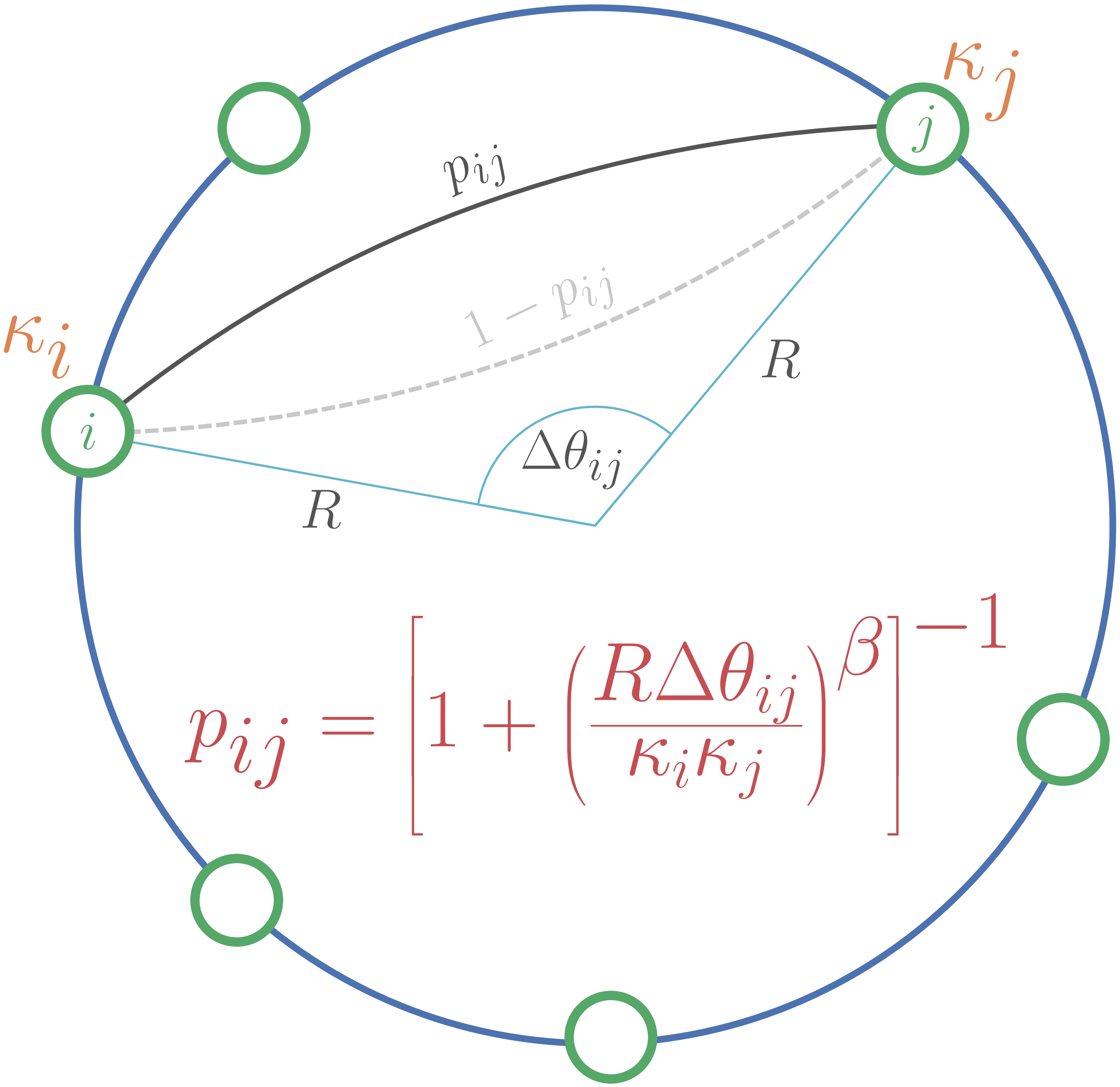
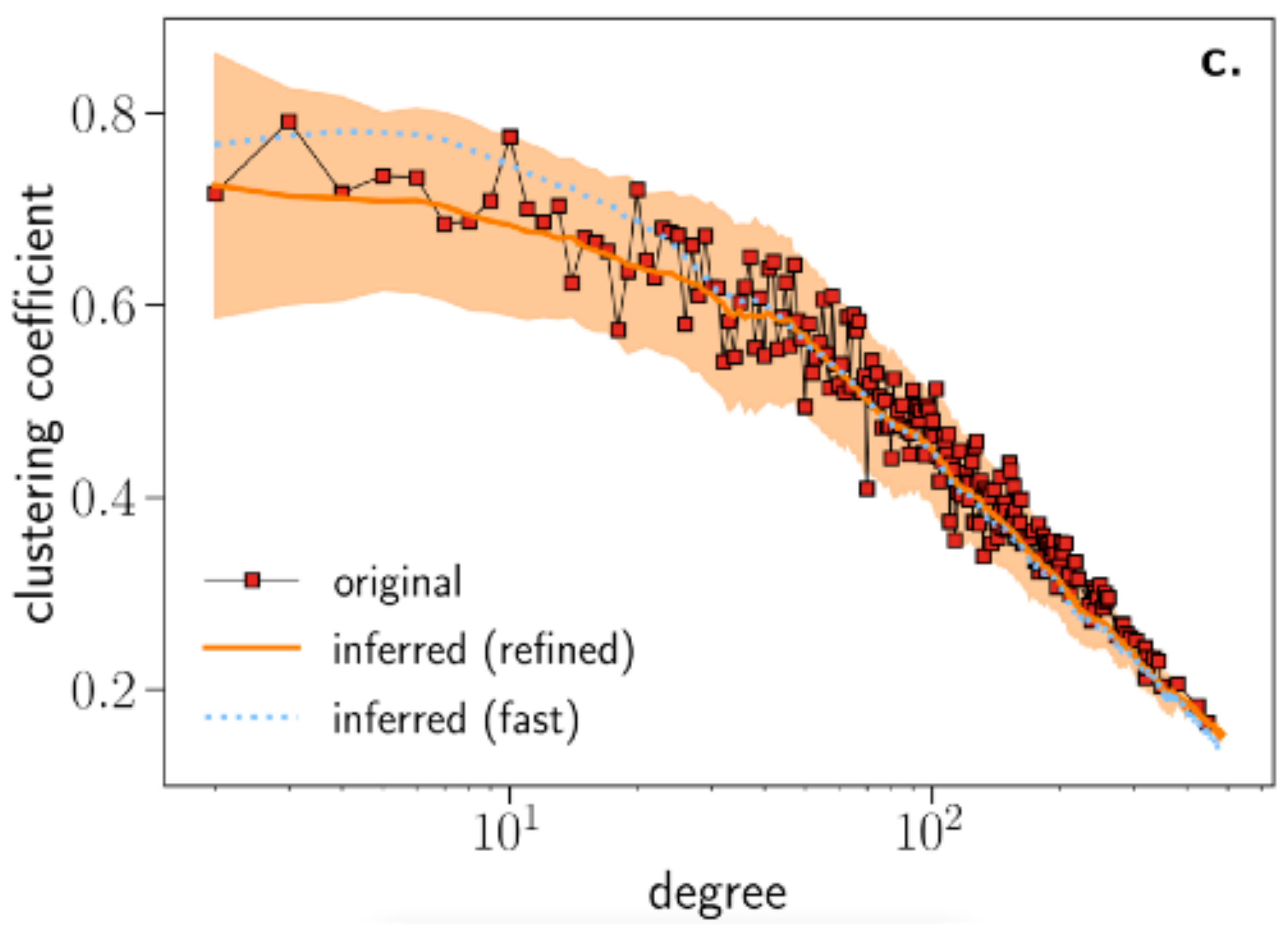
7


κ_j
j
p_{ij}
$1 - p_{ij}$
κ_i
i
R
$\Delta\theta_{ij}$
R
$p_{ij} = \left[1 + \left(\frac{R\Delta\theta_{ij}}{\kappa_i\kappa_j}\right)^{\beta}\right]^{-1}$



c.
0.8
0.6
0.4
0.2
clustering coefficient
original
inferred (refined)
inferred (fast)
10^1
10^2
degree

[15]

The $\mathbb{S}^1$ model

1. Sprinkle N nodes uniformly on a circle of radius R.
2. Assign an expected degree κ to each node according to some pdf $\rho(\kappa)$.
3. Draw a link between node i and node j with probability p_{ij}.

★ fixes the expected degree of nodes (κ) $\rightarrow$ soft configuration model (CM)

★ triangle inequality of the underlying metric space $\rightarrow$ triangles from pairwise interactions

★ level of clustering tuned with parameter β

Other properties and generalizations

- ▷ Amenable to many analytical calculations
- ▷ Geometric interpretation in terms of hyperbolic geometry (the $\mathbb{H}^2$ model) [1,2]
- ▷ Parsimonious explanation of self-similarity [3,4]
- ▷ Generalizable to weighted [5], bipartite [6,7,8], multiplex [9,10] and growing [11] networks
- ▷ Generalizable to networks with community structure [12,13,14]
- ▷ Mapping of real complex networks unto hyperbolic space [15,16]
- ▷ Identification of biochemical pathways in E. Coli [8]
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[2] Phys. Rev. E 82, 036106 (2010)

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[4] Nat. Rev. Phys. 3, 114 (2021)

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[6] Phys. Rev. E 84, 026114 (2011)

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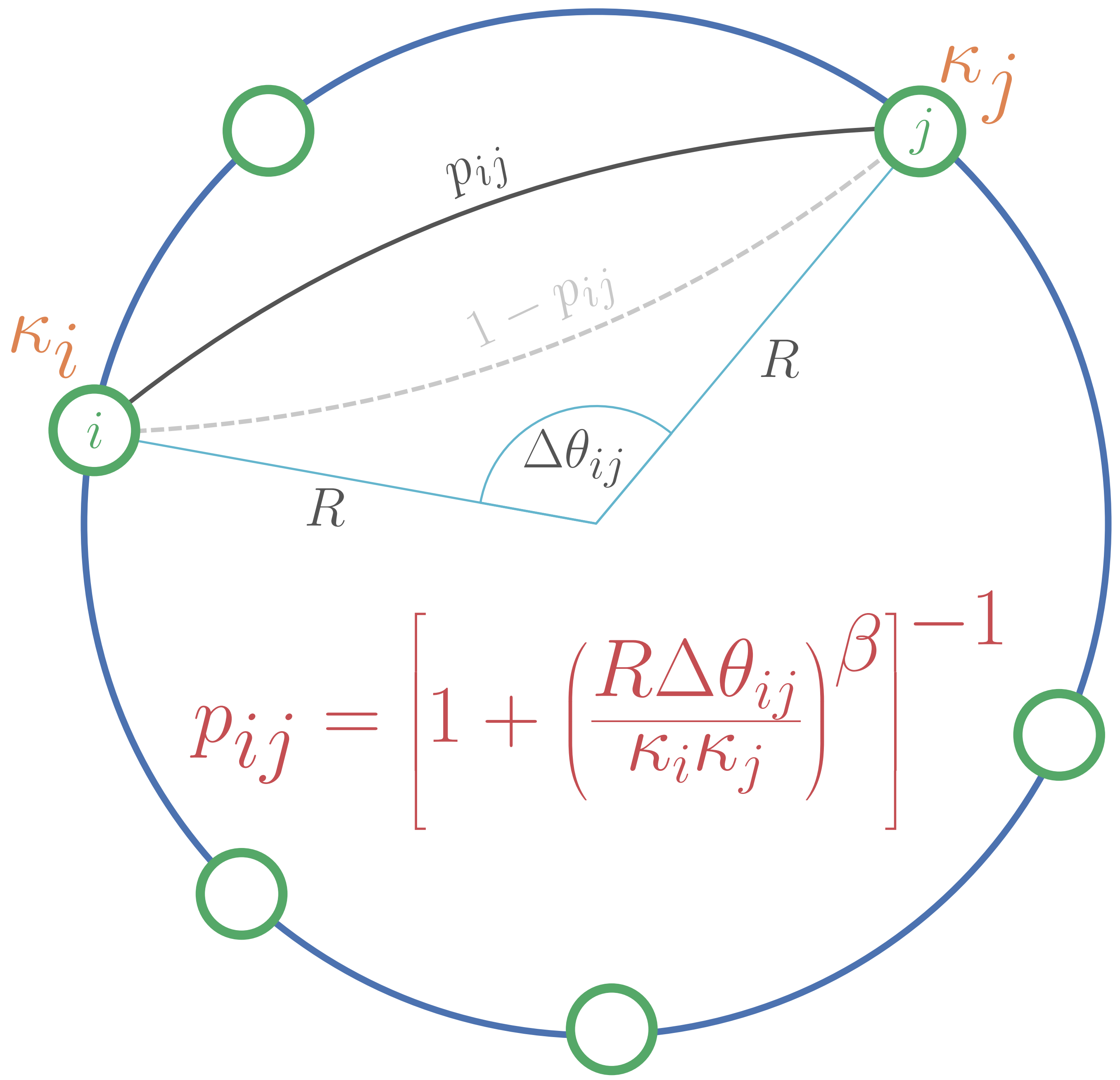
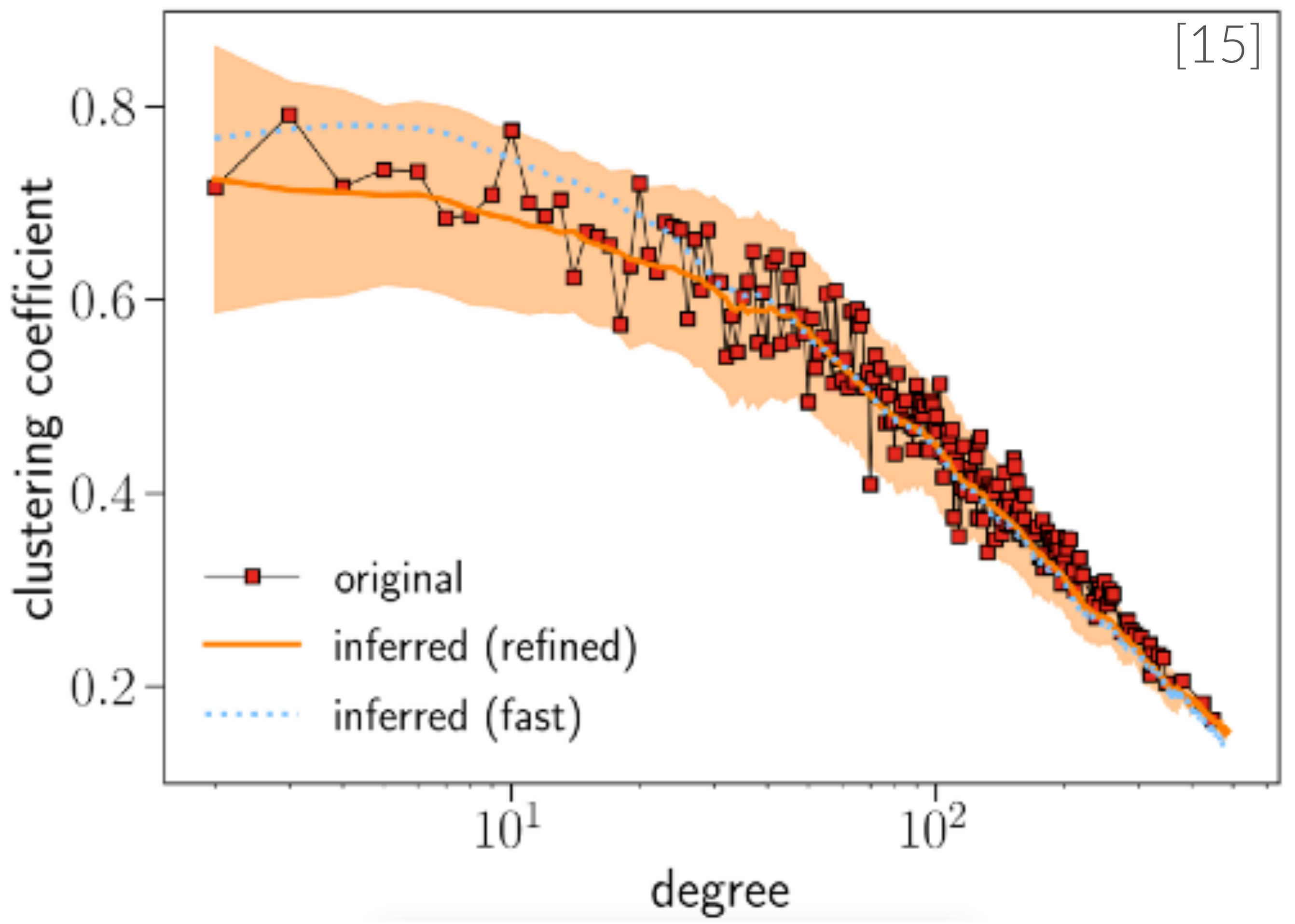
[17] Nat. Commun. 1, 62 (2010)

[18] PNAS 117, 20244 (2020)

A geometric approach to clustering : the $\mathbb{S}^1/\mathbb{H}^2$ model





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Three challenges in modeling directed networks

Properties of any metric space

Identity of indiscernibles	$d(x, y) = 0 \quad \Leftrightarrow \quad x = y$
Non-negativity	$d(x, y) \geq 0$
Symmetry	$d(x, y) = d(y, x)$
Triangle inequality	$d(x, y) \leq d(x, z) + d(z, y)$