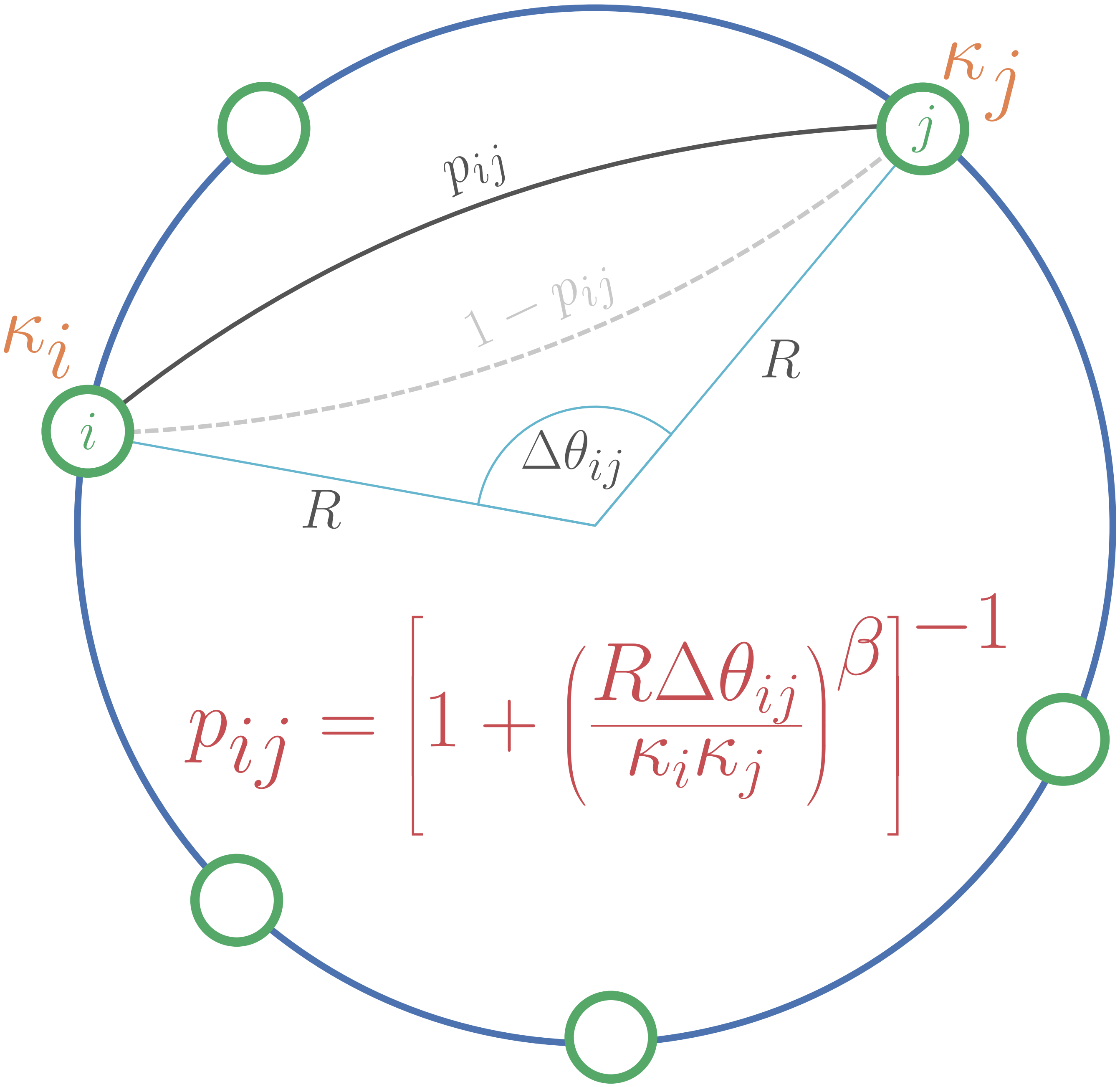
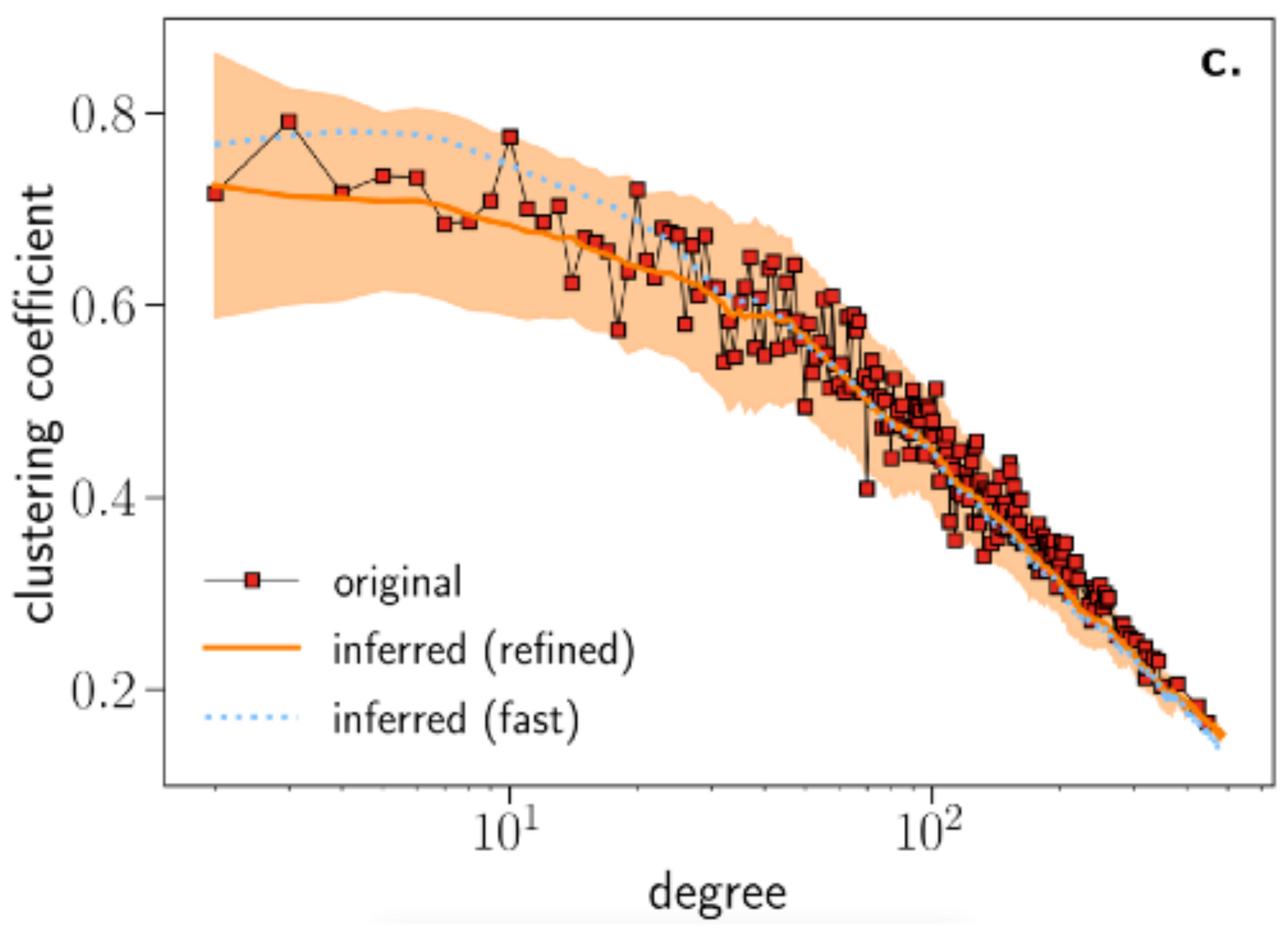
10



κ_j
j
p_{ij}
$1 - p_{ij}$
κ_i
i
R
$\Delta\theta_{ij}$
R
$p_{ij} = \left[1 + \left(\frac{R\Delta\theta_{ij}}{\kappa_i \kappa_j}\right)^{\beta}\right]^{-1}$



c.
0.8
0.6
0.4
0.2
clustering coefficient
10^1
10^2
degree
original
inferred (refined)
inferred (fast)

[15]

The $\mathbb{S}^1$ model

1. Sprinkle N nodes uniformly on a circle of radius R.
2. Assign an expected degree κ to each node according to some pdf $\rho(\kappa)$.
3. Draw a link between node i and node j with probability p_{ij}.

- ★ fixes the expected degree of nodes (κ) $\rightarrow$ soft configuration model (CM)
- ★ triangle inequality of the underlying metric space $\rightarrow$ triangles from pairwise interactions
- ★ level of clustering tuned with parameter β

Other properties and generalizations

- Amenable to many analytical calculations
- Geometric interpretation in terms of hyperbolic geometry (the $\mathbb{H}^2$ model) [1,2]
- Parsimonious explanation of self-similarity [3,4]
- Generalizable to weighted [5], bipartite [6,7,8], multiplex [9,10] and growing [11] networks
- Generalizable to networks with community structure [12,13,14]
- Mapping of real complex networks unto hyperbolic space [15,16]
- Identification of biochemical pathways in E. Coli [8]
- Efficient Internet routing protocols [17]
- Multiscale organization of the human connectome [18]
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- ...

[1] Phys. Rev. E 80, 035101 (2009)

[2] Phys. Rev. E 82, 036106 (2010)

[3] Phys. Rev. Lett. 100, 078701 (2008)

[4] Nat. Rev. Phys. 3, 114 (2021)

[5] Nat. Commun. 8, 14103 (2017)

[6] Phys. Rev. E 84, 026114 (2011)

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[13] J. Stat. Phys. 173, 775 (2018)

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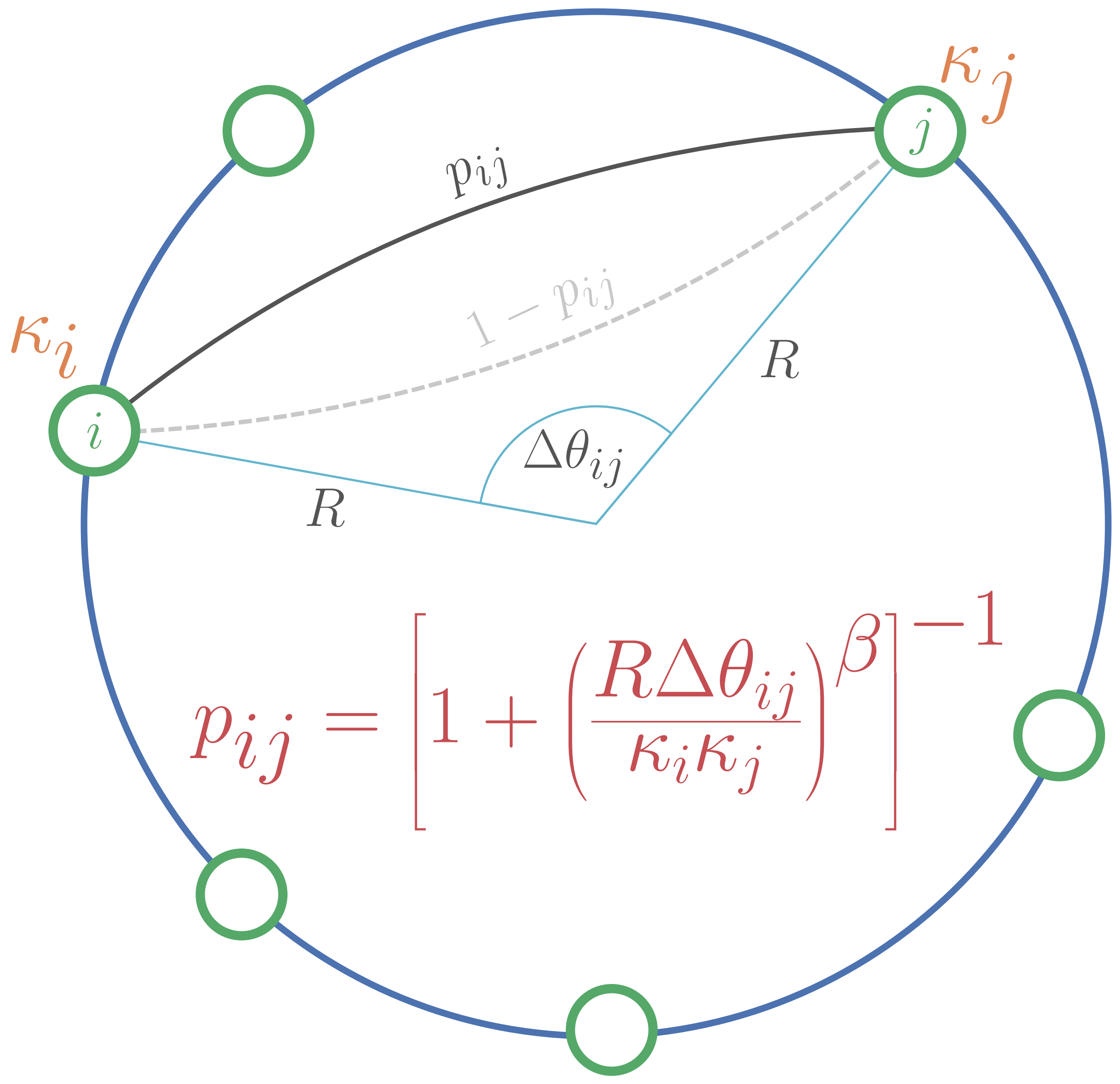
[15] New J. Phys. 21, 123033 (2019)

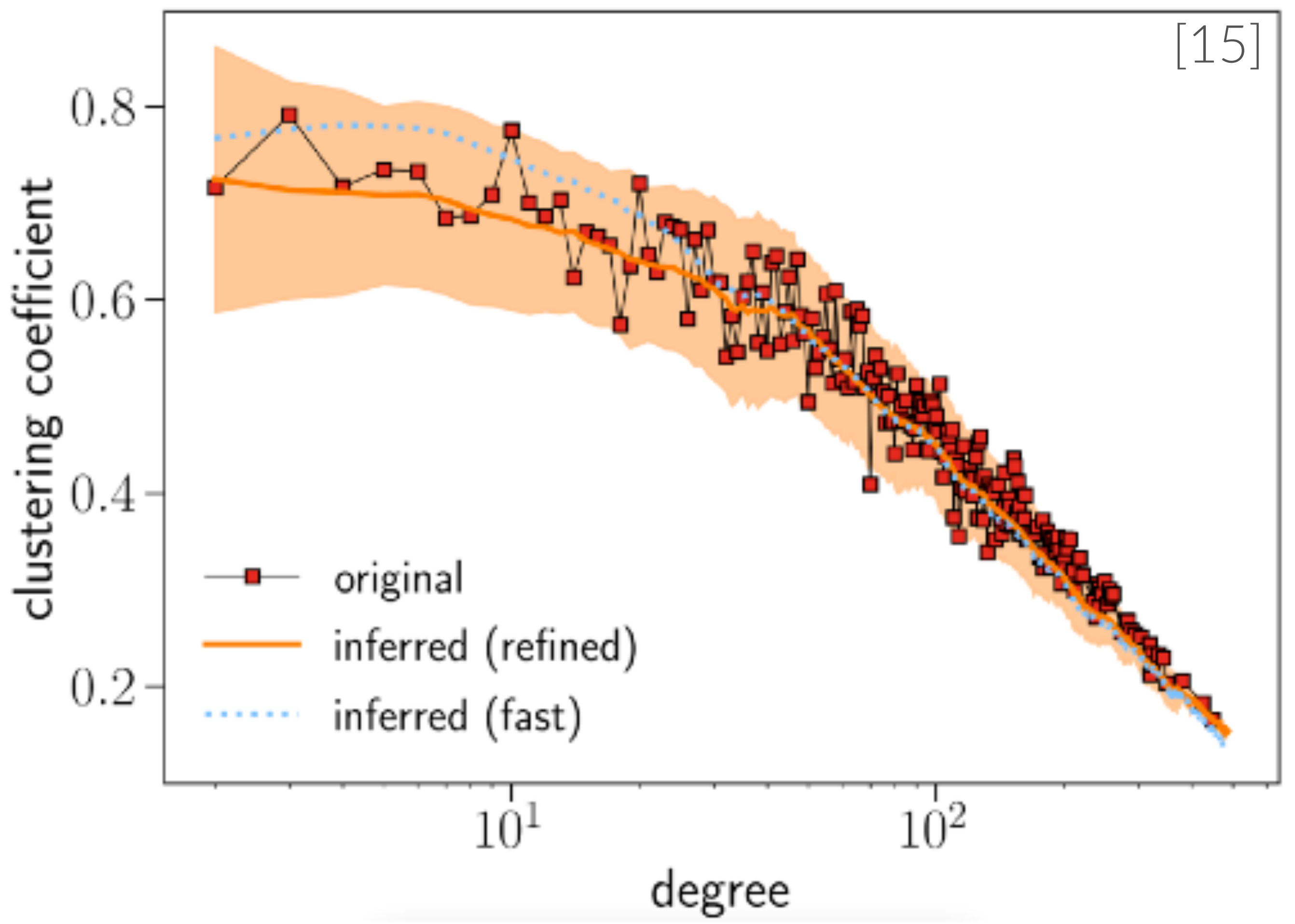
[16] Nat. Commun. 8, 1615 (2017)

[17] Nat. Commun. 1, 62 (2010)

[18] PNAS 117, 20244 (2020)

A geometric approach to clustering : the $\mathbb{S}^1/\mathbb{H}^2$ model





The $\mathbb{S}^1$ model

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Outline

1. Why models and the challenge of clustering
2. A geometric approach to clustering
3. Euclid and hyperbolic geometry
4. A hyperbolic solution to clustering
5. **Rethinking interactions: the case of directed graphs**
6. Rethinking interactions: the case of modular structure