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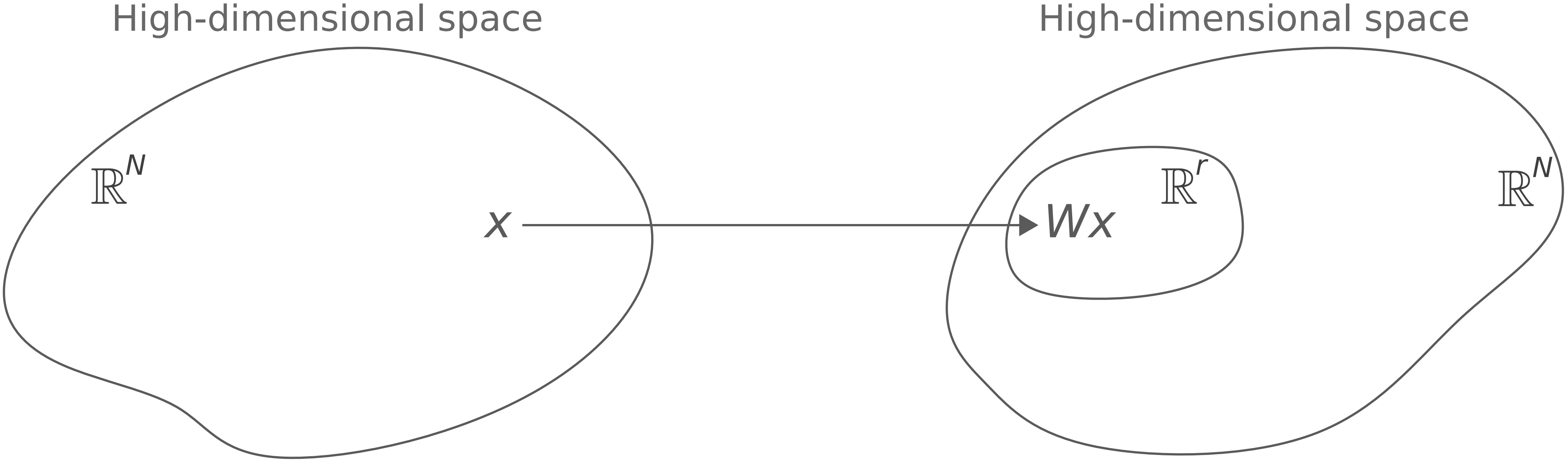
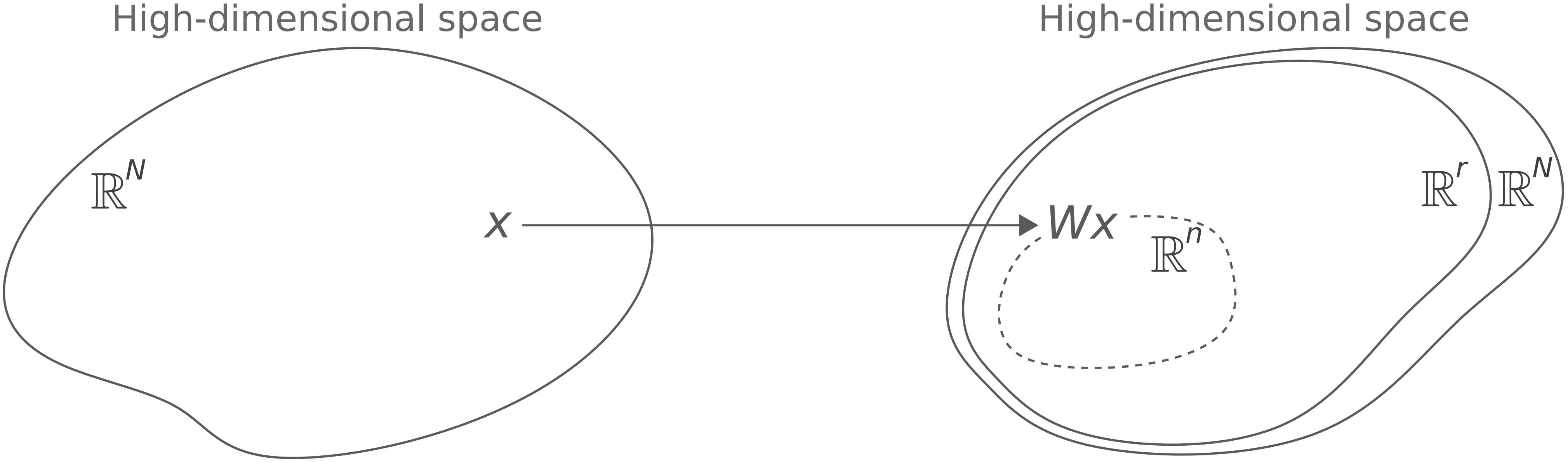

High-dimensional space
High-dimensional space
$\mathbb{R}^N$
x
Wx
$\mathbb{R}^r$
$\mathbb{R}^N$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The impact of effective low rank on the dynamics

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Many dynamics on networks have the form

$$\dot{\boldsymbol{x}} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \mathbf{g}(\boldsymbol{x}, \mathbf{W}\boldsymbol{x}) = \mathbf{g}(\boldsymbol{x}, \boldsymbol{y})$$

with $\boldsymbol{x} \in \mathbb{R}^N$.

Examples:

- SIS (mean-field) : $\dot{x}_i = -d_i x_i + \gamma(1 - x_i)\, y_i$
- Wilson-Cowan: $\dot{x}_i = -d_i x_i + (1 - a x_i)\frac{1}{1+e^{-b(\gamma\, y_i - c)}}$
- Recurrent Neural Networks (RNN): $\dot{x}_i = -d_i x_i + \tanh(\gamma y_i + c_i)$
- Kuramoto-Sakaguchi: $\dot{z}_j = \mathrm{i}\omega_j z_j + \gamma\, e^{-\mathrm{i}\alpha}\, y_j - \gamma\, e^{\mathrm{i}\alpha}\, z_j^2\, \bar{y}_j$ with $z_j = e^{\mathrm{i}\theta_j}$
- Population dynamics: $\dot{x}_i = -d x_i + \gamma x_i y_i$ (Lotka-Volterra)
  $\dot{x}_i = -d x_i - s x_i^2 + \gamma \frac{x_i y_i}{\alpha + y_i}$
  $\dot{x}_i = a - d\, x_i + b x_i^2 - c x_i^3 + \gamma\, x_i\, y_i$

for $i, j \in \{1, \ldots, N\}$ and $y_i = \sum_{j=1}^{N} W_{ij} x_j$.

Original dynamics

$$\dot{\boldsymbol{x}} = \mathbf{g}(\boldsymbol{x}, \mathbf{W}\boldsymbol{x})$$

Reduced dynamics (with $\boldsymbol{X} = \mathbf{M}\boldsymbol{x}$)

$$\dot{\boldsymbol{X}} = \mathbf{M}\mathbf{g}(\mathbf{M}^{+}\mathbf{X}, \mathbf{W}\mathbf{M}^{+}\mathbf{X})$$

where n is the dimension of the reduced system
$\mathbf{M}^{+}$ denotes the pseudoinverse of $\mathbf{M}$
$\mathbf{M} = \mathbf{V}_n^{\top}$ is n-truncated right singular vector matrix

The alignment error is

$$\mathcal{E}(x) \leq \frac{1}{\sqrt{n}}\left[\|\mathbf{V}_n^{\top}\mathbf{J}_x'(\mathbf{I} - \mathbf{V}_n\mathbf{V}_n^{\top})x\| + \sigma_{n+1}\|\mathbf{V}_n^{\top}\mathbf{J}_y'\|_2\|x\|\right]$$

where $\mathbf{J}_x', \mathbf{J}_y'$ are Jacobian matrices

Rapid singular value decrease can induce rapid alignment error decrease!

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A *workable* definition of "low" effective rank

Hint: the rapid decrease of the dominant singular values of the adjacency matrix implies a low effective rank

- low effective rank? $\Rightarrow$ effective rank scales at most sublinearly as the number of nodes, N, goes to infitnity ($N^{1-\varepsilon}$ with $\varepsilon \in (0, 1]$)