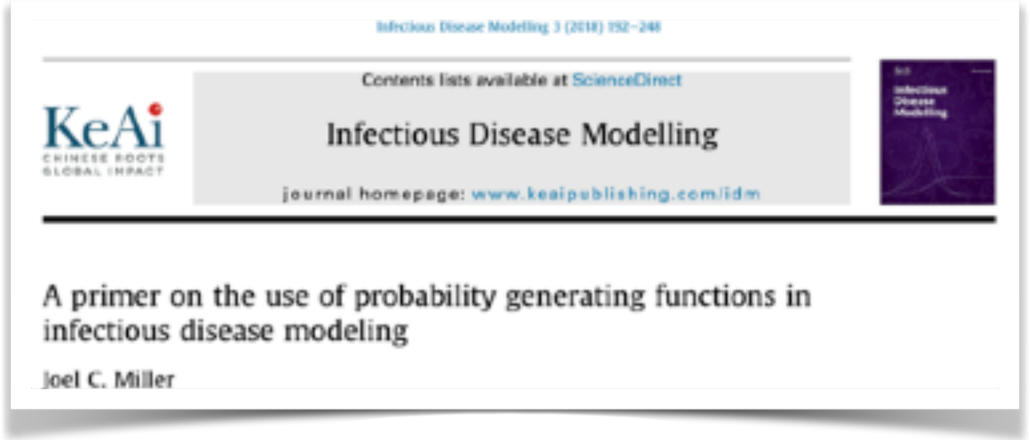
Contact network epidemiology


Infectious Disease Modelling 3 (2018) 192–248

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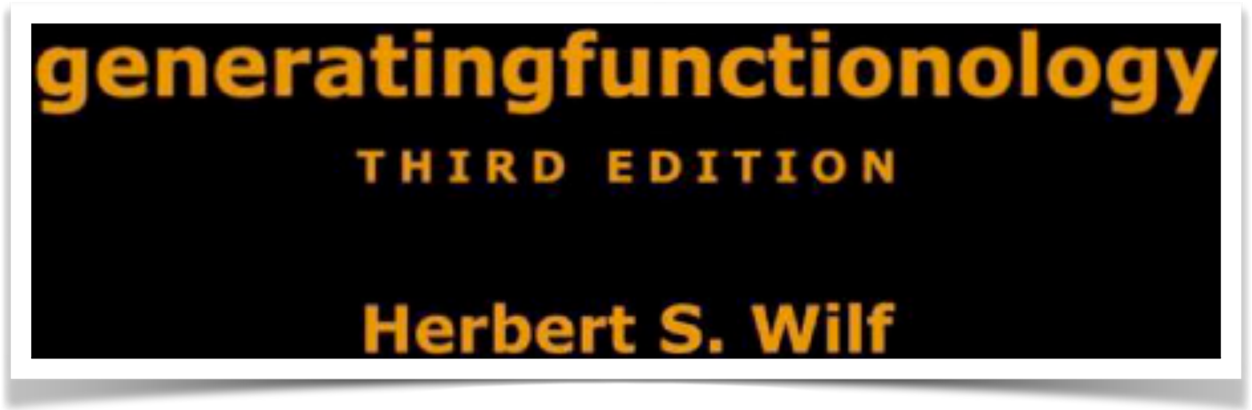
A primer on the use of probability generating functions in infectious disease modeling

Joel C. Miller



Probability generating functions (PGFs)

- a PGF is a formal power series whose coefficients are a probability mass function $\{a_n\}_{n\geq 0}$

$$A(x) = \sum_{n\geq 0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

- computing the moments

$$A(1) = \sum_{n\geq 0}^{\infty} a_n = 1 \, ; \qquad \langle n \rangle = \sum_{n\geq 0}^{\infty} n \, a_n = \left.\frac{dA(x)}{dx}\right|_{x=1} = A'(1) \, ; \qquad \langle n^p \rangle = \sum_{n\geq 0}^{\infty} n^p \, a_n = \left.\left(x\frac{d}{dx}\right)^p A(x)\right|_{x=1}$$

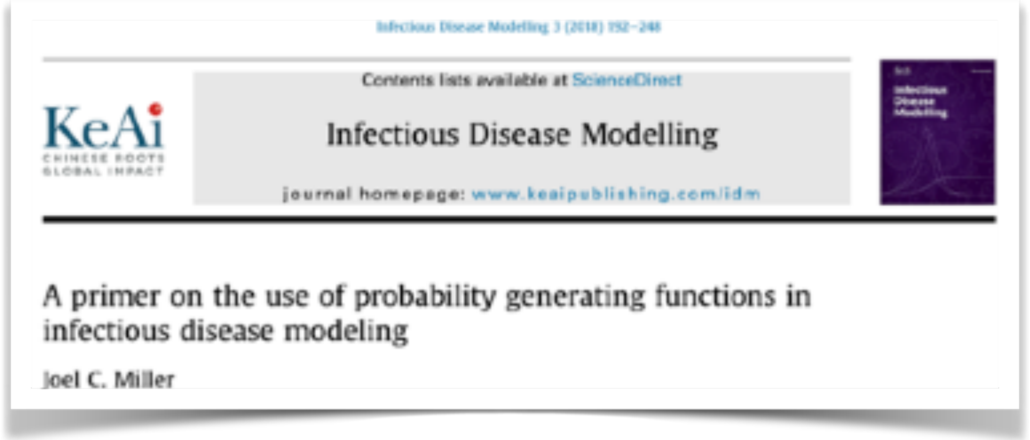
- extracting the coefficients

$$a_n = \frac{1}{n!} \left.\frac{d^n A(x)}{dx^n}\right|_{x=0} = \frac{1}{2\pi} \int_0^{2\pi} A(\mathrm{e}^{i\theta}) \mathrm{e}^{-in\theta} d\theta$$

- sum of a fix/random number of variables drawn independently

$$B_2^{\mathrm{fix}}(x) = \sum_{l\geq 0} b_l \, x^l = \sum_{l\geq 0} \sum_{n=0}^{l} a_n a_{l-n} x^l = \sum_{n\geq 0}^{\infty} a_n x^n \sum_{m\geq 0}^{\infty} a_m x^m = [A(x)]^2 \, ; \qquad B_p^{\mathrm{fix}}(x) = [A(x)]^p \, ; \qquad C^{\mathrm{rand}}(x) = \sum_{n\geq 0}^{\infty} a_n \big[A(x)\big]^n = A\big(A(x)\big)$$

Contact network epidemiology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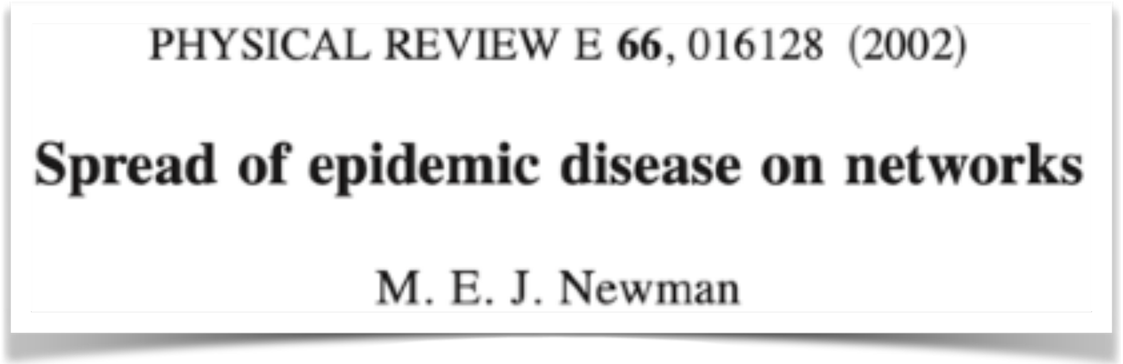
PHYSICAL REVIEW E **66**, 016128 (2002)

Spread of epidemic disease on networks

M. E. J. Newman

Probability generating functions (PGFs) formalism

- assuming a very, very large population (i.e. neglecting finite-size effects)
- patient zero causes k secondary infections with probability p_k (degree distribution of thfie network)

$$G_0(x) = \; + \; x + \; x^2 + \; x^3 + \ldots = \sum_{k\geq 0}^{\infty} p_k x^k \; ; \qquad \langle k \rangle = \sum_{k\geq 0}^{\infty} k\, p_k = G_0'(1) \; ; \qquad \langle k^2 \rangle = \sum_{k\geq 0}^{\infty} k^2\, p_k$$

- a newly infected individual causes k *new* infections with probability $(k+1)p_{k+1}/\langle k \rangle$ (*excess* degree distribution of the network)

$$G_1(x) = \; + \; x + \; x^2 + \; x^3 + \ldots = \sum_{k\geq 0}^{\infty} \frac{(k+1)p_{k+1}}{\langle k \rangle} x^k = \frac{G_0'(x)}{G_0'(1)}$$

- average number of secondary infections a newly infected individual causes

$$G_1'(1) = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} \equiv R_0$$

- *all* outbreaks will eventually die out when $R_0 < 1$
- *some* outbreaks will eventually die out when $R_0 > 1$