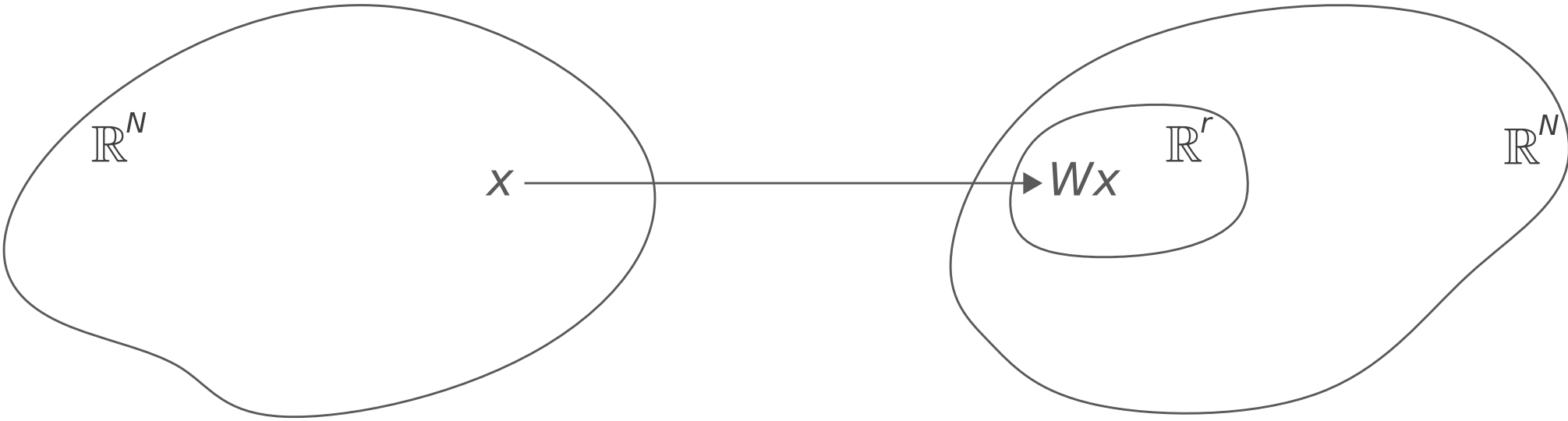


Low *effective* rank $\mathbf{W} \Rightarrow \mathbf{W}\mathbf{x}$ belongs to an *effectively* low-dimensional subspace

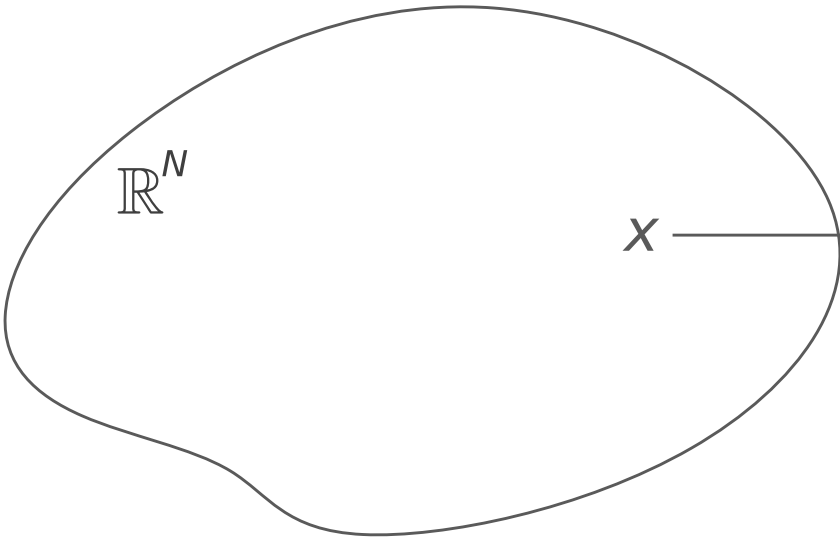
Low rank $W \Rightarrow Wx$ belongs to a low-dimensional subspace

High-dimensional space

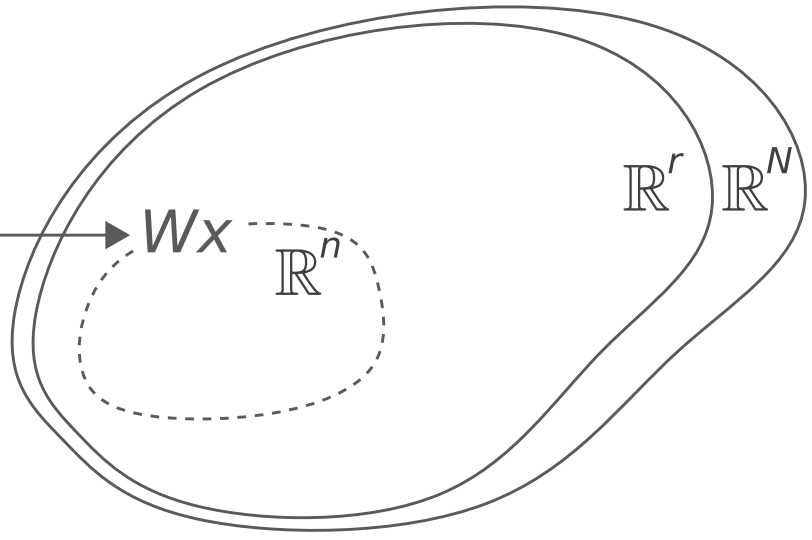
High-dimensional space



High-dimensional space



High-dimensional space



The impact of effective law on the dynamics



Many dynamics on networks have the form

$$\dot{\boldsymbol{x}} = \frac{d\boldsymbol{x}}{dt} = \mathbf{g}(\boldsymbol{x}, \mathbf{W}\boldsymbol{x}) = \mathbf{g}(\boldsymbol{x}, \boldsymbol{y})$$

with $\boldsymbol{x} \in \mathbb{R}^N$.

Examples:

- ▷ SIS (mean-field) : $\dot{x}_i = -d_i x_i + \gamma(1 - x_i) y_i$
- ▷ Wilson-Cowan: $\dot{x}_i = -d_i x_i + (1 - ax_i) \frac{1}{1 + e^{-b(\gamma y_i - c)}}$
- ▷ Recurrent Neural Networks (RNN): $\dot{x}_i = -d_i x_i + \tanh(\gamma y_i + c_i)$
- ▷ Kuramoto-Sakaguchi: $\dot{z}_j = i\omega_j z_j + \gamma e^{-i\alpha} y_j - \gamma e^{i\alpha} z_j^2 \bar{y}_j$ with $z_j = e^{i\theta_j}$
- ▷ Population dynamics: $\dot{x}_i = -d x_i + \gamma x_i y_i$ (Lotka-Volterra)
 $\dot{x}_i = -d x_i - s x_i^2 + \gamma \frac{x_i y_i}{\alpha + y_i}$
 $\dot{x}_i = a - d x_i + b x_i^2 - c x_i^3 + \gamma x_i y_i$

for $i, j \in \{1, \dots, N\}$ and $y_i = \sum_{j=1}^N W_{ij} x_j$.

Original dynamics

$$\dot{x} = g(x, \mathbf{W}x)$$

Reduced dynamics (with $\mathbf{X} = \mathbf{M}x$)

$$\dot{\mathbf{X}} = \mathbf{M}g(\mathbf{M}^+ \mathbf{X}, \mathbf{W}\mathbf{M}^+ \mathbf{X})$$

where n is the dimension of the reduced system

\mathbf{M}^+ denotes the pseudoinverse of \mathbf{M}

$\mathbf{M} = \mathbf{V}_n^\top$ is n -truncated right singular vector matrix

The alignment error is

$$\mathcal{E}(x) \leq \frac{1}{\sqrt{n}} \left[\|\mathbf{V}_n^\top \mathbf{J}'_x (\mathbf{I} - \mathbf{V}_n \mathbf{V}_n^\top) x\| + \sigma_{n+1} \|\mathbf{V}_n^\top \mathbf{J}'_y\|_2 \|x\| \right]$$

where $\mathbf{J}'_x, \mathbf{J}'_y$ are Jacobian matrices

Rapid singular value decrease can induce
rapid alignment error decrease!

The impact of effective low rank on the dynamics

Many dynamics on networks have the form

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \mathbf{g}(\mathbf{x}, \mathbf{W}\mathbf{x}) = \mathbf{g}(\mathbf{x}, \mathbf{y})$$

with $\mathbf{x} \in \mathbb{R}^N$.

Examples:

- ▷ SIS (mean-field) : $\dot{x}_i = -d_i x_i + \gamma(1 - x_i) y_i$
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 $\dot{x}_i = -dx_i - sx_i^2 + \gamma \frac{x_i y_i}{\alpha + y_i}$
 $\dot{x}_i = a - dx_i + bx_i^2 - cx_i^3 + \gamma x_i y_i$

for $i, j \in \{1, \dots, N\}$ and $y_i = \sum_{j=1}^N W_{ij} x_j$.

Original dynamics

$$\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, \mathbf{W}\mathbf{x})$$

Reduced dynamics (with $\mathbf{X} = \mathbf{M}\mathbf{x}$)

$$\dot{\mathbf{X}} = \mathbf{M}\mathbf{g}(\mathbf{M}^+ \mathbf{X}, \mathbf{W}\mathbf{M}^+ \mathbf{X})$$

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where $\mathbf{J}'_x, \mathbf{J}'_y$ are Jacobian matrices

Rapid singular value decrease can induce rapid alignment error decrease!

Reproduction of the dynamics with increasing accuracy

