



Macroscopic (large scale)  
properties

Microscopic (local)  
properties

$n = 1$



$n = 3$



$n = 5$



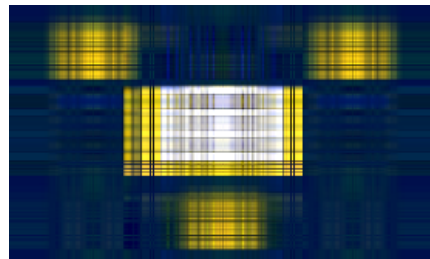
$n = 10$



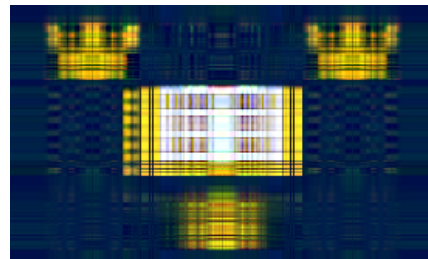
$n = 1$



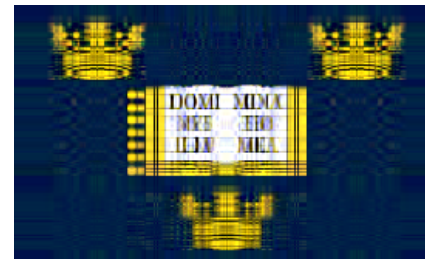
$n = 3$



$n = 5$



$n = 10$



$n = 25$



Lower  
effective ranks



Higher  
effective ranks

The effective ranks of adjacency matrices

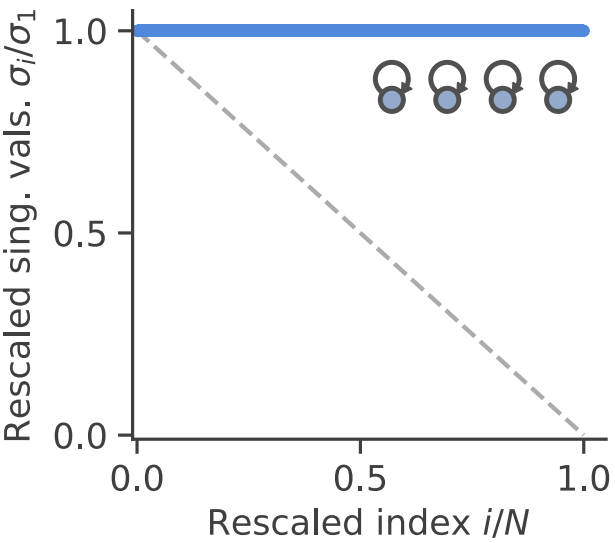
$$\mathbf{W} = \sum_{i=1}^{\textcolor{red}{r}} \sigma_i \mathbf{U}_i \mathbf{V}_i^\top \simeq \sum_{i=1}^{\textcolor{red}{n}} \sigma_i \mathbf{U}_i \mathbf{V}_i^\top$$

Effective rank  $n$  of a matrix

- ▷ number of “significant” nonzero singular values

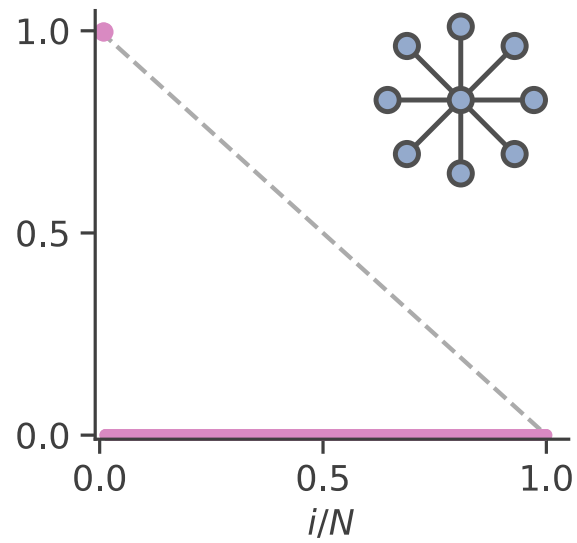
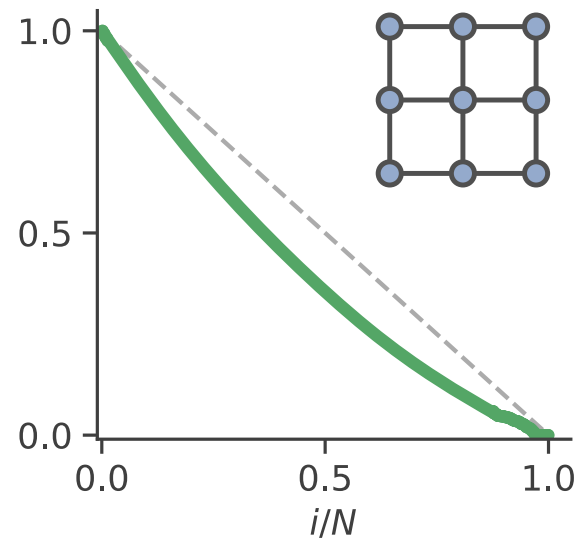
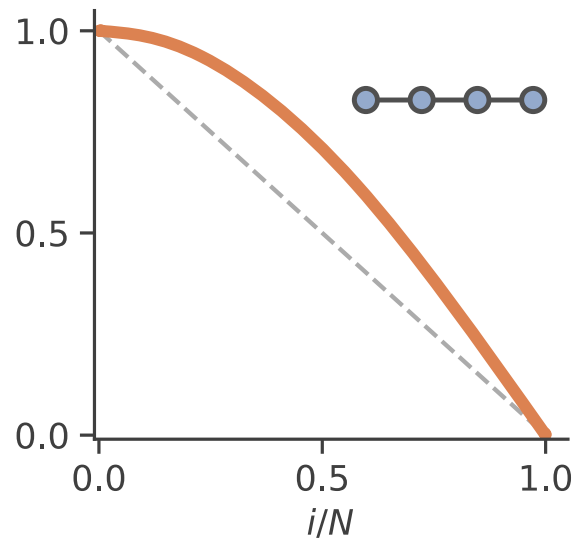
- ▷ term after which it is “reasonable” to truncate the sum

High effective ranks



→

Low effective ranks



Abbreviation	Expression
srnk	$\sum_{i=1}^r \sigma_i^2 / \sigma_1^2$
nrank	$\sum_{i=1}^r \sigma_i / \sigma_1$
energy	$\min \left[ \arg \max_{\ell \in \{1, \dots, N\}} \left( \sum_{i=1}^{\ell} \sigma_i^2 / \sum_{j=1}^r \sigma_j^2 > \tau \right) \right]$
elbow	$\frac{1}{\sqrt{2}} \arg \max_{i \in \{1, \dots, N\}} \left  \frac{i-1}{N-1} + \frac{\sigma_i - \sigma_N}{\sigma_1 - \sigma_N} - 1 \right  - 1$
erank	$\exp \left[ - \sum_{i=1}^r \frac{\sigma_i}{\sum_{j=1}^r \sigma_j} \log \frac{\sigma_i}{\sum_{j=1}^r \sigma_j} \right]$
thrank	$\# \left\{ \sigma_i \mid i \in \{1, \dots, N\} \text{ and } \sigma_i > \frac{4\sigma_{\text{med}}}{\sqrt{3}\mu_{\text{med}}} \right\}$
shrank	$\# \{s^*(\sigma_i) \mid i \in \{1, \dots, N\} \text{ and } s^*(\sigma_i) > 0\}$

The **singular values** (and the effective rank) encode information about the network's **topology**.



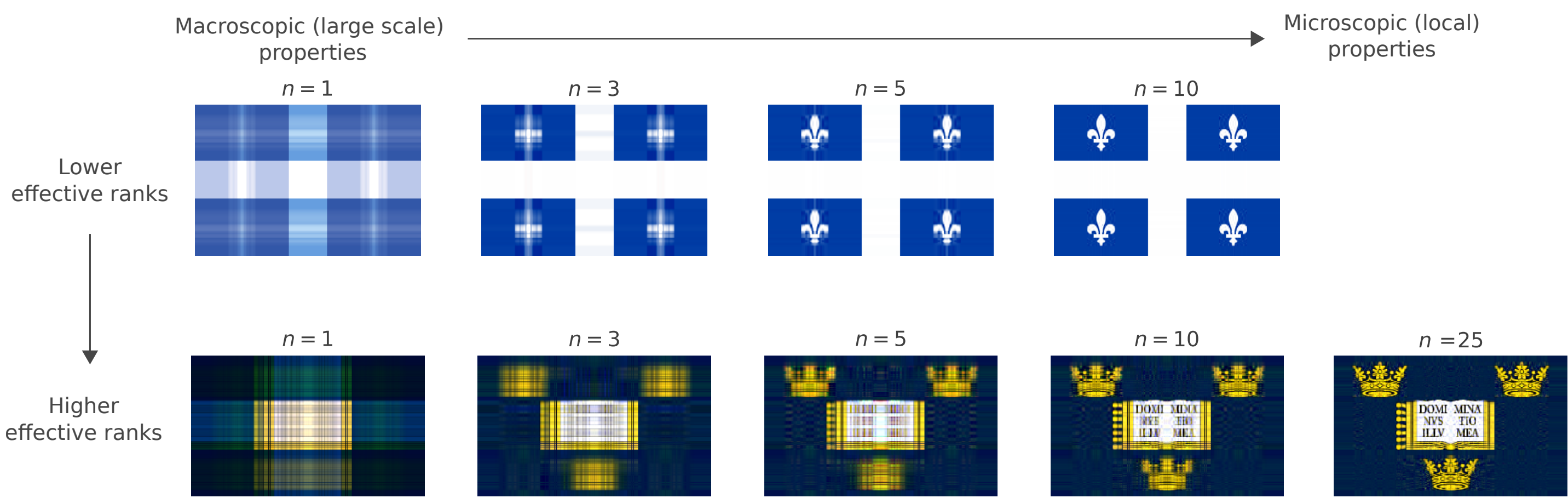




# The effective ranks of adjacency matrices

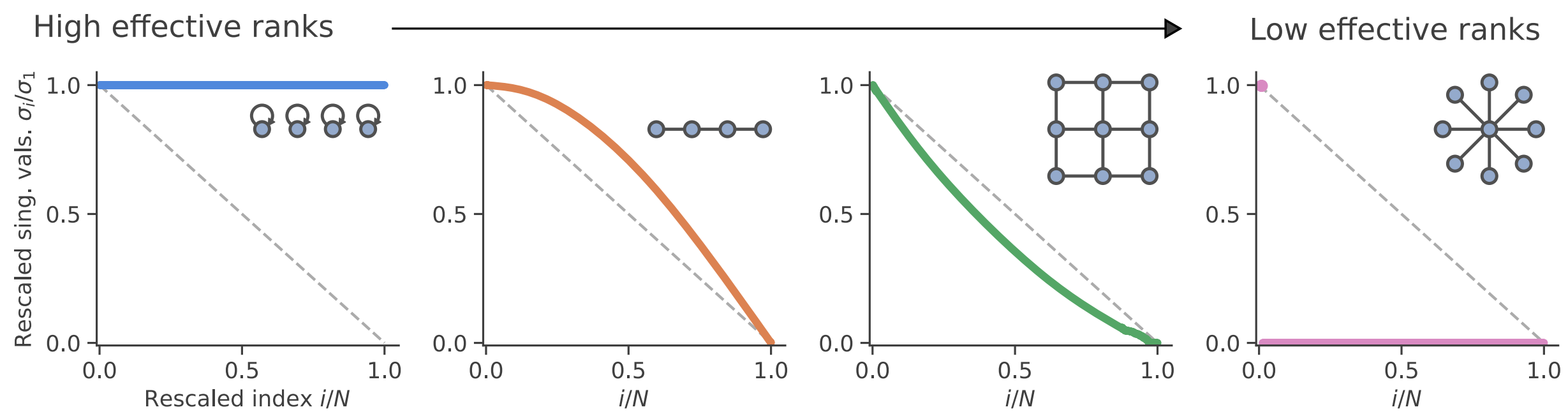
$$\mathbf{W} = \sum_{i=1}^r \sigma_i \mathbf{u}_i \mathbf{v}_i^\top \simeq \sum_{i=1}^n \sigma_i \mathbf{u}_i \mathbf{v}_i^\top$$

- Effective rank  $n$  of a matrix
- ▷ number of “significant” nonzero singular values
  - ▷ term after which it is “reasonable” to truncate the sum



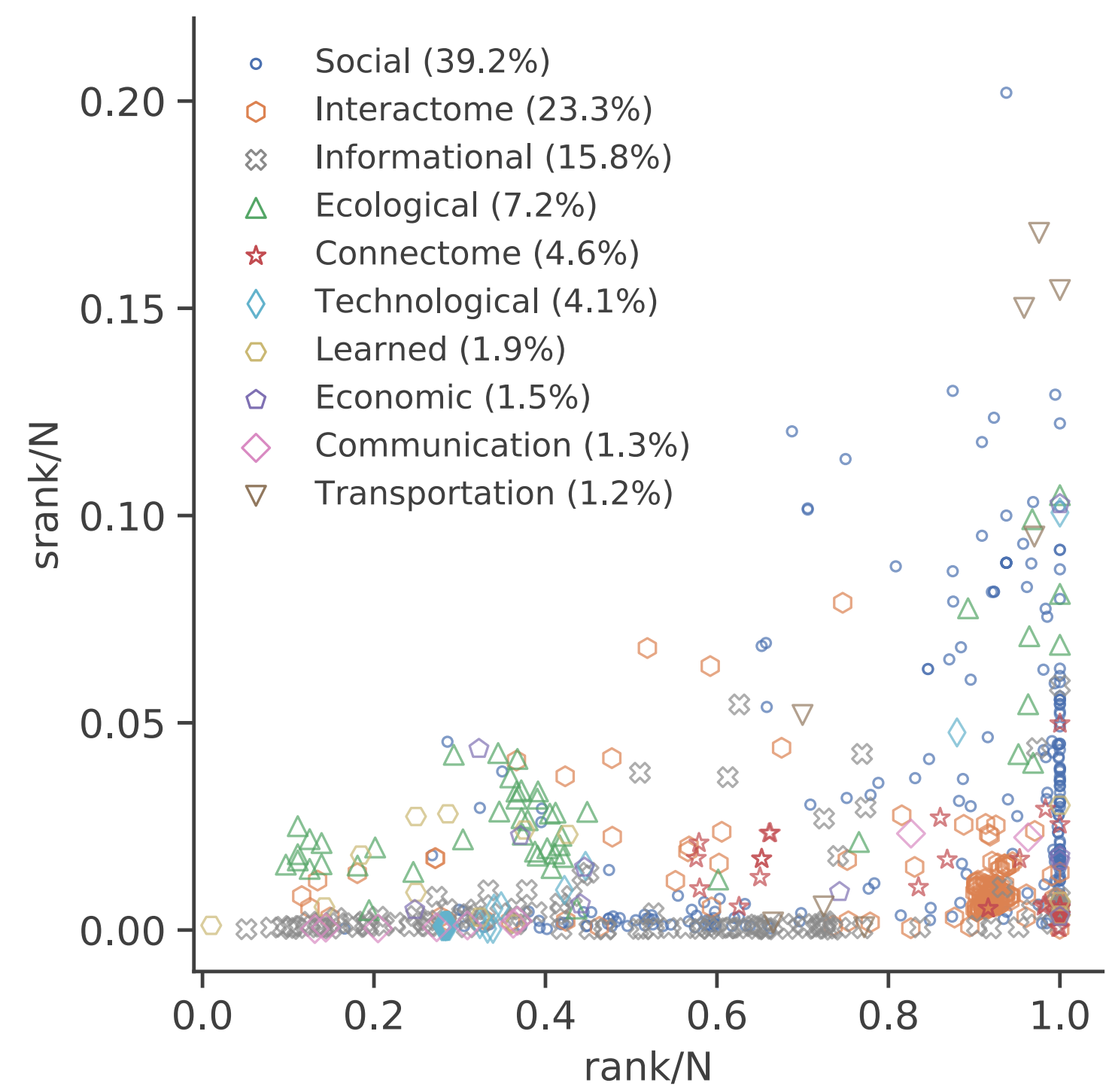
Abbreviation	Expression
srank	$\sum_{i=1}^r \sigma_i^2 / \sigma_1^2$
nrank	$\sum_{i=1}^r \sigma_i / \sigma_1$
energy	$\min \left[ \arg \max_{\ell \in \{1, \dots, N\}} \left( \sum_{i=1}^{\ell} \sigma_i^2 / \sum_{j=1}^r \sigma_j^2 > \tau \right) \right]$
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erank	$\exp \left[ - \sum_{i=1}^r \frac{\sigma_i}{\sum_{j=1}^r \sigma_j} \log \frac{\sigma_i}{\sum_{j=1}^r \sigma_j} \right]$
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shrank	$\# \{s^*(\sigma_i) \mid i \in \{1, \dots, N\} \text{ and } s^*(\sigma_i) > 0\}$

The singular values (and the effective rank) encode information about the network’s topology.



# The effective ranks of adjacency matrices

Many empirical networks appear to have a low effective rank!



Results for 679 empirical networks (502 unweighted networks and 177 weighted networks) downloaded from Netzschleuder.