

Emergence of higher-order interactions

Original Kuramoto-Sakaguchi dynamics

$$\dot{z}_j = i\omega_j z_j + \sum_{k=1}^N W_{jk} [z_k e^{-i\alpha} - z_j^2 \bar{z}_k e^{i\alpha}], \quad j \in \{1, \dots, N\}$$

Reduced dynamics

$$\dot{Z}_\mu = i \sum_{\nu=1}^n \Omega_{\mu\nu} Z_\nu + \sum_{\nu=1}^n \mathcal{W}_{\mu\nu}^{(2)} Z_\nu e^{-i\alpha} - \sum_{\alpha,\beta,\gamma=1}^n \mathcal{W}_{\mu\alpha\beta\gamma}^{(4)} Z_\alpha Z_\beta \bar{Z}_\gamma e^{i\alpha}$$

Original Quenched mean-field SIS dynamics

$$\dot{x}_i = -\alpha x_i + \beta(1 - x_i) \sum_{j=1}^N W_{ij} x_j, \quad i \in \{1, \dots, N\}$$

Reduced dynamics

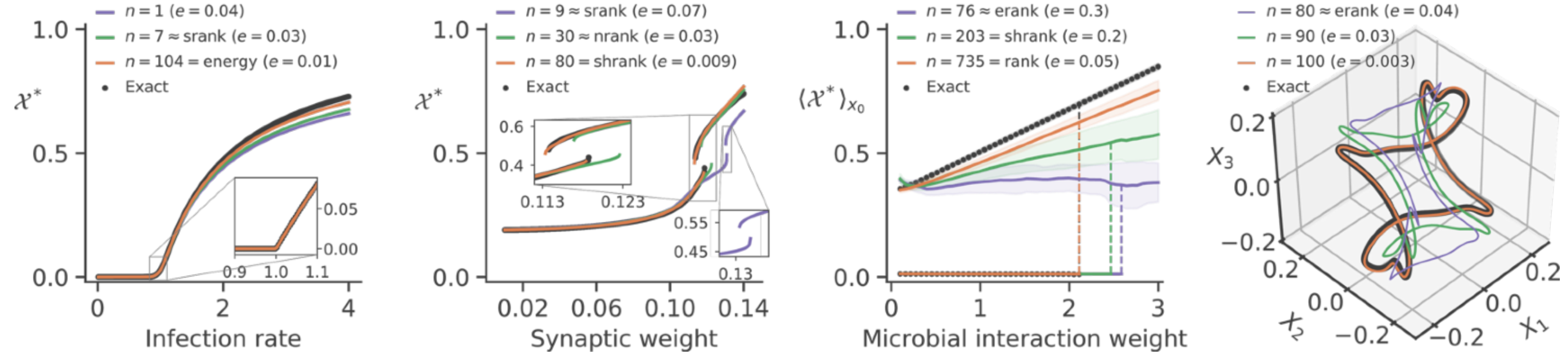
$$\dot{X}_\mu = -\alpha X_\mu + \beta \sum_{\nu=1}^n \mathcal{W}_{\mu\nu} X_\nu - \beta \sum_{\nu, \tau=1}^n \mathcal{W}_{\mu\nu\tau}^{(3)} X_\nu X_\tau, \quad \mu \in \{1, \dots, n\}$$

Reduction of the dynamics with increasing accuracy

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## Reproduction of the dynamics with increasing accuracy



## Emergence of higher-order interactions

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Reduced dynamics

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Reduced dynamics

$$\dot{Z}_\mu = i \sum_{\nu=1}^n \Omega_{\mu\nu} Z_\nu + \sum_{\nu=1}^n \mathcal{W}_{\mu\nu}^{(2)} Z_\nu e^{-i\alpha} - \sum_{\alpha, \beta, \gamma=1}^n \mathcal{W}_{\mu\alpha\beta\gamma}^{(4)} Z_\alpha Z_\beta \bar{Z}_\gamma e^{i\alpha}$$



# Main takeaways

- ▷ The rapid decrease of the singular values of adjacency matrices (i.e. low effective rank) offers a *justification for low-dimensional mathematical models* beyond mathematical and/or conceptual convenience.
- ▷ A *large proportion of real networks* can be considered as having a low effective rank.
- ▷ The *higher-order interactions* observed in some systems could be a *byproduct of a low-dimensional representation* used to analyze them.

# Challenges and open questions

- ▷ Could we *measure* the effective dimension *independently*?
- ▷ Could we design a *random graph model* based on *observed singular values* (singular vectors)?
- ▷ Are some of the *higher-order interactions* inferred from time series *artifacts of coarse-grained observations*?
- ▷ Could we designed more *interpretable observables*, perhaps nonlinear ones?



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