



Example 4: fixing the expected number of edges and the expected total energy

$$\bar{F}_l = \sum_{j=1}^N a_{lj} = k_l \quad (l = 1, \dots, N)$$

$$\bar{F}_{N+1} = \sum_{i=1}^N \sum_{j=i+1}^N \varepsilon_{ij} a_{ij} = \sum_{i=1}^N \sum_{j=i+1}^N f(x_{ij}) a_{ij} = E$$

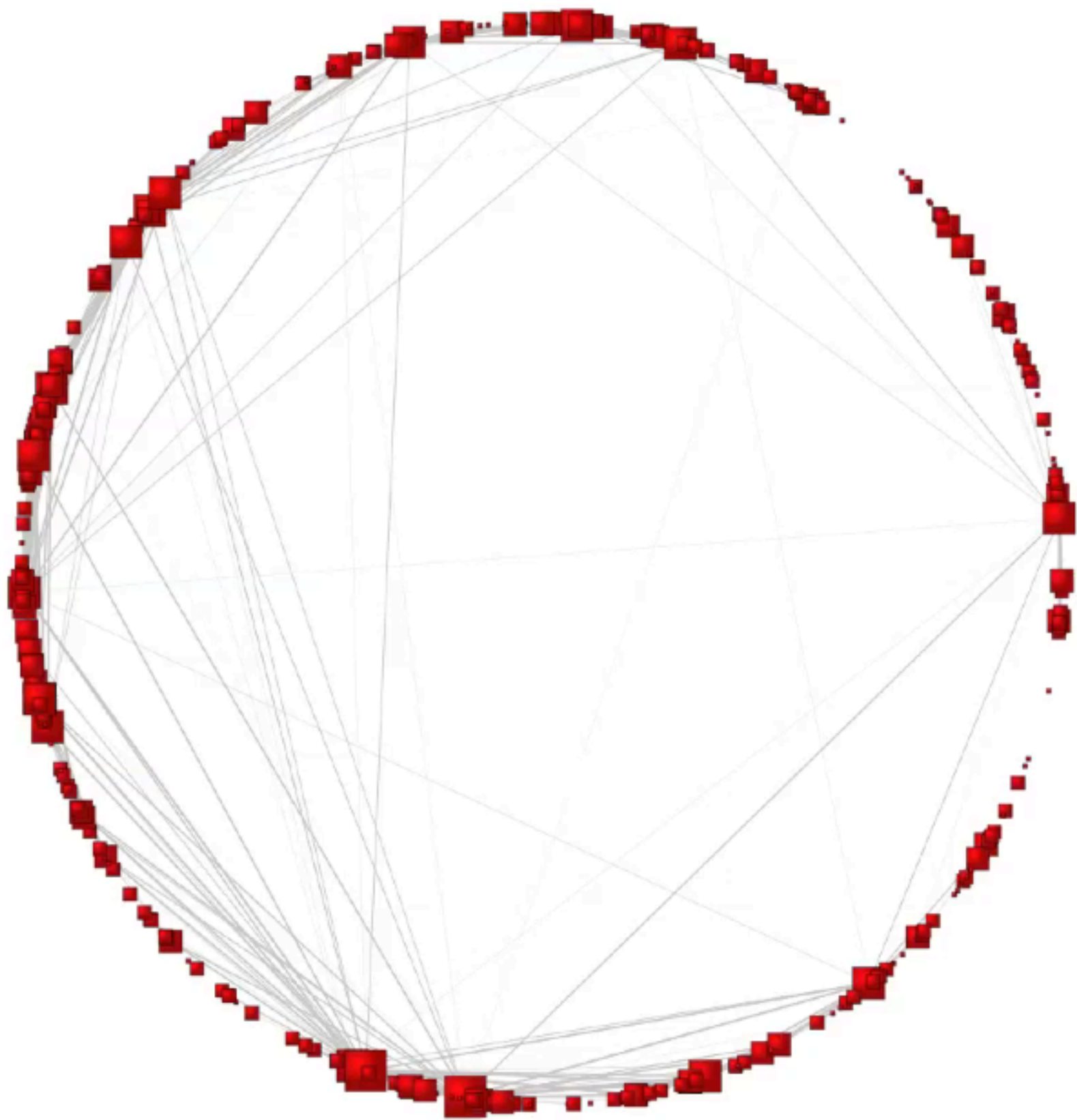
yields the heterogeneous random geometric graph ensemble

$$P(\mathbb{A}) = \prod_{i=1}^N \prod_{j=i+1}^N p_{ij}^{a_{ij}} (1 - p_{ij})^{1-a_{ij}} \quad \text{with} \quad p_{ij} = \frac{1}{e^{\beta \varepsilon_{ij} - \alpha_i - \alpha_j} + 1}.$$

The graphs will be sparse, highly clustered, small-world and devoid of non-structural degree-degree correlation iif $f(x_{ij}) = \ln x_{ij}$ and $\beta \in [D, D + 2]^a$. Redefining $\alpha_l = -(\beta/D) \ln(\sqrt{\mu} \kappa_l)$ yields

$$p_{ij} = \frac{1}{e^{\beta(\varepsilon_{ij} - \mu)} + 1} \quad \text{with} \quad \varepsilon_{ij} = \ln \left(\frac{x_{ij}}{(\kappa_i \kappa_j)^{\frac{1}{D}}} \right).$$

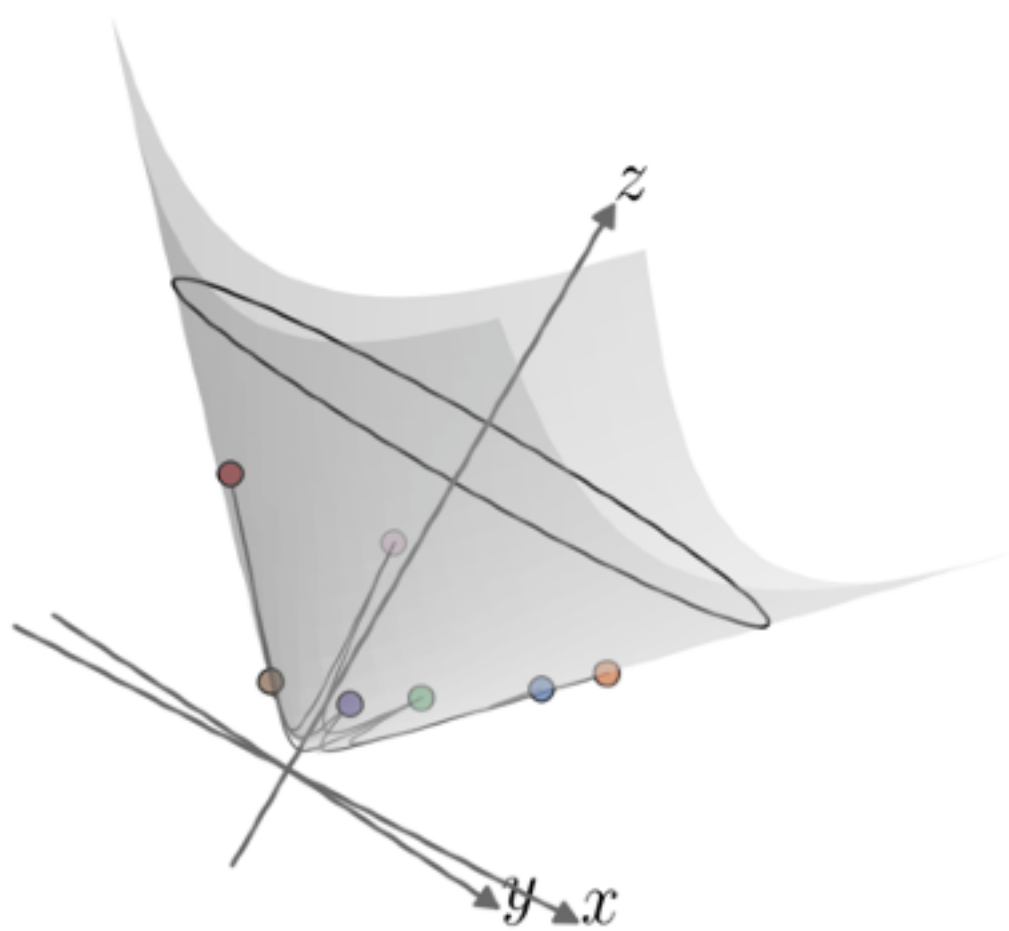
^a No upper bound if expected degree sequence is scale-free.



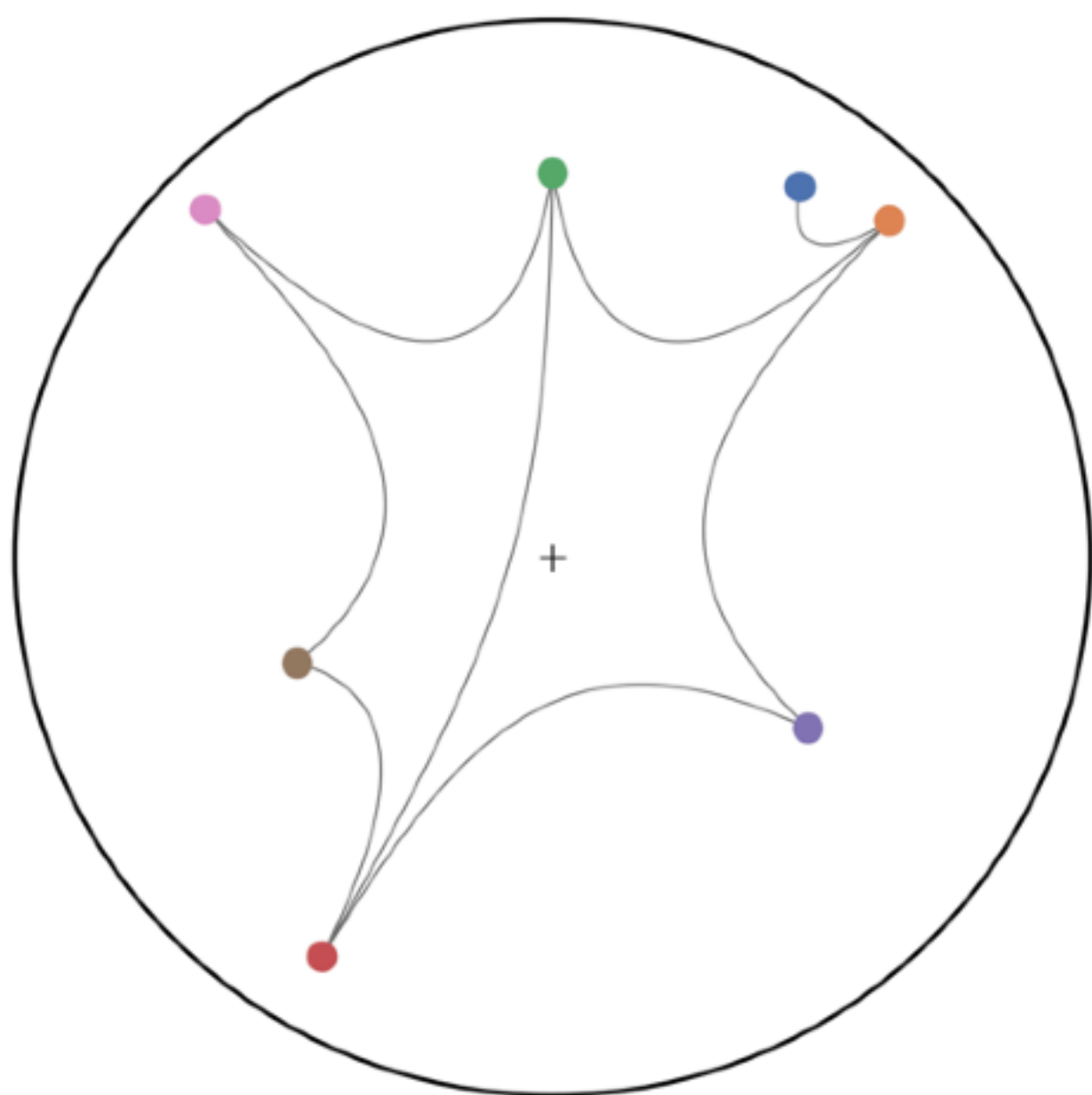
Phys. Rev. Research 2, 023040 (2020)

Phys. Rev. E 80, 035101 (2009)

Phys. Rev. E 82, 036106 (2010)



hyperboloid in $\mathbb{R}^{2,1}$

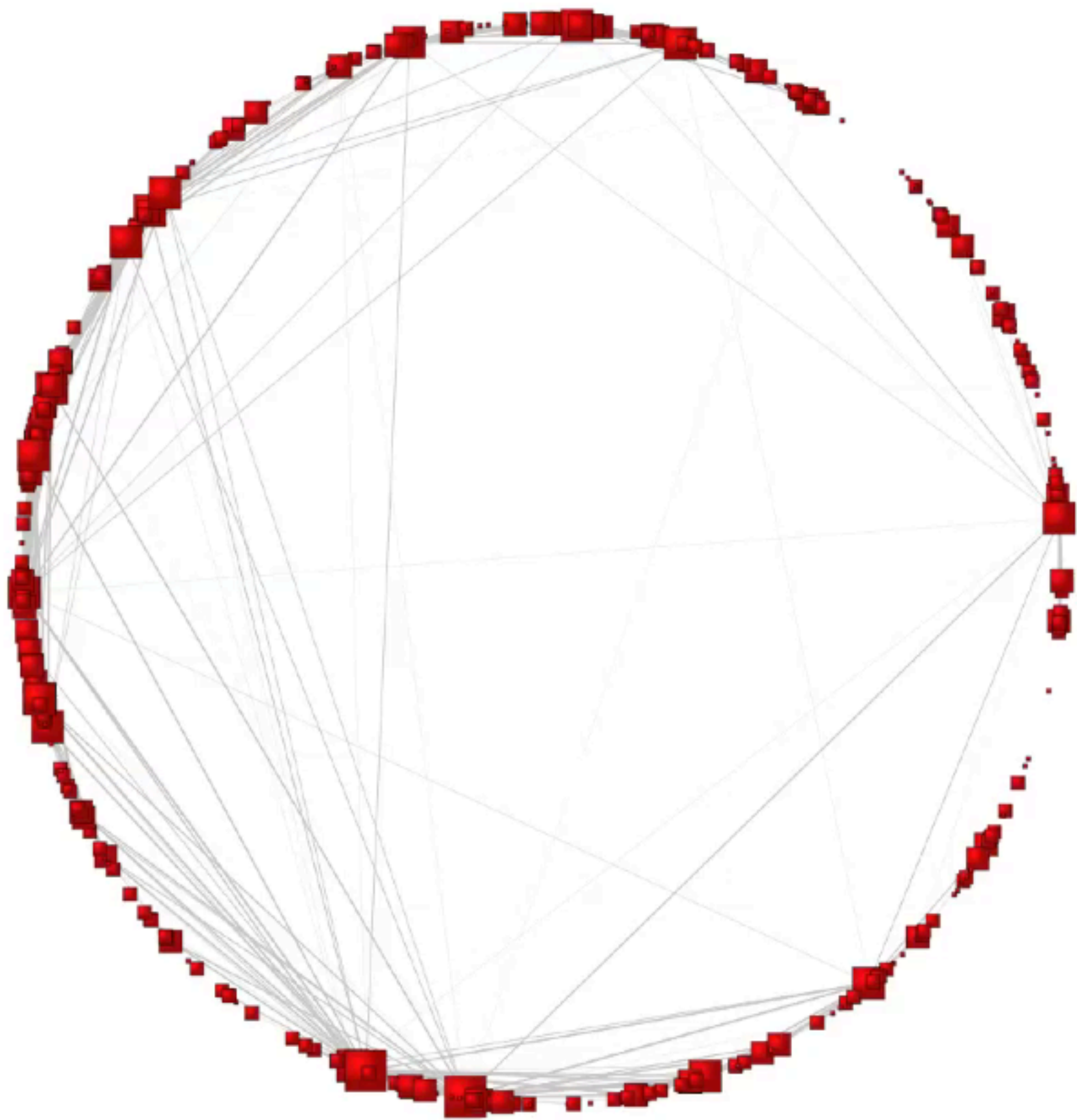


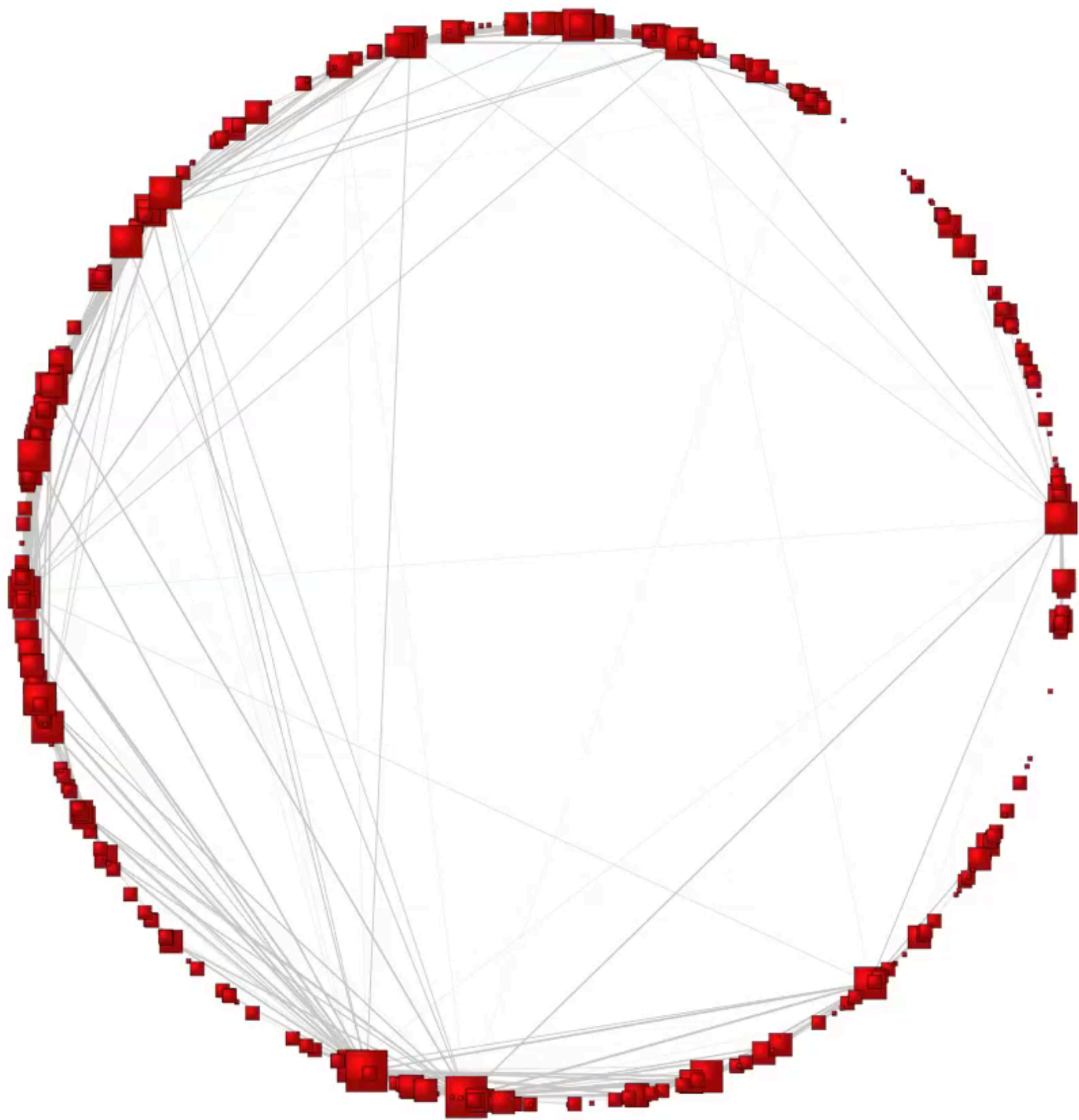
hyperbolic disk (r, θ)

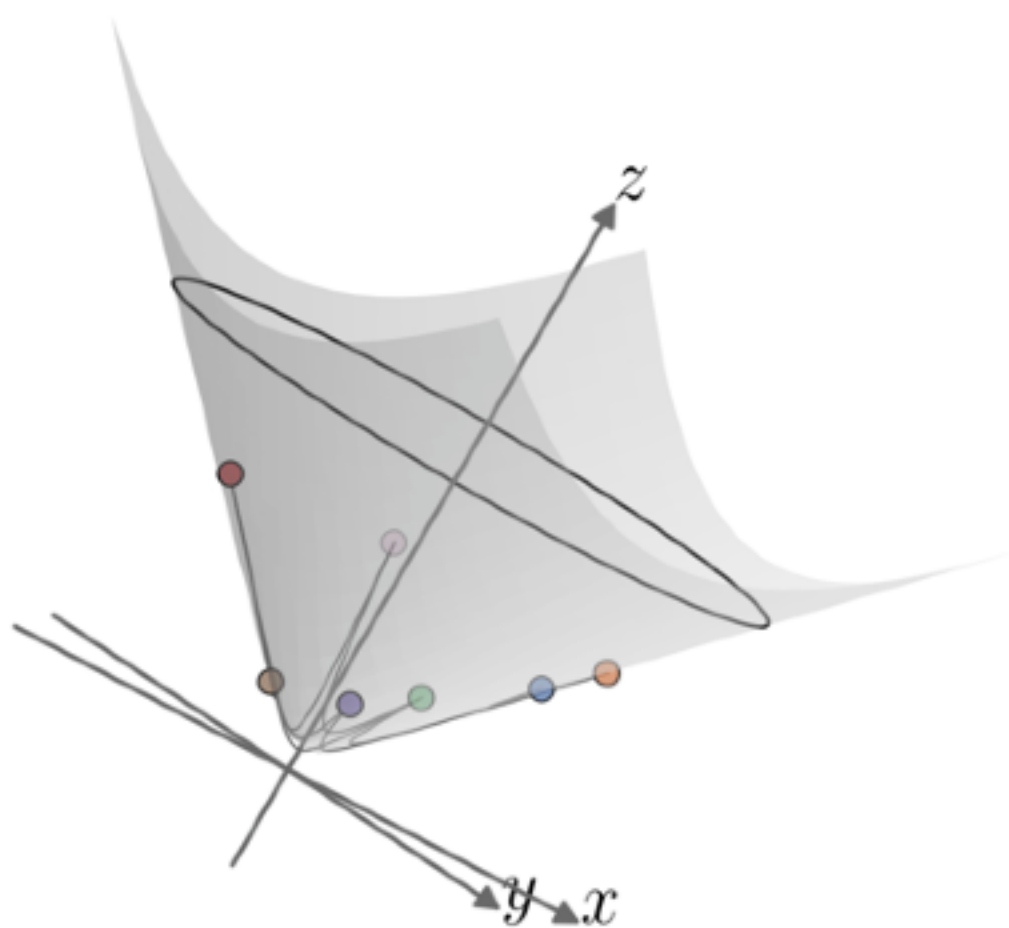
Maximally random geometric graph ensembles

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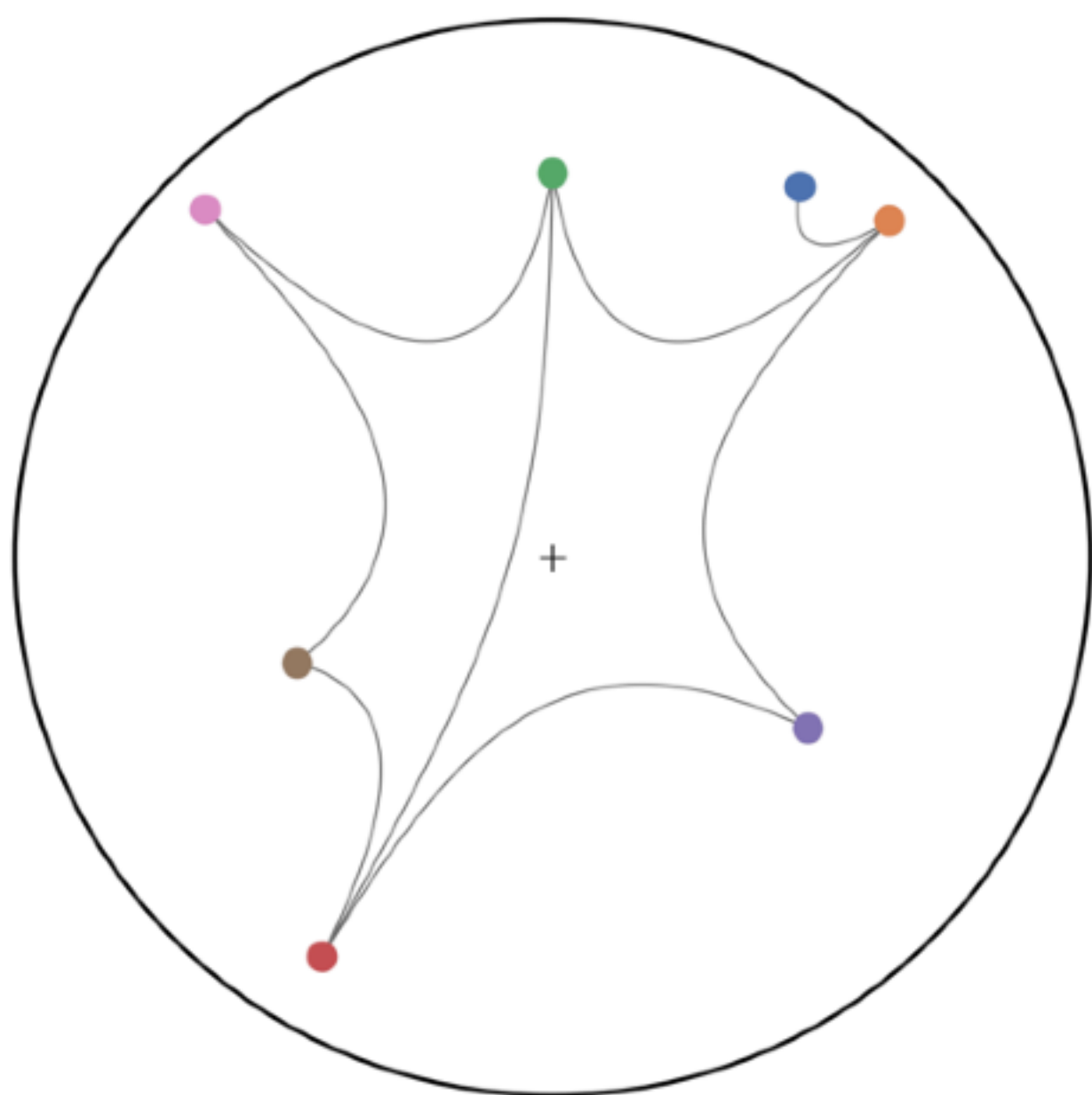
When the geometry is a D -dimensional sphere, S^D the model can be mapped to a purely geometric model in hyperbolic space \mathbb{H}^{D+1} .



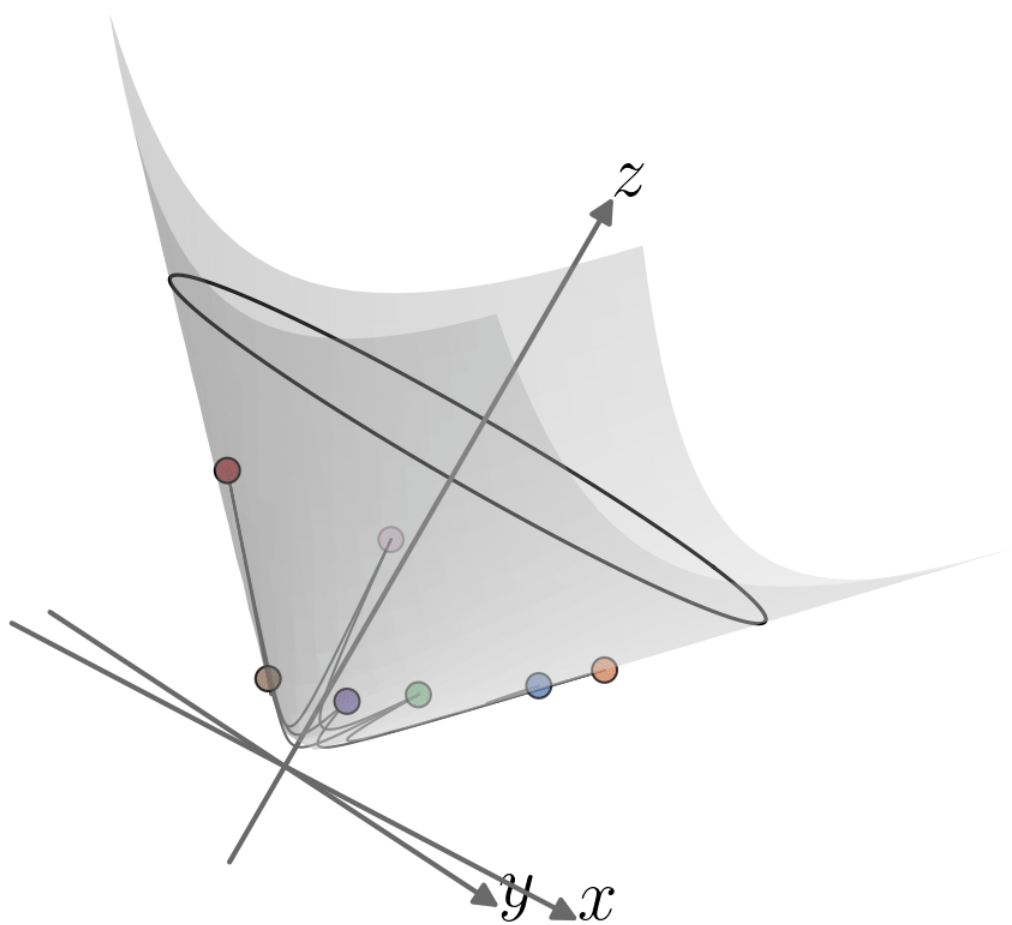




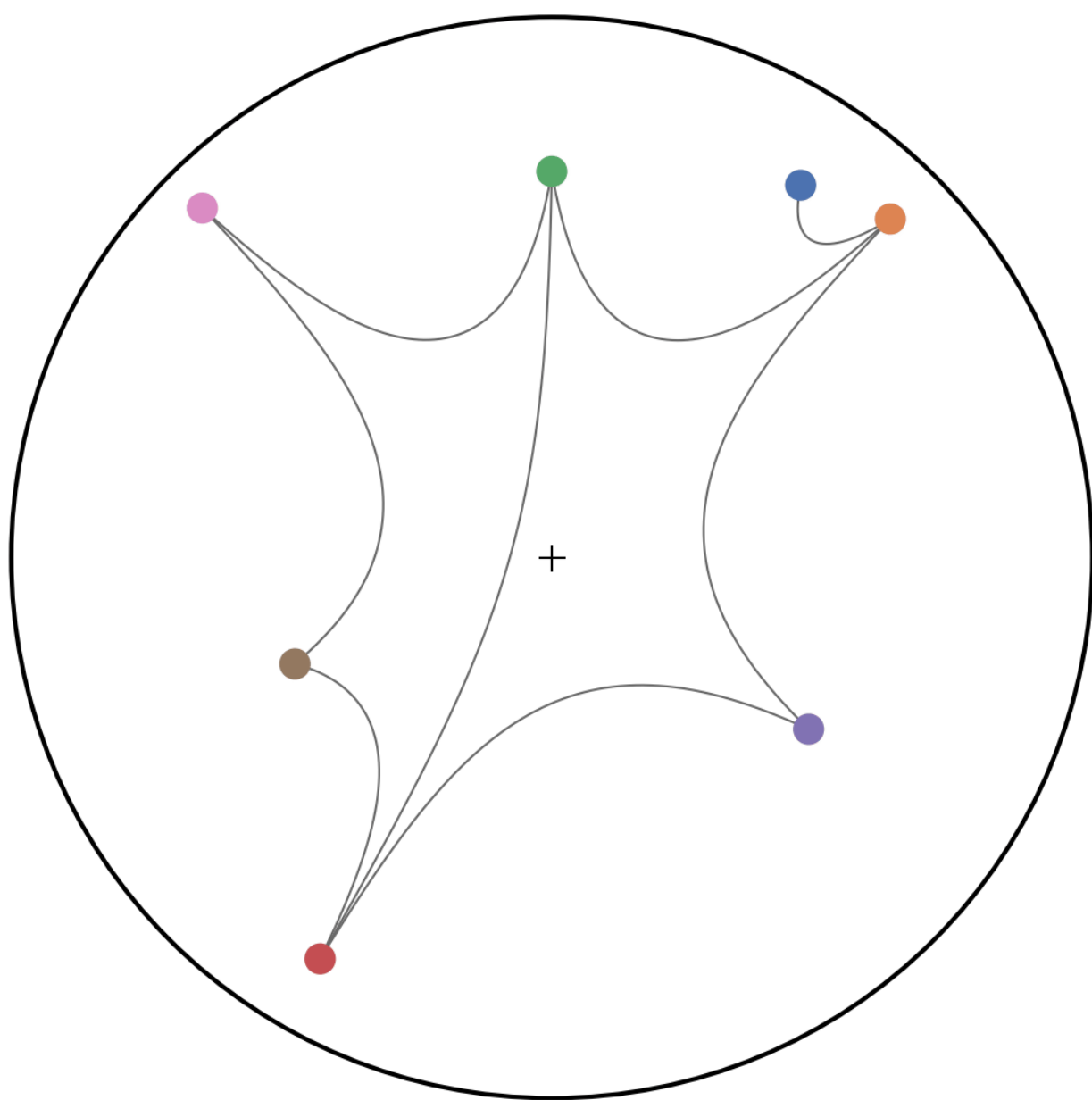
hyperboloid in $\mathbb{R}^{2,1}$



hyperbolic disk (r, θ)

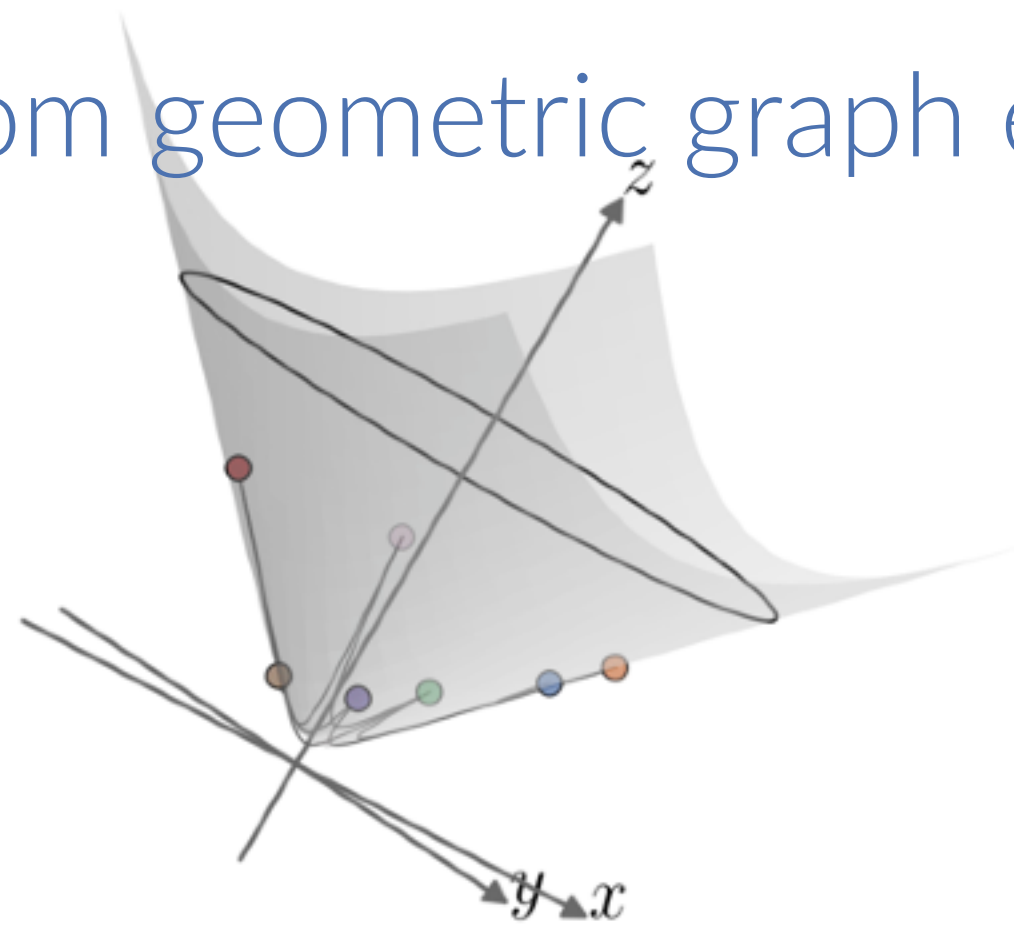


hyperboloid in $\mathbb{R}^{2,1}$

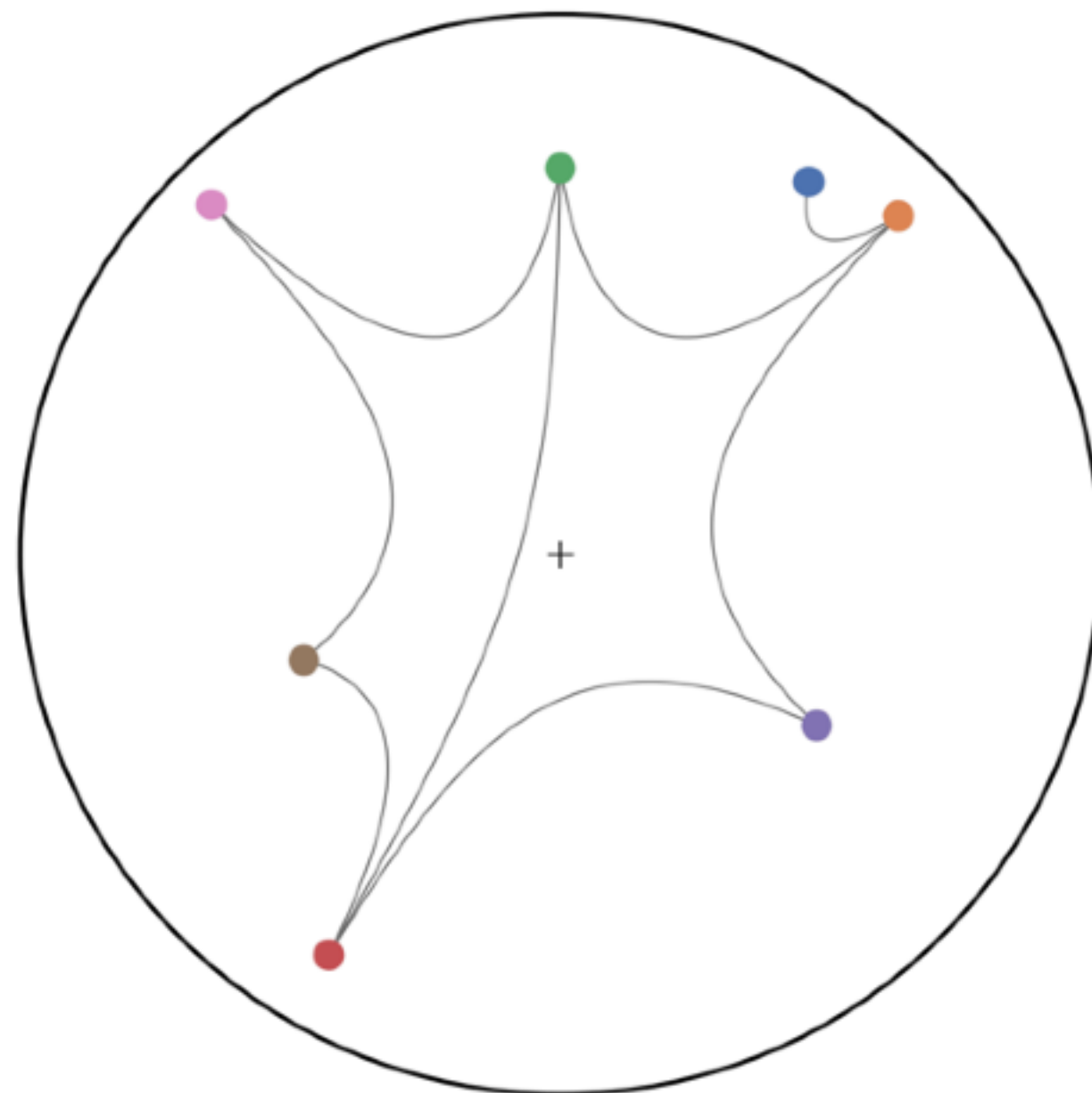


hyperbolic disk (r, θ)

Maximally random geometric graph ensembles

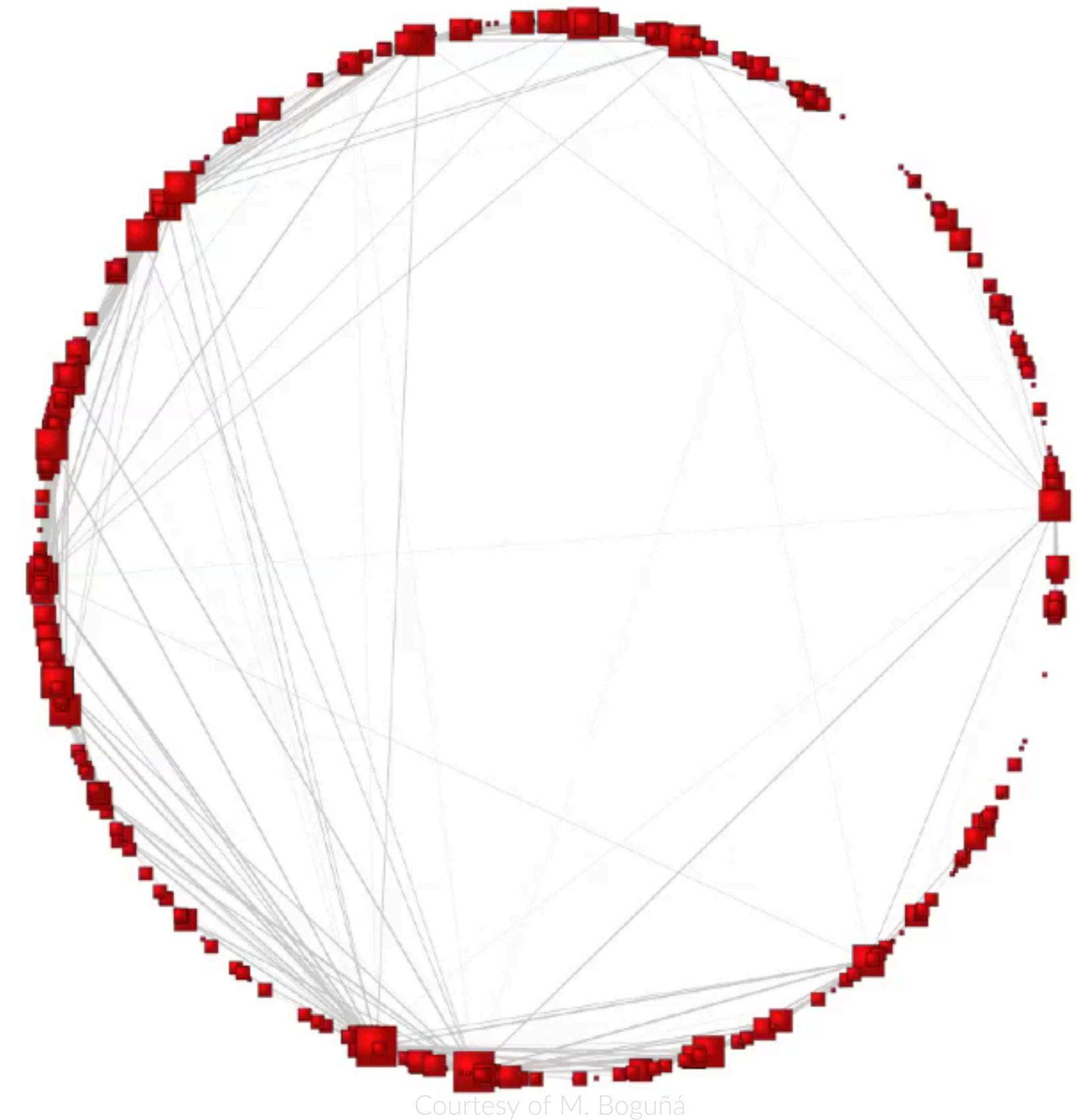


hyperboloid in $\mathbb{R}^{2,1}$



hyperbolic disk (r, θ)

When the geometry is a D -dimensional sphere, \mathbb{S}^D the model can be mapped to a **purely geometric model** in hyperbolic space \mathbb{H}^{D+1} .



Courtesy of M. Boguñá

A powerful and versatile framework

- ▷ Amenable to many **analytical calculations** [1,2]
- ▷ Generalizable to **weighted** [5], **bipartite** [6,7,8], **multiplex** [9,10], **directed** [4] and **growing** [11] networks
- ▷ Geometrical interpretation of preferential attachment [11]
- ▷ Parsimonious explanation of **self-similarity** [3]
- ▷ Generalizable to networks with **community structure** [12,13,14]
- ▷ **Mapping of real complex networks** unto hyperbolic space [15,16]
 - Reproduction of additional properties than the ones used to fit the parameters [4,15].
 - Identification of biochemical pathways in E. Coli [8]
 - Efficient Internet routing protocols [17]
 - Organization of the human connectome [18,20]
 - Self-similar architecture [19]
 - Evolution of hierarchy in international trade [21]
 - ...
- ▷ ...

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