Option Pricing

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Theory

& associated context

- Options
- Assumptions
- Black Schole pricing

Options ?

- Options are a type of *derivative;* They are related to other instruments.
- Contracted between a *writer* and a *holder*
- Right to buy (*call*) or sell (*put*) an asset (the *underlying*) at a given time (*exercise date*) for a given price (*strike price*).

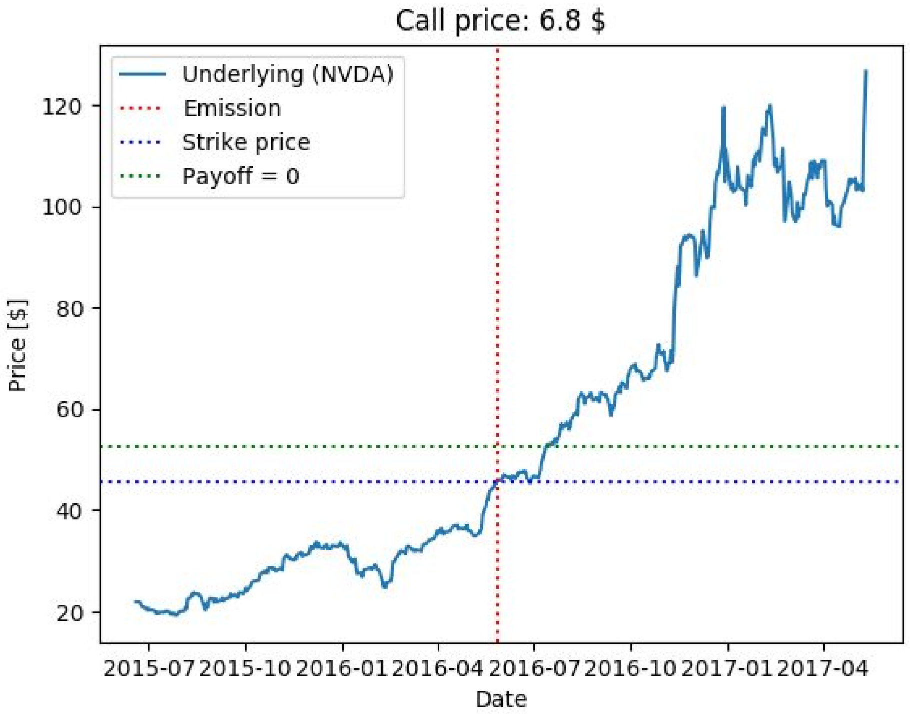
An asset is sold at the same price everywhere.

An asset’s price fully reflects all information.

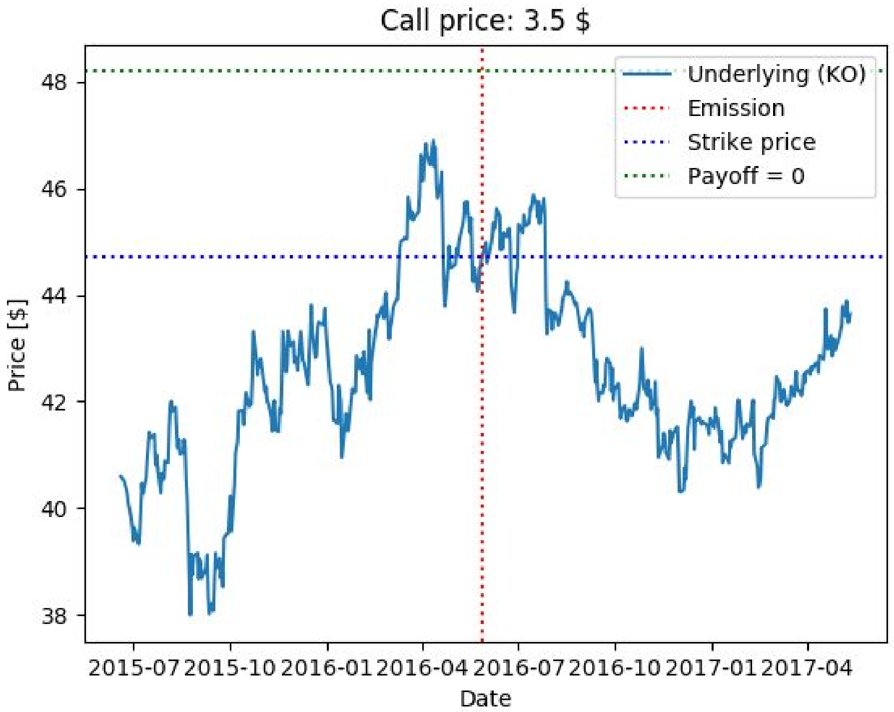
The rate of the price's fluctuation is a Wiener process.

Payoff

- If the underlying rises above the strike price, the buyer of a call has a positive payoff
- If the underlying goes below the strike price, the buyer of a put has a positive payoff.
- Otherwise the buyer will not use his option and buy or sell at the market rate, incurring only a loss equal to the option’s price.



Example of a positive payoff for a buyer



Example of a negative payoff for a buyer

Itô rule of calculus

Given an Itô process

$$\mathrm{d}S = a(S,t)\,\mathrm{d}t + b(S,t)\,\mathrm{d}W(t)$$

and a function $f(S,t)$, we have

$$\mathrm{d}f(S,t) = \left(\frac{\partial f}{\partial t} + a(S,t)\frac{\partial f}{\partial S} + \frac{1}{2}b(S,t)^2\frac{\partial^2 f}{\partial S^2}\right)\mathrm{d}t + b(S,t)\frac{\partial f}{\partial S}\,\mathrm{d}W(t)$$

Notation

$$
\begin{aligned}
S(t) &:= \text{Underlying asset price} \\
T &:= \text{Exercise date} \\
r &:= \text{Risk free interest rate} \\
\mathcal{C}(S,T) &:= \text{Payoff of a call option for asset S at time T} \\
\mathcal{P}(S,T) &:= \text{Payoff of a put option for asset S at time T} \\
\mathcal{O}(S,t) &:= \text{Option price}
\end{aligned}
$$

Asset price as an Itô process

$$\mathrm{d}S = \mu S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t)$$

- σ is the **volatility.**
- μ is the **drift**.

Delta hedging

- Base idea: The writer has some of the underlying and some cash to reduce his risk.
- She uses the money from the option sale to fund this.

$$
\begin{aligned}
\Delta(t) &= \text{Fraction of the underlying kept as safety.} \\
\mathit{\Pi}(t) &= \text{Cash amount kept as safety at time t.} \\
\mathcal{W}(t) &= \text{Wealth possessed by the writer at time } t. \\
&= \Delta(t)S(t) + \mathit{\Pi}(t) = \mathcal{O}(t)
\end{aligned}
$$

Itô process:

$$\begin{aligned}
\mathrm{d}\mathcal{O} &= \left(\frac{\partial \mathcal{O}}{\partial t} + \mu S(t)\frac{\partial \mathcal{O}}{\partial S} + \frac{1}{2}\left(\sigma S(t)\right)^2 \frac{\partial^2 \mathcal{O}}{\partial S^2}\right)\mathrm{d}t + \sigma S(t)\frac{\partial O}{\partial S}\,\mathrm{d}W(t) \\
&= \left(\frac{\partial O}{\partial t} + \frac{1}{2}\left(\sigma S(t)\right)^2 \frac{\partial^2 \mathcal{O}}{\partial S^2}\right)\mathrm{d}t + \frac{\partial \mathcal{O}}{\partial S}\,\mathrm{d}S
\end{aligned}$$

Delta hedging:

$$\mathrm{d}\mathcal{O} = \Delta(t)\,\mathrm{d}S + \mathrm{d}\mathit{\Pi} = \Delta(t)\,\mathrm{d}S + r\mathit{\Pi}\,\mathrm{d}t$$

Therefore:

$$\begin{aligned}
\Delta(t) &= \frac{\partial \mathcal{O}}{\partial S} \\
r\mathit{\Pi} &= \frac{\partial \mathcal{O}}{\partial t} + \frac{1}{2}\left(\sigma S(t)\right)^2 \frac{\partial^2 \mathcal{O}}{\partial S^2}
\end{aligned}$$

Finally:

$$\underbrace{\frac{\partial \mathcal{O}}{\partial t} + \frac{1}{2}\left(\sigma S(t)\right)^2 \frac{\partial^2 \mathcal{O}}{\partial S^2}}_{r\mathit{\Pi}} + \underbrace{r\frac{\partial \mathcal{O}}{\partial S}S}_{r\Delta(t)S} - r\mathcal{O} = 0$$

Edge conditions (call)

$$
\begin{aligned}
t = T: \quad & \mathcal{C}(S,T) & = max\left(S(T) - K, 0\right) \\
S = 0: \quad & \mathcal{C}(0,t) & = 0 \\
S \to \infty: \quad & \mathcal{C}(S,t) & \sim S
\end{aligned}
$$

Black Scholes solution

$$
\begin{aligned}
\mathcal{C}(S,t) &= SN(d_1) - Ke^{-r(T-t)}N(d_2) \\
d_1 &= \frac{\ln\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\
d_2 &= d_1 - \sigma\sqrt{T-t}
\end{aligned}
$$

Put call parity

The price of a put and a call are **linked**.

To show this, we have the following portfolio:

- The underlying S
- A put P on S
- We sold a call C on S

Both options are at strike K

The portfolio has the value:

$$\Pi(t) = S(t) + \mathcal{P}(S,t) - \mathcal{C}(S,t)$$

At expiry, the options yield:

$$\begin{aligned} \mathcal{C}(S,T) &= \max(S(T) - K, 0) \\ \mathcal{P}(S,T) &= \max(K - S(T), 0) \end{aligned}$$

So the payoff of the portfolio is:

$$\begin{aligned} \text{if } S \geq K: \quad & S \quad +0 - (S - K) = K \\ \text{if } S \leq K: \quad & S \quad +(K - S) - 0 = K \end{aligned}$$

The portfolio is risk free, therefore:

$$\mathit{\Pi}(t) = Ke^{-r(T-t)}$$

Replacing in previous equations gives:

$$\mathcal{P}(S,t) = \mathcal{C}(S,t) - S(t) + Ke^{-r(T-t)}$$

What if arbitrage exists?

Suppose $\mathit{\Pi}(t) = \epsilon K e^{-r(T-t)}$ with $\epsilon \neq 1$ (arbitrage).

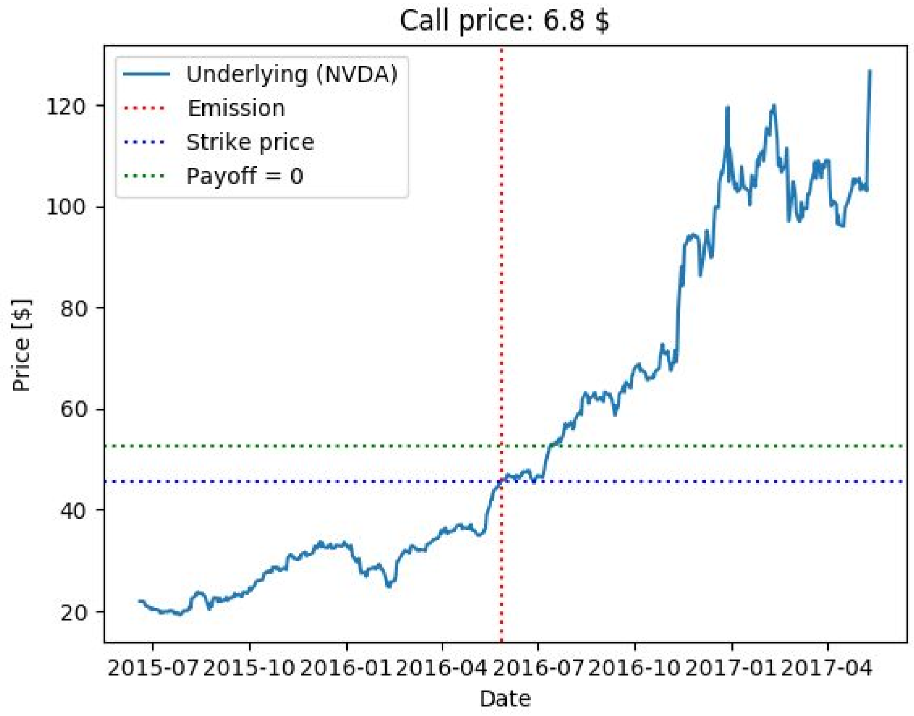
$$
\begin{aligned}
\epsilon < 1: \quad K - \mathit{\Pi}(t)e^{r(T-t)} &= K - (\epsilon K e^{-r(T-t)})e^{r(T-t)} \\
&= K(1-\epsilon) \\
\epsilon > 1: \quad \mathit{\Pi}(t) - K e^{-r(T-t)} &= (\epsilon - 1)K e^{-r(T-t)}
\end{aligned}
$$

We make more money than the risk free placement without taking any risk!

Testing

Testing approach

- Download historical price data (2015 - 2017)
- Price an option based on first half of the time serie (1 year)
- Compute payoff at the end of the second half.
- Average on the 500 largest US companies (S&P 500)



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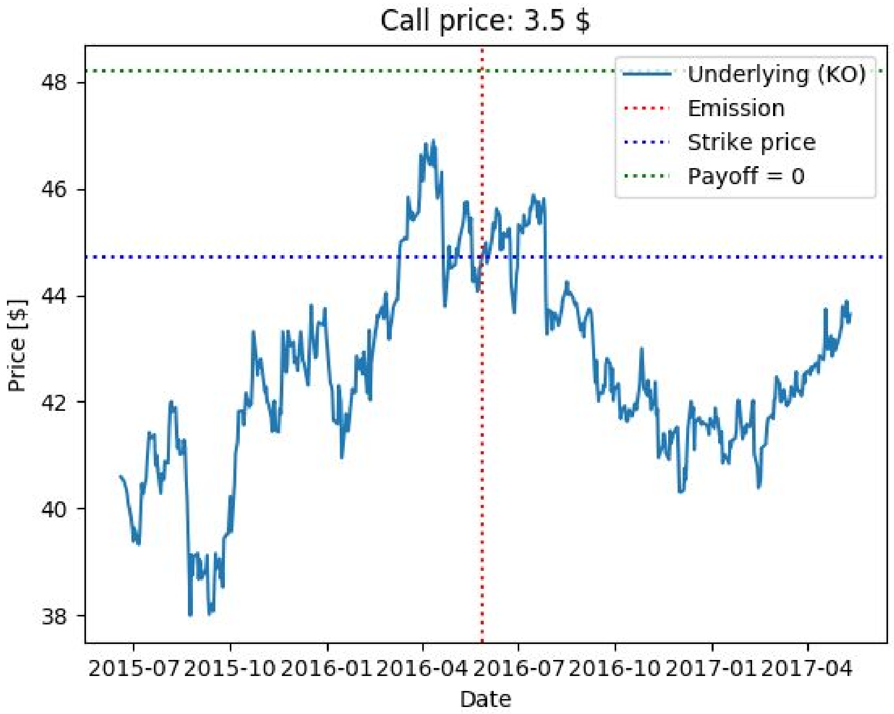

Example of a negative payoff for a buyer

Results & issues

- Estimating the parameters (volatility, risk free rate) is difficult; our model is imperfect.
- The program can calculate price for any European option.
- With correct parameters, the algorithm gives us a **fair price** on an average market.

Other possibilities

- Black Scholes on arbitrary derivatives (Jones, S. P., Eber, J. M., & Seward, J. (2000))
- Reverse Black Scholes: getting information about the market from option prices.
- Other models: LIBOR pricing model, Montecarlo pricing, etc.

Thank you!