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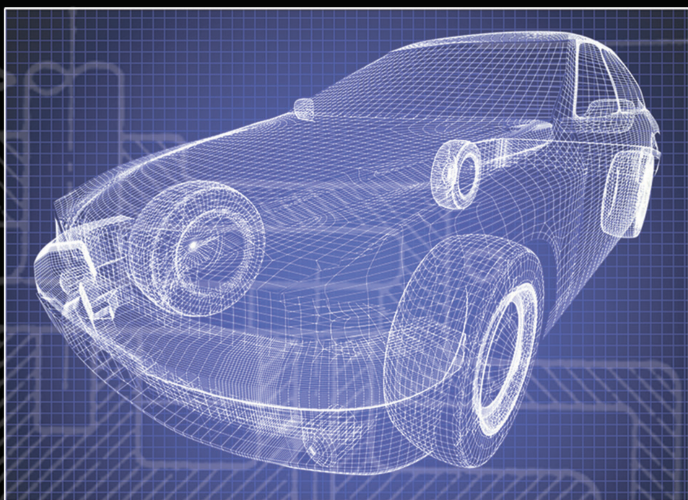
# VEHICLE DYNAMICS, STABILITY, AND CONTROL



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# VEHICLE DYNAMICS, STABILITY, AND CONTROL

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# Contents

# Preface

This book is the second edition of a book originally titled *Vehicle Stability.* The new title, *Vehicle Dynamics, Stability, and Control,* better describes this revised and extended version.

This book is the result of two activities that have given the author a great deal of pleasure and satisfaction over a period of more than forty years. The first was the initiation and teaching of a course for seniors and first-year graduate students in mechanical and aerospace engineering at the University of California, Davis. The course is intended to illustrate the application of techniques the students had learned in courses dealing with such topics as kinematics, rigid body dynamics, system dynamics, automatic control, stability theory, and aerodynamics to the study of the dynamic behavior of a number of vehicle types. In addition, specialized topics dealing specifically with vehicle dynamics such as the force generation by pneumatic tires, railway wheels and wings are also presented.

The second activity was a short course entitled "Vehicle Dynamics and Active Control," given by the author and his colleague, Professor Donald Margolis, numerous times in the United States and in several European countries. This short professional course was intended primarily for engineers in the automotive industry.

Although the short course for engineering professionals contained much of the material found in the academic course and in the present book, it was specialized in that it dealt only with automotive topics. The unique feature of the present book lies in its treatment of the dynamics and stability aspects of a variety of vehicle types. Anyone who has experience with vehicles knows that stability (or instability) is one of the most intriguing and mysterious aspects of vehicle dynamics. Why does a motorcycle sometimes exhibit a wobble of the front wheel when ridden "no hands" or a dangerous weaving motion at high speed? Why would a trailer suddenly begin to oscillate over several traffic lanes just because its load distribution is different from the usual? Why does a locomotive begin "hunting" back and forth on the tracks when traveling a high speed? Why is an airplane hard to fly when the passenger and luggage load is too far to the rear? Could it be that a car or truck could behave in an unstable way when driven above a critical speed? In addition, there are control questions such as "How can humans control an inherently unstable vehicle such as a bicycle?"

Many of these questions are answered in the book using the analysis of linear vehicle dynamic models. This allows the similarities and critical differences in the stability properties of different vehicle types to be particularly easily appreciated. Although analysis based on linearized mathematical models cannot answer all questions, general rigid body dynamics and nonlinear relations relating to force generation are discussed for several cases. Furthermore, many of the nonlinear aspects of vehicle dynamics are discussed, albeit often in a more quantitative manner.

It is possible, of course, to extend the models beyond the range of small perturbation inherent in the analysis of stability. Through the use of computer simulation, for example, one can discover the behavior of unstable vehicles when the perturbation variables grow to an extent that the linearized equations are no longer valid.

An aspect of the professional course that was of particular interest to working engineers was the discussion of active means of influencing the dynamics of automobiles. It is now fairly common to find active or semi-active suspension systems, active steering systems, electronically controlled braking systems, torque vectoring drive systems, and the like that actively control the dynamics of automotive vehicles. Many of these control systems follow the lead of similar systems first applied in aircraft. Throughout the book, whenever it is appropriate, the idea that active means might be used to improve the dynamics of a vehicle is presented. In particular, Chapter 11 discusses some of the active means used to improve the dynamics of vehicles such as cars and airplanes.

Since all the studies of vehicle dynamics begin with the formulation of mathematical models, there is a great deal of emphasis in this book on the use of methods for formulating equations of motion. Chapter 2 deals with the description of rigid body motion. In basic dynamics textbooks, the use of a body-fixed coordinate system that proves to be very useful to describe the dynamics of many vehicle types is typically not discussed at all. Chapter 2 describes the fundamental principles of mechanics for rigid bodies as well as the general equations when expressed in a body-fixed coordinate system.

In other chapters, the more conventional derivations of the equations of motion using Newton's laws directly or Lagrange equations with inertial coordinate systems are illustrated by a series of examples of increasing levels of complexity. This gives the reader the opportunity to compare several ways to formulate equations of motion for a vehicle.

For those with some familiarity with bond graph methods for system dynamics, Chapter 2 contains a section that shows how rigid body dynamics using a body-fixed coordinate frame can be represented in a graphical form using bond graphs. The Appendix gives complete bond graph representations for an automobile model used in Chapter 6 and for a simplified aircraft model used in Chapter 9. A number of bond graph processing programs are available. This opens up the possibility of using computer automated equation formulation and simulation for nonlinear versions of the linearized models used for stability analysis.

The academic course has proved to be very popular over the years for several reasons. First of all, many people are inherently interested in the dynamics of vehicles such as cars, bicycles, motorcycles, airplanes, and trains. Second, the idea that vehicles can exhibit unstable and dangerous behavior for no obvious reason is in itself fascinating. These instabilities are particularly obvious in racing situations or in speed record attempts, but in everyday life it is common to see trailers swaying back and forth or to see cars slewing around on icy roads and to wonder why this happens. Third, for those with some background in applied mathematics, it is always satisfying to see that relatively simple mathematical models can often illuminate dynamic behavior that would otherwise be baffling.

The book does not attempt to be a practical guide to the design or modification of vehicles. The reader will have an appreciation of how an aircraft designer goes about designing a statically stable aircraft but will not find here a complete discussion of the practical knowledge needed to become an expert. The reader will gain an appreciation of the automotive terms understeer, oversteer, and critical speeds, for example, but there are no rules of thumb given for modifying an autocross racer. The References do, however, include books and papers that will prove helpful in building up a practical knowledge base relating to a particular vehicle dynamics problem.

For those interested in using the book as a text, it is highly recommended that experiments and demonstrations be used in parallel with the classroom lectures. The University of California, Davis is fortunate to be located near the California Highway Patrol Academy, and their driving instructors have been generous in giving demonstrations of automobile dynamics on their high-speed track and their skid pans. The students are always impressed that the instructors use the same words to describe the handling of patrol cars that the engineers use such as "understeer" and "oversteer" but without using the formal definitions that engineers prefer. Also, a trip to a local dirt track provides a demonstration of the racers' terms "tight" and "loose" as well as some spectacular demonstrations of unstable dynamic behavior.

Furthermore, our university has its own airport. This has permitted the students to experience aircraft dynamics personally as well as analytically in the discussions of Chapter 9. The relation between stability and control is much more obvious as a passenger in a small airplane than it is in commercial aircraft. A good pilot can easily demonstrate several oscillatory modes of motion (these are all stable for production small airplanes except possibly for the low-frequency phugoid mode) and, if the passengers are willing, can show the beginnings of some divergent modes of motion for extreme attitudes.

In addition, it is possible to design laboratory experiments to illustrate many of the analyses in the book. As examples, we have designed a demonstration of trailer instability using a moving belt and a stationary model trailer as well as a small trailer attached to a three-wheeled bicycle. These models show that changes in center of mass location and moment of inertia do indeed influence stability just as the theory in Chapter 5 predicts. Model gliders have been designed to illustrate static stability and instability as discussed in Chapter 9. Even a rear-steering bicycle was fabricated to illustrate the control difficulties described in Chapter 7.

A number of exercises are included that may be assigned if the book is used as a text. (A solutions manual for instructors is available.) Some of these problems are included to help students appreciate assumptions behind derivations given in the book. Other problems extend the analyses of the corresponding chapter to new situations or relate topics in one chapter to other chapters. Still other problems are of a much more extensive nature and can form the basis of small projects. They are intended to illustrate how mathematical models of varying degrees of complexity can be used to suggest design rules for improving the dynamics, stability, and control of vehicles.

The author hopes that the readers of this book will be as fascinated with vehicle dynamics, stability, and control as he is and will be inspired to learn even more about these topics.

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# Author

**Dean Karnopp** is professor emeritus of mechanical and aerospace engineering at the University of California, Davis. The author of more than 178 professional publications including multiple editions of six textbooks, and he has worked as a consultant in various industries including extensive stays at the Daimler-Benz Corporation and the Robert Bosch Company. He is a fellow of the American Society of Mechanical Engineers and a life honorary member of the Franklin Institute and received the Senior U.S. Scientist award from the Alexander von Humbolt Stiftung of the German government. He received his B.S., M.S. (1957), and Ph.D. (1961) degrees in mechanical engineering from the Massachusetts Institute of Technology, Cambridge, where he also served as a faculty member.

# 1

## *Introduction: Elementary Vehicles*

Vehicles such as cars, trains, ships, and airplanes are intended to move people and goods from place to place in an efficient and safe manner. This book deals with certain aspects of the motion of vehicles usually described using the terms "dynamics, stability, and control." Although most people have a good intuitive idea what these terms mean, this book will deal with mathematical models of vehicles and with more precise and technical meaning of the terms. As long as the mathematical models reasonably represent the real vehicles, the results of analysis of the models can yield insight into the actual problems that vehicles sometimes exhibit, and in the best case, can suggest ways to cure vehicle problems by modification of the physical aspects of the vehicle or the introduction of automatic control techniques.

All vehicles represent interesting and often complex dynamic systems that require careful analysis and design to make sure that they behave properly. In particular, the stability aspect of vehicle motion has to do with assuring that the vehicle does not depart spontaneously from a desired path. It is possible for an automobile to start to spin out at high speed or a trailer to begin to oscillate back and forth in ever-wider swings seemingly without provocation. Most of this book deals with how the physical parameters of a vehicle influence its dynamic characteristics in general and its stability properties in particular.

The control aspect of vehicle motion has to do with the ability of a human operator or an automatic control system to guide the vehicle along a desired trajectory. In the case of a human operator, this means that the dynamic properties of such vehicles should be tailored to allow humans to control them with reasonable ease and precision. A car or an airplane that requires a great deal of attention to keep it from deviating from a desired path would probably not be considered satisfactory. Modern studies of the Wright brother's 1903 Flyer indicate that, while the brothers learned to control the airplane, it apparently was inherently so unstable that modern pilots are reluctant to fly an exact replica. The Wright brothers, of course, did not have the benefit of the understanding of aircraft stability that aeronautical engineers now have. Modern light planes can now be designed to be stable enough that flying them is not the daunting task that it was for the Wrights.

For vehicles using electronic control systems, the dynamic properties of the vehicle must be considered in the design of the controller to assure that the controlled vehicles are stable and have desirable dynamics. Increasingly, human operators exercise supervisory control of vehicles with automatic control systems. In some cases, the control system must stabilize an inherently unstable vehicle so that it is not difficult for the human operator to control its trajectory. This is the case for some modern fighter aircraft. In other cases, the active control system simply aids the human operator in controlling the vehicle. Aircraft autopilots, fly-by-wire systems, antilock braking systems, and electronic stability enhancement systems for automobiles are all examples of systems that modify the stability properties of vehicles with active means to increase the ease with which they can be controlled. Active stability enhancement techniques will be discussed in Chapter 11 after the dynamic properties of several types of vehicles have been analyzed.

In some cases, vehicle motion is neither actively controlled by a human operator nor by an automatic control system but yet the vehicle may exhibit very undesirable dynamic behavior under certain conditions. A trailer being pulled by a car, for example, should obviously follow the path of the car in a stable fashion. As we will see in Chapter 5, a trailer that is not properly designed or loaded may however exhibit growing oscillations at high speeds, which could lead to a serious accident. Trains that use tapered wheelsets are intended to self-center on the tracks without any active control, but above a critical speed, an increasing oscillatory motion called hunting can develop that may lead to derailment in extreme cases. For many vehicles, this type of unstable behavior may suddenly appear as a critical speed is exceeded. This type of unstable behavior is particularly insidious since the vehicle will appear to act in a perfectly normal manner until the first time the criti-cal speed is exceeded, at which time the unstable motion can have serious consequences.

This book will concentrate on the mathematical description of the dynamics of vehicles. An important topic that can be treated with some generality involves the stability of the motion of vehicles. Other aspects of vehicle dynamics to be discussed fall into the more general category of "vehicle handling" or "vehicle controllability" and are of particular importance under extreme conditions associated with emergency maneuvers.

Obviously, vehicle stability is of interest to anyone involved in the design or use of vehicles, but the topic of stability is of a more general interest. Since the dynamic behavior of vehicles such as cars, trailers, and airplanes is to some extent familiar to almost everybody, these systems can be used to introduce a number of concepts in system dynamics, stability, and control. To many people, these concepts seem abstract and difficult to understand when presented as topics in applied mathematics without some familiar physical examples. Everyday experience with a variety of vehicles can provide examples of these otherwise abstract mathematical concepts.

Engineers involved in the design and construction of vehicles typically use mathematical models of vehicles in order to understand the fundamental dynamic problems of real vehicles and to devise means for controlling vehicle motion. Unfortunately, it can be a formidable task to find accurate mathematical descriptions of the dynamics of a wide variety of vehicles. Not only do the descriptions involve nonlinear differential equations that seem to have little similarity from vehicle type to vehicle type, but in particular, the characterization of the force-producing elements can be quite disparate. One can easily imagine that rubber tires, steel wheels, boat hulls, or airplane wings act in quite different ways to influence the motions of vehicles operating on land, on water, or in the air.

On the other hand, all vehicles have some aspects in common. They are all usefully described for many purposes as essentially rigid bodies acted upon by forces that control their motion. Some of the forces are under control of a human operator, some may be under active control of an automatic control system, but all are influenced by the very nature of the force-generating mechanisms inherent to the particular vehicle type. This means that not very many people claim to be experts in the dynamics of a large number of types of vehicles.

When describing the stability of vehicle motion, however, the treatment of the various types of force-generating elements exhibits a great deal of similarity. In stability analysis, it is often sufficient to consider small deviations from a steady state of motion. The basic idea is that a stable vehicle will tend to return to the steady motion if it has been disturbed while an unstable vehicle will deviate further from the steady state after a disturbance. The mathematical description of the vehicle dynamics for stability analysis typically uses a linearized differential equation form based on the nonlinear differential equations that generally apply. The linearized equations show more similarity among vehicle types than the more accurate nonlinear equations. Thus, a focus on stability allows one to appreciate that there are interesting similarities and differences among the dynamic properties of a variety of vehicle types without being confronted with the complexities of nonlinear differential equation models.

In this chapter, stability analyses will be performed for two extremely simplified vehicle models to illustrate the approach. In later chapters more realistic vehicle models will be introduced and it will become clear that despite some analogous effects among vehicle types, ultimately the differences among the force-generating mechanisms for various vehicle types determine their behavior particularly under more extreme conditions than are considered in stability analyses. Complicated mathematical models are routinely used in the design of many vehicle types and are studied using computer simulation. Such models often contain so many parameters that it is not easy to see how to solve dynamic problems that may arise. In this book we will restrict the discussion mainly to relative simple but insightful models that are particularly good at illuminating stability problems.

In this introductory chapter, the two examples that will be analyzed require essentially no discussion of force-generating elements such as tires or wings. They do, however, introduce the basic ideas of vehicle stability analysis. The first example is actually kinematic rather than dynamic in the usual sense, since Newton's laws are not needed. The second example is truly dynamic but kinematic constraints take the place of force-generating elements. Subsequent Chapters 2, 3, and 4 provide a basis for the more complete dynamic analyses to follow in Chapters 5–11.

## 1.1 Tapered Wheelset on Rails

Although the ancient invention of the wheel was a great step forward for the transportation of heavy loads, when soft or rough ground was to be covered, wheels still required significant thrust to move under load. The idea of a railway was to provide a hard, smooth surface on which the wheels could roll and thus to reduce the effort required to move a load.

The first rail vehicles were used to transport ore out of mines. Cylindrical wheels with flanges on the inside to keep the wheels from rolling off the rails were used. To the casual observer, these early wheelsets resemble closely those used today. In fact, modern wheelsets differ in an important way from those earliest versions.

Using cylindrical wheels, which seemed logical at the time, it was found that the flanges on the wheels were often rubbing in contact with the sides of the rails. This not only increased the resistance of the wheels to rolling, but more importantly, also caused the flanges to wear quite rapidly. Eventually it was discovered that if slightly tapered rather than cylindrical wheels were used, the wheelsets would automatically tend to self-center and the flanges would hardly wear at all since they rarely touched the sides of the rails. Since this time almost all rail cars use tapered wheels.

The actual analysis of the stability of high-speed trains is quite complicated since multiple wheels, trucks on which the wheels are mounted, and the car bodies are involved. Also, somewhat surprisingly, steel wheels rolling on steel rails at high speeds do not simply roll as rigid bodies without slipping, as one might imagine. There are lateral and longitudinal "creeping" motions between the wheels and the rails that make the stability of the entire vehicle dependent upon the speed. Above a so-called critical speed a railcar will begin to "hunt" back and forth between the rails. At high enough speeds, the flanges will contact the rails and in the extreme case, the wheels may derail. The dynamic analysis of rail vehicles is discussed in more detail in Chapter 10.

The following analysis will be as simple as possible. Only a single wheelset is involved and the analysis is not even dynamic but rather kinematic. The two wheels will be assumed to roll without slip, which is a reasonable assumption at low speeds. The point of the exercise is to show that at very low speeds when the accelerations and hence the forces are very low, a wheelset with tapered wheels will tend to steer itself automatically towards the center of the rails. If the taper were to be in the opposite sense, the wheelset would be unstable and tent to veer toward one side of the rails or the other until the flanges would contact the rails. This example will serve to show how a small perturbation from a steady motion leads to linearized equations of motion that can be analyzed for their stability properties.

Figure 1.1 shows the wheelset in two conditions. At the left in Figure 1.1, the wheelset is in a centered position rolling at a constant forward speed on straight rails. The wheelset consists of two wheels rigidly attached to a common axle so both wheels rotate about their common axle at the same rate. The wheels are assumed to have point contact with the two straight lines that represent the rails. In the centered position shown at the left, the radius of each wheel at the contact point on the rails is the same, $r_0$.

The second part of the figure to the right shows the wheelset in a slightly perturbed position. In this part of the figure, the wheelset has moved off its centered position and its axle has assumed a slight angle with respect to a line perpendicular to the rails. The forward speed is assumed to remain constant.

In all of the examples to follow, we will distinguish between *parameters*, which are constants in the analysis, and *variables*, which change over time and are used to describe the motion of the system. In this case, the parameters are $l$, the separation of the rails, $\Psi$, the taper angle, $r_0$, the rolling radius when the wheelset is centered, and $\Omega$, the angular velocity of the wheelset about its axle, which will be assumed to be constant.

The main variables for this problem are $y$, the deviation of the center of the wheelset from the center of the rails and $\theta$, the angle between the wheelset axis and a line perpendicular to the rails. Other variables that may be of interest include $x$, the distance the wheelset has moved down the track, and $\phi$, the angle around the axle that the wheelset has turned when it has rolled a distance $x$. Still other variables such as the velocities of the upper and lower wheels in the sketch of Figure 1.1, $V_U$ and $V_L$, will be eliminated in the final equations describing the system.

Generally for a vehicle stability analysis, a possible steady motion will be defined and then small deviations from this steady motion will be assumed. In this book, this steady motion will often be called the *basic motion*. The essence of a stability analysis is to determine whether the deviations from the steady basic motion will tend to increase or decrease in time. If the deviations increase in time, the system is called unstable. If the deviations tend to decrease and the vehicle system returns to the basic motion, the system is called stable. These concepts will be made more precise later.

In this example, the basic motion occurs when the wheelset is perfectly centered with $\theta \cong \dfrac{dy}{dx}$, and $y$ both vanishing. Since $\Omega$ is assumed to be constant, the velocities $\dot{x} = r_0{}^{\circ} = V_U = V_L$ also are constant for the basic motion.

The deviated motion is called the *perturbed motion* and is characterized by the assumption that the deviation variables are small in some sense. In the present case, it is assumed that the variable $y$ is small with respect to $l$ and

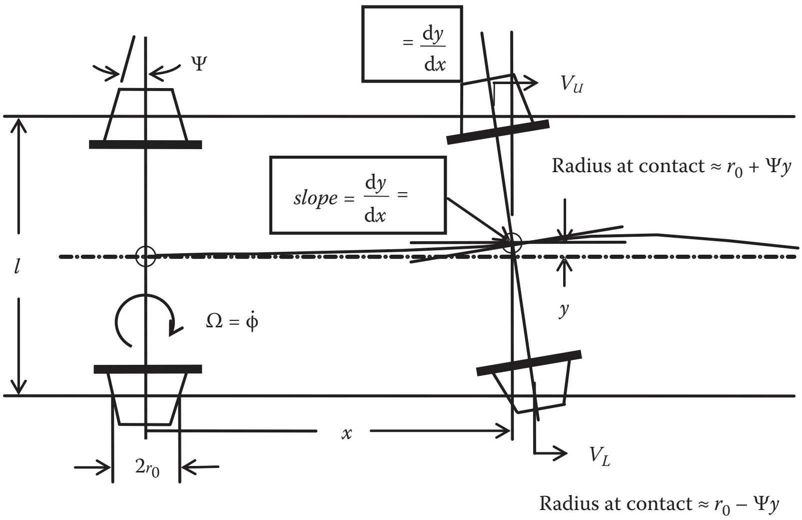



**FIGURE 1.1**
A tapered wheelset.

that the angle θ is also small. (Remember that angles are dimensionless, and small in this case means small with respect to 2π radians. We will encounter angles that must be considered small frequently in the subsequent chapters. Small angle approximations for trigonometric functions will then be used.)

For an *unstable* system, the small, perturbed variables such as $y$ and θ will grow spontaneously in time. For a *stable* system, the perturbed variables decrease in time and the motion approaches the basic motion. In this example, $y$ and θ will tend to return toward zero if the wheelset is stable.

For the perturbed motion, with $y$ and θ small but not zero, the upper and lower wheel velocities are slightly different because the rolling radii of the two wheels are no longer the same. The taper angle is assumed to be small so the change in radius is proportional to the offset distance and the taper angle. The magnitude of the change is thus approximately $\Psi y$, with the upper radius increasing and the lower radius decreasing (see Figure 1.1.) (Can you see that if the angle θ is very small, it has only a negligible effect on the radii?)

The rotational rate Ω of the wheelset is assumed to be constant even when the wheelset is not perfectly centered. The forward speeds of the upper and lower wheels are then

$$V_U = (r_0 + \Psi y)\Omega, \tag{1.1}$$

$$V_L = (r_0 - \Psi y)\Omega. \tag{1.2}$$

Note that when $y = 0$, the two wheel speeds are equal and have the value $r_0\Omega$, which is the forward speed of the wheelset.

In fact, the forward velocity of the center point of the wheelset, which was also $r_0\Omega$ for the basic motion, does not change appreciably for the perturbed motion. Simple kinematic considerations show that the speed of the center point of the wheelset is essentially the average of the motion of the two ends of the axle.

$$\dot{x} = \left(V_U + V_L\right)/2 = r_0\Omega, \tag{1.3}$$

when Equations 1.1 and 1.2 are used. Now one can compute the rate of change of the angle $\theta$, which has to do with the *difference* between the velocities of the two wheels. This is again a matter of simple kinematics. Using Equations 1.1 and 1.2 again, we find that

$$\dot{\theta} = \left(V_L - V_U\right)/l = -2\Psi y\Omega/l. \tag{1.4}$$

The rate of change of $y$ can also be found using the idea that $\theta \cong \frac{dy}{dx}$, and using Equation 1.3,

$$\frac{\mathrm{d}y}{\mathrm{d}t} \equiv \dot{y} \cong \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\dot{x} \circ \theta\left(r_0\Omega\right). \tag{1.5}$$

By differentiating Equation 1.5 with respect to time and substituting $\dot{\theta}$ from Equation 1.4, a single second-order equation for $y(t)$ results.

$$\ddot{y} + \left(\frac{r_0\Omega^2 2\Psi}{l}\right) y = 0. \tag{1.6}$$

Alternatively, Equations 1.4 and 1.5 can also be expressed in state space form (Ogata 1970).

$$\begin{bmatrix} \dot{\theta} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -2\Psi\Omega/l \\ r_0\Omega & 0 \end{bmatrix} \begin{bmatrix} \theta \\ y \end{bmatrix}. \tag{1.7}$$

In either form, Equation 1.6 or 1.7, these linear differential equations are easily solved. (The solution and stability analysis of linear equations will be discussed in some detail in Chapter 3.)

In this section, we will simply note the analogy of Equation 1.6 to the familiar equation of motion for a mass-spring vibratory system,

$$m\ddot{x} + kx = 0 \text{ or } \ddot{x} + (k/m)x = 0, \tag{1.8}$$

in which $x$ is the position of the mass, $k$ is a spring constant, and $m$ is the mass.

Common experience with a mass attached to a normal spring shows that after a disturbance, the spring pulls the mass back towards the equilibrium position, $x = 0$. Thereafter the mass oscillates back and forth, and if there is any friction at all, the mass will eventually come closer and closer to the equilibrium position. This is almost obviously a stable situation.

The general solution of Equation 1.8, for the position $x(t)$, assuming both $m$ and $k$ are positive parameters, is $x = A\sin(\omega_n t + const.)$ and $\omega_n = \sqrt{k/m}$ is the *undamped natural frequency*. Note that in Equation 1.8, no term representing friction is present, so the oscillation persists indefinitely.

This analogy between Equations 1.6 and 1.8 makes clear that the tapered wheelset (with the taper sense positive as shown in Figure 1.1) does tend to steer itself toward the center when disturbed and will follow a sinusoidal path down the track after being disturbed. Note that the solution of Equation 1.8 is assumed to be correct if $(k/m)$ is positive. The analogy holds because the corresponding term $(r_0\Omega^2 2\Psi/l)$ in Equation 1.6 is also positive for the taper situation shown in Figure 1.1.

It is easy and instructive to build a demonstration device that will show the self-centering properties of a wheelset using, for example, wooden dowels for the rails and paper cups taped together to make the wheelset with tapered wheels.

If the taper angle, $\Psi$, were to be negative, the wheelset equation would be analogous to a mass-spring system but with a spring with a negative spring constant rather than a positive one. A spring with a negative spring constant would create a force tending to push the mass away from the equilibrium position $x = 0$ rather than tending to pull it back as a normal spring does.

Such a system would not return to an equilibrium position after a disturbance but would rather accelerate ever farther away from equilibrium. Thus, one can see that a wheelset with a negative taper angle will tend to run off-center until stopped by one of the flanges. In this case, one would surely conclude that the motion of a wheelset with a negative taper angle is unstable. (The demonstration device with the paper cups taped together such that the taper is the opposite of the taper shown in Figure 1.1 will demonstrate this clearly.)

On the other hand, the wheelset with a positive taper angle while not unstable also is not strictly stable in this analysis since it continues to oscillate sinusoidally as it progresses down the rails. Just as a mass-spring oscillator with no damping would oscillate forever, the wheelset after an initial disturbance would not return to the basic motion but rather would continue to wander back and forth with a constant amplitude of motion as it traveled along.

In the case of the wheelset with positive taper angle, the natural frequency is found by analogy to the mass spring system to be

$$\omega_n = \Omega\sqrt{(2r_0\Psi/l)}. \tag{1.9}$$

The period of the oscillation is

$$T = \frac{2^\circ}{\omega_n} = \frac{2^\circ}{\Omega}\sqrt{\frac{l}{2r_0\Psi}} \tag{1.10}$$

and the wavelength of the oscillating motion is just the distance traveled in the time of one oscillation,

$$\lambda = \dot{x}T = r_0\Omega T = 2\pi r_0\sqrt{\frac{l}{2r_0\Psi}}\,. \tag{1.11}$$

It is interesting to note that the wavelength is independent of $\Omega$ and hence of the speed at which the wheelset is rolling along the track for this kinematic model.

For cylindrical wheels with $\Psi = 0$, there would be neither a self-centering effect nor a tendency to steer away from the center and $\ddot{y}$ would equal zero. In this case, the wheelset would roll in a straight line until one of the flanges encountered the side of one of the rails if the initial value of $\theta$ were not precisely zero. Furthermore, if the track had any curvature, the wheelset would certainly have to rely on the flanges to keep it on the rails. The tapered wheels follow a slightly curved track without slipping by shifting off-center, which adjusts the rolling radii as in Equations 1.1 and 1.2.

This example assumed rolling without any relative motion between the wheel and the rail at the contact point, which turns out to be unrealistic at high speed. What this simple example has shown is merely that tapered wheels tend to steer themselves toward the center of the tracks, but this mathematical model does not reveal whether the resulting oscillation would actually die down or build up in time. The tiniest change in the assumption of rolling without slip changes the purely sinusoidal motion to one that decays or grows slightly in amplitude.

This leaves open the possibility that the tapered wheelset, even with the positive taper angle, might actually be either truly stable or actually unstable. A more complicated model of the wheel–rail interaction is required to answer this question and it will be presented in Chapter 10.

Although this first example was analyzed in the time domain, Equation 1.11 suggests that time is not inherently a part of this problem. The sinusoidal motion is the same in space regardless of the speed of the wheelset so it might be more logical to view the problem as a function of space, $x$, rather than time, $t$. The next introductory example actually uses a dynamic model based on rigid body dynamics that is inherently time-based. In this sense it is more typical of the vehicle models to follow.

## 1.2 The Dynamics of a Shopping Cart

A large part of the rest of this book will be devoted to ground vehicles using pneumatic tires. It is not easy to start immediately to analyze such vehicles without beginning with a fairly long discussion of the means of characterizing the generation of lateral forces by tires. This is the subject of Chapter 4. On the other hand, it is possible to introduce some of the basic ideas of vehicle dynamics and stability if a ground vehicle can be idealized in such a way that the tire characteristics do not have to be described in any detail. In the introductory case to be discussed below, an idealized shopping cart, we will replace actual tire characteristics with simple kinematic constraints. This idealization actually works quite well for the hard rubber tires typically used on shopping carts up to the point at which the wheels actually are forced to slide sideways. This type of idealization allows one to focus on the dynamic model of the vehicle itself without worrying about details of the tire force-generating mechanism.

Most courses in dynamics mainly treat inertial coordinate systems when applying the laws of mechanics to rigid bodies. As will be demonstrated in Chapter 5, this approach is logical, for example when studying the stability of trailers, but there is another way to write equations for freely moving vehicles such as automobiles and airplanes that turns out to be simpler and is commonly used in vehicle dynamics studies. The description of the vehicle motion involves the use of a *body-fixed coordinate system* (i.e., a coordinate system attached to the body and moving with it). This type of description is not commonly discussed in typical mechanics textbooks so it will be presented in a general framework in Chapter 2. In the present example, the two ways of writing dynamic equations will be presented in a simplified form.

Finally, the shopping cart example introduces the analysis of stability for dynamic vehicle models in the simplest possible way. The final dynamic equation of motion is only of first order so the eigenvalue problem that is at the heart of stability analysis is almost trivial. The general concepts and theory behind stability analysis for linearized systems is discussed in some detail in Chapter 3.

A supermarket shopping cart has casters in the front that are supposed to swivel so that the front can be pushed easily in any direction and a back axle with fixed wheels that are supposed to roll easily and resist sideways motion. Anyone who has actually used a shopping cart realizes that real carts often deviate significantly from these ideals. The casters at the front often do not pivot well so the cart is hard to turn and one is often forced to skid the back wheels to make a sharp turn. Furthermore, the wheels often do not roll easily so quite a push is required to keep the cart moving. On the other hand, a mathematically ideal cart serves as a good introduction to the type of analysis used to study ground vehicles in general.

After analyzing the stability properties of the ideal cart, one can experiment with a nonideal real cart and find that the basic conclusions do hold true in the main even when the idealizations are not strictly true. (This might be better done in a parking lot than in the aisles of a supermarket.) This example provides a preview of the much more complex analysis of the lateral stability of automobiles in Chapter 6.

The interaction of the tires of the shopping cart with the ground will be highly simplified. It will be assumed that the front wheels generate no side force at all because of the pivoting casters. The front of the cart can be moved sideways with no side force required because the casters are supposed to swivel without friction and the wheels are also assumed to roll without friction.

At the rear, the wheels are assumed to roll straight ahead with no resistance to rolling but to allow no sideways motion at all. Obviously, if the rear wheels are pushed sideways hard enough they will slide sideways, but if the side forces are small enough, the wheels roll essentially only in the direction that they are pointed.

### 1.2.1 Inertial Coordinate System

The first analysis will consider motion of the cart to be described in an $x - y$ inertial reference frame. Figure 1.2 shows a view of the shopping cart seen from above. The coordinates $x$ and $y$ locate the center of mass of the cart with respect to the ground and $\phi$ is the angle of the cart centerline with respect to a line on the ground parallel to the $x$-axis. The $x - y$ axes are supposed to be neither accelerating nor rotating and thus they are an inertial frame in which Newton's laws are easily written. The cart is assumed to move only in plane motion.

In Figure 1.2, the parameters of the cart are $a$ and $b$, the distance from the center of mass to the front and rear axles, respectively, $m$, the mass, and $I_z$, the moment of inertia about the mass center and with respect to the vertical axis.

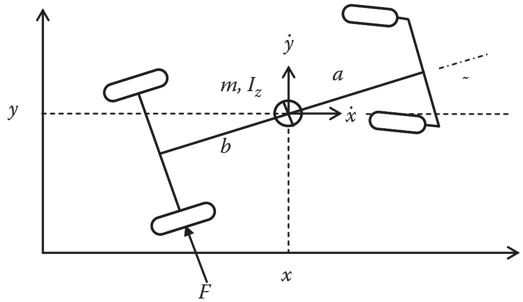



**FIGURE 1.2**
Shopping cart described in an inertial coordinate frame.

(As we will see, the *track* or width between the front or rear wheels turns out not to play a role in this analysis.)

The *basic motion* of the center of mass will be considered to be along the $x$-axis so the variables needed to describe the perturbed motion are primarily $y(t)$ and $\phi(t)$. (In principle, $x(t)$ is also needed, but for the basic as well as the perturbed motion, it will be assumed that $\dot{x} = U$, a constant, so $x(t) = Ut +$ constant. Thus, the speed $U$ will play the role of a parameter in the analysis. As we will see, this assumption is reasonable as long as the variables describing the perturbed motion remain small.) In the basic motion, $\dot{y}$, $\phi$, and $\dot{\phi}$ all vanish and the cart proceeds along the $x$-direction at a strictly constant speed $U$ since there are no forces in the $x$-direction.

For the perturbed motion, $\dot{y}$ is assumed to be small compared with $U$, and $\phi$ and $\dot{\phi}$ are also small. (To be precise, $b\dot{\phi}$ is considered to be small compared to $U$.) If $\phi$ is a small angle, the small angle approximations $\cos\phi \cong 1$ and $\sin\phi \cong \phi$ can be used when considering the perturbed motion. The side force at the rear wheels is $F$, and since the rear wheels are assumed to roll without friction, there is no force in the direction the rear wheels are rolling. At the front axle there is no force at all in the $x - y$ plane because of the casters. There are, of course, vertical forces at both axles that are necessary to support the weight of the cart, but they play no role in the lateral dynamics of the cart.

Because the cart is described in an inertial coordinate system and is executing plane motion, three equations of motion are easily written. The equations state that the force equals mass times acceleration of the center of mass in the $x$- and $y$-directions and moment about the center of mass equals rate of change of angular momentum around the $z$-axis. Using the fact that the angle $\phi$ is assumed to be very small for the perturbed motion, the equations of motion are

$$m\ddot{x} = -F\sin\phi \cong -F\phi \cong 0, \tag{1.12}$$

$$m\ddot{y} = F\cos\phi \circ F, \tag{1.13}$$

$$I_z\ddot{\phi} = -Fb. \tag{1.14}$$

In Equation 1.12, the acceleration in the $x$-direction is seen to be not exactly zero. However, we consider that the forward velocity $U$ is initially large and does not change much as long as the angle $\phi$ remains small. From Equations 1.13 and 1.14 we see that the acceleration in the $y$-direction and the angular acceleration are large compared to the $x$-direction acceleration when $\phi$ is small.

Considering the velocity of the center of the rear axle, one can derive the condition of zero sideways velocity for the rear wheels. The basic kinematic velocity relation for two points on the same rigid body (see, for example, Crandall et al. 1968) is

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{AB}, \tag{1.15}$$

where $\vec{\omega}$ is the angular velocity of the body and $\vec{r}_{AB}$ is the vector distance between $A$ and $B$.

In the case at hand, let $\vec{v}_A$ represent the velocity of the mass center and let $\vec{v}_B$ represent the velocity of the center of the rear axle. Then $\vec{\omega}$ is a vector perpendicular to $x$ and $y$ in the $z$-direction. The magnitude of $\vec{\omega}$ is $\dot{\phi}$. The magnitude of $\vec{r}_{AB}$ is just the distance, $b$, between the center of gravity and the center of the rear axle.

Figure 1.3 shows the velocity components involved when Equation 1.15 is evaluated. The component $b\dot{\phi}$ represents the term $\vec{\omega} \times \vec{r}_{AB}$ in Equation 1.15.

From Figure 1.3, one can see that the side velocity (with respect to the center line of the cart) at the center of the rear axle is

$$\dot{y}\cos\phi - b\dot{\phi} - U\sin\phi = 0. \tag{1.16}$$

Using the small angle approximations, the final relation needed is simply a statement that this sideways velocity of the center of the rear axle should vanish.

$$\dot{y} - b\dot{\phi} - U\phi = 0. \tag{1.17}$$

(It is true but perhaps not completely obvious at first that if the two rear wheels cannot move sideways but can only roll in the direction that they are pointed instantaneously, then any point on the rear axle also cannot have any sideways velocity. This means that the kinematic constraint of Equation 1.17 derived for the center of the axle properly also constrains the variables such that the two wheels also have no sideways velocity.)

Now combining Equations 1.13 and 1.14, one can eliminate $F$, yielding a single dynamic equation involving $\ddot{y}$ and $\ddot{\phi}$. Then, after differentiating

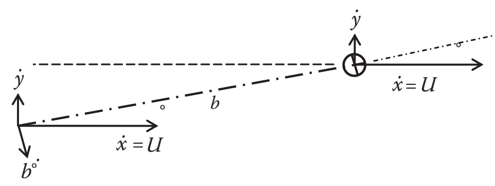



**FIGURE 1.3**
Velocity components at the center of the rear axle.

Equation 1.17 with respect to time, $\ddot{y}$ can be eliminated from this dynamic equation to yield a single equation for $\phi$. The result is

$$\left(I_z + mb^2\right)\ddot{\phi} + mbU\dot{\phi} = 0. \tag{1.18}$$

Equation 1.18 may appear to be of second order since it involves a term containing $\ddot{\phi}$, but because $\phi$ itself is missing, a first-order version of Equation 1.18 can be studied instead.

The angle $\phi$ actually has no particular significance. It is just the angle the cart makes with the $x$-axis, which itself is a line on the ground in an arbitrary direction. Since the angle $\phi$ itself does not appear in Equation 1.18, it is logical to consider the angular rate $\dot{\phi}$ as the basic variable rather than $\phi$. The angle $\phi$ is called the *yaw angle* and it is common to call the angular rate $\dot{\phi}$ the *yaw rate* and to give it the symbol $r$ in vehicle dynamics. Standard symbols usually used in vehicle dynamics will be presented later in Chapter 2.

In terms of yaw rate, Equation 1.18 can be rewritten in first-order form. Noting that $\dot{\phi} = r,\ \ddot{\phi} = \dot{r}$, the result is

$$\dot{r} + \left[mbU/\left(I_z + mb^2\right)\right]r = 0. \tag{1.19}$$

This equation is a linear ordinary differential equation with constant coefficients and is of the general form

$$\dot{r} + Ar = 0, \tag{1.20}$$

with $A = mbU/(I_z + mb^2)$.

The consideration of small perturbations from a basic motion for a stability analysis generally leads to linear differential equations. In the present case, the nonlinear equations, Equations 1.13 and 1.16, became linear in approximation because of the small perturbation assumption.

The analysis of the stability of the first-order equation, Equation 1.20, is elementary. The solution to this linear equation can be assumed to have an exponential form such as $r = \mathrm{R}e^{st}$, where both $R$ and $s$ need to be determined somehow. Then $\dot{r} = s\,\mathrm{R}e^{st}$, which when substituted into Equation 1.20, yields $s\ \mathrm{R}e^{st} + A\ \mathrm{R}e^{st} = 0$ or

$$(s + A)\mathrm{R}e^{st} = 0. \tag{1.21}$$

This result is the basis of an *eigenvalue* analysis. The general theory of eigenvalues and their use in stability analysis will be presented in Chapter 3. For now it is enough to note that when three factors must multiply to zero, as in Equation 1.21, the equation will be satisfied if any one of the three factors vanishes.

For example, if $R$ were to be zero, the product in Equation 1.21 would certainly be zero. This represents the so-called *trivial solution*. The assumed solution would then simply imply $r(t) = 0$, meaning that the yaw rate simply remains zero if it starts at zero. This should have been an obvious possibility from the beginning because it represents the basic motion. Another possible factor to vanish in Equation 1.21 is $e^{st}$. Not only is the vanishing of this factor not really possible, but even if it were, the result would again be the trivial solution. The only important condition for Equation 1.21 is the vanishing of the factor in the parentheses which happens when

$$s = -A. \tag{1.22}$$

Thus, we have determined that the$s$ takes on the value $-A$, the *eigenvalue,* and the only nontrivial solution has the form

$$r = \mathrm{R}e^{-At}, \tag{1.23}$$

in which $R$ is an arbitrary constant determined, for example, by the initial value of the yaw rate $r$ at time $t = 0$. Eigenvalues will be discussed at length in Chapter 3.

Figure 1.4 shows the nature of the solutions for the two cases when $A$ is positive or negative. When A is positive, the system is stable and the yaw rate will return to zero after a disturbance. If A is negative, the yaw rate increases in time, the cart begins to spin faster and faster, and the system is unstable.

In the case of the shopping cart with

$$A = mbU/(I_z + mb^2), \tag{1.24}$$

it is clear that $A$ will be positive as long as $U$ is positive in the direction shown in Figure 1.3 since the parameters $m$, $b$, and $I_z$ are inherently positive. This means that if a shopping cart is given a push in the forward direction but

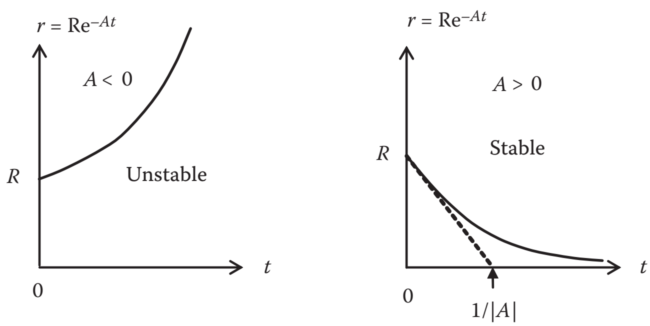



**FIGURE 1.4**
Responses for first-order unstable and stable systems.

with an initial yaw rate, the yaw rate will decline towards zero exponentially as time goes on. Ultimately, the cart will roll in essentially a straight line.

On the other hand, Equations 1.17, 1.18, and 1.19 remain valid whether $U$ is considered to be positive in the sense shown in Figure 1.3 or negative, and the same is true of Equation 1.24. So if the cart is pushed backwards, one can simply consider $U$ to be negative. Then the combined parameter $A$ in Equation 1.24 will be negative and Equation 1.23 will represent an *increasing* exponential response. If the yaw rate is given any initial value, however small, the yaw rate will increase exponentially in time. In fact, a cart pushed backward will eventually turn around and travel in the forward direction, even though the linearized dynamic equation we have been using cannot predict this, since the small angle approximations no longer are valid after the cart has turned through a large angle.

Another interesting aspect of first-order equation response is the speed with which the stable version returns to zero. If the response is written in the form

$$r = \mathrm{R}e^{-t/\tau} \tag{1.25}$$

where $\tau$ is defined to be the *time constant*,

$$\tau = 1/|A|, \tag{1.26}$$

it is clear that the time constant is just the time at which the initial response decays to $1/e$ times the initial value for the stable case.

For the shopping cart,

$$\tau = (I_z + mb^2)/mbU. \tag{1.27}$$

Figure 1.4 shows that the time constant is readily shown on a plot of the response in the stable case by extending the initial slope of the response from an initial point (assuming $t = 0$ at the initial point) to the zero line. The time constant is, in fact, the time at which the extended slope line intersects the zero line.

From Equation 1.27 it can be seen that the time constant depends on the physical parameters of the cart and the speed. Increasing the speed decreases the time constant and increases the quickness with which the yaw rate declines toward zero for the stable case. Pushing the cart faster in the backwards direction also quickens the unstable growth of the yaw rate.

### 1.2.2 Body-Fixed Coordinate System

The use of an inertial coordinate system to describe the dynamics of a vehicle may seem reasonable at first, but, in fact, many analyses of vehicles use a moving coordinate system attached to the vehicle itself. The vehicle motion is described by considering linear and angular velocities rather than positions. (In the previous analysis of the shopping cart, the angular position of the cart turned out to be less important than the angular rate.)

To introduce this idea we will now repeat the previous stability analysis using a coordinate system attached to the center of mass of the cart and rotating with the cart. This will require a restatement of the laws of dynamics, taking into consideration the noninertial moving coordinate system. The general case of rigid body motion described in a coordinate system attached to the body itself and executing three-dimensional motion will be presented in Chapter 2. Here we will present the simpler case of plane motion appropriate for the shopping cart.

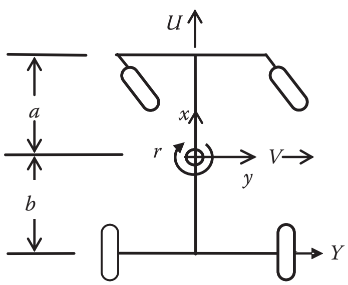
Figure 1.5 shows the moving coordinate system and the velocity components associated with it. In this description of the motion, the basic motion consists of a constant forward motion with velocity $U$, which again will function as a parameter. The variables for the cart are now the side or lateral velocity $V$ and the yaw rate $r$. This notation is in conformity with the general notation to be introduced in Chapter 2. For the basic motion, the side velocity and angular velocity are zero, $V = r = 0$.

The perturbed motion has small lateral velocity and yaw rate, $V \ll U$, $r \ll U/b$. Again the parameters are $b$, $m$, $I_z$, and $U$.

The $x - y$ coordinate system moves with the cart and the side force $Y$ at the rear axle always points exactly in the $y$-direction. By our assumption of freely rolling wheels, a possible force in the $x$-direction, $X$, is zero.

We are now in a position to write equations relating force to the mass times the acceleration of the center of mass and the moment to the change of angular momentum. Because the $x$-$y$-$z$ is rotating with the body, we must first properly find the absolute acceleration in terms of components in a rotating frame.

The fundamental way to find the absolute rate of change of any vector $\vec{v}$ measured in a frame rotating with angular velocity $\vec{\omega}$ is given by the well-known formula




$x - y - z$ coordinates are attached at the center of mass and move with the cart

$U$, $V$ are velocity components with respect to $x$, $y$

$r$ is the yaw rate (angular velocity around the $z$-axis, pointed down)

$Y$ is the force on the rear axle

**FIGURE 1.5**
Coordinate system attached to the shopping cart.