# Exploring the hidden metric space of complex networks

## — overview of the principal models and results —

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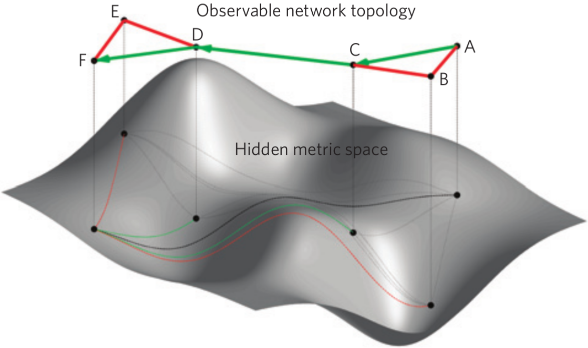
# Motivation and outline

We conjecture that complex networks are embedded in a metric space in which the distances between nodes represent intrinsic similarities and determine whether two nodes are connected or not.

- How can we model networks embedded in a hidden metric space ?
- What effect has a hidden metric space on the properties of the networks ?
- Can we infer a plausible hidden metric space for real networks ?




## Outline

- Random networks with hidden metric space
- Self-similarity
- Hyperbolic geometry
- Navigability
- Mapping of real networks

# Random networks with hidden metric space

- N nodes distributed in homogeneous and isotropic $D$-dimensional space
- Each node is assigned a hidden variable $\kappa$

$$\rho(\kappa) = (\gamma - 1)\kappa_0^{\gamma-1}\kappa^{-\gamma}$$

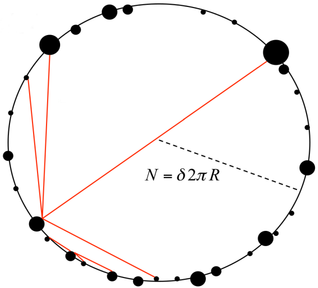
- Nodes are connected with probability $p(\chi)$ where

$$\chi = \frac{d}{d_c(\kappa, \kappa')} \propto \frac{d}{(\kappa\kappa')^{1/D}}$$

- If $p(\chi)$ is integrable over $\chi \in [0, \infty)$

$$\langle k(\kappa) \rangle \propto \kappa \qquad \Rightarrow \qquad P(k) \sim k^{-\gamma}$$

- Small-world: high degree nodes likely to be connected even at long distance.
- Triangle inequality implies strong clustering controlled by the specifics of $p(\chi)$.

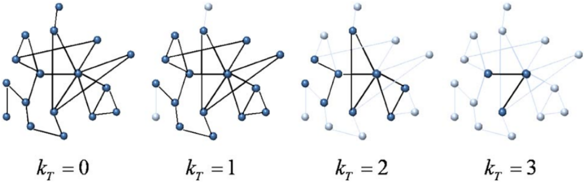


# Self-similarity

- Self-similarity is not well-defined in a proper geometric sense since most real networks are not explicitly embedded in any physical space.

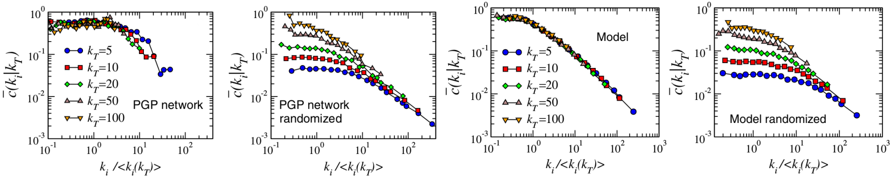
- How does being embedded in a metric space affect the self-similarity?

- Degree-thresholding renormalization
  - remove nodes with degree $k < k_T = 0, 1, 2, \ldots$
  - compute internal degree, $k_i$, and rescale according to $k_i / \langle k_i(k_T) \rangle$
  - generates a sequence of subnetworks $G(k_T)$ from a given network $G$



# Self-similarity

- In many real networks, the degree distribution, the degree-degree correlations and the degree-dependent clustering coefficient, $\bar{c}(k)$, are self-similar under the degree-thresholding renormalization.
- Only the degree distribution and the degree-degree correlations are self-similar for the randomized networks.




- Hidden metric space offers a natural explanation for the degree-thresholding self-similarity.
- Clustering could be a topological reflection of the triangle inequality in the hidden metric space.

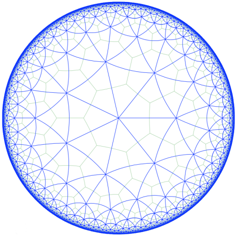
# Hyperbolic geometry

- Effective geometry of scale-free topology (consequence of the negative curvature)
- Exponential expansion of space ("continuous tree")
- Mapping unto the hyperbolic disk

$$\kappa = \kappa_0 \mathrm{e}^{\zeta(R-r)/2}; \quad \chi = \mathrm{e}^{\zeta(x-R)/2}$$

$\zeta$ : curvature parameter ; $x$ : hyperbolic distance

$R$ : radius of the disk ; $r$ : radial coordinate



- The choice $p(\chi) = 1/(1+\chi^{\beta})$ casts the model into the exponential random networks family (maximal randomness given the degree distribution).
- Level of clustering controlled via $\beta$ (coupling between the hidden metric space and the topology of the network).
- Configuration Model and Erdős-Rényi networks are limiting cases.

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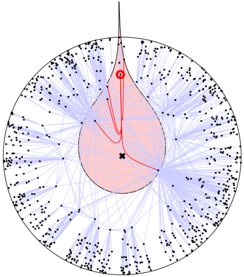

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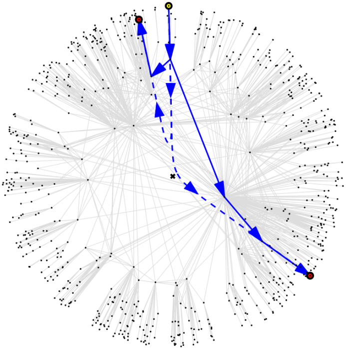
# Navigability

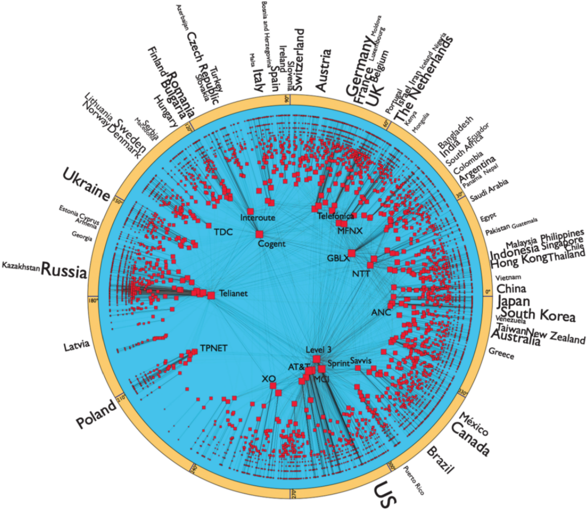
- Finding shortest paths requires a global knowledge of the network.
- Shortest paths are congruent with geodesics in hyperbolic plane if
  - power-law degree distribution
  - strong clustering.
- Greedy routine: forward information to the neighbor the closest to the target.



- Optimal navigation without global information (greedy paths are the geodesic paths when $N \gg 1$).
- Structural conditions found in real complex networks.

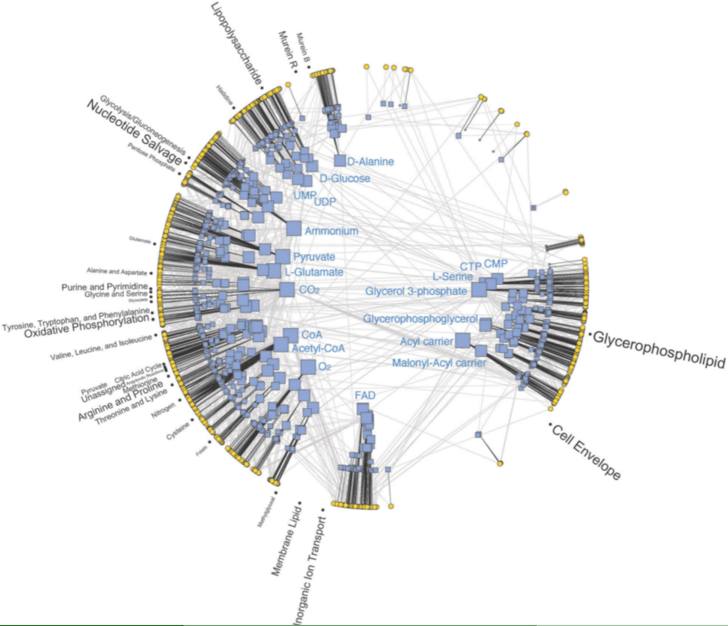
# Mapping of real networks

- The mapping of the Internet at the Autonomous Systems level allows to address the scalability limitations of today's Internet routing architecture.



# Mapping of real networks

- The mapping of E. coli's metabolism allows to identify biochemical pathways.



# Partial bibliography


– M. Á. Serrano, D. Krioukov, and M. Boguñá, *Self-similarity of complex networks and hidden metric spaces*, Phys. Rev. Lett. **100**, 078701 (2008)
– M. Boguñá, D. Krioukov, and K. C. Claffy *Navigability of complex networks*, Nat. Phys. **5**, 74 (2008)
– M. Boguñá and D. Krioukov, *Navigating ultrasmall worlds in ultrashort time*, Phys. Rev. Lett. **102**, 058701 (2009)
– D. Krioukov, F. Papadopoulos, A. Vahdat, and M. Boguñá, *Curvature and temperature of complex networks*, Phys. Rev. E **80**, 035101(R) (2009)
– M. Boguñá, F. Papadopoulos, and D. Krioukov, *Sustaining the Internet with hyperbolic mapping*, Nat. Commun. **1**, 62 (2010)
– D. Krioukov, F. Papadopoulos, M. Kitsak, A. M. Vahdat, and M. Boguñá, *Hyperbolic geometry of complex networks*, Phys. Rev. E **82**, 036106 (2010)
– F. Papadopoulos, M. Kitsak, M. Á. Serrano, M. Boguñá, and D. Krioukov, *Popularity versus similarity in growing networks*, Nature **489**, 537 (2012)
– M. Á. Serrano, M. Boguñá, and F. Sagués, *Uncovering the hidden geometry behind metabolic networks*, Mol. Biosyst. **8**, 843 (2012)
– D. Krioukov, M. Kitsak, R. S. Sinkovits, D. Rideout, D. Meyer, and M. Boguñá *Network cosmology*, Sci. Rep. **2**, 793 (2012)
– D. Krioukov, M. Kitsak, and M. Boguñá *Cosmological networks*, New J. Phys. **16**, 093031 (2014)
– K. Zuev, M. Boguñá, G. Bianconi, and D. Krioukov, *Emergence of soft communities from geometric preferential attachment*, to appear in Sci. Rep. (2015)
– A. Allard, G. García-Pérez, M. Á. Serrano, and M. Boguñá, *Weighted networks in hidden metric spaces*, to be submitted. (2015)