereby can describe more precisely meristems, which is a crucial system for cell division in plants.

This report is mainly meant to be the key of understanding of the work I have done in the lab, especially as this work is to be continued. Provided the computational and geometrical core of what I present, this report could therefore become a little heavy in details and explanations. Forewarned is forearmed…

Part I:
Simulation of tilings

1. Quantitative comparison of tissues

1.I. Topology tool

Comparing tissues is like comparing faces. But when it is very difficult to make an objective comparison, the pattern nature of tissues facilitates the findings of criteria that could be used to compare them. The main criterion that is used in the literature is the topological one. The distribution of the number of edges for each cell.

2. Dividing tissues, faster

Part II:
Study of division on plant meristems

1. Abstract

1.I. Introduction

In order to test the predictive power of the division rules stated above, we must try to compare them to biological systems more and more diverse. Many studies have been carried out on epidermal, flat tissues but here we propose to expand these studies on a crucial area of the plant: the meristem. The meristem of plants is a very interesting zone to study, since it is the main area of division, featuring undifferentiated cells that divide at a constant and high rate. Yet, contrary to the epidermal tissues, these meristems are not planar tissue, but rather have shapes in between cone and hemisphere which vary from a system to another. We want to compare the shape of observed cells on these meristems with some shape that would be predicted by Errera's rules. We will use two different approaches, one analytical research of key solutions, and one numerical approach with simulation relying on the division program we developed on Matlab.

1.II. General method

To carry out these comparisons, we will assume that a good representation of a meristem is a perfect cone, and we will focus on the central cell, which is at the heart of the division process. The cone approximation enables us to do all calculation of length and area on a diminished 2D plane. This 2D plane is the result of cutting the cone along a generating line and laying it on a flat surface. We will use the angle gamma of the resulting 'forbidden area' on this 2D plane to characterize the cone. The relation between this angle gamma and the geometrical angle of the cone is illustrated on Figure 1. I will only detail in this abstract the first step, i.e. the analytical approach.

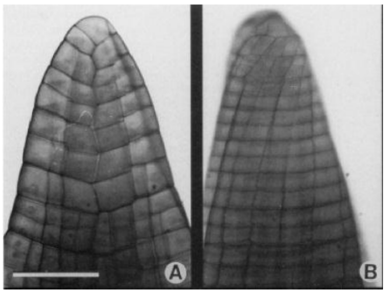
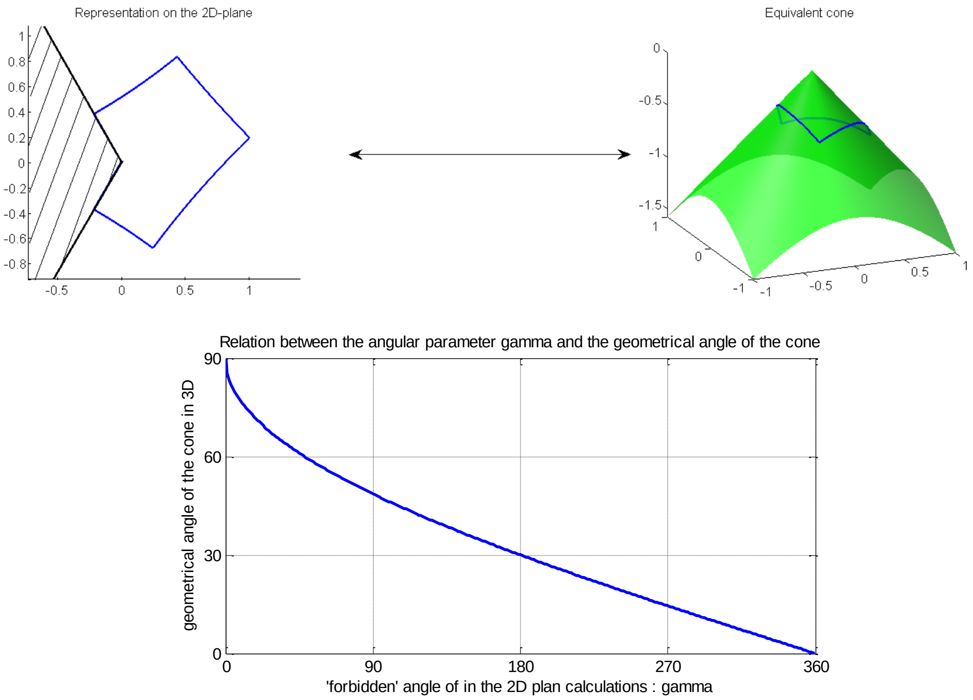


Figure 2. Example of pseudo-conic meristem with the Funaria apex

Figure 1 Relation between the 2D-gamma plane used for computation and the real cone

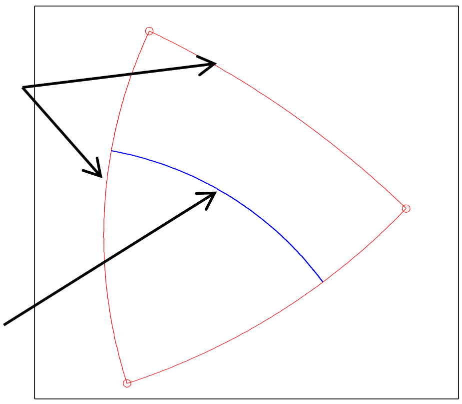


1.III. Fixed point solution on the cone

This analytical approach consists on finding cell shapes that satisfy some hypothesis which characterize the division on the center of meristems. Since a meristem can undergo between n and n REF division during its characteristic period of growth, the cell shapes that we observe are the result of a great number of these divisions. We will therefore look for 'fixed-point' solutions of the dynamic system defined by the central cell of the meristem, undergoing transformations that are framed by two major hypotheses. We want to figure out analytically what these solutions could be, and see how the shape of these solutions change with the parameter of the cone, gamma.

The *first hypothesis* is the fact that on the center of meristems, the growth between divisions is isotropic: shapes of the cells don't change between two divisions, only the characteristic length increases. The *second hypothesis* is the fact that the division which occurs on the center of this cone follows non-deterministic Errera's rules: the plane of division has the Errera shape, and the area of the mother cell is divided by two. Within the framework of these two hypotheses, we decide to look for solutions that are 'fixed-point solutions' *in one step*, which are preserved by one division and growth process. This has direct implications that are very useful to determine the solutions:

- The cell itself is made of walls that are products of previous divisions, whose shape is not altered during growth. Therefore, the walls of the cell are all arc of circles that connect each other with right angles ("Errera shape").
- Since growth is isotropic, the division should produce one of the daughter cell with the exact same shape than the shape of the mother cell. The self-similar daughter cell has half the area of the mother cell.



These two constraints define exactly our problem of "fixed-point solution on the cone" of which we seek the solutions. The interested reader is advised to have a look at the chapter 2 and 3 of this part to grasp thoroughly the concepts behind this research.

1.IV. Results

Having defined the proper geometrical parameters to describe the cell shape and set the right geometrical constraints that translate these hypotheses, we can implement a numerical resolution of our simple equations (see Section 2 of this part). These different solutions are then represented on the cone after proper transformations that enable us to jump from the 2D plane to the real cone. All these tasks are performed by a Matlab set of functions and scripts presented in the section "*3.Explanation of the resolution process*

Documentation of Matlab program". The first important result is the quasi-uniqueness of solution per number n of edges of this solution and per value of γ, the parameter of the cone. For each value of n and γ, there is no more than two solutions ($*$ 2 with the axial symmetry) if we consider the shape of the cell (size of the cell itseld and orientation around the cone may change).

1.IV.a Shape of solutions

The main results are the shape of the solutions depending on gamma, for any number of edges of this solution. There are two different solutions with five edges, one solution with 1 to 4 edges, and there is no solution with more than 6 edges for positive values of gamma. These positive values of gamma define a cone whereas negative values of gamma define a shape that can be interesting for some application but are not tackled yet - even though the computations remain valid for gamma< 0.

The results have been gathered into a dynamic GUI (Graphic User Interface) that can be provided upon asking*. This GUI displays in 3D the fixed-point solutions of the division problem for cells from 2 to 5 edges. These solutions do not exist for all values of gamma, and their existences are limited by a maximum value of gamma, which decreases with the number of edges. (see Table 1: "Geometric limitations and Existence domains of the cell shapes regarding the parameter gamma of the cone"). Several of these shapes are shown below.

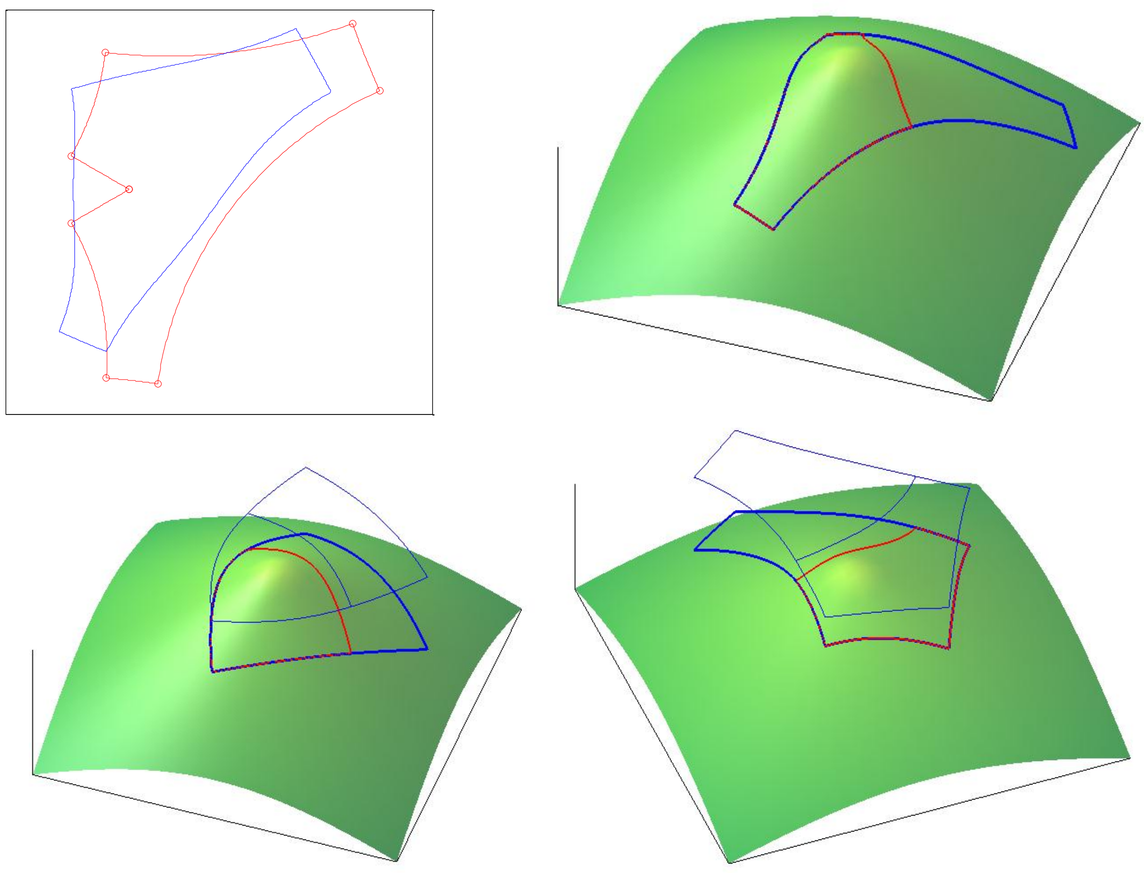


Figure 3 Examples of 3D shape of the solutions

From upper left to bottom right: 2D representation on the gamma plane of the solution shape for γ=60° et five-edged cell Type 2 (red line = as the result of the resolution; blue line = stretched for the projection on the cone); 3D representation of the latter; 3d representation with projection and dividing wall of 3 edged cell on a cone with γ=60°, and the same representation for γ=30 and a five edged cell Type 1.

* Upon asking the creator and writer of this prose at antoine.lizee@polytechnique.edu or the contact and supervising professor, Jacques Dumais, at jdumais@oeb.harvard.edu). The M-files of the GUI are working on any recent version of MATLAB. They may be included in the numerical package containing this report.

1.IV.b Existence domains of the solutions

The knowledge of the complete shape of the different solutions enables us to do the analytical computation for the research of the transition angle that would discriminate the existence domain of each solution. Indeed, the cell shapes that we have found are not complete solutions of the problem, since they are not necessarily the result of the optimal division of themselves. Indeed, even if the division plane which leads to an auto-similar daughter cell follows the general rules of divisions (arc of circles + right angles), it is not necessarily the shortest division plane among the candidates. “Self-stability” of the solution is granted when the deterministic cell division leads to the auto-similar daughter.

Instead of long exact analytical computation, I have chosen to take advantage of the strong division program which computes the possible division planes of a given cell (see Part I, section 2 of this report). This highly reliable program can compute with a great precision the geometrical characteristics of all the division planes; therefore, we are able to compute these limit angles for which there is a division plane that become larger than the one which lead to the auto-similar daughter cell. These results are summed up in below.

Shape of the cell	Gamma max (Geometrical limitations to self-similarity)	Existence domains (“self-stability” of the solution)
1 edge	360	234.5 / 360
2 edges	321	164.7 / 279.4
3 edges	266.7	- / 228.6
4 edges	187.3	-/ 135.5
5 edges	92.8	-/ 37.3
5 edges Type 2	133	- / 25.5
6 edges	-9.3	
7 edges	-114	
7 edges Type 2	-40.9	

Table 1 Geometric limitations and Existence domains of the cell shapes regarding the parameter gamma of the cone

These very valuable results should be compared to numerical simulations, which will be able to describe the dynamic properties of the system. Indeed, these transitions values are strict boundaries in gamma that cannot be exceeded by the solution of interest, because a greater or smaller gamma would led the system to another cell shape. Nevertheless, this domain of existence for the solutions could never be reached by real systems. Division process from other cell shape could lead to solutions (like periodic ones, with different shapes alternating) that would not allow to 'drop' on the solution we have considered. This issue must be tackled by complete simulations I did not have the time to carry out.

We can go further in the discussion around these solutions and their use for understanding real division, and this discussion is developed in Section 4 below.

2. How to get to grips with the analytical finding of solutions on the cone

In this section, I try to provide the reader some elements to grasp easily the complexity in the resolution of this simple problem. I have personally been through different examples "hands-on" (2-edeged-cell, 3-edged cell on the complete plane), but they would be long to present here, and I will directly explain the proper way to solve the problem that embrace all the solutions, regardless the shape of the cell.

2.I. Explanation of the geometrical problem

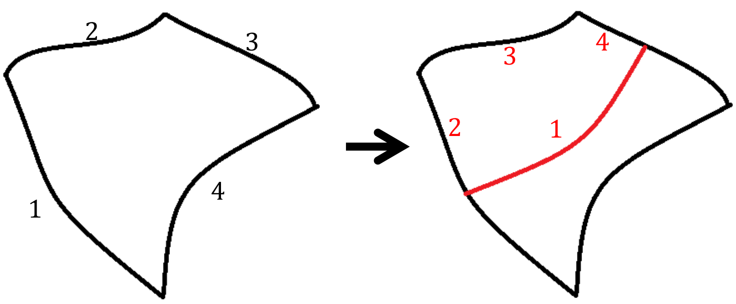
If it is not already done, the reader is advised to read the section '1.III Fixed point solution on the cone' not to miss the starting point.

2.I.a Periodicity

The first important thing to understand is the periodicity implied by the first hypothesis within the framework of a fixed point solution. Regardless the aspect of the edges, the isotropic growth imposes that the daughter cell which have the same number of edges has also the exact same shape. Since n-3 edges are strictly preserved and 2 other edges have many characteristics in common, the shape of the mother cell is already very constrained, as shown in Figure 4.

2.I.b Semantic rotations

In addition, we know that every edge must be shrunk by a $\sqrt{2}$ ratio during the division, because the final area in the daughter cell is half of the area of the mother cell. This leads to a more accurate definition of the geometrical consequences of our hypotheses. Indeed, the periodicity is not only needed, but every edge must be accounted in this periodicity, because if one edge is left behind in this process of transformation, it cannot be shrunk and keep the same shape at the same time – except for the straight edge exception.



Numbers label edges. Black numbers label edges in the mother cell, red numbers label edges of the daughter cell.

Considering that the new wall (red) have the role in the daughter cell that the wall "1" had in the mother cell, all the walls are rotated to preserve the global shape between the mother and the daughter cell. A lot of constraints arise concerning the shape of the mother cell. For instance the wall 2 should have the exact same shape of the wall 3, with only a difference in size – which is obviously not the case in this example.

Figure 4. Illustration of the constraints that stem from cell division with a dummy example

This periodicity should be understood deeply, through the concept of the "semantic rotations" that could occur to the edges during the division. Let see that with examples: in the four edges case we have two possible "semantic rotations". The first one, represented in Figure 4, can be described as follow: edge number 1 of the mother cell takes the role in the daughter cell that edge number 2 had in the mother cell, edge 2 takes the role of edge 3, edge with role number 4 in the daughter cell is located on the edge 3 of the mother cell, and edge 4 is replaced by the new edge, as a result of the division, which take the role of edge 1. If each arrow "->" means "becomes", this semantic transformation can be summed up with this formula:

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 4$$

whereas the representation of the other semantic rotation can be summed up by:

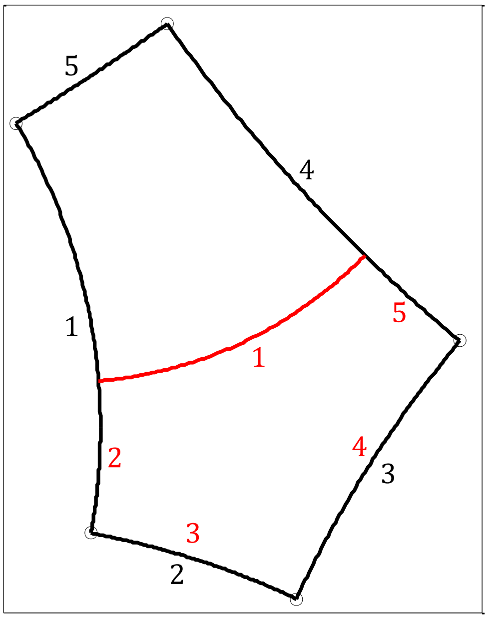
$$3 \rightarrow 2 \rightarrow 1 \rightarrow 4$$

The second transformation is the symmetric of the first one, so we will only resolve the problem for transformation characterized by increasing numbers of edges, and the solution could be flipped to get another solution, which is quite natural. Let us see right now why the semantic rotation of "Type 2", obtained by skipping one edge when rotating, does not lead to an self-similar daughter for the 4-edged cell. The formula would be:

$$1 \rightarrow 3 + \ 2 \leftrightarrow 4$$

the old edge 3 being replaced by the new edge, which has the role of edge 1. This is not an acceptable transformation, because the edges number 2 and 4 are flipped in their roles during the division. After two divisions, the edge which has been number 4 must have again the same shape has it has before, but with a smaller size. This is only possible for few shapes of edges, and on the particular case of arc of circles, only the straight edge could satisfy this paradigm (but anyway arise other problems). In the Type 1 rotation, all the edges become at one time the dividing wall and have the chance to completely change before they will have to assume again the role they had once before.

On the contrary, cell with odd numbers of edges beginning at 5 can undergo a semantic rotation of Type 2 during the division process. For instance, five-edged cells can undergo these two semantic rotations:



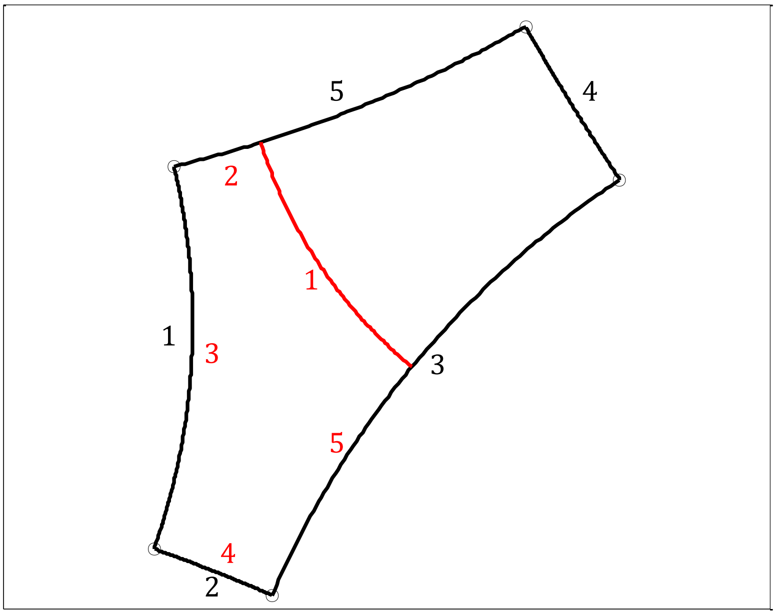


Figure 5. Difference between the Type 1and Type 2 semantic rotations
with the 5-edged example on the complete plane (γ=0deg)

$$1 \to 2 \to 3 \to 4 \to 5 \text{ and } 1 \to 3 \to 5 \to 2 \to 4.$$

In the first transformation, of Type 1, the edge number 5 is replaced by the dividing wall whereas in the rotation of Type 2, the fourth edge is replaced. You can see both examples of rotations in Figure 5.
We have gathered all the different rotations that could be undergone during a division by a solution cell in the following table. We have kept only the transformations that are not redundant with the symmetric transformations of lower rotations. For instance, the Type 2 rotation in a 3-edged cell equals the "Type -1" rotation – which is the symmetric of the Type 1 rotation. It is eventually basic algebra. To get all the possible solutions of the problem, we should determine all the shapes of the solutions for each possible transformation, depending on the number of edges of the final solution we want to get. Fortunately, we will see that there are no more than a few solutions for a cone, which are limited in their number of edges.

Number of Edges	2	3	4	5	6	7	8	9	10	11	12	13	14
Type	1=-1	1	1	1,2	1	1,2,3	1,3	1,2,4	1,3	1,2,3,4,5	1,3,5	1,2,3,4,5,6	1,3,5

Table 2. Existing Types of semantic rotation that could be undergone by the cell during the division, and which are not redundant with symmetric rotations.

The first important result is already at hand: if we make the hypothesis that each semantic rotation leads to only one solution (plus the symmetric), there is no more solutions than the number of semantic rotations. This hypothesis can be proven right now but we will prove it anyway through the resolution of the problem.

2.II. The good set of parameters

From each semantic rotation stem relations between the edges. These relations should characterize the shape of the mother cell, and give us the solution we seek. We do not know now if the geometrical constraints set by the definition of the semantic rotations are strong enough to determine one solution for each angle, nor do we know whether there will be a solution for each angle – a cell with a shape that will satisfy the relations set by the semantic rotation we consider. Nevertheless, we know that this transposition, from the isotropic growth hypothesis to the relations between edges imposed by the semantic rotation, is lossless in terms of constraints.

Let us see an example of what these relations are. If we consider the resolution of the problem for the 4-edged cell, there are only two set of solutions, one being the symmetric of the other. Considering the Type 1 semantic rotation give us all the solutions. Concerning the solution of this rotation, and minus the $\sqrt{2}$ ratio, we know that:

- all the characteristics of the edge 2 are these of the edge 3
- some of the characteristics of edge 1 are inherited from edge 2
- some of the characteristics of edge 3 are inherited from edge 4

We still need to add one last element to get the full equivalent of the perfect division problem: the edges have Errera shape. They are all arc of circles. Thus we need a good parameterization of the edges in order to translate the relations between them efficiently. The major challenge concerns the 3 edges that change partly (the new one, and the two adjacent), whereas the n-3 others are strictly preserved. We want the parameters which will characterize these 3 edges to change the less possible. In addition, these parameters must locate the cell regarding the center of the cone, which is also the center of the gamma plane. After many examples and trials, I found the golden set for each edge:

R_i The radius of curvature R_i is strictly maintained between the edges, regardless they are cut or not during the division. The $\sqrt{2}$ ratio is directly multiplied to R_i. This is the most convenient and obvious parameter.

h_i The distance hi from the center of the center to the edge has also the precious characteristic to be kept when an edge is cut in parts. The $\sqrt{2}$ ratio is also directly multiplied to h_i.

θ_i We need now to locate these edges around the point. Therefore, the choice of hi impose to take as the third parameter for each edge the angle between the segments [h_i].

This set of parameter is sufficient to completely define the solution, and the entire set is needed.

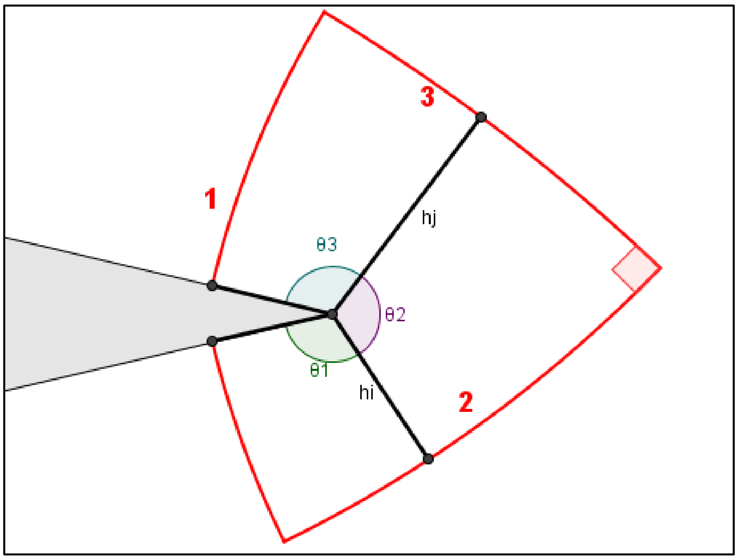
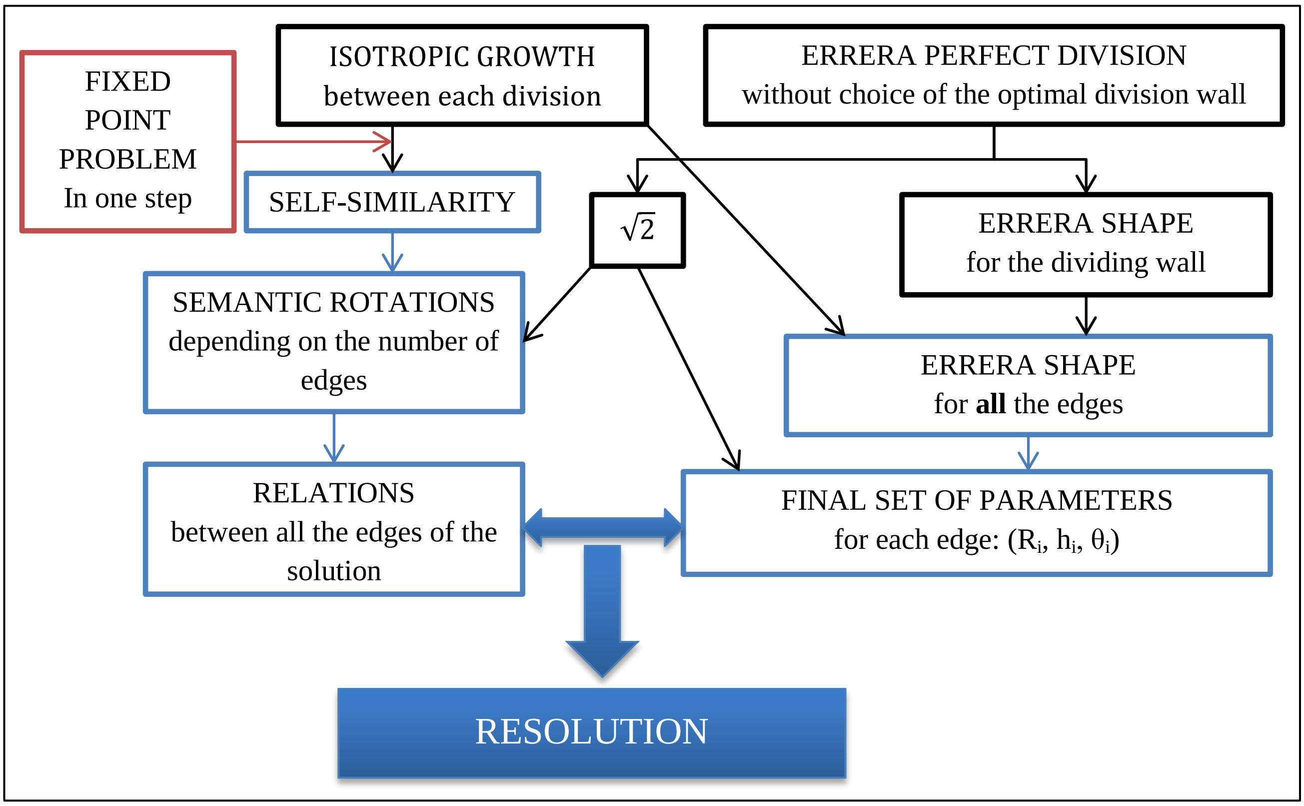


Figure 6. Illustration of the parameters h_i and θ_i

The black segments are the shortest distances between each edge in arc of circle and the center of the cone. These segments intersect the edges with right angles and will not change if an edge is cut. They define the distances h_i. In this case 'h_i' and 'h_j' are the first and second distances that define the second quarter between edge 2 and edge 3 (see next) but they should be named 'h_2' and 'h_3'.

2.III. Summary of the steps of reasoning

I try to summarize in the following figure the path we took throughout this section and the sub-section *"1.III Fixed point solution on the cone"*.



3. Explanation of the resolution process

Documentation of Matlab program

Provided the good set of parameters to describe each edge of the solution, provided the relations between all the edges, we must now carry on the resolution of the problem, with the aim of seeing and handling the solutions easily. In this section, I will describe the different steps carried out by my MATLAB program to solve the problem and then display it. It is the documentation that must be provided in order to fully understand the organization of the MATLAB program and the role of its components. In addition, the computing reader must refer to the lengthy help sections located at the beginning of each function of the program.

3.I. Birth of the geometrical quarters of resolution

We can now express the relations in the set of parameters we have chosen. We first draw the segments $[h_i]$ which link the center of the cone and the edge with the shortest distance. These segments are therefore attached to the edges with a right angle, as shown in Figure 6. They define quarters that will help us to get the final solution. We will go through the two main cases of Type 1 and Type 2, to show that this resolution process is a general method.

3.I.a Type 1 case

Let us consider the five-edged case. The formula to describe the Type 1 division process and the related semantic rotation of the edges would be:

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5$$

Given the conservation properties of the radius R_i, we can infer:

$$R_2 = \sqrt{2} * R_1$$
$$R_3 = \sqrt{2} * R_2 = 2 * R_1$$
$$R_4 = \sqrt{2} * R_3 = 2\sqrt{2} * R_1$$
$$R_5 = \sqrt{2} * R_4 = 4 * R_1$$

We get a similar result for h_i, because this parameter has the same properties of conservation (no change from an edge to a 'daughter' edge located on the previous one) and size modification (multiplication by $\sqrt{2}$ of the parameter for the daughter cell with half the area) compared to R_i:

$$h_i = \sqrt{2}^{i-1} * h_1$$

Concerning the θ angles, we must consider not only the length but the segments $[h_i]$, and we can see that only the angles around the new segment "$h_{1'}$", corresponding to the new dividing wall, change during the division process. To mark this, we will use the notation " ' " to denote the parameter of the edge in the daughter cell. By convention, θ_i is the angle between $[h_i]$ and $[h_{i+1}]$. The size modification for the theta angles is 1, and the angles that have similar roles between the mother and the daughter cells must have the same values. We get:

$$\theta_{i'} = \theta_i$$

Then:

$$\theta_{2'} = \boxed{\theta_1 = \theta_2}$$
$$\theta_{3'} = \boxed{\theta_2 = \theta_3}$$
$$\theta_{4'} = \boxed{\theta_3 = \theta_4}$$
$$\theta_5' \neq \theta_4$$
$$\theta_1' \neq \theta_5$$

This gives the definition of two angles θ_A and θ_B, and these relations do not depend on the number of edges. Actually, we see emerge two different quarters, A and B, at the heart of the cell. All the n-1 quarters defined by $[h_i]$ and $[h_{i+1}]$ for i=1…n-1, are all similar within a size difference*. They are the quarters of Type A whereas the quarter of Type B is only found between the two extreme $[h_n]$ and $[h_1]$. All the divisions of Type 1 have only these two Types of quarters, both of them with the same structure. The root of the differentiation between these two Types of quarters is the ratio r between the two lengths of the segments that define them. As a convention, we take the values for r that are greater than one, which implies that we flip the Quarter of Type B. In the Type 1 rotations we have:

$$r_A = \sqrt{2}$$
$$r_B = (\sqrt{2})^{n-1}$$

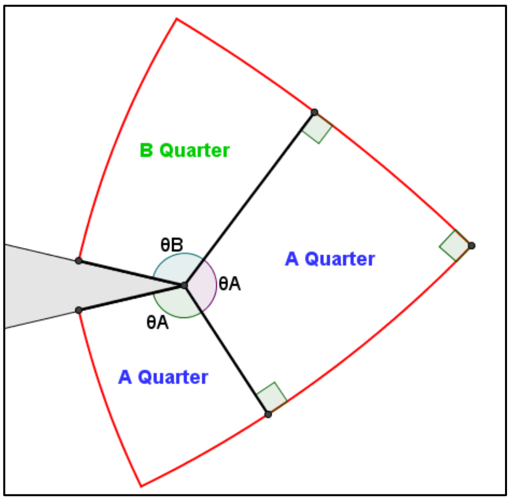


Figure 7. Displaying of the two Types of quarters for a cell with n=3

* The fact that the quarters are auto-similar comes from the shared angle θ, the shared ratio r, and the other properties that define a quarter (right angles, radius of the edges proportional to h, arc of circles).

Now that we have reduced the problem to these two quarters, we see finally how we have reached some general resolution of the problem, since it does not depend on the number of edges nor the angle gamma. These two constraints just impose the following relation, which link the two angles we need to determinate:

$$\gamma = 2\pi - (n-1) * \theta_A - \theta_B$$

3.I.b Type 2 case

The Type 2 semantic rotation is represented by:

$$1 \rightarrow 3 \rightarrow 5 \rightarrow 2 \rightarrow 4$$

Being this time the edge 4 which is replaced by the new edge. This led to these relations by a similar token than before:

$$R_3 = \sqrt{2}R_1$$
$$R_5 = 2R_1$$
$$R_2 = 2\sqrt{2}\,R_1$$
$$R_4 = 4R_1$$

With similar relations for h_i. By drawing Figure 8, we see now that we can extend our resolution implying two Types A and B of quarters to the semantic rotation of Type 2, and actually to any semantic rotation of Type N. We must only change the ratios r_A and r_B which defines the two quarters. Here we have:

$$r_A = \left(\sqrt{2}\right)^{E\left(\frac{n}{2}\right)}$$
$$r_B = \frac{r_A}{\sqrt{2}}$$

Which for five edges gives: $r_A = 2\sqrt{2}$ and $r_B = 2$.

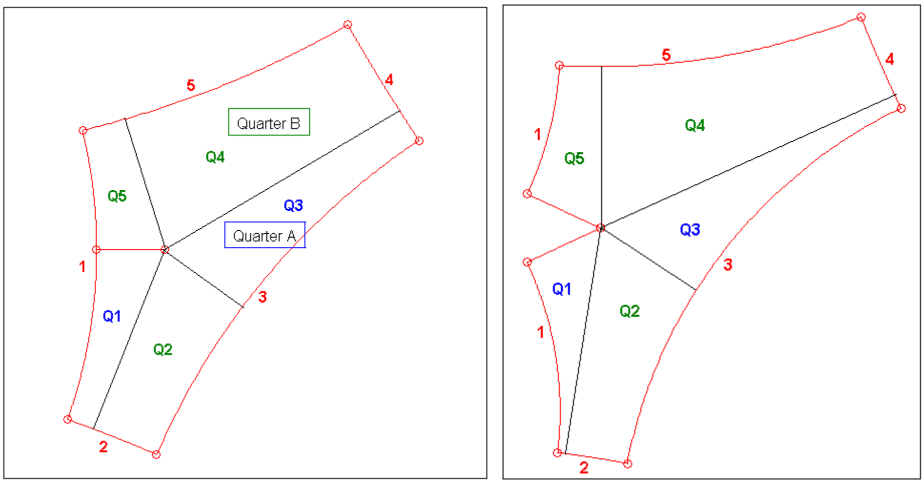


Figure 8. Quick draw of a 5-edged solution to a Type 2 rotation

The first draw is for γ=0° (full plane) and the second for γ=50°.
$[h_1]$ is in red whereas the other $[h_i]$ are in black. The B Quarters are qritten in green, the A quarters in blue.
Let us notice that the size of the segments $[h_i]$ increases by step of $\sqrt{2}$ with skipping one each time.

3.I.c What next? Summary

We have reduced the problem to these two different quarters. Before going further, we need a little recap of what we know about the cell. As seen just above, the semantic rotation gives us a lot of information on the set of parameters, combined with the particular geometry of the cell. If we set R_1=R as the value for normalization of the figure, we know:

R_i We know all the values of $(R_i)_{i=1...n}$ from R_1.

h_i We know with similar relation all the values $(h_i)_{i=1...n}$ given h_1. Actually, for $i = 1 ... n$, we know that $h_i = R_i * \delta$, because the relations that link the (h_i) family and the (R_i) family are exactly the same. Thus, we will use δ instead of h_1 as the unknown parameter to get the values of $(h_i)_{i=1...n}$.

θ_i We know that there are only two different values: θ_A and θ_B. But we know that these two unknowns are related by the equation:

$$\gamma = 2\pi - (n-1) * \theta_A - \theta_B$$

There is now only two unknowns left: δ and θ_A(or θ_B). We know that the rest of the information we need lies in these two quarters we have defined, and particularly in the structure of these quarters: the right angles and the edges in arc of circles.

3.II. Global sketch of the resolution

The figure presented here shows all the steps of the resolution. You see in the blue frames the working variables, and in red frames the functions that handle them.

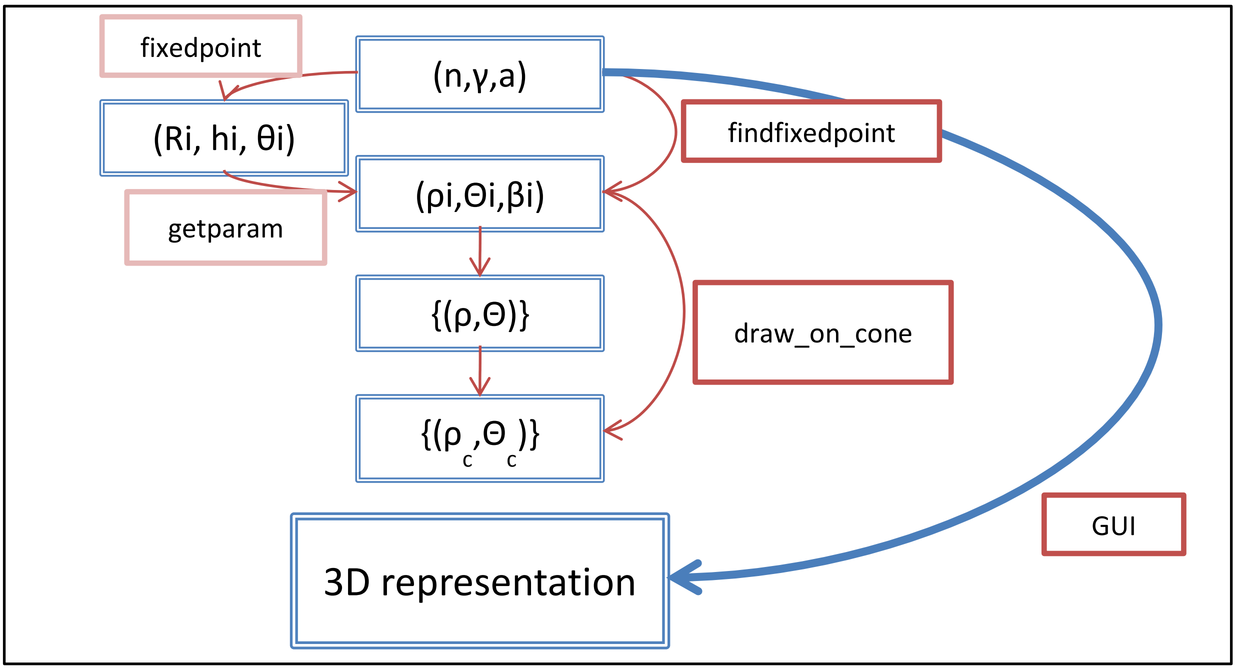
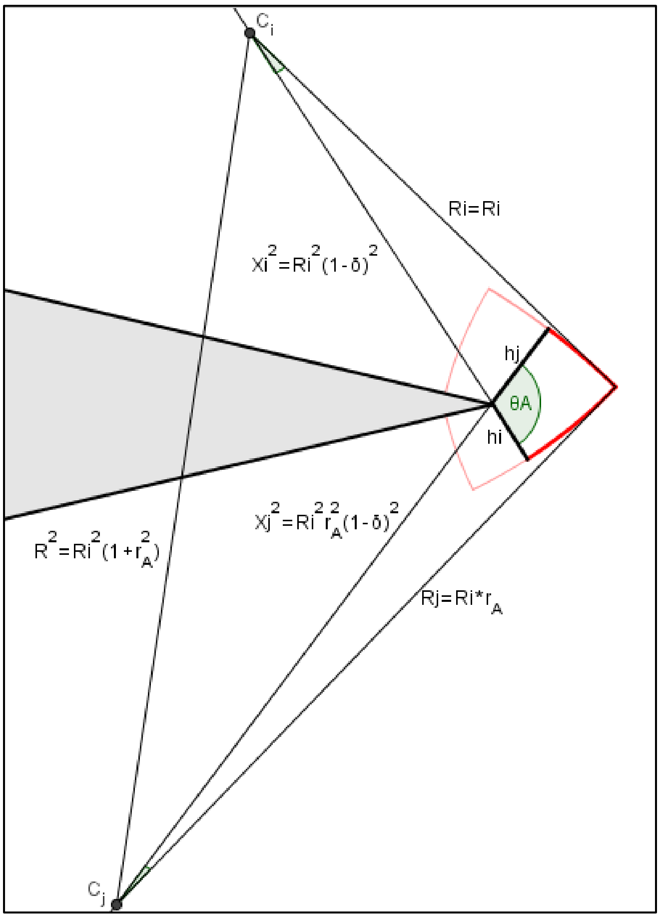


Figure 9. Global sketch of the resolution

3.III. Details of the resolution process

3.III.a Core resolution

The core resolution consists on the finding and characterizing the solution. Granted n, the number of edges, gamma, the angular parameter of the cone, and a, the normalizing area, we must find some values that entirely characterize the shape of the solution. This work is done by '`fixedpoint`', which solve equations in the working set of parameters (Ri, hi, θi) we have discussed above. This resolution takes its roots in the two quarters A and B that are defined by the drawing of the segments [hᵢ]. As stated above, we only need to determine the two unknowns δ and one angle θ. We need one equation by quarter, and this equation lies in the figure below.



The outer triangle is a right triangle, coming from the fact that the edges of the cell intersect with a right angle. This gives us a first relation:

$$R^2 = R_i^2(1 + r_A^2)$$

Afterwards, a simple Al Kashi formula applied to the inner triangle leads to the relation we are looking for:

$$R^2 = X_j^2 + X_i^2 - 2X_iX_jcos\theta_A$$

Soit, en simplifiant par R_i:

$$(1 + r_A^2) = (1 - \delta)^2(1 + r_A^2) - 2(1 - \delta)^2 r_A cos\theta_A$$

$i.e.:$

$$(1 + r_A^2) = (1 - \delta)^2(1 - 2r_A cos\theta_A + r_A^2)$$

Finally:

$$\boxed{\delta = \sqrt{\frac{1 + r_A^2}{1 - 2r_A cos\theta_A + r_A^2}}}$$

With r_A given by the semantic rotation we consider, this is a relation that we can write as: $\delta = f(\theta_A)$. Applying the same token to the B quarter give us the exact same relation $\delta = f(\theta_B) = g(\theta_A)$ using the relation we know between θ_A and θ_B. Resolving the system gives us all the values for the entire set of parameters (R_i, h_i, θ_i).

3.III.b Transformation of the set of parameters

We must note that the working set of parameters leads to the complete solution, since it enables anyone to draw the solution with a ruler, a compass and a measure of angles. Nevertheless it is not really convenient for computer drawing or analyzing of the shape. Thus, we need to get the solution in another set of parameters, which is basically the set used to describe any cell in the cellNetwork environment: the vertices coordinates (here, polar) plus the angles between the cord lengths and the curved edges. This task is carried out by the '`getparam`' function, used in the broader '`findfixedpoint`' function that give the solution in the convenient set of parameters from the input of n, gamma and a. It uses both '`fixedpoint`' and '`getparam`'.

3.III.c Displaying of solutions

Given the final values to describe the cell which is the solution of the problem, we need to put it back on the cone on two steps. First we will numerically create a big number of points to "map" the boundaries of the cell, translating the curvature in a big set of polar coordinates which describes the edges. Then we will "close" the gamma plane by to recreate the cone. This is done by a simple transformation described by the following equations:

$$\begin{cases} \theta' = \theta * \alpha \\ r' = \dfrac{r}{\alpha} \\ z = r * \sqrt{1 - \dfrac{1}{\alpha^2}} \end{cases}$$

With:

$$\alpha = \frac{\pi}{\pi - \frac{\gamma}{2}}$$

After setting up some visual elements and drawn the cone, we can display it in a nice 3D window of Matlab, to observe it accurately. REF is a snapshot of what is the result of this resolution process.

In order for the final user to easily get and manipulate these solutions, I have developed a GUI (Graphical User Interface) that displays dynamically the solutions for any angle and any number of edges. I have not yet succeeded in making this GUI portable and not platform dependent as a standalone application, but the Matlab files are available upon asking*.

3.IV. Remarks on the limitations of the resolution process

3.IV.a Geometrical limitations

We could spend a few lines and a figure to see why there are geometrical limitations to the existence of the solutions. The cell

3.IV.b Computational limitations

4. Results – Discussion

4.I. Topological results

4.II. Static vs. Dynamic

* Upon asking the creator and writer of this prose at antoine.lizee@polytechnique.edu or the contact and supervising professor, Jacques Dumais, at jdumais@oeb.harvard.edu). The M-files of the GUI are working on any recent version of MATLAB. They may be included in the numerical package containing this report.