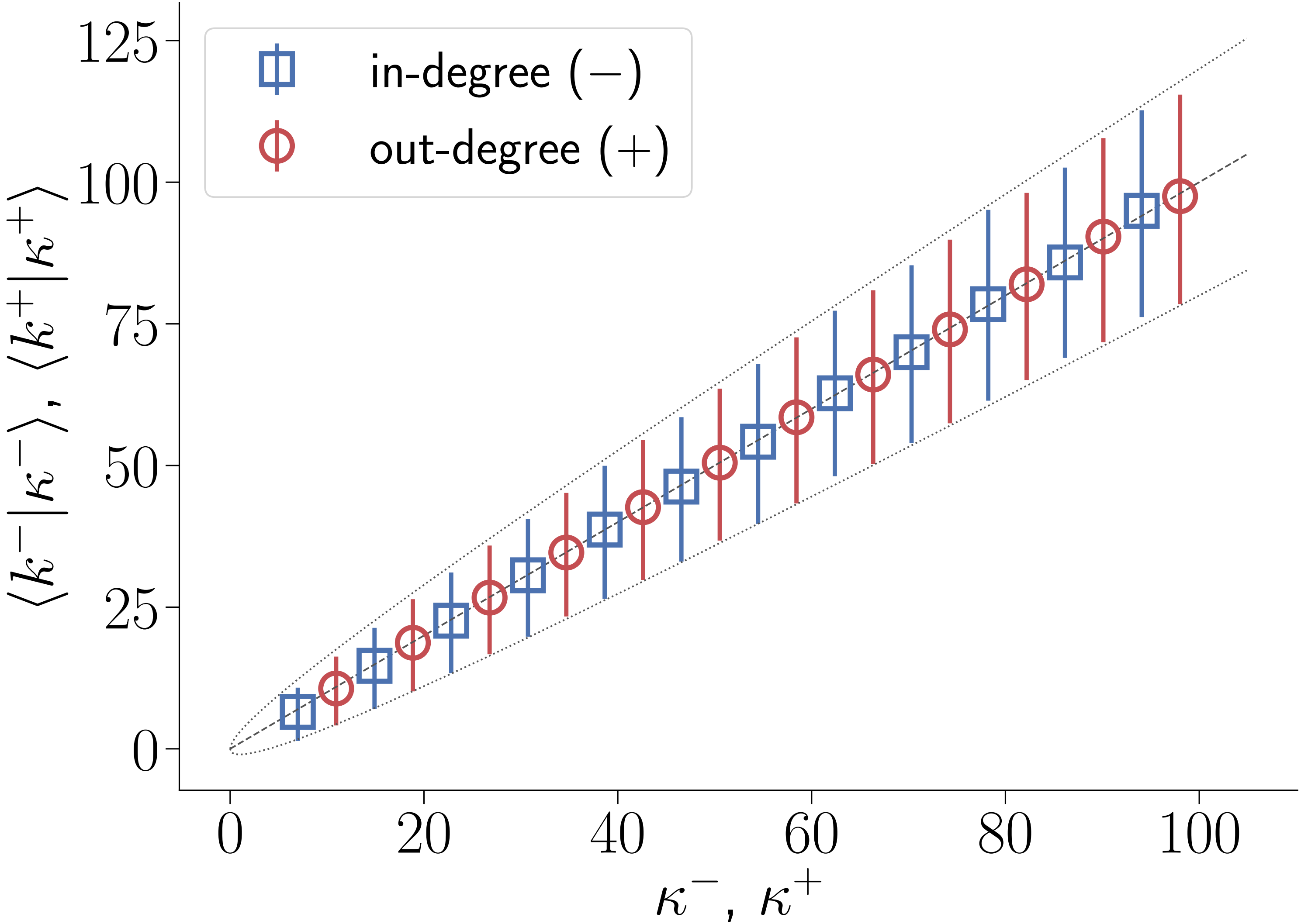
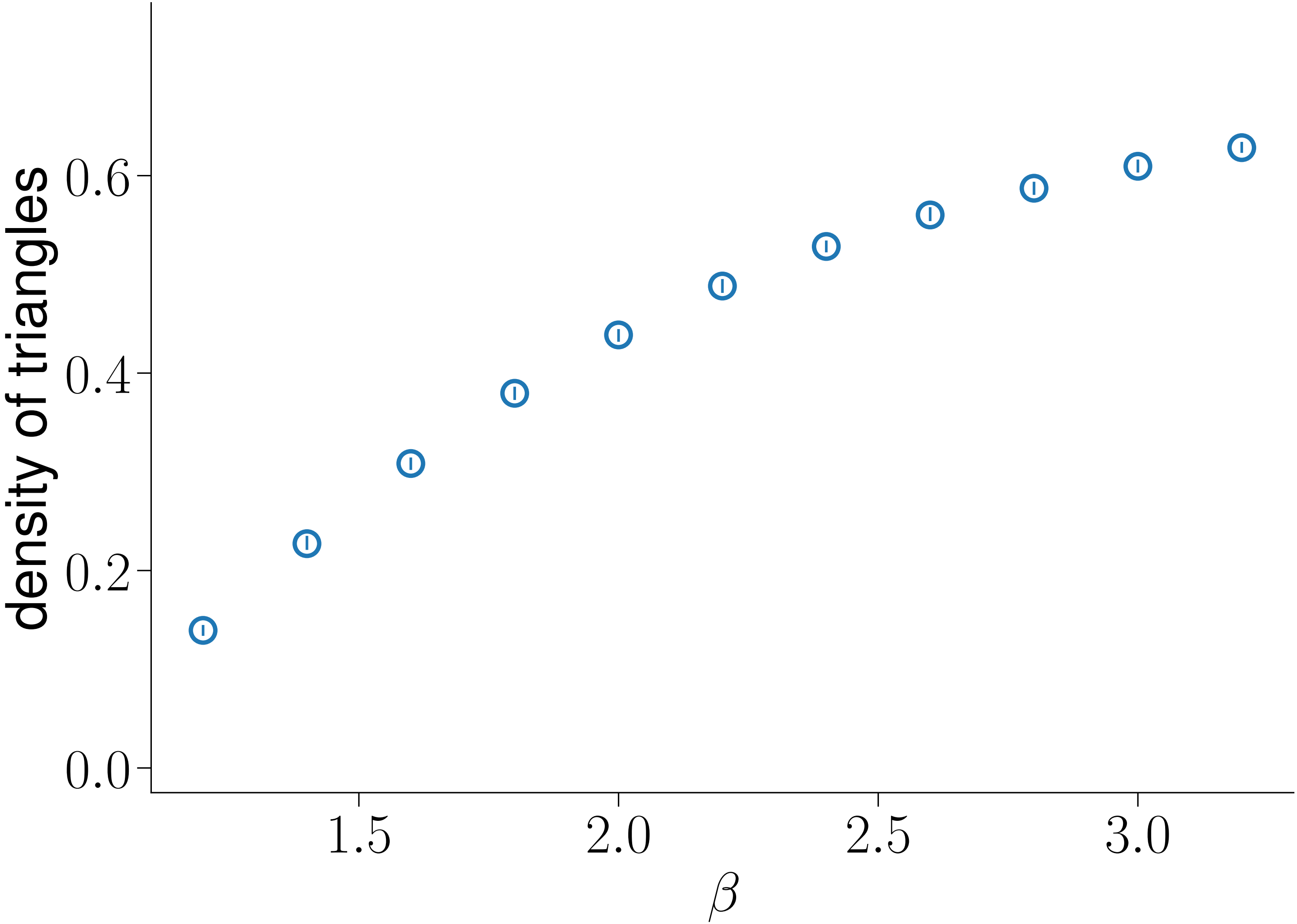
13


in-degree $(-)$
out-degree $(+)$
$\langle k^- | \kappa^- \rangle$, $\langle k^+ | \kappa^+ \rangle$
κ^-, κ^+
0
25
50
75
100
125
0
20
40
60
80
100


density of triangles
0.0
0.2
0.4
0.6
1.5
2.0
2.5
3.0
β

$$\mathbb{E}\left[k^{\mathrm{in}}\middle|\kappa^{\mathrm{in}}\right] \simeq \kappa^{\mathrm{in}}$$

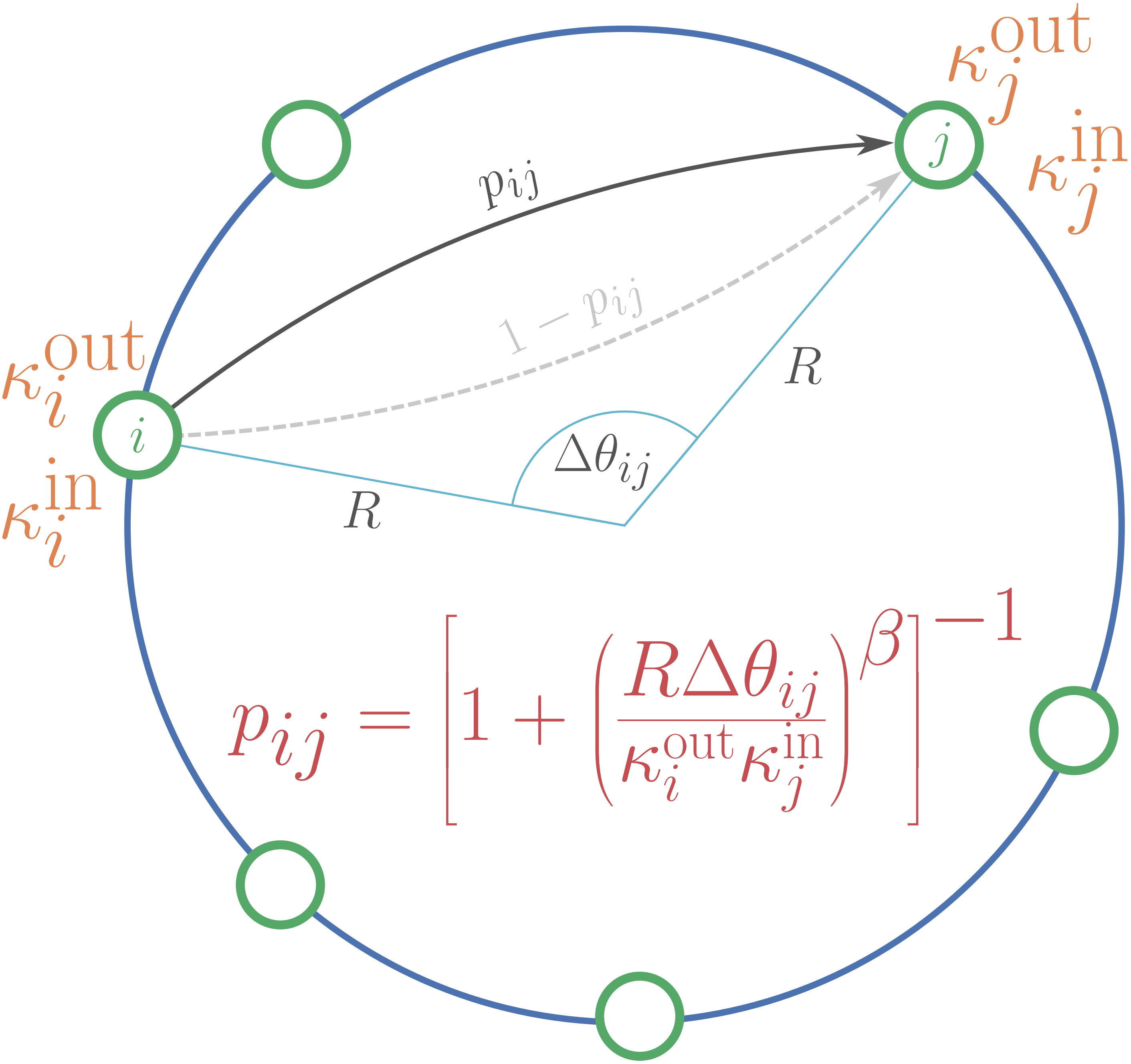
$$\mathbb{E}\left[k^{\mathrm{out}}\middle|\kappa^{\mathrm{out}}\right] \simeq \kappa^{\mathrm{out}}$$

$$P(k^{\mathrm{in}}, k^{\mathrm{out}}) \simeq \iint \frac{[\kappa^{\mathrm{in}}]^{k^{\mathrm{in}}} \mathrm{e}^{-\kappa^{\mathrm{in}}}}{k^{\mathrm{in}}!} \frac{[\kappa^{\mathrm{out}}]^{k^{\mathrm{out}}} \mathrm{e}^{-\kappa^{\mathrm{out}}}}{k^{\mathrm{out}}!} \times \rho(\kappa^{\mathrm{in}}, \kappa^{\mathrm{out}}) d\kappa^{\mathrm{in}} d\kappa^{\mathrm{out}}$$

The directed $\mathbb{S}^1$ model

1. Sprinkle N nodes uniformly on a circle of radius R.
2. Assign an expected in-degree κ^{in} and out-degree κ^{out} to each node according to some pdf $\rho(\kappa^{\mathrm{in}}, \kappa^{\mathrm{out}})$.
3. Draw a link from node i to node j with probability p_{ij}.

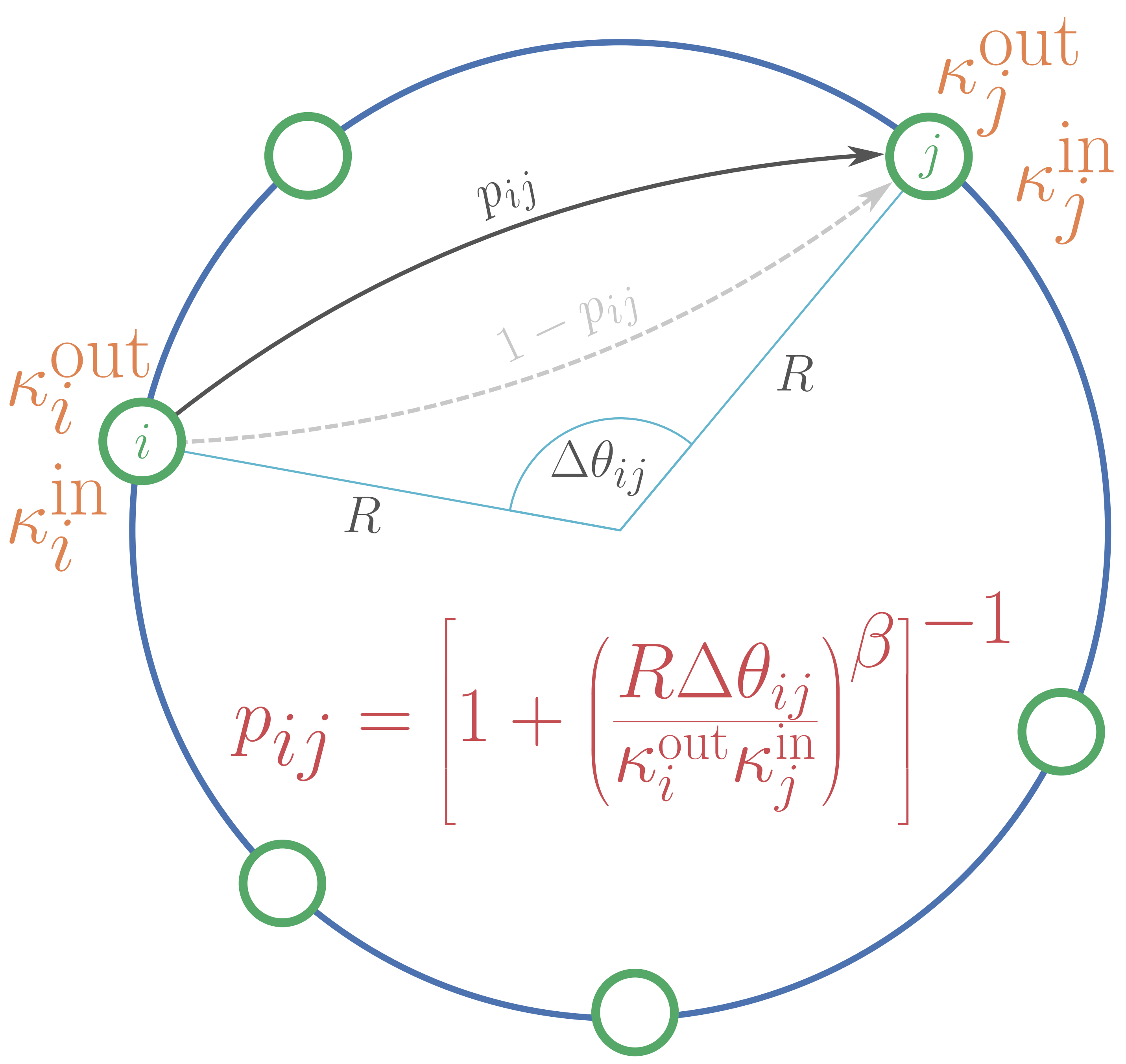
- ★ fixes the expected in-degree and out-degree of nodes $(\kappa^{\mathrm{in}}, \kappa^{\mathrm{out}}) \rightarrow$ soft directed CM
- ★ triangle inequality of the underlying metric space $\rightarrow$ triangles from pairwise interactions
- ★ level of clustering tuned with parameter β



κ_j^{out}
κ_j^{in}
j
p_{ij}
$1 - p_{ij}$
R
κ_i^{out}
i
κ_i^{in}
$\Delta\theta_{ij}$
R
$p_{ij} = \left[1 + \left(\frac{R\Delta\theta_{ij}}{\kappa_i^{\text{out}}\kappa_j^{\text{in}}}\right)^{\beta}\right]^{-1}$

The directed $\mathbb{S}^1$ model

The directed $\mathbb{S}^1$ model



$$p_{ij} = \left[1 + \left(\frac{R\Delta\theta_{ij}}{\kappa_i^{\mathrm{out}}\kappa_j^{\mathrm{in}}}\right)^{\beta}\right]^{-1}$$

$$\mathbb{E}\left[k^{\mathrm{in}} \middle| \kappa^{\mathrm{in}}\right] \simeq \kappa^{\mathrm{in}}$$

$$\mathbb{E}\left[k^{\mathrm{out}} \middle| \kappa^{\mathrm{out}}\right] \simeq \kappa^{\mathrm{out}}$$

$$P(k^{\mathrm{in}}, k^{\mathrm{out}}) \simeq \iint \frac{[\kappa^{\mathrm{in}}]^{k^{\mathrm{in}}} \mathrm{e}^{-\kappa^{\mathrm{in}}}}{k^{\mathrm{in}}!} \frac{[\kappa^{\mathrm{out}}]^{k^{\mathrm{out}}} \mathrm{e}^{-\kappa^{\mathrm{out}}}}{k^{\mathrm{out}}!} \times \rho(\kappa^{\mathrm{in}}, \kappa^{\mathrm{out}}) d\kappa^{\mathrm{in}} d\kappa^{\mathrm{out}}$$

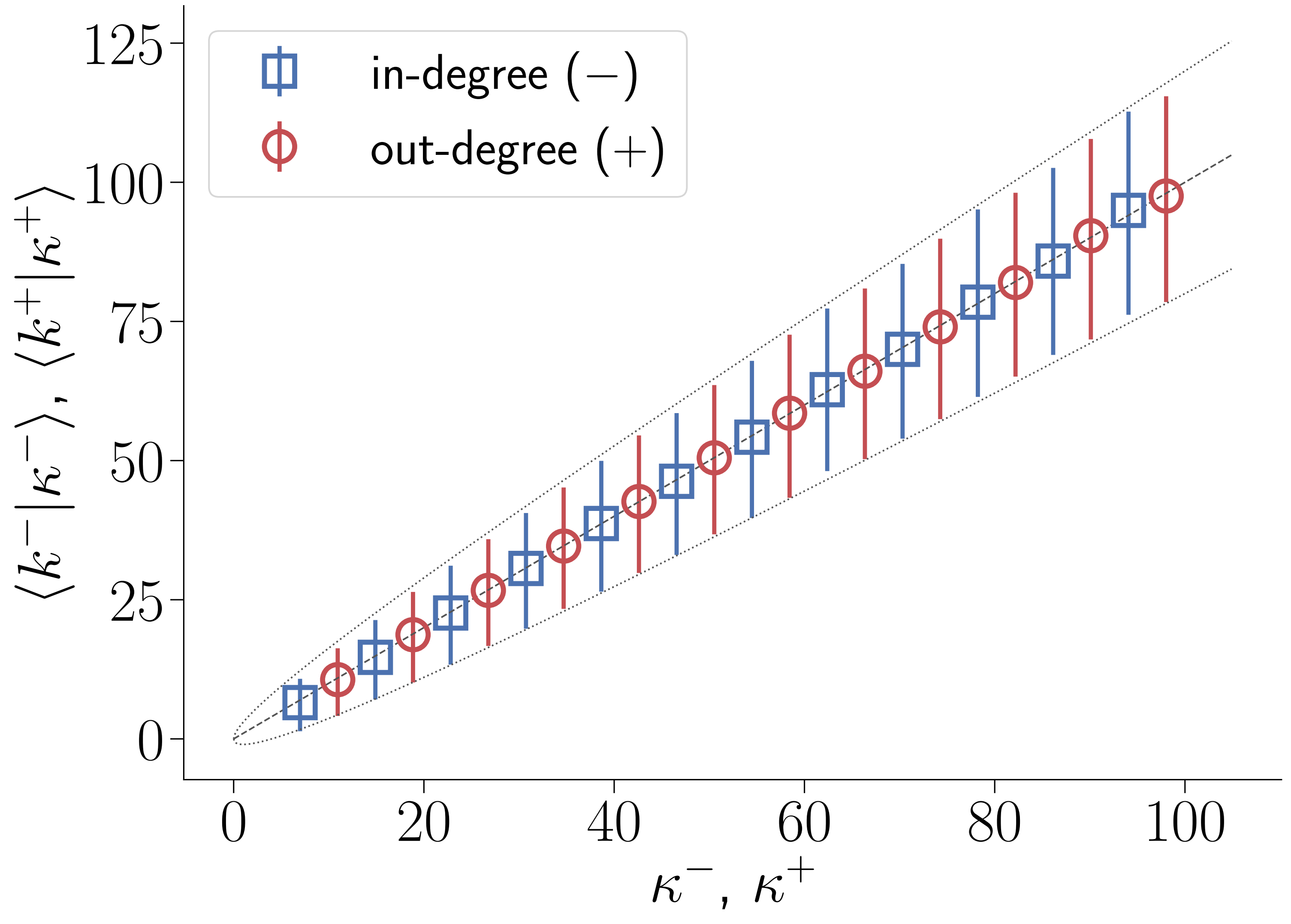
The directed $\mathbb{S}^1$ model

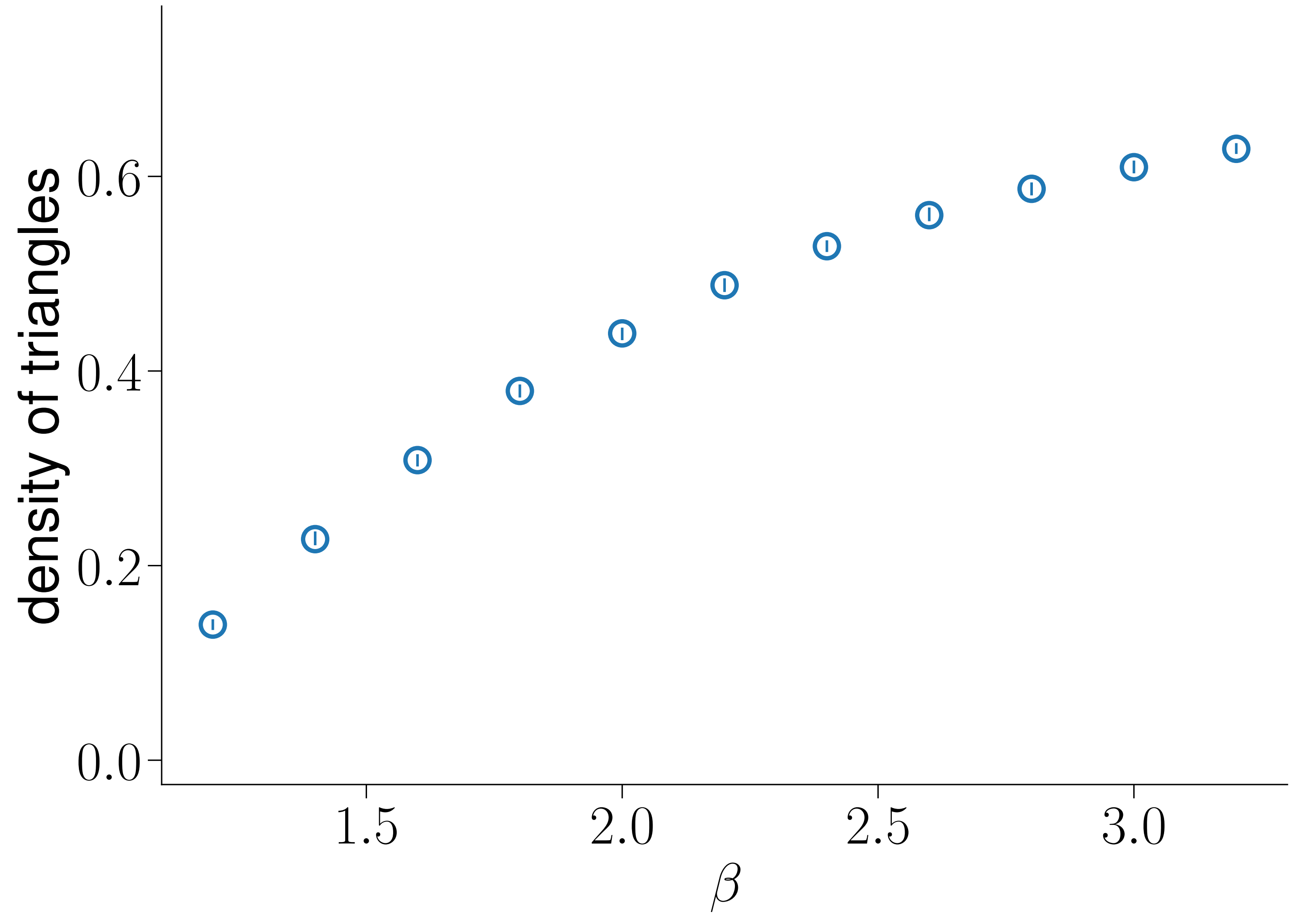
1. Sprinkle N nodes uniformly on a circle of radius R.
2. Assign an expected in-degree κ^{in} and out-degree κ^{out} to each node according to some pdf $\rho(\kappa^{\mathrm{in}}, \kappa^{\mathrm{out}})$.
3. Draw a link from node i to node j with probability p_{ij}.

★ fixes the expected in-degree and out-degree of nodes (κ^{in}, κ^{out}) → soft directed CM

★ triangle inequality of the underlying metric space → triangles from pairwise interactions

★ level of clustering tuned with parameter β

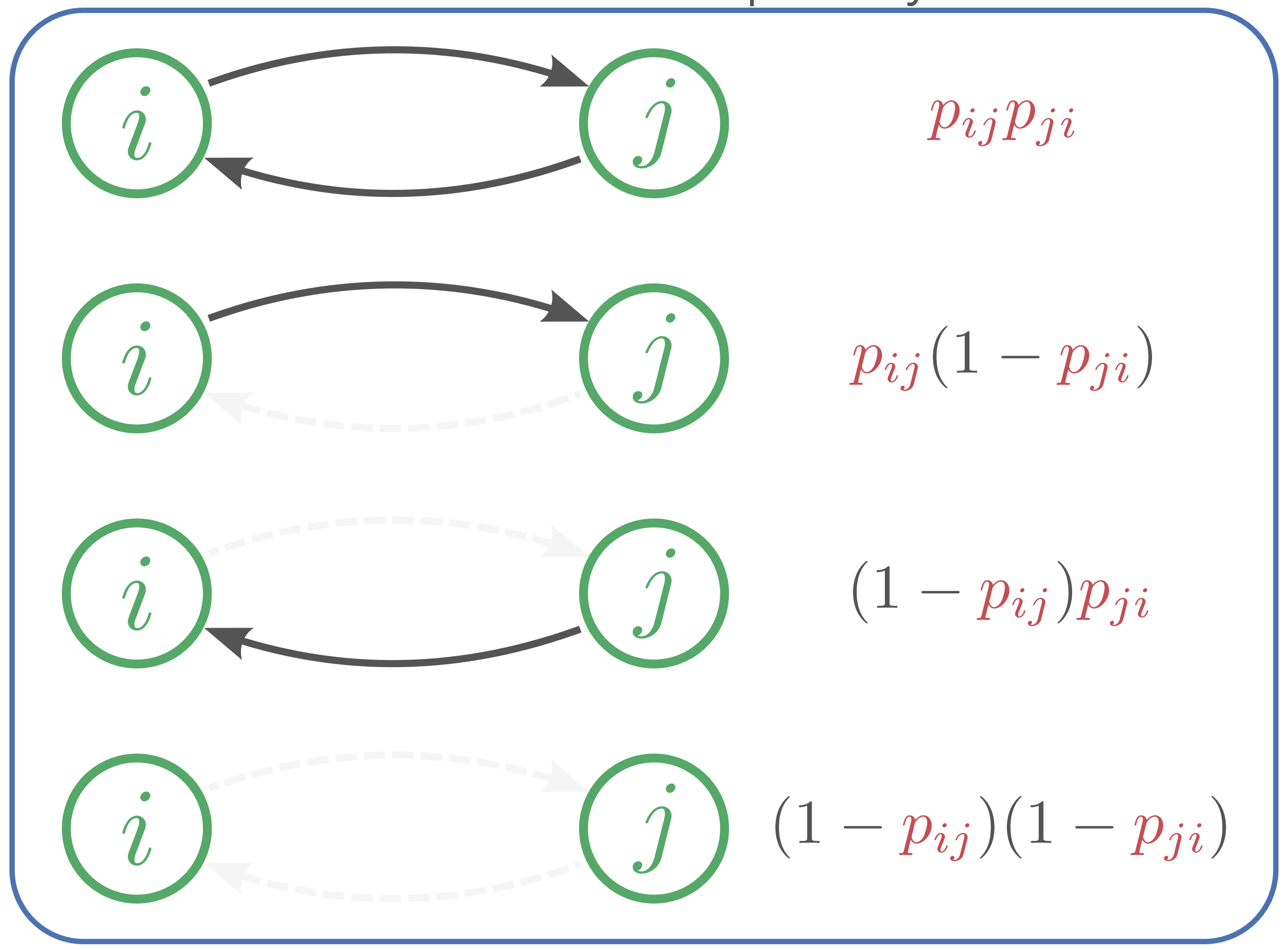




Reciprocity in the directed $\mathbb{S}^1$ model

A reciprocal connection between node i and node j occurs with probability $p_{ij}p_{ji}$.

Fortuitous reciprocity



$$r = \mathbb{E}\left[\frac{L^{\leftrightarrow}}{L}\right] = \mathbb{E}\left[\frac{k^{\leftrightarrow}}{k^{\text{out}}}\right] \approx \frac{\mathbb{E}\left[k^{\leftrightarrow}\right]}{\mathbb{E}\left[k^{\text{out}}\right]}$$

$$\simeq \iiiint \frac{\kappa_i^{\text{out}}\kappa_j^{\text{in}}}{\langle\kappa\rangle^2} \frac{1-\left(\frac{\kappa_i^{\text{out}}}{\kappa_i^{\text{in}}}\frac{\kappa_j^{\text{in}}}{\kappa_j^{\text{out}}}\right)^{\beta-1}}{1-\left(\frac{\kappa_i^{\text{out}}}{\kappa_i^{\text{in}}}\frac{\kappa_j^{\text{in}}}{\kappa_j^{\text{out}}}\right)^{\beta}}$$

$$\times\, \rho(\kappa_i^{\text{in}},\kappa_i^{\text{out}})\rho(\kappa_j^{\text{in}},\kappa_j^{\text{out}})d\kappa_i^{\text{in}}\kappa_i^{\text{out}}d\kappa_j^{\text{in}}\kappa_j^{\text{out}}$$

κ^{in} : in-degree

κ^{out} : out-degree

β : density of triangles