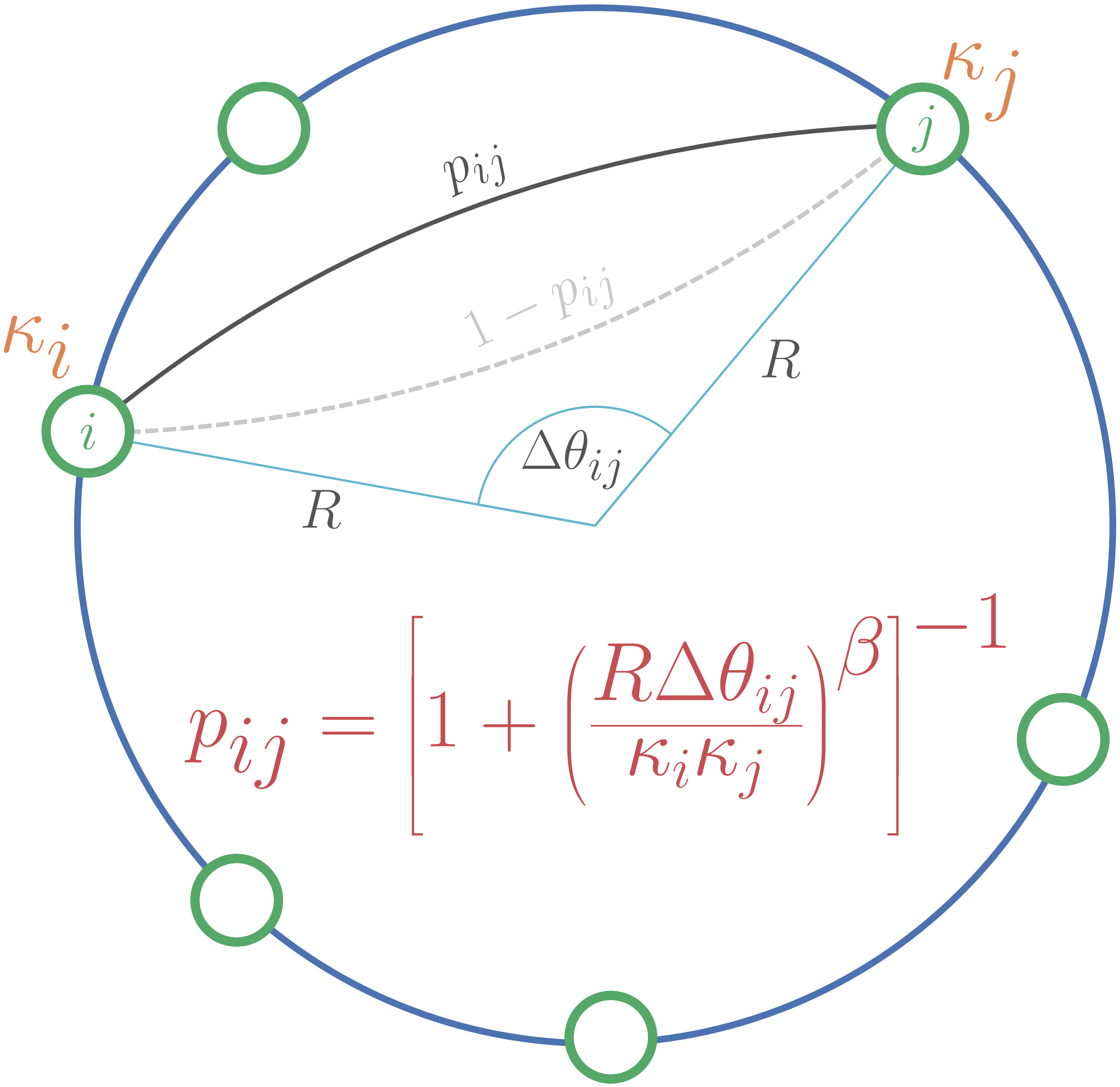
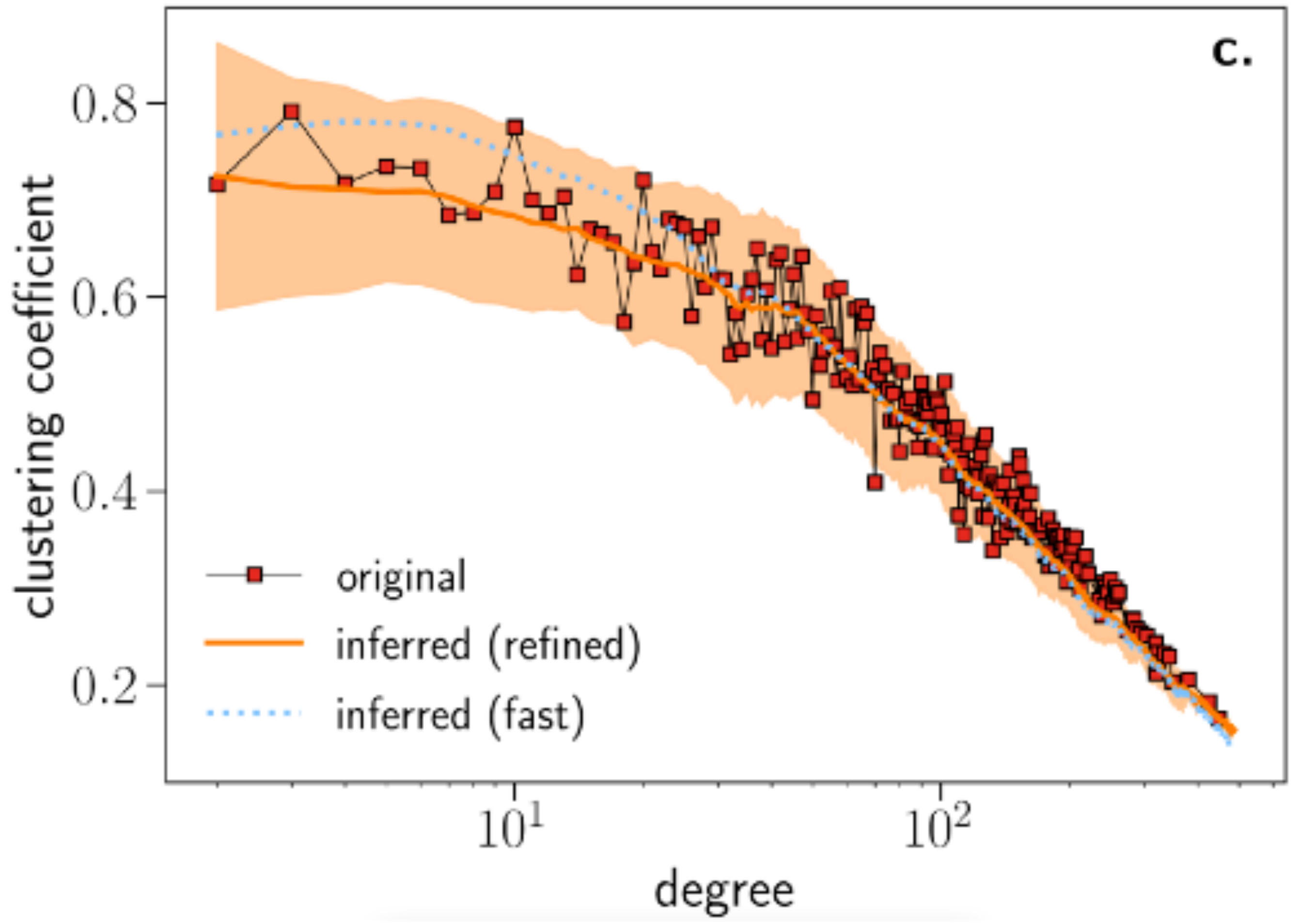

κ_j
j
p_{ij}
$1 - p_{ij}$
κ_i
R
i
$\Delta\theta_{ij}$
R
$p_{ij} = \left[1 + \left(\frac{R\Delta\theta_{ij}}{\kappa_i\kappa_j}\right)^{\beta}\right]^{-1}$



c.
0.8
0.6
0.4
0.2
clustering coefficient
original
inferred (refined)
inferred (fast)
10^1
10^2
degree

[15]

The $\mathbb{S}^1$ model

1. Sprinkle N nodes uniformly on a circle of radius R.
2. Assign an expected degree κ to each node according to some pdf $\rho(\kappa)$.
3. Draw a link between node i and node j with probability p_{ij}.

★ fixes the expected degree of nodes (κ) $\rightarrow$ soft configuration model (CM)

★ triangle inequality of the underlying metric space $\rightarrow$ triangles from pairwise interactions

★ level of clustering tuned with parameter β

Other properties and generalizations

- Amenable to many analytical calculations
- Geometric interpretation in terms of hyperbolic geometry (the $\mathbb{H}^2$ model) [1,2]
- Parsimonious explanation of self-similarity [3,4]
- Generalizable to weighted [5], bipartite [6,7,8], multiplex [9,10] and growing [11] networks
- Generalizable to networks with community structure [12,13,14]
- Mapping of real complex networks unto hyperbolic space [15,16]
- Identification of biochemical pathways in E. Coli [8]
- Efficient Internet routing protocols [17]
- Multiscale organization of the human connectome [18]
- Geometrical interpretation of preferential attachment [11]
- ...

[1] Phys. Rev. E 80, 035101 (2009)

[2] Phys. Rev. E 82, 036106 (2010)

[3] Phys. Rev. Lett. 100, 078701 (2008)

[4] Nat. Rev. Phys. 3, 114 (2021)

[5] Nat. Commun. 8, 14103 (2017)

[6] Phys. Rev. E 84, 026114 (2011)

[7] Phys. Rev. E 95, 032309 (2017)

[8] Mol. Biosyst. 8, 843 (2012)

[9] Nat. Phys. 12, 1076 (2016)

[10] Phys. Rev. Lett. 118, 218301 (2017)

[11] Nature 489, 537 (2012)

[12] Sci. Rep. 5, 9421 (2015)

[13] J. Stat. Phys. 173, 775 (2018)

[14] New J. Phys. 20, 052002 (2018)

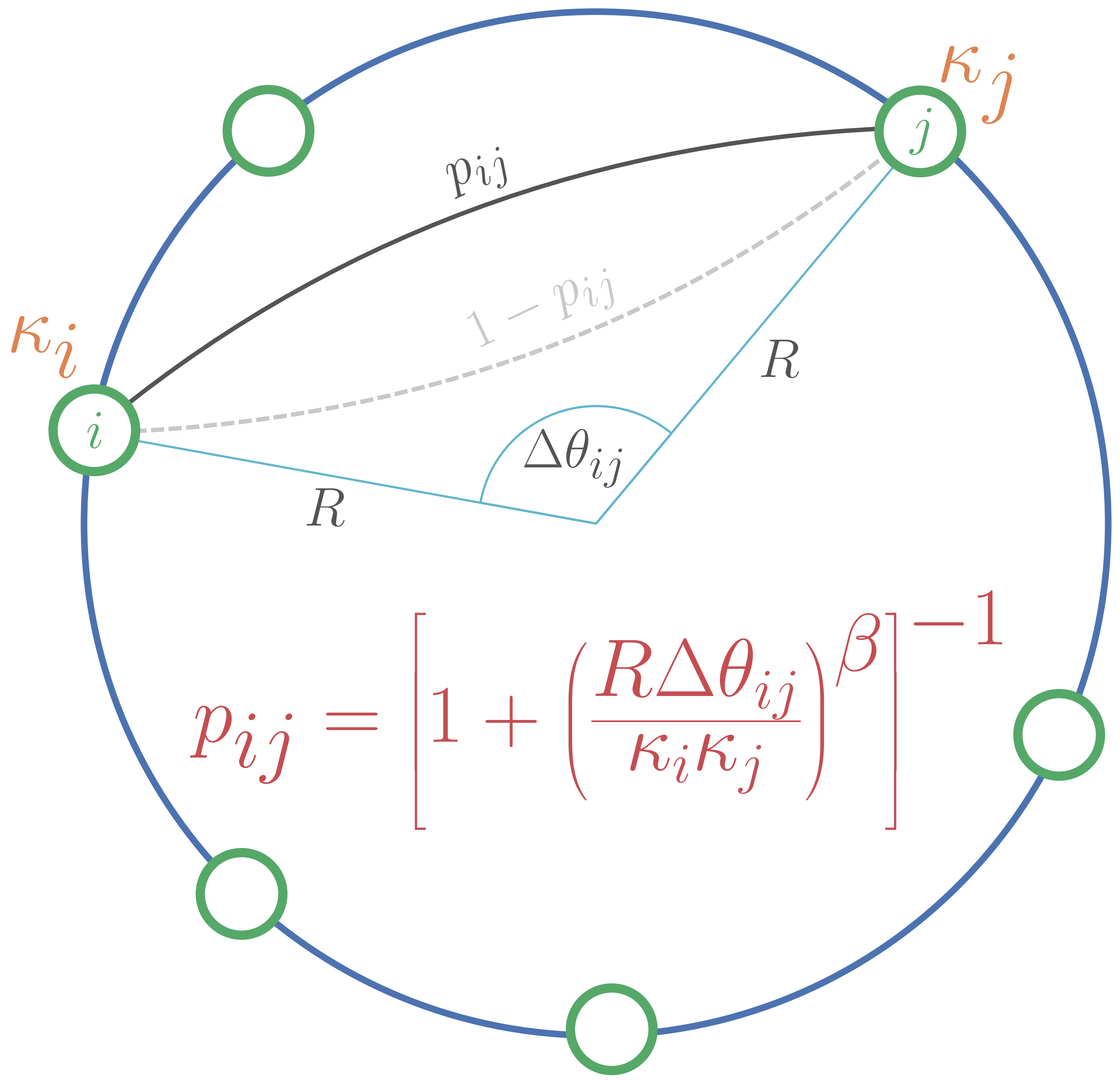
[15] New J. Phys. 21, 123033 (2019)

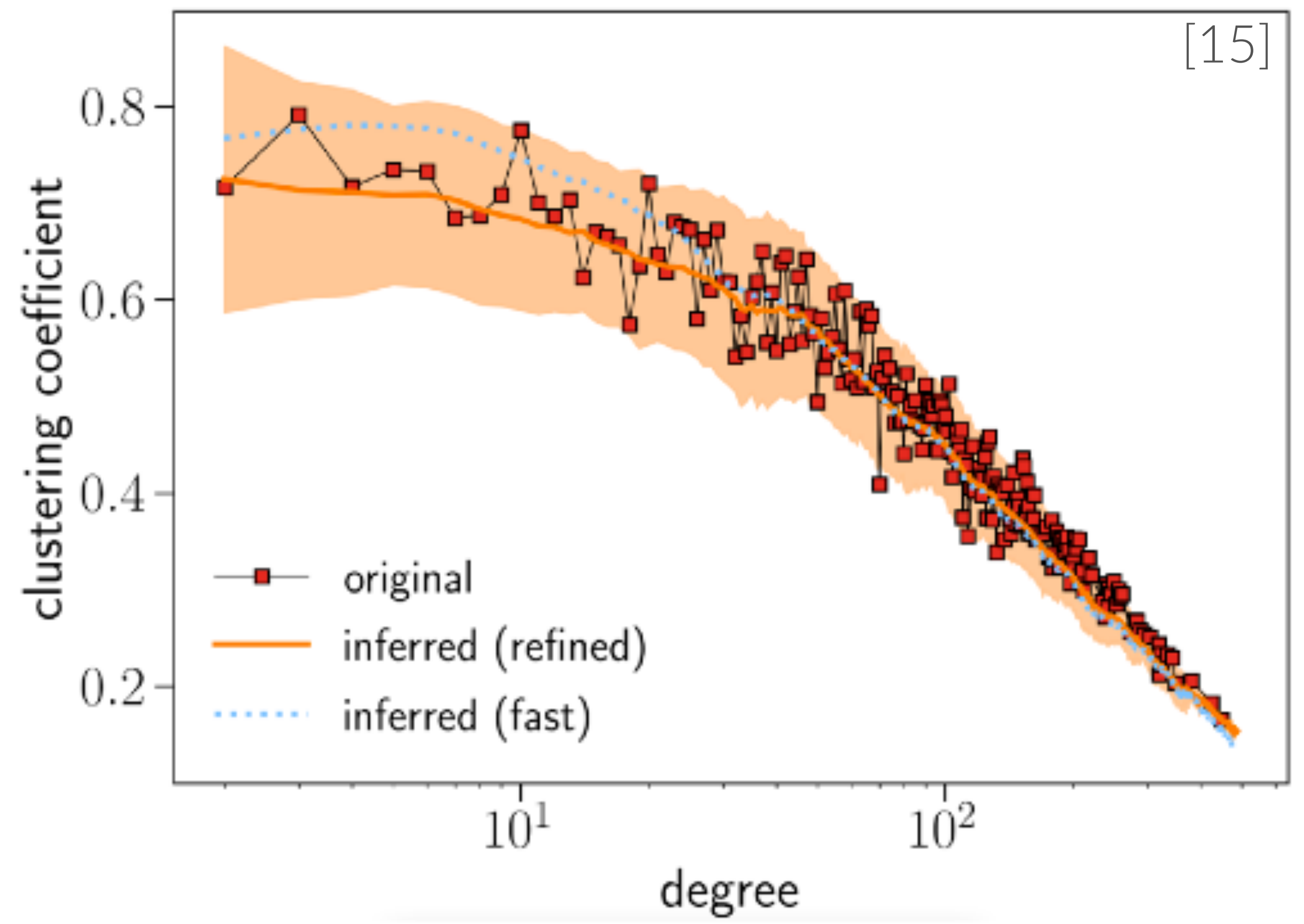
[16] Nat. Commun. 8, 1615 (2017)

[17] Nat. Commun. 1, 62 (2010)

[18] PNAS 117, 20244 (2020)

A geometric approach to clustering: the $\mathbb{S}^1/\mathbb{H}^2$ model





The $\mathbb{S}^1$ model

1. Sprinkle N nodes uniformly on a circle of radius R.
2. Assign an expected degree κ to each node according to some pdf $\rho(\kappa)$.
3. Draw a link between node i and node j with probability p_{ij}.

★ fixes the expected degree of nodes (κ) $\rightarrow$ soft configuration model (CM)

★ triangle inequality of the underlying metric space $\rightarrow$ triangles from pairwise interactions

★ level of clustering tuned with parameter β

Other properties and generalizations

▷ Amenable to many analytical calculations

▷ Geometric interpretation in terms of hyperbolic geometry (the $\mathbb{H}^2$ model) [1,2]

▷ Parsimonious explanation of self-similarity [3,4]

▷ Generalizable to weighted [5], bipartite [6,7,8], multiplex [9,10] and growing [11] networks

▷ Generalizable to networks with community structure [12,13,14]

▷ Mapping of real complex networks unto hyperbolic space [15,16]

▷ Identification of biochemical pathways in E. Coli [8]

▷ Efficient Internet routing protocols [17]

▷ Multiscale organization of the human connectome [18]

▷ Geometrical interpretation of preferential attachment [11]

▷ . . .

[1] Phys. Rev. E 80, 035101 (2009)
[2] Phys. Rev. E 82, 036106 (2010)
[3] Phys. Rev. Lett. 100, 078701 (2008)
[4] Nat. Rev. Phys. 3, 114 (2021)
[5] Nat. Commun. 8, 14103 (2017)
[6] Phys. Rev. E 84, 026114 (2011)
[7] Phys. Rev. E 95, 032309 (2017)
[8] Mol. Biosyst. 8, 843 (2012)
[9] Nat. Phys. 12, 1076 (2016)
[10] Phys. Rev. Lett. 118, 218301 (2017)
[11] Nature 489, 537 (2012)
[12] Sci. Rep. 5, 9421 (2015)
[13] J. Stat. Phys. 173, 775 (2018)
[14] New J. Phys. 20, 052002 (2018)
[15] New J. Phys. 21, 123033 (2019)
[16] Nat. Commun. 8, 1615 (2017)
[17] Nat. Commun. 1, 62 (2010)
[18] PNAS 117, 20244 (2020)

Three challenges in modeling directed networks

Properties of any metric space

Identity of indiscernibles	$d(x, y) = 0 \quad \Leftrightarrow \quad x = y$
Non-negativity	$d(x, y) \geq 0$
Symmetry	$d(x, y) = d(y, x)$
Triangle inequality	$d(x, y) \leq d(x, z) + d(z, y)$