MULTIVARIATE EXTREME VALUE DISTRIBUTION AND ITS FISHER INFORMATION MATRIX*

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Abstract

The paper is concerned with the basic properties of multivariate extreme value distribution (in the Logistic model). We obtain the characteristic function and recurrence formula of the density function. The explicit algebraic formula for Fisher information matrix is indicated. A simple and accurate procedure for generating random vector from multivariate extreme value distribution is presented.

Key words. Characteristic function, Fisher information matrix, Gumbel distribution, multivariate extreme value distribution

1. Introduction

Univariate extreme value theory and its successful applications promote the development of multivariate extreme value theory. The books[1-3] have summarized it on the probabilistic side. There are some papers[4-7] on multivariate extreme value distribution.

Let $(Y_{1i},\cdots,Y_{pi})$, $i=1,\cdots,n$ denote independent and identically distributed random vectors in R^p and define $M_n=(M_{1n},\cdots,M_{pn})$, where $M_{jn}=\max(Y_{j1},\cdots,Y_{jn})$, $j=1,\cdots,p$. Suppose there exist vectors $a_n=(a_{1n},\cdots,a_{pn})$ with each $a_{jn}>0$, $b_n=(b_{1n},\cdots,b_{pn})$ such that

$$\lim_{n\to\infty} P\{(M_{jn}-b_{jn})/a_{jn}<x_j,\ j=1,\cdots,p\}=G(x_1,\cdots,x_p),$$

where G is a nondegenerate p-dimensional distribution function. Then G is called a multivariate extreme value distribution.

If G is a multivariate extreme value distribution function, then each of its marginal distributions is of univariate extreme value and hence one of the classical three types. Different authors have assumed different marginal distributions, e.g. [8] has assumed the Gumbel distribution

$$H(x;\mu,\sigma)=\exp\Big\{-\exp\Big(-\frac{x-\mu}{\sigma}\Big)\Big\},\qquad -\infty<x<+\infty,$$

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where $-\infty < \mu < +\infty$, $\sigma > 0$.

As no finite parametric family exists for general dependence structure of $G(x_1, \cdots, x_p)$ it must be modelled in some other way. A simple model which appears to be widely applicable is the Logistic model. This is given by the joint distribution function

$$G(x_1, \cdots, x_p) = \exp\left\{ -\left[\sum_{i=1}^{p} \exp\left(-\frac{x_i - \mu_i}{\alpha\sigma_i}\right)\right]^{\alpha}\right\}. \tag{1.1}$$

$0 \le \alpha \le 1$ is called the dependence parameter, it measures the dependence between X_j, $j = 1 \cdots, p$, the extremes $\alpha \to 1$, $\alpha \to 0$ correspond respectively to independence and complete dependence.

The study of the basic properties of distribution (1.1) has received little attention. The purpose of this paper is to consider these properties. Section 2 presents the density function and the characteristic function of distribution (1.1). Also we illustrate a simple and accurate procedure for generating random vectors with distribution (1.1), which makes it possible to solve various statistical problems of the multivariate extreme value distribution. Section 3 is concerned with the explicit algebraic formulae for the Fisher information matrix.

2. Basic Properties

In this section we discuss the basic properties of the multivariate extreme value distribution (1.1). In order to avoid a complicated expression, let

$$y_i = \exp\left(-\frac{x_i - \mu_i}{\sigma_i}\right), \qquad i = 1, \cdots, p, \tag{2.1}$$

$$z_p = \left(\sum_{i=1}^{p} y_i^{\frac{1}{\alpha}}\right)^{\alpha}. \tag{2.2}$$

The subscript p may be omitted when there is no risk of confusion, and (1.1) can be written as

$$G(x_1, \cdots, x_p) = \exp\left\{ -\left(\sum_{i=1}^{p} y_i^{\frac{1}{\alpha}}\right)^{\alpha}\right\} = e^{-z}. \tag{2.3}$$

In the following, the four theorems are presented, and it is easy to prove them by means of induction. When $p = 2$ the corresponding results have been given by [9].

Theorem 1. The p-variate extreme value distribution has density

$$g_p(x_1, \cdots, x_p) = \frac{\partial^p G}{\partial x_1 \cdots \partial x_p} = \left(\prod_{i=1}^{p} \frac{y_i^{\frac{1}{\alpha}}}{\sigma_i}\right) z_p^{1-p/\alpha} Q_p(z_p, \alpha) e^{-z_p}, \tag{2.4}$$

where

$$Q_p(z, \alpha) = \left(\frac{p-1}{\alpha} - 1 + z\right) Q_{p-1}(z, \alpha) - z\frac{\partial Q_{p-1}(z, \alpha)}{\partial z}, \tag{2.5}$$

$$Q_1(z, \alpha) = 1.$$

We can write the density for $p=2,\cdots,5$, especially, from Theorem 1:

$$
\begin{aligned}
Q_2(z,\alpha) =& z+\frac{1}{\alpha}-1,\\
Q_3(z,\alpha) =& z^2+3\Big(\frac{1}{\alpha}-1\Big)z+\Big(\frac{1}{\alpha}-1\Big)\Big(\frac{2}{\alpha}-1\Big),\\
Q_4(z,\alpha) =& z^3+6\Big(\frac{1}{\alpha}-1\Big)z^2+\Big(\frac{1}{\alpha}-1\Big)\Big(\frac{11}{\alpha}-7\Big)z+\Big(\frac{1}{\alpha}-1\Big)\Big(\frac{2}{\alpha}-1\Big)\Big(\frac{3}{\alpha}-1\Big),\\
Q_5(z,\alpha) =& z^4+10\Big(\frac{1}{\alpha}-1\Big)z^3+5\Big(\frac{1}{\alpha}-1\Big)\Big(\frac{7}{\alpha}-5\Big)z^2\\
&+5\Big(\frac{1}{\alpha}-1\Big)\Big(\frac{2}{\alpha}-1\Big)\Big(\frac{5}{\alpha}-3\Big)z+\Big(\frac{1}{\alpha}-1\Big)\Big(\frac{2}{\alpha}-1\Big)\Big(\frac{3}{\alpha}-1\Big)\Big(\frac{4}{\alpha}-1\Big)
\end{aligned}
$$

and so on. In general, $Q_p(z,\alpha)$ is a $(p-1)$-order polynomial of z.

Let

$$
\begin{cases}
T_1=\cos^2\theta_1,\\
T_i=\Big(\prod\limits_{j=1}^{i-1}\sin^2\theta_j\Big)\cos^2\theta_i, & i=2,\cdots,p-1,\\
T_p=\prod\limits_{j=1}^{p-1}\sin^2\theta_j, & 0\le\theta_j\le\frac{\pi}{2},\ j=1,\cdots,p-1.
\end{cases}
\tag{2.6}
$$

The following transformation is very useful:

$$
y_i=zT_i^{\alpha},\qquad i=1,\cdots,p,
\tag{2.7}
$$

or

$$
x_i=\mu_i-\sigma_i(\log z+\alpha\log T_i),\qquad i=1,\cdots,p.
\tag{2.8}
$$

Theorem 2. The Jacobian of the transformation (2.8) is

$$
J_p=\frac{\partial(x_1,\cdots,x_p)}{\partial(z,\theta_1,\cdots,\theta_{p-1})}=\frac{\prod\limits_{i=1}^{p}(-2\alpha\sigma_i)}{2\alpha z\prod\limits_{i=1}^{p-1}(\cos\theta_i\sin\theta_i)}.
$$

We can obtain immediately the joint distribution of the random vector $(Z,\Theta_1,\cdots,\Theta_{p-1})$ by Theorem 1 and Theorem 2.

Theorem 3. The random vector $(Z,\Theta_1,\cdots,\Theta_{p-1})$ has distribution density

$$
f(z,\theta_1,\cdots,\theta_{p-1})=h(z)\prod_{j=1}^{p-1}f_j(\theta_j),
\tag{2.9}
$$

where

$$
h(z)=\frac{\alpha^{p-1}}{(p-1)!}Q_p(z,\alpha)e^{-z},\qquad z\ge 0,
\tag{2.10}
$$

$$
f_j(\theta_j)=2(p-j)\cos\theta_j(\sin\theta_j)^{2(p-j)-1},\qquad 0\le\theta_j\le\frac{\pi}{2},\ j=1,\cdots,p-1.
\tag{2.11}
$$

Hence $Z, \Theta_1, \cdots, \Theta_{p-1}$ are mutually independent.

Because $Q_p(z,\alpha)$ is a $(p-1)$-order polynomial of z, Z has a mixed Gamma distribution. Put

$$h(z) = \sum_{j=1}^{p} q_{p,j}\Gamma(z,j),$$

where

$$\Gamma(z,k) = \frac{1}{\Gamma(k)} z^{k-1} e^{-z}, \qquad z > 0$$

is the density function of Gamma distribution with parameter k, the weights, $q_{p,j}$, are decided by $Q_p(z,\alpha)$. It is easy to obtain the recurrence relations from (2.5):

$$\begin{aligned}
&q_{p,1} = \frac{\Gamma(p-\alpha)}{\Gamma(p)\Gamma(1-\alpha)}, \\
&(p-1)q_{p,j} = (p-1-j\alpha)q_{p-1,j} + \alpha(j-1)q_{p-1,j-1}, \qquad j = 2,\cdots,p-1, \\
&q_{p,p} = \alpha^{p-1}.
\end{aligned}$$

In fact we have shown an easy way to generate pseudorandom vectors from the distribution (1.1). Applying the inverse transform method, $\sin^2\Theta_j$, $\cos^2\Theta_j$ can be obtained from $U^{1/(p-j)}$, $1 - U^{1/(p-j)}$, respectively, where U is from the uniform distribution $U(0,1)$. By using the composition method, the random variate Z with the mixed Gamma distribution, $h(z)$, can be generated.

We now return to the characteristic function.

Theorem 4. Let the random vector $X = (X_1, \cdots, X_p)'$ has the distribution (1.1). Then the characteristic function of X is

$$\phi_x(t) = e^{it'\mu} \frac{\Gamma(1-it'\sigma)}{\Gamma(1-i\alpha t'\sigma)} \prod_{j=1}^{p} \Gamma(1 - i\alpha t_j \sigma_j), \tag{2.12}$$

where i is the imaginary unit and $\mu = (\mu_1, \cdots, \mu_p)'$, $\sigma = (\sigma_1, \cdots, \sigma_p)'$ are vectors of location and scale parameter in the marginal distribution, respectively, and $t = (t_1, \cdots, t_p)'$.

3. Fisher Information Matrix

By the theory of parameter estimation (for example, see [10]) the asymptotic covariance matrix of maximum likelihood estimators of parameters (α, ϕ_p) is $\frac{1}{n} I^{-1}(\alpha, \phi_p)$, where

$$I(\alpha,\phi_p) = E\begin{pmatrix} \left(\dfrac{\partial \ell}{\partial \alpha}\right)^2 & \dfrac{\partial \ell}{\partial \alpha}\dfrac{\partial \ell}{\partial \phi_p'} \\ \dfrac{\partial \ell}{\partial \alpha}\dfrac{\partial \ell}{\partial \phi_p} & \dfrac{\partial \ell}{\partial \phi_p}\dfrac{\partial \ell}{\partial \phi_p'} \end{pmatrix} \tag{3.1}$$

is the Fisher information matrix, $\phi_p' = (\mu', \sigma')$ is the marginal parameter vector and

$$\ell = \log g(x_1, \cdots, x_p) = -\sum_{i=1}^{p} \log \sigma_i + \frac{1}{\alpha}\sum_{i=1}^{p} \log y_i + \left(1 - \frac{p}{\alpha}\right)\log z - z + \log Q_p(z,\alpha) \tag{3.2}$$

is the log likelihood function for a single observation, and n is the sample size. From (2.1) and (2.2) we have these equalities:

$$\frac{\partial y_i}{\partial \mu_i} = \frac{y_i}{\sigma_i},$$

$$\frac{\partial y_i}{\partial \sigma_i} = -\frac{y_i}{\sigma_i}\log y_i,$$

$$\frac{\partial z}{\partial \alpha} = -2z\sum_{j=1}^{p-1}\sin^2\theta_1\ldots\sin^2\theta_{j-1}\,(\sin^2\theta_j\,\log\sin\theta_j + \cos^2\theta_j\,\log\cos\theta_j),$$

$$\frac{\partial z}{\partial \mu_i} = \frac{1}{\sigma_i}y_i^{\frac{1}{\alpha}}z^{1-\frac{1}{\alpha}},$$

$$\frac{\partial z}{\partial \sigma_i} = -\frac{1}{\sigma_i}y_i^{\frac{1}{\alpha}}z^{1-\frac{1}{\alpha}}\log y_i.$$

After a considerable manipulation, we can obtain the following score statistics:

$$\begin{cases} -\dfrac{\partial \ell}{\partial \alpha} = \dfrac{U}{\alpha} + RW - \dfrac{B}{Q}, \\ \sigma_i\dfrac{\partial \ell}{\partial \mu_i} = \dfrac{1}{\alpha} + WT_i, \\ -\sigma_i\dfrac{\partial \ell}{\partial \sigma_i} = 1 + (\log z + \alpha\log T_i)\left(\dfrac{1}{\alpha} + WT_i\right), \qquad 1 \le i \le p, \end{cases} \tag{3.3}$$

where

$$\begin{aligned} R =& R(\theta_1,\cdots,\theta_{p-1}) = \sum_{j=1}^{p-1}\sin^2\theta_1\cdots\sin^2\theta_{j-1}(\sin^2\theta_j\,\log\sin^2\theta_j + \cos^2\theta_j\,\log\cos^2\theta_j) \\ =& \sum_{j=1}^{p} T_j\,\log T_j, \\ U =& U(\theta_1,\cdots,\theta_{p-1}) = \sum_{j=1}^{p-1}\left[(p-j)\log\sin^2\theta_j + \log\cos^2\theta_j\right] = \sum_{j=1}^{p}\log T_j, \\ W =& W(z,\alpha) = 1 - \frac{p}{\alpha} - z + z\frac{A}{Q}, \\ A =& A(z,\alpha) = \frac{\partial Q_p(z,\alpha)}{\partial z}, \\ B =& B(z,\alpha) = \frac{\partial Q_p(z,\alpha)}{\partial \alpha}, \\ Q =& Q_p(z,\alpha). \end{aligned} \tag{3.4}$$

Theorem 3 shows that Z, $\Theta_1,\cdots,\Theta_{p-1}$ are mutually independent and gives the distribution density functions individually; it provides convenience for obtaining the explicit algebraic formula for elements of the Fisher information matrix. We see that (3.1) are the expectations of the squares and crossproducts of (3.3), continuous function of Z, $\Theta_1,\cdots,\Theta_{p-1}$ alone. By a complicated calculation we have

$$\begin{aligned} E\left(\frac{\partial \ell}{\partial \alpha}\right)^2 =& \frac{p^2}{\alpha^2}[e_0^2 - f_0 - 2(e_1^2 - f_1)] + \left\{e_2^2 - f_2 + \frac{p-1}{p+1}\left(e_2 + \frac{3}{2} - \frac{\pi^2}{6}\right)\right\}M_1 \\ & - \frac{p(p-1)}{\alpha^2}\left(4e_0 + \frac{2}{p} - \frac{\pi^2}{6}\right) + L + 2e_1N_1, \end{aligned} \tag{3.5}$$

$$\sigma_i E\left(\frac{\partial \ell}{\partial \alpha}\frac{\partial \ell}{\partial \mu_i}\right) = -\frac{pe_0}{\alpha^2} + \frac{e_1}{p}M_1 + \frac{N_1}{p}, \tag{3.6}$$

$$\sigma_i E\left(\frac{\partial \ell}{\partial \alpha}\frac{\partial \ell}{\partial \sigma_i}\right) = \frac{p}{\alpha}\left(e_1 - \frac{\gamma}{\alpha}e_0 + f_1\right) + \frac{p-1}{\alpha}\frac{\pi^2}{6} + \frac{\alpha}{p}\left\{e_1^2 - f_1 + \frac{1-p}{p+1}\left(\frac{\pi^2}{6} - 1\right)\right\}M_1 + \frac{\alpha e_1}{p}N_1 - \frac{e_1}{p}M_2 - \frac{1}{p}N_2, \tag{3.7}$$

$$\sigma_i^2 E\left(\frac{\partial \ell}{\partial \mu_i}\right)^2 = -\frac{1}{\alpha^2} + \frac{2M_1}{p(p+1)}, \tag{3.8}$$

$$\sigma_i\sigma_j E\left(\frac{\partial \ell}{\partial \mu_i}\frac{\partial \ell}{\partial \mu_j}\right) = -\frac{1}{\alpha^2} + \frac{M_1}{p(p+1)}, \tag{3.9}$$

$$\sigma_i^2 E\left(\frac{\partial \ell}{\partial \mu_i}\frac{\partial \ell}{\partial \sigma_i}\right) = \frac{2}{\alpha} - \frac{\gamma}{\alpha^2} + \frac{2}{p(p+1)}(\alpha e_2 M_1 - M_2), \tag{3.10}$$

$$\sigma_i\sigma_j E\left(\frac{\partial \ell}{\partial \mu_i}\frac{\partial \ell}{\partial \sigma_j}\right) = \frac{1}{\alpha} - \frac{\gamma}{\alpha^2} + \frac{1}{p(p+1)}\left\{\alpha\left(e_2 + \frac{1}{2}\right)M_1 - M_2\right\}, \tag{3.11}$$

$$\sigma_i^2 E\left(\frac{\partial \ell}{\partial \sigma_i}\right)^2 = \frac{4\gamma}{\alpha} - 1 - \frac{1}{\alpha^2}\left(\frac{\pi^2}{6} + \gamma^2\right) + \frac{2}{p(p+1)}\{\alpha^2(e_2^2 - f_2)M_1 - 2\alpha e_2 M_2 + M_3\}, \tag{3.12}$$

$$\sigma_i\sigma_j E\left(\frac{\partial \ell}{\partial \sigma_i}\frac{\partial \ell}{\partial \sigma_j}\right) = \frac{2\gamma}{\alpha} - 1 - \frac{1}{\alpha^2}\left(\frac{\pi^2}{6} + \gamma^2\right) + \frac{\pi^2}{6} + \frac{1}{p(p+1)}\cdot \left\{\alpha^2\left(e_2^2 - f_2 + e_2 - \frac{\pi^2}{6} + \frac{3}{2}\right)M_1 - \alpha(2e_2+1)M_2 + M_3\right\}, \tag{3.13}$$

where

$$\begin{cases} b_k = \prod_{i=1}^{p-1}(i + k\alpha), & c_k = \sum_{i=1}^{p-1}\frac{1}{i + k\alpha}, \\ d_k = -\sum_{i=1}^{p-1}\frac{1}{(i+k\alpha)^2}, & e_k = \sum_{i=1}^{p-1}\frac{1}{i+k}, \\ f_k = -\sum_{i=1}^{p-1}\frac{1}{(i+k)^2}, & k = 0, 1, 2, \end{cases} \tag{3.14}$$

$$M_1 = \frac{2B_1}{(p-1)!} + \frac{p^2 - \alpha^2}{\alpha^2} + I_1,$$

$$M_2 = \frac{1}{(p-1)!}[(3-2\gamma)B_1 + 2\alpha B_2] + \frac{p^2-\alpha^2}{\alpha^2}(\alpha c_0 - \gamma) + \frac{2}{\alpha}(p - \alpha) + I_2,$$

$$M_3 = \frac{1}{(p-1)!}\left\{2B_1\left(\frac{\pi^2}{6} + \gamma^2 - 3\gamma + 1\right) + 2\alpha B_2(3 - 2\gamma) + 2\alpha^2(2b_1D_1 - b_2D_2)\right\} + 4\frac{p-\alpha}{\alpha}(\alpha c_0 - \gamma) + \frac{p^2-\alpha^2}{\alpha^2}\left(\frac{\pi^2}{6} + \gamma^2 - 2\alpha\gamma c_0 + \alpha^2 D_0\right) + I_3,$$

$$N_1 = \frac{b_1}{(p-1)!}\left(\frac{p-1}{\alpha} - c_1\right) + \frac{(p-1)(p-\alpha)}{\alpha^2} + V_1,$$

$$N_2 = \frac{(2-p)(\alpha-p)}{\alpha}c_0 + \frac{(p-1)(\alpha-p)}{\alpha^2}\gamma, - \frac{b_1}{(p-1)!}\left(\alpha D_1 + (3 - p - \gamma)c_1 + \frac{(1-p)(1-\gamma)}{\alpha}\right) + V_2,$$

$$B_1 = 2b_1 - b_2, \qquad B_2 = 2b_1c_1 - b_2c_2,$$

$$D_k = c_k^2 + d_k, \qquad k = 0, 1, 2,$$

and

$$V_k = E_p\left(\frac{ZAB}{Q^2}\log^{k-1} Z\right), \qquad I_k = E_p\left(\frac{Z^2A^2}{Q^2}\log^{k-1} Z\right), \qquad k = 1, 2, 3,$$
$$L = E_p\left(\frac{B^2}{Q^2}\right),$$

which must be evaluated numerically; it is easy by standard routines for univariate numerical integration. $\gamma = 0.5772\cdots$ is Euler's constant, other notations are as in (3.4).

From (3.5)–(3.13) and the obvious symmetries of distribution (1.1) we can write all of the elements of Fisher information matrix of p-variate extreme value distribution with Gumble's margins in Logistic model.

Especially, when $p = 2$, the expressions reduce to

$$E\left(\frac{\partial \ell}{\partial \alpha}\right)^2 = \frac{1}{\alpha^2} + \left(\frac{5}{6} - \frac{\pi^2}{18}\right)\left(I_1 - \frac{2}{\alpha^2} + 1\right) + L + V_1,$$
$$\sigma_i E\left(\frac{\partial \ell}{\partial \alpha}\frac{\partial \ell}{\partial \mu_i}\right) = \frac{1}{4} + \frac{1}{4}I_1 + \frac{1}{2}V_1,$$
$$\sigma_i E\left(\frac{\partial \ell}{\partial \alpha}\frac{\partial \ell}{\partial \sigma_i}\right) = \left(\frac{5}{6} - \frac{\pi^2}{18}\right)\left(\frac{\alpha}{2} - \frac{1}{\alpha} + \frac{\alpha}{2}I_1\right) - \frac{1}{4}\left(1 - \gamma + \alpha - \frac{4}{\alpha} - \alpha V_1 + 2V_2 + I_2\right),$$
$$\sigma_i^2 E\left(\frac{\partial \ell}{\partial \mu_i}\right)^2 = \frac{1}{3}\left(1 + \frac{1}{\alpha^2} + I_1\right),$$
$$\sigma_i\sigma_j E\left(\frac{\partial \ell}{\partial \mu_i}\frac{\partial \ell}{\partial \mu_j}\right) = \frac{1}{6}\left(1 - \frac{2}{\alpha^2} + I_1\right),$$
$$\sigma_i^2 E\left(\frac{\partial \ell}{\partial \mu_i}\frac{\partial \ell}{\partial \sigma_i}\right) = \frac{1}{9}\left(\alpha - \frac{2}{\alpha} + \alpha I_1\right) + \frac{1}{3}\left(\frac{\gamma}{\alpha^2} - 1 + \gamma - \alpha - I_2\right),$$
$$\sigma_i\sigma_j E\left(\frac{\partial \ell}{\partial \mu_i}\frac{\partial \ell}{\partial \sigma_j}\right) = \frac{1}{6}\left(\frac{4}{3\alpha} - 1 + \gamma - \frac{\alpha}{6} - \frac{2\gamma}{\alpha^2} + \frac{5\alpha}{6}I_1 - I_2\right),$$
$$\sigma_i^2 E\left(\frac{\partial \ell}{\partial \sigma_i}\right)^2 = \frac{1}{18}\left\{\pi^2\left(1 + \frac{1}{\alpha^2}\right) - \frac{8\gamma}{\alpha} + \frac{28}{3} + \frac{6\gamma^2}{\alpha^2} + 6(\gamma - 1)^2 + 8\alpha(1-\gamma) - \frac{8}{3}\alpha^2 + \frac{4}{3}\alpha^2 I_1 - 4\alpha I_2 + 6I_3\right\},$$
$$\sigma_i\sigma_j E\left(\frac{\partial \ell}{\partial \sigma_i}\frac{\partial \ell}{\partial \sigma_j}\right) = \frac{1}{6}\left\{\frac{7}{18}\alpha^2 - \frac{19}{9} + \frac{1}{3}\left(\alpha - \gamma\alpha + \frac{8\gamma}{\alpha}\right) + \frac{\pi^2}{6}(3 - \alpha^2) + (\gamma - 1)^2 - \frac{2}{\alpha^2}\left(\frac{\pi^2}{6} + \gamma^2\right) + \left(\frac{37}{18} - \frac{\pi^2}{6}\right)\alpha^2 I_1 - \frac{5\alpha}{3}I_2 + I_3\right\}.$$

These agree with the bivariate case given by [9]. When $p = 1$, we have

$$b_k = 1, \qquad c_k = d_k = e_k = f_k = 0, \qquad I_1 = I_2 = I_3 = 0$$

and

$$M_1 = 1 + \frac{1}{\alpha^2},$$
$$M_2 = 1 - \gamma - \frac{\gamma}{\alpha^2} + \frac{2}{\alpha},$$
$$M_3 = 2 - \frac{4\gamma}{\alpha} - 2\gamma + \left(1 + \frac{1}{\alpha^2}\right)\left(\frac{\pi^2}{6} + \gamma^2\right).$$

We derive the well-known results again:

$$\sigma^2 E\left(\frac{\partial \ell}{\partial \mu}\right)^2 = 1, \qquad \sigma^2 E\left(\frac{\partial \ell}{\partial \mu}\frac{\partial \ell}{\partial \sigma}\right) = \gamma - 1,$$
$$\sigma^2 E\left(\frac{\partial \ell}{\partial \sigma}\right)^2 = \frac{\pi^2}{6} + (1-\gamma)^2.$$

We see that the elements of the Fisher information matrix can be expressed in terms of elementary functions only involving α; it is clear that they are independent of the marginal location parameters μ_i and are connected with the scale parameters σ_i in scale factor. Mainly $I(\alpha, \phi_p)$ is dependent on the dependence parameter α. We are sure that the approach followed in the present paper might be extended to Generalized Extreme Value distribution margins.

References

[1] Galambos J. The Asymptotic Theory of Extreme Order Statistics (2nd ed.). Krieger, Melbourne, Florida, 1987.

[2] Leadbetter, M.R., Lindgren, G. and Rootzen, H. Extremes and Related Properties of Random Sequences and Series. Springer-Verlag, New York, 1983.

[3] Resnick, S.I. Extreme Values, Point Processes and Regular Variation. Springer-Verlag, New York, 1987.

[4] Tawn, J.A. Bivariate Extreme Value Theory—Models and Estimation. *Biometrika*, 1988, 75: 397–415.

[5] Tawn, J.A. Modelling Multivariate Extreme Value Distributions. *Biometrika*, 1990, 77: 245–253.

[6] Smith, R.L., Tawn, J.A. and Yuen, H.K. Statistics of Multivariate Extreme. *Internat. Statist. Review*, 1990, 58: 47–58.

[7] Coles, S.G. and Tawn, J.A. Modelling Extreme Multivariate Events. *J. R. Statist. Soc.*, 1991, B 53: 377–392.

[8] Tiago de Oliveira, J. Bivariate Models foı Extremes. In Statistical Extremes and Applications, Reidel, Dordrecht, 131–153, 1984.

[9] Shi Daoji, Smith, R.L. and Coles, S.G. Joint Versus Marginal Estimation for Bivariate Extremes. Techn. Rep. 2074, Department of Statistics, University of North Carolina at Chapel Hill, 1992.

[10] Serfling, R.J. Approximation Theorems of Mathematical Statistics. Wiley, New York, 1980.