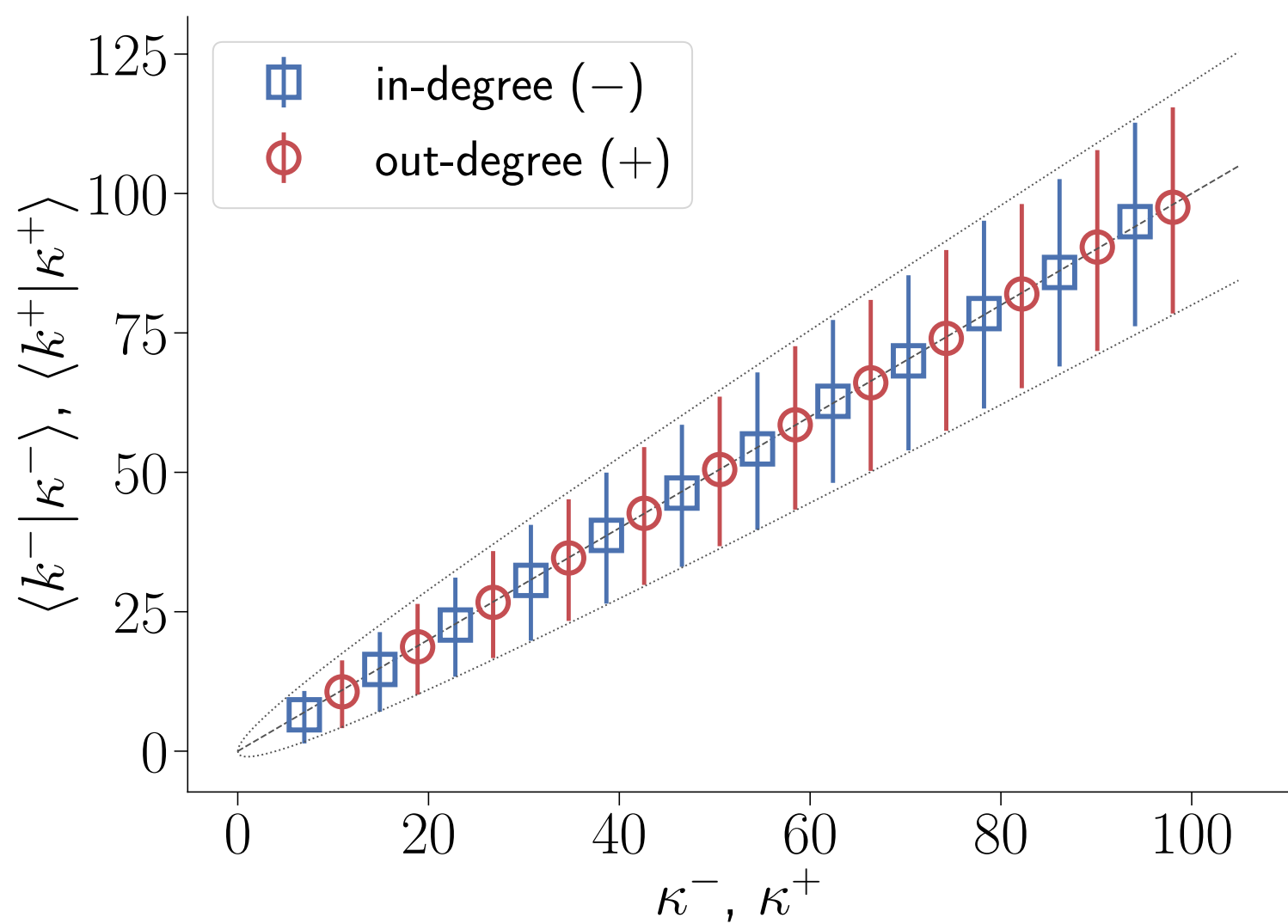


1

3



density of triangles

0.6
0.4
0.2
0.0

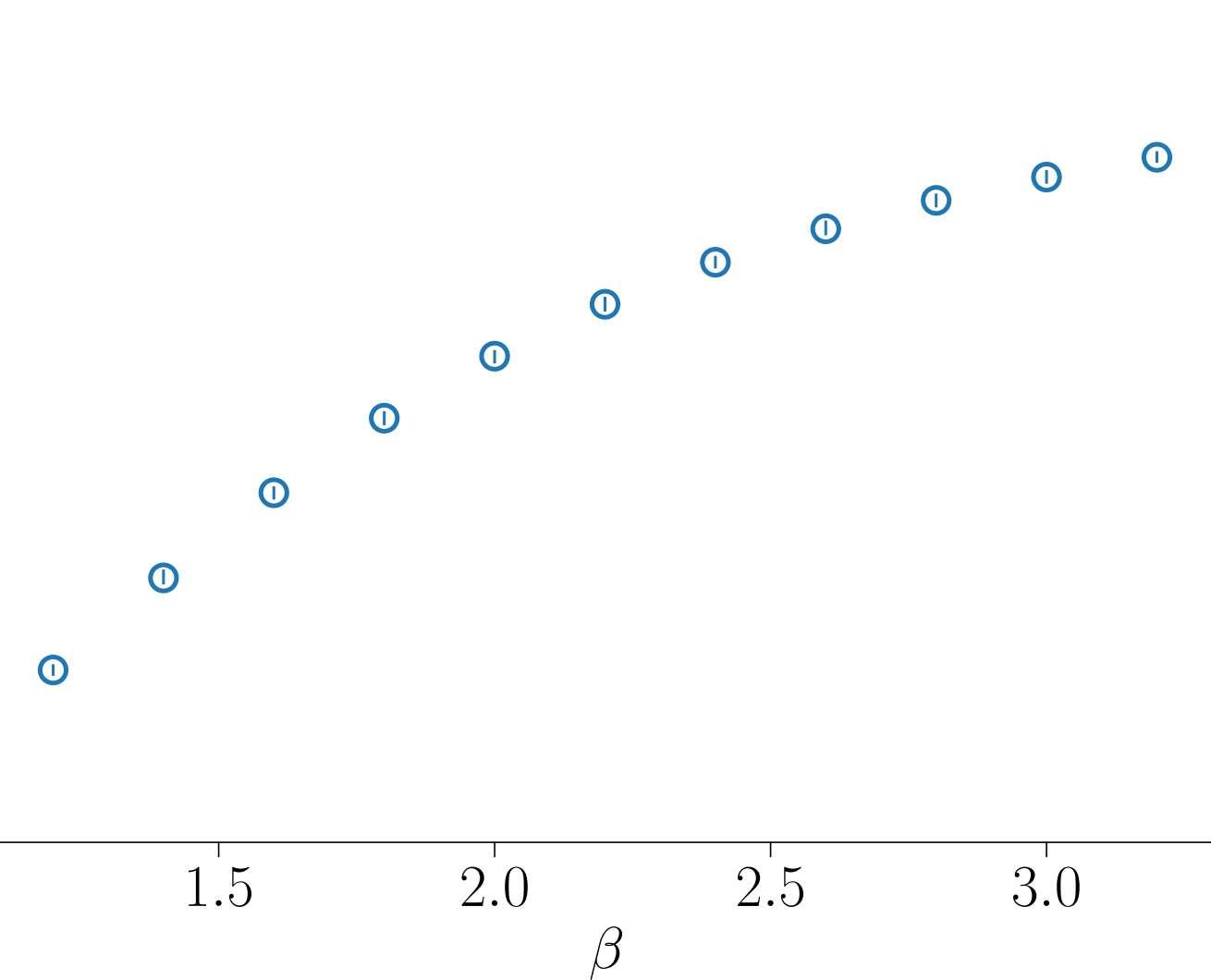
1.5

2.0

2.5

3.0

β



$$\mathbb{E} \left[k^{\text{in}} \mid \kappa^{\text{in}} \right] \simeq \kappa^{\text{in}}$$

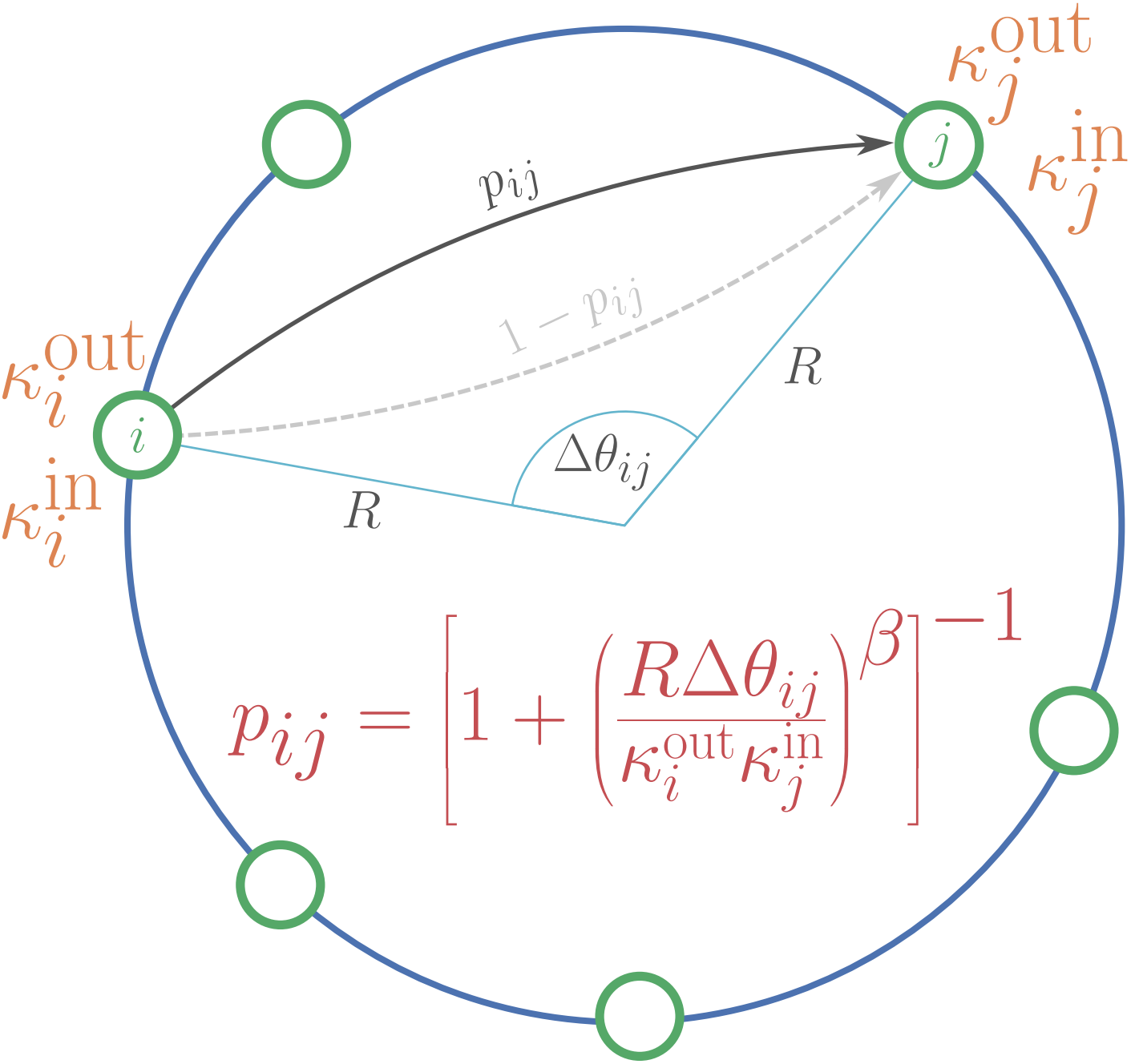
$$\mathbb{E} \left[k^{\text{out}} \mid \kappa^{\text{out}} \right] \simeq \kappa^{\text{out}}$$

$$P(k^{\text{in}}, k^{\text{out}}) \simeq \iint \frac{[\kappa^{\text{in}}]^{k^{\text{in}}} e^{-\kappa^{\text{in}}}}{k^{\text{in}}!} \frac{[\kappa^{\text{out}}]^{k^{\text{out}}} e^{-\kappa^{\text{out}}}}{k^{\text{out}}!} \\ \times \rho(\kappa^{\text{in}}, \kappa^{\text{out}}) d\kappa^{\text{in}} d\kappa^{\text{out}}$$

The directed \mathbb{S}^1 model

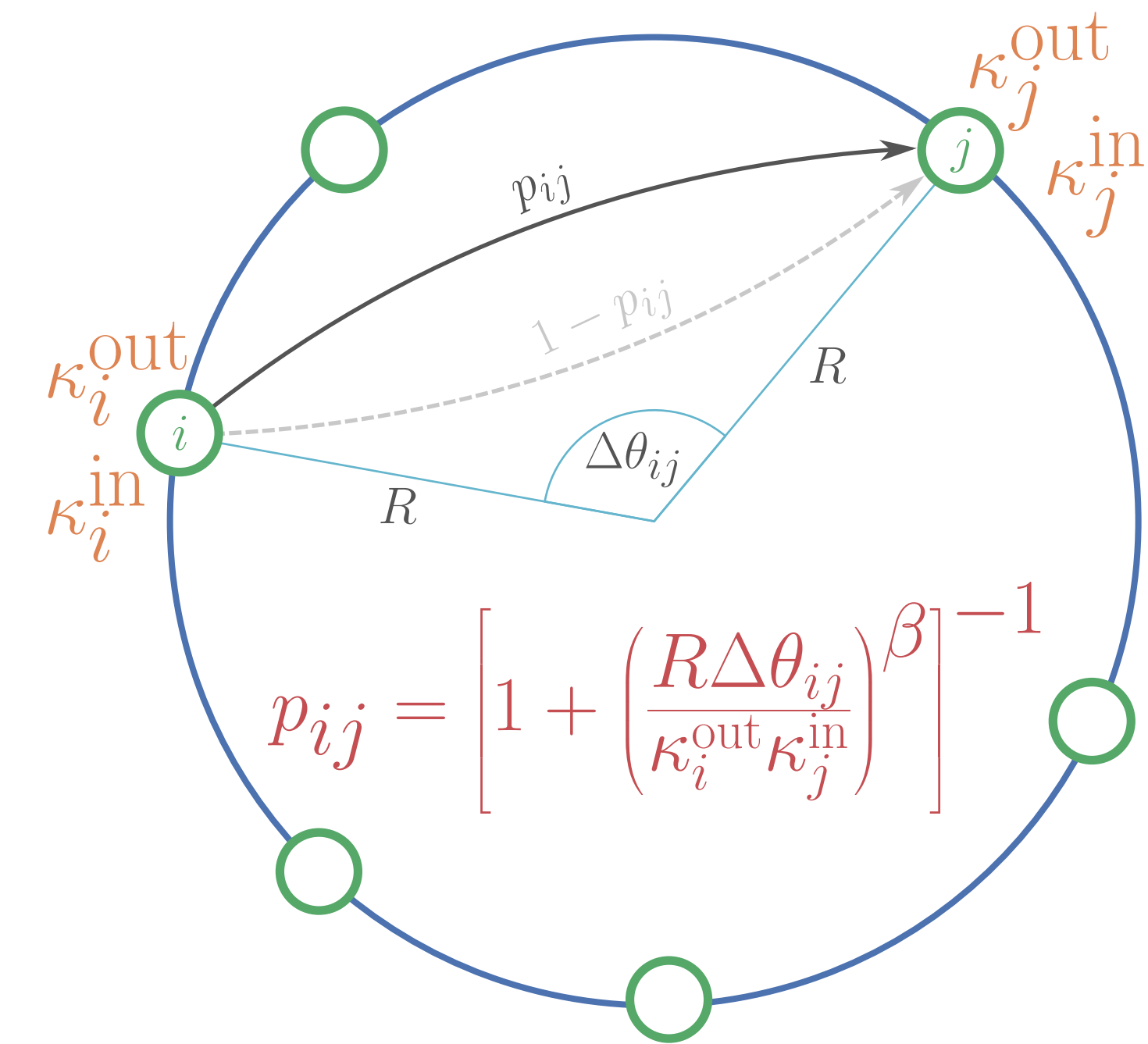
1. Sprinkle N nodes uniformly on a circle of radius R .
2. Assign an expected in-degree κ^{in} and out-degree κ^{out} to each node according to some pdf $\rho(\kappa^{\text{in}}, \kappa^{\text{out}})$.
3. Draw a link from node i to node j with probability p_{ij} .

- ★ fixes the expected in-degree and out-degree of nodes $(\kappa^{\text{in}}, \kappa^{\text{out}}) \rightarrow$ soft directed CM
- ★ triangle inequality of the underlying metric space \rightarrow triangles from pairwise interactions
- ★ level of clustering tuned with parameter β



The directed s_1 model

The directed \mathbb{S}^1 model



$$p_{ij} = \left[1 + \left(\frac{R\Delta\theta_{ij}}{\kappa_i^{\text{out}}\kappa_j^{\text{in}}} \right)^\beta \right]^{-1}$$

$$\mathbb{E}[k^{\text{in}} | \kappa^{\text{in}}] \simeq \kappa^{\text{in}}$$

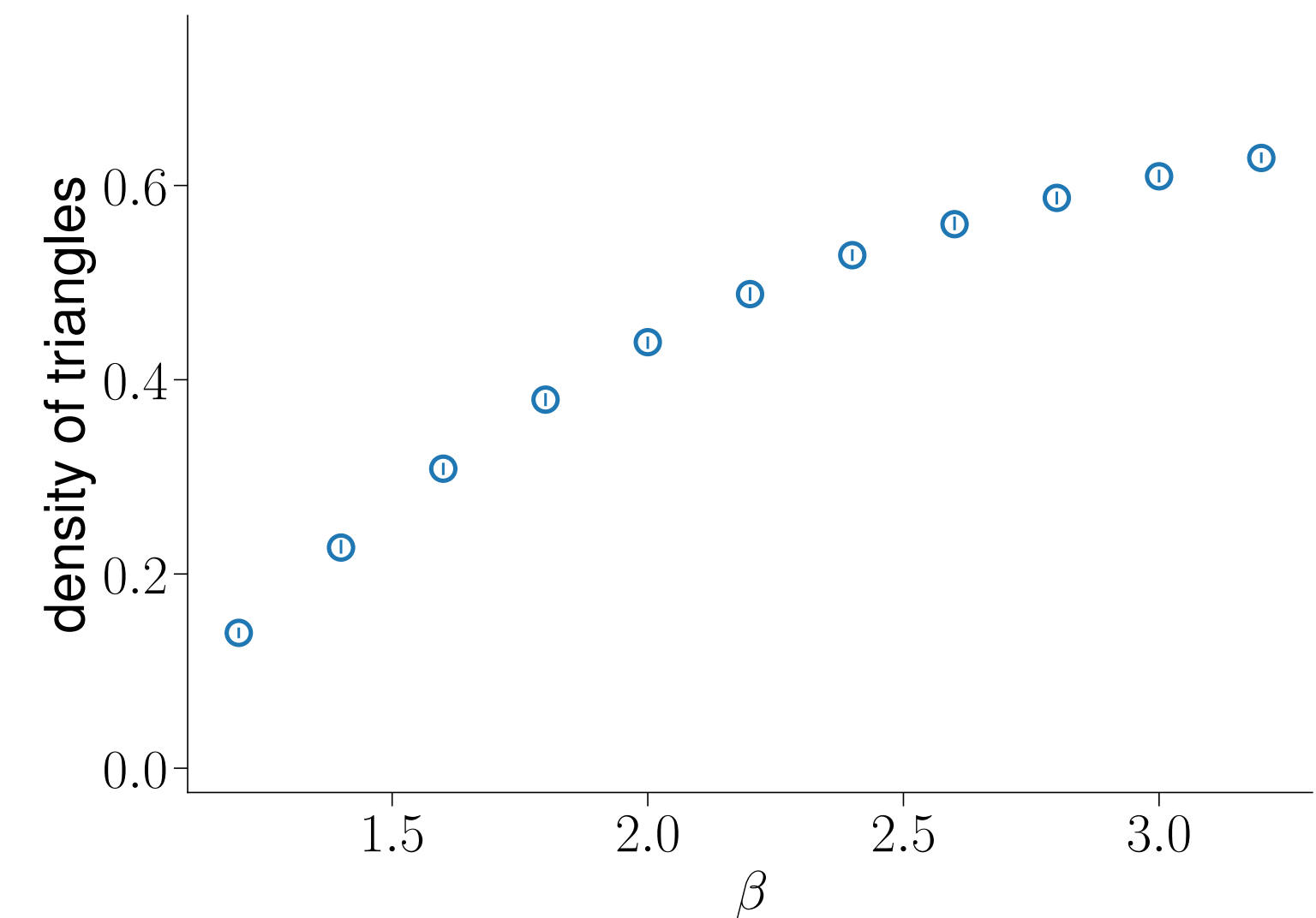
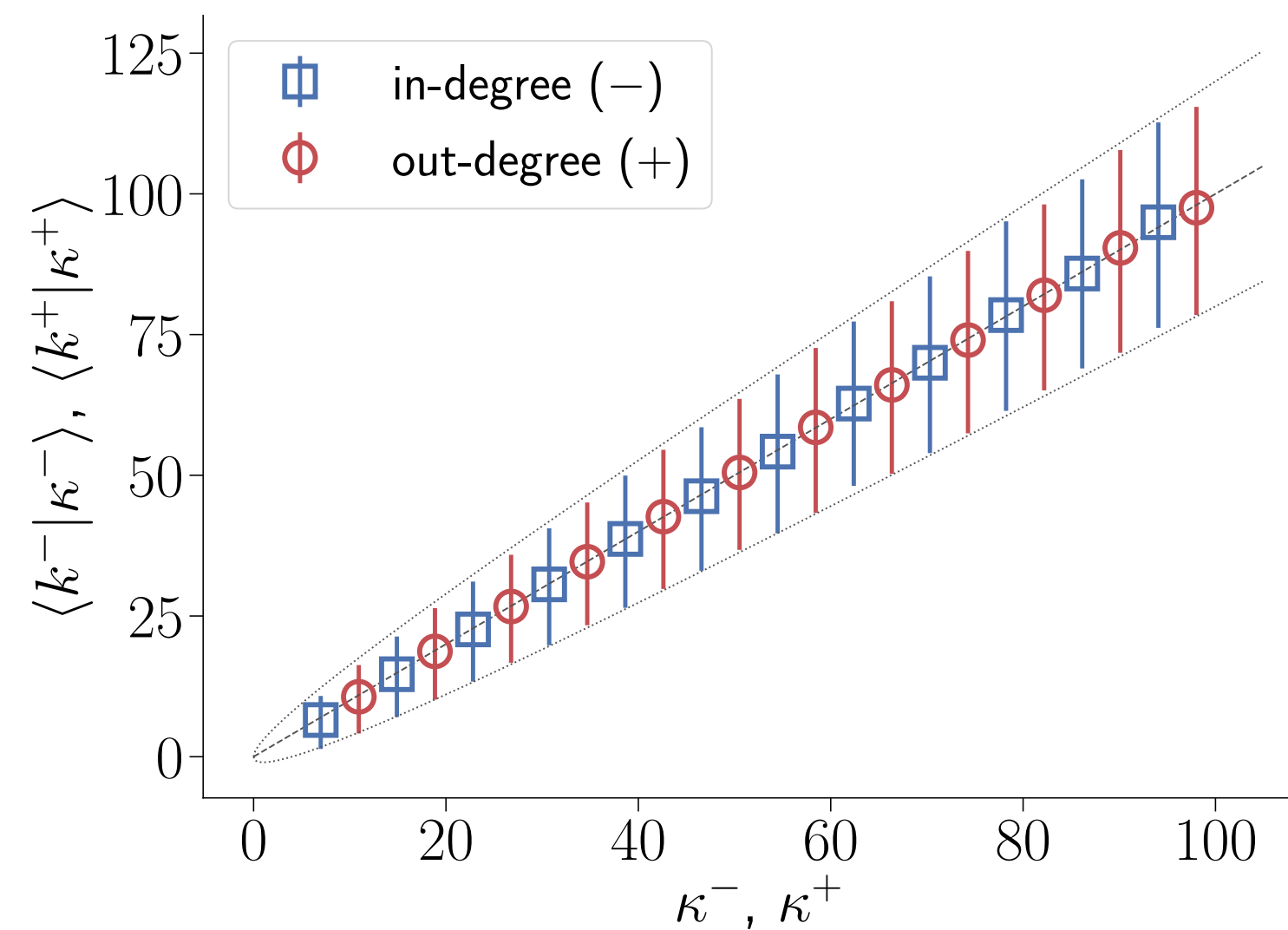
$$\mathbb{E}[k^{\text{out}} | \kappa^{\text{out}}] \simeq \kappa^{\text{out}}$$

$$P(k^{\text{in}}, k^{\text{out}}) \simeq \iint \frac{[\kappa^{\text{in}}]^{k^{\text{in}}} e^{-\kappa^{\text{in}}}}{k^{\text{in}}!} \frac{[\kappa^{\text{out}}]^{k^{\text{out}}} e^{-\kappa^{\text{out}}}}{k^{\text{out}}!} \times \rho(\kappa^{\text{in}}, \kappa^{\text{out}}) d\kappa^{\text{in}} d\kappa^{\text{out}}$$

The directed \mathbb{S}^1 model

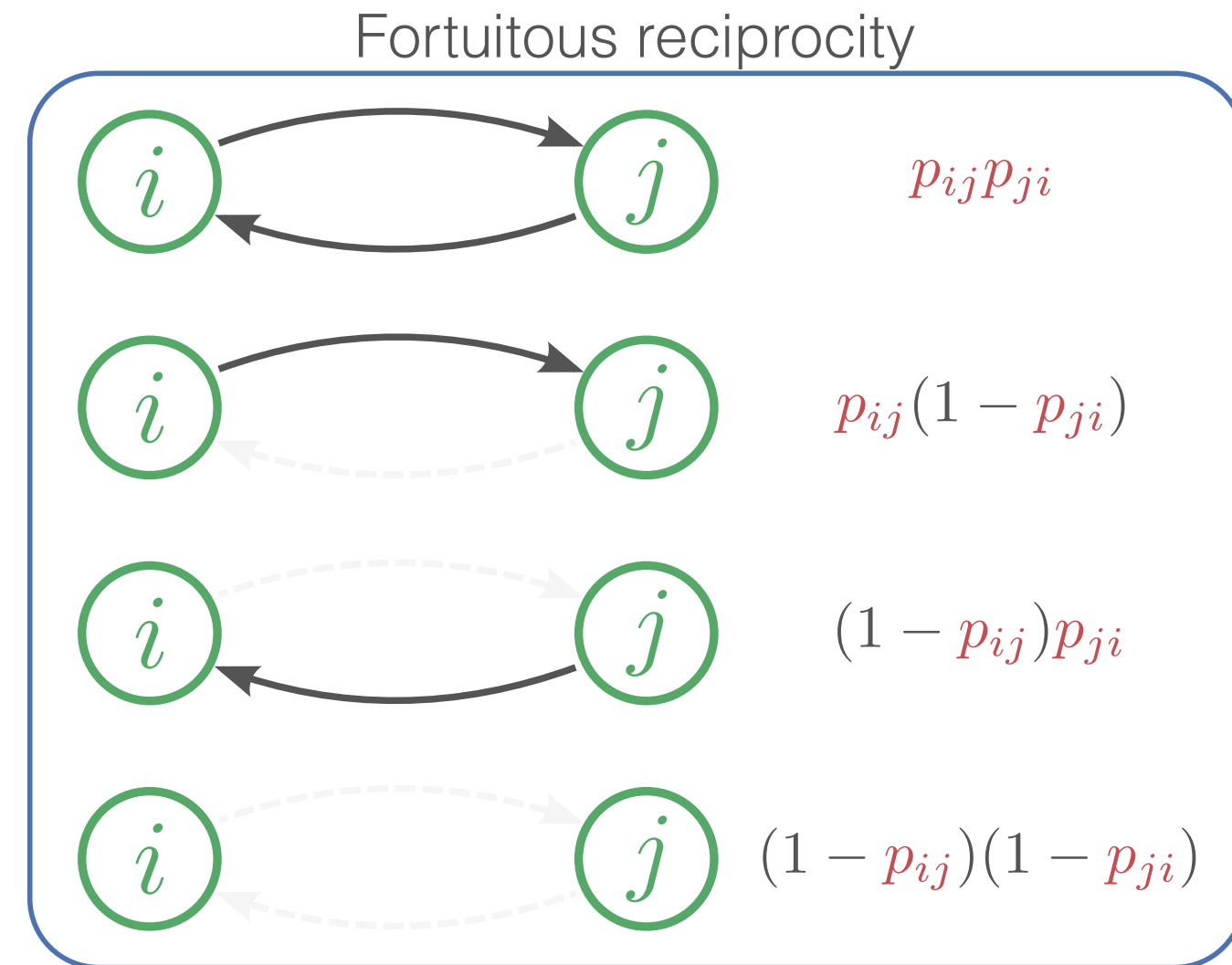
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Reciprocity in the directed \mathbb{S}^1 model

A reciprocal connection between node i and node j occurs with probability $p_{ij}p_{ji}$.



$$r = \mathbb{E} \left[\frac{L^{\leftrightarrow}}{L} \right] = \mathbb{E} \left[\frac{k^{\leftrightarrow}}{k^{\text{out}}} \right] \approx \frac{\mathbb{E} [k^{\leftrightarrow}]}{\mathbb{E} [k^{\text{out}}]}$$

$$\simeq \iiint \frac{\kappa_i^{\text{out}} \kappa_j^{\text{in}}}{\langle \kappa \rangle^2} \frac{1 - \left(\frac{\kappa_i^{\text{out}}}{\kappa_i^{\text{in}}} \frac{\kappa_j^{\text{in}}}{\kappa_j^{\text{out}}} \right)^{\beta-1}}{1 - \left(\frac{\kappa_i^{\text{out}}}{\kappa_i^{\text{in}}} \frac{\kappa_j^{\text{in}}}{\kappa_j^{\text{out}}} \right)^{\beta}}$$

$$\times \rho(\kappa_i^{\text{in}}, \kappa_i^{\text{out}}) \rho(\kappa_j^{\text{in}}, \kappa_j^{\text{out}}) d\kappa_i^{\text{in}} \kappa_i^{\text{out}} d\kappa_j^{\text{in}} \kappa_j^{\text{out}}$$

κ^{in} : in-degree

κ^{out} : out-degree

β : density of triangles