11

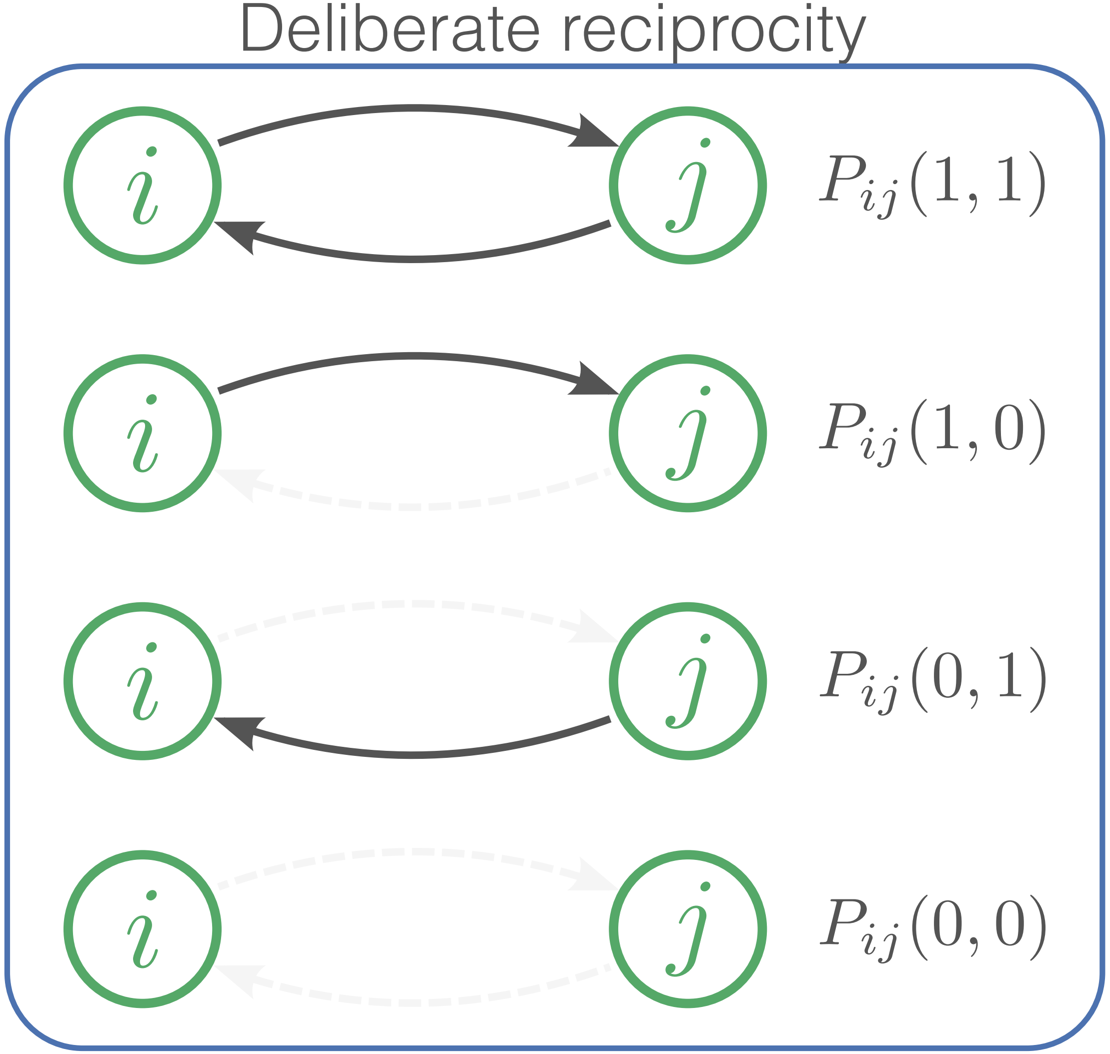
Level of reciprocity controlled with parameter $-1 \leq \nu \leq 1$

$$P_{ij}(1,1) = \begin{cases} (1-\nu)p_{ij}p_{ji} + \nu \min\{p_{ij}, p_{ji}\} & 0 \leq \nu \leq 1 \\ (1+\nu)p_{ij}p_{ji} + \nu(1 - p_{ij} - p_{ji})H(p_{ij} + p_{ji} - 1) & -1 \leq \nu \leq 0 \end{cases}$$

$\nu = 1:$ maximal reciprocity

$\nu = 0:$ fortuitous reciprocity

$\nu = -1:$ minimal reciprocity

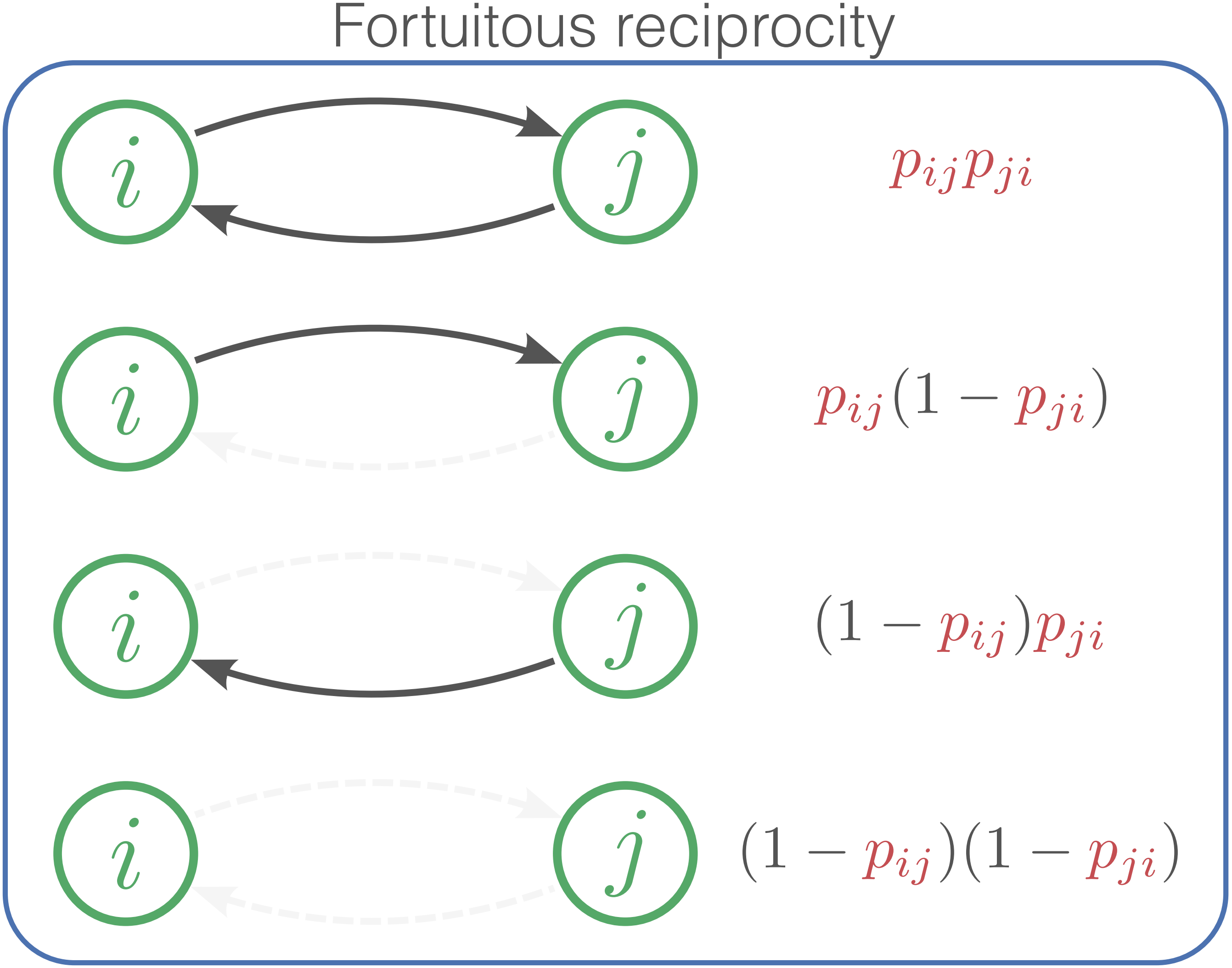

Deliberate reciprocity
i
j
$P_{ij}(1,1)$
$P_{ij}(1,0)$
$P_{ij}(0,1)$
$P_{ij}(0,0)$

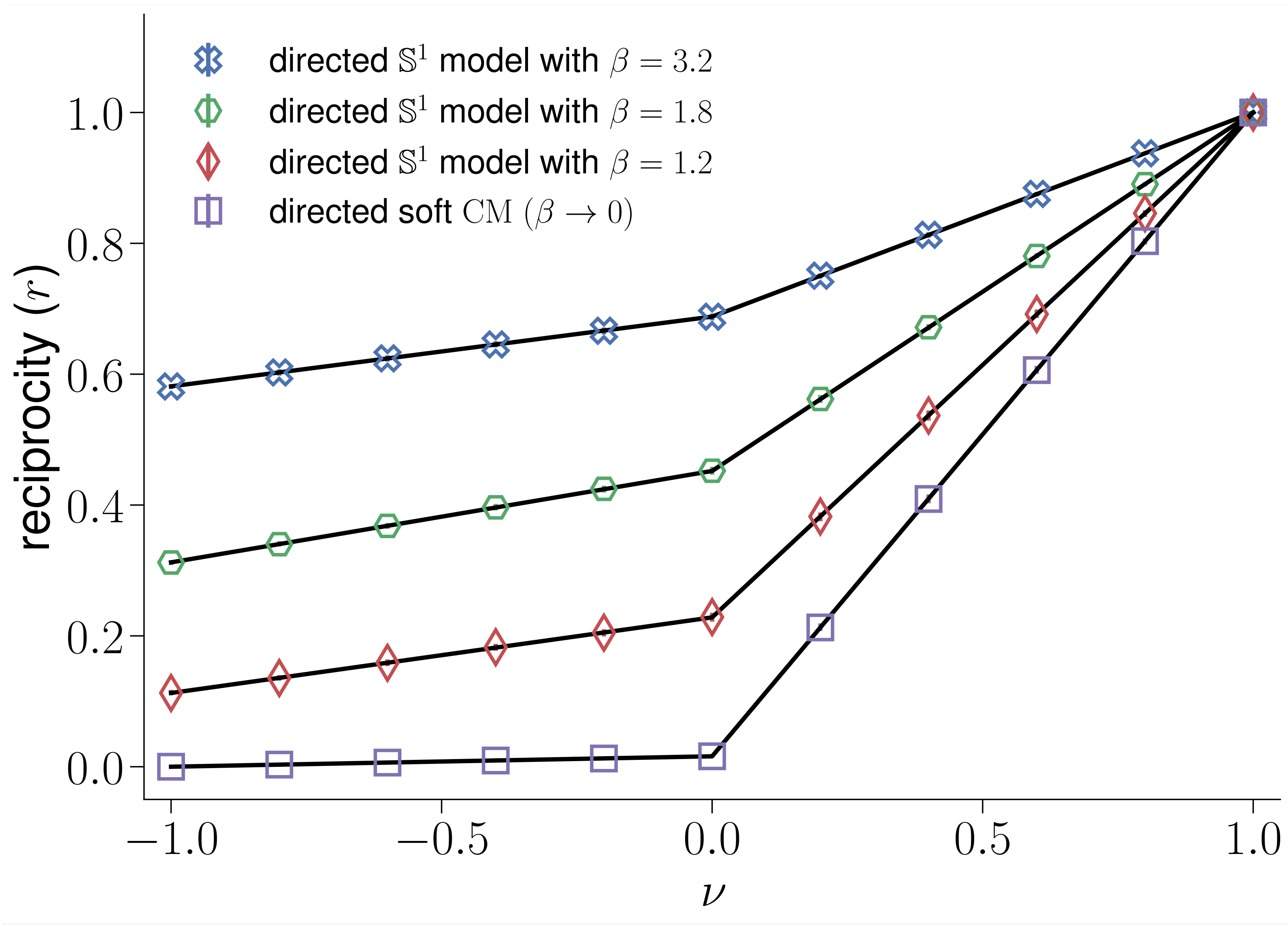
Condition 1: Preserves marginal probabilities

$$P_{ij}(1,0) + P_{ij}(1,1) = p_{ij}$$
$$P_{ij}(0,1) + P_{ij}(1,1) = p_{ji}$$

Condition 2: Normalized

$$\sum_{a_{ij}=0}^{1} \sum_{a_{ji}=0}^{1} P_{ij}(a_{ij}, a_{ji}) = 1$$


Fortuitous reciprocity
$p_{ij}p_{ji}$
$p_{ij}(1-p_{ji})$
$(1-p_{ij})p_{ji}$
$(1-p_{ij})(1-p_{ji})$


directed $\mathbb{S}^1$ model with $\beta = 3.2$
directed $\mathbb{S}^1$ model with $\beta = 1.8$
directed $\mathbb{S}^1$ model with $\beta = 1.2$
directed soft CM $(\beta \to 0)$
reciprocity (r)
ν
1.0
0.8
0.6
0.4
0.2
0.0
−1.0
−0.5
0.0
0.5
1.0

Deliberate reciprocity in random directed networks

A random network model defines the probability p_{ij} for a directed link to exist from node i to node j.

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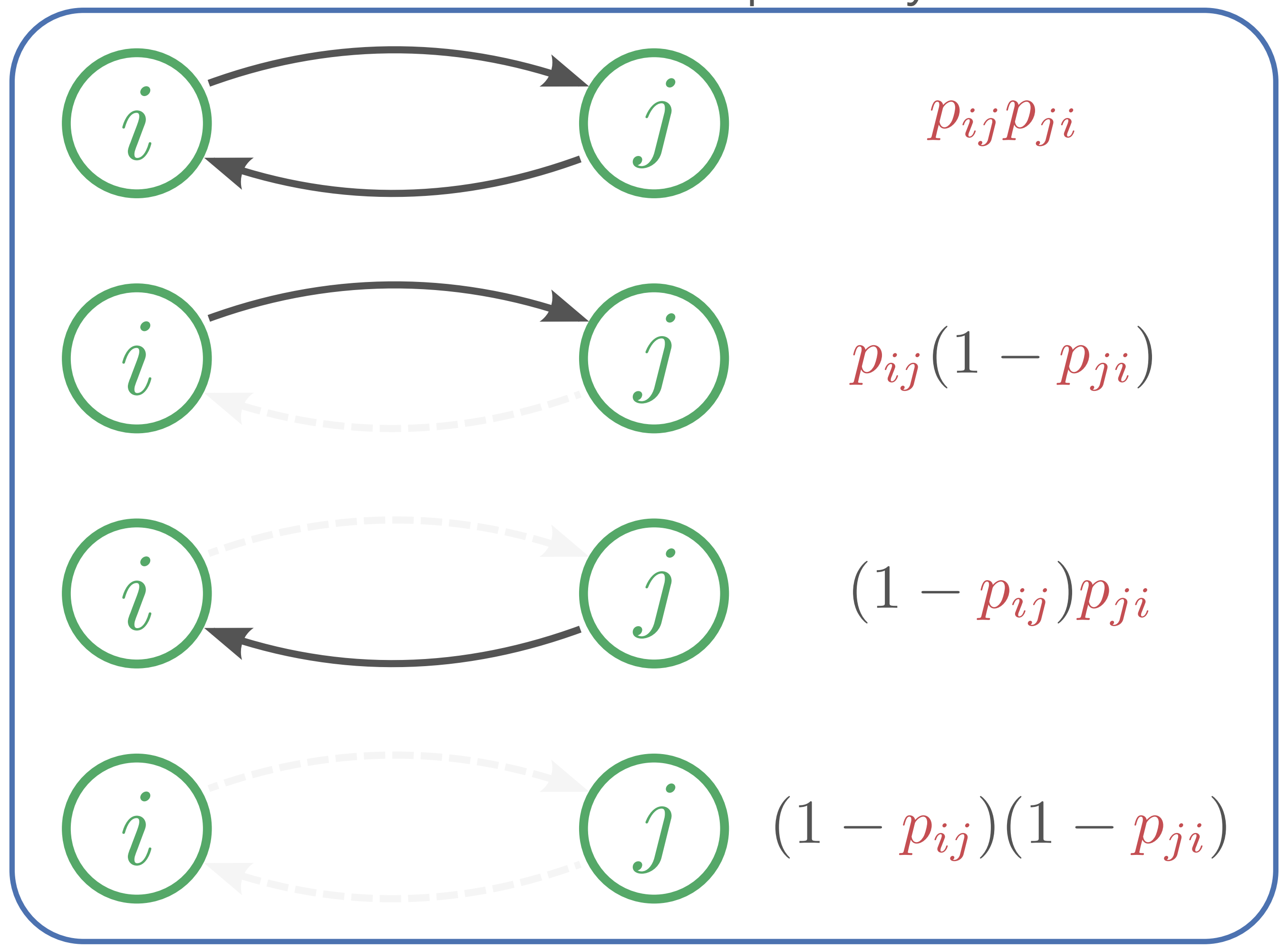
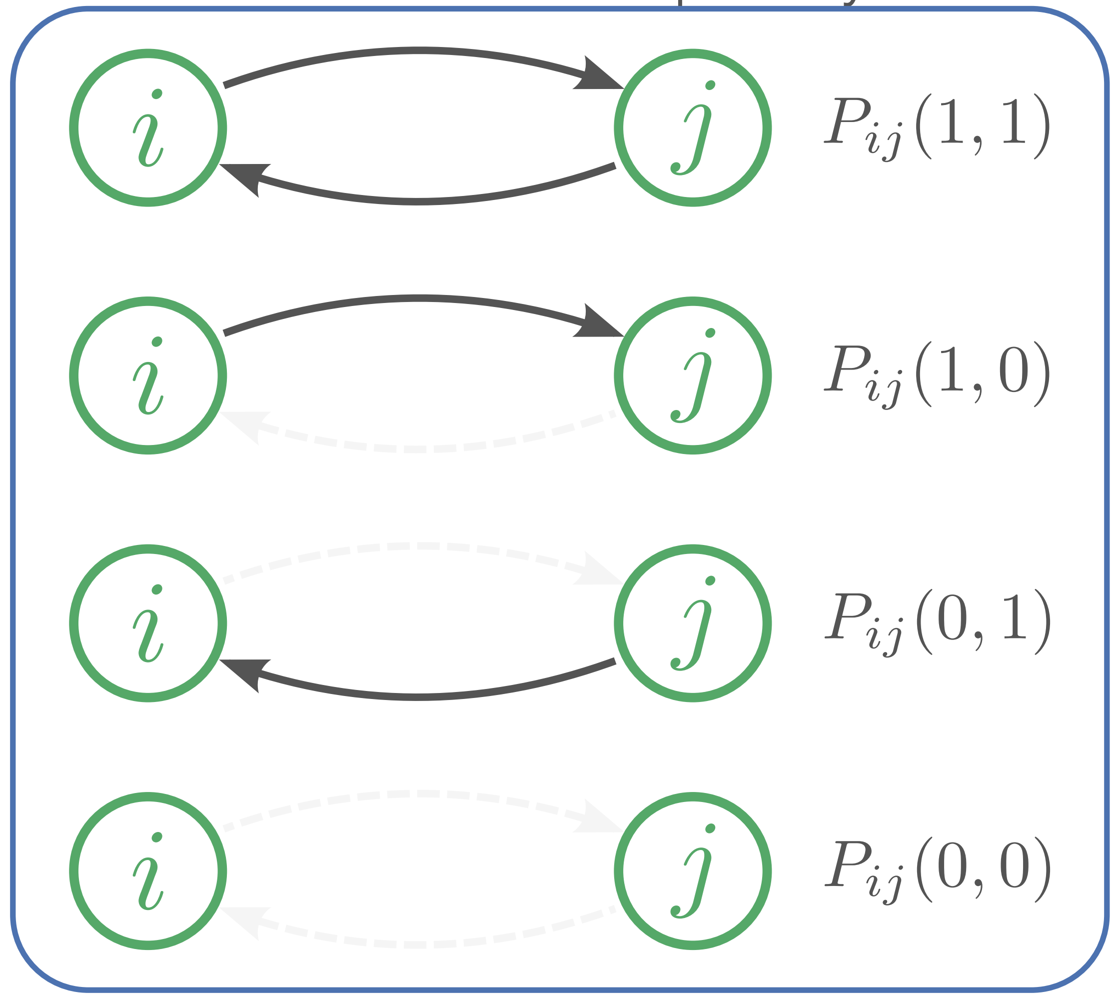
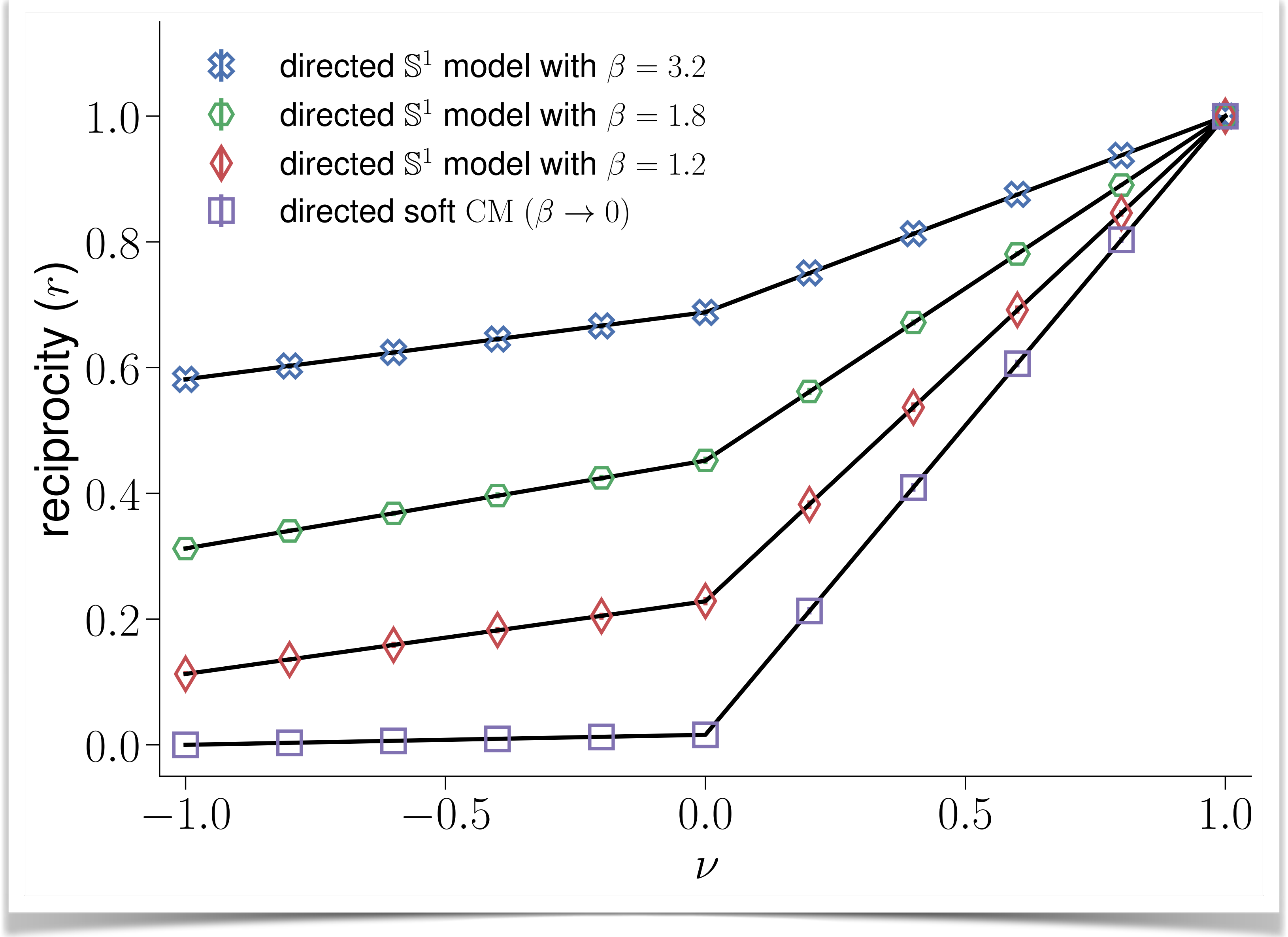
Fortuitous reciprocity



Deliberate reciprocity





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Level of reciprocity controlled with parameter $-1 \leq \nu \leq 1$

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$\nu = 1$: maximal reciprocity

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Fitting the directed $\mathbb{S}^1$ model to real networks

Inputs from a real network :

1. joint degree distribution $P(k^{\text{in}}, k^{\text{out}})$
2. reciprocity r
3. density of triangles

Assuming uniform angular positions for nodes,

1. infer $(\kappa^{\text{in}}, \kappa^{\text{out}})$ to replicate $P(k^{\text{in}}, k^{\text{out}})$ on average (analytical)
2. set ν to reproduce r (analytical)
3. adjust β to recreate the density of triangles (semi-analytical)

Generate a sample of random directed networks :

1. assign angular positions randomly
2. draw directed links using the probabilities defined by the framework for deliberate reciprocity

Inference algorithm adapted from New J. Phys. 21, 123033 (2019).