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Simulating Multivariate Extreme Value Distributions of Logistic Type

ALEC STEPHENSON
Department of Mathematics and Statistics, Lancaster University, Lancaster, LA1 4YF, U.K.
E-mail: a.stephenson@lancaster.ac.uk

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Abstract. Methods are given for simulating from symmetric and asymmetric versions of the multivariate logistic distribution, and from other multivariate extreme value distributions based on the well known logistic model. We consider two general approaches. The first approach uses transformations to derive random variables with a joint distribution function from which it is easy to simulate. The second approach derives from a specification of conditionally independent marginal components, conditioning on positive stable random variables. This specification extends to models of nested or hierarchical type and leads to an efficient way of incorporating marginal censoring. The algorithms presented in Sections 2 and 3 are available on request from the author. They are also included in the R (Ihaka and Gentleman, 1996) package evd (Stephenson, 2002), which is available from http://www.maths.lancs.ac.uk/~stephena/.

Key words. multivariate extreme value distribution, positive stable distribution, simulation

1. Introduction

The class of multivariate extreme value distributions can be characterized by the property of max-stability (e.g. Resnick, 1987). Let G be a d-dimensional distribution function with non-degenerate margins. All subsequent vector operations are applied componentwise. Suppose that, for all positive integers m, there exists normalizing vectors ϕ_m and ψ_m such that $\phi_m > 0$ and

$$G^m(\phi_m x + \psi_m) = G(x), \tag{1}$$

for all $x = (x_1, \ldots, x_d)$. Then G is a multivariate extreme value distribution function. The jth univariate marginal distribution of G is given by

$$G_j(z) = \exp\left[-\left\{1 + \xi_j\left(z - \mu_j\right)/\sigma_j\right\}_+^{-1/\xi_j}\right],$$

the generalized extreme value distribution, where (μ_j, σ_j, ξ_j) are the marginal parameters for the jth margin, $\sigma_j > 0$ and $h_+ = \max(h, 0)$. This will be denoted by GEV(μ_j, σ_j, ξ_j). The Gumbel distribution is obtained in the limit as the shape parameter $\xi_j \to 0$. The Fréchet and Weibull distributions are obtained when $\xi_j > 0$ and $\xi_j < 0$ respectively.

Multivariate extreme value distributions are important in extreme value theory since they form the complete class of limiting distributions of normalized componentwise maxima with non-degenerate margins. We consider simulating only from distributions with standard Fréchet univariate margins, where $\mu_j = \sigma_j = \xi_j = 1$ for each $j = 1, \ldots, d$, so that $G_j(z) = \exp(-1/z)$. This causes no loss in generality since an appropriate transformation in each margin yields the desired marginal form. No finite parameterization exists for the dependence structure of G. Sub-classes of differentiable parametric models are typically assumed and subsequently used for inference. In many cases it is easier to represent these subclasses in terms of the exponent measure, which we define as $V(\cdot) = -\log G(\cdot)$. Any multivariate distribution function whose exponent measure is homogeneous of order -1 (so that $V(tx) = V(x)/t$ for all $t > 0$ and all $x = (x_1, \ldots, x_d) > 0$) is multivariate extreme value, with Fréchet univariate margins.

There has been relatively little work concerned with the simulation of multivariate extreme value distributions. Shi et al. (1992) suggest a method for simulating from the bivariate (symmetric) logistic model, apparently based on a representation due to Lee (1979). The result on which this method is based is also given in Genest and Rivest (1993, Proposition 1.1). Ghoudi et al. (1998) derive simulation algorithms for a number of bivariate extreme value distribution functions. Shi (1999) gives a method for simulating from a simple trivariate nested logistic model (defined subsequently in equation 9). Shi (1995b) suggests a method for simulating from the multivariate (symmetric) logistic distribution (defined subsequently in equation 2). The simulation of simple multivariate nested logistic forms is considered by Shi et al. (1997). Shi (1995b) and Shi et al. (1997) are the only publications known to the author referring to the simulation of extreme value distributions in general dimensions. The simulation of Poisson processes related to multivariate extreme value models is considered by Nadarajah (1999).

Here simulation methods are presented for multivariate extreme value distributions of logistic type, which comprise the most flexible existing parametric forms for general dimensions. Simulation is increasingly used as part of statistical inference, for example, prediction, goodness-of-fit tests and confidence set construction. In particular, simulation provides a universal approach to the construction of confidence sets for critical extreme combinations of variables. In the Bayesian paradigm, the estimation of posterior predictive distributions for features of interest requires efficient simulation methods.

The remainder of this paper is organized as follows. In Section 2 we introduce symmetric and asymmetric versions of the multivariate logistic model, and present algorithms from which they can be simulated. The algorithm for the symmetric model is (using slightly different transformations) reproduced from Shi (1995b). In Section 3 we present alternative algorithms, based on the specification of conditionally independent marginal components, conditioning on positive stable random variables. Section 4 compares the algorithms given in Sections 2 and 3. The specification of conditionally independent marginal components has practical advantages when extending the algorithms to models that incorporate marginal censoring and models of nested or hierarchical type. We consider these extensions in Section 5.

The algorithms presented in Sections 2 and 3 are available on request from the author.

They are also included in the R (Ihaka and Gentleman, 1996) package evd (Stephenson, 2002), which is available from http://www.maths.lancs.ac.uk/~stephena/.

2. Transformation methods

Shi (1995b) suggests the following method for simulating from the multivariate logistic model

$$V_L(x) = \left(\sum_{j=1}^{d} x_j^{-1/\alpha} \right)^{\alpha} \tag{2}$$

of Gumbel (1960), where $\alpha \in (0,1]$. Suppose $X = (X_1, \ldots, X_d)$ is distributed as multivariate logistic. Consider the transformations

$$Z = \left(\sum_{j=1}^{d} X_j^{-1/\alpha} \right)^{\alpha} \tag{3}$$

$$T_i = (X_i Z)^{-1/\alpha} = \frac{X_i^{-1/\alpha}}{\sum_{j=1}^{d} X_j^{-1/\alpha}} \tag{4}$$

of Shi (1995a), so that $\sum_{i=1}^{d} T_i = 1$. Let $\mathscr{S}_d = \{(\omega_1, \ldots, \omega_d) \in \mathbb{R}_+^d : \sum_j \omega_j = 1\}$ denote the $d-1$ dimensional unit simplex. Shi (1995a) shows that $T_1, \ldots, T_d$ are independent of Z, $(T_1, \ldots, T_d)$ is distributed uniformly on $\mathscr{S}_d$, and that Z is a mixture of gamma distributions with density

$$f_d(z) = \sum_{j=1}^{d} p_{d,j} \Gamma(z, j),$$

where

$$\Gamma(z, k) = \frac{1}{\Gamma(k)} z^{k-1} e^{-z}, \quad z > 0,$$

is the density function of a gamma distribution with unit scale and shape parameter k. The mixture probabilities can be calculated using the recurrence relations

$$\begin{aligned} p_{d,1} &= \frac{\Gamma(d-\alpha)}{\Gamma(d)\Gamma(1-\alpha)}, \\ (d-1)p_{d,j} &= (d-1-\alpha j)p_{d-1,j} + \alpha(j-1)p_{d-1,j-1}, \quad j = 2, \ldots, d-1, \\ p_{d,d} &= \alpha^d, \end{aligned}$$

given in Shi (1995b). The first few values are

$$\begin{aligned} &p_{2,1} = 1-\alpha, \quad p_{2,2} = \alpha \\ &p_{3,1} = (1-\alpha/2)(1-\alpha), \quad p_{3,2} = 3\alpha(1-\alpha)/2, \quad p_{3,3} = \alpha^2 \\ &p_{4,1} = \textstyle\prod_{i=1}^{3}(1-\alpha/i), \quad p_{4,2} = \alpha(1-\alpha)(11-7\alpha)/6, \quad p_{4,3} = 2\alpha^2(1-\alpha), \quad p_{4,4} = \alpha^3. \end{aligned}$$

This provides a simple numerical method for the calculation of the d mixture probabilities $p_{d,1}, \ldots, p_{d,d}$ of $f_d(z)$ at any given dimension d and any $\alpha \in (0,1]$. Let $W_1, \ldots, W_d$ be independent standard exponential random variables. Denoting the cumulative probabilities by $P_m = \sum_{i=1}^{m} p_{d,i}$ for $m = 1, \ldots, d$ and setting $P_0 = 0$ yields the following algorithm, which is (using slightly different transformations) reproduced from Shi (1995b).

Algorithm 1.1: Multivariate logistic

1. Set $(T_1, \ldots, T_d) = (W_1/\sum_{j=1}^{d} W_j, \ldots, W_d/\sum_{j=1}^{d} W_j)$.
2. Generate U uniformly over $(0,1)$ and find $k \in \{1, \ldots, d\}$ such that $P_{k-1} \leq U < P_k$.
3. Generate Z from a gamma distribution with shape parameter k and unit scale.
4. Set $X = (X_1, \ldots, X_d) = (1/ZT_1^{\alpha}, \ldots, 1/ZT_d^{\alpha})$.

For the bivariate model ($d = 2$) this is the algorithm advocated by Shi et al. (1992). Ghoudi et al. (1998) successfully generalize Algorithm 1.1 to a number of bivariate extreme value distribution functions G by considering (an equivalent version of) the transformations $Z = V(X_1, X_2)$ and $T_1 = X_1/(X_1 + X_2)$, where $V = -\log G$ is the exponent measure of G.

Step 1 of Algorithm 1.1 specifies one method for simulating uniformly on $\mathscr{S}_d$ (e.g. Devroye, 1986). The method used in Step 2 has little or no effect on generation times (at least for $d \leq 100$). We use straight searching (e.g. Algorithm 3.10 in Ripley, 1987) in Step 2 to derive the generation times presented in Section 4. A number of alternative simulation algorithms for the gamma distribution are available (Tadikamalla and Johnson, 1981). When $k \leq 3$ we use the sum of k independent standard exponentials. When $k > 3$ we use a ratio-of-uniforms method (Kinderman and Monahan, 1977) presented as Algorithm GKM3 in Cheng and Feast (1979) and reproduced as Algorithm 3.20 in Ripley (1987). The expected number of uniform random variables required within the algorithm of Cheng and Feast (1979) is approximately invariant to the shape parameter k.

Let B be the set of all non-empty subsets of $\{1, \ldots, d\}$, and let $B_1 = \{b \in B : |b| = 1\}$, where $|b|$ denotes the number of elements in the set b. The multivariate asymmetric logistic model, introduced in full generality by Tawn (1990), is given by

$$V_{AL}(x) = \sum_{b\in B}\left(\sum_{i\in b}\left(\frac{\theta_{i,b}}{x_i}\right)^{1/\alpha_b}\right)^{\alpha_b}, \tag{5}$$

where the dependence parameters $\alpha_b \in (0, 1]$ for all $b \in B \setminus B_1$, and the asymmetry parameters $\theta_{i,b} \in [0, 1]$ for all $b \in B$ and $i \in b$. Let $B_{(i)} = \{b \in B : i \in b\}$. The constraints $\sum_{b\in B_{(i)}} \theta_{i,b} = 1$ for $i = 1, \ldots, d$ ensure that the univariate marginal distributions are standard Fréchet. The model contains $2^d - d - 1$ dependence parameters and $d(2^{d-1} - 1)$ asymmetry parameters, given the constraints. Further constraints arise from the possible redundancy of asymmetry parameters in the expansion of the exponent measure. Specifically, if $\alpha_b = 1$ for some $b \in B \setminus B_1$ then $\theta_{i,b} = 0$ for all $i \in b$. Let $b_{-i_0} = \{i \in b : i \neq i_0\}$. If, for some $b \in B \setminus B_1$, $\theta_{i,b} = 0$ for all $i \in b_{-i_0}$ then $\theta_{i_0,b} = 0$.

The simulation algorithm for the asymmetric logistic model requires the following theorem.

Theorem 1: *Let B be the set of all non-empty subsets of $\{1, \ldots, d\}$, and let $B_{(i)} = \{b \in B : i \in b\}$. For each $b \in B$, let $\{Z_{i,b} : i \in b\}$ be a set of random variables having a $|b|$-dimensional multivariate extreme value distribution with standard Fréchet margins and exponent measure V_b (if $|b| = 1$, the distribution is taken to be standard Fréchet). For different $b \in B$ the sets are mutually independent.*

The distribution of $(\max_{b\in B_{(1)}}\{\theta_{1,b}Z_{1,b}\}, \ldots, \max_{b\in B_{(d)}}\{\theta_{d,b}Z_{d,b}\})$ is multivariate extreme value, with standard Fréchet margins and exponent measure

$$V(z) = \sum_{b\in B} V_b(\{z_i/\theta_{i,b} : i \in b\}),$$

where $z = (z_1, \ldots, z_d)$, $\theta_{i,b} \geq 0$ for all $b \in B$ and $i \in b$, and $\sum_{b\in B_{(i)}} \theta_{i,b} = 1$ for $i = 1, \ldots, d$.

The theorem follows trivially using the following results.

$$\begin{aligned}
&\Pr\left(\max_{b\in B_{(1)}}\{\theta_{1,b}Z_{1,b}\} \leq z_1, \ldots, \max_{b\in B_{(d)}}\{\theta_{d,b}Z_{d,b}\} \leq z_d\right) = \Pr\big(\theta_{i,b}Z_{i,b} \leq z_i \ \forall i \in b, b \in B\big)\\
&\quad = \prod_{b\in B}\Pr\big(Z_{i,b} \leq z_i/\theta_{i,b} \ \forall i \in b\big) = \prod_{b\in B}\exp\big(-V_b\big(\{z_i/\theta_{i,b} : i \in b\}\big)\big)\\
&\quad = \exp\left(-\sum_{b\in B} V_b\big(\{z_i/\theta_{i,b} : i \in b\}\big)\right).
\end{aligned}$$

It is easily shown that V is homogeneous of order -1, using the same homogeneity property of each V_b. The constraints $\sum_{b\in B_{(i)}} \theta_{i,b} = 1$ for $i = 1, \ldots, d$ ensure that the margins are standard Fréchet.

Each term within the outer sum of the asymmetric logistic exponent measure (5) is of logistic form, subject to marginal transformations involving the asymmetry parameters. We can simulate from the asymmetric logistic distribution using Theorem 1 by taking maxima over functions of independent random vectors distributed as symmetric logistic. Specifically, we have the following algorithm.

Algorithm 1.2: Multivariate Asymmetric Logistic

1. For each $b \in B$
   a. Generate $\{Z_{i,b} : i \in b\}$ from a symmetric logistic distribution of dimension $|b|$, with dependence parameter α_b, using Algorithm 1.1.
   b. For each $i \in b$ set $X_{i,b} = \theta_{i,b} Z_{i,b}$.
2. Set $X = (X_1, \ldots, X_d) = (\max_{b \in B_{(1)}} \{X_{1,b}\}, \ldots, \max_{b \in B_{(d)}} \{X_{d,b}\})$.

If $|b| = 1$ in Step 1(a) simulation from the symmetric logistic distribution reduces to that of a univariate margin, namely the standard Fréchet distribution.

3. Conditional methods

The following method for simulating from the multivariate asymmetric logistic distribution is motivated by the physical interpretations of Tawn (1990). Define $\alpha_b = 1$ if $b \in B_1$ and let S_b for each $b \in B$ be independent random variables with positive stable distributions having characteristic exponent $\alpha_b \in (0, 1]$. This will be denoted by PS(α_b). The density of PS(α) for $\alpha \in (0, 1)$ is

$$f_{PS}(z) = -\frac{1}{\pi z} \sum_{k=1}^{\infty} \frac{\Gamma(k\alpha + 1)}{k!} (-z^{-\alpha})^k \sin(\alpha k \pi), \quad z > 0.$$

(Ibragimov and Chernin, 1959). When $\alpha = 1$ the distribution is degenerate at one. The inverse Gaussian distribution is the special case at $\alpha = 1/2$. If $Z \sim \mathrm{PS}(\alpha)$ the Laplace transform of Z, as a function of t, is $\exp(-t^{\alpha})$. Let $X_{i,b}$, for each $b \in B$ and each $i \in b$, be random variables with conditional distributions

$$\Pr(X_{i,b} < z | S_b = s_b) = \exp\{-s_b(\theta_{i,b}/z)^{1/\alpha_b}\}, \tag{6}$$

so that, for each $b \in B$, $\{X_{i,b} : i \in b\}$ are conditionally independent given S_b. Setting $X_i = \max_{b \in B_{(i)}} \{X_{i,b}\}$ yields random variables $X_1, \ldots, X_d$ which are conditionally independent given $\{S_b : b \in B\}$, with

$$\Pr(X_i < x_i | S_b = s_b, b \in B) = \exp\left\{ -\sum_{b \in B_{(i)}} s_b \left(\frac{\theta_{i,b}}{x_i} \right)^{1/\alpha_b} \right\}, \qquad i = 1, \ldots, d. \tag{7}$$

The conditional distribution of $X = (X_1, \ldots, X_d)$ is therefore

$$\Pr(X < x | S_b = s_b, b \in B) = \exp\left\{ - \sum_{b \in B} s_b \sum_{i \in b} \left(\frac{\theta_{i,b}}{x_i} \right)^{1/\alpha_b} \right\}.$$

The unconditional distribution of X is the expectation of this term with respect to the joint distribution of the S_b, namely the multivariate asymmetric logistic distribution, with exponent measure (5).

Simulation from a positive stable distribution is straightforward. Let U be a uniformly distributed over the interval $(0, \pi)$ and let W be standard exponential, independent of U. Then

$$\left[\frac{\sin\{(1-\alpha)U\}}{W} \right]^{(1-\alpha)/\alpha} \frac{(\sin \alpha U)}{(\sin U)^{1/\alpha}} \sim \mathrm{PS}(\alpha),$$

a representation due to Kanter (1975). In the deterministic case $(\alpha = 1)$ simulation can be avoided. Chambers et al. (1976) present simulation algorithms for general stable laws.

These results lead directly to the following algorithms. Let $W_1, \ldots, W_d$ be independent standard exponential random variables.

Algorithm 2.1: Multivariate logistic

1. Generate $S \sim \mathrm{PS}(\alpha)$.
2. Set $X = (X_1, \ldots, X_d) = ((S/W_1)^{\alpha}, \ldots, (S/W_d)^{\alpha})$.

Define $\alpha_b = 1$ when $b \in B_1$. Let $\{W_{i,b} : b \in B, i \in b\}$ be a set of independent standard exponential random variables.

Algorithm 2.2: Multivariate asymmetric logistic

1. For each $b \in B$
   a. Generate $S_b \sim \mathrm{PS}(\alpha_b)$.
   b. For each $i \in b$ set $X_{i,b} = \theta_{i,b}(S_b/W_{i,b})^{\alpha_b}$.
2. Set $X = (X_1, \ldots, X_d) = (\max_{b \in B_{(1)}} \{X_{1,b}\}, \ldots, \max_{b \in B_{(d)}} \{X_{d,b}\})$.

Step 1a implies that $S_b = 1$ when $b \in B_1$ because of the degeneracy of PS(α) at $\alpha = 1$. Step 1b uses the inverse transform method to simulate from the distribution (6).

4. Comparisons

All computations presented subsequently were carried out on a Pentium III 450 MHz machine with 512 Mb of RAM using the GNU C compiler with C routines called from the

Table 1. Marginal generation times in units of $(ms \times 10^{-2})$ for the algorithms presented in Sections 2 and 3, and for dimensions $d \leq 10$. The dependence parameters were fixed at 0.49. The asymmetry parameters in the asymmetric logistic model were fixed at $1/2^{d-1}$.

	Dimension								
Model	2	3	4	5	6	7	8	9	10
Logistic									
Algorithm 1.1	0.63	0.85	1.07	1.28	1.49	1.71	1.93	2.11	2.31
Algorithm 2.1	0.96	1.15	1.34	1.52	1.71	1.90	2.08	2.28	2.46
Asymmetric logistic									
Algorithm 1.2	0.7	2.6	7.3	18.4	44.4	104.1	237.1	531.5	1189.4
Algorithm 2.2	1.1	4.0	10.7	26.2	60.6	135.0	300.3	652.4	1415.3

R statistical computation system (Ihaka and Gentleman, 1996). The (default) random uniform generator provided with R was used throughout.

Table 1 gives the CPU time for the generation of one vector random variable (the marginal generation time) using the algorithms presented in Sections 2 and 3 for dimensions $d \leq 10$, at given parameter values. The marginal generation times are given in units of $(ms \times 10^{-2})$, so that the values represent the number of seconds taken to generate 100,000 random vectors. We ignore set-up times which are of little importance in this context. The marginal generation times obtained from implementations specifically targeted to bivariate models were approximately two thirds of those given in Table 1 for $d = 2$, since they permit a reasonable degree of simplification.

The empirical growth rate of the generation time for asymmetric logistic models of increasingly higher dimensions is seen to be rapid. This is expected, since in higher dimensions the asymmetric logistic model is inherently over-parameterized. For statistical modeling the dependence structure within a particular application is typically used to derive a more parsimonious subclass, which will inevitably result in faster generation times.

Algorithms 1.1 and 1.2 are consistently faster than their counterparts in Section 3 for $d \leq 10$, although the (relative) advantage decreases in higher dimensions. This decrease occurs as a consequence of the gamma random variable generated in Step 3 of Algorithm 1.1, which is faster (relative to Algorithm 2.1) in lower dimensions. Further comparisons have shown that for large d the generation times of competing algorithms are approximately equivalent.

For most simulation studies the speed advantage in low dimensions of Algorithms 1.1 and 1.2 over Algorithms 2.1 and 2.2 is small enough to be of little practical consequence. Furthermore, Algorithms 2.1 and 2.2 extend more easily to alternative models. In particular, extensions to models of nested or hierarchical type and extensions incorporating marginal censoring are sketched in the next section.

5. Generalizations

We briefly consider extensions of the multivariate asymmetric model that can be simulated using the techniques presented in Sections 2 and 3.

For each $j = 1, \ldots, r$ let V_{γ_j} be the exponent measure of the asymmetric logistic model (5), with parameter vector $\gamma_j = \{\theta_{i,b}^{(j)} : b \in B, i \in b\} \cup \{\alpha_b^{(j)} : b \in B \setminus B_1\}$. Let

$$V_M(x) = \omega_1 V_{\gamma_1}(x) + \omega_2 V_{\gamma_2}(x) + \cdots + \omega_r V_{\gamma_r}(x),$$

where $\omega = (\omega_1, \ldots, \omega_r) \in \mathscr{S}_r$, the $r-1$ dimensional unit simplex. It can be shown that $V_M(x)$ is the exponent measure of an extreme value distribution, with parameter vector $(\omega, \gamma_1, \ldots, \gamma_r)$. The application of the methods of Sections 2 and 3 to the simulation of this distribution yields no extra theoretical difficulties. Finite mixtures of exponent measures are often taken to induce marginal closure, so that the parametric form of the d-dimensional distribution function coincides with each d-dimensional marginal distribution of higher dimensional models.

Suppose that we wish to incorporate marginal censoring into the multivariate asymmetric logistic model. The censored distribution can be simulated from using standard rejection sampling techniques, rejecting samples that fall within the appropriate union of (possible infinite) hypercubes. The representation of Section 3 provides a more efficient method. Algorithm 2.2 can be extended by generating a set of positive stable random variables, as in Step 1, and then simulating from each margin in turn (using rejection sampling) from appropriately censored versions of the conditional distribution (7).

Less obviously, the conditioning methods of Section 3 extend to models of nested or hierarchical logistic type. Let C_b denote the set of non-empty subsets of $b \in B$, and let $C_{1,b} = \{c \in C_b : |c| = 1\}$. Consider the nested asymmetric logistic model (Tawn, 1990)

$$V_{NL}(x) = \sum_{b \in B} \left(\sum_{c \in C_b} \left[\sum_{i \in b \setminus c} \left(\frac{\theta_{i,c,b}}{x_i} \right)^{1/\alpha_b} + \left\{ \sum_{i \in c} \left(\frac{\theta_{i,c,b}}{x_i} \right)^{1/\alpha_b \alpha_{c,b}} \right\}^{\alpha_{c,b}} \right] \right)^{\alpha_b}, \tag{8}$$

where the dependence parameters α_b and $\alpha_{c,b}$ are contained in $(0, 1]$ for all $b \in B \setminus B_1$ and $c \in C_b \setminus C_{1,b}$, and the asymmetry parameters $\theta_{i,c,b} \in [0, 1]$ for all $b \in B$, $c \in C_b$ and $i \in b$. The constraints $\sum_{b \in B_{(i)}} (\sum_{c \in B: c \subset b} \theta_{i,c,b}^{1/\alpha_b})^{\alpha_b} = 1$ for $i = 1, \ldots, d$ ensure that the univariate marginal distributions are standard Fréchet. If any of the asymmetry parameters become redundant in the expansion of the exponent measure they must be set to zero.

Simple models which can be expressed in the form of model (8) include (omitting the subscript b on each dependence parameter, which remains constant)

$$V_{NL}^{(1)}(x_1, x_2, x_3) = \left[\left(x_1^{1/\alpha\alpha_{\{1,2\}}} + x_2^{1/\alpha\alpha_{\{1,2\}}} \right)^{\alpha_{\{1,2\}}} + x_3^{1/\alpha} \right]^{\alpha}, \tag{9}$$

which arises upon setting $\theta_{i,\{1,2\},\{1,2,3\}} = 1$ for $i = 1, 2, 3$ (which implies, using the sum constraints, that all other asymmetry parameters equal zero), and

$$V_{NL}^{(2)}(x_1, x_2, x_3, x_4) = \left[\left(x_1^{1/\alpha\alpha_{\{1,2\}}} + x_2^{1/\alpha\alpha_{\{1,2\}}}\right)^{\alpha_{\{1,2\}}} + \left(x_3^{1/\alpha\alpha_{\{3,4\}}} + x_4^{1/\alpha\alpha_{\{3,4\}}}\right)^{\alpha_{\{3,4\}}}\right]^{\alpha},$$

which arises upon setting $\theta_{i,\{1,2\},\{1,2,3,4\}} = 1$ for $i = 1, 2$ and $\theta_{i,\{3,4\},\{1,2,3,4\}} = 1$ for $i = 3, 4$. We now sketch an extension of the methods in Section 3 that facilitates simulation from the nested logistic model.

Define $\alpha_b = 1$ when $b \in B_1$ and $\alpha_{c,b} = 1$ when $b \in B$ and $c \in C_{1,b}$. Consider conditioning with respect to the set of independent positive stable random variables $\{S_b : b \in B\} \cup \{S_{c,b} : b \in B, c \in C_b\}$. Specifically, suppose $S_b \sim \mathrm{PS}(\alpha_b)$ and $S_{c,b} \sim \mathrm{PS}(\alpha_{c,b})$, and let $X_{i,c,b}$ be random variables such that

$$\Pr(X_{i,c,b} < z | S_b = s_b, S_{c,b} = s_{c,b}) = \begin{cases} \exp\{-s_b(\theta_{i,c,b}/z)^{1/\alpha_b}\}, & \text{for} \quad i \in b \setminus c \\ \exp\{-s_{c,b}s_b^{1/\alpha_{c,b}}(\theta_{i,c,b}/z)^{1/\alpha_b\alpha_{c,b}}\}, & \text{for} \quad i \in c. \end{cases}$$

Letting $X_i = \max\{X_{i,c,b}\}$, where the maximization is taken over $b \in B_{(i)}$ and $c \in C_b$, the conditional distribution of $X = (X_1, \ldots, X_d)$ becomes

$$\Pr(X < x | E) = \exp\left(-\sum_{b \in B} s_b \left[\sum_{c \in B : c \subset b} \left\{\sum_{i \in b \setminus c} \left(\frac{\theta_{i,c,b}}{x_i}\right)^{1/\alpha_b} + s_{c,b}\left(\sum_{i \in c}\left(\frac{\theta_{i,c,b}}{x_i}\right)^{1/\alpha_{c,b}\alpha_b}\right)\right\}\right]\right),$$

conditioning on the event $E = \{S_b = s_b, b \in B\} \cap \{S_{c,b} = s_{c,b}, b \in B, c \in C_b\}$. The unconditional distribution of X is the nested asymmetric logistic model, with exponent measure (8).

It appears to be more difficult to extend the transformation methods of Section 2 to the nested asymmetric logistic model other than in simple cases, such as $V_{NL}^{(1)}$ and $V_{NL}^{(2)}$, where transformations of the type given in Shi (1999) can be used.

The model presented here incorporates only one level of nesting. In theory it is possible to construct and simulate from an extreme value distribution of logistic form that contains any number of nested levels.

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