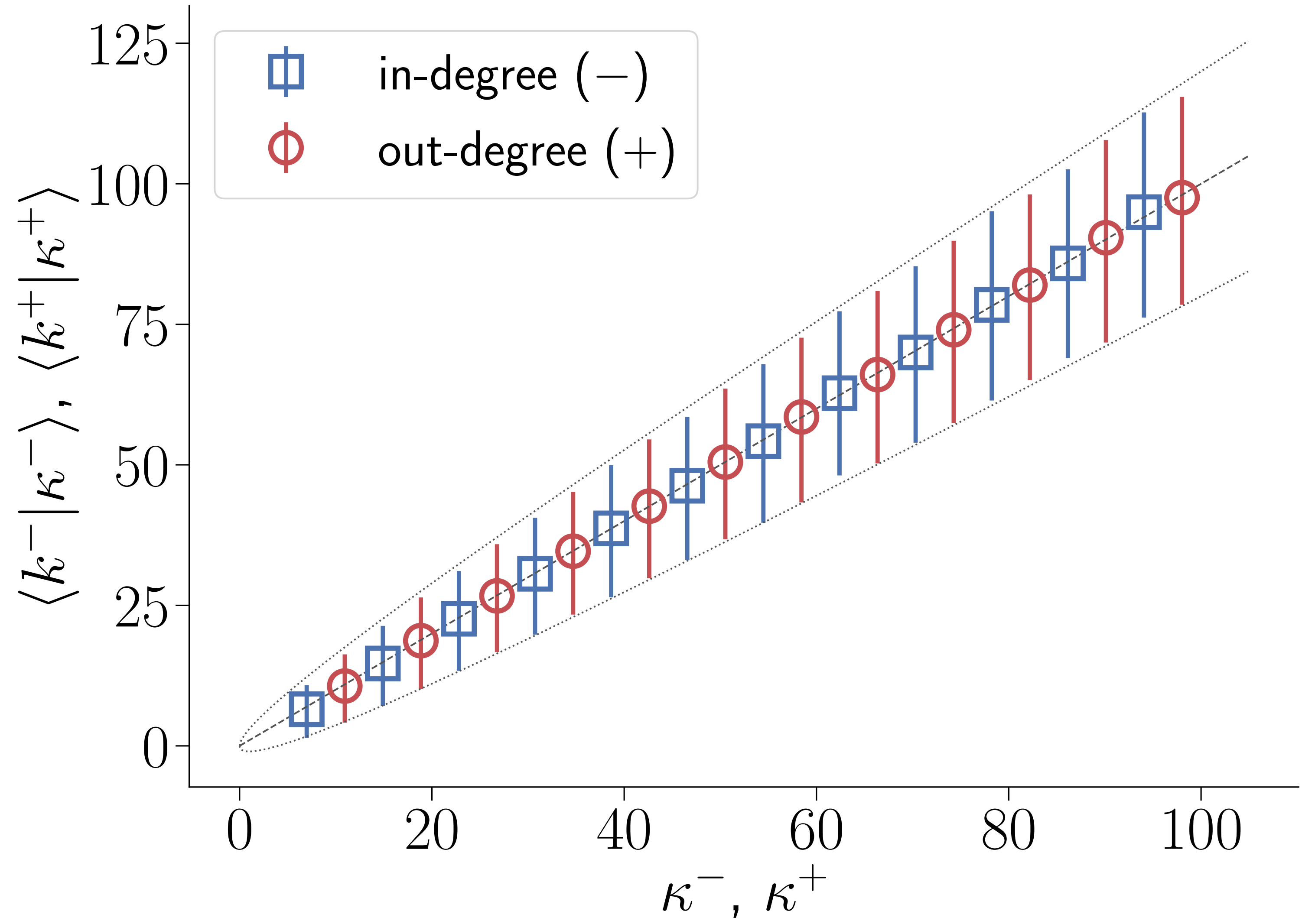
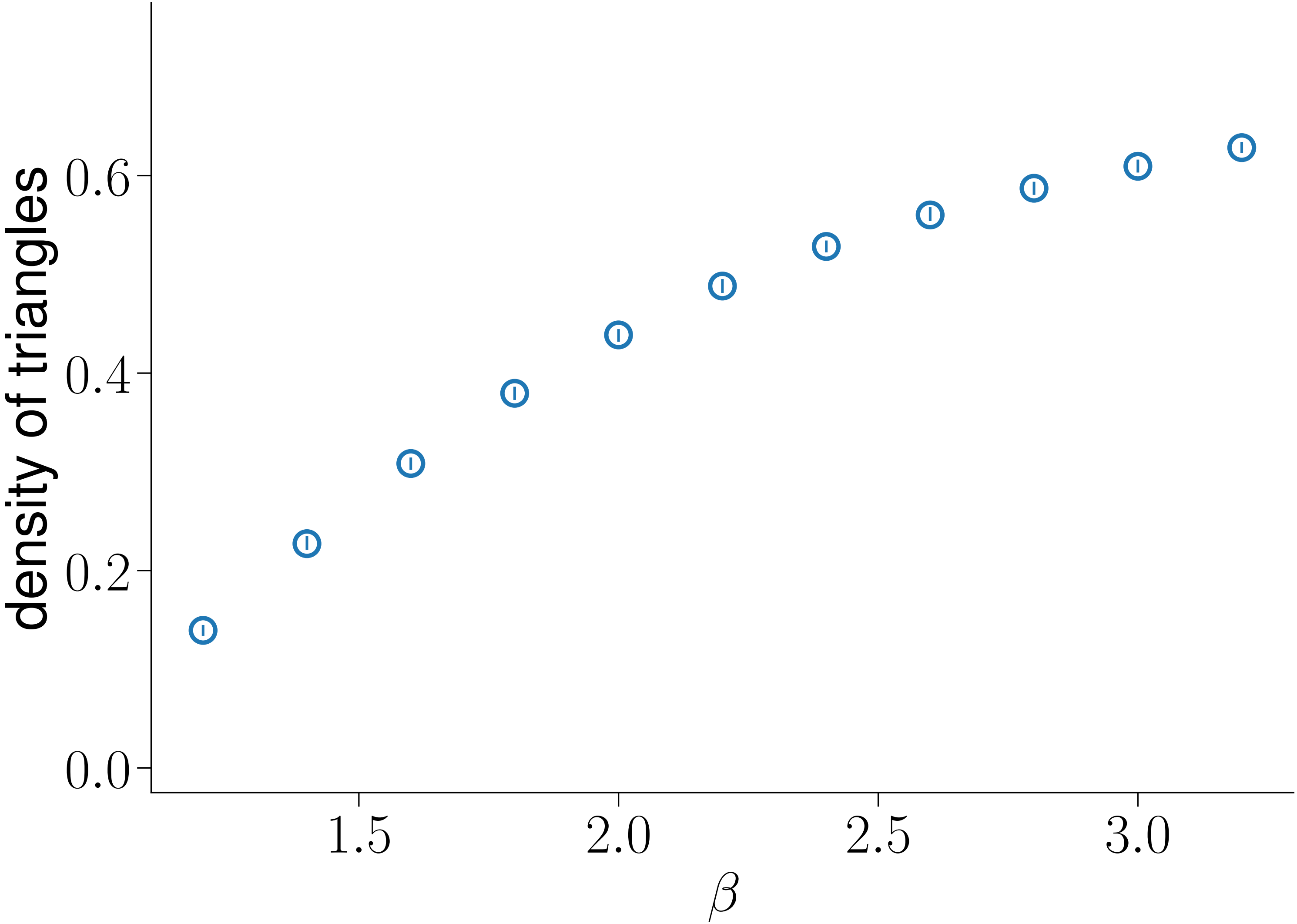
The directed $\mathbb{S}^1$ model: A straightforward generalization



in-degree (−)
out-degree (+)
$\langle k^- | \kappa^- \rangle$, $\langle k^+ | \kappa^+ \rangle$
κ^-, κ^+
0
25
50
75
100
125
0
20
40
60
80
100


density of triangles
0.0
0.2
0.4
0.6
1.5
2.0
2.5
3.0
β

$$\mathbb{E}\left[k^{\mathrm{in}}\middle|\kappa^{\mathrm{in}}\right] \simeq \kappa^{\mathrm{in}}$$

$$\mathbb{E}\left[k^{\mathrm{out}}\middle|\kappa^{\mathrm{out}}\right] \simeq \kappa^{\mathrm{out}}$$

$$P(k^{\mathrm{in}}, k^{\mathrm{out}}) \simeq \iint \frac{[\kappa^{\mathrm{in}}]^{k^{\mathrm{in}}} \mathrm{e}^{-\kappa^{\mathrm{in}}}}{k^{\mathrm{in}}!} \frac{[\kappa^{\mathrm{out}}]^{k^{\mathrm{out}}} \mathrm{e}^{-\kappa^{\mathrm{out}}}}{k^{\mathrm{out}}!} \times \rho(\kappa^{\mathrm{in}}, \kappa^{\mathrm{out}}) d\kappa^{\mathrm{in}} d\kappa^{\mathrm{out}}$$

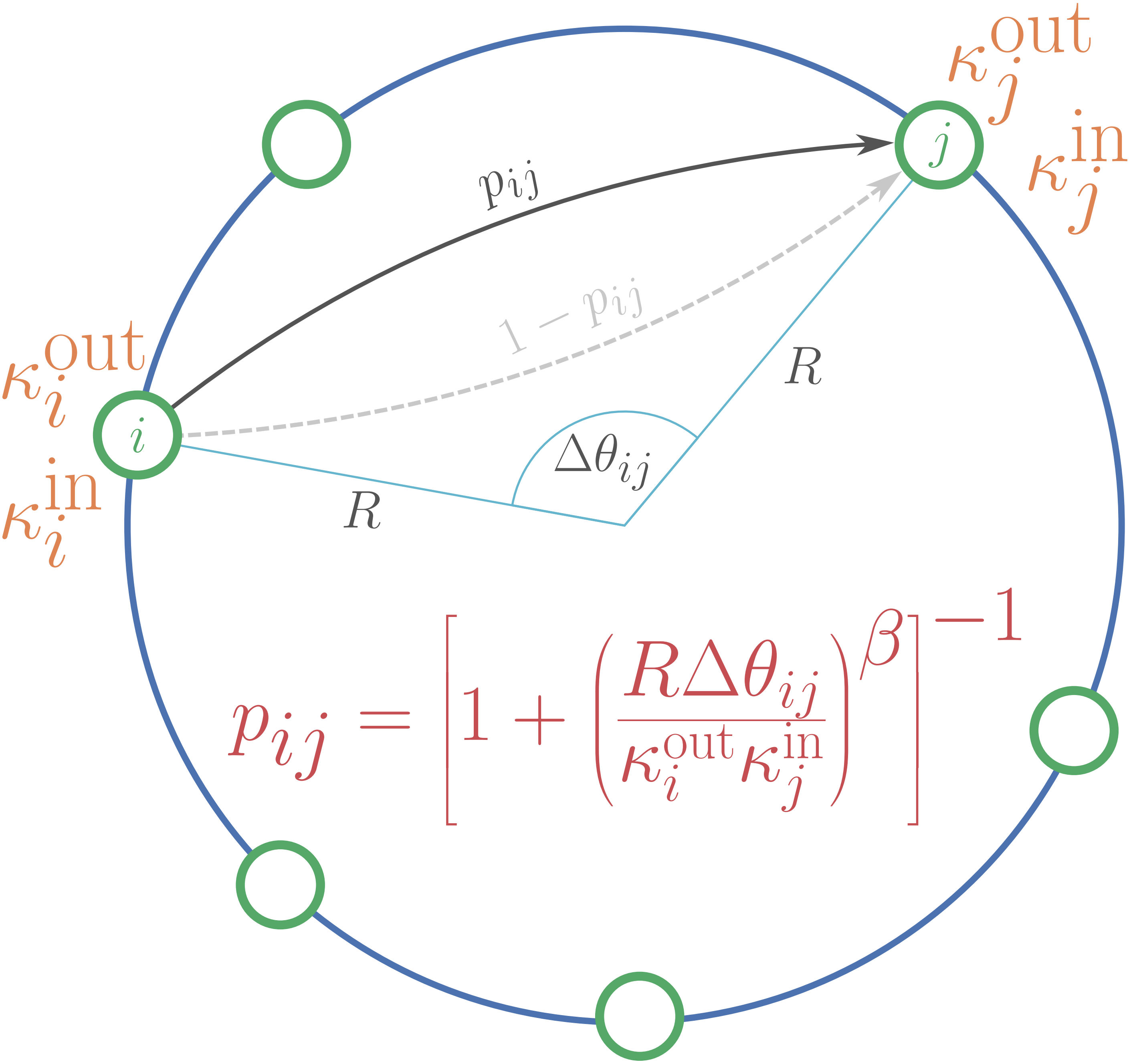
The directed $\mathbb{S}^1$ model

1. Sprinkle N nodes uniformly on a circle of radius R.
2. Assign an expected in-degree κ^{in} and out-degree κ^{out} to each node according to some pdf $\rho(\kappa^{\text{in}}, \kappa^{\text{out}})$.
3. Draw a link from node i to node j with probability p_{ij}.

★ fixes the expected in-degree and out-degree of nodes $(\kappa^{\text{in}}, \kappa^{\text{out}}) \to$ soft directed CM

★ triangle inequality of the underlying metric space $\to$ triangles from pairwise interactions

★ level of clustering tuned with parameter β



κ_i^out
κ_i^in
κ_j^out
κ_j^in
i
j
p_ij
1 − p_ij
R
R
Δθ_ij
p_ij = [1 + (RΔθ_ij / κ_i^out κ_j^in)^β]^−1

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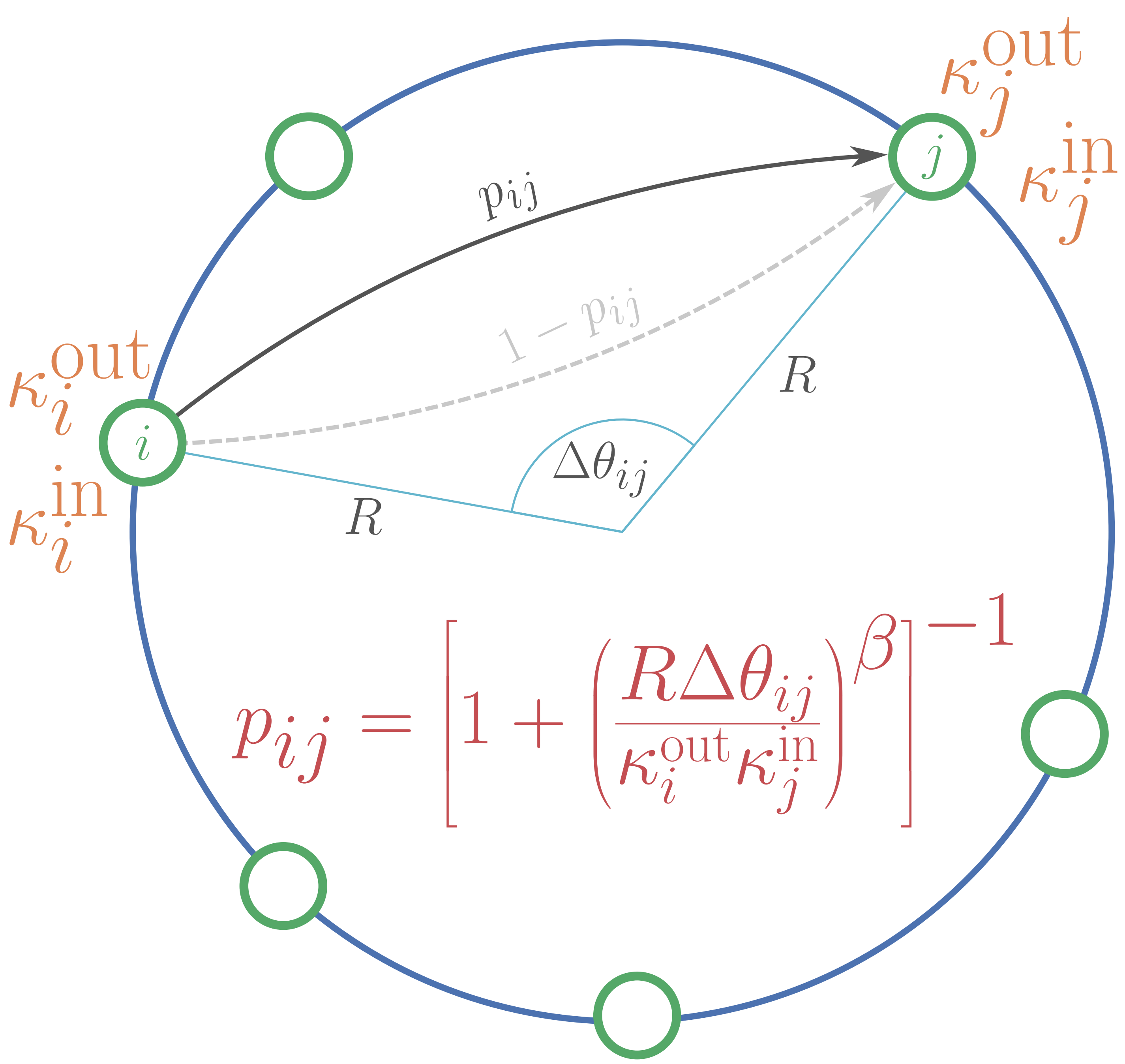
The directed $\mathbb{S}^1$ model: A straightforward generalization



$$p_{ij} = \left[1 + \left(\frac{R\Delta\theta_{ij}}{\kappa_i^{\mathrm{out}}\kappa_j^{\mathrm{in}}}\right)^{\beta}\right]^{-1}$$

$$\mathbb{E}\left[k^{\mathrm{in}} \middle| \kappa^{\mathrm{in}}\right] \simeq \kappa^{\mathrm{in}}$$

$$\mathbb{E}\left[k^{\mathrm{out}} \middle| \kappa^{\mathrm{out}}\right] \simeq \kappa^{\mathrm{out}}$$

$$P(k^{\mathrm{in}}, k^{\mathrm{out}}) \simeq \iint \frac{[\kappa^{\mathrm{in}}]^{k^{\mathrm{in}}} e^{-\kappa^{\mathrm{in}}}}{k^{\mathrm{in}}!} \frac{[\kappa^{\mathrm{out}}]^{k^{\mathrm{out}}} e^{-\kappa^{\mathrm{out}}}}{k^{\mathrm{out}}!} \times \rho(\kappa^{\mathrm{in}}, \kappa^{\mathrm{out}}) d\kappa^{\mathrm{in}} d\kappa^{\mathrm{out}}$$

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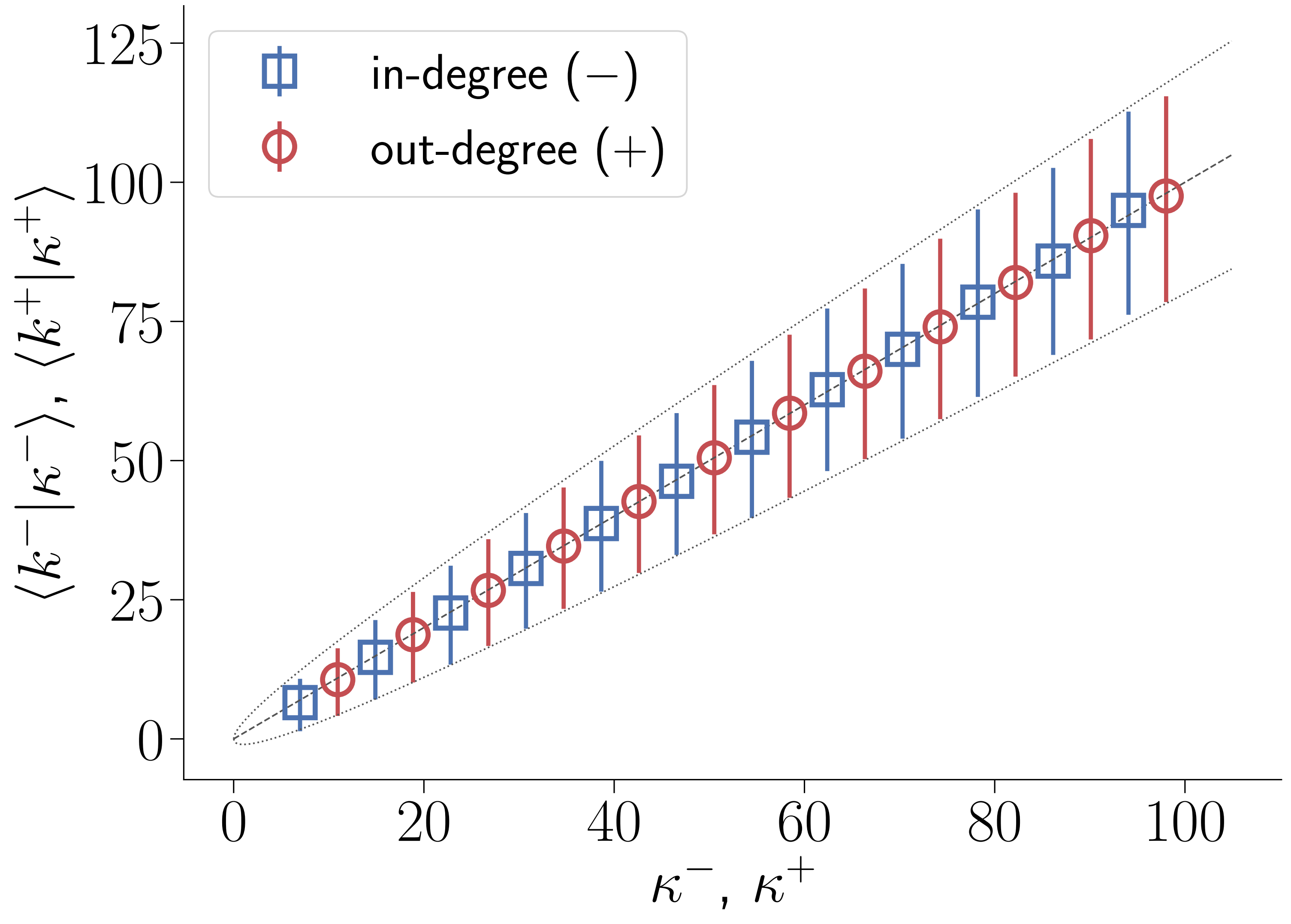
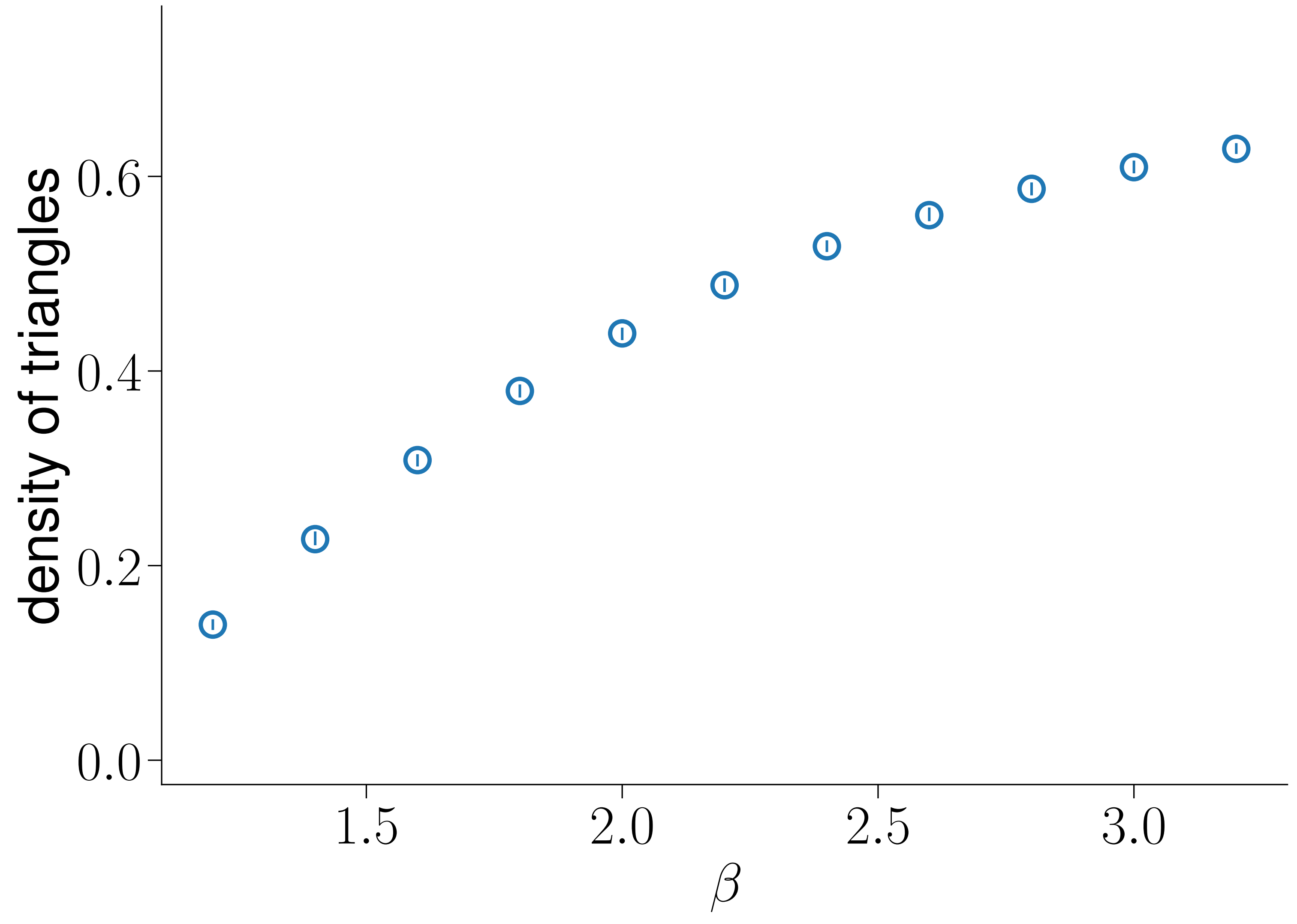
★ triangle inequality of the underlying metric space → triangles from pairwise interactions

★ level of clustering tuned with parameter β





The directed-reciprocal $\mathbb{S}^1$ model: A new link between connections and distances

A random network model defines the probability p_{ij} for a directed link to exist from node i to node j.