

Macroscopic (large scale)
properties

Microscopic (local)
properties

$n = 1$



$n = 3$



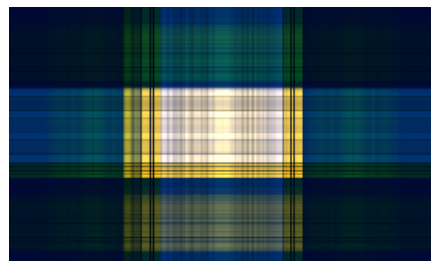
$n = 5$



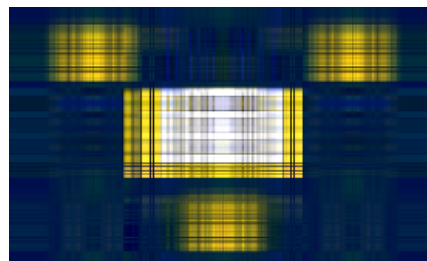
$n = 10$



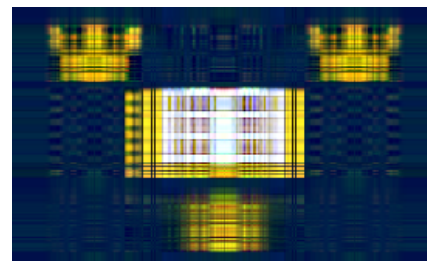
$n = 1$



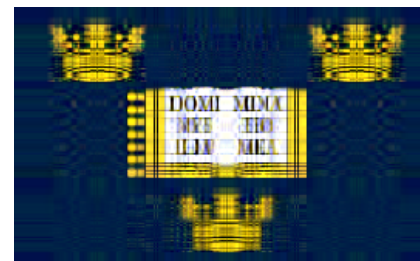
$n = 3$



$n = 5$



$n = 10$



$n = 25$



Lower
effective ranks



Higher
effective ranks

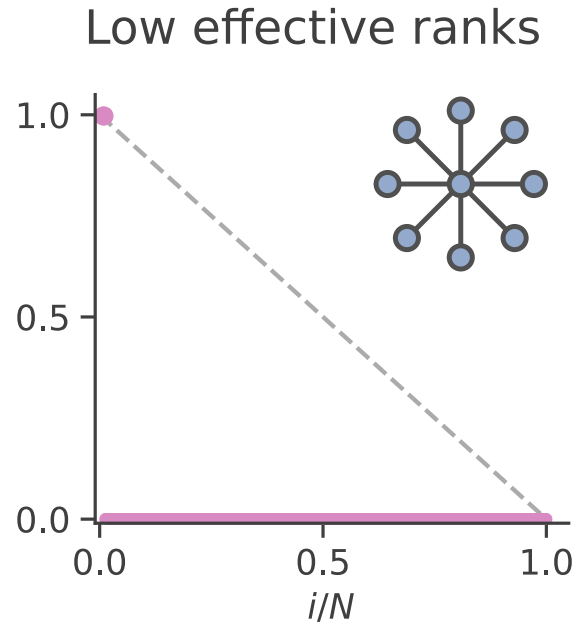
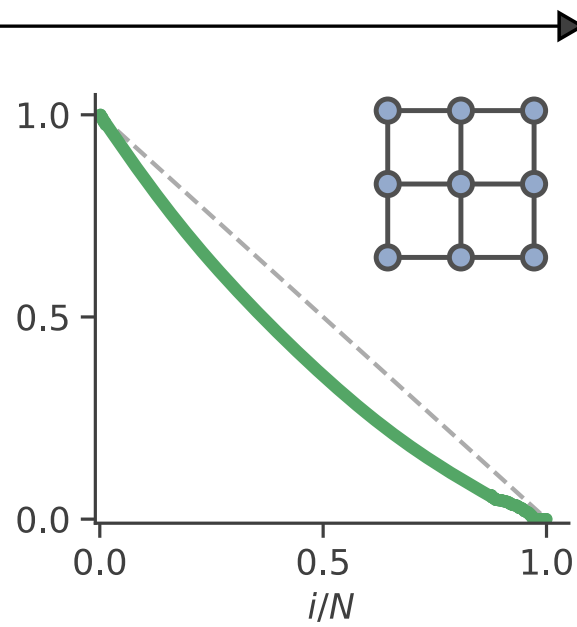
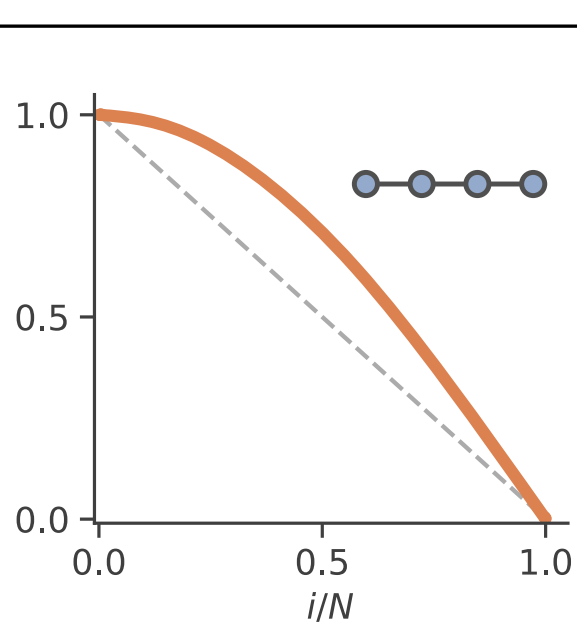
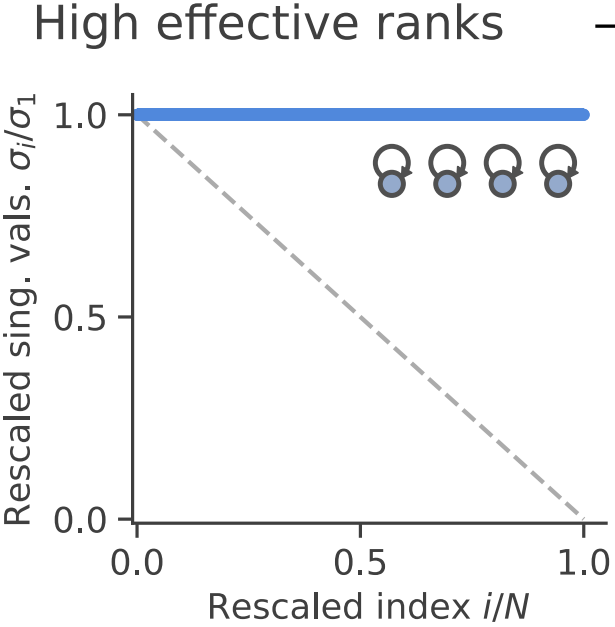
The effective ranks of adjacency matrices

$$\mathbf{W} = \sum_{i=1}^{\textcolor{red}{r}} \sigma_i \mathbf{U}_i \mathbf{V}_i^\top \simeq \sum_{i=1}^{\textcolor{red}{n}} \sigma_i \mathbf{U}_i \mathbf{V}_i^\top$$

Effective rank n of a matrix

- ▷ number of “significant” nonzero singular values

- ▷ term after which it is “reasonable” to truncate the sum



| Abbreviation | Expression |
|--------------|---|
| srank | $\sum_{i=1}^r \sigma_i^2 / \sigma_1^2$ |
| nrank | $\sum_{i=1}^r \sigma_i / \sigma_1$ |
| energy | $\min \left[\arg \max_{\ell \in \{1, \dots, N\}} \left(\sum_{i=1}^{\ell} \sigma_i^2 / \sum_{j=1}^r \sigma_j^2 > \tau \right) \right]$ |
| elbow | $\frac{1}{\sqrt{2}} \arg \max_{i \in \{1, \dots, N\}} \left \frac{i-1}{N-1} + \frac{\sigma_i - \sigma_N}{\sigma_1 - \sigma_N} - 1 \right - 1$ |
| erank | $\exp \left[- \sum_{i=1}^r \frac{\sigma_i}{\sum_{j=1}^r \sigma_j} \log \frac{\sigma_i}{\sum_{j=1}^r \sigma_j} \right]$ |
| thrank | $\# \left\{ \sigma_i \mid i \in \{1, \dots, N\} \text{ and } \sigma_i > \frac{4\sigma_{\text{med}}}{\sqrt{3}\mu_{\text{med}}} \right\}$ |
| shrank | $\# \{s^*(\sigma_i) \mid i \in \{1, \dots, N\} \text{ and } s^*(\sigma_i) > 0\}$ |

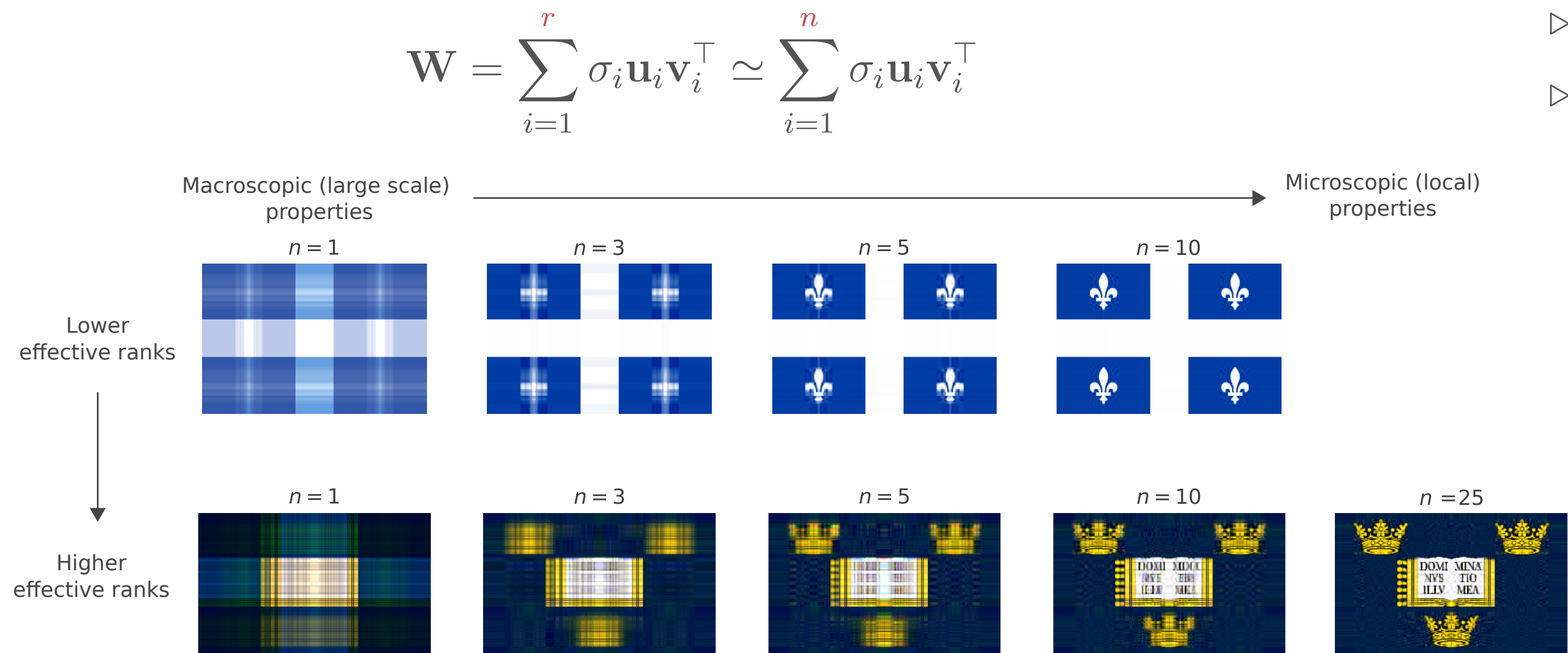
The **singular values** (and the effective rank) encode information about the network's topology.





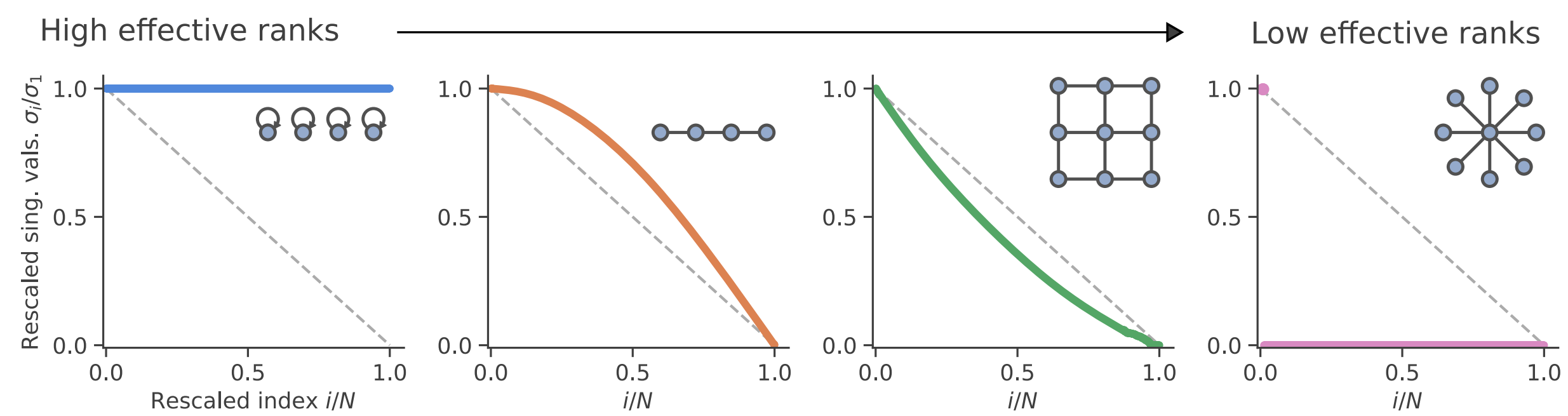
The effective ranks of adjacency matrices

- Effective rank n of a matrix
- ▷ number of “significant” nonzero singular values
 - ▷ term after which it is “reasonable” to truncate the sum



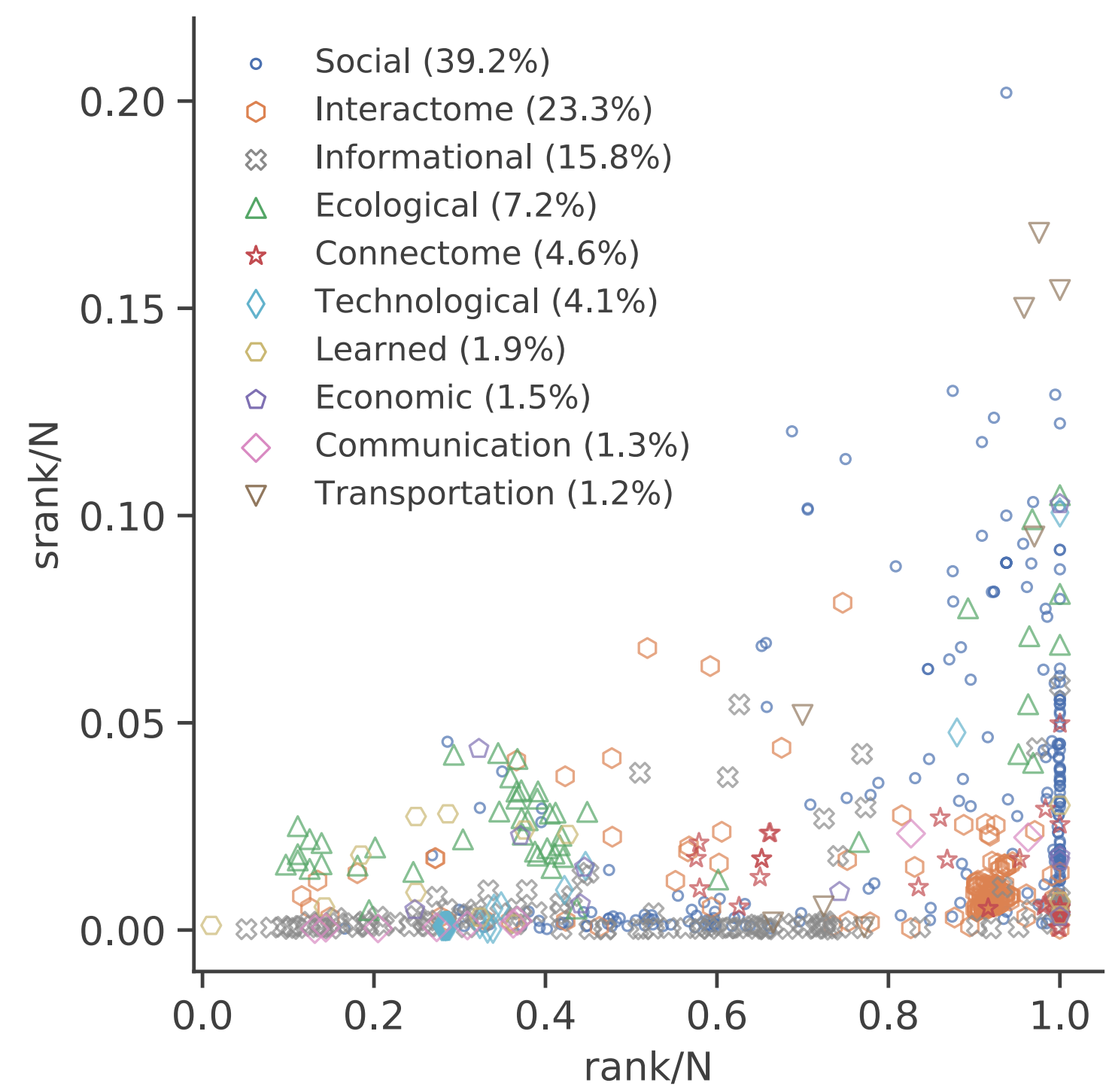
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The **singular values** (and the effective rank) encode information about the network’s **topology**.



The effective ranks of adjacency matrices

Many empirical networks appear to have a low effective rank!



Results for 679 empirical networks (502 unweighted networks and 177 weighted networks) downloaded from Netzscheuler.