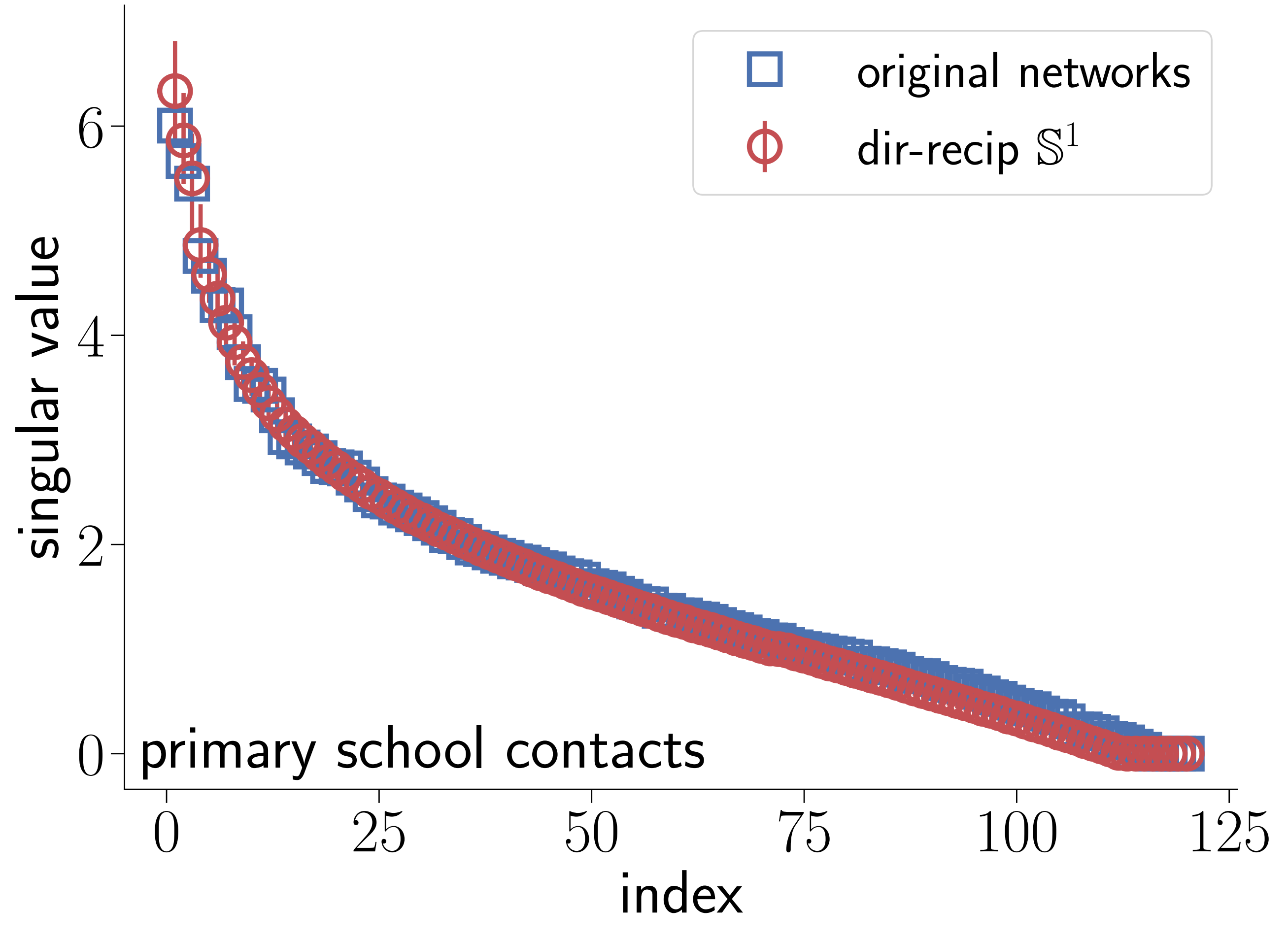
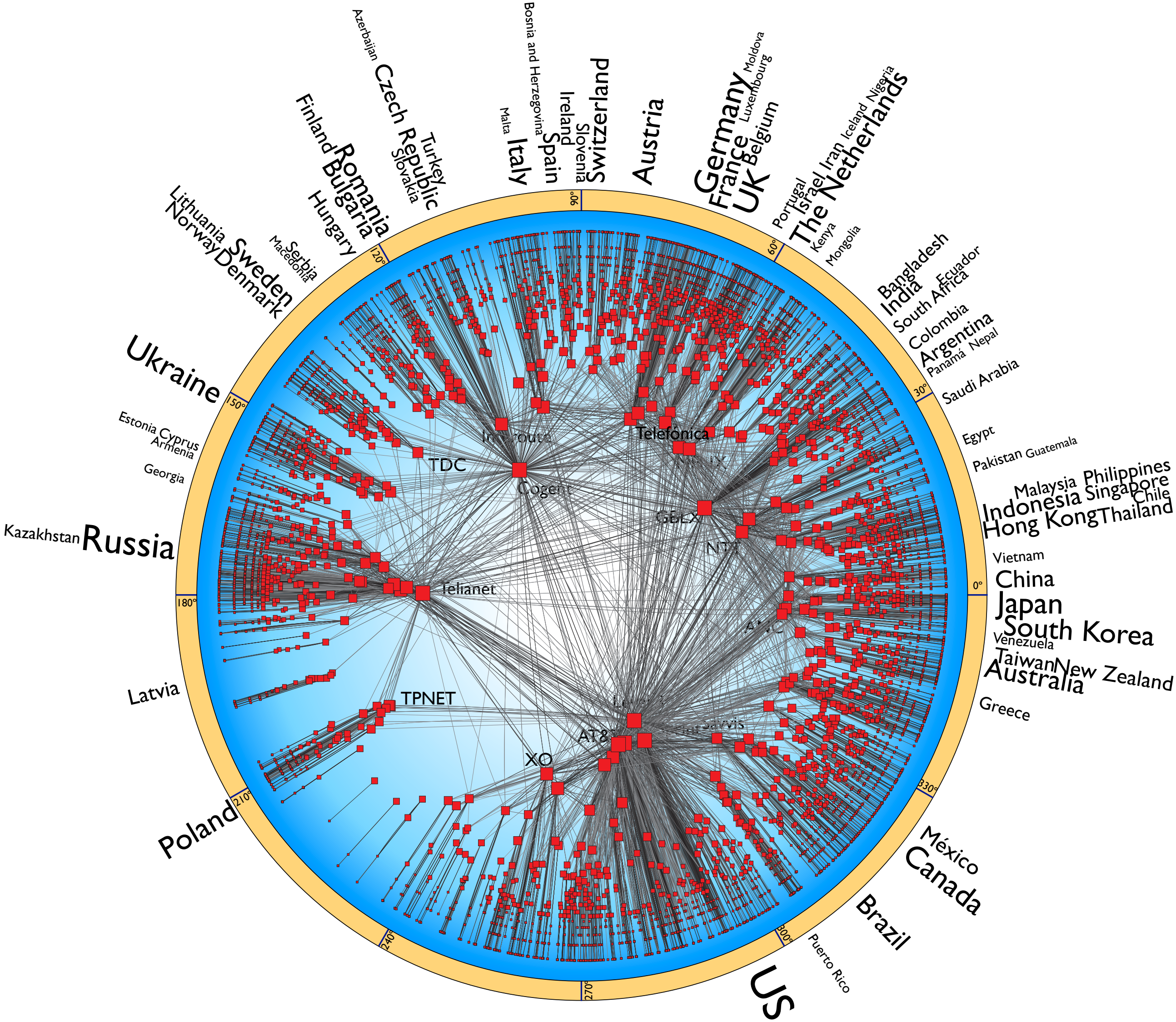
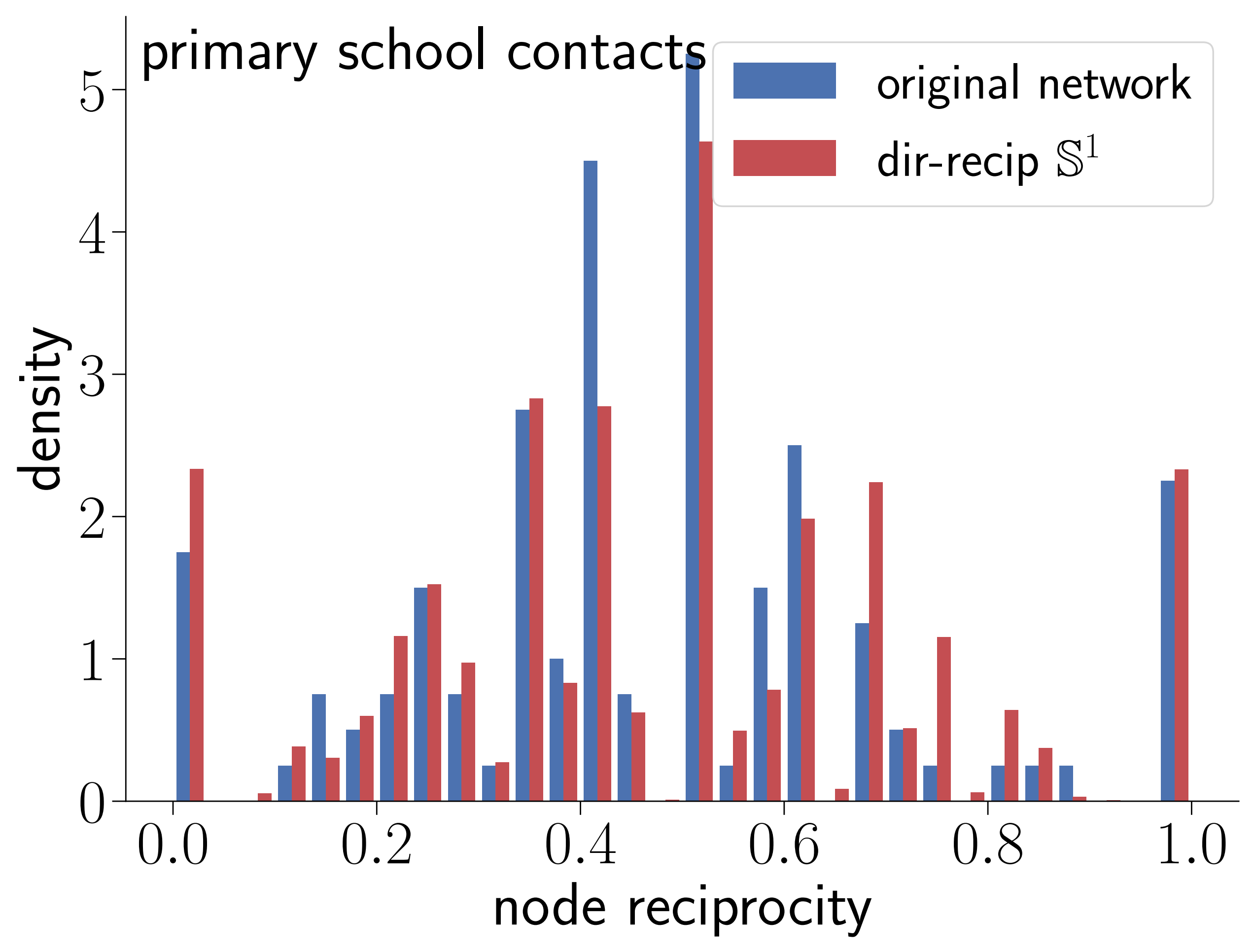
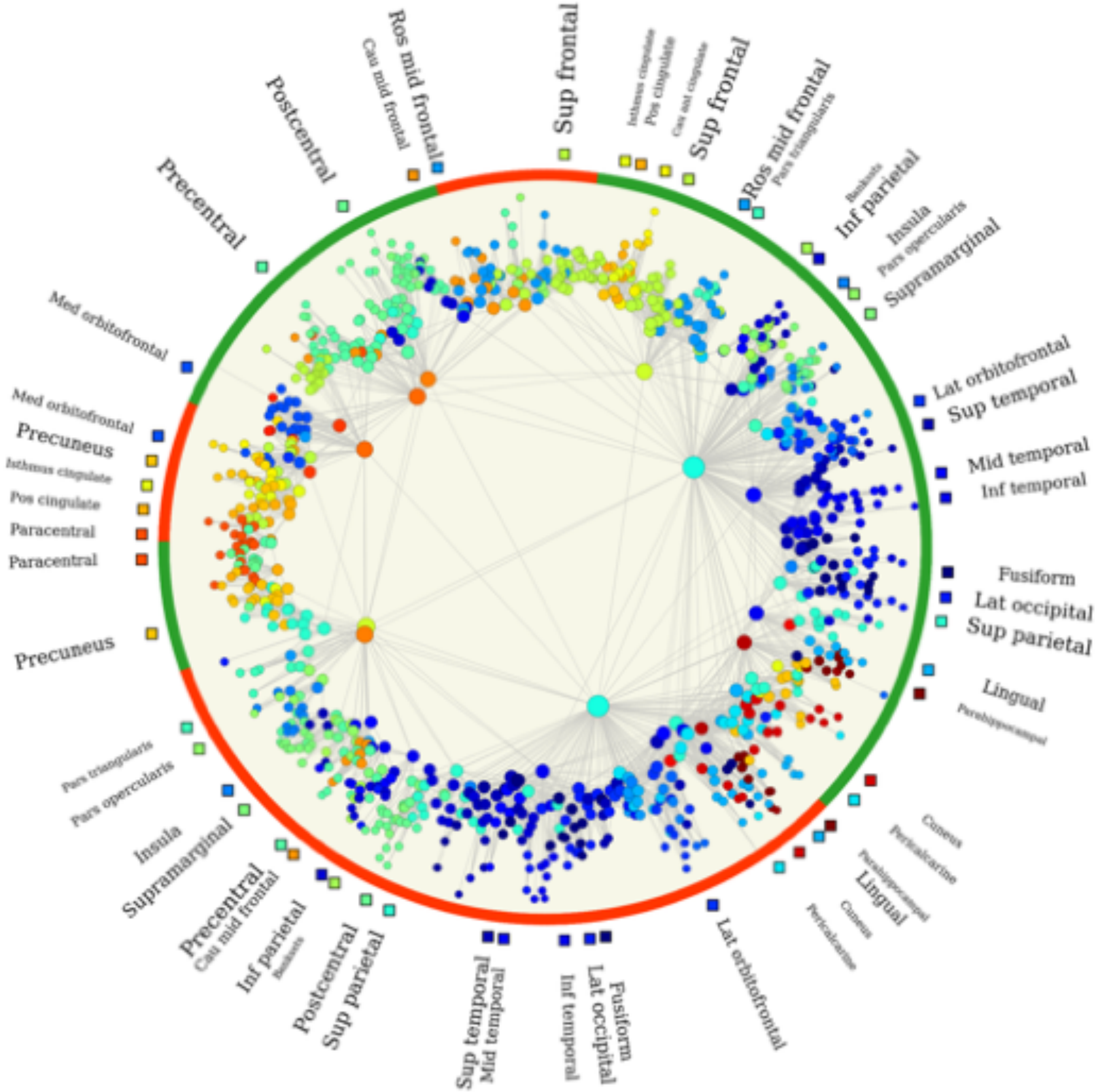

original networks
dir-recip $\mathbb{S}^1$
singular value
index
primary school contacts
0
2
4
6
0
25
50
75
100
125



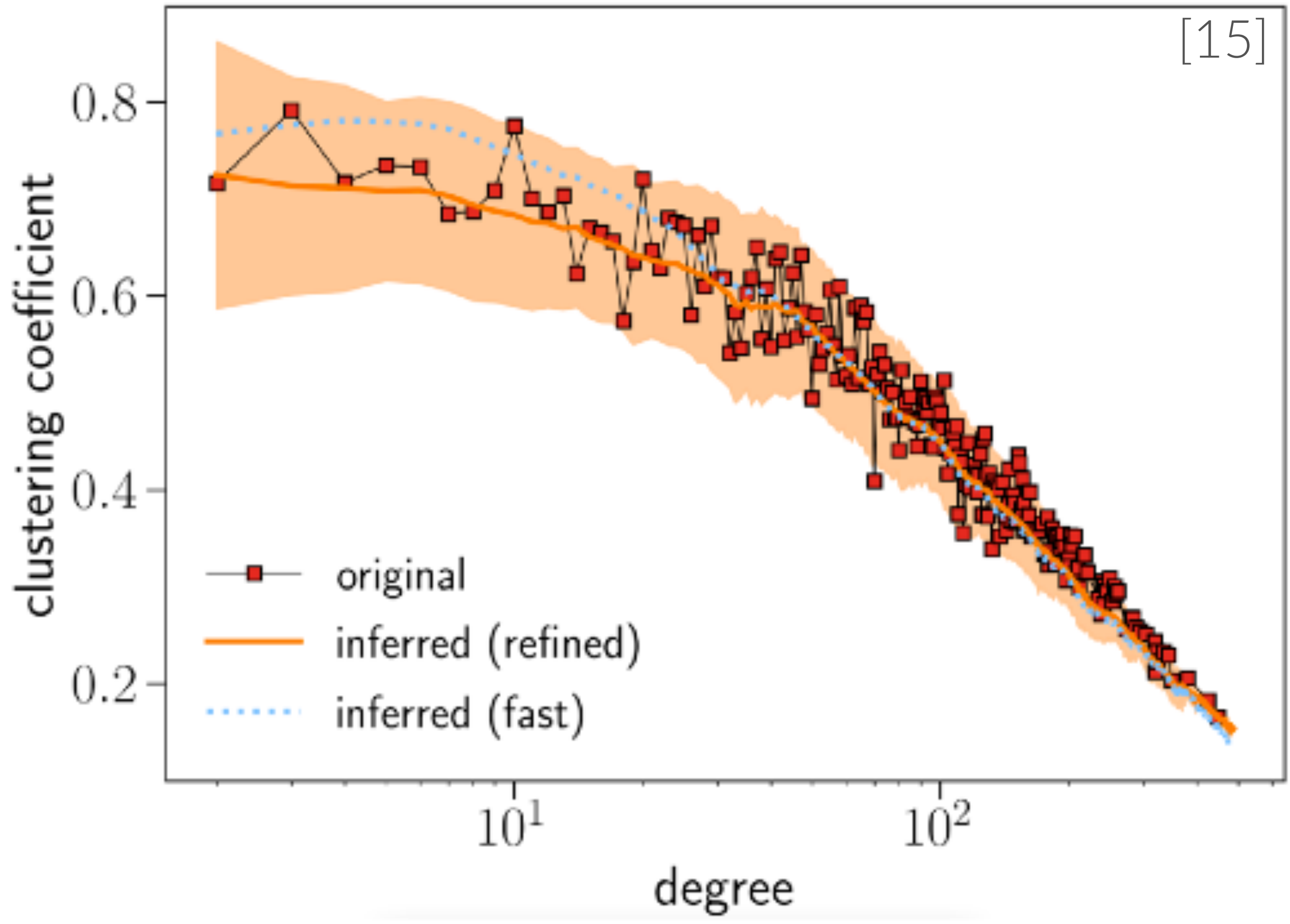
Russia
Kazakhstan
Ukraine
Estonia
Cyprus
Armenia
Georgia
Lithuania
Norway
Sweden
Denmark
Serbia
Macedonia
Finland
Hungary
Romania
Bulgaria
Azerbaijan
Czech Republic
Turkey
Slovakia
Malta
Italy
Bosnia and Herzegovina
Ireland
Spain
Slovenia
Switzerland
Austria
Germany
France
UK
Moldova
Luxembourg
Belgium
Portugal
Israel
Iran
Iceland
Nigeria
The Netherlands
Kenya
Mongolia
Bangladesh
India
Ecuador
South Africa
Colombia
Argentina
Panamá
Nepal
Saudi Arabia
Egypt
Pakistan
Guatemala
Malaysia
Philippines
Indonesia
Singapore
Chile
Hong Kong
Thailand
Vietnam
China
Japan
South Korea
Venezuela
Taiwan
New Zealand
Australia
Greece
México
Canada
Brazil
Puerto Rico
US
Poland
Latvia
0°
30°
60°
90°
120°
150°
180°
210°
240°
270°
300°
330°
TDC
Telefónica
Cogent
GBLX
NTT
Telianet
TPNET
XO
Savvis


primary school contacts
original network
dir-recip $\mathbb{S}^1$
density
node reciprocity
0
1
2
3
4
5
0.0
0.2
0.4
0.6
0.8
1.0



Sup frontal
Ros mid frontal
Cau mid frontal
Postcentral
Precentral
Med orbitofrontal
Med orbitofrontal
Precuneus
Isthmus cingulate
Pos cingulate
Paracentral
Paracentral
Precuneus
Isthmus cingulate
Pos cingulate
Cau ant cingulate
Sup frontal
Ros mid frontal
Pars triangularis
Bankssts
Inf parietal
Insula
Pars opercularis
Supramarginal
Lat orbitofrontal
Sup temporal
Mid temporal
Inf temporal
Fusiform
Lat occipital
Sup parietal
Lingual
Parahippocampal
Cuneus
Pericalcarine
Parahippocampal
Lingual
Cuneus
Pericalcarine
Lat orbitofrontal
Fusiform
Lat occipital
Inf temporal
Sup temporal
Mid temporal
Postcentral
Sup parietal
Inf parietal
Bankssts
Precentral
Cau mid frontal
Supramarginal
Insula
Pars opercularis
Pars triangularis

18



[15]
0.8
0.6
0.4
0.2
clustering coefficient
10^1
10^2
degree
original
inferred (refined)
inferred (fast)

[1] Phys. Rev. E 80, 035101 (2009)

[2] Phys. Rev. E 82, 036106 (2010)

[3] Phys. Rev. Lett. 100, 078701 (2008)

[5] Nat. Commun. 8, 14103 (2017)

[6] Phys. Rev. E 84, 026114 (2011)

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[12] Sci. Rep. 5, 9421 (2015)

[13] J. Stat. Phys. 173, 775 (2018)

[14] New J. Phys. 20, 052002 (2018)

[15] New J. Phys. 21, 123033 (2019)

[16] Nat. Commun. 8, 1615 (2017)

[17] Nat. Commun. 1, 62 (2010)

[18] PNAS 117, 20244 (2020)

A powerful and versatile framework

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- Generalizable to weighted [5], bipartite [6,7,8], multiplex [9,10], directed [4] and growing [11] networks
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- Parsimonious explanation of self-similarity [3]
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  - Self-similar architecture [19]
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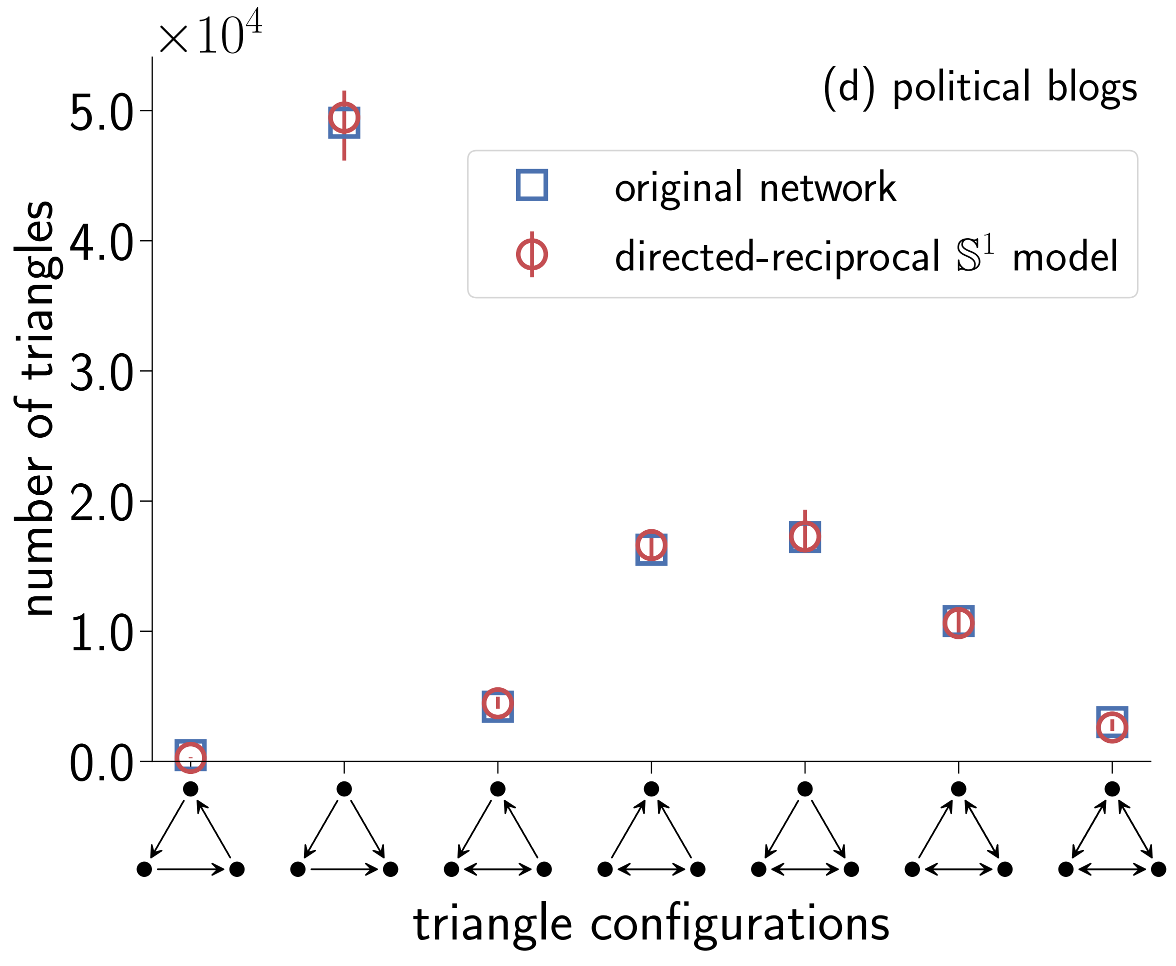
  - ...
- ...

[4] Nat. Phys. 20, 150 (2024)

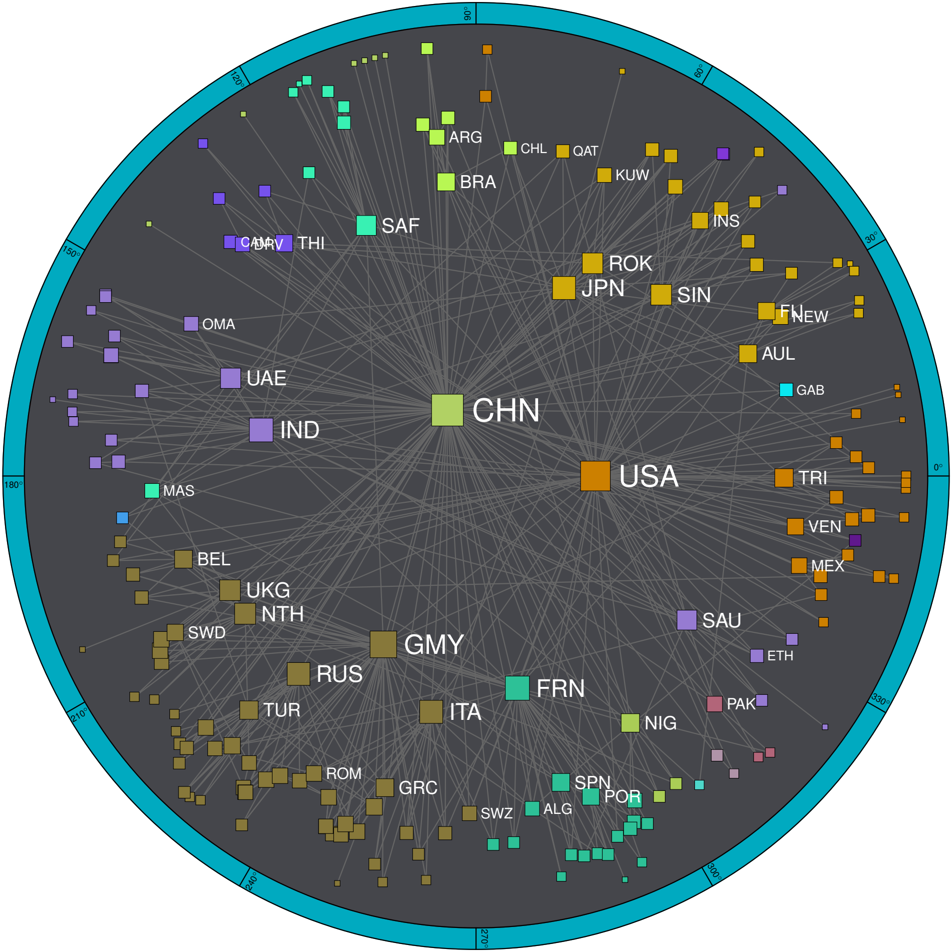
[19] Nat. Phys. 14, 583 (2018)

[20] PLOS Comput. Biol. 16, e1007584 (2020)

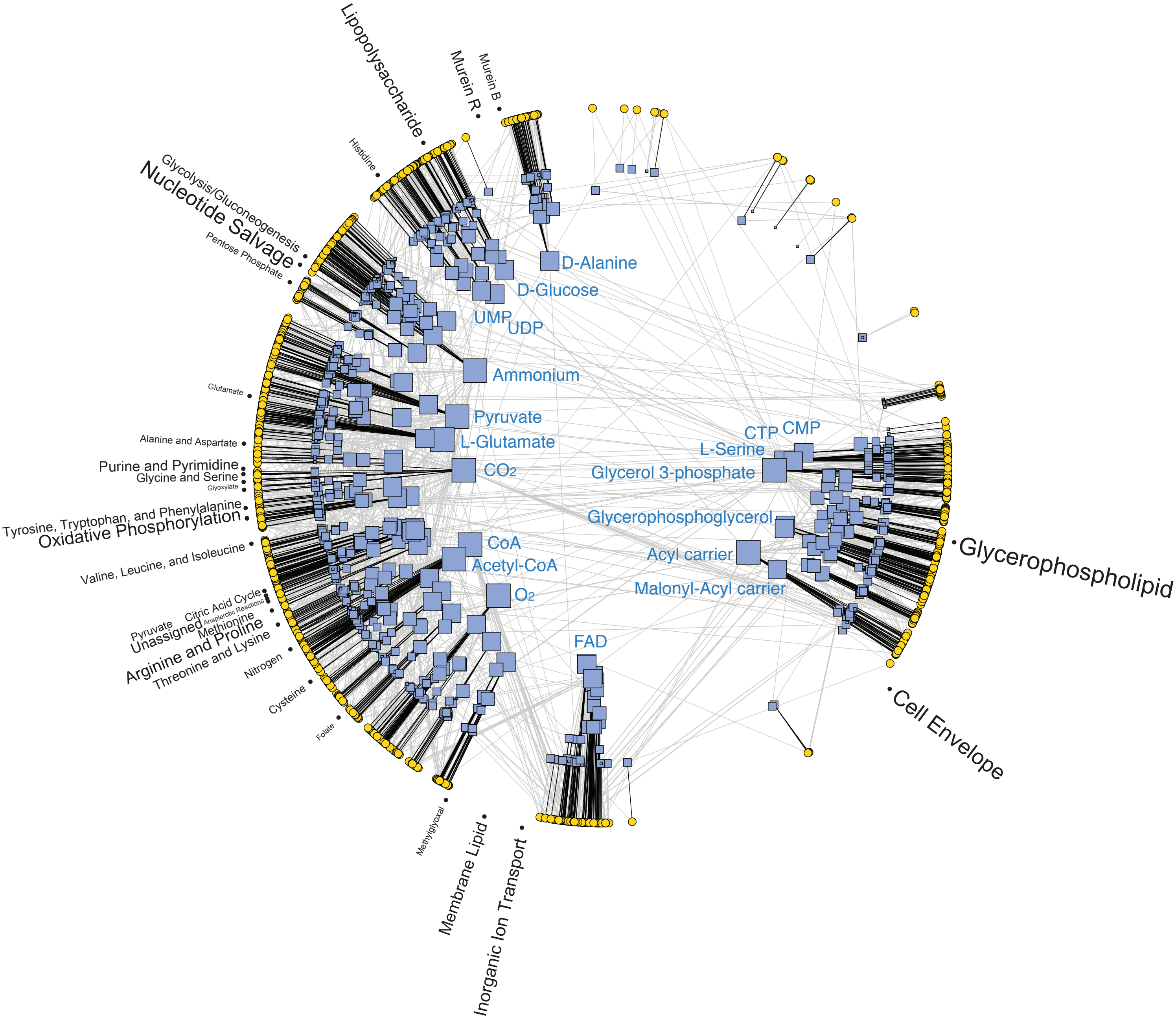
[21] Sci. Rep. 6, 33441 (2016)



$\times 10^4$
(d) political blogs
original network
directed-reciprocal $\mathbb{S}^1$ model
number of triangles
5.0
4.0
3.0
2.0
1.0
0.0
triangle configurations



90°
60°
30°
0°
330°
300°
270°
240°
210°
180°
150°
120°
CHN
USA
GMY
IND
UAE
OMA
THI
CAM
DRV
SAF
ARG
BRA
CHL
QAT
KUW
ROK
JPN
SIN
INS
FIJ
NEW
AUL
GAB
TRI
VEN
MEX
SAU
ETH
PAK
NIG
FRN
SPN
POR
ALG
SWZ
ITA
GRC
ROM
RUS
TUR
SWD
NTH
UKG
BEL
MAS


Lipopolysaccharide
Murein R.
Murein B.
Histidine
Glycolysis/Gluconeogenesis
Nucleotide Salvage
Pentose Phosphate
Glutamate
Alanine and Aspartate
Purine and Pyrimidine
Glycine and Serine
Glyoxylate
Tyrosine, Tryptophan, and Phenylalanine
Oxidative Phosphorylation
Valine, Leucine, and Isoleucine
Citric Acid Cycle
Anaplerotic Reactions
Pyruvate
Methionine
Unassigned
Arginine and Proline
Threonine and Lysine
Nitrogen
Cysteine
Folate
Methylglyoxal
Membrane Lipid
Inorganic Ion Transport
Cell Envelope
Glycerophospholipid
D-Alanine
D-Glucose
UMP
UDP
Ammonium
Pyruvate
L-Glutamate
CO_2
CoA
Acetyl-CoA
O_2
FAD
CTP
CMP
L-Serine
Glycerol 3-phosphate
Glycerophosphoglycerol
Acyl carrier
Malonyl-Acyl carrier

Review Article | Published: 29 January 2021

Network geometry

Marián Boguñá, Ivan Bonamassa, Manlio De Domenico, Shlomo Havlin, Dmitri Krioukov & M. Ángeles Serrano

Nature Reviews Physics **3**, 114–135 (2021)

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Challenges

Heterogeneous random geometric graph models are prime candidate to model real networked complex systems.

But they rely heavily on our capacity to find high quality embeddings of the original datasets.

- ▷ Difficult optimization problem
  - rugged landscape
  - numerous symmetries (rotation, reflection, graph automorphisms)
  - gradient not always well defined
- ▷ Out-of-the-box solutions do not work well
  - Hamiltonian Monte-Carlo
  - gradient descent
- ▷ Current state-of-the-art embedding methods
  - rely on heuristics
  - do not provide uncertainties (loglikelihood maximization)