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MEASURING OMEGA AND THE REAL CORRELATION FUNCTION FROM THE REDSHIFT CORRELATION FUNCTION

A. J. S. HAMILTON[1]

Joint Institute for Laboratory Astrophysics; and Department of Astrophysical, Planetary and Atmospheric Sciences; Box 440, University of Colorado, Boulder CO 80309

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ABSTRACT

Peculiar velocities distort the correlation function of galaxies in redshift space. In the linear regime, the distortion has a characteristic quadrupole plus hexadecapole form. The amplitude of the distortion depends on the cosmological density parameter Ω. Practical formulae are derived here which can be applied to redshift galaxy catalogs to measure Ω in the linear regime. The formulae also yield the real underlying correlation function in the linear regime, corrected for peculiar velocities.

Subject headings: cosmology: observations — cosmology: theory — galaxies: clustering

1. INTRODUCTION

Peculiar velocities distort the correlation function of galaxies observed in redshift space. Kaiser (1987) pointed out that in the linear regime in standard cosmology this distortion takes a particularly simple form in Fourier space. He showed that a wave of amplitude δ_k appears in redshift space to be amplified by a factor $(1 + f\mu_k^2)$, where μ_k is the cosine of the angle between the wavevector $\boldsymbol{k}$ and the line of sight, and

$$f \approx \Omega^{0.6} \tag{1}$$

is the dimensionless growth rate of growing modes in linear theory in a universe with density Ω relative to the critical density[2] (Peebles 1980, eq. [14.8]; see also Lahav et al. 1991). As a consequence, the power spectrum $|\boldsymbol{\delta}_k|^2$ observed in redshift space appears amplified over the true power spectrum $|\delta_k|^2$ by a factor $(1 + f\mu_k^2)^2$:

$$|\boldsymbol{\delta}_k|^2 = (1 + f\mu_k^2)^2 |\delta_k|^2 . \tag{2}$$

The power spectrum is the Fourier transform of the correlation function (Peebles 1980, § 41). Kaiser suggested that it should become possible to measure the cosmological density parameter Ω from the anisotropy of the redshift correlation function as redshift data become more extensive.

The purpose of this *Letter* is to complete the translation, begun by Kaiser, of his result (2) from Fourier space into real space. The result is a set of practical formulae which can be applied to redshift galaxy catalogs to measure Ω in the linear regime. The formulae also yield the real, deprojected correlation function in the linear regime.

2. THE FORWARD PROBLEM: FROM THE REAL TO THE REDSHIFT CORRELATION FUNCTION

Following Kaiser (1987), I assume the linear regime of the standard gravitational instability picture in the standard pressureless Friedmann cosmology (e.g., Peebles 1980). I imagine an observer observing structure in redshift space in the z-direction. I suppose that the structure is far enough away that the apparent line-of-sight displacements induced by peculiar velocities are effectively parallel, and I further assume that the structure being observed represents a fair and unbiased sample of matter in the universe.

The simplest way to translate Kaiser's (1987) formula (2) from Fourier space into real space is to recognize that the cosine μ_k, just a number in Fourier space, becomes an operator in real space

$$\mu_k^2 \equiv k_z^2/k^2 = (\partial/\partial z)^2(\nabla^2)^{-1} , \tag{3}$$

where $(\nabla^2)^{-1}$ denotes the inverse of the Laplacian operator. Thus equation (2) becomes in real space an operator equation relating the redshift correlation function $\boldsymbol{\xi}(\boldsymbol{r})$ to the true correlation function $\xi(r)$ (the redshift correlation function is printed in bold face to distinguish it from the true correlation function)

$$\boldsymbol{\xi}(\boldsymbol{r}) = [1 + f(\partial/\partial z)^2(\nabla^2)^{-1}]^2\xi(r) . \tag{4}$$

The solution of the inverse Laplacian is well known (it is the "potential" generated by a given "density"). Equation (4) thus reduces straightforwardly to an expression which is conveniently posed as a sum of spherical harmonics

$$\boldsymbol{\xi}(\boldsymbol{r}) = \xi_0(r)P_0(\mu) + \xi_2(r)P_2(\mu) + \xi_4(r)P_4(\mu) \tag{5}$$

with

$$\xi_0(r) = (1 + \tfrac{2}{3}f + \tfrac{1}{5}f^2)\xi(r) \tag{6}$$

$$\xi_2(r) = (\tfrac{4}{3}f + \tfrac{4}{7}f^2)[\xi(r) - \bar{\xi}(r)] \tag{7}$$

$$\xi_4(r) = \tfrac{8}{35}f^2[\xi(r) + \tfrac{5}{2}\bar{\xi}(r) - \tfrac{7}{2}\bar{\bar{\xi}}(r)] . \tag{8}$$

Here $\mu \equiv \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{z}}$ is the cosine of the angle, now in real space, between the pair separation $\boldsymbol{r}$ and the line of sight z, the $P_l(\mu)$ are Legendre polynomials [$P_0 = 1$, $P_2 = (3\mu^2 - 1)/2$, $P_4 = (35\mu^4 - 30\mu^2 + 3)/8$], and

$$\bar{\xi}(r) \equiv 3r^{-3}\int_0^r \xi(s)s^2\,ds \quad \text{and} \quad \bar{\bar{\xi}}(r) \equiv 5r^{-5}\int_0^r \xi(s)s^4\,ds . \tag{9}$$

Equations (5)–(8) express the redshift correlation function $\boldsymbol{\xi}(\boldsymbol{r})$ explicitly in terms of (integrals over) the true correlation function $\xi(r)$ in the linear regime. Equation (6), which predicts that in the linear regime the angle-averaged redshift correlation

[1] Presidential Young Investigator.

[2] I thank Jerry Ostriker for observing that f, whose value in linear theory comes from the continuity equation, is just the ratio v/Hr of the instantaneous peculiar velocity v, divided by the Hubble constant H, of a galaxy or structure to its peculiar displacement r since the universe began, and that nonstandard cosmologies may also predict a specific f which may have nothing to do with the cosmological density parameter Ω.

function ξ_0 should simply be amplified over the true correlation function ξ by the factor $1 + \frac{2}{3}f + \frac{1}{5}f^2$, was already pointed out by Kaiser (1987).

In the case of a pure power-law correlation function $\bar{\xi} \propto r^{-3-n}$ (note that, for a power law, the volume-averaged correlation function $\bar{\xi}$ is always positive, even if ξ is not), equations (6)–(8) become

$$\xi_0(r) = -\left(1 + \frac{2f}{3} + \frac{f^2}{5}\right)\frac{n}{3}\,\bar{\xi}(r) \tag{10}$$

$$\xi_2(r) = -\left(\frac{4f}{3} + \frac{4f^2}{7}\right)\frac{(3+n)}{3}\,\bar{\xi}(r) \tag{11}$$

$$\xi_4(r) = \frac{8f^2}{35}\,\frac{(3+n)(5+n)}{3(2-n)}\,\bar{\xi}(r)\,. \tag{12}$$

As another example, Figure 1 shows the harmonics ξ_l of the redshift correlation function predicted by equations (6)–(8), along with the real correlation function ξ and its integrals $\bar{\xi}$ and $\bar{\bar{\xi}}$, equation (9), for the case of the standard cold dark matter (CDM) power spectrum in the linear regime (Bardeen et al. 1986, eq. [G3]; Hamilton et al. 1991, eq. [5]). The figure illustrates the fact that in the linear regime, for reasonably well-behaved power spectra like CDM, the quadrupole harmonic ξ_2 is generically negative, the hexadecapole harmonic ξ_4 is generically positive (provided $n < 2$, eq. [12]), while the monopole harmonic ξ_0 can go through zero.

Physically, the negativity of the quadrupole harmonic ξ_2 of the redshift correlation function in the linear regime reflects the line-of-sight compression of clusters caused by infalling galaxies outside the turnaround radius. At the turnaround radius, the compression nominally goes to infinity, forming caustics in redshift space, but this nonlinear effect is not accounted for in linear theory. Inside the turnaround radius, the compression turns inside out, producing at smaller radii the "finger-of-god" effect and, presumably, a positive quadrupole harmonic ξ_2.

The hexadecapole harmonic ξ_4, and indeed all terms proportional to f^2 in all the equations above, arise from the dispersion of pair infall velocities.

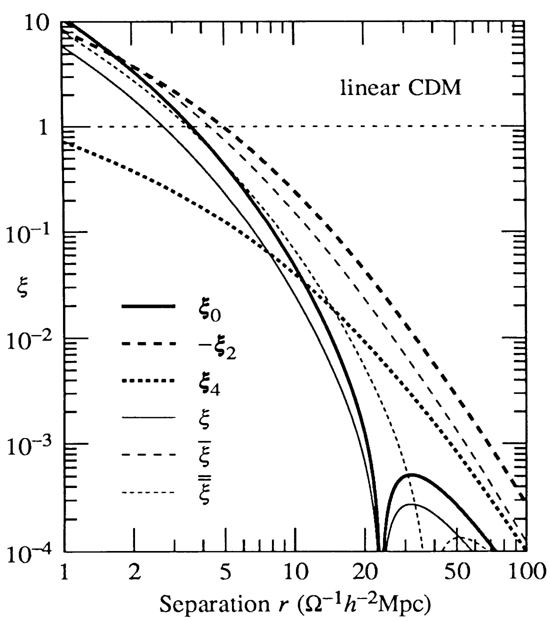

FIG. 1.—Harmonics $\xi_l(r)$ of the redshift correlation function (*thick lines*), along with the real correlation function $\xi(r)$ and its integrals $\bar{\xi}(r)$ and $\bar{\bar{\xi}}(r)$, eq. (9), (*thin lines*), for the standard cold dark matter power spectrum in the linear regime (Hamilton et al. 1991, eq. [5], normalized upward by a factor of 3). The amplitudes of the redshift functions are appropriate for $\Omega = 1$, i.e., $f = 1$; for other values of f the redshift functions should be scaled according to eqs. (6)–(8). Absolute values are plotted where the functions are negative. Note that ξ_2 is everywhere negative, ξ_0 and ξ are negative above about $23.5\Omega^{-1}\,h^{-2}$ Mpc, while $\bar{\bar{\xi}}$ is negative above about $38\Omega^{-1}\,h^{-2}$ Mpc.

3. THE INVERSE PROBLEM: FROM THE REDSHIFT TO THE REAL CORRELATION FUNCTION

In the previous section, a formula, equations (5)–(8), was obtained which expressed the observed redshift correlation function in the linear regime in terms of the true underlying correlation function. In this section, inverse formulae, equations (20)–(22) or alternatively (23)–(25), are derived which yield f and the true correlation function directly from the redshift correlation function, again in the linear regime.

The starting point is again Kaiser's (1987) result, equation (2), that the redshift power spectrum at wavevector $\boldsymbol{k}$ is amplified by $(1 + f\mu_k^2)^2$ over the true power spectrum. Notice that the amplification depends only on the *direction* of the wavevector $\boldsymbol{k}$ relative to the line of sight, not on its absolute value k. One way to project out all wavectors of a certain direction is to integrate the correlation function over planes normal to that direction. Let $\Xi(r)$ and $\boldsymbol{\Xi}(\boldsymbol{r})$ respectively denote the real and redshift correlation functions integrated over planes separated by distance $r \equiv |\boldsymbol{r}|$ and with normal in the direction $\hat{\boldsymbol{r}} \equiv \boldsymbol{r}/r$:

$$\Xi(r) \equiv \int \xi[(r^2 + s^2)^{1/2}]d^2s = \int_r^\infty \xi(s)2\pi s\,ds\,, \tag{13}$$

$$\boldsymbol{\Xi}(\boldsymbol{r}) \equiv \int \xi(\boldsymbol{s})\delta(\boldsymbol{s}\cdot\hat{\boldsymbol{r}} - r)d^3s\,. \tag{14}$$

Since the plane correlation function contains waves only in the direction normal to the plane, the plane correlation function Ξ in redshift space will simply be amplified by

$$\boldsymbol{\Xi}(\boldsymbol{r}) = (1 + f\mu^2)^2\Xi(r)\,, \tag{15}$$

where $\mu \equiv \hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{z}}$ is the real space cosine of the angle between the plane normal $\boldsymbol{r}$ and the line of sight $\boldsymbol{z}$. Equation (15) suggests that one way to proceed might be to measure the plane correlation function $\boldsymbol{\Xi}(\boldsymbol{r})$ directly from redshift data, and to look for the expected $(1 + f\mu^2)^2$ anisotropy. This is fine in principle, but impractical, for with the integrand of equation (14) for $\boldsymbol{\Xi}$ going as $\xi s\,ds$ at large separation s, there is little hope that the integral to infinity will converge in real, noisy data. To achieve faster convergence, first expand $\boldsymbol{\Xi}(\boldsymbol{r})$, equation (14), in spherical harmonics:

$$\boldsymbol{\Xi}(\boldsymbol{r}) = \sum_{l\ \mathrm{even}} \Xi_l(r)P_l(\mu)\,, \quad \Xi_l(r) = \int_r^\infty \xi_l(s)P_l(r/s)2\pi s\,ds\,. \tag{16}$$

The nonzero azimuthal ($m \neq 0$) harmonics vanish by symmetry about the line of sight, while the odd l harmonics vanish by pair exchange symmetry, $\boldsymbol{r} \rightarrow -\boldsymbol{r}$. Next, differentiate with respect to r:

$$-\frac{\partial\Xi_l(r)}{2\pi r\,\partial r} = \xi_l(r) - \int_r^\infty \xi_l(s)P_l'(r/s)ds/r\,, \tag{17}$$

where $P_l'(x) \equiv dP_l(x)/dx$ denotes the derivative of the Legendre polynomial. Except for $l = 0$, where the integral is identically zero, the integrands in equation (17) go as $\xi_l(s)ds/s$ at large separation s, two powers of s more convergent than equation (14) for Ξ. Differentiating the real plane correlation function $\Xi(r)$, equation (13), similarly yields

$$-\frac{\partial \Xi(r)}{2\pi r\, \partial r} = \xi(r) . \tag{18}$$

Combining the two equations (17) and (18) into the relation (15) gives

$$(1 + f\mu^2)^2 \xi(r) = \sum_{l \text{ even}} P_l(\mu)\left[\xi_l(r) - \int_r^\infty \xi_l(s) P_l'(r/s) ds/r\right] \tag{19}$$

which is now an explicit expression for f and the real correlation function ξ in terms of the harmonics ξ_l of the redshift correlation function. Although the summation in equation (19) is nominally over an infinite number of even l, the prediction is that only the $l = 0$, 2, and 4 harmonics should survive in the linear regime. These harmonics are, explicitly,

$$(1 + \tfrac{2}{3} f + \tfrac{1}{5} f^2)\xi(r) = \xi_0(r) \tag{20}$$

$$(\tfrac{4}{3} f + \tfrac{4}{7} f^2)\xi(r) = \xi_2(r) - 3 \int_r^\infty \xi_2(s) ds/s \tag{21}$$

$$\tfrac{8}{35} f^2 \xi(r) = \xi_4(r) - \tfrac{5}{2} \int_r^\infty \xi_4(s)[7(r/s)^2 - 3] ds/s . \tag{22}$$

Equations (20)–(22) are in effect the inverse of equations (6)–(8). While these equations (20)–(22) formally yield the real correlation function ξ in terms of the redshift correlation function ξ, they are probably not the equations of choice if the aim is to measure f, hence Ω. This is because $\xi(r)$ is liable to go through zero (as it does in CDM), or at any rate to be "unnaturally" small, in the range 20–100 h^{-1} Mpc of separations where ξ is likely to be both linear and measurable in existing or soon-to-exist redshift data. Thus the ratio of any pair of the three equations (20)–(22), which in principle gives f, in practice may be quite uncertain. The presence of terms of alternating sign on the right-hand sides of equations (21) and (22) hints that there is a cancellation of terms going on, which suggests that to measure f one would do better to manipulate the equations so as to reduce the level of cancellation.

A set of equations with fewer cancellations can be gotten from equations (20)–(22) by taking their volume average:

$$(1 + \tfrac{2}{3} f + \tfrac{1}{5} f^2)\bar{\xi}(r) = 3 \int_0^r \xi_0(s)(s/r)^3 ds/s \tag{23}$$

$$(\tfrac{4}{3} f + \tfrac{4}{7} f^2)\bar{\xi}(r) = -3 \int_r^\infty \xi_2(s) ds/s \tag{24}$$

$$\tfrac{8}{35} f^2 \bar{\xi}(r) = -\tfrac{3}{2} \int_r^\infty \xi_4(s)[7(r/s)^2 - 5] ds/s . \tag{25}$$

Unlike ξ, the volume average $\bar{\xi}$ should be everywhere positive for most reasonably well-behaved power spectra. Note that even though the integration from 0 to r extends into the nonlinear regime, the volume integrals $\bar{\xi}$, equation (9), and likewise $\int_0^r \xi_0(s)(s/r)^3 ds/s$ remain linear (Peebles 1980, § 70).

4. MEASURING OMEGA

Of the various possible equations which can be derived from the results of the previous section, the most rapidly convergent, with the least amount of cancellation, hence perhaps the most promising for measuring the cosmological density parameter Ω, that I have been able to find, are the following two pairs:

$$(1 + \tfrac{2}{3} f + \tfrac{1}{5} f^2)[\xi(r) - \bar{\xi}(r)] = \xi_0(r) - 3 \int_0^r \xi_0(s)(s/r)^3 ds/s \tag{26}$$

$$(\tfrac{4}{3} f + \tfrac{4}{7} f^2)[\xi(r) - \bar{\xi}(r)] = \xi_2(r) \tag{27}$$

and

$$(\tfrac{4}{3} f + \tfrac{4}{7} f^2)[\bar{\xi}(r) - \bar{\bar{\xi}}(r)] = -2 \int_0^r \xi_2(s)(s/r)^5 ds/s \tag{28}$$

$$\tfrac{8}{35} f^2 [\bar{\xi}(r) - \bar{\bar{\xi}}(r)] = 2 \int_r^\infty \xi_4(s)(r/s)^2 ds/s . \tag{29}$$

The ratio of equation (26) to (27) yields a value for f, hence Ω through equation (1), from the ratio of the monopole and quadrupole harmonics ξ_0 and ξ_2 of the redshift correlation function, while the ratio of equation (28) to (29) yields a value for f from the ratio of the quadrupole and hexadecapole harmonics ξ_2 and ξ_4. The equations thus provide redundant measures of f. The equations are further redundant in that they are true for any pair separation r large enough to be in the linear regime. A sensible idea would be to measure f at many r, and to use the consistency among the various values of f as one clue to the accuracy of its determination.

The only integral in equations (26)–(29) which extends to infinite separation s is equation (29), but it converges with encouraging rapidity, being two powers of s more convergent than equations (21) and (22) or (24) and (25), and four powers of s more convergent than equation (14) for the plane correlation function Ξ.

The only cancellation of terms on the right-hand sides of equations (26)–(29) is in the monopole equation (26), but this particular cancellation is to be regarded as a positive asset, since its main effect is to cancel out the mean galaxy density, which would normally have to be subtracted from neighbor counts to determine ξ_0 from observations. Indeed, the values of f determined from the ratios of equations (26) to (27) or (28) to (29) are entirely decoupled from the value of the mean galaxy density. This is nice, because uncertainty in the mean galaxy density is normally a plague on the measurement of the correlation function in the linear regime (e.g., de Lapparent, Geller, & Huchra 1988).

The harmonics ξ_l of the redshift correlation function are straightforward to measure as (suitably weighted) sums over galaxy pairs of the harmonics of each pair. The orthogonality of spherical harmonics makes it easy to project out the harmonics of a single pair. The only constraint is the obvious one that the redshift catalog must extend to large scales in the transverse as well as the line-of-sight direction. Thus a pencil beam redshift survey is no good for measuring Ω using equations (26)–(29), but a slice redshift survey is fine.

Real redshift data are complicated by noise, nonlinearity, biasing, and unfairness. In spite of these difficulties, the method of measuring Ω proposed here comes equipped with several tests of self-consistency, so that there is some hope of deciding whether what one is measuring is Ω or something else entirely. First, the higher order harmonics of the redshift correlation function, ξ_l for $l \geq 6$, should be small (nominally zero) compared to the $l = 0$, 2, 4 harmonics (the odd l harmonics vanish by pair exchange symmetry). Second, as long as the power spectrum is reasonably well behaved (for example, if the spectral index n varies not too rapidly), then ξ_2 should be negative, ξ_4 should be positive (provided $n < 2$, eq. [12]), and both should be monotonically decreasing in absolute value. This is unlike ξ_0, which can change sign. The test for negative, monotonic ξ_2 should be a particularly useful indicator, for fingers of god will probably make ξ_2 positive in the nonlinear regime. Finally, the values of f redundantly derived from the ratios of equations (26) to (27) and (28) to (29) at various separations r should of course agree.

The results of this *Letter* raise the hope that the cosmological density parameter Ω may be measured from the anisotropy of the redshift correlation function on scales 20–100 h^{-1} Mpc in the not too distant future.

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