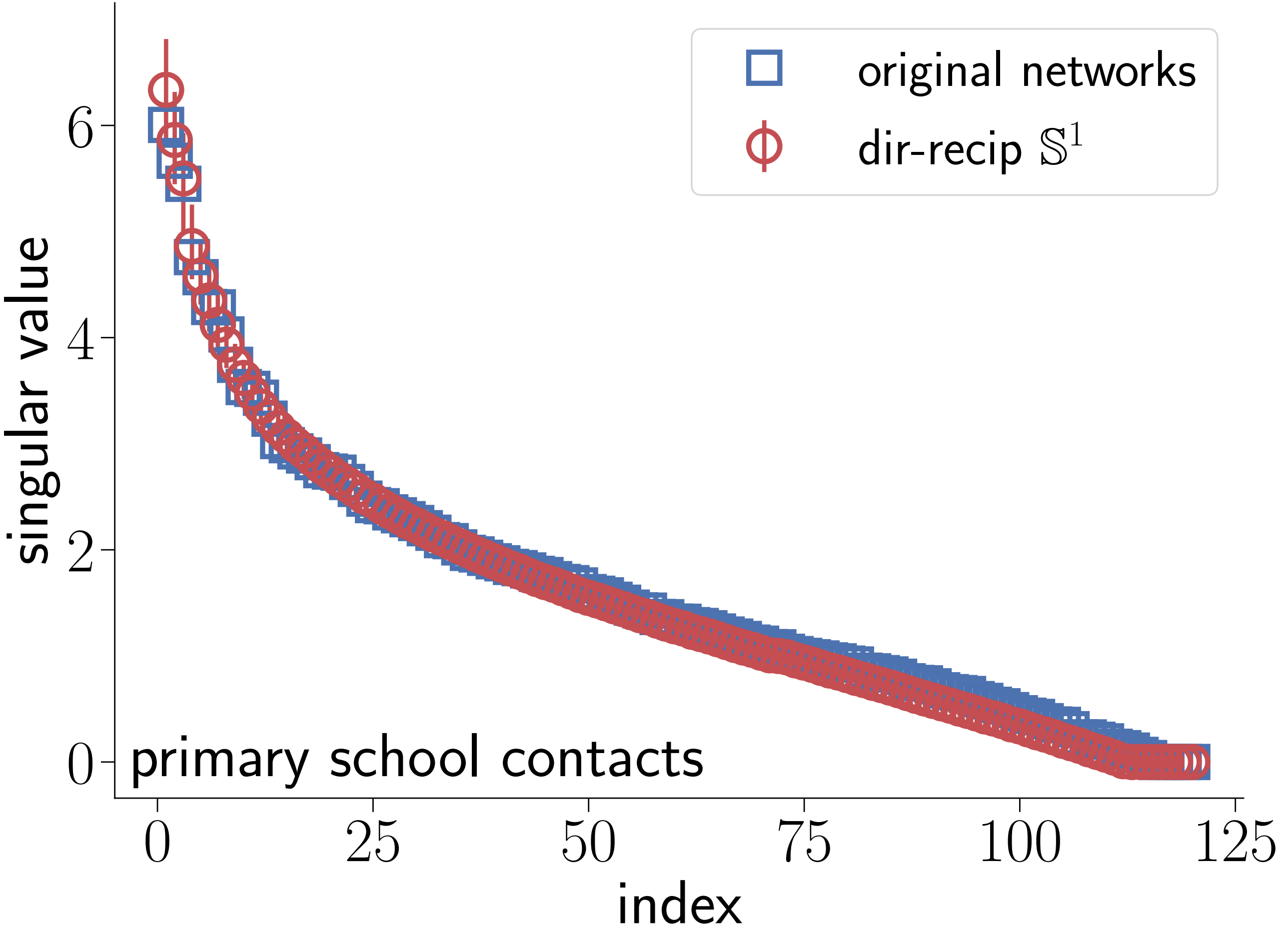
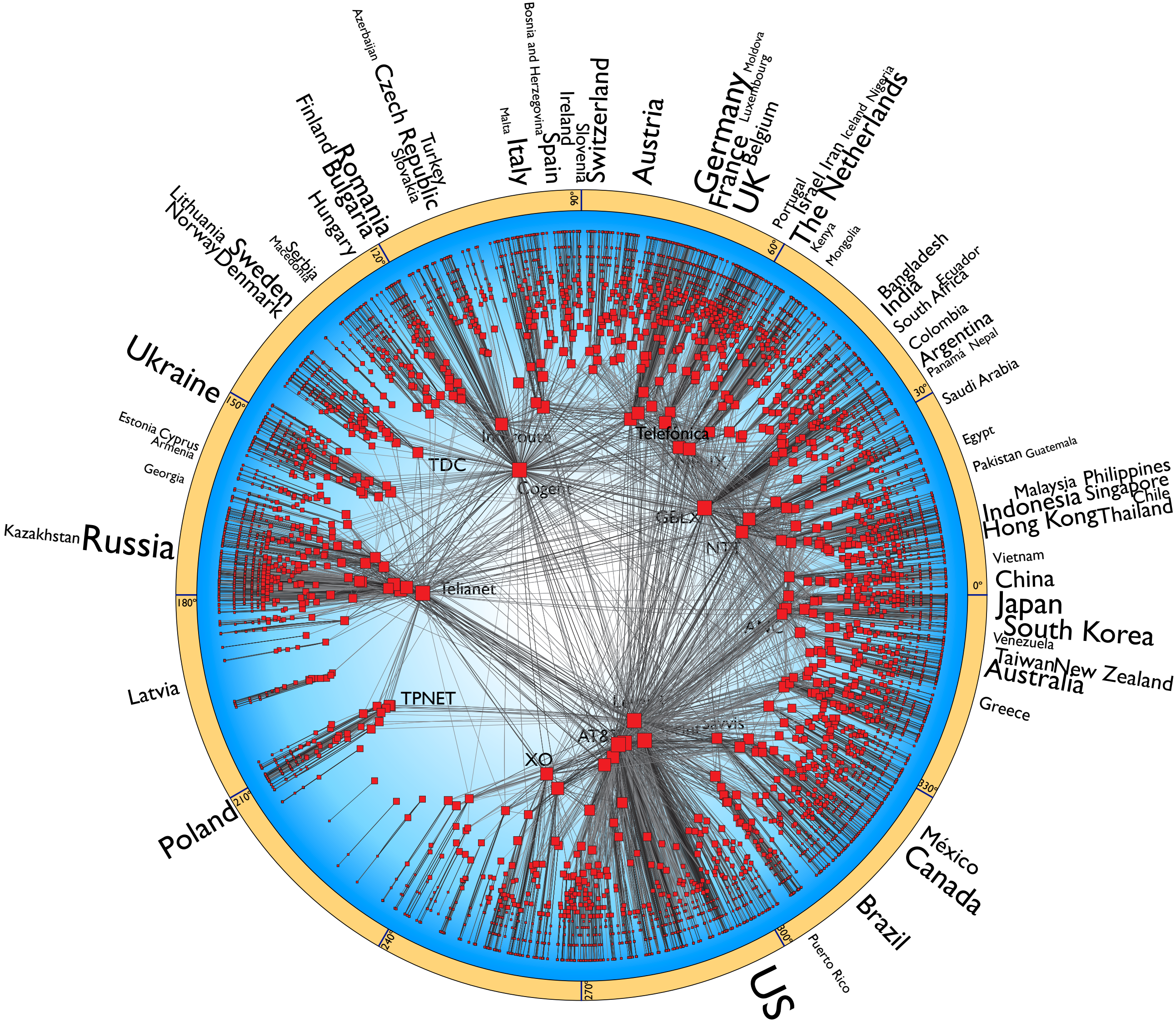
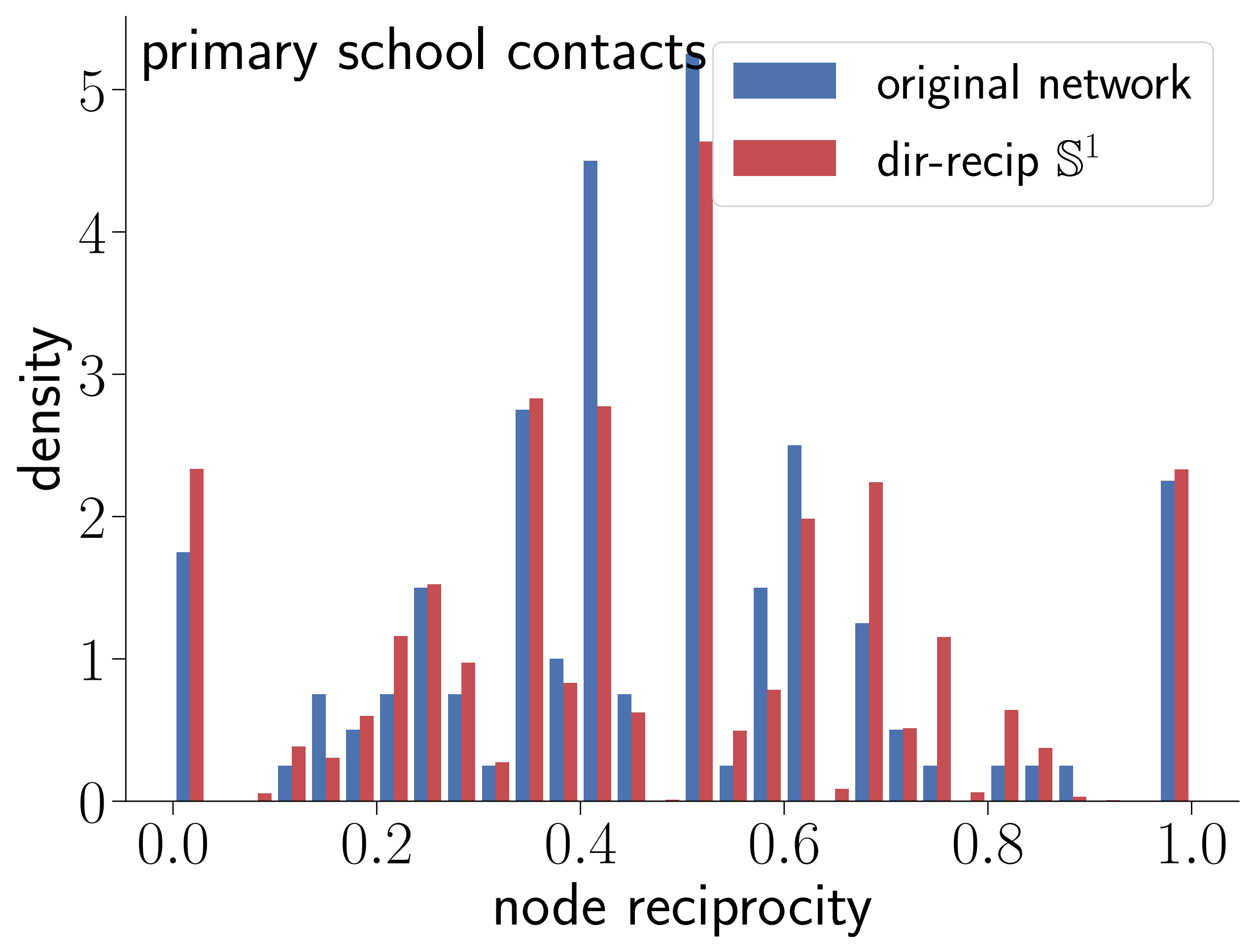


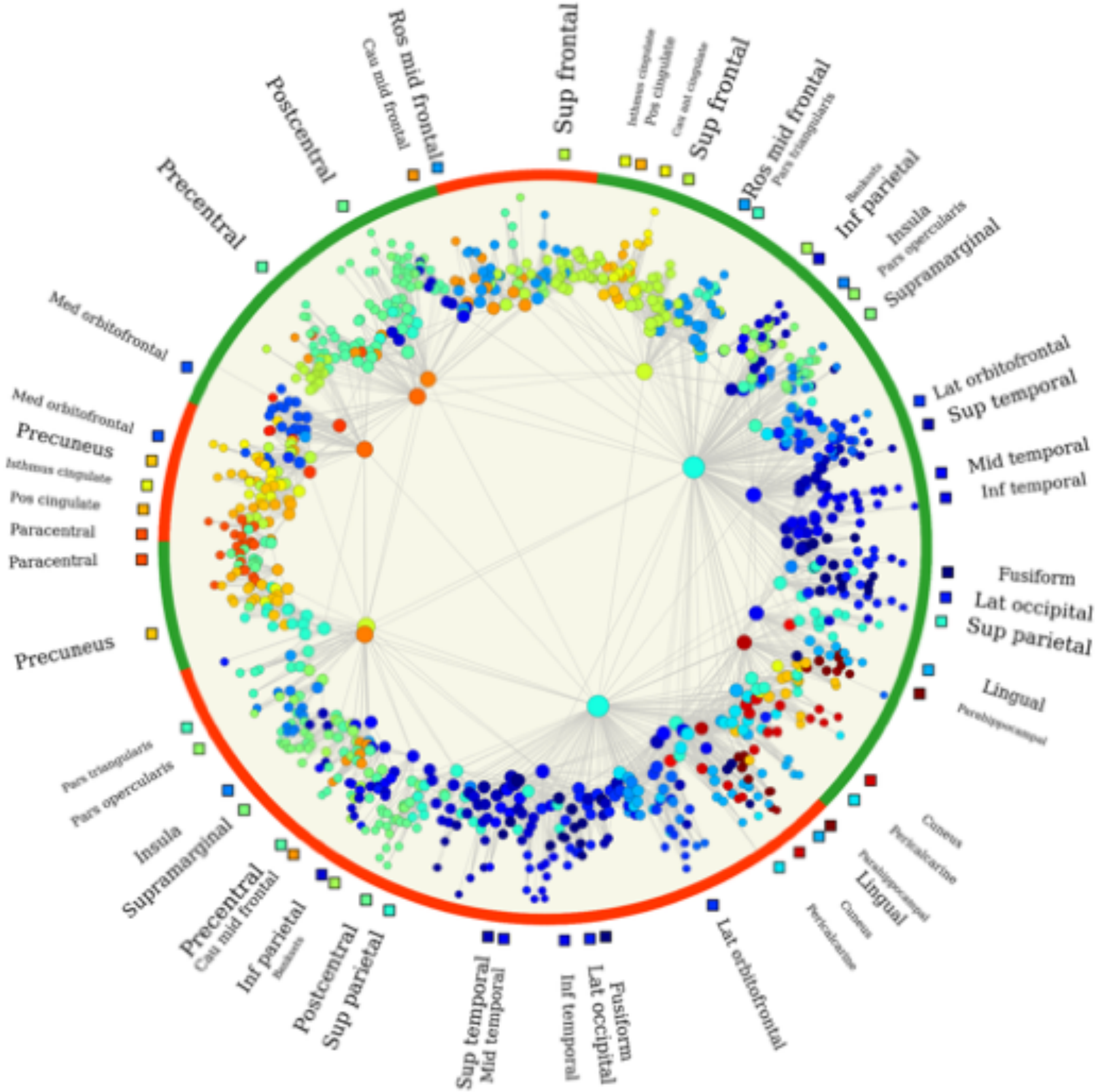
original networks
dir-recip $\mathbb{S}^1$
singular value
6
4
2
0
primary school contacts
0
25
50
75
100
125
index



0°
30°
60°
90°
120°
150°
180°
210°
240°
270°
300°
330°
Telefónica
TDC
Cogent
GBLX
NTT
Telianet
TPNET
XO
Savvis
Azerbaijan
Czech Republic
Turkey
Slovakia
Malta
Italy
Bosnia and Herzegovina
Spain
Ireland
Slovenia
Switzerland
Austria
Germany
Moldova
Luxembourg
France
UK
Belgium
Portugal
Israel
Iran
Iceland
Nigeria
The Netherlands
Kenya
Mongolia
Bangladesh
India
Ecuador
South Africa
Colombia
Argentina
Panamá
Nepal
Saudi Arabia
Egypt
Pakistan
Guatemala
Malaysia
Philippines
Indonesia
Singapore
Chile
Hong Kong
Thailand
Vietnam
China
Japan
South Korea
Venezuela
Taiwan
New Zealand
Australia
Greece
México
Canada
Brazil
Puerto Rico
US
Poland
Latvia
Russia
Kazakhstan
Georgia
Estonia
Cyprus
Armenia
Ukraine
Lithuania
Norway
Denmark
Sweden
Serbia
Macedonia
Hungary
Finland
Romania
Bulgaria

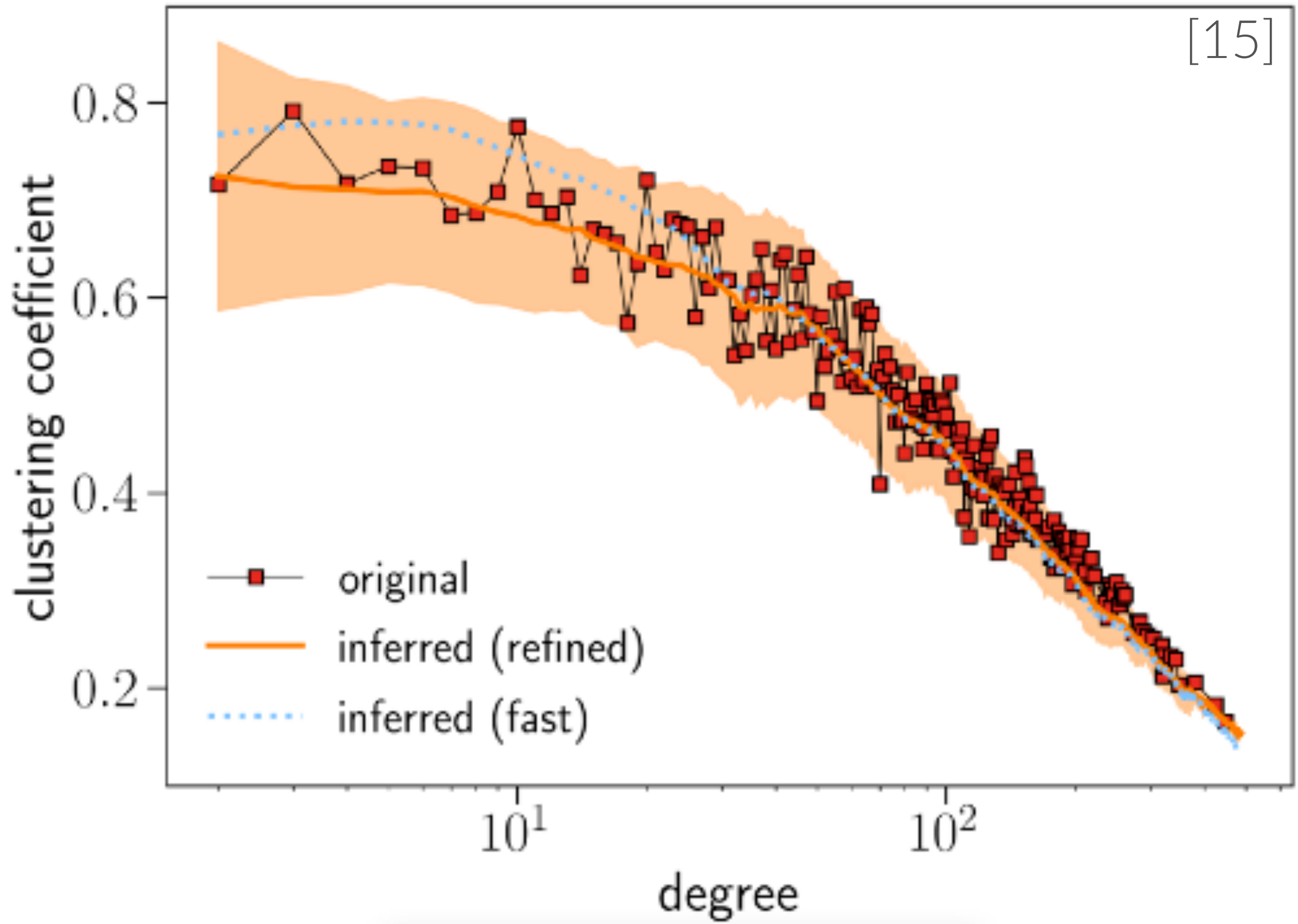


primary school contacts
original network
dir-recip $\mathbb{S}^1$
density
node reciprocity
0
1
2
3
4
5
0.0
0.2
0.4
0.6
0.8
1.0



Ros mid frontal
Cau mid frontal
Postcentral
Precentral
Med orbitofrontal
Sup frontal
Isthmus cingulate
Pos cingulate
Cau ant cingulate
Sup frontal
Ros mid frontal
Pars triangularis
Bankssts
Inf parietal
Insula
Pars opercularis
Supramarginal
Lat orbitofrontal
Sup temporal
Mid temporal
Inf temporal
Fusiform
Lat occipital
Sup parietal
Lingual
Parahippocampal
Cuneus
Pericalcarine
Parahippocampal
Lingual
Cuneus
Pericalcarine
Lat orbitofrontal
Fusiform
Lat occipital
Inf temporal
Sup temporal
Mid temporal
Postcentral
Sup parietal
Inf parietal
Bankssts
Precentral
Cau mid frontal
Supramarginal
Insula
Pars opercularis
Pars triangularis
Precuneus
Paracentral
Paracentral
Pos cingulate
Isthmus cingulate
Precuneus
Med orbitofrontal

18



[15]
0.8
0.6
0.4
0.2
clustering coefficient
original
inferred (refined)
inferred (fast)
10^1
10^2
degree

[1] Phys. Rev. E 80, 035101 (2009)

[2] Phys. Rev. E 82, 036106 (2010)

[3] Phys. Rev. Lett. 100, 078701 (2008)

[5] Nat. Commun. 8, 14103 (2017)

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[13] J. Stat. Phys. 173, 775 (2018)

[14] New J. Phys. 20, 052002 (2018)

[15] New J. Phys. 21, 123033 (2019)

[16] Nat. Commun. 8, 1615 (2017)

[17] Nat. Commun. 1, 62 (2010)

[18] PNAS 117, 20244 (2020)

A powerful and versatile framework

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- Generalizable to weighted [5], bipartite [6,7,8], multiplex [9,10], directed [4] and growing [11] networks
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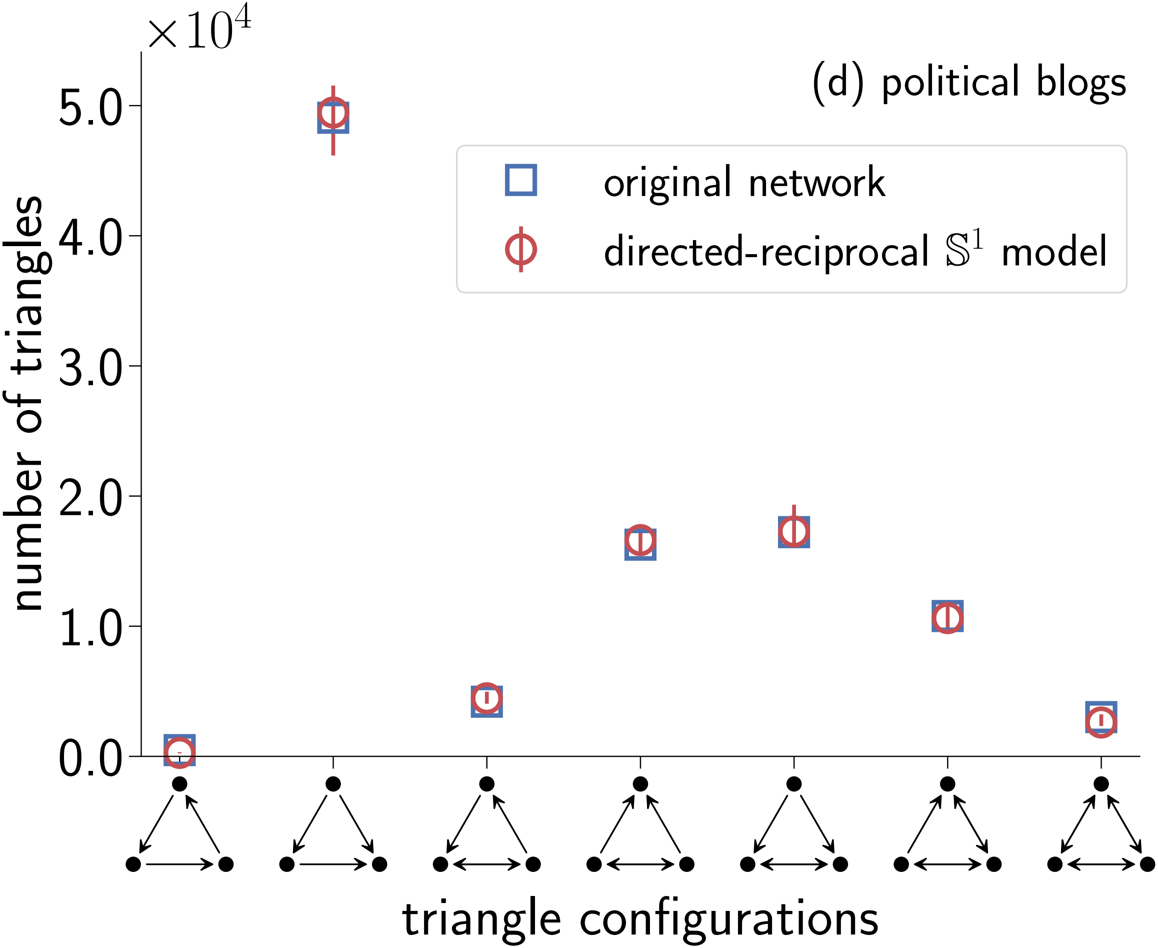
  - ...
- ...

[4] Nat. Phys. 20, 150 (2024)

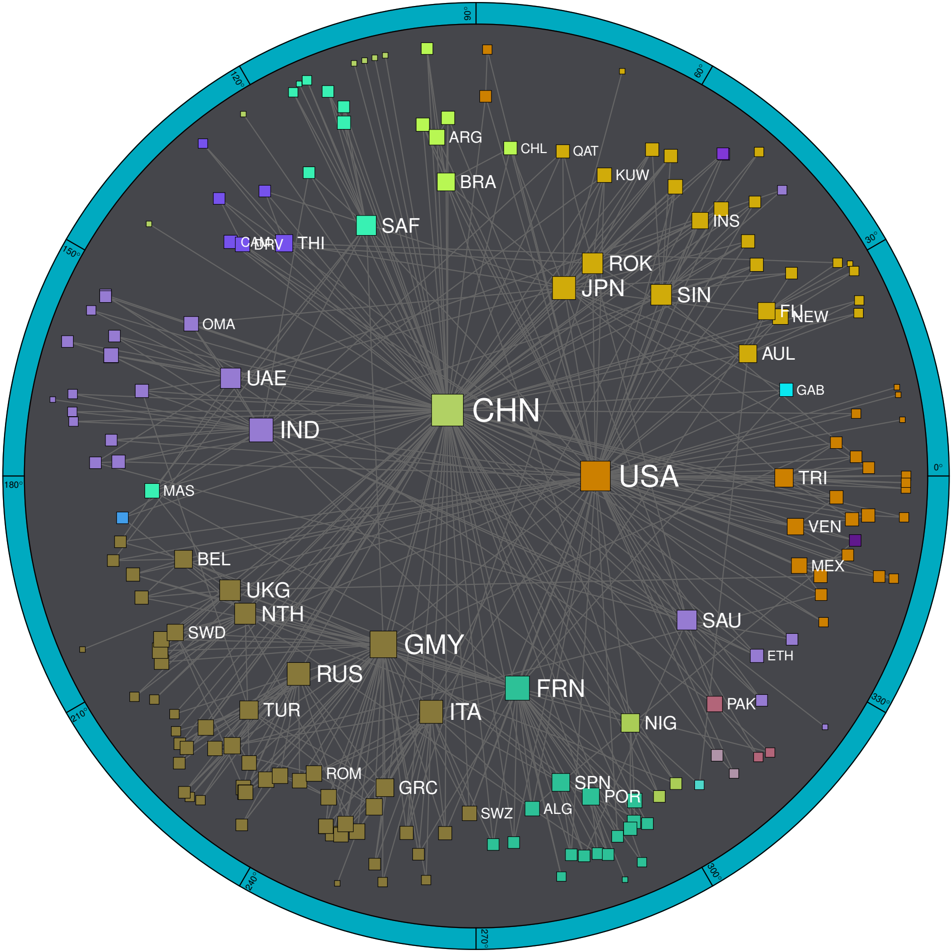
[19] Nat. Phys. 14, 583 (2018)

[20] PLOS Comput. Biol. 16, e1007584 (2020)

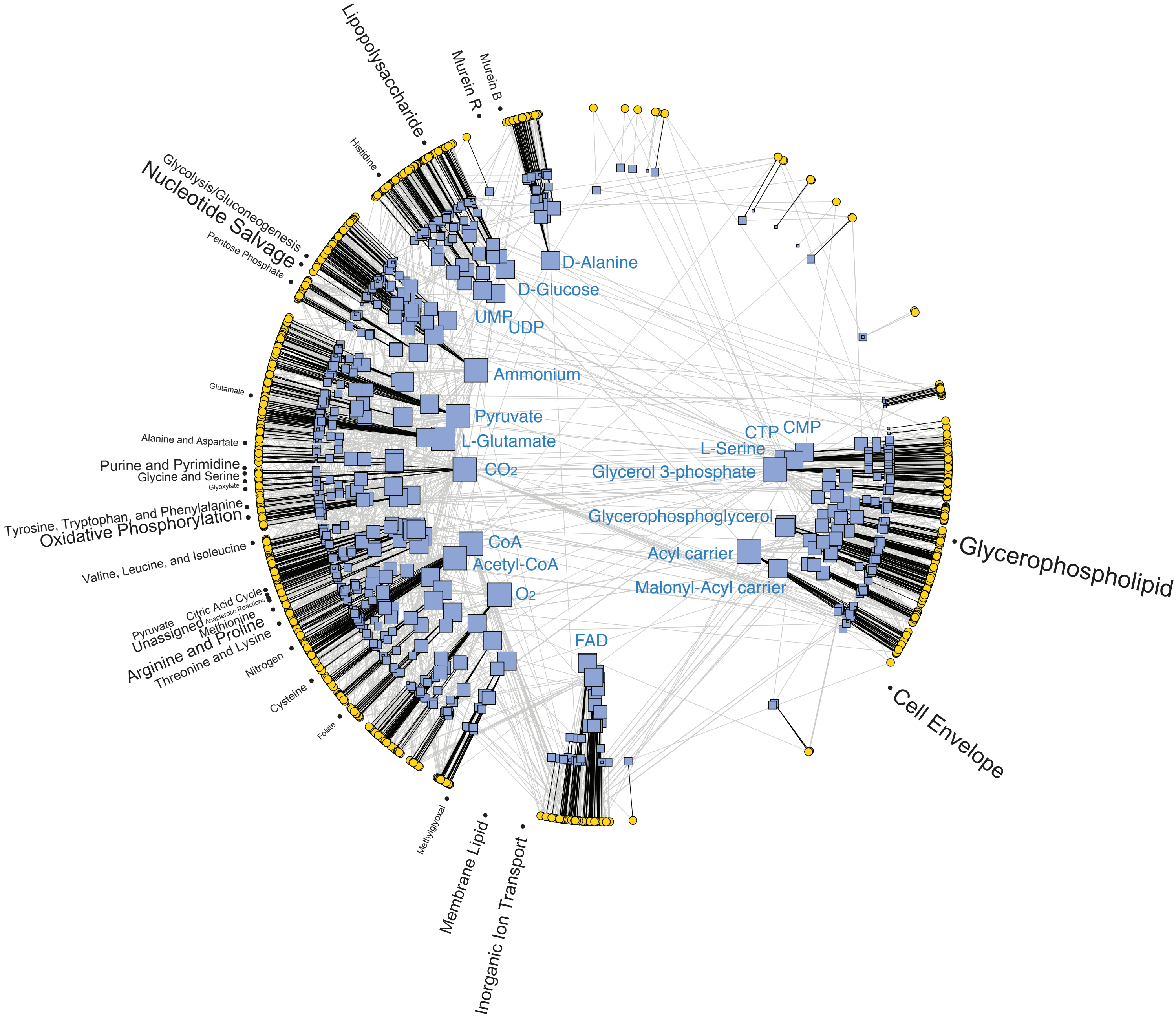
[21] Sci. Rep. 6, 33441 (2016)



×10^4
(d) political blogs
5.0
4.0
3.0
2.0
1.0
0.0
number of triangles
original network
directed-reciprocal $\mathbb{S}^1$ model
triangle configurations



90°
60°
30°
0°
330°
300°
270°
240°
210°
180°
150°
120°
CHN
USA
GMY
FRN
ITA
RUS
TUR
UKG
NTH
BEL
SWD
ROM
GRC
SWZ
ALG
SPN
POR
NIG
PAK
ETH
SAU
MEX
VEN
TRI
GAB
AUL
FIJ
NEW
SIN
JPN
ROK
INS
KUW
QAT
CHL
BRA
ARG
SAF
THI
CAM
DRV
OMA
UAE
IND
MAS



Lipopolysaccharide
Murein R.
Murein B.
Histidine
Glycolysis/Gluconeogenesis
Nucleotide Salvage
Pentose Phosphate
Glutamate
Alanine and Aspartate
Purine and Pyrimidine
Glycine and Serine
Glyoxylate
Tyrosine, Tryptophan, and Phenylalanine
Oxidative Phosphorylation
Valine, Leucine, and Isoleucine
Citric Acid Cycle
Anaplerotic Reactions
Pyruvate
Unassigned
Methionine
Arginine and Proline
Threonine and Lysine
Nitrogen
Cysteine
Folate
Methylglyoxal
Membrane Lipid
Inorganic Ion Transport
Cell Envelope
Glycerophospholipid
D-Alanine
D-Glucose
UMP
UDP
Ammonium
Pyruvate
L-Glutamate
CO_2
CoA
Acetyl-CoA
O_2
FAD
CTP
CMP
L-Serine
Glycerol 3-phosphate
Glycerophosphoglycerol
Acyl carrier
Malonyl-Acyl carrier

Review Article | Published: 29 January 2021

Network geometry

Marián Boguñá, Ivan Bonamassa, Manlio De Domenico, Shlomo Havlin, Dmitri Krioukov & M. Ángeles Serrano

Nature Reviews Physics **3**, 114–135 (2021)

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Challenges

Heterogeneous random geometric graph models are prime candidates to model real networked complex systems.

But they rely heavily on our capacity to find high-quality embeddings of the original datasets.

- ▷ Difficult optimization problem
  - rugged landscape
  - numerous symmetries (rotation, reflection, graph automorphisms)
  - gradient not always well defined
- ▷ Out-of-the-box solutions do not work well
  - Hamiltonian Monte Carlo
  - gradient descent
- ▷ Current state-of-the-art embedding methods
  - rely on heuristics
  - do not provide uncertainties (loglikelihood maximization)