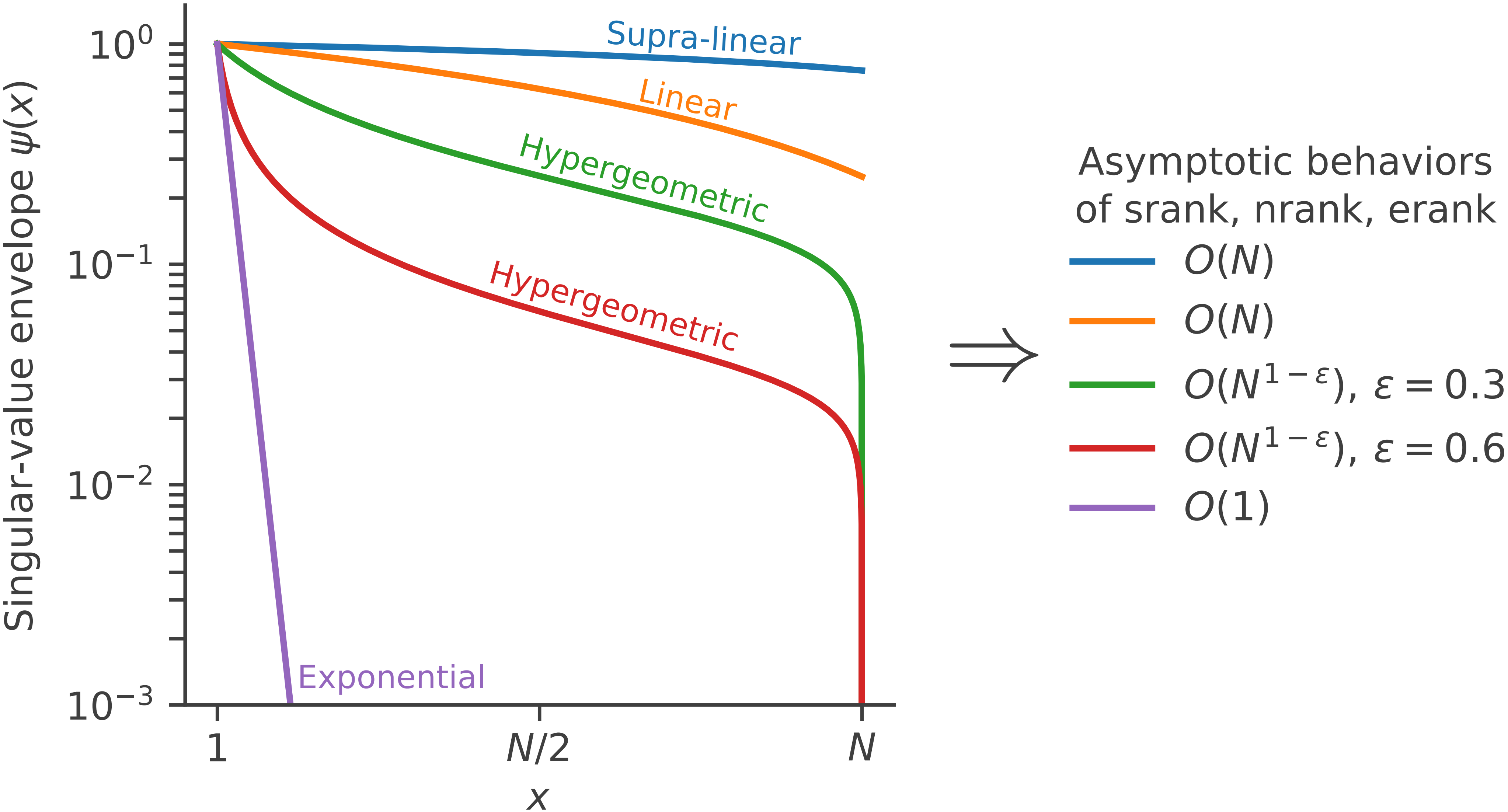
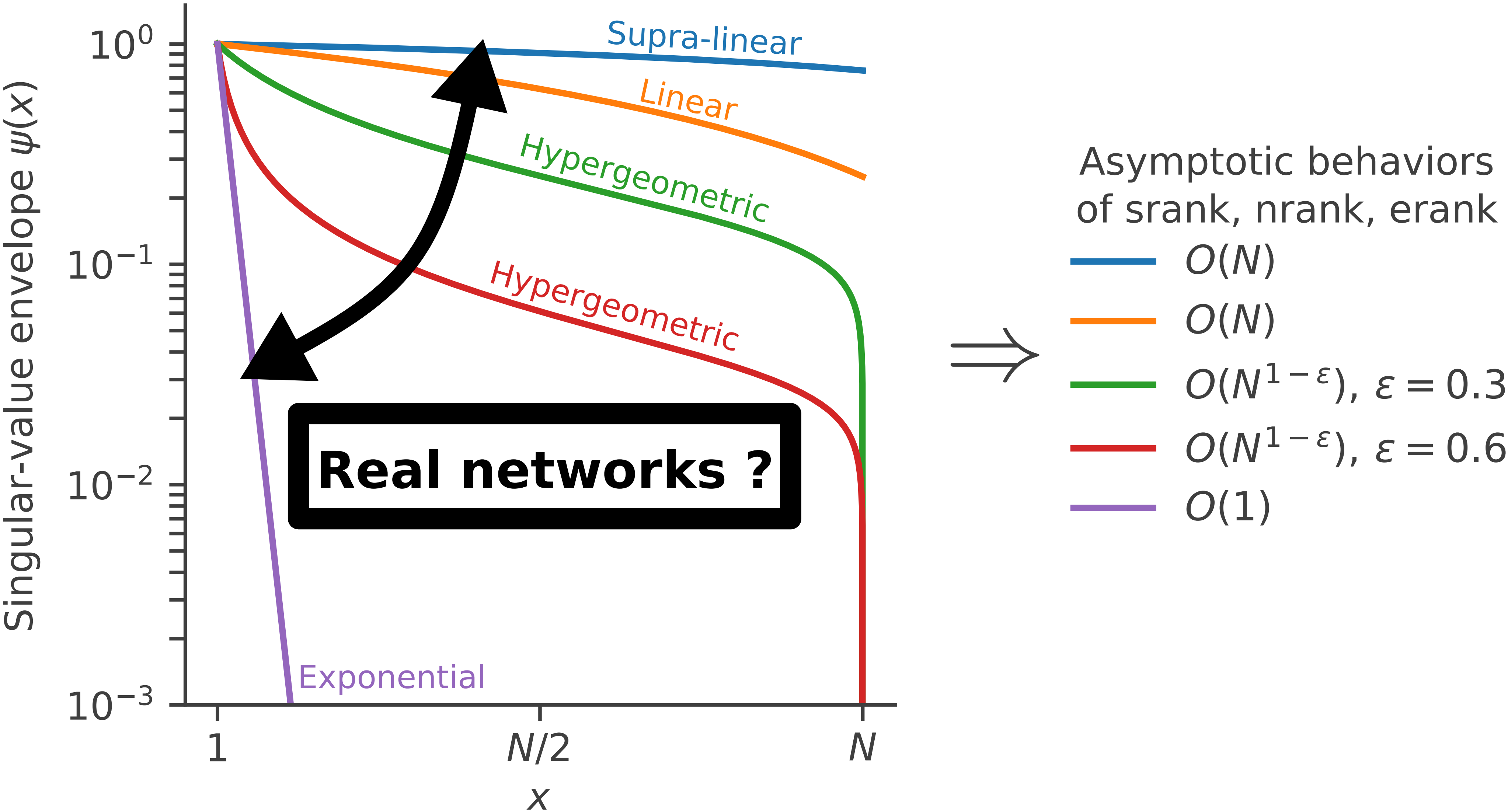


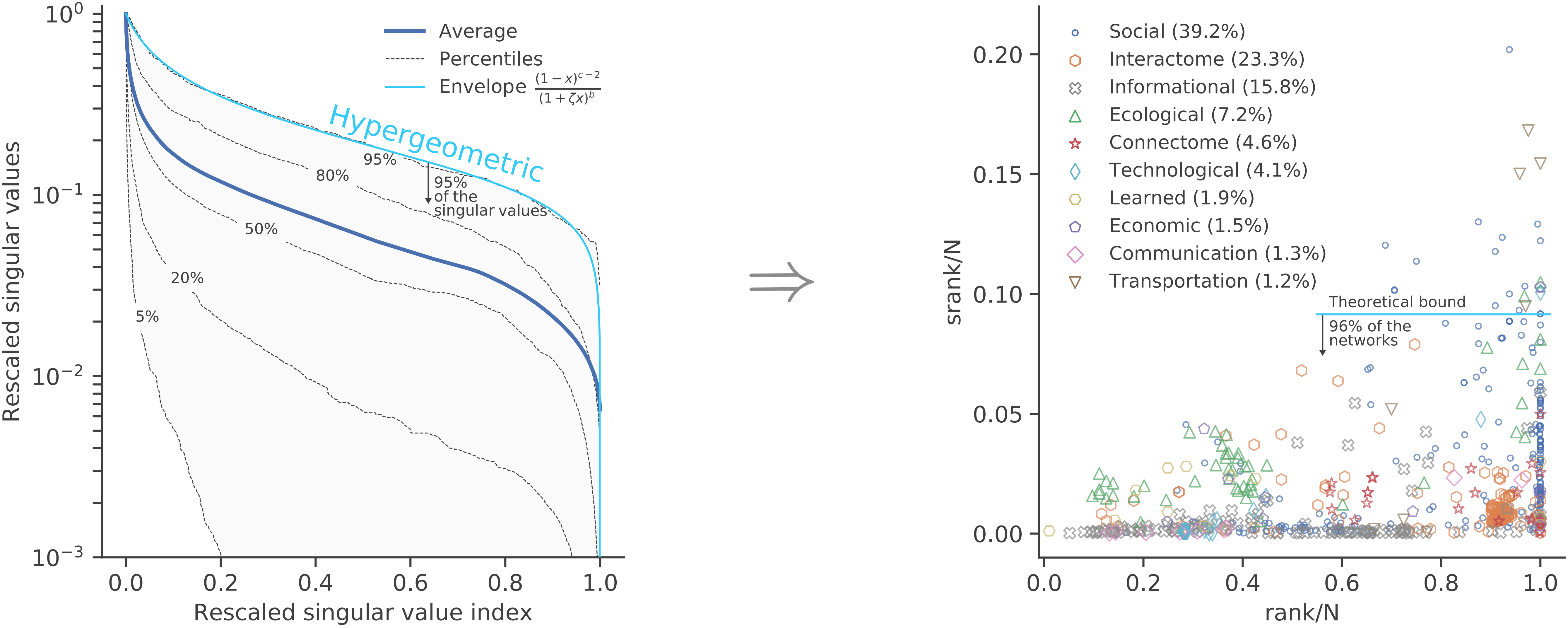
Supra-linear
Linear
Hypergeometric
Hypergeometric
Exponential
10^0
10^{-1}
10^{-2}
10^{-3}
Singular-value envelope $\psi(x)$
1
$N/2$
N
x
$\Rightarrow$
Asymptotic behaviors of srank, nrank, erank
$O(N)$
$O(N)$
$O(N^{1-\varepsilon})$, $\varepsilon = 0.3$
$O(N^{1-\varepsilon})$, $\varepsilon = 0.6$
$O(1)$



10^0
10^{-1}
10^{-2}
10^{-3}
Singular-value envelope $\psi(x)$
1
$N/2$
N
x
Supra-linear
Linear
Hypergeometric
Hypergeometric
Exponential
Real networks ?
$\Rightarrow$
Asymptotic behaviors of srank, nrank, erank
$O(N)$
$O(N)$
$O(N^{1-\varepsilon})$, $\varepsilon = 0.3$
$O(N^{1-\varepsilon})$, $\varepsilon = 0.6$
$O(1)$

A *workable* definition of “low” effective rank

8



10^0
10^{-1}
10^{-2}
10^{-3}
Rescaled singular values
Average
Percentiles
Envelope $\frac{(1-x)^{c-2}}{(1+\zeta x)^b}$
Hypergeometric
95%
80%
95% of the singular values
50%
20%
5%
0.0
0.2
0.4
0.6
0.8
1.0
Rescaled singular value index
Social (39.2%)
Interactome (23.3%)
Informational (15.8%)
Ecological (7.2%)
Connectome (4.6%)
Technological (4.1%)
Learned (1.9%)
Economic (1.5%)
Communication (1.3%)
Transportation (1.2%)
0.20
0.15
0.10
0.05
0.00
srank/N
Theoretical bound
96% of the networks
0.0
0.2
0.4
0.6
0.8
1.0
rank/N

The singular values of approx. 96% of considered networks are bounded from above by an hypergeometric envelope $\Rightarrow$ sublinear effective ranks!

Workable definition of low effective rank: $\sim$10% of the number of nodes N

Approx. 96% of the 679 networks qualify for having a low effective rank!

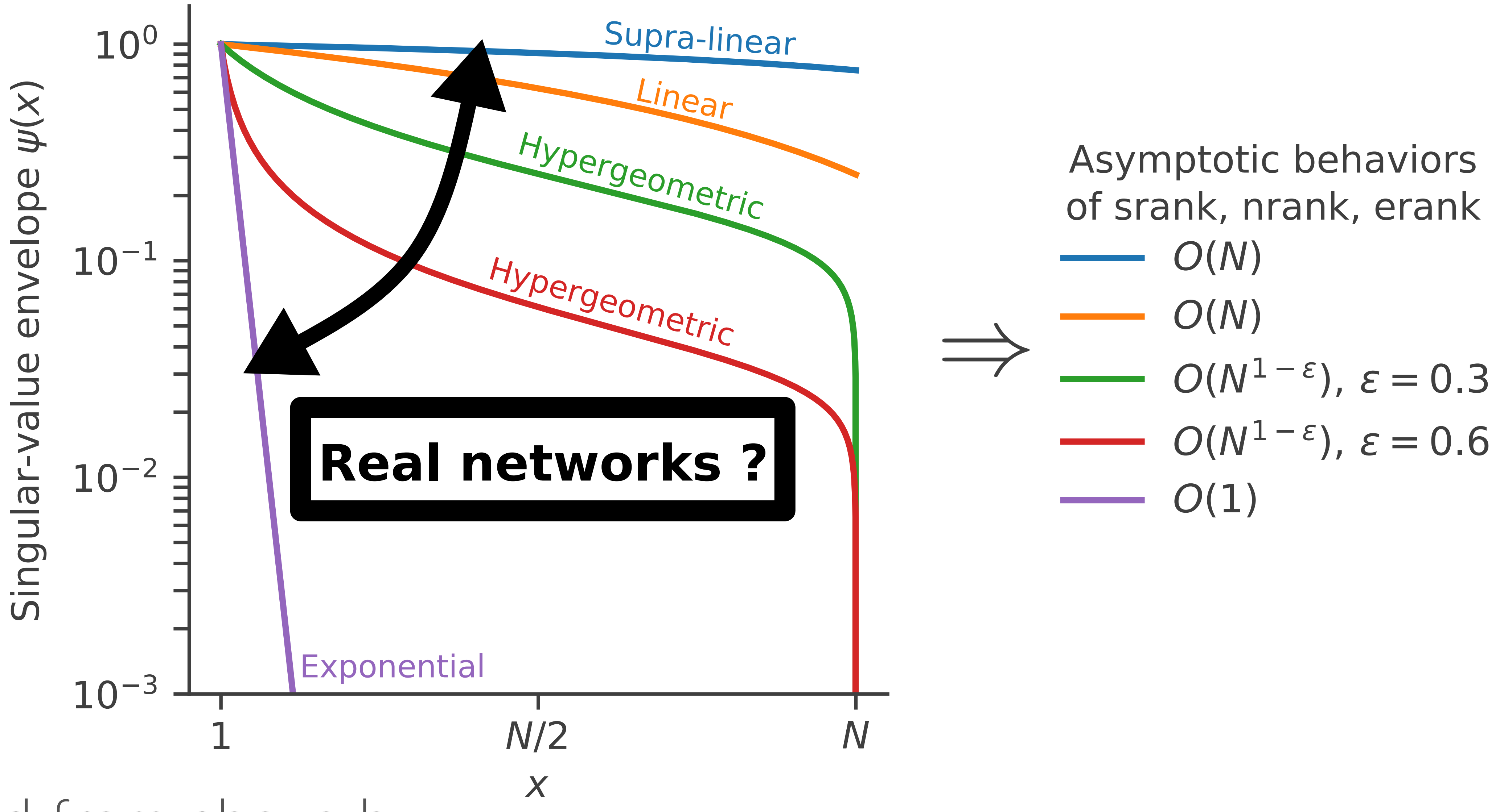
Hint: the rapid decrease of the dominant singular values of the adjacency matrix implies a low effective rank

- low effective rank? $\Rightarrow$ effective rank scales at most sublinearly as the number of nodes, N, goes to infitnity ($N^{1-\varepsilon}$ with $\varepsilon \in (0, 1]$)

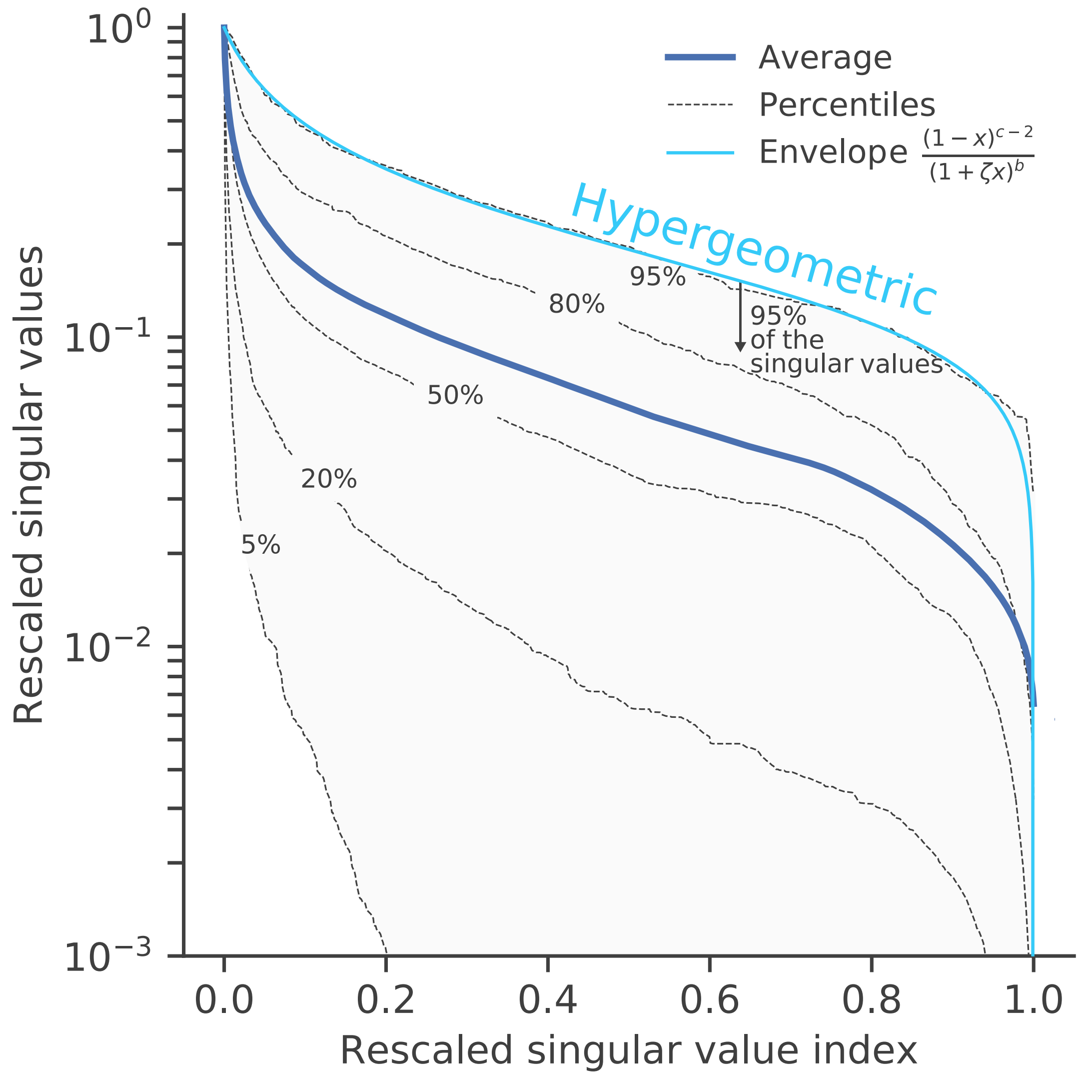
A *workable* definition of “low” effective rank

Hint: the rapid decrease of the dominant singular values of the adjacency matrix implies a low effective rank

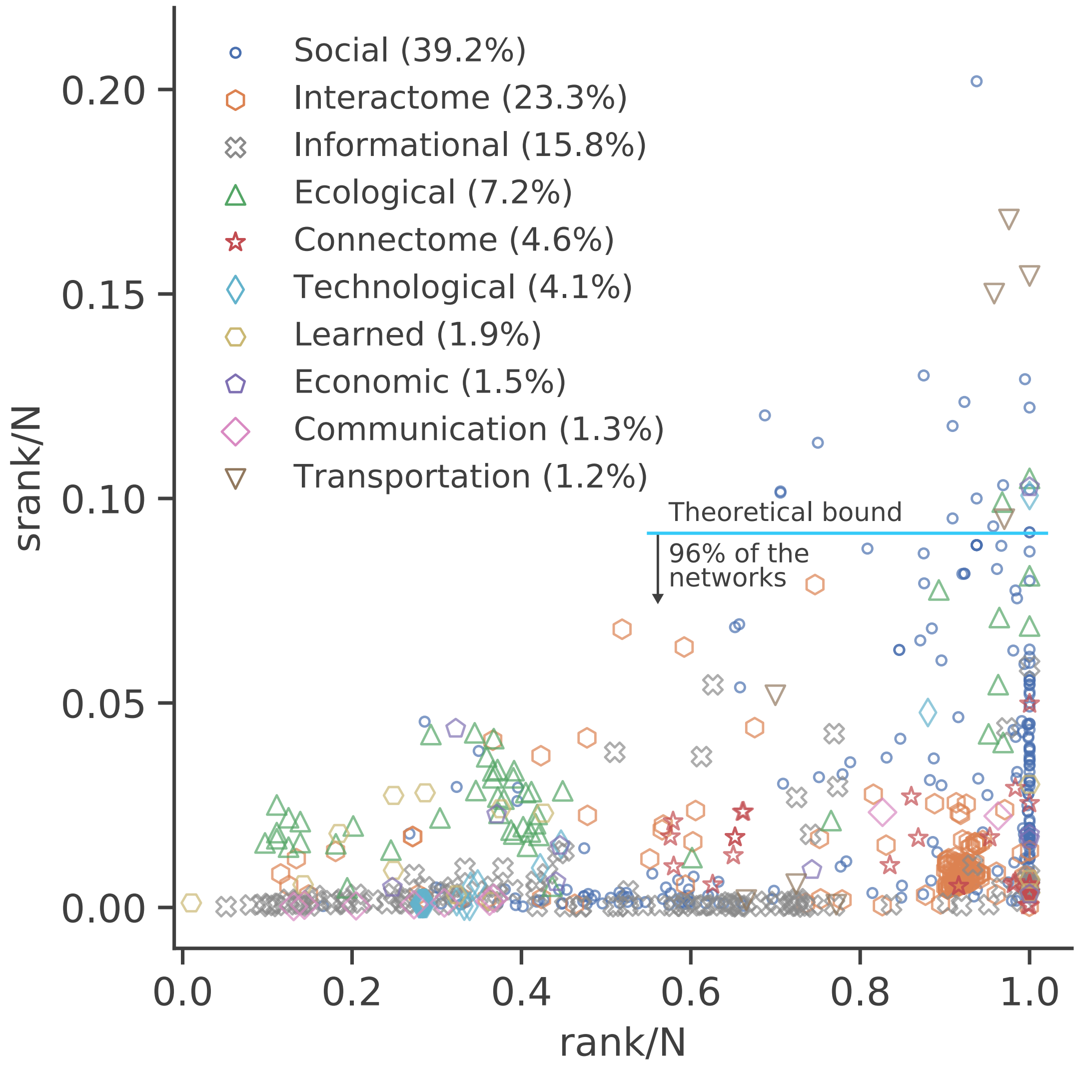
- ▷ low effective rank? ⇒ effective rank scales at most sublinearly as the number of nodes, N, goes to infitnity ($N^{1-\varepsilon}$ with $\varepsilon \in (0, 1]$)



The singular values of approx. 96% of considered networks are bounded from above by an hypergeometric envelope ⇒ sublinear effective ranks!



⇒



Workable definition of low effective rank: ∼10% of the number of nodes N

Approx. 96% of the 679 networks qualify for having a low effective rank!

The impact of effective low rank on the dynamics

Many dynamics on networks have the form

$$\dot{\boldsymbol{x}} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \mathbf{g}(\boldsymbol{x}, \mathbf{W}\boldsymbol{x}) = \mathbf{g}(\boldsymbol{x}, \boldsymbol{y})$$

with $\boldsymbol{x} \in \mathbb{R}^N$.

Examples:

- SIS (mean-field) : $\dot{x}_i = -d_i x_i + \gamma(1 - x_i)\, y_i$
- Wilson-Cowan: $\dot{x}_i = -d_i x_i + (1 - a x_i)\frac{1}{1+e^{-b(\gamma\, y_i - c)}}$
- Recurrent Neural Networks (RNN): $\dot{x}_i = -d_i x_i + \tanh(\gamma y_i + c_i)$
- Kuramoto-Sakaguchi: $\dot{z}_j = \mathrm{i}\omega_j z_j + \gamma\, e^{-\mathrm{i}\alpha}\, y_j - \gamma\, e^{\mathrm{i}\alpha}\, z_j^2\, \bar{y}_j$ with $z_j = e^{\mathrm{i}\theta_j}$
- Population dynamics: $\dot{x}_i = -d x_i + \gamma x_i y_i$ (Lotka-Volterra)
  $\dot{x}_i = -d x_i - s x_i^2 + \gamma \frac{x_i y_i}{\alpha + y_i}$
  $\dot{x}_i = a - d\, x_i + b x_i^2 - c x_i^3 + \gamma\, x_i\, y_i$

for $i, j \in \{1, ..., N\}$ and $y_i = \sum_{j=1}^{N} W_{ij} x_j$.