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Example 4: fixing the expected number of edges and the expected total energy

$$\bar{F}_l = \sum_{j=1}^{N} a_{lj} = k_l \quad (l = 1, \ldots, N)$$

$$\bar{F}_{N+1} = \sum_{i=1}^{N} \sum_{j=i+1}^{N} \varepsilon_{ij} a_{ij} = \sum_{i=1}^{N} \sum_{j=i+1}^{N} f(x_{ij}) a_{ij} = E$$

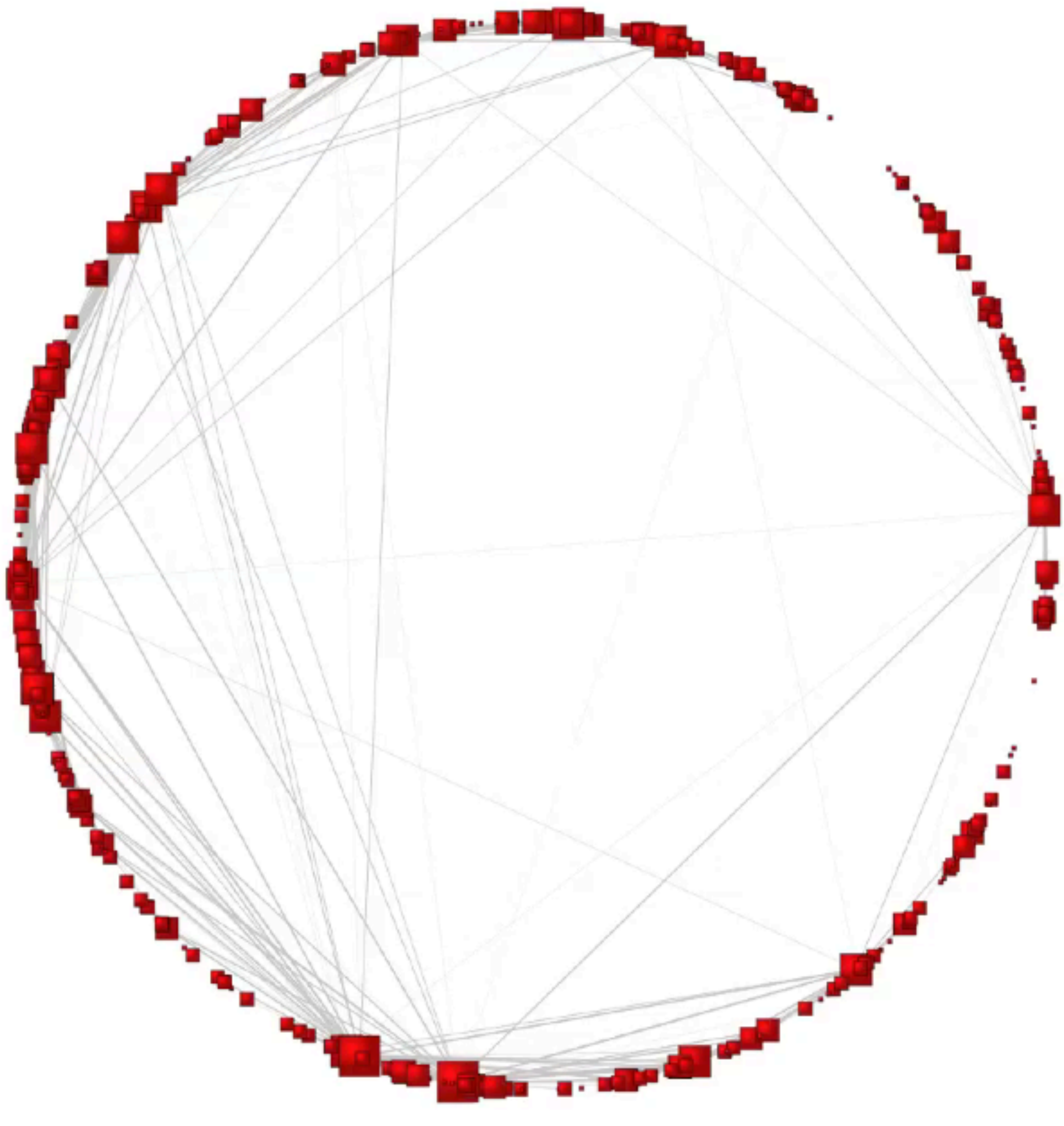
yields the heterogeneous random geometric graph

$$P(\mathbb{A}) = \prod_{i=1}^{N} \prod_{j=i+1}^{N} p_{ij}^{a_{ij}} (1 - p_{ij})^{1-a_{ij}} \quad \text{with} \quad p_{ij} = \frac{1}{e^{\beta \varepsilon_{ij} - \alpha_i - \alpha_j} + 1} .$$

The graphs will be sparse, highly clustered, small-world and devoid of non-structural degre-degre correlation iif $f(x_{ij}) = \ln x_{ij}$ and $\beta \in [D, D+2]$[a]. Redefining $\alpha_l = -(\beta/D) \ln(\sqrt{\mu} \kappa_l)$ yields

$$p_{ij} = \frac{1}{e^{\beta(\varepsilon_{ij} - \mu)} + 1} \quad \text{with} \quad \varepsilon_{ij} = \ln \left(\frac{x_{ij}}{(\kappa_i \kappa_j)^{\frac{1}{D}}} \right) .$$

[a] No upper bound if expected degree sequence is scale-free.



When the geometry is a D-dimensional sphere, $\mathbb{S}^D$ the model can be mapped to a purely geometric model in the hyperbolic disk $\mathbb{H}^{D,1}$.

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Phys. Rev. E 80, 035101 (2009)

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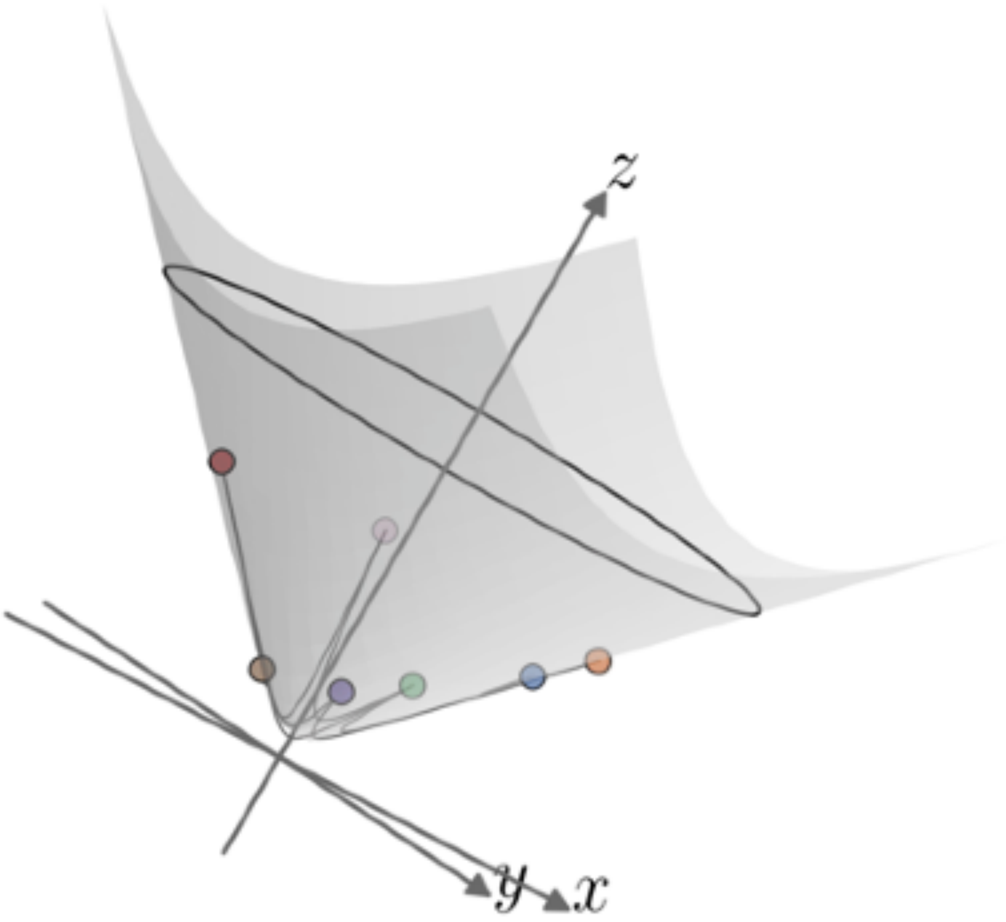
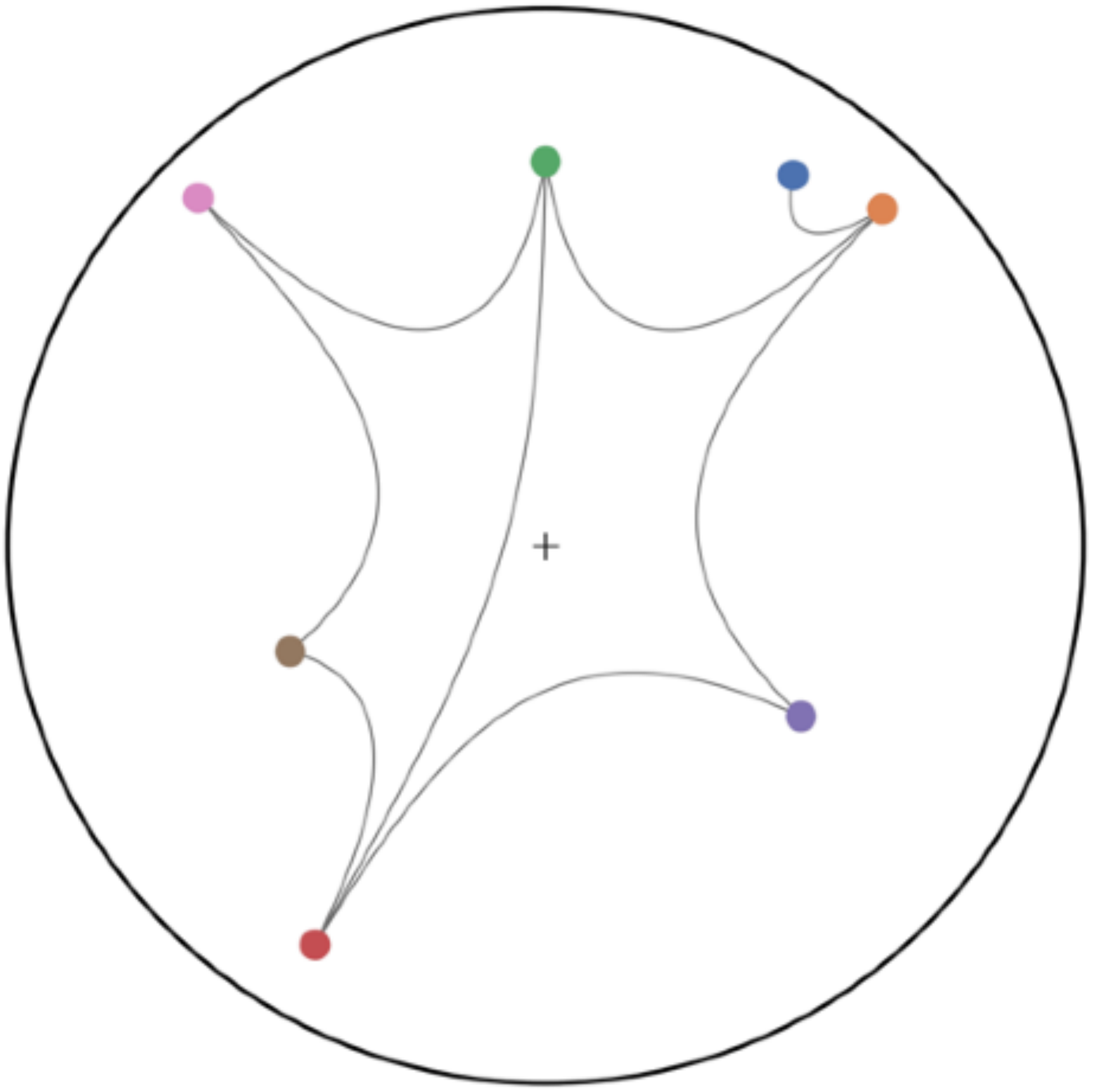
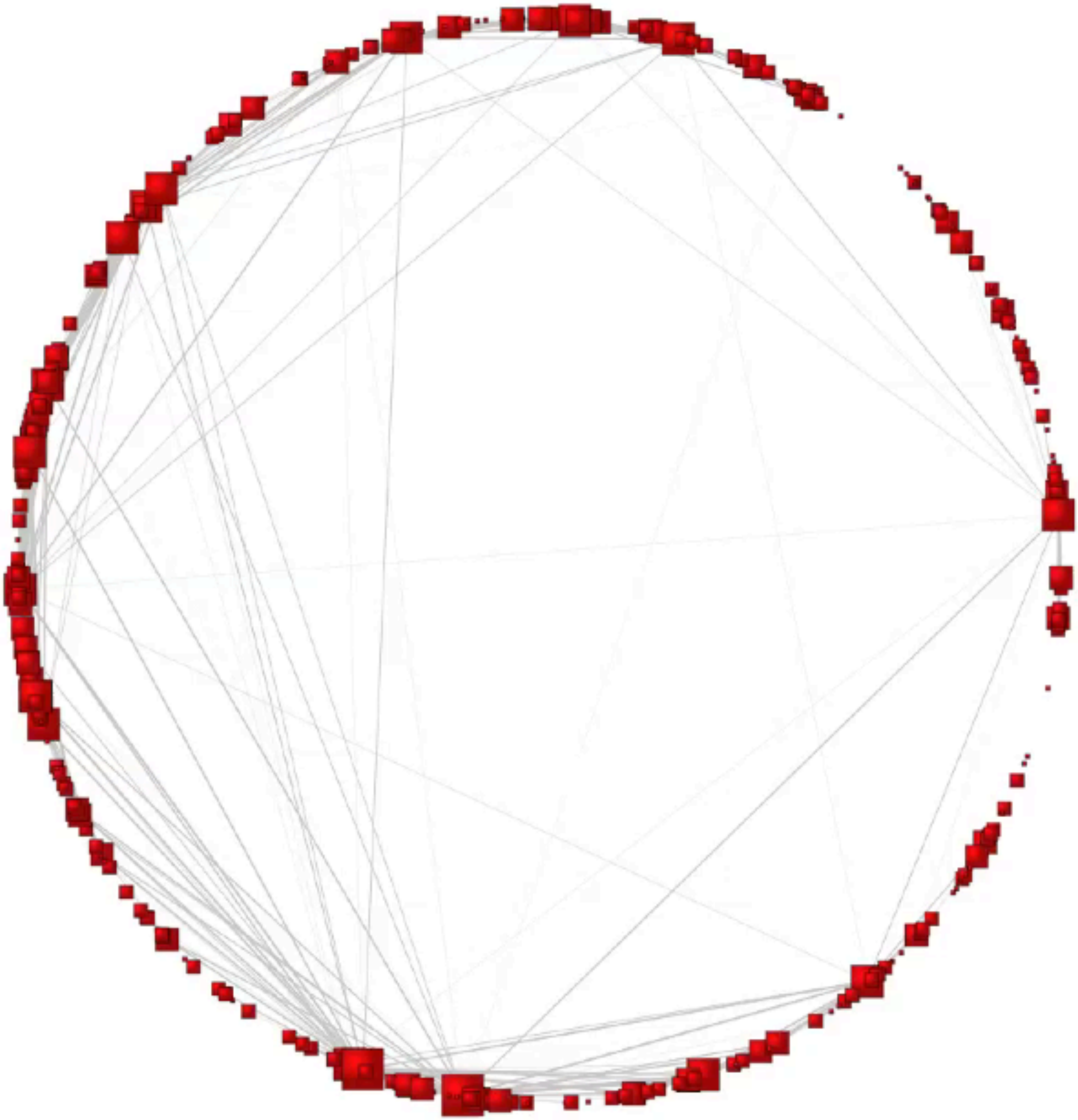


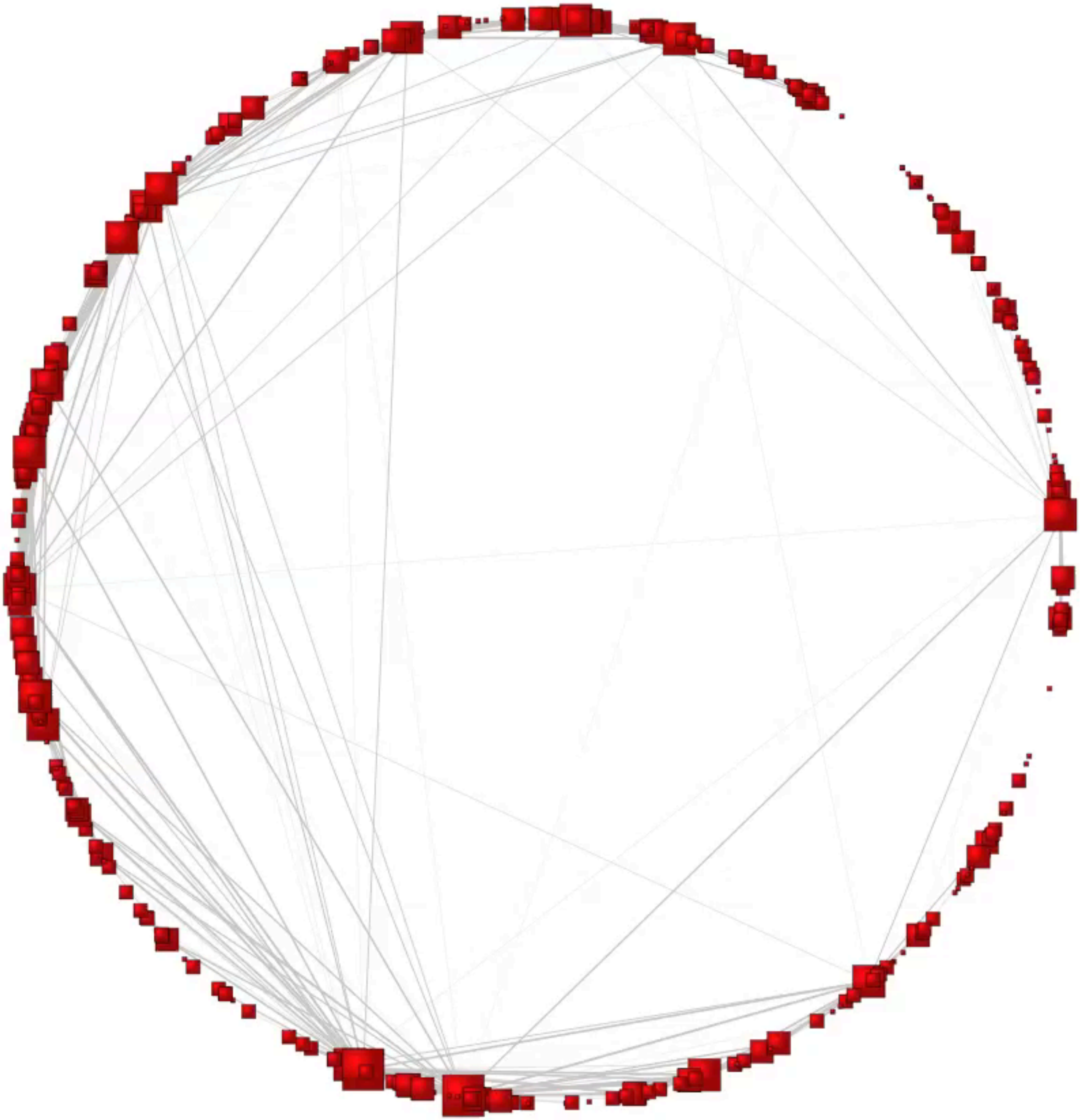
hyperboloid in $\mathbb{R}^{2,1}$

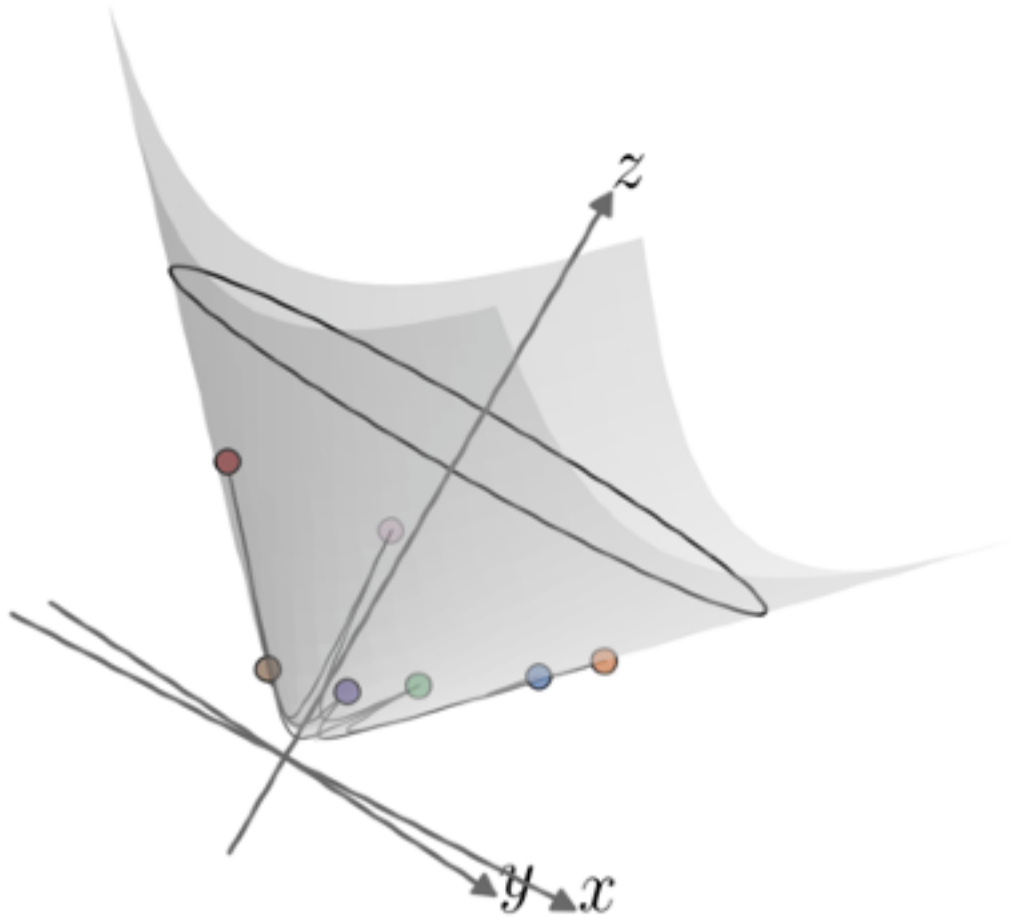


hyperbolic disk (r, θ)

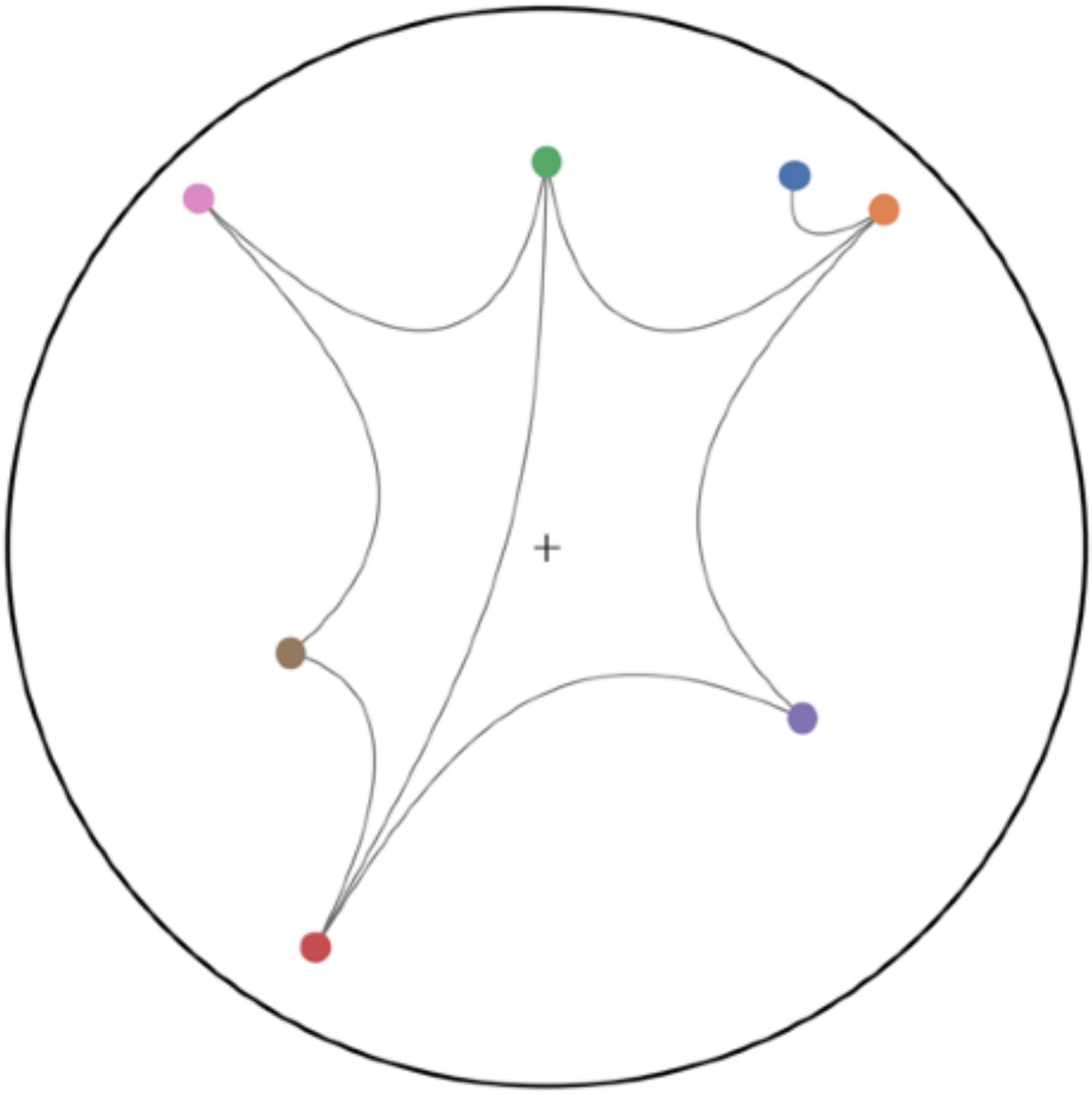
Courtesy of M. Boguñá



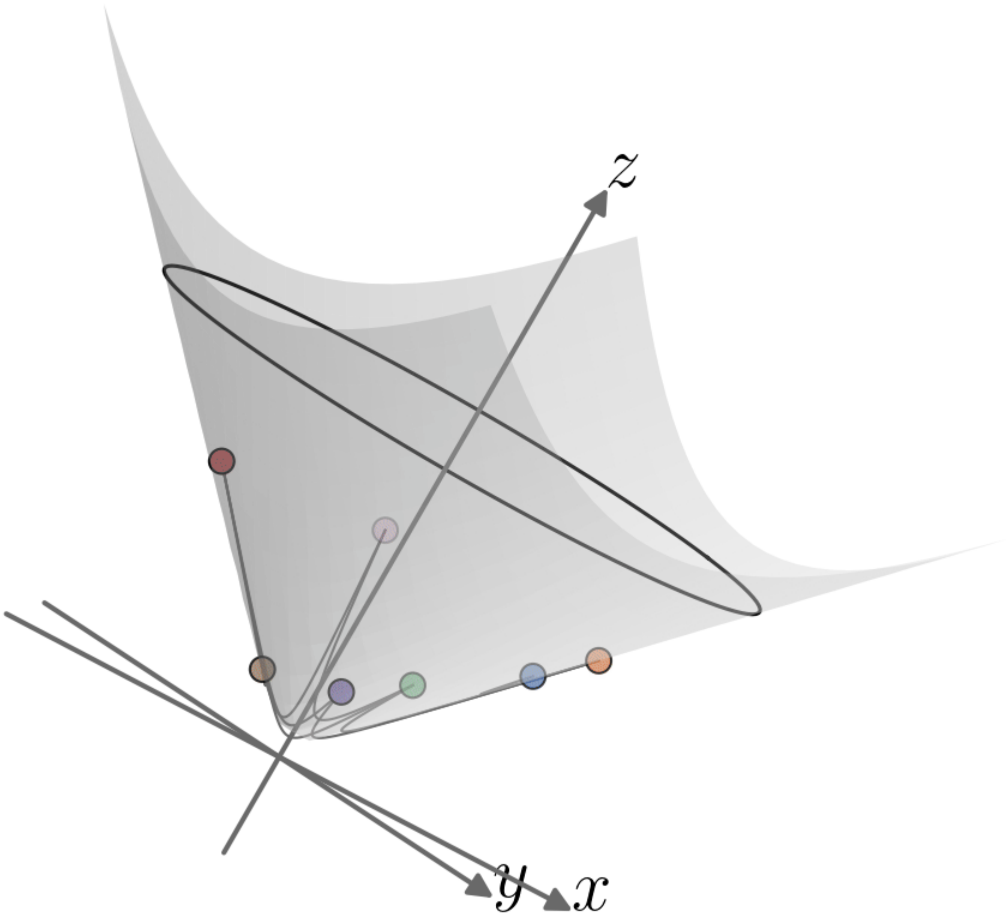




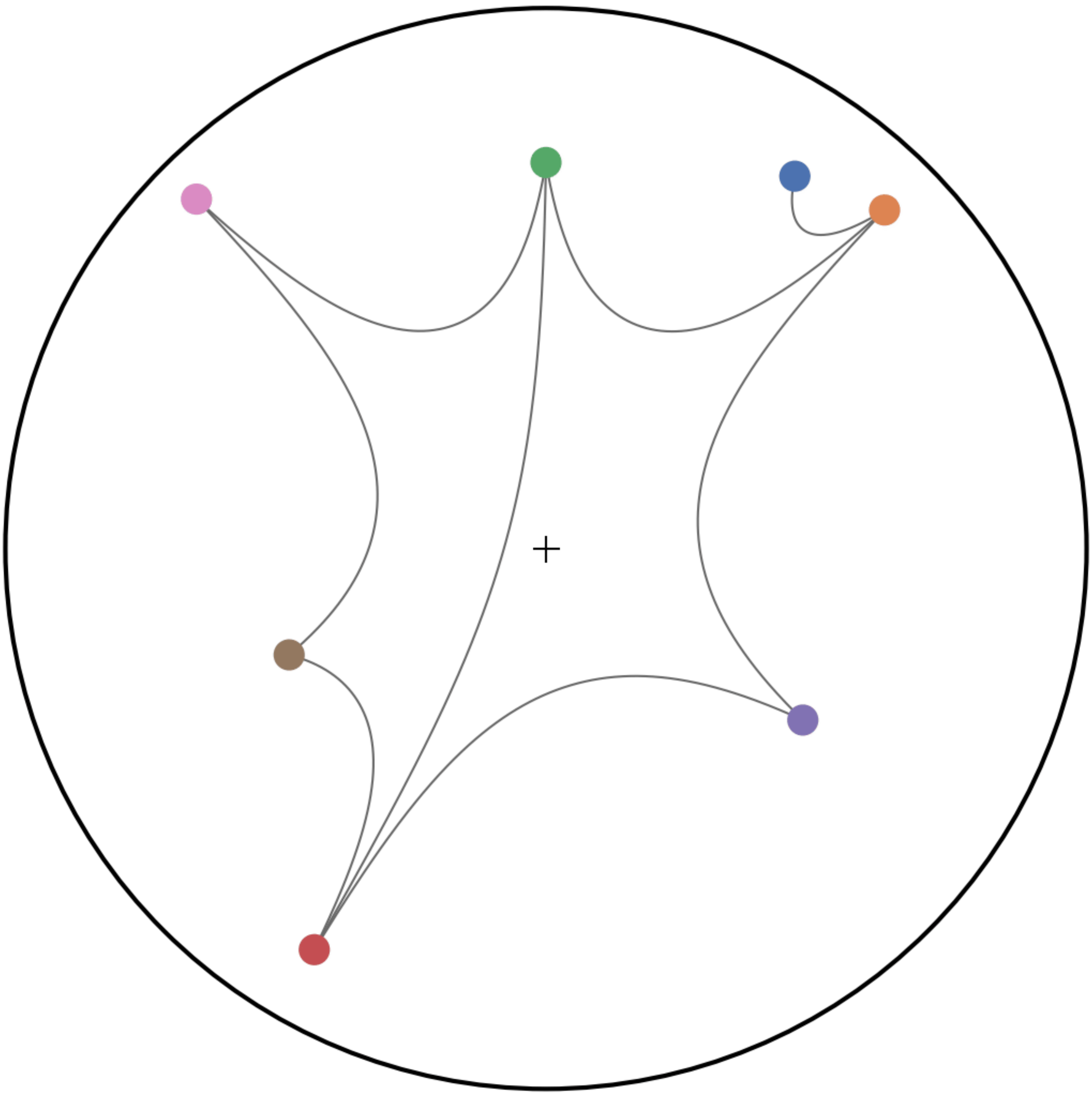
hyperboloid in $\mathbb{R}^{2,1}$



hyperbolic disk (r, θ)



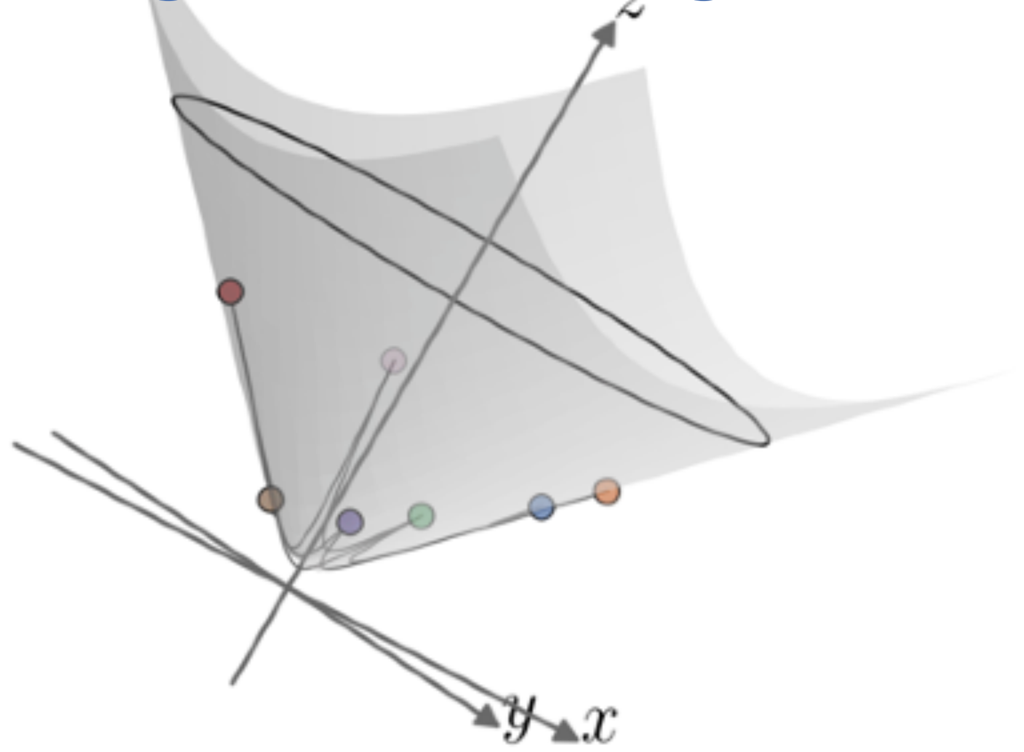
hyperboloid in $\mathbb{R}^{2,1}$



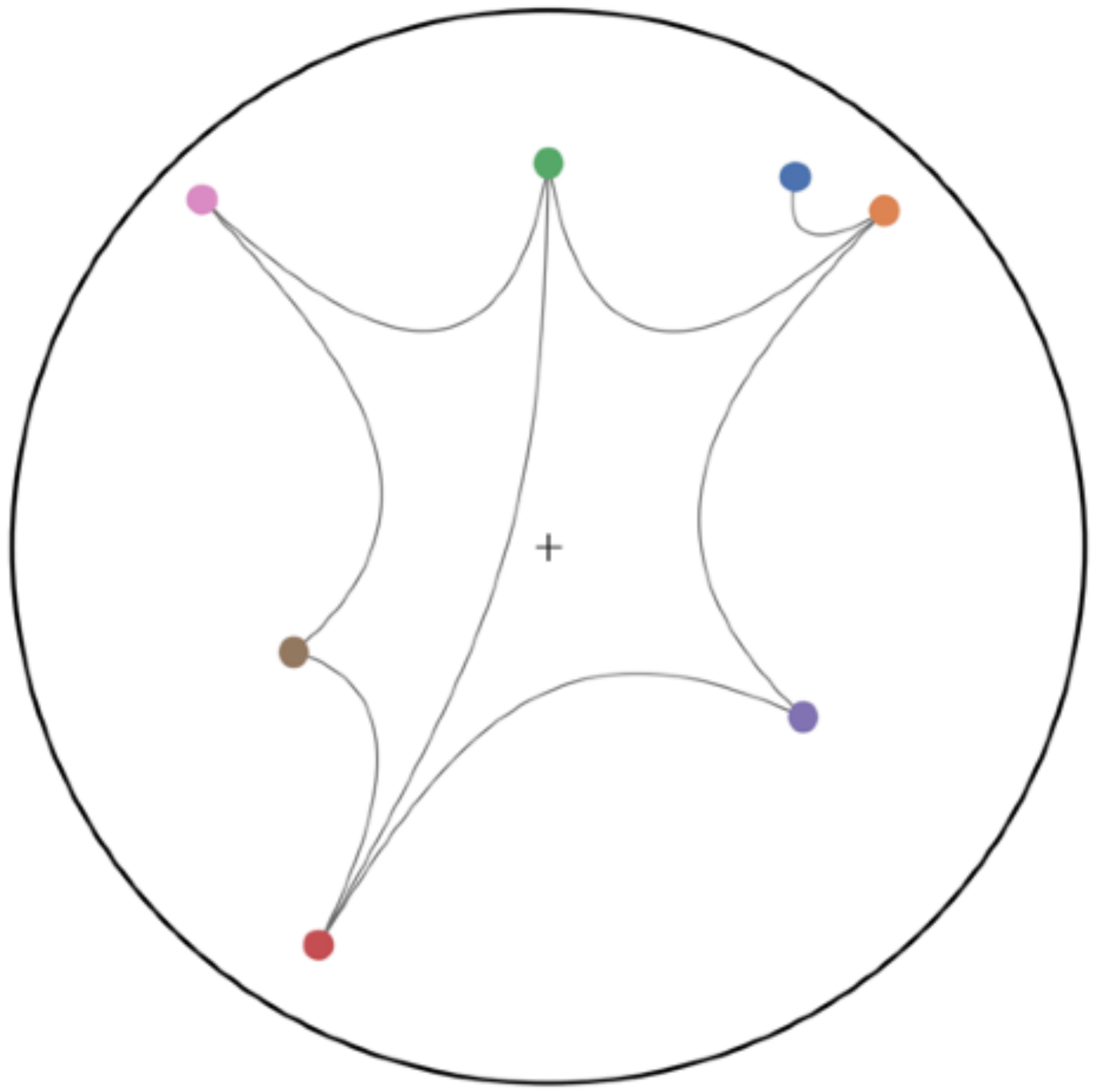
hyperbolic disk (r, θ)

Maximally random geometric graph ensembles

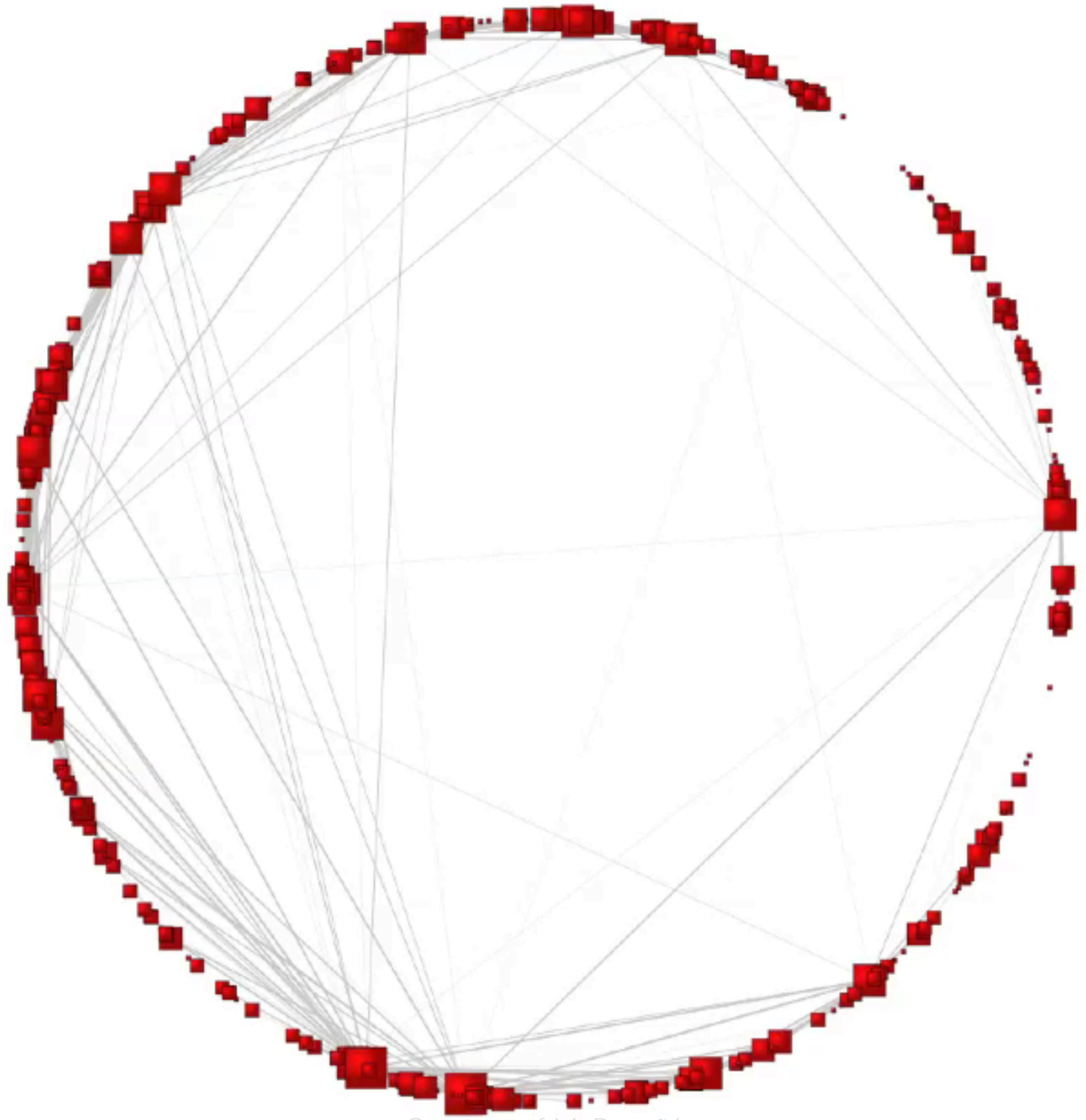
When the geometry is a D-dimensional sphere, $\mathbb{S}^D$ the model can be mapped to a purely geometric model in the hyperbolic disk $\mathbb{H}^{D,1}$.



hyperboloid in $\mathbb{R}^{2,1}$



hyperbolic disk (r, θ)



Courtesy of M. Boguñá

Phys. Rev. E 80, 035101 (2009)
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Phys. Rev. Research 2, 023040 (2020)

A powerful and versatile framework

- Amenable to many analytical calculations [1,2]
- Generalizable to weighted [5], bipartite [6,7,8], multiplex [9,10], directed [4] and growing [11] networks
- Geometrical interpretation of preferential attachment [11]
- Parsimonious explanation of self-similarity [3]
- Generalizable to networks with community structure [12,13,14]
- Mapping of real complex networks unto hyperbolic space [15,16]
  - Reproduction of addtionnal properties than the ones used to fit the parameters [4,15].
  - Identification of biochemical pathways in E. Coli [8]
  - Efficient Internet routing protocols [17]
  - Organization of the human connectome [18,20]
  - Self-similar architecture [19]
  - Evolution of hierarchy in international trade [21]
  - . . .
- . . .

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